Application of Galerkin Method in Deflection, Stability and Vibration of Rectangular Clamped Plates of Variable Thickness.

by

Araar Yamine

A thesis

submitted in partial fulfillment
of the requirements for the degree of
Master of applied science in Civil Engineering

Department of Civil Engineering, University of Ottawa, Ottawa, Ontario

ISBN 0-315-46708-8

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

Abstract

The variational method is an effective tool for solving complicated differential equations for which exact solutions are not available. The Galerkin method is known to be one of the most rapidly converging methods and has been applied successfully in the past for the solution of many types of linear differential equations in applied mechanics. In the present study, the suitability of the method for solution of deflection, stability and free vibration analysis of plates of variable thickness is investigated

The effects of the taper parameter and the plate aspect ratio on behaviour of plates of variable thickness are presented. It is shown that the stiffness of the plate of variable thickness tends to increase with increase in taper parameter and aspect ratio. The parameters taken into consideration for the analysis of the problem under investigation are the plate aspect ratio, taper parameter and load ratios. To determine the center deflection of plates of variable thickness on elastic foundation, various foundation moduli were also taken into consideration.

The deflection of plates of variable thickness is expressed in term of polynomials satisfying the boundary conditions. The definite integrals involved in the formulation of the Galerkin algebraic equations from the governing differential equations are evaluated by using trapezoidal rule. This results in a set of simultaneous linear homogeneous algebraic equations in terms of the coefficients C_i. The algebraic eigen-value problem is then solved for

eigen-values and eigen-vectors by using the Jacobi method. Convergence is examined in many typical cases and is found to be excellent.

Versatility, simplicity in formulation, quick convergence and efficiency are the obvious advantages of the method in comparison with other numerical methods which are in use. Furthermore, the Galerkin method requires very little computer memory space and computer time, thus making it most suitable for microcomputers.

Acknowledgements

The author feels extremely honored and wishes to express his deepest gratitude to Dr. S. F. Ng for his supervision, constructive criticism and advice throughout the investigation of this work. The author is also indebted to the computer center and Vanier library for their cooperation.

Moreover, the author would like to express his sincere appreciation to his family and friends. The financial assistance provided by the government of Algeria to continue this research is highly appreciated.

Dedication

To my father and my little brother Mohammed.

Nomenclature

x,y,z	rectangular cartisian coordinates
E	modulus of elasticity of isotropic material
ν	Poisson's ratio of isotropic material
h_0	plate thickness at x = 0
h	plate thickness
c	taper parameter
D_0	flexural rigidity of the plate at x = 0
	$D_0 = \frac{Eh_0^3}{12(1-\nu^2)}$
D	flexural rigidity
	$D = D_0(1+\xi c)^3$
q	lateral load per unit area
2a, 2b	dimension of the plate in x and y direction respectively
R	aspect ratio of the plate (a/b)
K	dimensionless foundation modulus $K = \frac{ka^4}{D_0}$
ξ, η	dimensionless parameters in directional coordinate for rectangular plate of variable thickness $\xi = x/a$, $\eta = y/b$
N_x, N_y	normal forces in x and y directions
N_{xy}	shear force parallel to x and y direction
M_x, M_y	bending moment in x and y directions respectively
M_{xy}	torsional moment
Q_x, Q_y	torsional shear force per unit width in the direction xz and yz planes respectively.

∇^2	Laplacian operator
r	load ratio $r = \frac{N_y}{N_x}$
ω	angular frequency of the plate
ρ	mass per unit surface area of the plate
t	time
W	lateral deflection of the plate

Contents

Abstract i

Acknowledgements i

Nomenclature i

1 INTRODUCTION 1

1.1 General 1

1.2 Objective and Scope 4

1.3 Outline of thesis 4

2 LITERATURE REVIEW 6

2.1 Introduction 6

2.2 Previous work 7

2.3 Summary 12

3 GALERKIN METHOD 13

3.1 Introduction 13

3.2 Galerkin method 14

3.3 Galerkin method in eigen values problems. 16

3.4 Trapezoidal rule 19

3.4.1 trapezoidal rule in one dimension 19

3.4.2 trapezoidal in two dimensions 20

4 DEFLECTION OF PLATES OF VARIABLE THICKNESS 22

4.1 Governing differential equations of plates of variable thickness 22

4.1.1 Introduction 22

4.1.2 Plates of variable thickness on elastic foundations . . 27

4.1.3 Boundary Conditions 28

4.2 Method of solution 29

4.3 Discussions and Conclusion 30

5 BUCKLING OF PLATES OF VARIABLE THICKNESS 32

5.1 Governing differential equation 32

5.2 Boundary conditions and the approximate displacement function 35

5.3 Method of solution 37

5.4 Discussion and Conclusions 38

6 VIBRATION ANALYSIS OF PLATES OF VARIABLE THICKNESS 40

6.1 Introduction 41

6.2 Differential equation 42

6.3 Boundary condition 45

6.4 Method of solution 46

6.5 Results and Conclusions 47

7 CONCLUSIONS 50

A GALERKIN METHOD 61

B DEFLECTION ANALYSIS 63

C BUCKLING ANALYSIS 80

D VIBRATION ANALYSIS 89

E COMPUTER PROGRAMS 103

List of Figures

A.1 Trapezoidal rule in one dimension 62

A.2 Trapezoidal rule in one dimension, general case 62

A.3 Trapezoidal rule in two dimensions 62

B.1 Inplane forces on plate element 64

B.2 Moment and shear forces on plate element 65

B.3 Rectangular coordinate system 66

B.4 Variation of central deflection with aspect ratio of clamped rectangular plates of variable thickness for various taper parameter c: 67

B.5 Variation of central deflection with taper parameter (c) of clamped plates of variable thickness for various plate aspect ratios 68

B.6 Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 0.5 69

B.7 Variation of central deflection with taper parameter (c) and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 0.8 70

B.8 Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 1 71

B.9 Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 1.5 72

B.10 Variation of central deflection with taper parameter (c) and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 2 73

B.11 Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.2 74

B.12 Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.4 75

B.13 Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.6 76

B.14 Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.8 77

C.1 Variation of buckling load with taper parameter for various aspect ratios R for clamped rectangular plates of variable thickness for $N_y = 0$. 81

C.2 Variation of buckling load with aspect ratio R for various taper parameter for clamped rectangular plates of variable thickness for $N_y = 0$ 82

C.3 Variation of buckling load with aspect load $\frac{N_y}{N_x}$ for various taper parameter c for clamped square plate of variable thickness 83

C.4 Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 2 84

C.5 Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 0.8 85

C.6 Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 1 86

D.1 Rectangular plate of variable thickness element subjected to an inertia force 90

D.2 Variation of circular frequency ω of clamped rectangular plates of variable thickness with taper parameter c for various aspect ratios R in the first mode category. 91

D.3 Variation of circular frequency ω of clamped rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the first mode category. 92

D.4 Variation of circular frequencies for three modes of clamped square plates of variable thickness with taper parameter in first mode category 93

D.5 Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratios R in the second mode category. 94

D.6 Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the second mode category. 95

D.7 Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratio R in the third mode category. 96

D.8 Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the third mode category. 97

D.9 Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratio R in the fourth mode category. 98

D.10 Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the fourth mode category. 99

D.11 Variation of circular frequencies for 3 modes of clamped square plates of variable thickness with taper parameter in the fourth mode category. 100

List of Tables

B.1 Comparison of center deflection α of rectangular plates of variable thickness for R = a/b = 1 78

B.2 Variation of max small deflection coeff α of clamped rectangular plate of variable thickness with dimensionless foundation modulus K for taper parameter c = 0 79

C.1 Convergence table for the buckling load coefficient C_r of clamped rectangular plates of variable thickness with various taper parameters, for various aspect ratios and $\frac{N_y}{N_x} = 1$ 87

C.2 Comparison of buckling load coefficient C_r for clamped rectangular homogeneous plate (c = 0)with aspect ratio 88

D.1 Convergence table of fundamental frequencies mode category 1 of clamped rectangular plates of variable thickness for various taper parameter c and aspect ratios. 101

D.2 Comparison of fundamental frequencies for each of four mode category for clamped rectangular plates of uniform thickness 102

Chapter 1

INTRODUCTION

1.1 General

Plates of variable thickness are widely used in engineering structures, such as bunkers, reinforced concrete breast walls, rectangular reservoirs, buttress dams, reinforced concrete pavement of highways and airport runways. Moreover, in ship design, the ship bottom is frequently considered as a complex plate of variable thickness. There was great interest in the application of plates of variable thickness to engineering structures after the second world war in the aeronautic and aerospace industry where those plates are frequently found in construction of modern high speed swept wing aeroplanes and missiles.

While the case of static bending is difficult to solve, solution to problems of vibration or buckling of plates of variable thickness is even more difficult to obtain. One important practical case of application is when the modulus of elasticity E is constant and the thickness of plates varies linearly along one of the coordinate axis (x-axis) and this variation is taken to be symmetric with respect to the middle surface.

Very often specifications ensuring that plates would withstand applied lateral loads are not sufficient criterion for design. In addition, the designer must keep in mind the effect of vibration and buckling. Such structures are often excited by wind load, moving traffic and operating machinery. To avoid resonance of plates in which the driving force frequency is equal to the plate natural frequency, we must first establish the natural frequency of the plate and design the plate such that its natural frequency will not be close to the driving force frequency. Therefore free vibration analysis is an essential first step in obtaining results for forced vibration of plates of variable thickness. To conclude, it is highly desirable to employ an accurate numerical method to predict the static, dynamic and stability behavior of plates of variable thickness.

The deflection of plates of variable thickness is generally small in comparison to its thickness. The middle plane of the plate remains neutral during bending. The normal stresses in the transverse direction are generally neglected along with the shear strain of the plate. The energy desissipated can also be disregarded, as no allowance is being made for the damping force. The plate material is homogeneous, isotropic and linearly elastic and obeys Hooke's law. In other words, a linear relationship between stress and strain is assumed.

The variational method is an effective tool for solving complicated differential equations; also, it is the most fruitful field of application in mathematical physics. The application of this method is based on equilibrium conditions and the principle of minimum potential energy.

Due to development of high speed electronic computers, many numerical methods can now be applied with success in solving complicated problems in mechanics. The Galerkin method is one of such methods known and is recognized as one of the most rapidly converging variational techniques in solving complicated differential equations. Moreover, it is a simple, versatile and efficient method allowing the solution of complicated problems using a relatively small number of unknowns compared to other numerical methods. This method is also a simple way of deriving the Lagrangian dynamical equation. According to this method the choice of the displacement functions must satisfy the boundary conditions of the problems. After substituting the displacement functions into the differential equation an error will be obtained requiring, the integral of weighted error over the domain to be set to zero. The coefficients obtained in the Galerkin equation are always definite integrals. These integrals are readily evaluated by employing the trapezoidal rule.

1.2 Objective and Scope

The main objective of this thesis is to apply the Galerkin method to deflection, eigen values and stability problems of isotropic clamped rectangular plates of variable thickness. The thickness of the plates varies linearly along one of coordinate x axis and this variation is taken to be symmetric with respect to the middle surface. The problems of plates of variable thickness resting on an elastic foundation are also studied.

1.3 Outline of thesis

Since the main scope of this thesis is concerned with the application of the Galerkin method to obtain results for deflection, vibration and stability of plates of variables thickness, a literature review is presented in Chap 2.

Chapter 3 discusses the derivation of the Galerkin method and the development of algebraic equations from the governing differential equation. Since the coefficients obtained by this method are always definite integrals, the trapezoidal rule is explained and developed in the same chapter.

Chapter 4 contains the derivation and the development of differential equation of plates of variable thickness in order to calculate the deflection of these plates due to lateral load with or without elastic foundations.

In Chapter 5, the Galerkin method is applied to solve the buckling of clamped, rectangular isotropic plates of variable thickness. Solution of eigen values problem is presented in Chapter 6, where the Jacobi method is used in solving both the buckling and vibration problems

The last chapter contains discussions and conclusions of the results obtained. The results are either tabulated or graphically presented. Whenever applicable, the results are checked with those of other investigors. The computer programs are also included for reference in the appendix.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

The subject of analysis of plates has roots which extend back at least three hundred years. According to Szilard [1], Euler was the first who solved the problem of free vibration of elastic membranes using the analogy of superposition of strings. Later J Bernouilli extended the theory of his teacher Euler and conducted several experiments of vibration of membranes. According to Soedel [10], Chladni explained the behavior of vibration of plates showing the various modes. In 1915 Sophie Germain used calculus of variations and by neglecting the bending stiffness, she obtained almost the correct equation for the bending of plates. In 1811 Lagrange developed the fundamental small deflection theory of plates and corrected the error made earlier by Sophie Germain. In 1824 Navier proposed an analytical equation and solved plate problems by using trigonometric series for the case of simply supported plates. Kirchhoff(1824–1887) extended the plate theory by introducing the additional effect due to stretching and subsequently Saint Venant formulated the correct mathematical differential equation of thin plates due to the combined effect of both bending and stretching.

Due to the natural complexities of plates of variable thickness, there were no studies of the problem until the problem of bending of circular plates of variable thickness was first solved in 1918 by Hozler according to Timoshenko [2]. Results for vibrations of rectangular plates of variable thickness were not available in the technical literature until 1960 by Mazurkiewicz according to Appl [3].

2.2 Previous work

In 1950, Young [4] utilized the Rayleigh Ritz method to solved the problem of vibration of a square homogeneous plate of uniform thickness, with all edge clamped and plates with two adjacent edges free and two adjacent edge clamped.

In 1951, Conway [5] used the method of variation of parameters and solved the problem of bending of circular plates of variable thickness, symmetrically loaded,with thickness of plate decreasing linearly with the distance from the center.

In 1953, Warburton [8] studied the transverse free vibration of rectangular plates of constant thickness with various boundary conditions by using the Rayleigh Ritz method employing characteristic beam functions in two directions. In the same year, Conway [6] used the method of variation of parameters to analyze the bending of circular plates of variable thickness symmetrically loaded.

Conway [58] in 1958 used the Levy approach to solve the bending of rectangular plates of variable thickness, uniformly loaded. The plates were simply supported in two opposite edges and having arbitrary boundary conditions on the other two edges with the thickness varying exponentially.

In 1965, Appl [3]extended the method in his previous paper [7] to find the upper and lower bound fundamental frequency for the case of simply supported rectangular plates with linear varying thickness. The deflection is assumed by a Levy type solution. In 1969, Ashton [12] presented a solution of the lowest four natural frequencies for simply supported tapered plates by using the Ritz method.

In 1970, The finite strip method was used by Cheung [14] for analysis of vibration of elastic orthotropic as well as isotropic plates of variable thickness with two opposite edges simply supported by assuming a simple polynomial in the transverse direction and a series function in the longitudinal direction. This method was found to give frequencies lower in magnitude than the exact values.

In 1971, Chopra [17] approximated the natural frequencies and mode of simply supported skew plates with the variation in plate thickness taken in one direction. The vibration is analyzed by using Lagrange's equations employing double series as oblique coordinates. The mode shape is presented in double sine series. In the same year, Cheung [15] used again the finite strip method to analyse the flexural vibration of rectangular plates by using also a polynomial beam function in the transverse direction and a series function in the longitudinal direction. The advantage of this method is to reduce a two dimensional problem to a one dimensional problem with the result that smaller number of variables are involved but the frequencies obtained by this method are lower than the actual values. Soong [19] approached the rectangular plate of variable thickness by a non-linear iteration procedure. This procedure is the modification of the Kantorovich method applicable to nonlinear multifunctional boundary value problems.

In 1972, Bastin [25] proposed a solution in closed form to calculate the moments of rectangular plates of variable thickness. The coefficients are determined by using Fourier series. An example was shown in the case of simply supported plates subjected by hydrostatic pressure. In further study Petrina [21] reported data for deflection and moment for plates with various boundary conditions for rectangular plates of variable thickness.

In 1973, Chehil [32] discussed the problem of buckling of simply supported rectangular plates with general variable thickness in one direction. Critical buckling is presented by using the perturbation technique, with the compression force assumed only along the x direction. Nair [27] approximated a solution for frequencies by using the Ritz method using also products of appropriate beam characteristic functions in the case of skew plates of uniform thickness with all combinations of simply supported, clamped and free conditions. Dokainish [29] analyzed the vibration of clamped orthotropic parallelogramic plates with variable thickness by employing the Galerkin method.The deflection in dimensionless oblique co-ordinates is expressed as a polynomial.

In 1974, Durvassula [35] proposed the repartition method for vibration of clamped rectangular and skew plates of constant thickness. The convergence of this method depends of the choice of the subdivision along two directions. Soni [34] examined the transversal vibration of plates of variable thickness using the quintic spline interpolation technique to find the frequencies and the moment . The plates are simply supported on two opposite edges and having arbitrary boundary conditions on the other edges with the thickness varying exponentially.

In 1975, Chen [38] tackled the problem of vibration of plates of variable thickness in a general manner. In fact he modified the Rayleigh Ritz method, by employing a trial function which satisfied any specific boundary conditions for plates varying linearly and exponentially. The integrals are evaluated analytically. Unfortunately, frequencies obtained by this method is not sufficiently accurate, when compared to actual frequencies obtained by other numerical methods.

The collocation least square method is a modification of the least square method. This method was applied by Ng [39] in 1977 to find the large deflection of isotropic plates of constant thickness as well as plates on elastic foundations. the accuracy of this method depends of the choice of collocation points.

In 1978, Gupta [45] studied the problem of transverse vibration of rectangular plates of exponentially varying thickness for the case of three combinations of boundary conditions at the edges. The problem was approached by quintic spline interpolation. Laura [41]determined the fundamental frequencies of clamped rectangular plates by using polynomials.

In 1979, Banerjee [43] derived the frequency equation for skew plates of variable thickness but his results were not accurate when compared with actual values for plates of uniform thickness. Nagaya [52] proposed an approach for free vibration of plates with straight line boundaries.

In 1982, Tomar [51] solved the problem of vibration of non homogeneous rectangular plates of variable thickness, with the thickness varying parabolically in one direction. The plates are simply supported in two opposite edges and having arbitrary boundary conditions for the other edges. The non homogeneity in the plate is due to the variation of the modulus of elasticity along x direction as well as the density. The mathematical analysis is based on Frobanius method and the transversal displacement is expressed by an infinite series. Sakata [48] approximated the natural frequencies of clamped orthotropic skew plates of constant thickness by using the reduction method. Valerga [49] applied the Ritz method with coordinate function on the buckling of circular plates of variable thickness.

In 1984, Chaudhuri [54] investigated the large amplitude free vibration of square plates of exponentially varying thickness by using the Von Karman equations.

In 1985, Murza [53] applied the finite element technique to solve the problem of vibration of cantilivered triangular plates of variable thickness and arbitrary planform. Recently in 1986, Ng [56] analyzed the free vibration and buckling of clamped skew sandwich plates of uniform thickness by employing the Galerkin method.

2.3 Summary

Results for bending and vibration of plates of variable thickness are scarce in the technical literature, probably due to the difficulty and complexity of this problem. To date no result is available in case of buckling or vibration of clamped rectangular plates of variable thickness.

Chapter 3

GALERKIN METHOD

3.1 Introduction

Boundary value problems in solid mechanics are often solved by variational methods such as Galerkin and Rayleigh-Ritz. These methods are especially applicable to problems where the deformation of the elastic body can be represented by means of functions of independent variables satisfying the existing boundary conditions. The Galerkin method, in particular, has been quite effective in analyzing problems pertaining to the bending, buckling and vibration of plates and shells, governed by partial differential equations.

In the small displacement theory of elasticity, the variational principle based on the theorem of virtual work was first introduced by Lagrange. The main objective of the variational method is to find from a group of admissible functions those which represent the deflection of the elastic body pertinent to its equilibrium conditions.

In 1915, B G Galerkin, utilizing the variational principle, first introduced the Galerkin method for solving differential equations in his treatise " Rods and Plates " (Vsterik, Ingeneroff, 1915, p.897) [11] . Galerkin generalizes and simplifies the virtual work principle which states that for an elastic body in equilibrium, the total virtual work performed by both the internal and external forces must vanish. It was pointed out by Grossman [59] that the Galerkin method, in application to mechanics, leads to the same result as the Lagrange principle of virtual work but employs a special co-ordinate system [11].

After the publication of the Galerkin method large amount of work has been published in which the method was used extensively for practical solutions of extremely diverse types of problems.

3.2 Galerkin method

The Galerkin method belongs to the same general class as the Rayleigh-Ritz method in that it seeks to obtain an approximate solution of differential equations with given boundary conditions by choosing functions which satisfy the boundary conditions exactly and then proceeds to specialize the chosen functions in such a way as to secure approximate satisfaction of the differential equations. Consider the equilibrium of a small infinitesimal element represented by the pertinent differential equation

$$\mathcal{L}(w) - p = 0 \tag{3.1}$$

where $\mathcal{L}$ is the differential operator, p is the external force. The equilibrium of the structural system will be obtained by integrating the differential equation over the domain A.

Let us express δw as a virtual displacement. The new position $w + \delta w$ produces an increase in strain energy stored.

The virtual work of the internal and external forces must vanish; hence

$$\delta W_i + \delta W_e = 0 \tag{3.2}$$

Thus we can write

$$\iint (\mathcal{L}(w) - p)\, \delta w \, dA = 0 \tag{3.3}$$

where A is the domain of integration.

Equation (3.3) is true only if the displacement function w is the exact solution of the problem under investigation; this function will be written in the form;

$$w = \sum_{i=1}^{n} C_i f_i(x, y) \tag{3.4}$$

Where $f_i(x, y)$ are functions satisfying all boundary conditions of the problem and C_i are undetermined coefficients. The functions $f_i(x, y)$ are considered linearly independent in the region.

The variation of small displacements will be expressed by:

$$\delta w = \sum_{i=1}^{n} f_i(x,y)\delta C_i \tag{3.5}$$

Substituting equation (3.5) into equation (3.3)

$$\sum_{i=1}^{n} \delta C_i \iint (\mathcal{L}(w) - p)\, f_i(x,y)\, dA = 0 \tag{3.6}$$

Equation (3.6) must satisfied for any small variation δw_i. Thus, the variations of δC_i are arbitrary; therefore, we arrive at the system of equations:

$$\iint (\mathcal{L}(w) - p)\, f_i(x,y)\, dA = 0 \tag{3.7}$$

Where $dA = dx\, dy$

Equation (3.7) can be written in the form;

$$\iint (\mathcal{L}(w) - p)\, f_i(x,y)\, dx\, dy = 0 \tag{3.8}$$

These are n Galerkin equations with n unknown coefficients for the static case. These coefficients can be determined by integrating the function over the entire domain A.

3.3 Galerkin method in eigen values problems.

Galerkin Method assumes a solution in the form of series of functions for the case of eigen value problem. In general this method requires that these functions satisfy all boundary conditions, not merely those which are imposed by the geometry of the supports.

$$\mathcal{L}(w) - \lambda\mathcal{M}(w) = 0 \tag{3.9}$$

Where $\mathcal{L}$ and $\mathcal{M}$ are differential operators. They are considered to be continuous in the domain A. The function w is subject to the boundary condition which in general do not depend on the eigen value λ.

The equilibrium of the structure is obtained by integrating these differential functions over the entire structure.

δw is the small arbitrary variation of the displacement. The virtual work of external and internal forces will be expressed by.

$$\delta W_i + \delta W_e = \delta(W_i + W_e) \tag{3.10}$$

Thus we can write;

$$\iint (\mathcal{L}(w) - \lambda\mathcal{M}(w))\delta w \, dA = 0 \tag{3.11}$$

The function w is subject to the boundary conditions which in general do not depend on the eigen value λ. We assume the displacement function in the form:

$$w = \sum_{i=1}^{n} C_i f_i(x, y) \tag{3.12}$$

where C_i are coefficients to be determined and $f_i(x, y)$ are functions satisfying the boundary conditions.

Expressing the small arbitrary variation of displacement by

$$\delta w = \sum_{i=1}^{n} f_i(x, y)\delta C_i \tag{3.13}$$

and by substituting equation (3.13) into equation 3.11

$$\sum_{i=1}^{n} \delta C_i \iint (\mathcal{L}(w) - \lambda \mathcal{M}(w)) f_i(x,y)\, dA = 0 \tag{3.14}$$

Equation (3.14) have to be satisfied for any small variation δw, thus the variations of δC are arbitrary, and equation (3.14) can be written in the form:

$$\iint (\mathcal{L}(w) - \lambda \mathcal{M}(w)) f_i(x,y)\, dA = 0 \tag{3.15}$$

where $dA = dx\, dy$

Equation (3.15) can be expressed in new form as:

$$\iint (\mathcal{L}(w) - \lambda \mathcal{M}(w)) f_i(x,y)\, dx\, dy = 0 \tag{3.16}$$

These are called Galerkins equations representing eigen value problem of a n degree of freedom system. This procedure leads to a set of homogeneous algebraic equations and to an eigen value eigenvector problem. The lowest eigen value ($\lambda_{min} = \lambda_{cr}$) determines the critical load for the case of stability.

The virtual work of external and internal forces is obtained from the differential equations of equilibrium without determining the potential energy of the system. This characteristic makes this method more universal than other numerical methods, and it is considered to be one of the most efficient computational method suitable for microcomputers. The merit of this technique for the solution of complex plate problems is reduced to the evaluation of certain definite integrals, which can be evaluated numerically. This method can be applied also to different type of differential equations such as hyperbolic, elliptical, parabolic etc... The accuracy of this method depends on the selection of the shape function.

3.4 Trapezoidal rule

The trapezoidal rule is needed in order to obtain the coefficients in the Galerkin equations. The rapid development of computers has made possible development of many numerical integration methods such as the trapezoidal rule in which the domain is divided into finite parts. In this section the trapezoidal rule will be described.

3.4.1 trapezoidal rule in one dimension

Considering an integrable function in the case of a one dimensional function f(x). We wish to evaluate the integral:

$$I = \int_a^b f(x)\,dx \tag{3.17}$$

The region of integration $a \leq x \leq b$ is divided into n equal subintervals of width Δx, where $\Delta x = \frac{b-a}{n}$

Each of those subintervals will be referred to as a panel. Considering now an expanded view for only one panel as illustrated in Figure A.1. The straight line approximates the function f(x), and thus the approximate area under the straight line is

$$\int_a^b f(x)\,dx = (b-a)\frac{(f(b)+f(a))}{2} \tag{3.18}$$

Where f(a) is the value of the function at the extreme left and f(b) is the value of the function at the extreme right.

By extending the trapezoidal rule over the entire interval Figure A.2.
The area over the first panel $=\Delta x \frac{(f(a)+f(1))}{2}$
The area over the i th panel $=\Delta x \frac{(f(i-1)+f(i))}{2}$
The area over the last panel $=\Delta x \frac{(f(n-1)+f(b))}{2}$
Thus the total area will be

$$\int_a^b f(x)\, dx = \frac{\Delta x}{2}(f(a) + 2\sum_{i=1}^{n-1} f(i) + f(b)) \tag{3.19}$$

3.4.2 trapezoidal in two dimensions

In the same manner, the integral of function f(x,y) will be evaluated. The determination of the integral represents the volume under the function f(x,y).

let us divide the domain A into four panels Figure A.3.
The volume over A1= $(f(1) + f(2) + f(4) + f(5))\frac{\Delta x \Delta y}{4}$
The volume over A2= $(f(2) + f(3) + f(5) + f(6))\frac{\Delta x \Delta y}{4}$
The volume over A3= $(f(4) + f(5) + f(7) + f(8))\frac{\Delta x \Delta y}{4}$
The volume over A4= $(f(5) + f(6) + f(8) + f(9))\frac{\Delta x \Delta y}{4}$
where A= A1 + A2 + A3 + A4
By extending this procedure for a general case the integral will be written in the following form:

$$\iint f(x,y)\, dx\, dy = \frac{\Delta x \Delta y}{4}(\sum_{i=1}^{n1} f(M_i) + 2\sum_{j}^{n2} f(M_j) + 4\sum_{k}^{n3} f(M_k)) \tag{3.20}$$

where; M_i is a corner point.
M_j is a boundary point except the corner point.

M_k is an interior point.

n1 is the number of corner points.

n2 is the number of boundary points except the corner points.

n3 is the number of interior points.

The trapezoidal method has been found to be very simple and accurate and it does not need extensive computer time or memory even for a fine mesh width.

Chapter 4

DEFLECTION OF PLATES OF VARIABLE THICKNESS

4.1 Governing differential equations of plates of variable thickness

4.1.1 Introduction

The classical theory of elasticity governing the small deflection theory of plates assumes a linear relationship between stress and strain. The great advantages of this assumption are:

- Rigorous mathematical presentation of the relationship between external and internal forces
- The law of superposition holds

The classical theory of bending of thin plates is expressed by the relation between the transverse deflection of the middle surface of the plate and the lateral loading q. This theory is applicable only when the deflection is small when compared to the thickness of the plate. The theory neglects the effect of membrane stresses which must be appreciable when the displacement w has a value comparable with the thickness of the plate. In other words, for ($w \leq 0.2h$) Kirchoff plate theory is accurate enough. By increasing the deflection further ($w \geq 0.3h$), the transverse deflection w is accompanied by stretching of the middle surface and the membrane forces produced by such stretching can help carry a significant amount of the lateral loads. The design of plates of variable thickness following the linear theory of elasticity can be over conservative.

Basic assumptions:

1. The middle surface of the plate remains unstrained during bending; thus, it is a neutral surface.
2. Points which lie on a normal to the middle surface of plate lie on a normal to the same surface after deformation.
3. Normal stresses in the direction transverse to the plate are negligeable when compared with other stresses in the plane of the plate.

4. The slope of the deflected plane in any direction is small so that its square may be neglected in comparison with unity.

The condition of equilibrium is met by considering a small element as shown in Figure B.1. For simplicity, only the middle surface of the plate is shown with side dx and dy.

First equilibrium

Consider now the effect of the middle surfaces per unit length N_x, N_y *and* N_{xy} as shown in Figure B.1. By neglecting the body forces, the equilibrium of an element of plate in x and y direction yields the conditions:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$
$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \qquad (4.1)$$

The second equilibrium of the plate element in two directions

a) forces in two directions due to inplane forces

The net contribution of the inplane forces N_x , N_y and N_{xy} (taking downward positive) in the plane is given by

$$(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}) dx dy \qquad (4.2)$$

b) forces in two directions due to lateral loads;

Let us assume now that Q_x and Q_y are the shear force per unit length. Also shown in Figure B.2 are for the directions of the bending and twisting moments acting per unit length, q is the intensity of the load on the upper surface of the plate. The condition of a vanishing resultant force in Figure

B.2 in the downward direction results in the equation;

$$\frac{\partial Q_x}{\partial x}dxdy + \frac{\partial Q_y}{\partial y}dxdy + qdxdy = 0$$

or

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (4.3)$$

Taking the resultant moment about the x axis and neglecting higher terms due to the moments of the load q and the moments due to the change of the force Q_y give the following equation:

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \qquad (4.4)$$

Similarly, by taking the resultant moment about the y axis gives:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \qquad (4.5)$$

Substituting the values of Q_y and Q_x from equation (4.4) and (4.5) respectively into equation (4.3), the resultant equation is:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \qquad (4.6)$$

By substituting the well known moment and curvature relationships

$$M_x = -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})$$

$$M_y = -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y})$$

into equations (4.4) and (4.5) gives the expressions for Q_x and Q_y in terms of deflection of the middle surface as follows:

$$Q_x = -D(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}) \qquad (4.7)$$

$$Q_y = -D(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}) \tag{4.8}$$

The rigidity D is not constant but is a function of the coordinates x and y. Substituting the well known moment and curvature relationship in to equation (4.6) in terms of deflection of the middle surface gives:

$$D \nabla^2 \nabla^2 w + 2\frac{\partial D}{\partial x}\frac{\partial \nabla^2 w}{\partial x} + 2\frac{\partial D}{\partial y}\frac{\partial \nabla^2 w}{\partial y} + \nabla^2 D \nabla^2 w$$
$$-(1-\nu)(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}) = q \tag{4.9}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *is the two dimensional laplacian operator.*

Equation (4.9) is the general expression for analysis of small deflection of plates of variable thickness

For the case when the thickness of plate varies linearly along x axis, such that $\frac{\partial D}{\partial y} = \frac{\partial^2 D}{\partial y^2} = 0$, equation (4.9) transforms into the following differential equation for the analysis of small deflection of plates of variable thickness.

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^4}) + 2\frac{\partial D}{\partial x}(\frac{\partial w^3}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})$$
$$+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) = q \tag{4.10}$$

where

ν = Poisson's ratio.

D = Flexural rigidity of the plate dependent on variable x.

q = intensity of the load of the plate.

If we include the effect of inplane forces given by equation (4.2) into the above equation of equilibrium in the vertical direction due to lateral loads, we obtain:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^4}) + 2\frac{\partial D}{\partial x}(\frac{\partial w^3}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})$$
$$+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) = q + N_x\frac{\partial^2 w}{\partial x^2} + 2\ N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} \quad (4.11)$$

4.1.2 Plates of variable thickness on elastic foundations

Often, a plate is placed on a continuous elastic foundation, such as reinforced concrete pavements, airport runways and foundation slabs of buildings. In the analysis of these plates of variable thickness on elastic foundations it is assumed that the restoring pressure is everywhere proportional to the deflection at that point. In essence, we assume that the support is isotropic, homogeneous and linearly elastic and such a type of subbase is called Winkler type with the relation $q^*(x, y) = kw$ where k = modulus of elastic support reaction per unit area per unit deflection: The constant k has a wide range of values and can be determined through bearing tests of the actual foundation material.

When the plate of variable thickness is supported by a continuous elastic foundation, the lateral load acting in the plate q and the reaction of the elastic foundation is $q^* = kw$. Thus the resultant distributed load is q - kw so that the partial differential equation (4.10) becomes:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^4}) + 2\frac{\partial D}{\partial x}(\frac{\partial w^3}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})$$
$$+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) = q - kw \qquad (4.12)$$

4.1.3 Boundary Conditions

The plate is considered thin and is clamped all round. A sketch of the plate and system coordinate system is shown in Figure B.3. The thickness of the plate varies linearly along the coordinate x axis and this variation is taken to be symmetric with respect to the middle surface.

For ease of computation, these equations will be transformed into dimensionless form by introducing the following non dimensional coordinates

$$R = \frac{a}{b} \qquad \xi = \frac{x}{a} \qquad \eta = \frac{y}{b}$$
$$W = \frac{w}{h_0} \qquad Q = \frac{qa^4}{D_0 h_0}$$

where 2a and 2b are the lengths of plate in x and y directions respectively The plate of variable thickness is clamped; therefore the deflection and the slope of the middle surface must be set to zero at the boundary

$$W = \frac{\partial W}{\partial \xi} = 0 \qquad at \qquad \xi = \pm 1$$
$$W = \frac{\partial W}{\partial \eta} = 0 \qquad at \qquad \eta = \pm 1$$

The thickness of the plate varies linearly in one direction such that

$$h(\xi, \eta) = h_0(1 + c\xi)$$

and the flexural rigidity is expressed as

$$D(\xi,\eta) = D_0(1+c\xi)^3$$

where c is a taper parameter and h_0 is the thickness of the plate at x = 0 After introducing the non-dimensional variables, equation (4.10) will be transformed into the following expression:

$$D(\frac{\partial^4 W}{\partial \xi^4} + 2R^2\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + R^4\frac{\partial^4 W}{\partial \eta^4}) + 2\frac{\partial D}{\partial \xi}(\frac{\partial^3 W}{\partial \xi^3} + R^2\frac{\partial^3 W}{\partial \xi \partial \eta^2}) + \frac{\partial^2 D}{\partial \xi^2}(\frac{\partial^2 W}{\partial \xi^2} + \nu R^2\frac{\partial^2 W}{\partial \eta^2}) = \frac{qa^4}{h_0} \quad (4.13)$$

The partial differential equation for plates of variable thickness resting on elastic foundation, in non-dimensional coordinates can be obtained by adding $\bar{K}W$ to the left hand side of equation (4.13) where $\bar{K} = ka^4$

4.2 Method of solution

The solution of equation (4.13) is obtained by using the Galerkin method, described in chapter 3. The displacement W is supposed to be a linearly independent continuous function capable of representing the lateral deflection.

$$W = (1-\xi^2)^2(1-\eta^2)^2(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4) \quad (4.14)$$

This function W is selected such that each term of this expression must satisfy all boundary conditions of the problem under investigation. This approximate displacement expressions are then substituted into the governing differential equation (4.13). The left hand side of equation (4.13) represents the operator $\mathcal{L}(w)$ of equation (3.8) in Chapter 3. After substituting equation (4.13) into equation (3.8), the solution of the governing differential equation of clamped rectangular plates of variable thickness would be reduced to the evaluation of definite integrals of simple functions. By using the trapezoidal rule for numerical integration mentioned in the previous Chapter, resulting linear equations of the undetermined coefficient (C_1, , C_2, ..., C_6) can be determined .

4.3 Discussions and Conclusion

For the purpose of demonstrating the accuracy and the convergence of the present method, results for deflection analysis of clamped isotropic homogeneous plates and plates of variable thickness are shown in table B.1. It is found that in both cases the results are in excellent agreement with these obtained by other investigators. Also, the present method shows excellent convergence even for high taper parameters.

In table B.2 numerical comparison is shown for center deflection of clamped homogeneous plates on a continuous elastic foundation between the present method and those obtained Ng[39] who used the method of collocation least squares. For the case of clamped rectangular plates of variable thickness on elastic foundation no comparison of results can be made as no data are as yet available in the technical literature.

Figures B.4 and B.5 show the center deflection for clamped rectangular plates of variable thickness for various taper parameters. It is interesting to observe that the center deflection decreases with an increase of the taper parameter and the aspect ratio. This can be attributed to the fact that both increases in aspect ratio and taper parameter increase significantly the stiffness of the plate.

The center deflection of clamped rectangular plate of variable thickness on a continuous elastic foundation are shown graphically in figures B.6 to B.14. Figures B.6, B.7, B.8, B.9 and B.10 show the center deflection of plates of variable thickness on elastic foundations for different aspect ratios. Figures B.11, B.12, B.13 and B.14 show the center deflection of clamped rectangular plates of variable thickness on a elastic foundation for different taper parameters. As can be expected the center deflection decreases with an increase in the foundation modulus. It can also be found that the curve tend to become constant with the increase in the modulus of the elastic foundation. This can be attributed to the fact that an increase in the intensity of the foundation reaction is equivalent to an increase in plate stiffness.

From the present study it can be concluded that Galerkin's method is both effective and very simple to apply. The method yields very accurate results for deflection analysis of clamped rectangular plates of variable thickness.

Chapter 5

BUCKLING OF PLATES OF VARIABLE THICKNESS

This chapter treats the application of the Galerkin method to the problem of clamped rectangular plates of variable thickness under inplane loads. The formulation of the Galerkin method for eigen values problems was presented in chapter 3.

5.1 Governing differential equation

All equilibrium states of a given plate are characterized either as normal or critical. The term buckling is generally used to describe the loss of stability of an engineering structure. The load producing such a condition of buckling is called the critical load. When a plate is compressed in its middle plane by inplane loads and the inplane loads may increase to such magnitude that the plate may become unstable and begin to buckle. If the load is further increased, the deflection becomes large and eventually the failure of the plate of variable thickness occurs

The mathematical formulations of linear elastic stability are based on the fact that the classical buckling path leading from a stable to unstable equilibrium always passes through a neutral state of equilibrium. It assumes a bifurcation of deformations at the critical load , two paths of deformations, one associated with the stable equilibrium and the other one pertinent to the unstable equilibrium condition.

To investigate the stability problems, we assume that the plate of variable thickness buckles slightly under compression by inplane loads acting in its middle plane. It is of basic importance that the differential equation of equilibrium be written for such a slightly buckled shape. Thus, the differential equation is obtained by considering the simultaneous bending and stretching of the plate of variable thickness obtained from equation (4.11) in the previous chapter by putting the lateral load equal to zero. Thus the differential equation of buckling of a plate of variable thickness, where the thickness of the plate varies linearly along the x axis and the shearing force acting in the plane to be everywhere equal to zero becomes:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) + 2\frac{\partial D}{\partial x}(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})$$
$$+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) = N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} \tag{5.1}$$

Where

D is the flexural rigidity of the plate and is dependent on the variable x

ν is Poisson's ratio

N_x is the inplane load in x direction

N_y is the inplane load in y direction

Suppose that there is a relationship between the inplane loads N_x and N_y for the plate of variable thickness such that

$$N_y = rN_x \tag{5.2}$$

then the assumed buckling of the plate of variable thickness occurs only for certain definite value of N_x.

For ease of computation it is convenient to put equation(5.1) in dimensionless form by using

$$\xi = \frac{x}{a} \qquad \eta = \frac{y}{b} \qquad R = \frac{a}{b}$$
$$N_\xi = N_x a^2 \qquad N_\eta = N_y a^2 \qquad W = \frac{w}{h_0}$$

After substituting the above dimensionless ratios into Equation(5.1) the governing differential equation for the buckling of plates with variable thickness becomes:

$$D(\frac{\partial^4 W}{\partial \xi^4} + 2R^2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + R^4 \frac{\partial^4 W}{\partial \eta^4}) + 2\frac{\partial D}{\partial \xi}(\frac{\partial^3 W}{\partial \xi^3} + R^2 \frac{\partial^3 W}{\partial \xi \partial \eta^2})$$
$$+\frac{\partial^2 D}{\partial \xi^2}(\frac{\partial^2 W}{\partial \xi^2} + \nu R^2 \frac{\partial^2 W}{\partial \eta^2}) = N_\xi(\frac{\partial^2 W}{\partial \xi^2} + rR^2 \frac{\partial^2 W}{\partial \eta^2}) \quad (5.3)$$

5.2 Boundary conditions and the approximate displacement function

For the purpose of demonstrating the buckling analysis of rectangular plates of variable thickness, Galerkin method will first be adopted. The plate is assumed thin and clamped all around, the thickness of the plates of variable thickness varies linearly along the coordinate x axis and the variation is taken to be symmetric with respect to the middle surface such that:

$$W = \frac{\partial W}{\partial \xi} \quad at \quad \xi = \pm 1$$
$$W = \frac{\partial W}{\partial \eta} \quad at \quad \eta = \pm 1$$
$$h = h_0(1 + c\xi)$$

Where:

c is the taper parameter of the plate of variable thickness

h_0 is the thickness of the plate at $x = 0$

The chosen functions are selected such that they satisfy the boundary conditions of the plates under investigation.

$$W = (1-\xi^2)^2(1-\eta^2)^2(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

It must be understood that the expressions W are symmetric about both variable ξ and η. As a result, when these expressions are used it can be expected that the plate of variable thickness buckles in one half wave in both directions for only the case when the aspect ratio R = 1. For aspect ratios greater or less than one, the plate of variable thickness can be expected to buckle into more than one half waves in the longer direction.

The following expressions are chosen for the displacement W to approximate the buckled shape of plate of variable thickness for aspect ratios greater than one as:

$$W = (1-\xi^2)^2\xi(1-\eta^2)^2(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

Similarly the following expressions are chosen for the displacement W to approximate the buckled shape of plate of variable thickness for aspect ratios less than one

$$W = (1-\xi^2)^2\eta(1-\eta^2)^2(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

Since it is not known in advance for what aspect ratio the plate of variable thickness will buckle more than a single half wave in the longer direction, for this investigation, for all aspect ratios greater or less than one, three expressions of W are used to find the buckling load of plates of variable thickness. It is obvious that the lowest buckling load obtained from the above three expressions is the critical buckling load for the plate under investigation.

5.3 Method of solution

The solution of equation(5.3) can be obtained by using the Galerkin method. Displacement expressions are supposed to be a linearly independent continuous function. These approximate functions are then substituted into the governing differential equation (5.3). The left hand side of equation (5.3) represents the operator $\mathcal{L}$ (w) of equation (3.11) and the right hand side without N_ξ represents the operator $\mathcal{M}$ (w) of the same equation described in Chapter 3 under the eigen-value problem section.

After substituting equation (5.3) into equation (3.11), the solution of the governing differential equation of clamped rectangular plates of variable thickness in buckling analysis would be reduced to the evaluation of definite integrals of simple functions by using the trapezoidal rule for numerical integration. This requirement leads to n equations which can be written in matrix form:

$$[A]\{C\} = N_\xi[B]\{C\} \tag{5.4}$$

defining an eigen-value problem. The eigen values are determined by using the Jacobi method. The lowest eigen value determines the critical load of the problem under investigation.

5.4 Discussion and Conclusions

To determine the accuracy and convergence of the present method for buckling analysis of both clamped isotropic homogeneous plates and plates of variable thickness, numerical experimentation was carried out by varying the number of terms in the displacement W. Results computed for different number of terms in the displacement W are shown in table C.1. It is found that as the number of terms is refined, the results obtained all converge to a stationary value.

In table C.2 numerical comparison is shown for the critical load coefficients C_r for clamped homogeneous plates under various combination of load and aspect ratios. The results obtained by the present method are found to be in excellent agreement with those obtained by Sa[60] who used the method of collocation least squares. For the case of clamped rectangular plates of variable thickness in buckling analysis no comparison of the results can be made as no data are yet available in the technical literature.

Figures C.1 and C.2 show the critical buckling load for rectangular plates of variable thickness for various taper parameters. It can be seen clearly from those figures in the buckling load increases with an increase of the taper parameters and aspects ratios. This is because of the fact that both increases in aspect ratio and taper parameters increase of the rigidity of the plate.

Figure C.3 shows the critical buckling load for clamped square plates of variable thickness for various load ratios. It can be observed from this figure that, for a given taper parameter the critical buckling load decreases with an increase of the load ratio, but it should be noted that the buckling load increases with an increase of taper parameter for a fixed load ratio.

The critical buckling load of clamped rectangular plates of variable thickness under various combination of load ratios are shown graphically in figure C.4 to C.6. It is interesting to note that the critical buckling load increases with an increase of the taper parameters c. This can be attributed to the fact that an increase of taper parameter increases the stiffness of the plate. It can be seen also that the curve tend to become constant when the load ratio becomes large. It can be concluded that when the load ratio becomes large, plates of variable thickness behave close to the buckling of homogeneous plates.

For the present study it can also be concluded that the suggested method of Galerkin is a very simple and accurate method for calculating the critical buckling load of plates of variable thickness.

Chapter 6

VIBRATION ANALYSIS OF PLATES OF VARIABLE THICKNESS

In this chapter, the Galerkin method is applied to the vibration analysis of clamped rectangular plates of variable thickness following the solution of the eigen-value problems discussed in Chapter 3.

6.1 Introduction

Very often specification ensuring that plates of variable thickness can withstand applied lateral loads is not sufficient criterion for design. In addition the designer must be concerned with the possibility of large cyclic displacements which may be induced by periodic or random time varying forces acting on the plate. Random forces are to be expected for example, on the surface of the plate exposed to tangential gas flow, as in aircraft components or in stationary structures exposed to high wind velocities. Such forces can also be expected when plates of variable thickness are subjected to tangential liquid flow, for example when rectangular plates of variable thickness are used as hulls of ships or submarines. Regular or periodic excitation forces are also likely to be experienced when plates of variable thickness form part of a structure housing rotating or reciprocating machinery. The source of such periodic forces could, for example, be reciprocating engines or compressors. The forces often excite the plates of variable thickness by imparting rapid oscillation.

It is known that associated with each plate natural frequency there is a distinct characteristic or mode shape which the plate of variable thickness acquires at it vibrates. The most used technique for resolving design problems is to avoid matching the frequencies of the driving forces with the natural freqency of the plate

It will be apparent in design that the ability to conduct an accurate free vibration analysis of rectangular plates of variable thickness is absolutely essential if the designer is to avoid the problem of resonance. In fact, the free vibration analysis is an essential first step toward obtaining solutions for the forced vibration of rectangular plates of variable thickness.

6.2 Differential equation

When a plate of variable thickness is loaded statically the elastic reaction of the plate is everywhere equal and opposite to the applied loading q as discussed in detail in Chapter 4. If there is no external applied loading but the plate of variable thickness is vibrating, the elastic reaction acting on each element of the plate (measured in the direction of negative w) produces an acceleration of each plate element in the same direction as shown in figure D1. The magnitude of the elastic reaction $\rho h dA\frac{\partial^2 w}{\partial t^2}$continues to act on the plate element and opposes acceleration. In the elastic reaction expression, area dA is the the product of the lengths dx and dy and ρh is the mass per unit surface. The differential equation(4.11) for deflection analysis which is developed in Chapter 4 may now be obtained from the preceding analysis by substituting $\rho h dA\frac{\partial^2 w}{\partial t^2}$ for q in order to develop the governing differential equation of free vibration analysis of rectangular plates of variable thickness. Thus, we obtain

$$D \nabla^2 \nabla^2 w + 2\frac{\partial D}{\partial x}\frac{\partial \nabla^2 w}{\partial x} + 2\frac{\partial D}{\partial y}\frac{\partial \nabla w}{\partial y} + \nabla^2 D \nabla^2 w$$
$$-(1-\nu)(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}) = -\rho h\frac{\partial^2 w}{\partial t^2} \qquad (6.1)$$

It is assumed that the thickness varies only along the x axis. With this assumption the above differential equation becomes:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^4}) + 2\frac{\partial D}{\partial x}(\frac{\partial w^3}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2})$$
$$+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) = -\rho h\frac{\partial^2 w}{\partial t^2} \quad (6.2)$$

where the displacement w is expressed as a function of the coordinate axes x and y and time t.

Let us assume that the displacement w(x,y,t) is expressed as a product of two functions one including only the coordinates x,y and the other one including the time t, thus the displacement w will be written as:

$$w(x,y,t) = w(x,y)T(t) \quad (6.3)$$

By substituting equation(6.3) into equation(6.2), we get the following differential equation.

$$\frac{1}{\rho h w(x,y)}\left(D\left(\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial x^4}\right) + 2\frac{\partial D}{\partial x}\left(\frac{\partial^3 w(x,y)}{\partial x^3} + \frac{\partial^3 w(x,y)}{\partial x \partial y^2}\right)\right)$$
$$+\frac{1}{\rho h w(x,y)}\left(\frac{\partial^2 D}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w(x,y)}{\partial y^2}\right)\right) = -\frac{1}{T(t)}\left(\frac{\partial^2 T(t)}{\partial t^2}\right) \quad (6.4)$$

It is interesting to note that the left hand side of equation(6.4) is a function of the coordinates x and y but the right hand side is a function of time t only; therefore this equation can be true only if both sides of the equation are equal to a constant.

Let us denote this constant by a positive real quantity ω^2. From the right hand side of the above equation we obtain:

$$\frac{\partial^2 T(t)}{\partial t^2} + \omega^2 T(t) = 0$$

The well known solution of this ordinary differential equation is

$$T(t) = A\sin(\omega t + \phi)$$

It is to be noted that T(t) represents simple harmonic motion. The quantity ω is the circular frequency which dictates the frequency of these sinusoidal oscillation. The quantity A is known as the amplitude and the quantity ϕ is called the phase angle which can be neglected when conducting the free vibration analysis. In fact the reference can be always selected such that it is always zero. From the left hand side of equation (6.4) we obtain

$$D(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial x^4})+2\frac{\partial D}{\partial x}(\frac{\partial w^3}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2})+\frac{\partial^2 D}{\partial x^2}(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2})-\omega^2 \rho h w = 0 \tag{6.5}$$

The above differential equation is a homogeneous partial differential equation for free vibration analysis of plates of variable thickness involving the properties of the plates, the circular frequency and the shape function describing the mode of vibration w(x,y).

It is convenient to introduce the following non-dimensional form

$$\xi = \frac{x}{a} \qquad \eta = \frac{y}{b} \qquad R = \frac{a}{b} \qquad W = \frac{w}{h_0}$$

By putting the dimensionless ratios into equation(6.5). The following differential equation can be obtained:

$$D\left(\frac{\partial^4 W(\xi,\eta)}{\partial \xi^4} + 2R^2\frac{\partial^4 W(\xi,\eta)}{\partial \xi^2 \partial \eta^2} + R^4\frac{\partial^4 W(\xi,\eta)}{\partial \eta^4}\right) + 2\frac{\partial D}{\partial \xi}\left(\frac{\partial^3 W(\xi,\eta)}{\partial \xi^3} + R^2\frac{\partial^3 W(\xi,\eta)}{\partial \xi \partial \eta^2}\right)$$

$$+\frac{\partial^2 D}{\partial \xi^2}\left(\frac{\partial^2 W(\xi,\eta)}{\partial \xi^2} + \nu R^2\frac{\partial^2 W(\xi,\eta)}{\partial \eta^2}\right) = \omega^2 a^4 h \rho W \tag{6.6}$$

6.3 Boundary condition

The plates are clamped on all four edges and are of variable thickness, the thickness of plates varies linearly along the x direction and the variation is taken to be symmetric with respect to the middle surface. Thus the boundary conditions used in the previous chapter will be valid for the present chapter.

All modes of vibration of the clamped rectangular plate of variable thickness will be placed into 4 categories:

1. Mode symmetric about both axis.
2. Mode antisymmetric with ξ axis.
3. Mode antisymmetric with η axis.
4. Mode antisymmetric about both axis

It is evident that the mode associated with any eigen-value will be composed of an integral number of half sine waves running in the direction of each axis. The higher eigen-values with associated higher frequencies, will have more complicated shapes involving higher number of these half sine waves. The simplest mode of all free vibration of plates of variable thickness with the lowest frequencies will involve half sine wave running in each of the two coordinate directions. This is called the fundamental mode.

The assuming shape functions W are appropriate displacement functions which individually satisfy at least the geometrical boundary condition. Again the satisfaction of the differential equation of motion is not required. Thus the chosen functions satisfying the geometrical boundary condition of the problem under investigation are:

Mode category 1

$$W(\xi,\eta) = (1-\xi^2)^2(1-\eta^2)^2(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

Mode category 2

$$W(\xi,\eta) = (1-\xi^2)^2(1-\eta^2)^2\xi(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

Mode category 3

$$W(\xi,\eta) = (1-\xi^2)^2(1-\eta^2)^2\eta(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

Mode category 4

$$W(\xi,\eta) = (1-\xi^2)^2(1-\eta^2)^2\xi\eta(C_1 + C_2\xi^2 + C_3\eta^2 + C_4\xi^2\eta^2 + C_5\xi^4 + C_6\eta^4)$$

6.4 Method of solution

The Galerkin method is used to obtain a solution of equation (6.5). The merit of this method is that the solution of vibration of plates of variable thickness is reduced to evaluation of certain definite integrals which can be evaluated numerically by using the trapezoidal rule. This procedure leads to a set of homogeneous algebraic equations and to an eigen-value eigen-vector problem. The procedure of the application of the Galerkin method in the previous chapter will also be valid for the present investigation

6.5 Results and Conclusions

To demonstrate the accuracy and convergence characteristics of the method, numerical experimentation was carried out by varying the number of terms in the displacement function . The results for the case of modes symmetric about both ξ and η axes for different number of terms, aspect ratios and taper parameters are tabuled in table D.1. From this table, it can be seen that the fundamental frequency converges to a stationary value even for high taper parameters.

Table D.2 shows the results for the fundamental frequency of different mode categories in free vibration analysis of a homogeneous plate for different aspect ratios. The results obtained by the present method are compared with Odman [61]. Excellent agreements are found.

Figure D.2 shows the variation of the fundamental mode category 1 for clamped rectangular plates of variable thickness with taper parameter c for different aspect ratios. Figure D.3 shows the variation of fundamental frequency category 1 with aspect ratios for different taper parameter c. It is interesting to observe that the fundamental mode category 1 of clamped rectangular plates of variable thickness increase with an increase in taper parameter and aspect ratios.

When the plate of variable thickness is vibrating in category 2 (Figures D.5 and D.6) the frequency increases continuously with the increase in taper parameter c or aspect ratio. This can be attributed to the fact that an increase in aspect ratio and taper parameter amounts to an increase in stiffness of the plate. The same is also true for plate vibration in category 4 as shown in Figures D.9 and D.10. Due to lack of data in the technical literature, no comparison of these results can be made.

Figures D.7 and D.8 show the variation in the fundamental mode category 3 of clamped rectangular plates of variable thickness with taper parameter c for various aspect ratios and with aspect ratios for various taper parameter c respectively. It is interesting to note that the curves tend to become linear with increase of the taper parameter or aspect ratio. This is due to the fact that plate vibration in category 3 is close to the vibration of homogeneous plates in the same category.

In Figures D.4 and D.11, it can be concluded that the frequencies in mode categories 2 and 3 are the same and degenerate into the same mode for uniform square plates. It is interesting to observe that, even for a small change in the taper parameter from zero, this degeneracy disappears and they become modes with distinct frequencies.

7

The Galerkin method has been shown to be accurate and simple to apply for free vibration analysis. The computer time and memory requirements are also found to be small. Again the Galerkin method has proved to be an extremely powerful and versatile tool for free vibration analysis of plates of variable thickness.

Chapter 7

CONCLUSIONS

By examination of analytical results obtained, the following conclusions can be drawn:

1. For small deflection, buckling and vibration analysis the Galerkin method is found to yield results which are in excellent agreement with those obtained by more laborious method for the case of uniform plates.

2. The centre deflection decreases with an increase of taper parameter c and aspect ratio; this can be attributed to the fact that both increases in taper parameter and aspect ratio increase in the give rise to a significant increase in the stiffness of the plate.

3. With increasing of taper parameter, plates of variable thickness become much more stiff. As a result of this, the plate of variable thickness tends to deflect less at high taper parameter c under the same lateral load.

4. For a given aspect ratio R = a/b and taper parameter c, the maximum center deflection of the plate of variable thickness decreases with increasing value of the foundation modulus. This is to be expected as the effect of the foundation modulus is to reduce the centre deflection.

5. With increasing value of foundation modulus the centre deflection curves tend to become linear. This is due to the increasing rigidity of the plate contributed by the elastic foundation.

6. Due to the increasing rigidity of the plate of variable thickness with increase in the taper parameter c, it has been found that the critical buckling load is very sensitive to the increase in taper parameter c.

7. With increasing load ratio $\frac{N_y}{N_x}$ the buckling load curves tend to become constant. This can be attributed to the fact that when the load ratio becomes large, the plates of variable thickness behave close to buckling of uniform plates.

8. For vibration analysis, the frequency increases with increase of taper parameter c for each of the four categories of vibration.

9. In mode category 3, plates of variable thickness vibrate close to the vibration of uniform plates.

10. For a square uniform plate the frequencies in the 2^{nd} And 3^{rd} mode categories are found to be of the same magnitude. These are called degenerated modes. The taper parameter c is found to split the degenerate frequencies of uniform thickness square plates to distinct frequencies.

Finally, for the present study, it can be concluded that, the Galerkin method proves to be an extremely powerful and versatile tool for all type of analysis considered. The computing time and memory requirements are small; thus making it specially suited for desk top microcomputers.

Bibliography

[1] R. Szilar. *Theory and Analysis of Plates, Classical and Numerical Method.* Prentice Hall, Inc. Englewood cliff, New Jersey, 1974

[2] Timoshenko S. and Woinowsky Krieger S. *Theory of Plates and Shells.* 2nd edition, Mc Graw Hill, New York 1959

[3] Appl f.c and Bayers N.R. *Fundamental Frequency of Simply Supported Rectangular Plate with Linearly thickness.* Journal of Mechanics,TRANS, ASME, Series E, Vol 32, Mai 1965, pp 163–168.

[4] Young D. *Vibration of Rectangular Plates by Ritz Method.* Journal of Applied Mechanics, Vol. 24, No 4, Dec. 1950, pp 448–453.

[5] H. D. Conway and Ithaca. N. Y. *Axilly Symmetrical Plates with Linearly Varying Thickness.* Journal of Applied Mechanics, June. 1951, pp 140–142.

[6] Conway H. D. *Closed Form Solution for Plates of variable thickness.* Journal of Applied Mechanics, TRANS. ASME, Series E, Vol. 20, 1953, pp 564–564

[7] F. C. Appl and C. F Zorowski. *Upper and Lower Bounds for Special eigen values*. Journal of Applied Mechanics, TRANS. ASME, Vol. 81, Series E, 1959, pp 246–250.

[8] Warburton G. B. *The Vibration of Rectangular Plates*. Proceeding of The Institution of Mechanical Engineers, Vol 168, No 12, 1954, pp 371–384.

[9] Klaus Jurgen Bath.*Finite Element Procedures In Engineering Analysis*. Prentice Hall–Inc ,New Jersey 1982.

[10] W. Soedel. *Vibrations of Shells and Plates*. Marcel Dekker,Inc , New York and Basel 1981.

[11] Ducan W. J. *Galerkin's Method in Mechanics and Differential Equations*. Reports and Memorandum No 1768, Aug. 3, 1937.

[12] J. E Ashton. *Natural Frequency and Modes of Vibration of Tapered Plates*.Journal of The Structural Division, Americain Society of Civil Engineering,Vol 95, 1969, pp 787–790.

[13] Timoshenko S and Gere. *Theory of Elastic Stability*. Mc Graw Hill, New York 1961.

[14] Y. K. Cheung and M. S. Cheung. *Free Vibration of Orthotropic, Variable Thickness, Continuous Rectangular plates*. Rev. Roum Sci Tech, Mechn Appl, Tome 15, No 3, pp 565–578, Bucarest 1970.

[15] Y. K. Cheung and M. S. Cheung. *Flexural Vibration of Rectangular and Other Polygonal Plates*. Journal of The Engineering Mechanics

Division , Americain Society of Engineering, Vol 97, No EM2, April 1971, pp. 391–411.

[16] D. J. Gorman. *Free Vibration Analysis of Rectangular Plates.* Elsevier New York, Amsterdam, Oxford 1982.

[17] I. Chopra and Durvasula. *Natural Frequencies and Modes of Tapered Skew Plates.*International Journal of Mechanical Sciences,Vol 13, 1971, pp 935–944.

[18] S. M. Dugar'yan. *Theory of Shells and Plates* . Proceeding of The 4th All Union Conferance On Shell and Plates Held at Ereran 24–31 October 1962, Printed S.Monson Jurusalem.

[19] Soon. T. C. *A Non Linear Method Applied To Variable Thickness Plates.* AIAA Journal, Vol. 10, No 2, Feb. 1972,pp 148–154.

[20] Illoyd Hamilton Donnell. *Beams, Plates, and Shells.* Mc Graw Hill 1976.

[21] Petrina P. and Conway. H. D. *Deflection and Moment Data For Rectangular Plates of Variable Thickness.* Journal of Applied Mechanics, TRANS ASME, Series E, Vol. 39, Sept. 1972, pp 814–815.

[22] K. Washizu. *Variational Methods In Elasticity And Plasticity.* Third Edition Pergamon Press 1982.

[23] P. S. Nair and Durvasula. *Vibration of Skew Plates.* Journal of Sound and Vibration, Vol. 26, 1973, pp 1–19

[24] P. G. Lowe. *Basic Principles of Plates Theory.*Surrey University Press 1982.

[25] Bastin H., Delchanbre M.,and Janssens P. *Bending of Rectangular Plates of Linearly Varying Thickness.* ZAM, Vol. 23, No 1, Jan. 1972, pp. 35–42.

[26] David Mc Farland, Bert L. Smith Walter D. Bernhard. *Analysis of Plates.* Spartan Books New York. Washington 1972.

[27] P. S. Nair and Durvasula. *Vibration of Generally Orthotropic Skew Plates.* Journal of Acoustical Society Of America, Vol. 55, No 5, May. 1974, pp. 998–1002.

[28] Cook. Roberd. D. *Concepts and Applications For Finite Element Analysis.* 2nd Edition John Wiley & Sons. New York 1981.

[29] M. A Dokainish and K. Kumar. *Vibration of Orthotropic Parallelogramic Plates With variable thickness.* Vol. 11, No. 12, AIAA Journal, 1973, pp. 1618–1621.

[30] Leonard Meirovitch. *Analytical Methods In Vibration.* The Mac Millan Company 1967.

[31] M. Hetenyi. *Beams On Elastic Foundation.* Ann Arbor, The University Of Michigan Press.

[32] Chehil, D. S., and Dua, S. S. *Buckling of Rectangular Plates With General Variation In thickness.* Journal of Applied Mechanics, TRANS.ASME Series E., Paper. No. 73 APM–10.

[33] Lam K. K. D. *Static and Dynamic Analysis of Skew Sandwich Plates By Finite Element Method.* M. Engg. Report, University of Ottawa, 1982.

[34] S. R. Soni and K. Sankara Rao. *Vibration of Non Uniform Rectangular Plates: A Spline Technique Method of Solution.* Journal of Sound and Vibration, Vol. 35, No. 1, 1974, pp. 35–45.

[35] S. Durvasula and P. S. Nair. *Application of Partition Method to Vibration Problems of Plates.* Journal of Sound and Vibration, Vol. 37, No. 3, 1974, pp. 429–445.

[36] H. L. Cox. *The Buckling of Plates and Shells.* The Macmillan Company New York 1963.

[37] E. H. Mansfield Sc.D. *The Bending and Stretching of Plates.* Pergamon Press 1964.

[38] Chen S. S. H. *Bending and Vibration of Plates of Variable Thickness.* Journal of engineering For Industry, TRANS.ASME, Series B, Vol. 98, No.1, Feb. 1976,pp. 166–170.

[39] Ng S. F. and Chan M.Y. *Solution of Some Boundary Value Problems In Applied Mechanics By Collocation Least Square Method.* Computer Methods In Applied Mechanics and Ingg, Vol. 11, 1977, pp.137-150.

[40] R. W. Hornberck. *Numerical Methods.*Prentice–hall,Inc,Englewood Cliff, New Jersey 1975.

[41] P. A. A Laura and L. E Luisini. *On The Determination of The Fundamental Frequency of Vibrations of Clamped Rectangular Plates of Variable Thickness.* Journal of Mechanical Engineers, Paper No.77/WA/DE–1, 1977, pp.1–3.

[42] P. A. A Laura and C. Filipich. *Fundamental Frequency of Vibration of Stepped Thickness Plates.* Journal of Sound and Vibration, Vol. 50, 1977, pp. 157–158.

[43] M. M. Banerjee. *On The Vibration Of Skews Plates Of Variable Thickness.* Journal of Sound And Vibration, Vol. 63, No. 3, 1979, pp. 377–383.

[44] Y. W. Keung. *Application Of Vlasov's Method to Static, Dynamic and stability Problems In Plates structures.* M. A. Thesis, University of Ottawa 1987.

[45] U. S. Gupta and R. Lal. *Transverse Vibrations of Non Uniform Rectangular Plates On Elastic Foundation.* Journal of Sound and Vibration, Vol. 61, No. 1, 1978, pp. 127–133.

[46] A. W Al Khafaji and J. R. Tooley.*Numerical Methods In Engineering Practice.*Holt Richard & Winston Inc 1986.

[47] C. Massalas, K. soldatos, and G. Tzivanidis. *Vibration and Staility of Thin Elastic Plate Resting On a Non Linear Elastic Foundation When The Deformation Is Large.* Journal of Sound and Vibration, Vol. 67, No. 2, 1979, pp. 284–288.

[48] T. Sakata and T. Hayashi.*Natural Frequencies of Clamped Orthotropic Skew Plates.* Journal of Sound and Vibration, Vol. 81, No. 2, 1982, pp. 287–298.

[49] B. Velerga De Greco and P. A. A. Laura. *Vibration and Buckling of Circular Plates of Variable Thickness.* Journal of Acoustical Society of America, Vol. 72, No. 3, Sept. 1982, pp.856–858.

[50] J. R. Kuttler and V. G Sigillito. *Vibrational Frequencies of Clamped Plates of Variable thickness.* Journal of Sound and Vibration, Vol. 86, No. 2, 1983, pp.181–189.

[51] J. S. Tomar, D. C. Gupta and N. C. Jain. *Vibration of Nonhomogeneous Plates of Variable Thichness.* Journal of Acoustical Society of America, Vol. 72, No. 3, Sep 1982, pp. 851–855.

[52] K. Nagaya. *Vibration of Plate With Straight Line Boundaries.* Journal of Sound and Vibration, Vol. 68, No. 1, 1980, pp. 35–43.

[53] S. Murza and M. Bijlani.*Vibration of Triangular Plates of Variable Thickness.* Computer & structures, Vol. 21, No. 6, 1985, pp. 1129–1135.

[54] S. K. Chaudhuri. *Large Amplitude Free Vibration of Square Plate of Variable Thickness.* Journal of Sound and Vibration, Vol. 92, No. 1, 1984, pp. 143–147.

[55] Osumu Majima and K. Hayashi.*Elastic Buckling and Flexural Vibration of Variable Thickness Annular Plates Under Uniform in Plane

Forces Along The Inner and or Outer Edges. Bulletin JSME, Vol.28, No.243, Sept. 1985, pp. 1830–1835.

[56] S. F. Ng and B. Das. *Free Vibration and Buckling of Clamped Skew Sandwich Plates by the Galerkin Method.* Journal of Sound and Vibration, Vol. 107, No. 1, 1986, pp. 97–106.

[57] Jack Vinson.*Structural Mechanics, The Behavior of Plates and Shells.* John Wiley & Sons 1974.

[58] Conway, H. D. *A Levy Type Solution For Rectangular Plate Of Variable Thickness.* Journal of Applied Mechanics, TRANS.ASME, Vol. 80, July. 1958, pp. 297–298.

[59] Grossman, E. P.*Oscillations of the Tail Units of Aeroplanes.* Reports and Memorandum No 2095, Transactions of The Center Hydrodynamics Institute, No 186.

[60] Sa, T.A *The Treatment of Deflection, Vibration and Stability of Plates using the Collocation Least Square Method.* M. A. Sc thesis University of Ottawa, 1980.

[61] Odman S. T.A *Studies of Boundary Value Problems, Part 2, Characteristic Functions of Rectangulars Plates.* Proc Nr 24 Swedish Cement and Concrete Res. Inst., Roy Inst. Tech. Stockholm, 1955, pp 7-62.

Appendix A

GALERKIN METHOD

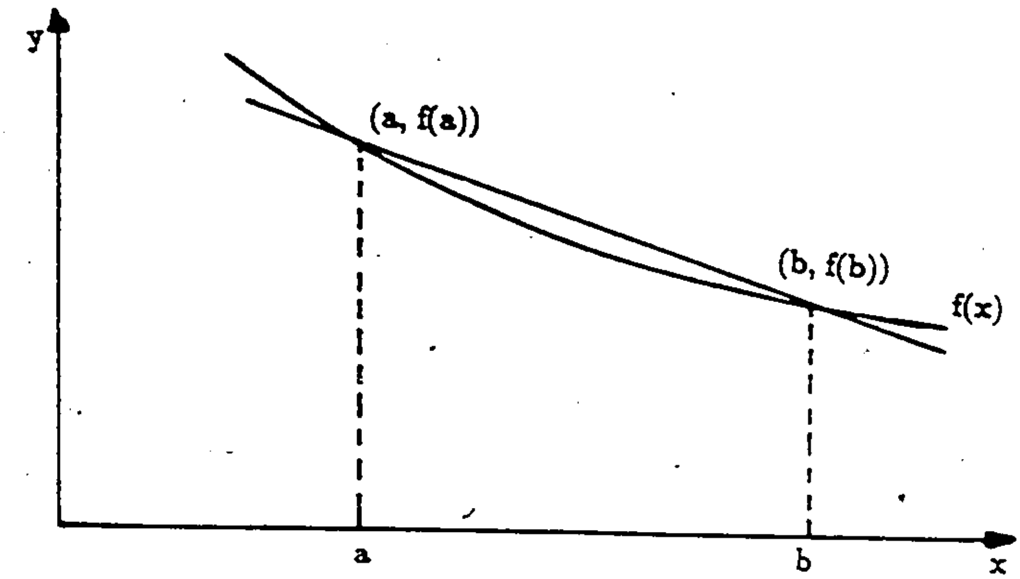


Figure A.1: Trapezoidal rule in one dimension

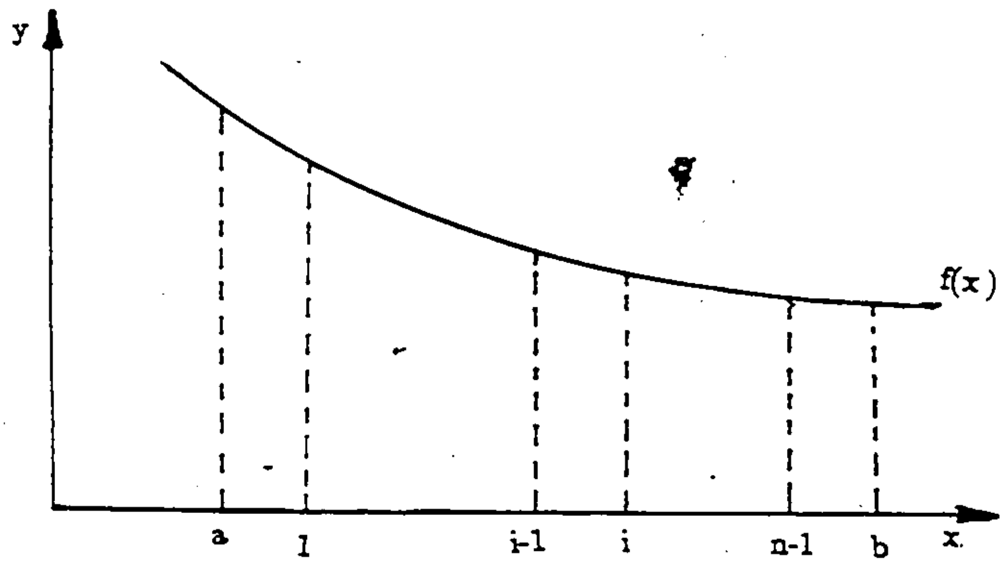


Figure A.2: Trapezoidal rule in one dimension, general case

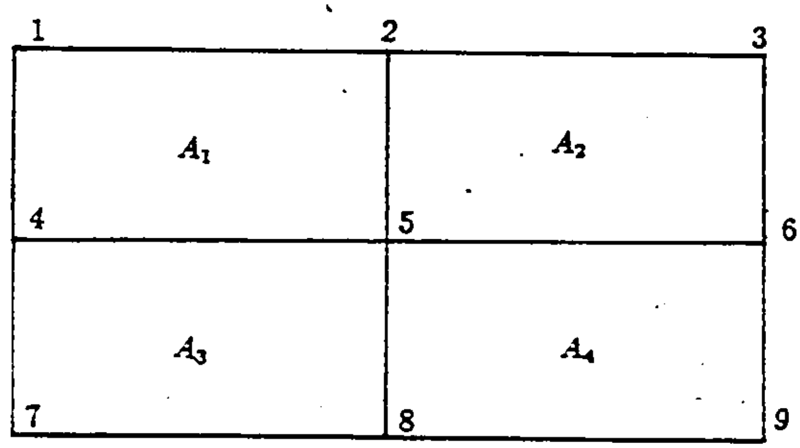


Figure A.3: Trapezoidal rule in two dimensions

Appendix B

DEFLECTION ANALYSIS

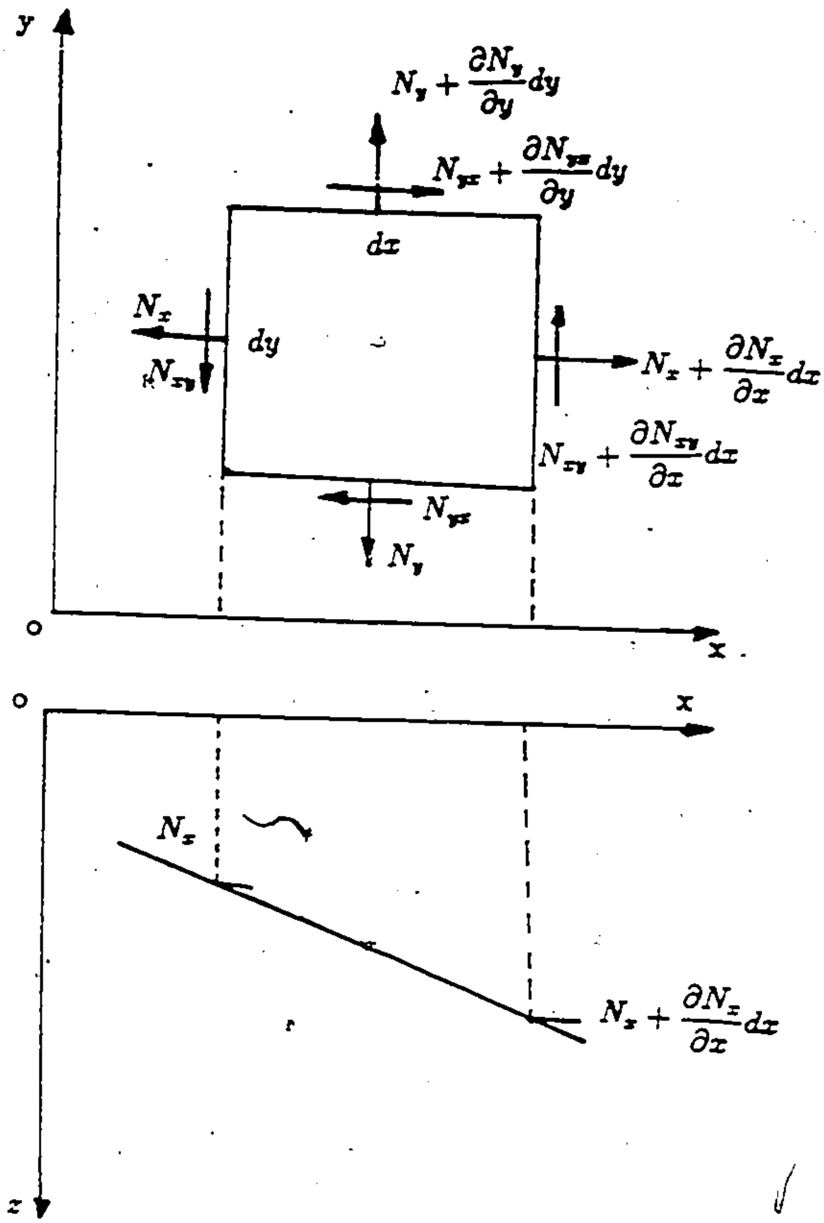


Figure B.1: Inplane forces on plate element

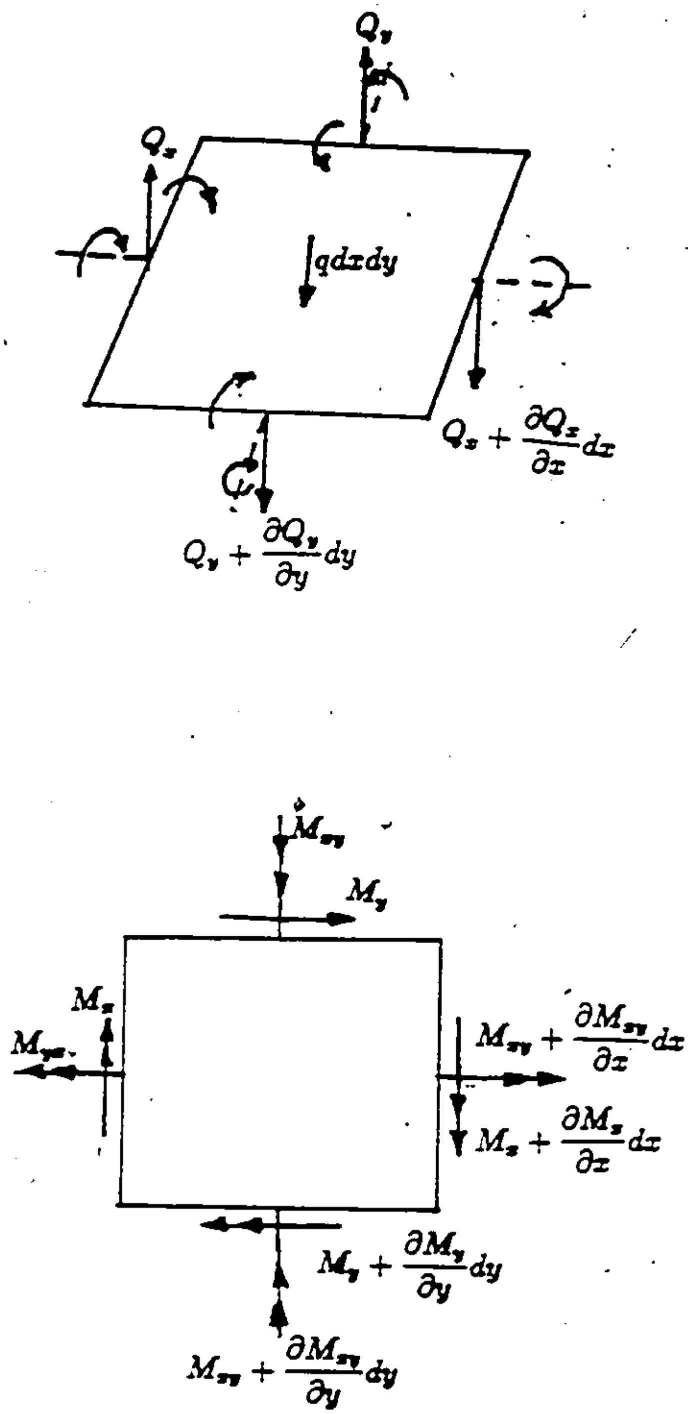
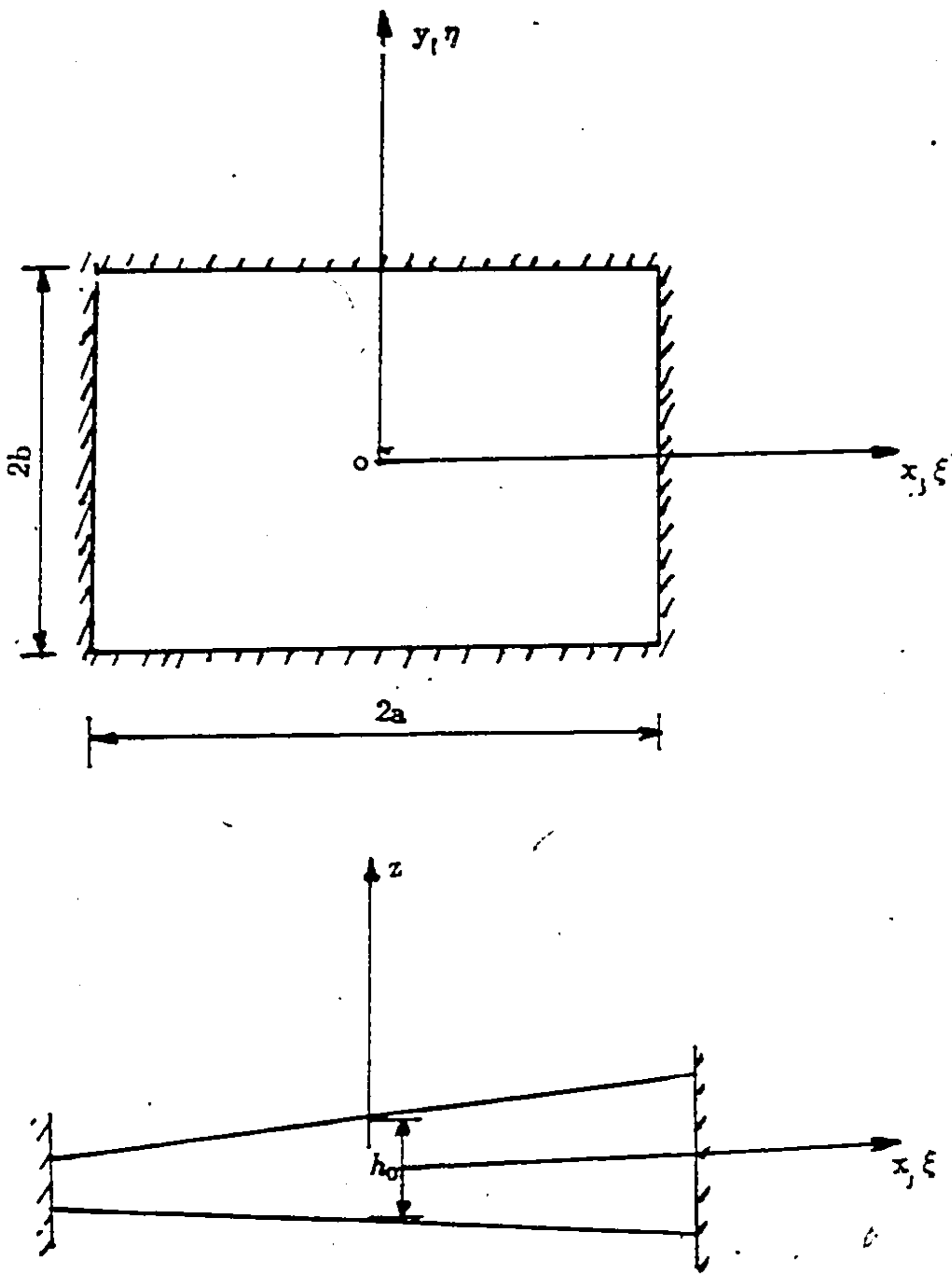


Figure B.2: Moment and shear forces on plate element



$h = h_0(1 + c\xi)$

Figure B.3: Rectangular coordinate system

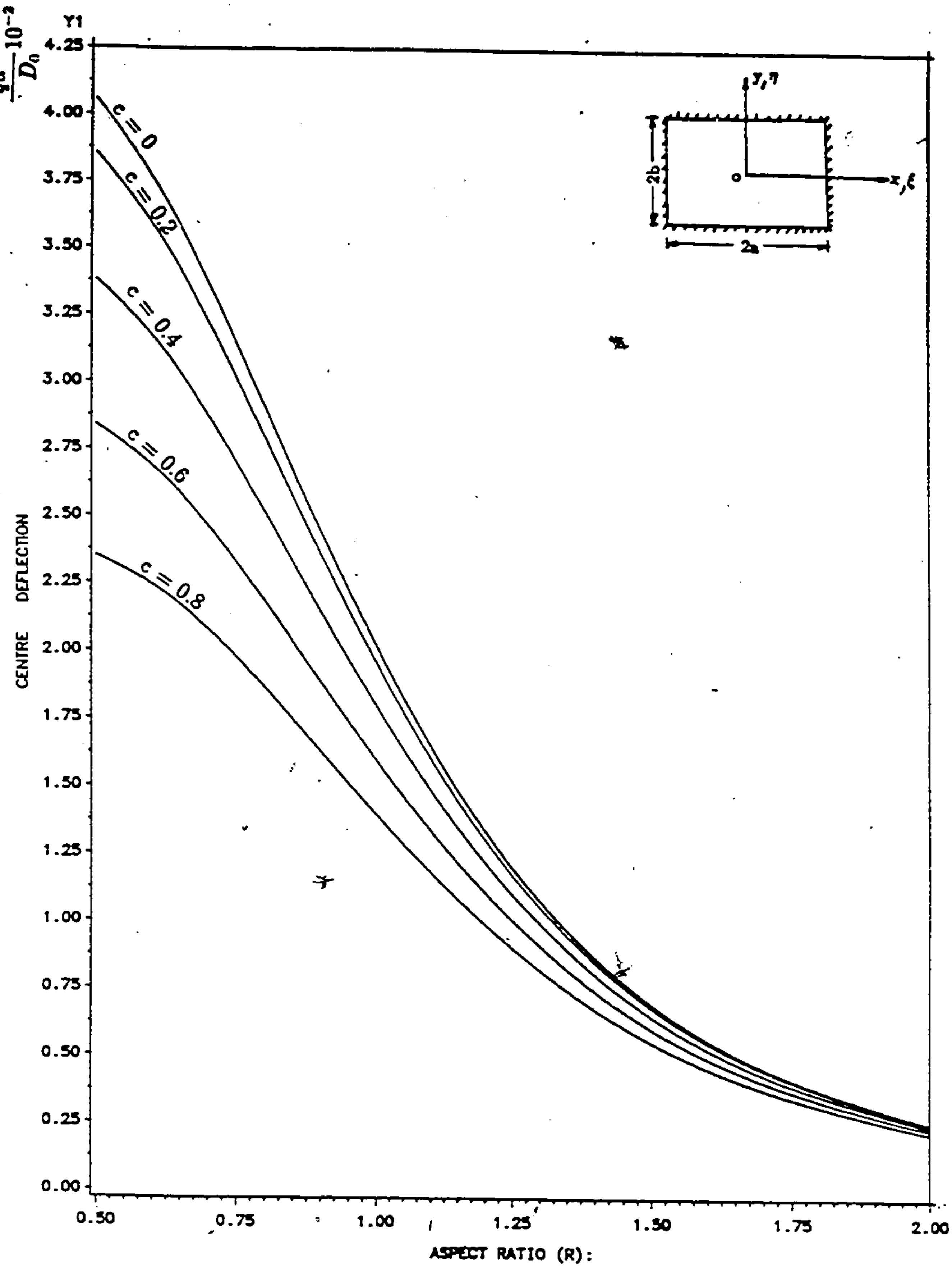


Figure B.4: Variation of central deflection with aspect ratio of clamped rectangular plates of variable thickness for various taper parameter c:

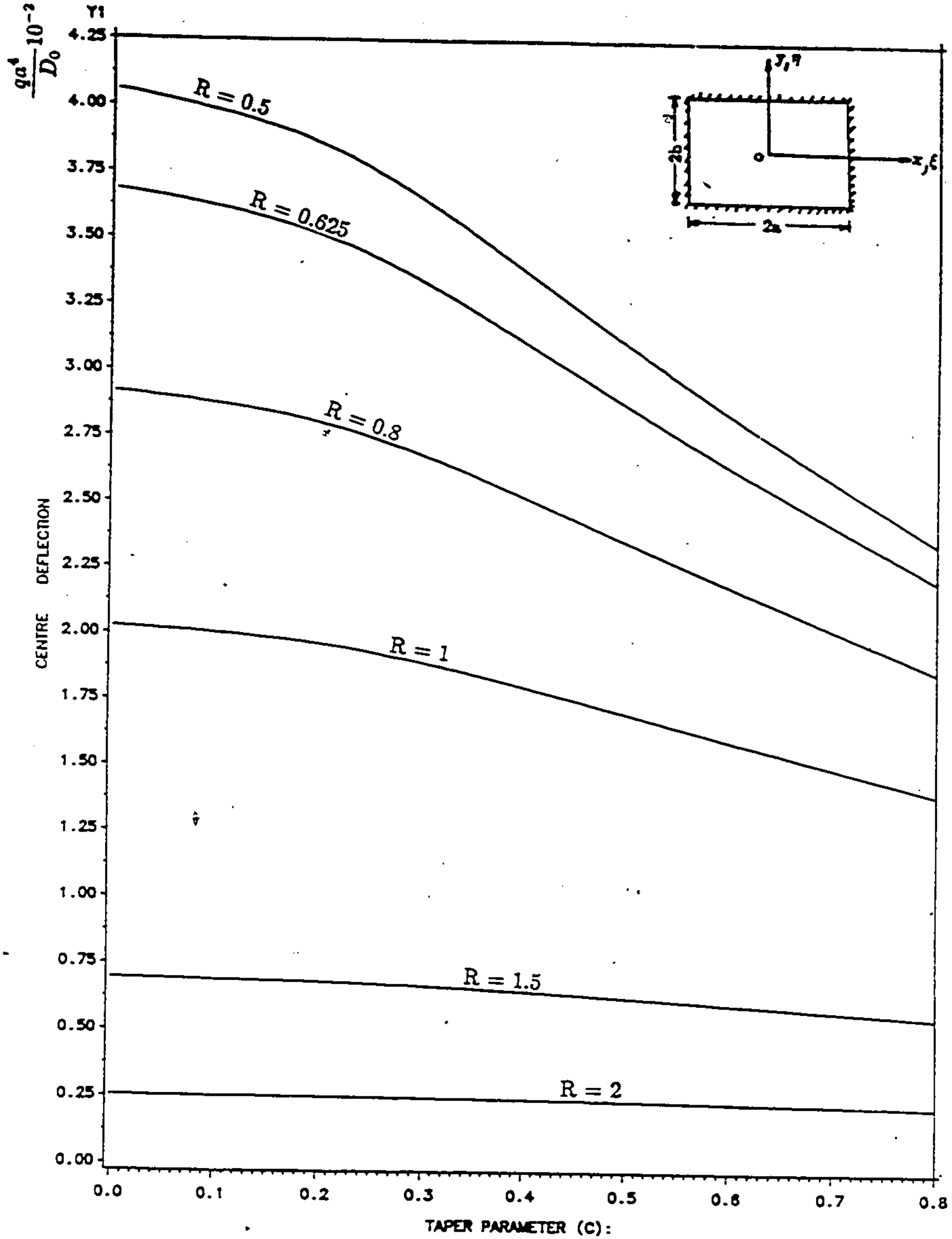


Figure B.5: Variation of central deflection with taper parameter (c) of clamped plates of variable thickness for various aspect ratio

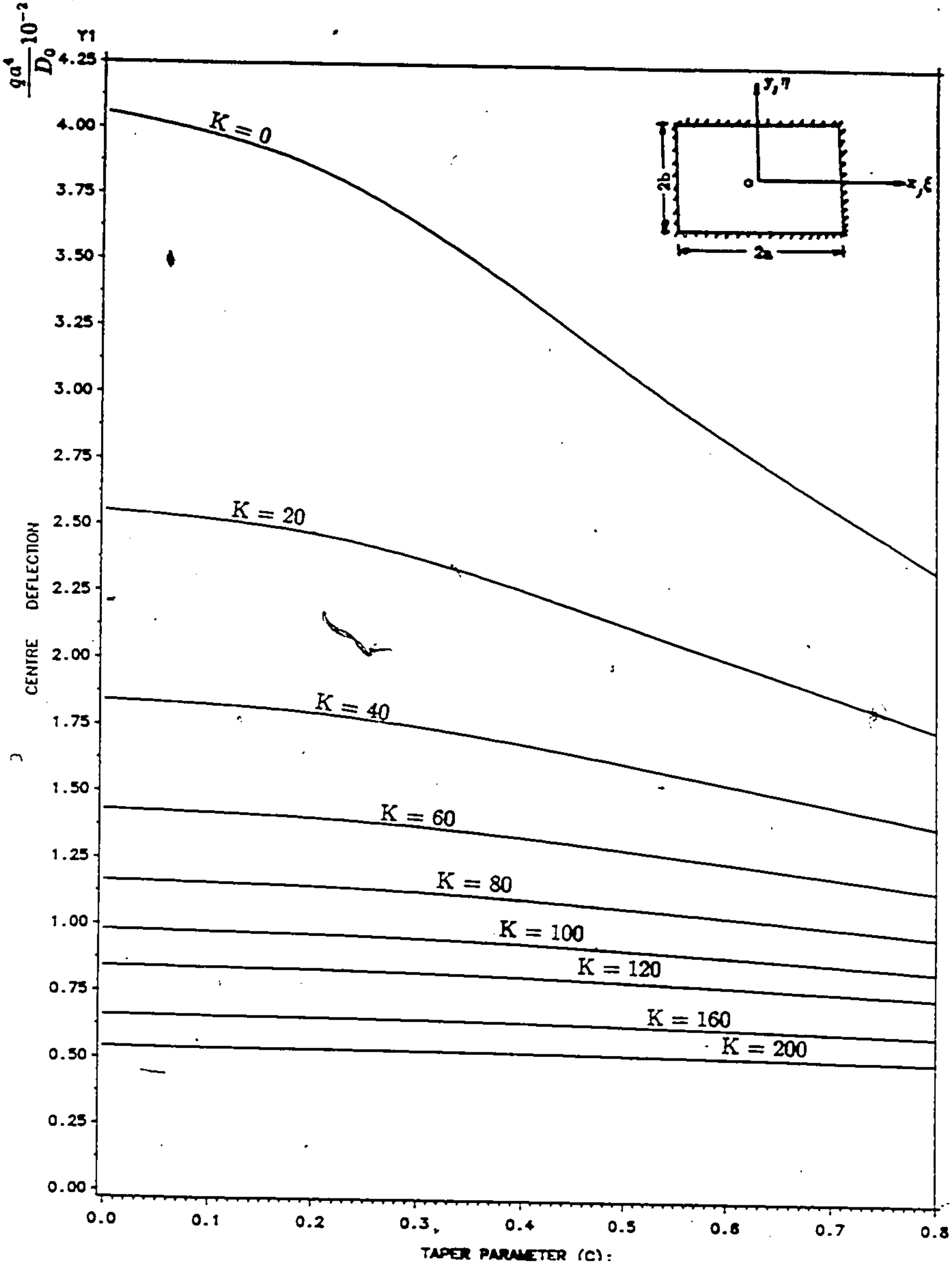


Figure B.6: Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 0.5

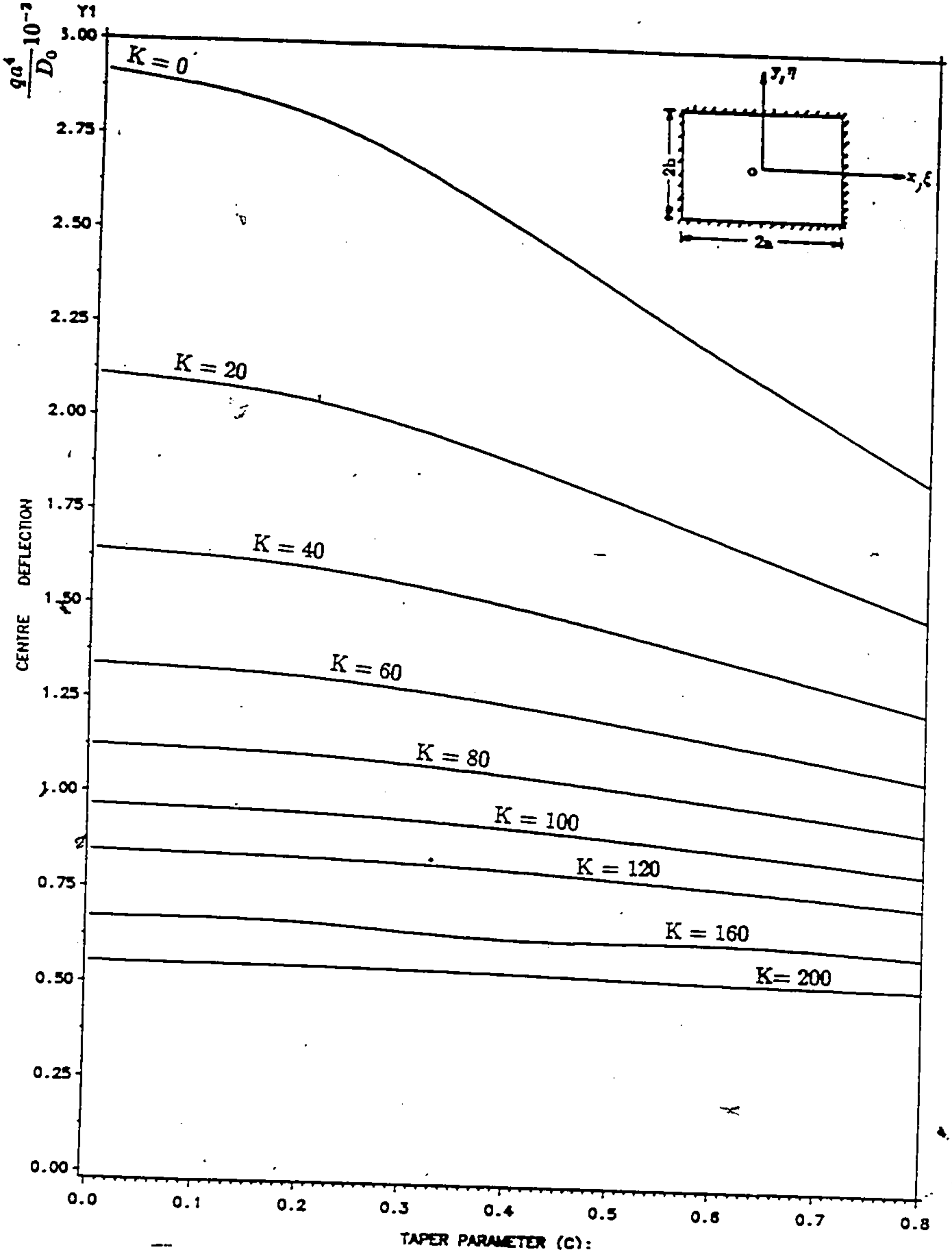


Figure B.7: Variation of central deflection with taper parameter (c) and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 0.8

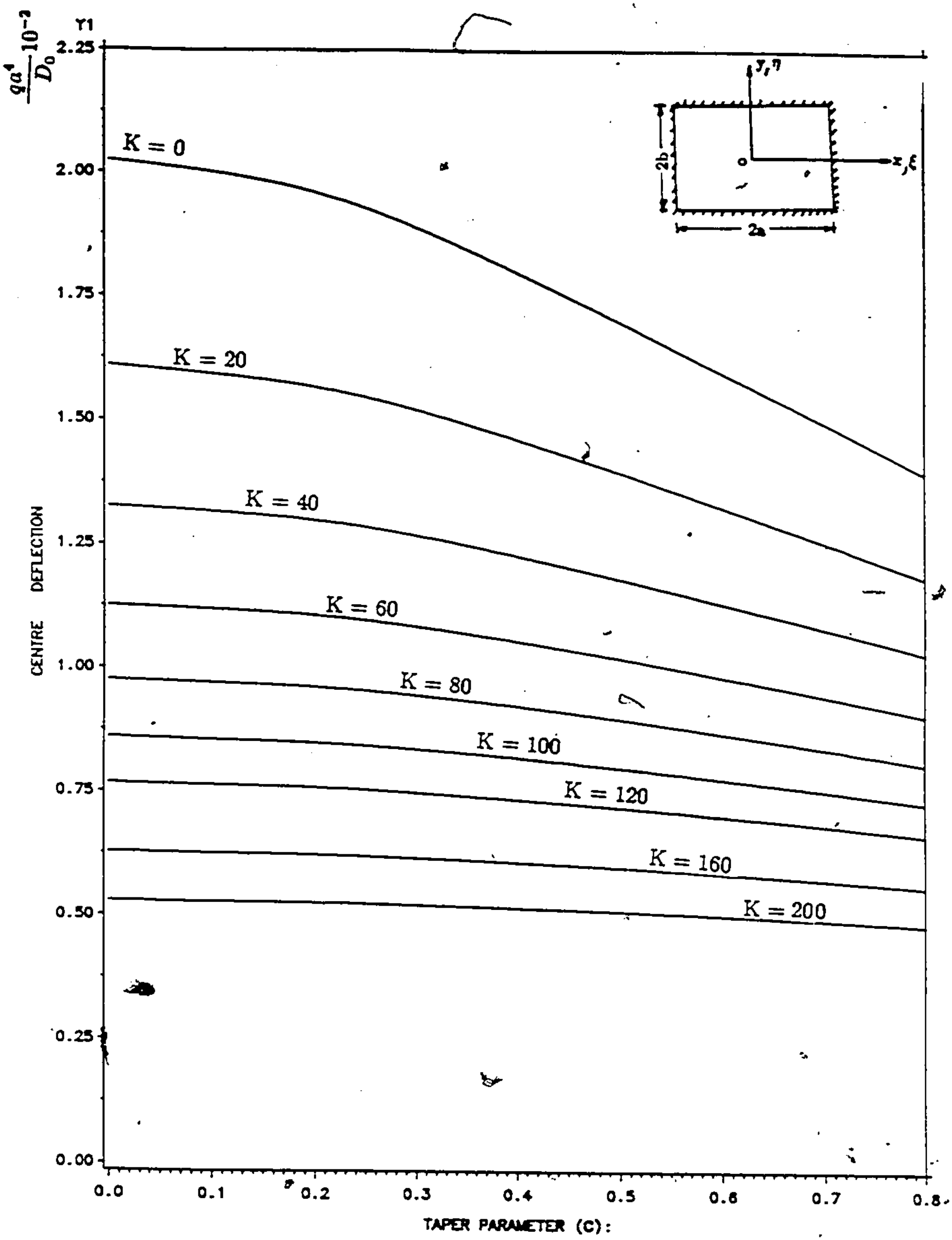


Figure B.8: Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 1

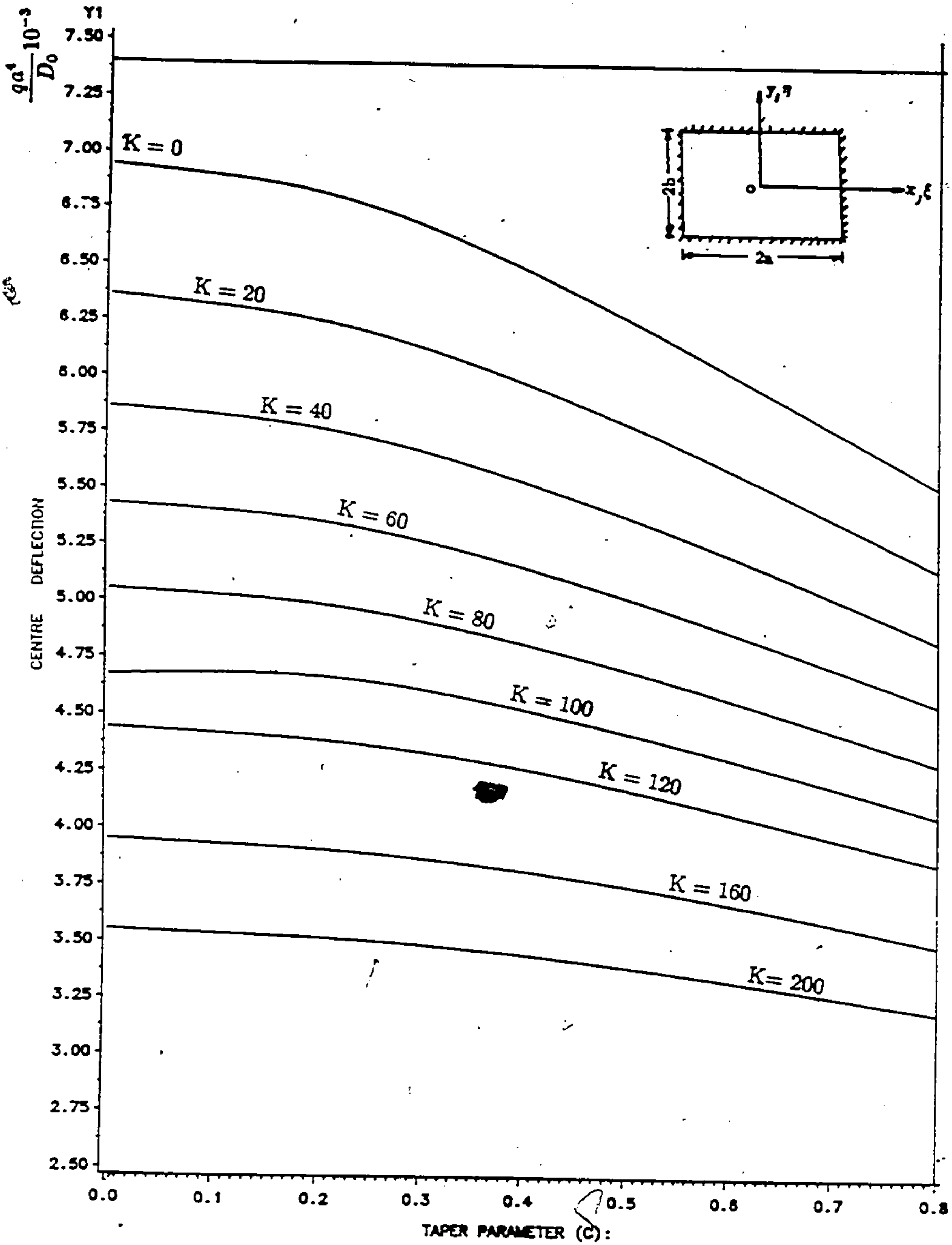


Figure B.9: Variation of central deflection with taper parameter (c)and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 1.5

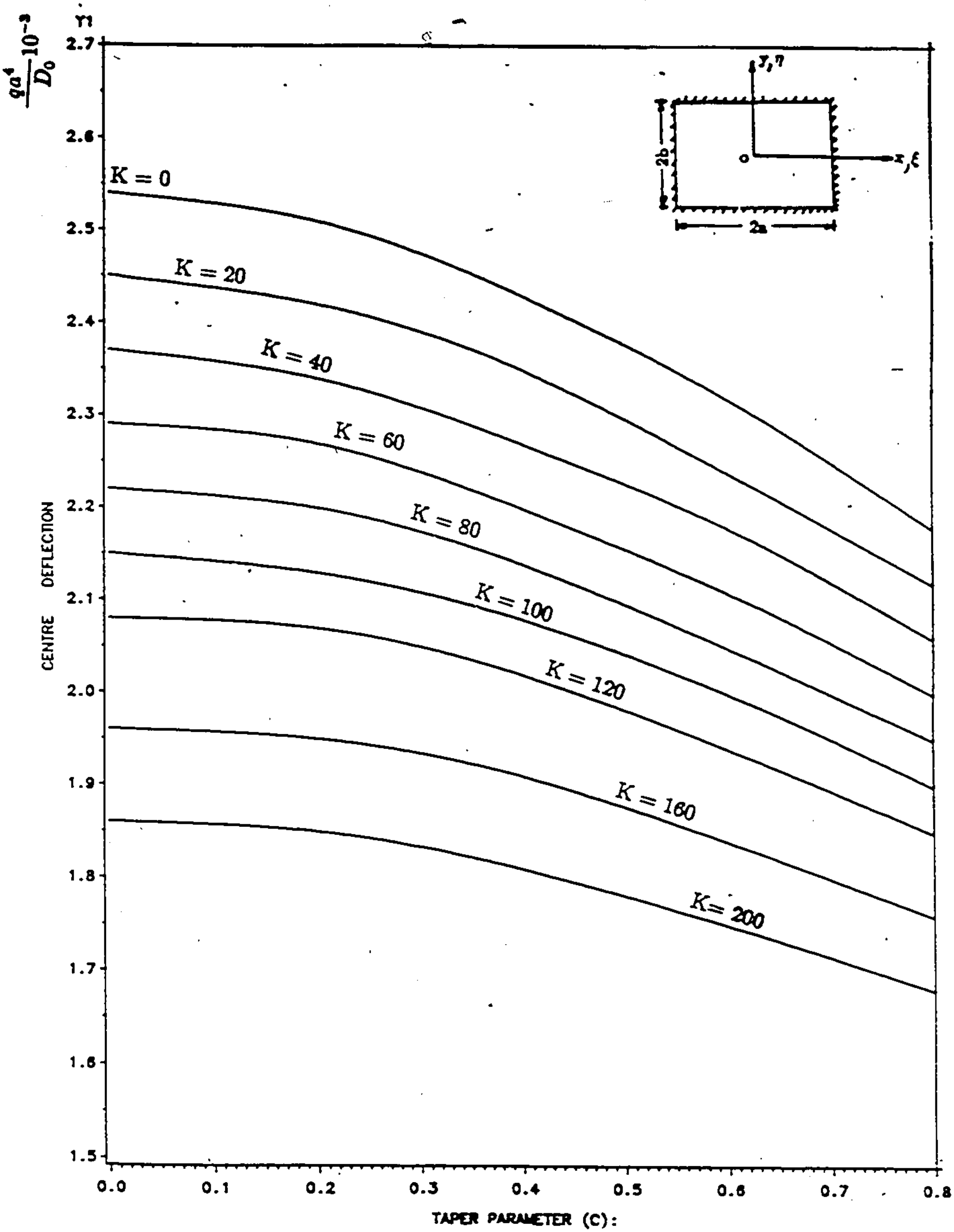


Figure B.10: Variation of central deflection with taper parameter (c) and foundation modulus for clamped rectangular plates of variable thickness with aspect ratio R = 2

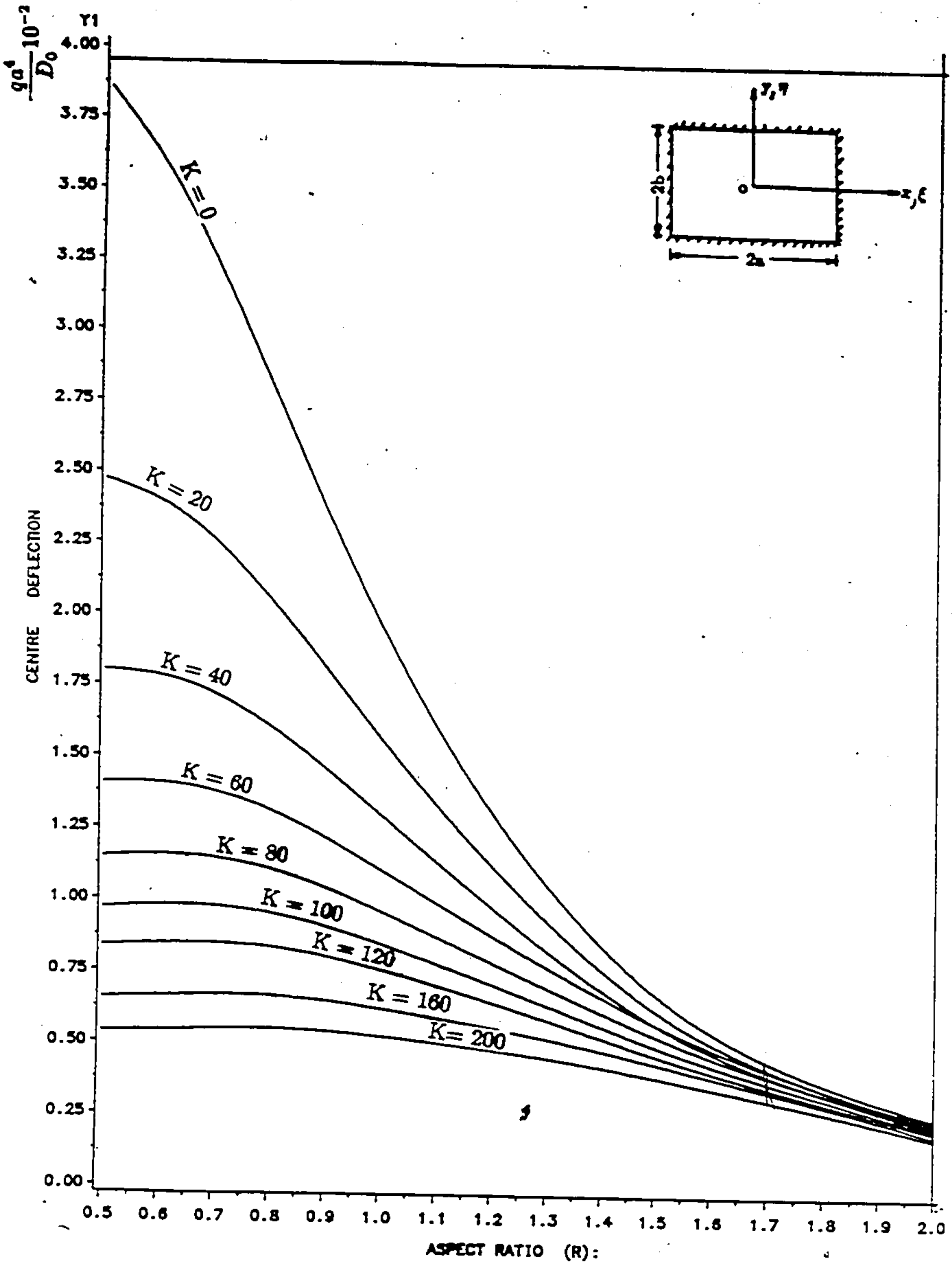


Figure B.11: Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.2

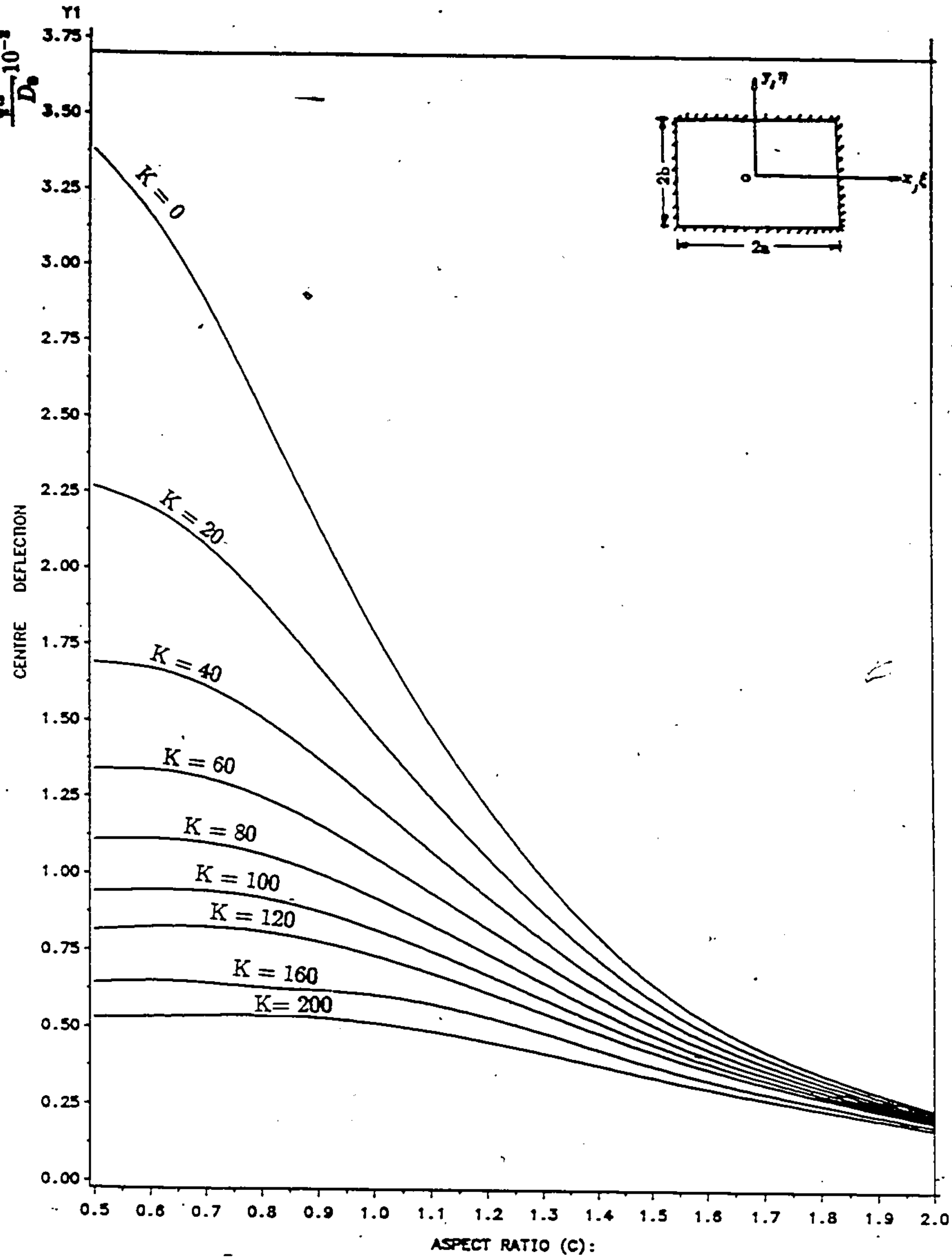


Figure B.12: Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.4

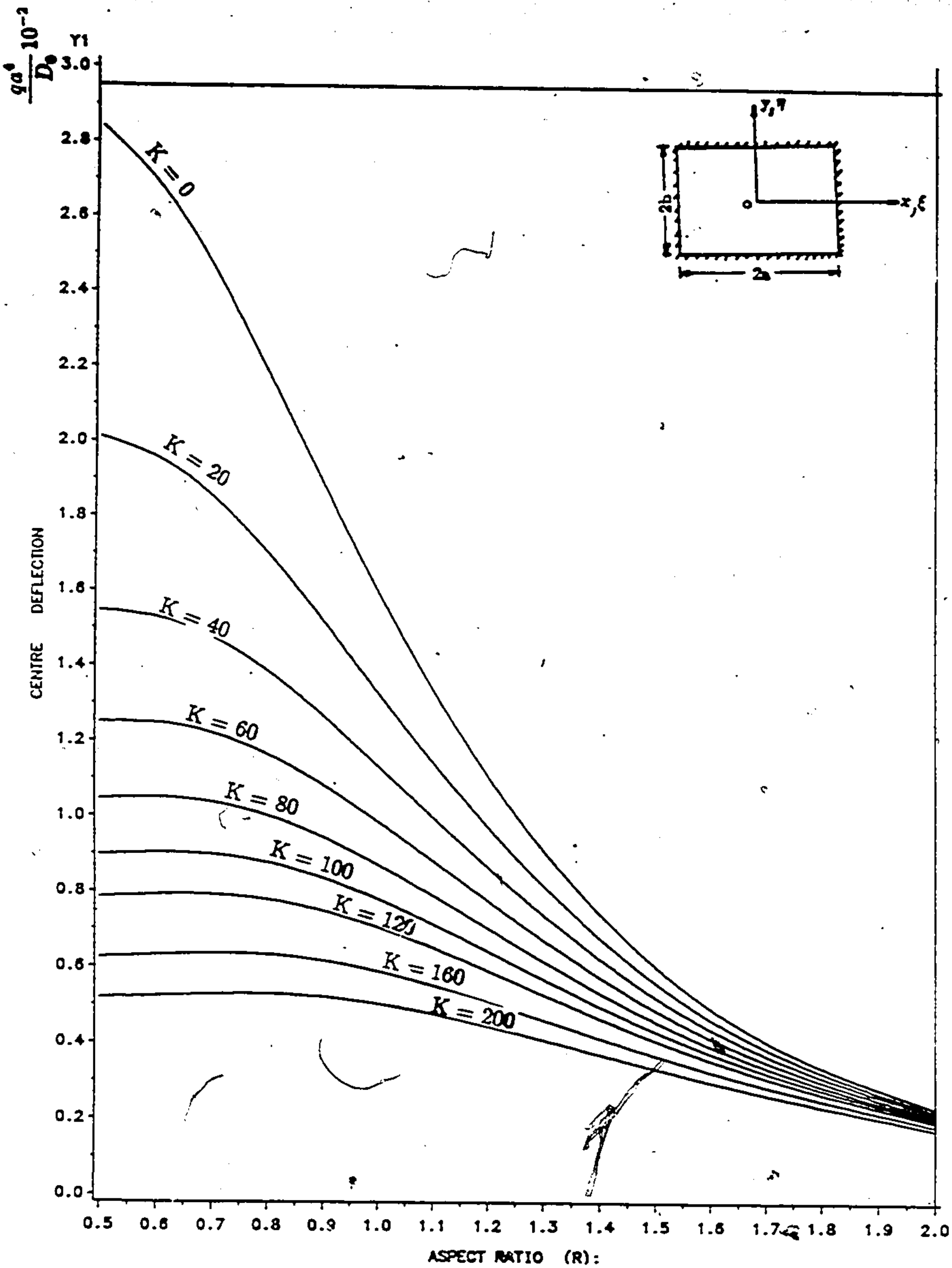


Figure B.13: Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.6

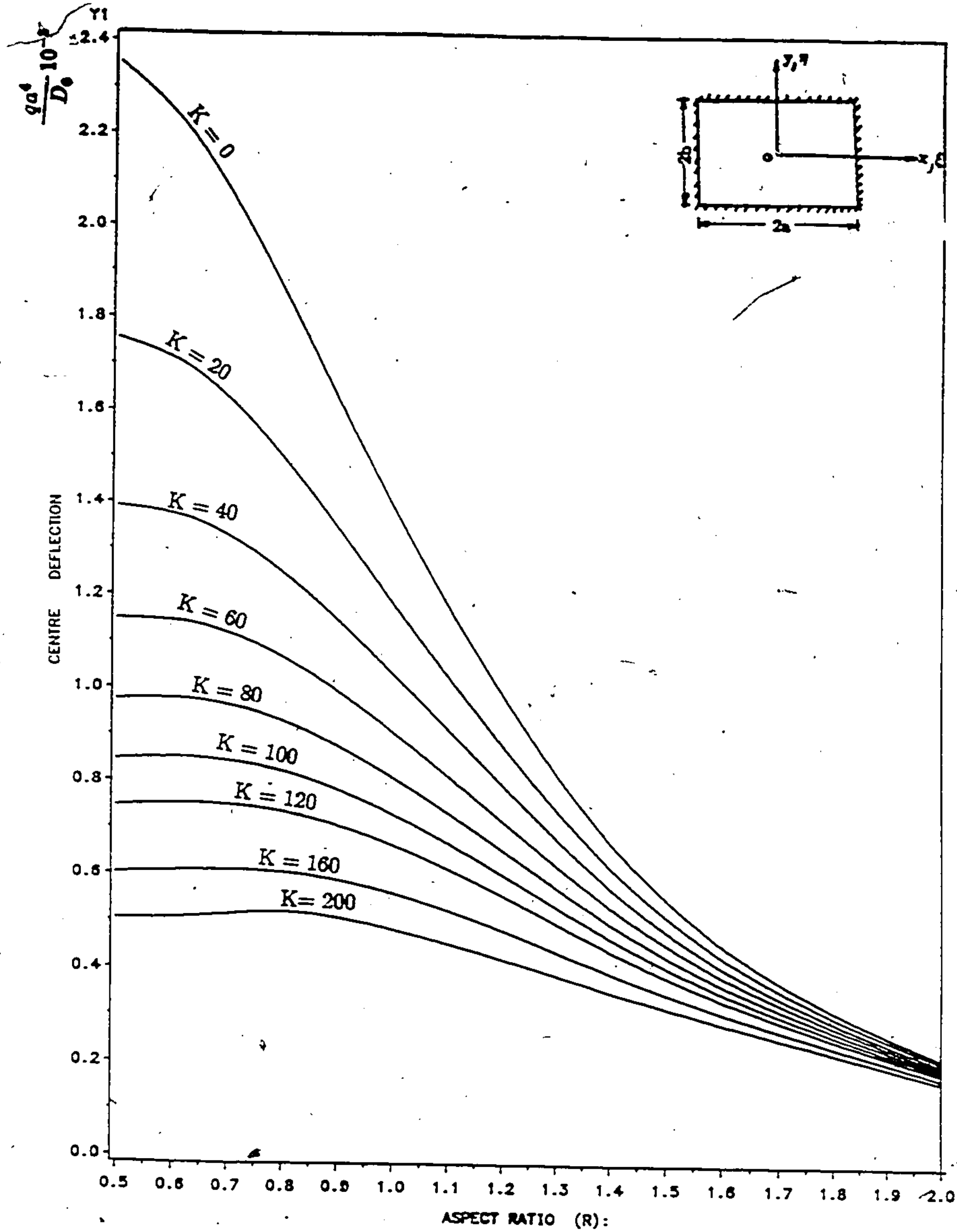


Figure B.14: Variation of central deflection with aspect ratio and foundation modulus for clamped rectangular plates of variable thickness with taper parameter c = 0.8

	TAPER PARAMETER C				
	0.0	0.2	0.4	0.6	0.8
1 term solution	0.02082	0.02055	0.01866	0.01618	0.01365
2 terms solution	0.02073	0.02010	0.01848	0.01640	0.01428
3 terms solution	0.02020	0.01957	0.01798	0.01593	0.01383
4 terms solution	0.02024	0.01960	0.01800	0.01594	0.01384
5 terms solution	0.02424	0.01960	0.01800	0.01800	0.01397
6 terms solution	0.02025	0.01961	0.01801	0.01600	0.01398
Ref[39]	0.020201				
Ref[43]	0.02062	0.01986	0.01785	0:01530	0.01274

Table B.1: Comparison of center deflection α of rectangular plates of variable thickness for R = a/b = 1

$$W = \alpha \frac{qa^4}{D_0}$$

K	R = a/b = 1		R = a/b = 0.75		R = a/b = 0.5	
	Present	Ref[39]	Present	Ref[39]	present	Ref[39]
0	0.020250	0.020201	0.031480	0.031412	0.040570	0.040545
20	0.016090	0.016033	0.022230	0.022201	0.025530	0.025525
40	0.013260	0.013245	0.017050	0.017036	0.018420	0.018411
60	0.011260	0.011251	0.013750	0.013741	0.014310	0.014307
80	0.009760	0.009756	0.011470	0.011463	0.011650	0.011656
100	0.008600	0.008594	0.009800	0.009797	0.009800	0.009812
120	0.007670	0.007665	0.008540	0.008530	0.008400	0.008459
140	0.006910	0.006903	0.007540	0.007534	0.007410	0.007427
160	0.006280	0.006377	0.006740	0.006734	0.006590	0.006616
180	0.005750	0.005745	0.006090	0.006076	0.005900	0.005963
200	0.005290	0.005291	0.005540	0.005528	0.005400	0.005425

Table B.2: Variation of max small deflection coeff α of clamped rectangular plate of variable thickness with dimensionless foundation modulus K for taper parameter c = 0 $\quad W = \alpha \frac{qa^4}{D_0}$

Appendix C

BUCKLING ANALYSIS

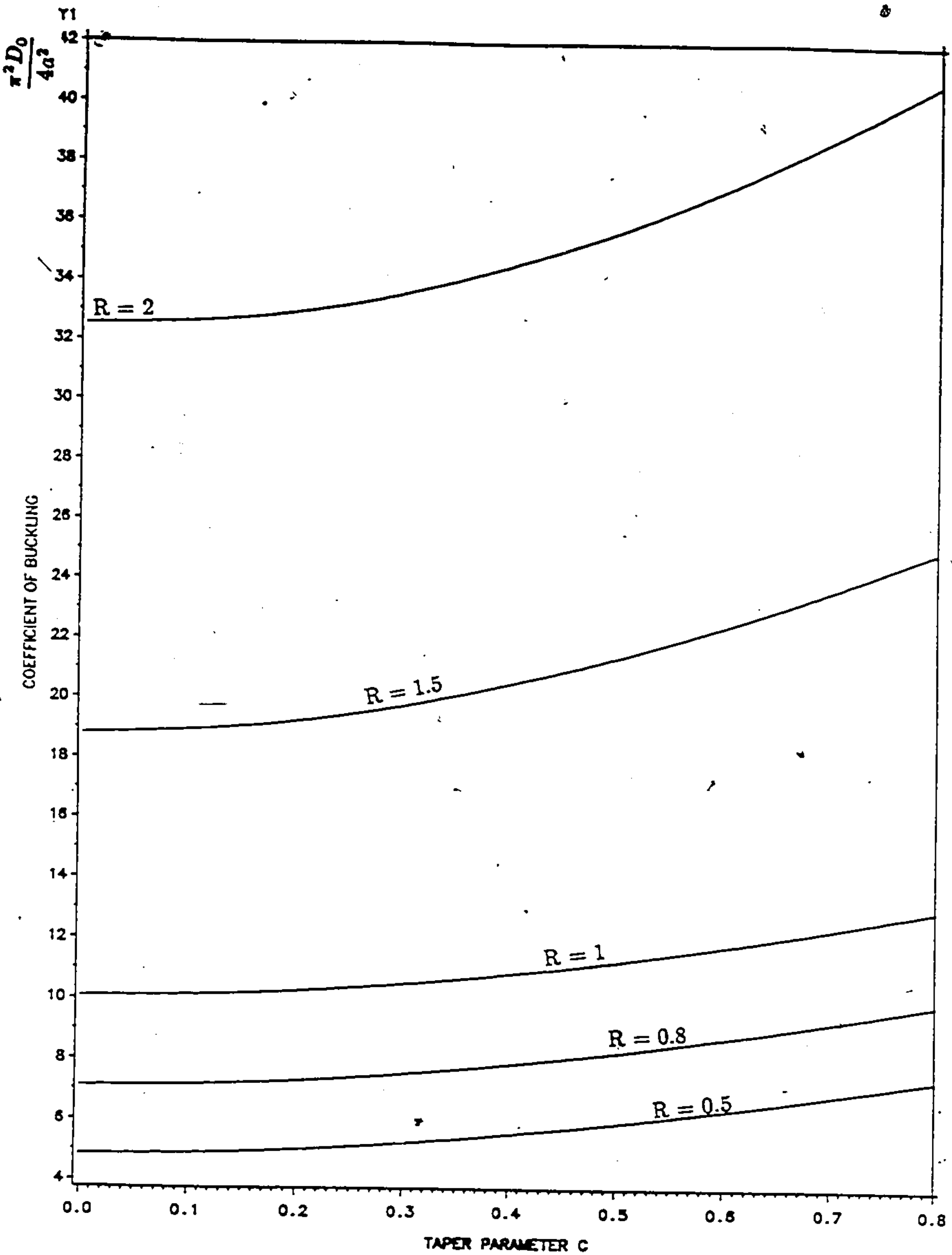


Figure C.1: Variation of buckling load with taper parameter for various aspect ratios R for clamped rectangular plates of variable thickness for $N_y = 0$.

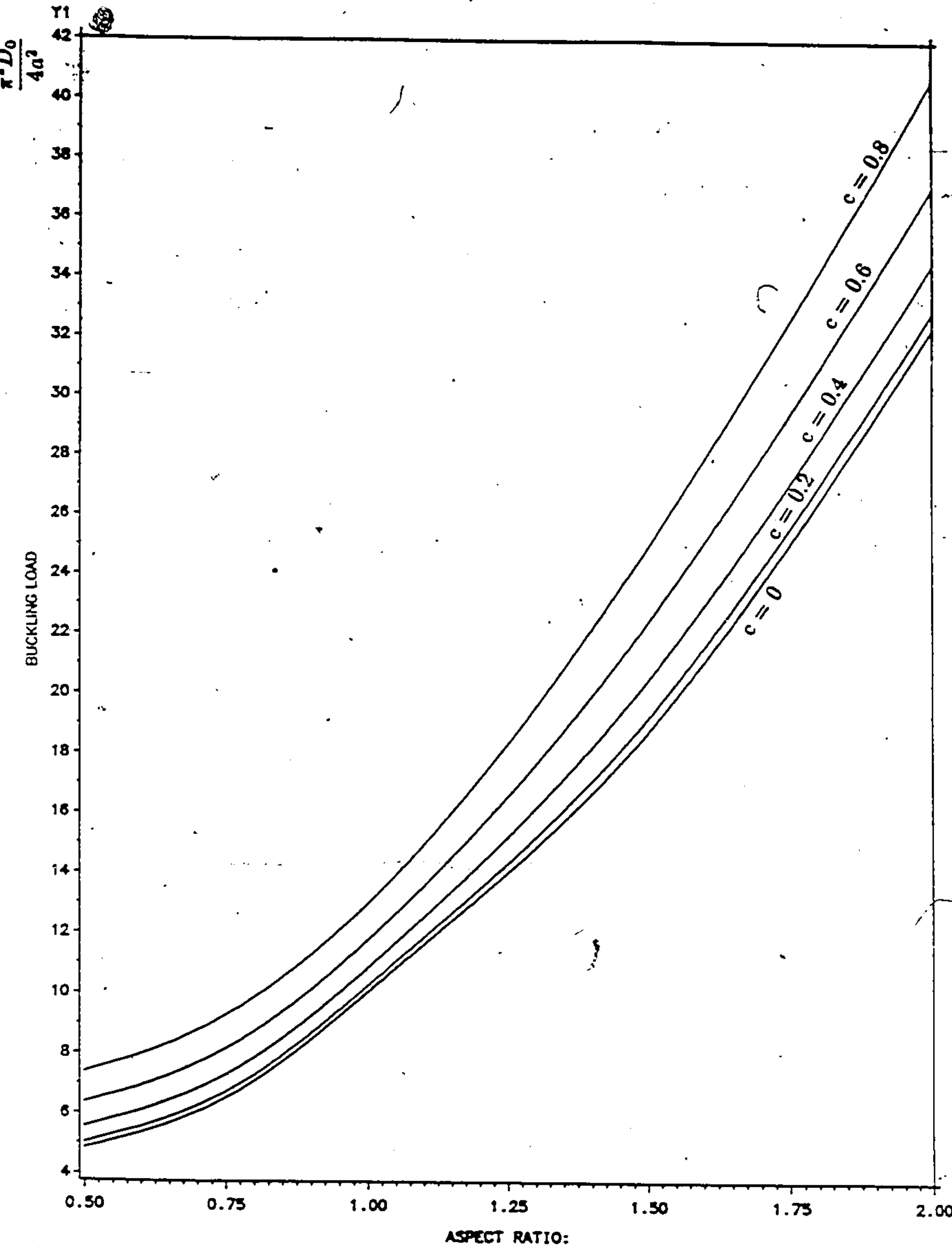


Figure C.2: Variation of buckling load with aspect ratio R for various taper parameter for clamped rectangular plates of variable thickness for $N_y = 0$.

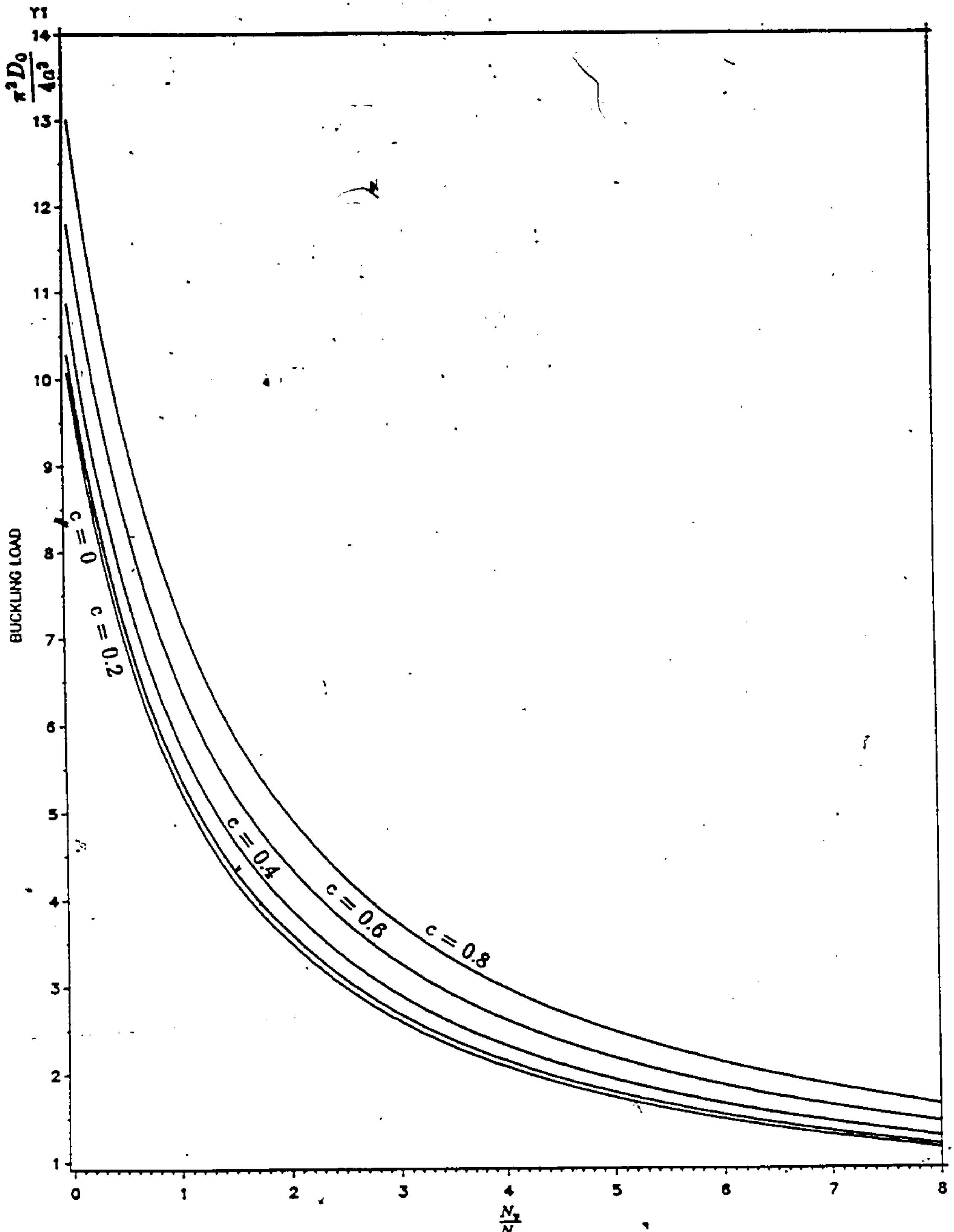


Figure C.3: Variation of buckling load with aspect load $\frac{N_y}{N_x}$ for various taper parameter c for clamped square plate of variable thickness

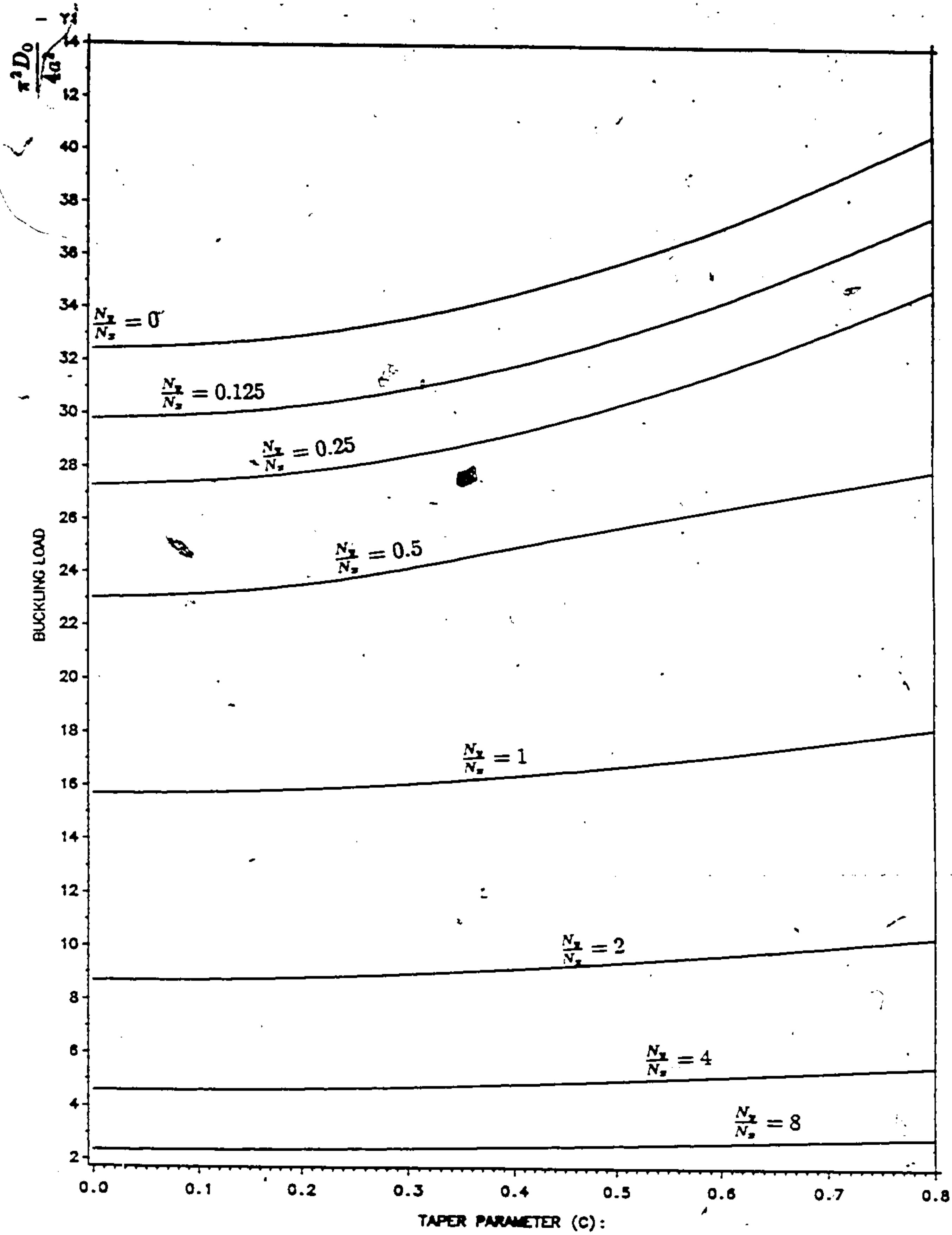


Figure C.4: Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 2

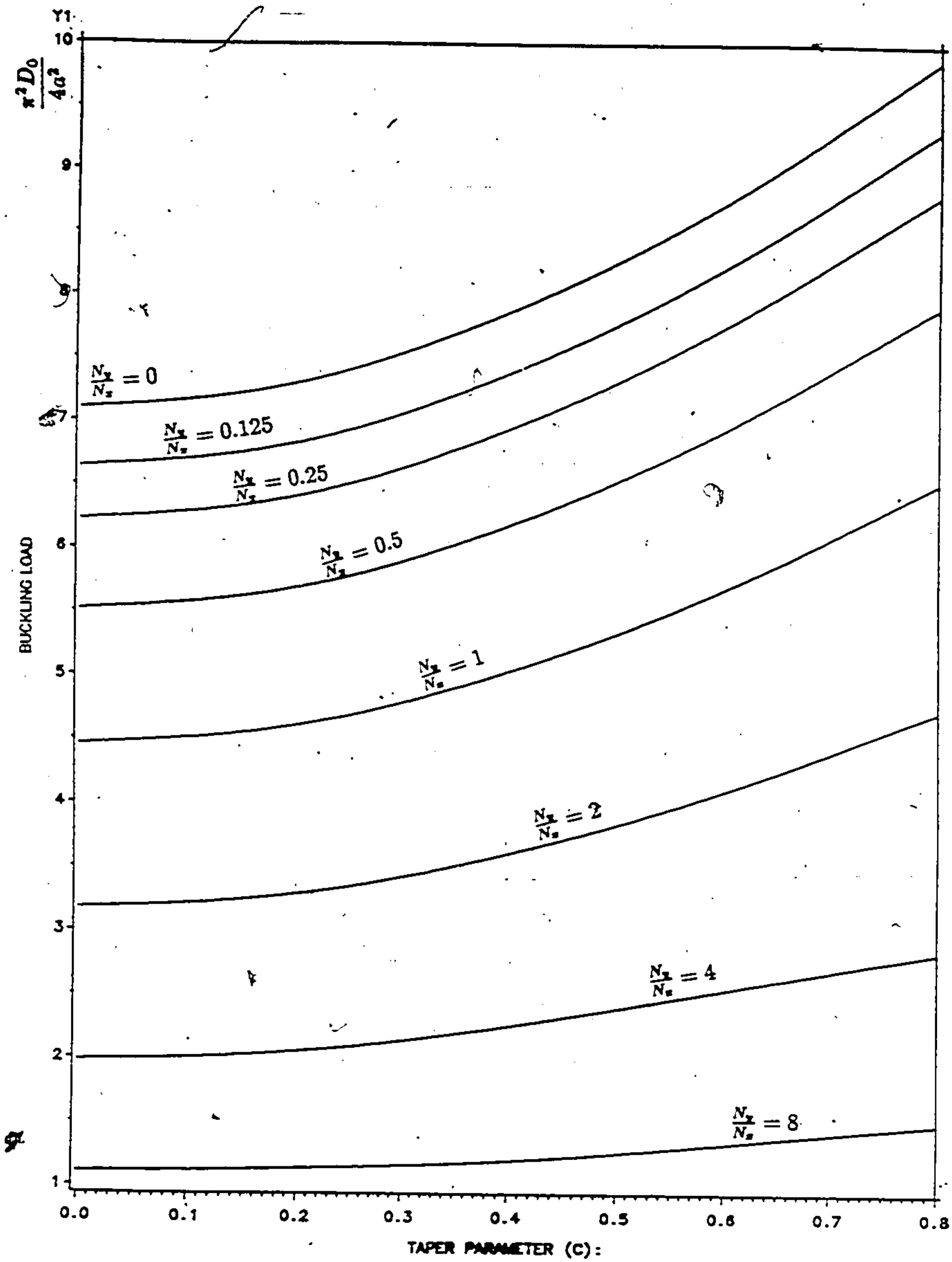


Figure C.5: Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 0.8

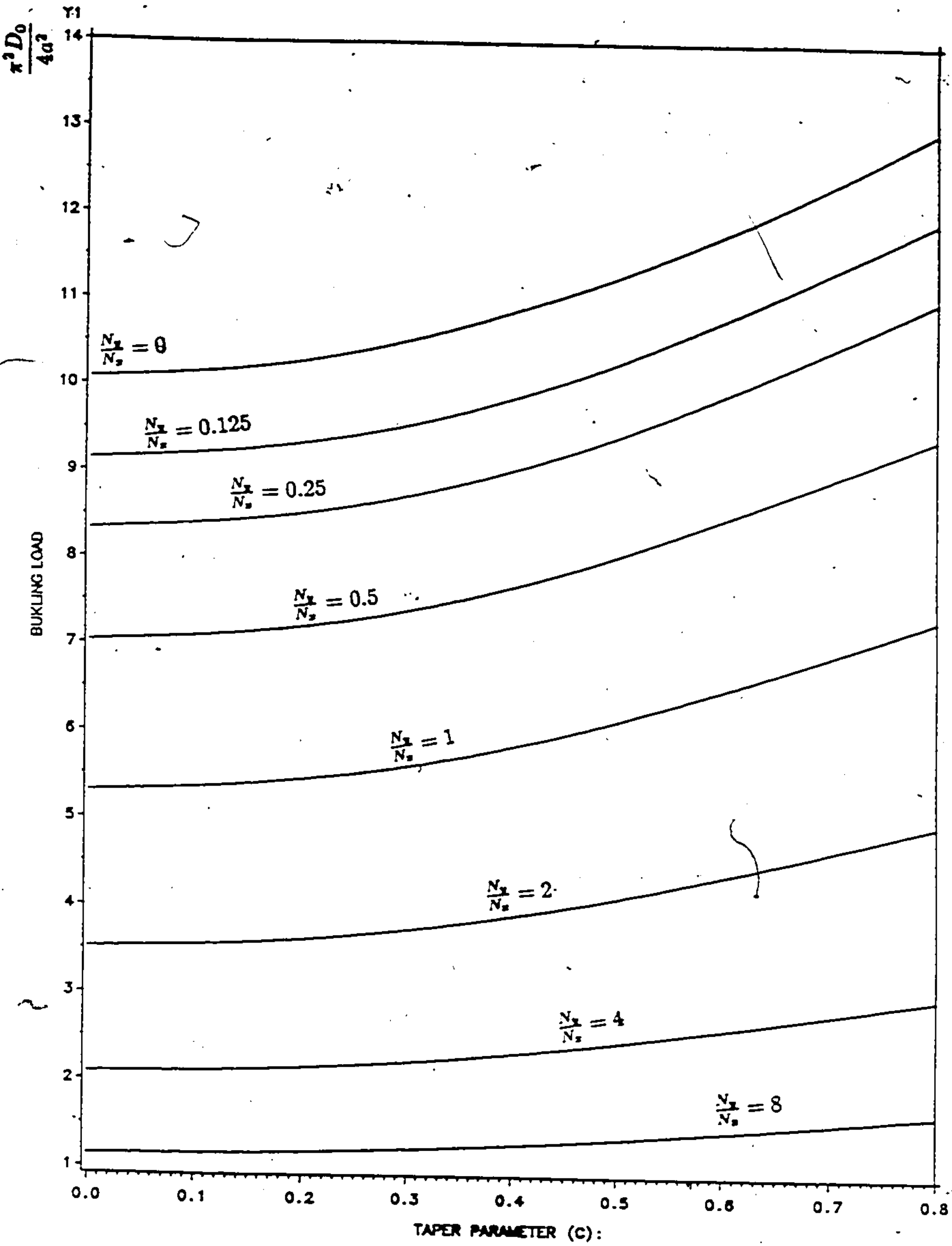


Figure C.6: Variation of buckling load with taper parameter for various load ratio for clamped rectangular plates of variable thickness for R = 1

		TAPER PARAMETER C				
		0.0	0.2	0.4	0.6	0.8
R = 0.5	1 term solution	4.10344	4.33335	5.02310	6.17262	7.78211
	2 terms solution	3.94725	4.11657	4.59739	5.34697	6.34630
	3 terms solution	3.92654	4.09405	4.56967	5.31109	6.29948
	4 terms solution	3.92561	4.09349	4.56966	5.31007	6.29291
	5 terms solution	3.92443	4.09045	4.55419	5.25047	6.13317
	6 terms solution	3.92405	4.09003	4.55361	5.24960	6.13181
	Sa Ref [60]	3.962				
R = 1	1 term solution	5.47128	5.66226	6.23525	7.19023	8.52715
	2 terms solution	5.38812	5.54031	5.97505	6.65821	7.57466
	3 terms solution	5.31273	5.46330	5.89337	6.56891	7.47457
	4 terms solution	5.30576	5.45719	5.88938	6.56720	7.47457
	5 terms solution	5.30516	5.45531	5.87800	6.52034	7.34437
	6 terms solution	5.30457	5.45471	5.87743	6.51978	7.34382
	Sa Ref [60]	5.333				
R = 2	1 term solution	16.41378	16.67027	17.43985	18.72248	21.51816
	2 terms solution	16.29578	16.49290	17.06182	17.97748	19.23604
	3 terms solution	15.70612	15.90434	16.47308	17.38171	18.62427
	4 terms solution	15.70243	15.90213	16.47293	17.38069	18.61797
	5 terms solution	15.70089	15.89570	16.43341	17.24188	18.28896
	6 terms solution	15.69617	15.89104	16.42894	17.23766	18.28498
	Sa Ref [60]	15.769				

Table C.1: Convergence table for the buckling load coefficient C_r of clamped rectangular plates of variable thickness with various taper parameters, for various aspect ratios and $\frac{N_x}{N_r} = 1$ $N_x = C_r \frac{\pi^2 D_0}{4a^2}$

$\frac{N_y}{N_x}$	R= a/b = 1		R= a/b = 1.5		R = a/b =2	
	Present solution	Sa Ref [60]	Present solution	Sa Ref [60]	Present solution	Sa Ref [60]
0	10.077	9.957	18.796	18.802	32.434	31.462
0.125	9.139	9.109	17.648	17.679	29.802	29.533
0.25	8.331	8.339	16.610	16.652	27.292	27.236
0.5	7.032	7.063	13.954	13.932	23.0276	23.118
1	5.305	5.333	9.275	9.326	15.696	15.769
2	3.516	3.532	5.461	5.478	8.703	8.721
4	2.082	2.085	2.984	2.987	4.565	4.584
8	1.142	1.139	1.565	1.562	2.354	2.351

Table C.2: Comparison of buckling load coefficient C_r for clamped rectangular homogeneous plate (c = 0)with aspect ratio

$$N_x = \frac{C_r \pi^2 D_0}{4a^2}$$

Appendix D

VIBRATION ANALYSIS

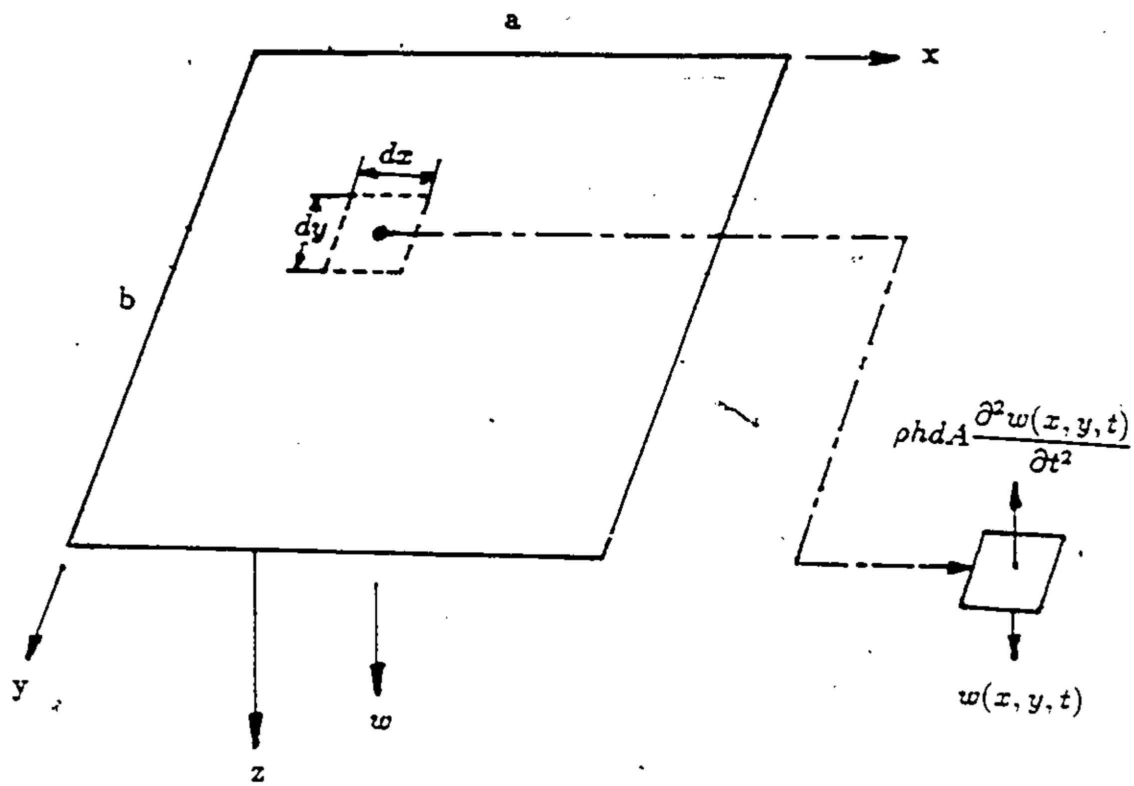


Figure D.1: Rectangular plate of variable thickness element subjected to an inertia force

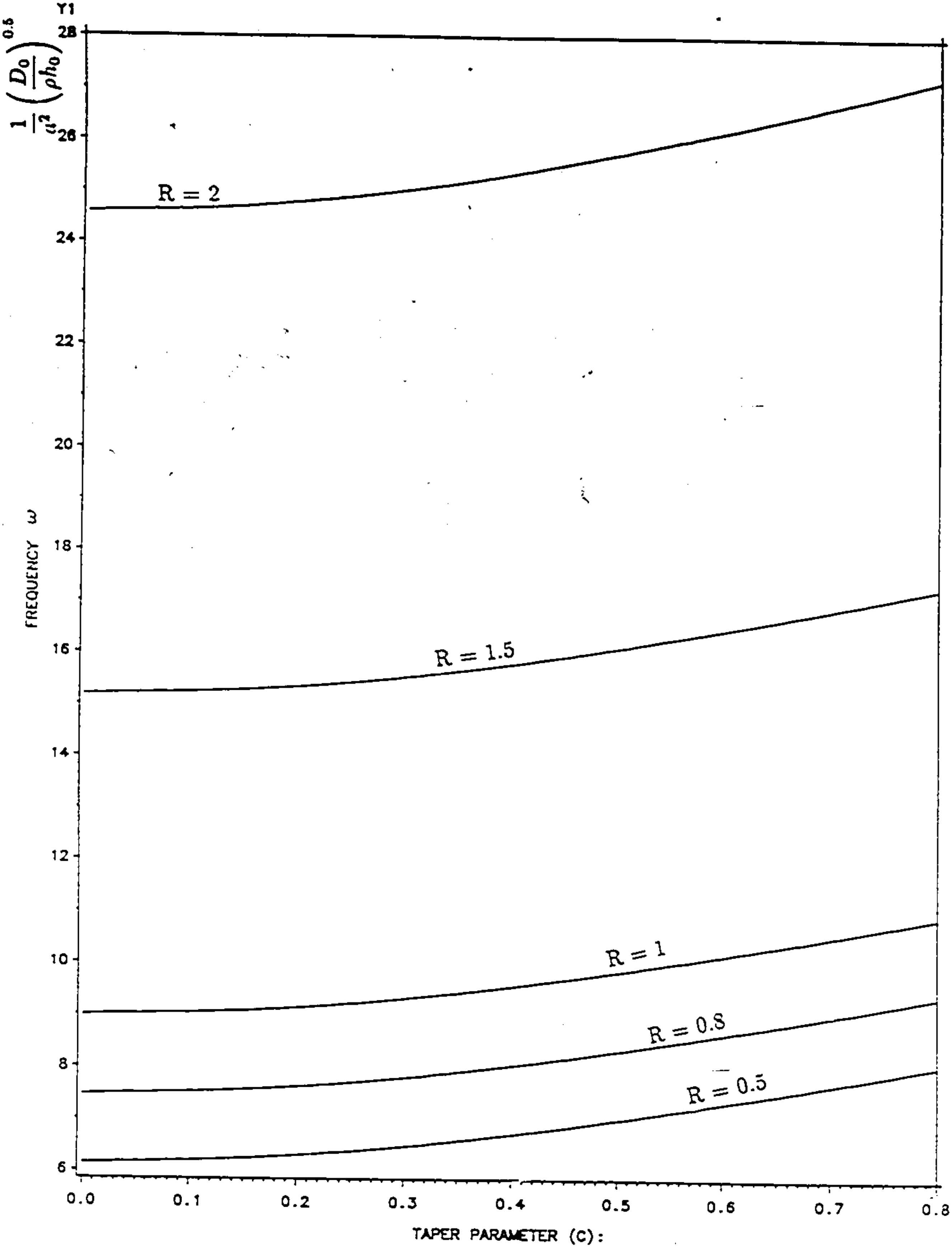


Figure D.2: Variation of circular frequency ω of clamped rectangular plates of variable thickness with taper parameter c for various aspect ratios R in the first mode category.

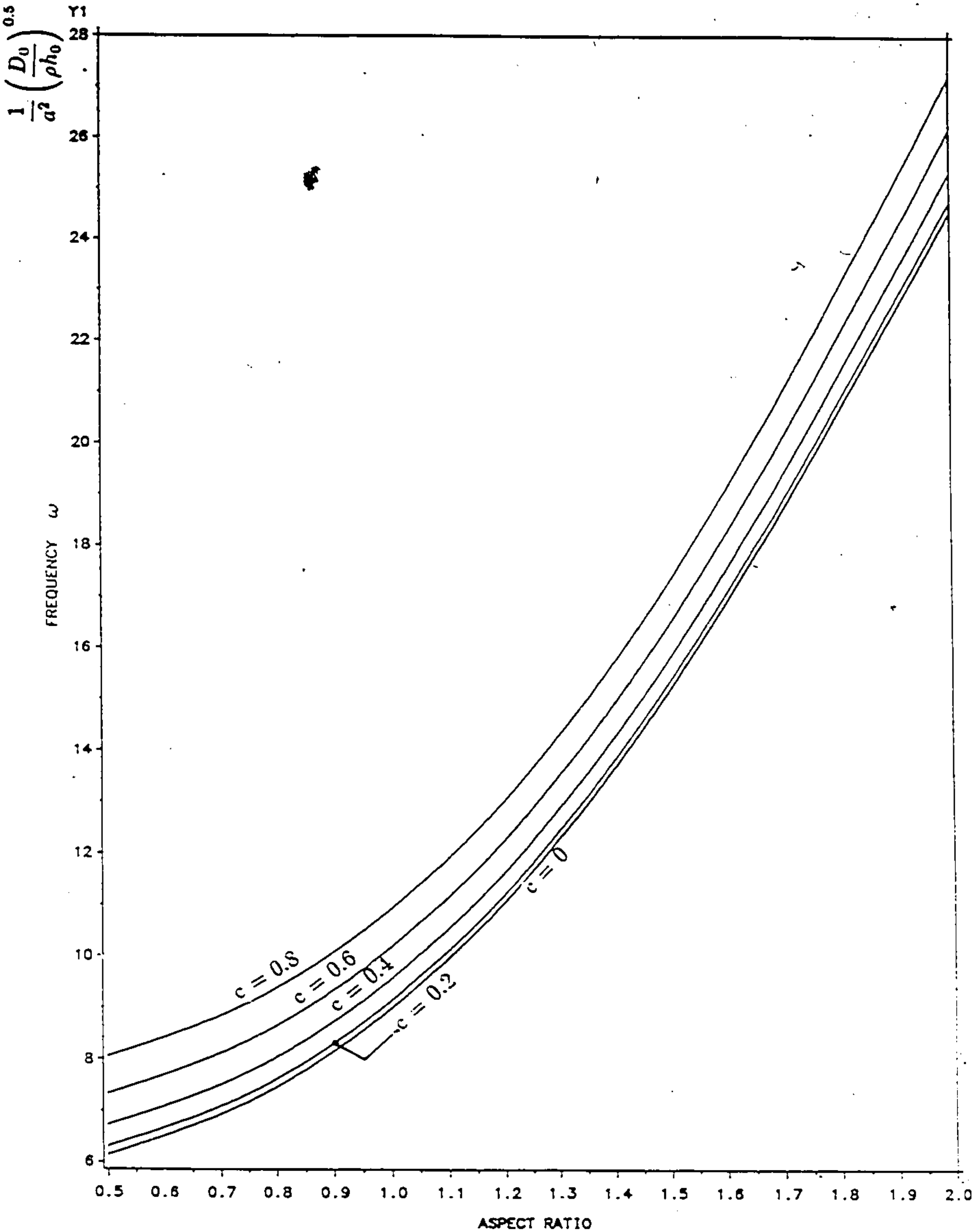


Figure D.3: Variation of circular frequency ω of clamped rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the first mode category.

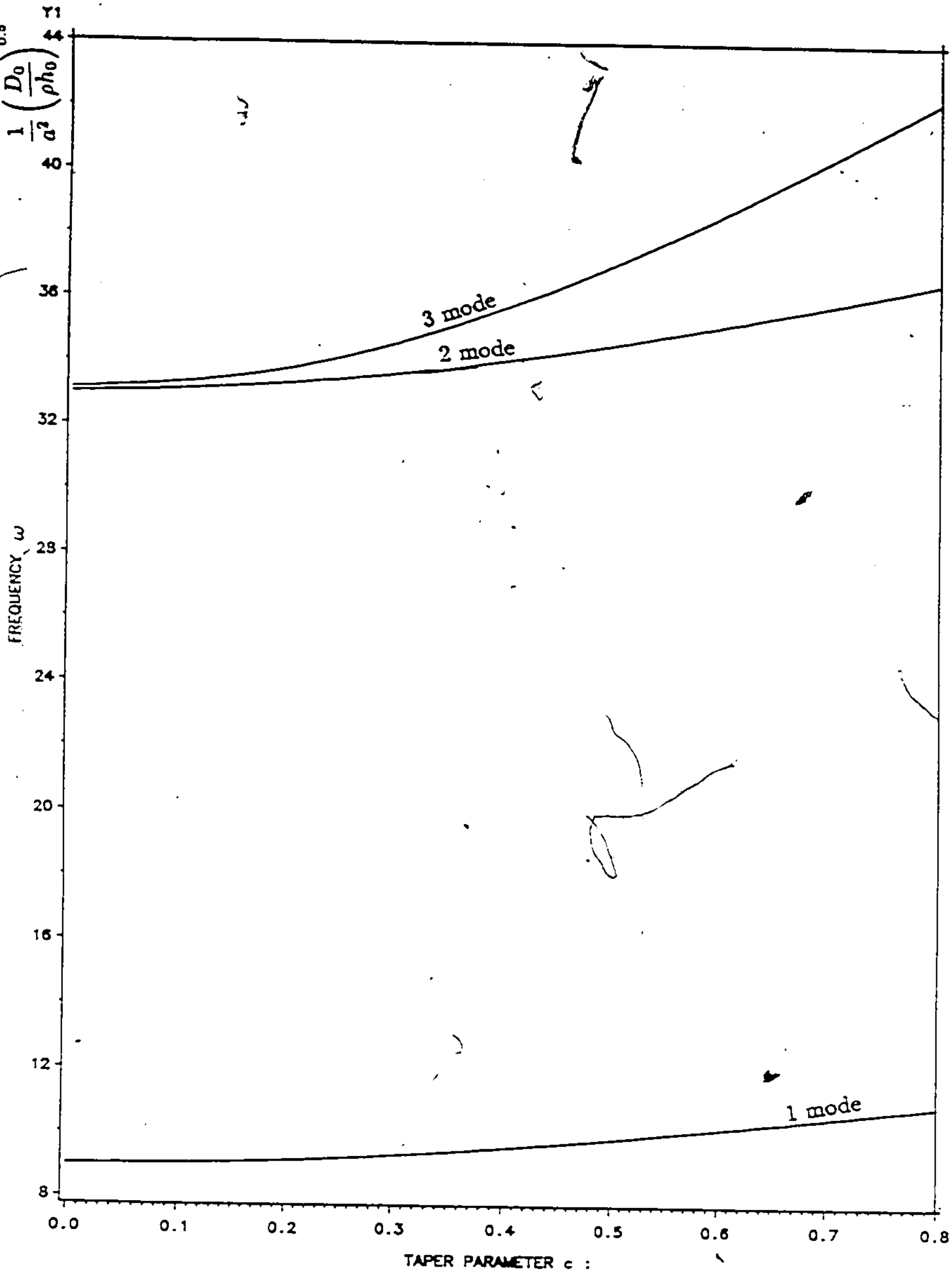


Figure D.4: Variation of circular frequencies for three modes of clamped square plates of variable thickness with taper parameter in first mode category

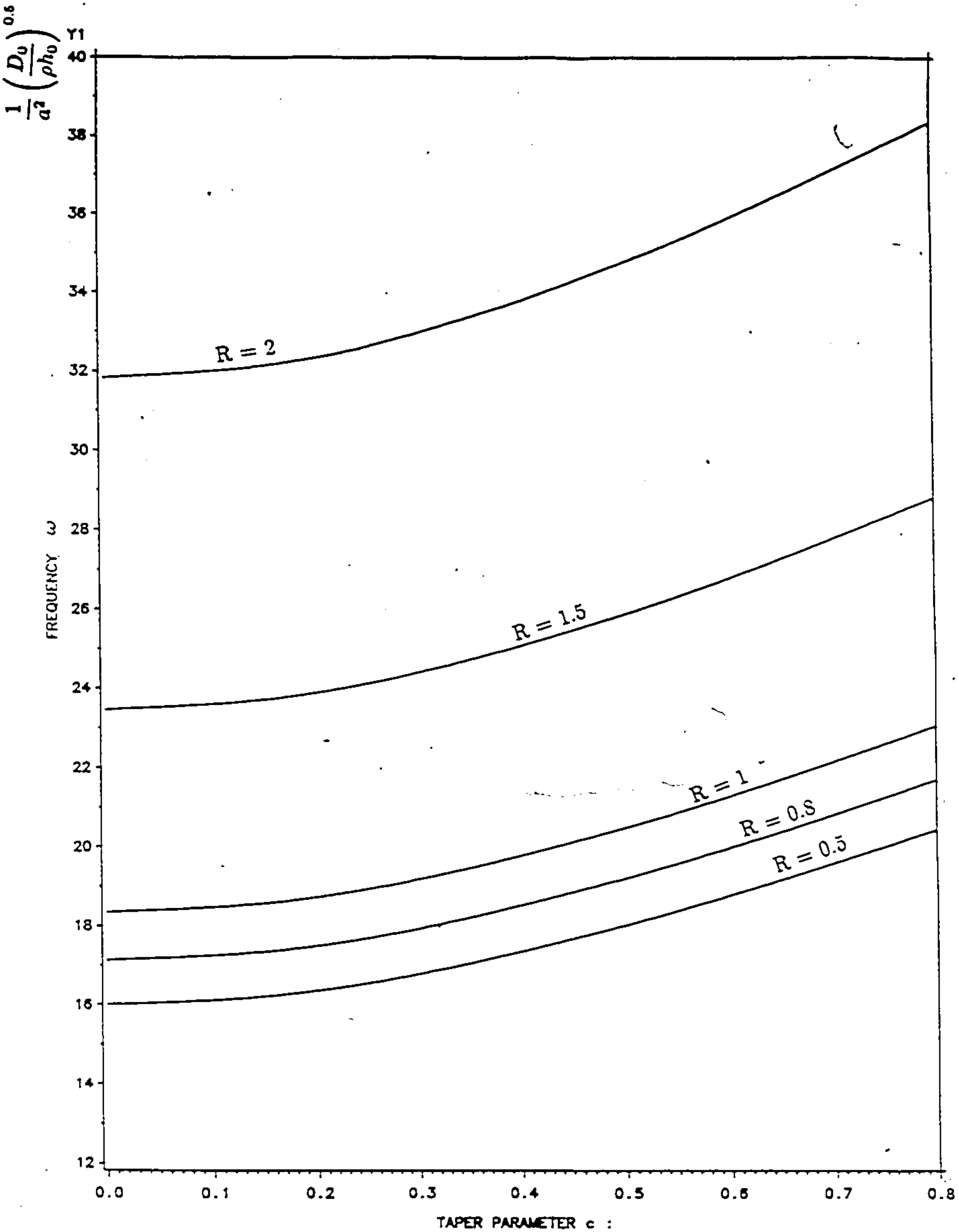


Figure D.5: Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratios R in the second mode category.

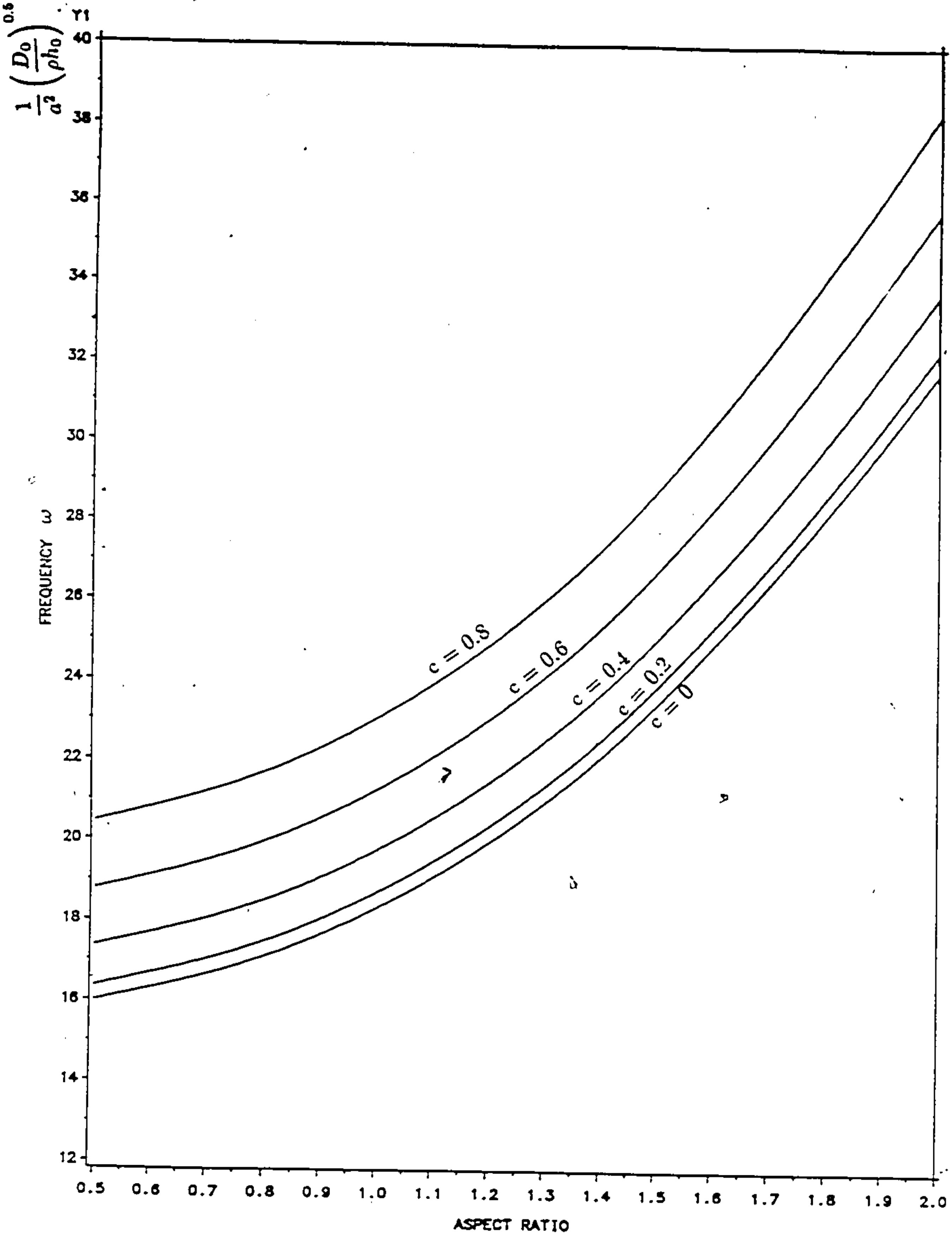


Figure D.6: Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the second mode category.

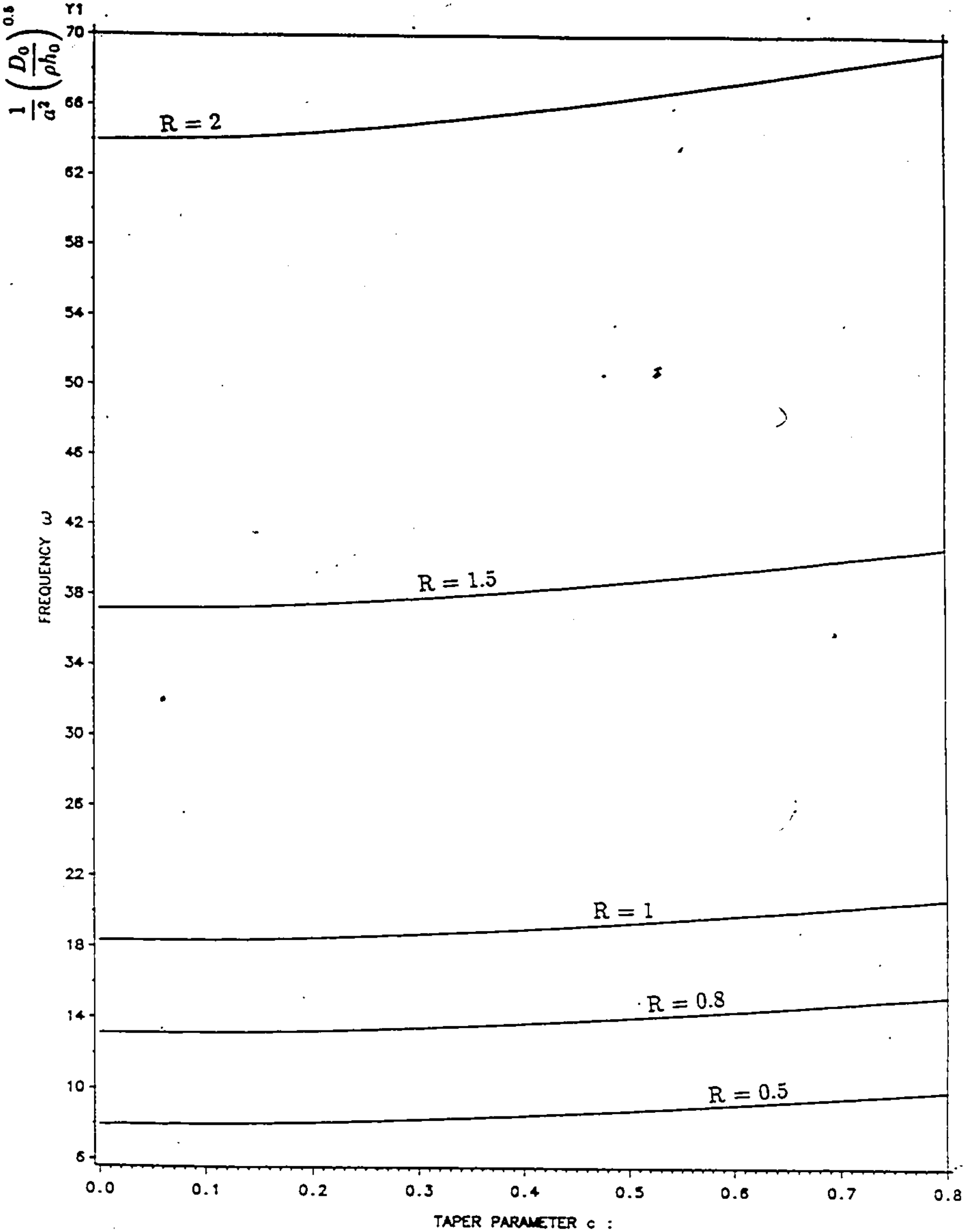


Figure D.7: Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratio R in the third mode category.

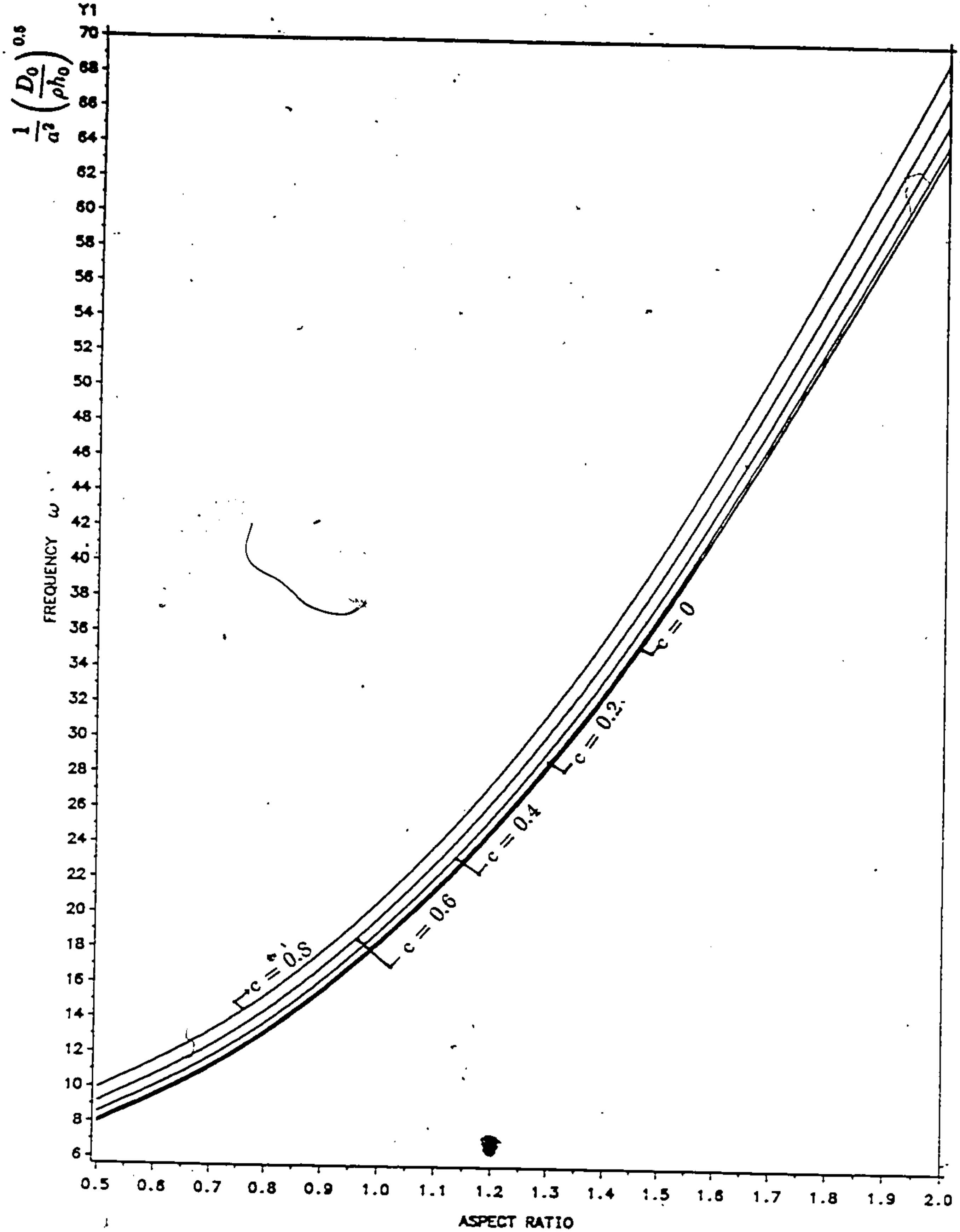


Figure D.8: Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the third mode category.

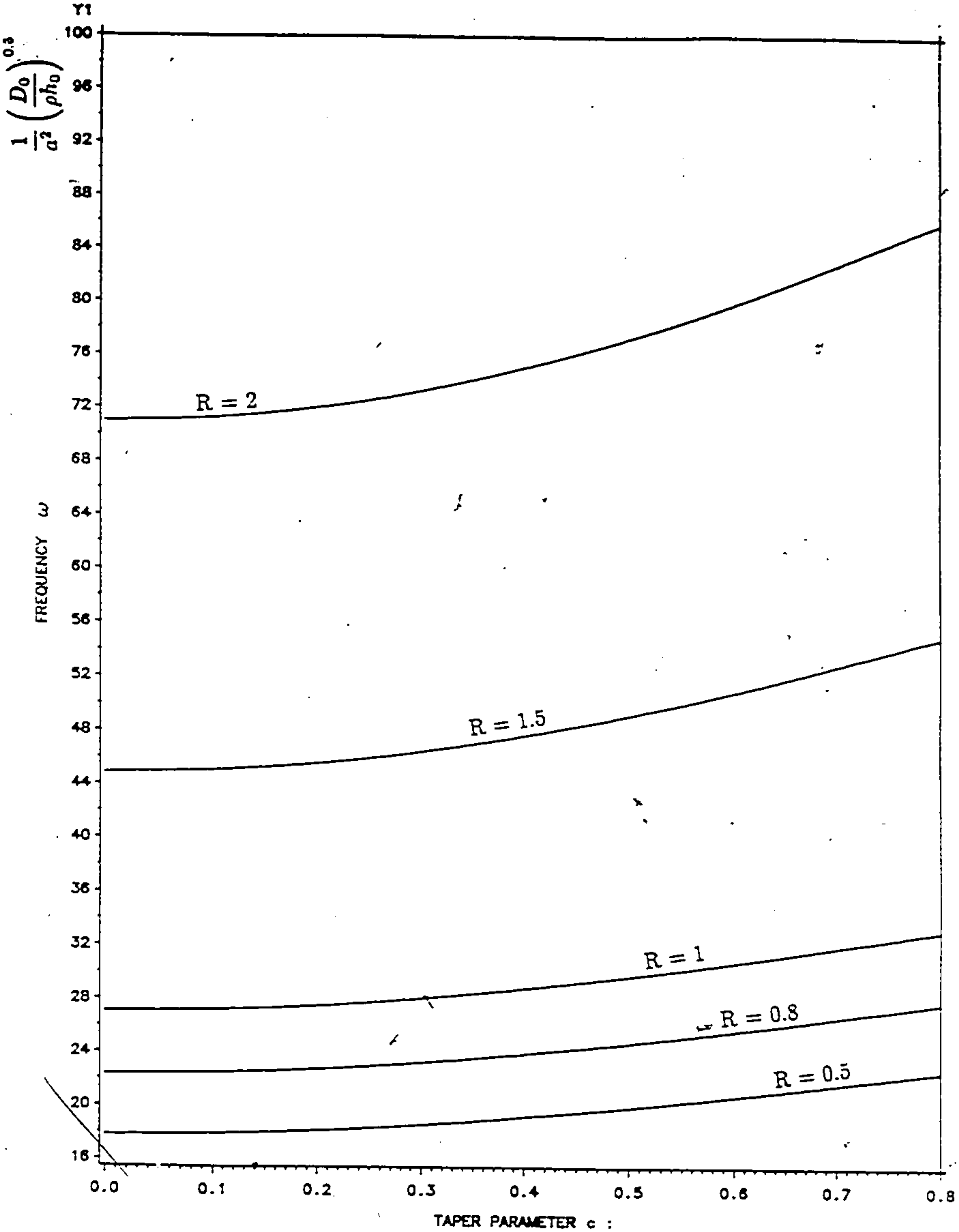


Figure D.9: Variation of circular frequency ω of rectangular plates of variable thickness with taper parameter c for various aspect ratio R in the fourth mode category.

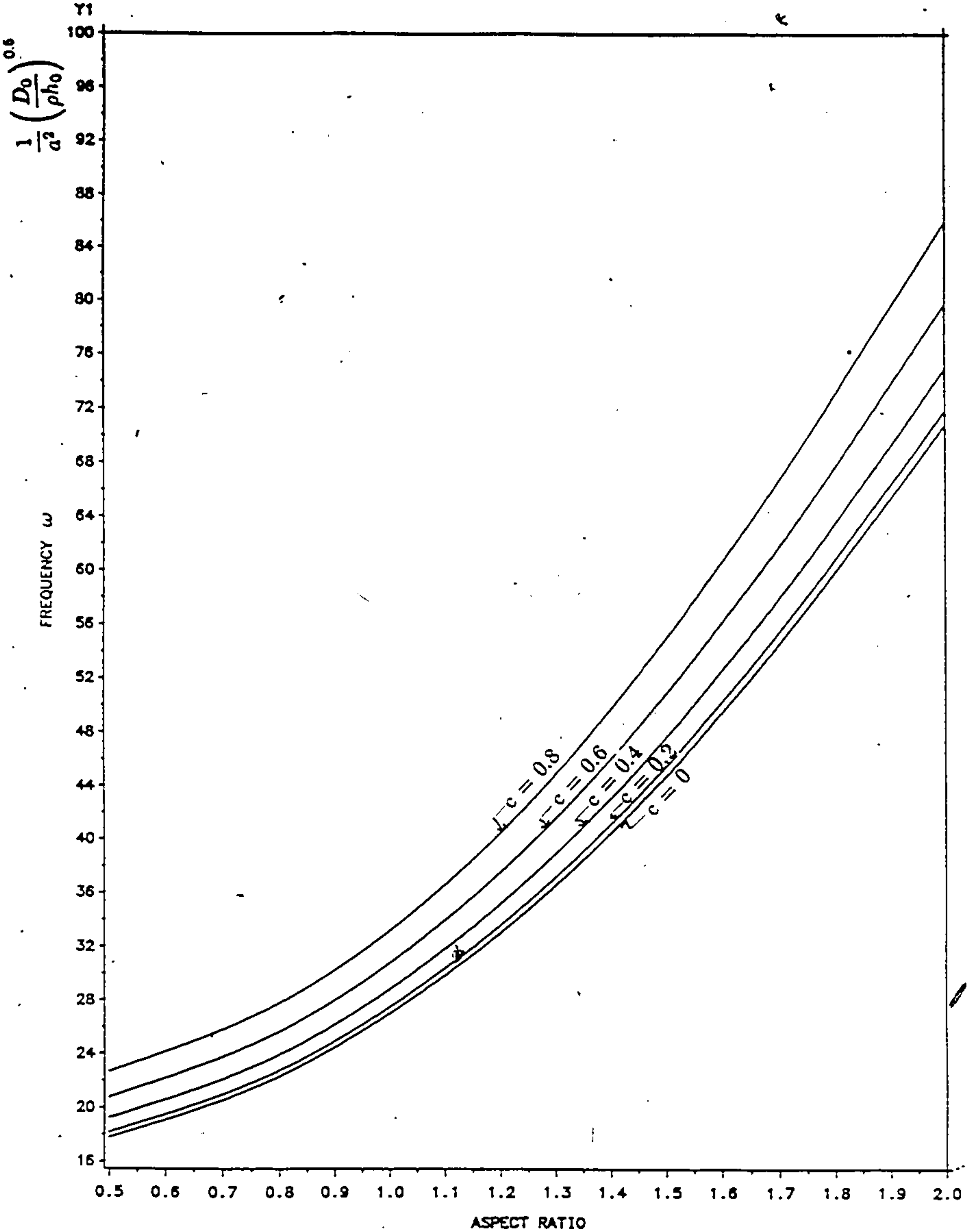


Figure D.10: Variation of circular frequency ω of rectangular plates of variable thickness with aspect ratio R for various taper parameter c in the fourth mode category.

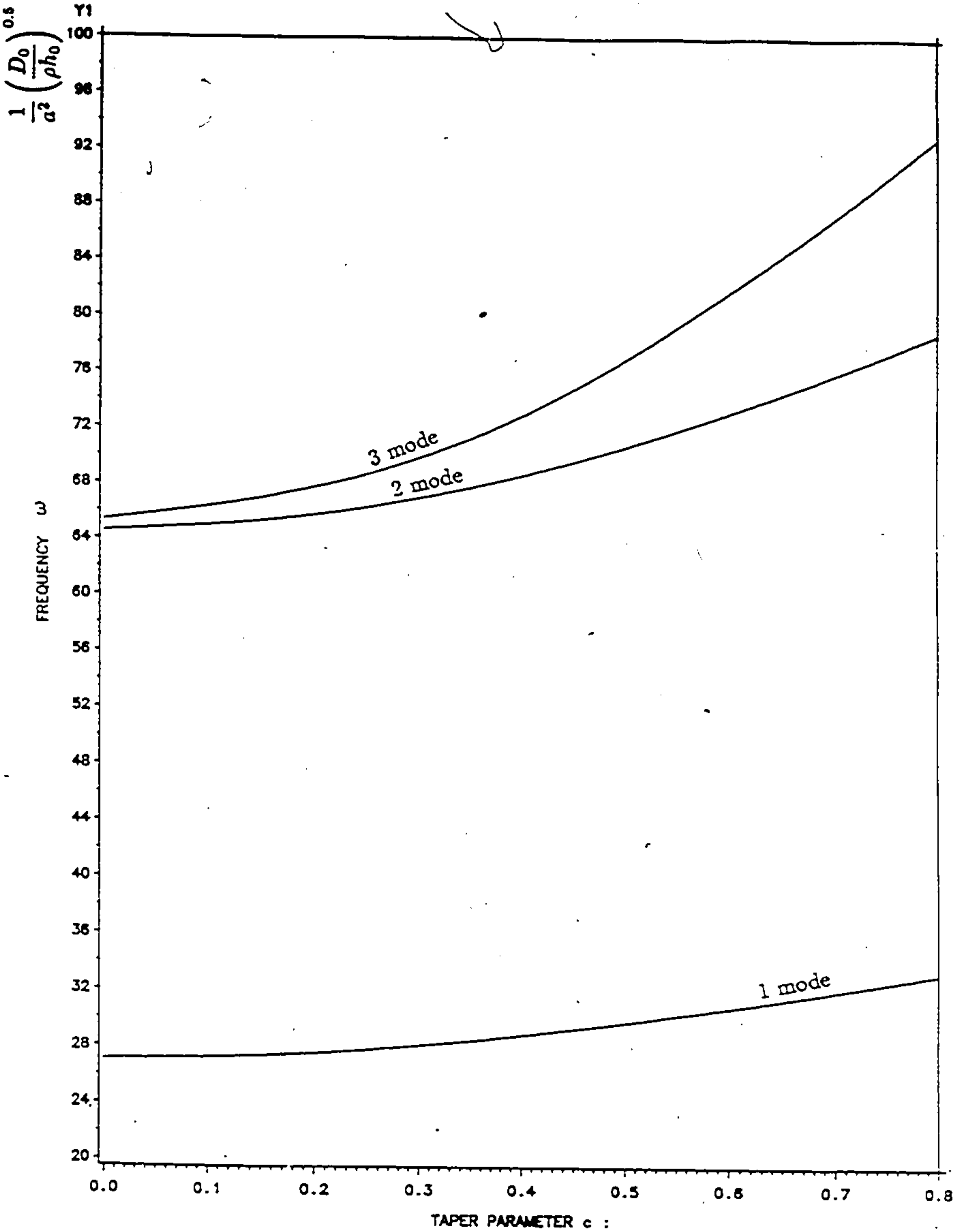


Figure D.11: Variation of circular frequencies for 3 modes of clamped square plates of variable thickness with taper parameter in the fourth mode category.

		TAPER PARAMETER C				
		0.0	0.2	0.4	0.6	0.8
R = 0.5	1 term solution	6.16188	6.33215	6.81750	7.55747	8.48570
	3 terms solution	6.15351	6.30710	6.72092	7.31072	8.01890
	4 terms solution	6.15531	6.30854	6.72181	7.31126	8.01982
	6 terms solution	6.14486	6.30199	6.72832	7.33617	8.05576
	Odman Ref [61]	6.145				
R = 1	1 term solution	9.00002	9.15576	9.60785	10.31741	11.23573
	3 terms solution	9.00002	9.15169	9.55746	10.13738	10.84698
	4 terms solution	9.01824	9.17124	9.58209	10.18032	10.81012
	6 terms solution	8.99716	9.15029	9.57150	10.18490	10.92722
	Odman Ref [61]	8.9997				
R = 1.5	1 term solution	15.21412	15.37280	15.83933	16.58774	17.58207
	3 terms solution	15.19500	15.36089	15.82175	16.50742	17.36514
	4 terms solution	15.19244	15.35831	15.81913	16.50477	17.36245
	6 terms solution	15.19166	15.35739	15.81669	16.49414	17.32661
	Odman Ref [61]	15.1931				

Table D.1: Convergence table of fundamental frequencies mode category 1 of clamped rectangular plates of variable thickness for various taper parameter c and aspect ratios.

R = 0.5		R = 1		R = 1.5		R = 2	
Present Solution	Odman Ref(51)	Present Solution	Odman Ref(61)	Present Solution	Odman Ref(61)	present Present	Odman Ref (61)
First eigen-value for doubly symmetric mode (category 1)							
6.144486	6.145	8.99716	8.9997	15.19166	15.1931	24.57942	24.580
First eigen-value for antisymmetric mode about x axis (category 2)							
15.99744	-	18.35053	18.351	23.46127	23.468	31.83031	31.830
First eigen-value for antisymmetric mode about y axis (category 3)							
7.95758	7.9575	18.35055	18.353	37.19886	37.423	63.98991	-
First eigen-value for doubly antisymmetric modes (category 4)							
17.77841	17.770	27.06378	27.059	44.85075	44.893	70.96305	71.08

Table D.2: Comparison of fundamental frequencies for each of four mode category for clamped rectangular plates of uniform thickness

Appendix E

COMPUTER PROGRAMS

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                               C
C       SUBROUTINE FOR THE INVERSION OF MATRIX                  C
C                                                               C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
        SUBROUTINE MATINV(A,NP)
C     A:   MATRIX TO BE INVERTED
C     NP:  ORDER OF THE MATRIX A
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION INDEX(20,2),A(20,20)
        DO 108 I=1,NP
  108 INDEX(I,1)=0
        II=0
  109 AMAX=-1.D0
        DO 110 I=1,NP
        IF(INDEX(I,1))110,111,110
  111 DO 112 J=1,NP
        IF(INDEX(J,1))112,113,112
  113 TEMP=DABS(A(I,J))
        IF(TEMP-AMAX)112,112,114
  114 IROW=I
        ICOL=J
        AMAX=TEMP
  112 CONTINUE
  110 CONTINUE
        IF(AMAX)225,115,116
  116 INDEX(ICOL,1)=IROW
        IF(IROW-ICOL)119,118,119
  119 DO 120 J=1,NP
        TEMP=A(IROW,J)
        A(IROW,J)=A(ICOL,J)
  120 A(ICOL,J)=TEMP
        II=II+1
        INDEX(II,2)=ICOL
  118 PIVOT=A(ICOL,ICOL)
        A(ICOL,ICOL)=1.D0
        PIVOT=1.D0/PIVOT
        DO 121 J=1,NP
  121 A(ICOL,J)=A(ICOL,J)*PIVOT
        DO 122 I=1,NP
        IF(I-ICOL)123,122,123
  123 TEMP=A(I,ICOL)
        A(I,ICOL)=0
        DO 124 J=1,NP
  124 A(I,J)=A(I,J)-A(ICOL,J)*TEMP
  122 CONTINUE
        GO TO 109
  125 ICOL=INDEX(II,2)
```

```
      IROW=INDEX(ICOL,1)
      DO 126 I=1,NP
      TEMP=A(I,IROW)
      A(I,IROW)=A(I,ICOL)
126   A(I,ICOL)=TEMP
      II=II-1
225   IF(II)125,127,125
115   WRITE(6,101)
101   FORMAT(1H0,2X,11H ZERO PIVOT)
127   CONTINUE
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                              C
C     SUBROUTINE FOR INTEGRATION OF FUNCTIONS                  C
C                                                              C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
      SUBROUTINE INTRGL (N,L,DX,DY,R,T,AREA)
C     N: NUMBER OF INTEGRATIONS
C     L: NUMBER OF NODES IN X AND Y DIRECTIONS
C     DX: SUBINTERVAL OF WIDTH IN X DIRECTION
C     DY: SUBINTERVAL OF WIDTH IN Y DIRECTION
C     R:  ASPECT RATIO A/B
C     T:  TAPER PARAMETER
      IMPLICIT REAL*8(A-H,O-Z)
      M=L-1
      SUM=0.0
      X=-1.0
      Y=-1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      X=-1.0
      Y=1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      X=1.0
      Y=1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      X=1.0
      Y=-1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      SUM=SUM*0.25
      X=-1.0+DX
      DO 1 I=2,M
      Y=-1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      Y=1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      Y=X+DX
    1 CONTINUE
      Y=-1.0+DY
      DO 2 I=2,M
      X=-1.0
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      X=1.0
```

```
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
      Y=Y+DY
    2 CONTINUE
      SUM=SUM*0.5
      X=-1.0
      DO 3 I=2,M
      Y=-1.0
      X=X+DX
      DO 3 J=2,M
      Y=Y+DY
      CALL F(N,X,Y,R,T,FA)
      SUM=SUM+FA
    3 CONTINUE
      AREA=SUM*DX*DY
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                              C
C     SUBROUTINE TO SOLVE THE GENERALIZED EIGENPROBLEM USING   C
C     THE GENERALIZED JACOBI ITERATION                         C
C                                                              C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
C
      SUBROUTINE JACOBI(A,B,X,EIGV,D,N,RTOL,NSMAX,IFPR,IOUT)
C --- INPUT VARIABLES ---
C      A(N,N)     = STIFFNESS MATRIX (ASSUMED POITIVE DEFINITE)
C      B(N,N)     = MASS MATRIX (ASSUMED POSITIVE DEFINITE)
C      X(N,N)     = MATRIX STORING EIGENVECTORS ON SOLUTION EXIT
C      EIGV(N)    = VECTOR STORING EIGENVALUES ON SOLUTION EXIT
C      D(N)       = WORKING VECTOR
C      N          = ORDER OF MATRICES A AND B
C      RTOL       = CONVERGENCE TOLERANCE (USUALLY SET TO 10.**-12)
C      NSMAX      = MAXIMUM NUMBER OF SWEEPS ALLOWED (USUALLY 15)
C      IFPR       = FLAG FOR PRINTING DURING ITERATION
C                   EQ 0  NO PRINTING   EQ 1 INTERMEDIATE PRINTING
C      IOUT       = OUTPUT DEVICE NUMBER (USUALLY 6)
C --- OUTPUT VARIABLE ---
C      A(N,N)     = DIAGONALIZED STIFFNESS MATRIX
C      B(N,N)     = DIAGONALIZED MASS MATRIX
C      X(N,N)     = EIGENVECTORS STORED COLUMNWISE
C      EIGV(N)    = EIGENVALUES
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(10,10),B(10,10),X(10,10),EIGV(10),D(10)
C     INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES
      DO 10 I=1,N
      IF(A(I,I).GT.0.AND.B(I,I).GT..0) GO TO 4
      WRITE(IOUT,2020)
      STOP
    4 D(I)=A(I,I)/B(I,I)
   10 EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
   20 X(I,J)=.0
   30 X(I,I)=1.
      IF(N.EQ.1) RETURN
C     INITIALIZE SWEEP COUNTER AND BEGIN ITERATION
      NSWEEP=0
      NR=N-1
   40 NSWEEP=NSWEEP+1
      IF(IFPR.EQ.1) WRITE(IOUT,2000)NSWEEP
C     CHECK IF PRESENT OFF-DIAG ELEM IS LARGE ENOUGH TO NEED ZEROING
      EPS=(.01**NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
```

```
      DO 210 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS)) GO TO 210
C    IF ZEROING IS NEEDED, CALCULATE ROTATION MATRIX ELEMS CA AND CG
      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.*AKK*AJJ)/4.
      IF(CHECK)50,60,60
   50 WRITE(IOUT,2020)
      STOP
   60 SQCH=DSQRT(CHECK)
      D1=AB/2.+SQCH
      D2=AB/2.-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=.0
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C    PERFORM THE GENERALIZED ROTATION TO ZERO PRESENT OFF-DIAG ELEM
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1)130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
      BJ=B(I,J)
      AK=A(I,K)
      BK=B(I,K)
      A(I,J)=AJ+CG*AK
      B(I,J)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  120 B(I,K)=BK+CA*BJ
  130 IF(KP1-N)140,140,160
  140 DO 150 I=KP1,N
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(K,I)
      BK=B(K,I)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(K,I)=AK+CA*AJ
  150 B(K,I)=BK+CA*BJ
  160 IF(JP1-KM1)170,170,190
```

```
  170 DO 180 I=JP1,KM1
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(I,K)
      BK=B(I,K)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  180 B(I,K)=BK+CA*BJ
  190 AK=A(K,K)
      BK=B(K,K)
      A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
      B(K,K)=BK+2.*CA*B(J,K)+CA*CA*B(J,J)
      A(J,J)=A(J,J)+2.*CG*A(J,K)+CG*CG*AK
      B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
      A(J,K)=.0
      B(J,K)=.0
C     UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
      DO 200 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+CG*XK
  200 X(I,K)=XK+CA*XJ
  210 CONTINUE
C     UPDATE THE EIGENVALUES AFTER EACH SWEEP
      DO 220 I=1,N
      IF(A(I,I).GT..0.AND.B(I,I).GT..0) GO TO 220
      WRITE(IOUT,2020)
      STOP
  220 EIGV(I)=A(I,I)/B(I,I)
      IF(IFPR.EQ.0.) GO TO 230
      WRITE(IOUT,2030)
      WRITE(IOUT,2010) (EIGV(I),I=1,N)
C     CHECK FOR CONVERGENCE
230   DO 240 I=1,N
      TOL=RTOL*D(I)
      DIF=DABS(EIGV(I)-D(I))
      IF(DIF.GT.TOL) GO TO 280
  240 CONTINUE
C     CHECK ALL OFF-DIAG ELEMS TO SEE IF ANOTHER SWEEP IS NEEDED
      EPS=RTOL**2
      DO 250 J=1,NR
      JJ=J+1
      DO 250 K=JJ,N
      EPSA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS)) GO TO 250
      GO TO 280
  250 CONTINUE
C     FILL OUT BOTTOM TRIANGLE OF RESULTANT MATRICES AND SCALE EIGENVEC
```

```
  255 DO 260 I=1,N
      DO 260 J=1,N
      A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
      DO 270 J=1,N
      BB=DSQRT(ABS(B(J,J)))
      DO 270 K=1,N
  270 X(K,J)=X(K,J)/BB
      RETURN
C     UPDATE  D  MATRIX AND START NEW SWEEP, IF ALLOWED
  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
      IF(NSWEEP.LT.NSMAX) GO TO 40
      GO TO 255
 2000 FORMAT(/' SWEEP NUMBER IN *JACOBI* = ',I4)
 2010 FORMAT(5X,4E18.8)
 2020 FORMAT(' *** ERROR  SOLUTION  STOP  '/
     1  ' MATRICES ARE NOT POSITIVE DEFINITE')
 2030 FORMAT(' CURRENT EIGENVALUES IN *JACOBI* ARE')
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                             C
C          PROGRAM FOR DEFINITE INTEGRATION                   C
C                                                             C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
C      M1: NUMBER OF INTEGRATION
C      N2: NUMBER OF NODES IN X AND Y DIRECTIONS
C      F FUNCTION TO BE INTEGRATED
C      R ASPECT RATIO A/B
C      T TAPER PARAMETER
C
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION ARA(100)
        READ(5,*) M1,N2,R
        DO 40  J=1,5
        READ(5,*) T
        WRITE(6,200)R,T
200   FORMAT('1',4X,'R=',F10.4,4X,'T=',F10.5,///)
        DO 10 I=1,M1
        DX=2/(FLOAT(N2-1))
        DY=2/(FLOAT(N2-1))
        CALL INTRGL(I,N2,DX,DY,R,T,AREA)
        ARA(I)=AREA
        WRITE(6,111)(ARA(I),I=1,M1)
 111  FORMAT(3X,F12.4,/)
   10 CONTINUE
  40  CONTINUE
        STOP
        END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                 C
C          PROGRAM FOR VIBRATION ANALYSIS                         C
C                                                                 C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
C      A AND B ARE MATRICES DEFINING TWO SIDES OF EIGEN-VALUE
C      PROBLEM WHICH ARE FORMED FROM THE RESULTS OF DEFINING
C      INTEGRATION.
C      N: ORDER OF MATRICES A AND B
C      EIGEN: EIGEN-VALUES
C
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION A(20,20),B(20,20),X(20,20),EIGV(20),D(20)
        READ(5,*) N
        READ(5,*)((A(I,J),J=1,N),I=1,N)
        READ(5,*)((B(I,J),J=1,N),I=1,N)
        WRITE(6,2)
    2  FORMAT('1',T10,'MATRIX STIFFNESS',//)
        WRITE(6,3)((A(I,J),J=1,N),I=1,N)
    3  FORMAT(6(3X,F10.5),/)
        WRITE(6,4)
    4  FORMAT(T10,'MASS MATRIX ',/)
        WRITE(6,7)((B(I,J),J=1,N),I=1,N)
    7  FORMAT(6(3X,F10.5),/)
C RTOL CONVERGENCE TOLERANCE
        RTOL=0.000000000001
C NSMAX = MAXIMUM NUMBER OF SWEEPS ALLOWED
        NSMAX=15
C IFPR FLAG FOR PRINTING DURING ITERATION
        IFPR=0
C IOUT OUTPUT DEVICE NUMBER
        IOUT =6
        CALL JACOBI (A,B,X,EIGV,D,N,RTOL,NSMAX,IFPR,IOUT)
        WRITE (6,9)
    9  FORMAT ('1','MATRIX (A)',//)
        WRITE(6,155)((A(I,J),J=1,N),I=1,N)
  155  FORMAT(6(3X,F10.5),/)
        PRINT 156
  156  FORMAT(T5,'MATRIX (B)',/)
        WRITE (6,160)((B(I,J),J=1,N),I=1,N)
  160  FORMAT(6(3X,F10.5),/)
        PRINT 170
  170  FORMAT(T5,'EIGEN VECTORS STORED COLUMNWISE X(I,J)',//)
        WRITE (6,180)((X(I,J),J=1,N),I=1,N)
  180  FORMAT(6(3X,F10.5),/)
        PRINT 190
  190  FORMAT(T5,'EIGEN VALUES X',//)
        DO 14 I=1,N
```

```
      EIGV(I)=EIGV(I)**0.5
 14   CONTINUE
      WRITE(6,20)(EIGV(I),I=1,N)
 20   FORMAT(3X,F15.5,/)
      STOP
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                    C
C  FUNCTIONS TO BE INTERGRATED TO FORM THE MATRIX A OF               C
C  THE  EIGEN-VALUE PROBLEM FOR THE MODE CATEGORY 1                  C
C                                                                    C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
      FUNCTION F(N,X,Y,R,T,FA)
C    N: NUMBER OF FUNCTIONS TO BE INTEGRATED
      R: ASPECT RATIO R=A/B
      T: TAPER PARAMETER
      X2=X*X
      X3=X2*X
      X4=X3*X
      X5=X4*X
      X6=X5*X
      X7=X6*X
      X8=X7*X
      X9=X8*X
      Y2=Y*Y
      Y3=Y2*Y
      Y4=Y3*Y
      Y5=Y4*Y
      Y6=Y5*Y
      Y7=Y6*Y
      Y8=Y7*Y
      Y9=Y8*Y
      AAX1=1-2*X2+X4-2*Y2+4*X2*Y2-2*Y2*X4+Y4-2*X2*Y4+X4*Y4
      AAX2=AAX1*X2
      AAX3=AAX1*Y2
      AAX4=AAX1*X2*Y2
      AAX5=AAX2*X2
      AAX6=AAX3*Y2
      TX=(1+T*X)
      R2=R*R
      R4=R2*R2
      BX1=((TX)**3*((24-48*Y2+24*Y4)+2*R2*(16-48*X2-48*Y2+144*X2*Y2)+
     1R4*(24-48*X2+24*X4)))
      BX2=6*T*(TX)**2*((24*X-48*Y2*X+24*Y4*X)+R2*(16*X-16*X3-48*X*Y2+48
     1*X3*Y2))
      BX3=6*T**2*(TX)*((-4+8*Y2-24*X2*Y2-4*Y4+12*X2*Y4+12*X2)+R2*0.30*
     1(-4+8*X2-4*X4+12*Y2-24*X2*Y2+12*X4*Y2))
C
      AX1=BX1+BX2+BX3
C
      BX4=(TX)**3*((-48+360*X2+96*Y2-720*X2*Y2-48*Y4+360*X2*Y4)+2*R2*(-8
     1+96*X2-120*X4+24*Y2-288*X2*Y2+360*X4*Y2)+R**4*(24*X2-48*X4+24*X6))
      BX5=6*T*(TX)**2*((-48*X+120*X3+96*X*Y2-240*X3*Y2-48*X*Y4+120*X3
     1*Y4)+R2*(-8*X+32*X3-24*X5+24*X*Y2-96*X3*Y2+72*X5*Y2))
```

```
      BX6=6*T**2*(TX)*((2-24*X2+30*X4-4*Y2+48*X2*Y2-60*X4*Y2+2*Y4-24*X2*
     1Y4+30*X4*Y4)+R2*0.30*(-4*X2+8*X4-4*X6+12*X2*Y2-24*X4*Y2-12*X6*Y2))
C
      AX2=BX4+BX5+BX6
C
      BX7=(TX)**3*((24*Y2-48*Y4+24*Y6)+2*R2*(-8+24*X2+96*Y2-288*X2*Y2-
     1120*Y4+360*X2*Y4)+R4*(-48+96*X2-48*X4+360*Y2-720*X2*Y2+360*X4*Y2))
      BX8=6*T*(TX)**2*((24*X*Y2-48*X*Y4+24*X*Y6)+R2*(-8*X+8*X3+96*X*Y2-
     1-96*X3*Y2-120*X*Y4+120*X3*Y4))
      BX9=6*T**2*(TX)*((-4*Y2+12*X2*Y2+8*Y4-24*X2*Y4-4*Y6+12*X2*Y6)+0.30
     1*R2*(2-4*X2+2*X4-24*Y2+48*X2*Y2-24*X4*Y2+30*Y4-60*X2*Y4+30*X4*Y4))
C
      AX3=BX7+BX8+BX9
C
      BX10=(TX)**3*((-48*Y2+360*X2*Y2+96*Y4-720*X2*Y4-48*Y6+360*X2*Y6)+2
     1*R2*(4-48*X2+60*X4-48*Y2+576*X2*Y2-720*X4*Y2+60*Y4-720*X2*Y4+900*
     2X4*Y4)+R4*(-48*X2+96*X4-48*X6+360*X2*Y2-720*X4*Y2+360*X6*Y2))
      BX11=6*T*(TX)**2*((-48*X*Y2+120*X3*Y2+96*X*Y4-240*X3*Y4-48*X*Y6+
     1120*X3*Y6)+R2*(4*X-16*X3+12*X5-48*X*Y2+192*X3*Y2-144*X5*Y2+60*X*Y4
     2-240*X3*Y4+180*X5*Y4))
      BX12=6*T**2*(TX)*((2*Y2-24*X2*Y2+30*X4*Y2-4*Y4+48*X2*Y4-60*X4*Y4+2
     1*Y6-24*X2*Y6+30*X4*Y6)+0.30*R2*(2*X2-4*X4+2*X6-24*X2*Y2+48*X4*Y2-
     224*X6*Y2+30*X2*Y4-60*X4*Y4+30*X6*Y4))
C
      AX4=BX10+BX11+BX12
C
      BX13=(TX)**3*((24-720*X2+1680*X4-48*Y2+1440*X2*Y2-3360*Y2*X4+24*Y4
     1-720*X2*Y4+1680*X4*Y4)+2*R**2*(-48*X2+240*X4-224*X6+144*X2*Y2-720
     2*X4*Y2+672*X6*Y2)+R**4*(24*X4-48*X6+24*X8))
      BX14=6*T*(TX)**2*((24*X-240*X3+336*X5-48*Y2*X+480*X3*Y2-672*Y2*X5
     1+24*Y4*X-240*X3*Y4+336*X5*Y4)+R2*(-16*X3+48*X5-32*X7+48*Y2*X3-144
     2*X5*Y2+96*X7*Y2))
      BX15=6*T**2*(TX)*((12*X2-60*X4+56*X6-24*Y2*X2+120*X4*Y2-112*Y2*X6+
     112*Y4*X2-60*X4*Y4+56*X6*Y4)+0.30*R2*(-4*X4+8*X6-4*X8+12*X4*Y2-24*
     2X6*Y2+12*X8*Y2))
C
      AX5=BX13+BX14+BX15
C
      BX16=(TX)**3*((24*Y4-48*Y6+24*Y8)+2*R**2*(-48*Y2+144*X2*Y2+240*Y4-
     1720*Y4*X2-224*Y6+672*X2*Y6)+R4*(24-48*X2+24*X4-720*Y2+1440*X2*Y2-
     2720*Y2*X4+1680*Y4-3360*X2*Y4+1680*X4*Y4))
      BX17=6*T*(TX)**2*((24*X*Y4-48*Y6*X+24*X*Y8)+R2*(-48*X*Y2+48*X3*Y2+
     1240*X*Y4-240*X3*Y4-224*X*Y6+224*X3*Y6))
      BX18=6*T**2*(TX)*((-4*Y4+12*X2*Y4+8*Y6-24*Y6*X2-4*Y8+12*X2*Y8)+0.3
     1*R2*(12*Y2-24*Y2*X2+12*X4*Y2-60*Y4+120*X2*Y4-60*Y4*X4+56*Y6-112*X2
     2*Y6+56*X4*Y6))
C
      AX6=BX16+BX17+BX18
C
      GOTO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
```

```
     124,25,26,27,28,29,30,31,32,33,34,35,36) N
C
1     F1=AX1*AAX1
      FA=F1
      RETURN
C
2     F2=AX2*AAX1
      FA=F2
      RETURN
C
3     F3=AX3*AAX1
      FA=F3
      RETURN
C
4     F4=AX4*AAX1
      FA=F4
      RETURN
C
      F5=AX5*AAX1
5     FA=F5
      RETURN
C
6     F6=AX6*AAX1
      FA=F6
      RETURN
C
7     F7=AX1*AAX2
      FA=F7
      RETURN
C
8     F8=AX2*AAX2
      FA=F8
      RETURN
C
9     F9=AX3*AAX2
      FA=F9
      RETURN
C
10    F10=AX4*AAX2
      FA=F10
      RETURN
C
11    F11=AX5*AAX2
      FA=F11
      RETURN
C
12    F12=AX6*AAX2
      FA=F12
      RETURN
C
```

```
C
C
C
C
C
C
C
C
C
C
C
C
```

```
      RETURN
C
 26   F26=AX2*AAX5
      FA=F26
      RETURN
C
27    F27=AX3*AAX5
      FA=F27
      RETURN
C
 28   F28=AX4*AAX5
      FA=F28
      RETURN
C
 29   F29=AX5*AAX5
      FA=F29
      RETURN
C
 30   F30=AX6*AAX5
      FA=F30
      RETURN
C
 31   F31=AX1*AAX6
      FA=F31
      RETURN
C
  32  F32=AX2*AAX6
      FA=F32
      RETURN
C
 33   F33=AX3*AAX6
      FA=F33
      RETURN
C
 34   F34=AX4*AAX6
      FA=F34
      RETURN
C
 35   F35=AX5*AAX6
      FA=F35
      RETURN
C
 36   F36=AX6*AAX6
      FA=F36
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                             C
C  FUNCTIONS TO BE INTERGRATED TO FORM THE MATRIX A OF        C
C  THE  EIGEN-VALUE PROBLEM FOR THE MODE CATEGORY 2           C
C                                                             C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
      FUNCTION F(N,X,Y,R,T,FA)
C    N: NUMBER OF FUNCTIONS TO BE INTEGRATED
      R: ASPECT RATIO R=A/B
      T: TAPER PARAMETER
      X2=X*X
      X3=X2*X
      X4=X3*X
      X5=X4*X
      X6=X5*X
      X7=X6*X
      X8=X7*X
      X9=X8*X
      Y2=Y*Y
      Y3=Y2*Y
      Y4=Y3*Y
      Y5=Y4*Y
      Y6=Y5*Y
      Y7=Y6*Y
      Y8=Y7*Y
      Y9=Y8*Y
      AAX1=X*(1-2*X2+X4-2*Y2+4*X2*Y2-2*Y2*X4+Y4-2*X2*Y4+X4*Y4)
      AAX2=AAX1*X2
      AAX3=AAX1*Y2
      AAX4=AAX1*X2*Y2
      AAX5=AAX2*X2
      AAX6=AAX3*Y2
      TX=(1+T*X)
      R2=R*R
      R4=R2*R2
      BX1=(TX)**3*((120*X-240*Y2*X+120*X*Y4)+2*R2*(48*X-80*X3-144*X*Y2+
     1240*X3*Y2)+R4*(24*X-48*X3+24*X5))
      BX2=6*T*(TX)**2*(-12+60*X2+24*Y2-120*Y2*X2-12*Y4+60*X2*Y4+R2*(-4+2
     14*X2-20*X4+12*Y2-72*X2*Y2+60*X4*Y2))
      BX3=6*T**2*(TX)*((-12*X+20*X3+24*X*Y2-40*Y2*X3-12*X*Y4+20*X3*Y4)+0
     1.3*R2*(-4*X+8*X3-4*X5+12*X*Y2-24*X3*Y2+12*X5*Y2))
C
      AX1=BX1+BX2+BX3
C
      BX4=TX**3*((-240*X+840*X3+480*X*Y2-1680*X3*Y2-240*X*Y4+840*X3*Y4)+
     12*R2*(-24*X+160*X3-168*X5+72*Y2*X-480*X3*Y2+504*X5*Y2)+R4*(24*X3-4
     28*X5+24*X7))
      BX5=6*T*TX**2*((6-120*X2+210*X4-12*Y2+240*X2*Y2-420*X4*Y
```

```
     12+6*Y4-120*X2*Y4+210*X4*Y4)+R2*(-12*X2+40*X4-28*X6+36*X2*Y2-120*X4
     2*Y2+84*X6*Y2))
      BX6=6*T**2*TX*((6*X-40*X3+42*X5-12*X*Y2+80*X3*Y2-84*Y2*X5+6*Y4*X-4
     10*X3*Y4+42*X5*Y4)+0.3*R2*(-4*X3+8*X5-4*X7+12*X3*Y2-24*X5*Y2+12*Y2*
     2X7))
C
      AX2=BX4+BX5+BX6
C
      BX7=TX**3*((120*X*Y2-240*X*Y4+120*X*Y6)+2*R2*(-24*X+40*X3+288*Y2*X
     1-480*X3*Y2-360*X*Y4+600*X3*Y4)+R4*(-48*X+96*X3-48*X5+360*Y2*X-720
     2*X3*Y2+360*X5*Y2))
      BX8=6*T*TX**2*((-12*Y2+60*X2*Y2+24*Y4-120*X2*Y4-12*Y6+60*X2*Y6)+R2
     1*(2-12*X2+10*X4-24*Y2+144*Y2*X2-120*X4*Y2+30*Y4-180*X2*Y4+150*X4*
     2Y4))
      BX9=6*T**2*TX*((-12*X*Y2+20*X3*Y2+24*Y4*X-40*X3*Y4-12*X*Y6+20*X3*Y
     16)+0.3*R2*(2*X-4*X3+2*X5-24*Y2*X+48*Y2*X3-24*X5*Y2+30*Y4*X-60*X3*Y
     24+30*X5*Y4))
      AX3=BX7+BX8+BX9
C
      BX10=TX**3*((-240*X*Y2+840*X3*Y2+480*X*Y4-1680*X3*Y4-240*X*Y6+840*
     1X3*Y6)+2*R2*(12*X-80*X3+84*X5-144*X*Y2+960*X3*Y2-1008*
     2X5*Y2+180*X*Y4-1200*X3*Y4+1260*X5*Y4)+R4*(-48*X3+96*X5-48*X7+360*X
     33*Y2-720*X5*Y2+360*X7*Y2))
      BX11=6*T*TX**2*((6*Y2-120*X2*Y2+210*X4*Y2-12*Y4+240*X2*Y4-420*X4*Y
     14+6*Y6-120*X2*Y6+210*X4*Y6)+R2*(6*X2-20*X4+14*X6-72*X2*Y2+240*X4*Y
     22-168*X6*Y2+90*X2*Y4-300*X4*Y4+210*X6*Y4))
      BX12=6*T**2*TX*((6*X*Y2-40*X3*Y2+42*X5*Y2-12*X*Y4+80*X3*Y4-84*X5*Y
     14+6*X*Y6-40*X3*Y6+42*X5*Y6)+0.3*R2*(2*X3-4*X5+2*X7-24*X3*Y2+48*X5*
     2Y2-24*X7*Y2+30*X3*Y4-60*X5*Y4+30*X7*Y4))
C
      AX4=BX10+BX11+BX12

      BX13=TX**3*((120*X-1680*X3+3024*X5-240*Y2*X+3360*X3*Y2-6048*Y2*X5+
     1120*Y4*X-1680*X3*Y4+3024*X5*Y4)+2*R2*(-80*X3+336*X5-288*X7+240*Y2*
     2X3-1008*X5*Y2+864*X7*Y2)+R4*(24*X5-48*X7+24*X9))
      BX14=6*T*TX**2*((60*X2-420*X4+504*X6-120*Y2*X2+840*X4*Y2-1008*Y2*X
     16+60*Y4*X2-420*X4*Y4+504*X6*Y4)+R2*(-20*X4+56*X6-36*X8-168*X6*Y2+1
     208*X8*Y2+60*X4*Y2))
      BX15=6*T**2*TX*((20*X3-84*X5+72*X7-40*Y2*X3+168*X5*Y2-144*Y2*X7+20
     1*Y4*X3-84*X5*Y4+72*X7*Y4)+0.3*R2*(-4*X5+8*X7-4*X9+12*X5*Y2-24*X7*Y
     22+12*X9*Y2))
C
      AX5=BX13+BX14+BX15
C
      BX16=TX**3*((120*X*Y4-240*X*Y6+120*X*Y8)+2*R2*(-144*X*Y2+240*X3*Y2
     1+720*Y4*X-1200*X3*Y4-672*X*Y6+1120*X3*Y6)+R4*(24*X-48*X3+24*X5-720
     2*Y2*X+1440*Y2*X3-720*X5*Y2+1680*Y4*X-3360*X3*Y4+1680*X5*Y4))
      BX17=6*T*TX**2*((-12*Y4+60*X2*Y4+24*Y6-120*X2*Y6-12*Y8+60*X2*Y8)+R
     12*(12*Y2-72*Y2*X2+60*X4*Y2-60*Y4+360*Y4*X2-300*X4*Y4+56*Y6-336*X2*
     2Y6+280*X4*Y6))
```

```
      BX18=6*T**2*TX*((-12*X*Y4+20*X3*Y4+24*Y6*X-40*X3*Y6-12*X*Y8+20*X3*
     1Y8)+0.3*R2*(12*X*Y2-24*X3*Y2+12*X5*Y2-60*X*Y4+120*Y4*X3-60*X5*Y4+5
     26*Y6*X-112*X3*Y6+56*X5*Y6))
C
      AX6=BX16+BX17+BX18
C
      GOTO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
     124,25,26,27,28,29,30,31,32,33,34,35,36) N
C
1     F1=AX1*AAX1
      FA=F1
      RETURN
C
2     F2=AX2*AAX1
      FA=F2
      RETURN
C
3     F3=AX3*AAX1
      FA=F3
      RETURN
C
4     F4=AX4*AAX1
      FA=F4
      RETURN
C
      F5=AX5*AAX1
5     FA=F5
      RETURN
C
6     F6=AX6*AAX1
      FA=F6
      RETURN
C
7     F7=AX1*AAX2
      FA=F7
      RETURN
C
8     F8=AX2*AAX2
      FA=F8
      RETURN
C
9     F9=AX3*AAX2
      FA=F9
      RETURN
C
10    F10=AX4*AAX2
      FA=F10
      RETURN
C
11    F11=AX5*AAX2
```

```
      FA=F11
      RETURN
C
 12   F12=AX6*AAX2
      FA=F12
      RETURN
C
 13   F13=AX1*AAX3
      FA=F13
      RETURN
C
 14   F14=AX2*AAX3
      FA=F14
      RETURN
C
 15   F15=AX3*AAX3
      FA=F15
      RETURN
C
 16   F16=AX4*AAX3
      FA=F16
      RETURN
C
 17   F17=AX5*AAX3
      FA=F17
      RETURN
C
 18   F18=AX6*AAX3
      FA=F18
      RETURN
C
 19   F19=AX1*AAX4
      FA=F19
      RETURN
C
 20   F20=AX2*AAX4
      FA=F20
      RETURN
C
 21   F21=AX3*AAX4
      FA=F21
      RETURN
C
 22   F22=AX4*AAX4
      FA=F22
      RETURN
C
 23   F23=AX5*AAX4
      FA=F23
      RETURN
```

```
C
 24    F24=AX6*AAX4
         FA=F24
         RETURN
C
 25    F25=AX1*AAX5
         FA=F25
         RETURN
C
 26    F26=AX2*AAX5
         FA=F26
         RETURN
C
27     F27=AX3*AAX5
         FA=F27
         RETURN
C
 28    F28=AX4*AAX5
         FA=F28
         RETURN
C
 29    F29=AX5*AAX5
         FA=F29
         RETURN
C
 30    F30=AX6*AAX5
         FA=F30
         RETURN
C
 31    F31=AX1*AAX6
         FA=F31
         RETURN
C
   32  F32=AX2*AAX6
         FA=F32
         RETURN
C
 33    F33=AX3*AAX6
         FA=F33
         RETURN
C
 34    F34=AX4*AAX6
         FA=F34
         RETURN
C
 35    F35=AX5*AAX6
         FA=F35
         RETURN
C
 36    F36=AX6*AAX6
         FA=F36
         RETURN
         END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                              C
C  FUNCTIONS TO BE INTERGRATED TO FORM THE MATRIX A OF         C
C  THE  EIGEN-VALUE PROBLEM FOR THE MODE CATEGORY 4            C
C                                                              C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
      FUNCTION F(N,X,Y,R,T,FA)
C    N: NUMBER OF FUNCTIONS TO BE INTEGRATED
      R: ASPECT RATIO R=A/B
      T: TAPER PARAMETER
      X2=X*X
      X3=X2*X
      X4=X3*X
      X5=X4*X
      X6=X5*X
      X7=X6*X
      X8=X7*X
      X9=X8*X
      Y2=Y*Y
      Y3=Y2*Y
      Y4=Y3*Y
      Y5=Y4*Y
      Y6=Y5*Y
      Y7=Y6*Y
      Y8=Y7*Y
      Y9=Y8*Y
      AAX1=X*Y*(1-2*X2+X4-2*Y2+4*X2*Y2-2*Y2*X4+Y4-2*X2*Y4+X4*Y4)
      AAX2=AAX1*X2
      AAX3=AAX1*Y2
      AAX4=AAX1*X2*Y2
      AAX5=AAX2*X2
      AAX6=AAX3*Y2
      TX=(1+T*X)
      R2=R*R
      R4=R2*R2
C
      BX1=(TX)**3*((120*X*Y-240*Y3*X+120*Y5*X)+2*R2*(144*X*Y-240*X3*Y-
     1240*X*Y3+400*X3*Y3)+R4*(120*X*Y-240*X3*Y+120*X5*Y))
C
      BX2=6*T*(TX)**2*(-12*Y+60*Y*X2+24*Y3-120*Y3*X2-12*Y5+60*X2*Y5+R2*
     1(-12*Y+72*X2*Y-60*Y*X4+20*Y3-120*X2*Y3+100*X4*Y3))
C
      BX3=6*T**2*(TX)*((-12*Y*X+20*Y*X3+24*X*Y3-40*Y3*X3-12*X*Y5+20*X3*Y
     15)+0.3*R2*(-12*X*Y+24*X3*Y-12*Y*X5+20*X*Y3-40*X3*Y3+20*X5*Y3))
      AX1=BX1+BX2+BX3
C
      BX4=TX**3*((-240*X*Y+840*Y*X3+480*X*Y3-1680*X3*Y3-240*X*Y5+840*X3*
     1Y5)+2*R2*(-72*X*Y+480*X3*Y-504*X5*Y+120*X*Y3-800*X3*Y3+840*X5*Y3)
```

```
     2+R4*(120*X3*Y-240*X5*Y+120*X7*Y))
C
      BX5=6*T*TX**2*((6*Y-120*Y*X2+210*Y*X4-12*Y3+240*X2*Y3-420*X4*Y3+6
     1*Y5-120*X2*Y5+210*X4*Y5)+R2*(-36*Y*X2+120*X4*Y-84*X6*Y+60*X2*Y3-20
     20*X4*Y3+140*X6*Y3))
C
      BX6=6*T**2*TX*((6*Y*X-40*Y*X3+42*Y*X5-12*Y3*X+80*X3*Y3-84*X5*Y3+6*
     1X*Y5-40*X3*Y5+42*X5*Y5)+0.3*R2*(-12*X3*Y+24*X5*Y-12*Y*X7+20*X3*Y3-
     240*X5*Y3+20*X7*Y3))
C
      AX2=BX4+BX5+BX6
C
      BX7=TX**3*((120*X*Y3-240*X*Y5+120*X*Y7)+2*R2*(-72*X*Y+120*X3*Y+480
     1*Y3*X-800*Y3*X3-504*X*Y5+840*X3*Y5)+R4*(-240*X*Y+480*Y*X3-240*X5*Y
     2+840*X*Y3-1680*X3*Y3+840*X5*Y3))
C
      BX8=6*T*TX**2*((-12*Y3+60*X2*Y3+24*Y5-120*X2*Y5-12*Y7+60*X2*Y7)+R
     12*(6*Y-36*X2*Y+30*X4*Y-40*Y3+240*Y3*X2-200*X4*Y3+42*Y5-252*X2*Y5+
     2210*X4*Y5))
C
      BX9=6*T**2*TX*((-12*X*Y3+20*X3*Y3+24*Y5*X-40*X3*Y5-12*X*Y7+20*X3*Y
     17)+0.3*R2*(6*X*Y-12*X3*Y+6*Y*X5-40*X*Y3+80*Y3*X3-40*X5*Y3+42*X*Y5-
     284*X3*Y5+42*X5*Y5))
C
      AX3=BX7+BX8+BX9
C
      BX10=TX**3*((-240*X*Y3+840*X3*Y3+480*X*Y5-1680*X3*Y5-240*X*Y7+840*
     1X3*Y7)+2*R2*(36*X*Y-240*X3*Y+252*X5*Y-240*X*Y3+1600*X3*Y3-1680*X5*
     2Y3+252*X*Y5-1680*X3*Y5+1764*X5*Y5)+R4*(-240*X3*Y+480*X5*Y-240*X7*Y
     3+840*X3*Y3-1680*X5*Y3+840*Y3*X7))
C
      BX11=6*T*TX**2*((6*Y3-120*X2*Y3+210*X4*Y3-12*Y5+240*X2*Y5-420*X4*Y
     15+6*Y7-120*X2*Y7+210*X4*Y7)+0.3*R2*(18*X2*Y-60*X4*Y+42*X6*Y-120*X2
     2*Y3+400*X4*Y3-280*X6*Y3+126*X2*Y5-420*X4*Y5+294*X6*Y5))
C
      BX12=6*T**2*TX*((6*X*Y3-40*X3*Y3+42*X5*Y3-12*X*Y5+80*X3*Y5-84*X5*Y
     15+6*X*Y7-40*X3*Y7+42*X5*Y7)+0.3*R2*(6*X3*Y-12*X5*Y+6*X7*Y-40*X3*Y3
     2+80*X5*Y3-40*X7*Y3+42*X3*Y5-84*X5*Y5+42*X7*Y5))
C
      AX4=BX10+BX11+BX12
C
      BX13=TX**3*((120*X*Y-1680*Y*X3+3024*Y*X5-240*Y3*X+3360*X3*Y3-6048*
     1X5*Y3+120*X*Y5-1680*X3*Y5+3024*X5*Y5)+2*R2*(-240*X3*Y+1008*X5*Y-86
     24*X7*Y+400*X3*Y3-1680*X5*Y3+1440*X7*Y3)+R4*(120*X5*Y-240*X7*Y+120*
     3X9*Y))
C
      BX14=6*T*TX**2*((60*Y*X2-420*Y*X4+504*X6*Y-120*Y3*X2+840*X4*Y3-100
     18*X6*Y3+60*X2*Y5-420*X4*Y5+504*X6*Y5)+R2*(-60*Y*X4+168*X6*Y-108*X8
     2*Y+100*X4*Y3-280*X6*Y3+180*X8*Y3))
C
```

```
      BX15=6*T**2*TX*((120*Y*X3-84*Y*X5+72*Y*X7-40*Y3*X3+168*X5*Y3-144*X7
     1*Y3+20*X3*Y5-84*X5*Y5+72*X7*Y5)+0.3*R2*(-12*Y*X5+24*X7*Y-12*Y*X9+2
     20*X5*Y3-40*X7*Y3+20*X9*Y3))
C
      AX5=BX13+BX14+BX15
C
      BX16=TX**3*((120*X*Y5-240*X*Y7+120*X*Y9)+2*R2*(-240*X*Y3+400*X3*Y3
     1+1008*Y5*X-1680*X3*Y5-864*X*Y7+1440*X3*Y7)+R4*(120*X*Y-240*X3*Y+
     2120*X5*Y-1680*X*Y3+3360*Y3*X3-1680*X5*Y3+3024*X*Y5-6048*X3
     3*Y5+3024*X5*Y5))
C
      BX17=6*T*TX**2*((-12*Y5+60*X2*Y5+24*Y7-120*X2*Y7-12*Y9+60*X2*Y9)+R
     12*(20*Y3-120*X2*Y3+100*X4*Y3-84*Y5+504*Y5*X2-420*X4*Y5+72*Y7-432*X
     22*Y7+360*X4*Y7))
C
      BX18=6*T**2*TX*((-12*X*Y5+20*X3*Y5+24*Y7*X-40*X3*Y7-12*X*Y9+20*X3*
     1Y9)+0.3*R2*(20*X*Y3-40*X3*Y3+20*X5*Y3-84*X*Y5+168*Y5*X3-84*X5*Y5+7
     22*X*Y7-144*X3*Y7+72*X5*Y7))
C
      AX6=BX16+BX17+BX18
C
      GOTO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
     124,25,26,27,28,29,30,31,32,33,34,35,36) N
C
1     F1=AX1*AAX1
      FA=F1
      RETURN
C
 2    F2=AX2*AAX1
      FA=F2
      RETURN
C
 3    F3=AX3*AAX1
      FA=F3
      RETURN
C
 4    F4=AX4*AAX1
      FA=F4
      RETURN
C
      F5=AX5*AAX1
 5    FA=F5
      RETURN
C
 6    F6=AX6*AAX1
      FA=F6
      RETURN
C
 7    F7=AX1*AAX2
      FA=F7
```

```
      RETURN
C
    8 F8=AX2*AAX2
      FA=F8
      RETURN
C
    9 F9=AX3*AAX2
      FA=F9
      RETURN
C
   10 F10=AX4*AAX2
      FA=F10
      RETURN
C
   11 F11=AX5*AAX2
      FA=F11
      RETURN
C
   12 F12=AX6*AAX2
      FA=F12
      RETURN
C
   13 F13=AX1*AAX3
      FA=F13
      RETURN
C
   14 F14=AX2*AAX3
      FA=F14
      RETURN
C
   15 F15=AX3*AAX3
      FA=F15
      RETURN
C
   16 F16=AX4*AAX3
      FA=F16
      RETURN
C
   17 F17=AX5*AAX3
      FA=F17
      RETURN
C
   18 F18=AX6*AAX3
      FA=F18
      RETURN
C
   19 F19=AX1*AAX4
      FA=F19
      RETURN
C
```

```
 20   F20=AX2*AAX4
      FA=F20
      RETURN
C
 21   F21=AX3*AAX4
      FA=F21
      RETURN
C
 22   F22=AX4*AAX4
      FA=F22
      RETURN
C
 23   F23=AX5*AAX4
      FA=F23
      RETURN
C
 24   F24=AX6*AAX4
      FA=F24
      RETURN
C
 25   F25=AX1*AAX5
      FA=F25
      RETURN
C
 26   F26=AX2*AAX5
      FA=F26
      RETURN
C
27    F27=AX3*AAX5
      FA=F27
      RETURN
C
 28   F28=AX4*AAX5
      FA=F28
      RETURN
C
 29   F29=AX5*AAX5
      FA=F29
      RETURN
C
 30   F30=AX6*AAX5
      FA=F30
      RETURN
C
 31   F31=AX1*AAX6
      FA=F31
      RETURN
C
  32  F32=AX2*AAX6
      FA=F32
```

```
      RETURN
C
 33   F33=AX3*AAX6
      FA=F33
      RETURN
C
 34   F34=AX4*AAX6
      FA=F34
      RETURN
C
 35   F35=AX5*AAX6
      FA=F35
      RETURN
C
 36   F36=AX6*AAX6
      FA=F36
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                 C
C  FUNCTIONS TO BE INTERGRATED TO FORM THE MATRIX A OF            C
C  THE  EIGEN-VALUE PROBLEM FOR THE MODE CATEGORY 3               C
C                                                                 C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
      FUNCTION F(N,X,Y,R,T,FA)
C    N: NUMBER OF FUNCTIONS TO BE INTEGRATED
      R: ASPECT RATIO R=A/B
      T: TAPER PARAMETER
      X2=X*X
      X3=X2*X
      X4=X3*X
      X5=X4*X
      X6=X5*X
      X7=X6*X
      X8=X7*X
      X9=X8*X
      Y2=Y*Y
      Y3=Y2*Y
      Y4=Y3*Y
      Y5=Y4*Y
      Y6=Y5*Y
      Y7=Y6*Y
      Y8=Y7*Y
      Y9=Y8*Y
      AAX1=Y*(1-2*X2+X4-2*Y2+4*X2*Y2-2*Y2*X4+Y4-2*X2*Y4+X4*Y4)
      AAX2=AAX1*X2
      AAX3=AAX1*Y2
      AAX4=AAX1*X2*Y2
      AAX5=AAX2*X2
      AAX6=AAX3*Y2
      TX=(1+T*X)
      R2=R*R
      R4=R2*R2
      BX1=(TX)**3*((24*Y-48*Y3+24*Y5)+2*R2*(48*Y-144*Y*X2-80*Y3+240*X2*
     1Y3)+R4*(120*Y-240*X2*Y+120*X4*Y))
C
      BX2=6*T*(TX)**2*(24*X*Y-48*Y3*X+24*X*Y5+R2*(48*X*Y-48*X3*Y-80*X*Y3
     1+80*X3*Y3))
      BX3=6*T**2*(TX)*((-4*Y+12*Y*X2+8*Y3-24*Y3*X2-4*Y5+12*X2*Y5)+0.3*R2
     1*(-12*Y+24*X2*Y-12*Y*X4+20*Y3-40*X2*Y3+20*X4*Y3))
C
      AX1=BX1+BX2+BX3
C
      BX4=TX**3*((-48*Y+360*X2*Y+96*Y3-720*X2*Y3-48*Y5+360*X2*Y5)+2*R2*(
     1-24*Y+288*X2*Y-360*X4*Y+40*Y3-480*X2*Y3+600*X4*Y3)+R4*(120*Y*X2-24
     20*Y*X4+120*X6*Y))
```

```
      BX5=6*T*TX**2*((-48*X*Y+120*X3*Y+96*X*Y3-240*X3*Y3-48*Y5*X+120*X3*
     1Y5)+R2*(-24*X*Y+96*X3*Y-72*X5*Y+40*X*Y3-160*X3*Y3+120*X5*Y3))
      BX6=6*T**2*TX*((2*Y-24*X2*Y+30*X4*Y-4*Y3+48*X2*Y3-60*X4*Y3+2*Y5-24
     1*X2*Y5+30*X4*Y5)+0.3*R2*(-12*X2*Y+24*X4*Y-12*Y*X6+20*X2*Y3-40*X4*Y
     23+20*X6*Y3))
C
      AX2=BX4+BX5+BX6
C
      BX7=TX**3*((24*Y3-48*Y5+24*Y7)+2*R2*(-24*Y+72*X2*Y+160*Y3-480*X2*Y
     13-168*Y5+504*X2*Y5)+R4*(-240*Y+480*Y*X2-240*X4*Y+840*Y3-1680*X2*Y3
     2+840*X4*Y3))
      BX8=6*T*TX**2*((24*X*Y3-48*X*Y5+24*X*Y7)+R2*(-24*X*Y+24*X3*Y+160*
     1Y3*X-160*X3*Y3-168*X*Y5+168*X3*Y5))
      BX9=6*T**2*TX*((-4*Y3+12*X2*Y3+8*Y5-24*X2*Y5-4*Y7+12*X2*Y7)+0.3*R2
     1*(6*Y-12*X2*Y+6*Y*X4-40*Y3+80*Y3*X2-40*X4*Y3+42*Y5-84*X2*Y5+42*X4*
     2Y5))
C
      AX3=BX7+BX8+BX9
C
      BX10=TX**3*((-48*Y3+360*X2*Y3+96*Y5-720*X2*Y5-48*Y7+360*X2*Y7)+2*R
     12*(12*Y-144*X2*Y+180*X4*Y-80*Y3+960*X2*Y3-1200*X4*Y3+84*Y5-1008*X2
     2*Y5+1260*X4*Y5)+R4*(-240*X2*Y+480*X4*Y-240*X6*Y+840*X2*Y3-1680*X4*
     3Y3+840*Y3*X6))
      BX11=6*T*TX**2*((-48*X*Y3+120*X3*Y3+96*X*Y5-240*X3*Y5-48*X*Y7+120*
     1X3*Y7)+0.3*R2*(12*X*Y-48*X3*Y+36*Y*X5-80*X*Y3+360*X3*Y3-240*X5*Y3+
     284*X*Y5-336*X3*Y5+252*X5*Y5))
      BX12=6*T**2*TX*((2*Y3-24*X2*Y3+30*X4*Y3-4*Y5+48*X2*Y5-60*X4*Y5+2*Y
     17-24*X2*Y7+30*X4*Y7)+0.3*R2*(6*X2*Y-12*X4*Y+6*X6*Y-40*Y3*X2+80*Y3*
     2X4-40*X6*Y3+42*X2*Y5-84*X4*Y5+42*X6*Y5))
      AX4=BX10+BX11+BX12
C
      BX13=TX**3*((24*Y-720*Y*X2+1680*Y*X4-48*Y3+1440*X2*Y3-3360*X4*Y3+2
     14*Y5-720*X2*Y5+1680*X4*Y5)+2*R2*(-144*X2*Y+720*X4*Y-672*X6*Y+240*X
     22*Y3-1200*X4*Y3+1120*X6*Y3)+R4*(120*X4*Y-240*X6*Y+120*X8*Y))
      BX14=6*T*TX**2*((24*Y*X-240*X3*Y+336*X5*Y-48*Y3*X+480*X3*Y3-672*X5
     1*Y3+24*X*Y5-240*X3*Y5+336*X5*Y5)+R2*(-48*Y*X3+144*X5*Y-96*X7*Y+80*
     2X3*Y3-240*X5*Y3+160*X7*Y3))
      BX15=6*T**2*TX*((12*Y*X2-60*Y*X4+56*Y*X6-24*Y3*X2+120*X4*Y3-112*X6
     1*Y3+12*X2*Y5-60*X4*Y5+56*X6*Y5)+0.3*R2*(-12*Y*X4+24*X6*Y-12*Y*X8+2
     20*X4*Y3-40*X6*Y3+20*X8*Y3))
C
      AX5=BX13+BX14+BX15
C
      BX16=TX**3*((24*Y5-48*Y7+24*Y9)+2*R2*(-80*Y3+240*X2*Y3+336*Y5-1008
     1*X2*Y5-288*Y7+864*X2*Y7)+R4*(120*Y-240*X2*Y+120*X4*Y-1680*Y3+3360*
     2Y3*X2-1680*X4*Y3+3024*Y5-6048*X2*Y5+3024*X4*Y5))
      BX17=6*T*TX**2*((24*X*Y5-48*X*Y7+24*X*Y9)+R2*(-80*X*Y3+80*X3*Y3+33
     16*Y5*X-336*X3*Y5-288*X*Y7+288*X3*Y7))
      BX18=6*T**2*TX*((-4*Y5+12*X2*Y5+8*Y7-24*X2*Y7-4*Y9+12*X2*Y9)+0.3*R
     12*(20*Y3-40*X2*Y3+20*X4*Y3-84*Y5+168*Y5*X2-84*X4*Y5+72*Y7-144*X2*Y
```

```
     27+72*X4*Y7))
C
      AX6=BX16+BX17+BX18
C
      GOTO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
     124,25,26,27,28,29,30,31,32,33,34,35,36) N
C
1     F1=AX1*AAX1
      FA=F1
      RETURN
C
2     F2=AX2*AAX1
      FA=F2
      RETURN
C
3     F3=AX3*AAX1
      FA=F3
      RETURN
C
4     F4=AX4*AAX1
      FA=F4
      RETURN
C
      F5=AX5*AAX1
5     FA=F5
      RETURN
C
6     F6=AX6*AAX1
      FA=F6
      RETURN
C
7     F7=AX1*AAX2
      FA=F7
      RETURN
C
8     F8=AX2*AAX2
      FA=F8
      RETURN
C
9     F9=AX3*AAX2
      FA=F9
      RETURN
C
10    F10=AX4*AAX2
      FA=F10
      RETURN
C
11    F11=AX5*AAX2
      FA=F11
```

```
      RETURN
C
   12 F12=AX6*AAX2
      FA=F12
      RETURN
C
   13 F13=AX1*AAX3
      FA=F13
      RETURN
C
   14 F14=AX2*AAX3
      FA=F14
      RETURN
C
   15 F15=AX3*AAX3
      FA=F15
      RETURN
C
   16 F16=AX4*AAX3
      FA=F16
      RETURN
C
   17 F17=AX5*AAX3
      FA=F17
      RETURN
C
   18 F18=AX6*AAX3
      FA=F18
      RETURN
C
   19 F19=AX1*AAX4
      FA=F19
      RETURN
C
   20 F20=AX2*AAX4
      FA=F20
      RETURN
C
   21 F21=AX3*AAX4
      FA=F21
      RETURN
C
   22 F22=AX4*AAX4
      FA=F22
      RETURN
C
   23 F23=AX5*AAX4
      FA=F23
      RETURN
C
```

```
   24   F24=AX6*AAX4
          FA=F24
          RETURN
C
   25   F25=AX1*AAX5
          FA=F25
          RETURN
C
   26   F26=AX2*AAX5
          FA=F26
          RETURN
C
  27    F27=AX3*AAX5
          FA=F27
          RETURN
C
   28   F28=AX4*AAX5
          FA=F28
          RETURN
C
   29   F29=AX5*AAX5
          FA=F29
          RETURN
C
   30   F30=AX6*AAX5
          FA=F30
          RETURN
C
   31   F31=AX1*AAX6
          FA=F31
          RETURN
C
     32  F32=AX2*AAX6
          FA=F32
          RETURN
C
   33   F33=AX3*AAX6
          FA=F33
          RETURN
C
   34   F34=AX4*AAX6
          FA=F34
          RETURN
C
   35   F35=AX5*AAX6
          FA=F35
          RETURN
C
   36   F36=AX6*AAX6
          FA=F36
          RETURN
          END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C  FUNCTIONS TO BE INTERGRATED TO FORM THE MATRIX B OF                C
C  THE  EIGEN-VALUE PROBLEM FOR THE MODE CATEGORY 1                   C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C
        FUNCTION F(N,X,Y,T,FA)
C     N: NUMBER OF FUNCTIONS TO BE INTEGRATED
C     T: TAPER PARAMETER
        X2=X*X
        X3=X2*X
        X4=X3*X
        Y2=Y*Y
        Y3=Y2*Y
        Y4=Y3*Y
        AAX1=1-2*X2+X4-2*Y2+4*X2*Y2-2*Y2*X4+Y4-2*X2*Y4+X4*Y4
        AAX2=AAX1*X2
        AAX3=AAX1*Y2
        AAX4=AAX1*X2*Y2
        AAX5=AAX2*X2
        AAX6=AAX3*Y2
        TX=(1+T*X)
C
        GOTO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
       124,25,26,27,28,29,30,31,32,33,34,35,36) N
C
1     F1=TX*AAX1*AAX1
        FA=F1
        RETURN
C
   2    F2=TX*AAX2*AAX1
        FA=F2
        RETURN
C
   3    F3=TX*AAX3*AAX1
        FA=F3
        RETURN
C
   4    F4=TX*AAX4*AAX1
        FA=F4
        RETURN
C
        F5=TX*AAX5*AAX1
   5    FA=F5
        RETURN
C
   6    F6=TX*AAX6*AAX1
        FA=F6
```

```
      RETURN
C
 7    F7=TX*AAX1*AAX2
      FA=F7
      RETURN
C
 8    F8=TX*AAX2*AAX2
      FA=F8
      RETURN
C
 9    F9=TX*AAX3*AAX2
      FA=F9
      RETURN
C
 10   F10=TX*AAX4*AAX2
      FA=F10
      RETURN
C
 11   F11=TX*AAX5*AAX2
      FA=F11
      RETURN
C
 12   F12=TX*AAX6*AAX2
      FA=F12
      RETURN
C
 13   F13=TX*AAX1*AAX3
      FA=F13
      RETURN
C
 14   F14=TX*AAX2*AAX3
      FA=F14
      RETURN
C
 15   F15=TX*AAX3*AAX3
      FA=F15
      RETURN
C
 16   F16=TX*AAX4*AAX3
      FA=F16
      RETURN
C
 17   F17=TX*AAX5*AAX3
      FA=F17
      RETURN
C
 18   F18=TX*AAX6*AAX3
      FA=F18
      RETURN
C
```

```
 19   F19=TX*AAX1*AAX4
        FA=F19
        RETURN
C
 20   F20=TX*AAX2*AAX4
        FA=F20
        RETURN
C
 21   F21=TX*AAX3*AAX4
        FA=F21
        RETURN
C
 22   F22=TX*AAX4*AAX4
        FA=F22
        RETURN
C
 23   F23=TX*AAX5*AAX4
        FA=F23
        RETURN
C
 24   F24=TX*AAX6*AAX4
        FA=F24
        RETURN
C
 25   F25=TX*AAX1*AAX5
        FA=F25
        RETURN
C
 26   F26=TX*AAX2*AAX5
        FA=F26
        RETURN
C
27    F27=TX*AAX3*AAX5
        FA=F27
        RETURN
C
 28   F28=TX*AAX4*AAX5
        FA=F28
        RETURN
C
 29   F29=TX*AAX5*AAX5
        FA=F29
        RETURN
C
 30   F30=TX*AAX6*AAX5
        FA=F30
        RETURN
C
 31   F31=TX*AAX1*AAX6
        FA=F31
```

```
      RETURN
C
   32 F32=TX*AAX2*AAX6
      FA=F32
      RETURN
C
   33 F33=TX*AAX3*AAX6
      FA=F33
      RETURN
C
   34 F34=TX*AAX4*AAX6
      FA=F34
      RETURN
C
   35 F35=TX*AAX5*AAX6
      FA=F35
      RETURN
C
   36 F36=TX*AAX6*AAX6
      FA=F36
      RETURN
      END
```