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Measure Theoretic Optimization in Information Theory and Stochastic Control Subject to Uncertainty

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Measure Theoretic Optimization in Information Theory and Stochastic Control Subject to Uncertainty

by

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Abstract

In the first chapter, two general classes of optimization problems are discussed. The application of these problems in information theory and stochastic control contexts is the main goal of this thesis. In chapters two and three, applications in source coding are discussed, then in the last two chapters, the applications of these problems in control of uncertain continuous time systems are explored.

Chapter two is concerned with lossless source coding for a family of source distributions whose relative entropy with respect to a given nominal distribution is bounded above by some fixed number. First the minimax average length source coding problem subject to the relative entropy constraint is considered. The minimizing players are the codeword lengths, while the maximizing players are the uncertain source distributions. This leads to coding with respect to the exponential pay-off. Second the minimax redundancy problem subject to relative entropy constraints is considered. It is shown that this problem reduces to an extension of the exponential Huffman coding.

In the third chapter, the problem of rate distortion formulation for abstract sources is considered. We introduce a general framework for rate distortion theory in Polish spaces. Previous well known results for abstract alphabets are extended to the more general case in which marginal measures on the reproduction space may not be absolutely continuous with respect to the optimal marginal measure. Moreover, the question of existence of a solution to the implicit equation of optimal distribution is addressed by proving a fixed-point theorem.

In chapter four, the problem of stochastic control for uncertain systems is considered. Here we aim at establishing explicit connections between the Legendre-Fenchel transform, robustness of uncertain systems, and risk-seeking and risk-averse optimization. The connections are obtained by introducing new classes of optimal stochastic uncertain systems in which the uncertainty is described by a family of probability measures, which satisfy a certain fidelity criteria. With respect to this formulation, the minimization of the relative entropy over the set of uncertain measures, subject to the fidelity criteria is shown to be equivalent to the Legendre-Fenchel transform.

In the last chapter, another class of stochastic optimal uncertain control systems are considered, in which uncertainty is described by a relative entropy constraint between the nominal and uncertain measures, while the pay-off is a linear functional of the uncertain measure. The theory is developed in an abstract setting and then applied to nonlinear partially observable continuous-time uncertain controlled systems.

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Contributions

Chapter Two

• Formulation of lossless coding for a class of uncertain sources with uncertainty characterized by a relative entropy constraint.
• Robust Shannon and Huffman Coding for a class of uncertain sources
• Minmax Redundancy Coding

Chapter Three

• Formulation of Rate Distortion Problem for Abstract Sources
• Gateaux Differentiability of Mutual Information
• Necessary Conditions of Optimality for the Abstract Rate Distortion Problem
• Relations of Abstract Rate Distortion to the Fixed Point Problems

Chapter Four

• Formulation of Stochastic Control Problem for a Class of Uncertain Systems with Average Energy Constraints, as a Max-Min Problem
• Existence of Minimizing Measures
• Equivalence of Constrianed and Unconstrained Problems
• Application of the Max-Min Approach to Partially Observable Uncertain Control Systems Driven by Stochastic Differential Equations(SDE)
• Relations of the Max-Min Problem to Disturbance Attenuation Problem

Chapter Five

• Formulation of Stochastic Control Problem for a Class of Uncertain Systems with Relative Entropy Constraints, as a Min-Max Problem
• Existence of Maximizing Measures
• Equivalence of Constrianed and Unconstrained Problems
• Application of the Min-Max Approach to Partially Observable Uncertain Control Systems Driven by SDEs
• Relations of the Min-Max Problem to Disturbance Attenuation Problem

Notations

- $\Re$ represents the set of real numbers.
- $H(\nu)$ is the entropy of the probability mass function ν.
- $H(\nu|\mu)$ is the relative entropy between probability measures ν and μ.
- $\mathcal{C}(\mathcal{X})$ is the class of all uniquely decodable codes defined on the source space $\mathcal{X}$.
- $||\nu - \mu||_{tv}$ is the variational distance between measures ν and μ.
- $\mathcal{B}(\Sigma)$ is the σ-algebra generated by open sets in Σ and $\mathcal{M}(\Sigma)$ is the space of all probability measures defined on $(\Sigma, \mathcal{B}(\Sigma))$.
- $\xrightarrow{w^*}$ denotes weak* covergence for a sequence of measures.
- $BC(\Sigma)$ is the space of bounded continuous real valued functions defined on Σ.
- $\delta f(x_0, x - x_0)$ is the Gateaux differential of the function f at the point x_0 in the direction $x - x_0$.
- $\mu - ess\sup_{x \in X} f(x) \triangleq \inf\left\{\alpha; \mu\{x \in X; f(x) > \alpha\} = 0\right\} \triangleq \inf_{\Delta \in \mathcal{N}} \sup_{x \in \Delta^c} f(x)$ is the essential supremum of function $f : X \to \Re$ with respect to the measure μ (defined over the measure space $(X, \mathcal{B}(X))$). Here $\mathcal{N}$ denotes the set of all μ-null sets in $\mathcal{B}(X)$.
- $\mathcal{L}(\Re^d; \Re^d)$ is the space of linear operators from $\Re^d$ to $\Re^d$.
- $L_2([0,T]; \Re^d)$ is the space of square integrable functions $f : [0,T] \to \Re^d$, i.e., $\int_0^T f'(t) f(t) dt < \infty$, where f' denotes transpose of vector f.
- $C([0,T]; \Re^m)$ is the space of continuous functions $g : [0,T] \to \Re^m$.

Contents

Chapter 1

Two Classes of Optimization Problems

In this chapter, the basic mathematical optimization problems used throughout the thesis are discussed. These problems include two related classes. The first problem involves the maximization of the average value of a given cost function. The average is taken with respect to an uncertain distribution ν which satisfies a relative entropy constraint subject to a given nominal distribution μ. The second problem is the minimization of relative entropy between an uncertain distribution ν and a nominal distribution μ, subject to constraints on average with respect to ν. The required background material and mathematical definitions can be found in Appendix A.

Let (Σ, d) denote a complete separable metric space (Polish Space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space in which $\mathcal{B}(\Sigma)$ is identified as the σ-algebra of Borel sets generated by open sets in Σ. Let $\mathcal{M}(\Sigma)$ denote the set of countably additive probability measures on $(\Sigma, \mathcal{B}(\Sigma))$, $\ell : \Sigma \rightarrow \Re$ be a real-valued measurable function bounded from below, and $H(\nu|\mu)$ be the relative entropy or Kullback-Leibler distance between measure $\nu \in \mathcal{M}(\Sigma)$ and measure $\mu \in \mathcal{M}(\Sigma)$, defined as follows:

$$H(\nu|\mu) \triangleq \begin{cases} \int_\Sigma \log(\frac{d\nu}{d\mu})d\nu & \text{, if } \nu << \mu; \text{ and } \log\frac{d\nu}{d\mu} \in L_1(\nu) \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

where $\nu << \mu$ denotes absolute continuity of ν with respect to μ, i.e., if $\mu(E) = 0$ for $E \in \mathcal{B}(\Sigma)$, then $\nu(E) = 0$. If $\nu << \mu$, then $f = \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative of the measure ν with respect to μ, which is the function $f : \Sigma \rightarrow [0, \infty)$ such that $\nu(E) = \int_E f d\mu$ for any $E \in \mathcal{B}(\Sigma)$. If Σ is a finite set then the integral in the above definition can be simplified and written in terms of summations. Here $H(\nu|\mu)$ is considered as a measure of discrepancy between measure $\nu \in \mathcal{M}(\Sigma)$ and measure $\mu \in \mathcal{M}(\Sigma)$. Now we define two classes of optimization problems as follows.

1.1 First Class of Problems

Suppose the nominal measure μ and a set of uncertain stochastic systems described by the uncertainty set $\mathcal{M}_R$ are given. The uncertainty set is defined as $\mathcal{M}_R \triangleq \{\nu \in \mathcal{M}(\Sigma); H(\nu|\mu) \leq R\}$, in which $\mu \in \mathcal{M}(\Sigma)$ is fixed, $R \in (0, \infty)$, and $E_\nu(\ell) \equiv \int_\Sigma \ell d\nu$ is the average cost under uncertain distribution ν. Now define the following optimization problems.
For a fixed nominal measure $\mu \in \mathcal{M}(\Sigma)$, let $\nu^* \in \mathcal{M}_R$ denote the measure which achieves the supremum of the average pay-off functional subject to the relative entropy constraint. Then the first optimization problem is defined by

$$J(\nu^*) = \sup_{\nu \in \mathcal{M}_R} E_\nu(\ell), \quad R \in (0, \infty) \tag{1.2}$$

1.2 Second Class of Problems

Again assuming that the nominal measure is given, define the following uncertainty sets:

$$M_o \equiv \{\nu \in M(\Sigma) : E_\nu(\ell) \leq \gamma\}$$

for some $\gamma < m$, and

$$M_p \equiv \{\nu \in M(\Sigma) : E_\nu(\ell) \geq \gamma\}$$

for $\gamma > m$. Here $m \equiv \int \ell d\mu$ is the nominal energy or cost and $\gamma \in \Re$ is a given constant. Two optimization problems that can be considered are the following.

$$J_o = \inf_{\nu \in M_o} H(\nu|\mu)$$

and

$$J_p = \inf_{\nu \in M_p} H(\nu|\mu)$$

The first problem will be used in the context of robust source coding for a class of sources in Chapter 2. The same setup will also be used in Chapter 5, for optimization of uncertain dynamical systems. The second problem will be considered in the context of rate distortion function for abstract sources as explained in Chapter 3, and its application in uncertain dynamical systems is also studied in Chapter 4.

1.3 Optimal Measures

The measures, which attain the supremum in the first class and the infimum in the second class have a similar structure. The following family describes these measures:

$$\nu^*(F) = \frac{\int_F e^{\xi\ell} d\mu}{\int_\Sigma e^{\xi\ell} d\mu}, \quad \forall F \in \mathcal{B}(\Sigma), \quad \xi \in \Re$$

This exponential form will be shown to be useful in deriving the properties of the optimal solutions. In the context of rate distortion problem, the above equation describes an implicit relationship between the marginal distribution on the reproduction space and the conditional distribution. This implicit relationship will be studied further in Chapter 3 in the context of fixed point problems.

Chapter 2

Lossless Source Coding for an Uncertain Class of Sources

2.1 Introduction

The well-known problem of finding a uniquely decodable code with minimum average code word length, gives rise to the Shannon code. However, if the true distribution of the source is unknown and the code is designed with respect to a nominal distribution, then the relative entropy between the nominal distribution and the true distribution appears in the bounds of average code word length [1](Theorem 5.4.3, pp. 89-90). Therefore, under such uncertainty description, the resulting codes will not be robust neither in the sense of average length nor redundancy. Suppose the nominal or approximate distribution of the source is μ, while the unknown or true probability distribution of the source is any ν, which is absolutely continuous with respect to μ, and satisfies the relative entropy constraint, $H(\nu|\mu) \leq R$, where R is a fixed positive real number. Assume the source space is denoted by $\mathcal{X}$ and let $\mathcal{C}(\mathcal{X})$ denote the class of all uniquely decodable codes defined on the space $\mathcal{X}$. Under such assumptions, one approach to the source coding problem for the whole class of uncertain sources is formulated as follows:

$$\inf_{C \in \mathcal{C}(\mathcal{X})} \sup_{\{\nu; H(\nu|\mu) \leq R\}} E_\nu(\ell(C)) \tag{2.1}$$

where $\ell(C)$ is a real-valued random variable representing the code word lengths of the code C, E_ν denotes expectation with respect to the distribution ν. The precise minimax formulation is found in Section 2.2. The main objective of the minimax formulation is to encode the output of uncertain sources using the worst-case distribution ν^*, i.e. the distribution which maximizes the average length, as a function of the nominal distribution μ. Thus, by encoding the uncertain source using ν^*, the resulting code will be robust in the sense of average length over the set of uncertain sources which satisfy the relative entropy constraint. The second approach is to consider the following minimax redundancy problem:

$$\inf_{C \in \mathcal{C}(\mathcal{X})} \sup_{\{\nu; H(\nu|\mu) \leq R\}} \Big(E_\nu(\ell(C)) - H(\nu) \Big) \tag{2.2}$$

where $H(\nu)$ denotes the entropy of the source with distribution ν. This encoding approach leads to codes, which are robust in the sense of average redundancy. In the following subsections, we give some motivations for the choice of the cost functional as average code length and the choice of uncertainty set (relative entropy constraint).

2.1.1 Robustness of Average Code Length

The robustness of the average code length has application in transmission of variable length codes, where there is a buffer at the output of the encoder [6]. Suppose one wants to place one encoder (followed by a buffer) at the output of sources belonging to an uncertain family. The distribution of these sources can also change with time. Since the size of the buffers should be fixed in advance, and since it is preferable to have an equal buffer size for all sources, then it would be natural to expect that the average length of the codes should not change too much as the source distributions change from place to place and from time to time. This motivates us to consider the problem of minimax average length in presence of uncertainty.

2.1.2 Relative Entropy and Total Variation Norm

In the context of uncertainty description, total variation norm is a natural choice, since it measures the normed distance between two probability distributions. On the other hand relative entropy lacks the properties of a norm. However, for small uncertainties, one can show that relative entropy can be quite close to the variational distance. From the Pinsker inequality [14], it follows that $||\nu - \mu||_{tv}^2 \leq 2H(\nu|\mu)$, where $||\nu - \mu||_{tv} \triangleq \sum_i |\nu_i - \mu_i|$. We give two examples in which for small deviations from the nominal μ, relative entropy and total variation distance remain close to each other.

1) Consider a small deviation from the nominal distribution described by the convex combination $\nu^\epsilon = \epsilon\eta + (1-\epsilon)\mu$, where η is an arbitrary fixed distribution such that $\eta << \mu$. Then it can be easily shown that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} H(\nu^\epsilon|\mu) = 0 \tag{2.3}$$

Also we have

$$||\nu^\epsilon - \mu||_{tv} = \sum_i |\nu_i^\epsilon - \mu_i| = \epsilon . ||\eta - \mu||_{tv} \tag{2.4}$$

From (2.3) and (2.4), we see that both sides of the Pinsker inequality are of the order $O(\epsilon^2)$.
2) As another example, consider a small deviation described by $\nu_i^\epsilon = \mu_i + (-1)^i\epsilon$, when the number of source symbols (M) is even. Here ϵ is small enough so that $\mu_i > \epsilon$ for all i. Then we have $\lim_{\epsilon\to 0}\frac{1}{\epsilon}H(\nu^\epsilon|\mu) = 0$, and $||\nu^\epsilon - \mu||_{tv} = M\epsilon$. Again we can see that both sides of the Pinsker inequality are of the same order $O(\epsilon^2)$. These two examples show that for small uncertainties, in some cases, the Pinsker inequality can be tight.

2.1.3 Organization of the Chapter and Literature Review

In Section 2.3, it is shown that the minimax coding problem in (2.1), leads to coding with respect to an exponential pay-off. Shannon coding for such a pay-off was considered in [2] and it was shown to be related to the Rényi entropy. Based on these results, in Section 2.4, a numerical method is presented for computation of robust Shannon codes which is based on a monotonicity property of the relative entropy. In Section 2.5, the robust property of the robust Shannon code is studied, and it is shown that the amount of change in average length due to uncertainty is much smaller for robust Shannon codes than the Shannon code.

In Section 2.6, minimax Huffman coding is presented, by first finding the worst case measure and then encoding based on the exponential pay-off. Huffman coding for exponential pay-off was first done in [6]. We discuss this method in context of the relative entropy uncertainty modeling. In Section 2.7, simulations are presented for minimax Shannon and Huffman coding, which illustrate the robustness of these codes. In Section 2.8, the minimax average redundancy coding problem is formulated [4, 11, 15, 16], when the uncertainty is modeled via a relative entropy constraint. It is shown that this formulation leads to an extension of exponential Huffman coding as considered in [17].

There is a significant number of papers dealing with the coding problem for an unknown source, when either the empirical distribution of the source is available, or the uncertainty is modeled through certain unknown parameters [3, 4, 7]. A good survey of the literature and more recent results are found in [12]. Uncertainty modeling using relative entropy has been considered in [11], in which the problem of coding for only one unknown source is addressed. However, here unlike [11] and [12], we are dealing with the problem of source coding for a class containing many sources, which satisfy the relative entropy constraint. In our modeling, knowledge of the nominal distribution and the uncertainty radius R, is assumed. A universal modeling using relative entropy has been discussed in [10], where the tightest upper bound for the relative entropy between empirical distributions (of available training sequences), and a nominal distribution is found. The nominal distribution is itself computed as part of a search algorithm.

Although, in this chapter we consider minimax average length and average redundancy, it should be noted that maximal cost functions have also been studied in the literature, for instance the maximal redundancy in [15] and [16]. Finally, we point out that the current minimax source coding formulation may be generalized to applications in which the nominal distribution is parameterized as in [12]. In this case, the methods found in [12], which employ maximum likelihood techniques, can be invoked to estimate the parameters of the nominal distribution. Additional generalizations may include situations in which the source is described by ergodic finite state Markov chains.

2.2 Problem Formulation

Consider a source generating outputs from a finite set of symbols, denoted by Σ, according to an unknown probability measure, $\nu \in \mathcal{M}(\Sigma)$, where $\mathcal{M}(\Sigma)$ is the set of all probability measures defined on the measurable space $(\Sigma, \mathcal{B}(\Sigma))$[1]. Suppose certain modeling assumptions or access to limited empirical data lead to the belief that the nominal (approximate) source probability measure denoted by $\mu \in \mathcal{M}(\Sigma)$ can be constructed, and that the true unknown probability measures are absolutely continuous with respect to the nominal measure μ in $\mathcal{M}(\Sigma)$ [2]. Then the uncertainty associated with the unknown source measures can be described through a set of probability measures which satisfy a relative entropy constraint, and it is characterized by $\mathcal{M}_R = \{\nu \in \mathcal{M}(\Sigma); H(\nu|\mu) \leq R\}$. Here, $H(\nu|\mu)$ denotes the relative entropy between the true measure ν and the nominal measure μ, while R is a known non-negative real number, which describes the set of all admissible uncertain measures. The larger the value of R, the larger the uncertainty set of admissible measures. Under the above assumptions, the source coding problem for the entire class of measures $\mathcal{M}_R$ is defined as follows. Given the nominal source measure $\mu \in \mathcal{M}(\Sigma)$, find the code word lengths $\{\ell_j^*\}$ of a uniquely dec $\in \mathcal{M}_R$ which solve the following minimax sourc

$$\begin{cases} J(\ell^*, \nu^*) = \inf_{(\ell_1,\dots,\ell_M)} \sup_{\nu \in M(\Sigma)} E_\nu\big(\ell\big) \\ \text{Subject to } \; H(\nu|\mu) \leq R, \quad \sum_{i=1}^M D^{-\ell_i} \leq 1. \end{cases}$$

$$\begin{cases} J(\ell^*, \nu^*) = \inf_{(\ell_1,\dots,\ell_M)} \sup_{\nu \in M(\Sigma)} E_\nu\big(\ell\big) \\ \text{Subject to } \; H(\nu|\mu) \leq R, \quad \sum_{i=1}^M D^{-\ell_i} \leq 1. \end{cases} \tag{2.5}$$

Here parameter M denotes the source alphabet size, "$\ell : \Sigma \to \{\ell_1, ..., \ell_M\}$" represents the lengths of the codeword, and it is assumed that the code is D-ary. Clearly, if one encodes

[1] $\mathcal{B}(\Sigma)$ is the smallest σ-algebra containing Σ.

[2] this means if $\mu(B) = 0$ for any set $B \in \mathcal{B}(\Sigma)$, then $\nu(B) = 0$. In this case we use the notation $\nu << \mu$

based on the worst case measure $\nu^* \in \mathcal{M}_R$, then the average code word length will be less sensitive to different choices of source distributions from the set $\mathcal{M}_R$.

2.3 The Maximizing Measure

Using the theory of Lagrangian functionals, the minimax strategies of the constrained optimization problem can be found, as follows.
For every $s \in \Re, \lambda \in \Re$ define the Lagrangian as

$$L^{\lambda,s}(\ell,\nu) \triangleq E_\nu\big(\ell\big) - s\big(H(\nu|\mu) - R\big) + \lambda\Big(\sum_{i=1}^{M} D^{-\ell_i} - 1\Big), \tag{2.6}$$

and the associated dual functional

$$L^{\lambda,s}(\ell,\nu^*) = \sup_{\nu\in\mathcal{M}_R} L^{\lambda,s}(\ell,\nu)$$

It is important to notice that, the Kraft inequality does not depend on the probability measure $\nu \in \mathcal{M}_R$. The problem of maximization over $\mathcal{M}_R$ can be re-written in the unconstrained form. The equivalence of constrained and unconstrained problems can be shown as follows. Let $X = \big\{$countably additive signed measures $\nu : \mathcal{B}(\Sigma) \to \Re, \nu(\Sigma) < \infty, \nu << \mu\big\}$. It can easily be shown that X forms a vector space over the ground field $\Re$ and $\mathcal{M}_R$ is a convex set. Take $G(\nu) = H(\nu|\mu) - R,$ where $\nu \in X$ and $R \neq 0$. Then $G : X \to \Re$, is a convex mapping from X into the ordered vector space $(\Re, \prec)$ with natural ordering. If we take $\nu = \mu$, then $G(\nu) = -R < 0$. Hence there exists a measure $\nu \in X$ such that $G(\nu) < 0$. Also, $\sup_{\nu\in\mathcal{M}_R} \sum_{i=1}^{M} \ell_i\nu_i$ is finite, since all code lengths are finite. Let $f(\nu) = E_\nu(\ell)$. Then the conditions of the Lagrange duality theorem in [18] (page 224-225) are satisfied, and the constrained and unconstrained problems (corresponding to the supremum over $\mathcal{M}_R$) are equivalent [3].

Theorem 2.3.1 *For $s > 0$, the dual function $L^{\lambda,s}(\ell,\nu^*)$ is given by*

$$L^{\lambda,s}(\ell,\nu^*) = sR + s\log\Big(\sum_{i=1}^{M} e^{\frac{\ell_i}{s}}\mu_i\Big) + \lambda\Big(\sum_{i=1}^{M} D^{-\ell_i} - 1\Big). \tag{2.7}$$

where

$$\nu_i^* = \frac{e^{\frac{\ell_i}{s}}\mu_i}{\sum_{j=1}^{M} e^{\frac{\ell_j}{s}}\mu_j}, \qquad \forall i \in \{1, ..., M\}. \tag{2.8}$$

[3]The theorem in [18] deals with minimization of a convex functional. Multiplying the equation (4) on page 224, in a minus sign, converts the problem to maximization of a concave functional. This can be applied to maximization of $E_\nu(\ell)$, since it is linear hence concave.

Proof. Since $H(\nu|\nu^*) \geq 0$ we have,

$$H(\nu|\mu) = H(\nu|\nu^*) + \sum_{i=1}^{M} \nu_i \log \frac{\nu_i^*}{\mu_i} \geq \sum_{i=1}^{M} \frac{\ell_i}{s}\nu_i - \log\Big(\sum_{i=1}^{M} e^{\frac{\ell_j}{s}} \mu_j\Big)$$

Where the inequality is achieved for $\nu = \nu^*$. Since $s > 0$, then

$$\sum_{i=1}^{M} \ell_i \nu_i - sH(\nu|\mu) \leq s \log\Big(\sum_{i=1}^{M} e^{\frac{\ell_j}{s}} \mu_j\Big)$$

and the upper bound is achieved for $\nu = \nu^*$. •

Next, define $f(s,\ell) = s\log\Big(\sum_{i=1}^{M} e^{\frac{\ell_i}{s}} \mu_i\Big) + sR$; then $f(s,\ell)$ has the following properties.

Lemma 2.3.2 *(Convexity)*

The function $f(s,\ell)$ has the following properties.

a) f(s,ℓ) is convex as a function of ℓ ($\forall s > 0$);

b) f(s,ℓ) is convex as a function of s ($\forall \ell > 0$).

Proof. It follows from a simple application of Hölder inequality ([81], p.80). For details see Appendix B.•

Next, we address the properties of the optimal of the Lagrange multiplier s.

Theorem 2.3.3 *For a given set of code lengths, let $s_0 = arg \ \min_{s>0} f(s,\ell)$. Then s_0 occurs on the boundary of the constraint, that is*

$$H(\nu^{*,s_0}|\mu) = R \tag{2.9}$$

where

$$\nu^{*,s_0}{}_i = \frac{e^{\frac{\ell_i}{s_0}} \mu_i}{\sum_{j=1}^{M} e^{\frac{\ell_j}{s_0}} \mu_j}, \qquad \forall i \in \{1,...,M\}. \tag{2.10}$$

Proof. Since $f(s,\ell)$ is a differentiable convex function of s, the minimum exists and is found by

$$\frac{\partial f}{\partial s}|_{s=s_0} = R + \left(\log\Big(\sum_{i=1}^{M} e^{\frac{\ell_i}{s}} \mu_i\Big) + s.\frac{\sum_{i=1}^{M} \frac{-\ell_i}{s^2} e^{\frac{\ell_i}{s}} \mu_i}{\sum_{i=1}^{M} e^{\frac{\ell_i}{s}} \mu_i}\right)|_{s=s_0} = 0$$

Which yields

$$R + \log\Big(\sum_{i=1}^{M} e^{\frac{\ell_i}{s_0}} \mu_i\Big) = \frac{\sum_{i=1}^{M} \frac{\ell_i}{s_0} e^{\frac{\ell_i}{s_0}} \mu_i}{\sum_{i=1}^{M} e^{\frac{\ell_i}{s_0}} \mu_i}. \tag{2.11}$$

By Theorem 2.3.1, we know that the probability distribution which attains the supremum of $L^{\lambda,s_0}(\ell,\nu)$ is given by

$$\nu_i^{*,s_0} = \frac{e^{\frac{\ell_i}{s_0}}\mu_i}{\sum_{j=1}^{M} e^{\frac{\ell_j}{s_0}}\mu_j}, \qquad \forall i \in \{1,...,M\}.$$

Moreover, the relative entropy between μ and ν^{*,s_0} is

$$H(\nu^{*,s_0}|\mu) = \frac{\sum_{i=1}^{M} \frac{\ell_i}{s_0} e^{\frac{\ell_i}{s_0}}\mu_i}{\sum_{i=1}^{M} e^{\frac{\ell_i}{s_0}}\mu_i} - \log\Big(\sum_{j=1}^{M} e^{\frac{\ell_j}{s_0}}\mu_j\Big).$$

Using (2.11), we obtain $H(\nu^{*,s_0}|\mu) = R$.•

Next, we find the solution to the second optimization problem, by minimizing $f(s,\ell)$ subject to the Kraft inequality constraint.

2.4 Robust Shannon Source Coding

Let

$$L^{\lambda,s}(\ell,\nu^{*,s}) = sR + s\log\Big(\sum_{i=1}^{M} e^{\frac{\ell_i}{s}}\mu_i\Big) + \lambda\Big(\sum_{i=1}^{M} D^{-\ell_i} - 1\Big).$$

This is a convex function in both s and ℓ_i's, which is also differentiable with respect to the ℓ_i's and s. The parameter M denotes number of source letters to be encoded. We would like to minimize $L^{\lambda,s}(\ell,\nu^{*,s})$ with respect to s and ℓ_i's. Let s^*, λ^* and $\ell^* = (\ell_1^*,...,\ell_M^*)$ be the solutions of the above constrained minimization, then by the Kuhn-Tucker conditions, we have

$$\begin{aligned}
\frac{\partial L^{\lambda,s}(\ell,\nu^{*,s})}{\partial s}|_{\ell=\ell^*,s=s^*,\lambda=\lambda^*} &= 0 \\
\frac{\partial L^{\lambda,s}(\ell,\nu^{*,s})}{\partial \ell_j}|_{\ell=\ell^*,s=s^*,\lambda=\lambda^*} &= 0, \qquad \forall j \in \{1,...,M\} \\
\sum_{i=1}^{M} D^{-\ell_i^*} - 1 &\leq 0 \\
\lambda^*.(\sum_{i=1}^{M} D^{-\ell_i^*} - 1) &= 0 \\
\lambda^* &\geq 0 \\
s^* &\geq 0.
\end{aligned} \tag{2.12}$$

where $\ell = (\ell_1,...,\ell_M)$ and $\ell^* = (\ell_1^*,...,\ell_M^*)$. Since the Kraft inequality does not depend on s, then the result of Theorem 2.3.3 holds, hence we have

$$H(\nu^{*,s^*}|\mu) = R \tag{2.13}$$

where ν^{*,s^*} is given by

$$\nu_i^{*,s^*} = \frac{e^{\frac{\ell_i^*}{s^*}} \mu_i}{\sum_{j=1}^{M} e^{\frac{\ell_i^*}{s^*}} \mu_j}, \qquad \forall i \in \{1, ..., M\}. \tag{2.14}$$

The solution to this problem is given in [2]. The codeword lengths are given by the following

$$\ell_j^* = \lceil \log_D \left(\frac{1}{\nu^{*,s^*}} \right) \rceil, \qquad \forall j \in \{1, ..., M\}. \tag{2.15}$$

Where ν^{*,s^*} is defined as

$$\nu_i^{*,s} \triangleq \frac{\mu_i{}^{\alpha}}{\sum_{j=1}^{M} \mu_j{}^{\alpha}}, \quad \alpha \triangleq \frac{s^* \ln D}{s^* \ln D + 1}, \qquad \forall i \in \{1, ..., M\} \tag{2.16}$$

Notice that we have to find $(M+1)$ unknown values ($\ell_1^*, ..., \ell_M^*$ and s^*); these are found from (2.15) which gives M equations, and the relative entropy given by (2.13). Finally, using this set of $(M+1)$ equations, the lengths and s^* can be found. Also notice that,

$$\begin{aligned} \lim_{s^* \to \infty} \nu^{*,s^*} &= \mu \\ \lim_{s^* \to \infty} \ell_j^* &= \lceil \log_D \left(\frac{1}{\mu_j} \right) \rceil \end{aligned}$$

as expected. •

Remark 2.4.1 *Assume the nominal distribution μ is uniform, in other words $\mu_i = \frac{1}{M}$, then from (2.15) and (2.16) it is seen that all code word lengths are equal and ν^{*,s^*} is uniform, hence $\nu^{*,s^*} = \mu$, which corresponds to the solution of (2.15) when $s^* \to \infty$. This shows that we gain nothing by using robust coding method if the nominal distribution is itself uniform. In this case, the best option to encode an unknown distribution from the relative entropy constraint set is to encode the nominal distribution, i.e. the worst case distribution is the uniform one, and this result holds for any $R > 0$.*

Theorem 2.4.2 *Assume $0 \leq R \leq H(\eta|\mu)$, where η is a uniform distribution, (e.g., $\eta_i = \frac{1}{M}$ for $1 \leq i \leq M$). Then $G(R) \triangleq \sup_{\nu \in \mathcal{M}_R} H(\nu)$ is attained at $\nu = \nu^*$ given by*

$$\nu_i^{*,\beta} = \frac{\mu_i^{\frac{\beta}{1+\beta}}}{\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}}} \tag{2.17}$$

where $\beta \geq 0$ is chosen such that $H(\nu^{*,\beta}|\mu) = R$.

Proof. see Appendix B.

Remark 2.4.3 *The form of the maximizing distribution in (2.17) is the same as found in (2.16). Moreover since $H(\nu^{*,s}|\mu)$ for ν^* given in (2.16), is a continuous non-increasing function of s, the equation $H(\nu^{*,s}|\mu) = R$ has only one solution. The same argument is true for $H(\nu^{*,\beta}|\mu) = R$ found above. So, indeed the distributions $\nu^{*,s}$ and $\nu^{*,\beta}$ must be the same and $\alpha = \frac{\beta}{1+\beta}$.*

2.4.1 An Iterative Algorithm for Finding the Robust Shannon Code

In this section, we will develop a simple iterative method for finding s^* , ν^{*,s^*} and ℓ_i^*'s for a given nominal distribution μ and a given uncertainty radius R. The numerical algorithm is based on the non-increasing property of $H(\nu^{*,s}|\mu)$ a function of s, which enables us to find the maximum allowable radius of uncertainty. The upper bound on R is important because it gives some a priori information on the amount of uncertainty admitted by a given nominal measure.

Theorem 2.4.4 *$H(\nu^{*,s}|\mu)$ is a non-increasing function of s.*

Proof. See Appendix B. •

In the next theorem, we provide a necessary condition for the existence of a solution to the Robust Shannon Code in terms of the maximum allowable uncertainty radius admitted by R.

Theorem 2.4.5 *The necessary condition for existence of a solution to the robust Shannon coding problem for a nominal distribution μ and a given radius of uncertainty R is*

$$R \leq R_{\max} \triangleq \frac{1}{M}\sum_{i=1}^{M} \ln\left(\frac{1}{\mu_i}\right) - \ln M \tag{2.18}$$

Proof. Using the non-increasing property of $H(\nu^{*,s}|\mu)$ as proven in Theorem 2.4.4, we have,

$$H(\nu^{*,s_1}|\mu) \geq H(\nu^{*,s_2}|\mu), \qquad for \quad s_1 \leq s_2.$$

Now, let $s_2 = s^*$, then by (2.13), $H(\nu^{*,s_2}|\mu) = R$. Substituting this in the above inequality gives

$$H(\nu^{*,s}|\mu) \geq R, \qquad \forall s \leq s^*.$$

Taking limit, as $s \to 0$, and then using (B.2) in the proof of Theorem 2.4.4, we have

$$\lim_{s\to 0}\sum_{i=1}^{M}\left(\frac{{\mu_i}^{\alpha}}{\sum_{j=1}^{M}{\mu_j}^{\alpha}}\right)\log\left(\frac{{\mu_i}^{\alpha-1}}{\sum_{j=1}^{M}{\mu_j}^{\alpha}}\right) \geq R.$$

Since $\alpha = \frac{s \ln D}{1 + s \ln D}$, as $s \to 0$, then $\alpha \to 0$ as well, hence,

$$R \leq \frac{1}{M} \sum_{i=1}^{M} \ln\left(\frac{1}{\mu_i}\right) - \ln M.$$

Remark 2.4.6 *The upper bound in (2.18) can be written as $R \leq H(\eta|\mu)$, where η is a uniform distribution i.e. $\eta_i = \frac{1}{M}$ for $i \in \{1, ..., M\}$. Also using (B.1) coding with respect to the distribution ν^* leads to an average length close to Rényi entropy. This is similar to the result given in [9].*

Next, an upper bound on the maximum code length is derived followed by a bound on the maximum value the Lagrange multipliers $s > 0$ can assume. Both bounds are important in the derivation of the numerical algorithm for finding the optimal minimax code.

Lemma 2.4.7 *Suppose $\{\ell_1^*, ..., \ell_M^*\}$ and s^* be the solution to the robust Shannon coding problem. Then*

$$\ell_{\max}^* \leq \left\lceil \log_D\left(\frac{1}{\mu_{\min}}\right) \right\rceil$$

where $\mu_{\min} = \min_{1 \leq i \leq M}\{\mu_i\}$ and $\ell_{\max}^* = \max_{1 \leq i \leq M}\{\ell_i^*\}$

Proof. See Appendix B.

Next an upper bound on the Lagrange multiplier s^* is found.

Lemma 2.4.8 *Suppose $\{\ell_1^*, ..., \ell_M^*\}$ and s^* be the solutions to the robust Shannon coding problem. Then*

$$s^* \leq \frac{1}{R}\left(\left\lceil \log_D\left(\frac{1}{\mu_{\min}}\right) \right\rceil - H_D(\mu)\right).$$

Proof. See Appendix B.

Iterative Algorithm

Now, using the results of Lemma 2.4.7 and Lemma 2.4.8, and the monotonicity property of $H(\nu^{*,s}|\mu)$ as shown in Theorem 2.4.4, an iterative algorithm for finding the optimal solution $(s^*, \ell_1^*, ..., \ell_M^*)$ is proposed. By the non-increasing property of $H(\nu^{*,s}|\mu)$ and (2.13), s^* is the lower bound of all s which belong to the set characterized by Lemma 2.4.8. Hence, it is obvious that for an $s \geq s^*$, $H(\nu^{*,s}|\mu) < R$, and in order to reach the boundary R, we have to decrease s. First we pick an initial s equal to the upper bound of Lemma 2.4.8, then find codeword lengths using (2.15) with s^* replaced by s. Then calculate $H(\nu^{*,s}|\mu)$. If this is not on the boundary (R), decrease s by a fixed step size δ. The step size should be chosen such that $\delta << s$. Here we have used $\delta = 0.01s$.

2.5 Robustness of Robust Shannon Codes

In this section, an inequality is derived which explains the relative insensitivity or robustness property of the average codeword length for the robust Shannon code comparing to that of the ordinary Shannon code. Let ν_1 and ν_2 be any two distributions in the relative entropy constraint set $\mathcal{M}_R$. We claim that for a robust Shannon code, changes in the average code length, when the average is computed with respect to different distributions within the set $\mathcal{M}_R$, are smaller than similar changes for the ordinary Shannon code. First, consider the Shannon code for the nominal distribution. Then

$$\sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\mu_i}\right) \leq E_{\nu_1}\big(\ell(\mu)\big) < \sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\mu_i}\right) + 1. \tag{2.19}$$

For the robust code we have

$$\sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) \leq E_{\nu_1}\big(\ell(\nu^*)\big) < \sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) + 1. \tag{2.20}$$

By (2.15) and (2.16), we have

$$\sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) = \log_D \Big(\sum_{j=1}^{M} \mu_j^{\alpha}\Big) + \alpha \sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\mu_i}\right) \tag{2.21}$$

where $\alpha \triangleq \frac{s^* \ln D}{1+s^* \ln D}$. Similar relations hold for any $\nu_2 \in \mathcal{M}_R$. Using (2.20) and the similar inequality for ν_2, the difference in the average length of the robust Shannon code with respect to ν_1 and ν_2 can be bounded as follows.

$$E_{\nu_1}\big(\ell(\nu^*)\big) - E_{\nu_2}\big(\ell(\nu^*)\big) \leq \sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) + 1 - \sum_{i=1}^{M} \nu_{2,i} \log_D \left(\frac{1}{\nu_i^*}\right)$$
$$E_{\nu_1}\big(\ell(\nu^*)\big) - E_{\nu_2}\big(\ell(\nu^*)\big) \geq \sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) - \sum_{i=1}^{M} \nu_{2,i} \log_D \left(\frac{1}{\nu_i^*}\right) - 1$$

Combining the above two inequalities leads to the following inequality.

$$\sum_{i=1}^{M} (\nu_{1,i} - \nu_{2,i}) \log_D \left(\frac{1}{\nu_i^*}\right) - 1 \leq E_{\nu_1}\big(\ell(\nu^*)\big) - E_{\nu_2}\big(\ell(\nu^*)\big) \leq \sum_{i=1}^{M} (\nu_{1,i} - \nu_{2,i}) \log_D \left(\frac{1}{\nu_i^*}\right) + 1$$

This can be rewritten using the obvious inequality $-|X| \leq X \leq |X|$:

$$-\left|\sum_{i=1}^{M} (\nu_{1,i} - \nu_{2,i}) \log_D \left(\frac{1}{\nu_i^*}\right)\right| - 1 \leq E_{\nu_1}\big(\ell(\nu^*)\big) - E_{\nu_2}\big(\ell(\nu^*)\big) \leq \left|\sum_{i=1}^{M} (\nu_{1,i} - \nu_{2,i}) \log_D \left(\frac{1}{\nu_i^*}\right)\right| + 1$$

which finally leads to the upper bound.

$$\left|E_{\nu_1}\big(\ell(\nu^*)\big) - E_{\nu_2}\big(\ell(\nu^*)\big)\right| \leq \left|\sum_{i=1}^{M} \nu_{1,i} \log_D \left(\frac{1}{\nu_i^*}\right) - \sum_{i=1}^{M} \nu_{2,i} \log_D \left(\frac{1}{\nu_i^*}\right)\right| + 1.$$

The above can be simplified using (2.21):

$$\left|E_{\nu_1}\left(\ell(\nu^*)\right) - E_{\nu_2}\left(\ell(\nu^*)\right)\right| \leq \alpha.\left|\sum_{i=1}^{M} \nu_{1,i} \log_D\left(\frac{1}{\mu_i}\right) - \sum_{i=1}^{M} \nu_{2,i} \log_D\left(\frac{1}{\mu_i}\right)\right| + 1. \tag{2.22}$$

The upper bound for the difference of the average code length for any two distributions $\nu_1, \nu_2 \in \mathcal{M}_R$ of the Shannon code can be derived in a similar way.

$$\left|E_{\nu_1}\left(\ell(\mu)\right) - E_{\nu_2}\left(\ell(\mu)\right)\right| \leq \left|\sum_{i=1}^{M} \nu_{1,i} \log_D\left(\frac{1}{\mu_i}\right) - \sum_{i=1}^{M} \nu_{2,i} \log_D\left(\frac{1}{\mu_i}\right)\right| + 1. \tag{2.23}$$

Since $s^* < 1$, then $\alpha < 1$ and this implies that the difference of the average code length for any two distributions $\nu_1, \nu_2 \in \mathcal{M}_R$ of the robust Shannon code, namely (2.22) is smaller than that of the Shannon code, namely (2.23). This illustrates the robustness property of the robust Shannon code.

2.6 Robust Huffman Coding

Due to the restriction on the maximum allowable uncertainty imposed by R in Lemma 2.4.5, the robust Shannon coding cannot be applied in all cases. In this section, we discuss the Huffman coding for uncertain sources belonging to the relative entropy constraint set. The problem is formulated as follows. Assume a source has nominal source distribution $\mu \in \mathcal{M}(\Sigma)$ and the set of possible distributions for this source is $\{\nu \in \mathcal{M}(\Sigma); H(\nu|\mu) \leq R\}$. Suppose $\mathcal{C}$ is the set of all prefix codes defined for this source. Find $c^* \in \mathcal{C}$ and $\nu^* \in \mathcal{M}_R$ such that

$$\inf_{c \in \mathcal{C}} \sup_{\nu \in \mathcal{M}_R} \sum_{i=1}^{M} \ell_i(c)\nu_i = \sum_{i=1}^{M} \ell_i(c^*)\nu_i^*. \tag{2.24}$$

Here, the inner supremum is found by using the result of Theorem 2.3.1:

$$\sup_{\nu \in \mathcal{M}_R} \sum_{i=1}^{M} \ell_i(c)\nu_i = \sup_{\nu \in \mathcal{M}_R} \left(\sum_{i=1}^{M} \ell_i(c)\nu_i - s(H(\nu|\mu) - R)\right) = sR + s\log\left(\sum_{i=1}^{M} e^{\frac{\ell_i(c)}{s}}\mu_i\right) \tag{2.25}$$

The supremum is attained by

$$\nu_i^* = \frac{e^{\frac{\ell_i(c)}{s}}\mu_i}{\sum_{j=1}^{M} e^{\frac{\ell_j(c)}{s}}\mu_j}, \quad 1 \leq i \leq M.$$

Next, we find optimal prefix codes, which minimize (2.25), similar to the classical Huffman codes. First, we consider the case when s is fixed, to develop the optimal coding method based on the pay-off function (2.25). This problem has been solved in [6] as an extension of the ordinary Huffman coding method. If we want to solve the optimal coding problem with relative entropy constraints given by (2.24), we have to find the optimal s as well. The optimal solution $\{\ell_1^*, ..., \ell_M^*\}$ and s^*, is such that $H(\nu^{*,s^*}|\mu) = R$, which follows from Theorem 2.3.3.

2.7 Numerical Examples

In this section, we present robust coding for some numerical examples with a particular choice of nominal source distribution. Take the nominal distribution to be

$$\mu = [0.1 \quad 0.2 \quad 0.005 \quad 0.004 \quad 0.001 \quad 0.02 \quad 0.07 \quad 0.6]$$

The upper bound for R in robust Shannon coding is $R_{\max} = 1.5108$. We take $R = 1.0$ and $D = 2$. The approximate solution of the robust Shannon code is

$$s^* = 0.2013 \qquad \ell_{\text{robust-shannon}} = [3 \quad 3 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3]$$

Also the robust Huffman code is found to be $\ell_{\text{robust-huffman}} = [3 \quad 3 \quad 3 \quad 4 \quad 4 \quad 3 \quad 3 \quad 2]$. The Shannon and Huffman codes for the nominal distribution are

$$\ell_{\text{shannon}} = [4 \quad 3 \quad 8 \quad 8 \quad 10 \quad 6 \quad 4 \quad 1], \ \ell_{\text{huffman}} = [3 \quad 2 \quad 6 \quad 7 \quad 7 \quad 5 \quad 4 \quad 1]$$

Now, in order to check the robustness of each of the above codes, we take different distributions from the relative entropy constraint set $\mathcal{M}_R$, for $R = 1.0$. Distributions which belong to the uncertainty set are found using methods 1) and 2) below.

1) Use robust Shannon coding algorithm for different values of R_0 between 0 and 1.0, to find different ν's such that $H(\nu|\mu) = R_0 < 1.0$.

2) Use robust Huffman coding algorithm for different values of R_0 between 0 and 1.0, to find different ν's such that $H(\nu|\mu) = R_0 < 1.0$. Denote $S = E_{\nu_0}[\ell_{\text{shannon}}(\mu)]$, $H = E_{\nu_0}[\ell_{\text{huffman}}(\mu)]$, $RS = E_{\nu_0}[\ell_{\text{robust-shannon}}]$ and $RH = E_{\nu_0}[\ell_{\text{robust-huffman}}(\mu)]$.

A) **Simulations for different distributions, which has the same structure as in the robust Shannon coding method**

A-1) Take $\nu_0 = [0.1405 \quad 0.2167 \ 0.0216 \ 0.0188 \ 0.0079 \ 0.0514 \ 0.1125 \ 0.4306]$, then $H(\nu_0|\mu) = 0.1013 < R$.

$$S = 2.8036 \quad RS = 3.0997 \quad H = 2.3091 \quad RH = 2.5962$$

A-2) Take $\nu_0 = [0.153 \quad 0.2065 \quad 0.0419 \quad 0.0381 \quad 0.0209 \quad 0.0763 \quad 0.1312 \quad 0.3321]$, then $H(\nu_0|\mu) = 0.2982 < R$.

$$S = 3.3951 \quad RS = 3.1772 \quad H = 2.7746 \quad RH = 2.7269$$

A-3) Take $\nu_0 = [0.1542 \quad 0.1916 \quad 0.0602 \quad 0.0562 \quad 0.0364 \quad 0.0931 \quad 0.1379 \quad 0.2705]$, then $H(\nu_0|\mu) = 0.5087 < R$.

$$S = 3.867 \quad RS = 3.2459 \quad H = 3.1424 \quad RH = 2.8221$$

A-4) Take $\nu_0 = [0.1514 \quad 0.1785 \quad 0.0745 \quad 0.0706 \quad 0.0509 \quad 0.1034 \quad 0.1392 \quad 0.2315]$, then $H(\nu_0|\mu) = 0.6915 < R$.

$$S = 4.2196 \;\; RS = 3.2994 \;\; H = 3.414 \;\; RH = 2.89$$

A-5) Take $\nu_0 = [0.1435 \quad 0.1578 \quad 0.0951 \quad 0.0923 \quad 0.0763 \quad 0.115 \quad 0.1366 \quad 0.1834]$, then $H(\nu_0|\mu) = 0.99 < R$.

$$S = 4.7292 \;\; RS = 3.3787 \;\; H = 3.8016 \;\; RH = 2.9851$$

B) **Simulations for different distributions, which has the same structure as in the robust Huffman coding method**

B-1) Take $\nu_0 = [0.1388 \quad 0.1989 \quad 0.0189 \quad 0.0210 \quad 0.0053 \quad 0.054 \quad 0.1356 \quad 0.4275]$, then $H(\nu_0|\mu) = 0.1116 < R$.

$$S = 2.818 \;\; RS = 3.0992 \;\; H = 2.3516 \;\; RH = 2.5988$$

B-2) Take $\nu_0 = [0.1943 \quad 0.1279 \quad 0.0295 \quad 0.0717 \quad 0.0179 \quad 0.0389 \quad 0.136 \quad 0.3838]$, then $H(\nu_0|\mu) = 0.3277 < R$.

$$S = 3.3107 \;\; RS = 3.158 \;\; H = 2.7652 \;\; RH = 2.7058$$

B-3) Take $\nu_0 = [0.2054 \quad 0.0984 \quad 0.0428 \quad 0.0343 \quad 0.0086 \quad 0.1714 \quad 0.1438 \quad 0.2953]$, then $H(\nu_0|\mu) = 0.5245 < R$.

$$S = 3.7185 \;\; RS = 3.2571 \;\; H = 3.0976 \;\; RH = 2.7476$$

B-4) Take $\nu_0 = [0.2063 \quad 0.0670 \quad 0.0636 \quad 0.0509 \quad 0.0127 \quad 0.2542 \quad 0.1444 \quad 0.2009]$, then $H(\nu_0|\mu) = 0.9306 < R$.

$$S = 4.3729 \;\; RS = 3.3814 \;\; H = 3.6292 \;\; RH = 2.8627$$

The above simulations illustrate that as the unknown distribution moves away from the nominal one, the expected length of the Shannon and Huffman codes changes rapidly, while on the other hand the expected length of the robust Shannon and robust Huffman codes changes very slowly. Also, as the relative entropy between the unknown and nominal distributions increases, the expected code word length for the robust Shannon and the robust Huffman codes are becoming closer to each other, while for ordinary codes, these averages are far apart.

2.8 Minimax Redundancy Coding

Minimax redundancy is one of the most widely used measures of performance for designing codes when the source is subject to uncertainty (e.g., [3], [4] and [5]). In this section, we formulate the minimax coding problem using redundancy as a pay-off, subject to a relative entropy constraint model of uncertainty. As it turns out, the minimax solution to this problem leads to encoding with an exponential pay-off similar to (2.25). Let $\{\ell_1, ..., \ell_M\}$ denote the code word lengths for the source symbols $\{x_1, ..., x_M\}$. Assume $\nu \in \mathcal{M}_R$, which implies $H(\nu|\mu) \leq R$. Let $r(\ell, \nu)$ denote the redundancy of the code. Formulate the problem of minimax redundancy as follows.

$$\inf_{(\ell_1,...,\ell_M)} \sup_{\nu \in \mathcal{M}_R} \Big(E_\nu(\ell) - H_D(\nu)\Big) = \inf_{(\ell_1,...,\ell_M)} \sup_{\nu \in \mathcal{M}_R} r(\ell, \nu).$$

Redundancy can be written as follows.

$$r(\ell, \nu) = \frac{1}{\log D} H(\nu|\theta) = \frac{1}{\log D}\Big(H(\nu|\mu) + \sum_{i=1}^{M} \nu_i \log\Big(\frac{\mu_i}{\theta_i}\Big)\Big) \tag{2.26}$$

Where $\theta_i = D^{-\ell_i}$ for all $i \in \{1, ..., M\}$. Now consider the following probability distribution.

$$\nu_i^\circ = \frac{\left(\frac{\mu_i}{\theta_i}\right)^\beta \mu_i}{\sum_{k=1}^{M} \left(\frac{\mu_k}{\theta_k}\right)^\beta \mu_k} \qquad \beta > 0, \;\; i \in \{1, ..., M\}$$

The relative entropy between ν and ν° can be shown to be.

$$H(\nu|\nu^\circ) = H(\nu|\mu) + \log\Big(\sum_{k=1}^{M} \Big(\frac{\mu_k}{\theta_k}\Big)^\beta \mu_k\Big) - \beta \sum_{i=1}^{M} \nu_i \log\Big(\frac{\mu_i}{\theta_i}\Big) \tag{2.27}$$

Now substitute $\sum_{i=1}^{M} \nu_i \log \frac{\mu_i}{\theta_i}$ from (2.27) into (2.26). Then

$$r(\ell, \nu) = \frac{1}{\log D}\Big(\frac{\beta+1}{\beta} H(\nu|\mu) - \frac{1}{\beta} H(\nu|\nu^\circ) + \frac{1}{\beta} \log\Big(\sum_{k=1}^{M} \Big(\frac{\mu_k}{\theta_k}\Big)^\beta \mu_k\Big)\Big) \tag{2.28}$$

Since $\beta > 0$, the supremum in (2.28) is attained for $\nu = \nu^\circ$, for the value of β such that $H(\nu^\circ|\mu) = R$. Hence,

$$\sup_{\nu \in \mathcal{M}_R} r(\ell, \nu) = r(\ell, \nu^\circ) \tag{2.29}$$

Therefore minimizing the worst-case redundancy over the codeword lengths leads to the following problem.

$$\inf_{(\ell_1,...,\ell_M)} \sum_{k=1}^{M} \Big(\frac{\mu_k}{\theta_k}\Big)^\beta \mu_k = \inf_{(\ell_1,...,\ell_M)} \sum_{k=1}^{M} e^{\beta \log D(\ell_k - \ell_k')} \mu_k$$

Where $\ell'_k = \log_D \frac{1}{\mu_k}$. This is a special case of the general problem considered in [17]. This problem can be solved using the exponential Huffman coding as described in Section 2.6 for the following cost function.

$$\inf_{(\ell_1,\ldots,\ell_M)} \sum_{k=1}^{M} e^{s\ell_k} \xi_k$$

Where $s = \beta \log D$ and ξ is a probability distribution given by

$$\xi_k = \frac{\mu_k^{\beta+1}}{\sum_{i=1}^{M} \mu_i^{\beta+1}}$$

2.9 Conclusion

In this chapter a class of source coding problems for uncertain sources was studied. The uncertainty is described by a relative entropy constraint between the true and the nominal distributions. The source coding problem for this class of uncertainties, is formulated and solved using minimax strategies, giving rise to robust versions of Shannon and Huffman coding methods. The minimax coding problem is solved for two different cost functions, namely average length and redundancy. An important extension is to consider sources which are described by ergodic finite state Markov chains.

Chapter 3

On the Rate Distortion Formulation for Abstract Sources

3.1 Introduction

The conventional rate distortion formulation for finite-alphabet sources as well as continuous sources has been studied thoroughly in the literature [1], [21], [22] and [33]. For a recent survey of rate distortion theory see [29]. In this chapter, we consider the problem of characterization of the rate distortion function in Polish spaces for abstract sources. The formulation of the rate distortion function for abstract alphabets has been studied by Csiszár [23]. The question of existence of a solution in Polish spaces under some continuity assumptions on the distortion function and compactness of the reproduction space, was resolved under the topology of weak convergence. The formulation in [23] is based on two important assumptions, namely, 1) compactness of the reproduction space, and 2) absolute continuity of all marginal distributions with respect to the optimal marginal distribution. The compactness assumption in [23] is crucial in order to formulate the problem using countably additive measures and to show existence of a minimizing measure using tightness arguments and the Prohorov theorem [80]. The absolute continuity assumption in [23] is vital throughout the derivations for the optimal solution, because it enables one to apply the chain rule of Radon-Nikodym derivatives. Under these assumptions, the optimal solution will be shown to be

$$q^*(x,dy) = \frac{e^{s\rho(x,y)}\nu^*(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu^*(dz)} \tag{3.1}$$

where ρ is the distortion function, q^* is the optimal conditional distribution, ν^* is the optimal marginal distribution and $\hat{A}$ is the reproduction space. Next, we elaborate more on the above assumptions and the restrictions imposed by them on the formulation of rate distortion problems.

Compactness. The compactness assumption is too restrictive. In some applications such as compression using wavelet transforms [28], the reproduction space is the linear span of an infinite basis, which is not compact. More discussion about the issue of compactness in this application can be found in Section 3.5. In order to relax the compactness assumption, one has to use a weaker topology. In this chapter, the weak* topology is considered. This topology has recently been used in the literature in the context of channel capacity for continuous alphabets [32]. Using this topological framework, the compactness assumption on the reproduction space is relaxed. However, in order to formulate the problem in the weak* topology, identifying the appropriate dual Banach spaces is crucial. This will be discussed in Section 3.2.

Absolute Continuity. Csiszár [23] uses an assumption which implies that all marginal distributions on the reproduction space are absolutely continuous with respect to the marginal distribution corresponding to the optimal solution. This crucial assumption, enables one to use the chain rule for Radon-Nikodym derivatives [[23], Lemmas 1.3-1.4, p.60]. In general, such an assumption may not be valid, hence necessary condition of optimality must be derived using calculus of variations on suitable space of functions with values in the space of measures. One of the main results of this chapter, is to show that the mutual information is Gateaux differentiable with respect to the conditional distribution defined on the reproduction space. This property relaxes the absolute continuity assumption. The Gateaux differentiability is then employed to derive necessary and sufficient conditions for optimality. The forms of the optimal distributions are the same as in the abstract alphabet case in [23].

Implicit form of the minimizer. One of the fundamental issues that is yet to be addressed, is whether the nonlinear equation in (3.1) has a solution. For the case of finite alphabet, the analog of (3.1), has a solution due to the Blahut algorithm [22], because in the limit the algorithm leads to an equation like (3.1). It should be pointed out that in this problem we are essentially dealing with two different existence problems. The first one is concerned with the existence of a conditional measure on the reproduction space, which minimizes the mutual information. However, it does not provide us with any specific minimizing candidate. Such a candidate is found later using the variational method in Section 3.4 (or chain rule and inequalities as in [23]). This candidate does not have an explicit form, rather it is an implicit equation describing a nonlinear relationship between the conditional and marginal distributions on the reproduction space. Such an equation may not have solutions in general, and therefore the second existence problem appears, i.e., existence of a solution to the nonlinear equation in (3.1). If this equation does not have a solution, then the minimizing measure exists but does not have the form given in that equation.

There is an extension of the Blahut algorithm in the general case, in which $\lim_{n\to\infty} I(\mu; q_n) = R(D)$, where $\{q_n\}$ is a sequence constructed via the alternating minimization method. Moreover, one can find a measure q_0 such that $q_n \to q_0$ weakly, and by the lower semi-continuity of mutual information show that, $I(\mu; q_0) = R(D)$ [[26], p.217-218] . However, unlike the finite dimensional case, it is not possible to show that q_0 and its marginal ν_0 will satisfy (3.1), since the convergence $q_n \to q_0$ is not in the strong topology[1]. In this chapter, existence of a solution to the implicit nonlinear equation (3.1), is proved using Tihonov Fixed Point theorem which holds for locally convex topological vector spaces. Without this existence result, the necessary conditions of optimality do not hold, since in that proof the solution is an implicit one, and if the implicit relationship does not have a solution, then the optimality condition would be meaningless.

Source coding theorem with fidelity criterion for abstract sources has been addressed in many papers. For separable metric spaces results in this direction can be found in [24]. This result can be applied to the set up considered in this chapter. Alternative approaches based on Large Deviation techniques are found in [27], while methods based on generalized AEP (asymptotic equipartition property) are given in [31]. A source coding theorem for stationary source is given in [25].

The rest of the chapter is organized as follows. In Section 3.2, the general rate distortion problem is formulated and appropriate function spaces are introduced. The question of existence of solution of the problem is treated in Section 3.3. Moreover, within the same section, the equivalence of the constrained and unconstrained problems is established. In other words, the conditions required for using Lagrange multipliers method are rigorously justified and proved. Necessary conditions of optimality are presented in Section 3.4. In the same section, existence of solution to a fixed point problem arising from the necessary conditions is presented. In Section 3.5, some examples of concrete Polish spaces of practical interest are provided.

3.2 Problem Formulation

In this section, the appropriate topologies and function spaces are identified and the weak* compactness of the constrained set is shown. Let $(A, \mathcal{A})$ be a measurable space denoting the source space with $\mathcal{A}$ being an algebra of subsets of the set A generated by closed sets. Similarly let $(\hat{A}, \hat{\mathcal{A}})$ be another measurable space denoting the reproduction space with $\hat{\mathcal{A}}$

[1] $q_n \to q_0$ strongly iff $q_n(x, F) \to q_0(x, F)$ for all x and all F in the σ-algebra on the reproduction space.

being an algebra of subsets of the set $\hat{A}$. The reproduction space may be a proper subset of the source space $\hat{A} \subseteq A$.

Assume $q : A \times \hat{\mathcal{A}} \to [0,1]$ is a mapping satisfying the following two properties:

1) For every $x \in A$, the set function $q(x,\cdot)$ is a probability measure (possibly finitely additive) on $\hat{\mathcal{A}}$.

2) For every $F \in \hat{\mathcal{A}}$, the function $q(\cdot, F)$ is $\mathcal{A}$-measurable.

Any such map q is called a stochastic kernel or transition probability. Let Q denote the class of all such stochastic kernels.

Let $\mathcal{M}_1(A)$ denote the space of probability measures (possibly finitely additive) on the source space A and let $\mu \in \mathcal{M}_1(A)$ be fixed. For the given pair $\{q, \mu\}$ we may introduce three probability measures as follows:

(P1): the joint (or compound) probability measure $P \in \mathcal{M}_1(A \times \hat{A})$ given by

$$P(G) = (\mu \otimes q)(G) = \int_A q(x, G_x)\mu(dx) \qquad \forall G \in \mathcal{A} \times \hat{\mathcal{A}}$$

where $\otimes$ denotes the convolution, G_x is the x-section of G defined by $G_x = \{y \in \hat{A} : (x,y) \in G\}$;

(P2): the marginal probability measure $\nu \in \mathcal{M}_1(\hat{A})$ corresponding to $q \in Q$ is given by

$$\nu(F) = P(A \times F) = \int_A q(x, (A \times F)_x)\mu(dx) = \int_A q(x, F)\mu(dx) \qquad \forall F \in \hat{\mathcal{A}};$$

and finally

(P3): the product measure π of μ and ν is given by

$$\pi(G) = (\mu \times \nu)(G) = \int_A \nu(G_x)\mu(dx) \qquad \forall G \in \mathcal{A} \times \hat{\mathcal{A}}.$$

Let $\rho : A \times \hat{A} \to [0,\infty)$ be a $\mathcal{A} \times \hat{\mathcal{A}}$-measurable function. For each $D \in [0,\infty)$, define the set $Q(D)$ as

$$Q(D) = \{q \in Q : \int_A \int_{\hat{A}} \rho(x,y)q(x,dy)\mu(dx) \leq D\}.$$

For the given ρ and μ we assume that $Q(D)$ is nonempty. Necessary and sufficient conditions for this are given in the sequel.

In the following definition, the mutual information is defined using Radon-Nikodym derivatives. An alternative way is to define it by taking supremum over finite partitions over the product space [34](p.89 and p.122). This alternative definition is used in Appendix C to show Gateaux differentiability of the mutual information.

Definition 3.2.1 *(Rate Distortion Function) Corresponding to any pair* $(\mu, q) \in M_1(A) \times Q$, *the relative entropy of the associated joint probability measure* P *with respect to the product measure* $\pi = \mu \times \nu$ *of its marginals is called the mutual information which is denoted by*

$$I(\mu; q) \equiv H(P||\pi) = \int_{A\times\hat{A}} \log\Big(\frac{q(x,dy)}{\nu(dy)}\Big) q(x,dy)\mu(dx),$$

where the argument of the logarithmic function denotes the Radon-Nikodym derivative of $q(x,\cdot)$ *with respect to its marginal* $\nu(\cdot)$. *The rate distortion function is then given by,*

$$R(D) \equiv \inf_{q\in Q(D)} I(\mu; q). \tag{3.2}$$

Remark 3.2.2 *The interpretation of* $R(D)$ *is that it specifies the minimum amount of information required to reproduce the source with average distortion (fidelity) equal or less than* D.

Throughout the rest of this chapter we assume that the function $I_\mu(q)$ is well defined for each $q \in Q$. This of course requires that for every $q \in Q$, $q(x,\cdot)$ has Radon-Nikodym derivative with respect to its marginal $\nu(\cdot)$. Necessary and sufficient conditions for existence of RND for finitely additive measures can be found in [44]. Next we use a theorem known as chain rule for relative entropy.

Lemma 3.2.3 *(Chain Rule) Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be polish*[2] *spaces and* α *and* β *two probability measures on* $\mathcal{X} \times \mathcal{Y}$. *Let* α_1 *and* β_1 *denote the first marginals of* α *and* β *(i.e.,* $\alpha_1(E) = \alpha(E \times \mathcal{Y})$ *and* $\beta_1(E) = \beta(E \times \mathcal{Y})$, $\forall E \in \mathcal{B}(\mathcal{X})$*) respectively, and* $\alpha_2(x, dy)$ *and* $\beta_2(x, dy)$ *the stochastic kernels on* $\mathcal{Y}$ *given* $\mathcal{X}$ *such that the following decompositions hold,* $\alpha = \alpha_1 \otimes \alpha_2$ *and* $\beta = \beta_1 \otimes \beta_2$. *Then the function mapping* $x \mapsto H(\alpha_2(x,\cdot)||\beta_2(x,\cdot))$ *is measurable and the relative entropy of* α *relative to* β *is given by*

$$H(\alpha||\beta) = H(\alpha_1||\beta_1) + \int_{\mathcal{X}} H(\alpha_2(x,\cdot)||\beta_2(x,\cdot)))\alpha_1(dx)$$

Proof. See [[41], Theorem B.2.1, p 401].

By applying the chain rule to $H(P||\pi)$ as defined earlier, we find that

$$H(P||\pi) = H(\mu||\mu) + \int_A H(q(x,\cdot)||\nu(\cdot))\mu(dx) = \int_A H(q(x,\cdot)||\nu(\cdot))\mu(dx)$$

[2] A complete separable metric space

and hence we have

$$H(P||\pi) = \int_A \int_{\hat{A}} \log\big(\frac{q(x,dy)}{\nu(dy)}\big) q(x,dy)\mu(dx)$$

which coincides with the expression given in Definition 3.2.1. Assuming that $Q(D) \equiv Q_\rho(D)$ is nonempty, it follows from the Definition 3.2.1 that the rate distortion function is given by

$$R(D) = \inf_{q \in Q(D)} I_\mu(q) = \inf_{q \in Q(D)} \int_A \int_{\hat{A}} \log\big(\frac{q(x,dy)}{\nu(dy)}\big) q(x,dy)\mu(dx). \tag{3.3}$$

Next we introduce the appropriate topologies and function spaces used in this chapter. Throughout the rest of the chapter we assume that both A and $\hat{A}$ are Polish spaces (complete separable metric spaces) and so normal topological spaces. In fact, it is not necessary to restrict to Polish spaces; it suffices if both A and $\hat{A}$ are regular topological spaces. For applications however (see section 3.5), it is necessary to use metrizable spaces and hence Polish spaces are good choices. Let $BC(\hat{A})$ denote the vector space of bounded continuous real valued functions defined on the Polish space $\hat{A}$. Furnished with the sup norm topology, this is a Banach space. Let $\big(BC(\hat{A})\big)^*$ denote its topological dual. It is known [[37], IV.6.2, p 262] that $\big(BC(\hat{A})\big)^*$ is isometrically isomorphic to the Banach space of finitely additive regular bounded measures on $\hat{A}$. Denote this by $M_{rba}(\hat{A})$ and let $\Pi_{rba}(\hat{A}) \subset M_{rba}(\hat{A})$ denote the set of regular bounded finitely additive probability measures on $\hat{A}$. Clearly if A is compact, then $\big(BC(\hat{A})\big)^*$ will be the space of countably additive measures, as in [23]. Now consider the space $L_1(\mu, BC(\hat{A}))$, i.e. the space of all μ integrable functions defined on A with values in $BC(\hat{A})$. In other words, for each $\phi \in L_1(\mu, BC(\hat{A}))$ we have

$$\| \phi \|_\mu \equiv \int_A ||\phi(x)(\cdot)||_{BC(\hat{A})} \mu(dx) < \infty.$$

With respect to this norm topology, $L_1(\mu, BC(\hat{A}))$ is a Banach space. Since the Banach spaces $BC(\hat{A})$ and its dual $M_{rba}(\hat{A})$ do not satisfy RNP(Radon Nikodym property), the dual of $L_1(\mu, BC(\hat{A}))$ is not $L_\infty(\mu, M_{rba}(\hat{A}))$. However, it follows from the theory of "lifting" [[40],Theorem 7, p94, Theorem 9, p97] that the dual of the above space is $L^w_\infty(\mu, M_{rba}(\hat{A}))$, i.e., the space of all $M_{rba}(\hat{A})$ valued functions $\{q\}$ which are weak star measurable in the sense that for each $\phi \in BC(\hat{A})$, $x \longrightarrow q_x(\phi) \equiv \int_{\hat{A}} \phi(z) q(x,dz)$ is μ measurable and μ-essentially bounded. Now define the admissible set as follows

$$Q_{ad} \equiv L^w_\infty(\mu, \Pi_{rba}(\hat{A})) \subset L^w_\infty(\mu, M_{rba}(\hat{A})).$$

In other words, Q_{ad} is the unit sphere in the space $L^w_\infty(\mu, M_{rba}(\hat{A}))$. Clearly, for each $\phi \in L_1(\mu, BC(\hat{A}))$ we may define a linear functional on $L^w_\infty(\mu, M_{rba}(\hat{A}))$ by

$$\ell_\phi(q) = \int_A \Big(\int_{\hat{A}} \phi(x,y) q(x,dy) \Big) \mu(dx).$$

This is certainly a bounded linear w^*-continuous functional on $L_\infty^w(\mu, M_{rba}(\hat{A}))$. Now let $\rho : A \times \hat{A} \longrightarrow [0, \infty]$ be any $\mathcal{A} \times \hat{\mathcal{A}}$ measurable (cost) function from the class $L_1(\mu, BC(\hat{A}))$ and introduce the constraint set

$$Q(D) = \{q \in Q_{ad};\ \ \ell_\rho(q) \le D\}.$$

We assume that this set is nonempty. It is clear that it is convex, bounded and w^*-closed and hence it is w^*-compact (as a w^*-closed subset of the w^*-compact set Q_{ad}). Compactness of Q_{ad} follows from the Alaoglu theorem [[37],Theorem V.4.2, 424] see also [42]].
There are certain important cases in which ρ may not be bounded, for instance when ρ is a metric of a metric space. We state a lemma, which is crucial to extend the w^*-closedness property of $Q(D)$ to those cases.

Lemma 3.2.4 *Let $A, \hat{A}$ be two Polish spaces and $\rho : A \times \hat{A} \longrightarrow [0, \infty]$, is measurable, nonnegative, extended real valued function and also $y \to \rho(x, y)$ is continuous on $\hat{A}$, for μ-almost all $x \in A$. For any $D \in [0, \infty)$, introduce the set*

$$Q_\rho(D) \equiv \left\{ q \in Q_{ad} : \ell_\rho(q) \equiv \int_A \Big(\int_{\hat{A}} \rho(x, y) q(x, dy) \Big) \mu(dx) \le D \right\}$$

and suppose it is nonempty. Then $Q_\rho(D)$ is a bounded w^* closed convex subset of Q_{ad} and hence w^* compact.

Proof. Clearly the set $Q_\rho(D)$ is bounded and convex. We prove that it is weak star closed. Let $\{q_\alpha\} \in Q_\rho(D) \subset Q_{ad}$ be a net. Since Q_{ad} is weak* compact, there exists a subnet of the net $\{q_\alpha\}$, relabeled as the original net, and an element $q \in Q_{ad}$ such that $q_\alpha \xrightarrow{w^*} q$ [3]. We must show that $q \in Q_\rho(D)$. Considering the sequence $\{\rho_k \equiv \rho \wedge k, k \in N\}$, which are bounded measurable functions (continuous in the second argument), it follows from the weak star convergence of the net $\{q_\alpha\}$ to q that

$$\int_A \Big(\int_{\hat{A}} \rho_k(x, y) q(x, dy) \Big) \mu(dx) = \lim_\alpha \int_A \Big(\int_{\hat{A}} \rho_k(x, y) q_\alpha(x, dy) \Big) \mu(dx) \tag{3.4}$$

for each $k \in N$. Since ρ is non-negative and $\rho_k \uparrow \rho$ as $k \to \infty$ and $q_\alpha \in Q_\rho(D)$, we have

$$\lim_\alpha \int_A \Big(\int_{\hat{A}} \rho_k(x, y) q_\alpha(x, dy) \Big) \mu(dx) \le \lim_\alpha \int_A \Big(\int_{\hat{A}} \rho(x, y) q_\alpha(x, dy) \Big) \mu(dx) \le D. \tag{3.5}$$

Combining (3.4) and (3.5) we arrive at the following inequality

$$\int_A \Big(\int_{\hat{A}} \rho_k(x, y) q(x, dy) \Big) \mu(dx) \le D,$$

[3] i.e. $\left| \int_A \int_{\hat{A}} \phi(x, y) q_\alpha(x, dy) \mu(dx) - \int_A \int_{\hat{A}} \phi(x, y) q(x, dy) \mu(dx) \right| \to 0$ for any $\phi \in L_1(\mu; BC(\hat{A}))$.

which is valid for all $k \in N$. Since $\rho_k \uparrow \rho$ and they are nonnegative, it follows from Lebesgue monotone convergence theorem and nonnegativity of stochastic kernels that

$$\int_A \Big(\int_{\hat{A}} \rho(x,y) q(x,dy) \Big) \mu(dx) \leq D.$$

This shows that the weak star limit $q \in Q_\rho(D)$ and hence we have proved that the set $Q_\rho(D)$ is a weak star closed subset of Q_{ad}. Being a weak star closed subset of a weak star compact set, it is weak star compact. This completes the proof. •

3.3 Existence of Solutions

In this section we study the question of existence of solution to the rate distortion problem 3.2 as stated in the preceding section. The methodology is based on the classical lower semi continuity of relative entropy and compactness of the set $Q(D)$. This approach is also used in [23] under stronger topologies and compactness assumption for the reproduction space. However, here we do not require compactness of $\hat{A}$, and hence the method of [23] is not applicable. In addition, our approach covers both countably additive and finitely additive measures.

Following this we demonstrate the equivalence of constrained and unconstrained problems which is used later to develop necessary conditions. Since ρ is fixed, for convenience of notation, from now on we set $Q_\rho(D) = Q(D)$.

Theorem 3.3.1 *Suppose the assumptions of Lemma 3.2.4 hold. Then the problem* $R(D) = \inf_{q \in Q(D)} I_\mu(q)$ *has a minimum.*

Proof. First we prove that $q \longrightarrow I_\mu(q)$ is weak star lower semi continuous. Let $\{q_\alpha\}$ be a net from Q_{ad} and suppose it is weak star convergent to q. Define the net $P_\alpha \in \Pi_{rba}(A \times \hat{A})$ given by the convolution product $P_\alpha \equiv \mu \otimes q_\alpha$. Take any $\varphi \in BC(A \times \hat{A})$ and consider the expression

$$\int_{A \times \hat{A}} \varphi(x,y) P_\alpha(dx \times dy) \equiv \int_{A \times \hat{A}} \varphi(x,y) q_\alpha(x,dy) \mu(dx).$$

Since $q_\alpha \xrightarrow{w^*} q$ in $L_\infty^w(\mu, \Pi_{rba}(\hat{A}))$, it is clear from the above expression that

$$P_\alpha \xrightarrow{w^*} P \equiv \mu \otimes q \ \text{ in } \ \Pi_{rba}(A \times \hat{A}).$$

Similarly one can easily verify that the net of the product measures $\{\pi_\alpha\}$ converges to the product measure π,

$$\pi_\alpha \equiv \nu_\alpha \times \mu \xrightarrow{w^*} \pi \equiv \nu \times \mu,$$

where $\{\nu_\alpha\}$ are the marginals of $\{P_\alpha\}$ on $\hat{A}$ and ν is its weak star limit. Now we use the lower semi continuity property of relative entropy [[41], Lemma 1.4.3, p36]. Examining the proof in [41] one can easily verify that the same procedure holds true not only for countably additive measures but also for finitely additive ones. Using this fact we conclude that

$$H(P||\pi) \leq \underline{\lim} H(P_\alpha||\pi_\alpha).$$

By definition 3.2.1, this is equivalent to

$$I_\mu(q) \leq \underline{\lim}_\alpha I_\mu(q_\alpha).$$

This proves weak star lower semi continuity of $I_\mu(\cdot)$ on Q_{ad}. We have already observed in Lemma 3.2.4 that the set $Q(D) \equiv Q_\rho(D)$ is weak* compact, and we have just seen that I_μ is w^*-lower semi continuous. Hence $I_\mu(q)$ attains its infimum on $Q(D)$. So there exists a $q^* \in Q(D)$ such that $R(D) = I_\mu(q^*)$. •

Remark 3.3.2 *The measures considered here are finitely additive. However, if $\{A, \hat{A}\}$ are compact Polish spaces, these measures are countably additive. In the noncompact case, we may use Stone-Čeck compactification turning A and $\hat{A}$ into compact Hausdorff spaces βA and $\beta\hat{A}$, respectively [39]. Using these spaces one can extended the finitely additive measures to countably additive ones. In this case one may replace $L^w_\infty(\mu, M_{rba}(\hat{A}))$ by $L^w_\infty(\mu, M_{rca}(\beta\hat{A}))$.*

In summary, we have shown that the rate distortion problem as stated in this chapter has a solution. The next step will be to show the equivalence of constrained and unconstrained optimization problems and develop necessary conditions of optimality.

Theorem 3.3.3 *Suppose $\rho : A \times \hat{A} \longrightarrow \overline{R}_0 \equiv [0, \infty]$ is continuous in the second argument and the set $\Gamma \equiv \{(x,y) \in A \times \hat{A} : \rho(x,y) < D\}$ is nonempty. Then the constrained problem as stated in Theorem 3.3.1, is equivalent to an unconstrained problem as stated below:*

$$\begin{aligned}\inf_{q \in Q(D)} I_\mu(q) &= \max_{z\geq 0} \inf_{q\in Q(D)} \{I_\mu(q) + zG(q)\} \\ &= \max_{z\geq 0} \inf_{q\in Q(D)} \Big\{I_\mu(q) + z\Big(\int_A \int_{\hat{A}} \rho(x,y)q(x,dy)\mu(dx) - D\Big)\Big\}.\end{aligned}$$

Further the infimum occurs on the boundary of the set $Q(D)$.

Proof. Our proof is based on Lagrange Duality theorem [18], Theorem 1, p224. We choose $X \equiv L^w_\infty(\mu, M_{rba}(\hat{A}))$ which is clearly a vector space. For the set Ω the natural choice is the set $\Omega = Q_{ad} \equiv L^w_\infty(\mu, \Pi_{rba}(\hat{A})) \subseteq X$. Define

$$G(q) = \ell_\rho(q) - D \equiv \int_A \Big(\int_{\hat{A}} \rho(x,y)q(x,dy)\Big)\mu(dx) - D, \qquad q \in L^w_\infty(\mu, M_{rba}(\hat{A})).$$

It is clear that $G(\cdot)$ is a convex mapping from $L_\infty^w(\mu, M_{rba}(\hat{A}))$ into the real line with the natural ordering $(\Re, \prec) \equiv Z$. Also recall that $q \longrightarrow I_\mu(q)$ is convex and well defined on Ω and that, by Theorem 3.3.1, $\inf\{I_\mu(q), q \in Q(D)\}$ exists and is finite. Thus, according to the Lagrange duality theorem referred to above, it suffices to show that there exists a $q_1 \in \Omega$ such that

$$G(q_1) = \int_A \{\int_{\hat{A}} \rho(x,y) q_1(x,dy)\}\mu(dx) - D < 0.$$

Introduce the sets $A_1 \equiv \{x \in A : \Gamma_x \neq \emptyset\}$ and $A_0 \equiv A \setminus A_1$, with Γ_x denoting the x-section of Γ. Define the measure valued function q_1 as follows

$$q_1(x, \Gamma_x) = 0, \ \ \forall \ \ x \in A_0; \ \ q_1(x, \hat{A}) = 1, \ \ \forall \ \ x \in A$$

$$0 \leq q_1(x, B) \leq 1, B \subset \Gamma_x, \ \ q_1(x, \Gamma_x) = 1, \ \ \forall \ \ x \in A_1$$

where $B \in \hat{\mathcal{A}}$. Since by hypothesis $\Gamma \neq \emptyset$ we have $\mu(A_1) > 0$ and thus the kernel q_1 is well defined and it belongs to $L_\infty^w(\mu, \Pi_{rba}(\hat{A}))$. Using this kernel in the expression for $\ell_\rho(q)$, one can easily verify that $\ell_\rho(q_1) < D$ and hence $G(q_1) < 0$. Then, by the Lagrange Duality theorem, we arrive at the conclusion of the theorem as stated. Also it follows from the same duality theorem that if the infimum is achieved by some $q^* \in L_\infty^w(\mu, \Pi_{rba}(\hat{A}))$, then

$$z\Big(\int_A \int_{\hat{A}} \rho(x,y) q^*(x,dy)\mu(dx) - D\Big) = 0.$$

In other words, for non-zero $z \in [0, \infty)$, solution occurs on the boundary. This completes the proof. •

Remark 3.3.4 *In the next section in Theorem 3.4.2, it will be shown that the case $z = 0$ is trivial since it leads to a situation where the optimum q is independent of $x \in A$.*

3.4 Necessary Conditions of Optimality

The following result is essential in deriving the necessary conditions of optimality. Since we did not assume absolute continuity of marginal measures as in [23], an alternative method based on calculus of variations on the space of measures is developed to find the necessary conditions of optimality.

Theorem 3.4.1 *Suppose $I_\mu(q) \equiv I(\mu; q)$ is well defined for every $q \in L_\infty^w(\mu, \Pi_{rba}(\hat{A}))$ possibly taking values from the set $[0, \infty]$. Then $q \to I_\mu(q)$ is Gateaux differentiable at every point* *in* $L^w_\infty(\mu, \Pi_{rba}(\hat{A}))$, *and the Gateaux derivative at the point* q_0 *in the direction* $q - q_0$ *is given by*

$$\delta I_\mu(q_0, q - q_0) = \int_A \int_{\hat{A}} \log\left(\frac{q_0(x, dy)}{\nu_0(dy)}\right)(q - q_0)(x, dy)\mu(dx)$$

where ν_0 *is the marginal measure on* $\hat{\mathcal{A}}$ *corresponding to* q_0.

Proof. See Appendix C.

In view of Theorem 3.3.3, the constrained problem defined by (3.3) can be reformulated using Lagrange multipliers as follows:

$$R(D) = \inf_{q \in Q_{ad}} \{I_\mu(q) - s(\ell_\rho(q) - D)\}, \tag{3.6}$$

where I_μ and ℓ_ρ are as defined in the preceding section and $s \in (-\infty, 0]$ is the Lagrange multiplier. Note that Q_{ad} is a proper subset of the vector space $L^w_\infty(\mu, M_{rba}(\hat{A}))$. Thus the problem (3.6) is not completely free of constraints. To obtain a fully unconstrained problem we must introduce yet another Lagrange multiplier so that we have an optimization problem on the vector space $L^w_\infty(\mu, M_{rba}(\hat{A}))$ without constraints. This is presented in the proof of the following theorem.

Theorem 3.4.2 *The infimum in problem* (3.6) *is attained at* $q^* \in L^w_\infty(\mu, \Pi_{rba}(\hat{A}))$ *given by*

$$q^*(x, F) = \frac{\int_F e^{s\rho(x,y)}\nu^*(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu^*(dz)}, \qquad s \leq 0 \tag{3.7}$$

where $F \in \hat{\mathcal{A}}$ *and* ν^* *is the marginal of* $P^* = \mu \otimes q^*$ *on* $\hat{A}$. *The corresponding rate distortion function has the following form*

$$R(D) = sD - \int_A \log\Big(\int_{\hat{A}} e^{s\rho(x,y)}\nu^*(dy)\Big)\mu(dx).$$

If $R(D) > 0$ *then* $s < 0$ *and hence*

$$\int_A \int_{\hat{A}} \rho(x, y)q^*(x, dy)\mu(dx) = D.$$

Proof. First note that $s \leq 0$ comes from the discussion about equivalence of constrained and unconstrained problems in section 3.3 (Theorem 3.3.3). Using a pair of Lagrange multipliers $\{s, \lambda(\cdot)\}$, introduce the extended cost functional $J_{\mu,D}(q)$ as follows:

$$J_{\mu,D}(q) = I_\mu(q) - s(\ell_\rho(q) - D) + \int_A \lambda(x)\Big(\int_{\hat{A}} q(x, dy) - 1\Big)\mu(dx).$$

This is a fully unconstrained problem defined on the vector space $L^w_\infty(\mu, M_{rba}(\hat{A}))$. Since both μ and D are fixed for the given problem, for simplicity of notation we may suppress them and set $J_{\mu,D}(q) \equiv J(q)$. By use of the same technique as in Theorem 3.4.1, we can show that the Gateaux derivative of J on $L^w_\infty(\mu, M_{rba}(\hat{A}))$ at any point q^* in the direction $q - q^*$ is given by the following expression,

$$\begin{aligned}
\delta J(q^*; q-q^*) &= \int_A \int_{\hat{A}} \log\Big(\frac{q^*(x,dy)}{\nu^*(dy)}\Big)(q-q^*)(x,dy)\mu(dx) - s\int_A\int_{\hat{A}} \rho(x,y)(q-q^*)(x,dy)\mu(dx) \\
&\quad + \int_A\int_{\hat{A}} \lambda(x)(q-q^*)(x,dy)\mu(dx) \\
&= \int_A\int_{\hat{A}} \log\Big(e^{\lambda(x)-s\rho(x,y)}\frac{q^*(x,dy)}{\nu^*(dy)}\Big)(q-q^*)(x,dy)\mu(dx), \quad \forall q \in L^w_\infty(\mu, M_{rba}(\hat{A})).
\end{aligned}$$

Since $J(q)$ is convex in q, it follows from the basic principle of calculus of variation that a necessary and sufficient condition for q^* to be the minimizer is that $\delta J(q^*; q-q^*) = 0$ for all $q \in L^w_\infty(\mu, M_{rba}(\hat{A}))$. Since this equality holds for all $q \in L^w_\infty(\mu, M_{rba}(\hat{A}))$ (which is a linear vector space), the corresponding Gateaux gradient must vanish which requires that q^* must satisfy the following identity,

$$\frac{q^*(x,dy)}{\nu^*(dy)} = e^{-\lambda(x)+s\rho(x,y)}. \tag{3.8}$$

This result and the required constraint $q^* \in L^w_\infty(\mu, \Pi_{rba}(\hat{A}))$, imply that $\lambda(x)$ must satisfy the relation $\lambda(x) = \log(\int_{\hat{A}} e^{s\rho(x,y)}\nu^*(dy))$. Hence q^* is given by the following expression,

$$q^*(x,F) = \frac{\int_F e^{s\rho(x,y)}\nu^*(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu^*(dz)}, \qquad \forall\ x \in A \text{ and } F \in \hat{\mathcal{A}}.$$

Since $s \leq 0$ and $\rho \geq 0$, it is evident that $q^* \in L^w_\infty(\mu, \Pi_{rba}(\hat{A}))$. Substituting this in the expression for $J(q)$ we obtain

$$R(D) = J(q^*) = sD - \int_A \log\Big(\int_{\hat{A}} e^{s\rho(x,y)}\nu^*(dy)\Big)\mu(dx). \tag{3.9}$$

Clearly, for $s = 0$, we have $R(D) = 0$ and $q^*(x,F) = \nu^*(F)$ for μ-almost all $x \in A$. This is trivial and so we must have $s < 0$. Thus, it follows from the result of Thoerem 3.3.3, that the solution occurs on the boundary of $Q(D)$ giving

$$\int_A\int_{\hat{A}} \rho(x,y)q^*(x,dy)\mu(dx) = D$$

for some $s < 0$. We denote the corresponding value of s by s^*. This completes the proof of all the necessary (and sufficient) conditions for optimality as stated in the theorem. •

Remark 3.4.3 *It should be pointed out that the solution given in Theorem 3.4.2, is a nonlinear non-trivial relationship between q^* and ν^*. Equation (3.7) is essentially a new nonlinear* *equation connecting those two distributions, which in turn is required to be solved. Such an equation may not have solutions in general, i.e., the form of q^* is just a candidate for the optimal minimizing measure. So far from Theorem 3.3.1, we already know that there exists a minimizing measure. If (3.7) does not have a solution, then the minimizing measure exists but does not have the form given in that equation. In the finite alphabet case, Blahut algorithm proves existence of a pair (q^*, ν^*) which satisfies the above equation. However in the abstract alphabet case, this can not be deduced from the extension of Blahut algorithm, since the convergence of measures in the extended algorithm is not in the strong topology [26]. Hence the question of existence of a pair (q^*, ν^*) satisfying (3.7) should be investigated.*

Note that the expression (3.7) giving the optimal solution as stated in Theorem 3.4.2 is not explicit. This is only an implicit relationship between q^* and its marginal ν^*. So this gives rise to a fixed point problem. We must prove that this fixed point problem has a solution. In view of the expression (3.7), for fixed but arbitrary $s < 0$, we define the operator T_s as follows:

$$T_s(q)(x,F) \equiv \frac{\int_F e^{s\rho(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu(dz)} = \frac{\int_F e^{s\rho(x,y)} \int_A q(\xi, dy)\mu(d\xi)}{\int_{\hat{A}} e^{s\rho(x,z)} \int_A q(\xi, dz)\mu(d\xi)} \qquad x \in A, F \in \hat{\mathcal{A}}. \quad (3.10)$$

Clearly the operator T_s maps $X \equiv L^w_\infty(\mu, M_{rba}(\hat{A}))$ into the unit ball $B_1(X)$ of X. We prove that it has a fixed point in $Q_{ad} \equiv L^w_\infty(\mu, \Pi_{rba}(\hat{A})) \subset B_1(X)$. Define

$$\rho(x,\hat{A}) \equiv \inf\{\rho(x,y), y \in \hat{A}\}, x \in A.$$

Theorem 3.4.4 *Suppose the triple $\{\hat{A}, \rho, \mu\}$ satisfy the following inequality*

$$\int_A \rho(x,\hat{A})\mu(dx) \le D,$$

and further, for μ almost all $x \in A$, $y \longrightarrow \rho(x,y)$ is continuous on $\hat{A}$. Then there exists an $\hat{s} \in (-\infty, 0]$ such that for each $s \in (-\infty, \hat{s}]$, T_s has a fixed point in $Q(D) \subset X$.

Proof. Take $X = L^w_\infty(\mu, M_{rba}(\hat{A}))$, and $K = Q(D) = \{q \in Q_{ad} : \quad \ell_\rho(q) \le D\}$. We note that X with respect to its weak star topology is a locally convex topological vector space. Clearly the set K is convex. The assumption related to ρ is necessary for the set K to be nonempty. In fact this condition is also sufficient if the set

$$\{y \in \hat{A} : \rho(x,y) = \rho(x,\hat{A}),\ \forall\ x \in A\}$$

is nonempty. We have shown before that K is compact with respect to the weak star topology. Thus K is a nonempty, compact convex set in the locally convex space X. Given $x \in A$,

$q \in X$ and $s \in (-\infty, 0]$, define the measure-valued function $T_s(q)(x, \cdot)$ as follows

$$T_s(q)(x, dy) = \frac{e^{s\rho(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu(dz)}, \qquad \text{where } \nu(F) = \int_A q(x, F)\mu(dx), \quad F \in \hat{\mathcal{A}}, \quad x \in A.$$

It is obvious that $T_s(q)(x, \cdot) \in \Pi_{rba}(\hat{A})$ for each $x \in A$. Hence

$$\mu - ess \sup_{x \in A} ||T_s(q)|| = 1 < \infty, \qquad \forall s \in (-\infty, 0].$$

Clearly, then $T_s(q) \in L^w_\infty(\mu, \Pi_{rba}(\hat{A}))$. We show that for suitable $s \leq 0$, the operator T_s maps K into itself. In other words, $T_s(q)$ satisfies the inequality constraint, $\ell_\rho(T_s(q)) \leq D$ for each $q \in K$. Take any $q \in K \equiv Q(D)$. Then

$$\ell_\rho(T_s(q)) \equiv \int_A \int_{\hat{A}} \rho(x, y) T_s(q)(x, dy)\mu(dx) = \int_A \frac{\int_{\hat{A}} \rho(x, y) e^{s\rho(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu(dz)} \mu(dx). \tag{3.11}$$

For any finite positive number r define the functions $\rho_r \equiv \rho \wedge r$ and h_r by

$$h_r(s, x) \equiv \frac{\int_{\hat{A}} \rho_r(x, y) e^{s\rho_r(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho_r(x,z)}\nu(dz)}, \quad s \in (-\infty, 0], x \in A.$$

Then consider the function

$$s \longrightarrow \ell_{\rho_r}(T_s(q)) = \int_A h_r(s, x)\mu(dx)$$

on the semi infinite interval $(-\infty, 0]$. Taking the derivative of the function $h_r(s, x)$ with respect to the variable s one can verify that

$$\begin{aligned}
\frac{d}{ds} h_r(s, x) &= \frac{d}{ds}\left(\frac{\int_{\hat{A}} \rho_r(x, y) e^{s\rho_r(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho_r(x,z)}\nu(dz)}\right) \\
&= \left(\frac{\left(\int_{\hat{A}} \rho_r(x, y)^2 e^{s\rho_r(x,y)}\nu(dy)\right)\left(\int_{\hat{A}} e^{s\rho_r(x,z)}\nu(dz)\right) - \left(\int_{\hat{A}} \rho_r(x, z) e^{s\rho_r(x,z)}\nu(dz)\right)^2}{\left(\int_{\hat{A}} e^{s\rho_r(x,z)}\nu(dz)\right)^2}\right) \geq 0,
\end{aligned}$$

where non-negativity comes from the Hölder inequality. Since the function ρ_r is positive and bounded above by r, all the integrals in the numerator and denominator of the above expression are finite and positive. Thus for fixed $x \in A$ and $r > 0$, the real valued function

$$s \longrightarrow h_r(s, x)$$

is a nonnegative monotone non decreasing continuous function on $(-\infty, 0)$ having finite limit (from the left) at $s = 0$. Note that as $r \to \infty$,

$$h_r(s, x)) \to h(s, x) \equiv \frac{\int_{\hat{A}} \rho(x, y) e^{s\rho(x,y)}\nu(dy)}{\int_{\hat{A}} e^{s\rho(x,z)}\nu(dz)}$$

for each $s \in (-\infty, 0]$ and $x \in A$. In fact the convergence is uniform on compact subsets of the set $(-\infty, 0)$. Therefore $s \longrightarrow h(s,x)$ is also monotone, nondecreasing, continuous and, since μ is a positive measure, the function

$$s \longrightarrow \int_A h(s,x)\mu(dx) = \ell_\rho(T_s(q)) \equiv H(s)$$

is also a monotone nondecreasing continuous function. From this one can easily verify that

$$H(s) \equiv \int_A \int_{\hat{A}} \rho(x,y) T_s(q)(x,dy)\mu(dx) \leq \int_A \int_{\hat{A}} \rho(x,y)\nu(dy)\mu(dx) = H(0).$$

Using similar approach as given in [[30], property 1, p1097] one can show that

$$\lim_{s \downarrow -\infty} h(s,x) = \nu - ess \inf_{y \in \hat{A}} \rho(x,y). \tag{3.12}$$

Using this expression it follows from dominated convergence theorem that

$$\lim_{s \to -\infty} H(s) = \int_A \nu - ess \inf_{y \in \hat{A}} \rho(x,y)\mu(dx).$$

Clearly, we have the inequality

$$\int_A \int_{\hat{A}} \rho(x,y) q(x,dy)\mu(dx) \geq \int_A (\nu - ess \inf_{y \in \hat{A}} \rho(x,y))\mu(dx) = \lim_{s \to -\infty} H(s).$$

We have two possible situations: (A): q is an interior point of $Q(D)$. Then it follows from the above expression that $\lim_{s \to -\infty} H(s) < D$ and hence by the monotonicity of $s \to H(s)$, there exists an $\hat{s} \in (-\infty, 0]$ such that $T_s(q) \in Q(D)$ for $s \in (-\infty, \hat{s}]$. (B): q is on the boundary of $Q(D)$. Then $\lim_{s \to -\infty} H(s) \leq D$, and we may face either of two possibilities:

(B-1) If $\lim_{s \to -\infty} H(s) < D$, then again there exists an $\hat{s} \in (-\infty, 0]$ such that $T_s(q) \in Q(D)$ for $s \in (-\infty, \hat{s}]$.

(B-2) If $\lim_{s \to -\infty} H(s) = D$, then we combine this with $\int_A \int_{\hat{A}} \rho(x,y) q(x,dy)\mu(dx) = D$ to deduce the following

$$\int_A \Big\{ \int_{\hat{A}} \rho(x,y) q(x,dy) - (\nu - ess \inf_{y \in \hat{A}} \rho(x,y)) \Big\} \mu(dx) = 0.$$

Since the argument within the parenthesis is non-negative and μ is a positive measure, it follows from this that

$$\int_{\hat{A}} \rho(x,y) q(x,dy) = (\nu - ess \inf_{y \in \hat{A}} \rho(x,y))$$

which leads to the following

$$\rho(x,y) = (\nu - ess \inf_{y \in \hat{A}} \rho(x,y)) \;\; \nu\text{-a.s.}$$

and this means that $\rho(x,y)$ must be constant in y, ν-a.s., but then for this case it follows from the definition of the operator T_s that $T_s(q)(x,F) = \nu(F)$ for all $x \in A$ and $F \in \hat{A}$. Then from (3.11)

$$H(s) = \ell_\rho(T_s(q)) = \int_A \int_{\hat{A}} \rho(x,y)\nu(dy)\mu(dx) = \int_A \rho(x,y)\mu(dx); \quad \forall s \in (-\infty, 0]$$

On the other hand, from the hypothesis that $\lim_{s\to-\infty} H(s) = D$, we have

$$\int_A \rho(x,y)\mu(dx) = D$$

Hence $T_s(q) \in Q(D)$. This shows that under any of the conditions, (A) and (B-1) and also the trivial case (B-2), $T_s(q) \in K \equiv Q(D)$ for $s \in (-\infty, \hat{s}]$ whenever $q \in K$. Therefore for such s, the operator T_s maps K into subsets of K. Now we fix $s \in (-\infty, \hat{s}]$ and show that T_s is a continuous operator. Notice that

$$T_s(q)(x,dy) = \frac{\int_A e^{s\rho(x,y)} q(\xi,dy)\mu(d\xi)}{\int_A \int_{\hat{A}} e^{s\rho(x,z)} q(\xi,dz)\mu(d\xi)} \qquad x \in A. \tag{3.13}$$

Define

$$\begin{aligned} V_x(q) &\equiv \int_{\hat{A}} \int_A e^{s\rho(x,y)} q(\xi,dy)\mu(d\xi), \\ U_x(\phi,q) &\equiv \int_{\hat{A}} \int_A \phi(x,y) e^{s\rho(x,y)} q(\xi,dy)\mu(d\xi) \ \text{ for } \phi \in L_1(\mu; BC(\hat{A})), \end{aligned}$$

and let $q_n \xrightarrow{w*} q$ as $n \to \infty$ in $L_\infty^w(\mu, M_{rba}(\hat{A}))$. Then, for each $\phi \in L_1(\mu, BC(\hat{A}))$, it follows from (3.13) and the above notations that

$$\begin{aligned} &\Big| \int_A \int_{\hat{A}} \phi(x,y) T_s(q_n)(x,dy)\mu(dx) - \int_A \int_{\hat{A}} \phi(x,y) T_s(q)(x,dy)\mu(dx) \Big| \\ &= \left| \int_A \left\{ \frac{U_x(\phi,q_n)}{V_x(q_n)} - \frac{U_x(\phi,q)}{V_x(q)} \right\} \mu(dx) \right|. \end{aligned} \tag{3.14}$$

Since $q_n \xrightarrow{w^*} q$ it follows from continuity of ρ over $\hat{A}$ and negativity of s that $\lim_{n\to\infty} V_x(q_n) = V_x(q)$ and $\lim_{n\to\infty} U_x(\phi,q_n) = U_x(\phi,q)$ for every $x \in A$. Hence $\frac{U_x(\phi,q_n)}{V_x(q_n)} - \frac{U_x(\phi,q)}{V_x(q)} \to 0$ as $n \to \infty$ for every $x \in A$. Now notice that

$$\Big| \frac{U_x(\phi,q_n)}{V_x(q_n)} - \frac{U_x(\phi,q)}{V_x(q)} \Big| \le 2||\phi||_{BC(\hat{A})}(x), x \in A.$$

Since $\phi \in L_1(\mu; BC(\hat{A}))$, this shows that the integrand in (3.14) is dominated by a μ-integrable function. Thus it follows from the Lebesgue Dominated convergence theorem that the expression given by (3.14) converges to zero as $n \to \infty$. This proves that the map $q \longrightarrow T_s(q)$ is continuous in the weak star topology. Therefore all the conditions of the

Tihonov Fixed-Point Theorem, [[38], Corollary 9.6, p452], are satisfied. Hence $T_s(q)$ has a fixed point, i.e. there exists a $q \in Q(D)$ such that $T_s(q) = q$. •

Remark 4.5. Some interesting and useful properties of the rate distortion function R are as stated below.

(P1) $D \to R(D)$ is a convex, non-increasing function of D on $[0, +\infty)$ to $[0, +\infty]$.

(P2) There exists a number $D_{\max} > 0$ such that $R(D) > 0$ for all $D < D_{\max}$ and $R(D) = 0$ for all $D \geq D_{\max}$. The number $D_{\max}$ is given by

$$D_{\max} = \inf_{y \in \hat{A}} \int_A \rho(x, y)\mu(dx).$$

(P3) The solution q^* (with its marginal denoted by ν^*) of the optimization problem (3.3) must satisfy the following inequality

$$\int_A e^{s^* \rho(x,y)} \beta(x)\mu(dx) \leq 1, \qquad \forall y \in \hat{A}$$

where $\beta(x) = \left(\int_{\hat{A}} e^{s^* \rho(x,y)} \nu^*(dy) \right)^{-1}$.
For proof of these facts the reader is referred to [23].

3.5 Examples of A and $\hat{A}$ in Applications

In this section, we introduce some examples for the spaces used before. Suppose the source space A is a weakly compact subset of a separable Banach space E. The weak topology is metrizable [[37],Theorem V.6.3, 434] and with respect to this metric, A is a compact Polish space. For the reproduction space $\hat{A}$, one may choose any closed subset of A. Since a closed subset of a compact Polish space is also a compact Polish space, $\hat{A}$ is a compact Polish space. In this situation, the source distribution μ belongs to $M_1(A) \equiv \Pi_{rca}(A) \subset M_{rca}(A)$ and, the admissible stochastic kernels will be the set $Q_{ad} \equiv L^w_\infty(\mu, \Pi_{rca}(\hat{A})) \subset L^w_\infty(\mu, M_{rca}(\hat{A}))$, where $\Pi_{rca}(A), \Pi_{rca}(\hat{A})$ are countably additive regular measures rather than finitely additive ones as considered in the general theory. More specifically, one may take A to be the closed unit ball of a separable reflexive Banach space E. This is weakly compact and so the weak topology is metrizable and with respect to this metric topology A is compact, hence a compact Polish space. Recall that for μ to be a regular countably additive measure on a Banach space E (with separable dual E^*) it is necessary and sufficient that the corresponding covariance operator Q_μ be positive and nuclear, that is

$$Q_\mu \geq 0, \text{ and } Tr(Q_\mu) \equiv \sum (Q_\mu e_i^*, e_i^*) \equiv \int_E \sum (e_k^*(x - m))^2 \mu(dx) < \infty, \tag{3.15}$$

where $\{e_i, e_i^*\}$ is a biorthogonal basis of the pair $\{E, E^*\}$. Here $m \in E$ is the mean given by

$$e^*(m) \equiv \int_E e^*(x)\mu(dx)$$

provided it exists. The operator Q_μ has the form

$$Q_\mu \equiv \sum \lambda_i e_i \otimes e_i, \quad \text{and} \quad TrQ_\mu = \sum \lambda_i < \infty.$$

In practical communication problems involving image compression, an appropriate choice of source and reproduction spaces is important. We present a brief discussion of this here. Let Γ be a bounded measurable subset of R^n ($n = 2$ for image compression) and consider the Lebesgue spaces $\{L_p(\Gamma), 1 < p < \infty\}$. These are separable reflexive Banach spaces with the corresponding duals given by $\{L_q(\Gamma), q = (p/p-1), 1 < p < \infty\}$. Suppose we have $A \equiv B_1(E)$, the closed unit ball of $E \equiv L_p(\Gamma)$, which is weakly compact and so metrizable giving a compact Polish space. Let $\{e_i\}$ be a normalized Schauder basis for E with $\{e_i^*\} \subset E^*$ the associated dual basis. For the reproduction space $\hat{A}$, one may choose

$$\hat{A} \equiv \overline{span}\{e_i, 1 \leq i \leq N\} \cap B_1(E)$$

for any finite positive integer N. Here one natural choice for the distortion measure ρ is

$$\rho(x, y) \equiv \| x - y \| = \left(\int_\Gamma |x(\xi) - y(\xi)|^p d\xi\right)^{1/p}.$$

An approach similar to the above has been used in image compression using wavelet transforms [28], where images are considered as elements of $L_p(\Gamma)$, with $\Gamma \subset \Re^2$ a rectangle. The space of reproduced images is the linear span of a finite family of wavelets.

Remark 6.1 In case $p = 2$, E is a Hilbert space and in this case the nuclearity of the covariance operator means that

$$Tr(Q_\mu) \equiv \int_E \| (x - m) \|_E^2 \mu(dx) < \infty. \tag{3.16}$$

In other words, the source is a second order random field with mean field $m \in E$ and covariance operator $Q_\mu \in \mathcal{L}_n^+(E)$. Clearly if μ is a Gaussian measure, the pair $\{m, Q_\mu\}$ completely characterizes the source.

Remark 3.5.1 *If we do not fix the number (N) of elements of the basis used in reproduction a priori, and only require that finitely many of them be used in the compression, then the reproduction space is not finite dimensional. In this case, the reproduction space will not be compact anymore and weak* topology used in Theorem 3.3.1 should be taken into consideration.*

3.6 Conclusion

In this chapter, a general framework for the rate distortion problem in spaces of measure valued functions has been introduced. The questions of existence of optimal solutions and equivalence of constrained and unconstrained problems have been addressed. The optimal solution is given by an implicit relation, which give rise to a fixed point problem. Existence of a solution to this fixed point problem has been demonstrated. Some examples of source and reproduction spaces are presented illustrating the relevance of the abstract results.

Chapter 4

Optimization of Stochastic Uncertain Systems Subject to Energy Constraints

4.1 Introduction

The main objective of this chapter is to identify and explore specific relations between robustness and information theoretic concepts. In particular, we seek to determine the exact relations between the Legendre-Fenchel transform (which is often used to identify entropy rate functionals in Large Deviations), dissipation (which is used to address robustness of uncertain systems) and risk-seeking and risk-averse optimization problems. Further, these relations are also explored in the context of robustness of uncertain nonlinear partially observable systems. The main contribution of this chapter is the derivation of several properties associated with robustness of uncertain systems, in an abstract setting, which are then shown to be applicable to uncertain stochastic nonlinear control systems.

The connections discussed above are established by introducing a new class of stochastic uncertain systems, in which the pay-off is described by the relative entropy or Kullback-Leibler distance between an unknown probability measure and a known (fixed) probability measure, subject to a fidelity constraint expressed in terms of the unknown probability measure.

The abstract formulation of the new class of stochastic systems is introduced next. The abstract results are derived in sections 4.2, 4.3 and 4.4. Section 4.5 is devoted to the application of the results to nonlinear uncertain stochastic systems described by stochastic differential equations.

A. Abstract Formulation Let $\left(\Sigma, d\right)$ denote a complete separable metric space (a Polish space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space, in which $\mathcal{B}(\Sigma)$ is the σ-algebra generated by open sets in Σ. Let $\mathcal{M}(\Sigma)$ denote the set of probability measures on $(\Sigma, \mathcal{B}(\Sigma))$ and $\mathcal{U}_{ad}$ the set of admissible controls available to Player 1. By choosing a control policy u for the nominal system (system perfectly known and assumed to be free from unknown external disturbances), Player-1 can induce a probability measure $\mu^u \in \mathcal{M}(\Sigma)$ in order to meet its performance objectives. Player-2, representing an adversary, tries to thwart the objective of Player-1 by trying to gain knowledge of the measure μ^u through jamming signals perturbing the system. Such information may be used by Player-2 to diminish the effectiveness of control actions of the Player-1.

For a given $u \in \mathcal{U}_{ad}$, let $M(u) \subset \mathcal{M}(\Sigma)$ denote the set of probability measures induced by the perturbed system while Player-1 exercises control u. We assume that in the absence of perturbation the set $M(u)$ reduces to the singleton $\{\mu^u\}$. Let $\ell^u : \Sigma \rightarrow \Re$ be a real-valued measurable function bounded from below; and $H(\nu|\mu)$ denote the entropy of ν relative to the measure μ both from $M(u)$(see Definition 4.2.1). The relative entropy $H(\nu|\mu)$ is also called the Kulback-Leibler distance between the two measures. In fact it is a measure of divergence between the two probability measures. Throughout this chapter, "<<" will denote absolute continuity of one measure with respect to another. Let $\mu \in M(u)$ denote the measure induced by the nominal system. In order to gain knowledge of the measure $\mu(\equiv \mu^u)$, Player-2 tries to minimize the relative entropy $H(\nu|\mu)$ by choosing a measure $\nu \in M(u)$ subject to some additional constraints. Let $M_c(u) \subset M(u)$ denote such a constraint. Then clearly the objective of Player-2 is to find a ν that matches as closely as possible with μ. This leads to the following minimization problem:

$$\inf_{\nu \in M_c(u)} H(\nu|\mu),$$

for every control $u \in \mathcal{U}_{ad}$ exercised by the Player-1. Since such knowledge gained by Player-2 may be damaging to the Player-1, it tries to choose a control policy to maximize this. This gives rise to the maximin problem

$$\sup_{u \in \mathcal{U}_{ad}} \inf_{\nu \in M_c(u)} H(\nu|\mu).$$

This is the general problem we consider in this chapter for different constraints $M_c(u)$.
Another situation that leads to similar maximin problem can be described as follows: Let μ^0 denote the measure induced by an uncontrolled nonlinear stochastic system. Suppose this system without control is intrinsically unstable. The objective of Player-1 is to stabilize the system (stay away from μ^o) by choosing an appropriate control policy while Player-2 tries to destabilize the process using its limited resources (energy limitations). This leads to the

Maximin problem

$$\sup_{u \in \mathcal{U}_{ad}} \inf_{\nu \in M_c(u)} H(\nu|\mu^0).$$

This is the class of problems considered in this chapter. There are two possible physically meaningful choices of the constraint set M_c. These are

$$M_c(u) = M_o(u) \equiv \{\nu \in M(u) : \int \ell^u d\nu \leq \gamma\}$$

for some $\gamma < \int \ell^u d\mu^0 \equiv m^0$, and

$$M_c(u) = M_p(u) \equiv \{\nu \in M(u) : \int \ell^u d\nu \geq \gamma\}$$

for $\gamma > m^0$. In Appendix D, under Theorem 4.4.1, it is shown that if $\inf_{x \in \Sigma} \ell^u(x) < \gamma$, then $M_o(u)$ is non-empty and if $\sup_{x \in \Sigma} \ell^u(x) > \gamma$ then $M_p(u)$ is also non-empty.

Primal Problems.

(PP1): For a given control law $u \in \mathcal{U}_{ad}$, and a fixed nominal measure $\mu \in M(u)$, define the functional

$$J_o(u) \equiv \inf\{H(\nu|\mu) : \nu \in M_o(u)\}. \tag{4.1}$$

The ultimate objective is to find a control policy that maximizes this functional over the set $\mathcal{U}_{ad}$. As an intermediate step, we prove the existence of a $\nu^* \in M_o(u)$ at which the infimum indicated above is attained. Then we present an explicit characterization of this measure.

(PP2): Similarly, for a given control law $u \in \mathcal{U}_{ad}$, and a fixed measure $\mu \in M(u)$, define the functional

$$J_p(u) \equiv \inf\{H(\nu|\mu) : \nu \in M_p(u)\}. \tag{4.2}$$

Again we derive similar results for this problem as in (PP1). Clearly, the minimizing measures in (4.1), (4.2) will be a function of the nominal measure $\mu \in M(u)$ and the energy function ℓ^u. We point out that the generalization of (PP1), (PP2) to multiple energy constraints $\int_\Sigma \ell_j^u d\nu \leq \gamma_j$, $\int_\Sigma \ell_j^u d\nu \geq \gamma_j$, respectively, $1 \leq j \leq M$ is a natural extension, and can be dealt with while the conclusions derived throughout the chapter remain unaffected.

We proceed further to re-cast the problems (PP1) and (PP2) in terms of games. Here, we assume that the objectives of the control and uncertainty are conflicting (as it is usually done in robust control problems), hence the following games are introduced.

Primal Minimax Problems.

(PMP1) For a given $\mu \in M(u)$, find a control law $u^* \in \mathcal{U}_{ad}$ so that

$$J_o(u^*) = \sup_{u \in \mathcal{U}_{ad}} J_o(u) \tag{4.3}$$

(PMP2) For a given $\mu \in M(u)$, find a control law $u^* \in \mathcal{U}_{ad}$ so that

$$J_p(u^*) = \sup_{u \in \mathcal{U}_{ad}} J_p(u) \tag{4.4}$$

The control objective is to keep the system away from the undesirable situation as far as possible. This is done through maximizing over all admissible controls. This is an evasion problem, in which uncertainty pushes the system toward an unwanted state (such as unstable) and the control tries to evade this.

The rest of this chapter is organized as follows. In Section 4.2, the basic definitions and duality relations are introduced between relative entropy, cumulant generating functions and Legendre-Fenchel transforms. In addition, in Section 4.3 existence of the minimizing measures associated with (PP1), (PP2) is proved under the topology of weak* convergence. In Section 4.4, the equivalence between the constrained and unconstrained problems is derived, and the basic results leading to the relations between information theoretic concepts of large deviations, dissipation inequalities of robustness, and risk-sensitive optimization summarized above are derived (Lemma 4.4.2 and Theorem 4.4.3). In Section 4.5, the abstract formulation of primal problems (PP1), (PP2) are generalized to nonlinear continuous time stochastic partially observable control systems. Identification of the entropy rate functional associated with stochastic differential equations driven by Brownian Motion which is scaled by a small parameter can be found in [19]. Related material are also found in [75, 76]. Moreover, relations to the sub-optimal disturbance attenuation problem are discussed. Existence of optimal control laws and worst case measure are proved by reformulating the problem using conditional distributions, which satisfy certain Partial Differential Equations (PDE's).

4.2 Duality Relations and Tilted Measures

In this section, the basic definitions and duality relations are introduced between relative entropy, cumulant moment generating functions, and Legendre-Fenchel transforms. The concepts introduced in the following definition are employed extensively in subsequent sections to characterize the optimal strategies, establish the various relations discussed earlier in an abstract setting and apply to nonlinear stochastic control of partially observable systems.

Definition 4.2.1 *Let $\nu, \mu \in \mathcal{M}(\Sigma)$ and $\ell : \Sigma \to \Re$ a measurable function.*
1) The moment generating function of ℓ with respect to μ is defined by

$$M_\mu(s) \triangleq E_\mu\Big(e^{s\ell}\Big) = \int_\Sigma e^{s\ell} d\mu, \quad s \in \Re \tag{4.5}$$

2) The cumulant generating function of ℓ with respect to μ is defined by

$$\Psi_\mu(s) \triangleq \log M_\mu(s) = \log \int_\Sigma e^{s\ell} d\mu, \quad s \in \Re \tag{4.6}$$

3) The Legendre-Fenchel Transform of $\Psi_\mu(s)$ is defined by

$$\Psi_\mu^\star(x) \triangleq \sup_{s\in\Re} \Big\{ sx - \Psi_\mu(s) \Big\}, \quad x \in \Re \tag{4.7}$$

4) The relative entropy of ν with respect to μ is defined by

$$H(\nu|\mu) \triangleq \begin{cases} \int_\Sigma \log(\frac{d\nu}{d\mu}) d\nu & \text{if } \nu << \mu \text{ and } \log \frac{d\nu}{d\mu} \in L_1(\nu) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

It can be shown that $\Psi_\mu(s)$ as a function of ℓ is convex, $H(\nu|\mu)$ as a function of $\nu, \mu \in \mathcal{M}(\Sigma)$ is convex in both the arguments, $H(\nu|\mu) \geq 0$, and $H(\nu|\mu) = 0$, if and only if $\nu = \mu$. Thus, $H(\nu|\mu)$ can be considered as a measure of discrepancy between the two measures. Moreover, $M_\mu(s), \Psi_\mu(s)$ are convex functions of $s \in \Re$.

The moment generating function (4.5) and cumulant generating function (4.6), when employed in the context of stochastic control and filtering, represent the so-called risk-sensitive pay-off [49, 53, 54, 56]. For linear quadratic problems, the solution of risk-sensitive problems is equivalent to the solution of the minimax game formulation of the disturbance attenuation problem [49, 56]. Similarly, connections between risk-sensitive pay-off functionals and deterministic and stochastic minimax games with square integrable disturbances are also established in [58, 60, 61, 65].

The Legendre-Fenchel Transform (4.7) is employed in Large Deviations Theory to identify the entropy rate functional, which describes the exponential rate of convergence to zero of rare events.

In addition, the cumulant generating function and the relative entropy are in duality with respect to a Legendre-Fenchel transform, and the following result is a variant of a theorem found in [65].

Theorem 4.2.2 *For a given $s \in \Re$, and $\ell : \Sigma \to \Re$ a measurable function such that $s\ell$ is bounded below*

$$-\Psi_\mu(s) = -\log E_\mu(e^{s\ell}) = \inf_{\nu\in\mathcal{M}(\Sigma)} \Big\{ H(\nu|\mu) - \int_\Sigma s\ell d\nu \Big\} \tag{4.9}$$

Moreover, if $\ell e^{s\ell} \in L_1(\mu)$, then the infimum in (4.9) is attained by the tilted probability measure ν^* given by

$$d\nu^* = \frac{e^{s\ell} d\mu}{\int_\Sigma e^{s\ell} d\mu} \tag{4.10}$$

Proof. The proof is given in [65].

4.3 Existence of Minimizing Measures

This section is concerned with establishing the existence of a minimizing measures associated with the problems (4.1) and (4.2). For a given $u \in \mathcal{U}_{ad}$, existence of a minimizing measure $\nu^* \in M(u)$ for these problems, is proved under the topology of weak* convergence. Without loss of generality, the explicit dependence of measures and functions on u is suppressed.
Let Σ be a complete separable metric space with metric d and sigma field $\mathcal{B}(\Sigma)$. Let $BC(\Sigma)$ denote the space of bounded continuous real valued functions defined on Σ. Furnished with the sup norm topology, this is a Banach space. It is known that $\left(BC(\Sigma)\right)^*$ is isometrically isomorphic to the Banach space of finitely additive regular bounded measures on $\mathcal{B}(\Sigma)$ [37]. Let $\mathcal{M}(\Sigma) \subset \left(BC(\Sigma)\right)^*$ be the space of regular finitely additive probability measures on $\mathcal{B}(\Sigma)$. Let $\{\nu_n\}, n \geq 1$ be a sequence in $\mathcal{M}(\Sigma)$. Then by definition $\nu_n \xrightarrow{w^*} \nu$ if $\int_\Sigma f d\nu_n \to \int_\Sigma f d\nu$ for every $f \in BC(\Sigma)$.
Next, define the following minimization problems associated with (4.1) and (4.2).

Problem 4.3.1 ***(Basic Extremal Problems)*** *Let $\ell \in BC(\Sigma)$, $\mu \in \mathcal{M}(\Sigma)$ a fixed nominal measure, and $m \triangleq E_\mu(\ell) = \int_\Sigma \ell d\mu$, $\gamma \in \Re$. Define the functional $J(\nu) \triangleq H(\nu|\mu)$, and the (constraint) sets $M_o = \{\nu \in \mathcal{M}(\Sigma);\ \int_\Sigma \ell d\nu \leq \gamma\}$ and $M_p = \{\nu \in \mathcal{M}(\Sigma);\ \int_\Sigma \ell d\nu \geq \gamma\}$. Then*

P1) for $\gamma < m$, find $\nu^* \in M_o$ so that

$$J(\nu^*) = \inf_{\nu \in M_o} J(\nu) \triangleq J_o(\nu^*) \tag{4.11}$$

P2) for $\gamma > m$, find $\nu^* \in M_p$ so that

$$J(\nu^*) = \inf_{\nu \in M_p} J(\nu) \triangleq J_p(\nu^*). \tag{4.12}$$

Since $J : \mathcal{M}(\Sigma) \to [0, \infty]$ is a convex functional and the sets M_o and M_p (which are defined by linear functionals) are convex, the problems as stated above are convex optimization problems. The next theorem establishes existence of a minimizing measure under the topology of weak* convergence.

Theorem 4.3.2

1. Let (Σ, d) be any complete separable metric space, $\mathcal{M}(\Sigma)$ the space of probability measures defined on Σ and $\mathcal{K} \triangleq \{(\nu, \mu) \in \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma) : \nu << \mu\}$. Then the map $(\nu, \mu) \to H(\nu|\mu)$ from $\mathcal{K} \longrightarrow [0, \infty]$ is jointly weak lower semi continuous.*

2. For each $\ell \in BC(\Sigma)$ and $\gamma \in R$, the sets M_o and M_p are compact in the weak topology. Moreover, the same result holds if ℓ is continuous and bounded from below.*

3. For each problem of 4.3.1, there exists a minimizing measure $\nu^* \in \mathcal{M}(\Sigma)$. Moreover, the same result holds if ℓ is continuous and bounded from below.

Proof. (1). An elegant proof based on the duality between relative entropy and Free energy can be found in [41]. (2). For $\ell \in BC(\Sigma)$, it is easy to verify that these sets are weak* closed. Being weak star closed subsets of the weak star compact set $\mathcal{M}(\Sigma)$, they are weak star compact. Next we consider the case of ℓ continuous and bounded from below. Suppose we have a net $\{\nu_\alpha\}$ in M_o, where $\alpha \in (\mathcal{D}, \leq)$ (a directed set). Then by Alaoglu's theorem there exists a measure $\nu \in \mathcal{M}(\Sigma)$ and a subnet $\{\nu_{\alpha_k}\}$ such that $\nu_{\alpha_k} \overset{w^*}{\to} \nu$. We will show that $\nu \in M_o$. Since ℓ is bounded below, there exists $L \in \Re$ such that $g(x) \triangleq \ell(x) - L \geq 0$, where L denotes the lower bound. We have

$$\overline{\lim} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) = \int_\Sigma g(x) \wedge m \;\; d\nu(x) \leq \int_\Sigma g(x) d\nu(x)$$

where $m \in N$ is arbitrary. Then there exists $\alpha_1 \in \mathcal{D}$ such that

$$\begin{aligned} &\sup_{\alpha \geq \alpha_1} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) \leq \int_\Sigma g(x) d\nu(x) \\ &\sup_m \sup_{\alpha \geq \alpha_1} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) \leq \int_\Sigma g(x) d\nu(x). \end{aligned} \tag{4.13}$$

Now we have the following

$$\begin{aligned} \overline{\lim} \int_\Sigma g(x) d\nu_\alpha(x) \quad &= \overline{\lim} \sup_{m \in N} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) = \inf_{\alpha_0} \sup_{\alpha \geq \alpha_0} \sup_m \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) \\ &= \inf_{\alpha_0} \sup_m \sup_{\alpha \geq \alpha_0} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x) \leq \sup_m \sup_{\alpha \geq \alpha_1} \int_\Sigma g(x) \wedge m \;\; d\nu_\alpha(x). \end{aligned}$$

Combine the above with (4.13) to obtain

$$\overline{\lim} \int_\Sigma g(x) d\nu_\alpha(x) \leq \int_\Sigma g(x) d\nu(x). \tag{4.14}$$

For the reverse inequality, notice that for every $m \in N$ we have

$$\underline{\lim}_\alpha \int_\Sigma g(x) d\nu_\alpha(x) \geq \underline{\lim}_\alpha \int_\Sigma g_m(x) d\nu_\alpha(x) = \int_\Sigma g_m(x) d\nu(x), \quad \forall m \in N$$

where $g_m = g \wedge m$. The equality in the above holds because g_m is bounded continuous. Let $m \to \infty$, then $g_m \uparrow g$, and by the Lebesgue monotone convergence theorem we have

$$\underline{\lim}_\alpha \int_\Sigma g(x) d\nu_\alpha(x) \geq \int_\Sigma g(x) d\nu(x). \tag{4.15}$$

By combining (4.14) and (4.15), we have $\lim_\alpha \int_\Sigma g(x) d\nu_\alpha(x) = \int_\Sigma g(x) d\nu(x)$, and the same result holds for ℓ. Hence $\nu \in M_o$, proving the weak* compactness of the set M_o. The proof for M_p is similar.

3. Existence of minimizing measure follows from parts (1), (2) and Weirstrass theorem. This completes the proof. •

Remark. According to Theorem 4.3.2, there exists a minimizing measure. Moreover, for a fixed $\mu \in \mathcal{M}(\Sigma)$, the real valued function $\nu \longrightarrow H(\nu|\mu)$ defined on the set $\{\nu \in \mathcal{M}(\Sigma); H(\nu|\mu) < \infty\}$, is strictly convex [1] and therefore the minimizing measure is unique.

4.4 Equivalence of Constrained and Unconstrained Optimization and Legendre-Fenchel Transform

This section is concerned with reformulating the constrained optimization problem 4.3.1, as an unconstrained one using the theory of Lagrange functionals, and then showing the equivalence of the two problems. Connections are obtained between the rate functional associated with Large Deviations theory, Legendre-Fenchel transform and the problem of disturbance attenuation for a fixed control law. Through the control version of Legendre-Fenchel transform several properties of the minimizing measure are established including the range of values of the Lagrange multiplier for which the constrained and unconstrained problems are equivalent.

In addition, relations with optimization of stochastic systems in which the pay-off is an exponential functional, known as risk-sensitive problems, are also obtained. In particular, it is shown that problem $J_o(\nu^*)$ in (4.11) corresponds to the optimistic scenario (emphasizing the best cases) in which strategies are risk-seeking, while problem $J_p(\nu^*)$ in (4.12) corresponds

[1] We have $\frac{d^2}{dx^2}\left(x \log x\right) > 0$ for $x > 0$, hence $x \log x$ is strictly convex on $(0, \infty)$. Also $f = \frac{d\nu}{d\mu} \geq 0$. Since $0.\log 0 = 0$, then one can redefine $H(\nu|\mu) = \int_{\{x \in \Sigma; f(x) > 0\}} f \log f d\mu$, which is strictly convex.

to the pessimistic scenario (emphasizing the worst cases) in which strategies are risk-averse. The constrained problems 4.3.1 can be converted into unconstrained ones by introducing the Lagrangian functionals and the dual functionals. In Theorem 4.4.3 the equivalence of this dual functional with the two problems in 4.3.1 is established by specifying the range of values of the Lagrange multiplier $s \in \Re$.

For each $\gamma \in R$ and every $s \in \Re$, define the Lagrangian

$$J^{s,\gamma}(u,\nu) \triangleq H(\nu|\mu) - s\Big(E_\nu(\ell^u) - \gamma\Big) \tag{4.16}$$

Define the unconstrained problem

$$J^\gamma(u,\nu^*) \equiv \sup_{s\in\Re} \inf_{\nu\in M(u)} J^{s,\gamma}(u,\nu)$$

and the constrained problems as follows:

$$\begin{aligned} J_o^\gamma(u,\nu^*) &\equiv \inf_{\nu\in M_o(u)} H(\nu|\mu) \\ J_p^\gamma(u,\nu^*) &\equiv \inf_{\nu\in M_p(u)} H(\nu|\mu). \end{aligned} \tag{4.17}$$

Note that these are the problems (P1) and (P2). The following theorem establishes the equivalence of the constrained and unconstrained optimization problems.

Theorem 4.4.1 *Consider the problems P1 and P2 (see Problem 4.3.1) corresponding to a given control policy $u \in \mathcal{U}_{ad}$. Suppose they are finite.* [2] *Then*
(1) if $\inf_{x\in\Sigma} \ell^u(x) < \gamma$, we have

$$J_o^\gamma(u,\nu^*) = J^\gamma(u,\nu^*). \tag{4.18}$$

(2)if $\sup_{x\in\Sigma} \ell^u(x) > \gamma$, we have

$$J_p^\gamma(u,\nu^*) = J^\gamma(u,\nu^*). \tag{4.19}$$

Moreover, the same conclusions hold if ℓ^u is continuous and bounded below.

Proof. We shall employ the Lagrange Duality theorem in [18](pp.224-225). It follows from the properties of relative entropy that $\nu \longrightarrow H(\nu|\mu)$ is a convex functional. Let

$$\begin{aligned} X &\equiv \Big\{\text{finitely additive signed measures } \nu : \mathcal{B}(\Sigma) \rightarrow \Re, \nu(\Sigma) < \infty\Big\} \\ \textit{and} \quad \Omega &\equiv \Big\{\nu \in X; \nu \text{ is a finitely additive probability measure } \Big\}. \end{aligned}$$

[2] i.e., there exists at least one measure $\nu_o \in M_o(u)$ and one measure $\nu_p \in M_p(u)$ for which $H(\nu_o|\mu) < \infty$ and $H(\nu_p|\mu) < \infty$.

Clearly X furnished with total variation norm is a normed vector space. Also Ω is a convex set, since if $\nu_1, \nu_2 \in \Omega$ we have $\nu(\Sigma) = \lambda\nu_1(\Sigma) + (1-\lambda)\nu_2(\Sigma) = 1$ so $\nu = \lambda\nu_1 + (1-\lambda)\nu_2 \in \Omega$ for any $\lambda \in [0,1]$. Define the mapping

$$\nu \longrightarrow G(\nu) = \int_\Sigma \ell^u d\nu - \gamma$$

Clearly this is a convex map from X into the ordered vector space $(\Re, \prec)$. In Appendix D, under the assumptions of Theorem 4.4.1, it is shown that if $\inf_{x\in\Sigma} \ell^u(x) < \gamma$, then there exists a measure ν in Ω such that $\int_\Sigma \ell^u d\nu < \gamma$. Similarly, if $\sup_{x\in\Sigma} \ell^u(x) > \gamma$, there exists a measure ν in Ω such that $\int_\Sigma \ell^u d\nu > \gamma$. Therefore, the equivalence of constrained and unconstrained problems follows from the Lagrange Duality theorem in [18](pp.224-225). Since the existence of solution also holds for ℓ^u which is continuous and bounded from below, equivalence of the constrained and unconstrained problems also holds for this class of functions. •

Lemma 4.4.2 presents several properties of the unconstrained problems, including monotonicity properties of the dual functional with respect to γ and conditions for finding the Lagrange multiplier.

Lemma 4.4.2 *For a given $u \in \mathcal{U}_{ad}$, assume $\ell^u : \Sigma \to \Re$ is a measurable function which is bounded below. Define the Legendre-Fenchel transform of $\Psi_\mu(s)$ by*

$$\Psi_\mu^\star(\gamma) = \sup_{s\in\Re} \left\{ s\gamma - \Psi_\mu(s) \right\}$$

Then the following statements hold.

(1) $\inf_{\nu\in M(u)} J^{s,\gamma}(u,\nu) = s\gamma - \Psi_\mu(s), \quad \gamma \in \Re, \; s \in \Re.$

(2) Letting $\nu^{*,s}$ denote the minimizer in (1), the functional $J^{s,\gamma}(u,\nu^{*,s})$ is concave in $s \in \Re$.

(3) $\Psi_\mu^\star(\gamma) = \sup_{s\in\Re} J^{s,\gamma}(u,\nu^{*,s})$ *is a convex function of $\gamma \in \Re$.*

(4) $\Psi_\mu^\star(\gamma) \geq 0, \forall \gamma \in \Re$.

(5) If $\ell^u \in L_1(\mu)$ and $E_\mu(\ell^u) = m^u$ then

(a) $\Psi_\mu^\star(\gamma) = 0$ *for* $\gamma = m^u$.

(b) $\Psi_\mu^\star(\gamma)$ *is non-decreasing for* $\gamma \in [m^u, \infty)$, *that is,*

$$\Psi_\mu^\star(\gamma_1) \leq \Psi_\mu^\star(\gamma_2), \quad m^u \leq \gamma_1 \leq \gamma_2 < \infty.$$

(c) $\Psi_\mu^\star(\gamma)$ *is non-increasing for* $\gamma \in (-\infty, m^u]$, *that is,*

$$\Psi_\mu^\star(\gamma_2) \geq \Psi_\mu^\star(\gamma_1), \quad -\infty < \gamma_2 \leq \gamma_1 \leq m^u.$$

(d) Suppose there exists an open connected set $I \subset \Re$ such that $\ell^u e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$, then $J^{s,\gamma}(u, \nu^{*,s})$ is continuously differentiable with respect to $s \in I$

$$\frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s}) = \gamma - \frac{E_\mu\left(\ell^u e^{s\ell^u}\right)}{E_\mu\left(e^{s\ell^u}\right)} = \gamma - E_{\nu^{*,s}}\left(\ell^u\right) \tag{4.20}$$

where

$$d\nu^{*,s} = \frac{e^{s\ell^u} d\mu}{\int_\Sigma e^{s\ell^u} d\mu} \tag{4.21}$$

In addition, if $(\ell^u)^2 e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$, then $J^{s,\gamma}(u, \nu^{*,s})$ is twice continuously differentiable with respect to $s \in I$ and

$$\frac{d^2}{ds^2} J^{s,\gamma}(u, \nu^{*,s}) = -\left\{E_{\nu^{*,s}}\left((\ell^u)^2\right) - \left(E_{\nu^{*,s}}(\ell^u)\right)^2\right\} < 0 \tag{4.22}$$

(e) If the first assumption of (d) holds then the supremum is attained on the boundary of the constraint sets M_o, M_p for some $s^* \in I$, i.e.,

$$E_{\nu^{*,s}}\left(\ell^u\right)|_{s=s^*} = \gamma$$

where $\nu^{*,s}$ is given by (4.21) and it is unique if the second assumption holds.

Proof. The proof for (1)-(5a-5c) is standard [19, 52]. For the proof of (5d) and (5e) see Appendix D.

Next, the regions over which $J^{s,\gamma}(u, \nu^{*,s})$ is maximized are identified, and conditions for finding the Lagrange multipliers are derived for the problems P1 and P2 (see Problem 4.3.1).

Theorem 4.4.3 *Recall the Problems in 4.17, and suppose $J_o^\gamma(u, \nu^*), J_p^\gamma(u, \nu^*)$ are finite.*
1) Risk-Seeking Scenario. When $m^u \triangleq E_\mu(\ell^u) > \gamma$, there exists a minimizing measure $\nu^* \in M(u)$ which satisfies

$$J_o^\gamma(u, \nu^*) = \sup_{s \le 0} \left\{J_o^{s,\gamma}(u, \nu^*)\right\} = \sup_{s \le 0} \left\{s\gamma - \Psi_\mu(\gamma)\right\} = \Psi_\mu^\star(\gamma) = \inf_{\nu \in M_o(u)} H(\nu|\mu) \tag{4.23}$$

and if there exists an open connected set $I \subset \Re$ such that $\ell^u e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$ then $\nu^* \in M(u)$ is given by

$$d\nu^* = \frac{e^{s^* \ell^u} d\mu}{\int_\Sigma e^{s^* \ell^u} d\mu}, \qquad \text{for certain } s^* \le 0. \tag{4.24}$$

Moreover, if in addition $(\ell^u)^2 e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$ then the supremum over $s \le 0$ in (4.23) is attained at $s^* < 0$ and

$$\gamma = E_{\nu^*}(\ell^u)|_{s=s^*} \le E_{\nu^*}(\ell^u) < E_\mu(\ell^u) = m^u, \quad \forall s \in [s^*, 0] \tag{4.25}$$

2) Risk-Averse Scenario. When $m^u \triangleq E_\mu(\ell^u) < \gamma$, *then there exists a minimizing measure* $\nu^* \in M(u)$ *which satisfies*

$$J_p^\gamma(u,\nu^*) = \sup_{s\geq 0}\left\{J_p^{s,\gamma}(u,\nu^*)\right\} = \sup_{s\geq 0}\left\{s\gamma - \Psi_\mu(\gamma)\right\} \overset{\nabla}{=} \Psi_\mu^*(\gamma) \tag{4.26}$$

$$= \inf_{\nu\in M_p(u)} H(\nu|\mu) \tag{4.27}$$

and for some $s \geq 0$, *if there exists an open connected set* $I \subset \Re$ *such that* $\ell^u e^{\eta\ell^u} \in L_1(\mu)$ *for all* $\eta \in I$, *then* $\nu^* \in M(u)$ *is given by*

$$d\nu^* = \frac{e^{s^*\ell^u}d\mu}{\int_\Sigma e^{s^*\ell^u}d\mu}, \text{ for certain } \quad s^* \geq 0. \tag{4.28}$$

Moreover, if in addition $(\ell^u)^2 e^{\eta\ell^u} \in L_1(\mu)$ *for all* $\eta \in I$ *then the supremum over* $s \geq 0$ *in (4.26) is attained at* $s^* > 0$ *given by*

$$\gamma = E_{\nu^*}(\ell^u)|_{s=s^*} \geq E_{\nu^*}(\ell^u) > E_\mu(\ell^u) = m^u, \quad \forall s \in [0, s^*] \tag{4.29}$$

Moreover, the above statements also hold for those ℓ^u *which are continuous and bounded from below.*

Proof. See Appendix D.

Remark 4.4.4 *In view of part 4 of Lemma 4.4.2 the following inequality holds.*

$$\left\{E_{\nu^*}\left\{\ell^u\right\} - \frac{1}{s}H(\nu^*|\mu)\right\}|_{s=s^*} \leq \gamma. \tag{4.30}$$

For $s^* > 0$ *corresponding to the risk-averse scenario (e.g., PP2), inequality (4.30), when applied to dynamical systems, is equivalent to a dissipation inequality. Thus, the risk-averse scenario leads to strategies of the type which arise in optimal* H^∞ *disturbance attenuation problems [72]. This connection to disturbance attenuation of dynamical systems is further discussed in Section 4.5.2.*

4.5 Partially Observed Uncertain Systems with Wide Sense Controls

In this section, the abstract formulation of Problem 4.3.1 leading to Lemma 4.4.2 and Theorem 4.4.3 is applied to partially observable problems. However, for these results to be applicable, certain transformation of the partially observable problems are required in order to describe the uncertainty and the nominal model through conditional distributions, rather than a priori distribution. Once this is done, existence of the minimax strategies (from the class of wide sense controls [69])[3], and several properties of the optimal solution are presented, which are equivalent to those stated under Lemma 4.4.2 and Theorem 4.4.3.

4.5.1 Problem Formulation

Let $\{x(t)\}_{t\geq 0}$ denote the state process subject to control, $\{y(t)\}_{t\geq 0}$ the observation process, and $\{u(t)\}_{t\geq 0}$ the control process, all defined for a fixed and finite time interval $[0,T]$.
For each $u \in \mathcal{U}_{ad}$ (set of admissible controls to be defined shortly) the nominal state and the observation process, giving rise to a nominal probability measure P, are governed by the following system of stochastic differential equations:

$$\big(\Sigma, \mathcal{B}(\Sigma), P\big) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ \\ dy(t) = h(x(t))dt + Ndv(t), \quad y(0) = 0 \end{cases} \tag{4.31}$$

Here $x(t) \in \Re^n, y(t) \in \Re^d, u(t) \in \mathcal{U} \subset \Re^k$, $\{w(t)\}_{t\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are independent Brownian motions taking values in $\Re^m, \Re^d$, respectively, which are also independent of the initial state $x(0) = \xi$. Given the nominal measure P, find a $u^* \in \mathcal{U}_{ad}$ and a probability measure Q^{u^*} which solves the following constrained optimization problem:

$$J(u^*, Q^{u^*}) = \sup_{u\in\mathcal{U}_{ad}} \inf_{Q\in M(u)} H(Q|P) \tag{4.32}$$

$$\text{subject to fidelity } E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\} \leq \gamma, \quad \gamma \in \Re, \tag{4.33}$$

or

$$\text{subject to fidelity } E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\} \geq \gamma, \quad \gamma \in \Re. \tag{4.34}$$

The following assumptions are introduced.

Assumptions 4.5.1 *The nominal system satisfies the following assumptions:*
1) The control $\{u(t); t \in [0,T]\}$ is non anticipative and takes values in $\mathcal{U} \subset \Re^k$ which is compact and convex.
2) $f : \Re^n \times \mathcal{U} \to \Re^n, \sigma : \Re^n \to \mathcal{L}(\Re^n;\Re^m), f(x,u) = f_0(x) + f_1(x)u$, and f_0, f_1, σ are bounded and Lipschitz continuous.
3) $h : \Re^n \to \Re^d, h \in BC(\Re^n)$ and twice continuously differentiable.
4) $N \in \mathcal{L}(\Re^d;\Re^d)$ and $\exists \beta > 0$ such that $NN' \geq \beta I_d$.

[3]Wide sense controls are independent of future increments of the observations and the process noise. This is a wider class than strict sense controls, which are measurable with respect to the σ-algebra generated by the observations.

5) The initial state ξ has distribution Π_0.

6) $\lambda : \Re^n \times \mathcal{U} \rightarrow \Re, \kappa : \Re^n \rightarrow \Re$, are continuous, bounded and $u \longrightarrow \lambda(x,u)$ is convex for all $x \in \Re^n$.

Next, the problem statement is made precise by identifying the spaces on which the nominal system is defined and introducing the precise definition of admissible controls.

Consider the sample space

$$\Omega = \Omega^w \times \Omega^x \times \Omega^y \times \Omega^u$$

where

$$\Omega^w = C([0,T];\Re^m), \quad \Omega^x = C([0,T];\Re^n), \quad \Omega^y = C([0,T];\Re^d), \quad \Omega^u = L_2([0,T];\mathcal{U})$$

and $y(0) = 0$ is assumed throughout. Here, $\Omega^w, \Omega^x, \Omega^y$ are endowed with the usual sup-norm topology, while Ω^u is endowed with the weak topology (which is metrizable and separable). A typical element of Ω is $\omega(t) = \Big(w(t,\omega), x(t,\omega), y(t,\omega), u(t,\omega)\Big), 0 \leq t \leq T$. Let $\Omega^{w,x} = \Omega^w \times \Omega^x, \Omega^{y,u} = \Omega^y \times \Omega^u$. Then Ω is provided with a filtration $\{\mathcal{F}_t; t \in [0,T]\}$ which is defined as follows.

Let $\mathcal{F}_t^w = \sigma\{w(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^x = \sigma\{x(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^y = \sigma\{y(s); 0 \leq s \leq t\}$, which may be regarded as the Borel $\sigma-$algebras on $C([0,T];\Re^q), q = m,n,d$, respectively, and $\mathcal{F}_t^u = \sigma\{\int_0^s u(\tau)d\tau; 0 \leq s \leq t\}$, which is the Borel $\sigma-$algebra on Ω^u.

Introduce the product $\sigma-$algebras

$$\mathcal{F}_t \triangleq \mathcal{F}_t^{w,x} \times \mathcal{F}_t^{y,u}, \quad \mathcal{F}_t^{w,x} \triangleq \mathcal{F}_t^w \times \mathcal{F}_t^x, \quad \mathcal{F}_t^{y,u} \triangleq \mathcal{F}_t^y \times \mathcal{F}_t^u$$

where $\mathcal{F}_t^{w,x}$, $\mathcal{F}_t^{y,u}$, $\mathcal{F}_t$ are the Borel $\sigma-$algebras on $\Omega^{w,x}, \Omega^{y,u}, \Omega$, respectively.

Fix a sample path for the observation and control process $\{y(\cdot,\omega), u(\cdot,\omega)\}$. Given the initial data $x(0) = \xi, y(0) = 0, w(0) = 0$, Assumptions 4.5.1 imply existence of a unique probability measure $\bar{P}_\xi^{y,u}$ on $\Big(\Omega^{w,x}, \mathcal{F}_T^{w,x}\Big)$, which coincides with the distribution measure of $\{x(t), w(t); t \in [0,T]\}$ given $\{(y(\cdot,\omega), u(\cdot,\omega)\}$, such that $\{w(t); t \in [0,T]\}$ is a Wiener process and

$$\Big(\Omega^{w,x}, \mathcal{F}_T^{w,x}, \bar{P}_\xi^{y,u}\Big) : x(t) = \xi + \int_0^t f(x(s), u(s))ds + \int_0^t \sigma(x(s))dw(s). \tag{4.35}$$

In addition, Assumptions 4.5.1 also guarantee that the map $(\xi, u) \longrightarrow \bar{P}_\xi^{y,u}$ is continuous in the weak sense, that is, for each $\varphi \in BC(\Omega^{x,w})$, $(\xi, u) \longrightarrow \bar{P}_\xi^{y,u}(\varphi)$ is continuous on $\Re^n \times \Omega^u$ endowed with the weak topology.

Definition 4.5.2 *The set of admissible controls denoted by $\mathcal{U}_{ad}$ consists of probability measures π on $\left(\Omega^{y,u}, \mathcal{F}_T^{y,u}\right)$, that is, $\pi \in \mathcal{M}(\Omega^{y,u})$, such that $\{y(t); t \in [0,T]\}$ is $\mathcal{F}_T^{y,u} - \pi$-a.s. Brownian motion.(see [69], p.264)*

The projection $\{y(\cdot,\omega), u(\cdot,\omega)\} \mapsto y(\cdot,\omega)$ maps $\pi \in \mathcal{M}(\Omega^{y,u})$ onto a Wiener measure, and for all $t \in [0,T]$, $u(t)$ and $\sigma\{y(r) - y(t); 0 \leq t \leq r \leq T\}$ are independent under π. Given the measure $\Pi_0 \in \mathcal{M}(\Re^n)$ corresponding to the initial state $x(0) = \xi$, by Bayes rule we have

$$\bar{P}^{y,u}(A) = \int_{\Re^n} \bar{P}_\xi^{y,u}(A) d\Pi_0(\xi), \quad A \in \mathcal{F}_T^{w,x} \tag{4.36}$$

which is the unique joint distribution measure of $\{x(t), w(t); t \in [0,T]\}$ given $\{y(\cdot,\omega), u(\cdot,\omega)\}$. For each $\pi \in \mathcal{U}_{ad}$ define the joint distribution measure $\tilde{P}^\pi$ on $\left(\Omega, \mathcal{F}_T\right)$ by

$$\tilde{P}^\pi(dw, dx, du, dy) \triangleq \bar{P}^{y,u}(dw, dx) \times \pi(dy, du) \in \mathcal{M}(\Omega). \tag{4.37}$$

Notice that the projection $\{w(\cdot,\omega), x(\cdot,\omega), y(\cdot,\omega), u(\cdot,\omega)\} \mapsto \{y(\cdot,\omega), u(\cdot,\omega)\}$ under $\tilde{P}^\pi \in \mathcal{M}(\Omega)$ is $\pi \in \mathcal{M}(\Omega^{y,u})$.

Finally, define the nominal measure P^π as follows:

Let $\{y(t) = Nv(t); t \in [0,T]\}$, be an $\left(\mathcal{F}_t, \tilde{P}^\pi\right)$-Wiener process with covariance $NN't$, and introduce the $\left(\mathcal{F}_t, \tilde{P}^\pi\right)$-adapted martingale process

$$\Lambda^u(t) = \exp\Big\{ \int_0^t h'(x(s))(NN')^{-1} dy(s) - \frac{1}{2}\int_0^t h'(x(s))\big(NN'\big)^{-1} h(x(s))ds \Big\} \tag{4.38}$$

Define the nominal measure through the Radon-Nikodym derivative

$$\frac{dP^\pi(w,x,y,u)}{d\tilde{P}^\pi(w,x,y,u)}|_{\mathcal{F}_T} \triangleq \Lambda^u(T) \tag{4.39}$$

Then, Assumptions 4.5.1 imply that $P^\pi(\Omega) = E_{\tilde{P}^\pi}\left\{\Lambda^u(t)\right\} = 1, \forall t \in [0,T]$, and thus $P^\pi \in \mathcal{M}(\Omega)$. Moreover, by Girsanov's theorem, $Nv^\pi(t) \triangleq y(t) - \int_0^t h(x(s))ds$ is a Wiener process with covariance $NN't$ under $P^\pi \in \mathcal{M}(\Omega)$, and the distribution of $\{w(t), x(t); t \in [0,T]\}$ is invariant under the measure change of (4.39). Thus, under the measure $P^\pi \in \mathcal{M}(\Omega)$, $\{v^\pi(t); t \in [0,T]\}$ and $\{w(t); t \in [0,T]\}$ are independent Wiener processes. Hence, for each $\pi \in \mathcal{U}_{ad}$, there exists a unique nominal measure $P^\pi \in \mathcal{M}(\Omega)$ under which the state $\{x(t); t \in [0,T]\}$ and observation process $\{y(t); t \in [0,T]\}$ satisfy (4.31).

The following result is given in [69].

Lemma 4.5.3 *The set of admissible controls $\mathcal{U}_{ad}$ is compact under weak sequential convergence (with respect to the Prohorov metric).*

This follows from the compactness and convexity assumptions for $\mathcal{U}$ and the assumption 4.5.1. (see [69], Lemma 2.3, p.265)

Now we can state the problem precisely as follows:

Problem 4.5.4 *Suppose the nominal system given by (4.31) is perturbed by an unknown (measurable) process $\{\gamma, \delta\}$ taking values from a fixed set $\mathcal{D}$ in a function space (to be identified later). This gives rise to the uncertain system described by*

$$\left(\Omega, \mathcal{F}_t, Q^{\gamma,\delta}\right) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))\gamma(t)dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ dy(t) = h(x(t))dt + N\delta(t)dt + Ndv(t), \quad y(0) = 0, \end{cases} \tag{4.40}$$

where $Q^{\gamma,\delta}$ denotes the probability measure on Ω induced by the system (5.40) corresponding to a given realization $\{\gamma, \delta\} \in \mathcal{D}$ and a given control $\pi \in \mathcal{U}_{ad}$. For a given $\pi \in \mathcal{U}_{ad}$, let

$$M(\pi) \equiv \{Q^{\gamma,\delta} \in \mathcal{M}(\Omega) : \{\gamma, \delta\} \in \mathcal{D}\}$$

denote the family of probability measures induced by the uncertain system. Clearly the graph of this multi-measure $M(\pi)$ is given by

$$\Xi \equiv \{(\pi, Q) \in \mathcal{U}_{ad} \times \mathcal{M}(\Omega) : Q \in M(\pi)\}.$$

Define $J(\pi, Q) = H(Q|P)$ on Ξ. Then given the nominal measure $P \in \mathcal{M}(\Omega)$, find a pair $(\pi^*, Q^*) \in \Xi$ which solves the following constrained optimization problem.

$$J(\pi^*, Q^*) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{Q \in M(\pi)} J(\pi, Q) \tag{4.41}$$

$$\textit{subject to fidelity } E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \leq \alpha, \quad \alpha \in \Re \tag{4.42}$$

or

$$\textit{subject to fidelity } E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \geq \alpha, \quad \alpha \in \Re. \tag{4.43}$$

4.5.2 Relations to Disturbance Attenuation Problem

Here, the relation between problem 4.5.4 and the usual formulation of disturbance attenuation in robust control is established. The robust control problem is defined as follows [71]. Consider an uncertain partially observable system given by

$$\left(\Omega, \mathcal{F}_t, Q^{\gamma,\delta}\right) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))\gamma(t)dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ dy(t) = h(x(t))dt + N\delta(t)dt + Ndv(t), \quad y(0) = 0 \end{cases} \tag{4.44}$$

where $Q^{\gamma,\delta} \in M(\pi)$. Define $\mathcal{H} = L_2([0,T];\Re^m) \times L_2([0,T];\Re^d)$ and $B_r = \{(\gamma,\delta) \in \mathcal{H}; ||(\gamma,\delta)||_{\mathcal{H}} < r\}$ and in problem 4.5.4, let $\mathcal{D} = \mathcal{H}$. Define the pay-off functional $J : \mathcal{U}_{ad} \to \Re$ by [4]

$$J(\pi) = \sup_{(\gamma,\delta)\in B_r^c} \frac{E_Q\Big\{ \int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big\}}{\frac{1}{2}E_Q(||(\gamma,\delta)||_{\mathcal{H}}^2)} \tag{4.45}$$

where $r > 0$ is a fixed constant, B_r^c is the complement of the set B_r and $Q = Q^{\gamma,\delta}$. The optimal robust control problem is to find control law $\pi^* \in \mathcal{U}_{ad}$ such that

$$J(\pi^*) = \inf_{\pi\in\mathcal{U}_{ad}} J(\pi). \tag{4.46}$$

It is clear from the above description that the objective of the controller is to minimize the cost functional, while the uncertainty tries to maximize it. Since (4.46) is rather difficult to solve, an alternative sub-optimal control problem is introduced in which the cost $J(\pi)$ is required to be below a given level (larger than the infimum). In other words the objective is to find a control law $\pi \in \mathcal{U}_{ad}$ such that

$$J(\pi) \leq \frac{1}{s}, \qquad s > 0. \tag{4.47}$$

This is equivalent to minimizing over $\pi \in \mathcal{U}_{ad}$ the pay-off functional

$$\begin{aligned} J^s(\pi) = \sup_{(\gamma,\delta)\in B_r^c} \Big\{ sE_Q\Big[\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big] \\ -\frac{1}{2}E_Q\Big[\int_0^T \Big(||\gamma(t)||^2 + ||\delta(t)||^2\Big)dt\Big]\Big\} \end{aligned} \tag{4.48}$$

and ensuring the pay-off is non-positive. This problem can be reformulated in an unconstrained form as follows

$$\begin{aligned} J^s(\pi) = \sup_{(\gamma,\delta)\in\mathcal{H}} \Big\{ sE_Q\Big[\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big] \\ -\frac{1}{2}E_Q\Big[\int_0^T \Big(||\gamma(t)||^2 + ||\delta(t)||^2\Big)dt\Big] + \beta\,\Big(E_Q\Big[\int_0^T \Big(||\gamma(t)||^2 + ||\delta(t)||^2\Big)dt\Big] - r\Big)\Big\} \end{aligned} \tag{4.49}$$

where in the non-trivial case of finite $J^s(\pi)$, $\beta \in (0, \frac{1}{2})$. Therefore, the actual robust control problem is to find a $\pi^* \in \mathcal{U}_{ad}$ so that

$$J^s(\pi^*) = \inf_{\pi\in\mathcal{U}_{ad}} J^s(\pi). \tag{4.50}$$

[4] It is assumed that there is always an ambient disturbance present in the system, with energy equal or greater than r.

By use of Girsanov's theorem, in particular the density

$$\Upsilon \equiv \exp\Big(\int_0^T \gamma'(s)dw(s) - \frac{1}{2}\int_0^T ||\gamma(s)||^2 ds\Big) \times \exp\Big(\int_0^T \delta'(s)dv(s) - \frac{1}{2}\int_0^T ||\delta(s)||^2 ds\Big),$$

giving $dQ = \Upsilon dP$, one can verify (see [75]) that the entropy of Q relative to P is given by

$$H(Q|P) = \frac{1}{2}E_Q\Big\{\int_0^T \Big(||\gamma(t)||^2 + ||\delta(t)||^2\Big)dt\Big\}. \tag{4.51}$$

Hence, (4.50) is equivalent to the unconstrained form of problem (4.41) multiplied by a fixed constant, i.e.,

$$\begin{aligned}&\sup_{u\in\mathcal{U}_{ad}} \inf_{Q\in M(\pi)} \Big\{H(Q|P) - s_0 E_Q\Big(\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big)\Big\}\\ &= -\frac{1}{\frac{1}{2}-\beta} \inf_{u\in\mathcal{U}_{ad}} \sup_{Q\in M(\pi)} \Big\{sE_Q\Big(\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big) - (\frac{1}{2}-\beta)H(Q|P)\Big\}\end{aligned}$$

where $s_0 = \frac{s}{\frac{1}{2}-\beta}$. In view of (4.49), this is equivalent to (4.50).

4.5.3 Duality of Uncertain Systems with Wide Sense Controls

Following similar procedure as in Section 4.4, Problem 4.5.4 can be reformulated using the dual functional as follows:

For every $s \in \Re$ define the Lagrangian

$$J^{s,\alpha}(\pi, Q) \triangleq H(Q|P) - s\Big(E_Q\Big\{\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big\} - \alpha\Big) \tag{4.52}$$

and its associated dual functional

$$J^{s,\alpha}(\pi, Q^*) = \inf_{Q\in M(\pi)} J^{s,\alpha}(\pi, Q) \tag{4.53}$$

In addition define the quantity

$$\Psi_\pi^*(\alpha) \triangleq \sup_{s\in\Re} J^{s,\alpha}(\pi, Q^*) \tag{4.54}$$

which may or may not exist.

The statements of the next Corollary follow directly from Lemma 4.4.2 and Theorem 4.4.3.

Corollary 4.5.5 *For an arbitrary but fixed $\pi \in \mathcal{U}_{ad}$, the statements of Lemma 4.4.2 and Theorem 4.4.3 hold for the functionals (4.53) and (4.54).*

Proof. Identifying,

$$\begin{aligned}&\Sigma \mapsto \Omega;\quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u});\quad \mu \mapsto P \in M(\pi);\quad \nu \mapsto Q \in M(\pi)\\ &\ell^u \mapsto \int_0^T \lambda(x(t),u(t))dt + \kappa(x(T)),\end{aligned}$$

the proof is identical to that of Lemma 4.4.2 and Theorem 4.4.3.

Remark 4.5.6 *Notice that in Corollary 4.5.5, the minimizing measure Q^* is defined on $\left(\Omega, \mathcal{F}_T\right)$. In partially observable problems it is crucial to reformulate them as completely observable problems, using separated laws, in which the minimizing and maximizing players are functionals of conditional distributions, rather than a priori distributions. For the specific problem under investigation, the minimizing measure Q^* should be restricted on $\left(\Omega^{y,u}, \mathcal{F}_T^{y,u}\right)$, because this is the only information available to both controller and minimizing measure. Towards this end, substituting $dP = \Lambda^u(T)d\tilde{P}$, in (4.38), and then restricting $\frac{dQ^*}{d\tilde{P}}$ onto $\mathcal{F}_T^{y,u}$ gives*

$$dQ^*|_{\mathcal{F}_T^{y,u}} = \frac{E_{\tilde{P}^{y,u}}(e^{s\ell^u}\Lambda^u(T))d\pi}{\int_{\Omega^{y,u}} E_{\tilde{P}^{y,u}}(e^{s\ell^u}\Lambda^u(T))d\pi} \tag{4.55}$$

This approach is considered in the next section.

4.5.4 Duality of Separated Uncertain Systems with Wide Sense Controls

In this section, separated strategies are introduced by describing the nominal systems using conditional distributions rather than a priori distributions. In the theory of partially observable systems such strategies are called separated strategies. They are important in answering questions on existence of optimal policies and in reformulating partially observable problems as fully observable ones in which the role of a state variable is played by conditional distributions.

For each $\left\{u(\cdot,\omega), y(\cdot,\omega)\right\}$, introduce the functional

$$\chi^u(t) \triangleq \exp\left(s\int_0^t \lambda(x(\tau), u(\tau))d\tau\right) \tag{4.56}$$

and define the measure-valued process $\{M_t^{y,u}\}_{t\geq 0}$ by

$$M_t^{y,u}(\phi) \triangleq E_{\tilde{P}^{y,u}}\left\{\phi(x(t))\chi^u(t)\Lambda^u(t)\right\}, \quad \phi \in BC(\Re^n) \tag{4.57}$$

where $\Lambda^u(t)$ is defined in (4.38) and (4.39). Denote the kernel associated with $M_t^{y,u}$ by $(y,u) \mapsto M_t(dx|y,u) \triangleq M_t^{y,u}(dx)$.

Then

$$M_t^{y,u}(\phi) = \int_{\Re^n} \phi(z)M_t^{y,u}(dz), \qquad \phi \in BC(\Re^n) \tag{4.58}$$

and moreover, the measure-valued process $\{M_t^{y,u}\}_{t\geq 0}$ satisfies the following stochastic partial differential equation written in the weak form

$$M_t^{y,u}(\phi) = \Pi_0(\phi) + \int_0^t M_\tau^{y,u}(L(u))\phi)d\tau + s\int_0^t M_\tau^{y,u}(\lambda^u\phi)d\tau$$

$$+\int_0^t M_\tau^{y,u}(h'(NN')^{-1}\phi)dy(\tau) \tag{4.59}$$

where $L(u)$ is the Backward Kolmogorov operator associated with the state process of the nominal system given by

$$L(u) = \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma')_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n f_i(x,u(t))\frac{\partial}{\partial x_i}$$

corresponding to any given control policy u. Introduce the normalized probability measure

$$d\mu_T^\pi(x,y,u) \triangleq \frac{d\mu_{T,un}^\pi(x,y,u)}{\int_{\Re^n \times \Omega^{y,u}} d\mu_{T,un}^\pi(x,y,u)} = \frac{d\mu_{T,un}^\pi(x,y,u)}{\mu_{T,un}^\pi(\Re^n \times \Omega^{y,u})} \tag{4.60}$$

where

$$d\mu_{T,un}^\pi(x,y,u) = dM_T^{y,u}(x)d\pi(y,u)$$

and $\pi \in \mathcal{M}(\Omega^{y,u})$ is a probability measure.

Remark 4.5.7 *For an arbitrary but fixed $\pi \in \mathcal{U}_{ad}$, the dual functional, $J^{s,\alpha}(\pi, Q^*)$, given in (4.53), can be expressed in terms of the measures $M_T^{y,u}$ and π as follows.*

$$J^{s,\alpha}(\pi,Q^*) = s\alpha - \log\int_{\Omega^{y,u}} M_T^{y,u}(e^{s\kappa})d\pi(y,u) = s\alpha - \Psi_P(s)$$

where κ is as defined in Assumption 4.5.1.

Corollary 4.5.8 *The dual functional $J^{s,\alpha}(\pi,Q^*)$ can be expressed as an optimization involving measures on $\Re^n$ as follows.*

$$\begin{aligned}
J^{s,\alpha}(\pi,Q^*) &= s\alpha - \log\Big(\int_{\Re^n\times\Omega^{y,u}} e^{s\kappa(z)}d\mu_{T,un}^\pi(z,y,u)\Big) \\
&= s\alpha - \log\Big(\int_{\Re^n\times\Omega^{y,u}} e^{s\kappa(z)}d\mu_T^\pi(z,y,u)\Big) - \log\Big(\mu_{T,un}^\pi(\Re^n\times\Omega^{y,u})\Big) \\
&= \inf_{\nu\in\mathcal{M}(\Re^n\times\Omega^{y,u})}\Big\{H(\nu|\mu_T^\pi) - s\Big(\int_{\Re^n\times\Omega^{y,u}}\kappa(z)d\nu(z,y,u) - \alpha\Big)\Big\} - \log\Big(\mu_{T,un}^\pi(\Re^n\times\Omega^{y,u})\Big) \\
&= J^{s,\alpha}(\pi,\nu^{\pi,*})
\end{aligned}$$

where the infimum is attained by $\nu^{\pi,*} \in \mathcal{M}(\Re^n \times \Omega^{y,u})$ given by

$$d\nu^{\pi,*}(x,y,u) = \frac{e^{s\kappa(x)}d\mu_T^\pi(x,y,u)}{\int_{\Re^n\times\Omega^{y,u}} e^{s\kappa(z)}d\mu_T^\pi(z,y,u)} \tag{4.61}$$

Here the parameter s is the same as in equation (4.59).

Proof. Identifying

$$\Sigma \mapsto \Re^n \times \Omega^{y,u}; \quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u}); \quad \mu \mapsto \mu^{\pi} \in \mathcal{M}(\Re^n \times \Omega^{y,u});$$
$$\ell^u \mapsto \kappa; \quad \lambda = 0.$$

the proof is identical to that of Lemma 4.4.2.

Existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ was proved in [69] [Theorem 4.1,p269; Theorem 7.2, p279].

Theorem 4.5.9 *For any admissible s, (a):* $\pi \longrightarrow J^{s,\alpha}(\pi, \nu^{\pi,*})$ *is weakly upper-semi continuous on* $\mathcal{U}_{ad}$, *(b): There exists a* $\pi^* \in \mathcal{U}_{ad}$ *such that* $J^{s,\alpha}(\pi^*, \nu^{\pi^*,*}) \geq J^{s,\alpha}(\pi, \nu^{\pi,*}), \forall \pi \in \mathcal{U}_{ad}$.

Proof. The derivation is similar to that given in [69].

Remark 4.5.10 *There is a smaller class of control laws (strict sense controls) denoted by* $\mathcal{U}^s_{ad}$ *which is the subset of those* $\pi \in \mathcal{U}_{ad}$ *in which the measure* π *can be written as*

$$\pi(dy, du) = \delta_{\underline{u}(y)}(du) \times W_y(dy) \tag{4.62}$$

for some mapping $\underline{u} : \Omega^y \rightarrow \Omega^u$, *which is* $\left(\mathcal{F}^y_t, \mathcal{F}^u_t\right)$*-measurable, where* $\delta_{\underline{u}(y)}(\cdot)$ *denotes the Dirac measure concentrated at* $\underline{u}(y)$. *It is very difficult to prove existence of optimal control policies from this class, although from the practical point of view these are the most desirable ones. Nevertheless, the expressions derived for* $\pi \in \mathcal{U}_{ad}$ *apply to the class of strict-sense controls* $\mathcal{U}^s_{ad}$ *provided the following integration mapping is introduced* $\int_{\Omega^{y,u}} \left(\cdot \right) d\pi(y,u) \mapsto \int_{\Omega^y} \left(\cdot \right)|_{u(t)=\underline{u}(y)} dW_y(dy)$.

4.5.5 Evolution of the Density of the Minimizing Measure

Some additional regularities on σ, Π_0 would imply that the measure valued process $\{M^{y,u}_t, t \geq 0\}$ has a density $\{p^{y,u}(t,\cdot) \equiv e^{y \cdot h(\cdot)} q^{y,u}(t,\cdot), t \geq 0\}$. The following conditions are sufficient to guarantee this.

7) $n = m, a(x) \triangleq \sigma(x)\sigma'(x) \geq \alpha I_n, \ \alpha > 0, \ \forall \ x \in \Re^n \ \ \frac{\partial}{\partial x_j} a_{i,j} \in L^{\infty}(\Re^n), \forall i,j.$

8) Π_0 has a density $p_0(x)$ and $p_0 \in L_2(\Re^n)$. Then,

$$M^{y,u}_t(\phi) = \int_{\Re^n} \phi(z) e^{y'(t)h(z)} q^{y,u}(t,z)dz, \quad \phi \in BC(\Re^n) \tag{4.63}$$

Moreover, $q^{y,u}(\cdot, z)$ is the solution of the following partial differential equation

$$\frac{\partial}{\partial t} q^{y,u}(t,x) = A^*(y(t)) q^{y,u}(t,x) + e(x, y(t), u(t)) q^{y,u}(t,x)$$
$$+ s\lambda(x, u(t)) q^{y,u}(t,x), \quad (t,x) \in (0,T] \times \Re^n \tag{4.64}$$
$$q^{y,u}(0,x) = p_0(x), \quad x \in \Re^n \tag{4.65}$$

where $A^*(y)$ is the adjoint operator of $A(y)$ given by

$$A(y) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,y,u)\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}(ay\cdot\nabla h)_i\frac{\partial}{\partial x_i}$$

$$\begin{aligned} e(x,y,u) &= \frac{1}{2}(ay\cdot\nabla h, y\cdot\nabla h) - \hat{f}\cdot(y\cdot\nabla h) - \|h\|^2_{\Re^d} \\ \hat{f}_i &\equiv f_i - (1/2)\sum_{j\geq 1}\frac{\partial}{\partial x_j}a_{i,j} \end{aligned}$$

in which $\cdot$, $(,)$ are the dot products in $\Re^d$, $\Re^n$, respectively. Using the density of the measure valued process the pay-off functional can be expressed as follows:

$$J^{s,\alpha}(\pi, Q^*) = s\alpha - \log\int_{\Omega^{y,u}}\Big\{\int_{\Re^n} e^{s\kappa(z)}e^{y'(T)h(z)}q^{y,u}(T,z)dz\Big\}\times d\pi(y,u). \quad (4.66)$$

Existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ follows from Theorem 4.5.9.

4.5.6 Partially Observable Uncertain LQ Problem

For the purpose of illustrating the concepts presented earlier, we consider the following partially observed linear LQG problem.

Assumptions 4.5.11 *The system parameters $\{f,\sigma,h\}$ of (4.31), the density of the initial state x_0, and the cost integrands $\{\lambda,\kappa\}$ are given by*

$$f(x,u) = Fx + Bu, \quad \sigma(t,x) = G, \quad h(x) = Hx, \quad 2\lambda(x,u) = x'Qx + u'Ru, \quad 2\kappa(x) = x'Mx$$

$$p_0(x) = \frac{\exp(-\frac{1}{2}\|P_0^{-\frac{1}{2}}(x-\xi)\|^2)}{(2\pi)^{\frac{n}{2}}|P_0|^{\frac{1}{2}}}, \quad P_0 = P_0' > 0.$$

Under Assumptions 4.5.11, it can be shown that $M_t^{y,u}$ has a density $m^{y,u}(t,x)$ given by

$$dM^{y,u}(t,x) = m^{y,u}(t,x)dx = \nu^u_{0,t}\times\frac{\exp\left(-\frac{1}{2}\|P(t)^{-\frac{1}{2}}(x-r(t))\|^2\right)}{(2\pi)^{\frac{n}{2}}|P(t)|^{\frac{1}{2}}}\times\exp\frac{s}{2}\left(\mathcal{C}^u_{0,t}+\mathcal{I}_{0,t}\right)dx,$$

where

$$\nu^u_{0,t} = \exp\left\{\textstyle\int_0^t (Hr(s))'\left(NN'\right)^{-1}dy - \frac{1}{2}\int_0^t\|N^{-1}Hr(s)\|^2 ds\right\},$$

$$\mathcal{C}^u_{0,t} \triangleq \textstyle\int_0^t\{r'Qr + u'(s)Ru(s)\}ds, \quad \mathcal{I}_{0,t}\triangleq\int_0^t Tr(PQ)ds$$

and $P:[0,T]\to\mathcal{L}(\Re^n,\Re^n)$, $P = P' \geq 0$, $r:[0,T]\times\Omega\to\Re^n$, are given by

$$\left.\begin{aligned} &\dot{P} = FP + PF' + sPQP + GG' - PH'\big(NN'\big)^{-1}HP, \quad P(0) = P_0, \\ &dr = (F + sPQ)\,rdt + Budt + PH'\big(NN'\big)^{-1}\big(dy - Hrdt\big), \\ &r(0) = \xi, \quad y(\cdot) \text{ is an } \{\mathcal{F}^{y,u}_t; t\in T\}\text{ -adapted Wiener process} \\ &\qquad\qquad \text{with correlation } NN'. \end{aligned}\right\} \quad (4.67)$$

Denote by $\tilde{\rho}(AB)$ the spectral radius of AB (A, B are matrix-valued functions), and define $\tilde{s}^* \triangleq \sup\{s : P(t) \geq 0, \ \ , \tilde{\rho}(P(t)M) < \frac{1}{s} \ \forall t \in [0,T]\}$, where P is given by the solution of (4.67). Then

$$\int_{\Re^n} e^{\frac{s}{2}x'Mx} dM_T^{y,u}(x) = \frac{1}{|I - sP(T)M|^{\frac{1}{2}}} \times \exp\frac{s}{2}\Big\{r'(T)(I - sP(T)M)^{-1}Mr(T)\Big\}$$
$$\times \exp\frac{s}{2}\left(\mathcal{C}_{0,T}^u + \mathcal{I}_{0,T}\right) \times \nu_{0,T}^u, \ s \in [0, \tilde{s}^*]. \tag{4.68}$$

Therefore, by Corollary 4.5.8, the resulting pay-off which should be minimized with respect to $\pi \in \mathcal{U}_{ad}$ is given by

$$J^{s,\alpha}(\pi, \nu^{\pi,*}) = s\alpha + \frac{1}{2}\log|I - sP(T)M| - \log\int_{\Omega^{y,u}} \Big\{\exp\frac{s}{2}\Big\{r'(T)(I - sP(T)M)^{-1}Mr(T)\Big\}$$
$$\times \exp\frac{s}{2}\left(\mathcal{C}_{0,T}^u + \mathcal{I}_{0,T}\right)\Big\} \times d\pi(y,u), \quad s \in [0, \tilde{s}^*] \tag{4.69}$$

If we further restrict the class of control laws to strict-sense then the optimal control law can be found, and it is given in [56, 62, 64].

4.6 Conclusion

This chapter is concerned with a new class of stochastic control problems, in which the pay-off is described by the relative entropy between the nominal and the uncertain measures while the uncertain measures satisfy certain energy inequality constraints. The controller seeks to maximize the pay-off while the uncertainty seeks to minimize it. Throughout, it is shown that this problem is equivalent to maximin games in which the strategies observe the conditional distribution associated with the nominal model. Moreover, this problem incorporates both pessimistic and optimistic strategies, depending on the constraint chosen. Certain monotonicity properties of the optimal solution are discussed, while relations to the well-known Legendre-Fenchel transform are introduced. In addition, connections to minimax games of partially observable stochastic systems and to risk-sensitive control problems are investigated.

Chapter 5

Optimization of Stochastic Uncertain Systems Subject to Relative Entropy Constraints

5.1 Introduction

This chapter is concerned with non-parametric robust control techniques applied to uncertain dynamical systems. The uncertainty description of these systems is characterized by the class of measures, which satisfy a relative entropy constraint, with respect to a nominal measure. These measures are quite general; they may correspond to random processes, which are solutions of linear or nonlinear stochastic dynamical systems.

The problem of robust control is formulated by minimizing the worst case (maximum) cost over the set of admissible controls. The cost is considered to be a linear functional of the uncertain measure and the worst case is identified by maximum cost over the relative entropy constraint set. In the context of robustness, this approach leads to minimax techniques, in which the worst case estimate of the uncertain measure subject to the uncertainty description is sought.

At the abstract level, a general framework is put forward in which the basic ideas are explained, and the fundamental results are derived. At this level, systems are represented by measures on measurable spaces, energy signals by linear functionals on the space of measures, and uncertainty by sets described by bounded relative entropy between the true measure and the nominal measure.

At the application level, the theory is applied to a class of uncertain nonlinear partially observable stochastic control systems described by stochastic differential equations, and connections with optimal disturbance attenuation and dissipation of dynamical systems are sought.

Optimal sensitivity, H^∞ spaces and induced norms were introduced by Zames [45], in 1981, to address robustness of deterministic control problems, which are subject to uncertainties. This formulation leads to the minimization of an induced norm, using a minimax game formulation. Since the publication of Zames seminal paper [45], several approaches have been proposed to extend robustness of linear control systems, with respect to unknown disturbances and unmodeled dynamics, to nonlinear deterministic and stochastic systems. Three pay-off functionals, which received significant attention are deterministic minimax games, risk-sensitivity pay-offs, and stochastic minimax games, because of their relations to attenuating disturbances to error signals. Earlier work is found in [49, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65]. The attenuation of disturbances is also described in the context of dissipation inequalities [46], which imply that the output power is less than or equal the input power. In the context of an induced norm, this is equivalent to ensuring that the controller should be designed in order to ensure that the induced norm is always less than or equal to one. A system, which enjoys such a property is called dissipative. A review of these techniques and their connections is found in [46, 47].

The deterministic formulation of minimax games is based on the assumption that the noises have finite energy, while the dissipation inequality is established through the value function of the dynamic games [47, 48]. The stochastic analog of the deterministic minimax games is based on the assumption that the noises consist of color (or white noise) and finite energy disturbances. For fully observed dynamic games (both deterministic and stochastic), analysis and synthesis questions are discussed, in many places, for example, in [60, 77], using the Isaacs equation. Unfortunately, this is not the case for stochastic partially observed nonlinear minimax games. In fact, very little work has been done in formulating and analyzing such classes of stochastic minimax problems from the control theoretic point of view. This is due to the difficulty in determining the worst case uncertainty, showing existence of minimax solutions, with respect to the class of control laws and disturbances, which are described through output feedback control laws.

However, the problem of computing or characterizing the optimal sensitivity reduction for general nonlinear fully observed or partially observed (output feedback) problems has not yet been addressed at a satisfactory level of generality. For linear problems the induce norm is often expressed in terms of the solution to certain Ricatti equations. Unfortunately, for nonlinear problems such Ricatti equations are not available and thus techniques, which can be used to grade the performance of sub-optimal minimax solutions might be of interest.

One of the main contributions of this chapter is to characterize the optimal sensitivity reduction or disturbance attenuation level for stochastic uncertain systems, and to provide an explicit equation for the worst case measure, including an evolution equation for this measure, for general classes of stochastic systems. In addition, to derive several properties of the dual functional which are important in comparing the sub-optimal disturbance attenuation problems to the optimal one.

The abstract formulation of the problem is described as follows.

A. Abstract Formulation

Let (Σ, d) denote a complete separable metric space (Polish Space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space in which $\mathcal{B}(\Sigma)$ are identified as the Borel sets generated by open sets in Σ. Let $\mathcal{M}(\Sigma)$ denote the set of probability measures on $(\Sigma, \mathcal{B}(\Sigma))$, $\mathcal{U}_{ad}$ the set of admissible controls. For a given $u \in \mathcal{U}_{ad}$, let $M(u) \subset \mathcal{M}(\Sigma)$ denote the set of probability measures induced by the uncertain system, while control u is applied. In the absence of uncertainty, the probability measure induced by the system is denoted by $\{\mu^u\}$ (nominal measure), and the set $M(u)$ reduces to the singleton $\{\mu^u\}$. Let $\ell^u : \Sigma \rightarrow \Re$ a real-valued measurable function bounded from below; and $H(\nu|\mu)$ denote the relative entropy or Kullback-Leibler distance between measures ν and μ both from $M(u)$. Throughout this chapter, " $<<$" will denote absolute continuity of one measure with respect to another. Let $\mu \in M(u)$ denote the measure induced by the nominal system, and $\nu \in M(u)$ the measure induced by the uncertain system. Define the average cost by the integral of ℓ^u with respect to the measure ν. Suppose that the choice of uncertainty measures is restricted by a relative entropy constraint between the true and nominal measures. In the worst case scenario, uncertainty would seek to maximize the cost, and the controller naturally tries to minimize the average cost. This leads to the following Minimax problem

$$\inf_{u \in \mathcal{U}_{ad}} \sup_{\nu \in M_R(u)} \int_\Sigma \ell^u d\nu$$

Where $R \in (0, \infty)$ is fixed and the constraint set is defined as follows

$$M_R(u) \equiv \{\nu \in M(u) : H(\nu|\mu) \leq R\}$$

The Primal Problem.

(PP) For a fixed control law $u \in \mathcal{U}_{ad}$, and a fixed nominal measure $\mu \in M(u)$, define the functional

$$J(u) \equiv \sup_{\nu \in M_R(u)} \int_\Sigma \ell^u d\nu \tag{5.1}$$

The objective is to find a control policy that minimizes the above average cost over the set $\mathcal{U}_{ad}$. Towards this end, we prove the existence of a measure $\nu^* \in M_R(u)$ at which the supremum is attained.

The Primal Minimax Problem.
(PMP) The robust control problem is defined as follows. For a given $\mu \in M(u)$, find a control law $u^* \in \mathcal{U}_{ad}$ such that

$$J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u) \tag{5.2}$$

The solution to this optimization problem returns the worst case measure among those, which satisfy the constraint, as a function of the nominal measure.

The constraint $M_R(u)$ describes the set of all admissible measures induced by the uncertain stochastic system, relative to the nominal measure. The pay-off $E_\nu(\ell^u)$ represents average energy with respect to the unknown measure ν, such as integral quadratic constraints, tracking errors, etc. The parameter R controls the size of uncertainty set associated with the family of stochastic uncertain systems with respect to the nominal stochastic system.

The rest of this chapter is organized as follows. In Section 5.2 existence of the maximizing measures associated with (PP) and (PMP) is proved under the topology of weak* convergence. In Section 5.3, the equivalence between the constrained and unconstrained problems is derived and basic properties of the problem are discussed. In Section 5.4, the abstract formulations in (PP) and (PMP) problems are applied to nonlinear continuous time stochastic partially observable control systems. In subsection 5.4.6, an explicit expression is presented for the maximizing measure when the inequality constraints have a quadratic form, and the nominal measure is Gaussian.

5.2 Existence of Maximizing Measure

The basic definitions and duality relations between relative entropy, free energy, and cumulant moment generating functions were introduced in section 4.2 of chapter 4. Here we show the existence of the maximizing measure associated with the problem (5.1). Without loss of generality, the explicit dependence of measures and functions on u is suppressed.
Let Σ be a complete separable metric space with metric d and sigma field $\mathcal{B}(\Sigma)$. Let $BC(\Sigma)$ denote the space of bounded continuous real valued functions defined on Σ. Furnished with the sup norm topology, this is a Banach space. It is known that $\left(BC(\Sigma)\right)^*$ is isometrically iso-

morphic to the Banach space of finitely additive regular bounded measures on $\mathcal{B}(\Sigma)$ [37]. Let $\mathcal{M}(\Sigma) \subset \left(BC(\Sigma)\right)^*$ be the space of regular finitely additive probability measures on $\mathcal{B}(\Sigma)$. Let $\{\nu_n\}, n \geq 1$ be a sequence in $\mathcal{M}(\Sigma)$. Then by definition $\nu_n \stackrel{w^*}{\to} \nu$ if $\int_\Sigma f d\nu_n \to \int_\Sigma f d\nu$ for every $f \in BC(\Sigma)$.

Problem 5.2.1 ***(Basic Extremal Problem)*** *Let $\ell \in BC(\Sigma)$, $\mu \in \mathcal{M}(\Sigma)$ a fixed nominal measure, and $R \in (0, \infty)$. Define the functional $J(\nu) \triangleq \int_\Sigma \ell d\mu$, and the (constraint) set $M_R = \{\nu \in \mathcal{M}(\Sigma);\ \ H(\nu|\mu) \leq R\}$. Then find $\nu^* \in M_R$ so that*

$$J(\nu^*) = \sup_{\nu \in M_R} J(\nu) \tag{5.3}$$

The next theorem establishes existence of a maximizing measure under the topology of weak* convergence.

Theorem 5.2.2 *Let (Σ, d) be any complete separable metric space, $\mathcal{M}(\Sigma)$ the space of probability measures defined on Σ.*
1) For each $R \in (0, \infty)$, the set M_R is compact in the weak topology.*
2) For problem of 5.2.1, there exists a maximizing measure $\nu^* \in \mathcal{M}(\Sigma)$. Moreover, the same result holds if ℓ is continuous and bounded from below.

Proof. 1) Weak* sequential compactness of M_R is shown in [41].
2) If $\ell \in BC(\Sigma)$, then the functional $\int_\Sigma \ell d\nu$ is continuous in the weak* sense and existence follows from this and weak* compactness of the set M_R (by Weirstrass theorem). Next we consider the case of ℓ continuous, bounded from below. In part 2 of the proof of theorem 4.3.2 in chapter 4, weak* continuity of the functional $\int_\Sigma \ell d\nu$ was proved. Hence existence of maximizing measures follows again from Weirstrass theorem. $\bullet$

5.3 Equivalence of the Constrained and Unconstrained Problems

This section is concerned with reformulating the constrained optimization problem 5.2.1, as an unconstrained one using the theory of Lagrange functionals, and then showing the equivalence of the two problems. Subsequently, many properties of the maximizing measure and the dual functional are derived. Connections to the optimal sensitivity reduction associated with the standard H^∞ disturbance attenuation problem are established, and lower and upper bounds on the optimal solution are derived. In addition, certain monotonicity properties of the relative entropy with respect to the Lagrange multiplier are derived. The constrained problem 5.2.1 can be converted into an unconstrained one by introducing the Lagrangian functional and the dual functional. In Theorem 5.3.1 the equivalence of this dual functional with problem 5.2.1 is established. For each $R \in (0, \infty)$ and every $s \in \Re$, define the Lagrangian

$$J^{s,R}(u, \nu) \triangleq E_\nu(\ell^u) - s\Big(H(\nu|\mu) - R\Big) \tag{5.4}$$

Define the unconstrained problem

$$J_u^R(u, \nu^*) = \inf_{s \geq 0} \sup_{\nu \in M(u)} J^{s,R}(u, \nu) \tag{5.5}$$

and the constrained one as follows

$$J^R(u, \nu^*) = \sup_{\nu \in M_R(u)} E_\nu(\ell^u) \tag{5.6}$$

This is the same as problem 5.2.1. The next theorem establishes the equivalence of constrained and unconstrained optimization problems.

Theorem 5.3.1 *Consider problem 5.2.1 with a given $u \in \mathcal{U}_{ad}$. Suppose it is finite. Then we have*

$$J^R(u, \nu^*) = J_u^R(u, \nu^*) \tag{5.7}$$

Moreover, the same conclusions hold if ℓ^u is continuous and bounded below.

Proof. We shall employ the Lagrange Duality theorem in [18](pp.224-225). It is obvious that $\nu \longrightarrow \int_\Sigma \ell^u d\nu$ is a convex functional. Let

$$\begin{aligned} X &= \Big\{\text{finitely additive signed measures } \nu : \mathcal{B}(\Sigma) \rightarrow \Re, \nu(\Sigma) < \infty\Big\} \\ \Omega &= \Big\{\nu \in X; \nu \text{ is a finitely additive probability measure }\Big\} \end{aligned}$$

The theorem in [18] deals with minimization of a convex functional. Multiplying both sides of equation (4) in Theorem (1) in [18] (page 224), in a minus sign, converts the problem to maximization of a concave functional over the set Ω. This can be applied to maximization of $E_\nu(\ell)$ over the constrained set, since $\int_\Sigma \ell^u d\nu$ is a linear functional of ν (hence concave). It can easily be shown that X forms a vector space over the ground field $\Re$. Also, it is obvious that Ω is a convex set. Define the mapping

$$\nu \longrightarrow G(\nu) = H(\nu|\mu) - R$$

This is a convex map from X into the ordered vector space $(\Re, \prec)$ with natural ordering. If we take $\nu = \mu$, then $G(\nu) = -R < 0$. Hence there exists a measure $\nu \in X$ such that $G(\nu) < 0$. Also by assumption the supremum is finite, hence the equivalence of constrained and unconstrained problems follows from the Lagrange Duality theorem in [18](pp.224-225). Since the existence of solution also holds for ℓ^u continuous and bounded from below, equivalence of the constrained and unconstrained problems also hold for this class of functions. •

Lemma 5.3.2 presents several properties of the unconstrained problem.

Lemma 5.3.2 *For a given $u \in \mathcal{U}_{ad}$, assume $\ell^u : \Sigma \to \Re$ is a measurable function which is bounded from below. Then the following statements hold.*

(1) $\sup_{\nu \in M(u)} J^{s,R}(u, \nu) = s\Psi_\mu(\frac{1}{s}) + sR, \quad R \in (0, \infty), s \geq 0$.

(2) Letting $\nu^{*,s}$ denote the maximizer in (1), the functional $J^{s,R}(u, \nu^{*,s})$ is convex in $s > 0$.

(3) The function $\Gamma_\mu(s) \triangleq s\Psi_\mu(\frac{1}{s})$ is a non-increasing function of $s \in (0, \infty)$.

(4) Assume for some $\eta > 0$, $\ell^u e^{\eta \ell^u} \in L_1(\mu)$. Define the infimum of the dual functional $\Phi^*_\mu(R) \triangleq \inf_{s>0} J^{s,R}(u, \nu^*)$. Then

$$\Phi^*_\mu(0) = \lim_{s \to \infty} s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu = E_\mu(\ell^u) \tag{5.8}$$

(5) Under the assumptions of (4), the supremum of the dual functional $J^{s,R}(u, \nu^{*,s})$ over $s > 0$ is bounded above and from below as follows.

$$E_\mu\Big(\ell^u\Big) \leq \Phi^*_\mu(R) \leq R + \log E_\mu\Big(e^{\ell^u}\Big) \tag{5.9}$$

Moreover if ℓ^u is bounded, then

$$E_\mu\Big(\ell^u\Big) \leq \Phi^*_\mu(R) \leq \min\Big\{R + \log E_\mu(\ell^u), ||\ell^u||_\infty\Big\}$$

(6) If there exists an open connected set $I \subset \Re$ such that $\ell^u e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$, then the infimum of the functional $J^{s,R}(u, \nu^{*,s})$ over $s > 0$ is uniquely attained at

$$H(\nu^{*,s}|\mu)|_{s=s^*} = R \tag{5.10}$$

where

$$d\nu^{*,s} = \frac{e^{\frac{\ell^u}{s}} d\mu}{\int_\Sigma e^{\frac{\ell^u}{s}} d\mu} \tag{5.11}$$

(7) If there exists an open connected set $I \subset \Re$ such that $\ell^u e^{\eta \ell^u} \in L_1(\mu)$ and $(\ell^u)^2 e^{\eta \ell^u} \in L_1(\mu)$ for all $\eta \in I$, then the relative entropy $H(\nu^{*,s}|\mu)$ is a non-increasing function of $s > 0$, that is,

$$0 \leq H(\nu^{*,s}|\mu^u)|_{s=s_2} \leq H(\nu^{*,s}|\mu)|_{s=s_1} \leq H(\nu^{*,s}|\mu)|_{s=s^*} = R, \;\; 0 < s^* \leq s_1 \leq s_2 \tag{5.12}$$

Proof. See Appendix E.

5.4 Partially Observed Uncertain Systems With Wide Sense Controls

In this section, the abstract formulation of Problem 5.2.1 and the results derived in Lemma 5.3.2 are employed to address partially observable stochastic systems. However, before these results can be applied, a specific transformation is introduced to convert the partially observable problem into a problem of complete information, in which the nominal and uncertain measures are described by a conditional distribution.

5.4.1 Problem Formulation

Let $\{x(t)\}_{t\geq 0}$ denote the state process which is subject to control, $\{y(t)\}_{t\geq 0}$ the observation process, and $\{u(t)\}_{t\geq 0}$ the control process, all defined for a fixed and finite time $[0,T]$.
For each $u \in \mathcal{U}_{ad}$ (set of admissible controls to be defined shortly) the nominal state and observation process, giving rise to a nominal probability measure P, are governed by the following system of stochastic differential equations.

$$\Big(\Sigma, \mathcal{B}(\Sigma), P\Big) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ \\ dy(t) = h(x(t))dt + Ndv(t), \quad y(0) = 0 \end{cases} \tag{5.13}$$

Here $x(t) \in \Re^n, y(t) \in \Re^d, u(t) \in \mathcal{U} \subset \Re^k$, $\{w(t)\}_{t\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are independent Brownian motions taking values in $\Re^m, \Re^d$, respectively, which are also independent of the initial state $x(0) = \xi$. Given the nominal measure P, find a $u^* \in \mathcal{U}_{ad}$ and a probability measure Q^{u^*} which solve the following constrained optimization problem.

$$\begin{cases} J(u^*, Q^{u^*}) = \inf_{u\in\mathcal{U}_{ad}} \sup_{Q\in M(u)} E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\} \\ \\ \text{subject to fidelity } H(Q|P) \leq R, \quad R \in (0,\infty) \end{cases} \tag{5.14}$$

The basic assumption about the controls, functions like f, h, λ and κ, initial state ξ, N are the same as those in Assumption 4.5.1 in Chapter 4. Moreover the required topological spaces and σ-algebras are the same as those defined in Chapter 4.

Fix a sample path for the observation and control process $\{y(\cdot,\omega), u(\cdot,\omega)\}$. Consider the state process $\{x(s); 0 \leq s \leq T\}$ generated by (5.13) for a given initial data $x(0) = \xi, w(0) = 0, y(0) = 0$. Then Assumptions 4.5.1 in Chapter 4, imply a path wise unique solution of the state process given in (5.13), for a given $x(0) = \xi, w(0) = 0$, and hence the solution is also unique in probability law [82]. Thus, existence of a unique probability measure $\bar{P}_\xi^{y,u}$ on $\left(\Omega^{w,x}, \mathcal{F}_T^{w,x}\right)$ which coincides with the distribution of $\{w(t), x(t); t \in [0,T]\}$ given $\{y(\cdot,\omega), u(\cdot,\omega)\}$, such that $\{w(t); t \in [0,T]\}$ is a Wiener process and

$$\left(\Omega^{w,x}, \mathcal{F}_T^{w,x}, \bar{P}_\xi^{y,u}\right) : x(t) = \xi + \int_0^t f(x(s), u(s))ds + \int_0^t \sigma(x(s))dw(s) \qquad (5.15)$$

Definition 5.4.1 *The set of admissible controls denoted by $\mathcal{U}_{ad}$ consists of probability measures π on $\left(\Omega^{y,u}, \mathcal{F}_T^{y,u}\right)$, that is, $\pi \in \mathcal{M}(\Omega^{y,u})$, such that $\{y(t); t \in [0,T]\}$ is $\mathcal{F}_T^{y,u}$ – π-a.s. Brownian motion.*

The projection $\big(y(\cdot,\omega), u(\cdot,\omega)\big) \mapsto y(\cdot,\omega)$ maps $\pi \in \mathcal{M}(\Omega^{y,u})$ onto a Wiener measure, and for all $t \in [0,T]$, $u(t)$ and $\sigma\{y(r) - y(t); 0 \le t \le r \le T\}$ are independent under π. Given the measure $\Pi_0 \in \mathcal{M}(\Re^n)$ corresponding to the initial state $x(0) = \xi$, by Bayes rule we have

$$\bar{P}^{y,u}(A) = \int_{\Re^n} \bar{P}_\xi^{y,u}(A) d\Pi_0(\xi), \qquad A \in \mathcal{F}_T^{w,x}$$

which is the unique joint distribution measure of $\{x(t), w(t); t \in [0,T]\}$ given $\{y(\cdot,\omega), u(\cdot,\omega)\}$. For each $\pi \in \mathcal{U}_{ad}$, define the joint distribution measure $\tilde{P}^\pi$ on $\big(\Omega, \mathcal{F}_T\big)$ by

$$\tilde{P}^\pi(dw, dx, du, dy) \triangleq \bar{P}^{y,u}(dw, dx) \times \pi(dy, du) \in \mathcal{M}(\Omega) \qquad (5.16)$$

Notice that the projection $\{w(\cdot,\omega), x(\cdot,\omega), y(\cdot,\omega), u(\cdot,\omega)\} \mapsto \{y(\cdot,\omega), u(\cdot,\omega)\}$ under $\tilde{P}^\pi \in \mathcal{M}(\Omega)$ is $\pi \in \mathcal{M}(\Omega^{y,u})$. Finally, define the nominal measure P^π as follows. Let $\{y(t) = Nv(t);\ t \in [0,T]\}$, be an $(\mathcal{F}_t, \tilde{P}^\pi)$-Wiener process with covariance $NN't$. Introduce the $(\mathcal{F}_t, \tilde{P}^\pi)$-adapted exponential martingale process

$$\Lambda^u(t) = \exp\Big\{ \int_0^t h'(x(s))\big(NN'\big)^{-1} dy(s) - \frac{1}{2}\int_0^t h'(x(s))\big(NN'\big)^{-1} h(x(s))ds \Big\} \qquad (5.17)$$

Define the nominal measure through the Radon-Nikodym derivative

$$\frac{dP^\pi(w,x,y,u)}{d\tilde{P}^\pi(w,x,y,u)}\Big|_{\mathcal{F}_T} \triangleq \Lambda^u(T) \qquad (5.18)$$

Under the measure $P^\pi \in \mathcal{M}(\Omega)$, the processes $\{v^\pi(t); t \in [0,T]\}$ and $\{w(t); t \in [0,T]\}$ are independent Wiener processes, where $Nv^\pi(t) \triangleq y(t) - \int_0^t h(x(s))ds; t \in [0,T]$. Hence, for each $\pi \in \mathcal{U}_{ad}$, there exists a unique nominal measure $P^\pi \in \mathcal{M}(\Omega)$, under which the state $\{x(t); t \in [0,T]\}$ and observation process $\{y(t); t \in [0,T]\}$ satisfy (5.13). The following result is given in [69].

Lemma 5.4.2 *The set of admissible controls $\mathcal{U}_{ad}$ is compact under weak sequential convergence (with respect to the Prohorov metric).*

This follows from the compactness and convexity assumptions for $\mathcal{U}$ and the assumption 4.5.1 in chapter 4. Next, we state the precise formulation of the uncertain stochastic problem.

Problem 5.4.3 *Suppose the nominal system given by* (5.13) *is perturbed by an unknown (measurable) process* $\{\gamma, \delta\}$ *taking values from a fixed set* $\mathcal{D}$ *in a function space (to be identified later). This gives rise to the uncertain system described by*

$$\left(\Omega, \mathcal{F}_t, Q^{\gamma,\delta}\right) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))\gamma(t)dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ dy(t) = h(x(t))dt + N\delta(t)dt + Ndv(t), \quad y(0) = 0, \end{cases} \tag{5.19}$$

where $Q^{\gamma,\delta}$ *denotes the probability measure on* Ω *induced by the system* (5.19) *corresponding to a given realization* $\{\gamma, \delta\} \in \mathcal{D}$ *and a given control* $\pi \in \mathcal{U}_{ad}$*. For a given* $\pi \in \mathcal{U}_{ad}$*, let*

$$M(\pi) \equiv \{Q^{\gamma,\delta} \in \mathcal{M}(\Omega) : \{\gamma, \delta\} \in \mathcal{D}\}$$

denote the family of probability measures induced by the uncertain system. Clearly the graph of this multi-measure $M(\pi)$ *is given by*

$$\Xi \equiv \{(\pi, Q) \in \mathcal{U}_{ad} \times \mathcal{M}(\Omega) : Q \in M(\pi)\}.$$

Define $J(\pi, Q) = E_Q\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\}$ *on* Ξ*. Then given the nominal measure* $P \in \mathcal{M}(\Omega)$*, find a pair* $(\pi^*, Q^*) \in \Xi$ *which solves the following constrained optimization problem.*

$$J(\pi^*, Q^*) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{Q \in M(\pi)} J(\pi, Q) \tag{5.20}$$

$$\textit{subject to fidelity } \; H(Q|P) \le R, \quad R \in (0, \infty) \tag{5.21}$$

5.4.2 Relations to Robust Control and Minimax Games

Here, the exact relation between the problems of Problem 5.4.3 and the usual formulation of the robust disturbance attenuation control problem is established.

Optimal Disturbance Attenuation Formulation

The robust control problem is defined as follows. Consider an uncertain partially observable system given by

$$\left(\Omega, \mathcal{F}_t, Q^{\gamma,\delta}\right) : \begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))\gamma(t)dt + \sigma(x(t))dw(t), \quad x(0) = \xi \\ dy(t) = h(x(t))dt + N\delta(t)dt + Ndv(t), \quad y(0) = 0 \end{cases} \tag{5.22}$$

where $Q^{\gamma,\delta} \in M(\pi)$. Let $\mathcal{H} = L_2([0,T];\Re^m)\times L_2([0,T];\Re^d)$ and $B_r = \{(\gamma,\delta) \in \mathcal{H}; ||(\gamma,\delta)||_{\mathcal{H}} < r\}$ and in problem 5.4.3, let $\mathcal{D} = \mathcal{H}$. Define the pay-off functional $J : \mathcal{U}_{ad} \to \Re$ by [1]

$$J(\pi) = \sup_{(\gamma,\delta)\in B_r^c} \frac{E_Q\big\{ \int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\big\}}{\frac{1}{2}E_Q(||(\gamma,\delta)||^2_{\mathcal{H}})} \tag{5.23}$$

where $r > 0$ is a fixed constant, B_r^c is the complement of the set B_r and $Q = Q^{\gamma,\delta}$. The optimal robust control problem is to find control law $\pi^* \in \mathcal{U}_{ad}$ such that

$$J(\pi^*) = \inf_{\pi\in\mathcal{U}_{ad}} J(\pi). \tag{5.24}$$

Similar to Chapter 4, the sub-optimal problem is defined as follows.

$$J(\pi) \le s, \quad s > 0. \tag{5.25}$$

This is equivalent to minimizing over $\pi \in \mathcal{U}_{ad}$ the pay-off functional

$$J^s(\pi) = \sup_{(\gamma,\delta)\in B_r^c} \Big\{E_Q\Big[\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big] - s\frac{1}{2}E_Q\Big[\int_0^T \big(||\gamma(t)||^2 + ||\delta(t)||^2\big)dt\Big]\Big\} \tag{5.26}$$

and ensuring the pay-off is non-positive. This problem can be reformulated in an unconstrained form as follows

$$J^s(\pi) = \sup_{(\gamma,\delta)\in \mathcal{H}} \Big\{E_Q\Big[\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big] + (\beta - \frac{1}{2}s)E_Q\Big[\int_0^T \big(||\gamma(t)||^2 + ||\delta(t)||^2\big)dt\Big]\Big\} - \beta r \tag{5.27}$$

where in the non-trivial case of finite $J^s(\pi)$, $\beta \in (0, \frac{1}{2}s)$. Therefore, the actual robust control problem is to find a $\pi^* \in \mathcal{U}_{ad}$ so that

$$J^s(\pi^*) = \inf_{\pi\in\mathcal{U}_{ad}} J^s(\pi). \tag{5.28}$$

By use of Girsanov's theorem, in particular the density

$$\Upsilon \equiv \exp\Big(\int_0^T \gamma'(s)dw(s) - \frac{1}{2}\int_0^T ||\gamma(s)||^2 ds\Big) \times \exp\Big(\int_0^T \delta'(s)dv(s) - \frac{1}{2}\int_0^T ||\delta(s)||^2 ds\Big),$$

giving $dQ = \Upsilon dP$, one can verify (see [75]) that the entropy of Q relative to P is given by

$$H(Q|P) = \frac{1}{2}E_Q\Big\{\int_0^T \big(||\gamma(t)||^2 + ||\delta(t)||^2\big)dt\Big\}. \tag{5.29}$$

Hence, (5.28) is equivalent to the unconstrained form of problem (5.20), i.e.,

$$\sup_{u\in\mathcal{U}_{ad}} \inf_{Q\in M(\pi)} \Big\{E_Q\Big(\int_0^T \lambda(x(t),u(t))dt + \kappa(x(T))\Big) - s_0 H(Q|P)\Big\}$$

where $s_0 = \frac{1}{2}s - \beta$. In view of (5.27), this is equivalent to (5.28).

[1] It is assumed that there is always an ambient disturbance present in the system, with energy equal or greater than r.

5.4.3 Duality of Wide Sense Uncertain Systems

Similar to Section 5.3, Problem 5.4.3 can be reformulated using the dual functional as follows: For every $s \in \Re$ define the Lagrangian

$$J^{s,R}(\pi, Q) \triangleq E_Q\Big\{\int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\} - s\Big(H(Q|P) - R\Big) \tag{5.30}$$

and its associated dual functional

$$J^{s,R}(\pi, Q^*) = \sup_{Q \in M(\pi)} J^{s,R}(\pi, Q) \tag{5.31}$$

In addition define the quantity

$$\Phi_\pi^*(R) \triangleq \inf_{s>0} J^{s,R}(\pi, Q^*) \tag{5.32}$$

The statements of the next Corollary follow directly from Theorem 5.3.1 and Lemma 5.3.2.

Corollary 5.4.4 *For an arbitrary but fixed $\pi \in \mathcal{U}_{ad}$, the statements of Theorem 5.3.1 and Lemma 5.3.2 hold for the functionals (*5.31*) and (*5.32*).*

Proof. Identifying,

$$\begin{aligned} &\Sigma \mapsto \Omega; \quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u}); \quad \mu \mapsto P \in M(\pi); \quad \nu \mapsto Q \in M(\pi) \\ &\ell^u \mapsto \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \end{aligned}$$

the proof is similar to that of Theorem 5.3.1 and Lemma 5.3.2.

Remark 5.4.5 *Notice that the results of Corollary 5.4.4, state that the maximizing measure Q^* is defined on $\big(\Omega, \mathcal{F}_T\big)$. For the specific problem under investigation, the maximizing measure Q^* should be restricted on $\big(\Omega^{y,u}, \mathcal{F}_T^{y,u}\big)$, because this is the only information available to both controller and maximizing measure. Towards this end, substituting $dP = \Lambda^u(T)d\tilde{P}$, in (5.17), and then restricting $\frac{dQ^*}{dP}$ onto $\mathcal{F}_T^{y,u}$ gives*

$$dQ^*|_{\mathcal{F}_T^{y,u}} = \frac{E_{\tilde{P}^{y,u}}(e^{s\ell^u}\Lambda^u(T))d\pi}{\int_{\Omega^{y,u}} E_{\tilde{P}^{y,u}}(e^{s\ell^u}\Lambda^u(T))d\pi} \tag{5.33}$$

This approach is considered in the next section.

5.4.4 Duality of Separated Uncertain Systems with Wide Sense Controls

Similar to Chapter 4, in this section, separated strategies are introduced by describing the nominal systems using conditional distributions rather than a priori distributions. Such strategies are called separated strategies. For each sample path $\{(u(\cdot,\omega),y(\cdot,\omega)\}$, introduce the functional

$$\chi^u(t) \triangleq \exp\Big(\frac{1}{s}\int_0^t \lambda(x(\tau),u(\tau))d\tau\Big) \tag{5.34}$$

and define the measure-valued process $\{M_t^{y,u}\}_{t\geq 0}$ by

$$M_t^{y,u}(\phi) \triangleq E_{\bar{P}^{y,u}}\Big\{\phi(x(t))\chi^u(t)\Lambda^u(t)\Big\}, \quad \phi \in BC(\Re^n) \tag{5.35}$$

Where $\Lambda^u(t)$ is defined in (5.17). Denote the kernel associated with $M_t^{y,u}$ by $(y,u) \mapsto M_t(dx|y,u) \triangleq M_t^{y,u}(dx)$. Then

$$M_t^{y,u}(\phi) = \int_{\Re^n} \phi(z) M_t^{y,u}(dz), \qquad \phi \in BC(\Re^n) \tag{5.36}$$

and moreover, the measure-valued process $\{M_t^{y,u}\}_{t\geq 0}$ satisfies the following stochastic partial differential equation written in the weak form

$$\begin{aligned} M_t^{y,u}(\phi) &= \Pi_0(\phi) + \int_0^t M_\tau^{y,u}(L(u)\phi)d\tau + \frac{1}{s}\int_0^t M_\tau^{y,u}(\lambda^u\phi)d\tau \\ &+ \int_0^t M_\tau^{y,u}(h'(NN')^{-1}\phi)dy(\tau) \end{aligned} \tag{5.37}$$

where $L(u)$ is the Backward Kolmogorov operator associated with the state process of the nominal system given by

$$L(u) = \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma')_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n f_i(x,u(t))\frac{\partial}{\partial x_i}$$

corresponding to any given control policy u. Introduce the normalized probability measure

$$d\mu_T^\pi(x,y,u) \triangleq \frac{d\mu_{T,un}^\pi(x,y,u)}{\int_{\Re^n\times\Omega^{y,u}} d\mu_{T,un}^\pi(x,y,u)} = \frac{d\mu_{T,un}^\pi(x,y,u)}{\mu_{T,un}^\pi(\Re^n\times\Omega^{y,u})}$$

where

$$d\mu_{T,un}^\pi(x,y,u) = dM_T^{y,u}(x)d\pi(y,u)$$

and $\pi \in \mathcal{M}(\Omega^{y,u})$ is a probability measure.

Remark 5.4.6 *For an arbitrary but fixed $\pi \in \mathcal{U}_{ad}$, the dual functional, $J^{s,R}(\pi, Q^*)$, given in (5.31) can be expressed in terms of the measures $M_T^{y,u}$ and π as follows.*

$$J^{s,R}(\pi, Q^*) = s \log \int_{\Omega^{y,u}} M_T^{y,u}(e^{\frac{\kappa}{s}}) d\pi(y,u) + sR$$

where κ is as defined in Assumption 4.5.1.

Corollary 5.4.7 *The dual functional $J^{s,R}(\pi, Q^*)$ can be expressed as an optimization involving measures on $\Re^n$ as follows.*

$$\begin{aligned}
&J^{s,R}(\pi, Q^*) = sR + s\log\Big(\int_{\Re^n \times \Omega^{y,u}} e^{\frac{\kappa(z)}{s}} d\mu^{\pi}_{T,un}(z,y,u)\Big) \\
&= sR + s\log\Big(\int_{\Re^n \times \Omega^{y,u}} e^{\frac{\kappa(z)}{s}} d\mu^{\pi}_{T}(z,y,u)\Big) + s\log\left(\mu^{\pi}_{T,un}(\Re^n \times \Omega^{y,u})\right) \\
&= \sup_{\nu \in \mathcal{M}(\Re^n \times \Omega^{y,u})} \Big\{\int_{\Re^n \times \Omega^{y,u}} \kappa(z) d\nu(z,y,u) - s(H(\nu|\mu^{\pi}_T) - R)\Big\} + s\log\left(\mu^{\pi}_{T,un}(\Re^n \times \Omega^{y,u})\right)
\end{aligned}$$

where the infimum is attained by $\nu^{\pi,*} \in \mathcal{M}(\Re^n \times \Omega^{y,u})$ given by

$$d\nu^{\pi,*}(x,y,u) = \frac{e^{s\kappa(x)} d\mu^{\pi}_T(x,y,u)}{\int_{\Re^n \times \Omega^{y,u}} e^{s\kappa(z)} d\mu^{\pi}_T(z,y,u)} \tag{5.38}$$

Here the parameter s is the same as in equation (5.37).

Proof. Identifying

$$\begin{aligned}
&\Sigma \mapsto \Re^n \times \Omega^{y,u}; \quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u}); \quad \mu \mapsto \mu^{\pi} \in \mathcal{M}(\Re^n \times \Omega^{y,u}); \\
&\ell^u \mapsto \kappa; \quad \lambda = 0.
\end{aligned}$$

the proof is identical to that of Lemma 5.3.2.

Existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ was proved in [69] [Theorem 4.1,p269; Theorem 7.2, p279].

Theorem 5.4.8 *For any admissible s, (a): $\pi \longrightarrow J^{s,R}(\pi, \nu^{\pi,*})$ is weakly lower-semi continuous on $\mathcal{U}_{ad}$, (b): There exists a $\pi^* \in \mathcal{U}_{ad}$ such that $J^{s,R}(\pi^*, \nu^{\pi^*,*}) \leq J^{s,R}(\pi, \nu^{\pi,*}), \forall \pi \in \mathcal{U}_{ad}$.*

Proof. The derivation is similar to that given in [69].

5.4.5 Evolution of the Density of the Maximum Measure

Some additional regularities on σ, Π_0 would imply that the measure $\{M_t^{y,u}, t \geq 0\}$ has a density. The following conditions are to guarantee this.

7) $n = m, a(x) \triangleq \sigma(x)\sigma'(x) \geq I_n\alpha, \alpha > 0, \forall x \in \Re^n$, $\frac{\partial}{\partial x_j}a_{i,j} \in L^\infty(\Re^n), \forall i, j$.

8) Π_0 has a density $p_0(x)$ and $p_0 \in L_2(\Re^n)$. Then

$$M_t^{y,u}(\phi) = \int_{\Re^n} \phi(z) e^{y'(t)h(z)} q^{y,u}(t,z)dz, \quad \phi \in BC(\Re^n) \tag{5.39}$$

Moreover, $q^{y,u}(\cdot, z)$ is the solution of the following partial differential equation

$$\frac{\partial}{\partial t} q^{y,u}(t,x) = A^*(y(t))q^{y,u}(t,x) + e(x, y(t), u(t))q^{y,u}(t,x)$$

$$+\frac{1}{s}\lambda(x, u(t))q^{y,u}(t,x), \quad (t,x) \in (0,T] \times \Re^n \tag{5.40}$$

$$q^{y,u}(0,x) = p_0(x), \quad x \in \Re^n \tag{5.41}$$

where $A^*(y)$ is the adjoint operator of $A(y)$ given by

$$A(y) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,y,u)\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}(ay \cdot \nabla h)_i \frac{\partial}{\partial x_i}$$

$$\begin{aligned} e(x,y,u) &= \frac{1}{2}(ay \cdot \nabla h, y \cdot \nabla h) - \hat{f} \cdot (y \cdot \nabla h) - ||h||^2_{\Re^d} \\ \hat{f}_i &\equiv f_i - (1/2)\sum_{j \geq 1} \frac{\partial}{\partial x_j} a_{i,j} \end{aligned}$$

in which $\cdot, (,)$ are the dot products in $\Re^d$, $\Re^n$, respectively. Using the density of the measure valued process, the pay-off functional can be expressed as follows:

$$J^{s,R}(\pi, Q^{\pi,*}) = sR + s\log\int_{\Omega^{y,u}}\Big\{\int_{\Re^n} e^{\frac{1}{s}\kappa(z)} e^{y'(T)h(z)} q^{y,u}(T,z)dz\Big\} \times d\pi(y,u) \tag{5.42}$$

Existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ follows from Theorem 5.4.8.

5.4.6 Partially Observable Uncertain LQ Problem

For the purpose of illustrating the concepts presented earlier, we consider the following partially observed linear LQG problem.

Assumptions 5.4.9 *The system parameters $\{f, \sigma, h\}$ of (5.13), the density of the initial state $x(0)$, and the cost inetgrands $\{\lambda, \kappa\}$ are given by $f(x,u) = Fx + Bu, \sigma(t,x) = G, h(x) = Hx$, $2\lambda(x,u) = x'Qx + u'Ru, 2\kappa(x) = x'Mx$, $p_0(x) = \frac{\exp(-\frac{1}{2}||P_0^{-\frac{1}{2}}(x-\xi)||^2)}{(2\pi)^{\frac{n}{2}}|P_0|^{\frac{1}{2}}}$, $P_0 = P_0' > 0$, each element having appropriate dimensions.*

Under Assumptions 5.4.9, it can be shown that $M_t^{y,u}$ has a density $m^{y,u}(x,t)$ given by

$$dM^{y,u}(t,x) = m^{y,u}(x,t)dx = \nu_{0,t}^u \times \frac{\exp\big(-\frac{1}{2}||P(t)^{-\frac{1}{2}}(x - r(t))||^2\big)}{(2\pi)^{\frac{n}{2}}|P(t)|^{\frac{1}{2}}} \times \exp\frac{1}{2s}(\mathcal{C}_{0,t}^u + \mathcal{I}_{0,t})dx,$$

where

$$\nu^u_{0,t} = \exp\left\{ \int_0^t (Hr(s))' \left(NN'\right)^{-1} dy(s) - \frac{1}{2}\int_0^t ||N^{-1}Hr(s)||^2_{\Re^d} ds \right\},$$

$$\mathcal{C}^u_{0,t} \triangleq \int_0^t \{r'Qr + u'(s)Ru(s)\}\, ds, \quad \mathcal{I}_{0,t} \triangleq \int_0^t Tr(PQ)ds.$$

and

$P : [0, T] \to \mathcal{L}(\Re^n; \Re^n), P = P' \geq 0, r : [0, T] \times \Omega \to \Re^n$, are given by

$$\left.\begin{array}{c} \dot{P} = FP + PF' + \frac{1}{s}PQP + GG' - PH'\left(NN'\right)^{-1}HP, P(0) = P_0, \\ dr = \left(F + \frac{1}{s}PQ\right) rdt + Budt + PH'\left(NN'\right)^{-1}\left(dy - Hrdt\right), \\ \\ r(0) = \xi, \quad y(\cdot) \text{ is an } \{\mathcal{F}^{y,u}_t; t \in T\} \text{ -adapted Wiener process} \\ \text{with correlation } NN'. \end{array}\right\} \tag{5.43}$$

Denote by $\tilde{\rho}(AB)$ the spectral radius of AB (A, B are matrix-valued functions), and define $\tilde{s}^* \triangleq \inf\{s; P \geq 0, \ \forall t \in [0, T], \ \tilde{\rho}(PM) < s \ \forall t \in [0, T]\}$, where P is defined in (5.43). Then

$$\int_{\Re^n} e^{\frac{1}{s}x'Mx} dM^{y,u}(T, x) = \frac{1}{|I - \frac{1}{s}P(T)M|^{\frac{1}{2}}} \times \exp\frac{1}{2s}\left\{r'(T)(I - \frac{1}{s}P(T)M)^{-1}Mr(T)\right\}$$
$$\times \exp\frac{1}{2s}\left(\mathcal{C}^u_{0,T} + \mathcal{I}_{0,T}\right) \times \nu^u_{0,T}, \quad s \in [\tilde{s}^*, \infty) \tag{5.44}$$

Therefore, by Corollary 5.4.7, the resulting pay-off which should be minimized with respect to $\pi \in \mathcal{U}_{ad}$ is given by

$$J^{s,R}(\pi, Q^{\pi,*}) = sR - \frac{s}{2}\log|I - \frac{1}{s}P(T)M| + s\log\int_{\Omega_{y,u}} \left\{ \exp\frac{1}{2s}\left\{r'(T)(I - \frac{1}{s}P(T)M)^{-1}Mr(T)\right\}\right.$$
$$\left.\times \exp\frac{1}{2s}\left(\mathcal{C}^u_{0,T} + \mathcal{I}_{0,T}\right)\right\} \times d\pi(y, u), \quad s \in [\tilde{s}^*, \infty) \tag{5.45}$$

If we further restrict the class of control laws to strict-sense then the optimal control law can be found, and it is given in [56, 62].

5.5 Conclusion

This chapter is concerned with a new class stochastic control systems, in which the pay-off is described by the relative entropy between the nominal measure and the uncertain measure, while the uncertain measures satisfy certain energy inequality constraints. The controller seeks to minimize the worst-case cost (maximum) over the set of admissible control laws. The uncertainty seeks to maximize the pay off over the set of unknown measures which satisfy a relative entropy constraint with respect to the nominal measure. Certain monotonicity properties of the optimal solution are discussed, lower and upper bounds on the optimal performance are derived. In addition, connections to minimax games of partially observable stochastic systems and risk-sensitive control problems and and the optimal disturbance attenuation problem are established.

Future Work and Open Problems

Several possible applications and extensions of the topics discussed in this thesis, are presented here.

• The problem of lossless source coding can be extended for an uncertain class of sources characterized by a variational norm constraint. This problem was discussed briefly at the end of Chapter 2 in the context of universal coding. As mentioned before, Pinsker inequality shows that this class covers a much larger set of sources than the relative entropy constraint class.

• The problem of lossless source coding can also be extended in another direction. It can be applied to uncertain sources with memory, such as sources characterized by Markov chains. The uncertainty of the Markon chains may occur in their initial state distribution or in their transition probability matrix. The second choice, can cover Markov chains with stuctural uncertainties.

• The rate distortion theorem for a general class of sources was first proved by Sakrison [35]. Also, the rate distortion function for a class of sources, is formulated as a min-max problem, where the maximum is taken over all source distributions within the class. The issue of computation of the rate distortion function has been addressed in [36] for a class of parametric sources. It may be interesting to study this computation for more general classes of sources. Two main examples can be 1) class of sources characterized by relative entropy constraints and 2) class of sources characterized by variational norm constraints.

• In the context of stochastic control problems, one imporatnt extension would be to consider the diffusion control problems posed in Chapters 4 and 5 without the bounded condition on the cost function. Also the problem of min-max control for uncertain systems characterized by variational norm constraints, requires further research.

Publications

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Appendix A

In this section some of the mathematical concepts used in the thesis, are explained in detail.

Definition A.0.1 *A family $\mathcal{B}(\Sigma)$ of subsets of a space Σ is called a σ-algebra $\mathcal{B}(\Sigma)$ if*

1) $\emptyset, \Sigma \in \mathcal{B}(\Sigma)$.
2) *For any sequence* $\{A_n\}_{n=1}^{\infty}$ *of sets in* $\mathcal{B}(\Sigma), \cup_{n=1}^{\infty} A_n \in \mathcal{B}(\Sigma)$.
3) $\forall A \in \mathcal{B}(\Sigma), A^c \in \mathcal{B}(\Sigma)$.

Definition A.0.2 *A measurable space is a pair $(\Sigma, \mathcal{B}(\Sigma))$ where Σ is a set(space) and $\mathcal{B}(\Sigma)$ is a σ-algebra of subsets of Σ.*

Definition A.0.3 *Let Σ be a space and $\mathcal{B}(\Sigma)$ a σ-algebra of subsets of Σ. A countably additive measure on $\mathcal{B}(\Sigma)$ is a function $\mu : \mathcal{B}(\Sigma) \to [0, \infty]$ such that:*
1) $\mu(\emptyset) = 0$.
2) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ *for any sequence* $\{A_n\}_{n=1}^{\infty}$ *of pairwise disjoint members of* $\mathcal{B}(\Sigma)$.

Definition A.0.4 *A measure space is a triple $(\Sigma, \mathcal{B}(\Sigma), P)$ where $(\Sigma, \mathcal{B}(\Sigma))$ is a measurable space and P is a measure on $\mathcal{B}(\Sigma)$. When P is a probability measure(i.e., $P(\Sigma)=1$)), then the triple $(\Sigma, \mathcal{B}(\Sigma), P)$ is called a probability measure space, or simply a probability space.*

Definition A.0.5 *Let Σ be a space and $\mathcal{B}(\Sigma)$ a σ-algebra of subsets of Σ. A finitely additive measure on $\mathcal{B}(\Sigma)$ is a function $\mu : \mathcal{B}(\Sigma) \to [0, \infty]$ such that:*
1) $\mu(\emptyset) = 0$.
2) $\mu(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n)$ *for any finite sequence* $\{A_n\}_{n=1}^{N}$ *of pairwise disjoint members of* $\mathcal{B}(\Sigma)$*. This property may not hold for an infinite sequence.*

Definition A.0.6 *Let X be a normed vector space with the ground field F. A linear operator $T : X \to F$ is called a bounded linear functional if there exists $N \geq 0$ such that*

$$||Tx|| \leq N.||x|| \qquad \forall x \in X$$

Definition A.0.7 *A metric space $(\mathcal{X}, d)$ is called complete if every cauchy sequence from X converges to a limit in X.*

Definition A.0.8 *A normed space X is called a Banach space, if it is complete*[1].

Definition A.0.9 *A metric space $(\mathcal{X}, d)$ is separable if it contains a countable dense subset. In other words, $(\mathcal{X}, d)$ is called separable, if X contains a countable subset Y such that its closure is equal to X, i.e., $\overline{Y} = X$.*

Definition A.0.10 *A complete separable metric space $(\mathcal{X}, d)$ is called a Polish space.*

Definition A.0.11 *Let X be a normed vector space with the ground field F. The set of all bounded linear functionals from X to F is called the dual of the normed space X and is denoted by X^*.*

Definition A.0.12 *Let X be a normed vector space and X^* be its dual. A sequence $\{x_n\}$ in X is said to converge weakly to the element $x \in X$ if $x^*(x_n) \longrightarrow x^*(x)$ for every $x^* \in X^*$* [2].

Definition A.0.13 *Let X be a normed vector space and X^* be its dual. A sequence $\{x_n^*\}$ in X^* is said to converge in the weak* sense to the element $x^* \in X^*$ if $x_n^*(x) \longrightarrow x^*(x)$ for every $x \in X$.*

Definition A.0.14 *Let X be a Polish space with metric d and $\mathcal{P}(X)$ be the set of probability measures on X. For any pair ν and μ in $\mathcal{P}(X)$, Lévy-Prohorov metric is defined as follows.*

$$\mathcal{L}(\nu, \mu) \triangleq \inf\{\epsilon > 0 \; : \; \nu(F) \leq \mu(F^\epsilon) + \epsilon, \quad \textit{For all closed subsets } F \textit{ of } X\}$$

where $F^\epsilon \triangleq \{x \in X : \; d(x, F) < \epsilon\}$.

Definition A.0.15 *Two measures μ and ν defined on $(\Sigma, \mathcal{B}(\Sigma))$ (finite or infinite) are called mutually singular if there exist disjoint measurable sets A and B such that $\Sigma = A \cup B$ and $\mu(B) = \nu(A) = 0$.*

Theorem A.0.16 *Jordan Decomposition*

For any signed measure ν on the measurable space $(\Sigma, \mathcal{B}(\Sigma))$ there exists a unique pair ν^+, ν^- of finite mutually singular measures such that $\nu = \nu^+ - \nu^-$.

[1] complete with respect to the metric $d(x, y) = ||x - y||$, $x, y \in X$ where $|| \cdot ||$ is the norm defined on X.

[2] Here $x^*(x)$ denotes the value of the functional x^* at a point $x \in X$.

For proof see [43].

Definition A.0.17 *Given probability measures ν and μ defined on $(\Sigma, \mathcal{B}(\Sigma))$, the variational norm or total variation norm is defined by*

$$\|\nu - \mu\|_v \triangleq (\nu - \mu)^+(\Sigma) + (\nu - \mu)^-(\Sigma)$$

where $(\nu - \mu)^+$ and $(\nu - \mu)^-$ are the measures appearing in the Jordan decomposition of $\nu - \mu$.

Appendix B

Proof of Lemma 2.3.2.

a) Let $\ell_1 = (\ell_{11}, \ell_{12}, ..., \ell_{1M})$ and $\ell_2 = (\ell_{21}, \ell_{22}, ..., \ell_{2M})$ be the codeword lengths for two arbitrary codes. Then

$$\begin{aligned} f(s, \lambda\ell_1 + (1-\lambda)\ell_2) &= sR + s\log(\sum_{i=1}^{M} e^{\frac{\lambda\ell_{1i}+(1-\lambda)\ell_{2i}}{s}}.\mu_i) \\ &= sR + s\log(\sum_{i=1}^{M} e^{\frac{\lambda\ell_{1i}}{s}}.e^{\frac{(1-\lambda)\ell_{2i}}{s}}.\mu_i) \end{aligned}$$

An application of Hölder Inequality, $E_\mu[XY] \le E_\mu[X^p]^{\frac{1}{p}}.E_\mu[Y^q]^{\frac{1}{q}}$, (where $\frac{1}{p} + \frac{1}{q} = 1$, with $X_i = e^{\frac{\lambda.\ell_{1i}}{s}}$, $Y_i = e^{\frac{(1-\lambda).\ell_{2i}}{s}}$, $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$) gives,

$$\begin{aligned} E_\mu[e^{\frac{\lambda.\ell_1}{s}}.e^{\frac{(1-\lambda).\ell_2}{s}}] &\le (E_\mu[(e^{\frac{\lambda.\ell_1}{s}})^{\frac{1}{\lambda}}])^{\lambda}.(E_\mu[(e^{\frac{(1-\lambda).\ell_2}{s}})^{\frac{1}{1-\lambda}}])^{(1-\lambda)} \\ E_\mu[e^{\frac{\lambda.\ell_1+(1-\lambda).\ell_2}{s}}] &\le (E_\mu[e^{\frac{\ell_1}{s}}])^{\lambda}.(E_\mu[e^{\frac{\ell_2}{s}}])^{(1-\lambda)} \\ \sum_{i=1}^{M} e^{\frac{\lambda.\ell_{1i}+(1-\lambda).\ell_{2i}}{s}}.\mu_i &\le (\sum_{i=1}^{M} e^{\frac{\ell_{1i}}{s}}.\mu_i)^{\lambda}.(\sum_{i=1}^{M} e^{\frac{\ell_{2i}}{s}}.\mu_i)^{(1-\lambda)} \\ \log(\sum_{i=1}^{M} e^{\frac{\lambda.\ell_{1i}+(1-\lambda).\ell_{2i}}{s}}.\mu_i) &\le \lambda\log(\sum_{i=1}^{M} e^{\frac{\ell_{1i}}{s}}.\mu_i) + (1-\lambda).log(\sum_{i=1}^{M} e^{\frac{\ell_{2i}}{s}}.\mu_i) \\ f(s, \lambda\ell_1 + (1-\lambda)\ell_2) &\le \lambda.f(s,\ell_1) + (1-\lambda).f(s,\ell_2) \qquad \forall\lambda \in [0,1] \end{aligned}$$

Hence $f(s,\ell)$ is a convex function of ℓ.

b) $\forall s_1, s_2 > 0$

$$f(\lambda s_1 + (1-\lambda)s_2, \ell) = (\lambda s_1 + (1-\lambda)s_2)R + (\lambda s_1 + (1-\lambda)s_2)\log E_\mu[e^{\frac{\ell}{\lambda s_1+(1-\lambda)s_2}}]$$

$$E_\mu[e^{\frac{\ell}{\lambda s_1+(1-\lambda)s_2}}] = E_\mu[e^{\frac{\ell(\lambda+(1-\lambda))}{\lambda s_1+(1-\lambda)s_2}}] = E_\mu[e^{\frac{\lambda\ell}{\lambda s_1+(1-\lambda)s_2}}.e^{\frac{(1-\lambda)\ell}{\lambda s_1+(1-\lambda)s_2}}]$$

Letting $X = e^{\frac{\lambda\ell}{\lambda s_1+(1-\lambda)s_2}}$ and $Y = e^{\frac{(1-\lambda)\ell}{\lambda s_1+(1-\lambda)s_2}}$, $p = \frac{\lambda s_1+(1-\lambda)s_2}{\lambda s_1}$, $q = \frac{\lambda s_1+(1-\lambda)s_2}{\lambda s_2}$, by Hölder Inequality we have:

$$E_\mu[e^{\frac{\lambda\ell}{\lambda s_1+(1-\lambda)s_2}}.e^{\frac{(1-\lambda)\ell}{\lambda s_1+(1-\lambda)s_2}}] = E_\mu[X.Y] \le E_\mu[X^p]^{\frac{1}{p}}.E_\mu[Y^q]^{\frac{1}{q}}$$

$$
\begin{aligned}
E_\mu[e^{\frac{\lambda \ell}{\lambda s_1+(1-\lambda)s_2}}.e^{\frac{(1-\lambda)\ell}{\lambda s_1+(1-\lambda)s_2}}] &\leq (E_\mu[e^{\frac{\ell}{s_1}}])^{\frac{\lambda s_1}{\lambda s_1+(1-\lambda)s_2}}.(E_\mu[e^{\frac{\ell}{s_2}}])^{\frac{(1-\lambda)s_2}{\lambda s_1+(1-\lambda)s_2}} \\
(\lambda s_1+(1-\lambda)s_2)\log E_\mu[e^{\frac{\ell}{\lambda s_1+(1-\lambda)s_2}}] &\leq \lambda s_1 \log E_\mu[e^{\frac{\ell}{s_1}}]+(1-\lambda)s_2 log E_\mu[e^{\frac{\ell}{s_2}}] \\
f(\lambda s_1+(1-\lambda)s_2,\ell) &\leq \lambda.f(s_1,\ell)+(1-\lambda).f(s_2,\ell) \qquad \forall\lambda\in[0,1]
\end{aligned}
$$

Proof of Theorem 2.4.4. From Section 2.4, we know that for a given $s>0$, the result of the optimization problem over codeword lengths is given by

$$
\begin{aligned}
\ell_i &= \left\lceil \log_D\left(\frac{1}{\nu^{*,s}}\right)\right\rceil \\
\nu_i^{*,s} &= \frac{{\mu_i}^\alpha}{\sum_{j=1}^M {\mu_j}^\alpha}, \qquad \forall i\in\{1,...,M\}.
\end{aligned} \tag{B.1}
$$

Where $\alpha=\frac{s\ln D}{s\ln D+1}$. By substituting $(B.1)$ in $H(\nu^{*,s}|\mu)$, we deduce

$$
\begin{aligned}
H(\nu^{*,s}|\mu) &= \sum_{i=1}^M \nu_i^{*,s}\log\left(\frac{\nu_i^{*,s}}{\mu_i}\right) \\
H(\nu^{*,s}|\mu) &= \sum_{i=1}^M \left(\frac{{\mu_i}^\alpha}{\sum_{j=1}^M {\mu_j}^\alpha}\right)\log\left(\frac{{\mu_i}^{\alpha-1}}{\sum_{j=1}^M {\mu_j}^\alpha}\right).
\end{aligned} \tag{B.2}
$$

By differentiating H with respect to s, we obtain

$$
\frac{d}{ds}H(\nu^{*,s}|\mu)=\frac{dH}{d\alpha}.\frac{d\alpha}{ds} \tag{B.3}
$$

where

$$
\frac{d\alpha}{ds}=\frac{\ln D}{(1+s\ln D)^2}=\ln D.(1-\alpha)^2 \tag{B.4}
$$

and

$$
\frac{dH}{d\alpha}=(\alpha-1).\frac{\left(\sum_{i=1}^M \mu_i^\alpha(\ln\mu_i)^2\right)\left(\sum_{i=1}^M \mu_i^\alpha\right)-\left(\sum_{i=1}^M \mu_i^\alpha\ln\mu_i\right)^2}{\left(\sum_{i=1}^M \mu_i^\alpha\right)^2}. \tag{B.5}
$$

Hence, (B.3) can be written as

$$
\frac{dH}{ds}=-\ln D.(1-\alpha)^3.\frac{\left(\sum_{i=1}^M \mu_i^\alpha(\ln\mu_i)^2\right)\left(\sum_{i=1}^M \mu_i^\alpha\right)-\left(\sum_{i=1}^M \mu_i^\alpha\ln\mu_i\right)^2}{\left(\sum_{i=1}^M \mu_i^\alpha\right)^2}. \tag{B.6}
$$

By the Cauchy-Schwarz inequality we have

$$
|\sum_{i=1}^M \mu_i^\alpha\ln\mu_i| = |\sum_{i=1}^M \mu_i^{\frac{\alpha}{2}}\mu_i^{\frac{\alpha}{2}}\ln\mu_i| \leq \left(\sum_{i=1}^M (\mu_i^{\frac{\alpha}{2}})^2\right)^{\frac{1}{2}}.\left(\sum_{i=1}^M(\mu_i^{\frac{\alpha}{2}}\ln\mu_i)^2\right)^{\frac{1}{2}}.
$$

Therefore, we have the following inequality

$$
\left(\sum_{i=1}^M \mu_i^\alpha\ln\mu_i\right)^2 \leq \left(\sum_{i=1}^M \mu_i^\alpha\right).\left(\sum_{i=1}^M \mu_i^\alpha(\ln\mu_i)^2\right). \tag{B.7}
$$

Since $0 < \alpha < 1$, then from (B.6), (B.7) we deduce that

$$\frac{d}{ds} H(\nu^{*,s} \mid \mu) \leq 0.$$

Hence $H(\nu^{*,s} \mid \mu)$, is a non-decreasing function of s.

Proof of Lemma 2.4.7. From (2.15) and (2.16), we know that the longest codeword $\ell^*_{\max}$ corresponds to the smallest nominal source probability $\mu_{\min}$, hence using (2.14), we have

$$\ell^*_{\max} = \left\lceil \log_D \left(\frac{1}{\nu^{*,s^*}_{\min}}\right) \right\rceil = \left\lceil \log_D \left(\frac{\sum_{i=1}^M e^{\frac{\ell^*_i}{s^*}} \mu_i}{e^{\frac{\ell^*_{\max}}{s^*}} \mu_{\min}}\right) \right\rceil \leq \left\lceil \log_D \left(\frac{\sum_{i=1}^M e^{\frac{\ell^*_{\max}}{s^*}} \mu_i}{e^{\frac{\ell^*_{\max}}{s^*}} \mu_{\min}}\right) \right\rceil = \left\lceil \log_D \left(\frac{1}{\mu_{\min}}\right) \right\rceil$$

Proof of Lemma 2.4.8. Using the notation in Section 2.4, we have

$$\begin{aligned}
L^{\lambda,s^*}(\ell, \nu^{*,s^*}) &= \inf_{s>0} \inf_{(\ell_1,\ldots,\ell_M)} L^{\lambda}(\ell, \nu^{*,s}) \\
&= \inf_{s>0} \inf_{(\ell_1,\ldots,\ell_M)} \left(sR + s\log\left(\sum_{i=1}^M e^{\frac{\ell_i}{s}} \mu_i\right) + \lambda.\left(\sum_{i=1}^M D^{-\ell_i} - 1\right)\right) \\
&\leq \inf_{s>0} \inf_{(\ell_1,\ldots,\ell_M)} \left(sR + s\log\left(\sum_{i=1}^M e^{\frac{\ell_i}{s}} \mu_i\right)\right) && \text{(B.8)} \\
&\leq \inf_{s>0} \left(sR + s\log\left(\sum_{i=1}^M e^{\frac{\ell^*_i}{s}} \mu_i\right)\right) \\
&\leq \inf_{s>0}(sR) + \sup_{s>0} \left(s\log\left(\sum_{i=1}^M e^{\frac{\ell^*_i}{s}} \mu_i\right)\right) \\
&\leq \|\ell^*\|_\infty = \ell^*_{\max} && \text{(B.9)}
\end{aligned}$$

where $(B.8)$ follows from Kraft inequality and non-negativity of λ, and $(B.9)$ follows from the fact that $s\log\left(\sum_{i=1}^M e^{\frac{\ell^*_i}{s}} \mu_i\right) \leq \ell^*_{\max}$ Now, from $(B.9)$ we have

$$s^*R + s^* \log\left(\sum_{i=1}^M e^{\frac{\ell^*_i}{s^*}} \mu_i\right) \leq \ell^*_{\max}.$$

Applying Jensen inequality to the left hand side, we obtain

$$s^*R + \sum_{i=1}^M \ell^*_i \mu_i \leq \ell^*_{\max}. \tag{B.10}$$

Moreover, we have

$$E_\mu(\ell^*) \geq H_D(\mu). \tag{B.11}$$

By $(B.10)$ and $(B.11)$, we deduce

$$s^*R + H_D(\mu) \leq \ell^*_{\max}.$$

Now, using the result of Lemma 2.4.7 we obtain

$$s^* R + H_D(\mu) \leq \left\lceil \log_D \left(\frac{1}{\mu_{\min}}\right) \right\rceil.$$

Hence, the upper bound for s^* is

$$s^* \leq \frac{1}{R}\left(\left\lceil \log_D \left(\frac{1}{\mu_{\min}}\right) \right\rceil - H_D(\mu) \right).$$

Proof of Theorem 2.4.2 Suppose $\hat{\nu} \in \mathcal{M}_R$. Then

$$\begin{aligned}
H(\hat{\nu}) &= \sum_{i=1}^{M} \hat{\nu}_i \log \frac{1}{\hat{\nu}_i} = -\sum_{i=1}^{M} \hat{\nu}_i \log \left(\frac{\hat{\nu}_i}{\nu_i^*} \nu_i^*\right) = -H(\hat{\nu}|\nu^*) - \sum_{i=1}^{M} \hat{\nu}_i \log \nu_i^* \\
&= -H(\hat{\nu}|\nu^*) - \sum_{i=1}^{M} \hat{\nu}_i \log \left(\frac{\mu_i^{\frac{\beta}{1+\beta}}}{\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}}} \right) \\
&= -H(\hat{\nu}|\nu^*) + \frac{\beta}{1+\beta} \sum_{i=1}^{M} \hat{\nu}_i \log \left(\frac{1}{\mu_i}\right) + \log \left(\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}} \right) \\
&= -H(\hat{\nu}|\nu^*) + \frac{\beta}{1+\beta} \sum_{i=1}^{M} \hat{\nu}_i \log \left(\frac{\hat{\nu}_i}{\mu_i} . \frac{1}{\hat{\nu}_i}\right) + \log \left(\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}} \right) \\
&= -H(\hat{\nu}|\nu^*) + \frac{\beta}{1+\beta} \big(H(\hat{\nu}|\mu) + H(\hat{\nu}) \big) + \log \left(\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}} \right).
\end{aligned}$$

Then,

$$\frac{1}{1+\beta} H(\hat{\nu}) = -H(\hat{\nu}|\nu^*) + \frac{\beta}{1+\beta} H(\hat{\nu}|\mu) + \log \left(\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}} \right). \tag{B.12}$$

It is clear that the right hand side in $(B.12)$, is maximized when $\hat{\nu} = \nu^*$ and $H(\nu^*|\mu) = R$, so

$$G(R) = \sup_{\nu \in \mathcal{M}_R} H(\nu) = \beta R + (1+\beta) \log \left(\sum_{j=1}^{M} \mu_j^{\frac{\beta}{1+\beta}} \right)$$

where β is chosen such that $H(\nu^*(\beta)|\mu) = R$. That is, we proved that $G(R)$ is attained for $\nu = \nu^*$. It is clear that $H(\nu^*(\beta)|\mu)$ is continuous in β, and

$$\lim_{\beta \to 0} H(\nu^*(\beta)|\mu) = \sum_{i=1}^{M} \frac{1}{M} \log \left(\frac{\frac{1}{M}}{\mu_i} \right) = H(\eta|\mu)$$

where η is uniform distribution. Also,

$$\lim_{\beta \to \infty} H(\nu^*(\beta)|\mu) = \sum_{i=1}^{M} \frac{\mu_i}{\sum_{j=1}^{M} \mu_j} \log \left(\frac{1}{\sum_{j=1}^{M} \mu_j} \right) = 0.$$

Since $H(\nu^*(\beta)|\mu)$ is continuous in β, and goes to $H(\eta|\mu)$ as $\beta \to 0$ and goes to zero as $\beta \to \infty$, then if $0 \leq R \leq H(\eta|\mu)$, there exists a non-negative value of β, for which $H(\nu^*(\beta)|\mu) = R$.

Appendix C

Proof of Theorem 3.4.1. Denote the marginal measures on $\hat{A}$ corresponding to q and q_0, by ν and ν_0, respectively. Let $q_1 = q_0 + \epsilon(q - q_0)$ and call its marginal measure ν_1. Consider the following limit,

$$\begin{aligned}
L &= \lim_{\epsilon \downarrow 0} \Big\{ \frac{I(\mu; q_1) - I(\mu; q_0)}{\epsilon} - \int_A \int_{\hat{A}} \log\Big(\frac{q_0(x, dy)}{\nu_0(dy)}\Big)(q - q_0)(x, dy)\mu(dx) \Big\} \\
&= \lim_{\epsilon \downarrow 0} \Big\{ \frac{1}{\epsilon}\Big(\int_A \int_{\hat{A}} \log\Big(\frac{q_1(x, dy)}{\nu_1(dy)}\Big) q_0(x, dy)\mu(dx) - \int_A \int_{\hat{A}} \log\Big(\frac{q_0(x, dy)}{\nu_0(dy)}\Big) q_0(x, dy)\mu(dx)\Big) \\
&+ \int_A \int_{\hat{A}} \log\Big(\frac{q_1(x, dy)}{\nu_1(dy)}\Big)(q - q_0)(x, dy)\mu(dx) - \int_A \int_{\hat{A}} \log\Big(\frac{q_0(x, dy)}{\nu_0(dy)}\Big)(q - q_0)(x, dy)\mu(dx) \Big\}.
\end{aligned} \tag{C.1}$$

Consider the first two terms of $(C.1)$. Using a generalized definition of mutual information [34], we have

$$\begin{aligned}
I &= \frac{1}{\epsilon}\Big(\int_A \int_{\hat{A}} \log\Big(\frac{q_1(x, dy)}{\nu_1(dy)}\Big) q_0(x, dy)\mu(dx) - \int_A \int_{\hat{A}} \log\Big(\frac{q_0(x, dy)}{\nu_0(dy)}\Big) q_0(x, dy)\mu(dx)\Big) \\
&= \frac{1}{\epsilon} \int_A \Big(\sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_1(x, E)}{\nu_1(E)}\Big) q_0(x, E) - \sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_0(x, E)}{\nu_0(E)}\Big) q_0(x, E)\Big)\mu(dx)
\end{aligned} \tag{C.2}$$

where $\mathcal{P}(\hat{A})$ denotes the collection of all finite partitions of $\hat{A}$. Now, given $\delta > 0$ and $x \in A$, there exists a partition P' of $\hat{A}$ such that

$$\sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_1(x, E)}{\nu_1(E)}\Big) q_0(x, E) \leq \sum_{E \in P'} \log\Big(\frac{q_1(x, E)}{\nu_1(E)}\Big) q_0(x, E) + \delta\epsilon.$$

Hence $(C.2)$ can be upperbounded as follows,

$$I \leq \frac{1}{\epsilon} \int_A \Big(\sum_{E \in P'} \log\Big(\frac{q_1(x, E)}{\nu_1(E)}\Big) q_0(x, E) - \sum_{E \in P'} \log\Big(\frac{q_0(x, E)}{\nu_0(E)}\Big) q_0(x, E)\Big)\mu(dx) + \delta \tag{C.3}$$

In general, q_1 may not be absolutely continuous with respect to q_0 [1]. So there may exist a nontrivial $P_1 \subset P'$ such that $q_0(x, E) = 0$ and $q_1(x, E) \neq 0$ for any $E \in P_1$, and $q_1 << q_0$(or

[1] or equivalently, q may not be absolutely continuous with respect to q_0

equivalently $q << q_0$) while restricted to $P' - P_1$. Using this decomposition, the inequality in $(C.3)$ can be written as follows,

$$I \leq \frac{1}{\epsilon}\int_A\Big\{\sum_{E\in P'-P_1}\log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)q_0(x,E) - \sum_{E\in P'-P_1}\log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)q_0(x,E)\Big)\mu(dx)$$
$$+\frac{1}{\epsilon}\int_A\sum_{E\in P_1}\log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)q_0(x,E)\mu(dx) - \frac{1}{\epsilon}\int_A\sum_{E\in P_1}\log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)q_0(x,E)\mu(dx) + \delta.$$

The third term in the above is zero, since $q_0(x,E) = 0$ for any $E \in P_1$. Using the convention $0\log\frac{0}{0} = 0$, as used in the definition of relative entropy [1], the fourth term is also zero, so we have

$$I \leq \frac{1}{\epsilon}\int_A\Big(\sum_{E\in P'-P_1}\log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)q_0(x,E) - \sum_{E\in P'-P_1}\log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)q_0(x,E)\Big)\mu(dx) + \delta$$

Using the definition of ν_0, the above inequality reduces to the following one,

$$I \leq \sum_{E\in P'-P_1}\int_A\frac{1}{\epsilon}\log\Big(\frac{q_1(x,E)}{q_0(x,E)}\Big)q_0(x,E)\mu(dx) - \sum_{E\in P'-P_1}\frac{1}{\epsilon}\nu_0(E)\log\Big(\frac{\nu_1(E)}{\nu_0(E)}\Big) + \delta \quad (C.4)$$

Now we have

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Big(\log q_1(x,E) - \log q_0(x,E)\Big) = \frac{(q-q_0)(x,E)}{q_0(x,E)},\quad \forall x\in A. \quad (C.5)$$

Since $q_1 = q_0 + \epsilon(q - q_0)$, we have $\nu_1 = \nu_0 + \epsilon(\nu - \nu_0)$ and therefore

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Big(\log \nu_1(E) - \log \nu_0(E)\Big) = \frac{(\nu-\nu_0)(E)}{\nu_0(E)}. \quad (C.6)$$

In order to use $(C.5)$, we need to show that as $\epsilon \to 0$, the integral and the limit in $(C.4)$ can be interchanged. Toward this end, consider the following inequalities,

$$\frac{1}{\epsilon}\log\Big(\frac{(1-\epsilon)q_0(x,E)+\epsilon q(x,E)}{q_0(x,E)}\Big)q_0(x,E) \geq \frac{1}{\epsilon}\log(1-\epsilon)q_0(x,E), \quad (C.7)$$
$$\frac{1}{\epsilon}\log\Big(\frac{(1-\epsilon)q_0(x,E)+\epsilon q(x,E)}{q_0(x,E)}\Big)q_0(x,E) \leq \frac{1}{\epsilon}\log(1-\epsilon).q_0(x,E) + \frac{1}{1-\epsilon}\frac{q(x,E)}{q_0(x,E)}q_0(x,E), \quad (C.8)$$

in which we have used the classical inequality, $\log x \leq x - 1$ where $x > 0$. By combining $(C.7)$ and $(C.8)$ we arrive at the following inequality,

$$\Big|\frac{1}{\epsilon}\log\Big(\frac{(1-\epsilon)q_0(x,E)+\epsilon q(x,E)}{q_0(x,E)}\Big)q_0(x,E)\Big| \leq \frac{1}{\epsilon}\log\Big(\frac{1}{1-\epsilon}\Big).q_0(x,E) + \frac{1}{1-\epsilon}q(x,E). \quad (C.9)$$

Now let $g_\epsilon(x) = \frac{1}{\epsilon}\log\big(\frac{1}{1-\epsilon}\big)q_0(x,E) + \frac{1}{1-\epsilon}q(x,E)$. It can be easily seen that limit and integration can be interchanged for $g_\epsilon(x)$. Thus it follows from $(C.9)$ and a general form of the

Lebesgue Dominated Convergence theorem [43][2] that we can do the same for the left side of $(C.9)$. Using the results of $(C.5)$ and $(C.6)$ and letting $\epsilon \rightarrow 0$ in $(C.4)$ we obtain

$$\lim_{\epsilon \rightarrow 0} I \leq \sum_{E \in P'-P_1} \Big\{ \int_A \Big(\frac{q(x,E)-q_0(x,E)}{q_0(x,E)}\Big) q_0(x,E)\mu(dx) - \Big(\frac{\nu(E)-\nu_0(E)}{\nu_0(E)}\Big)\nu_0(E) \Big\} + \delta \tag{C.10}$$

Since ν and ν_0 are the marginals of $\mu \otimes q$ and $\mu \otimes q_0$ respectively, finally we have [3]

$$\lim_{\epsilon \rightarrow 0} I \leq \delta. \tag{C.11}$$

Now going back to $(C.2)$, given $\delta > 0$, there exists another partition P'' of $\hat{A}$ such that

$$\sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big) q_0(x,E) \leq \sum_{E \in P''} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big) q_0(x,E) + \delta\epsilon.$$

Then $(C.2)$ can be written as

$$I \geq \frac{1}{\epsilon} \int_A \Big(\sum_{E \in P''} \log\Big(\frac{q_1(x,E)}{q_0(x,E)}\Big) q_0(x,E) - \sum_{E \in P''} \log\Big(\frac{\nu_1(E)}{\nu_0(E)}\Big) q_0(x,E)\Big)\mu(dx) - \delta;$$

and by taking the limit, we arrive at the following result

$$\lim_{\epsilon \rightarrow 0} I \geq -\delta. \tag{C.12}$$

Since $\delta > 0$ is arbitrary, it follows from $(C.11)$ and $(C.12)$ that

$$\lim_{\epsilon \rightarrow 0} I = 0. \tag{C.13}$$

Now in $(C.1)$, the last two terms can be written as follows.

$$\begin{aligned} Y = \int_A \Bigg(& \sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)(q(x,E) - q_0(x,E)) \\ & - \sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)(q(x,E) - q_0(x,E))\Bigg)\mu(dx). \end{aligned} \tag{C.14}$$

For any $\delta > 0$, there exists a partition $P' \in \mathcal{P}(\hat{A})$ such that

$$\sup_{P \in \mathcal{P}(\hat{A})} \sum_{E \in P} \log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)(q(x,E) - q_0(x,E)) \leq \sum_{E \in P'} \log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)(q(x,E) - q_0(x,E)) + \delta.$$

[2] If $|f_n(x)| \leq g_n(x)$, $\forall n$, for g_n integrable, $g_n \rightarrow g$ and $f_n \rightarrow f$ as $n \rightarrow \infty$, and if $\lim_{n\rightarrow\infty} \int_A g_n(x)\mu(dx) = \int_A g(x)\mu(dx)$, then $\lim_{n\rightarrow\infty} \int_A f_n(x)\mu(dx) = \int_A f(x)\mu(dx)$.

[3] Note that in general $\sum_{E \in P'-P_1} \nu(E) < 1$ and $\sum_{E \in P'-P_1} q(x,E) < 1$ but $\sum_{E \in P'-P_1} \nu_0(E) = 1$ and $\sum_{E \in P'-P_1} q_0(x,E) = 1$, so the first term on the right hand side of $(C.10)$ is zero only because ν and ν_0 are marginals of $\mu \otimes q$ and $\mu \otimes q_0$.

Then $(C.14)$ can be upper bounded as follows.

$$Y \leq \int_A \Big\{ \sum_{E\in P'} \log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)(q(x,E) - q_0(x,E))$$
$$- \sum_{E\in P'} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)(q(x,E) - q_0(x,E))\Big\}\mu(dx) + \delta.$$

The above inequality can re-written as follows.

$$Y \leq \int_A \Bigg\{ \sum_{E\in P'} \log\Big(\frac{q_1(x,E)}{q_0(x,E)}\Big)(q(x,E) - q_0(x,E))\Bigg\}\mu(dx)$$
$$- \sum_{E\in P'} \log\Big(\frac{\nu_1(E)}{\nu_0(E)}\Big)(\nu(E) - \nu_0(E)) + \delta. \tag{C.15}$$

In order to interchange the limit and integration, we use inequalities similar to $(C.7)$ and $(C.8)$, giving

$$\Bigg|\log\Big(\frac{q_1(x,E)}{q_0(x,E)}\Big)(q(x,E) - q_0(x,E))\Bigg| \leq \Bigg(\log\Big(\frac{1}{1-\epsilon}\Big) + \frac{\epsilon}{1-\epsilon}.\frac{q(x,E)}{q_0(x,E)}\Bigg).(q(x,E) + q_0(x,E)).$$

Let $f_\epsilon(x) = \Big(\log\big(\frac{1}{1-\epsilon}\big) + \frac{\epsilon}{1-\epsilon}.\frac{q(x,E)}{q_0(x,E)}\Big).(q(x,E)+q_0(x,E))$. Then we can obviously interchange the limit and integration for this function. Hence we can do the same in $(C.15)$ and this leads to

$$\lim_{\epsilon\longrightarrow 0} Y \leq \delta. \tag{C.16}$$

Similarly, given $\delta > 0$, there exists a partition $P'' \in \mathcal{P}(\hat{A})$ such that

$$\sup_{P\in\mathcal{P}(\hat{A})} \sum_{E\in P} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)(q(x,E) - q_0(x,E)) \leq \sum_{E\in P''} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)(q(x,E) - q_0(x,E)) + \delta.$$

Then $(C.14)$ can be lower bounded as follows

$$Y \geq \int_A \Big\{ \sum_{E\in P''} \log\Big(\frac{q_1(x,E)}{\nu_1(E)}\Big)(q(x,E) - q_0(x,E))$$
$$- \sum_{E\in P''} \log\Big(\frac{q_0(x,E)}{\nu_0(E)}\Big)(q(x,E) - q_0(x,E))\Big\}\mu(dx) - \delta.$$

Then by taking limits we obtain

$$\lim_{\epsilon\longrightarrow 0} Y \geq -\delta \tag{C.17}$$

By combining $(C.16)$ and $(C.17)$ and letting $\delta \longrightarrow 0$ we arrive at the following result,

$$\lim_{\epsilon\longrightarrow 0} Y = 0. \tag{C.18}$$

It follows from $(C.13)$,$(C.18)$ and $(C.1)$ that

$$L = \lim_{\epsilon \to 0} (I + Y) = 0.$$

Thus we may now conclude that $q \longrightarrow I_\mu(q) \equiv I(\mu; q)$ is Gateaux differentiable on Q_{ad} and that its Gateaux differential at $q_0 \in Q_{ad}$ in the direction $q - q_0$ is given by

$$\begin{aligned} I_\mu(q_0, q - q_0) &\equiv \lim_{\epsilon \downarrow 0} \frac{I(\mu; q_0 + \epsilon(q - q_0)) - I(\mu; q_0)}{\epsilon} \\ &= \int_A \int_{\hat{A}} \log\left(\frac{q_0(x, dy)}{\nu_0(dy)}\right)(q - q_0)(x, dy)\mu(dx). \end{aligned}$$

Moreover, since the form of the initial logarithm function has not changed, $I(\mu; q)$ is continuously Gateaux differentiable. •

Appendix D

Proof of Theorem 4.4.1.

1) First consider the first problem as in (4.18). We have to show that there exists a measure $\nu \in \Omega$ such that $\int_\Sigma \ell^u d\nu < \gamma$, i.e., existence of an interior point. By assumption, the set $\Gamma = \{x \in \Sigma; \quad \ell^u(x) < \gamma\}$ is nonempty. Define measure ν_1 as follows

$$\nu_1(\Gamma) = 1 \qquad \nu_1(\Gamma^c) = 0$$

Then one can easily verify that $\int_\Sigma \ell^u d\nu_1 < \gamma$. The inequality is strict since ν_1 is a probability measure.

2) Now consider the second problem as in (4.19). We have to show that there exists a measure $\nu \in \Omega$ such that $\int_\Sigma \ell^u d\nu > \gamma$, i.e., existence of an interior point. By assumption, the set $\Delta = \{x \in \Sigma; \quad \ell^u(x) > \gamma\}$ is nonempty. Define measure ν_2 as follows

$$\nu_2(\Delta) = 1 \qquad \nu_2(\Delta^c) = 0$$

Then one can easily verify that $\int_\Sigma \ell^u d\nu_2 > \gamma$.

Proof of Lemma 4.4.2. It is sufficient to prove the statements over those ν which satisfy $H(\nu|\mu) < \infty$.

5)

d) By assumptions we can interchange integration and differentiation, hence

$$\begin{aligned} \frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s}) \,|_{s=s_0} &= \gamma - \frac{d}{ds}\Big(\log \int_\Sigma e^{s\ell^u} d\mu \Big) \,|_{s=s_0} = \gamma - \frac{\int_\Sigma \ell^u e^{s_0 \ell^u} d\mu}{\int_\Sigma e^{s_0 \ell^u} d\mu} \\ &= \gamma - \frac{E_\mu\Big(\ell^u e^{s_0 \ell^u}\Big)}{E_\mu\Big(e^{s_0 \ell^u}\Big)} = \gamma - E_{\nu^{*,s}}\Big(\ell^u\Big) \end{aligned}$$

where

$$d\nu^{*,s_0} = \frac{e^{s_0 \ell^u} d\mu}{\int_\Sigma e^{s_0 \ell^u} d\mu}$$

In addition since $\exists \xi \in \Re$ in the neighborhood of s_0 such that $(\ell^u)^2 e^{\xi \ell^u} \in L_1(\mu)$, then $E_\mu\Big(\ell^u e^{s\ell^u}\Big)$ is differentiable at $s = s_0$, and so $J^{s,\gamma}(u, \nu^{*,s})$ is twice differentiable at $s = s_0$ and

$$\frac{d^2}{ds^2} J^{s,\gamma}(u, \nu^{*,s}) \mid_{s=s_0} = -\Big\{E_{\nu^{*,s}}\Big((\ell^u)^2\Big) - \Big(E_{\nu^{*,s}}(\ell^u)\Big)^2\Big\}_{s=s_0} < 0$$

e) The first part is obvious. Since $\exists \eta \in \Re$ in the neighborhood of s^* such that $\ell^u e^{\eta \ell^u} \in L_1(\mu)$, then $J^{s,\gamma}(u, \nu^{*,s})$ is differentiable at $s = s^*$, and

$$\frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s}) \mid_{s=s^*} = \gamma - E_{\nu^{*,s}}\Big(\ell^u\Big) = 0 \tag{D.1}$$

where

$$d\nu^{*,s^*} = \frac{e^{s^* \ell^u} d\mu}{\int_\Sigma e^{s^* \ell^u} d\mu}$$

Then we get

$$E_{\nu^{*,s}}\Big(\ell^u\Big) \mid_{s=s^*} = \gamma \tag{D.2}$$

Since $J^{s,\gamma}(u, \nu^{*,s})$ is concave in s, then $(D.1)$ gives the maximum point, and therefore the supremum is attained at $s = s^*$, and it occurs on the boundary by $(D.2)$. Also if $(\ell^u)^2 e^{s\ell^u} \in L_1(\mu)$, then $J^{s,\gamma}(u, \nu^{*,s})$ is twice differentiable at $s \in \Re$, and then the supremum is uniquely attained at $s = s^*$ at which $(D.1)$ is satisfied.

Proof of Theorem 4.4.3.

1) Risk-Seeking Scenario

The existence of minimizing measure was shown in Theorem 4.3.2. Also using the result of Theorems 4.2.2, 4.4.1 and part 1) of Lemma 4.4.2, the constrained problem and unconstrained problem are equivalent. Then using "s" as the Lagrange multiplier, we have

$$\begin{aligned} \inf_{\nu \in M_o(u)} H(\nu \mid \mu) &= \sup_{s \in \Re} \inf_{\nu \in M(u)} \Big(H(\nu \mid \mu) - s(E_\nu(\ell^u) - \gamma)\Big) \\ &= \sup_{s \in \Re} \inf_{\nu \in M(u)} J^{s,\gamma}(u, \nu) = \sup_{s \in \Re} J^{s,\gamma}(u, \nu^{*,s}) = \sup_{s \in \Re} \Big(s\gamma - \log \int_\Sigma e^{s\ell^u} d\mu\Big) \end{aligned} \tag{D.3}$$

where for the second infimum, it is sufficient to consider only those ν which satisfy $H(\nu|\mu) < \infty$. Assume $m^u = E_\mu\{\ell^u\} > \gamma$ [1]. The supremum in right hand side of (D.3), can be split into two parts, as follows

$$\sup_{s \in \Re} J^{s,\gamma}(u, \nu^{*,s}) = \max\Big\{\sup_{s>0} J^{s,\gamma}(u, \nu^{*,s}) \ , \ \sup_{s \le 0} J^{s,\gamma}(u, \nu^{*,s})\Big\} \tag{D.4}$$

[1] the other case, i.e. $m^u \le \gamma$ is trivial.

If $s > 0$, then $s\gamma < sm^u$. Also Jensen inequality implies that

$$\log \int_\Sigma e^{s\ell^u} d\mu \geq s \int_\Sigma \ell^u d\mu = sm^u > s\gamma$$

By combining the above two inequalities, we deduce

$$J^{s,\gamma}(u, \nu^{*,s}) = s\gamma - \log \int_\Sigma e^{s\ell^u} d\mu < 0 \quad \forall s \in (0, \infty) \tag{D.5}$$

Also by part (4) of Lemma 4.4.2, $\sup_{s \in \Re} J^{s,\gamma}(u, \nu^{*,s})$ is non-negative and similarly $\sup_{s \leq 0} J^{s,\gamma}(u, \nu^{*,s}) \geq 0$. Hence by $(D.4)$, we have

$$\sup_{s \in \Re} J^{s,\gamma}(u, \nu^{*,s}) = \sup_{s \leq 0} J^{s,\gamma}(u, \nu^{*,s}) \quad \text{if } m^u > \gamma$$

Combine this result with (D.3), to get equation (4.23). By Theorem 4.2.2 and the above conclusion, the minimizing measure ν^* is given by

$$d\nu^* = \frac{e^{s^* \ell^u} d\mu}{\int_\Sigma e^{s^* \ell^u} d\mu}, \qquad s^* \leq 0$$

Where s^* is the point at which the supremum $\sup_{s \leq 0} J^{s,\gamma}(u, \nu^{*,s})$ is attained. This establishes (4.24).

If $\ell^u e^{s\ell^u} \in L_1(\mu)$ and $(\ell^u)^2 e^{s\ell^u} \in L_1(\mu)$, then by Lemma 4.4.2, $J^{s,\gamma}(u, \nu^{*,s})$ is twice differentiable, and $\frac{d^2}{ds^2} J^{s,\gamma}(u, \nu^{*,s}) \leq 0$, hence $\frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s}) = \gamma - E_{\nu^{*,s}}(\ell^u)$ is a non increasing function of s, Therefore

$$\frac{d}{ds} J^{0,\gamma}(u, \nu^{*,0}) \leq \frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s}) \leq \frac{d}{ds} J^{s^*,\gamma}(u, \nu^*), \qquad \forall s \in [s^*, 0]$$

By Lemma 4.4.2, part e), $\frac{d}{ds} J^{s,\gamma}(u, \nu^{*,s})|_{s=s^*} = 0$, and for $s = 0$, $\nu^{*,s} = \mu$. Thus,

$$\gamma - E_\mu(\ell^u) \leq \gamma - E_{\nu^{*,s}}(\ell^u) \leq 0, \qquad \forall s \in [s^*, 0]$$

By re-arranging the inequalities, we obtain

$$\gamma = E_{\nu^{*,s}}(\ell^u) \mid_{s=s^*} \leq E_{\nu^{*,s}}(\ell^u) \leq E_\mu(\ell^u), \qquad \forall s \in [s^*, 0]$$

This establishes equation (4.25).

2) Risk-Averse Scenario

The existence of minimizing measure was shown in Theorem 4.3.2. Also using the result of Theorems 4.2.2, 4.4.1 and part 1) of Lemma 4.4.2, constrained problem and unconstrained problem are equivalent. Then using "s" as the Lagrange multiplier, we have

$$\begin{aligned} \inf_{\nu \in M_p(u)} H(\nu \mid \mu) &= \sup_{s \in \Re} \inf_{\nu \in M(u)} \Big(H(\nu \mid \mu) - s(E_\nu(\ell^u) - \gamma) \Big) \\ &= \sup_{s \in \Re} \inf_{\nu \in M(u)} J^{s,\gamma}(u, \nu) = \sup_{s \in \Re} J^{s,\gamma}(u, \nu^{*,s}) = \sup_{s \in \Re} \Big(s\gamma - \log \int_\Sigma e^{s\ell^u} d\mu \Big) \end{aligned} \tag{D.6}$$

where for the second infimum, it is sufficient to consider only those ν which satisfy $H(\nu|\mu) < \infty$. Assume $m^u = E_\mu\{\ell^u\} < \gamma$.[2]

The supremum in right hand side of (D.6), can be split into two parts, as follows

$$\sup_{s\in\Re} J^{s,\gamma}(u,\nu^{*,s}) = \max\{\sup_{s\geq 0} J^{s,\gamma}(u,\nu^{*,s}) \ , \ \sup_{s<0} J^{s,\gamma}(u,\nu^{*,s})\} \tag{D.7}$$

If $s<0$, then $s\gamma < sm^u$. Also, by Jensen inequality

$$\log \int_\Sigma e^{s\ell^u} d\mu \geq s\int_\Sigma \ell^u d\mu = sm^u > s\gamma$$

By combining the two above inequalities, we obtain

$$J^{s,\gamma}(u,\nu^{*,s}) = s\gamma - \log\int_\Sigma e^{s\ell^u} d\mu < 0, \quad \forall s\in(-\infty,0) \tag{D.8}$$

Also, by Lemma 4.4.2, part (4), $\sup_{s\in\Re} J^{s,\gamma}(u,\nu^{*,s})$ is nonnegative and similarly $\sup_{s\geq 0} J^{s,\gamma}(u,\nu^{*,s}) \geq 0$. Therefore from $(D.7)$ we deduce the following

$$\sup_{s\in\Re} J^{s,\gamma}(u,\nu^{*,s}) = \sup_{s\geq 0} J^{s,\gamma}(u,\nu^{*,s}), \quad \text{if } m^u < \gamma$$

This establishes equality in (4.26). Combine this result with (D.6), to get equation (4.27). By Theorem 4.2.2, the minimizing measure ν^* is given by

$$d\nu^* = \frac{e^{s^*\ell^u}d\mu}{\int_\Sigma e^{s^*\ell^u}d\mu}, \quad s^*\geq 0$$

Where s^* is the point at which the supremum $\sup_{s\geq 0} J^{s,\gamma}(u,\nu^{*,s})$ is attained and this establishes (4.28).

If $\ell^u e^{s\ell^u} \in L_1(\mu)$ and $(\ell^u)^2 e^{s\ell^u} \in L_1(\mu)$, then by Lemma 4.4.2, $J^{s,\gamma}(u,\nu^{*,s})$ is twice differentiable, and $\frac{d^2}{ds^2}J^{s,\gamma}(u,\nu^{*,s}) \leq 0$, hence $\frac{d}{ds}J^{s,\gamma}(u,\nu^{*,s}) = \gamma - E_{\nu^{*,s}}(\ell^u)$ is a non increasing function of s. Therefore

$$\frac{d}{ds}J^{s^*,\gamma}(u,\nu^*) \leq \frac{d}{ds}J^{s,\gamma}(u,\nu^{*,s}) \leq \frac{d}{ds}J^{0,\gamma}(u,\nu^{*,0}), \quad \forall s\in[0,s^*]$$

By Lemma 4.4.2, part e), $\frac{d}{ds}J^{s,\gamma}(u,\nu^{*,s})|_{s=s^*} = 0$, and for $s=0$, $\nu^{*,s}=\mu$. Thus

$$0 \leq \gamma - E_{\nu^{*,s}}(\ell^u) \leq \gamma - E_\mu(\ell^u), \quad \forall s\in[0,s^*]$$

By re-arranging the inequalities,

$$E_\mu(\ell^u) \leq E_{\nu^{*,s}}(\ell^u) \leq \gamma = E_{\nu^{*,s}}(\ell^u)\ |_{s=s^*}, \quad \forall s\in[0,s^*]$$

and this establishes equation (4.29).

[2] the other case, i.e. $m^u \geq \gamma$ is trivial.

Appendix E

Proof of Lemma 5.3.2.

1) This is an immediate consequence of Theorem 4.2.2.

2) For any s_1 , s_2 $\in [0,\infty)$ and $\forall\lambda \in [0,1]$, let $s = \lambda s_1 + (1-\lambda)s_2$, then we have

$$J^{s,R}(u,\nu^{*,s}) = (\lambda s_1 + (1-\lambda)s_2)R + (\lambda s_1 + (1-\lambda)s_2)\log E_\mu[e^{\frac{\ell^u}{\lambda s_1+(1-\lambda)s_2}}]$$

The expectation on the right hand side can be written as

$$E_\mu[e^{\frac{\ell^u}{\lambda s_1+(1-\lambda)s_2}}] = E_\mu[e^{\frac{\ell^u(\lambda+(1-\lambda))}{\lambda s_1+(1-\lambda)s_2}}] = E_\mu[e^{\frac{\lambda\ell^u}{\lambda s_1+(1-\lambda)s_2}}.e^{\frac{(1-\lambda)\ell^u}{\lambda s_1+(1-\lambda)s_2}}]$$

Let $X = e^{\frac{\lambda\ell^u}{\lambda s_1+(1-\lambda)s_2}}$, $Y = e^{\frac{(1-\lambda)\ell^u}{\lambda s_1+(1-\lambda)s_2}}$, $p = \frac{\lambda s_1+(1-\lambda)s_2}{\lambda s_1}$ and $q = \frac{\lambda s_1+(1-\lambda)s_2}{(1-\lambda)s_2}$, e.g, $1/p+1/q = 1$. Then by Hölder's Inequality we have

$$\begin{aligned}
E_\mu[e^{\frac{\lambda\ell^u}{\lambda s_1+(1-\lambda)s_2}}.e^{\frac{(1-\lambda)\ell^u}{\lambda s_1+(1-\lambda)s_2}}] &= E_\mu[X.Y] \le E_\mu[X^p]^{\frac{1}{p}}.E_\mu[Y^q]^{\frac{1}{q}} \\
(\lambda s_1 + (1-\lambda)s_2)\log E_\mu[e^{\frac{\ell^u}{\lambda s_1+(1-\lambda)s_2}}] &\le \lambda s_1 \log E_\mu[e^{\frac{\ell^u}{s_1}}] + (1-\lambda)s_2 \log E_\mu[e^{\frac{\ell^u}{s_2}}] \\
J^{s,R}(u,\nu^{*,s}) &\le \lambda.J^{s_1,R}(u,\nu^{*,s_1}) + (1-\lambda).J^{s_2,R}(u,\nu^{*,s_2}),\ \forall\lambda\in[0,1]
\end{aligned}$$

This shows convexity of $J^{s,R}(u,\nu^{*,s})$ with respect to s.

3) We shall apply the Lyapounov inequality [81], which states that, given a random variable X defined on the probability space $(\Sigma, \mathcal{B}(\Sigma), \mu)$, and two real numbers α and β such that $0 < \alpha \le \beta$, then $\left(E[|X|^\alpha]\right)^{\frac{1}{\alpha}} \le \left(E[|X|^\beta]\right)^{\frac{1}{\beta}}$.

Next we apply the Lyapounov inequality by letting $X = e^{\ell^u}$, $\alpha = \frac{1}{s_1}$ and $\beta = \frac{1}{s_2}$, where $s_1 > 0, s_2 > 0,\ s_1 \ge s_2$, which implies $0 < \alpha \le \beta$. Then

$$\left(E_\mu\left[\left(e^{\ell^u}\right)^{\frac{1}{s_1}}\right]\right)^{s_1} \le \left(E_\mu\left[\left(e^{\ell^u}\right)^{\frac{1}{s_2}}\right]\right)^{s_2}, \quad \forall s_1 \ge s_2 > 0$$

Taking the logarithm of both sides

$$s_1 \log E_\mu\left[e^{\frac{\ell^u}{s_1}}\right] \le s_2 \log E_\mu\left[e^{\frac{\ell^u}{s_2}}\right], \quad \forall s_1 \ge s_2 > 0$$

Therefore $\Gamma_\mu(s)$ is a non-increasing function of $s \in (0, \infty)$.

4) Since by part (3), $s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu$ is a non-increasing function of s, hence

$$\begin{aligned}
\Phi^*_\mu(0) &= \inf_{s>0} \Big\{ s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu \Big\} \\
&= \lim_{s\to\infty} \Big\{ s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu \Big\} = \lim_{\epsilon\to 0} \frac{\log \int_\Sigma e^{\epsilon \ell^u} d\mu}{\epsilon} \\
&= \frac{d}{d\eta} \log \int_\Sigma e^{\eta \ell^u} d\mu \Big|_{\eta=0} = \frac{\frac{d}{d\eta}\Big(\int_\Sigma e^{\eta\ell^u} d\mu \Big)}{\int_\Sigma e^{\eta \ell^u} d\mu} \Big|_{\eta=0}
\end{aligned} \tag{E.1}$$

The numerator of the last expression can be evaluated as follows. By assumption, there exists an $\eta_0 > 0$ such that $\ell^u e^{\eta_0 \ell^u} \in L_1(\mu)$. Also for any $\eta \in (0, \eta_0]$, we have

$$\left| \frac{d}{d\eta} e^{\eta \ell^u} \right| \leq |\ell^u| e^{\eta \ell^u}.I_{\ell\geq 0\}} + |\ell^u| e^{\eta \ell^u}.I_{\{\ell<0\}} \leq |\ell^u| e^{\eta_0 \ell^u}.I_{\{\ell\geq 0\}} + |\ell^u| I_{\{\ell<0\}} \ , \forall \eta \in (0, \eta_0] \tag{E.2}$$

Suppose $L \in \Re$ is the lower bound of ℓ^u then,

$$|\ell^u| e^{\eta_0 \ell^u} = e^{\eta_0 L}.|\ell^u| e^{\eta_0(\ell^u - L)} \geq |\ell^u| e^{\eta_0 L}$$

So from the above inequality and the assumption $\ell^u e^{\eta_0 \ell^u} \in L_1(\mu)$, it follows that $\ell^u \in L_1(\mu)$. Hence the upperbound in $(E.2)$ is integrable and independent of η. Using this and mean value theorem, one can interchange differentiation and integraion in $(E.1)$ [1]. Finally we get the desired result.

$$\Phi^*_\mu(0) = \frac{\int_\Sigma \ell^u e^{\eta \ell^u} d\mu}{\int_\Sigma e^{\eta \ell^u} d\mu} \Big|_{\eta=0} = \int_\Sigma \ell^u d\mu$$

5) By nonnegativity of s and R, we have

$$J^{s,R}(u, \nu^{*,s}) = sR + s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu \geq s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu$$

Taking infimum of both sides over $s \in (0, \infty)$

$$\Phi^*_\mu(R) = \inf_{s>0} \Big\{ J^{s,R}(u, \nu^{*,s}) \Big\} \geq \inf_{s>0} \Big\{ s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu \Big\}, \quad \forall R \geq 0 \tag{E.3}$$

Then by part (4), the right hand side will be $E_\mu(\ell^u)$. On the other hand, we have

$$\begin{aligned}
&\Phi^*_\mu(R) = \inf_{s>0} \Big\{ J^{s,R}(u, \nu^{*,s}) \Big\} \leq J^{1,R}(u, \nu^{*,s}) \\
&\Phi^*_\mu(R) \leq R + \log E_\mu\{e^{\ell^u}\}
\end{aligned} \tag{E.4}$$

From $(E.3)$ and $(E.4)$ we get

$$E_\mu\{\ell^u\} \leq \Phi^*_\mu(R) \leq R + \log E_\mu\{e^{\ell^u}\}, \quad \forall R \geq 0 \tag{E.5}$$

[1]Mean Value theorem can be applied since $e^{\eta \ell^u}$ is differentiable and continuous in η.

Moreover if we assume that ℓ^u is μ essentially bounded, then one can prove that

$$\sup_{s>0} s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu = \lim_{s\to 0}\left(s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu\right) = ||\ell^u||_\infty \tag{E.6}$$

Now we have

$$\Phi^*_\mu(R) = \inf_{s>0}\left\{sR + s\log\int_\Sigma e^{\frac{\ell^u}{s}} d\mu\right\} \le \inf_{s>0}(sR) + \sup_{s>0}\left(s\log\int_\Sigma e^{\frac{\ell^u}{s}} d\mu\right)$$

Using $(E.6)$ we obtain

$$\Phi^*_\mu(R) \le ||\ell^u||_\infty \tag{E.7}$$

Combining $(E.5)$ and $(E.7)$ we deduce $E_\mu\{\ell^u\} \le \Phi^*_\mu(R) \le \min\left\{R + \log E_\mu\{\ell^u\}, ||\ell^u||_\infty\right\}$.

6) The assumption ensure that $\int_\Sigma e^{\frac{\ell^u}{s}} d\mu$ is differentiable with respect to s. Also by part 2), $J^{s,R}(u,\nu^{*,s})$ is convex in s. The derivative of $J^{s,R}(u,\nu^{*,s})$ is

$$\frac{d}{ds}J^{s,R}(u,\nu^{*,s}) = R + \log\int_\Sigma e^{\frac{\ell^u}{s}} d\mu + s\left(\frac{\int_\Sigma(\frac{-\ell^u}{s^2})e^{\frac{\ell^u}{s}} d\mu}{\int_\Sigma e^{\frac{\ell^u}{s}} d\mu}\right)$$

Assume that the minimum of $J^{s,R}(u,\nu^{*,s})$ is attained at $s = s^*$. Then

$$\log\int_\Sigma e^{\frac{\ell^u}{s^*}} d\mu + R = -\frac{\int_\Sigma(\frac{-\ell^u}{s^*})e^{\frac{\ell^u}{s^*}} d\mu}{\int_\Sigma e^{\frac{\ell^u}{s^*}} d\mu} = \frac{1}{s^*}\int_\Sigma \ell^u d\nu^{*,s^*} \tag{E.8}$$

where ν^{*,s^*} is the maximizing measure, as given in (5.11), at point $s = s^*$. Hence we can find $H(\nu^{*,s^*}|\mu)$

$$\begin{aligned} H(\nu^{*,s}|\mu)\mid_{s=s^*} &= \int_\Sigma \log\Big(\frac{d\nu^{*,s^*}}{d\mu}\Big) d\nu^{*,s^*} \\ &= \int_\Sigma \log\left(\frac{e^{\frac{\ell^u}{s^*}}}{\int_\Sigma e^{\frac{\ell^u}{s^*}} d\mu}\right) d\nu^{s^*,*} \\ &= \int_\Sigma \frac{1}{s^*}\ell^u d\nu^{s^*,*} - \log\Big(\int_\Sigma e^{\frac{\ell^u}{s^*}} d\mu\Big) \end{aligned} \tag{E.9}$$

By $(E.8)$ and $(E.9)$ we have

$$H(\nu^{*,s}|\mu)\mid_{s=s^*} = R$$

This indicates that infimum of $J^{s,R}(u,\nu^{*,s})$ occurs on the boundary of the relative entropy constraint set.

7) By part (6), if $\nu^{*,s}$ is defined as in (5.11), then

$$H(\nu^{*,s}|\mu) = \int_\Sigma \frac{1}{s}\ell^u d\nu^{*,s} - \log\Big(\int_\Sigma e^{\frac{\ell^u}{s}} d\mu\Big)$$

The assumption ensures that $s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu$ is twice differentiable with respect to $s \in (0, \infty)$, hence

$$H(\nu^{*,s}|\mu) = -\frac{d}{ds}\left(s \log \int_\Sigma e^{\frac{\ell^u}{s}} d\mu\right) = -\frac{d}{ds}\Gamma_\mu(s) \tag{E.10}$$

Also by parts 2) and 3), $\Gamma_\mu(s)$ is a convex non-increasing function of $s \in (0, \infty)$. Hence $\frac{d}{ds}\Gamma_\mu(s) \leq 0$, which in turn gives $H(\nu^{*,s}|\mu) \geq 0$, as expected. Also, convexity of $\Gamma_\mu(s)$ implies that $\frac{d^2}{ds^2}\Gamma_\mu(s) \geq 0$. By our assumptions, the second derivative is well defined. Hence by $(E.10)$, we have

$$\frac{d}{ds}H(\nu^{*,s}|\mu) \leq 0, \quad \forall s > 0$$

Therefore $H(\nu^{*,s}|\mu)$ is a non-increasing function of s over $s \in (0, \infty)$. Thus,

$$0 \leq H(\nu^{*,s}|\mu)\ |_{s=s_2} \leq H(\nu^{*,s}|\mu)\ |_{s=s_1} \qquad \text{where } 0 < s_1 \leq s_2$$

Also, by part (6), we know that the upper bound is achieved at $s = s^*$. Hence,

$$0 \leq H(\nu^{*,s}|\mu)\ |_{s=s_2} \leq H(\nu^{*,s}|\mu)\ |_{s=s_1} \leq H(\nu^{*,s}|\mu)\ |_{s=s^*} = R, \quad \text{where } 0 < s^* \leq s_1 \leq s_2$$

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