# LOCALLY NILPOTENT DERIVATIONS AND THEIR QUASI-EXTENSIONS

By

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# Abstract

In this thesis, we introduce the theory of locally nilpotent derivations and use it to compute certain ring invariants. We prove some results about quasi-extensions of derivations and use them to show that certain rings are non-rigid. Our main result states that if $\mathbf{k}$ is a field of characteristic zero, $C$ is an affine $\mathbf{k}$-domain and $B = C[T,Y]/\langle T^nY - f(T)\rangle$, where $n \geq 2$ and $f(T) \in C[T]$ is such that $\delta^2(f(0)) \neq 0$ for all nonzero locally nilpotent derivations $\delta$ of $C$, then $\mathrm{ML}(B) \neq \mathbf{k}$. This shows in particular that the ring $B$ is not a polynomial ring over $\mathbf{k}$.

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# Dedication

I would like to dedicate this work to my mother Ilana and my father Ezra.

# Notation

Throughout this paper we use the following notation:

• $\mathbb{N}$ is the set of non-negative integers.

• $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are, as usual, the sets of integers, rational numbers, real numbers, and complex numbers respectively.

• $U \subset V$ means that $U$ is a subset of $V$. $U \subsetneq V$ means that $U$ is a proper subset of $V$.

• All rings are assumed to be **commutative** with multiplicative identity 1.

• All subrings of a ring $B$ contain $1_B$ and $-1_B$.

• If $B$ is a ring, $B^*$ is the set of units of $B$.

• All ring homomorphisms $f : A \to B$ satisfy $f(1_A) = 1_B$.

• $I \lhd R$ means that $I$ is an ideal of $R$.

• $\mathbf{k}$ is a field.

• We abbreviate "Unique Factorization Domain" by "UFD".

• $R^{[n]} = R[X_1, X_2, ..., X_n]$ is a polynomial ring in $n$ variables with coefficients in the ring $R$.

• Given a ring $A$, an $A$-algebra is a pair $(B, f)$ such that $B$ is ring and $f : A \to B$ is a ring homomorphism.

• If $B$ is an $A$-algebra and $S$ is a subset of $B$, then we write $A[S]$ for the subalgebra of $B$ generated by $S$. If $S = \{b_1, \cdots, b_n\}$, we write $A[b_1, \cdots, b_n]$ rather than $A[\{b_1, \cdots, b_n\}]$.

• If $(B, f)$ and $(C, g)$ are $A$-algebras, a homomorphism of $A$-algebras from $(B, f)$ to $(C, g)$ is a ring homomorphism $h : B \to C$ satisfying $h \circ f = g$. Homomorphisms of $A$-algebras are also called "$A$-homomorphisms".

- Given a field **k**, a **k**-domain is a **k**-algebra that is also an integral domain.
- When applied to a ring, the word “affine” means finitely generated as an algebra over a field. An affine **k**-domain is an integral domain that is also finitely generated as a **k**-algebra.

# Contents

# Introduction

Let $B$ be a ring (recalling that in this thesis, all rings are assumed to be commutative—see Notation). A *derivation* of $B$ is a map $D : B \to B$ that satisfies $D(x + y) = D(x) + D(y)$ and $D(xy) = D(x)y + xD(y)$ for all $x, y \in B$. A derivation $D$ is *locally nilpotent* if for every $b \in B$, there exists $n \in \mathbb{N}$ such that $D^n(b) = 0$.

We write $\mathrm{LND}(B)$ for the set of all locally nilpotent derivations of a ring $B$. The *Makar-Limanov invariant* of B, denoted $\mathrm{ML}(B)$, is the intersection of the kernels of all locally nilpotent derivations of $B$. Namely,

$$\mathrm{ML}(B) = \bigcap_{D \in \mathrm{LND}(B)} \ker(D).$$

This invariant was introduced by Makar-Limanov in his 1996 paper [14] (he called it the ring of absolute constants of $B$, and denoted it $\mathrm{AK}(B)$). In that paper, Makar-Limanov solved an open problem posed by Peter Russell several years before. The problem was to prove that the $\mathbb{C}$-algebra $R = \mathbb{C}[X, Y, Z, T]/\langle X + X^2Y + Z^2 + T^3\rangle$ (known as Russell's Cubic) is not isomorphic to a polynomial ring in 3 variables over $\mathbb{C}$. Several specialists had tried to prove this as it was the only missing step in the proof of a result of major importance (which states that all algebraic $\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable). In [14], Makar-Limanov proved that $\mathrm{ML}(R) \neq \mathbb{C}$. It is straightforward to verify that if $P$ is a polynomial ring in $n$ variables over a field $\mathbf{k}$ of characteristic zero, then $\mathrm{ML}(P) = \mathbf{k}$. So $R \ncong \mathbb{C}[X, Y, Z]$.

This made it clear that the theory of locally nilpotent derivations can be used for classifying rings, and in particular for proving that certain rings are not polynomial rings. This idea turned out to be quite fruitful and, in the last 20 years, locally nilpotent derivations have become a standard tool in the study of commutative rings.

It should be noted that the invariant $\mathrm{ML}(B)$ is often difficult to calculate. One of the aims of this thesis is to present some of the techniques that have been developed for calculating it. We shall also apply those techniques to calculate $\mathrm{ML}(B)$ for $B$ in a particular class of rings.

Throughout the thesis, we assume that the reader is familiar with basic definitions, facts, and results about rings, fields and modules that are typically taught at the advanced undergraduate level. Results that are not discussed in depth in the text can typically be found in any algebra textbook such as Dummit and Foote's *Abstract Algebra* [5].

Chapter 1 is a review of some of the algebraic concepts that are frequently used in the theory of locally nilpotent derivations. In particular, we discuss degree functions, graded rings and localization.

Chapter 2 begins our discussion of derivations starting directly from the definitions. Some basic results are proven, and the basic tools that are commonly used are introduced. The technique of homogenizing derivations and the relationship between a derivation and its homogenization are discussed. We also introduce the Derksen and Makar-Limanov ring invariants, which will be particularly useful in Chapter 4 where we prove our main results.

Chapter 3 considers extensions and quasi-extensions of derivations. We begin with some general results about extensions and quasi-extenstions. We then consider the special case of quasi-extensions of derivations from $R$ to $R[x]$ where $x$ is a root of a prime element of the polynomial ring $R[X]$ and prove what we believe are some new results. Namely, we provide a description of the set of quasi-extensions of derivations from $R$ to $R[x]$. We also generalize a result of Freudenburg and Moser-Jauslin, which allows us to show that certain rings are not rigid.

Chapter 4 applies the results from Chapters 1-3 and discusses classes of rings with non-trivial Makar-Limanov invariant. While, in general, the Makar-Limanov invariant is difficult to calculate, we do manage the feat for certain classes of rings. The main result of this thesis is Theorem 4.2.23, which asserts that if $\mathbf{k}$ is a field of characteristic zero, $C$ is an affine $\mathbf{k}$-domain, and $B = C[T,Y]/\langle T^nY - f(T)\rangle$ where $n \geq 2$ and $f(T) \in C[T]$ is such that $\delta^2(f(0)) \neq 0$ for all nonzero locally nilpotent derivations $\delta$ of $C$, then $\mathrm{ML}(B) \neq \mathbf{k}$. In particular, this shows that $B$ is not a polynomial ring over $\mathbf{k}$.

The last section of Chapter 4 discusses the relationship between the results we have developed and Zariski's Cancellation Problem, a problem introduced by Oscar Zariski over 60 years ago for which only partial results have been established. For a recent survey on this problem, we refer the reader to [10].

# Chapter 1

# Some Algebra

This chapter presents some basic definitions and results that will be used repeatedly in this thesis. We introduce degree functions, which are used directly and indirectly to prove a number of results in the theory of locally nilpotent derivations. We list briefly some facts about localization which will be used later on in the text. All results in this chapter are well known; most can be found as remarks or simple exercises in any textbook that treats the subject.

## 1.1 Degree Functions

**1.1.1 Definition.** A *total order* on a set $S$ is a binary relation '$\leq$' on $S$ such that for all $a, b, c \in S$,

- If $a \leq b$ and $b \leq a$, then $a = b$
- If $a \leq b$ and $b \leq c$, then $a \leq c$
- $a \leq b$ or $b \leq a$

**1.1.2 Definition.** Let $(G, +)$ be an abelian group. Suppose $\leq$ is a total order on $G$ satisfying:

$$\forall a, b, c \in G, \text{if } a \leq b \text{ then } a + c \leq b + c.$$

Then $(G, +, \leq)$ is called a *totally ordered abelian group.*

**1.1.3 Notation.** We will write $G$ instead of $(G, +)$ or $(G, +, \leq)$ when there is no ambiguity.

**1.1.4 Definition.** Let $G$ be a totally ordered abelian group. A *degree function* on a ring $B$ is a map $\deg : B \to G \cup \{-\infty\}$ such that:

a. $\forall x \in B$, $\deg(x) = -\infty \iff x = 0$

b. $\forall x, y \in B$, $\deg(x + y) \leq \max(\deg(x), \deg(y))$

c. $\forall x, y \in B$, $\deg(xy) = \deg(x) + \deg(y)$

**1.1.5 Definition.** Let $B$ and $G$ be as in Definition 1.1.4 and consider the particular case where $G = \mathbb{Z}$. Suppose furthermore that $\mathrm{im}(\deg) \subset \mathbb{N} \cup \{-\infty\}$. In this special case, we call the map $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ *a degree function with values in* $\mathbb{N}$ and we write $\deg : B \to \mathbb{N} \cup \{-\infty\}$ instead. Moreover, when we write explicitly "$\deg : B \to \mathbb{N} \cup \{-\infty\}$" we may abbreviate the phrase "degree function with values in $\mathbb{N}$" by simply "degree function".

**1.1.6 Lemma.** *Suppose $B$ is a nonzero ring, and* $\deg : B \to \mathbb{N} \cup \{-\infty\}$ *is a degree function on $B$. Then $B$ is an integral domain.*

*Proof.* If $x, y \in B$ are nonzero elements, then $\deg(xy) = \deg(x) + \deg(y) \geq 0$ by parts $(a)$ and $(c)$ of Definition 1.1.4 and so $xy$ is nonzero. □

**1.1.7 Lemma.** *Let $B[T]$ be a polynomial ring in one variable over a nonzero ring $B$. Given $f \in B[T]$, let* $\deg_T(f) \in \mathbb{N} \cup \{-\infty\}$ *denote the usual degree of $f$ as a polynomial in $T$, where we use the convention that* $\deg_T(0) = -\infty$*. Then*

$$\deg_T : B[T] \to \mathbb{N} \cup \{-\infty\}, \quad f \mapsto \deg_T(f)$$

*is a degree function as defined in Definition* 1.1.5 *if and only if $B$ is an integral domain.*

*Proof.* $(\Leftarrow)$ Recalling that $\deg_T(0) = -\infty$, the three requirements are easy to verify.

($\Rightarrow$) By contrapositive, suppose $B$ is not an integral domain and choose $f = \sum_{i=0}^{n} b_i T^i$, $g = \sum_{j=0}^{m} c_j T^j$ such that $b_n \neq 0$, $c_m \neq 0$, and $b_n c_m = 0$. Then $\deg_T(f) = m$, $\deg_T(g) = n$, but $\deg_T(fg) < m + n$, which contradicts part (c) of Definition 1.1.4.

□

## 1.2 Graded Rings

The results in this section are well known in the case of $\mathbb{Z}$-graded rings. We discuss the general case of $G$-graded rings where $G$ is any totally ordered abelian group. Everything in this section is known and can be found in [18].

**1.2.1 Definition.** Let $S$ be a ring and let $(G, +)$ be an abelian group. A *$G$-grading* of $S$ is a family $(S_d)_{d \in G}$ such that:

- $S_d$ is a subgroup of $S$ for every $d$ in $G$.
- $S = \bigoplus_{d \in G} S_d$
- $\forall d, e \in G, \quad S_d \cdot S_e \subset S_{d+e}$

The ring $S$ together with the collection $(S_d)_{d \in G}$ is called a *$G$-graded ring*. When the group $G$ is obvious, we simply say that $S$ is a *graded ring*.

**1.2.2 Definition.** In the context of the above definition, let $s \in S_d \setminus \{0\}$. We then say that $s$ is *homogeneous of degree $d$*.

**1.2.3 Remark.** The element $0_S$ is always defined to be homogeneous of degree $-\infty$.

**1.2.4 Remark.** If $S$ is a $\mathbb{Z}$-graded ring such that $S_d = 0$ for all $d < 0$, we say that $S$ is an $\mathbb{N}$-graded ring, in which case $S = \bigoplus_{d \in \mathbb{N}} S_d$.

**1.2.5 Example.** Let $S = \mathbf{k}[X]$ and let $S_d = \{aX^d : a \in \mathbf{k}\}$ for each $d \in \mathbb{N}$. Then $S_0 = \mathbf{k}, \quad S_1 = \mathbf{k} \cdot X, \quad S_2 = \mathbf{k} \cdot X^2$ etc.
It is straightforward to verify that $S$ is an $\mathbb{N}$-graded ring.

**1.2.6 Example.** Let $S = \mathbf{k}[X_1, ..., X_n]$ and let $S_d = \text{span}_{\mathbf{k}}\{{X_1}^{i_1}{X_2}^{i_2}\cdots{X_n}^{i_n} : i_1 + i_2 + \cdots + i_n = d\}$ where $d \in \mathbb{N}$. We remark that $S_0 = \mathbf{k}$ and $S_1 = \sum_{i=1}^{n} \mathbf{k} \cdot X_i$. The ring $S$ together with the collection $(S_d)_{d\in\mathbb{N}}$ is an $\mathbb{N}$-graded ring.

**1.2.7 Remark.** One can generalize the above grading of a polynomial ring $R$ as follows. Let $R = \mathbf{k}[X_1, ..., X_n]$ and let $d_1, \cdots, d_n$ be elements of an abelian group $G$. For each $d \in G$, define $S_d$ as follows:

$$S_d = \text{span}_{\mathbf{k}}\{{X_1}^{i_1}{X_2}^{i_2}...{X_n}^{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n, i_1d_1 + i_2d_2 + \cdots + i_nd_n = d\}.$$

It is then straightforward to check that the collection $(S_d)_{d\in G}$ defines a $G$-grading of $R$ in which each $X_i$ is homogeneous of degree $d_i$ and each element of $\mathbf{k} \setminus \{0\}$ is homogeneous of degree 0. The case where $G = \mathbb{Z}$ will be particularly useful throughout this thesis.

**1.2.8 Remark.** Let $S = \bigoplus_{d\in G} S_d$ be a graded ring. Any nonzero $f \in S$ is a finite sum of homogeneous elements and so we can write

$$f = f_{d_1} + f_{d_2} + \cdots + f_{d_r}$$

where $d_1, ..., d_r$ are distinct elements of $G$ and $0 \neq f_{d_i} \in S_{d_i}$ for each $f_{d_i}$ in the above sum. We remark that this decomposition of $f$ is unique because the graded ring is a direct sum by definition.

**1.2.9 Definition.** In the setting of Remark 1.2.8, for nonzero $f \in S$, the *support of f* is defined as:

$$\text{supp}(f) = \left\{\, d_i \in G \,:\, f_{d_i} \neq 0 \,\right\}.$$

These $f_{d_i}$ are called the *homogeneous components of f*. If $f = 0$, we define $\text{supp}(f) = \varnothing$.

**1.2.10 Proposition.** *Let $S = \bigoplus_{d\in G} S_d$ be a graded ring. Then $S_0$ is a subring of $S$.*

*Proof.* By Remark 1.2.8, since $S$ is a direct sum, we can write 1 as a finite sum of homogeneous components $u_d$, so $1 = \sum_{d\in F} u_d$ where $F$ is a nonempty finite subset of $G$ and $u_d \in S_d \setminus \{0\}$ for each $d \in F$.

Let $h$ be homogeneous of degree $m$. Then, $h = 1h = \sum_{d \in F} u_d h$ where the $u_d h$ are either 0 or homogeneous of degree $m + d$. Since $h$ is homogeneous of degree $m$, we must have $h = u_0 h$. So in fact $h = u_0 h$ for any homogeneous element $h$.

Next, let $f \in S$. Writing $f$ as a sum of its homogeneous components, we also have that $f = \sum_{d \in G} f_d = \sum_{d \in G} u_0 f_d = u_0 \sum_{d \in G} f_d = u_0 f$. In particular, $1 = u_0 \cdot 1 = u_0 \in B_0$.

Since $1 \in S_0$ and $S_0$ is a subgroup of $(S, +)$, we must have that $-1 \in S_0$. Since $S_0$ is closed under addition and multiplication by the first and third requirements in the definition of a graded ring, it follows that $S_0$ is a subring of $S$.

□

**1.2.11 Definition.** Let $G$ be an abelian group and let $A = \bigoplus_{i \in G} A_i$ and $B = \bigoplus_{i \in G} B_i$ be two graded rings. A map $\varphi : A \to B$ is an *isomorphism of graded rings* if

- $\varphi : A \to B$ is an isomorphism of rings
- $\varphi(A_i) = B_i$ for all $i$

**1.2.12 Definition.** Let $B = \bigoplus_{i \in G} B_i$ be a graded ring where $G$ is an abelian group and let $\mathcal{G}$ denote the grading $(B_i)_{i \in G}$ of $B$. Let $f = \sum_{j \in G} f_j$ where $f \in B$ and $f_j \in B_j$. If $B$ is an integral domain and $G$ is totally ordered, we can define a function

$$\deg_{\mathcal{G}} : B \to G \cup \{-\infty\} \text{ where}$$

$$\deg_{\mathcal{G}}(f) = \begin{cases} \max\{\, j \in G \,:\, f_j \neq 0 \,\} & \text{if } f \neq 0 \\ -\infty, & \text{if } f = 0. \end{cases}$$

Note that $\deg_{\mathcal{G}}$ is a degree function as defined in Definition 1.1.4. We call it *the degree function determined by the grading* $\mathcal{G}$.

## 1.3 Facts About Localization

In this section, we discuss some facts about localization. For an introduction to localization from basic principles, the reader can refer to Chapter 15 of [5] or Chapter 3 of [2].

**1.3.1 Notation.** In this section, we assume $R$ is a ring, $S \subset R$ is a multiplicative set (a set containing 1 that is closed under mulplication), $M$ is an $R$-module and $\ell : R \to S^{-1}R$ is the localization map. (Note that $\ell(x) = \frac{x}{1}$ for all $x \in R$.)

**1.3.2 Proposition.** *The localization map $\ell : R \to S^{-1}R$ is injective if and only if $S$ does not contain zero divisors.*

*Proof.* We begin with ($\Leftarrow$). Let $r \in R$ and suppose $\ell(r) = 0_{S^{-1}R}$. Then $\ell(r) = \frac{r}{1} = \frac{0}{1}$ in $S^{-1}R$ and so $\exists s \in S$ such that $sr = 0$. We conclude that $r$ must equal 0 since $S$ does not contain zero divisors and so $\ell$ is injective.

For ($\Rightarrow$), let $s \in S, 0 \neq r \in R$ be such that $sr = 0$. Then $\ell(r) = \ell(0) = 0_{S^{-1}R}$ and so $\ell$ is not injective. □

**1.3.3 Corollary.** *Let $R$ be an integral domain. Then $\ell$ is injective if and only if $S$ does not contain zero. Furthermore, if $0 \notin S$, then $S^{-1}R$ is an integral domain.*

*Proof.* The first statement follows immediately from 1.3.2. The second statement is obvious since $S^{-1}R \subset \operatorname{Frac} R$ which is a field and hence an integral domain. □

The following proposition can be found in [2] as Proposition 3.3.

**1.3.4 Proposition.** *Suppose $0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0$ is a short exact sequence of $R$-modules. Then $0 \to S^{-1}L \xrightarrow{i'} S^{-1}M \xrightarrow{\pi'} S^{-1}N \to 0$ is a short exact sequence of $S^{-1}R$ modules.*

*Proof.* Recall that $i'(\frac{l}{s}) = \frac{i(l)}{s}$ for every $l \in L$ and every $s \in S$. Similarly, $\pi'(\frac{m}{s}) = \frac{\pi(m)}{s}$ for every $m \in M$ and every $s \in S$.

We first show that $i'$ is injective. Let $\frac{l}{s} \in \ker(i')$. We show $\frac{l}{s} = \frac{0}{1} \in S^{-1}L$. We have,

$$\begin{aligned}
\frac{l}{s} \in \ker(i') &\iff \frac{i(l)}{s} = \frac{0}{1} \in S^{-1}M \\
&\iff \exists x \in S \text{ such that } x \cdot i(l) = 0_M \\
&\iff \exists x \in S \text{ such that } i(xl) = 0_M \\
&\iff \exists x \in S \text{ such that } xl \in \ker(i) \\
&\iff \exists x \in S \text{ such that } x(l-0) = 0_L \qquad \text{(since } i \text{ is injective)} \\
&\iff \exists x \in S \text{ such that } x(l \cdot 1 - 0 \cdot 1) = 0_L \\
&\iff \frac{l}{1} = \frac{0}{1} \text{ in } S^{-1}L \\
&\iff \frac{l}{s} = \frac{0}{s} = \frac{0}{1} \in S^{-1}L.
\end{aligned}$$

The surjectivity of $\pi'$ follows immediately from the surjectivity of $\pi$.

Lastly we show by double inclusion that $\ker(\pi') = \operatorname{im}(i')$. For ($\supset$), suppose $\frac{l}{s} \in S^{-1}L$. It follows directly from the definitions of $i'$ and $\pi'$ that $\pi'(i'(\frac{l}{s})) = 0_{S^{-1}N}$ giving the required inclusion.

For ($\subset$), suppose $\frac{m}{s} \in S^{-1}M$ and that $\pi'(\frac{m}{s}) = 0_{S^{-1}N}$. Then $\frac{\pi(m)}{s} = 0_{S^{-1}N}$ implies that there exists $x \in S$ such that $x \cdot \pi(m) = 0_N$. For such an $x$, it follows that

$$\begin{aligned}
x \cdot \pi(m) = 0_N &\Rightarrow \pi(xm) = 0_N \\
&\Rightarrow xm \in \ker(\pi) \\
&\Rightarrow xm \in \operatorname{im}(i) \\
&\Rightarrow \exists l \in L \text{ such that } i(l) = xm.
\end{aligned}$$

For this $l$, we have that

$$\begin{aligned}
\frac{m}{1} = \frac{i(l)}{x} &\Rightarrow \frac{m}{1} = i'\left(\frac{l}{x}\right) \\
&\Rightarrow \frac{m}{s} = i'\left(\frac{l}{sx}\right) \\
&\Rightarrow \frac{m}{s} \in \operatorname{im} i'.
\end{aligned}$$

□

**1.3.5 Remark.** If $I$ is a ideal of $R$ then the inclusion map $i : I \to R$ is an injective homomorphism of $R$-modules, so $S^{-1}I \to S^{-1}R$ is an injective homomorphism of $S^{-1}R$-modules. One can verify that the image of $S^{-1}I \to S^{-1}R$ is the ideal of $S^{-1}R$ generated by $\ell(I)$ where $\ell$ is the localization map, so we may identify $S^{-1}I$ with this ideal.

**1.3.6 Corollary.** *Let $I \lhd R$ and let $S^{-1}I$ be the ideal generated by $I$ in $S^{-1}R$. Then,*

$$(S^{-1}R)/(S^{-1}I) \cong S^{-1}(R/I)$$

*Proof.* Since $I, R$ and $R/I$ are $R$-modules,

$$0 \to I \to R \to R/I \to 0$$

is a short exact sequence of $R$-modules. It then follows by Proposition 1.3.4 that

$$0 \to S^{-1}I \to S^{-1}R \to S^{-1}(R/I) \to 0$$

is a short exact sequence of $S^{-1}R$ modules.

Since $S^{-1}I \lhd S^{-1}R$, we have that

$$0 \to S^{-1}I \to S^{-1}R \to (S^{-1}R)/(S^{-1}I) \to 0$$

is also a short exact sequence with the same first three terms. The result follows since in a short exact sequence

$$0 \to L \to M \to N \to 0,$$

the elements $L$ and $M$ determine $N$. □

**1.3.7 Lemma.** *Let $R$ be an integral domain and let $f : R \to S$ be an injective homomorphism. Then the image of $f$, denoted $f(R) \subset S$, is an integral domain.*

*Proof.* Since $f$ is injective, $f(R)$ is isomorphic to $R$ and is hence an integral domain. □

**1.3.8 Remark.** Let $A$ and $B$ be rings, let $\varphi : A \to B$ be a homomorphism and let $S \subset A$ be a multiplicative set. Regarding $B$ as an $A$-module, it makes sense to consider the $S^{-1}A$-module $S^{-1}B$. We note also that $\varphi(S)$ is a multiplicative set of $B$, so we may consider the ring $\varphi(S)^{-1}B$. It is easy to see that there is an isomorphism

$$\varphi(S)^{-1}B \cong S^{-1}B$$
$$b/\varphi(s) \leftrightarrow b/s.$$

The following proposition can be found in [2] as Proposition 3.11 (iv).

**1.3.9 Proposition.** *Let $I \lhd R$ be a prime ideal and suppose that $I \cap S = \varnothing$. Then $S^{-1}I$ is a prime ideal of $S^{-1}R$.*

*Proof.* By contrapositive, we suppose $S^{-1}I$ is not prime and show that either $I$ is not prime or $I \cap S \neq \varnothing$.

We have $(S^{-1}R)/(S^{-1}I) \cong S^{-1}(R/I) \cong \pi(S)^{-1}(R/I)$ where the first equality follows from Corollary 1.3.6 and the second from Remark 1.3.8 (where $\pi : R \to R/I$ is the projection map). Since by assumption $S^{-1}I$ is not prime, it follows that $(S^{-1}R)/(S^{-1}I) \cong \pi(S)^{-1}(R/I)$ is not an integral domain.

If $I$ is not prime, the proof is complete. If $I$ is prime, $R/I$ is an integral domain. Since $\pi(S)^{-1}(R/I)$ is not an integral domain, by Corollary 1.3.3 we must have that $\pi(S)$ contains $0_{R/I}$, and so $I \cap S \neq \varnothing$. □

# Chapter 2

# Derivations

## 2.1 The Basics

Everything in this section is well known. Daigle's lecture notes [3] provide a good introduction. Freudenburg's book [6] offers a more complete and advanced treatment of the subject.

**2.1.1 Definition.** Let $B$ be a ring. A *derivation* of $B$ is a map $D : B \to B$ satisfying the following two conditions:

(1) $\forall f, g \in B$, $D(f+g) = D(f) + D(g)$

(2) $\forall f, g \in B$, $D(fg) = D(f)g + fD(g)$

The set of derivations of $B$ is denoted $\text{Der}(B)$. Condition (2) is often called either the *product rule* or the *Leibniz rule*.

**2.1.2 Example.** Let $B$ be any ring. Then the zero map from $B$ to $B$ is a derivation.

**2.1.3 Example.** Let $B$ be the ring of smooth functions $f : \mathbb{R} \to \mathbb{R}$. Given $f \in B$, let $f'$ be the derivative of $f$ as defined in terms of limits. Then $D : B \to B$ given by $D(f) = f'$ is a derivation.

**2.1.4 Definition.** Let $B[T]$ be a polynomial ring in one variable with coefficients in a ring $B$. For any $f \in B[T]$, we write $f$ as the formal sum $f = \sum_{i=0}^{n} b_i T^i$ where each $b_i \in B$.

The *derivative*, denoted $\frac{d}{dT}$, is the map $\frac{d}{dT} : B[T] \to B[T]$ given by $\frac{d}{dT}(\sum_{i=0}^{n} b_i T^i) = \sum_{i=1}^{n} i b_i T^{i-1}$.

More generally, given the ring $B[X_1, X_2, \cdots, X_n]$ we define the *partial derivative with respect to* $X_j$

$$\frac{\partial}{\partial X_j} : B[X_1, X_2, \cdots, X_j, \cdots, X_n] \to B[X_1, X_2, \cdots, X_j, \cdots, X_n] \tag{1}$$

in the following way.

Let $m = a{X_1}^{i_1} \cdots {X_j}^{i_j} \cdots {X_n}^{i_n}$ (where $a \in B$) be a monomial in $B[X_1, X_2, \cdots, X_j, \cdots, X_n]$. Define

$$\begin{aligned} \frac{\partial}{\partial X_j}(m) &= \frac{\partial}{\partial X_j}(a{X_1}^{i_1} \cdots {X_j}^{i_j} \cdots {X_n}^{i_n}) = i_j a{X_1}^{i_1} \cdots {X_j}^{i_j-1} \cdots {X_n}^{i_n} && \text{if } i_j > 0, \\ \frac{\partial}{\partial X_j}(m) &= \frac{\partial}{\partial X_j}(a{X_1}^{i_1} \cdots {X_j}^{i_j} \cdots {X_n}^{i_n}) = 0 && \text{if } i_j = 0. \end{aligned}$$

Extending $\frac{\partial}{\partial X_j}$ linearly to all of $B[X_1, X_2, \cdots, X_j, \cdots, X_n]$ defines the map (1).

**2.1.5 Remark.** Let the setup be as described in Definition 2.1.4. It is easy to check that the derivative is a derivation in the ring $B[T]$ and that each of the partial derivatives is a derivation in the ring $B[X_1, \cdots, X_n]$.

**2.1.6 Notation.** Given a ring $B$, a derivation $D : B \to B$ and a polynomial $f = \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} {X_1}^{i_1} \cdots {X_n}^{i_n} \in B[X_1, \cdots, X_n]$, define

$$f^{(D)} = \sum D(a_{i_1 \cdots i_n}) {X_1}^{i_1} \cdots {X_n}^{i_n} \in B[X_1, \cdots, X_n].$$

**2.1.7 Example.** Given a ring $B$ and a derivation $D : B \to B$, the map $\hat{D} : B[X_1, \cdots, X_n] \to B[X_1, \cdots, X_n]$ defined by $\hat{D}(f) = f^{(D)}$ for all $f \in B[X_1, \cdots, X_n]$ is a derivation of $B[X_1, \cdots, X_n]$.

**2.1.8 Definition.** Given a ring $B$, the *kernel* of a derivation $D \in \mathrm{Der}(B)$, denoted $\ker(D)$, is defined as follows:

$$\ker(D) = \{b \in B : D(b) = 0\}.$$

**2.1.9 Lemma.** *Let $B$ be a ring and let $D \in \mathrm{Der}(B)$. Then $\ker(D)$ is a subring of $B$. Moreover if $B$ is a field, then $\ker(D)$ is a subfield of $B$.*

*Proof.* We have $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1)$. Subtracting $D(1)$ from both sides gives $D(1) = 0$.

Before showing $-1 \in \ker(D)$ we show $0 \in \ker(D)$. Indeed, $D(0) = D(0 \cdot 0) = 0$ by the product rule.

Since $0 = D(0) = D(-1 + 1) = D(-1) + D(1) = D(-1) + 0 = D(-1)$ we have that $-1 \in \ker(D)$.

Last, we show that $\ker(D)$ is closed under addition and multiplication. Let $f, g \in \ker(D)$. Then $D(fg) = D(f)g + fD(g) = 0 \cdot g + f \cdot 0 = 0$ so $fg \in \ker(D)$. Similarly, $D(f + g) = D(f) + D(g) = 0 + 0 = 0$ so $f + g \in \ker(D)$ as well.

Now suppose $B$ is a field. It suffices to show that if $0 \neq b \in \ker(D)$, then $b^{-1} \in \ker(D)$ as well. Let $0 \neq b \in \ker(D)$. Then

$$0 = D(1) = D(b^{-1}b) = b^{-1}D(b) + bD(b^{-1}) = bD(b^{-1}).$$

Since $b$ is nonzero and $B$ is a field, we must have $D(b^{-1}) = 0$ as required.

□

**2.1.10 Remark.** Note that the calculations showing that $\ker(D)$ is a subring of $B$ do not depend on $B$ being a commutative ring. Hence, the result holds for non-commutative rings as well.

**2.1.11 Remark.** Given $D_1, D_2 \in \mathrm{Der}(B)$, the map $D_1 + D_2 : B \to B$

$$(D_1 + D_2)(x) = D_1(x) + D_2(x) \tag{2}$$

is a derivation. Similarly, given $b \in B$ and $D \in \mathrm{Der}(B)$, the map $bD : B \to B$

$$(bD)(x) = b \cdot D(x) \tag{3}$$

is a derivation. These facts are straightforward to verify, and imply that $\mathrm{Der}(B)$ is a $B$-module.

**2.1.12 Definition.** Let $A \subset B$, where $A$ and $B$ are both rings. A derivation $D : B \to B$ such that $A \subset \ker(D)$ is called an *$A$-derivation of $B$*. The set of all $A$-derivations is denoted $\mathrm{Der}_A(B)$ and is a submodule of the $B$-module $\mathrm{Der}(B)$.

**2.1.13 Remark.** The derivative $\frac{d}{dT} : B[T] \to B[T]$ is a B-derivation of $B[T]$, and each partial derivative $\frac{\partial}{\partial X_j}$ is a $B$-derivation of $B[X_1, \cdots, X_n]$.

**2.1.14 Lemma.** *Let $B$ be a ring, let $f(X) \in B[X]$ be a polynomial and let $b \in B$. Then for any $D \in \mathrm{Der}(B)$,*

$$D(f(b)) = f^{(D)}(b) + f'(b) \cdot D(b)$$

*where $f' \in B[X]$ is the derivative of $f$ and $f^{(D)}$ is defined as in* 2.1.6*.*

*Proof.* The proof is a straightforward computation and is left to the reader. □

The following lemma can be found in [6] as Proposition 18(d).

**2.1.15 Lemma.** *Let $B$ be an integral domain of characteristic zero and let $D \in \mathrm{Der}(B)$. Then $A = \ker(D)$ is algebraically closed in $B$.*

*Proof.* Suppose $b \in B$ is algebraic over $A$. We show $D(b) = 0$. Let $f \in A[X] \setminus \{0\}$ be a polynomial with coefficients in $A$ of minimal degree satisfying $f(b) = 0$. We then have that

$$0 = D(f(b)) = f^{(D)}(b) + f'(b) \cdot D(b) = f'(b) \cdot D(b)$$

Note that the third equality follows since the coefficients of $f$ are contained in $\ker(D)$. Also, $f'(b) \neq 0$ since $f$ has minimal degree and $B$ is of characteristic zero, and so we conclude that $D(b) = 0$ since $B$ is an integral domain.

□

## 2.2 Derivations and localization

Again, the results of this section are well known and can be found in Daigle's lecture notes [3] as well as in Freudenburg's book [6].

**2.2.1 Definition.** Given a ring $B$, a multiplicative set $S \subset B$ and a derivation $D : B \to B$, consider the ring $S^{-1}B$ and define $S^{-1}D : S^{-1}B \to S^{-1}B$ by:

$$S^{-1}D(\tfrac{b}{s}) = \frac{sD(b) - bD(s)}{s^2} \text{ where } b \in B, s \in S.$$

Then $S^{-1}D$ is a derivation of $S^{-1}B$ called the *extension of* $D$ from $B$ to $S^{-1}B$. We leave the tedious but straightforward verification to the reader. We have the following commutative diagram:

$$\begin{array}{ccc} S^{-1}B & \xrightarrow{S^{-1}D} & S^{-1}B \\ \ell \uparrow & & \uparrow \ell \\ B & \xrightarrow[D]{} & B \end{array}$$

**2.2.2 Notation.** When considering elements of the ring $S^{-1}B$, we will sometimes abuse notation by writing the element $\frac{b}{1} \in S^{-1}B$ simply as $b$.

**2.2.3 Example.** Consider the derivation $\frac{\partial}{\partial Y}$ on the ring $B = \mathbb{C}[X, Y, Z]$. By Definition 2.2.1, we can extend $\frac{\partial}{\partial Y}$ to $\mathbb{C}(X, Y, Z)$, the fraction field of $B$ by letting $S$ be the multiplicative set of nonzero elements of $B$. As an example, we compute

$$\frac{\partial}{\partial Y}\left(\frac{XYZ}{Y^3 + Z^3}\right) = \frac{(Y^3 + Z^3)(XZ) - XYZ(3Y^2)}{(Y^3 + Z^3)^2} = \frac{XY^3Z + XZ^4 - 3XY^3Z}{(Y^3 + Z^3)^2}.$$

**2.2.4 Corollary.** *Let $B$ be a $\mathbb{Q}$-algebra. Then* $\mathrm{Der}(B) = \mathrm{Der}_{\mathbb{Q}}(B)$.

*Proof.* ($\supset$) is obvious. For ($\subset$), we show that for every derivation $D : B \to B$, $\mathbb{Q} \subset \ker(D)$.

If $B = 0$, the result is obvious so assume $B$ is nonzero. Let $D \in \mathrm{Der}(B)$. First note that since $B$ is a $\mathbb{Q}$-algebra and any nonzero homomorphic image of a field is isomorphic to itself, we have that $\mathbb{Q} \subset B$ and so $B$ is a ring of characteristic 0.

Since $B$ has characteristic 0 and $\ker(D)$ is a subring of $B$, $\mathbb{Z} \subset \ker(D)$.

Let $\frac{m}{n} \in \mathbb{Q}$, and note that $n \in B^*$. We have

$$0 = D(m) = D\left(n \cdot \frac{m}{n}\right) = nD\left(\frac{m}{n}\right) + \frac{m}{n}D(n) = nD\left(\frac{m}{n}\right)$$

which implies that $D(\frac{m}{n}) = 0$ since $n \in B^*$. Thus $\mathbb{Q} \subset \ker(D)$.

∎

## 2.3 Locally Nilpotent Derivations

### 2.3.1 Definitions and Basic Examples

**2.3.1 Definition.** Let $B$ be a ring and $D : B \to B$ a derivation. Let $D^n : B \to B$ be the composition of $D$ with itself $n$-times. The derivation $D$ is called *locally nilpotent* if for each $b \in B$, there exists $n \in \mathbb{N}$ such that $D^n(b) = 0$. The set of all locally nilpotent derivations of $B$ is denoted $\mathrm{LND}(B)$.

**2.3.2 Remark.** In general, it is not the case that $D^n$ is a derivation. For example, if $B$ is the ring $\mathbf{k}[T]$ where $\mathbf{k}$ does not have characteristic 2, and $D = \frac{d}{dT}$ is the derivative, we have $\frac{d^2}{dT^2}(T^2) = 2$ but $T \cdot \frac{d^2}{dT^2}(T) + \frac{d^2}{dT^2}(T) \cdot T = 0$ and so the Leibniz rule fails.

**2.3.3 Remark.** The above definition does not imply that there exists $n \in \mathbb{N}$ such that $D^n(b) = 0$ for all $b \in B$. The following example illustrates this point.

**2.3.4 Example.** Consider the polynomial rings $B[T]$ and $B[X_1, \dots, X_n]$ over a ring $B$, and the derivations $\frac{d}{dT}$ of $B[T]$ and $\frac{\partial}{\partial X_j}$ of $B[X_1, \dots, X_n]$ as in Definition 2.1.4. Then

$$\tfrac{d}{dT} \in \mathrm{LND}\big(B[T]\big) \quad \text{and} \quad \tfrac{\partial}{\partial X_j} \in \mathrm{LND}\big(B[X_1, \dots, X_n]\big).$$

We verify that $\frac{d}{dT} \in \mathrm{LND}(B[T])$ and leave the verification that $\frac{\partial}{\partial X_j} \in \mathrm{LND}\big(B[X_1, \dots, X_n]\big)$ to the reader.

Let $f \in B[T]$. If $f = 0$ then clearly $\frac{d}{dT}(f) = 0$. If $f \neq 0$, $f$ can be written as a finite sum of nonzero monomials of distinct degrees. Let $f = \sum\limits_{i=1}^{n} f_i$ where each $f_i$ is a monomial of standard $T$-degree $d_i$. Let $d = \max\limits_{1 \le i \le n} d_i$. Then

$$\frac{d}{dT}^{d+1}(f) = \frac{d}{dT}^{d+1} \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \frac{d}{dT}^{d+1}(f_i) = 0.$$

Note that if $B$ has characteristic zero then for each $n$ we have $\frac{d^n}{dT^n}(T^n) = n! \neq 0$, illustrating the point made in Remark 2.3.3.

**2.3.5 Definition.** Let $B$ be a ring and let $D \in \mathrm{Der}(B)$. We define the set $\mathrm{Nil}(D)$ as follows:

$$\mathrm{Nil}(D) = \left\{ b \in B \,:\, \exists n \in \mathbb{N} \text{ such that } D^n(b) = 0 \right\}.$$

**2.3.6 Remark.** Let $B$ be a ring. Let $D \in \mathrm{Der}(B)$ and let $x, y \in B$. Then $\forall n \in \mathbb{N}$

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(x) D^i(y) \tag{4}$$

This fact can be proven straightforwardly by induction.

**2.3.7 Remark.** Let $D \in \mathrm{Der}(B)$. When $n = 0$, we use the convention that $D^n : B \to B$ is the identity map on $B$, even when $D$ is the zero derivation.

**2.3.8 Lemma.** *Let $B$ be a ring and $D \in \mathrm{Der}(B)$. The set $\mathrm{Nil}(D)$ is a subring of $B$.*

*Proof.* First note that $\ker(D) \subset \mathrm{Nil}(D)$. We have already shown in Lemma 2.1.9 that $1 \in \ker(D), -1 \in \ker(D)$, so both are contained in $\mathrm{Nil}(D)$ as well. Next, let $a, b \in \mathrm{Nil}(D)$ be such that $D^m(a) = D^n(b) = 0$. We then have that $D^{\max(m,n)}(a + b) = 0$, and, from Remark 2.3.6, that $D^{m+n}(ab) = 0$. It follows that $\mathrm{Nil}(D)$ is closed under addition and multiplication, completing the proof. □

**2.3.9 Lemma.** *Let $B$ be a ring, let $D \in \mathrm{LND}(B)$ and let $g \in \ker(D)$. Then $gD \in \mathrm{LND}(B)$.*

*Proof.* Let $b \in B$. We note that since $g \in \ker(D)$, $gD(b) = D(gb)$. Also note that $g^n \in \ker(D)$ for all $n \in \mathbb{N}$. It then follows inductively that

$$(gD)^n(b) = D^n(g^n b) = g^n D^n(b) \qquad \text{for all } n \in \mathbb{N}.$$

For $n$ large enough, $D^n(b) = 0$ so $(gD)^n(b) = 0$. We conclude that $gD \in \mathrm{LND}(B)$. □

The following proposition can be found in [6] as Principle 4.

**2.3.10 Proposition.** *Let $I \lhd R$, where $R$ is a ring, and suppose $D \in \mathrm{Der}(R)$.*

a. *There exists at most one set map* $D_I : R/I \to R/I$ *satisfying*

$$D_I(f + I) = D(f) + I \quad \forall f \in R$$

b. *The map* $D_I$ *exists if and only if* $D(I) \subset I$.

c. *If* $D(I) \subset I$ *then* $D_I \in \mathrm{Der}(R/I)$.

d. *If* $D(I) \subset I$ *and* $D \in \mathrm{LND}(R)$, *then* $D_I \in \mathrm{LND}(R/I)$.

*Proof.* We begin by proving (b). To prove ($\Leftarrow$), let $f, f' \in R$ be such that $f + I = f' + I$. Then there exists $i \in I$ such that $f' = f + i$. We then have that $D(f') + I = D(f + i) + I = D(f) + I$ where the last equality follows since $D(I) \subset I$. This shows that $D_I$ is a well-defined map.

We prove ($\Rightarrow$) by contradiction. Assume the map $D_I$ exists and that $D(I) \not\subset I$. It follows that there exists $i \in I$ such that $D(i) \notin I$. But then we have $0_{R/I} = D_I(0 + I) = D_I(i + I) = D(i) + I \neq 0_{R/I}$ since $D(i) \notin I$, a contradiction.

It is clear that if $D_I$ exists, then it is unique, proving (a).

For (c), we note that $D_I$ exists by part (b), and leave the straightforward verification that $D_I$ is a derivation to the reader.

Finally for part (d), suppose $D : R \to R$ is locally nilpotent. Again, $D_I$ is well-defined by (b). Since $D$ is locally nilpotent, for any $r \in R$, there exists $n \in \mathbb{N}$ such that $D^n(r) = 0$. The same value of $n$ ensures that $D_I^n(r + I) = 0 + I = 0_{R/I}$ and so $D_I$ is locally nilpotent as well. □

The following proposition can be found in [3] as Exercise 1.15 (d).

**2.3.11 Proposition.** *Let* $B$ *be a ring, let* $D \in \mathrm{LND}(B)$ *and let* $A = \ker(D)$. *Then, given a multiplicatively closed subset* $S$ *of* $A$, *the map* $S^{-1}D : S^{-1}B \to S^{-1}B$ *given by*

$$S^{-1}D(\frac{x}{s}) = \frac{D(x)}{s}, \quad x \in B, s \in S \tag{5}$$

*is a locally nilpotent derivation of* $S^{-1}B$ *and coincides with the derivation defined in Definition* 2.2.1. *Moreover,* $\ker(S^{-1}D) = S^{-1}A$.

*Proof.* Note first that the proposition is trivial if $0 \in S$, since in that case $S^{-1}A = S^{-1}B = 0$. As such, we may assume henceforth that $0 \notin S$. As $S \subset A$, the map $S^{-1}D : S^{-1}B \to S^{-1}B$ as defined in Equation (5) coincides with the derivation defined in Definition 2.2.1. It follows that $S^{-1}D \in \mathrm{Der}(S^{-1}B)$. For any $x \in B$ and $s \in S$ we have

$$(S^{-1}D)^n(\tfrac{x}{s}) = \frac{D^n(x)}{s} \quad \text{for all } n \geq 0,$$

so $S^{-1}D$ is locally nilpotent, since $D$ is.

For the second part, let $\frac{a}{s} \in S^{-1}A$, where $a \in A, s \in S$. Then $S^{-1}D(\frac{a}{s}) = \frac{D(a)}{s} = 0_{S^{-1}A}$ since $a \in \ker(D)$, and so $S^{-1}A \subset \ker(S^{-1}D)$.

For the other inclusion, suppose $\frac{x}{s} \in \ker(S^{-1}D)$, $x \in B, s \in S$. Then $\frac{D(x)}{s} = \frac{0}{1}$ and so there exists $s' \in S$ such that $s'D(x) = 0$. Since $s' \in S \subset \ker(D)$, $s' \in \ker(D)$ and so $D(s'x) = 0$. Since $\frac{x}{s} = \frac{s'x}{s's}$, $s'x \in A$ and $s's \in S$, we have $\frac{x}{s} \in S^{-1}A$.

□

### 2.3.2 The Induced Degree Function of a Locally Nilpotent Derivation

**2.3.12 Definition.** Let $B$ be a $\mathbb{Q}$-algebra and let $D \in \mathrm{LND}(B)$. We define $\zeta_D$, the *exponential map associated to D*, as follows:

$$\zeta_D : B \to B[T], \quad b \mapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b) T^n \tag{6}$$

The following proposition can be found in [3] as Theorem 3.3.

**2.3.13 Proposition.** *Let $B$ be a $\mathbb{Q}$-algebra. Let $D \in \mathrm{LND}(B)$, and let $A = \ker(D)$. The exponential map $\zeta_D$ defined above is an injective $A$-homomorphism.*

*Proof.* We first show that $\zeta_D$ is an $A$-homomorphism. It is clear that for all $a, b \in B$ we have that $\zeta_D(a + b) = \zeta_D(a) + \zeta_D(b)$ and $\zeta_D(1) = 1$. So $\zeta_D$ preserves addition and maps $1_B$ to $1_{B[T]}$. Next, let $b_1, b_2 \in B$. We have to show that

$$(\sum_{i \in \mathbb{N}} \frac{1}{i!} D^i(b_1) T^i)(\sum_{j \in \mathbb{N}} \frac{1}{j!} D^j(b_2) T^j) = \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b_1 b_2) T^n. \tag{7}$$

It suffices to show that the coefficients of $T^n$ on each side of (7) match for all $n \in \mathbb{N}$. Now, starting from the left hand side, for each $n \in \mathbb{N}$ the coefficient of $T^n$ is

$$\sum_{i+j=n} \frac{1}{i!j!} D^i(b_1) D^j(b_2) = \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!j!} D^i(b_1) D^j(b_2) = \frac{1}{n!} D^n(b_1 b_2) \qquad (8)$$

which is exactly the coefficient of $T^n$ from the right hand side of (7). (Note that we used Remark 2.3.6 in the second equality of (8).) It follows that $\zeta_D$ is a multiplicative map.

Since $A = \ker(D)$ embeds both in $B$ and $B[T]$, to show that $\zeta_D$ is an $A$-homomorphism, we must show that $\forall a \in A, \zeta_D(a) = a$. Given that $\forall a \in A, \forall n \geq 1, D^n(a) = 0$, it follows that $\zeta_D(a) = \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(a) T^n = \frac{1}{0!} D^0(a) T^0 = a$, as required.

To show injectivity, it suffices to show that $\zeta_D(b) = 0 \Rightarrow b = 0$ since $\zeta_D$ is an additive map. Observe next that $\zeta_D(b) = 0$ if and only if $\frac{1}{n!} D^n(b) T^n = 0$ for every $n \in \mathbb{N}$. In particular, when $n = 0$ we must have, $0 = \frac{1}{0!} D^0(b) T^0 = b$. Thus $\zeta_D(b) = 0$ implies $b = 0$ so $\zeta_D$ is an injective map.

□

**2.3.14 Definition.** Let $B$ be a ring and let $D \in \mathrm{LND}(B)$. For $b \in B$, $b \neq 0$, define $\deg_D(b) = \max\{n \in \mathbb{N} : D^n(b) \neq 0\}$. Define $\deg_D(0) = -\infty$. This defines a set map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$. Note that this map is not necessarily a degree function as defined in Definition 1.1.4.

Observe that $\ker(D) = \{b \in B : \deg_D(b) \leq 0\}$.

**2.3.15 Remark.** Suppose $\varphi : B \to B'$ is an injective ring homomorphism and $\deg : B' \to \mathbb{N} \cup \{-\infty\}$ is any degree function on $B'$, then $\deg \circ \varphi : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.

The following proposition can be found in [3] as Proposition 4.8.

**2.3.16 Proposition.** *Let $B$ be an integral domain of characteristic 0 and let $D \in \mathrm{LND}(B)$. Then $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.*

*Proof.* We first consider the case where $\mathbb{Q} \subset B$. If $\mathbb{Q} \subset B$, then $\zeta_D : B \to B[T]$ is defined and is an injective homomorphism of rings. Let $\deg_T : B[T] \to \mathbb{N} \cup \{-\infty\}$ be the standard degree function on $B[T]$. (Note that $\deg_T$ is a degree function since $B$ is an integral domain.) By the Remark 2.3.15, $\deg_T \circ \zeta_D$ is a degree function as well. Since $\deg_D = \deg_T \circ \zeta_D$, $\deg_D$ is a degree function.

Now consider the general case. Since $B$ is a nonzero ring of characteristic 0 and $\ker(D)$ is a subring of $B$, $\mathbb{Z} \subset \ker(D)$. Let $S = \mathbb{Z} \backslash \{0\}$. Then $S^{-1}D \in \mathrm{LND}(S^{-1}B)$. Since $\mathbb{Q} \subset S^{-1}B$, $S^{-1}B$ is a $\mathbb{Q}$-algebra and so $\deg_{S^{-1}D}$ is a degree function as a consequence of the special case proven above.

Since $B$ is an integral domain and $0 \notin S$ the localization $B \to S^{-1}B$ is injective. We have the following diagram:

$$\begin{array}{ccc} S^{-1}B & \xrightarrow{S^{-1}D} & S^{-1}B \\ \uparrow & & \uparrow \\ B & \xrightarrow{D} & B \end{array}$$

Injectivity implies that $D$ is a restriction of $S^{-1}D$ and it follows that $\deg_D$ is the restriction of $\deg_{S^{-1}D}$. Since $\deg_{S^{-1}D}$ is a degree function, its restriction $\deg_D$ must be as well.

□

**2.3.17 Definition.** Let $A \subset B$, where both $A$ and $B$ are integral domains. The ring $A$ is *factorially closed in* $B$ if:

$$\forall x, y \in B\backslash\{0\},\ \ xy \in A \Rightarrow x \in A \text{ and } y \in A$$

The following useful corollary is attributable to Rentschler. It can be found in [19] as Remark 2.

**2.3.18 Corollary.** *If $B$ is an integral domain of characteristic 0 and $D \in \mathrm{LND}(B)$, then* $\ker(D)$ *is factorially closed in $B$.*

*Proof.* Let $a, b \in B \setminus \{0\}$ be such that $ab \in \ker(D)$. Consider the degree function $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ as in Proposition 2.3.16. Since $0 = \deg_D(ab) = \deg_D(a) + \deg_D(b)$, it follows that $0 = \deg_D(a) = \deg_D(b)$ and hence that $a, b \in \ker(D)$. □

**2.3.19 Corollary.** *Let $B$ be an integral domain of characteristic zero and let $D \in$* $\mathrm{LND}(B)$. *Then,* *(i)* $B^* \subset \ker(D)$

*(ii) If* $\mathbf{k}$ *is a field included in* $B$ *then* $\mathbf{k} \subset \ker(D)$.

*Proof.* For (i), let $b \in B^*$ and write $1 = b^{-1}b$. Since $1 \in \ker(D)$ and $\ker(D)$ is factorially closed by Corollary 2.3.2, we have that $b \in \ker(D)$ as required.

For (ii), let $\mathbf{k} \subset B$ and let $k \in \mathbf{k}$. If $k = 0$, then $k \in \ker(D)$ is clear, and if $k \neq 0$, then $k \in B^*$ and the result follows from part (i). □

**2.3.20 Remark.** Let $B$ be an integral domain of characteristic 0, let $b \in B$, let $D \in \text{LND}(B)$ and let $n \in \mathbb{N}$. Then it is easy to see that

$$\deg_D\left(D^n(b)\right) = \begin{cases} \deg_D(b) - n, & \text{if } D^n(b) \neq 0, \\ -\infty, & \text{if } D^n(b) = 0. \end{cases}$$

In particular note that $\deg_D\left(D^n(b)\right) \leq \deg_D(b) - n$ is true in all cases.

The following two Corollaries will be used regularly throughout the thesis. They are both slight generalizations of Corollary 1.20 in [6].

**2.3.21 Corollary.** *Let* $B$ *be an integral domain of characteristic 0, let* $b \in B$*, let* $D \in \text{LND}(B)$ *and let* $n \geq 1$*. If* $b$ *divides* $D^n(b)$*, then* $D^n(b) = 0$.

*Proof.* We assume that $b$ is nonzero, otherwise the result is trivial. Since $b$ divides $D^n(b)$, there exists $x \in B$ such that $D^n(b) = bx$. Suppose, by contradiction, that $D^n(b) \neq 0$; then $x \neq 0$.

Consider the degree function $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$. By Proposition 2.3.16 we have that $\deg_D(D^n(b)) = \deg_D(bx) = \deg_D(b) + \deg_D(x)$ where $\deg_D(D^n(b))$, $\deg_D(b)$, $\deg_D(x)$ are all natural numbers. Also, $\deg_D(D^n(b)) = \deg_D(b) - n$ so $\deg_D(b) - n = \deg_D(b) + \deg_D(x)$. This implies that $\deg_D(x)$ is a negative number which is impossible. □

**2.3.22 Corollary.** *Let* $B$ *be an integral domain of characteristic 0, let* $b \in B$*, let* $D \in \text{LND}(B)$*. If* $0 \leq m < n$ *and* $D^m(b)$ *divides* $D^n(b)$*, then* $D^n(b) = 0$.

*Proof.* This follows from Corollary 2.3.21. □

### 2.3.3 Absolute Degree

**2.3.23 Definition.** A ring $B$ is said to be *rigid* if $\mathrm{LND}(B) = \{0\}$.

**2.3.24 Definition.** Let $B$ be an integral domain containing $\mathbb{Q}$ and let $b \in B$. For each $D \in \mathrm{LND}(B)$, let $\deg_D$ be defined as in Definition 2.3.14. The *absolute degree* of $b$, denoted $|b|_B$ is defined as follows:

- If $B$ is rigid, then $|b|_B = \infty$ for all $b \in B \setminus \{0\}$ and $|0|_B = -\infty$.
- If $B$ is not rigid then

$$|b|_B = \min\left\{ \deg_D(b) \,:\, D \in \mathrm{LND}(B), D \neq 0 \right\}. \tag{9}$$

**2.3.25 Remark.** If $B$ is not rigid, then

$$|\cdot|_B : B \to \mathbb{N} \cup \{-\infty\}$$

is a set map, but we do not claim that $|\cdot|_B$ defines a degree function as defined in Definition 1.1.4.

## 2.4 The Degree of a Derivation

**2.4.1 Definition.** Let $B$ be an integral domain, $G$ a totally ordered abelian group, and $\deg : B \to G \cup \{-\infty\}$ a degree function. Given $D \in \mathrm{Der}(B)$, define the non-empty set $U \subset G \cup \{-\infty\}$ as follows:

$$U = \left\{ \deg(D(b)) - \deg(b) \,:\, b \in B \setminus \{0\} \right\}.$$

If $U$ has a greatest element $u$, we say that $\deg(D)$ exists and is equal to $u$.

**2.4.2 Remark.** We note that $\deg(D) = -\infty$ if and only if $D = 0$.

**2.4.3 Definition.** Let $B, G$ and deg be as in Definition 2.4.1 and let $D \in \mathrm{Der}(B)$. Define the *defect function associated to* $D$, $\mathrm{def}_D : B \to G \cup \{-\infty\}$, as follows:

$$\operatorname{def}_D(b) = \begin{cases} \deg(D(b)) - \deg(b) & \text{if } b \neq 0 \\ -\infty & \text{if } b = 0 \end{cases}$$

We simply write $\operatorname{def}(b)$ instead of $\operatorname{def}_D(b)$ when it is obvious from the context which derivation we are specifying.

The following lemma can be found in [18] as Lemma 2.2.3.

**2.4.4 Lemma.** *Let* $B, G$ *and* deg *be as in Definition* 2.4.1. *Let* $D \in \operatorname{Der}(B)$ *and let* $\operatorname{def}_D : B \to G \cup \{-\infty\}$ *be the defect function of* $D$. *Then,*

*a.* $\operatorname{def}_D(fg) \leq \max\{\operatorname{def}_D(f), \operatorname{def}_D(g)\}$ *for all* $f, g \in B$.

*b. If* $f_1, f_2, \cdots, f_n \in B$ *satisfy* $\deg(\sum_{i=1}^{n} f_i) = \max_{1 \leq i \leq n}\{\deg(f_i)\}$, *then*

$$\operatorname{def}_D(\sum_{i=1}^{n} f_i) \leq \max_{1 \leq i \leq n}\{\operatorname{def}_D(f_i)\}$$

*Proof.* For (a), we may assume that $f$ and $g$ are nonzero. We have

$$\begin{aligned} \operatorname{def}_D(fg) &= \deg(D(fg)) - \deg(fg) \\ &= \deg(fD(g) + gD(f)) - (\deg(f) + \deg(g)) \\ &\leq \max\{\deg(fD(g)), \deg(gD(f))\} - (\deg(f) + \deg(g)) \\ &\leq \max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} - (\deg(f) + \deg(g)) \end{aligned}$$

If $\max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} = \deg(f) + \deg(D(g))$, it follows that $\operatorname{def}_D(fg) \leq \deg(D(g)) - \deg(g) = \operatorname{def}_D(g)$.

If $\max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} = \deg(g) + \deg(D(f))$, it follows that $\operatorname{def}_D(fg) \leq \deg(D(f)) - \deg(f) = \operatorname{def}_D(f)$. Either way, $\operatorname{def}_D(fg) \leq \max\{\operatorname{def}_D(f), \operatorname{def}_D(g)\}$. This completes the proof of (a).

For (b), we have

$$\begin{aligned}
\operatorname{def}_D\Big(\sum_{i=1}^n f_i\Big) &= \deg\Big(D\Big(\sum_{i=1}^n f_i\Big)\Big) - \deg\Big(\sum_{i=1}^n f_i\Big) \\
&= \deg\Big(\sum_{i=1}^n D(f_i)\Big) - \max_{1\le i\le n}\{\deg(f_i)\} \\
&\le \max_{1\le i\le n}\{\deg(D(f_i))\} - \max_{1\le i\le n}\{\deg(f_i)\} \\
&\le \max_{1\le i\le n}\{\operatorname{def}_D(f_i)\}
\end{aligned}$$

□

The following can be found in [18] as Lemma 2.2.4.

**2.4.5 Lemma.** *Let $B$ be an integral domain and $0 \notin S$ a multiplicative set in $B$. Let* $\deg : S^{-1}B \to G \cup \{-\infty\}$ *and* $\deg : B \to G \cup \{-\infty\}$ *be degree functions where the former degree function restricts to the latter. Let $D \in \mathrm{Der}(B)$ and let $S^{-1}D : S^{-1}B \to S^{-1}B$ be its localization as defined in Definition* 2.2.1*. Then* $\deg(D)$ *exists if and only if* $\deg(S^{-1}D)$ *exists. Moreover, if* $\deg(D)$ *and* $\deg(S^{-1}D)$ *exist, then* $\deg(D) = \deg(S^{-1}D)$*.*

*Proof.* Consider the following two sets:

$$\begin{aligned}
U &= \{\ \deg(D(f)) - \deg(f)\ :\ f \in B \setminus \{0\}\ \} \\
U' &= \{\ \deg(S^{-1}D(f)) - \deg(f)\ :\ f \in S^{-1}B \setminus \{0\}\ \}
\end{aligned}$$

Clearly $U \subset U'$. To prove the result it suffices to show that for each $u' \in U'$, there exists an element $u \in U$ such that $u' \le u$. Let $\delta : S^{-1}B \to G \cup \{-\infty\}$ denote the defect function of $S^{-1}D$. By Lemma 2.4.4, for every $x, y$ in $S^{-1}B$ we have $\delta(xy) \le \max\{\delta(x), \delta(y)\}$. Let $u' \in U'$. Then $u' = \delta(\frac{x}{s})$ for some nonzero $x \in B$ and some $s \in S$. We have $\delta(\frac{x}{s}) \leq \max\{\delta(x), \delta(\frac{1}{s})\}$, where $\delta(x) \in U$ and

$$\begin{aligned}\delta\left(\frac{1}{s}\right) &= \deg\left(S^{-1}D\left(\frac{1}{s}\right)\right) - \deg\left(\frac{1}{s}\right)\\ &= \deg\left(\frac{1}{s^2}D(s)\right) - \deg\left(\frac{1}{s}\right)\\ &= \deg(D(s)) - \deg(s)\\ &= \delta(s) \in U\end{aligned}$$

so there exists $u \in U$ such that $u' \leq u$. □

We require the following result established by Levi in 1942 [13].

**2.4.6 Lemma.** *An abelian group $G$ can be totally ordered if and only if it is torsion free.*

The following result is well known. It can be found in [18] as Lemma 1.3.9.

**2.4.7 Lemma.** *Let $G$ be a torsion-free abelian group and let $B$ be a $G$-graded integral domain. Then,*

- *a. If $0 \neq h \in B$ is homogeneous and $h = fg$ with $f, g \in B$, then both $f$ and $g$ are homogeneous.*
- *b. If $\mathbf{k}$ is a field that is a subring of $B$, then $\mathbf{k} \subset B_0$.*

*Proof.* For part (a), since $G$ is torsion-free, by Lemma 2.4.6, we can choose a total ordering for $G$.

Let $G_+ = \{\, g \in G \,:\, g \geq 0 \,\}$ and let $f = \sum_{i \in G} f_i$ be the homogeneous decomposition of $f \in B \setminus \{0\}$. Define the map $\omega : B \setminus \{0\} \to G_+$ as follows:

$$\omega(f) = \max\{\, i \in G \,:\, f_i \neq 0 \,\} - \min\{\, i \in G \,:\, f_i \neq 0 \,\}$$

Note that $f$ is homogeneous if and only if $\omega(f) = 0$ and that for all $f, g \in B \setminus \{0\}$, $\omega(fg) = \omega(f) + \omega(g)$ (where we use that $B$ is an integral domain). Now if $h = fg$ is homogeneous where $f, g \in B \setminus \{0\}$, we have $0 = \omega(h) = \omega(fg) = \omega(f) + \omega(g)$ from which we conclude that $\omega(f) = \omega(g) = 0$ since $\mathrm{im}(\omega) \subset G_+$. This completes (a).

For (b), let $k \in \mathbf{k}$. By Proposition 1.2.10, $B_0$ is a subring of $B$ so $1, 0, -1 \in B_0$. We may assume henceforth that $k \neq 1, 0, -1$. Since $1 = kk^{-1}$, $k$ and $k^{-1}$ must be homogeneous by part (a). This shows that every element of $\mathbf{k}$ is homogeneous.

Now, for every $k \in \mathbf{k} \setminus \{-1\}$, the element $1 + k \in \mathbf{k}$ is homogeneous, from which it follows that $k \in B_0$. Since we already know $-1 \in B_0$, we have that $\mathbf{k} \subset B_0$. □

The following can be found in [18] as Proposition 2.2.5.

**2.4.8 Lemma.** *Let $G$ be a totally ordered abelian group and let $B$ be a $G$-graded integral domain such that $B = \mathbf{k}[h_1, ..., h_n]$ where $\mathbf{k}$ is a field of characteristic zero and the $h_i$ are homogeneous elements of $B$. Let* $\deg : B \to G \cup \{-\infty\}$ *be the degree function determined by the grading. Then, for all $D \in \operatorname{Der}_{\mathbf{k}}(B)$, $\deg(D)$ exists and*

$$\deg(D) = \max \left\{ \operatorname{def}_D(h_i) \, : \, 1 \leq i \leq n \right\}.$$

*Proof.* Let $D \in \operatorname{Der}_{\mathbf{k}}(B)$ and let $M = \max\{\operatorname{def}(h_i)\}$. We show that $\operatorname{def}(b) \leq M$ for all $b \in B$. We begin by proving the result for monomials, then generalize to homogeneous elements, followed by arbitrary elements.

Recall that for all $b_1, b_2 \in B$, $\operatorname{def}(b_1 b_2) \leq \max\{\operatorname{def}(b_1), \operatorname{def}(b_2)\}$, by Lemma 2.4.4 (a). Also note that $\operatorname{def}(\lambda) = -\infty$ for all $\lambda \in \mathbf{k}$ since $D$ is a $\mathbf{k}$-derivation. It then follows that

$$\operatorname{def}(\lambda h_1^{e_1} ... h_n^{e_n}) \leq M \quad \text{for } \lambda \in \mathbf{k} \text{ and } e_i \in \mathbb{N}. \tag{10}$$

Equation (10) gives the result for monomials.
Next, observe that if $b_1, ..., b_m \in B$ satisfy $\deg(\sum_{i=1}^{m} b_i) = \max\{\deg(b_i)\}$, Lemma 2.4.4 (b) implies that

$$\operatorname{def}(\sum_{i=1}^{m} b_i) \leq \max\{\operatorname{def}(b_i)\} \tag{11}$$

We now show that for any homogeneous element $H$ in $B$, $\operatorname{def}(H) \leq M$. Since $H$ is homogeneous, we can find a finite collection of homogeneous elements $\{b_i\}$ such that $H = \sum_{i=1}^{m} b_i$ where the $b_i$ have form $\lambda h_1^{e_1} ... h_n^{e_n}$ and $\deg(H) = \deg(b_i)$ for every $i$. (Lemma 2.4.7 gives the result that all elements of $\mathbf{k}$ are homogeneous, which is required for the $b_i$ to be homogeneous.) Using (10), $\operatorname{def}(b_i) \leq M$ for all $i$. Then, since the hypothesis for (11) is satisfied, it follows that $\operatorname{def}(H) \leq M$.

Lastly, for arbitrary $b \in B$, write $b = \sum H_i$ where the $H_i$ are homogeneous of distinct degrees. Again, since the hypothesis of (11) is satisfied, it follows that $\text{def}_D(b) \le \max\{\text{def}(H_i)\} \le M$.

□

**2.4.9 Corollary.** *Let* $B$, $G$ *and* deg *be as in Lemma* 2.4.8*, then for all* $D \in \text{LND}(B)$*,*

$$\deg(D) = \max\left\{ \text{def}(h_i) \,:\, 1 \le i \le n \right\}.$$

*Proof.* This is immediate since $\text{LND}(B) \subset \text{Der}_{\mathbf{k}}(B)$ by Corollary 2.3.19 (ii). □

**2.4.10 Corollary.** *Let* $G$ *be a totally ordered abelian group, let* $B$ *be a* $G$*-graded integral domain and let* $\deg : B \to G \cup \{-\infty\}$ *be the degree function determined by the grading. Assume* $B$ *is finitely generated as an algebra over a field of characteristic zero. Then for every* $D \in \text{LND}(B)$*,* $\deg(D)$ *exists.*

*Proof.* Since $B$ is a finitely generated $\mathbf{k}$-algebra, $B$ can be generated by a finite collection of homogeneous elements. The result now follows from Corollary 2.4.9. □

## 2.5 Homogenizing Derivations

In this section, we assume that $B$ is an integral domain equipped with a degree function $\deg : B \to G \cup \{-\infty\}$ where $G$ is a totally ordered abelian group.

**2.5.1 Definition.** The following process allows us to construct a graded ring which we denote as $\text{Gr}(B)$. This ring $\text{Gr}(B)$ is called the *associated graded ring of* $B$. It is an integral domain and is graded by the same group $G$.

For each $g \in G$, define

$$B_{\le g} = \{b \in B : \deg(b) \le g\} \text{ and } B_{<g} = \{b \in B : \deg(b) < g\}.$$

Next define $\text{Gr}(B) = \bigoplus_{g \in G} (B_{\le g}/B_{<g})$. Multiplication is defined as follows:

Let $b_1, b_2 \in B \setminus \{0\}$ and let $d_1 = \deg(b_1)$ and $d_2 = \deg(b_2)$. Then, $(b_1 + B_{<d_1})(b_2 + B_{<d_2}) = b_1 b_2 + B_{<(d_1+d_2)} \in \text{Gr}(B)_{(d_1+d_2)}$.

**2.5.2 Definition.** Let $b \in B$ and let $g = \deg(b)$. Define $\mathrm{gr}(b) \in \mathrm{Gr}(B)$ as follows:

$$\mathrm{gr}(b) = \begin{cases} b + B_{<g} & \text{if } b \neq 0 \\ 0 & \text{if } b = 0. \end{cases}$$

This defines a set map $\mathrm{gr} : B \to \mathrm{Gr}(B)$.

**2.5.3 Remark.** The map $\mathrm{gr} : B \to \mathrm{Gr}(B)$ maps every element of $B$ to a homogeneous element of $\mathrm{Gr}(B)$. In fact, the image of the map gr is precisely the set of homogeneous elements of $\mathrm{Gr}(B)$. Moreover, if $b \in B$, then $b$ and $\mathrm{gr}(b)$ have the same degree.

**2.5.4 Remark.** The map $\mathrm{gr} : B \to \mathrm{Gr}(B)$ preserves multiplication but does not preserve addition and hence is not a homomorphism of rings.

**2.5.5 Definition.** Let $D \in \mathrm{Der}(B)$ be such that $\deg(D)$ exists as defined in Definition 2.4.1. Define a derivation $\mathrm{gr}\, D : \mathrm{Gr}(B) \to \mathrm{Gr}(B)$ as follows:

If $D = 0$ then $\mathrm{gr}\, D = 0$. Otherwise suppose $D \neq 0$ and let $d = \deg(D)$. For each $g \in G$ and each nonzero $f \in B$ satisfying $\deg(f) = g$, let

$$(\mathrm{gr}\, D)(f + B_{<g}) = D(f) + B_{<(g+d)}$$

and let $(\mathrm{gr}\, D)(0) = 0$. Extend $\mathrm{gr}\, D$ linearly to all of $\mathrm{Gr}(B)$. It is straightforward to check that $\mathrm{gr}\, D : \mathrm{Gr}(B) \to \mathrm{Gr}(B)$ is a derivation.

**2.5.6 Definition.** Let $B = \bigoplus_{g \in G} B_g$ be a $G$-graded ring and let $D \in \mathrm{Der}(B)$. The derivation $D$ is *homogeneous* if there exists an element $h \in G$ such that $D(B_g) \subset B_{g+h}$ for every $g \in G$. When $D$ is nonzero, $h$ is unique and we say that $D$ is *homogeneous of degree* $h$. If $D = 0$, we define $\deg(D) = -\infty$. The set of all homogeneous locally nilpotent derivations is denoted as $\mathrm{HLND}(B)$.

A discussion of the results in the following proposition can be found in Section 1.1.5 of [6].

**2.5.7 Proposition.** *Let* $D \in \mathrm{Der}(B)$ *be such that* $\deg(D)$ *exists and let* $d = \deg(D)$. *Then*

*a.* $\mathrm{gr}\, D$ *is a homogeneous derivation of* $\mathrm{Gr}(B)$*, also of degree* $d$

*b.* $\operatorname{gr} D = 0$ *if and only if* $D = 0$

*c.* $\operatorname{gr}(\ker D) \subset \ker(\operatorname{gr} D)$

*d. If $D$ is locally nilpotent, then so is* $\operatorname{gr} D$.

*Proof.* The proposition is trivial if $D = 0$ so we assume throughout that $D$ is nonzero. We have already mentioned that $\operatorname{gr} D$ is a derivation. It is homogeneous, as by definition $\operatorname{gr} D$ maps $\operatorname{Gr}(B)_g$ into $\operatorname{Gr}(B)_{g+d}$ for every $g \in G$. The fact that $\operatorname{gr} D$ sends $\operatorname{Gr}(B)_g$ into $\operatorname{Gr}(B)_{g+d}$ also implies that $\operatorname{gr} D = 0$ or that $\deg(\operatorname{gr} D) = d$. As such, the fact that $\operatorname{gr} D$ has degree $d$ will follow from the proof of (b).

To prove (b), we note first that ($\Leftarrow$) holds by definition. For ($\Rightarrow$), suppose $D$ is nonzero of degree $d \in G$. Then, since $\deg(D)$ exists and is equal to $d$, $d$ is the greatest element of the collection

$$U = \left\{ \deg(D(f)) - \deg(f) \,:\, f \in B \setminus \{0\} \right\}.$$

As such, there exists nonzero $f \in B$ such that $\operatorname{def}(f) = d$. For any such $f$, let $g = \deg(f)$. Then $\deg(D(f)) = \deg(f) + \operatorname{def}(f) = g + d$, so $(\operatorname{gr} D)(f + B_{<g}) = D(f) + B_{<g+d} \neq 0$, showing that $\operatorname{gr} D \neq 0$. This completes the proof of part (b) and that of (a) as well.

For part (c), let $f \in \ker D$ and suppose $f$ is nonzero and has degree $g$. We must show that $(\operatorname{gr} D)(\operatorname{gr} f) = 0$. Indeed, $(\operatorname{gr} D)(\operatorname{gr} f) = (\operatorname{gr} D)(f + B_{<g}) = D(f) + B_{<(g+d)} = 0$ since $f \in \ker D$.

Lastly we prove part (d). Suppose $0 \neq D \in \operatorname{LND}(B)$. Let $a \in B \setminus \{0\}$, and let $g = \deg(a)$. We then have $\operatorname{gr}(a) \in B_{\leq g}/B_{<g}$.

Let $n$ be such that $D^n(a) = 0$. Observing first that $(\operatorname{gr} D)(\operatorname{gr}(a)) \in \{0, \operatorname{gr}(D(a))\}$, it follows by induction that $(\operatorname{gr} D)^m(\operatorname{gr}(a)) \in \{0, \operatorname{gr}(D^m(a))\}$ for all $m \geq 1$. It follows that for $m$ large enough, $\operatorname{gr}(D^m(a)) = 0$ and so we conclude that $(\operatorname{gr} D)^m(\operatorname{gr}(a)) = 0$, showing that $(\operatorname{gr} D) \in \operatorname{LND}(\operatorname{Gr}(B))$. □

**2.5.8 Remark.** Let $B$ be a $G$-graded integral domain where $G$ is totally ordered, let $\deg : B \to G \cup \{-\infty\}$ be the degree function determined by the grading and let $\operatorname{Gr}(B)$ be the associated graded ring of $B$ as defined in Definition 2.5.1. There exists an isomorphism $\varphi : B \to \text{Gr}(B)$ and moreover, given a derivation $D \in \text{Der}(B)$ such that $\deg(D)$ exists, the map $\tilde{D} : B \to B$ defined by $\tilde{D} = \varphi^{-1}\,\text{gr}(D)\varphi$ is a homogeneous derivation satisfying $\deg(\tilde{D}) = \deg(\text{gr}(D)) = \deg(D)$.

## 2.6 The Makar-Limanov and Derksen Invariants

The Makar-Limanov invariant was introduced in 1996 by Leonid Makar-Limanov in order to show that Russell's cubic $\mathbb{C}[X,Y,Z,T]/\langle X+X^2Y+Z^2+T^3\rangle$ is not isomorphic to $\mathbb{C}^{[3]}$ as a $\mathbb{C}$-algebra [14]. One year later, Derksen introduced his own invariant and provided an alternative proof of the same result in [4]. In this short section we provide the definitions and some basic results. In Chapter 4, we will compute the Makar-Limanov and Derksen invariants of a few different rings and we will show that these rings are not polynomial rings over a field **k**.

**2.6.1 Definition.** Let $B$ be a ring of characteristic zero. The *Makar-Limanov invariant of* $B$, denoted $\text{ML}(B)$, is the intersection of all kernels of locally nilpotent derivations.

$$\text{ML}(B) = \bigcap_{D \in \text{LND}(B)} \ker(D) \tag{12}$$

**2.6.2 Remark.** Since $\ker(D)$ is a subring of $B$ for every $D$, it is clear that $\text{ML}(B)$ is a subring of $B$. If $\mathbf{k}$ is a field of characteristic zero and $B$ is a $\mathbf{k}$-domain then $\mathbf{k} \subset \text{ML}(B)$ by 2.3.19; when $\text{ML}(B) = \mathbf{k}$ we say that $B$ has *trivial* ML-invariant.

**2.6.3 Remark.** A ring $B$ is rigid if and only if $\text{ML}(B) = B$. (See Definition 2.3.23.)

**2.6.4 Example.** Let $B = \mathbb{C}[X,Y,Z]$, a polynomial ring in 3 variables. Then $\text{ML}(B) = \mathbb{C}$. We know, by Corollary 2.3.19, that since $\mathbb{C}$ is a field contained in $B$, $\mathbb{C} \subset \ker(D)$ for every $D \in \text{LND}(B)$. So $\mathbb{C} \subset \text{ML}(B)$. For the other inclusion, since $\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \in \text{LND}(B)$,

$$\text{ML}(B) \subset \ker(\frac{\partial}{\partial X}) \cap \ker(\frac{\partial}{\partial Y}) \cap \ker(\frac{\partial}{\partial Z}) = \mathbb{C}[Y,Z] \cap \mathbb{C}[X,Z] \cap \mathbb{C}[X,Y] = \mathbb{C}.$$

**2.6.5 Example.** A similar argument shows that if $B = \mathbf{k}^{[n]}$ where $\mathbf{k}$ is of characteristic 0 and $n \geq 1$, then $\mathrm{ML}(B) = \mathbf{k}$. Thus, it is clear that $B = \mathbf{k}^{[n]}$ is not rigid for $n \geq 1$.

**2.6.6 Definition.** Let $B$ be a $\mathbf{k}$-domain. The *Derksen invariant* of $B$, denoted $\mathrm{Derk}(B)$ is the subalgebra of $B$ generated by all elements included in the kernel of some $D$ in $\mathrm{LND}(B) \setminus \{0\}$. We can write,

$$\mathrm{Derk}(B) = \mathbf{k}[A] \text{ where } A = \bigcup_{D \in \mathrm{LND}\,(B) \setminus \{0\}} \ker(D).$$

We leave the following computation of the Derksen invariant of a polynomial ring over $\mathbf{k}$ as a simple exercise.

**2.6.7 Example.** Let $B = \mathbf{k}^{[n]}$, where $\mathbf{k}$ is of characteristic 0. If $n = 1$ then $\mathrm{Derk}(B) = \mathbf{k}$. If $n \geq 2$, then $\mathrm{Derk}(B) = B$.

We will demonstrate the case where $n = 2$. Let $B = \mathbf{k}^{[2]} = \mathbf{k}[X, Y]$. Then $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ are both locally nilpotent. Since $X \in \ker\left(\frac{\partial}{\partial Y}\right)$,$Y \in \ker\left(\frac{\partial}{\partial X}\right)$ (and $\mathbf{k}$ is contained in both kernels), it follows that $\mathrm{Derk}(B) = \mathbf{k}[X, Y] = B$.

# Chapter 3

# Extensions and Quasi-extensions

This chapter discusses extensions and quasi-extensions of derivations. Most of Section 3.1 is well known. Unless otherwise stated, we could not find references for the content of Sections 3.2-3.4 and believe it to be original research.

## 3.1 Generalities

Most of the results in this section are well known and we refer the reader to Sections 1.3 and 1.4 of [17]. For Proposition 3.1.8 and Lemma 3.1.9, we could not find a suitable reference but we believe the results are already known.

**3.1.1 Definition.** Let $A \subset B$ be rings, and suppose $(\delta, D) \in \mathrm{Der}(A) \times \mathrm{Der}(B)$ satisfies $D(a) = \delta(a)$ for all $a \in A$. Then we say that $D$ is an *extension* of $\delta$ and that $\delta$ *can be extended from* $A$ *to* $B$.

**3.1.2 Example.** Consider the polynomial ring $A[X] = A^{[1]}$ over a ring $A$, and let $\delta \in \mathrm{Der}(A)$. Given a polynomial $P = \sum_i a_i X^i \in A[X]$ (where $a_i \in A$ for all $i$), we write $P^{(\delta)} = \sum_i \delta(a_i) X^i \in A[X]$. Define the map $\hat{\delta} : A[X] \to A[X]$ by $\hat{\delta}(P) = P^{(\delta)}$ for all $P \in A[X]$. Then $\hat{\delta} \in \mathrm{Der}(A[X])$ is an extension of $\delta$ and $\hat{\delta}(X) = 0$. Note that $\ker(\hat{\delta}) = \ker(\delta)[X] = \ker(\delta)^{[1]}$.

The following lemma is well known. Part (a) can be found in [17] as Theorem 1.2.1, part (b) in [6] as Principle 6.

**3.1.3 Lemma.** *Consider the polynomial ring $A[X] = A^{[1]}$ over a ring $A$.*

*a. Given $\delta \in \operatorname{Der}(A)$ and $f \in A[X]$, there exists a unique extension $D \in \operatorname{Der}(A[X])$ of $\delta$ satisfying $D(X) = f$.*

*b. Assume that $A$ is an integral domain of characteristic zero, let $\delta \in \operatorname{Der}(A)$ and let $D \in \operatorname{Der}(A[X])$ be an extension of $\delta$. Then*

$$D \text{ is locally nilpotent} \iff \delta \text{ is locally nilpotent and } D(X) \in A.$$

*Proof.* For part (a), we begin by proving existence. Define the map $D : A[X] \to A[X]$ as $D = \hat{\delta} + f\frac{d}{dX}$, where $\hat{\delta}$ is defined as in Example 3.1.2. Observe that $D$ is a well-defined derivation since $\operatorname{Der}(A[X])$ is an $A[X]$-module and $\hat{\delta}, \frac{d}{dX} \in \operatorname{Der}(A[X])$. Clearly $D(X) = f$ and $D(a) = \hat{\delta}(a) = \delta(a)$ for all $a \in A$.

For uniqueness, suppose $D_1, D_2 \in \operatorname{Der}(A[X])$ are extensions of $\delta$ and that $D_1(X) = D_2(X) = f$. Note that $D_1 - D_2 \in \operatorname{Der}(A[X])$. We then have that, $D_1(a) = \delta(a) = D_2(a)$ for all $a \in A$ and so $A \subset \ker(D_1 - D_2)$. Next observe that $(D_1 - D_2)(X) = 0$ and so $X \in \ker(D_1 - D_2)$. Since $\ker(D_1 - D_2)$ is a subring of $A[X]$ containing both $A$ and $X$, it follows that $\ker(D_1 - D_2) = A[X]$. Thus, for all $p \in A[X], 0 = (D_1 - D_2)(p) = D_1(p) - D_2(p)$ and so $D_1 = D_2$.

For part (b) we begin with ($\Leftarrow$). Since $\delta$ is locally nilpotent and $D(a) = \delta(a)$ for all $a \in A$, it follows that $A \subset \operatorname{Nil}(D)$. Since $D(X) \in A$, there exists $n$ such that $\delta^n(D(X)) = 0$. Since $D$ is an extension of $\delta$, $D^{n+1}(X) = 0$ so $X \in \operatorname{Nil}(D)$. Since $\operatorname{Nil}(D)$ is a subring of $A[X]$ containing both $A$ and $X$, it follows that $\operatorname{Nil}(D) = A[X]$ and that $D$ is locally nilpotent.

For ($\Rightarrow$), since $D$ is locally nilpotent, $\delta$ obviously is as well. It suffices to show that if $D(X) \notin A$ then $D$ is not locally nilpotent.

Suppose $D(X) \notin A$ and let deg be the standard degree of a polynomial. Define $g = D(X)$ and consider $f = \sum_{i=0}^{k} a_i X^i \in A[X] \setminus A$ where $a_i \in A, a_k \neq 0$ and $k \geq 1$. Note that $\deg(f) = k$. We will prove that $\deg(D(f)) \geq k$ which will show that $D$ is not locally nilpotent. By Lemma 2.1.14, $D(f) = f^{(\delta)} + f' \cdot g$ where $f'$ is the derivative of $f$. We consider separately the cases where $\deg(g) \geq 2$ and $\deg(g) = 1$.

If $\deg(g) \geq 2$, then $\deg(D(f)) = \deg(f^{(\delta)} + f' \cdot g) \geq k+1$ since $f' \cdot g$ has degree at least $k+1$ (since $A$ is an integral domain of characteristic zero) and $f^{(\delta)}$ has degree at most $k$.

If $\deg(g) = 1$, $D(X) = g = bX + c$ where $b \in A \setminus \{0\}$ and $c \in A$. Suppose by contradiction that $\deg(D(f)) < \deg(f)$. It then follows that

$$\begin{aligned}\deg(f^{(\delta)} + f' \cdot g) \leq k-1 &\Rightarrow D(a_k) + k \cdot a_k \cdot b = 0\\ &\Rightarrow D(a_k) = -k \cdot a_k \cdot b\\ &\Rightarrow D(a_k) = 0\\ &\Rightarrow k \cdot a_k \cdot b = 0\end{aligned}$$

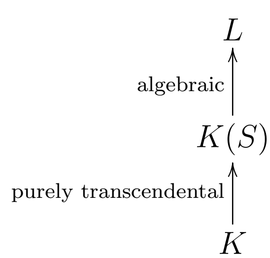
which is a contradiction since $k, a_k$ and $b$ are nonzero and $A$ is an integral domain. (Note that the first implication follows since the coefficient of $X^k$ of $D(f)$ must equal 0 and the third follows from Corollary 2.3.21.) We conclude that $\deg(D(f)) \geq \deg(f)$ in this case as well.

Having shown that $\deg(D(f)) \geq \deg(f)$ for all $f \in A[X] \setminus A$, we conclude $D$ is not locally nilpotent. □

The following result follows from Proposition 1.3.1 and Theorem 1.4.1 in [17].

**3.1.4 Lemma.** *Let $L/K$ be an extension of fields of characteristic zero. Then each derivation $\delta \in \mathrm{Der}(K)$ has at least one extension to a derivation $D \in \mathrm{Der}(L)$. Moreover, if $L$ is algebraic over $K$ then the extension $D$ is unique.*

*Proof.* We begin by noting that since every field extension has a transcendence basis, we can choose a transcendence basis $S$ such that we have the following diagram.




It is now clear that it suffices to prove the following two cases:

(1) The field extension $L/K$ is purely transcendental.

(2) The field extension $L/K$ is algebraic.

For (1), we let $L = K(S)$ and show that we can extend $\delta$ from $K$ to $K(S)$. Define $\hat{\delta} : K[S] \to K[S]$ by declaring that $\hat{\delta}(k) = \delta(k)$ for all $k \in K$ and that $\hat{\delta}(s) = 0$ for all $s \in S$ and extend $\hat{\delta}$ linearly to all of $K[S]$. It is straightforward to verify that $\hat{\delta}$ is derivation that extends $\delta$. We can further extend $\hat{\delta}$ from $K[S]$ to its fraction field $K(S)$ by the method described in Definition 2.2.1. We denote this derivation $\tilde{\delta}$. This completes (1).

For (2), we suppose $L/K$ is an algebraic field extension, and begin by proving that if an extension of $\delta$ exists, then it is unique. Suppose $D_1, D_2 \in \mathrm{Der}(L)$ are both extensions of $\delta$. Then $D_1 - D_2 \in \mathrm{Der}(L)$. Clearly $K \subset \ker(D_1 - D_2) \subset L$. Since by Lemmas 2.1.9 and 2.1.15, $\ker(D_1 - D_2)$ is a subfield that is algebraically closed in $L$, and $L$ is an algebraic extension of $K$, we have that $L \subset \ker(D_1 - D_2)$ and conclude that $L = \ker(D_1 - D_2)$. Hence $D_1(\ell) = D_2(\ell)$ for all $\ell \in L$ and so $D_1 = D_2$.

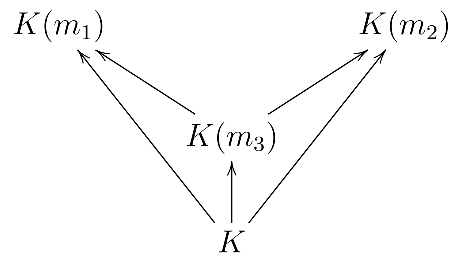
To prove existence, let $m \in L$, let $P \in K[X]$ be its minimal polynomial, and let $P'$ be the derivative of $P$. We will first extend $\delta \in \mathrm{Der}(K)$ to some $\delta_m \in \mathrm{Der}\big(K[m]\big)$. Since $K$ is a field of characteristic 0 and $P$ is irreducible, $P$ and $P'$ are relatively prime. Consequently, there exist $U, V \in K[X]$ such that $P^{(\delta)} = UP + VP'$.

Let $\Delta = -V\frac{d}{dX} + \hat{\delta}$ where $\hat{\delta} \in \mathrm{Der}(K[X])$ is defined as in 3.1.2. We then have, by a straightforward verification, that $\Delta \in \mathrm{Der}\big(K[X]\big)$ extends $\delta$ and satisfies $\Delta(X) = -V$. Then $\Delta(P) = P^{(\hat{\delta})} + P'\Delta(X) = P^{(\hat{\delta})} - VP' = UP$, showing that $\Delta(\langle P \rangle) \subset \langle P \rangle$. By Proposition 2.3.10, $\Delta$ induces a derivation $\Delta_P$ of $K[X]/\langle P \rangle = K[m]$. We define $\delta_m$ to be equal to this $\Delta_P$.

So $\delta$ has a unique extension $\delta_m \in \mathrm{Der}(K[m])$. Next, we show that if $m \in K(m_1) \cap K(m_2)$ then $\delta_{m_1}(m) = \delta_{m_2}(m)$. This shows that the members of the collection $\{\delta_m\}_{m \in L}$ agree on the intersection of their domains and hence define a map $D : L \to L$ that extends $\delta$.

Note that $K(m_1)$ and $K(m_2)$ are each finite extensions of $K$ and so $K(m_1) \cap K(m_2)$ is finite as well. By the Primitive Element Theorem, any finite extension of a field of characteristic zero is a simple extension, so there exists $m_3 \in K(m_1) \cap K(m_2)$ such that $K(m_3) = K(m_1) \cap K(m_2)$.

Let $m_3$ be such that $K(m_3) = K(m_1) \cap K(m_2)$. Observe that we have the following diagram of algebraic field extensions.




Now, since $K(m_1), K(m_2), K(m_3)$ are all algebraic over $K$, we can construct $\delta_{m_1} \in \mathrm{Der}(K(m_1)), \delta_{m_2} \in \mathrm{Der}(K(m_2)), \delta_{m_3} \in \mathrm{Der}(K(m_3))$ which are extensions of $\delta \in \mathrm{Der}(K)$. Note also that since $K(m_1)$ is algebraic over $K(m_3)$, we can extend $\delta_{m_3}$ to a derivation $\delta_{m_1'} \in \mathrm{Der}(K(m_1))$ which is an extension of $\delta$. Uniqueness of extensions implies that $\delta_{m_1} = \delta_{m_1'}$ and so $\delta_{m_1}$ must be an extension of $\delta_{m_3}$. By an identical argument, $\delta_{m_2}$ must also be an extension of $\delta_{m_3}$ and so we conclude that if $m \in K(m_3)$, $\delta_{m_1}(m) = \delta_{m_3}(m) = \delta_{m_2}(m)$.

Lastly, we show $D : L \to L$ is a derivation. Let $l_1, l_2 \in L$. It is clear that $l_1, l_2 \in K(l_1, l_2) = K(l_3)$ for some $l_3 \in L$ (again by the Primitive Element Theorem). Thus there exists $\delta_{l_3} \in \mathrm{Der}(K(l_3))$ which extends $\delta$. Also, since $D$ extends $\delta_{l_3}$, we have

$$D(l_1 + l_2) = \delta_{l_3}(l_1 + l_2) = \delta_{l_3}(l_1) + \delta_{l_3}(l_2) = D(l_1) + D(l_2).$$

A similar argument shows that $D$ also satisfies the Leibniz Rule, completing the proof.

□

**3.1.5 Remark.** Given $R \subset B$ where $R$ and $B$ are integral domains of chacteristic 0, it is not necessarily the case that every derivation of $R$ can be extended to $B$. We demonstrate this in the following example.

**3.1.6 Example.** Let $B = \mathbf{k}[X]$ and let $R = \mathbf{k}[X^2] \subset \mathbf{k}[X] = B$. Let $Y = X^2$ and observe that $R = \mathbf{k}[Y]$. The ring $R$ is clearly a polynomial ring in one variable. Next, let $\delta = \frac{d}{dY} \in \mathrm{Der}(R)$. We show that $\delta$ cannot be extended to $\mathbf{k}[X]$.

Suppose $D \in \mathrm{Der}(B)$ is an extension of $\delta$. Then

$$1 = \delta(Y) = D(X^2) = 2XD(X)$$

which would require that $D(X) = (2X)^{-1} \in B$, a contradiction.

However, there exist $D \in \mathrm{Der}(B)$ and $g \in B \setminus \{0\}$ such that $D(r) = g\delta(r)$ for all $r \in R$. For instance, let $D = \frac{d}{dX} \in \mathrm{Der}(B)$ and let $g = 2X$. Then, by a straightforward verification $D(r) = 2X\delta(r)$ for all $r \in R$.

Example 3.1.6 motivates the following definition.

**3.1.7 Definition.** Let $R \subset B$ be rings. If a pair $(\delta, D) \in \mathrm{Der}(R) \times \mathrm{Der}(B)$ satisfies the statement

$$\text{there exists } g \in B \setminus \{0\} \text{ such that } D(r) = g\delta(r) \text{ for all } r \in R$$

then we call $D$ a *quasi-extension* of $\delta$. One also says that the pair $(\delta, D)$ is a *quasi-extension*. We write $\mathrm{QE}(R, B)$ for the set of all pairs $(\delta, D) \in \mathrm{Der}(R) \times \mathrm{Der}(B)$ that are quasi-extensions.

**3.1.8 Proposition.** *Let $A \subset B \subset C$ be integral domains. Suppose $\delta \in \mathrm{Der}(A), \Delta \in \mathrm{Der}(B)$ and $D \in \mathrm{Der}(C)$. Suppose furthermore, that $\Delta$ is a quasi-extension of $\delta$ and that $D$ is a quasi-extension of $\Delta$. Then $D$ is a quasi-extension of $\delta$.*

*Proof.* Let $g \in C \setminus \{0\}$ be such that $D(b) = g\Delta(b)$ for every $b \in B$, and let $h \in B \setminus \{0\}$ be such that $\Delta(a) = h\delta(a)$ for all $a \in A$. We then have $D(a) = g\Delta(a) = gh\delta(a)$ for all $a \in A$ noting that $gh \in C \setminus \{0\}$ since $C$ is an integral domain. $\square$

**3.1.9 Lemma.** *Let $R \subset B$ be domains of characteristic zero such that $B$ is finitely generated as an $R$-algebra. Then each derivation $\delta \in \mathrm{Der}(R)$ has at least one quasi-extension $D \in \mathrm{Der}(B)$.*

*Proof.* By localization as defined in Definition 2.2.1, we can extend $\delta$ to a derivation $\delta'$ of $\mathrm{Frac}\, R$. By Lemma 3.1.4, we can then extend $\delta'$ to a derivation $\Delta$ of $\mathrm{Frac}\, B$; we obtain that $\Delta \in \mathrm{Der}(\mathrm{Frac}\, B)$ is an extension of $\delta$.

Note that for each $g \in B \setminus \{0\}$, $g\Delta \in \operatorname{Der}(\operatorname{Frac} B)$. We show that there exists $g \in B \setminus \{0\}$ such that $g\Delta(B) \subset B$, from which it follows that $g\Delta|_B \in \operatorname{Der}(B)$.

Let $B = R[b_1, ..., b_k]$ where $b_1, ..., b_k \in B$ are the generators of $B$ as an $R$-algebra. Let $\frac{n_i}{d_i} = \Delta(b_i) \in \operatorname{Frac}(B)$, where $n_i, d_i \in B$, $d_i \neq 0$, $1 \leq i \leq k$. Let $g = \prod_{i=1}^{k} d_i$. Since $\Delta(R) = \delta(R) \subset R \subset B$, we obviously have that $g\Delta(R) \subset B$. It is also clear that $g\Delta(b_i) \in B$ for every $i$ where $1 \leq i \leq k$. It follows that $g\Delta(B) \subset B$ and so we conclude that $g\Delta|_B \in \operatorname{Der}(B)$. Finally since $\Delta(r) = \delta(r)$ for all $r \in R$, it is clear that $g\Delta(r) = g\delta(r)$ for all $r \in R$, showing that $g\Delta|_B \in \operatorname{Der}(B)$ is a quasi-extension of $\delta$. □

We will see in Chapter 4 that quasi-extensions can be used for studying locally nilpotent derivations. The following lemma is useful as we try to determine the Makar-Limanov and Derksen invariants for particular classes of rings. It can be found in [6] as Lemma 5.38.

**3.1.10 Lemma.** *Let $R \subset B$ be domains of characteristic zero and let $\delta \in \operatorname{Der}(R)$ and $D \in \operatorname{Der}(B)$ be such that $D$ is a quasi-extension of $\delta$. If $D$ is locally nilpotent then so is $\delta$.*

*Proof.* Suppose that $D$ is locally nilpotent. Recall that $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function and that there exists $g \in B \setminus \{0\}$ such that $D(s) = g\delta(s)$ for all $s \in R$. Let $r \in R$. We shall prove that

$$\text{if } \deg_D(r) \geq 0 \text{ then } \deg_D(\delta(r)) < \deg_D(r). \tag{13}$$

Assume that $\deg_D(r) \geq 0$ (i.e., $r \neq 0$). The conclusion $\deg_D(\delta(r)) < \deg_D(r)$ is clear if $\delta(r) = 0$, so let us assume that $\delta(r) \neq 0$. Then $D(r) = g\delta(r) \neq 0$, so $\deg_D(D(r)) = \deg_D(r) - 1$ and consequently

$$\deg_D(r) - 1 = \deg_D(D(r)) = \deg_D(g\delta(r)) = \deg_D(g) + \deg_D(\delta(r)) \geq \deg_D(\delta(r))$$

(note that $\deg_D(g) \geq 0$, since $g \neq 0$). This proves (13). Since $\big(\deg_D(\delta^n(r))\big)_{n=0}^{\infty}$ is a sequence in $\mathbb{N} \cup \{-\infty\}$, (13) implies that there exists $n \geq 0$ such that $\deg_D(\delta^n(r)) = -\infty$. Thus $\delta^n(r) = 0$, which proves that $\delta$ is locally nilpotent. □

For the balance of this chapter, we turn our attention towards the following question.

**3.1.11 Question.** Given integral domains $R \subset B$, is it possible to describe all $(\delta, D) \in \mathrm{Der}(R) \times \mathrm{Der}(B)$ such that $D$ is a quasi-extension of $\delta$?

We shall consider this question in the case where $B$ is generated by one element as an $R$-algebra. We first consider general derivations, then consider locally nilpotent derivations.

## 3.2 Quasi-extensions where $B = R[x]$

From this point forward, we believe that all results, unless otherwise stated, are new.

In this section, we fix integral domains $R \subset B$ such that $\mathbb{Q} \subset R$. We also assume throughout this section that $B$ is generated by one element as an $R$-algebra, namely that $B = R[x]$. Consider the polynomial ring $R[X] = R^{[1]}$ and the $R$-homomorphism

$$\pi : R[X] \to B, \quad \pi(F) = F(x) \text{ for all } F \in R[X].$$

Let $\mathfrak{p} = \ker(\pi) \in \operatorname{Spec} R[X]$ and note that $\mathfrak{p} \cap R = \{0\}$. Also observe that $B = R^{[1]}$ if and only if $\mathfrak{p} = \{0\}$.

**3.2.1 Lemma.** *Given $\mathcal{D} \in \mathrm{Der}(R[X])$, there exists at most one $\overline{\mathcal{D}} \in \mathrm{Der}(B)$ such that the diagram*

$$\begin{array}{ccc} R[X] & \xrightarrow{\mathcal{D}} & R[X] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\overline{\mathcal{D}}} & B \end{array}$$

*commutes. Moreover, $\overline{\mathcal{D}}$ exists if and only if $\mathcal{D}(\mathfrak{p}) \subset \mathfrak{p}$.*

*Proof.* This follows from Proposition 2.3.10. □

**3.2.2 Definition.** We define a map $\tau : \mathrm{QE}(R, B) \to \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$ as follows. Given $(\delta, D) \in \mathrm{QE}(R, B)$, let $\tau(\delta, D) = (\delta, g, D(x))$ where $g$ is the unique element of $B \setminus \{0\}$ satisfying:

(i) if $\delta = 0$ then $g = 1$,

(ii) $D(r) = g\delta(r)$ for all $r \in R$.

Our goal in this section is to show that the map

$$\tau : \mathrm{QE}(R, B) \to \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$$

defined in Definition 3.2.2 is injective and to describe its image.

**3.2.3.** Let $K = \operatorname{Frac} R$, and note that any $\delta \in \mathrm{Der}(R)$ has a unique extension $\delta \in \mathrm{Der}(K)$ as described in Definition 2.2.1. (Note that we use the same symbol "$\delta$".) Given $F \in K[X]$ we may consider $F^{(\delta)}(X) \in K[X]$ and $F^{(\delta)}(x) \in \operatorname{Frac} B$.

Any triple $(\delta, g, h) \in \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$ determines a set map

$$\Delta_{(\delta,g,h)} : K[X] \to \operatorname{Frac} B, \quad F(X) \mapsto gF^{(\delta)}(x) + F'(x)h = \begin{vmatrix} F^{(\delta)}(x) & F'(x) \\ -h & g \end{vmatrix}$$

and a simple calculation shows that $\Delta = \Delta_{(\delta,g,h)}$ satisfies $\Delta(F+G) = \Delta(F) + \Delta(G)$ and $\Delta(FG) = \Delta(F)G(x) + F(x)\Delta(G)$ for all $F, G \in K[X]$.

Define $P(X) \in K[X]$ by stipulating that if $x$ is algebraic over $K$ then $P(X)$ is the minimal monic polynomial of $x$ over $K$; and that if $x$ is transcendental (or equivalently if $B = R^{[1]}$), then $P(X)$ is the zero polynomial. Recall that $\mathfrak{p}$ is an ideal of $R[X]$.

Note that for any $(\delta, g, h) \in \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$, the following conditions are equivalent:

(i) $\Delta_{(\delta,g,h)}(F) = 0$ for all $F \in \mathfrak{p}$,

(ii) $\Delta_{(\delta,g,h)}(P) = 0$.

We demonstrate the equivalence of (i) and (ii).

*Proof.* Throughout this proof we will write $\Delta$ as an abbreviation for $\Delta_{(\delta,g,h)}$.

To prove that (ii) $\Rightarrow$ (i), we note first that the case where $P$ is the zero polynomial is trivial and note that for any $F \in \mathfrak{p}$, one can view $F$ as an element of $K[X]$. Suppose (ii) holds and let $F \in \mathfrak{p} = \ker(\pi)$ where $\pi$ is the projection $R[X] \to B$. Since

$F(x) = \pi(F) = 0_B$ and $P(X)$ is the minimal monic polynomial of $x$ over $K$, it follows that $P$ divides $F$ (in $K[X]$).

Let $Q \in K[X]$ be such that $F = QP$ and note that if $F$ is nonzero, then $Q$ is nonzero. Observe that

$$\Delta(F) = \Delta(QP) = Q(x)\Delta(P) + P(x)\Delta(Q) = 0$$

since $\Delta(P) = 0$ by (ii) and $P(x) = 0$ since $P$ is the minimal polynomial of $x$. So (i) holds.

We now show that (i) $\Rightarrow$ (ii). Let $r \in R \setminus \{0\}$ be such that $rP \in R[X]$. Since $\pi(rP) = rP(x) = 0$, we have that $rP \in \mathfrak{p}$. By (i) we must have that $\Delta(rP) = 0$ from which it follows that $0 = \Delta(rP) = r\Delta(P) + P(x)\Delta(r) = r\Delta(P)$ which implies that $\Delta(P) = 0$ since $r$ is nonzero by assumption. $\square$

**3.2.4 Definition.** We shall say that a triple $(\delta, g, h) \in \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$ is *admissible* if it satisfies the condition "if $\delta = 0$ then $g = 1$" as well as the equivalent conditions (i) and (ii) of 3.2.3. We write $\mathrm{Adm}(R, B)$ for the set of all triples $(\delta, g, h) \in \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$ that are admissible. Given $\delta \in \mathrm{Der}(R)$, we write $\mathrm{Adm}_\delta(R, B)$ for the set of all pairs $(g, h) \in (B \setminus \{0\}) \times B$ such that $(\delta, g, h)$ is admissible.

**3.2.5 Proposition.** *The map* $\tau : \mathrm{QE}(R, B) \to \mathrm{Der}(R) \times (B \setminus \{0\}) \times B$ *is injective, and its image is* $\mathrm{Adm}(R, B)$.

*Proof.* Suppose that $(\delta_1, D_1), (\delta_2, D_2) \in \mathrm{QE}(R, B)$ have the same image by $\tau$, say $\tau(\delta_1, D_1) = (\delta, g, h) = \tau(\delta_2, D_2)$. Then $\delta_1 = \delta = \delta_2$, $D_1(x) = h = D_2(x)$, and $D_1(r) = g\delta(r) = D_2(r)$ for all $r \in R$. It follows that $D_1 = D_2$, showing that $\tau$ is injective.

Let $(\delta, D) \in \mathrm{QE}(R, B)$; then $\tau(\delta, D) = (\delta, g, h)$ where $h = D(x) \in B$, and where $g \in B \setminus \{0\}$ satisfies "$g = 1$ if $\delta = 0$" and $D(r) = g\delta(r)$ for all $r \in R$. Let $F \in \mathfrak{p}$, then

$$0 = D(0) = D(F(x)) = F^{(D)}(x) + F'(x)D(x) = gF^{(\delta)}(x) + F'(x)h = \Delta_{(\delta,g,h)}(F),$$

so $(\delta, g, h)$ satisfies condition (i) of 3.2.3 and consequently $(\delta, g, h) \in \mathrm{Adm}(R, B)$. This shows that $\mathrm{im}(\tau) \subset \mathrm{Adm}(R, B)$.

Let $(\delta, g, h) \in \mathrm{Adm}(R, B)$. Pick $G, H \in R[X]$ such that $\pi(G) = g$ and $\pi(H) = h$, and define $\mathcal{D} = G\hat{\delta} + H\frac{d}{dX} \in \mathrm{Der}(R[X])$ where $\hat{\delta}$ is defined as in Example 3.1.2. For any $F \in R[X]$, we have

$$\pi(\mathcal{D}(F)) = \pi\big(G\hat{\delta}(F) + F'(X)H\big) = gF^{(\delta)}(x) + F'(x)h = \Delta_{(\delta,g,h)}(F).$$

Since $(\delta, g, h)$ is admissible, this implies that $\pi(\mathcal{D}(F)) = 0$ for all $F \in \mathfrak{p}$; so $\mathcal{D}(\mathfrak{p}) \subset \mathfrak{p}$ and consequently $\overline{\mathcal{D}} \in \mathrm{Der}(B)$ is defined by Lemma 3.2.1. It is clear that $\overline{\mathcal{D}}(x) = h$ and that $\overline{\mathcal{D}}(r) = g\delta(r)$ for all $r \in R$; also note that the definition of $\mathrm{Adm}(R, B)$ implies that $g \in B \setminus \{0\}$ and that $g = 1$ if $\delta = 0$. So $(\delta, \overline{\mathcal{D}}) \in \mathrm{QE}(R, B)$ and $\tau(\delta, \overline{\mathcal{D}}) = (\delta, g, h)$, showing that $\mathrm{Adm}(R, B) \subset \mathrm{im}(\tau)$. □

**3.2.6 Corollary.** *Let* $\delta \in \mathrm{Der}(R)$*. Then*

$$\begin{array}{rcl} \mathcal{Q}_\delta \ : \ \mathrm{Adm}_\delta(R, B) & \longrightarrow & \mathrm{Der}(B) \\ (g, h) & \longmapsto & \text{the unique } D \in \mathrm{Der}(B) \text{ satisfying } D(x) = h \text{ and} \\ & & D(r) = g\delta(r) \text{ for all } r \in R \end{array}$$

*is a well-defined injective set map, and* $\mathrm{im}(\mathcal{Q}_\delta)$ *is the set of all quasi-extensions* $D \in \mathrm{Der}(B)$ *of* $\delta$*.*

*Proof.* Let $(g, h) \in \mathrm{Adm}_\delta(R, B)$. Then $(\delta, g, h) \in \mathrm{Adm}(R, B)$, so by Proposition 3.2.5 there exists a unique $D \in \mathrm{Der}(B)$ satisfying

$$D \text{ is a quasi-extension of } \delta \text{ and } \tau(\delta, D) = (\delta, g, h). \tag{14}$$

Now, for any $D \in \mathrm{Der}(B)$, condition (14) is equivalent to

$$D(x) = h \text{ and } D(r) = g\delta(r) \text{ for all } r \in R, \tag{15}$$

so there exists a unique $D \in \mathrm{Der}(B)$ satisfying condition (15), which shows that $\mathcal{Q}_\delta$ is well-defined.

For injectivity, suppose $(g, h)$ and $(g', h')$ are distinct elements of $\mathrm{Adm}_\delta(R, B)$ and let $D_1 = \mathcal{Q}_\delta(g, h), D_2 = \mathcal{Q}_\delta(g', h')$.

If $\delta = 0$ then $g = 1 = g'$ so we must have that $h \neq h'$. This implies that $D_1(x) \neq D_2(x)$ and hence that $D_1 \neq D_2$. Next, suppose $\delta \neq 0$. If $h \neq h'$, again we have $D_1(x) \neq D_2(x)$.

Finally, suppose $h = h'$ but $g \neq g'$. Since $\delta \neq 0$, there exists $r \in R \setminus \ker(\delta)$. For such an $r$, we have $(D_1 - D_2)(r) = (g - g')\delta(r) \neq 0$ since $g \neq g'$, $\delta(r) \neq 0$ and $B$ is an integral domain. Hence $(D_1 - D_2)(r) \neq 0$, which shows that $D_1 \neq D_2$ and that $\mathcal{Q}_\delta$ is injective.

We now show that $\mathrm{im}(\mathcal{Q}_\delta)$ is the set of quasi-extensions of $\delta$. Let $(g, h) \in \mathrm{Adm}_\delta(R, B)$; since $\mathcal{Q}_\delta(g, h)$ is the unique $D \in \mathrm{Der}(B)$ satisfying (14), $\mathcal{Q}_\delta(g, h) \in \mathrm{Der}(B)$ is a quasi-extension of $\delta$, showing that $\mathrm{im}(\mathcal{Q}_\delta)$ is included in the set of all quasi-extensions $D \in \mathrm{Der}(B)$ of $\delta$. Conversely, suppose that $D \in \mathrm{Der}(B)$ is a quasi-extension of $\delta$. Define $(g, h)$ by $\tau(\delta, D) = (\delta, g, h) \in \mathrm{Adm}(R, B)$; then $(g, h) \in \mathrm{Adm}_\delta(R, B)$ and $\mathcal{Q}_\delta(g, h) = D$. So all quasi-extensions $D \in \mathrm{Der}(B)$ of $\delta$ belong to $\mathrm{im}(\mathcal{Q}_\delta)$. □

**3.2.7 Remark.** Corollary 3.2.6 provides a satisfactory description of the set of all quasi-extensions $D \in \mathrm{Der}(B)$ of a given $\delta \in \mathrm{Der}(R)$.

**3.2.8 Lemma.** *Let $(\delta, D) \in \mathrm{QE}(R, B)$ and let $(\delta, g, h) = \tau(\delta, D)$. Then $D$ is locally nilpotent if and only if $\delta$ is locally nilpotent and $g, h \in \mathrm{Nil}(D)$.*

*Proof.* For ($\Rightarrow$), since $D$ is locally nilpotent $\delta$ is as well, by Lemma 3.1.10. It is obvious that $g, h \in \mathrm{Nil}(D)$ because $\mathrm{Nil}(D) = B$, completing the proof of ($\Rightarrow$).

For ($\Leftarrow$), since $h \in \mathrm{Nil}(D)$, and $D(x) = h$, $x \in \mathrm{Nil}(D)$ as well. Note that since $\delta$ is locally nilpotent, given any $r \in R$, there exists $k \geq 1$ such that $\delta^k(r) \in \mathrm{Nil}(D)$. It then suffices to prove for any $r \in R$ and $k \geq 1$, that $\delta^k(r) \in \mathrm{Nil}(D)$ implies $\delta^{k-1}(r) \in \mathrm{Nil}(D)$. It would then follow $\delta^0(r) = r \in \mathrm{Nil}(D)$ for any $r \in R$ and hence that $R \subset \mathrm{Nil}(D)$ which, together with $x \in \mathrm{Nil}(D)$ would prove the implication.

Let $k \geq 1$ be such that $\delta^k(r) \in \mathrm{Nil}(D)$. Since $g \in \mathrm{Nil}(D)$ and $\mathrm{Nil}(D)$ is a subring of $B$, $D(\delta^{k-1}(r)) = g\delta^k(r) \in \mathrm{Nil}(D)$ and hence $\delta^{k-1}(r) \in \mathrm{Nil}(D)$ as required. Since $R$ and $x$ are contained in $\mathrm{Nil}(D)$, it follows that $D$ is locally nilpotent. □

## 3.3 Applications to Algebraic Ring Extensions

Let $R$ be an integral domain containing $\mathbb{Q}$ and let $F(X) = a_n X^n + \cdots + a_1 X + a_0 \in R[X]$ (where $n \geq 1$, $a_i \in R$ and $a_n \neq 0$). Suppose that the principal ideal $\langle F \rangle$ of $R[X]$ is a prime ideal, and define $B = R[X]/\langle F \rangle$.

Note that the hypotheses of Section 3.2 are satisfied and that if we write $\mathfrak{p} = \ker \pi$ (as in Section 3.2) then $\mathfrak{p} = \langle F \rangle$.

**3.3.1 Proposition.** *Let $\delta \in \mathrm{Der}(R)$.*

a. *There exists exactly one $D \in \mathrm{Der}(B)$ satisfying*

$$D(x) = -F^{(\delta)}(x) \quad \textit{and} \quad D(r) = F'(x)\delta(r) \textit{ for all } r \in R. \tag{16}$$

b. *The derivation $D$ is a quasi-extension of $\delta$.*

c. *Suppose that $\delta$ is locally nilpotent and that $\delta^2(a_i) = 0$ for all $i \geq 0$. Then $D$ is locally nilpotent if and only if $\delta(a_i) = 0$ for all $i \geq 1$.*

*Proof.* The claim is trivial if $\delta = 0$, in which case $D = 0$ as well. Henceforth, we assume $\delta \neq 0$.

We begin with part (a). To prove the existence of $D$, we begin by verifying that $\Delta_{(\delta, F'(x), -F^{(\delta)}(x))}(F) = 0$. We shall write simply $\Delta = \Delta_{(\delta, F'(x), -F^{(\delta)}(x))}$. We have

$$\Delta(F) = F'(x)F^{(\delta)}(x) + F'(x)(-F^{(\delta)}(x)) = 0.$$

It follows that for any $G \in R[X]$,

$$\Delta(FG) = \Delta(F)G(x) + F(x)\Delta(G) = 0 \cdot G(x) + 0 \cdot \Delta(G) = 0,$$

so $(\delta, F'(x), -F^{(\delta)}(x))$ satisfies condition (i) of 3.2.3. So $(\delta, F'(x), -F^{(\delta)}(x)) \in \mathrm{Adm}(R, B)$ and hence $(F'(x), -F^{(\delta)}(x)) \in \mathrm{Adm}_\delta(R, B)$. By Corollary 3.2.6 we may define

$$D = \mathcal{Q}_\delta(F'(x), -F^{(\delta)}(x)) \in \mathrm{Der}(B).$$

It is clear that $D$ satisfies (16). Moreover, (16) determines $D(x)$ and $D|_R$, so the uniqueness of $D$ easily follows. This complete the proof of (a).

Since $D = \mathcal{Q}_\delta(F'(x), -F^{(\delta)}(x))$, part (b) follows from Corollary 3.2.6.

For part (c), suppose that $\delta$ is locally nilpotent and that $\delta^2(a_i) = 0$ for all $i \geq 0$. Note that we have

$$D(x) = -\sum_{i=0}^{n} \delta(a_i)x^i \quad \text{and} \quad D(r) = F'(x)\delta(r) \text{ for all } r \in R.$$

To prove ($\Rightarrow$), we argue by contrapositive. Suppose $\delta(a_i) \neq 0$ for some $i \geq 1$. We will show that $D \notin \mathrm{LND}(B)$.

We first claim that since $\delta(a_i) \neq 0$ for some $i \geq 1$, it follows that $D(x) \neq 0$. By contradiction, suppose $D(x) = 0$. It would then follow that $\sum_{i=0}^{n} \delta(a_i)X^i \in \ker(\pi) = \langle F \rangle$ and consequently that $\sum_{i=0}^{n} \delta(a_i)X^i = gF(X)$ for some $g \in R$ (since the degree of $F(X)$ is $n$ and the degree of $\sum_{i=0}^{n} \delta(a_i)X^i$ is at most $n$). This implies $a_i | \delta(a_i)$ for all $i$, which by Corollary 2.3.21 implies that $\delta(a_i) = 0$ for all $i$. This contradicts the assumption that $\delta(a_i) \neq 0$ for some $i$, so we have that $D(x) \neq 0$.

Next note that $\ker(\delta) \subset \ker(D)$. Observe also that

$$\begin{aligned}
D^2(x) &= -D(\sum_{i=0}^{n} \delta(a_i)x^i) \\
&= -\sum_{i=1}^{n} D(\delta(a_i)x^i) \\
&= -\sum_{i=1}^{n} \delta(a_i)D(x^i) \qquad \text{since } \delta(a_i) \in \ker(D) \\
&= -\sum_{i=1}^{n} \delta(a_i)ix^{i-1}D(x) \\
&= -D(x)\sum_{i=1}^{n} \delta(a_i)ix^{i-1} \qquad (17)
\end{aligned}$$

We show that $\sum_{i=1}^{n} \delta(a_i)ix^{i-1} \neq 0$.

By contradiction, suppose that $\sum_{i=1}^{n} \delta(a_i)ix^{i-1} = 0$. We would then have that $\sum_{i=1}^{n} \delta(a_i)iX^{i-1} = gF(X)$ for some $g \in R[X]$. Since the degree of $\sum_{i=1}^{n} \delta(a_i)iX^{i-1}$ is strictly less than the degree of $F(X)$, we must have $g = 0$. This implies that $\delta(a_i) = 0$ for all $i \geq 1$, contradicting the fact that $\delta(a_i) \neq 0$ for some $i \geq 1$.

By (17), we conclude that $D(x)$ divides $D^2(x)$. Since $D(x) \neq 0$ and $\sum_{i=1}^{n} \delta(a_i)ix^{i-1} \neq 0$ we have that $D^2(x) \neq 0$ by (17). By Corollary 2.3.22, it follows that $D$ is not locally nilpotent, which completes the proof of ($\Rightarrow$).

For ($\Leftarrow$), since $\delta(a_i) = 0$ for all $i \geq 1$ we have

$$D(x) = -\delta(a_0) \text{ and } D(r) = F'(x)\delta(r) \text{ for all } r \in R.$$

By Lemma 3.2.8, it suffices to show that $\delta(a_0)$ and $F'(x)$ belong to $\text{Nil}(D)$. By assumption, $\delta(a_0) \in \ker(\delta) \subset \text{Nil}(D)$.

Since $D(x) = -\delta(a_0)$, $x$ too is in $\text{Nil}(D)$. Similarly, all coefficients of $F'(x)$ are elements of $\ker(\delta)$ and hence belong to $\text{Nil}(D)$. Since $\text{Nil}(D)$ is a subring of $B$ containing $x$ along with the coefficients of $F'(x)$, it follows that $F'(x) \in \text{Nil}(D)$. $\square$

We can use the above proposition to show that certain rings are not rigid. The following corollary provides a set of examples.

**3.3.2 Corollary.** *Let $A$ be an integral domain containing $\mathbb{Q}$ and let $F = (aX + b) + a_1Y + ...a_nY^n \in A[X, Y]$ where $n \geq 1$, $a, b, a_1, ..., a_n \in A$ and $a_n \neq 0$.*

*Suppose that $\langle F \rangle$ is a prime ideal of $A[X, Y]$. Then $A[X, Y]/\langle F \rangle$ is not rigid.*

*Proof.* Since $\langle F \rangle$ is prime, $A[X, Y]/\langle F \rangle$ is an integral domain. Let $R = A[X]$ and observe that $\delta = \frac{d}{dX} \in \text{Der}(R)$ is nonzero and locally nilpotent. Then $F \in R[Y] = A[X, Y]$, $\delta(a_i) = 0$ for all $1 \leq i \leq n$, and $\delta^2(aX + b) = 0$. Setting $a_0 = aX + b$, it follows that the hypothesis of Proposition 3.3.1(c) is satisfied. By that result, $\delta$ has a quasi-extension $D \in \text{Der}(B)$ that is locally nilpotent. Since $\delta \neq 0$, we have that $D \neq 0$ and consequently $B$ is not rigid. $\square$

Proposition 3.3.1 also immediately yields the following result of Freudenburg and Moser-Jauslin, Lemma 4.1 of [7].

**3.3.3 Corollary.** *Suppose $B = R[X]/\langle X^n + r \rangle$ where $R$ is an integral domain containing $\mathbb{Q}$, $r \in R$, $n \geq 2$, and $\langle X^n + r \rangle$ is a prime ideal of $R[X]$. If $|r|_R \leq 1$, then $B = R[X]/\langle X^n + r \rangle$ is not rigid.*

**3.3.4 Remark.** In 3.3.2, if $A$ is a UFD, $a \neq 0$ and $\gcd(a, b, a_1, a_2, \cdots, a_n) = 1$ in $A$, then $\langle F \rangle$ is a prime ideal of $A[X, Y]$.

**3.3.5 Remark.** In 3.3.3, if $R$ is a UFD and $X^n + r$ is irreducible in $K[X]$ where $K = \text{Frac}\, R$, then $\langle X^n + r \rangle$ is a prime ideal of $R[X]$. In this regard, we include the following result, found as Theorem 8.1.6 in [12].

**3.3.6 Theorem.** *Let $K$ be a field, let $n \geq 1$ and let $a \in K$. Then, $X^n - a$ is irreducible in $K[X]$ if and only if $a \notin K^p$ for all primes $p$ dividing $n$ and $a \notin -4K^4$ whenever $4$ divides $n$.*

# Chapter 4

# Rings with Non-Trivial Makar-Limanov Invariant

Let $\mathbf{k}$ be a field of characteristic zero, let $C$ be an affine $\mathbf{k}$-domain and consider the ring

$$B = C[T,Y]/\langle T^n Y - f(T)\rangle, \tag{18}$$

where $n \geq 2$ and $f(T) \in C[T]$ is such that $f(0) \neq 0$. We will see that $B$ is an integral domain. The first two sections of this chapter are devoted to the proof of Theorem 4.2.23, which asserts that if $|f(0)|_C > 1$ then $\mathrm{ML}(B) \neq \mathbf{k}$ (so in particular $B$ is not a polynomial ring over $\mathbf{k}$).

Section 4.1 proves the special case where $f(T)$ is a constant polynomial. In this case there is a natural way to define a $\mathbb{Z}$-grading on $B$. We refer to this situation as "the graded case". Section 4.2 proves the general case by reducing it to the graded case. This reduction is accomplished by showing that the associated graded ring $\mathrm{Gr}(B)$ is the graded ring studied in Section 4.1 and by using the techniques studied in Section 2.4.

Section 4.3 explains the relation between the first two sections and the Cancellation Problem. We shall see that the family of rings studied in Sections 4.1–4.2 contains some potential counterexamples to the Cancellation Problem in characteristic zero. This is in fact one of the motivations for studying these rings.

## 4.1 The Graded Case: $C[T,Y]/\langle T^nY - f_0\rangle$

Throughout this section, we let $\mathbf{k}$ be a field of characteristic zero, $C$ an affine $\mathbf{k}$-domain, $f_0 \in C \setminus \{0\}$, $n \geq 2$, and

$$B = C[T,Y]/\langle T^nY - f_0\rangle.$$

We write $B = C[t,y]$ where $t$ and $y$ are the images of $T$ and $Y$ in $B$.

**4.1.1 Lemma.** *The ring $B$ is an integral domain.*

*Proof.* This is a special case of Lemma 4.2.2, which we will prove in the next section. □

**4.1.2 Remark.** Observe that $C[T,Y]$ has a $\mathbb{Z}$-grading defined by declaring that

- each element of $C \setminus \{0\}$ is homogeneous of degree 0,
- $T$ is homogeneous of degree $-1$,
- $Y$ is homogeneous of degree $n$.

Then $T^nY - f_0$ is a homogeneous element of $C[T,Y]$, so $\langle T^nY - f_0\rangle$ is a homogeneous ideal of $C[T,Y]$, so $B$ inherits a $\mathbb{Z}$-grading from $C[T,Y]$. The graded ring $B = \bigoplus_{n\in\mathbb{Z}} B_n$, satisfies:

$$C \subset B_0, \quad t \in B_{-1}, \quad y \in B_n.$$

Let $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ be the degree function determined by the grading as in 1.2.12.

**4.1.3 Lemma.** *Any element of $B$ can be written as a finite sum of the form*

$$\sum_{i\geq 0} a_i t^i + \sum_{l>0} b_{0,l} y^l + \sum_{l>0} b_{1,l} t y^l + \dots + \sum_{l>0} b_{n-1,l} t^{n-1} y^l \qquad (a_i, b_{j,l} \in C, i, l \in \mathbb{Z}). \quad (19)$$

*Proof.* Let $b \in B$, and observe that it can be written as a sum of monomials of form $dt^py^q$ where $d \in C$, $p, q \in \mathbb{N}$. Each monomial will satisfy one of the following two mutually exclusive conditions:

a. $p \geq nq$

b. $p < nq$

First, suppose $p \geq nq$. Then $dt^p y^q = d{f_0}^q t^{p-nq}$ which has form $at^i$ where $a \in C$ and $i \in \mathbb{N}$, corresponding to a term in the first summand of Equation (19).

Second, suppose $p < nq$. Then $dt^p y^q = d{f_0}^{\lfloor \frac{p}{n} \rfloor} t^{p-n\lfloor \frac{p}{n} \rfloor} y^{q-\lfloor \frac{p}{n} \rfloor}$ which has form $bt^j y^l$ where $b \in C, 0 \leq j \leq n-1$ and $l > 0$. This monomial can be represented as a term in one of the summands in (19) (other than the first one). Since any monomial of $b$ has the form of an element inside one of the summands of (19), we can write $b$ as a sum of elements of this form, which gives the result. □

**4.1.4 Example.** The above proof is heavy in notation. It is easier to demonstrate the method by means of an example.

Let $B = C[T,Y]/\langle T^3 Y - c \rangle$ where $c \in C$ is irreducible. We then have that $B$ is an integral domain. Let $b = 5t^{10}y^5 \in B$. To write $b$ in the form described in (19), we repeatedly replace $t^3 y$ with $c$ until we obtain the proper form. We have $b = 5t^{10}y^5 = 5ct^7y^4 = 5c^2t^4y^3 = 5c^3ty^2$ which has the proper form.

**4.1.5 Remark.** Observe that each term ($a^i t^i$ or $b_{j,l}t^i y^l$) in the sum (19) is homogeneous; each nonzero term $a_i t^i$ in the first summand of (19) has degree $\leq 0$ and that all other nonzero terms $b_{j,l}t^j y^l$ have positive degree. Moreover, it is clear that distinct nonzero terms in (19) have distinct degrees.

**4.1.6 Remark.** Consider the subset $\mathcal{B} = \mathcal{B}_- \cup \mathcal{B}_+$ of $B$, where

$$\mathcal{B}_- = \left\{ t^i \,:\, i \in \mathbb{N} \right\} \quad \text{and} \quad \mathcal{B}_+ = \left\{ t^i y^\ell \,:\, 0 \leq i \leq n-1,\ \ell > 0 \right\}.$$

Then Lemma 4.1.3 asserts that $\mathcal{B}$ is a generating set of the $C$-module $B$, and Remark 4.1.5 adds that the elements of $\mathcal{B}$ are homogeneous of distinct degrees. Since $C \subset B_0$, it follows that $B$ is a free $C$-module with basis $\mathcal{B}$.

**4.1.7 Remark.** From Lemma 4.1.3, we observe that if $b_+ \in \bigoplus_{m>0} B_m$ then

$$b_+ = \sum_{q>0} b_{0,q} y^q + \sum_{q>0} b_{1,q} t y^q + \ldots + \sum_{q>0} b_{n-1,q} t^{n-1} y^q \qquad (b_{j,q} \in C).$$

It is clear that $\bigoplus_{m>0} B_m \subset yB$ since $y$ divides all monomials of $b_+$. We remark that $\bigoplus_{m>0} B_m$ is closed under addition and multiplication, but is neither an ideal nor a subring of $B$.

**4.1.8 Remark.** Again from Lemma 4.1.3 it is immediately clear that $B_0 = C, B_1 = Ct^{n-1}y, B_n = Cy$. Note that for each $z \in \mathbb{Z}_+$ we can write $z = nj - i$ where $i, j \in \mathbb{N}$, $0 \leq i \leq n-1$, $j \geq 1$ and that this decomposition of $z$ is unique. As such we may write $B_z = CM_z$, where $M_z \in B_z \setminus \{0\}$ is chosen as follows

$$M_z = \begin{cases} t^{-z} & \text{if } z \leq 0 \\ t^i y^j & \text{if } z > 0, \text{ where } z = nj - i,\ 0 \leq i \leq n-1,\ j \geq 1. \end{cases}$$

Note that by Corollary 2.4.10, if $D \in \text{LND}(B)$, then $\deg(D)$ exists.

**4.1.9 Proposition.** *Suppose* $0 \neq D \in \text{LND}(B)$ *and let* $d = \deg(D)$ *where* $d \in \mathbb{Z}$. *Then,*

*a.* $D(C) \neq 0$.

*b. If $D$ is homogeneous, there exists $\delta \in \text{LND}(C) \setminus \{0\}$ such that $D$ is a quasi-extension of $\delta$. Moreover, the pair $(\delta, D)$ satisfies $D(c) = M_d\delta(c)$ for every $c \in C$.*

*Proof.* For part (a) we proceed by contradiction and assume that $D(C) = 0$. It then follows that $t^n y \in \ker(D)$. Since $\ker(D)$ is a factorially closed subring of $B$, $t$ and $y$ are elements of $\ker(D)$ as well. Since we now have that $C$, $t$ and $y$ are in $\ker(D)$, this implies that $B = C[t, y] \subset \ker(D)$ which contradicts the assumption that $D$ is nonzero. This proves (a).

For part (b), note that if $c \in C$, then $D(c) \in B_d = C \cdot M_d$, so ${M_d}^{-1}D(c) \in C$. It follows that $\delta : C \to C$ defined by

$$\delta(c) = M_d^{-1}D(c) \quad \forall c \in C$$

is a well-defined set map.

Note that $M_d$ is defined as in Remark 4.1.8. It is straightforward to verify that $\delta$ is a derivation of $C$. Multiplying both sides of the above equation by $M_d$ shows that

$D$ is a quasi-extension of $\delta$ satisfying $D(c) = M_d\delta(c)$. By part (a), $\delta \neq 0$. Since $D$ is locally nilpotent, Lemma 3.1.10 implies that $\delta$ is locally nilpotent as well. □

**4.1.10 Corollary.** *If $C$ is rigid, then $B$ is rigid.*

*Proof.* By contrapositive, suppose $B$ is not rigid. Let $D \in \text{LND}(B)$ be nonzero and by Remark 2.5.8, let $\tilde{D} \in \text{LND}(B)$ be its homogenization. Since $\tilde{D}$ is homogeneous, nonzero and locally nilpotent, it follows from Proposition 4.1.9 (b) that there exists $\delta \in \text{LND}(C) \setminus \{0\}$ so $C$ is not rigid. □

**4.1.11 Remark.** If $0 \neq D : B \to B$ is a homogeneous locally nilpotent derivation such that $\deg(D) \geq 1 - n$, then by Remark 4.1.7, $D(y) \in \bigoplus_{n>0} B_n \subset yB$. As such $y \mid D(y)$ and hence by Corollary 2.3.21, $D(y) = 0$.

**4.1.12 Proposition.** *Suppose $|f_0|_C > 1$. Then, no homogeneous locally nilpotent derivation of $B$ has positive degree.*

*Proof.* Suppose that $|f_0|_C > 1$ and that there exists a homogeneous locally nilpotent derivation $D \in \text{LND}(B)$ with $\deg(D) = d > 0$. We will derive a contradiction.

By Remark 4.1.11, $D(y) = 0$. By Proposition 4.1.9 there exists $\delta \in \text{LND}(C) \setminus \{0\}$ such that $D(c) = M_d\delta(c)$ for every $c \in C$. Since $d > 0$, we have $M_d = t^i y^j$ where $d = nj - i$, where $i, j \in \mathbb{N}$ are chosen as in Remark 4.1.8. Observe that

$$\begin{aligned} D(t^n y) &= nt^{n-1}yD(t) && \text{since } y \in \ker(D), \\ D(t^n y) &= D(f_0) = \delta(f_0)t^i y^j \end{aligned}$$

More succinctly we have:

$$nt^{n-1}yD(t) = D(t^n y) = D(f_0) = \delta(f_0)t^i y^j \tag{20}$$

Observe first that $D(t) = 0 \iff D(f_0) = 0 \iff \delta(f_0) = 0$. Since $\delta \neq 0$, the statement $\delta(f_0) = 0$ contradicts the assumption that $|f_0|_C > 1$, so it follows that $\delta(f_0) \neq 0, D(f_0) \neq 0$, and $D(t) \neq 0$.

Since $D(t) \neq 0$, $t$ does not divide $D(t)$ by Corollary 2.3.21. Since $D$ is homogeneous, it follows that, $D(t) \in B_{ng}$ for some $g \in \mathbb{N}$ (otherwise $t$ would divide $D(t)$ by

(19)), which implies that $\deg(D) = gn + 1$. As such, we must have $M_d = t^i y^j$ with $i = n - 1$ and $j > 0$. Consequently, Equation (20) gives

$$\begin{aligned} nt^{n-1}yD(t) &= \delta(f_0)t^{n-1}y^j \\ \iff t^{n-1}y(D(t) - \frac{1}{n}\delta(f_0)y^{j-1}) &= 0 \\ \iff D(t) &= \frac{1}{n}\delta(f_0)y^{j-1} \end{aligned}$$

where the last equality follows since $t^{n-1}y \neq 0$ and $B$ is an integral domain. Next, observe that

$$\begin{aligned} D^2(t) &= \frac{1}{n}\delta(f_0)D(y^{j-1}) + y^{j-1}D(\frac{1}{n}\delta(f_0)) \\ &= \frac{1}{n}\delta^2(f_0)t^{n-1}y^{2j-1} \qquad \text{since } D(y) = 0. \end{aligned}$$

Since $n \geq 2$, we obtain that $t|D^2(t)$ and so $D^2(t) = 0$ by Corollary 2.3.21. It then follows that $\frac{1}{n}\delta^2(f_0)t^{n-1}y^{2j-1} = 0$ and hence that $\delta^2(f_0) = 0$. This contradicts the assumption that $|f_0|_C > 1$, which completes the proof. □

**4.1.13 Corollary.** *Suppose $|f_0|_C > 1$. Then, no locally nilpotent derivation of $B$ has positive degree.*

*Proof.* Let $D \in \mathrm{LND}(B)$, and let $\tilde{D} \in \mathrm{LND}(B)$ be the homogenization of $D$ as described in Remark 2.5.8. Then $\deg(D) = \deg(\tilde{D}) \leq 0$ where the equality follows from Remark 2.5.8 and the inequality follows from Proposition 4.1.12. □

**4.1.14 Theorem.** *Suppose $|f_0|_C > 1$. Then $t \in \mathrm{ML}(B)$.*

*Proof.* Let $D \in \mathrm{LND}(B)$. We must show $D(t) = 0$. If $D = 0$ the result is obvious, so we assume henceforth that $D \neq 0$. Since $B$ is $\mathbf{k}$-affine, it follows from Corollary 2.4.9 that $\deg(D)$ exists and is finite. By Corollary 4.1.13, $\deg(D) \leq 0$.

It then follows that $D(t) = \sum_{i \geq 1} c_i t^i$ where $c_i \in C$ (otherwise $\deg(D)$ would be positive). This implies that $t$ divides $D(t)$ and so $D(t) = 0$ by Corollary 2.3.21. □

**4.1.15 Theorem.** *Suppose $|f_0|_C > 1$. Then for each $D \in \mathrm{LND}(B) \setminus \{0\}$, $\ker(D) \subsetneq C[t]$. Moreover, $\mathrm{Derk}(B) \subset C[t]$.*

*Proof.* Let $D \in \mathrm{LND}(B) \setminus \{0\}$ and suppose $\ker(D) \not\subset C[t]$. Then, there exists $g \in \ker(D) \setminus C[t]$. By Lemma 4.1.3, $\deg(g) > 0$. By Lemma 2.3.9, for any $k \in \mathbb{N}$ we have $g^k D \in \mathrm{LND}(B) \setminus \{0\}$. Moreover, one can verify that $\deg(g^k D) = \deg(g^k) + \deg(D)$, so for $k$ large enough we obtain $\deg(g^k D) > 0$. This contradicts the fact that no element of $\mathrm{LND}(B) \setminus \{0\}$ has positive degree as stated in Corollary 4.1.13. This proves that $\ker(D) \subsetneq C[t]$. The inclusion is proper since $D(C) \neq 0$ by Proposition 4.1.9. □

**4.1.16 Remark.** Note that the assumption $n \geq 2$ that was made at the start of the section is necessary for Proposition 4.1.12 and Theorem 4.1.14 to hold.

Note first that if $n = 0$, then $C[T, Y]/\langle T^0 Y - f_0 \rangle \cong C[t]$ and the $t$-derivative is locally nilpotent and homogeneous of degree 1. As $\frac{d}{dt}(t) \neq 0$, we have $t \notin \mathrm{ML}(B)$ so Theorem 4.1.14 fails.

If $n = 1$, we have $R = C[T, Y]/\langle TY - f_0 \rangle$. Let $\delta \in \mathrm{LND}(C)$. By Lemma 3.1.3 (a), we may extend $\delta$ to a derivation $D_1 \in \mathrm{Der}(C[T, Y])$ satisfying:

$D_1(c) = \delta(c)$ for all $c \in C$

$D_1(Y) = 0$

$D_1(T) = 0.$

Since $\mathrm{Der}(C[T, Y])$ is a $C[T, Y]$-module, $Y D_1$, $\delta(f_0)\frac{\partial}{\partial T}$ belong to $\mathrm{Der}(C[T, Y])$. Let $D_2 = Y D_1 + \delta(f_0)\frac{\partial}{\partial T} \in \mathrm{Der}\, C[T, Y]$. Observe that $D_2$ satisfies

$D_2(c) = Y\delta(c)$ for all $c \in C$

$D_2(Y) = 0$

$D_2(T) = \delta(f_0).$

Using the fact that $\delta$ is locally nilpotent and that $Y \in \ker(D_2)$, it is straightforward to verify that $C \subset \mathrm{Nil}(D_2)$. The elements $T$ and $Y$ are also obviously in $\mathrm{Nil}(D_2)$ and so $D_2$ is locally nilpotent.

Since $D_2(TY - f_0) = Y\delta(f_0) - Y\delta(f_0) = 0$, by Proposition 2.3.10 $D_2$ induces a well-defined derivation $D \in \mathrm{LND}(B)$ which maps $t$ to $\delta(f_0) \neq 0$. Since $D$ is homogeneous, nonzero and $\deg(D) = 1$, Proposition 4.1.12 fails. Since $D(t) \neq 0$, we have $t \notin \mathrm{ML}(B)$ so Theorem 4.1.14 fails.

## 4.2 The General Case: $C[T,Y]/\langle T^nY - f(T)\rangle$

The aim of this section is to prove the following fact, which is the main result of this thesis. The result will appear below as Theorem 4.2.23.

**Theorem.** *Let* $\mathbf{k}$ *be a field of characteristic zero,* $C$ *an affine* $\mathbf{k}$*-domain and*

$$B = C[T,Y]/\langle T^nY - f(T)\rangle,$$

*where* $n \geq 2$ *and* $f(T) \in C[T]$ *is such that* $|f(0)|_C > 1$*. Then* $B$ *is a domain,* $t \in \mathrm{ML}(B)$ *and* $\mathrm{Derk}(B) \subset C[t]$*, where* $t \in B$ *is the image of* $T$*. In particular,* $\mathrm{ML}(B) \neq \mathbf{k}$ *and* $\mathrm{Derk}(B) \neq B$*.*

The following assumptions will be in effect throughout this section: $\mathbf{k}$ is a field of characteristic zero, $C$ is an affine $\mathbf{k}$-domain and

$$B = C[T,Y]/\langle T^nY - f(T)\rangle,$$

where $n \geq 2$ and $f(T) \in C[T]$ is such that $f(0) \neq 0$. We write $B = C[t,y]$ where $t$ and $y$ are the images of $T$ and $Y$ in $B$. We shall also use the notation $f_0 = f(0)$.

Note that our assumption '$f(0) \neq 0$' is weaker than the assumption '$|f(0)|_C > 1$' of the theorem. The first steps of the proof are carried out under the weaker assumption that $f(0) \neq 0$, and we will assume that $|f(0)|_C > 1$ in the final parts of the argument.

**4.2.1 Lemma.** *The ring* $B_t$ *is an integral domain and* $B_t = C[T,T^{-1}] = C[T]_T$*.*

*Proof.* Let $S \subset C[T,Y]$ be the multiplicative set $S = \{1, T, T^2, \dots\}$. Observe that

$$\begin{aligned} B_t &\cong S^{-1}C[T,Y]/S^{-1}\langle T^nY - f(T)\rangle \cong C[T,T^{-1},Y]/\langle T^nY - f(T)\rangle \\ &\cong C[T,T^{-1},Y]/\langle Y - \tfrac{f(T)}{T^n}\rangle \cong C[T,T^{-1}], \end{aligned} \tag{21}$$

where we note that the second isomorphism in (21) follows from Corollary 1.3.6.

Since $C$ is an integral domain, so is $C[T]$. By Corollary 1.3.3, $C[T,T^{-1}] = C[T]_T$ is also an integral domain. □

**4.2.2 Lemma.** *The ring* $B = C[T,Y]/\langle T^nY - f(T)\rangle$ *is an integral domain.*

*Proof.* By Lemma 4.2.1, $B_t$ is an integral domain. We will show that the localization homomorphism $\ell : B \to B_t$ is injective, which implies that $B$ is an integral domain.

To show that $\ell : B \to B_t$ is injective, we will prove the equivalent result that $t$ is not a zero divisor in $B$. (This equivalence follows from Proposition 1.3.2.) To prove that $t$ is not a zero divisor in $B$, we must prove that if $F \in C[T,Y]$ satisfies $TF \in \langle T^nY - f(T)\rangle \lhd C[T,Y]$, then $F \in \langle T^nY - f(T)\rangle$.

Suppose that $TF \in \langle T^nY - f(T)\rangle$. Then,

$$TF = (T^nY - f(T))G \text{ for some } G \in C[T,Y]. \tag{22}$$

Observe that by setting $T = 0$ we obtain that $0 = -f(0) \cdot G(0,Y)$. Since $f(0) \neq 0$, $G(0,Y) = 0$, since $C$ and hence $C[T,Y]$ are integral domains. This implies that $T|G$ and so $G/T \in C[T,Y]$. Let $H = G/T \in C[T,Y]$. Since $C[T,Y]$ is an integral domain, (22) implies that $F = (T^nY - f(T))H$ and so $F \in \langle T^nY - f(T)\rangle$, as required. □

**4.2.3 Remark.** Since $B$ is an integral domain, it makes sense to consider the element $t^{-1}$ of the field of fractions of B. It then follows from Lemma 4.2.1 that

$$B_t = C[t, t^{-1}]$$

where t is transcendental over C. Note that $C[t,t^{-1}]$ is called the *ring of Laurent polynomials over C.*

**4.2.4.** As was the case in Section 4.1, we would like to show that if $|f(0)|_C > 1$ then $t \in \mathrm{ML}(B)$. This situation is more difficult than the one studied in Section 4.1 because in the present case we are unable to immediately define a non-trivial grading on $B$. However, we can define a grading on the larger ring $B_t$.

The following proposition is not needed to prove Theorem 4.2.23. We include it simply to add some perspective.

**4.2.5 Proposition.** *If $C$ is a UFD and $f(0)$ is irreducible in $C$, then $B$ is a UFD.*

To prove the above proposition, we require two lemmas. The first can be found as Exercise 6 in Chapter 3, Section 4 of [11]. The second is a special case of Theorem 20.2 in [16].

**4.2.6 Lemma.** *Let $R$ be a UFD and let $S$ be a multiplicative set of $R$ that does not contain 0. Then $S^{-1}R$ is a UFD.*

**4.2.7 Lemma.** *Let $R$ be a Noetherian integral domain. If there exists a prime element $p$ of $R$ such that $R_p$ is a UFD, then $R$ is a UFD.*

*Proof of Proposition* 4.2.5. Since $B$ is a Noetherian domain, by Lemma 4.2.7, it suffices to prove that $t$ is a prime element in $B$ and that $B_t$ is a UFD.

To show that $t$ is prime in $B$, we show that $B/\langle t\rangle$ is an integral domain. Indeed, we have

$$B/\langle t\rangle \cong C[T,Y]/\langle T^nY - f(T), T\rangle \cong C/\langle f(0)\rangle.$$

Since $f(0)$ is irreducible in $C$ and $C$ is a UFD, $f(0)$ is prime and so $B/\langle t\rangle \cong C/\langle f(0)\rangle$ is an integral domain. This shows that $t$ is prime in $B$.

By Proposition 4.2.2, $B_t = C[t,t^{-1}] = C[t]_t$ so it remains to show $C[t]_t$ is a UFD. Since $C$ is a UFD, $C[t]$ is a UFD. By Lemma 4.2.6, $C[t]_t$ is a UFD, as required. ☐

**4.2.8.** Let $\mathcal{B} = B_t$. We define a $\mathbb{Z}$-grading $\mathcal{B} = \bigoplus_{m\in\mathbb{Z}} \mathcal{B}_m$ on $\mathcal{B}$ by declaring that $\mathcal{B}_m = C \cdot t^{-m}$ for each $m \in \mathbb{Z}$. Observe that $\mathcal{B}_0 = C$ and $t$ is homogeneous of degree $-1$. This grading determines a degree function $\text{DEG} : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$.

Since $B \subset \mathcal{B}$ we can consider the restriction of DEG to $B$. Let $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ be the restriction of DEG. Then, deg is a degree function on $B$ and the pair $(B, \deg)$ determines the $\mathbb{Z}$-graded integral domain $\text{Gr}(B) = \bigoplus_{m\in\mathbb{Z}} B_{\leq m}/B_{<m}$ as discussed in depth in Section 2.4.

Let $D \in \text{LND}(B) \setminus \{0\}$. By Proposition 2.3.11, $S^{-1}D \in \text{LND}(\mathcal{B})$ (where $S = \{1, t, t^2, ...\}$). By Corollary 2.4.10, the degree of $S^{-1}D$ exists. Lemma 2.4.5 implies that the degree of $D$ exists as well (since deg is a restriction of DEG). By Proposition 2.5.7, $D \in \text{LND}(B) \setminus \{0\}$ determines a nonzero homogeneous locally nilpotent derivation $\text{gr}(D) : \text{Gr}(B) \to \text{Gr}(B)$, where $D$ and $\text{gr}(D)$ have the same degree.

**4.2.9 Remark.** The discussion in 4.2.8 shows that

$$\text{LND}(B) \to \text{HLND}(\text{Gr}(B)),$$
$$D \mapsto \text{gr}\, D,$$

is a well-defined map and preserves degree. (See Definition 2.5.6 for the definition of HLND.)

**4.2.10 Lemma.** *Any element of $B$ can be written as a finite sum of the form*

$$\sum_{i\geq 0} a_i t^i + \sum_{l>0} b_{0,l} y^l + \sum_{l>0} b_{1,l} t y^l + \dots + \sum_{l>0} b_{n-1,l} t^{n-1} y^l \qquad (a_i, b_{j,l} \in C, i, l \in \mathbb{Z}). \quad (23)$$

*Proof.* The proof is nearly identical to the proof of Lemma 4.1.3. □

**4.2.11 Remark.** Recall that since we can embed $B$ in $C[t, t^{-1}]$ we may write $y = f(t)t^{-n}$. We also have $f(t) = a_k t^k + \cdots + a_0$ where $a_i \in C, a_0 \neq 0$. So, $y = a_k t^{k-n} + \cdots + a_0 t^{-n}$ and so it follows that $\deg(y) = n$. Note that $y$ is in general not a homogeneous element of $C[t, t^{-1}]$.

**4.2.12 Remark.** Observe that the terms of form $a^i t^i$ in the sum (23) are homogeneous in $C[t, t^{-1}]$ but the terms of form $b_{j,l} t^i y^l$ need not be; each nonzero term $a_i t^i$ in the first summand of (23) has degree $\leq 0$ and all other nonzero terms $b_{j,l} t^j y^l$ have degree $nl - j > 0$. Moreover, it is clear that distinct nonzero terms in (23) have distinct degrees. As was the case in Section 4.1, it follows that $B$ is a free $C$-module with basis $\{ t^i \,:\, i \in \mathbb{N} \} \cup \{ t^i y^\ell \,:\, 0 \leq i \leq n-1,\ \ell > 0 \}$. We also point out that, for $b \in B$, $\deg(b) \leq 0$ if and only if $b \in C[t]$.

**4.2.13 Remark.** Recall that if $d \in \mathbb{Z}_+$, we can write $d = nl - j$ where $j, l \in \mathbb{N}$, $0 \leq j \leq n-1$, $l \geq 1$ and that this decomposition of $d$ is unique.

**4.2.14 Lemma.** *Let $x \in B \setminus \{0\}$ and let $d = \deg(x) \in \mathbb{Z}$. Write*

$$x = \sum_{i\geq 0} a_i t^i + \sum_{l>0} b_{0,l} y^l + \sum_{l>0} b_{1,l} t y^l + \dots + \sum_{l>0} b_{n-1,l} t^{n-1} y^l \qquad (a_i, b_{j,l} \in C, i, l \in \mathbb{Z})$$

*as in* (23).

a. *If $d \leq 0$, then $x = \sum_{i\geq -d} a_i t^i$ where $a_{-d} \neq 0$.*

b. *Suppose $d > 0$. Then $b_{j,l} = 0$ whenever $nl - j > d$ and $b_{j,l} \neq 0$ for the unique pair $(j, l)$ satisfying $nl - j = d$, where $0 \leq j \leq n-1$ and $l \geq 1$.*

*Proof.* We prove (b), and leave (a) as a similar exercise to the reader.

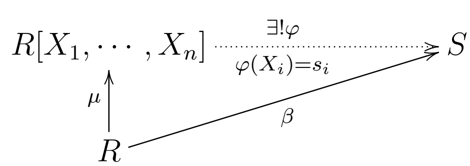
For (b), suppose $b_{j,l} \neq 0$ for some $j, l$ satisfying $nl - j > d$. Let $d' = \max\{ nl - j : nl - j > d \text{ and } b_{j,l} \neq 0 \}$. The set is non-empty by assumption, and the maximum $d'$ exists because the sum in (23) is finite. Let $j', l'$ be such that $d' = nl' - j'$ and $b_{j',l'} \neq 0$. It then follows that $\deg(x) = d' > d$ since all other nonzero terms in any of the summations have degree strictly less than $d'$. This is a contradiction, and so we conclude that $b_{j,l} = 0$ for every pair $j, l$ satisfying $nl - j > d$.

We show $b_{j,l} \neq 0$ when $nl - j = d$. We know $b_{j,l} = 0$ whenever $nl - j > d$. If $b_{j,l} = 0$ when $nl - j = d$, it would follow that $\deg(x) < d$ since $x$ must be a sum of terms where each term has degree strictly less than $d$. But this contradicts our assumption that $\deg(x) = d$ and so we must have $b_{j,l} \neq 0$.

□

We recall the following theorem regarding polynomial rings. See Chapter 3, Theorem 5.5 of [11] for a proof.

**4.2.15 Theorem.** *Let $R$ be a ring, let $n \geq 1$ and let $R[X_1, \cdots, X_n]$ be a polynomial ring in $n$ variables over $R$. For any choice of a ring $S$, of a homomorphism $\beta : R \to S$ and $n$-tuple $(s_1, \cdots, s_n) \in S^n$, there exists a unique homomorphism $\varphi : R[X_1, \cdots, X_n] \to S$ satisfying $\varphi(X_i) = s_i$ for all $i = 1, \cdots, n$ and $\varphi \circ \mu = \beta$.*




We also recall the following results relating the transcendence degree and dimension of affine rings. See for instance (14.G) and (14.H) of [15].

**4.2.16 Definition.** Let $\mathfrak{p}$ be a prime ideal of a ring $A$. The *height of* $\mathfrak{p}$ *in* $A$, denoted "$\mathrm{ht}(\mathfrak{p})$", is the largest number $n$ (or $\infty$ if no such $n$ exists) for which there exists a chain of prime ideals such that

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{n-1} \subsetneq \mathfrak{p}_n.$$

**4.2.17 Definition.** Let $A$ be a nonzero ring. The *Krull dimension of* $A$, denoted "$\dim(A)$" is defined as

$$\dim(A) = \sup\{\operatorname{ht}(\mathfrak{p}) \ : \mathfrak{p} \text{ is a prime ideal of } A\}.$$

**4.2.18 Theorem.** *Let* $\mathbf{k}$ *be a field and let* $A$ *be an affine* $\mathbf{k}$*-domain. Then,*

*a.* $\dim(A) = \operatorname{trdeg}_{\mathbf{k}}(A)$.

*b. For any prime ideal* $\mathfrak{p} \lhd A$, $\dim(A) = \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p})$.

The following result is well known. In the case of Noetherian rings, the result follows from Krull's Principal Ideal Theorem which can be found in [2] as Corollary 11.17. The proof provided below works for both Noetherian rings and Unique Factorization Domains.

**4.2.19 Lemma.** *Let* $R$ *be either a unique factorization domain or a Noetherian domain, and let* $I = \langle i \rangle$ *be a nonzero, principal and prime ideal of* $R$*. Then* $\operatorname{ht}(I) = 1$.

*Proof.* Clearly $\operatorname{ht}(I) \geq 1$ since $\langle 0 \rangle$ is prime and $I \neq \langle 0 \rangle$. It now suffices to show that if $J$ is prime and $I \supset J \supsetneq \langle 0 \rangle$, then $J = I$. Indeed, let $j \in J$ be nonzero. Since $J \subset I$, and since $R$ is either Noetherian or a UFD, we may write $j = i^k l$ where $k \geq 1$ and $i$ does not divide $l \in R$. Since $l \notin I$, it follows that $l \notin J$. Since $J$ is prime, we must have $i^k \in J$ from which it follows (again since $J$ is prime) that $i \in J$. So $J \supset I$ and thus $J = I$. □

**4.2.20 Lemma.** *The ideal* $\langle T^n Y - f(0) \rangle \lhd C[T, Y]$ *is a prime ideal of height* 1.

*Proof.* The ideal $\langle T^n Y - f(0) \rangle$ is prime by Lemma 4.1.1, and has height 1 by Lemma 4.2.19. □

**4.2.21 Proposition.** *Let* $B = C[T, Y]/\langle T^n Y - f(T) \rangle = C[t, y]$*. Then* $\operatorname{Gr}(B) \cong C[T', Y']/\langle T'^n Y' - f(0) \rangle$ *with the same grading that we considered in Section* 4.1.

*Proof.* To simplify notation, let $A = C[T', Y']/\langle T'^n Y' - f(0) \rangle$. It suffices to define an isomorphism of rings $\varphi : A \to \operatorname{Gr}(B)$ such that $\varphi(A_m) \subset \operatorname{Gr}(B)_m$ for every $m \in \mathbb{Z}$.

To obtain $\varphi$, we will define a surjective homomorphism of rings $\tilde{\varphi} : C[T', Y'] \to \mathrm{Gr}(B)$ and show that $\ker(\tilde{\varphi})$ is $\langle T'^m Y' - f(0)\rangle$. To obtain $\tilde{\varphi}$, we first define the map $\beta : C \to \mathrm{Gr}(B)$. We then extend it to $C[T', Y']$ using Theorem 4.2.15.

For each $c \in C$, let $\beta(c) = \mathrm{gr}(c) \in \mathrm{Gr}(B)$. We have that $\beta(1) = \mathrm{gr}(1) = 1 + B_{<0}$ which is the identity of $\mathrm{Gr}(B)$. Moreover, $\beta$ is multiplicative since gr is multiplicative. Lastly, if $c_1, c_2 \in C$, then $\beta(c_1 + c_2) = \mathrm{gr}(c_1 + c_2) = \mathrm{gr}(c_1) + \mathrm{gr}(c_2) = \beta(c_1) + \beta(c_2)$ where the second equality holds because $\mathrm{gr}(c_1)$ and $\mathrm{gr}(c_2)$ are homogeneous of the same degree. We conclude that $\beta$ is additive and hence is a homomorphism of rings.

Note that $\beta$ is injective, since if $\beta(c) = 0$, by Definition 2.5.2, it follows that $c = 0$. We will regard $C$ as a subring of $\mathrm{Gr}(B)$. Note also that $C = \mathrm{Gr}(B)_0$ by Lemma 4.2.14 (a).

We have $\beta : C \to \mathrm{Gr}(B)$, and let $i : C \hookrightarrow C[T', Y']$ be the inclusion map. Applying Theorem 4.2.15, there exists a unique homomorphism $\tilde{\varphi} : C[T', Y'] \to \mathrm{Gr}(B)$ such that

$$\begin{aligned} \tilde{\varphi}(T') &= \mathrm{gr}(t) = t + B_{<-1} \in \mathrm{Gr}(B)_{-1}, \\ \tilde{\varphi}(Y') &= \mathrm{gr}(y) = y + B_{<n} \in \mathrm{Gr}(B)_n, \\ \tilde{\varphi}(c) &= \mathrm{gr}(c) = c + B_{<0} \in \mathrm{Gr}(B)_0 \text{ for all } c \in C. \end{aligned}$$

(Note that $\tilde{\varphi}(c)$ is really $\tilde{\varphi}(i(c))$.)

To show surjectivity, it suffices to show that every homogeneous element of $\mathrm{Gr}(B)$ is in the image of $\tilde{\varphi}$. Let $x + B_{<d}$ be a nonzero homogeneous element of $\mathrm{Gr}(B)$ where $\deg(x) = d \in \mathbb{Z}$. Using Remark 4.2.10, we may write

$$x = \sum_{i \geq 0} a_i t^i + \sum_{l > 0} b_{0,l} y^l + \sum_{l > 0} b_{1,l} t y^l + \dots + \sum_{l > 0} b_{n-1,l} t^{n-1} y^l \qquad (a_i, b_{j,l} \in C, i, l \in \mathbb{Z}).$$

We consider the cases $d \leq 0$ and $d > 0$ separately. If $d \leq 0$, then by Lemma 4.2.14 (a), $x = \sum\limits_{i \geq -d} a_i t^i$ where the sum is finite and $a_{-d} \neq 0$. Moreover, $\tilde{\varphi}(a_{-d} T'^{-d}) = \mathrm{gr}(a_{-d} t^{-d}) = \mathrm{gr}(x) = x + B_{<d}$.

Now suppose $d > 0$ and let $d = nl - j$ where $l$ and $j$ are chosen as in Remark 4.2.13. By Lemma 4.2.14, we have $\tilde{\varphi}(b_{j,l} T'^j Y'^l) = \mathrm{gr}(b_{j,l} t^j y^l) = \mathrm{gr}(x) = x + B_{<d}$. Again, note

that the second equality holds because $b_{j,l}t^j y^l$ is the only monomial of degree $d$ in the summation above, and is the term of highest degree.

Having completed both cases, we have shown that $\tilde{\varphi}$ is surjective.

Next we will show that $\ker(\tilde{\varphi}) = \langle T'^n Y' - f(0)\rangle$ and conclude by the First Isomorphism Theorem for Rings that $A \cong \mathrm{Gr}(B)$. Observe that

$$\begin{aligned}\tilde{\varphi}(T'^n Y' - f(0)) &= \tilde{\varphi}(T'^n)\varphi(Y') - \tilde{\varphi}(f(0))\\ &= \mathrm{gr}(t^n)\,\mathrm{gr}(y) - \mathrm{gr}(f(0))\\ &= \mathrm{gr}(t^n y) - \mathrm{gr}(f(0))\\ &= \mathrm{gr}(f(t)) - \mathrm{gr}(f(0))\\ &= 0\end{aligned}$$

so it follows that $\langle T^n Y - f(0)\rangle \subset \ker\tilde{\varphi}$.

For the other inclusion, note first that since $\tilde{\varphi}(C[T', Y'])$ is contained in the integral domain $\mathrm{Gr}(B)$, $\tilde{\varphi}(C[T', Y'])$ is an integral domain and so $\ker(\tilde{\varphi})$ is prime. Recall that $\langle T'^n Y' - f(0)\rangle$ is prime by Lemma 4.2.20. Since $\ker(\tilde{\varphi}) \supset \langle T'^n Y' - f(0)\rangle$, it follows that $\mathrm{ht}(\ker(\tilde{\varphi})) \geq 1$. To show that $\ker(\tilde{\varphi}) \subset \langle T'^n Y' - f(0)\rangle$, it now suffices to show that $\mathrm{ht}(\ker(\tilde{\varphi})) = 1$. It will then follow by the First Isomorphism Theorem that $\tilde{\varphi}$ induces an isomorphism $\varphi : C[T', Y']/\langle T'^n Y' - f(0)\rangle \to \mathrm{Gr}(B)$.

To show that $\mathrm{ht}(\ker(\tilde{\varphi})) = 1$, we begin by showing that $\mathrm{gr}(t)$ is transcendental over $C$. Since $\mathrm{gr}(t)$ is a homogeneous element of nonzero degree in the graded integral domain $\mathrm{Gr}(B)$, it follows that $\mathrm{gr}(t)$ is transcendental over $\mathrm{Gr}(B)_0 = C$.

This implies that $\mathrm{trdeg}_C\,\mathrm{Gr}(B) \geq 1$. By Theorem 4.2.18, we have $\dim C[T', Y'] = \mathrm{trdeg}_{\mathbf{k}}(C[T', Y'])$ and $\dim(C[T', Y']) = \dim(C[T', Y']/\ker\tilde{\varphi}) + \mathrm{ht}(\ker\tilde{\varphi})$.

$$\begin{aligned}\mathrm{trdeg}_{\mathbf{k}}(C[T, Y]) &= \dim(C[T', Y']/\ker\tilde{\varphi}) + \mathrm{ht}(\ker\tilde{\varphi})\\ \mathrm{trdeg}_{\mathbf{k}}(C) + 2 &= \dim(\mathrm{Gr}(B)) + \mathrm{ht}(\ker\tilde{\varphi})\\ \mathrm{trdeg}_{\mathbf{k}}(C) + 2 &= \mathrm{trdeg}_{\mathbf{k}}(\mathrm{Gr}(B)) + \mathrm{ht}(\ker\tilde{\varphi})\\ \mathrm{trdeg}_{\mathbf{k}}\,C + 2 &= \mathrm{trdeg}_{\mathbf{k}}\,C + \mathrm{trdeg}_C\,\mathrm{Gr}(B) + \mathrm{ht}(\ker\tilde{\varphi})\\ 2 &= \mathrm{trdeg}_C\,\mathrm{Gr}(B) + \mathrm{ht}(\ker\tilde{\varphi})\end{aligned}$$

Since $\operatorname{trdeg}_C \operatorname{Gr}(B) \geq 1$, it follows that $\operatorname{ht}(\ker \tilde{\varphi}) \leq 1$ as required. So, $\tilde{\varphi}$ induces an isomorphism of rings $\varphi : A \to \operatorname{Gr}(B)$.

Finally, it remains to prove that $\varphi(A_m) \subset \operatorname{Gr}(B)_m$. Define a $\mathbb{Z}-$grading on $C[T', Y']$ by declaring that all elements $c \in C \setminus \{0\}$ are homogeneous of degree 0, and $T'$ and $Y'$ are homogeneous of degrees $-1$ and $n$ respectively. The map $\tilde{\varphi} : C[T', Y'] \to \operatorname{Gr}(B)$ is a homomorphism of graded rings which maps $C$ to $C$, and generators $T', Y'$ of $C[T', Y']$ to the generators $\operatorname{gr}(t)$ and $\operatorname{gr}(y)$ of $\operatorname{Gr}(B)$.

Next, observe that $\ker \tilde{\varphi} = \langle T'^n Y' - f(0) \rangle$ is a homogeneous ideal of $C[T', Y']$, and so we have that $A_m = \pi(C[T', Y']_m)$ where $\pi : C[T', Y'] \to A$ is the projection. Next observe that $\varphi(A_m) = \varphi(\pi(C[T', Y']_m)) = \tilde{\varphi}(C[T', Y']_m) \subset \operatorname{Gr}(B)_m$ which is the required result. This completes the proof.

□

**4.2.22 Proposition.** *If $|f(0)|_C > 1$, then no locally nilpotent derivation of $B$ has positive degree.*

*Proof.* Let $D \in \operatorname{LND}(B)$. By 4.2.8 and 4.2.9, $\deg(D)$ exists and $\operatorname{gr}(D) \in \operatorname{LND}(\operatorname{Gr}(B))$ is homogeneous of the same degree. By Proposition 4.1.12, no locally nilpotent homogeneous derivation of $\operatorname{Gr}(B)$ has positive degree. It follows that $\deg(D) \leq 0$. □

We now arrive at the main result of the thesis.

**4.2.23 Theorem.** *Let $\mathbf{k}$ be a field of characteristic zero, $C$ an affine $\mathbf{k}$-domain and*

$$B = C[T, Y]/\big(T^n Y - f(T)\big),$$

*where $n \geq 2$ and $f(T) \in C[T]$ is such that $|f(0)|_C > 1$. Then $B$ is an integral domain, $t \in \operatorname{ML}(B)$ and $\operatorname{Derk}(B) \subset C[t]$, where $t \in B$ is the image of $T$. In particular, $\operatorname{ML}(B) \neq \mathbf{k}$ and $\operatorname{Derk}(B) \neq B$.*

*Proof.* We have already proven that $B$ is an integral domain. We first show that $t \in \operatorname{ML}(B)$.

Let $D \in \operatorname{LND}(B) \setminus \{0\}$. Since $\deg(D) \leq 0$ by Proposition 4.2.22, it follows that $\deg(D(t)) \leq \deg(t) = -1$. By Equation (23), $D(t) = \sum_{i=1}^{k} a_i t^i$ where $k \geq 1$. So $t | D(t)$, which implies that $D(t) = 0$ by Corollary 2.3.22. This proves that $t \in \operatorname{ML}(B)$.

Next, we show $\mathrm{Derk}(B) \subset C[t]$. Let $D \in \mathrm{LND}(B)\setminus\{0\}$ and suppose $\ker(D) \not\subset C[t]$. Then, there exists $g \in \ker(D)\setminus C[t]$. By Remark 4.2.12, $\deg(g) > 0$. By Lemma 2.3.9, for any $k \in \mathbb{N}$ we have $g^k D \in \mathrm{LND}(B) \setminus \{0\}$. Moreover, one can verify that $\deg(g^k D) = \deg(g^k) + \deg(D)$, so for $k$ large enough we obtain $\deg(g^k D) > 0$. This contradicts Proposition 4.2.22. So every $D \in \mathrm{LND}(B) \setminus \{0\}$ satisfies $\ker(D) \subset C[t]$, and this proves that $\mathrm{Derk}(B) \subset C[t]$.

It is clear that $\mathrm{ML}(B) \neq \mathbf{k}$ and $\mathrm{Derk}(B) \neq B$. □

## 4.3 Relation with Zariski's Cancellation Problem

In 1949, Zariski proposed a number of questions at the Paris Colloquium on Algebra and the Theory of Numbers. The following related question later gained prominence and became known as "Zariski's Cancellation Problem".

**4.3.1 Question.** Let $R$ be a $\mathbf{k}$-algebra and let $m, n \in \mathbb{N}$. Suppose $R^{[m]} \cong \mathbf{k}^{[n+m]}$. Does it follow that $R \cong \mathbf{k}^{[n]}$?

Both in characteristic zero and in characteristic $p$, the question has an affirmative answer for $n + m \leq 3$. The results are well known but we will not discuss the details and refer the reader to [10].

In characteristic zero, when $n \geq 3$ the problem is still open and much of what we have done in this thesis is motivated by this problem. In characteristic $p$, the problem has recently been solved.

### 4.3.1 Gupta's Result

In characteristic $p$, Question 4.3.1 was open for $n \geq 3$ until 2011 when in [8], Gupta provided an example showing that Question 4.3.1 has a negative answer for $n = 3$. She later generalized her methods to show that for every $p$ and every $n \geq 3$, the question has a negative answer, settling Question 4.3.1 for every prime $p$ and every $n$. We briefly discuss her main results below. All results in this subsection are discussed in [9] and so we refer the reader there for details.

For the balance of this subsection, we assume that $\mathbf{k}$ is a field of characteristic $p > 0$.

**4.3.2 Definition.** An element $f(Z,T) \in \mathbf{k}[Z,T]$ is a *line* if $\mathbf{k}[Z,T]/\langle f \rangle \cong \mathbf{k}^{[1]}$.

**4.3.3 Definition.** A line $f$ is called a *non-trivial line* if $\mathbf{k}[Z,T] \neq \mathbf{k}[f]^{[1]}$.

**4.3.4 Remark.** It is not obvious that non-trivial lines exist. The first non-trivial lines were discovered by Beniamino Segre in [20]. We will later see that non-trivial lines only exist in $\mathbf{k}^{[2]}$ when $\mathbf{k}$ is field of characteristic $p > 0$.

To construct her counterexample, Gupta considers rings of the following type:

$$G_m = \mathbf{k}[X_1, \cdots, X_m, Y, Z, T]/\langle X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T) \rangle,$$

where $m \geq 1$, $r_i > 1$ for each $i$, $1 \leq i \leq m$ and $f(Z,T) \in \mathbf{k}[Z,T]$.

She then proves the following two theorems.

**4.3.5 Theorem.** *Suppose $f(Z,T)$ is a line in $\mathbf{k}[Z,T]$. Then $G_m^{[1]} \cong \mathbf{k}[X_1, \cdots, X_m]^{[3]} \cong \mathbf{k}^{[m+3]}$*

**4.3.6 Theorem.** *Suppose $f(Z,T)$ is a non-trivial line in $\mathbf{k}[Z,T]$. Then $G_m \ncong \mathbf{k}^{[m+2]}$*

Lastly, she obtains:

**4.3.7 Corollary.** *Question* 4.3.1 *has a negative answer for every field* $\mathbf{k}$ *of characteristic $p > 0$ and every $n \geq 3$.*

*Proof.* Fix $n \geq 3$. Let $m = n - 2$. Let $P_m = \mathbf{k}[X_1, \cdots, X_m, Y, Z, T]$ and let $f(Z,T)$ be a non-trivial line in $\mathbf{k}[Z,T] \subset P_m$. By Theorem 4.3.5, $G_m^{[1]} \cong \mathbf{k}^{[n+1]}$. By Theorem 4.3.6, $G_m \ncong \mathbf{k}^{[n]}$. □

### 4.3.2 Candidates in Characteristic Zero

In this subsection, we assume that $\mathbf{k}$ has characteristic zero and discuss how our results in Sections 4.1 – 4.2 relate to Zariski's Cancellation Problem in characteristic zero. We begin with the following statement which is a special case of an unpublished result of Daigle.

**4.3.8 Corollary.** *Let $m > 1$ be an integer and suppose $f \in \mathbf{k}[X_1, \cdots, X_m] = \mathbf{k}^{[m]}$ is such that $\mathbf{k}[X_1, \cdots, X_m]/\langle f \rangle \cong \mathbf{k}^{[m-1]}$. Let $T, Y$ be independent indeterminates over $\mathbf{k}[X_1, \cdots, X_m]$ and let $n \in \mathbb{N}$. Then the $\mathbf{k}[T]$-algebra*

$$A = \mathbf{k}[X_1, \cdots, X_m, T, Y]/\langle T^n Y - f \rangle$$

*satisfies $A^{[m-1]} = \mathbf{k}[T]^{[2m-1]} = \mathbf{k}^{[2m]}$.*

**4.3.9.** Let $C = \mathbf{k}[X_1, \cdots, X_m]$ and suppose there exists $f \in C$ such that $|f|_C > 1$ and $\mathbf{k}[X_1, \cdots, X_m]/\langle f \rangle \cong \mathbf{k}^{[m-1]}$. Let $A = C[T, Y]/\langle T^n Y - f \rangle$. Corollary 4.3.8 would imply that $A^{[m-1]} \cong \mathbf{k}^{[2m]}$ and Theorem 4.2.23 would imply that $A \ncong \mathbf{k}^{[m+1]}$. Such an $f$ would provide a negative answer to Question 4.3.1. Unfortunately, it is not yet known whether or not such an $f \in C = \mathbf{k}[X_1, \cdots, X_m]$ exists.

**4.3.10 Definition.** Let $P = \mathbf{k}[X_1, X_2, \cdots, X_n]$ be a polynomial ring. An element $f_1 \in P$ is called a *variable of $P$* if there exist elements $f_2, \cdots, f_n$ such that $\mathbf{k}[f_1, f_2, \cdots, f_n] \cong P$.

The following theorem, known as the Abhyankar-Moh-Suzuki Theorem, follows from Theorem 1.2 in [1].

**4.3.11 Theorem.** *Suppose $\mathbf{k}$ is a field of characteristic zero. If $f \in \mathbf{k}[X_1, X_2]$ satisfies $\mathbf{k}[X_1, X_2]/\langle f \rangle = \mathbf{k}^{[1]}$, then $f$ is a variable of $\mathbf{k}[X_1, X_2]$.*

**4.3.12 Remark.** The Abhyankar-Moh-Suzuki Theorem implies that if $\mathbf{k}$ is a field of characteristic zero, then non-trivial lines do not exist in $\mathbf{k}^{[2]}$.

**4.3.13 Remark.** Suppose $C = \mathbf{k}[X_1, X_2]$. To answer Question 4.3.1, by combining Corollary 4.3.8 with Theorem 4.2.23 it would be sufficient to find an $f$ satisfying $\mathbf{k}[X_1, X_2]/\langle f \rangle \cong \mathbf{k}^{[1]}$ and $|f|_C > 1$. However, by the Abhyankar-Moh-Suzuki Theorem, any $f$ satisfying $\mathbf{k}[X_1, X_2]/\langle f \rangle \cong \mathbf{k}^{[1]}$ is a variable of $\mathbf{k}[X_1, X_2]$. One can verify that for any variable $v$ in a polynomial ring $P = \mathbf{k}^{[n]}$ , $|v|_P = 0$ when $n \geq 2$, and $|v|_P = 1$ when $n = 1$. As such, one cannot find an $f$ to satisfy this pair of sufficient conditions.

At this point we leave the reader with the following natural question that arises from 4.3.9 which we have not yet been able to answer.

**4.3.14 Question.** Let $C = \mathbf{k}[X_1, \cdots, X_m]$ where $m \geq 3$. Does there exist an element $f \in C$ such that $|f|_C > 1$ and $C/\langle f \rangle \cong \mathbf{k}^{[m-1]}$?

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