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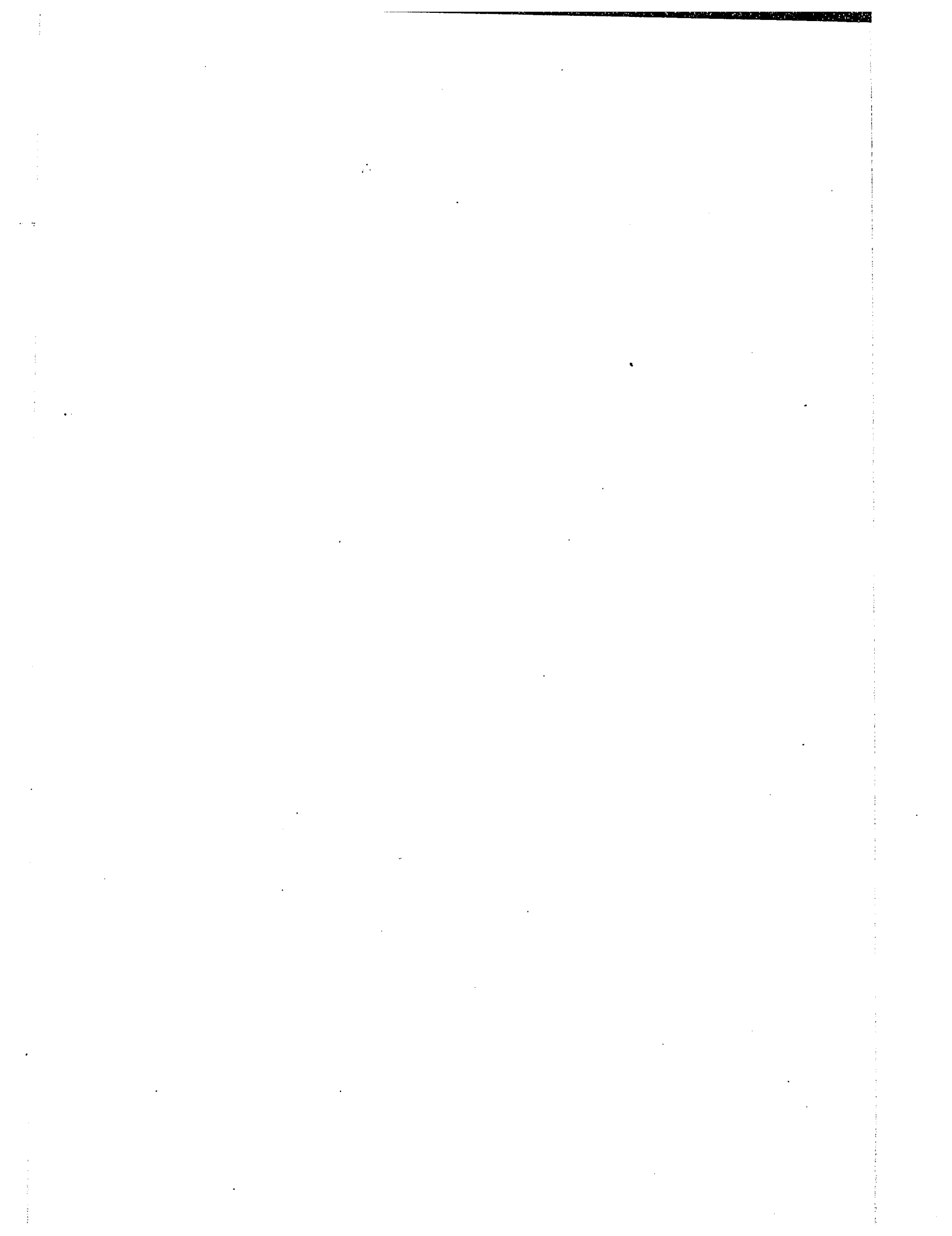
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ORDER IDEALS IN A C^* -ALGEBRA

A thesis submitted

by

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to

the Faculty of Pure and Applied Science

of the University of Ottawa

in partial fulfillment of the requirements

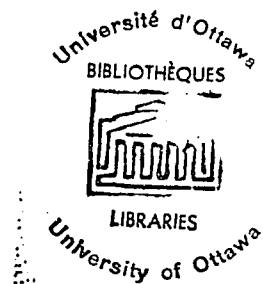
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ABSTRACT

Let A be a C^* -algebra. Since the bidual of A can be considered as a W^* -algebra, this enables us to prove the following duality theorems:

(i) There exists a bijection between the norm-closed 2-sided ideals of A and the norm-closed invariant order ideals of A .

(ii) There exists a bijection between the norm-closed left ideals of A and the norm-closed order ideals of A .

(iii) There exists an order inverting bijection between the norm-closed 2-sided ideals of A and the weak*-closed invariant faces of $S(A)$, where $S(A)$ is the state space of A .

The object of the thesis is to verify the above observations and to give Størmer's solution to J. Dixmier's problem: if N and M are norm-closed 2-sided ideals of A , then $(N + M)^+ = N^+ + M^+$, where N^+ and M^+ denote the positive parts of N and M respectively.

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INTRODUCTION

Let A be a C^* -algebra. Since the bidual of A can be considered as a W^* -algebra (§ 2.2), this enables us to employ W^* -algebra methods to study some properties of ideals of A and, in particular, to answer in the affirmative the following question asked by J. Dixmier in [3, p.20]: If M and N are norm-closed 2-sided ideals of A , is it true that $(M + N)^+ = M^+ + N^+$, where M^+ and N^+ denote the positive parts of M and N respectively? The solution to this problem given here is due to E. Størmer [14].

Chapter I deals with general properties of positive linear functionals on a C^* -algebra A which we shall frequently apply throughout the thesis. It is well known that for each positive linear functional on A there corresponds a cyclic representation of A which is unique up to equivalence, and conversely. This is described in detail in § 1.2. In Chapter II, we define the four basic topologies on $B(H)$, the algebra of all continuous linear operators on the complex Hilbert space, and establish some useful properties of W^* -algebras.

In Chapter III, we deal with the structure of norm-closed ideals in a C^* -algebra obtained by E. G. Effros in [5]. We show that there exists a bijection between the family of all norm-closed 2-sided ideals of A and the family of all norm-closed invariant order ideals of A [Theorem (3.2.5)];

similarly, there exists a bijection between the family of all norm-closed left ideals of A and the family of all norm-closed order ideals of A [Theorem (3.2.8)]. In § 3.3 we prove the existence and uniqueness of a polar decomposition for a normal linear functional on a W^* -algebra. More precisely, if f is a normal linear functional on a W^* -algebra A , then there exists a partial isometry U in A such that $f = U|f|$, where $|f| = U^*f$ is a positive linear functional and $\|U^*f\| = \|f\|$. Using this polar decomposition, we show in §3.4 that if N is a weak*-closed order ideal in the dual A' of a C^* -algebra A , then $N = N^{\perp\perp}$, where $N^\perp = \{T \in A': T \geq 0 \text{ and } f(T) = 0 \text{ if } T \in N\}$, and $N^{\perp\perp} = (N^\perp)^\perp = \{f \in A': f \geq 0 \text{ and } f(T) = 0 \text{ if } T \in N^\perp\}$. We apply the equality $N = N^{\perp\perp}$ in Chapter IV to prove the following duality theorem: There exists an order inverting bijection between norm-closed 2-sided ideals of A and weak*-closed invariant faces of $S(A)$, where $S(A)$ denotes the state space of A . We use this duality theorem to solve J. Dixmier's problem which is given in Theorem (4.2.2).

CHAPTER I

REPRESENTATION OF A C*-ALGEBRA

§ 1.1. Positive linear functionals.

Let A be an algebra over the field C of complex numbers. A mapping $x \rightarrow x^*$ from A into itself is called an involution if (i) $(x^*)^* = x$, (ii) $(\lambda x)^* = \bar{\lambda}x^*$, (iii) $(x + y)^* = x^* + y^*$, (iv) $(xy)^* = y^*x^*$ for all x, y in A and for all λ in C . An algebra with an involution defined on it is called a *-algebra. A Banach *-algebra is a Banach algebra with an involution, and a C*-algebra is a Banach *-algebra A such that $\|x\|^2 = \|x^*x\|$ for all $x \in A$.

Let A be a *-algebra. An element x of A is called hermitian if $x^* = x$, and it is called unitary if $x^*x = xx^* = 1$. An element x is called positive if $x = y^*y$ for some $y \in A$. The set of all hermitian elements of A will be denoted by A_h , the set of all unitary elements by A_u , and the set of all positive elements by A^+ .

Let A be a C*-algebra. For each $x \in A$, let $x_1 = \frac{1}{2}(x + x^*)$ and $x_2 = -\frac{1}{2}i(x - x^*)$. Then x_1, x_2 are hermitian and $x = \frac{1}{2}(x_1 + ix_2)$. If x is hermitian, then $x^2 = x^*x$ is positive and hence has a positive square root $(x^2)^{\frac{1}{2}}$ [3, p.12]. Let $x^+ = \frac{1}{2}((x^2)^{\frac{1}{2}} + x)$, $x^- = \frac{1}{2}((x^2)^{\frac{1}{2}} - x)$.

Then x^+ , x^- are positive and $x = x^+ - x^-$. Moreover, $x^+x^- = 0$. For each hermitian x with $\|x\| \leq 1$, let $u = x + i(1 - x^2)^{\frac{1}{2}}$. It is easy to verify that $u^*u = uu^* = 1$. Therefore u is unitary and $x = \frac{1}{2}(u + u^*)$. From these considerations we see that A is spanned by either A_h , A_u or A^+ .

Let A' be the dual (conjugate) space of A . An element $f \in A'$ is called positive if $f(x^*x) \geq 0$ for all $x \in A$. For each $f \in A'$, define f^* by $f^*(x) = \overline{f(x^*)}$ for all $x \in A$; f^* belongs to A' and is called the adjoint linear functional of f . If $f^* = f$, f is called hermitian. For an element $f \in A'$, let $f_1 = \frac{1}{2}(f + f^*)$ and $f_2 = -\frac{1}{2}i(f - f^*)$. Then f_1, f_2 are hermitian and $f = f_1 + f_2$. Let $f, g \in A'$. If $f - g$ is positive, we say that f majorizes g and write $f \geq g$. A positive element f of A' is called a state of A if $\|f\| = 1$.

THEOREM (1.1.1). If f is a positive linear functional on a C^* -algebra A , then for all $x, y \in A$, we have:

- (i) $|f(y^*x)|^2 \leq f(x^*x) f(y^*y)$;
- (ii) $f(y^*x) = \overline{f(x^*y)}$;
- (iii) $f(x^*) = \overline{f(x)}$;
- (iv) f is bounded and $|f(x)|^2 \leq \|f\| f(x^*x)$;
- (v) if A has an identity element, then $|f(x)|^2 \leq f(1)f(x^*x)$.

PROOF. [11, p.213] For all $x, y \in A$, and for all $\alpha, \beta \in \mathbb{C}$,

$$0 \leq f((\alpha x + \beta y)^*(\alpha x + \beta y))$$

$$(1) = |\alpha|^2 f(x*x) + \alpha \bar{\beta} f(y*x) + \bar{\alpha} \beta f(x*y) + |\beta|^2 f(y*y).$$

Since $f(x*x) \geq 0$ and $f(y*y) \geq 0$ are real, $\alpha \bar{\beta} f(y*x) + \bar{\alpha} \beta f(x*y)$ is real for all α, β . Take $\alpha = \beta = 1$, then $f(y*x) + f(x*y)$ is real and so $\text{Im}(f(y*x)) = -\text{Im}(f(x*y))$. Take $\alpha = 1, \beta = i$, then $-i f(y*x) + i f(x*y)$ is real and so $\text{Re}(f(y*x)) = \text{Re}(f(x*y))$. Therefore $f(y*x) = \overline{f(x*y)}$.

This proves (ii). To prove (i), let α be real and

$\beta = f(y*x)$, then by (1) we have

$$\begin{aligned} & \alpha^2 f(x*x) + |f(y*x)|^2 + \alpha f(y*x) f(x*y) + |f(y*x)|^2 f(y*y) \\ & = \alpha^2 f(x*x) + 2\alpha |f(y*x)|^2 + |f(y*x)|^2 f(y*y) \geq 0. \end{aligned}$$

Since this holds for all real α , the discriminant of the quadratic is negative or zero, whence $|f(y*x)|^2 \leq f(x*x)f(y*y)$.

This proves (i).

(iv): [11, p.246] We prove that f is bounded. For this it is sufficient to prove boundedness on A_h . Note that, if $h \in A_h$, then there exists two hermitian elements h_+, h_- with non-negative spectra such that $h = h_+ - h_-$, $h_+ h_- = h_- h_+ = 0$ [11, p.243]. Let $h_a = h_+ + h_-$. It follows that $\|h\| = \|h_a\|$, $\pm h \leq h_a$ and, since f is positive, $|f(h)| \leq f(h_a)$. Therefore, if f is not bounded on A_h , then there exists a sequence h_n of elements of A_h such that $h_n \geq 0$, $\|h_n\| = 1$, and $f(h_n) \geq 2^n$ for each n . Define $h = \sum_{k=1}^n 2^{-k} h_k$. Then $h \in A_h$ and $h - \sum_{k=1}^n 2^{-k} h_k = \sum_{k=n+1}^{\infty} 2^{-k} h_k = u_n$, say. Then, by [11, Lemma (4.7.10) and Corollary (4.7.13)], $u_n \geq 0$. Therefore, $f(h) \geq \sum_{k=1}^n 2^{-k} f(h_k) \geq n$ for all n . This is impossible since $f(h)$ is a finite

number; hence f must be bounded. Now, let $\{e_\lambda\}$ be an approximate identity in A consisting of hermitian elements [11, Theorem (4.8.14)]. Then, by (i), $|f(e_\lambda x)|^2 \leq f(e_\lambda^2) f(x^*x) \leq \|f\| f(x^*x)$. Since f is bounded, $\lim_\lambda f(e_\lambda x) = f(x)$. Hence $|f(x)|^2 \leq \|f\| f(x^*x)$. This proves (iv). (iii) follows from (ii) using (iv) and the approximate identity in A . If A has an identity element, put $y=1$ in (i) to obtain (v).

The inequality (i) in Theorem (1.1.1) is called the Cauchy-Schwartz inequality.

THEOREM (1.1.2). Let A be a C^* -algebra with identity. A linear functional f on A is positive if and only if $\|f\| = f(1)$.

PROOF. [10, pp.189-190; 11, p.247]. Let f be a positive linear functional on A , and let $x = x^*$ in A with $\|x\| < 1$. Then by [3, Lemma (2.1.3)], $1 - x$ is of the form y^*y for some $y \in A$. Hence $f(1 - x) = f(y^*y) \geq 0$. Consequently, $|f(1)| \geq f(x)$. Substituting here $-x$ in place of x , we have $|f(1)| \geq -f(x)$. Hence $|f(x)| \leq f(1)$. Next, let x be an arbitrary hermitian element and put $x_1 = (\|x\| + \varepsilon)^{-1} x$, where $\varepsilon > 0$. It is clear that $x_1 = x_1^*$ and $\|x_1\| < 1$, and so

$$|f(x_1)| \leq f(1) \quad \text{or} \quad |f(x)| \leq f(1)(\|x\| + \varepsilon).$$

Hence $|f(x)| \leq f(1)\|x\|$ since ε is arbitrary.

Now suppose that x is an arbitrary element of A . Then x^*x is a hermitian element; consequently,

$$(2) \quad f(x*x) \leq f(1) \|x*x\| \leq f(1) \|x\|^2.$$

But, as f is positive, by Theorem (1.1.1) (v) and (2), we have

$$|f(x)|^2 \leq f(1) f(x*x) \leq f(1)^2 \|x\|^2,$$

and so $|f(x)| \leq f(1) \|x\|$. Hence $\|f\| \leq f(1)$. On the other hand, by the definition of $\|f\|$, $f(1) \leq \|f\|$. Hence $\|f\| = f(1)$.

Conversely, suppose $\|f\| = f(1)$. Without loss of generality, we may assume that $f(1) = 1$. Write $f = f_1 + if_2$ where f_1, f_2 are hermitian. As $f(1), f_1(1)$ and $f_2(1)$ are real, $f_2(1) = 0$. We now show that $f_2 = 0$. To do this, let x be an arbitrary hermitian element of A , and set $u = \lambda - ix$ where λ is an arbitrary real number. Then

$$(3) \quad \|u\|^2 = \|u*u\| \leq \lambda^2 + \|x\|^2.$$

Also,

$$\begin{aligned} |f(u)|^2 &= |(f_1 + if_2)(\lambda + ix)|^2 \\ &= \lambda^2 + 2\lambda f_2(x) + f_2^2(x) + f_1^2(x). \end{aligned}$$

Therefore, from (3) we obtain

$$\begin{aligned} \|u\|^2 &\leq |f(u)|^2 - 2\lambda f_2(x) - f_2^2(x) - f_1^2(x) + \|x\|^2 \\ &\leq |f(u)|^2 - 2\lambda f_2(x) + \|x\|^2 \\ &\leq \|u\|^2 - 2\lambda f_2(x) + \|x\|^2. \end{aligned}$$

Hence $2\lambda f_2(x) \leq \|x\|^2$. Since λ and x are arbitrary, $f_2 = 0$, and so $f = f_1$. Thus f is a hermitian linear functional. Next, we shall show that f is positive.

Suppose on the contrary that there exists an element x of A such $f(x*x) < 0$. We can assume that $0 \leq x*x \leq 1$ so that $0 \leq 1 - x*x \leq 1$, and hence $\|1 - x*x\| \leq 1$. Then $1 = f(1) = f(1 - x*x + x*x) = f(1 - x*x) + f(x*x) < f(1 - x*x) \leq \|1 - x*x\| \leq 1$ since $f(x*x) < 0$. This is a contradiction and hence f is positive.

THEOREM (1.1.3). For each positive linear functional on a C^* -algebra A , we have:

- (i) $|f(y*xy)| \leq \|x\|f(y*y)$ for all $x, y \in A$;
- (ii) $\|f\| = \sup_{\|y\| \leq 1} f(y*y)$.

PROOF. [3, p.23] For each $y \in A$ define a mapping F on A by $F(x) = f(y*xy)$ for all $x \in A$. Then F is a continuous linear functional on A and it is positive since for all $x \in A$,

$$F(x*x) = f(y*(x*x)y) = f((xy)*(xy)) \geq 0.$$

Hence by [3, Proposition (2.1.5)], $\|F\| = f(y*y)$, and $|f(y*xy)| = |F(x)| \leq \|F\|\|x\| = f(y*y)\|x\|$ for all $x, y \in A$.

This proves (i).

Next, we shall prove (ii). It follows from Theorem (1.1.1) (iv) that $|f(y)|^2 \leq \|f\|f(y*y)$ for all $y \in A$.

By taking the supremum over all $y \in A$ with $\|y\| \leq 1$

on both sides of this inequality, we obtain

$$\|f\|^2 \leq \|f\| \sup_{\|y\| \leq 1} f(y*y), \text{ hence } \|f\| \leq \sup_{\|y\| \leq 1} f(y*y). \text{ On the other}$$

hand, for each $y \in \Lambda$ with $\|y\| \leq 1$, we have $|f(y^*y)| \leq \|f\| \|y^*y\| = \|f\| \|y\|^2 \leq \|f\|$. Consequently, we have $\|f\| = \sup \{f(y^*y) : \|y\| \leq 1\}$.

Let A be a Banach $*$ -algebra and A_h be the set of all hermitian elements of A . Then A_h is a real Banach space. Let f be a hermitian continuous linear functional on A and $f|_{A_h}$ denote the restriction of f to A_h . Then the mapping $f \rightarrow g = f|_{A_h}$ is a continuous isomorphism between the set of all hermitian continuous linear functionals on A and the set of all real-valued continuous linear functionals on A_h . In fact, we have $\|f\| = \|g\|$ ([3], p.5).

THEOREM (1.1.4). Let A be a C^* -algebra and g a hermitian continuous linear functional on A . Then there exist two positive linear functionals f and f' on A such that $g = f - f'$ and $\|g\| = \|f\| + \|f'\|$; the positive linear functionals f and f' are uniquely determined by these properties.

PROOF. [6, pp.98-99] Let B be the set of all positive linear functionals on A with norm ≤ 1 . Then, by ([3], Proposition (2.5.5), p.37), B is $\sigma(A', A)$ -compact (i.e., B is weak*-compact). Since $\sigma(A_h', A_h)$ is the topology on A_h' induced by the topology $\sigma(A', A)$ on A' , by the remark above, B can be considered as a $\sigma(A_h', A_h)$ -compact convex subset of the dual space A_h'

of A_h . The polar B° of B in A_h is equal to the closed unit ball S of A_h . In fact, let $C(B)$ be the set of all continuous real-valued functions on B . For each x in A_h , define a continuous real-valued function F_x on B by $F_x(f) = f(x)$ for all f in B . Then, by ([3], Proposition (2.6.3), p.39), the mapping $\varphi: x \rightarrow F_x$ is an isometric isomorphism of A_h into $C(B)$. Hence S is isometrically isomorphic to the set

$$\begin{aligned} \{F_x \in C(B): \|F_x\| \leq 1\} &= \{F_x \in C(B): \sup_{f \in B} |F_x(f)| \leq 1\} \\ &= \{F_x \in C(B): \sup_{f \in B} |f(x)| \leq 1\}. \end{aligned}$$

Now, by definition, $B^\circ = \{x \in A_h: \sup_{f \in B} |f(x)| \leq 1\}$.

Therefore $B^\circ = S$. The closed unit ball S' of A_h' is equal to the set $\{f' \in A_h': \sup_{x \in S} |f'(x)| \leq 1\}$, which is

the polar of S in A_h' , i.e., $S' = S^\circ$. Thus S' is the bipolar of B in A_h' . By ([13], Corollary 1, p.36), S' is the $\sigma(A_h', A_h)$ -closed absolutely convex envelope of B . Since B is $\sigma(A', A)$ -closed and convex then, by ([15], Theorem (3.4-F), p.132), S' is the absolutely convex envelope of B . Therefore, by ([15], Theorem (3.4-E), p.132), each element of S' is of the form $\alpha u - \beta v$ with u, v in B and α, β positive scalars

such that $\alpha + \beta \leq 1$ [15, p.132]. Let us assume that $\|g\| = 1$. Then the restriction g_h of g to A_h is in S' . Thus $g_h = f - f'$ with $f = \alpha u$, $f' = \beta v$. Hence $g = f - f'$ where $f \geq 0$, $f' \geq 0$. We also have $1 = \|g\| \leq \|f\| + \|f'\| \leq \alpha + \beta \leq 1$. Hence $\|g\| = \|f\| + \|f'\|$. The uniqueness of such f, f' follows from [6, Theorem 1, p.97].

§ 1.2. Representations associated with positive linear functionals.

Let H be a complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators on H into itself with the operator bounded norm. $B(H)$ is a C^* -algebra with the involution given by the adjoint.

A homomorphism h from a C^* -algebra A into a C^* -algebra is called a $*$ -homomorphism if $h(x^*) = h(x)^*$ for all $x \in A$. A $*$ -homomorphism of A into $B(H)$ is called a representation of A on H . Let $L: x \rightarrow L(x)$ be a representation of a C^* -algebra A on the Hilbert space H . Then L is called a cyclic representation if there exists a vector φ in H such that the set $\{L(x)\varphi : x \in A\}$ is dense in H . The vector φ is called a cyclic vector for L . A representation L of A on H is said to be faithful if it is one-to-one.

Let φ be a non-zero vector of H and L a representation of A on H . For each $x \in A$, let $p(x) = (L(x)\varphi, \varphi)$. Then p is a positive linear

functional on A since

$$\begin{aligned} p(x*x) &= (L(x*x)\varphi, \varphi) \\ &= (L(x)^*L(x)\varphi, \varphi) \\ &= (L(x)\varphi, L(x)\varphi) \geq 0 \end{aligned}$$

for all $x \in A$.

THEOREM (1.2.1). Let L be a representation of a C^* -algebra A on a Hilbert space H . Then $\|L(x)\| \leq \|x\|$ for all $x \in A$. If L is a faithful representation, then $\|L(x)\| = \|x\|$ for all $x \in A$.

PROOF. By [3, Proposition (1.3.7)], we have $\|L(x)\| \leq \|x\|$ for all $x \in A$. Now if L is faithful by [3, Proposition (1.8.1)], we have $\|L(x)\| = \|x\|$ for all $x \in A$.

THEOREM (1.2.2). Let L and L' be two cyclic representations of A on the Hilbert space H and H' , and let φ, φ' be cyclic vectors for L and L' , respectively. For each $x \in A$, let

(1) $p(x) = (L(x)\varphi, \varphi), \quad p'(x) = (L'(x)\varphi', \varphi')$.

If $p(x) = p'(x)$ for all $x \in A$, then L and L' are equivalent, i.e., there exists an isometric isomorphism U on H onto H' such that $UL(x) = L'(x)U$ for all $x \in A$.

PROOF. [10, p.242] Let u be the mapping of the set $\{L(x)\varphi : x \in A\}$ into the set $\{L'(x)\varphi' : x \in A\}$ defined as follows: for each $x \in A$, let

$$u(L(x)\varphi) = L'(x)\varphi'.$$

Thus, if we let $L(x)\varphi = \xi$ and $L'(x)\varphi' = \xi'$, we have $u(\xi) = \xi'$. It is easy to see that $\|u(\xi)\| = \|\xi\|$; in fact,

$$\begin{aligned}\|\xi\|^2 &= (\xi, \xi) = (L(x)\varphi, L(x)\varphi) \\ &= (L(x^*x)\varphi, \varphi) \\ &= (L'(x)\varphi', L'(x)\varphi') \\ &= (\xi', \xi') \\ &= \|\xi'\|^2 = \|u(\xi)\|^2.\end{aligned}$$

Since the sets $\{L(x)\varphi : x \in A\}$ and $\{L'(x)\varphi' : x \in A\}$ are dense in H and H' respectively, u can be extended to an isometric isomorphism U of H onto H' . Moreover, for each $y \in A$, we have

$$\begin{aligned}UL(y)\xi &= UL(y)(L(x)\varphi) \\ &= UL(yx)\varphi \\ &= L'(yx)\varphi' \\ &= L'(y)L'(x)\varphi' \\ &= L'(y)U(\xi)\end{aligned}$$

which shows that $UL(y) = L'(y)U$ for all $y \in A$. Hence L and L' are equivalent and this completes the proof.

The theorem just proved shows that a cyclic representation L of a C^* -algebra A is determined up to an equivalence by the positive linear functional $p(x) = (L(x)\xi, \xi)$. The question arises whether or not, for each positive linear functional p on A , there exists a representation L^p of A such that $p(x) = (L^p(x)\xi, \xi)$ for all $x \in A$ and for some non-zero vector ξ of H . The answer to this question is in the affirmative. We shall construct L^p as follows [10, pp.242-245]:

Let \tilde{A} be the C^* -algebra obtained from A by the adjunction of an identity and \tilde{p} the canonical extension of p to \tilde{A} [3, Propositions (1.3.8) and (2.1.4)]. For all $x, y \in \tilde{A}$, let

$$(x, y) = \tilde{p}(y^*x).$$

It is clear that (x, y) is linear in x and conjugate linear in y . Since \tilde{p} is positive, $\tilde{p}(y^*x) = \overline{\tilde{p}(x^*y)}$ by Theorem (1.1.1); i.e., $(x, y) = \overline{(y, x)}$. The set $N = \{x \in \tilde{A} : (x, x) = 0\}$ is a left ideal of \tilde{A} . Let $H' = \tilde{A}/N$ and, for each $x \in \tilde{A}$, let x' be the coset in H' determined by x . H' with the inner product defined by $(x', y') = (x, y)$ is a pre-Hilbert space. Let H be its completion; H is a Hilbert space. Since N is a left ideal of \tilde{A} , for each $x \in \tilde{A}$, we may define a linear operator $L^{\tilde{p}}(x)$ on H' by setting $L^{\tilde{p}}(x)(y') = (xy)'$ for all $y' \in H'$. We shall show that $L^{\tilde{p}}(x)$ is bounded. By definition of the inner product (\cdot, \cdot) on H' , we have

$$\begin{aligned}
 \|L^{\tilde{p}}(x)(y')\|^2 &= (L^{\tilde{p}}(x)(y'), L^{\tilde{p}}(x)(y')) = ((xy)', (xy)') \\
 &= \tilde{p}((xy)*(xy)) \leq \tilde{p}(1) \|(xy)*(xy)\| \\
 &= \tilde{p}(1) \|xy\|^2 \leq \tilde{p}(1) \|x\|^2 \|y\|^2 \\
 &= \tilde{p}(1) \|x\|^2 \|y'\|^2 \leq \|x\|^2 \|y'\|^2.
 \end{aligned}$$

Thus $L^{\tilde{p}}(x)$ is bounded and

$$(3) \quad \|L^{\tilde{p}}(x)\| \leq \|x\|.$$

Consequently, this operator can be uniquely extended to a bounded operator on H having the same operator bound.

We denote this extension by the same symbol $L^{\tilde{p}}(x)$.

Therefore, the inequality (3) also remains valid

for the norm of the operator $L^{\tilde{p}}(x)$ on H . The mapping $x \rightarrow L^{\tilde{p}}(x)$ is a representation of \tilde{A} on H . In fact,

$$\begin{aligned}
 L^{\tilde{p}}(\alpha x + \beta y)(z') &= ((\alpha x + \beta y)z)' = \alpha(xy)' + \beta(yz)' \\
 &= \alpha L^{\tilde{p}}(x)(z') + \beta L^{\tilde{p}}(y)(z').
 \end{aligned}$$

It remains to prove that $(L^{\tilde{p}}(x))^* = L^{\tilde{p}}(x^*)$, i.e., that $(L^{\tilde{p}}(x)y', z') = (y', L^{\tilde{p}}(x^*)z')$ for all $y', z' \in H$.

But

$$\begin{aligned}
 (L^{\tilde{p}}(x)y', z') &= (y', L^{\tilde{p}}(x^*)z') = \tilde{p}\{z^*xy\} = \tilde{p}((x^*z)^*y) \\
 &= (y, x^*z) = (y', (x^*z)') \\
 &= (y', L^{\tilde{p}}(x^*)z').
 \end{aligned}$$

Thus $L^{\tilde{p}}$ is a representation. Moreover, $L^{\tilde{p}}$ is cyclic.

To see this, let $\varphi = 1 + N \in H'$ and let x be any element of \tilde{A} . Then $L^{\tilde{p}}(x)(\varphi) = (x1)' = x'$. Consequently, the set of all vectors of the form $L^{\tilde{p}}(x)(\varphi)$ for all $x \in \tilde{A}$ coincides with the set H' of all equivalence classes.

Since H' is dense in H , it follows that φ is a cyclic vector and $(L^{\tilde{P}}(x)\varphi, \varphi) = ((x1)', \varphi) = (x1, 1) = p(x)$. The restriction of $L^{\tilde{P}}$ to A is clearly a representation of A on H ; we denote this representation by L^P . Since the set $L^P(A)$ is the canonical image of A in H' , by [3, Proposition (2.1.5)(vii)], $L^P(A)$ is dense in H' and hence dense in H . Thus φ is a cyclic vector for L^P restricted to A . Also, $\|L^P(x)\| \leq \|x\|$ for all $x \in A$. We have thus proved the following theorem:

THEOREM (1.2.3). Let A be a C^* -algebra. To every cyclic representation L of A , with cyclic vector φ , there corresponds a positive functional $p(x) = (L(x)\varphi, \varphi)$. The representation L is defined uniquely up to equivalence by the functional p . Conversely, to every positive linear functional p on A there corresponds a cyclic representation L^P and a cyclic vector φ such that $p(x) = (L^P(x)\varphi, \varphi)$ for all $x \in A$. Moreover, $\|L^P(x)\| \leq \|x\|$ for all $x \in A$.

Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a family of Hilbert spaces and let H be the family of all functions (ξ_λ) defined on Λ such that:

(i) for each $\lambda \in \Lambda$, $\xi_\lambda \in H_\lambda$;

(ii) (ξ_λ) contains at most a countable number of elements

which are different from zero;

(iii) $\sum_{\lambda} \|\xi_\lambda\|^2 < \infty$.

We define addition, multiplication by scalars and inner product in H by the following formulae:

$$\begin{aligned}(\xi_\lambda) + (\eta_\lambda) &= (\xi_\lambda + \eta_\lambda), \\ \alpha(\xi_\lambda) &= (\alpha \xi_\lambda), \\ ((\xi_\lambda), (\eta_\lambda)) &= \sum_\lambda (\xi_\lambda, \eta_\lambda).\end{aligned}$$

Then with these operations, H is a Hilbert space, called the Hilbert sum (or direct sum) of the Hilbert spaces H_λ and we denote it by $H = \bigoplus_{\lambda \in \Lambda} H_\lambda$.

Let A be a C^* -algebra, $\{H_\lambda\}_{\lambda \in \Lambda}$ a family of Hilbert spaces and $\{L_\lambda\}_{\lambda \in \Lambda}$ a family of representations of A such that L_λ is a representation of A on H_λ for each $\lambda \in \Lambda$. Let H be the Hilbert sum of H_λ . Let $(\xi_\lambda) \in H$ and $x \in A$. Then, since for each $x \in A$ and each $\lambda \in \Lambda$, $\|L_\lambda(x)\| \leq \|x\|$, it follows that $(L_\lambda(x)\xi_\lambda) \in H$; in fact $\|(L_\lambda(x)\xi_\lambda)\|^2 = ((L_\lambda(x)\xi_\lambda), (L_\lambda(x)\xi_\lambda)) = \sum_\lambda (L_\lambda(x)\xi_\lambda, L_\lambda(x)\xi_\lambda) = \sum_\lambda \|L_\lambda(x)\xi_\lambda\|^2 \leq \sum_\lambda \|L_\lambda(x)\|^2 \|\xi_\lambda\|^2 \leq \|x\|^2 \sum_\lambda \|\xi_\lambda\|^2$. Now, let $L(x)$ be the operator on H defined by $L(x)(\xi_\lambda) = (L_\lambda(x)\xi_\lambda)$. Then it is easy to see that $L(x)$ is a bounded linear operator on H for every $x \in A$, with $\|L(x)\| \leq \|x\|$. Since each L_λ is a representation of A on H , it is easily seen that L is a representation of A on H which we call the Hilbert sum (or direct sum) of the representations L_λ and denote it by $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$.

THEOREM (1.2.4). Let Q be the set of all positive linear functionals defined on a C^* -algebra A . For each $p \in Q$, let L^p be the corresponding representation of A , H_p its underlying Hilbert space and φ_p the cyclic vector. Let

$L = \bigoplus_{p \in Q} L^p$, the Hilbert sum of L_p , and $H = \bigoplus_{p \in Q} H_p$,

the Hilbert sum of H_p . Then the norm closure of $\{L(x)H: x \in A\}$ is H .

PROOF. [3, p.43] Considering \mathcal{G}_p as a vector of H , we have

$$\text{cl}(L(A)H) \supseteq \text{cl}(L(A)\mathcal{G}_p) = H_p \quad (p \in Q)$$

whence

$$H \supseteq \text{cl}(L(A)H) \supseteq \bigoplus_{p \in Q} H_p = H,$$

where $\text{cl}(L(A)H)$ and $\text{cl}(L(A)\mathcal{G}_p)$ denote the norm closures of $L(A)H$ in H and $L(A)\mathcal{G}_p$ in H_p respectively.

DEFINITION. The mapping L defined in Theorem (1.2.4) is called the universal representation of A , and H is called the underlying Hilbert space of L .

REMARK: By [3, Proposition (2.7.3)], $\|L(x)\| = \|x\|$ for all $x \in A$.

these relations, see [10, pp.441-444].

Since, for all ξ, η in H and $T \in B(H)$, we have

$$4(T\xi, \eta) = (T(\xi + \eta), \xi + \eta) - (T(\xi - \eta), \xi - \eta) \\ + i(T(\xi + i\eta), \xi + i\eta) - i(T(\xi - i\eta), \xi - i\eta),$$

it follows that the weak and ultraweak topologies are also defined by the seminorms of the form $T \rightarrow |(T\xi, \xi)|$

$(\xi \in H)$ and $T \rightarrow \left| \sum_{i=1}^{\infty} (T\xi_i, \xi_i) \right|$ ($\xi_i \in H, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$)

respectively.

The above topologies on $B(H)$ are all compatible with the vector space structure of $B(H)$, but not in general with the algebraic structure of $B(H)$. In particular, $(S, T) \rightarrow ST$ is not always strongly continuous, however, if we denote by $B_1(H)$ the unit ball of $B(H)$, the equality

$$ST - S_0T_0 = S(T - T_0) + (S - S_0)T_0$$

shows that the mapping $(S, T) \rightarrow ST$ on $B_1(H) \times B(H)$ into $B(H)$ is strongly continuous. This mapping is also ultrastrongly continuous. The mappings $S \rightarrow ST, T \rightarrow ST$ on $B(H)$ into itself are weakly, ultraweakly and ultrastrongly continuous. The mapping $T \rightarrow T^*$ is only weakly and ultraweakly continuous [7, Chapter 12].

THEOREM (2.1.1). Strong (resp. weak) and ultrastrong (resp. ultraweak) topologies coincide on each bounded

subset M of $B(H)$.

PROOF. [2, p.36] Without loss of generality, we may assume that $M = \{T \in B(H) : \|T\| < 1\}$. Let N be an ultrastrong neighbourhood of 0 in M . We may assume that

$$N = \left\{ T \in M : \left| \sum_{i=1}^{\infty} (T\xi_i, \xi_i) \right| < 1, \xi_i \in H, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty \right\}.$$

Since $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, there exists a positive integer n

such that $\sum_{i=n}^{\infty} \|\xi_i\|^2 < \frac{1}{2}$. Let $N' = \left\{ T \in M : \sum_{i=1}^{n-1} \|T\xi_i\|^2 < \frac{1}{2} \right\}$.

It is clear that N' is a strong neighbourhood of 0 in M . For each $T \in N'$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \|T\xi_i\|^2 &= \sum_{i=1}^{n-1} \|T\xi_i\|^2 + \sum_{i=n}^{\infty} \|T\xi_i\|^2 < \frac{1}{2} + \|T\| \sum_{i=n}^{\infty} \|\xi_i\|^2 \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Thus $T \in N$ and $N' \subseteq N$. Hence N is a strong neighbourhood of 0 in M and hence the strong topology coincides with the ultrastrong topology on the bounded set M .

Next, we shall show that the ultraweak and weak topologies coincide on M . Let

$$N = \left\{ T \in M : \left| \sum_{i=1}^{\infty} (T\xi_i, \xi_i) \right| < 1, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \xi_i \in H \right\}$$

be an ultraweak neighbourhood of 0 in M . Since

$\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, there exists an integer m such that

$$\sum_{i=m}^{\infty} \|\xi_i\|^2 < \frac{1}{2}. \text{ Let } N' = \left\{ T \in M : \left| \sum_{i=1}^m (T\xi_i, \xi_i) \right| < \frac{1}{2} \right\}.$$

Then N' is a weak neighbourhood of 0 in M . For each $T \in N'$, we have

$$\begin{aligned} \left| \sum_{i=1}^{\infty} (T \xi_i, \xi_i) \right| &\leq \left| \sum_{i=1}^n (T \xi_i, \xi_i) \right| + \left| \sum_{i=n+1}^{\infty} (T \xi_i, \xi_i) \right| \\ &< \frac{1}{2} + \|T\| \sum_{i=n+1}^{\infty} \|\xi_i\|^2 < \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Hence $T \in N$ and $N' \subseteq N$. This shows that ultraweak topology coincides with the weak topology on M .

NOTATION:

(i) For each pair of vectors ξ, η of H and each $T \in B(H)$, let $\omega_{\xi, \eta}(T) = (T \xi, \eta)$ and $\omega_{\xi, \xi} = \omega_{\xi}$. It is clear that ω_{ξ} is weakly continuous positive linear functional on $B(H)$.

(ii) Let X' denote the dual (conjugate) space of a locally convex topological vector space X .

(iii) For each *-subalgebra A of $B(H)$ let A_{\sim} denote the set of all weakly continuous linear functionals on A and A_{*} denote the (norm) closure of A_{\sim} in A' .

(iv) For each couple of vectors ξ, η of H , let $[\xi, \eta]$ denote linear mapping of rank one on H into itself given by $[\xi, \eta] \zeta = (\zeta, \xi) \eta$ for all $\zeta \in H$. Since $\|[\xi, \eta] \zeta\| \leq \|\xi\| \|\eta\| \|\zeta\|$, $[\xi, \eta]$ is continuous and hence it is completely continuous and $[\xi, \eta]^* = [\eta, \xi]$.

(v) We shall write B for $B(H)$ throughout this chapter.

THEOREM (2.1.2). Let f be a linear functional on $B(H)$.

Then the following statements are equivalent:

$$(i) \quad f = \sum_{i=1}^{\infty} \omega_{\varphi_i, \psi_i} \quad \text{with} \quad \sum_{i=1}^{\infty} \|\varphi_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\psi_i\|^2 < \infty.$$

(ii) f is ultraweakly continuous.

(iii) f is ultrastrongly continuous.

Moreover, if f is ultrastrongly continuous, then $f \in B_*$.

PROOF. [3, p.38] (i) \Rightarrow (ii) \Rightarrow (iii) is clear. We have only to prove that (iii) \Rightarrow (i). Suppose that f is ultrastrongly continuous. Since $B(H)$ with the ultrastrong topology is a locally convex space, then there exists a ultrastrongly continuous seminorm p on $B(H)$ such that $|f(T)| \leq p(T)$ for all $T \in B(H)$ [15, p.143]. That is, there exists a sequence $\{\varphi_i\}$ of vectors of H with $\sum_{i=1}^{\infty} \|\varphi_i\|^2 < \infty$ such that

$$|f(T)| \leq \left(\sum_{i=1}^{\infty} \|T\varphi_i\|^2 \right)^{\frac{1}{2}} \quad \text{for all } T \in B(H). \quad \text{Let } \bar{H} = \bigoplus_{i=1}^{\infty} H_i$$

where $H_i = H$ for all i . For each $T \in B(H)$, we define an operator \bar{T} on \bar{H} by $\bar{T}(\bar{\varphi}) = (T\varphi_i)$ for all $\bar{\varphi} = (\varphi_i)$

in \bar{H} . It is clear that the set $M = \{\bar{T}(\bar{\varphi}) : T \in B(H)\}$

is a subspace of \bar{H} . Define a linear functional θ on M by $\theta(\bar{T}(\bar{\varphi})) = f(T)$. Then as $|\theta(\bar{T}(\bar{\varphi}))| = |f(T)| \leq \left(\sum_{i=1}^{\infty} \|T\varphi_i\|^2 \right)^{\frac{1}{2}}$

$= \|(T\varphi_i)\| = \|\bar{T}(\bar{\varphi})\|$ for all $\bar{\varphi} = (\varphi_i) \in \bar{H}$. θ is continuous

on M . Then by Hahn-Banach Theorem, θ can be extended to

a continuous linear functional θ' on \bar{H} . By Riesz

representation theorem, there exists $\bar{\eta} = (\eta_i) \in \bar{H}$:

$$f(T) = \theta(\bar{T}(\bar{\varphi})) = (\bar{T}(\bar{\varphi}), \bar{\eta}) = \sum_{i=1}^{\infty} (T\varphi_i, \eta_i).$$

Hence, $f = \sum_{i=1}^{\infty} \omega_{\varphi_i, \eta_i}$. This proves (i).

Finally, suppose that f is ultrastrongly continuous,

i.e., $f = \sum_{i=1}^{\infty} \omega_{\varphi_i, \eta_i}$ and $f_n = \sum_{i=1}^n \omega_{\varphi_i, \eta_i}$ where (φ_i)

and (η_i) are sequences of vectors of H with $\sum_{i=1}^{\infty} \|\varphi_i\|^2$

and $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$. Then f_n is weakly continuous since

each $\omega_{\varphi_i, \eta_i}$ is weakly continuous. Now,

$$\begin{aligned} \|f - f_n\| &= \sup_{\|T\| \leq 1} |f(T) - f_n(T)| = \sup_{\|T\| \leq 1} \left| \sum_{i=n+1}^{\infty} (T\varphi_i, \eta_i) \right| \\ &\leq \sum_{i=n+1}^{\infty} \|\varphi_i\| \|\eta_i\| \leq \sum_{i=1}^{\infty} \|\varphi_i\| \|\eta_i\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

since $\sum_{i=1}^{\infty} \|\varphi_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$.

We observe that if X is a linear space and τ_1, τ_2 are two locally convex topologies on X such that (X, τ_1) and (X, τ_2) have the same continuous linear functionals, then a convex set is closed in (X, τ_1) iff it is closed in (X, τ_2) [4, p.418]. It follows from Theorem (2.1.2) that

COROLLARY (2.1.3). The ultraweak and ultrastrong closure of a convex set in $B(H)$ coincide.

THEOREM (2.1.4). Let $\xi_i, \eta_i, \xi'_j, \eta'_j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ be vectors of H and $g = \sum_{i=1}^m [\xi_i, \eta_i]$,

$$g' = \sum_{j=1}^n [\xi'_j, \eta'_j]. \quad \text{If } g = g', \text{ then } \sum_{i=1}^m \omega_{\xi_i, \eta_i} = \sum_{j=1}^n \omega_{\xi'_j, \eta'_j}.$$

PROOF. [2, p.39] For each couple of vector γ, γ' of H , we have

$$\begin{aligned} \sum_{i=1}^m ([\gamma, \gamma'] \xi_i, \eta_i) &= \sum_{i=1}^m ((\xi_i, \gamma) \gamma', \eta_i) = \sum_{i=1}^m (\gamma', (\gamma, \xi_i) \eta_i) \\ &= (\gamma', \sum_{i=1}^m [\xi_i, \eta_i] \gamma) = (\gamma', \sum_{j=1}^n [\xi_j', \eta_j'] \gamma) \\ &= \sum_{j=1}^n (\gamma', (\gamma, \xi_j') \eta_j') = \sum_{j=1}^n ([\gamma, \gamma'] \xi_j', \eta_j') \end{aligned}$$

Let F be the set of all operators of finite rank on H . Since the set of all operators which commutes with all elements of F is CI , where I is the identity operator in $B(H)$ and C is the field of complex numbers, F is weakly dense in $B(H)$ [2, p.44]. By the weak continuity and linearity, we have

$$\sum_{j=1}^m (T\xi_j, \eta_j) = \sum_{k=1}^n (T\xi_k', \eta_k')$$

for all $T \in B(H)$, i.e.,

$$\sum_{j=1}^m \omega_{\xi_j, \eta_j} = \sum_{k=1}^n \omega_{\xi_k', \eta_k'}$$

THEOREM (2.1.5). Let f be a weakly continuous linear functional on B . Then there exist two orthonormal systems (e_1, e_2, \dots, e_n) , $(e_1', e_2', \dots, e_n')$ and positive scalars λ_i ($i = 1, 2, \dots, n$) such that

$$f = \sum_{i=1}^n \lambda_i \omega_{e_i, e_i'} \quad \text{and} \quad \|f\| = \sum_{i=1}^n \lambda_i$$

PROOF. [3, p.39] Since f is weakly continuous,

$$f = \sum_{j=1}^m \omega_{\xi_j, \eta_j} \quad [4, p.421]. \quad \text{The mapping } g = \sum_{j=1}^m [\xi_j, \eta_j]$$

is continuous, and is of finite rank, hence it is completely continuous. Then there exist two orthonormal systems (e_i) , (e'_i) , $i = 1, 2, \dots, n$ and positive scalars λ_i such that

$$g = \sum_{i=1}^n \lambda_i [e_i, e'_i] = \sum_{i=1}^n [\lambda_i e_i, e'_i]$$

[13, p.18]. By Theorem (2.1.4), we have

$$f = \sum_{i=1}^n \omega_{\lambda_i e_i, e'_i} = \sum_{i=1}^n \lambda_i \omega_{e_i, e'_i}$$

Thus,

$$|f(T)| \leq \sum_{i=1}^n \lambda_i \|T\| \|e_i\| \|e'_i\| = \|T\| \sum_{i=1}^n \lambda_i$$

Hence $\|f\| \leq \sum_{i=1}^n \lambda_i$. Let $T' = \sum_{i=1}^n [e_i, e'_i]$. Then $\|T'\| \leq 1$.

In fact, let $(e_\alpha)_{\alpha \in \Delta}$ be an orthonormal basis of H , which contains (e_i) , $i = 1, 2, \dots, n$. For each ξ in H , write $\xi = \sum_{k=1}^{\infty} (\xi, e_k) e_k$, then $T'(\xi) = \sum_{i=1}^n (\xi, e_i) e'_i$.

$\|T'(\xi)\|^2 \leq \|\xi\|^2$. Therefore we have

$$f(T') = \sum_{i=1}^n \lambda_i, \text{ i.e., } \|f\| \geq \sum_{i=1}^n \lambda_i$$

THEOREM (2.1.6). Each element of B_* is ultraweakly continuous.

PROOF. [2, p.40] Let f be an element of B_* and let $\{f_k\}$ be a sequence of weakly continuous linear functionals such that $f = \sum_{k=1}^{\infty} f_k$ and $\|f_k\| \leq 2^{-k}$. By Theorem (2.1.5),

$$f_k = \sum_{i=1}^{n_k} \lambda_i^k \omega_{e_i^k}, \quad e_i^k \quad \text{with} \quad \|e_i^k\| = \|e_i^{k'}\| = 1, \quad \lambda_i^k > 0 \quad \text{and}$$

$$\sum_{i=1}^{n_k} \lambda_i^k = \|f_k\| \leq 2^{-k}. \quad \text{Hence} \quad f = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \omega_{(\lambda_i^k)^{\frac{1}{2}} e_i^k}, (\lambda_i^k)^{\frac{1}{2}} e_i^k \right),$$

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \|(\lambda_i^k)^{\frac{1}{2}} e_i^k\|^2 \right) \leq \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \lambda_i^k \right)$$

$$< \sum_{k=1}^{\infty} 2^{-k} < \infty. \quad \text{Similarly} \quad \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \|(\lambda_i^k)^{\frac{1}{2}} e_i^{k'}\|^2 \right) < \infty.$$

Hence f is ultraweakly continuous.

COLLARY (2.1.7). B_* coincides with the set of all ultraweakly continuous linear functionals on $B(H)$.

PROOF. By Theorem (2.1.2) and Theorem (2.1.6).

THEOREM (2.1.8). A linear functional on $B(H)$ is weakly continuous if and only if it is strongly continuous.

PROOF. [4, p.477] Since the strong topology on $B(H)$ is stronger than the weak topology on $B(H)$, a weakly continuous linear functional on $B(H)$ is strongly continuous.

Conversely, let f be a strongly continuous linear functional on $B(H)$. Then there exists a finite subset $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of H and an $\varepsilon > 0$ such that $\|T\varphi_i\| < \varepsilon$, $T \in B(H)$, $i = 1, 2, \dots, n$ implies $|f(T)| < 1$.

Consider the Hilbert space $H_n = H \oplus H \oplus \dots \oplus H$ of n -tuples $\Psi = (\psi_1, \psi_2, \dots, \psi_n)$ with $\psi_i \in H$, $i = 1, 2, \dots, n$; the norm in H_n is $\|\Psi\| = \max_{1 \leq i \leq n} \|\psi_i\|$. Define $S: B(H) \rightarrow H_n$

by $T \rightarrow (T\varphi_1, T\varphi_2, \dots, T\varphi_n)$ for all $T \in B(H)$. For each $\Phi = (T\varphi_1, T\varphi_2, \dots, T\varphi_n)$, put $F(\Phi) = f(T)$. Then F is

a functional on $S(B(H))$. Since $\|F\| = \sup_i \|T\varphi_i\| < \delta \mathcal{E}$ implies $|f(T)| < \delta$. Hence F is bounded and so continuous on $S(B(H))$.

By Hahn-Banach Theorem, F has a continuous extension F_1 defined on all of H_n . Now each Ψ in H_n can be uniquely in the form $\Psi = \psi_1 + \psi_2 + \dots + \psi_n$ with $\psi_i \in H$, $i = 1, 2, \dots, n$ and so

$$\begin{aligned} F_1(\Psi) &= F_1(\psi_1, 0, \dots, 0) + \dots + F_1(0, \dots, 0, \psi_n) \\ &= \psi_1^*(\psi_1) + \dots + \psi_n^*(\psi_n) \end{aligned}$$

where $\psi_i^* = F_1|_{H_i}$, $i = 1, 2, \dots, n$. By Riesz's representation theorem, there exist $\xi_1, \xi_2, \dots, \xi_n$ of H such that

$$\psi_i^*(\psi_i) = (\psi_i, \xi_i), \quad i = 1, 2, \dots, n; \text{ consequently,}$$

$$\begin{aligned} f(T) &= F_1(S(T)) \text{ has the form } f(T) = \sum_{i=1}^n \psi_i^*(T\varphi_i) = \\ &= \sum_{i=1}^n \psi_i^*(T\varphi_i, \xi_i) = \sum_{i=1}^n \omega_{\varphi_i, \xi_i}(T) \text{ and it is clear that } f \end{aligned}$$

is weakly continuous.

COROLLARY(2.1.9). Let M be an ultraweakly closed subspace of $B(H)$, and f a linear functional on M . Then the following statements are equivalent:

(i) f is weakly continuous.

(ii) f is strongly continuous.

(iii) $f = \sum_{i=1}^n \omega_{\varphi_i, \psi_i}$ where φ_i, ψ_i , $i = 1, 2, \dots, n$ are

vectors of H .

COROLLARY (2.1.10). The weak and strong closures of a convex set in $B(H)$ coincide.

PROOF. This follows from Theorem (2.1.8) and the remark preceding Corollary (2.1.3).

THEOREM (2.1.11). $B(H)$ is the dual of B_* (considered as Banach space).

PROOF. [2, p.37] Let E be the dual space of B_* . For each $T \in B(H)$, define \hat{T} in E by $\hat{T}(f) = f(T)$ for all $f \in B_*$. Let θ be a linear mapping on $B(H)$ into E such that $\theta(T) = \hat{T}$. Then θ is a continuous linear function on $(B(H), \sigma(B(H), B_*))$ into $(E, \sigma(E, B_*))$. In fact, for each $T_0 \in B(H)$, the set

$$N = \{G \in E: |\hat{T}_0(f) - G(f)| < \varepsilon, f \in B_*, \varepsilon > 0\}$$

is a neighbourhood of $\theta(T_0) = \hat{T}_0$ in $\sigma(E, B_*)$. Let

$$N' = \{T \in B(H): |f(T_0) - f(T)| < \varepsilon, f \in B_*\}.$$

Then N' is a neighbourhood of T_0 in $\sigma(B(H), B_*)$. Since $\hat{T}(f) = f(T)$, $\theta(N') \subset N$ and so θ is continuous. By

Corollary (2.1.7), $\sigma(B(H), B_*)$ is just the ultraweak

topology on $B(H)$. If $\theta(T) = \hat{T} = 0$ for some $T \in B(H)$,

then $f(T) = 0$ for all $f \in B_*$. As $(B(H), \sigma(B(H), B_*))$ is a

locally convex Hausdorff space, $T = 0$. Hence θ is one-to-one

Let B_1 and E_1 be the closed unit balls of $B(H)$ and E respectively. Then B_1 is $\sigma(B(H), B_*)$ -compact [1, pp.65-66]

Since θ is continuous and

$$\|\theta(T)\| = \|\hat{T}\| = \sup_{\|f\| \leq 1} |\hat{T}(f)| = \sup_{\|f\| \leq 1} |f(T)| \leq \|T\|.$$

$\theta(B_1)$ is $\sigma(E, B_*)$ -compact and contained in E_1 . Moreover

$\theta(B_1)$ is dense in E_1 . In fact, if it were not, there

would exist $e \in E_1$, $\varepsilon > 0$ and $f \in B_*$, $\|f\|=1$, such that $|e(f) - \hat{T}(f)| > \varepsilon$ for all $T \in B_1$. Since B_1 is ultraweakly compact, there is $T_0 \in B_1$ with $|f(T_0)| = 1$. We may assume that $e(f) - f(T_0) > \varepsilon$ (otherwise consider $-f$). Hence $e(f) > \varepsilon + f(T_0) > 1$. But $|e(f)| \leq \|e\| \|f\| \leq 1$, a contradiction. Thus $\theta(B_1) = E_1$ because $\theta(B_1)$ is closed. Moreover, θ is isometric; for if it were not, there would exist a $T \in B_1$ such that $\|\theta(T)\| < \|T\|$. Hence $\theta(\|T\|^{-1} T) \in E_1$ and $(\theta(T))^{-1} T \notin B_1$. Therefore θ is not one-to-one which is a contradiction. Thus $\theta(B(H)) = E$, that is, $B(H)$ is isometrically isomorphic to E .

THEOREM (2.1.12). Let A be a weakly closed $*$ -subalgebra of $B(H)$. Then A_* is the norm closure of A_\sim and A is the dual space of A_* .

PROOF. [2, p.41] By Theorem (2.1.2), each element of A_* is of the form $\sum_{i=1}^{\infty} \omega_{\alpha_i, \beta_i}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|\beta_i\|^2 < \infty$.

Also, each element of A_\sim is of the form $\sum_{i=1}^n \omega_{\alpha_i, \beta_i}$.

Hence A_* is the norm closure of A_\sim . By Corollary (2.1.7), $\sigma(B(H), B_*)$ -closed. By Theorem (2.1.11), $B(H)$ is the dual space of B_* . Hence A is just the vector subspace of $B(H)$ orthogonal to the set $A^\perp = \{f \in B_* : f(T) = 0 \text{ for all } T \in A\}$, that is, $A = (A^\perp)^\perp$ [15, p.232]. The mapping $T \rightarrow \bar{T}$ where $T \in (B_*)' = B(H)$ and \bar{T} is defined by $\bar{T}(f + A) = T(f)$ ($f \in B_*$), is an isometric isomorphism of $(A^\perp)^\perp = A$ onto

$(B_*/A^\perp)'$. We write $A \cong (B_*/A^\perp)'$. Thus $A' \cong (B_*/A^\perp)''$; this is to say B_*/A^\perp is a subspace of A' . Since every nonzero $f \in B_*/A^\perp \subseteq A'$ is ultraweakly continuous on A and $f(A) \neq 0$, we see that $f \in A_*$. On the other hand, for each $g \in A_*$, g can be extended to an ultraweakly continuous linear functional on $B(H)$. Hence $B_*/A^\perp \cong A_*$.

THEOREM (2.1.13). Let M be an ultraweakly closed subspace of $B(H)$, and K a convex subset of M , and for each real number $r > 0$, let $M_r = \{T \in M: \|T\| \leq r\}$. Then the following statements are equivalent:

- (i) K is ultraweakly closed.
- (ii) K is ultrastrongly closed.
- (iii) $K \cap M_r$ is ultraweakly closed for each r .
- (iv) $K \cap M_r$ is weakly closed for each r .
- (v) $K \cap M_r$ is ultrastrongly closed for each r .
- (vi) $K \cap M_r$ is strongly closed for each r .

PROOF. [2, p.41] (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (v) follow from Corollary (2.1.3). (iii) \Leftrightarrow (iv) and (v) \Leftrightarrow (vi) follow from Theorem (2.1.1) and the remark preceding Corollary (2.1.3).

(i) \Leftrightarrow (iii) Since M is ultraweakly closed in $B(H)$, it is norm-closed in $B(H)$. Hence M is a Banach space. By Theorem (2.1.12), M is the conjugate space of the Banach space M_* . It is easy to see that $\sigma(M, M_*)$ is equal to the ultraweak topology on M . But by Krein-Šmulian Theorem [4, p.429], K is weak*-closed if and only if $K \cap M_r$ is

weak*-closed for every $r > 0$. This completes the proof.

THEOREM (2.1.14). Let M be an ultraweakly closed subspace of $B(H)$, M_1 the closed unit ball in M and f a linear functional on M . Then the following statements are equivalent:

- (i) f is ultraweakly continuous.
- (ii) f is ultrastrongly continuous.
- (iii) $f = \sum_{i=1}^{\infty} \omega_{\varphi_i, \eta_i}$ with $\sum_{i=1}^{\infty} \|\varphi_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$,
 $\varphi_i, \eta_i \in H$, $i = 1, 2, \dots$

- (iv) The restriction of f to M_1 is ultraweakly continuous.
- (v) The restriction of f to M_1 is weakly continuous.
- (vi) The restriction of f to M_1 is ultrastrongly continuous.

(vii) The restriction of f to M_1 is strongly continuous.

PROOF. [2, pp.41-42] (i) \Leftrightarrow (ii) \Leftrightarrow (iii) and (iv) \Leftrightarrow (vi) follow from Theorem (2.1.2). (iv) \Leftrightarrow (v) and (vi) \Leftrightarrow (viii) follow from Theorem (2.1.1). To prove (vi) \Rightarrow (i), let f be a linear functional on M such that its restriction to M_1 is ultrastrongly continuous. Let K be the hyperplane generated by $f(T) = 0$. Then K is an ultrastrongly closed convex subset of M . Hence by Theorem (2.1.13)(iii), $K \cap M_r$ is ultraweakly closed and so f is ultraweakly continuous. Since (ii) \rightarrow (iv) is trivial, this completes the proof.

§ 2.2. Definition of a W^* -algebra.

Let A be a $*$ -subalgebra of $B(H)$. Then the weak, ultraweak, strong and ultrastrong closures of A coincide [2, p.43]. A W^* -algebra (von Neumann algebra, or ring of operators) is a $*$ -subalgebra of $B(H)$ with identity which is closed with respect to the weak topology (and hence closed with respect to the ultraweak, strong, and ultrastrong topologies) on $B(H)$. Note that every W^* -algebra is a C^* -algebra.

If A is a subset of $B(H)$, the set A^c consisting of all elements of $B(H)$ which commute with all elements of A is called the commutant of A . $(A^c)^c = A^{cc}$ is called the bicommutant of A . It is clear that A^{cc} is weakly closed and $A \subseteq A^{cc}$. A is a W^* -algebra if and only if $A^{cc} = A$ [10, p.448].

If A is a W^* -algebra, the Banach space A_* of ultraweakly continuous linear functionals on A is called the predual of A . By virtue of Theorem (2.1.14), A_* is also the Banach space of ultrastrongly continuous linear functionals on A . The elements of A_* are called normal linear functionals.

§ 2.3. The bidual of a C^* -algebra.

THEOREM (2.3.1). Let A be a C^* -algebra contained in $B(H)$, and let \tilde{A} be the smallest W^* -algebra containing A . Suppose that each continuous linear functional f on A is

ultrastrongly continuous and hence has a unique ultrastrong continuous extension \tilde{f} to \tilde{A} . Then the following statements are true:

- (i) The mapping $f \rightarrow \tilde{f}$ is an isometric isomorphism from the Banach space A' onto the predual \tilde{A}_* of \tilde{A} .
- (ii) For each $T \in \tilde{A}$, let \hat{T} be the linear functional $f \rightarrow \tilde{f}(T)$ on A' . Then $T \rightarrow \hat{T}$ is an isometric isomorphism on \tilde{A} onto the bidual A'' of A . The restriction of the mapping $T \rightarrow \hat{T}$ to A is a canonical injection on A into A'' .

PROOF. [3, pp.235-236] (i): If $f \in A'$, then by hypothesis, $\tilde{f} \in \tilde{A}_*$, the predual of \tilde{A} . Suppose that A_1 and \tilde{A}_1 are the closed unit balls of A and \tilde{A} , respectively. Then A_1 is ultrastrongly dense in \tilde{A}_1 by Kapłansky Density Theorem [2, p.46] and, since the weak and ultrastrong closures of A coincide [2, p.43], it follows that $\sup \{|f(T)| : T \in A_1\} = \sup \{|\tilde{f}(S)| : S \in \tilde{A}_1\}$, i.e., $\|f\| = \|\tilde{f}\|$. Thus $f \rightarrow \tilde{f}$ is an isometric isomorphism on A' into \tilde{A}_* . That it is onto \tilde{A}_* follows from the fact that every element of \tilde{A}_* is the continuous ultrastrong extension of its restriction to A .

(ii): By (i) and Theorem (2.1.12), $T \rightarrow \hat{T}$ is an isometric isomorphism on \tilde{A} onto A'' . If $T \in A$, then clearly \hat{T} is the continuous linear functional $f \rightarrow f(T)$ on A' . Thus $T \rightarrow \hat{T}$ maps A canonically into A'' .

THEOREM (2.3.2). If A is a W^* -algebra, then each element in the predual A_* of A is a linear combination of normal positive functionals on A .

PROOF. [2, p.54] Let f be an element of A_* . Then by Theorem (2.1.14), $f = \sum_{j=1}^{\infty} \omega_{\xi_j, \eta_j}$ for some sequences (ξ_j) ,

(η_j) of vectors of H with $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$, $\sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty$.

Now, for all $T \in B(H)$, we have

$$\begin{aligned} 4(T\xi_j, \eta_j) &= (T(\xi_j + \eta_j), \xi_j + \eta_j) - (T(\xi_j - \eta_j), \xi_j - \eta_j) \\ &\quad + i(T(\xi_j + i\eta_j), \xi_j + i\eta_j) - i(T(\xi_j - i\eta_j), \xi_j - i\eta_j). \end{aligned}$$

Thus,

$$\begin{aligned} f &= \frac{1}{4} \left(\sum_{j=1}^{\infty} \omega_{\xi_j + \eta_j} - \sum_{j=1}^{\infty} \omega_{\xi_j - \eta_j} + i \sum_{j=1}^{\infty} \omega_{\xi_j + i\eta_j} \right. \\ &\quad \left. - i \sum_{j=1}^{\infty} \omega_{\xi_j - i\eta_j} \right). \end{aligned}$$

As each functional of the form ω_{ξ} is positive, the theorem is proved.

THEOREM (2.3.3). Let A be a C^* -algebra and L the universal representation of A and \tilde{A} the weak closure of $L(A)$, which is a W^* -algebra on the underlying Hilbert space H of L . Then the following statements are true:

(i) Every positive normal functional on \tilde{A} is of the form ω_{φ} ($\varphi \in H$). Every ultraweakly continuous linear functional on \tilde{A} is weakly continuous.

(ii) If $f \in A^*$, then there exists a unique weakly continuous linear functional \tilde{f} on \tilde{A} such that $\tilde{f}(L(T)) = f(T)$ for all $T \in A$.

(iii) The mapping $f \rightarrow \tilde{f}$ is an isometric isomorphism of A^* onto the predual of \tilde{A} which transforms the set of all positive linear functionals on A into the set of all normal

positive linear functionals on \tilde{A} . We have $\tilde{f}^* = \tilde{f}^*$ for all $f \in A'$.

(iv) For each $S \in \tilde{A}$, let \hat{S} be the linear functional $f \rightarrow \tilde{f}(S)$ on A' . Then the mapping ψ defined by $\psi(S) = \hat{S}$ is an isometric isomorphism of \tilde{A} onto A'' whose composition with L (i.e., $\psi \circ L$) is the canonical injection of A into A'' .

(v) This isomorphism ψ is bicontinuous for the weak (operator) topology on A and the $\sigma(A'', A')$ -topology on A'' .

PROOF. [3, pp.236-237] For each $f \in A'$, define a functional f' on $L(A)$ by

$$f'(L(T)) = f(T) \quad (1)$$

for all $T \in A$. Then f' is linear and continuous since f is linear and $|f'(L(T))| = |f(T)| \leq \|f\| \|T\| = \|f\| \|L(T)\|$ for all $T \in A$. Then $\|f'\| \leq \|f\|$. Conversely, since $|f(T)| = |f'(L(T))| \leq \|f'\| \|L(T)\| = \|f'\| \|T\|$, we have $\|f\| \leq \|f'\|$ whence $\|f\| = \|f'\|$. Also, for each f' in $L(A)'$, the relation (1) defines a continuous linear functional f on A such that $\|f\| = \|f'\|$. Therefore the mapping ρ defined by $\rho(f) = f'$ is an isometric isomorphism of A' onto $L(A)'$. Since $L(T)$ is positive if and only if T is positive, it follows that f' is positive if and only if f is positive.

(i) Let \tilde{f} be a normal positive linear functional on \tilde{A} . Then the restriction $f' = \tilde{f}|_{L(A)}$ of \tilde{f} to $L(A)$ is positive on $L(A)$. Hence $f = \rho^{-1}(f')$ is positive on A . Let L^f and φ_f be the representation and vector defined by f . Then we have $f(T) = (L^f(T)\varphi_f, \varphi_f)$ for all T in A . Since $L = \bigoplus_{f \in Q} L^f$, where Q is the set of all positive linear functionals on A , we have $f(T) = (L(T)\varphi, \varphi)$, where $\varphi = \varphi_f$ is considered as a vector in H . Thus

$f'(L(T)) = f(T) = (L(T)\varphi, \varphi)$ for all T in A . Since, by Theorem (1.2.4) and [2, Corollary 1, p.44], $L(A)$ is ultraweakly dense in \tilde{A} , we have by the ultraweak continuity of f' that $f(S) = (S\varphi, \varphi)$ for all $S \in \tilde{A}$, i.e., $\tilde{f} = \omega_\varphi$. Hence \tilde{f} is weakly continuous. Therefore it follows from Theorem (2.3.2) that each ultraweakly continuous linear functional on \tilde{A} is weakly continuous. This proves (i).

(ii). If h' is a positive functional on $L(A)$, then $h = \mathcal{J}^{-1}(h')$ is positive on A and so $h \in Q$, where Q is the set of all positive linear functionals on A . Therefore $h'(L(T)) = h(T) = (L(T)\mathcal{S}_h, \mathcal{S}_h)$. Hence h' is weakly continuous on $L(A)$. Since each g in $L(A)'$ is of the form $g = g_1 + ig_2$, where g_1, g_2 are hermitian, by Theorem (1.1.7), g is a linear combination of positive functionals on $L(A)$ and therefore weakly continuous. It follows that if $f \in A'$, then $f' = \mathcal{J}(f)$ is weakly continuous on $L(A)$. Hence f' can be extended to a unique weakly continuous linear functional \tilde{f} on \tilde{A} . Moreover, we have $f(L(T)) = f'(L(T)) = f(T)$ for all $T \in A$. This proves (ii).

(iii). If $f' \in L(A)'$, then f' is weakly continuous and hence is ultraweakly continuous. Let \tilde{f} be the unique extension of f' to \tilde{A} by weak continuity. By Theorem (2.3.1), the mapping $f' \rightarrow \tilde{f}$ is an isometric isomorphism of $L(A)'$ onto the predual of \tilde{A} . Since the mapping $f \rightarrow f'$ is an isomorphism of A' onto $L(A)'$, $f \rightarrow \tilde{f}$ is an isometric isomorphism of A' onto the predual of \tilde{A} . If f is positive, then, by the same argument as in (ii), $f' = \mathcal{J}(f) = \omega_{\mathcal{S}_f}$ on $L(A)$. Hence by the weak continuity of f'

$\tilde{f} = \omega_{\varphi_f}$ on \tilde{A} . Therefore \tilde{f} is positive and normal. If $f \in A'$, then $f^* \in A'$. Since $(f^*)' = (f')^*$, $f'^* \in L(A)'$. We have $(f'^*)(L(T)) = f^*(T) = \overline{f(T^*)} = \overline{\tilde{f}(L(T)^*)} = \tilde{f}^*(L(T))$ for all $T \in A$. Hence, by the weak continuity, $\tilde{f}'^*(S) = \tilde{f}^*(S)$ for all $S \in \tilde{A}$. This proves (iii).

(iv) By (iii), $\psi: S \rightarrow \hat{S}$ is an isometric isomorphism of \tilde{A} into A'' . To see that ψ is onto A'' , let $\varphi \in A''$ and define a linear functional l_φ on \tilde{A}_* by $l_\varphi(\tilde{f}) = \varphi(f)$. It is clear that $l_\varphi \in (\tilde{A}_*)'$ with $\|l_\varphi\| = \|\varphi\|$. Now by [3, A23], l_φ can be uniquely and isometrically identified with an element S of \tilde{A} such that $l_\varphi(\tilde{f}) = \tilde{f}(S)$ for all $\tilde{f} \in \tilde{A}_*$. Hence $\hat{S}(f) = \varphi(f)$ for all $f \in A'$ and so $\hat{S} = \varphi$, which shows that ψ is onto A'' . Since $y \rightarrow L(y)$ is an isometric mapping of A onto $L(A)$, it follows that $\psi \circ L$ is the canonical injection of A into A'' .

(v) From (i) we know that the predual of \tilde{A} is also the set of all weakly continuous linear functionals on \tilde{A} . By (iii), every weakly continuous linear functional on \tilde{A} has the form \tilde{f} with $f \in A'$. Now a sub-base of 0-neighbourhoods N for the weak topology in \tilde{A} is the family of sets of the form

$$N = \{S \in \tilde{A}: |\hat{f}(S)| < \varepsilon, \varepsilon > 0, f \in A'\},$$

and a sub-base of 0-neighbourhoods N' for the topology $\sigma(A'', A')$ in A'' is the family of sets of the form

$$N' = \{S \in A'': |\hat{S}(f)| < \varepsilon, \varepsilon > 0, f \in A'\}.$$

Since the isomorphism $\psi: S \rightarrow \hat{S}$ transforms the elements of N into the elements of N' and its inverse transforms the

elements of N' into the elements of N , it follows that $\Psi: S \rightarrow \hat{S}$ is bicontinuous.

REMARK:

By (iv), we can identify A'' with \tilde{A} . Since \tilde{A} is a W^* -algebra, A'' can be considered as a W^* -algebra. Hence by (iii), the predual $(A'')^*$ of the W^* -algebra A'' can be identified with the dual A' of A . It follows from this identification that the weak*-topology $\sigma(A'', A')$ on A'' coincides with the ultraweak topology $\sigma(A'', (A'')^*)$ on A'' . Consequently, by (v); these topologies coincide with the weak (operator) topology on the W^* -algebra A'' . When A is regarded as imbedded in A'' , the restriction of $\sigma(A'', A')$ to A coincides with the weak topology $\sigma(A, A')$ on A . In particular, if K is a norm-closed convex subset of A , then it is relatively closed in A under the weak topology [4, p.422], and hence relatively closed in A under the ultraweak topology. A is weakly* dense in A'' [4, p.425], thus weakly dense in A'' . As the weak and strong closures of a convex set in a W^* -algebra coincide, A is strongly dense in A'' .

CHAPTER III

CLOSED ORDER IDEALS IN A C*-ALGEBRA AND ITS DUAL

§ 3.1. Definitions and basic concepts.

In what follows, for each closed subspace X of a complex Hilbert space H , let P_X be the (orthogonal) projection of H onto X , i.e., $P_X(H) = X$. Let T be an element of $B(H)$. It is clear that the kernel of T , $\ker T = \{h \in H: T(h) = 0\}$, is a closed subspace of H . If $X = (\ker T)^\perp$, the orthogonal complement of $\ker T$, then the projection $E = P_X$ is called the support of T . It is easy to see that E is the smallest projection in $B(H)$ such that $TE = ET = T$ [10, p.445]. Let M be the closure of $T(H)$ and let $F = P_M$, then F is the support of T^* since $(\ker T^*)^\perp = M$. Hence, if T is hermitian, then its support E_T is the projection on the closure of the range $T(H)$ of T . If T is an element of a C^* -algebra A , then E_T is the smallest projection in A'' (considered as a W^* -algebra) such that $TE_T = E_T T = T$.

Let p be a positive normal functional on a W^* -algebra A and let N be the set of all projections $G \in A$ such that $p(G) = 0$. Then, by [2, Proposition 3, p.61], N contains a projection F such that $G \leq F$ for all $G \in N$ and $p(TF) = p(FT) = 0$ for all $T \in A$. Clearly F is the only such projection in N . The projection $E = I - F$ is called

the support of p . It is easy to see that $p(T) = p(ETE)$ for all $T \in A$.

Let $U \in B(H)$ and let E be the support of U . Then U is called partially isometric if U is isometric on $X = E(H)$; $U(H) = U(X)$ is a closed vector subspace of H . Let $M = U(X)$ and $F = P_M$. We called E (resp. X) the initial projection (resp. initial subspace) of U , and F (resp. M) the final projection (resp. final subspace) of U . U^* is also partially isometric, having initial projection F and final projection E . We have $U^*U = E$ and $UU^* = F$. It follows that an element $V \in B(H)$ is a partial isometry iff $V = VV^*V$ [7, pp.62-63].

Let $T \in B(H)$, E the support of T , F the support of T^* , $X = E(H)$, $M = F(H)$, and $|T| = (T^*T)^{\frac{1}{2}}$. Then $|T|$ has the support E and the closure of $|T|(H)$ is equal to X . The linear mapping $|T|\ell \rightarrow T\ell$ ($\ell \in H$) is isometric from $|T|(H)$ onto $T(H)$ and therefore can be extended to an isometric linear operator V on X onto M . Let U be a partially isometric operator on H with the support E such that U coincides with V on X . Then $T = U|T|$. This representation of T as the product of the unique partially isometry U and the positive operator $|T| = (T^*T)^{\frac{1}{2}}$ is called the polar decomposition of T . The equality $T^* = U^*(U|T|U^*)$ is the polar decomposition of T^* . We also have $|T^*| = U|T|U^*$ and $|T| = U^*|T^*|U$ [7, pp.68-69].

For each subset N of a C^* -algebra A , let

$N^\perp = \{f \in A': f \geq 0, f(T) = 0, T \in N\}$. Then
 $N^{\perp\perp} = (N^\perp)^\perp = \{T \in A: T \geq 0, f(T) = 0, f \in N^\perp\}$. The set
 $N^\circ = \{f \in A': \operatorname{Re} f(T) \leq 1, T \in N\}$ is called the polar of N .
 It is easy to see that if N is balanced, and in particular
 a subspace of A , then $N^\circ = \{f \in A': |f(T)| \leq 1, T \in N\}$.

§ 3.2. Order ideals in a C*-algebra.

Let A be a C*-algebra, A_h the set of all hermitian elements of A and A^+ the set of all positive elements of A . It is easy to see that A_h is a partially ordered vector space with the partial order defined by the cone A^+ . A subset N of A^+ is called an order ideal of A if $N + N \subseteq N$, $\lambda N \subseteq N$ for all $\lambda \geq 0$ and such that, if $S \in N$ and $T \in A$ with $0 \leq T \leq S$, then $T \in N$.

Let A be a C*-algebra. For each positive operator T of A ; let (T) be the norm-closure of the smallest order ideal containing T , i.e., (T) is the norm-closure of all positive operators S of A such that $S \leq \lambda T$ for some positive scalar λ . For each $S \in A$ and $f \in A'$, let Sf and fS be defined by $(Sf)(T) = f(ST)$, $(fS)(T) = f(TS)$ for all $T \in A$. If A is a W*-algebra and if E_f denotes the support of the positive normal functional f on A , then $E_f f = f E_f = f$. A subspace M of the dual A' of A is said to be left invariant if for all $f \in M$ and $S \in A$, $Sf \in M$.

THEOREM(3.2.1). Let A be a C*-algebra and T a positive element of A . Then $(T) = \{T\}^{\perp\perp}$.

PROOF. [5, p.393] It is clear that $\{T\}^{\perp\perp}$ is a norm-closed order ideal containing T . Hence $(T) \subseteq \{T\}^{\perp\perp}$.

Conversely, let $S \in \{T\}^{\perp\perp}$ and let E_T, E_S be supports of T and S respectively. We wish to show that $E_S \leq E_T$. Suppose on the contrary that $E_S > E_T$. Then there exists a vector $\ell \in H$ such that $E_T(\ell) = 0$ and $E_S(\ell) \neq 0$. Since $TE_T = T$ and $T(\ell) = TE_T(\ell) = 0$, $\omega_\ell(T) = (T\ell, \ell) = 0$. Since ω_ℓ is weakly continuous, it is continuous. Therefore $\omega_\ell \in \{T\}^{\perp\perp}$. But $E_S(\ell) \neq 0$ and $\ker S = \ker E_S$; hence $\omega_\ell(S) = (S\ell, \ell) \neq 0$, which contradicts the fact that $S \in \{T\}^{\perp\perp}$. Therefore $E_S \leq E_T$. For each positive integer $n \geq 1$, define the function f_n on the reals by $f_n(t) = 0$ for $t \leq 0$, $f_n(t) = nt$ for $0 \leq t \leq n^{-1}$ and $f_n(t) = 1$ for $t \geq n^{-1}$. Then $\{f_n(T)\}$ is an increasing sequence of positive operators with supremum E_T . Hence $\{f_n(T)\}$ converges to E_T strongly [2, p.331]. As the map $(U, V) \rightarrow UV$ is strongly continuous on bounded sets [§ 2.1], we have $f_n(T)Sf_n(T) \rightarrow E_TSE_T$ strongly and hence weakly. It follows from the inequality

$$f_n(T)Sf_n(T) \leq \|S\|f_n(T)^2 \leq \|S\|f_n(T) \leq (\|S\|n) T,$$

that the sequence $\{f_n(T)Sf_n(T)\}$ lies in (T) . Since (T) is convex and norm-closed in A and hence weakly closed [4, p.422], $E_TSE_T \in (T)$. But $S = E_S S = E_S S E_S \leq E_T S E_T \in (T)$. Hence $\{T\}^{\perp\perp} \subset (T)$.

COROLLARY (3.2.2). Let A be a C^* -algebra and T a positive element of A . Then the following statements are true:

- (i) (T) is an order ideal.
- (ii) If $S \in (T)$, then $S^{\frac{1}{2}} \in (T)$.

(iii) If N is a norm-closed order ideal of A and $T \in N$, then $T^{\frac{1}{2}} \in N$.

PROOF. [5, p.393] (i) is clear. To prove (ii), by Theorem (3.2.1), it suffices to show that $S^{\frac{1}{2}} \in \{T\}^{\perp\perp}$. Let p be a positive element of A' with $p(T) = 0$. Then by Theorem (1.1.1)(iv) $p(T^{\frac{1}{2}})^2 \leq \|p\|p((T^{\frac{1}{2}})^*T^{\frac{1}{2}}) = \|p\|p(T) = 0$. Hence $p(T^{\frac{1}{2}}) = 0$. Consequently, we have $f(T^{\frac{1}{2}}) = 0$ for all $f \in \{T^{\frac{1}{2}}\}^{\perp}$, i.e., $T^{\frac{1}{2}} \in \{T\}^{\perp\perp}$. This proves (ii).

Finally, if N is a norm-closed order ideal in A such that $(T) \subseteq N$, then, by (ii), $T^{\frac{1}{2}} \in (T) \subseteq N$. This proves (iii).

LEMMA (3.2.3). Let S, T be positive elements of a W^* -algebra A such that $S \leq T$. Then there exists a unique element $D \in A$ with the following properties:

(i) $S^{\frac{1}{2}} = DT^{\frac{1}{2}}$.

(ii) The support of D is majorized by the support of T .

PROOF. [2, pp.11-12] For any $\ell \in H$, we have

$$\|S^{\frac{1}{2}}\ell\|^2 = (S\ell, \ell) \leq (T\ell, \ell) = \|T^{\frac{1}{2}}\ell\|^2;$$

in particular, $T^{\frac{1}{2}}\ell = 0$ implies $S^{\frac{1}{2}}\ell = 0$. The mapping $T^{\frac{1}{2}}\ell \longrightarrow S^{\frac{1}{2}}\ell$ is a continuous linear operator C which maps $T^{\frac{1}{2}}(H)$ into H . Let D be the unique continuous extension of C to $\text{cl}(T^{\frac{1}{2}}(H)) = \text{cl}(T(H))$ into H , where $\text{cl}(T^{\frac{1}{2}}(H))$ and $\text{cl}(T(H))$ denote the closures of $T^{\frac{1}{2}}(H)$ and $T(H)$.

We have $S^{\frac{1}{2}} = DT^{\frac{1}{2}}$. Next, we observe that $\ker S \supseteq \ker T$.

Hence the support of S is majorized by the support of T . Finally, let U be a unitary element in

A^c , the commutant of A [§2.2] and let $UDU^{-1} = G$. Then

$$GT^{\frac{1}{2}} = UDU^{-1}T^{\frac{1}{2}} = UDT^{\frac{1}{2}}U^{-1} = US^{\frac{1}{2}}U^{-1} = S^{\frac{1}{2}}UU^{-1} = S^{\frac{1}{2}}$$

and hence by uniqueness of D , $G = D$. Thus, $UDU^{-1} = D$ which gives $D \in A$.

THEOREM (3.2.4). If N is a norm-closed left ideal of a C^* -algebra A , then the positive part N^+ is a norm-closed order ideal of A .

PROOF. [5, p.394] It is clear that N^+ is norm-closed. Also, $N^+ + N^+ \subseteq N^+$ and $\lambda N^+ \subseteq N^+$ for all positive scalar λ . Suppose that $T \in N^+$ and $S \in A$ with $0 \leq S \leq T$. We shall show that $S \in N^+$. Since $0 \leq S \leq T$, by Lemma (3.2.3), there exists a unique element $D \in A''$ (considered as a W^* -algebra) such that $S^{\frac{1}{2}} = DT^{\frac{1}{2}}$ and $\|D\| \leq 1$. Then $S = (S^{\frac{1}{2}})(S^{\frac{1}{2}})^* = (DT^{\frac{1}{2}})(DT^{\frac{1}{2}})^* = DTD^*$. Since $N \cap N^*$ is a 2-sided ideal of A , $DTD^* \in N \cap N^*$, i.e. $S \in N \cap N^*$. Since $S \geq 0$, $S \in N^+$. Hence N^+ is a norm-closed order ideal of A .

DEFINITION. Let A be a C^* -algebra. A subset N of A is said to be invariant if for all $T \in N$, $S \in A$, $STS^* \in N$.

THEOREM (3.2.5). Let A be a C^* -algebra. Then the map $N \rightarrow N^+$ is a bijection between the norm-closed 2-sided ideals of A and the norm-closed invariant order ideals of A .

PROOF. [5, p.396] If N is a norm-closed 2-sided ideal of A , it is clear that N^+ is invariant.

Conversely, if M is an invariant norm-closed order ideal of A . Let $N = (A''M) \cap A$. Since A'' is a W^* -algebra [see

the remark to Theorem (2.3.3)], A'' contains the identity and therefore $M \subseteq N^+$. To prove the converse inclusion, let $S \in A''$ and $P \in M$ with $SP \in A$ and $SP \geq 0$. Then, as

$$0 \leq (SP)^2 = (SP)^*(SP) = PS^*SP \leq \|S\|^2 P^2 \leq (\|S\|^2 \|P\|)P,$$

we have $(SP)^2 \in M$ since M is an order ideal. Hence, by Corollary (3.2.3), $SP \in M$ and so $N^+ \subseteq M$. Thus $N^+ = M$.

We show next that N is a norm-closed ideal of A . Clearly $AN \subseteq N$. To prove that N is closed under addition, let $S, T \in N$. Then S^*S and T^*T are in $N^+ = M$. It follows from the inequality

$$0 \leq (S + T)^*(S + T) \leq 2(S^*S + T^*T)$$

that $(S + T)^*(S + T) \in M$ and thus, since M is an order ideal, $P = ((S + T)^*(S + T))^{\frac{1}{2}} \in M = N^+$. Let $S + T = UP$ be the polar decomposition of $S + T$ with $U \in A''$.

We see that $S + T \in (A''M) \cap A = N$.

To show that N is norm-closed, let $\{T_n\}$ be a sequence in N converging in the norm to an element T of A . Then the sequence $\{T_n^*T_n\}$ lies in $N^+ = M$ as N is a left ideal and $T_n^*T_n \geq 0$ for each n . Since M is norm-closed, $T^*T \in M$ and hence $(T^*T)^{\frac{1}{2}} \in M$. From the polar decomposition $T = U|T|$ of T , where $U \in A''$ and $|T| = (T^*T)^{\frac{1}{2}}$, we see that $T \in (A''M) \cap A = N$. Hence N is norm-closed. Thus N is a norm-closed left ideal.

Finally, we shall show that N is also a right ideal of A . Let $T \in N$ and let $T = U|T|$ with $U \in A''$ be the polar decomposition of T . Now $|T| = (T^*T)^{\frac{1}{2}} \in N^+$ and

A is strongly dense in A'' and $U \in A''$ (and hence $U^* \in A''$). Therefore there exists a net $\{U_\alpha^*\}$ on A converging strongly to U^* . The net $\{U_\alpha | T | U_\alpha^*\}$ lies in $M = N^+$ as M is invariant. Also, $U_\alpha | T | U_\alpha^*$ converges to $U | T | U^*$ weakly and since $U | T | U^* = |T^*|$, we have $U | T | U^* \in A$. As M is convex and norm-closed, it is relatively closed in A under the weak topology [4, p.422]. Thus $|T^*| \in M = N^+$. Since $T^* = V |T^*|$, for a partial isometry $V \in A''$, and $N = (A''M) \cap A$, it follows that $T^* \in N$. This shows that N is self-adjoint, i.e., $N^* = N$, and therefore is a 2-sided ideal of A .

From the proof of Theorem (3.2.5), we have the following:

COROLLARY (3.2.6). Let N be a norm-closed order ideal of a C^* -algebra A and let $A''N$ be the set of all SP with $S \in A''$ and $P \in N$. Then $M = (A''N) \cap A$ is a norm-closed left ideal such that $M^+ = N$.

THEOREM (3.2.7). If N is a norm-closed left ideal in a C^* -algebra, then $N = (A''N^+) \cap A$, where $A''N^+$ denotes the set of all elements of the form SP with $S \in A''$ and $P \in N^+$.

PROOF. [5, p.394] Let T be an element of A such that $T = SP$ with $S \in A''$ and $P \in N^+$. Since A is weakly dense in A'' , there exists a net $\{S_\alpha\}$ in A converging weakly to S . Since N is a left ideal of A , $\{S_\alpha P\}$ lies in N and converges weakly to $SP = T$. As N is convex and norm-closed, it is weakly closed in A [4, p.422] and hence $T \in N$. This proves that $(A''N^+) \cap A \subseteq N$.

Conversely, let $T \in N$. Then $|T|^2 = T^*T$ is in the C^* -algebra $N \cap N^*$ and hence $|T| \in (N \cap N^*)^+ = N^+$.

Since $T = U|T|$ with $U \in A''$, $T \in (A''N^+) \cap A$. Hence $N \subseteq (A''N^+) \cap A$ and therefore $N = (A''N^+) \cap A$.

Combining Theorems (3.2.4), (3.2.6) and (3.2.7), we obtain the following:

THEOREM (3.2.8). There exists a one-to-one correspondence $N \rightarrow N^+$ between norm-closed left ideals and norm-closed order ideals in a C^* algebra A .

§ 3.3. Polar decomposition for continuous linear functionals.

THEOREM (3.3.1). Let A be a W^* -algebra, E a projection in A and f in the predual A_* of A . Then we have the following relations:

- (i) $\|f\|^2 \geq \|Ef\|^2 + \|(I - E)f\|^2$.
- (ii) If $\|f\| = \|Ef\|$, then $f = Ef$.

PROOF. [3, p.239] (ii) follows from (i). We prove (i).

Let H be the Hilbert space on which A acts, and let $B = B(H)$ be the algebra of all bounded linear operators on H into itself. B is a Banach space under the operator bound norm. Since A is a weakly closed subspace of B , it is a norm-closed subspace of B [4, p.422]. If \bar{A} denotes the set of all continuous linear functionals on B which vanish on A , and if $g' + \bar{A}$ denotes the class of all elements $g \in B'$ such that $g - g' \in \bar{A}$, the mapping $\Psi: g' + \bar{A} \rightarrow g$,

where g is defined by $g(S) = g'(S)$ for all $S \in A$, is an isometric isomorphism of B'/\bar{A} onto A' [15, p.188]. Therefore, if $f \in A_*$, there exists an $f' + \bar{A} \in B'/\bar{A}$ such that $\Psi(f' + \bar{A}) = f$. Hence $\|f\| = \|f' + \bar{A}\| = \inf \{\|y\| : y \in f' + \bar{A}\}$. If $\varepsilon > 0$ is given, there exists $y \in B'$ such that $\|f\| + \varepsilon > \|y\|$. If we show that $\|y\|^2 \geq \|Ey\|^2 + \|(I - E)y\|^2$, then we will have $(\|f\| + \varepsilon)^2 \geq \|Ef\|^2 + \|(I - E)f\|^2$ from which (i) will follow since ε is arbitrary. It remains thus to prove (i) for the case where $A = B$. Since the set of all weakly continuous functionals on A is dense in A_* [Theorem (2.1.12)], we may assume that f is weakly continuous. Then, by Theorem (2.1.5), there exist orthonormal systems (e_1, e_2, \dots, e_n) , $(e'_1, e'_2, \dots, e'_n)$ and positive scalars $\lambda_i, i = 1, 2, \dots, n$, such that

$$\|f\| = \lambda_1 + \dots + \lambda_n \quad \text{and} \quad f(T) = \sum_{i=1}^n \lambda_i (Te_i, e'_i)$$

for all $T \in A$. Hence

$$(Ef)(T) = \sum_{i=1}^n \lambda_i (Te_i, Ee'_i),$$

$$(I - E)f(T) = \sum_{i=1}^n \lambda_i (Te_i, (I - E)e'_i).$$

Therefore

$$\begin{aligned} \|Ef\|^2 + \|(I - E)f\|^2 &\leq \left(\sum_{i=1}^n \lambda_i \|Ee'_i\| \right)^2 + \left(\sum_{i=1}^n \lambda_i \|(I - E)e'_i\| \right)^2 \\ &= \sum_{i=1}^n \lambda_i^2 (\|Ee'_i\|^2 + \|(I - E)e'_i\|^2) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j (\|Ee_i\| \cdot \|Ee_j\| \\
 & \quad + \|(I - E)e_i\| \|(I - E)e_j\|) \\
 & \leq \sum_{i=1}^n \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \\
 & = \left(\sum_{i=1}^n \lambda_i \right)^2 = \|f\|^2.
 \end{aligned}$$

THEOREM (3.3.2). Suppose that A is a W^* -algebra and f is an element of the predual A_* of A . Then

- (i) There exists a partial isometry $U \in A$ such that $p = U^*f$ positive, $Up = f$ and $\|p\| = \|f\|$.
- (ii) If D is any operator of A with $\|D\| \leq 1$, Df positive, and $\|Df\| = \|f\|$, then $Df = p$.
- (iii) There exists only one partial isometry U such that $UU^* = E_p$, where E_p is the support of p .

PROOF. [3, pp.240-242; 5, pp.298-400] (i) For simplicity,

we assume that $\|f\| = 1$. Let A_1 be the unit ball of A and $K = \{T \in A_1 : f(T) = 1\}$. As A_1 is weakly compact, it is ultraweakly compact. Since f is ultraweakly continuous, there exists an element T of A_1 such that $\|f(T)\| = 1$. Multiplying by a scalar, we may assume that $f(T) = 1$ so that $T \in K$ and hence K is nonempty. Since K is ultraweakly closed and contained in A_1 , K is ultraweakly compact. It is clear that K is also convex. Therefore, by Krein-Milman Theorem, K has an extremal point V say. V is also extremal in A_1 , for if $V = \frac{1}{2}S + \frac{1}{2}T$, where $\|S\| \leq 1$ and $\|T\| \leq 1$, then $\|f(S)\| \leq 1$,

$$\|f(T)\| \leq 1 \quad \text{and} \quad 1 = f(V) = \frac{1}{2} f(S) + \frac{1}{2} f(T).$$

Thus $f(S) = f(T) = 1$, i.e., S and T are in K and $V = S = T$. From Kadison's characterization of the extremal points of the unit ball of a C^* -algebra [8, p.325], V is also a partial isometry.

Define the linear functional p on A by $p(T) = f(VT)$ for all $T \in A$. Then $p \in A_*$, and $\|p\| \leq 1$. Since $p(I) = f(V) = 1$, p is positive by Theorem (1.1.4). Also $p(V^*V) = f(VV^*V) = f(V) = 1$, since V is a partial isometry; hence $V^*V \geq E_p$ where E_p is the support of p . Let $U = E_p V^*$. Then $UU^* = E_p$ and, for all $S \in A$, we have

$$(U^*f)(S) = f(VE_p S) = p(E_p S) = p(S).$$

Let $U^*U = E$. Then as $\|Ef\| \leq 1$ and

$$(Ef)(U^*) = f(U^*UU^*) = f(U^*) = f(VE_p) = p(E_p) = 1.$$

we have $\|Ef\| = 1$. From Theorem (3.3.1), $Ef = f$, and for all $S \in A$, we have

$$(Up)(S) = p(US) = f(U^*US) = (Ef)(S) = f(S),$$

i.e., $Up = f$.

(ii) Suppose that D is an operator in A_1 such that Df is positive, $\|Df\| = 1$ and $\|D\| \leq 1$. Then $f(D) = (Df)(I) = \|Df\| = 1$ by Theorem (1.1.4).

If L and φ are respectively the representation of A and cyclic vector defined by p , then $(\varphi, L(D^*U^*)\varphi) = p(UD) = f(D) = 1$ by definition of p . Since $(\varphi, \varphi) = (L(I)\varphi, \varphi) = p(I) = \|p\| = 1$, we have

$$(\varphi, \varphi) = (\varphi, L(D^*U^*)\varphi) \leq \|\varphi\| \|L(D^*U^*)\varphi\| \leq \|\varphi\|^2 \|UD\|$$

$$\leq \|g\|^2 \|D\| \|U\| \leq \|g\|^2.$$

It follows from the Cauchy-Schwartz's inequality that $L(D^*U^*)g = g$. Thus

$$\begin{aligned} (Df)(S) &= f(DS) = p(UDS) = (L(UDS)g, g) = (L(S)g, L(D^*U^*)g) \\ &= (L(S)g, g) = p(S). \end{aligned}$$

Hence $Df = p$.

(iii) Let Y be another partial isometry with Y^*f positive $YY^*f = f$ and $YY^* = E_p$. Since $\|f\| = \|YY^*f\| \leq \|Y^*f\| \leq \|f\|$, we have $1 = \|f\| = \|Y^*f\|$ and, by Theorem (3.3.1), $p = Y^*f$. We want to show that $Y = U$. Let $X = UY^*$. Then $p(X) = p(UY^*) = U_p(Y^*) = f(Y^*) = Y^*f(I) = p(I) = 1$. Similarly $p(X^*) = 1$. Since $1 = p(X)^2 \leq p(XX^*) \leq 1$, we have $p(XX^*) = 1$ and $p((E_p - X)(E_p - X)^*) = 1 - 1 - 1 + 1 = 0$. Now, since U, Y are partial isometries such that $UU^* = E_p$ and $YY^* = E_p$, $U = UU^*U = E_pU$ and $Y = E_pY$. Then $X = UY^* = (E_pU)(E_pY)^* = E_p(UY^*)E_p = E_pX E_p$, i.e., $X = E_pX E_p \in E_p A E_p$. Since $p((E_p - X)^*(E_p - X)) = 0$ and p is faithful on $E_p A E_p$ [2, p.61], we have $(E_p - X)^*(E_p - X) = 0$ whence $E_p - X = 0$ or $E_p = X$. Suppose that H is the underlying Hilbert space for A , E_1 is the projection of H onto $(\ker U)^\perp$, the initial space of U . For each $\xi \in E_p(H)$, we have $\|U(Y^*\xi)\| = \|E\xi\| = \|\xi\|$ and hence $\|U(Y^*\xi)\| \geq \|Y^*\xi\|$. Thus $Y^*\xi \in E_1(H)$. As $U : E_1(H) \rightarrow E_p(H)$ is an isometry, the equality $UY^*\xi = \xi = UU^*\xi$ implies that $Y^*\xi = U^*\xi$; moreover U^* and Y^* vanish on $(E_p(H))^\perp$. Hence $Y^* = U^*$ or $Y = U$.

If A is a W^* -algebra and f is an element of A_* we shall denote the function p defined in Theorem (3.3.2) by $|f|$. If A is a C^* -algebra and f belongs to A' , then, considering A' as the predual of A'' [Theorem (2.2.3)], we have that $|f|$ is defined and is also a member of A' . The equality $f = U|f|$, where $UU^* = E_f$, is called the polar decomposition of f .

THEOREM (3.3.3). Suppose that A is a W^* -algebra and $\{f_n\}$ is a sequence in the predual A_* of A , converging in the norm topology to a function f . Then there exists a subsequence of the $\{|f_n|\}$ converging in the weak topology to $|f|$.

PROOF. [5, p.401] Let $f_n = U_n|f_n|$ be the polar decompositions. Taking a subsequence, we may assume that the sequence U_n^* converges ultraweakly to some operator $D \in A$ with $\|D\| \leq 1$. For each $T \in A$, we have

$$\begin{aligned} \left| |f_n|(T) - Df(T) \right| &\leq |f_n(U_n^*T) - f(U_n^*T)| + |f(U_n^*T) - f(DT)| \\ &\leq \|f_n - f\| \|T\| + |f(U_n^*T) - f(DT)|. \end{aligned}$$

Since the latter expression tends to 0, the subsequence $|f_n|$ converges weakly to Df . It follows that Df is positive, and $\|f_n\| = |f_n|(I)$ tends to $\|Df\| = (Df)(I)$. But $\|f\|$ is the limit of the sequence $\|f_n\|$, hence $\|Df\| = \|f\|$ and we conclude from Theorem (3.3.2) that $Df = |f|$.

THEOREM (3.3.4). If f and g are in the predual A_* of a W^* -algebra A , then for all $T \in A$, we have

$$\| |f + g| (T) \|^2 \leq (\|f\| + \|g\|) (|f| (T^*T) + |g| (T^*T))$$

PROOF. [5, p.401] Let $f = U |f|$, $g = V |g|$, and $f + g = S |f + g|$ be the polar decompositions. Then for all $T \in A$,

$$\begin{aligned} \| |f + g| (T) \|^2 &= \| |f| (US^*T) + |g| (VS^*T) \|^2 \\ &\leq \left[\| |f| (US^*T) \| + \| |g| (VS^*T) \| \right]^2 \\ &\leq \left[\|f\| (US^*SU^*)^{\frac{1}{2}} |f| (T^*T)^{\frac{1}{2}} \right. \\ &\quad \left. + \|g\| (VS^*SV^*)^{\frac{1}{2}} |g| (T^*T)^{\frac{1}{2}} \right]^2 \\ &\leq \left[\|f\|^2 |f| (T^*T)^{\frac{1}{2}} + \|g\|^2 |g| (T^*T)^{\frac{1}{2}} \right]^2 \\ &\leq (\|f\| + \|g\|) (|f| (T^*T) + |g| (T^*T)). \end{aligned}$$

§ 3.4. Order ideals in the dual of a C*-algebra.

Let B be a W^* -algebra, and B_* its predual. If p is a positive element of B_* , it was shown in [5, p402] that $\{p\}^{\perp\perp}$ is the smallest norm-closed order ideal containing p , i.e., $\{p\}^{\perp\perp}$ is the norm-closure of the set of all $q \in B_*$ such that $0 \leq q \leq \lambda p$ for some positive scalar λ .

LEMMA (3.4.1). Let p and q be positive elements in the predual B_* of a W^* -algebra B such that there exists a constant k with $\|q(S)\|^2 \leq k p(S^*S)$ for all $S \in B$. Then $q \in \{p\}^{\perp\perp}$.

PROOF. [5, p.403] By Theorem (1.2.3), there exists a representation L of B and a cyclic vector φ corresponding to p such that $p(T) = (L(T)\varphi, \varphi)$ for all $T \in B$. To show that $q \in \{p\}^{\perp\perp}$, let $S \geq 0$ and $p(S) = 0$. Then $L(S)\varphi = L(S^{\frac{1}{2}})(L(S^{\frac{1}{2}})\varphi) = 0$ and $p(S^*S) = (L(S^*S)\varphi, \varphi)$

$= (L(S)^*L(S)\varphi, \varphi) = (L(S)\varphi, L(S)\varphi) = 0$. Hence $\|q(S)\|^2 \leq k p(S^*S) = 0$ and $q \in \{p\}^{\perp\perp}$.

THEOREM (3.4.2). For each norm-closed order ideal N of the predual B_* of a W^* -algebra B , there exists a norm-closed left invariant subspace M in B_* such that $M^+ = N$.

PROOF. [5, p.403] Let $M = BN$, where BN denotes the set of all Sp with $S \in B$ and $p \in N$. We shall show that M is a norm-closed left invariant subspace such that $M^+ = N$.

First of all, we shall show that $M^+ = N$. Since B is a W^* -algebra, it contains the identity operator and hence $M^+ \supseteq N$. Now, for any $q \in M^+$, $q = Up$ where $U \in B$, $p \in N$. If $C \geq 0$ is in B and $p(C) = 0$, then as shown in the proof of Lemma (3.4.1), $LP(C)\varphi = 0$ which implies $LP(SC)\varphi = 0$ and $0 = (LP(SC)\varphi, \varphi) = p(SC) = Sp(C)$ for all $S \in B$. Hence $q(C) = Up(C) = 0$ and $q \in \{p\}^{\perp\perp}$. Since N is a norm-closed order ideal and $\{p\}^{\perp\perp}$ is a subset of N , we have $q \in N$. Hence $M^+ = N$. Next, if $f \in M = BN$, then $f = Sp$ for some $S \in B$, $p \in N$. For each $T \in B$, $Tf = T(Sp) = (TS)p \in BN = M$, hence M is left invariant.

To show that M is closed under addition, let f and g be two elements of M with polar decompositions $f = V|f|$ and $g = U|g|$. Then $|f| = V^*f$ and $|g| = U^*g$ are in $M^+ = N$ since M is left invariant. By Theorem (3.3.4), if $S \in B$, then

$$\|(f + g)(S)\|^2 \leq (\|f\| + \|g\|) (|f| + |g|)(S^*S)$$

and hence by Lemma (3.4.1), $|f + g| \in \{|f| + |g|\}^{\perp\perp}$, which in turn is a subset of N . From its polar decomposition, it

is thus clear that $f + g \in M$. This shows that M is an invariant subspace of B_* .

To show that M is norm-closed, let $\{f_n\}$ be a sequence in M converging in the norm to a function f , the sequence $\{f_n\}$ lies in $M^\perp = N$ and, by Lemma (3.3.3), there exists a subsequence converging weakly to $|f|$. Since N is a norm-closed convex set, it is weakly closed, we have $|f| \in N$. By making use of the polar decomposition, we have $f \in M$.

THEOREM (3.4.3). Let A be a C^* -algebra and A' , A'' its dual and bidual respectively. If N is a norm-closed subspace of A' , then it is invariant under left multiplication by elements of A'' .

PROOF. [5, p.414] Let p be an element of N and $S \in A''$. We want to show that $Sp \in N$. Since A is weakly dense in A'' , there exists a net $\{S_\alpha\}$ in A converging to S under the topology $\sigma(A'', A')$ on A'' . Hence $f(S_\alpha) \rightarrow f(S)$ for all $f \in A'$; in particular, $(fT)S_\alpha \rightarrow (fT)S$ for all $T \in A''$. Since

$$\begin{aligned} (S_\alpha p)T - (Sp)T &= p(S_\alpha T) - p(ST) = (pT)S_\alpha - (pT)S \\ &= (pT)(S_\alpha - S), \end{aligned}$$

we see that $S_\alpha p \rightarrow Sp$ under $\sigma(A', A'')$. Since N is convex and norm-closed, it is weakly closed, and hence $Sp \in N$ as asserted.

THEOREM (3.4.4). If N is a norm-closed order ideal in the predual B_* of a W^* -algebra B , then $N^{\perp\perp} = N$.

PROOF. [5, p.405] By Theorem (3.4.2), there exists a norm-

closed left invariant subspace M in B_* such that $M^\perp = N$. We claim that $N^\perp = M^\perp$. It is clear that $N^\perp \supseteq M^\perp$. Now, let $S \in N^\perp$ and $f \in M$. Then by Theorem (3.3.2), $f = U f$ and hence $|f| = U^* f \in M^\perp = N$. Thus $|f|(S) = 0$. As shown in the proof of Lemma (3.4.1), we have $|f|(S^2) = |f|(S^*S) = 0$. But $|f(S)|^2 = ||f|(US)|^2 \leq |f|(U^*U) |f|(S^*S) = 0$ by Cauchy-Schwartz inequality. Hence $f(S) = 0$ and so $S \in M^\perp$. Thus $M^\perp \supset N^\perp$ and consequently $N^\perp = M^\perp$. On the other hand, we have $M^{\perp\perp} = M^{\circ\perp}$, where M° denotes the polar of M . For clearly, $M^\perp \subseteq M^\circ$ and $M^{\perp\perp} \supseteq M^{\circ\perp}$. Now, if $p \in M^{\perp\perp}$ and $S \in M^\circ$, then $S^*S \in M^\circ$ since M is left invariant, and from $p(S^*S) = 0$ and the Cauchy-Schwartz inequality, we obtain $p(S) = 0$. Hence $S \in M^{\circ\perp}$ and $M^{\circ\perp} \subseteq M^{\perp\perp}$. Hence $M^{\perp\perp} = M^{\circ\perp}$. Thus, we obtain $N^{\perp\perp} = M^{\perp\perp} = M^{\circ\perp} = M^{\circ\circ} \cap B_*^+$. Since M is a norm-closed subspace of B_* , by the Bipolar Theorem [12, Corollary 1, p.36], we have $M^{\circ\circ} = M$. Hence $M^{\circ\circ} \cap B_*^+ = M^\perp$ and so $N^{\perp\perp} = M^\perp = N$.

COROLLARY (3.4.5). If N is a norm-closed order ideal in the dual A' of a C^* -algebra A , then $N = ((N)_{A''}^\perp)^\perp$.

PROOF. [5, p.405]. Since A'' can be considered as a W^* -algebra, and since A' can be identified with the predual of A'' [p.36], the Corollary follows immediately from Theorem (3.4.4).

THEOREM (3.4.6). Let A' be the dual of a C^* -algebra A and N a weak*-closed order ideal of A' . Then there exists a weak*-closed left invariant subspace M of A'

such that $M^+ = N$.

PROOF. [5, pp.405-406] Since N is convex and weak*-closed, it is norm-closed, [4, p.422 and § 3.1]. By Theorem (3.4.2), there exists a norm-closed left invariant subspace M of A' such that $M^+ = N$. We shall show that M is also weak*-closed. By Krein-Šmulian Theorem [4, p.429], it suffices to show that the intersection of M and the unit ball A'_1 of A' is weak*-closed. Let $\{f_\alpha\}$ be a net in $M \cap A'_1$ converging weakly* to f , and $f_\alpha = U_\alpha |f|$ be the corresponding polar decompositions, $U_\alpha \in A''$. Since M is a left invariant norm-closed subspace of A' , $Tf \in M$ for all $T \in A$, $Sf \in M$ for all $S \in A''$ by Theorem (3.4.3). Hence $|f_\alpha| = U^* f_\alpha$ lie in $M^+ = N$. Since the unit ball of A' is weak*-compact, we may select a subset $\{|f_\beta|\}$ converging weakly* to a positive functional p , which must lie in N . Since $|f_\beta| \in M \cap A'_1$, we have

$$\begin{aligned} |f_\beta(T)|^2 &= |U_\beta |f_\beta| (T)|^2 = ||f_\beta| (U_\beta T)|^2 \\ &\leq |f_\beta| (T^*T) |f_\beta| (U^*U) \leq |f_\beta| (T^*T) \end{aligned}$$

for all $T \in A$. Hence $|f(T)|^2 \leq p(T^*T)$. Since f and p are in the predual of A'' [p.37], we have $|f(S)|^2 \leq p(S^*S)$ for all $S \in A''$. Letting $f = U|f|$ be the polar decomposition, we have for each $S \in A''$,

$$|f(S)|^2 = |f(U^*S)|^2 \leq p(S^*U^*US) \leq p(S^*S).$$

from Lemma (3.4.1), $|f| \in \{p\}^{\perp\perp}$, which in turn is a subset of N . Then $f = U|f|$ is in M , which shows that M

is weak*-closed.

THEOREM (3.4.7). If N is a weak*-closed order ideal in the dual of a C*-algebra A , then $N = ((N)_{A'}^{\perp})^{\perp}$.

PROOF. [5, p.406] By Theorem (3.4.6), there exists a weak*-closed left invariant subspace M with $M^{\perp} = N$. Since A' is the predual of A'' [p.35], the argument used in the proof of Theorem (3.4.4) shows that

$$N^{\perp\perp} = (M_{A''}^{\circ})^{\circ} \cap (A')^{\perp}.$$

Hence, since M is weak*-closed, $N^{\perp\perp} = M^{\perp} = N$.

CHAPTER IV

TWO SIDED IDEALS IN A C*-ALGEBRA

§ 4.1. Invariant faces of a state space.

Let A be a C^* -algebra and let A' be the conjugate space of A . The state space of A is the set of all states of A , topologized by regarding it as a subspace of A' in the weak*-topology, and shall be denoted by $S(A)$. By a face of $S(A)$ we mean a convex subset F of $S(A)$ such that if $f \in F$, $g \in S(A)$ and $\alpha g \leq f$ for some $\alpha > 0$, then $g \in F$. F is an invariant face if $f \in F$ implies the state $S \rightarrow f(T^*ST) f(T^*T)^{-1}$ belongs to F whenever $f(T^*T) \neq 0$ and $T \in A$. If F is an invariant face of $S(A)$, let F^\perp denote the set of all elements T such that $f(T) = 0$ for all $f \in F$. Also if N is a subset of A , let N_A^\perp denote the set of all $f \in A'$ with $f(T) = 0$ for all $T \in N$, and let N^\perp denote $N_A^\perp \cap S(A)$.

THEOREM (4.1.1). Let A be a C^* -algebra. The map $N \rightarrow N^\perp$ is an order inverting bijection between norm-closed 2-sided ideals of A and weak*-closed invariant faces of $S(A)$.

PROOF. If N is a norm-closed 2-sided ideal of A , then N^\perp is a weak*-closed invariant face. Moreover, if N_1 and N_2 are two norm-closed 2-sided ideals of A with $N_1 \subseteq N_2$, then $N_2^\perp \subseteq N_1^\perp$. Hence the mapping is order

inverting. To show that the mapping is also onto, let F be a weak*-closed invariant face of $S(A)$ and let G be the smallest weak*-closed order ideal in the conjugate space A' of A generated by F ; G is an invariant order ideal. By Theorem (3.4.7), $G = (G^\perp)_{A'}^\perp$. Let $N = G^\perp$, we see that N is norm-closed and a 2-sided ideal of A since G is invariant; furthermore $N_{A'}^\perp = (G^\perp)_{A'}^\perp$. Since $G \cap S(A) = F$, $N^\perp = N_{A'}^\perp \cap S(A) = (G^\perp)_{A'}^\perp \cap S(A) = G \cap S(A) = F$. Hence the map $N \rightarrow N^\perp$ is onto. As it is clear that it is also one-to-one, it is an order inverting bijection.

COROLLARY (4.1.2). Let A be a C*-algebra, N a norm-closed 2-sided ideal of A and F a weak*-closed invariant face of $S(A)$. Then $N^{\perp\perp} = N$ and $F^{\perp\perp} = F$.

PROOF. Note that if F is a weak*-closed invariant face of $S(A)$, and G is the weak*-closed order ideal in A' generated by F , then $F^\perp = G^\perp$. It follows that if $\rho : N \rightarrow N^\perp$ denotes the bijection in the preceding theorem then $\rho^{-1} : F \rightarrow F^\perp$; in particular, $N = \rho^{-1}(N^\perp) = N^{\perp\perp}$. As $F^\perp = G^\perp$ and, by Theorem (3.4.7), $(G^\perp)_{A'}^\perp = G$, we have $F^{\perp\perp} = (F^\perp)_{A'}^\perp \cap S(A) = (G^\perp)_{A'}^\perp \cap S(A) = G \cap S(A) = F$.

THEOREM (4.1.3). Let N and M be closed 2-sided ideals in a C*-algebra A with identity. Then

- (i) $(N + M)^\perp = N^\perp \cap M^\perp$.
- (ii) $(N \cap M)^\perp = \text{conv}(N^\perp, M^\perp)$ where $\text{conv}(N^\perp, M^\perp)$ is the convex hull of $N^\perp \cap M^\perp$.

PROOF. (1) Since N, M are norm-closed 2-sided ideals of the C^* -algebra A , $N + M$ is again a norm-closed 2-sided ideal of A . Since $N \rightarrow N^\perp$ is order inverting and $N, M \subseteq N + M$, $(N + M)^\perp \subseteq N^\perp \cap M^\perp$. Conversely, if $f \in N^\perp \cap M^\perp$, then $f(T) = 0$ and $f(S) = 0$ for all $T \in N$, $S \in M$. Hence $f(P) = 0$ for all $P \in N + M$ which proves that $f \in (N + M)^\perp$. Therefore $N^\perp \cap M^\perp \subseteq (N + M)^\perp$.

(2) Since $N \cap M \subseteq N, M$, we have $(N \cap M)^\perp \supseteq N^\perp, M^\perp$. Now, $(N \cap M)^\perp$ is convex and weak*-closed, and $\text{conv}(N^\perp, M^\perp)$ is the smallest weak*-closed convex subset of $S(A)$ containing N^\perp and M^\perp . Hence $(N \cap M)^\perp \supseteq \text{conv}(N^\perp, M^\perp)$. It remains to show that $(N \cap M)^\perp \subseteq \text{conv}(N^\perp, M^\perp)$. By Corollary (4.1.2), this is equivalent to show that $\{\text{conv}(N^\perp, M^\perp)\}^\perp \supseteq (N \cap M)^{\perp\perp} = N \cap M$. Let $T \in N \cap M$. Then $f(T) = 0$ and $g(T) = 0$ for all $f \in N^\perp$ and $g \in M^\perp$. Hence $h(T) = 0$ for all $h \in \text{conv}(N^\perp, M^\perp)$, and $T \in \{\text{conv}(N^\perp, M^\perp)\}^\perp$. This completes the proof.

DEFINITION. A mapping φ on a convex subset K of a vector space into another vector space is called an affine mapping if $\varphi(ak + (1 - a)k') = a\varphi(k) + (1 - a)\varphi(k')$ for all $k, k' \in K$ and $0 \leq a \leq 1$.

Let A be a C^* -algebra with identity. For each $T \in A$, let $\hat{T}(f) = f(T)$ for all f in A' , the conjugate space of A . Then \hat{T} is a weak*-continuous linear functional. The restriction of \hat{T} to $S(A)$ is an affine

mapping on $S(A)$, the state space of A . By definition of \hat{T} , $|\hat{T}(f)| = |f(T)| \leq \|f\| \|T\|$, then $\|\hat{T}\| \leq \|T\|$. Conversely, if $T \neq 0$, then by Hahn-Banach Theorem, there exists an $f \in A'$ with $\|f\| = 1$ and $f(T) = \|T\|$ [4, p.65]. Hence $\|T\| = |f(T)| = |\hat{T}(f)| \leq \|T\| \|f\| = \|T\|$ and so $\|\hat{T}\| = \|T\|$. Furthermore, $\widehat{T + S} = \hat{T} + \hat{S}$ and $\widehat{\alpha T} = \alpha \hat{T}$. If $\hat{T} = \hat{S}$ for some $T, S \in A$, then $\hat{T}(f) = \hat{S}(f)$ for all $f \in A'$, i.e., $f(T) = f(S)$, and $f(T - S) = 0$ for all $f \in A'$. Hence $T - S = 0$ since A' is total on A . Therefore $S = T$. This shows that the canonical mapping $T \rightarrow \hat{T}$, for all $T \in A$ is an isometric homomorphism on A into A'' .

THEOREM (4.1.4). Let A be a C^* -algebra with identity.

For each hermitian element T of A , let \hat{T}_S be the restriction of \hat{T} to $S(A)$. Then the mapping $\psi: T \rightarrow \hat{T}_S$ is an isometric order-isomorphism of the hermitian part A_h of A onto the Banach space of all weak*-continuous real affine functions on $S(A)$.

PROOF. It is clear that each \hat{T}_S is a real affine function on $S(A)$ and is weak*-continuous for each hermitian element T of A . Since ω_ξ with $\|\xi\| = 1$ is a state and $\sup\{|\hat{T}_S(\omega_\xi)| : \|\xi\| = 1\} = \sup\{|(T\xi, \xi)| : \|\xi\| = 1\} = \|T\|$, we see that $\|\hat{T}\| = \|\hat{T}_S\|$ and ψ is an isometry and therefore one-to-one. It remains to show that it is onto. Let φ_0 be a real affine weak*-continuous function on $S(A)$. It is well known that each element of A' can be written as the sum of two hermitian linear functionals on A , and each hermitian functional of A' is the difference of two positive linear functionals on A [Theorem (1.1.4)]. It follows that $S(A)$ is total on A , and in

particular $S(A)$ is total on the real Banach space A_h . Hence $S(A)$ is not contained in any hyperplane of A_h^* and therefore, by [9, Lemma (4.1)], there exists a unique linear functional φ on A_h^* into the real field R and a unique real number α_0 in R such that $\varphi_0(f) = \varphi(f) + \alpha_0$ for all $f \in S(A)$. Let $\varphi|_{S(A)}$ be the restriction of φ to $S(A)$. Since φ_0 and the constant function $T_{\alpha_0}: f \rightarrow \alpha_0$ are weak*-continuous on $S(A)$, it follows that $\varphi|_{S(A)}$ is also weak*-continuous since $\varphi|_{S(A)} = \varphi_0 - T_{\alpha_0}$. Since $S(A)$ is a weak*-compact convex set and the closed unit ball D of A_h^* is given by $D = Q - Q$, where $Q = \{\alpha f: \alpha \in [0, 1], f \in S(A)\}$, we have from [9, Lemma (4.2)] that φ is weak*-continuous on A_h^* . Therefore, by [15, Theorem (3.81-A)], there exists an element $T \in A_h$ such that $\varphi(f) = f(T)$ for all $f \in A_h^*$. Hence ψ is onto.

§ 4.2. Two-sided ideals in a C*-algebra.

The purpose of this section is to answer the question asked by J. Dixmier in [3, p.20]. This is done in Theorem (4.2.2). Let A be a C*-algebra with identity, N a norm-closed 2-sided ideal of A . If Φ is the canonical homomorphism of A onto A/N , then the map $f \rightarrow f \circ \Phi$ is an affine isomorphism of $S(A/N)$ onto N . Thus the map $\Phi(T) \rightarrow T|_N$ is an order-isomorphic isometry on the self-adjoint operators in A/N . We shall make use of this fact to prove the following THEOREM (4.2.1). Let A be a C*-algebra with identity I , and let M, N be norm-closed 2-sided ideals of A . Let $T \in (M + N)^+$, and let $0 < \varepsilon < 1$. Then there exist

$B \in M^+$ and $C \in N^+$ such that $0 \leq T - B - C \leq \varepsilon I$.

PROOF. [14, pp.255-256] Multiplying T by a scalar, we may assume that $T \leq I$. It follows that $\|T\| \leq 1$. Let ψ be the canonical homomorphism of A onto A/N . Then $\psi(M + N) = \psi(M)$. Now $\psi(T) \geq 0$. Then there exists $B_1 \in M^+$ such that $\psi(B_1) = \psi(T)$. Then $\hat{B}_1/M^\perp = 0$ and $\hat{T}/N^\perp = \hat{B}_1/N^\perp$. Since $(M + N)^\perp = \text{conv}(M^\perp, N^\perp)$ by Theorem (4.1.3), $\hat{B}_1/(M \cap N)^\perp \leq \hat{T}/(M \cap N)^\perp$. Let f be the canonical homomorphism of A onto $A/M \cap N$. Then $0 \leq f(B_1) \leq f(T)$. Let f be the real continuous function $f(x) = (3^{-1}\varepsilon)^2$ for $x \leq (3^{-1}\varepsilon)^2$ and $f(x) = x$ for $x > (3^{-1}\varepsilon)^2$. Then $f(T) = (3^{-1}\varepsilon)^2$ for $T \leq (3^{-1}\varepsilon)^2$ and $f(T) = T$ for $T > (3^{-1}\varepsilon)^2$. Since T is positive, it has a unique positive square root $T^{\frac{1}{2}}$ [7, p.58]. Since $\|I - T^{\frac{1}{2}}\| < 1$, $T^{\frac{1}{2}}$ has inverse $T^{-\frac{1}{2}}$ [11, p.12]. With this remark, let

$$S = f(T)^{-\frac{1}{2}} B_1 f(T)^{-\frac{1}{2}}$$

Then $S \geq 0$ and $S \in M^+$ since M is a 2-sided ideal.

Now,

$$\begin{aligned} (1) \quad 0 \leq f(S) &= f(f(T))^{-\frac{1}{2}} f(B_1) f(f(T))^{-\frac{1}{2}} \\ &\leq f(f(T))^{-\frac{1}{2}} f(T) f(f(T))^{-\frac{1}{2}} \\ &\leq f(I). \end{aligned}$$

Let g be the real continuous function $g(x) = x$ for $x \leq 1$, $g(x) = 1$ for $x > 1$. Since $g(0) = 0$, $g(S)$ is by the Stone-Weierstrass theorem a uniform limit of polynomials in S without constant terms. Since $S \in M^+$, and M is

uniformly closed, $g(S) \in M^+$. By (1), we have

$$(2) \quad f(g(S)) = g(f(S)) = f(S).$$

Let

$$B = (f(T)^{\frac{1}{2}} - 3^{-1}\varepsilon I) g(S) (f(T)^{\frac{1}{2}} - 3^{-1}\varepsilon I).$$

Since $g(S) \in M^+$ so is B . Now $(f(x)^{\frac{1}{2}} - 3^{-1}\varepsilon)^2 \leq x$ for $x \geq 0$, and $g(S) \leq I$. Hence $0 \leq B \leq T$. By (2)

$$\begin{aligned} f(B) &= (f(f(T))^{\frac{1}{2}} - 3^{-1}\varepsilon f(I)) f(g(S)) (f(f(T))^{\frac{1}{2}} - 3^{-1}\varepsilon f(I)) \\ &= f(B_1) - 3^{-1}\varepsilon [f(f(T))^{\frac{1}{2}} f(S) + f(S) f(f(T))^{\frac{1}{2}} - 3^{-1}\varepsilon f(S)]. \end{aligned}$$

Since $\|f(f(T))^{\frac{1}{2}}\| \leq 1$, $\|f(S)\| \leq 1$ and $\varepsilon < 1$,

$$\|\hat{B}|_{(M \cap N)^\perp} - \hat{B}_1|(M \cap N)^\perp\| = \|f(B) - f(B_1)\| \leq \varepsilon;$$

in particular,

$$(3) \quad \|\hat{B}|_{N^\perp} - T|_{N^\perp}\| = \|\hat{B}|_{N^\perp} - \hat{B}_1|_{N^\perp}\| < \varepsilon.$$

Applying the preceding argument to $T - B$ instead of T and to N instead of M . Choose $C_1 \in N^+$ such that $C_1 \leq T - B$, and

$$(4) \quad \|\hat{C}_1|_{M^\perp} - (\hat{T} - \hat{B})|_{M^\perp}\| < \varepsilon.$$

Since $\hat{C}_1|_{N^\perp} = 0$, (3) implies

$$(5) \quad \|\hat{C}_1|_{N^\perp} - (\hat{T} - \hat{B})|_{N^\perp}\| \leq \varepsilon.$$

By (4) and (5), we have

$$\|\rho(C_1) - \rho(T - B)\| = \|\hat{C}_1 | \text{conv}(M^+, N^+)\| \leq \varepsilon.$$

Let $D = T - (B + C_1)$. Then $D \geq 0$, and $\|\rho(D)\| \leq \varepsilon$. Let h be the real continuous function $h(x) = 0$, for $x \leq \varepsilon$, $h(x) = x - \varepsilon$ for $x > \varepsilon$. Then $\rho(h(D)) = h(\rho(D)) = 0$, and $h(D) \in (M \cap N)^+ \subseteq N^+$. Furthermore

$$(6) \quad D - \varepsilon I \leq h(D) \leq D.$$

Let $C = C_1 + h(D)$. Then $C \in N^+$, and by (6), we have $0 \leq B + C \leq B + C_1 + D = T \leq B + C_1 + h(D) + \varepsilon I = B + C + \varepsilon I$.

This completes the proof.

THEOREM (4.2.2). Let N and M be norm-closed 2-sided ideals in a C^* -algebra A with identity I . Then

$$(N + M)^+ = N^+ + M^+.$$

PROOF. [14, pp.256-257] If $T \in M^+ + N^+$, then $T = B + C$ with $B \in M^+$ and $C \in N^+$. Hence $T \in (M + N)^+$. Multiplying T by a scalar, we may assume that $0 \leq T \leq I$. By the foregoing theorem, choose $B_0 \in M^+$ and $C_0 \in N^+$ such that

$$0 \leq T - B_0 - C_0 \leq 2^{-1}I.$$

Then $\|B_0\| \leq \|T\| \leq 1$, $\|C_0\| \leq \|T\| \leq 1$. Suppose inductively B_0, B_1, \dots, B_{n-1} are chosen in M^+ and C_0, C_1, \dots, C_{n-1} are chosen in N^+ such that $\|B_j\| \leq 2^{-j}$, and $\|C_j\| \leq 2^{-j}$,

and

$$0 \leq T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leq 2^{-n} I.$$

Applying the foregoing theorem to $T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j$ and

to $\varepsilon = 2^{-n-1}$. Then there exist $B_n \in M^+$ and $C_n \in N^+$ such that

$$0 \leq T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j - B_n - C_n \leq 2^{-n-1} I$$

or

$$(7) \quad 0 \leq T - \sum_{j=0}^n B_j - \sum_{j=0}^n C_j \leq 2^{-n-1} I.$$

Moreover, by (7) $\|B_n\| \leq 2^{-n}$, $\|C_n\| \leq 2^{-n}$; the induction is complete. Let

$$B = \sum_{j=0}^{\infty} B_j, \quad C = \sum_{j=0}^{\infty} C_j.$$

Then $B \in M^+$ and $C \in N^+$, and

$$\|T - B - C\| = \lim_{n \rightarrow \infty} \|A - \sum_{j=0}^n B_j - \sum_{j=0}^n C_j\| \leq \lim_{n \rightarrow \infty} 2^{-n-1} = 0.$$

Thus $A = B + C \in M^+ + N^+$, and $(M + N)^+ \subseteq M^+ + N^+$.

This completes the proof.

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