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**Finitely asymptotic properties of powers of characters
of compact semisimple Lie groups**


Ellen Marie O'Brien

submitted to the school of Graduate Studies and Research
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in Mathematics*

University of Ottawa
Ottawa, Ontario
Canada

April 1994

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 Ellen Marie O'Brien, Ottawa, Canada, 1994



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Abstract

This thesis has to do with decomposing tensor powers of irreducible representations of compact semisimple Lie groups or their Lie algebras. We will be concerned only with the set of irreducible representations appearing in the decomposition. In particular, we determine whether, for a sufficiently high tensor power, this set is “maximal”, in a sense to be made more precise below. We rely on the theory of semisimple Lie algebras, and describe the results in terms of characters.

A (reducible) character of a finite dimensional complex semisimple Lie algebra is saturated if for each of its dominant weights, the corresponding irreducible character is a summand in its orthogonal decomposition. A character is eventually saturated if some power is saturated. In the first part of the thesis, we describe all eventually saturated irreducible characters. In particular, it is shown that any irreducible character whose highest weight is in the interior of the Weyl chamber is eventually saturated; if the Lie algebra is simply laced, then all irreducible characters are eventually saturated; if not, then there are irreducible characters (with highest weights on the boundary of the Weyl chamber) that are not eventually saturated. We use the PRV conjecture to derive necessary and sufficient conditions for eventual saturation. These conditions are expressed in terms of cones generated by the weights of the character. The convex hull of the weight diagram, and the weights adjacent to the highest weight along the edges of the convex hull, are described in detail.

We show in the second part of the thesis that if the Lie algebra is A_d , and $d \leq 5$, then for any integer $n \geq d + 1$ and any irreducible character χ_λ , the product χ_λ^n is saturated. We also establish this result for certain irreducible characters of A_d , $d \geq 5$. These results are proved by induction on the rank of the Lie algebra and on the highest weight of the character. The geometry of the convex hull of the set of weights comes in to play here as well, and the dominant faces of this set are described. A reduction result, relating the “restriction to a dominant face” of the decomposition of a product of characters to that of a corresponding product in an algebra of lower rank, is established. The Littlewood-Richardson rule is used to compare products in which the component irreducible characters have highest weights which differ by a small amount.

Similar induction arguments are used to describe all the irreducible characters appearing in the decomposition of a product of irreducible characters of A_2 . Some of the irreducible characters appearing in a product of characters of higher rank algebras can also be determined using this type of induction.

The questions considered here arise in the study of product type actions of compact groups. The results on eventual saturation of irreducible characters are useful in computing the equivariant ordered K-theory of certain fixed point C^* -algebras under the corresponding group actions.

Introduction

According to Weyl's "unitarian trick", any finite dimensional representation of a compact semisimple Lie group is completely reducible. That is, the representation decomposes as a direct sum of irreducible representations. In practice however, decomposing a representation is a bit of work (a headache, a big problem, a complicated procedure, a mystery, difficult, painful, all of the above). Even the basic problem of decomposing the tensor product of two irreducible representations into irreducible components is non-trivial. There are formulas to effect the decomposition: Steinberg's formula involves a double sum over the Weyl group, and as a result is not practical in many cases; the Littlewood-Richardson rule provides a combinatorial method for decomposing products when the Lie algebra is of type A_d (and, for the brave, there are combinatorial methods for the other types of Lie algebras); the Brauer-Weyl formula (sometimes referred to as Klimyk's formula) has a nice geometric flavour, but involves a knowledge of the weight multiplicities of one of the representations.

A (seemingly) less imposing problem is to determine which irreducible representations appear in the orthogonal decomposition of a given representation, without concern for the number of occurrences—we call this the multiplicity—of each component representation. Since every irreducible representation is determined by its highest weight, we may view the set of all irreducible components of a representation as a finite subset of a Euclidean space. For a given representation, there is a "largest" set of possible highest weights for which the corresponding irreducible representation could appear, namely the set of all dominant weights of the representation. We will be concerned with representations in which all the corresponding irreducible representations do appear. In particular, we will determine whether a sufficiently high tensor power of any irreducible representation has this property.

We will rely on the representation theory of semisimple complex Lie algebras ([Hu1] is one reference for this material). It is convenient, if unorthodox, to discuss the work in terms of characters of the Lie algebra of the group. Recall that a representation is completely determined by its character, and the character of a tensor product of two representations is the product of the characters of the component representations.

Let \mathcal{G} be a finite dimensional semisimple complex Lie algebra. A character χ of \mathcal{G} is *saturated* if, for any dominant weight of χ , the corresponding irreducible character appears as a summand in the orthogonal decomposition of χ . If χ_λ is the irreducible character of \mathcal{G} with highest weight λ , then χ_λ is saturated if and only if λ is its only dominant weight. When this holds, the weight λ is called *minimal* or *miniscule*; any fundamental weight of A_d is minimal for example. A list of minimal weights is given in [Hu1; Ex. 13.13].

We say that χ is *eventually saturated* if some power of χ is saturated. We describe all eventually saturated irreducible characters in the first part of the thesis. It turns out that most irreducible characters are eventually saturated, but if \mathcal{G} is not simply laced then there are examples of irreducible characters which are not eventually saturated.

It is natural to ask what power of an irreducible character is needed to guarantee saturation, and we address this question for irreducible characters of A_d in the second part of the thesis. In particular, we ask whether there exists an integer N such that for all integers $n \geq N$, χ^n is saturated.

This holds for all irreducible characters of A_1 with $N = 2$, and all characters of A_1 with $N = 3$. David Bernier, an undergraduate student of David Handelman's, showed in the mid-eighties that it also holds for all irreducible characters of A_2 with $N = 4$. We will show that if $n \geq d + 1$ then χ_λ^n is saturated for all irreducible characters of A_d when $d \leq 5$, and for certain irreducible characters of A_d when $d > 5$. It seems reasonable to expect that this holds for all irreducible characters of A_d , however the method used here is impractical for large d .

The question of eventual saturation arises in the study of certain group actions on C^* -algebras ([HR],[Ha6],[Ha1],[Ha3]). Suppose that G is a connected, locally compact Lie group of rank d , and $\pi : G \rightarrow GL_n(\mathbb{C})$ an n -dimensional representation of G . Let A be the infinite tensor product algebra, $A := \otimes M_n(\mathbb{C})$; then G acts on A via the representation α , where $\alpha(g) := \otimes \text{Ad } \pi(g)$. In order to classify this type of action, Handelman is interested in computing the equivariant ordered K_0 -theory of the fixed point algebra A^G , and the crossed product algebra $A \rtimes_\alpha G$. Determining the ordering on $K_0(A^G)$ and $K_0(A \rtimes_\alpha G)$ amounts to answering the following "eventual positivity" question:

If χ is the character of G corresponding to the representation π , and ψ is a difference of characters of G , does there exist an integer m such that the product $\chi^m \psi$ is a character?

In case G is a d -torus, the question reduces to one of determining whether all the coefficients in the product of Laurent polynomials are nonnegative. A complete solution to this problem was obtained in [Ha5]. If G is not a torus, suppose that T is a maximal torus of G , and let P and f be the restrictions of χ and ψ respectively to T . In [Ha3; Theorem 1.1], it is shown that, if G is semisimple, then for most characters χ , the product $\chi^m \psi$ is a character for some m if and only if, for some n the product $P^n f$ has no negative coefficients. An essential ingredient in the proof of this result is the fact that, if λ is a dominant weight in the interior of the fundamental Weyl chamber, then χ_λ is eventually saturated.

Chapters 1, 2, and 3 comprise Part I of the thesis. In Chapter 1, we rely on the PRV conjecture ([Ku]) to derive necessary and sufficient conditions for eventual saturation. These conditions are expressed in terms of certain cones of weights associated with χ_λ . Let $\Pi(\lambda)$ be the set of weights of χ_λ , and let P_λ be the cone generated additively by the set of differences $\lambda - \Pi(\lambda)$. Define \mathcal{E}_λ , the subset of elements $\lambda - \beta$ in $\lambda - \Pi(\lambda)$ such that β lies next to λ along an edge of the convex hull of $\Pi(\lambda)$, and let C_λ be the cone generated by the set \mathcal{E}_λ . Then χ_λ is eventually saturated if and only if $P_\lambda = C_\lambda$, and this holds if and only if C_λ includes all the elements of its convex hull belonging to the root lattice (Theorem 1.9). In case λ belongs to the interior of the fundamental Weyl chamber, it is easy to see that the two cones are equal, hence the corresponding irreducible characters are eventually saturated (Corollary 1.10).

In the proof of [Ha2; Proposition D2], there is a description of the edges, emanating from the vertex λ , of the convex hull of the set of weights of an irreducible character of A_d . A generalization of this description is given in Proposition 1.2, and from this it follows that the convex hulls of the two cones C_λ and P_λ are equal. In Chapter 3, the sets \mathcal{E}_λ are described in detail for the simple algebras. We use this description to check whether the cones C_λ and P_λ are equal, and consequently whether χ_λ is eventually saturated. If \mathcal{G} is simply laced, then all irreducible characters are eventually saturated; if \mathcal{G} is not simply laced, then eventual saturation fails if \mathcal{E}_λ consists entirely of long roots, or if all the elements of \mathcal{E}_λ lying on a face of dimension 2 or greater are long roots. All other irreducible characters are eventually saturated (Theorem 3.1).

Suppose that $\chi = \chi_\lambda + \sum \chi_\mu$ where the sum runs over a subset of the set of dominant

weights of χ_λ . Then χ is eventually saturated if and only if χ_λ is, and so by Corollary 1.10, χ is eventually saturated if λ belongs to the interior of the Weyl chamber. In general, it is not true that a sum of eventually saturated characters is eventually saturated—even for A_2 ([Ha3; Example 2.8]). However, using Theorem 1.9, it is proved in [op. cit.] that a (reducible) character is eventually saturated if no extreme point of the convex hull of its weight diagram lies on a boundary of the Weyl chamber.

By using the PRV conjecture, we reduce the problem of eventual saturation to one about sums of finite subsets of a lattice \mathcal{L} . Let D be a finite subset of \mathcal{L} , and let ND be the set of all sums of N elements of D . In Chapter 1, we ask whether all lattice points in the convex hull of ND can be expressed as sums of N elements of D . We show that this holds for all sufficiently large integers N provided the set D satisfies certain local conditions at each of its vertices (Lemma 1.1).

This type of problem is also considered in [Ha2], in particular for sets of the form $K \cap \mathcal{L}$ where K is a convex polytope with vertices in the lattice \mathcal{L} . Let Q be a Laurent polynomial in d variables with nonnegative coefficients whose set of exponents equals $K \cap \mathcal{L}$. In [Ha2], the author defines rings R_Q and R_K . The former is $K_0(A^T)$, where A is the infinite tensor product of matrix algebras, and the action of the d -torus T on A is that induced by the representation of T whose character is Q . The ring R_K is obtained by inverting certain elements of R_Q , and is an invariant for the polytope K . It is shown that the rings R_Q and R_K are integrally closed in their respective fields of fractions if and only if the set $K \cap \mathcal{L}$ satisfies local conditions similar to those of Lemma 1.1. The ring R_Q may be defined for any Laurent polynomial Q with nonnegative coefficients—that is, the set of exponents of Q need not have the form $K \cap \mathcal{L}$. In Chapter 2, we extend Handelman’s result to show that R_Q is integrally closed in its field of fractions if and only if its set of exponents satisfies the conditions of Lemma 1.1 (Theorem 2.2).

If K is the convex hull of the set of weights $\Pi(\lambda)$ of an irreducible character χ_λ , and Q the restriction of this character to the maximal torus, then it is shown in [Ha2] that R_K is factorial if and only if K is simple as a polytope, and the elements of \mathcal{E}_λ generate the root lattice. Such polytopes are called integrally simple; these are described for the algebra A_d in [Ha2; Proposition D2]. We describe all simple and integrally simple weight polytopes in Proposition 3.3.

Chapters 4 to 6 make up Part II of the thesis, and here we focus on quantitative properties of powers and other products of irreducible characters. Our aim is to determine the minimal exponent required for saturation of a power χ_λ^m . At the same time, we gain information about the geometry of the set of weights of an irreducible character, and the “geometry” of products of characters (that is, the geometry of the set of highest weights of the irreducible characters appearing in such products). This information can be applied in the analysis of the ring R_χ .

In Chapters 4 and 5 we develop an induction argument which depends on the structure of the set of weights $\Pi(\lambda)$. The first step in this argument is to relate the faces of the convex hull of $\Pi(\lambda)$ to certain lower rank algebras. A reduction result, (Proposition 4.6), then allows us to describe the “faces” of the set of highest weights of irreducible characters appearing in a product by induction on the rank of \mathcal{G} . The next step is to describe the remaining irreducible characters by induction on the weight $m\lambda$ (or, more generally, on the sum of the highest weights of the factors of the product). In order to do this, we compare a given product of irreducible characters with a “perturbed” product, whose component highest weights are close to those of the original product. Our main tool for computing products is the Littlewood-Richardson rule, thus we restrict our attention to irreducible characters of A_d for this investigation. In Chapter 6, we apply the induction argument to prove

saturation of certain powers of characters of A_d . A similar argument might also be applied to describe the set of highest weights of the irreducible characters appearing in other products of characters of A_d . We investigate this possibility in Appendix A.

The results in Chapter 4 apply to any finite dimensional representation of a semisimple finite dimensional Lie algebra. We describe those faces of the convex hull of $\Pi(\lambda)$, ($\text{cvx } \Pi(\lambda)$), which include dominant weights, and refer to these as the *dominant faces* of $\text{cvx } \Pi(\lambda)$. To each of the dominant faces, we can associate a semisimple subalgebra of \mathcal{G} . If λ and μ are dominant weights, then the set of highest weights of irreducible characters appearing in the product is a subset of $\Pi(\lambda + \mu)$. On the dominant faces of $\Pi(\lambda + \mu)$, the product $\chi_\lambda \chi_\mu$ reduces to a product of irreducible characters of the corresponding semisimple subalgebra (Proposition 4.6). This result, expressed algebraically, appears in a paper by Berenstein and Zelevinsky [BZ; Proposition 1.3], where the authors use it to justify certain conjectures about products of characters in B_d , C_d and D_d . They do not give a geometric interpretation of the result. In our proof, we rely on the geometry of the convex hull of $\Pi(\lambda)$, and the fact that on the dominant faces, the weight multiplicities equal those of a corresponding irreducible character of the associated subalgebra (Lemma 4.1). The information about the dominant faces given here should also be useful in the analysis of the positivity problem for irreducible characters whose highest weight lies on a boundary of the fundamental Weyl chamber. In that case, the reduction to the maximal torus result fails, but one may make use of the correspondence between the dominant faces of $\text{cvx } \Pi(\lambda)$ and families of pure unfaithful traces on the ring R_χ .

If χ and ψ are two characters, write $\chi \geq \psi$ if the difference $\chi - \psi$ is a sum of irreducible characters. If χ and ψ are products of irreducible characters, and the set of weights of ψ is included in that of χ , when does it follow that $\chi \geq \psi$? We address this question in Chapter 5; the principal result is Proposition 5.1: if $\mu_1, \mu_2, \dots, \mu_n$ are dominant weights of A_d and $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}$ are fundamental dominant weights (not necessarily distinct) whose sum belongs to the root lattice, then

$$\chi_{\mu_1} \chi_{\mu_2} \cdots \chi_{\mu_n} \leq \chi_{\mu_1 + \lambda_{i_1}} \chi_{\mu_2 + \lambda_{i_2}} \cdots \chi_{\mu_n + \lambda_{i_n}}.$$

Proposition 5.1 implies, for example, that $\chi_\lambda^{d+1} \leq \chi_{\lambda + \lambda_i}^{d+1}$ for any fundamental weight λ_i , and $\chi_\lambda \chi_\mu \leq \chi_{\lambda + \lambda_1} \chi_{\mu + \lambda_d}$. When the rank of \mathcal{G} is greater than three, Proposition 5.1 alone is not sufficient to describe the irreducible characters whose highest weights do not lie on a face of the set of weights of a product. We also need to compare products in which one of the component highest weights is perturbed by a small but not dominant weight. The arguments required for these comparisons are more intricate, and the results obtained—Lemmas 5.5 to 5.12—are weaker than those of Proposition 5.1.

Finally, we show in Chapter 6 that if $d \leq 5$, then for all irreducible characters of A_d , χ_λ^m is saturated whenever $m \geq d+1$ (Theorem 6.8). We also establish this result for irreducible characters of A_d ($d > 5$) with highest weights of the form $\lambda = l_1 \lambda_1 + l_2 \lambda_2$ or $\lambda = l_1 \lambda_1 + l_d \lambda_d$, (Propositions 6.4, 6.5, and 6.7 respectively).

Chapter 1

Necessary and sufficient conditions for eventual saturation

Let \mathcal{G} be a semisimple finite dimensional complex Lie algebra, and let χ_λ be the irreducible character of \mathcal{G} with highest weight λ . We will establish in this chapter necessary and sufficient conditions for the eventual saturation of χ_λ .

Denote the set of weights of χ_λ by $\Pi(\lambda)$. The PRV conjecture (proved, for example, in [Ku]) asserts that any irreducible character whose highest weight is the sum of a vertex of the convex hull of $\Pi(\lambda)$ and one of $\Pi(\mu)$ appears in the orthogonal decomposition of the character $\chi_\lambda \chi_\mu$. We can argue by induction that if μ_1, \dots, μ_m are dominant weights, any irreducible character whose highest weight is the sum of m vertices, one from each of the convex hulls of $\Pi(\mu_1), \Pi(\mu_2), \dots$, and $\Pi(\mu_m)$, appears in the decomposition of the product $\chi_{\mu_1} \chi_{\mu_2} \cdots \chi_{\mu_m}$. The details: suppose that the statement holds for $m-1$, and let $\tau_i, i = 0, 1, \dots, m$ be elements of the Weyl group such that $\lambda' = \tau_1 \lambda + \cdots + \tau_m \lambda$ is a dominant weight. Let $\omega = \tau_0(\tau_1 \lambda + \cdots + \tau_{m-1} \lambda)$ be the dominant weight conjugate to $\tau_1 \lambda + \cdots + \tau_{m-1} \lambda$. By our induction hypothesis, the irreducible character χ_ω appears in χ_λ^{m-1} . Therefore the irreducible character $\chi_{\lambda'}$, whose highest weight is conjugate to $\omega + \tau_0(\tau_m \lambda)$ appears in $\chi_\omega \chi_\lambda$ and so in χ_λ^m .

If all the dominant weights of χ_λ^N could be expressed as such a sum of vertices, then χ_λ^N would be saturated; this is the idea behind the argument we use to give sufficient conditions for eventual saturation.

We begin by considering a finite subset D of a lattice \mathcal{L} . In Lemma 1.1, we establish necessary and sufficient conditions so that, for all sufficiently large integers N , any lattice point in the convex hull of the set of all sums of N elements of D may be written as a sum of N elements of D . These conditions are expressed in terms of certain cones associated with the vertices of the convex hull of D .

In Section 2, we define two cones: the first, P_λ , is generated by the set $\lambda - \Pi(\lambda)$; the second, C_λ , is generated by a subset thereof. The generators of C_λ have the form $\lambda - \beta$, where β is a weight of χ_λ and lies next to λ on an edge of $\text{cvx } \Pi(\lambda)$ (Proposition 1.2), thus the convex hulls of the two cones are equal (Corollary 1.3). We give a description of these sets (and of the convex hull of $\Pi(\lambda)$) in terms of linear functionals exposing their faces in Proposition 1.4.

If there is a generator of P_λ which does not belong to C_λ , then χ_λ is not eventually saturated (Proposition 1.7). In Theorem 1.9 we show that χ_λ is eventually saturated if and only if the cones C_λ and P_λ are equal, and the latter holds if and only if C_λ satisfies the condition of Lemma 1.1, and the group generated by C_λ equals the root lattice. It follows readily that χ_λ is eventually saturated if λ belongs to the interior of the fundamental Weyl chamber (Corollary 1.10). We will use Theorem 1.9 in Chapter 3 to give a complete description of all eventually saturated irreducible characters.

For the reader unfamiliar with the elementary theory of convex sets, the reference [L] may be useful.

1. Normal lattice cones and relatively convex sets

If A and B are subsets of \mathbf{R}^d , write $A + B$ for the set of sums $\{a + b \mid a \in A, b \in B\}$, and

$A - B$ for the set of differences $\{a - b \mid a \in A, b \in B\}$; for a positive integer N , let NA be the set of all sums of N elements of A . Denote the convex hull of A by $\text{cvx } A$, and let $\mathcal{L}(A)$ be the abelian group generated by the set of differences of elements of A . We say that A is a (proper) cone if it is an additive semi-group and $A \cap -A = \{0\}$, and A is called a *lattice cone* if $A \subset \mathcal{L}$ for some lattice \mathcal{L} . Given a subset A of a lattice, when can we express any lattice point in $\text{cvx } A$ as a sum of elements of A ? In this section, we show that for subsets of the form ND where D is finite, it is enough to check that this property holds for certain cones generated by the elements of D .

Let D be a finite subset of a lattice $\mathcal{L} \subset \mathbb{Q}^d$. We will call D \mathcal{L} -convex (or *relatively convex* with respect to \mathcal{L}) if $D = \text{cvx } D \cap \mathcal{L}$. If D' is a finite set and $D' \subset (\mathcal{L} + w)$ for some $w \in \mathbb{Z}^d$, then D' is said to be \mathcal{L} -convex if $D' - w = \text{cvx } (D' - w) \cap \mathcal{L}$. Even if the set D is \mathcal{L} -convex, it is not necessarily true that any of the sets ND is relatively convex. (See [Ha2; III.3] and [Ha4; Chapter 3] for examples). Let $\mathcal{L}(D)$ be the lattice generated by the set of differences of elements in D . In Lemma 1.1, we show that the sets ND are $\mathcal{L}(D)$ -convex for all sufficiently large integers N if and only if certain cones associated with the vertices of $\text{cvx } D$ are $\mathcal{L}(D)$ convex.

For each vertex v of $\text{cvx } D$, let C_v be the cone generated by the set of differences $\{v - w \mid w \in D\}$, that is

$$C_v := \bigcup_{k \in \mathbb{N}} k(v - D) \quad (1.1.1).$$

We refer to C_v as the *local cone* of D at v . The local cones C_v are unchanged if we replace D by kD or by $D + w$; that is, the local cone of D at v is equal to that of kD at kv for any k , and also to that of $D + w$ at $v + w$ for any $w \in \mathcal{L}$. The lattice $\mathcal{L}(D)$ equals $\mathcal{L}(C_v)$.

A lattice cone C is *normal* if whenever mx belongs to C for some positive integer m and point x in $\mathcal{L}(C)$, then x belongs to C . Observe that if C is a lattice cone and $C \subset \mathbb{Q}^d$, then C is normal if and only if $C = \text{cvx } C \cap \mathcal{L}(C)$. (A proof: Suppose that C is normal and x belongs to $\text{cvx } C \cap \mathcal{L}(C)$. Then $x = \sum_{i=1}^{d+1} r_i c_i$ where the r_i are nonnegative real numbers and c_i belong to C . Since x belongs to $\mathcal{L}(C) \subset \mathbb{Q}^d$, we can find nonnegative rational coefficients r_i , so there is an integer M such that Mx belongs to C . Then x belongs to C , by normality; since C is contained in $\text{cvx } C \cap \mathcal{L}(C)$, the two sets are equal. For the reverse implication, suppose that $C = \text{cvx } C \cap \mathcal{L}(C)$, and for some x in $\mathcal{L}(C)$ and some integer m , mx belongs to C . Then $x = \frac{1}{m}(mx)$ belongs to $\text{cvx } C \cap \mathcal{L}(C) = C$.)

It is not difficult to see that if D is a finite subset of \mathbb{Q}^d and ND is saturated for all sufficiently large integers N , then each of the local cones C_v is normal. We show in Lemma 1.1 that the converse is also true.

We originally proved Lemma 1.1 using techniques established in [Ha2; Chapter III]; this result is included in Chapter 2 (Theorem 2.2).

When an earlier version of Part I of this thesis was accepted for publication in Crelles Journal ([O'B]), the referee suggested the proof of Lemma 1.1 presented below. It relies on the Shapley-Folkman Theorem ([S; Appendix 2]); we include a statement of the latter for the reader's convenience.

Theorem (Shapley-Folkman). Let D_i , ($i = 1, 2, \dots, n$) be non-empty subsets of \mathbb{R}^d and $z \in \text{cvx } (\sum_{i=1}^n D_i)$. Then there exist points $x_i \in \text{cvx } D_i$ such that $z = \sum_{i=1}^n x_i$ and at most d of the x_i do not belong to D_i .

Lemma 1.1. Let D be a finite subset of \mathbb{Z}^d . The set nD is $\mathcal{L}(D)$ -convex for all sufficiently large integers N if and only if, for each vertex v of the convex hull of D , the local cone C_v is normal.

Proof: Let $\mathcal{L} = \mathcal{L}(D) = \mathcal{L}(C_v)$. We may assume that $0 \in D$ so that $nD \subset \mathcal{L}$ for all integers N and let $\partial_e D$ be the set of vertices of $\text{cvx } D$. Suppose that x belongs to $\text{cvx } nD$ for some integer n . By the Shapley-Folkman Theorem, we may write $x = \sum_{v \in \partial_e D} a_v v + x'$ where the coefficients a_v are nonnegative integers summing to $n - d$ and $x' \in d\text{cvx } D$. If $x \in \mathcal{L}$, then x' belongs to the finite set $d\text{cvx } D \cap \mathcal{L}$. Normality of the local cones C_v implies that for each $v \in \partial_e D$, there exist nonnegative integers $b_y(x', v)$ indexed by $y \in D$, such that $(x' - dv) = \sum_{y \in D} b_y(x', v)(y - v)$. We therefore obtain, for each $v \in \partial_e D$ an expression:

$$x = \sum_{\substack{v' \in \partial_e D \\ v' \neq v}} a_{v'} v' + \sum_{y \in D} b_y(x', v) y + \left(a_v + d - \sum_{y \in D} b_y(x', v) \right) v.$$

Then $x \in nD$ provided that $\left(a_v + d - \sum_{y \in D} b_y(x', v) \right) > 0$ for some extreme point v . Since the set $d\text{cvx } D \cap \mathcal{L}$ is finite, there is an integer B such that, for all x' in this set and $v \in \partial_e D$, $\sum_{y \in D} b_y(x', v) \leq B$. Thus if we choose n sufficiently large, then $a_v + d - B > a_v + d - \sum_{y \in D} b_y(x', v) > 0$ holds for at least one of the elements $v \in \partial_e D$. (In particular, if $|\partial_e D| = k$, and $N > k(B - d)$, then we must have $a_v \geq B - d$ for at least one $v \in \partial_e D$, so $x \in nD$ if $n \geq N$).

To establish the converse, observe that if $y \in \text{cvx } C_v \cap \mathcal{L}(C_v)$ for some $v \in \partial_e D$, then for all sufficiently large integers M , $Mv - y \in \text{cvx } MD \cap \mathcal{L}(C_v) = MD \cap \mathcal{L}(D)$. Choose M so that MD is $\mathcal{L}(D)$ -convex. Then $Mv - y \in MD$, so $y \in M(v - D) \subset C_v$.*

Example: The set of weights $\Pi(\lambda)$ of the irreducible character χ_λ is relatively convex with respect to the root lattice Λ_r , since it is saturated ([Hu1; Proposition 21.3]). We claim that the set of weights of χ_λ^k is Λ_r -convex for any positive integer k . To see this, it is enough to show that for any positive integer k , the set of weights of χ_λ^k equals that of $\chi_{k\lambda}$. The former consists of sums of k weights of χ_λ . By [Hu1; Proposition 21.3], w is a weight of χ_λ if and only if the difference $\lambda - \sigma w$ is a sum of simple roots for all σ in the Weyl group. If w_1, \dots, w_k are weights of χ_λ then the difference $k\lambda - \sigma(w_1 + w_2 + \dots + w_k)$ is a sum of simple roots for each σ in the Weyl group, so $(w_1 + \dots + w_k)$ is a weight of $\chi_{k\lambda}$. On the other hand, $\chi_{k\lambda}$ is a summand in the orthogonal decomposition of χ_λ^k , so all the weights of $\chi_{k\lambda}$ belong to $k\Pi(\lambda)$. Thus for all integers k we have

$$k\Pi(\lambda) = \Pi(k\lambda) = k\text{cvx } \Pi(\lambda) \cap \Lambda_r. \quad (1.1.2)$$

2. Two lattice cones associated with an irreducible character

Here $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ will be a set of simple roots of \mathcal{G} . Recall that the simple roots span a d -dimensional real Euclidean space (\mathbb{R}^d) with bilinear form $(\ , \)$ inherited from the Killing form. For x and y in \mathbb{R}^d , set $\langle x, y \rangle = 2(x, y) / (y, y)$. The fundamental weights $\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ satisfy the conditions $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for all i and j in the set $I = \{1, 2, \dots, d\}$.

Denote by Λ_r and Λ the lattices generated (as abelian groups) by the simple roots and the fundamental weights, respectively. The set of dominant weights, that is, those λ in Λ for which $\langle \lambda, \alpha_i \rangle \geq 0$ for all i in I , is denoted Λ^+ .

For a dominant weight λ in Λ^+ , let P_λ be the cone generated additively by the set $\{\lambda - w \mid w \in \Pi(\lambda)\}$, that is

$$P_\lambda := \bigcup_{k \in \mathbb{N}} k(\lambda - \Pi(\lambda)).$$

We associate a second cone with λ as follows: set $I_\lambda := \{i \in I \mid \langle \lambda, \alpha_i \rangle > 0\}$, and let W_λ be the subgroup of the Weyl group W generated by the simple reflections $\sigma_j, j \notin I_\lambda$. W_λ is the isotropy group of λ , by [Hu2; Theorem 1.12]. Let C_λ be the cone generated by the set

$$\mathcal{E}_\lambda := \{\sigma \alpha_i \mid \sigma \in W_\lambda, i \in I_\lambda\},$$

that is

$$C_\lambda := \bigcup_{k \in \mathbb{N}} k \mathcal{E}_\lambda.$$

Observe that the elements of \mathcal{E}_λ are positive roots ([Hu1; lemma 10.2B]). Moreover, each of the weights $\lambda - \alpha_i, i \in I_\lambda$ belongs to $\Pi(\lambda)$ ([Hu1; Section 13]), as do all conjugates under W of these weights. In particular $\lambda - \mathcal{E}_\lambda$ is contained in $\Pi(\lambda)$, so C_λ is a subcone of P_λ . We shall see in Proposition 1.2 that the elements of $\lambda - \mathcal{E}_\lambda$ are the weights in $\Pi(\lambda)$ lying adjacent to λ along the edges of $\text{cvx } \Pi(\lambda)$, and consequently the convex hulls of C_λ and P_λ are equal.

The result in Proposition 1.2 is a generalization of the description, given in the proof [Ha2; Proposition D2], of the edges of weight diagrams for characters of $\text{SU}(d+1)$.

Proposition 1.2. (i) If v is a vertex of $\text{cvx } \Pi(\lambda)$, then there exist nonnegative integers κ_β such that

$$v = \lambda - \sum_{\beta \in \mathcal{E}_\lambda} \kappa_\beta \beta.$$

(ii) The weight w in $\Pi(\lambda)$ lies adjacent to λ along an edge of $\text{cvx } \Pi(\lambda)$ if and only if $\lambda - w$ belongs to \mathcal{E}_λ .

Proof: (i) The vertices of $\text{cvx } \Pi(\lambda)$ are the images $\tau\lambda$ of λ under the action of the Weyl group. We argue by induction on the length of τ , $l(\tau)$. If $l(\tau) = 1$, then τ is a simple reflection, say σ_j . Now $\sigma_j\lambda = \lambda - \langle \lambda, \alpha_j \rangle \alpha_j$ and $\langle \lambda, \alpha_j \rangle$ is a nonnegative integer; if it is not zero, then α_j belongs to \mathcal{E}_λ , so (i) will hold in this case.

Now suppose that $\tau = \sigma_j \tau'$ for some τ' in W with $0 < l(\tau') < l(\tau)$. By hypothesis, we may write $\tau'\lambda = \lambda - \sum_{\beta} \kappa_\beta \beta$ where the coefficients are nonnegative integers. If j does not belong to I_λ , then σ_j lies in W_λ , $\sigma_j \beta$ belongs to \mathcal{E}_λ for all β in \mathcal{E}_λ , and $\tau\lambda = \sigma_j \tau'\lambda = \lambda - \sum_{\beta \in \mathcal{E}_\lambda} \kappa_\beta (\sigma_j \beta)$. Thus (i) holds. There remains the case that j does belong to I_λ .

If j belongs to I_λ , then α_j lies in \mathcal{E}_λ , and

$$\tau\lambda = \left(\lambda - \sum_{\beta \in \mathcal{E}_\lambda} \kappa_\beta \beta \right) - \langle \tau'\lambda, \alpha_j \rangle \alpha_j.$$

Now $\langle \tau'\lambda, \alpha_j \rangle$ is an integer, and as $l(\sigma_j \tau') > l(\tau')$, it follows from [Hu2; Theorem 5.4] that $\langle \tau'\lambda, \alpha_j \rangle$ is nonnegative. Hence $\tau\lambda$ has the desired form, completing the induction.

(ii) By [Hu1; Proposition 21.3], each weight w in $\Pi(\lambda)$ can be expressed as $w = \lambda - \sum_{i=1}^d k_j \alpha_j$ where k_j are uniquely determined nonnegative integers. Consequently, for each i in I_λ , λ and

$\sigma_i \lambda = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i$ are distinct vertices of $\text{cvx } \Pi(\lambda)$ connected by an edge, and $\lambda - \alpha_i$ is the weight lying next to λ along this edge. As W_λ consists of invertible linear transformations that fix λ , it acts on the set of edges emanating from λ . Hence if i belongs to I_λ then σ in W_λ will map the edge between vertices λ and $\sigma_i \lambda$ to an edge between λ and $\sigma \sigma_i \lambda$. The weight lying adjacent to λ along this edge is $\sigma(\lambda - \alpha_i) = \lambda - \sigma \alpha_i$. It follows from (i) that these are all the edges at λ . *

Corollary 1.3.

$$\text{cvx } P_\lambda = \text{cvx } C_\lambda = \left\{ \sum_{\beta \in \mathcal{E}_\lambda} r_\beta \beta \mid r_\beta \in \mathbf{R}^+ \right\}$$

*

For $i \in I$, let $c_i := 2/(\alpha_i, \alpha_i)$ and define the linear functionals $f_i := c_i \lambda_i$, so that for τ in the Weyl group we have

$$\tau f_i(x) := c_i (\lambda_i, \tau^{-1} x).$$

If $x = \sum_{i=1}^d r_i \alpha_i$, then $f_i(x) = r_i$; if β belongs to \mathcal{E}_λ (the latter identified with a subset of \mathbf{Z}^d) then $f_i(\beta)$ is the i^{th} coordinate of β , and $f_i(\sigma \beta)$ is the i^{th} coordinate of $\sigma^{-1} \beta$. If $\tau \in W_{\lambda_i}$, then $\tau f_i = f_i$.

Proposition 1.4.

- (i)
$$\text{cvx } \Pi(\lambda) = \bigcap_{\substack{\tau \in W \\ i \in I}} \{x \mid \tau f_i(x) \leq f_i(\lambda)\}$$
- (ii)
$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in I}} \{x \mid \sigma f_i(x) \geq 0\} = \text{cvx } P_\lambda.$$

Proof: (i) Let K denote the right side of (i). By [Hu1; Proposition 21.3], $\Pi(\lambda)$ is the set of weights w in Λ such that for each τ in W , there exist nonnegative integers k_1, k_2, \dots, k_d such that

$$\lambda - \tau w = \sum_{i=1}^d k_i \alpha_i.$$

Thus a weight w lies in $\Pi(\lambda)$ if and only if $f_i(\lambda - \tau w)$ is a nonnegative integer for all τ in W and i in I . Hence $\Pi(\lambda) = K \cap (\Lambda_\tau + \lambda)$ and $\text{cvx } \Pi(\lambda) \subseteq K$. To obtain the reverse inclusion, define

$$C_K = \bigcap_{i \in I} \{x \in \mathbf{R}^d \mid f_i(x) \leq f_i(\lambda)\}.$$

We will show first that the intersection of C_K with the closure of the fundamental Weyl chamber, $\bar{\mathcal{C}}$, is contained in $\text{cvx } \Pi(\lambda)$. The cone C_K consists of all points in \mathbf{R}^d admitting the form $\lambda - \sum_{i=1}^d r_i \alpha_i$ with nonnegative real r_i , and is thus a convex set with the single extreme point, at λ , and extreme rays in the direction of $-\alpha_i$ for each of the simple roots α_i . We claim that the ray $\{\lambda - r_i \alpha_i, r_i \in \mathbf{R}^+\}$ meets the closed fundamental Weyl chamber at a point other than λ if and only if i belongs to I_λ .

To verify this claim, observe that if $j \notin I_\lambda$, then $\lambda \notin \mathcal{C}$, and for any positive real number r , $(\lambda - r\alpha_j, \alpha_j) = -r(\alpha_j, \alpha_j) < 0$, so the ray along $-\alpha_j$ does not meet $\bar{\mathcal{C}} \setminus \{\lambda\}$. On the other hand, if $i \in I_\lambda$, then $(\lambda, \alpha_i) > 0$, so for sufficiently small $r > 0$, $(\lambda - r\alpha_i, \alpha_i) = (\lambda, \alpha_i) - r(\alpha_i, \alpha_i) > 0$. If $k \neq i$, then $(\lambda - r\alpha_i, \alpha_k) = (\lambda, \alpha_k) - r(\alpha_i, \alpha_k) \geq (\lambda, \alpha_k) \geq 0$, so the ray along $-\alpha_i$ has a non-empty intersection with $\bar{\mathcal{C}} \setminus \{\lambda\}$. Now recall from Proposition 1.2(ii) that the edges of $\text{cvx } \Pi(\lambda)$ at the vertex λ have the form $\sigma\{\lambda - r_i\alpha_i \mid 0 \leq r_i \leq \langle \lambda, \alpha_i \rangle\}$ where $i \in I_\lambda$ and $\sigma \in W_\lambda$. We have seen that for any $i \in I_\lambda$, the edge along $-\alpha_i$ —we will call it R_i —meets $\bar{\mathcal{C}} \setminus \{\lambda\}$. Moreover, any element on this edge which does not belong to $\bar{\mathcal{C}}$ is mapped by σ_i to an element on the edge belonging to $\bar{\mathcal{C}}$. This implies that the distinct images of R_i under W_λ do not meet $\bar{\mathcal{C}} \setminus \{\lambda\}$, as we now demonstrate. Suppose that for some $\tau \in W_\lambda$ the weight $\tau(\lambda - r\alpha_i)$ belongs to $\bar{\mathcal{C}}$. Since $\lambda - r\alpha_i$ belongs to R_i , either $\lambda - r\alpha_i \in \bar{\mathcal{C}}$ or $\sigma_i(\lambda - r\alpha_i) \in \bar{\mathcal{C}}$. Thus, by [Hu1; Lemma 10.3 B], either $\lambda - r\alpha_i = \tau(\lambda - r\alpha_i) = \lambda - r\tau\alpha_i$ or $\sigma_i(\lambda - r\alpha_i) = \tau(\lambda - r\alpha_i)$. But either of these implies that $\tau R_i = R_i$. Consequently the distinct images under W_λ of R_i do not meet $\bar{\mathcal{C}} \setminus \{\lambda\}$. Thus $\text{cvx } \Pi(\lambda) \cap \bar{\mathcal{C}} = C_K \cap \bar{\mathcal{C}}$.

Since $K = \bigcap_{\sigma \in W} (\sigma C_K)$ we have $K \cap \bar{\mathcal{C}} \subseteq C_K \cap \bar{\mathcal{C}} \subseteq \text{cvx } \Pi(\lambda)$. If w belongs to K , then there exists τ in W such that τw is dominant; since K and $\text{cvx } \Pi(\lambda)$ are both stable under W , τw belongs to $K \cap \bar{\mathcal{C}}$ and so to $\text{cvx } \Pi(\lambda)$, and thus w belongs to $\text{cvx } \Pi(\lambda)$.

(ii) The two convex cones $\text{cvx } P_\lambda$ and $\text{cvx } C_\lambda$ are equal, and are spanned over \mathbb{R}^+ by \mathcal{E}_λ , as we observed in Corollary 1.3. Therefore, $\text{cvx } P_\lambda$ is the intersection of those images of the cone $\lambda - C_K$ which have vertex 0; hence

$$\begin{aligned} \text{cvx } P_\lambda &= \bigcap_{\sigma \in W_\lambda} \sigma(\lambda - C_K) = \bigcap_{\sigma \in W_\lambda} (\lambda - \sigma C_K) \\ &= \bigcap_{\substack{\sigma \in W_\lambda \\ i \in I}} \{x \mid \sigma f_i(x) \geq 0\}. \end{aligned}$$

✱

We noted above that \mathcal{E}_λ consists of positive roots. If all $\alpha_i, i \in I_\lambda$ have the same length, the all roots in \mathcal{E}_λ do as well. The next lemma shows that \mathcal{E}_λ generates the entire root lattice, provided \mathcal{E}_λ does not consist entirely of long roots in the non-simply laced case.

Lemma 1.5. Suppose that \mathcal{G} is simple and λ is a nonzero dominant weight.

(i) If \mathcal{G} is simply laced, then $\mathcal{L}(C_\lambda) = \Lambda_r$.

(ii) If \mathcal{G} is not simply laced and $\langle \lambda, \alpha_s \rangle \neq 0$ for some simple short root α_s , then $\mathcal{L}(C_\lambda) = \Lambda_r$.

(iii) In all cases, $\mathcal{L}(C_\lambda)$ has rank d .

Proof: For (i) and (ii), it suffices to show every simple root, α_j ($j \in I$), belongs to $\mathcal{L}(C_\lambda)$. Evidently α_j belongs if j lies in I_λ . Let Γ be the Dynkin diagram of \mathcal{G} . When computing elements in \mathcal{E}_λ , it is helpful to keep the following in mind. If the nodes on Γ labelled by k and k' are not connected, then $\sigma_k(\alpha_{k'}) = \alpha_{k'}$. If they are connected by a single edge, then $\sigma_k(\alpha_{k'}) = \alpha_{k'} + \alpha_k$. If the nodes are connected by m edges, then $\sigma_k(\alpha_{k'}) = \alpha_{k'} + \alpha_k$ if $\alpha_{k'}$ is short and α_k is long, and $\sigma_k(\alpha_{k'}) = \alpha_{k'} + m\alpha_k$ if $\alpha_{k'}$ is long and α_k is short.

(i) If j is not a member of I_λ , there is a subset $\{j_0 = j, j_1, j_2, \dots, j_k\}$ of I whose corresponding subgraph of Γ is connected and exactly one index, j_k , belongs to I_λ . By deleting some of the

indices if necessary, we may assume that the set $\{j_1, j_2, \dots, j_k\}$ satisfies these properties as well. Then both of the elements, $\alpha_j + \alpha_{j_1} + \alpha_{j_2} + \dots + \alpha_{j_k}$ and $\alpha_{j_1} + \alpha_{j_2} + \dots + \alpha_{j_k}$ belong to \mathcal{E}_λ , so α_j belongs to $\mathcal{L}(C_\lambda)$.

(ii) If j is not in I_λ , then there is an interval (a contiguous set of integers) of one of the forms, $\{j, j+1, j+2, \dots, k-1, k\}$ or $\{j, j-1, j-2, \dots, k+1, k\}$ whose intersection with I_λ is $\{k\}$ for some node k ; if α_j is short, we can choose the interval so that α_k is short. Consequently either the roots $\alpha_j + \alpha_{j+1} + \dots + \alpha_k = \sigma_j \sigma_{j+1} \dots \sigma_{k-1}(\alpha_k)$ and $\alpha_{j+1} + \dots + \alpha_k = \sigma_{j+1} \dots \sigma_{k-1}(\alpha_k)$ both belong to \mathcal{E}_λ or $\alpha_j + \alpha_{j-1} + \dots + \alpha_k$ and $\alpha_{j-1} + \dots + \alpha_k$ both belong to \mathcal{E}_λ . In either case, α_j is the difference, so lies in $\mathcal{L}(C_\lambda)$.

(iii) Suppose $\langle \lambda, \alpha_s \rangle = 0$ for all simple short roots α_s . Set $m = 3$ if \mathcal{G} is G_2 , $m = 2$ otherwise. Then \mathcal{E}_λ includes all roots of the form $\alpha_j + \dots + \alpha_l + m(\alpha_{l+1} + \dots + \alpha_k)$, (or $\alpha_j + \dots + \alpha_l + m(\alpha_{l-1} + \dots + \alpha_k)$), where $I_\lambda \cap \{j, \dots, k\} = \{n\}$ and α_n is long; the terms with coefficient 1 are long roots and those with coefficient m are short. The argument in (ii) then yields that long simple roots and m times each of the short simple roots lie in $\mathcal{L}(C_\lambda)$. Hence in all cases, $\mathcal{L}(C_\lambda)$ includes nonzero multiples of every simple root, and so has rank d . *

If $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \dots \times \mathcal{G}_n$ and $\Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_n$ is a corresponding partition of the set of simple roots of \mathcal{G} into orthogonal components, we assume in the following corollary that $\langle \lambda, \Delta_j \rangle \neq 0$ for any j (which implies in particular that $\lambda \neq 0$).

Corollary 1.6. The dimension of $\text{cvx } \Pi(\lambda)$ is equal to the rank of \mathcal{G} . *

Example: Consider the irreducible character of A_3 whose highest weight is $\lambda = \lambda_1$ (the standard character). The set \mathcal{E}_λ consists of the roots $\alpha_1, \alpha_1 + \alpha_2$, and $\alpha_1 + \alpha_2 + \alpha_3$, $\Pi(\lambda) = \{\lambda, \lambda - \alpha_1, \lambda - (\alpha_1 + \alpha_2), \lambda - (\alpha_1 + \alpha_2 + \alpha_3)\}$, and $\text{cvx } \Pi(\lambda)$ is a tetrahedron whose set of vertices equals $\Pi(\lambda)$. (See Figure 4(B) in Chapter 4.) The cones C_λ and P_λ are evidently equal in this case, and all weights of $\Pi(\lambda)$ (trivially) have the form $\lambda - \sum_{\beta \in \mathcal{E}_\lambda} k_\beta \beta$ for nonnegative integers k_β (the latter holds whenever the cones P_λ and C_λ are equal (Corollary 3.7)).

3. Necessary and sufficient conditions for eventual saturation

The cone C_λ depends only on the subset I_λ , which in turn depends only on the face of the fundamental Weyl chamber that contains λ . When λ belongs to the interior of the fundamental Weyl chamber, \mathcal{E}_λ is the set of simple roots, and C_λ consists of all weights of the form $\sum_{i=1}^d k_i \alpha_i$, where the coefficients k_i are nonnegative integers. Of course, all weights in $\lambda - \Pi(\lambda)$ have this form (by [Hu1; Proposition 21.3]) so $P_\lambda \subseteq C_\lambda$ and the two cones are therefore equal. This is not always the case when λ lies on a face of the fundamental Weyl chamber (if \mathcal{G} is not simply laced). We will show here that χ_λ is eventually saturated if and only if $P_\lambda = C_\lambda$ (Theorem 1.9) and as a consequence, that χ_λ is eventually saturated for regular weights λ (Corollary 1.10). Necessity follows from the next proposition.

We let C be the fundamental Weyl chamber, \bar{C} its closure, and for a characters χ write (χ_w, χ) for the multiplicity of the irreducible character χ_w in the decomposition of χ . An *atom* in a cone C is a nonzero element that cannot be expressed as a sum of two nonzero elements of C .

Proposition 1.7. If a is an atom of P_λ not in \mathcal{E}_λ , then

$$(\chi_{N\lambda-a}, \chi_\lambda^N) = 0$$

for all nonnegative integers N .

Proof: We show first that if λ' is any dominant weight lying on the face of $\bar{\mathcal{C}}$ generated by λ (that is, $\langle \lambda, \alpha_j \rangle = 0$ implies $\langle \lambda', \alpha_j \rangle = 0$), then $(\chi_{\lambda+\lambda'-a}, \chi_\lambda \chi_{\lambda'}) = 0$.

Since a is an atom of P_λ , $\lambda - a$ is a weight of χ_λ . If for some i in I_λ , $\lambda - a + \alpha_i$ were also a weight of χ_λ , then $a - \alpha_i$ would belong to P_λ , and we could write $a = (a - \alpha_i) + \alpha_i$. As a does not belong to \mathcal{E}_λ , neither summand can be zero, contradicting a being an atom. So $\lambda - a + \alpha_i$ cannot belong to $\Pi(\lambda)$ for any i in I_λ .

In terms of the \mathcal{G} -module V_λ affording χ_λ , this says that if x_i is the standard basis element of \mathcal{G} corresponding to α_i , then for any vector v in V_λ of weight $\lambda - a$, $x_i.v = 0$ whenever $\langle \lambda, \alpha_i \rangle \neq 0$. Suppose that $(\chi_{\lambda+\lambda'-a}, \chi_\lambda \chi_{\lambda'}) > 0$. By [PRV; Theorem 2.1], $(\chi_{\lambda+\lambda'-a}, \chi_\lambda \chi_{\lambda'})$ equals the dimension of the subspace of V_λ ,

$$V_\lambda(\lambda - a, \lambda') = \left\{ v \in V_\lambda \mid v \text{ has weight } \lambda - a \text{ and } x_j^{\langle \lambda', \alpha_j \rangle + 1}.v = 0 \text{ for all } j \right\}.$$

There is thus a nonzero vector v in V_λ of weight $\lambda - a$ such that $x_j.v = 0$ whenever $\langle \lambda', \alpha_j \rangle = 0$. If $\langle \lambda', \alpha_j \rangle \neq 0$, then $\langle \lambda, \alpha_j \rangle \neq 0$, and we have seen that $x_j.v = 0$ in this case as well, so $x_j.v = 0$ for all j . In other words, v is a maximal vector. As V_λ has a unique maximal vector and it has weight λ , we must have $a = 0$, a contradiction. Thus $(\chi_{\lambda+\lambda'-a}, \chi_\lambda \chi_{\lambda'}) = 0$.

Now suppose that for all n less than N , $(\chi_{n\lambda-a}, \chi_\lambda^n) = 0$. Recall that if μ_1 and μ_2 are dominant weights, and χ_ν appears in the product $\chi_{\mu_1} \chi_{\mu_2}$ then there is a weight $w \in \Pi(\mu_1)$ such that $\nu = w + \mu_2$. Thus if $N\lambda - a$ were to occur as a highest weight of χ_λ^N , then there would exist elements w_k in $\Pi(\lambda)$, such that

$$N\lambda - a = \lambda + \sum_{k=2}^N w_k, \quad (1.3.1)$$

where $(\chi_{\lambda+w_2}, \chi_\lambda^2) > 0$, and for each $k > 2$, $(\chi_{\lambda+w_2+\dots+w_k}, \chi_\lambda \chi_{\lambda+w_2+\dots+w_{k-1}}) > 0$. However, from (1.3.1), $a = \sum_{k=2}^N (\lambda - w_k)$; since a is an atom of P_λ and each $\lambda - w_k$ belongs to P_λ , we must have $w_k = \lambda$ for all but one k , and (1.3.1) becomes $N\lambda - a = (N-1)\lambda + \lambda - a$.

By assumption, $(\chi_{(n-1)\lambda+(\lambda-a)}, \chi_\lambda^n) = 0$ for $n < N$, thus $N\lambda - a$ could appear as a highest weight in χ_λ^N only if it occurred in the product $\chi_\lambda \chi_{(N-1)\lambda}$. But λ and $(N-1)\lambda$ generate the same face of $\bar{\mathcal{C}}$, so it follows from the first part of the proof (with $\lambda' = (N-1)\lambda$) that $(\chi_{N\lambda-a}, \chi_\lambda \chi_{(N-1)\lambda}) = 0$, thus $(\chi_{N\lambda-a}, \chi_\lambda^N) = 0$.*

Before presenting the main result, we mention the following fact about squares of irreducible characters.

Lemma 1.8. If $i \in I_\lambda$, then $(\chi_{2\lambda-\alpha_i}, \chi_\lambda^2) = 1$.

Proof: It suffices to show that $(\chi_{2\lambda-\alpha_i}, \chi_\lambda^2) > 0$, since the multiplicity of the weight $\lambda - \alpha_i$ in χ_λ equals 1. Let \mathcal{G}_i be the subalgebra of \mathcal{G} generated by the basis elements x_i and y_i corresponding to the roots α_i and $-\alpha_i$ respectively. Denote by $V_{\lambda,i}$ the \mathcal{G}_i submodule of V_λ generated by a highest weight vector. By the Clebsch-Gordon formula, there is a maximal vector v' for \mathcal{G}_i of weight

$2\lambda - \alpha_i$ in $V_{\lambda,i} \otimes V_{\lambda,i} \subset V_\lambda \otimes V_\lambda$. From (1.1.2) it follows that $x_j.v' = 0$ for all $j \neq i$, hence v' is a maximal vector for \mathcal{G} in $V_\lambda \otimes V_\lambda$. *

Theorem 1.9. The following are equivalent.

- (i) The irreducible character χ_λ is eventually saturated;
- (ii) $C_\lambda = P_\lambda$;
- (iii) C_λ is normal and $\mathcal{L}(C_\lambda) = \Lambda_r$.
- (iv) all sufficiently large powers of χ_λ are saturated.

Proof: Proposition 1.7 yields (i) implies (ii). Assume (ii) holds; we establish (iii). It follows from [Hu1; Proposition 21.3] that $P_\lambda = \text{cvx } P_\lambda \cap \Lambda_r$, so

$$C_\lambda \subseteq (\text{cvx } C_\lambda) \cap \mathcal{L}(C_\lambda) \subseteq (\text{cvx } C_\lambda) \cap \Lambda_r = (\text{cvx } P_\lambda) \cap \Lambda_r = P_\lambda.$$

Equality of the cones therefore implies that C_λ is normal and $\text{cvx } C_\lambda \cap \mathcal{L}(C_\lambda) = \text{cvx } C_\lambda \cap \Lambda_r$. The lattices Λ_r and $\mathcal{L}(C_\lambda)$ must be equal, as they have the same rank (Lemma 1.5).

Assume (iii). We establish (i) and (iv). Define the set D_λ as follows:

$$\begin{aligned} D_\lambda &= \bigcup_{\sigma \in W} (\cup_{i \in I_\lambda} \{\sigma(2\lambda - \alpha_i)\} \cup \{\sigma(2\lambda)\}) \\ &= \bigcup_{\sigma \in W/W_\lambda} (\cup_{\beta \in \mathcal{E}_\lambda} \{\sigma(2\lambda - \beta)\} \cup \{\sigma(2\lambda)\}). \end{aligned}$$

The dominant weights in D_λ are those of the form $2\lambda - \alpha_i$ and the weight 2λ . All the corresponding irreducible characters appear in the decomposition of χ_λ^2 (Lemma 1.8). By the PRV theorem, any irreducible character whose highest weight may be written as a sum of m elements of D_λ appears in χ_λ^{2m} . We will show that, for m sufficiently large, every dominant weight of χ_λ^{2m} belongs to mD_λ .

The local cones of D_λ have the form $\sigma(C_\lambda)$ for $\sigma \in W$, and $\mathcal{L}(D) = \mathcal{L}(C_\lambda) = \Lambda_r$ by hypothesis. As elements of W leave Λ_r stable, each of the local cones of D_λ is normal by hypothesis. It follows from Lemma 1.1 that ND_λ is Λ_r -convex for all N exceeding some integer n .

We observed in (1.1.2) that the set of weights of χ_λ^{2N} equals $\Pi(2N\lambda)$. If $N \geq n$, then the fact that ND_λ is Λ_r -convex implies that

$$\Pi(2N\lambda) = \text{cvx } \Pi(2N\lambda) \cap \Lambda_r = \text{cvx } ND_\lambda \cap \Lambda_r = ND_\lambda.$$

Thus any dominant weight of χ_λ^{2N} may be expressed as a sum of N elements of D_λ and (i) follows by the PRV conjecture, as mentioned above.

Now we show (iv). Choose $N \geq n$ such that $2N \geq d$. Pick $m \geq 2N$ and suppose that w is a dominant weight in $\Pi(m\lambda)$. By the Shapley-Folkman Theorem (with $D_i = \{\tau\lambda \mid \tau \in W\}$), w admits a decomposition, $w = \sum_{i=1}^d w_i + \sum_{i=d+1}^m \tau_i\lambda$ where w_i are weights in $\Pi(\lambda)$ and τ_i belong to the Weyl group. Select σ in W such that the weight $w' := \sigma(\sum_{i=1}^d w_i + \sum_{i=d+1}^{2N} \tau_i\lambda)$ is dominant. Then $\chi_{w'}$ appears in the product χ_λ^{2N} by saturation of χ_λ^{2N} . It now follows from the PRV conjecture that χ_w , whose highest weight is conjugate to $w' + \sum_{i=2N+1}^m \sigma\tau_i\lambda$, occurs in the decomposition of χ_λ^m , yielding (iv). *

Corollary 1.10. If λ is a dominant weight in the interior of the Weyl chamber, then all sufficiently high powers of χ_λ are saturated.

Proof: In this case, the cones C_λ and P_λ are equal (and simplicial).*

Chapter 2

The integral closure of R_Q

Suppose that Q is a Laurent polynomial with nonnegative real coefficients in the variables x_1, \dots, x_d ; we write $Q = \sum \lambda_w x^w$ where the sum runs over the lattice points in \mathbf{Z}^d , $x^w = x_1^{w_1} x_2^{w_2} \dots x_d^{w_d}$, and λ_w are nonnegative real numbers. Define the set of exponents of Q ,

$$\text{Log } Q := \left\{ w \in \mathbf{Z}^d \mid \lambda_w \neq 0 \right\},$$

and let $\mathcal{L}(Q)$ be the subgroup of \mathbf{Z}^d generated by the set of differences $\text{Log } Q - \text{Log } Q$. In this chapter, we relate the relative convexity of $\text{Log } Q$ with respect to \mathbf{Z}^d to the integral closure of the ring R_Q studied in [Ha1] and [Ha2].

Recall the definition of the algebra R_Q given in [Ha1; Chapter 1]: $R_Q := \mathbf{R}[x^w/Q]_{w \in \text{Log } Q}$. By [op. cit.; I.4], R_Q consists of the rational functions of the form f/Q^m where f is a real Laurent polynomial and there exists an integer n such that $\text{Log } f + n\text{Log } Q \subseteq (m+n)\text{Log } Q$. It is a partially ordered ring with positive cone generated over \mathbf{R}^+ by the elements x^w/Q where $w \in \text{Log } Q$. If Q is replaced by a power Q^k or by $x^w Q$ we obtain the same algebra.

It is shown in [Ha2; Propositions III.2, III.8A] that when $\text{Log } Q$ is \mathbf{Z}^d -convex, R_Q is integrally closed in its field of fractions if and only if the local cones of $\text{Log } Q$ are normal. It follows from Lemma 1.1 that R_Q is integrally closed if and only if $k\text{Log } Q$ is \mathbf{Z}^d -convex for all sufficiently large integers k . We extend this result to the rings R_Q corresponding to any Laurent polynomial with nonnegative coefficients (Theorem 2.2).

The definition of a normal cone given in Chapter 1 is a generalization of that given in [op. cit.]. (The latter, called “local solidity”, applies only to sets which are \mathbf{Z}^d -convex.) Otherwise, the proof of Theorem 2.2 given here is essentially the same as that of [Ha2; Propositions III.2 and III.8A]. Before presenting the proof, we recall some of the definitions and results of [Ha2; Chapter III].

A *lattice polytope* K is the convex hull of a finite subset of given lattice \mathcal{L} , or the translate of such a set by an element in \mathbf{Z}^d . It is a compact convex subset of \mathbf{R}^d whose set of vertices belongs to a coset of \mathbf{Z}^d by the lattice \mathcal{L} . For lattice polytopes, the set of vertices equals the set of extreme points; we denote this set by $\partial_e K$. The lattice polytope K is *solid* with respect to \mathcal{L} if for all positive integers k , $k(K \cap \mathcal{L}) = kK \cap \mathcal{L}$ —in other words, K is *solid* with respect to \mathcal{L} if $k(K \cap \mathcal{L})$ is \mathcal{L} -convex. The convex hull of the set of weights of χ_λ is solid with respect to Λ_τ for instance, by (1.1.2). It is shown in [Ha2; Proposition III.7] that dK is solid for any lattice polytope K in \mathbf{R}^d . Solidity implies the more general condition, “local solidity”, which we now define.

For each vertex v of K , form two local cones: the first,

$$\bigcup_{k \in \mathbf{N}} k((K - v) \cap \mathcal{L}) \tag{2.0.1}$$

generated additively by the elements of $(K - v) \cap \mathcal{L}$, and the second

$$\left(\bigcup_{k \in \mathbf{N}} k(K - v) \right) \cap \mathcal{L} \tag{2.0.2}$$

the lattice points lying in the cone generated by $K - v$. K is *locally solid* if the two cones (2.0.1) and (2.0.2) are equal for all vertices v of K .

The conclusion of [Ha2; Propositions III.2, III.8] is that if $\mathcal{L} = \mathbb{Z}^d$ and $\text{Log } Q$ is saturated, then R_Q is integrally closed in its field of fractions if and only if $\text{cvx } \text{Log } Q$ is locally solid. In this case, local solidity of $\text{cvx } \text{Log } Q$ is equivalent to C_v being normal at each vertex v : for any vertex v of $K := \text{cvx } \text{Log } Q$, C_v is the negative of (2.0.1), $\mathcal{L}(C_v) = \mathbb{Z}^d$, and $\text{cvx } C_v \cap \mathbb{Z}^d$ is the negative of (2.0.2).

We will use Lemma 2.1 in the proof of Theorem 2.2 to show that if all multiples of $M \text{Log } Q$ are relatively convex for some integer M , then all sufficiently large multiples of $\text{Log } Q$ are relatively convex. This fact may be deduced from the Shapley-Folkman Theorem; this was pointed out by the referee mentioned earlier. The proof given here requires no machinery, and is extracted from [Ha2; III.6].

Lemma 2.1. Let \mathcal{L} be a lattice in \mathbb{Q}^d , and let K be a compact convex polyhedron in \mathbb{R}^d whose extreme points belong to \mathcal{L} .

- (i) Suppose $|\partial_e K| = n$. Then for each integer $N > n - 1$, and x in $(NK) \cap \mathcal{L}$, there exist x_0 in $n'K \cap \mathcal{L}$ for some integer $n' \leq n - 1$ and x_1 a sum of exactly $N - n'$ extreme points of K (counting multiplicities) such that $x = x_0 + x_1$.
- (ii) If N is an integer exceeding d , then $NK \cap \mathcal{L} = (dK \cap \mathcal{L}) + (N - d)(K \cap \mathcal{L})$.

Proof: (i) We may write $x = \sum_{v \in \partial_e K} r_v v$ where r_v are all nonnegative real numbers, adding to N . Define $x_1 = \sum_{v \in \partial_e K} [r_v] v$ ($[\]$ denotes the greatest integer part). Then $n > \sum_{v \in \partial_e K} (r_v - [r_v]) = n'$. Note that $n' = N - \sum [r_v]$, so is an integer, and $x_0 := x - x_1 = \sum_{v \in \partial_e K} (r_v - [r_v]) v$ satisfies all the properties ascribed to it.

(ii) As in [Ha2; III.5], we may write K as a union of simplices with vertices in \mathcal{L} , so that NK is a union of N -fold multiples of these simplices. Hence it suffices to prove the assertion in the case K is a simplex (observe that we do not obtain extreme points, as in (i), when we return from the special case to the general case). Since a simplex has at most $d + 1$ vertices, we may apply (i) with $n = d + 1$. *

Theorem 2.2. Let Q be a Laurent polynomial, with no negative coefficients; suppose additionally that $\mathcal{L}(Q) = \mathcal{L}$ has rank d . The following are equivalent.

- (i) For each vertex v of $\text{cvx } \text{Log } Q$, C_v is normal;
- (ii) R_Q is integrally closed in its field of fractions;
- (iii) there exists an integer M such that $nM \text{Log } Q$ is \mathcal{L} -convex for all nonnegative integers n ;
- (iv) for all sufficiently large integers N , $N \text{Log } Q$ is \mathcal{L} -convex.

Proof: We may assume the origin belongs to $\text{Log } Q$ (on replacing Q by $x^w Q$ for suitable w), so that $\text{Log } Q \subset \mathcal{L}$.

(i) \implies (ii). We follow the proof by localization given in [Ha2; Proposition III.2]. As $\mathcal{L} \simeq \mathbb{Z}^d$, we may (temporarily) identify C_v with a normal lattice cone, C'_v , in $(\mathbb{Z}^+)^d$, and then define a semigroup, $M_v = \{x^w \mid w \in C'_v\}$. It follows from [Ho; Proposition 1] that the semigroup algebra $\mathbf{R}[M_v]$ is integrally closed in its field of fractions. If J is a prime ideal in R_Q , then the localization of R_Q at J is a localization of $\mathbf{R}[M_v]$ for some vertex v , by [Ha2; Lemma III.1]. Hence every localization of R_Q at a prime ideal is integrally closed; thus R_Q is itself integrally closed.

(ii) \implies (iii). Set $S = \text{cvx Log } Q^d \cap \mathcal{L}$. By [Ha2; Proposition III.4], $\sum_{w \in S} x^w / Q^d$ belongs to the integral closure of R_Q , hence to R_Q itself. By definition of R_Q , there is a positive integer N such that $S + N \text{Log } Q^d \subseteq (N+1) \text{Log } Q^d$; as $\text{Log } Q^d \subset S$ (since 0 is in $\text{Log } Q$), $S + N \text{Log } Q^d = (N+1) \text{Log } Q^d$. The convex hulls of S and $\text{Log } Q^d$ are the same, hence their sets of extreme points are identical. We can therefore apply [Ha3; Lemma 2.1] which asserts that there exists a positive integer m such that for all nonnegative integers p , $mS + p \text{Log } Q^d = (m+p)S$. Hence for all positive integers k ,

$$km(N+1) \text{Log } Q^d = k(mS + mN \text{Log } Q^d) = km(N+1)S.$$

By [Ha2; Proposition III.7], $\text{cvx Log } Q^d$ is solid; on setting $M = m(N+1)d$, we have

$$\begin{aligned} kM \text{Log } Q &= km(N+1) (\text{cvx Log } Q^d \cap \mathcal{L}) \\ &= (kM \text{cvx Log } Q) \cap \mathcal{L}, \end{aligned}$$

so $kM \text{Log } Q$ is \mathcal{L} -convex for all k .

(iii) \implies (iv). Let $K = \text{cvx Log } Q$, and observe that $\partial_e K$ is contained in $\text{Log } Q$. Suppose $|\partial_e K| = n$, and pick an integer t so that $Mt \geq n$. We will show that if $N > Mt$, then $N \text{Log } Q$ is \mathcal{L} -convex. Pick x in $NK \cap \mathcal{L}$; by Lemma 2.1(a), x belongs to $(n'K \cap \mathcal{L}) + (N - n')(\partial_e K)$. We can absorb $Mt - n'$ of the $N - n'$ extreme points appearing in the second summand into the first, so that x belongs to $(MtK \cap \mathcal{L}) + (N - Mt)(\partial_e K)$. Since $\partial_e K$ is contained in $\text{Log } Q$, the second summand is contained in $(N - Mt) \text{Log } Q$, and by hypothesis, the first is contained in $Mt \text{Log } Q$. Hence x belongs to $N \text{Log } Q = \text{Log } Q^N$. Thus $NK \cap \mathcal{L} \subseteq \text{Log } Q^N$, and the reverse inclusion is trivial.

(iv) \implies (i). We note that (iv) implies (iii), and prove that (iii) implies (i). For each k , $kN \text{Log } Q$ is \mathcal{L} -convex, so $\text{cvx Log } Q^N$ is solid. This implies that $\text{cvx Log } Q^N$ is locally solid, and therefore $C_{Nv} = C_v$ is normal for each vertex v of $\text{cvx Log } Q$. *

Chapter 3

Eventual saturation for irreducible characters of simple Lie algebras

Here we use Theorem 1.9 to characterize the eventually saturated characters for simple algebras; we will prove Theorem 3.1, below. Let α_{cs} be the (unique) simple short root for which there is a simple long root α_{cl} such that $\langle \alpha_{cs}, \alpha_{cl} \rangle \neq 0$.

Theorem 3.1. Let λ be a nonzero dominant weight of the simple Lie algebra \mathcal{G} . The irreducible character χ_λ is eventually saturated if and only if one of the following holds:

- (i) \mathcal{G} is simply laced;
 - (ii) $\langle \lambda, \alpha_{cs} \rangle \neq 0$;
 - (iii) \mathcal{G} is either of type C_d or F_4 and $\langle \lambda, \alpha_l \rangle = 0$ for all simple long roots α_l .
- Furthermore, when χ_λ is eventually saturated, all sufficiently large powers of it are saturated.

In Section 1, we describe in detail the sets \mathcal{E}_λ for simple algebras, and determine the lattice $\mathcal{L}(C_\lambda)$ when \mathcal{G} is not simply laced (Lemma 3.2). Combining this information with Lemma 1.5, we deduce that $\mathcal{L}(C_\lambda)$ equals the root lattice if and only if \mathcal{G} is simply laced or $\langle \lambda, \alpha_s \rangle \neq 0$ for some simple short root α_s . We then turn our attention to determining when the cones C_λ are normal. The simplest examples of normal cones are simplicial cones; we describe the cones C_λ which are simplicial in Section 2 (Proposition 3.3). Sufficient conditions for normality are then given in Lemma 3.5 and Lemma 3.6. The proof of Theorem 3.1 follows in Section 4. Finally, we observe that when χ_λ satisfies the conditions of Theorem 3.1, any weight in $\Pi(\lambda)$ can be written as a sum of elements in $\lambda - \mathcal{E}_\lambda$ (Corollary 3.9). Examples of irreducible characters which are not eventually saturated are given in Section 5.

1. Description of the sets \mathcal{E}_λ

We observed earlier that if λ lies in the interior of the Weyl chamber, then \mathcal{E}_λ is the set of simple roots. If λ lies on a proper face of the Weyl chamber, then \mathcal{E}_λ includes the simple roots α_j for j in I_λ ; the remaining elements are certain sums of simple roots which we list below. In each case, it is straightforward to verify that the given subset of \mathcal{E}_λ includes α_j if j belongs to I_λ , and is invariant under the generators σ_i , $i \notin I_\lambda$ of W_λ , and thus must equal \mathcal{E}_λ . The ordering on the roots is as in [Hu1; p. 58].

1. \mathcal{G} is of type A_d .

Here \mathcal{E}_λ consists of all sums of simple roots of the form

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_m \tag{*}$$

where $1 \leq j \leq m \leq d$ and exactly one of the indices in $\{j, j+1, \dots, m\}$ lies in I_λ .

For example, if $d = 4$ and $I_\lambda = \{1, 4\}$, then

$$\mathcal{E}_\lambda = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4\}.$$

2. \mathcal{G} is of type B_d .

(a) If d belongs to I_λ , then \mathcal{E}_λ consists of roots as in (*).

(b) If d does not belong to I_λ , then \mathcal{E}_λ consists of roots

(i) $\alpha_j + \alpha_{j+1} + \cdots + \alpha_m$ with $m < d$, and

(ii) $\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1} + 2\alpha_d = \sigma_d(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1})$ where exactly one of the indices in $\{j, \dots, d-1\}$ lies in I_λ , and

(iii) if none of $k, k+1, \dots, d-1, d$ belong to I_λ , all sums of the form

$$\begin{aligned} & (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1} + 2\alpha_d) + (\alpha_k + \alpha_{k+1} + \cdots + \alpha_{d-1}) \\ & = \sigma_k \sigma_{k+1} \cdots \sigma_{d-1} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1} + 2\alpha_d), \end{aligned}$$

with the first summand as in (ii), and $j < k < d$.

For example, if $d = 3$, and $I_\lambda = \{1\}$, then

$$\mathcal{E}_\lambda = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}.$$

3. \mathcal{G} is of type C_d .

(a) If $d-1$ belongs to I_λ , then \mathcal{E}_λ consists of roots as in (*).

(b) If $d-1$ and d do not belong to I_λ , then \mathcal{E}_λ consists of roots

(i) as in (*), and

(ii) if none of $k, k+1, \dots, d-1, d$ belong to I_λ , all sums of the form

$$\begin{aligned} & (\alpha_j + \alpha_{j+1} + \cdots + \alpha_d) + (\alpha_k + \alpha_{k+1} + \cdots + \alpha_{d-1}) \\ & = \sigma_k \sigma_{k+1} \cdots \sigma_{d-1} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1} + \alpha_d), \end{aligned}$$

with the first summand as in (i), and $j < k < d$.

(c) If $d-1$ does not belong to I_λ but d does, then \mathcal{E}_λ consists of roots

(i) $\alpha_j + \alpha_{j+1} + \cdots + \alpha_m$ with $\{j, j+1, \dots, m\} \cap I_\lambda = \{n\}$ where $n < d$, and

(ii) $2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{d-1} + \alpha_d = \sigma_j \sigma_{j+1} \cdots \sigma_{d-1}(\alpha_d)$, with $I_\lambda \cap \{j, j+1, \dots, d\} = \{d\}$.
So, for example, if $d = 3$ and $I_\lambda = \{1, 3\}$, then

$$\mathcal{E}_\lambda = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_3, 2\alpha_2 + \alpha_3\}.$$

4. \mathcal{G} is of type D_d .

(a) If either $d-2$ lies in I_λ or at least two of $d-3, d-1, d$ lie therein, then \mathcal{E}_λ consists of all sums of simple roots

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_m \quad \text{or} \quad \alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-2} + \alpha_d,$$

where exactly one index in $\{j, j+1, \dots, m\}$ or in $\{j, j+1, \dots, d-2, d\}$ respectively belongs to I_λ .

(b) Otherwise, \mathcal{E}_λ consists of roots as in (a) together with all sums of the form

$$\begin{aligned} & (\alpha_j + \alpha_{j+1} + \cdots + \alpha_d) + (\alpha_k + \alpha_{k+1} + \cdots + \alpha_{d-2}) \\ & = \sigma_k \sigma_{k+1} \cdots \sigma_{d-2} (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{d-1} + \alpha_d), \end{aligned}$$

where the first summand is as in (a), $j < k \leq d - 2$, and $I_\lambda \cap \{k, \dots, d - 2\} = \emptyset$
 If $d = 5$, and $I_\lambda = \{1, 5\}$, then \mathcal{E}_λ consists of the following roots:

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_5, \alpha_3 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5\}.$$

5. \mathcal{G} is one of E_6, E_7, E_8 .

Here we index the nodes along the straight line of length five, six, or seven in the Dynkin diagram as $1, 3, 4, 5, 6, \dots$.

- (a) If 4 lies in I_λ or at least two of 2, 3, or 5 belong, then \mathcal{E}_λ consists of sums $\alpha_j + \alpha_{j+1} + \dots + \alpha_m$ or $\alpha_1 + \alpha_3 + \alpha_4 + \dots + \alpha_m$ where exactly one index in $\{j, j + 1, \dots, m\}$ or in $\{1, 3, \dots, m\}$ respectively belongs to I_λ .
- (b) Otherwise, \mathcal{E}_λ consists of roots as in (a) as well as

$$(\alpha_j + \alpha_{j+1} + \dots + \alpha_m) + (\alpha_k + \dots + \alpha_4 + \dots + \alpha_n),$$

where the first summand is as in (a) with $j = 1$ or $j = 2$, $m \geq 5$, $k = 3$ or $k = 4$, and $I_\lambda \cap \{k, \dots, n\} = \emptyset$. If I_λ equals $\{1, 7\}$, $\{1, 7, 8\}$ or $\{1, 8\}$, \mathcal{E}_λ also includes the roots

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

and, if $I_\lambda = \{1, 8\}$, the roots

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.$$

6. \mathcal{G} is G_2 .

$$\mathcal{E}_\lambda = \begin{cases} \{\alpha_1, \alpha_2\} & \text{if } I_\lambda = \{1, 2\} \\ \{\alpha_1, \alpha_1 + \alpha_2\} & \text{if } I_\lambda = \{1\} \\ \{\alpha_2, 3\alpha_1 + \alpha_2\} & \text{if } I_\lambda = \{2\}. \end{cases}$$

7. \mathcal{G} is F_4 .

If 3 belongs to I_λ , then \mathcal{E}_λ consists of sums as in (*). Otherwise \mathcal{E}_λ is of the following form:

$$\left\{ \begin{array}{ll} \{\alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\} & \text{if } I_\lambda = \{1, 2\} \\ \{\alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_4, \alpha_3 + \alpha_4\} & \text{if } I_\lambda = \{1, 2, 4\} \\ \{\alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_4, \alpha_3 + \alpha_4\} & \text{if } I_\lambda = \{2, 4\} \\ \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4\} & \text{if } I_\lambda = \{1, 4\} \\ \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\} & \text{if } I_\lambda = \{1\} \\ \{\alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4\} & \text{if } I_\lambda = \{2\} \\ \{\alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\} & \text{if } I_\lambda = \{4\}. \end{array} \right.$$

Lemma 3.2. Define $m := 2$ if \mathcal{G} is one of B_d , C_d , or F_4 , and $m := 3$ if \mathcal{G} is G_2 . If $\langle \lambda, \alpha_s \rangle = 0$ for all simple short roots α_s , $\mathcal{L}(C_\lambda)$ is spanned as an abelian group by

$$\{ \alpha_i \mid \alpha_i \text{ is a simple long root} \} \cup \{ m\alpha_s \mid \alpha_s \text{ is a simple short root} \} .$$

Proof: It follows from Lemma 1.5(iii) that $\mathcal{L}(C_\lambda)$ includes the given set; the reverse inclusion follows from the description of the sets \mathcal{E}_λ given above.*

Lemma 1.5 and Lemma 3.2 together yield that $\mathcal{L}(C_\lambda) = \Lambda_r$ if and only if \mathcal{G} is simply laced or λ is not orthogonal to some simple short root.

2. Integrally simple polytopes and simplicial cones

A polytope K with vertices in a coset $\mathcal{L} + x$ of \mathbf{Z}^d relative to a lattice \mathcal{L} is called *integrally simple*—or *\mathcal{L} -integrally simple*—if it is simple, (that is, if there are exactly d edges at each vertex), and if at every vertex v the convex hull of the nearest points to v in $K \cap (\mathcal{L} + x)$ is a fundamental simplex for \mathcal{L} . In case $K = \text{cvx } \Pi(\lambda)$, the *weight polytope* of λ , and \mathcal{L} equals the root lattice, K will be Λ_r -integrally simple whenever \mathcal{E}_λ consists of exactly d vectors which generate Λ_r as an abelian group. Simplicity of the weight polytopes can therefore be determined using the description of the sets \mathcal{E}_λ given earlier. In general, all simple weight polytopes are \mathcal{L} -integrally simple, where $\mathcal{L} = \mathcal{L}(C_\lambda)$, the lattice generated by \mathcal{E}_λ . If \mathcal{G} is simply laced, then all simple weight polytopes are Λ_r -integrally simple, by Lemma 1.5. If \mathcal{G} is not simply laced, then a simple weight polytope $\text{cvx } \Pi(\lambda)$ will be Λ_r -integrally simple if and only if $\langle \lambda, \alpha_s \rangle \neq 0$ for some simple short root α_s , by Lemma 3.2. We list the simple and Λ_r -integrally simple weight polytopes in Proposition 3.3.

Proposition 3.3.

- (a) If \mathcal{G} is simply laced, then $\text{cvx } \Pi(\lambda)$ is simple and Λ_r -integrally simple if and only if one of the following holds
- (i) $\mathcal{G} = A_d$ and $I_\lambda = \{j, j+1, \dots, j+k\}$ where $j = 1, j = d$ or $k \geq 1$,
 - (ii) $\mathcal{G} = D_d$ and $I_\lambda = \{j, j+1, \dots, d-2, d\}$ or $I_\lambda = \{j, j+1, \dots, m\}$ and I_λ includes $d-2$ and at least one of $d-3, d-1, d$, or
 - (iii) $\mathcal{G} = E_6, E_7$, or E_8 and I_λ is an interval (that is, a contiguous set of integers) including 4 and at least one of 2, 3, or 5 (these are the nodes adjacent to the branch node “4”), or $I_\lambda = \{1, 3, \dots, 3+k\}$ for some $k \geq 1$.
- (b) If $\mathcal{G} = B_d$ or C_d , then $\text{cvx } \Pi(\lambda)$ is simple if either $I_\lambda = \{j, j+1, \dots, d\}$ for some $j \leq d$, or $I_\lambda = \{j, j+1, \dots, d-1\}$ with $j \leq d-2$.
- (i) If $\mathcal{G} = B_d$ and $\text{cvx } \Pi(\lambda)$ is simple, then $\text{cvx } \Pi(\lambda)$ is Λ_r -integrally simple if and only if d belongs to I_λ ;
 - (ii) If $\mathcal{G} = C_d$ and $\text{cvx } \Pi(\lambda)$ is simple, then $\text{cvx } \Pi(\lambda)$ is Λ_r -integrally simple if and only if $I_\lambda \neq \{d\}$.
- (c) If $\mathcal{G} = G_2$, then all weight polytopes are simple; they are Λ_r -integrally simple if and only if λ is not a multiple of λ_2 .

(iv) If $\mathcal{G} = F_4$, then $\text{cvx } \Pi(\lambda)$ is simple if and only if I_λ is an interval of length at least 2 including 2 or 3; it is Λ_r -integrally simple as well if and only if I_λ is an interval of length at least two and includes the index 3.

It is shown in [Ha2] that if $K = \text{cvx } \Pi(\lambda)$ and K is Λ_r -integrally simple, then the ring R_K (an invariant for K) is factorial (Theorem VI.1). The simple weight polytopes for A_d are also described, in an appendix (Proposition D2).

A proper lattice cone of dimension d is *simplicial* if it has exactly d edges; the cone C_λ is therefore simplicial whenever $\text{cvx } \Pi(\lambda)$ is simple.

Suppose that x belongs to $\text{cvx } C_\lambda \cap \mathcal{L}(C_\lambda)$. We may then write $x = \sum_{\beta \in \mathcal{E}_\lambda} r_\beta \beta$ where the coefficients r_β are nonnegative real numbers; we also have $x = \sum_{\beta \in \mathcal{E}_\lambda} k_\beta \beta$ where the k_β are integers. If C_λ is simplicial, then the set \mathcal{E}_λ consists of linearly independent vectors; thus for each β , the coefficients k_β and r_β are equal, and are non-negative integers. So x belongs to C_λ , and C_λ is normal. It follows from Theorem 1.9 that χ_λ is eventually saturated when $\text{cvx } \Pi(\lambda)$ is Λ_r -integrally simple.

3. Sufficient conditions for normality

We identify an element of Λ_r with the d -tuple of its coordinates with respect to the basis of simple roots; the simple roots are thus identified with the standard basis elements e_i of \mathbb{Z}^d .

A subset I' of $I = \{1, 2, \dots, d\}$ is an *interval* if $I' = \{i, i+1, \dots, i+k\}$ for some integers i, k . For any interval I' of I define the vector $e_{I'} := \sum_{i \in I'} e_i$. Let S be a set of intervals of I , C_S the cone generated (additively) by $\{e_J \mid J \in S\}$, and $H_S = \mathcal{L}(C_S)$, the abelian group generated by the same set. The next two lemmas indicate that the cones C_S satisfy a ‘‘Carathéodory-type’’ condition—that is, each element in $\text{cvx } C_S \cap \mathbb{Z}^d$ belongs to a simplicial cone $C_T \subset C_S$.

Lemma 3.4. $\mathbb{Z}^d / (H_S)$ is torsion free.

Proof. The proof is by induction on the rank of H_S . If the rank is 1, $|S| = 1$ and the result follows as all the coefficients of e_J are equal to 0 or 1. Assume that $\text{rank } H_S > 1$. We may also assume that 1 belongs to some interval in S , and let $J(1)$ be the shortest interval in S beginning at 1. If J is any interval in S beginning at 1, then $J(1) \subseteq J$. Let $S' = \{J \in S \mid 1 \notin J\} \cup \{J \setminus J(1) \mid J \in S, 1 \in J\}$. Then S' consists of intervals. If J is an interval in S and $1 \in J$, then $e_J = e_{J \setminus J(1)} + e_{J(1)}$ so H_S is a sum $H_{S'} + e_{J(1)}\mathbb{Z}$; since none of the intervals in S' contains 1, the sum is direct. Hence $\text{rank } H_{S'} < \text{rank } H_S$ and the induction assumption applies to the former subgroup.

Suppose x belongs to \mathbb{Z}^d and mx belongs to H_S . Write $mx = u + te_{J(1)}$ where $u \in H_{S'}$, $t \in \mathbb{Z}$. Comparing the first entries of both sides, we have $t = mt'$ for some integer t' . Hence every entry of $u = m(x - t'e_{J(1)})$ is divisible by m . Since $x - t'e_{J(1)} \in \mathbb{Z}^d$, and $u = m(x - t'e_{J(1)}) \in H_{S'}$ it follows from the induction assumption that $x - t'e_{J(1)}$ belongs to $H_{S'}$, so x belongs to H_S . *

Lemma 3.5.

(i) If T is a subset of S such that $\{e_J \mid J \in T\}$ is rationally independent, then $\text{cvx } C_T \cap H_S = C_T$.

(ii) For every x in $\text{cvx } C_S \cap H_S$ there exists a subset $T \subset S$ such that $\{e_J \mid J \in T\}$ is rationally independent and x belongs to C_T . As a consequence, C_S is normal.

Proof. (i) Suppose that x belongs to $\text{cvx } C_T \cap H_S$. Then there exist nonnegative real numbers r_J indexed by $J \in T$ such that $x = \sum_{J \in T} r_J e_J$. Since x and e_J , $J \in T$ belong to \mathbb{Z}^d , we can find coefficients r_J which are nonnegative rational numbers. Then mx belongs to H_T for

some positive integer m , so by Lemma 3.4, x belongs to H_T . We therefore have an expression $x = \sum_{J \in T} k_J e_J$ where the coefficients k_J are integers. By hypothesis, the set $\{e_J \mid J \in T\}$ is rationally independent, so for each $J \in T$, the coefficients r_J and k_J are equal, and are therefore nonnegative integers. Hence $x \in C_T$.

(ii) For any x in the convex cone $\text{cvx } C_S$ we have an expression $x = \sum_{I \in S} r_I e_I$ where the coefficients r_I are nonnegative real numbers. Following the proof of Carathéodory's Theorem ([L; Theorem 2.23]) we may reduce this expression to one of the form $x = \sum_{J \in T} r'_J e_J$ where the coefficients r'_J are nonnegative real numbers, and T is a subset of S such that $\{e_J \mid J \in T\}$ is rationally independent. (The reduction is achieved by considering expressions of dependence among the e_I : from the original expression for x , one subtracts an appropriate multiple of the expression $\sum_{I \in S} a_I e_I = 0$ thereby reducing the cardinality of the set $\{I \mid r_I \neq 0\}$ by at least one.) If x belongs to $\text{cvx } C_S \cap H_S$, then x belongs to $\text{cvx } C_T \cap H_S$, and by (i) x belongs to C_T which is contained in C_S . *

Combined with Theorem 1.9 and the description of the sets \mathcal{E}_λ in Section 1, Lemma 3.5 yields that every irreducible character of Lie algebra of type A_d is eventually saturated.

For sets \mathcal{E}_λ which are not generated by intervals, we use a different approach to prove normality. If C is a cone in \mathbf{Z}^d generated by a finite set D and F is a face of $\text{cvx } C$, then F is a convex cone generated (as an additive subsemigroup of \mathbf{R}^d) by the subset $F \cap D$. If the lattice cone C_F generated by $F \cap D$ is equal to $F \cap \mathcal{L}(C)$, then any element in $\text{cvx } C \cap \mathcal{L}(C)$ lying on F belongs to C . This is enough to guarantee normality of the cone C when certain conditions on the set D are imposed, as the next lemma shows.

Lemma 3.6. Let C be a cone in \mathbf{Q}^d generated by a finite set D . Suppose there are linear functionals on \mathbf{R}^d , g_j ($j \in J$), such that $\text{cvx } C = \bigcap_{j \in J} \{x \in \mathbf{R}^d \mid g_j(x) \geq 0\}$.

(i) If for all δ in D and j in J , $g_j(\delta) \in \{0, 1\}$ then C is normal.

(ii) Suppose that $g_j(\delta)$ is a nonnegative integer for each j in J , and δ in D , and that for some δ_0 in D we have $g_j(\delta_0) \in \{0, 1\}$ for all j in J . If for each relative face, $C^j := \{x \in C \mid g_j(x) = 0\}$ of C , it follows that $\text{cvx } C^j \cap \mathcal{L}(C) = C^j$, then C is normal.

Proof: (i) We may assume that the rank of $\mathcal{L}(C)$ is d . If $C = \{0\}$, there is nothing to do. Otherwise, it follows from the hypotheses that for each j , either $g_j(C)$ equals $\{0\}$ or it equals \mathbf{Z}^+ itself. Choose x in $\text{cvx } C \cap \mathcal{L}(C)$; then $g_j(x)$ is a nonnegative integer for every j . Define

$$J_x = \{j \in J \mid g_j(x) \neq 0\}.$$

We argue by induction on the cardinality of the set J_x . If J_x is empty, then $x = 0$, and we are done. Otherwise, J_x is not empty and x is not zero. Let D_x consist of those of the generators that lie on the smallest face of $\text{cvx } C$ containing x :

$$D_x = \{\delta \in D \mid g_j(\delta) = 0 \text{ for all } j \notin J_x\}.$$

Unless $x = 0$ the set D_x cannot be empty.

As x is not zero, we may choose δ in D_x such that for some k in J_x , $g_k(\delta) = 1$. Set $n = \min \{g_i(x) \mid i \in J_x \text{ and } g_i(\delta) = 1\}$. Then $x - n\delta$ belongs to $\mathcal{L}(C)$, and for all $j \in J_x$, $g_j(x - n\delta) \geq g_j(x) - n \geq 0$. Since $g_k(x - n\delta) = g_k(x) = 0$ for k not in J_x , x lies in $\text{cvx } C$. Moreover, there is at least one i in J_x such that $g_i(x - n\delta) = 0$, so the cardinality of the set $J_{x-n\delta}$ is strictly less than that of J_x . By induction, $x - n\delta$ belongs to C , so that x does as well.

(ii) Suppose x belongs to $\text{cvx } C \cap \mathcal{L}(C)$. By hypothesis, $g_j(x)$ is a nonnegative integer for each j in J . If $g_k(x) = 0$ for some k in J , then x belongs to $\text{cvx } C^k \cap \mathcal{L}(C) = C^k$, whence x belongs to C .

Otherwise $g_j(x) \neq 0$ for all j in J . Let $n = \min \{g_j(x) \mid j \in J, g_j(\delta_0) = 1\}$. Say $n = g_k(x)$. Then $g_j(x - n\delta_0) = g_j(x) - n \geq 0$ for all j in J , and $g_k(x - n\delta_0) = 0$. Hence $x - n\delta_0$ belongs to $\text{cvx } C^k \cap \mathcal{L}(C) = C^k$, so x belongs to C .*

It turns out that a cone C_λ fails to be normal only if the condition in (ii) does not hold for one of the faces of C_λ .

Distracting remarks:

1. Notice that the argument of Lemma 3.6 provides an algorithm for decomposing an element of C as a sum of generators.
2. Even if the generators of the cone C do not satisfy the conditions of Lemma 3.5 or Lemma 3.6, it is possible to use a combination of these ideas to prove normality. In particular, there are certain cones C_λ corresponding to irreducible characters of E_d in which normality holds on the faces; we argue that if $x \in \text{cvx } C_\lambda \cap \mathcal{L}(C_\lambda)$, then either there exists a generator $\beta \in E_\lambda$ such that $x - n\beta$ lies on a face of C_λ , or x belongs to the convex hull of a certain subcone C' of C . When the generators of C' have the form $\{e_I \mid I \in S\}$ or $\{e_I + e_J \mid I \in S, J \in S'\}$ for sets of intervals S and S' , Lemma 3.4 implies that $x \in \text{cvx } C' \cap \mathcal{L}(C')$. We can then establish normality by showing that the (smaller) cone C' is normal. (These arguments appear in Appendix C.)

4. The Proof of Theorem 3.1

We first make an observation about the faces of $\text{cvx } C_\lambda$, which will be useful in the proof of Theorem 3.1. Recall the description of the convex cones $\text{cvx } C_\lambda$ given in Proposition 1.4(ii):

$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in I}} \{x \mid \sigma f_i(x) \geq 0\}.$$

Let $F^j := \{x \in C_\lambda \mid f_j(x) = 0\} = \{\sum_{i \neq j} r_i \alpha_i \mid r_i \in \mathbb{R}^+\}$. For $J \subset I$ define $F^J := \bigcap_{j \in J} F^j$. Elements of C_λ belonging to F^j are sums involving only the simple roots α_i , where $i \neq j$; those belonging to F^J are sums involving only the simple roots α_i , with $i \notin J$. If $J' \subset J$, then $F^{J'}(\lambda)$ is a face of $F^J(\lambda)$; it is a proper face if the inclusion of J' in J is strict.

If λ belongs to the interior of the Weyl chamber, then the faces F^j are distinct $d-1$ -faces (facets) of $\text{cvx } C_\lambda$; this does not always hold for weights lying on the boundary of \bar{C} .

Lemma 3.7. Let λ be a dominant weight of \mathcal{G} .

(a) Suppose that $\mathcal{G} = A_d, B_d, C_d, F_4$, or G_2 .

(i) If for some $k > 1$, $I_\lambda \cap \{k, k+1, \dots, d\} = \emptyset$, then for $j = k, k+1, \dots, d-1$, F^j is a proper face of F^{j+1} .

(ii) If for some $k < d$, $I_\lambda \cap \{1, 2, \dots, k\} = \emptyset$, then for $j = 2, 3, \dots, k$, F^j is a proper face of F^{j-1} .

(b) Suppose that $\mathcal{G} = D_d$.

(i) If for some $k \geq d-2$, $I_\lambda \cap \{k, k+1, \dots, d\} = \emptyset$, then for $j = k, k+1, \dots, d-3$, F^j is a proper face of F^{j+1} , and F^{d-2} is a proper face of F^{d-1} and of F^d .

(ii) If for some $k \leq d-3$, $I_\lambda \cap \{1, 2, \dots, k\} = \emptyset$, then for $j = 2, \dots, k$, F^j is a proper face of F^{j-1} .

(c) Suppose that $\mathcal{G} = E_d$.

(i) If for some $k \geq 4$, $I_\lambda \cap \{1, 2, \dots, k\} = \emptyset$, then for $j = k, \dots, d-1$, F^j is a proper face of F^{j+1} . Also, if $2 \notin I_\lambda$, then F^4 is a proper face of F^2 .

Proof: We will prove (a); parts (b) and (c) are proved in a similar way.

Recall the description of the generating set \mathcal{E}_λ of C_λ given in Section 1: if $\beta \in \mathcal{E}_\lambda$, then the set of all indices j such that α_j is a summand of β includes exactly one element of I_λ , and the corresponding subgraph of the Dynkin diagram of \mathcal{G} is connected.

(i) If $j \geq k$, then the generators of C_λ belonging to F^j are sums involving simple roots from the set $\{\alpha_1, \alpha_2, \dots, \alpha_{j-1}\}$, so $F^j = F^{\{j, j+1, \dots, d\}} \subset F^{j+1}$.

(ii) In this case, if $j \leq k$, then the elements of \mathcal{E}_λ belonging to F^j are sums of simple roots from the set $\{\alpha_{j+1}, \alpha_{j+2}, \dots, \alpha_d\}$, so $F^j = F^{\{1, 2, \dots, j\}} \subset F^{j-1}$. *

We deduce from [Hu1; p. 58] that

$$\alpha_{cs} = \begin{cases} \alpha_d & \text{if } \mathcal{G} \text{ is of type } B_d \\ \alpha_{d-1} & \text{if } \mathcal{G} \text{ is of type } C_d \\ \alpha_3 & \text{if } \mathcal{G} \text{ is of type } F_4 \\ \alpha_1 & \text{if } \mathcal{G} \text{ is of type } G_2. \end{cases}$$

Proof of Theorem 3.1, sufficiency for normality: If any of conditions (i)–(iii) holds, then $\mathcal{L}(C_\lambda) = \Lambda_r$ by Lemma 1.5; we show that the cones are normal, and apply Theorem 1.9. We will refer to elements of the form $e_J = \sum_{i \in J} e_i$ as intervals if J is an interval.

1. If \mathcal{G} is A_d , or if $\langle \lambda, \alpha_{cs} \rangle \neq 0$, then with respect to the basis of simple roots, \mathcal{E}_λ consists of intervals (as in part (a) of the description for B_d , C_d , and G_2 , or whenever $3 \in I_\lambda$ in case \mathcal{G} is of type F_4). Now Lemma 3.5 implies C_λ is normal (and more) in these cases.

2. Suppose \mathcal{G} is C_d , and $\langle \lambda, \alpha_d \rangle = 0$. Let n be the largest index in I_λ . If $n = d-1$, then \mathcal{E}_λ consists of intervals and Lemma 3.5 applies. Otherwise, Lemma 3.7 implies that $F^n \subset F^{n+1} \subset \dots \subset F^d$; combining this with Proposition 1.4(ii) yields

$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in \{1, 2, \dots, n-1, d\}}} \{x \mid \sigma f_i(x) \geq 0\}.$$

As $\sigma f_i(\beta) = f_i(\sigma^{-1}\beta) \in \{0, 1\}$ for $i = 1, 2, \dots, n-1, d$, it follows from Lemma 3.6(i) that C_λ is normal.

3. If \mathcal{G} is F_4 and $I_\lambda = \{4\}$, then we may proceed as in 2. We obtain $F^4 \subset F^3 \subset F^2 \subset F^1$ and $\text{cvx } C_\lambda = \bigcap_{\sigma \in W_\lambda} \{x \mid \sigma f_1(x) \geq 0\}$. As the first coefficient of any element of \mathcal{E}_λ can only equal 0 or 1, Lemma 3.6(i) implies C_λ is normal.

4. Suppose \mathcal{G} is D_d .

If $d-2$ and one of $d-3$, $d-1$, or d belong to I_λ , then with respect to the basis of simple roots (possibly reordered), the elements of \mathcal{E}_λ are intervals, and by Lemma 3.5, C_λ is normal.

If $I_\lambda \subseteq \{1, 2, \dots, d-2\}$, suppose that n is the largest index in I_λ . Then $F^n \subset F^{n+1} \subset \dots \subset F^{d-1}$ by Lemma 3.7, so

$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in \{1, 2, \dots, n-1, d-1, d\}}} \{x \mid \sigma f_i(x) \geq 0\}.$$

Since $f_i(\beta) \in \{0, 1\}$ for $i \in \{1, 2, \dots, n, d-1, d\}$, Lemma 3.6(i) applies.

(Alternatively, note that C_λ is equivalent to the corresponding cone C_λ for D_d —we identify elements in C_λ with their coordinate vectors with respect to the basis of simple roots. The map $e_i \mapsto e_i$ ($i \leq d-1$), $e_d \mapsto e_{d-1} + e_d$ is an invertible linear transformation from the cone for D_d onto that for C_d . By 2, C_λ is normal.)

If $I_\lambda = \{d-1\}$, then from the description of \mathcal{E}_λ , $F^{d-1} \subset F^{d-2} \subset \dots \subset F^1$ and

$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in \{1, d\}}} \{x \mid \sigma f_i(x) \geq 0\},$$

and again Lemma 3.6(i) applies, since $\sigma f_i(\beta) \in \{0, 1\}$ for $i \in \{1, d\}$, σ in W_λ and β in \mathcal{E}_λ .

Similarly, if $I_\lambda = \{d\}$, $F^d \subset F^{d-2} \subset \dots \subset F^1$ and

$$\text{cvx } C_\lambda = \bigcap_{\substack{\sigma \in W_\lambda \\ i \in \{1, d-1\}}} \{x \mid \sigma f_i(x) \geq 0\},$$

and Lemma 3.6(i) applies.

Suppose I_λ contains $d-1$ or d , as well as some $n < d-2$. The root α_n in \mathcal{E}_λ satisfies $\sigma f_i(\alpha_n) = f_i(\sigma^{-1}\alpha_n) \in \{0, 1\}$, so Lemma 3.6(ii) will apply once we verify the condition on the faces. It is enough to check that $\text{cvx } F^s \cap \mathcal{L}(C_\lambda) = F^s$ for each of the faces F^s , $s \in I$; we do this by induction on the rank, d , as follows.

For s in $\{d-3, d-2, d-1, d\}$, the elements in \mathcal{E}_λ which belong to F^s are intervals with respect to the basis $\{\alpha_k\}_{k \neq s}$; Lemma 3.6 then implies $\text{cvx } F^s \cap \Lambda_r = F^s$. If $d = 4$, we may apply Lemma 3.6(ii) to conclude that C_λ is normal.

Otherwise, assume that for $d' < d$, all the cones C_λ for $D_{d'}$ are normal. If $s < d-3$, then the set $\{\beta \in \mathcal{E}_\lambda \mid f_s(\beta) = 0\}$ comprises two sets orthogonal to each other. The first consists of intervals $\alpha_j + \dots + \alpha_m$ with $I_\lambda \cap \{j, \dots, m\} = \{n\}$, $n \leq m < s$. The second consists of two types of elements: intervals $\alpha_j + \dots + \alpha_m$ where $I_\lambda \cap \{j, \dots, m\} = \{n\}$ and $n \geq j > s$, and sums $\alpha_j + \dots + 2\alpha_k + \dots + 2\alpha_{d-1} + \alpha_d$ with $I_\lambda \cap \{j, j+1, \dots, d\} = \{d-1\}$ or $\{d\}$ and $j > s$. The cone F^s is therefore the direct product of the cones generated by these two sets. That generated by the first set is normal as it consists of intervals. The cone generated by the second set is equal to the cone C_λ of D_{d-s} (when elements are identified with their coordinate vectors with respect to the basis $\{\alpha_{s+1}, \dots, \alpha_d\}$) with $I_\lambda = (I_\lambda \cap \{s+1, \dots, d\}) - s$. It is normal by the induction assumption, hence F^s is normal. By Lemma 3.2, $\mathcal{L}(F^s)$ is generated by the set $\{\alpha_j\}_{j \neq s}$. Thus if x belongs to $\text{cvx } F^s \cap \mathcal{L}(C_\lambda)$, then $f_s(x) = 0$, and x belongs to $\text{cvx } F^s \cap \mathcal{L}(F^s) = F^s$. Finally, we may apply Lemma 3.6(ii) to conclude that C_λ is normal.

5. Suppose \mathcal{G} is one of E_6, E_7 , or E_8 . If I_λ includes 4 and at least one of 2, 3, or 5, then with respect to the basis of simple roots (possibly reordered), the elements of \mathcal{E}_λ are intervals, and Lemma 3.3 implies that C_λ is normal.

If 4 does not belong to I_λ , but at least two of 2, 3, or 5 do, then the elements of \mathcal{E}_λ are sums of distinct simple roots and Lemma 3.6(i) applies. If 4 does not belong to I_λ but the latter includes at least one of 2, 3, or 5, together with another index, say $m \neq 4$, then $\sigma f_i(\alpha_m)$ belongs to $\{0, 1\}$ for all σ in W_λ and i in I_λ . The faces F^j satisfy $\text{cvx } F^j \cap \Lambda_r = F^j$ (we argue by induction for $j = 7, 8$), and Lemma 3.6(ii) applies.

If $I_\lambda \subseteq \{1, 2, \dots, 5\}$ or $I_\lambda \subseteq \{5, \dots, 8\}$, then we may argue as in part (4) that if n is the largest (respectively, the smallest) index in I_λ , then either $n = 1$ and $F^1 \subset F^3 \subset \dots \subset F^d$, or $n > 1$ and $F^n \subset F^{n+1} \subset \dots \subset F^d$ (respectively $F^n \subset F^{n-1} \subset \dots \subset F^3 \subset F^1$). It follows from Lemma 3.6(i) that C_λ is normal.

The remaining cases are those in which $1 \in I_\lambda \subseteq \{1, 6, 7, 8\}$, and at least one of 6, 7, or 8 belongs to I_λ . We have verified normality in all of these cases in Appendix C. This can also be checked by computer; if a cone C satisfies $\mathcal{L}(C) = \mathbf{Z}^d$, then it is normal if and only if the set of all sums of d generators of C is relatively convex with respect to \mathbf{Z}^d . (In fact, by an unpublished result of B. Reznick, d can be replaced by $(d - 1)$.)

Proof of necessity: If \mathcal{G} is B_d or G_2 , then α_{cs} is the only short simple root. If $\langle \lambda, \alpha_{cs} \rangle = 0$, then by Lemma 3.2, $\mathcal{L}(C_\lambda) \neq \Lambda_r$, so by Theorem 1.9, χ_λ is not eventually saturated.

If \mathcal{G} is C_d , suppose that $\langle \lambda, \alpha_d \rangle \neq 0$. If $\langle \lambda, \alpha_s \rangle = 0$ for all simple short roots α_s , then by Lemma 3.2 again, $\mathcal{L}(C_\lambda) \neq \Lambda_r$. Suppose instead that $\langle \lambda, \alpha_s \rangle \neq 0$ for some simple short root α_s , but $\langle \lambda, \alpha_{d-1} \rangle = 0$, and let j be the largest index less than d in I_λ . Set $a = \alpha_{j+1} + \dots + \alpha_{d-1} + \alpha_d$; then a belongs to $\mathcal{L}(C_\lambda) = \Lambda_r$, and $a = ((2\alpha_{j+1} + \dots + 2\alpha_{d-1} + \alpha_d) + \alpha_d) / 2$, so a belongs to $\text{cvx } C_\lambda \cap \mathcal{L}(C_\lambda)$. However, a does not lie in C_λ itself, as we deduce immediately from the form of elements of \mathcal{E}_λ . So C_λ is not normal in this case.

Finally, suppose \mathcal{G} is F_4 . If $\langle \lambda, \alpha_3 \rangle = \langle \lambda, \alpha_4 \rangle = 0$, then $\mathcal{L}(C_\lambda) \neq \Lambda_r$ by Lemma 3.2. If I_λ is one of $\{1, 4\}$, $\{2, 4\}$, or $\{1, 2, 4\}$, it is routine to verify that $a := \alpha_1 + \alpha_2 + \alpha_3$ belongs to $\text{cvx } C_\lambda \cap \mathcal{L}(C_\lambda)$, but not to C_λ , so the latter is not normal.*

Corollary 3.9. For \mathcal{G} and λ as in Theorem 3.1, all weights of χ_λ can be expressed as $\lambda - \sum_{\beta \in \mathcal{E}_\lambda} k_\beta \beta$, where k_β are nonnegative integers. *

5. Examples of characters which are not eventually saturated

We describe here the irreducible characters in rank 2 which are not eventually saturated. The examples (Figures 3(A) and 3(B)) were computed using the software *Simplie*.

If \mathcal{G} is of type B_2 , then the only irreducible characters which are not eventually saturated are those with highest weights $\lambda := l\lambda_1$. In that case, $\mathcal{E}_\lambda = \{\alpha_1, \alpha_1 + 2\alpha_2\}$, and $\lambda_1 = \alpha_1 + \alpha_2$ is an atom of P_λ which does not lie in C_λ . Consequently, $(\chi_{(Nl-1)\lambda_1}, \chi_{\lambda_1}^N) = 0$ for all integers N . Figure 3(A) shows the set of irreducible characters (plotted with respect to their highest weights) appearing in $\chi_{\lambda_1}^5$. The only weight of $\chi_{\lambda_1}^5$ for which the corresponding irreducible character does not appear is $4\lambda_1$. It can be deduced from [Ha3; Lemma 2.3] that if $N > 5$, then $\chi_{\lambda_1}^N + \chi_{(N-1)\lambda_1}$ is saturated (in other words, all irreducible characters but $\chi_{(N-1)\lambda_1}$ appear in the product).

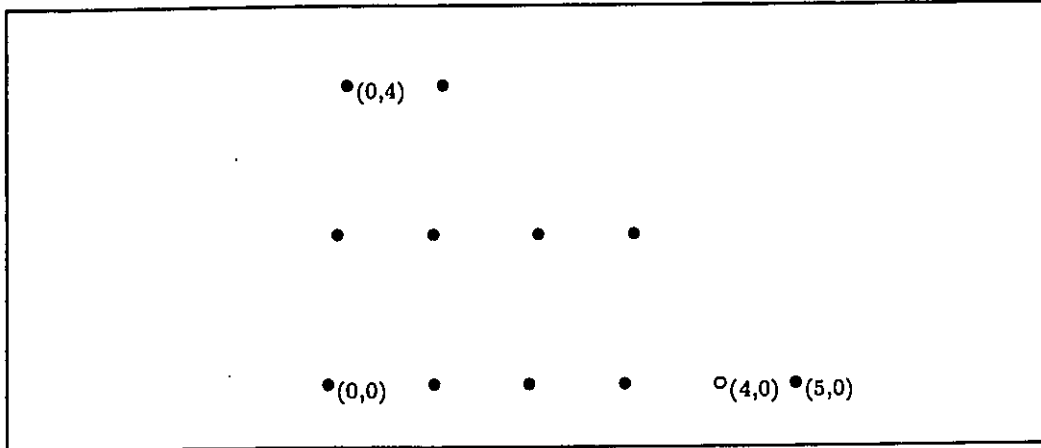


Figure 3(A)

In case $\mathcal{G} = G_2$, the irreducible characters which aren't eventually saturated have highest weights of the form $\lambda = l\lambda_2$. Here $\mathcal{E}_\lambda = \{\alpha_2, 3\alpha_1 + \alpha_2\}$, and in this case, $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$ are atoms of P_λ which do not lie in C_λ . For any positive integer N , the weight $N\lambda - (2\alpha_1 + \alpha_2)$ is not dominant; on the other hand, $N\lambda - (\alpha_1 + \alpha_2)$ is always dominant. By Proposition 1.7, $(\chi_{N\lambda - (\alpha_1 + \alpha_2)}, \chi_\lambda^N) = 0$ for all positive integers N . In case $\lambda = \lambda_2$, Figure 3(B) shows that for all other dominant weights of χ_λ^5 , the corresponding irreducible character appears. (In this plot, $\lambda_1 = (0, 1)$ and $\lambda_2 = (1, 0)$.)

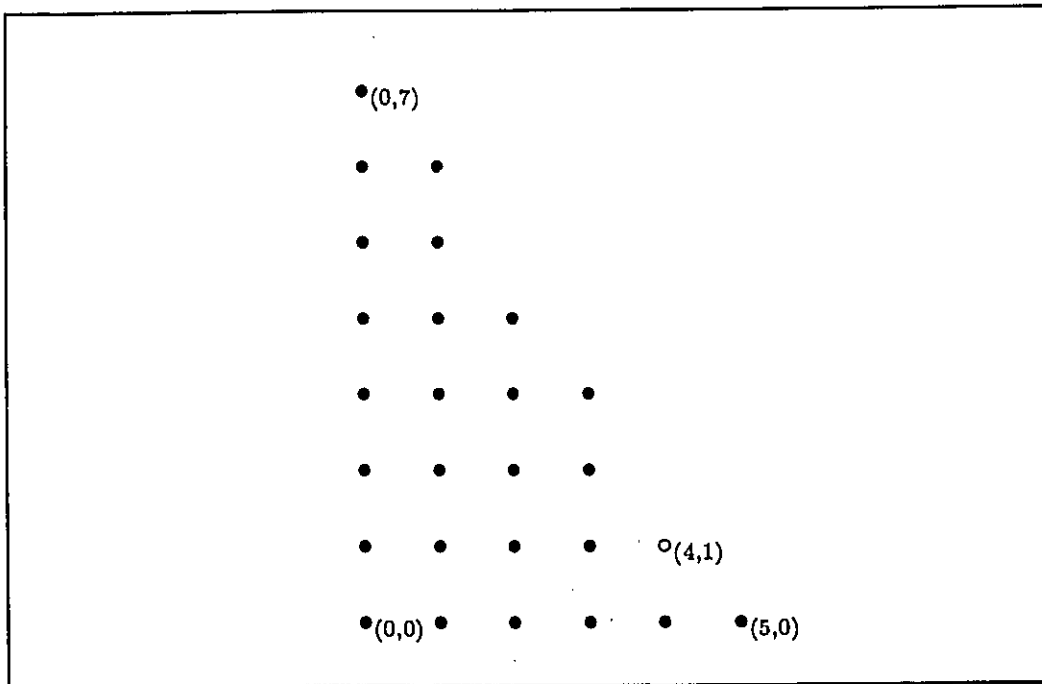


Figure 3(B)

Chapter 4

Dominant faces of $\text{cvx } \Pi(\lambda)$

After having determined which irreducible characters are eventually saturated, we would like to obtain information about the exponent required to give saturation of a product χ_λ^m . Our goal in the next three chapters is to set up an induction argument, based on a geometric description of the set of weights, to determine this exponent. (Such an argument might also be used to describe the decomposition of arbitrary products of irreducible characters—see Appendix A.) As a first step in this argument, we establish a reduction result (Proposition 4.6), which allows us to describe by induction on the rank of \mathcal{G} those irreducible characters whose highest weights lie on certain faces of the set of weights of the product.

Recall that in Chapter 1 we described the weight polytope $\text{cvx } \Pi(\lambda)$ of an irreducible character χ_λ in terms of linear functionals τf_j , where $\tau f_j(x) := 2(\lambda_j, \tau^{-1}x)/(\alpha_j, \alpha_j)$ (Proposition 1.4). We will consider here the faces $F^j(\lambda)$ of $\text{cvx } \Pi(\lambda)$ exposed by the functionals f_j :

$$F^j(\lambda) := \{x \mid f_j(x) = f_j(\lambda)\} .$$

In Section 1 of this chapter, we associate a semisimple subalgebra \mathcal{G}^j of \mathcal{G} with the face $F^j(\lambda)$, and to each weight w on $F^j(\lambda)$ a weight $p_j w$ of \mathcal{G}^j . We argue that the weight diagram of χ_λ , when restricted to $F^j(\lambda)$, equals (up to translation) the weight diagram of the irreducible character of \mathcal{G}^j whose highest weight is $p_j \lambda$ (Lemma 4.1). Using this fact, and an elementary geometric argument (Lemma 4.5), we show that on the face $F^j(\lambda + \nu)$, any product $\chi_\lambda \chi_\nu$ “behaves” as a product of irreducible characters of \mathcal{G}^j with highest weights $p_j \lambda$ and $p_j \nu$ respectively (Proposition 4.6). In other words, the sets of highest weights of the irreducible characters appearing in the products are equal, as are the multiplicities of the corresponding irreducible characters. If we set $F^J(\lambda) := \bigcap_{j \in J} F^j(\lambda)$, then an analogous result holds for the faces $F^J(\lambda)$ (Corollary 4.7).

We have seen in Chapter 3 (Theorem 3.1) that the eventual saturation of an irreducible character depends only on the smallest face of the fundamental Weyl chamber containing the highest weight. This face also determines the shape of the weight polytope of the character, by Proposition 1.2. Thus eventual saturation depends on the shape of the weight polytope, rather than the highest weight of the character. This is one reason why the convex structure of the weight polytope plays an important role in the study of the rings $K_0(A^G)$ and $K_0(A^T)$. Another is that there is a one to one correspondence between the faces $F^J(\lambda)$ —we call these the dominant faces of $\text{cvx } \Pi(\lambda)$ —and families of pure unfaithful states on $K_0(A^G)$. The “reduction to faces” result of this chapter (Proposition 4.6) may be used to describe a reduction for $K_0(A^G)$ when the underlying character is eventually saturated: for each J , there is an isomorphism between $K_0(A^{G^J})$ and the quotient of $K_0(A^G)$ by a certain ideal. The ideal here is described in terms of the face $F^J(\lambda)$. This information can be used to analyse the “eventual positivity problem” when reduction to the torus fails. (It is possible that the exponent required to give saturation also plays a role in this analysis.) We include in Section 2 a number of examples of 3-dimensional weight polytopes created using the software *Mathematica*.

1. A “reduction to faces” argument

Let λ be a dominant weight of \mathcal{G} , and fix $j \in I$. Observe that any weight in $\text{cvx } \Pi(\lambda)$ belonging to $F^j(\lambda)$ has the form $\lambda - \sum_{i \neq j} k_i \alpha_i$ for some nonnegative integers k_i , and arbitrary elements of $F^j(\lambda)$ have the form $\lambda - \sum_{i \neq j} r_i \alpha_i$ where the r_i are nonnegative real numbers. We define a subalgebra \mathcal{G}^j of \mathcal{G} corresponding to the face $F^j(\lambda)$ as follows.

As we mentioned in Chapter 1, the set of simple roots spans a d -dimensional Euclidean space, which we denote \mathbf{R}^d , with bilinear form $(,)$ induced by the Killing form. Let $\Delta_j := \Delta \setminus \{\alpha_j\}$, E_j the subspace of \mathbf{R}^d spanned by Δ_j , and Φ_j the set of roots $\Phi \cap E_j$. Then Φ_j is a root system in \mathbf{R}^d (or in E_j) with base Δ_j and Weyl group $W_j := W_{\Delta_j}$ (respectively W_j restricted to E_j) generated by the reflections σ_i , $i \neq j$ ([Hu2; Proposition 1.10 (a)]). If $\mathcal{G} = \mathcal{H} \oplus_{\alpha \in \Phi} \mathcal{G}_\alpha$ is a root space decomposition of \mathcal{G} , then we define a semisimple subalgebra \mathcal{G}^j of \mathcal{G} having root system Φ_j by setting \mathcal{H}^j equal to the \mathbf{R} span of the set $\{[\mathcal{G}_{\alpha_i}, \mathcal{G}_{-\alpha_i}] \mid i \neq j\}$ and $\mathcal{G}^j := \mathcal{H}^j \oplus_{\alpha \in \Phi_j} \mathcal{G}_\alpha$. In other words, \mathcal{G}^j is the subalgebra of \mathcal{G} generated by the basis elements x_i , $i \neq j$, of \mathcal{G} corresponding to the simple roots α_i , $i \neq j$. If Γ is the Dynkin diagram of \mathcal{G} , then the Dynkin diagram of \mathcal{G}^j can be obtained from Γ by deleting the node labelled by j .

Let p_j be the orthogonal projection from \mathbf{R}^d (the \mathbf{R} span of the set of simple roots Δ) onto E_j : $p_j(x) := x - (\lambda_j, x)\lambda_j / (\lambda_j, \lambda_j)$. Notice that if $i \neq j$ and $k \neq j$, then $\langle p_j \lambda_i, \alpha_k \rangle = \langle \lambda_i, \alpha_k \rangle$, so $\{p_j \lambda_i, i \neq j\}$ is the set of fundamental weights dual to the basis Δ_j of E_j . Thus $p_j \bar{C}$ is the fundamental dominant chamber for W_j in E_j , and p_j maps the weight lattice of \mathcal{G} onto that of \mathcal{G}^j ; if $\mu = (m_1, m_2, \dots, m_d)$ with respect to the basis $\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ of \mathbf{R}^d , then $p_j \mu = (m_1, \dots, m_{j-1}, m_{j+1}, \dots, m_d)$ with respect to the basis $\{p_j \lambda_1, \dots, p_j \lambda_d\}$ of E_j .

We also observe that, when restricted to $F^j(\lambda)$, the projection p_j is simply translation by a constant multiple of λ_j . To see this, recall that any element x on $F^j(\lambda)$ has the form $x = \lambda - \sum_{i \neq j} r_i \alpha_i$ where the coefficients r_i are nonnegative real numbers. Therefore

$$p_j x = p_j \lambda - \sum_{i \neq j} r_i \alpha_i = x - (\lambda_j, x)\lambda_j / (\lambda_j, \lambda_j).$$

Finally, note that if $i \neq j$ then $p_j \sigma_i(x) = p_j(x - \langle x, \alpha_i \rangle \alpha_i) = p_j x - \langle x, \alpha_i \rangle \alpha_i = p_j x - \langle p_j x, \alpha_i \rangle \alpha_i = \sigma_i(p_j x)$, so p_j commutes with all the elements of W_j .

The result of Lemma 4.1 may be known, although I do not know of a reference. (An example for A_2 appears in [AS]). It is a special case of Proposition 4.6.

Let $m(\lambda; \mu)$ be the multiplicity of the weight μ in the irreducible character χ_λ of \mathcal{G} , and let $m_j(\lambda; \mu)$ be the multiplicity of the weight $p_j \mu$ in the irreducible character of \mathcal{G}^j with highest weight $p_j \lambda$.

Lemma 4.1. $p_j(F^j(\lambda) \cap \Pi(\lambda))$ is the set of weights of the irreducible character of \mathcal{G}^j with highest weight $p_j \lambda$. For all weights μ in $F^j(\lambda)$, $m(\lambda; \mu) = m_j(\lambda; \mu)$.

Proof: Let V_λ be the \mathcal{G} -module affording the character χ_λ , and denote the subspace of vectors in V_λ having weight w by $V_\lambda(w)$. The space $V_\lambda^j := \bigoplus_{v \in F^j(\lambda)} V_\lambda(v)$ is a \mathcal{G}^j -module; we show that this \mathcal{G}^j module is irreducible with highest weight $p_j \lambda$. If $v \in V_\lambda(\lambda)$, then v is a maximal vector for \mathcal{G}^j with weight $p_j \lambda$, so the irreducible representation of \mathcal{G}^j with highest weight $p_j \lambda$ is a component of V_λ^j . Notice that if v belongs to $F^j(\lambda)$, then the weight $v + \alpha_j$ does not belong to $\Pi(\lambda)$ (by [Hu1; Proposition 21.3]). Thus $x_j \cdot v = 0$ for all weights $v \in F^j(\lambda)$, where x_j is the standard basis element of \mathcal{G} corresponding to α_j . If $v' \in V_\lambda^j$ were another maximal vector for \mathcal{G}^j , then $x_j \cdot v' = 0$

for all $i \neq j$, so v' would also be a maximal vector for \mathcal{G} . By [Hu1; Corollary 20.2] v' would be a multiple of v , so V_λ^j is irreducible. *

Recall that the face $F^j(\lambda)$ is invariant under W_j . The next lemma shows that if w is a weight on $F^j(\lambda)$, then it is conjugate under W_j to a dominant weight on $F^j(\lambda)$. As a corollary, we obtain a description of the faces of $\text{cvx } \Pi(\lambda)$ which contain dominant weights.

Lemma 4.2. If x belongs to $F^j(\lambda)$, then there is a $\sigma \in W_j$ such that σx belongs to $F^j(\lambda) \cap \bar{\mathcal{C}}$.

Proof: Suppose that x belongs to $F_j(\lambda)$. Then there is a $\sigma \in W_j$ such that $\langle \sigma(p_j x), \alpha_i \rangle \geq 0$ for all $i \neq j$. Since p_j commutes with W_j , we have $0 \leq \langle \sigma(p_j x), \alpha_i \rangle = \langle p_j(\sigma x), \alpha_i \rangle = \langle \sigma x, \alpha_i \rangle$ when $i \neq j$. On the other hand, σx belongs to $F_j(\lambda)$, so it has the form $\lambda - \sum_{i \neq j} r_i \alpha_i$ where $r_i \geq 0$, thus $\langle \sigma x, \alpha_j \rangle = \langle \lambda, \alpha_j \rangle - \sum_{i \neq j} r_i \langle \alpha_i, \alpha_j \rangle \geq \langle \lambda, \alpha_j \rangle \geq 0$ and σx therefore belongs to $\bar{\mathcal{C}}$. *

Corollary 4.3.

(i) If for some τ in W , $\tau F^j(\lambda) \cap \mathcal{C} \neq \emptyset$, then $\tau \in W_j$ and $\tau F^j(\lambda) = F^j(\lambda)$;

(ii) If w is a dominant weight lying on a face of $\Pi(\lambda)$, then w lies on one of the faces $F^j(\lambda)$.

Proof: (i) Suppose that x belongs to $\tau F^j(\lambda) \cap \mathcal{C}$. Then $\tau^{-1}x$ belongs to $F^j(\lambda)$, so by Lemma 4.2, there is an element $\sigma \in W_j$ such that $\sigma \tau^{-1}x$ belongs to $\bar{\mathcal{C}}$ and hence to \mathcal{C} . By [Hu1; Lemma 10.2 B], $\sigma \tau^{-1}x = x$, and $\sigma \tau^{-1}$ is the identity element of W , hence $\tau = \sigma \in W_j$ and $\tau F^j(\lambda) = \tau \sigma^{-1} F^j(\lambda) = F^j(\lambda)$.

(ii) If w lies on a face of $\text{cvx } \Pi(\lambda)$ then by Proposition 1.4, w lies on one of the faces $\sigma F^j(\lambda)$, where σ is an element of the Weyl group of \mathcal{G} . Thus $\sigma^{-1}w$ is conjugate to a dominant weight w' on $F^j(\lambda)$ by (i), and therefore $w = w'$ lies on $F^j(\lambda)$. *

We call the faces $F^j(\lambda)$ the *dominant faces* of $\text{cvx } \Pi(\lambda)$.

Corollary 4.4. If $\lambda \in \mathcal{C}$, then $(\lambda - \sigma\lambda, \lambda_j) = 0$ if and only if $\sigma \in W_j$.

Proof: By definition of $F^j(\lambda)$, $(\lambda - \sigma\lambda, \lambda_j) = 0$ if and only if $\sigma\lambda$ belongs to $F^j(\lambda)$. If $\sigma \in W_j$, then $\sigma\lambda$ lies on $F^j(\lambda)$, so $(\lambda - \sigma\lambda, \lambda_j) = 0$. Conversely, if $\sigma\lambda$ belongs to $F^j(\lambda)$, then λ belongs to $\sigma^{-1}F^j(\lambda) \cap \mathcal{C}$, so by Corollary 4.3(i), σ^{-1} belongs to W_j . *

In next lemma, suppose that $\lambda \in \bar{\mathcal{C}}$ and $\lambda' \in \mathcal{C}$.

Lemma 4.5. If w lies on a face of $\Pi(\lambda + \lambda')$ and w belongs to $\Pi(\lambda) + \lambda'$, then w belongs to $F^j(\lambda) + \lambda'$ for some $j \in I$.

Proof: Suppose that for some $x \in \Pi(\lambda)$ and $\sigma \in W$, $x + \lambda'$ belongs to $\sigma F^j(\lambda + \lambda')$. Then by Proposition 1.4, $\sigma f_j(x + \lambda') = f_j(\lambda) + f_j(\lambda')$. But $\sigma f_j(x) \leq f_j(\lambda)$ with equality only if x belongs to $\sigma F^j(\lambda)$, and $\sigma f_j(\lambda') = f_j(\sigma^{-1}\lambda') \leq f_j(\lambda')$, with equality if and only if $\sigma \in W_j$ (Corollary 4.4). Hence x belongs to $F^j(\lambda)$. The reverse inclusion is also true, obviously. *

Let $M(\lambda, \mu; \nu + \mu)$ be the multiplicity of the irreducible character $\chi_{\nu + \mu}$ of \mathcal{G} in the decomposition of the product $\chi_\lambda \chi_\mu$, and let $M_j(\lambda, \mu; \nu + \mu)$ be the multiplicity of the irreducible character of \mathcal{G}^j with highest weight $p_j(\nu + \mu)$ in the product of irreducible characters of \mathcal{G}^j with highest weights $p_j\lambda$ and $p_j\mu$ respectively.

Proposition 4.6. If ν is a weight of χ_λ belonging to the face $F^j(\lambda)$, then

$$M(\lambda, \mu; \nu + \mu) = M_j(\lambda, \mu; \nu + \mu).$$

Proof: By Racah's formula ([FH; Exercise 25.31], for example)

$$M(\lambda, \mu; \nu + \mu) = \sum_{\sigma \in W} \text{sgn} \sigma m(\lambda; \sigma(\nu + \mu + \delta) - \mu - \delta). \quad (4.1.1)$$

Since ν lies on the face $F^j(\lambda)$ of $\text{conv} \Pi(\lambda)$, $\nu + \mu + \delta$ lies on the face $F^j(\lambda + \mu + \delta)$ of $\text{conv} \Pi(\lambda + \mu + \delta)$. It follows from Lemma 4.5 that if $\sigma(\nu + \mu + \delta) - \mu - \delta$ belongs to $\Pi(\lambda)$ then it belongs to $F^j(\lambda)$ and σ belongs to W_j . In that case, the multiplicity of the weight $\sigma(\nu + \mu + \delta) - \mu - \delta$ would equal that of its image under p_j in the irreducible character of \mathcal{G}^j with highest weight $p_j \lambda$, by Lemma 4.1. Therefore

$$\begin{aligned} M(\lambda, \mu; \nu + \mu) &= \sum_{\sigma \in W_j} \text{sgn} \sigma m(\lambda; \sigma(\nu + \mu + \delta) - \mu - \delta) \\ &= \sum_{\sigma \in W_j} \text{sgn} \sigma m_j(\lambda; \sigma(\nu + \mu + \delta) - \mu - \delta) \\ &= M_j(\lambda, \mu; \nu + \mu). \end{aligned}$$

✱

Remark: Expressed algebraically, Proposition 4.5 says that if w is dominant weight of the form $\lambda + \mu - \sum_{i \neq j} \alpha_i$, then $M(\lambda, \mu; w) = M_j(\lambda, \mu; w)$. In [BZ; Proposition 1.3], the authors prove this result using [PRV; Theorem 2.1].

We can describe the weights on lower dimensional faces by defining, for subsets J of I , faces $F^J(\lambda) := \cap_{j \in J} F^j(\lambda)$; weights on $F^J(\lambda)$ have the form $\lambda - \sum_{i \notin J} k_i \alpha_i$. Let \mathcal{G}^J be the subalgebra of \mathcal{G} generated by the basis elements x_i , $i \notin J$, and let p_J be the projection from \mathbb{R}^d onto the span of the simple roots α_i , $i \notin J$. Using Lemma 4.1, we argue inductively that $F^J(\lambda) \cap \Pi(\lambda)$ is the translate of the set of weights of the irreducible character \mathcal{G}^J with highest weight $p_J \lambda$; then Proposition 4.6 holds for the faces $F^J(\lambda)$ as well. These observations are summarized Corollary 4.7 below.

Let $m_J(\lambda; w)$ be the multiplicity of the weight $p_J w$ in the irreducible character of \mathcal{G}^J with highest weight $p_J \lambda$, and $M_J(\lambda, \mu; \nu + \mu)$ the multiplicity of the irreducible character with highest weight $p_J(\nu + \mu)$ in the product of irreducible characters of \mathcal{G}^J with highest weights $p_J \lambda$ and $p_J \mu$ respectively.

Corollary 4.7.

- (i) $F^J(\lambda) \cap \Pi(\lambda)$ is the translate of the set of weights of the irreducible character of \mathcal{G}^J with highest weight $p_J \lambda$, and if $w \in F^J(\lambda) \cap \Pi(\lambda)$, then $m(\lambda; w) = m_J(\lambda; w)$.
- (ii) If ν is a weight of χ_λ belonging to $F^J(\lambda)$, and μ is a dominant weight, then

$$M(\lambda, \mu; \nu + \mu) = M_J(\lambda, \mu; \nu + \mu).$$

✱

The last item in this section is an immediate consequence of an exercise in [Hu1]. (The result in the exercise is attributed to Kostant.)

Let S_J be the set of vertices of the face $F^J(\lambda) + \mu$; that is,

$$S_J := \{ \sigma\lambda + \mu \mid \sigma \in W_J := \langle \sigma_i, i \notin J \rangle \} .$$

Corollary 4.8. If $S_J \subset \bar{C}$, then for any weight ν in $\Pi(\lambda)$ of the form $\nu = \lambda - \sum_{i \notin J} k_i \alpha_i$, the irreducible character $\chi_{\nu+\mu}$ appears in the product $\chi_\lambda \chi_\mu$ and $(\chi_\lambda \chi_\mu, \chi_{\nu+\mu}) = m(\lambda; \nu)$. The weights in S_J are vertices of the set of highest weights of irreducible characters appearing in the product, and $(\chi_\lambda \chi_\mu, \chi_{\sigma\lambda+\mu}) = 1$.

Proof: Suppose that w_1 and w_2 are dominant weights of \mathcal{G} , and all the weights in $\Pi(w_1) + w_2$ are dominant. Then it follows from the Brauer-Weyl formula that all the corresponding irreducible characters appear in the product $\chi_{w_1} \chi_{w_2}$, and the multiplicity of the irreducible character with highest weight $\nu + w_2$ equals $m(w_1; \nu)$ (Exercise 12, p.142 in [Hu1]).

If $S_J \subset \bar{C}$, then $\text{cvx } S_J \subset \bar{C}$, so all weights in $F^J(\lambda) + \mu$ are dominant. By Proposition 4.6 and the result in the preceding paragraph (for products of characters of \mathcal{G}^J), $(\chi_\lambda \chi_\mu, \chi_{\nu+\mu}) = m(\lambda; \nu)$ for any ν in $F^J(\lambda)$. *

2. Examples of three dimensional weight polytopes

In view of Proposition 1.4 and Lemma 4.1, one can inductively construct the weight polytope $\text{cvx } \Pi(\lambda)$ of an irreducible character (at least in principle). The images of three dimensional weight polytopes which follow were produced with the software *Simplie* and *Mathematica*.

Notice in figures 4(A) and 4(F) below that if λ belongs to the interior of the Weyl chamber, then $\text{cvx } \Pi(\lambda)$ (or more precisely, the undirected graph consisting of its vertices and edges) is the Cayley graph of the Weyl group of \mathcal{G} with respect to the generators $\{\sigma_i, i \in I\}$. (Recall that the vertices of the Cayley graph of a group G are the elements of G . If g_1, g_2, \dots, g_d are generators of G , then at each vertex of the graph there are d outgoing edges and d incoming edges labelled by the generators g_i ; the edge labelled g_i connects the vertex v to vg_i . In case a generator g_j has order 2 (as is the case for the Weyl group), the picture is modified so that there is a single (undirected) edge labelled by g_j at each vertex.)

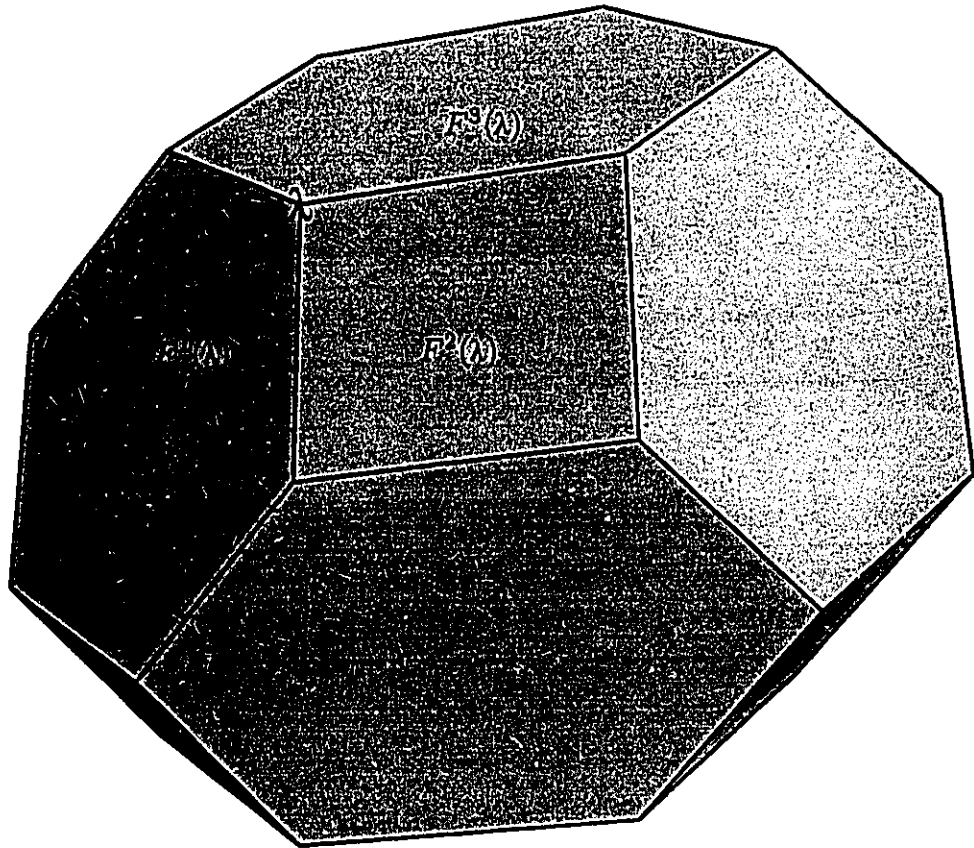


Figure 4(A): $A_3 \lambda = (1, 1, 1)$

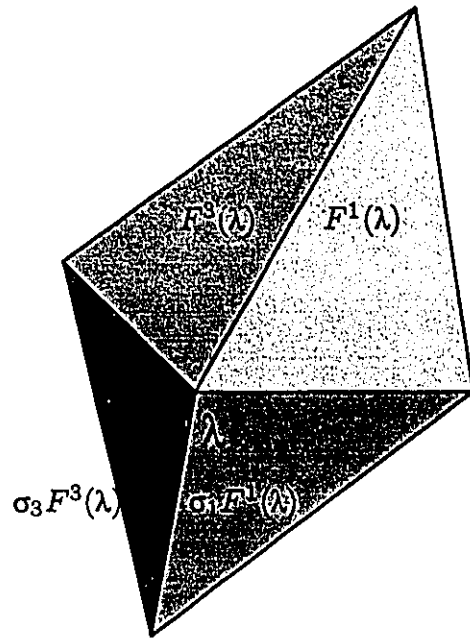


Figure 4(B): $A_3 \lambda = (0, 1, 0)$

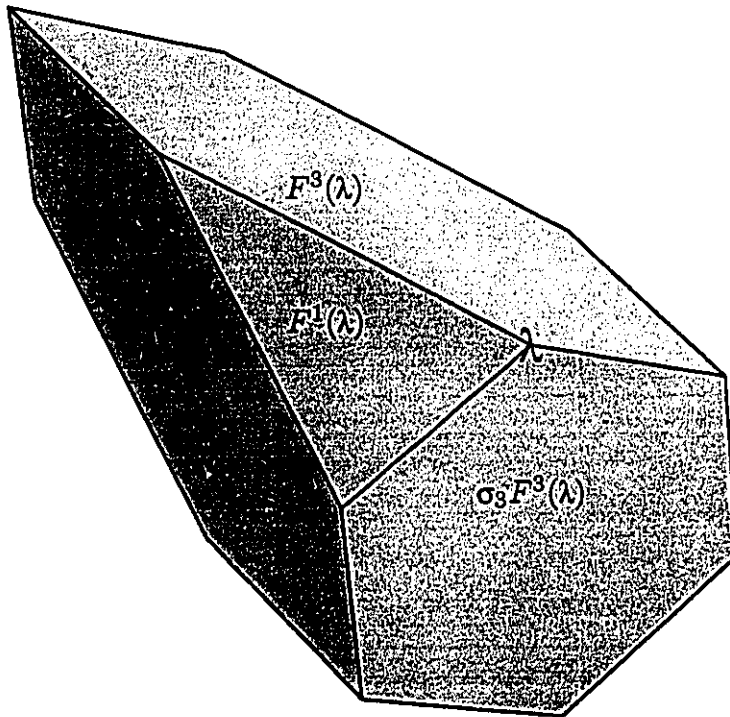


Figure 4(C): $A_3 \lambda = (1, 1, 0)$

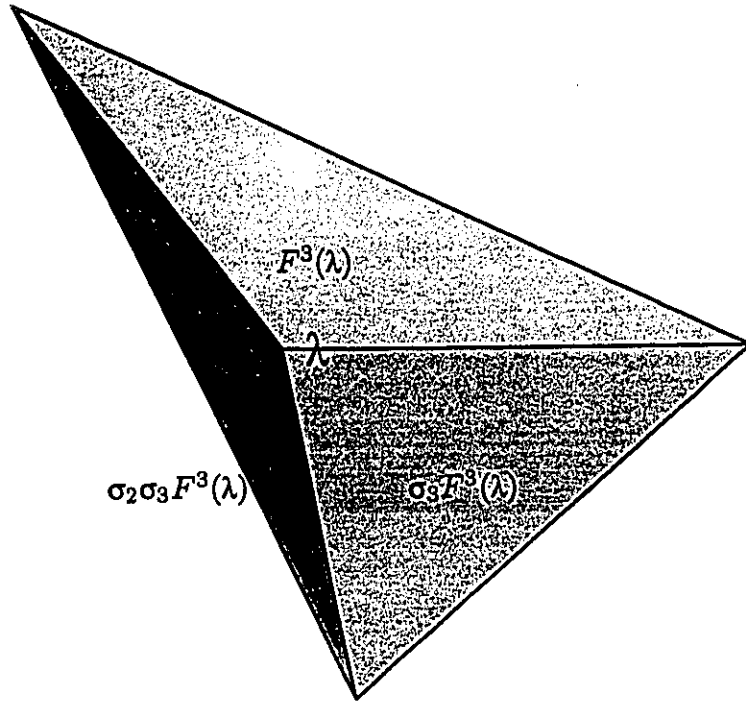


Figure 4(D): $A_3 \lambda = (1, 0, 0)$

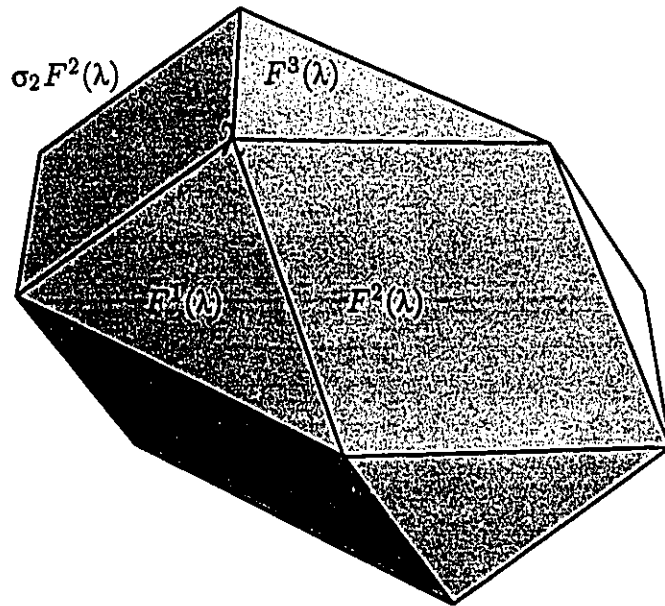


Figure 4(E): $A_3 \lambda = (1, 0, 1)$

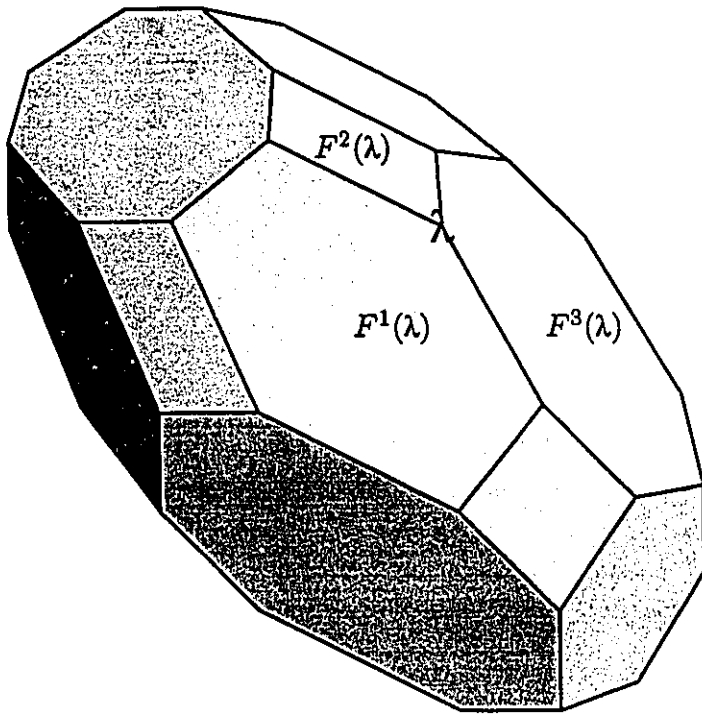


Figure 4(F): $B_3 \lambda = (1, 1, 1)$

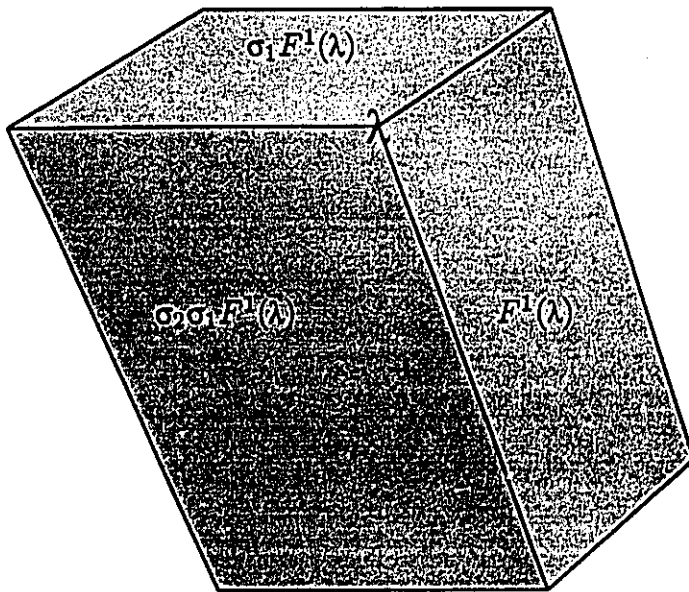


Figure 4(G): $B_3 \lambda = (0, 0, 1)$

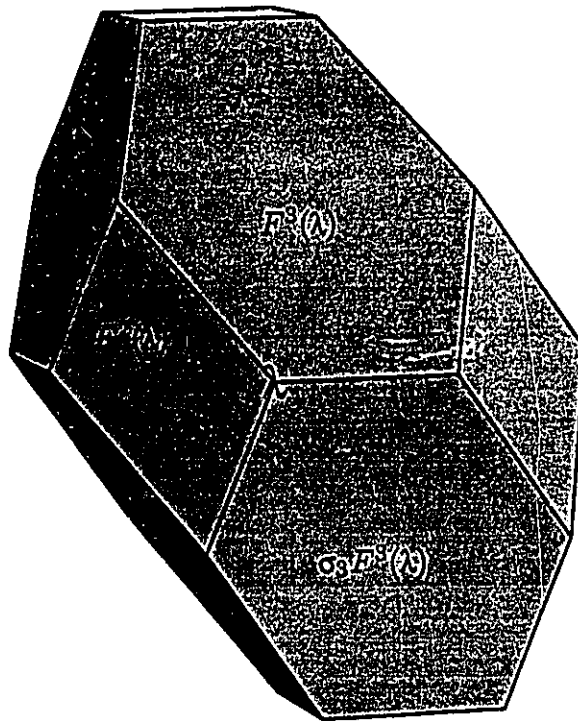


Figure 4(H): $B_3 \lambda = (1, 1, 0)$

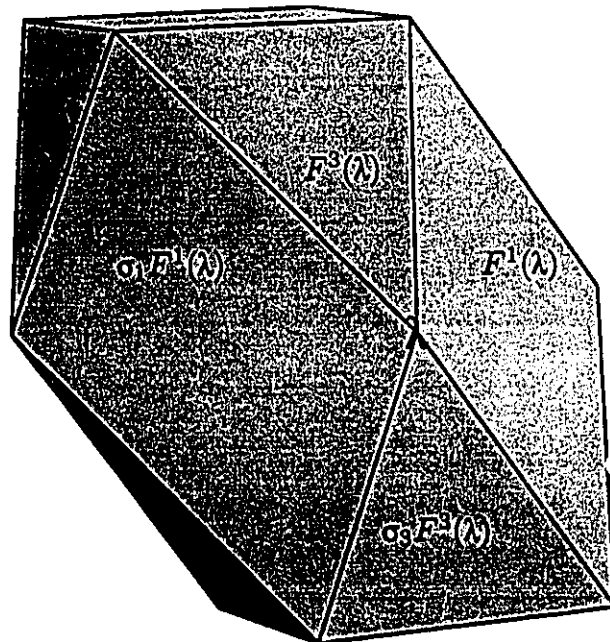


Figure 4(I): $B_3 \lambda = (0, 1, 0)$

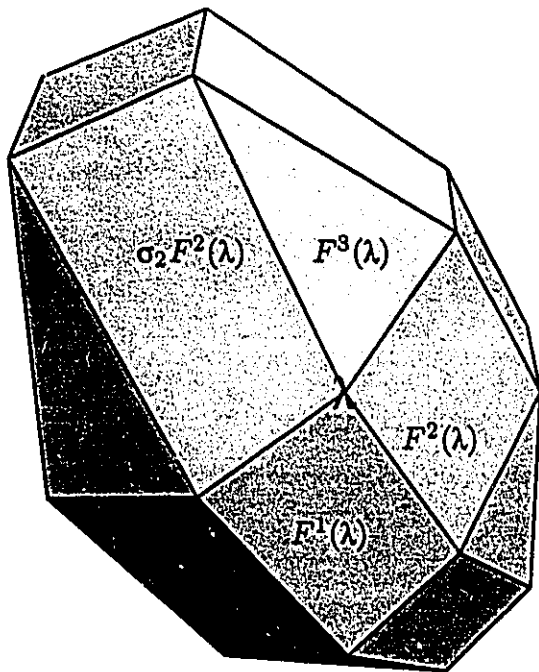


Figure 4(J): $B_3 \lambda = (1, 0, 1)$

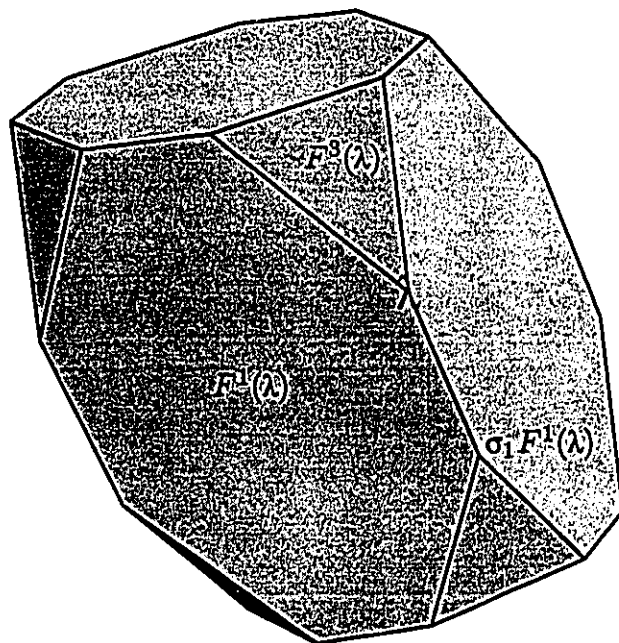


Figure 4(K): $B_3 \lambda = (0, 1, 1)$

Chapter 5

Comparing products of irreducible characters of A_d

Consider a product of irreducible characters $\chi_{\mu_1}\chi_{\mu_2}\cdots\chi_{\mu_n}$ with *total highest weight* defined to be $\mu_1+\mu_2+\cdots+\mu_n$. Using the results in Chapter 4 and induction on the rank of \mathcal{G} , we can describe certain irreducible characters appearing in this product; namely, those with highest weights on the dominant faces of $\Pi(\mu_1+\cdots+\mu_n)$. To describe the remaining irreducible characters appearing in the product, we would like to use induction on the total highest weight.

Recall the partial ordering on the weights, defined in [Hu1; Section 10.1]: $\mu \prec \lambda$ if the difference $\lambda - \mu$ is a sum of simple roots (with nonnegative coefficients). If λ and μ are dominant weights belonging to the same coset of the weight lattice relative to the root lattice, then $\mu \prec \lambda$ if and only if $\Pi(\mu) \subset \Pi(\lambda)$ ([A; Proposition 6.5]). We expect that if the total highest weight of a product $\chi_{\nu_1}\chi_{\nu_2}\cdots\chi_{\nu_n}$ is less than that of $\chi_{\mu_1}\chi_{\mu_2}\cdots\chi_{\mu_n}$, and the differences $\mu_j - \nu_j$ are sufficiently “small” (each equal to a fundamental weight for instance), then all the irreducible characters appearing in $\chi_{\nu_1}\chi_{\nu_2}\cdots\chi_{\nu_n}$ also appear in $\chi_{\mu_1}\chi_{\mu_2}\cdots\chi_{\mu_n}$.

We investigate this conjecture here by comparing products of irreducible characters of A_d using the Littlewood-Richardson rule. The most successful result is the following inequality, which we prove in Section 1.

Proposition 5.1. Let $\lambda_{i_0}, \lambda_{i_1}, \dots, \lambda_{i_t}$ be fundamental weights (not necessarily distinct) such that the sum $\sum_{j=0}^t \lambda_{i_j}$ belongs to the root lattice Λ_r . Then for dominant weights $\mu_0, \mu_1, \dots, \mu_t$,

$$\chi_{\mu_0}\chi_{\mu_1}\cdots\chi_{\mu_t} \leq \chi_{\mu_0+\lambda_{i_0}}\chi_{\mu_1+\lambda_{i_1}}\cdots\chi_{\mu_t+\lambda_{i_t}}. \quad (5.0.1)$$

We begin Section 1 by describing the Littlewood-Richardson rule [Mac] for computing the product of two irreducible characters of A_d using Young tableaux. We also describe a slightly modified algorithm to accommodate products of more than two irreducible characters. Our method in proving Proposition 5.1 will be to “transform” a tableau of weight w which satisfies the conditions of the Littlewood-Richardson rule for the product $\chi_{\mu_0}\cdots\chi_{\mu_t}$ to one of the same weight satisfying the conditions for the product $\chi_{\mu_0+\lambda_{i_0}}\cdots\chi_{\mu_t+\lambda_{i_t}}$. Lemmas 5.2 and 5.3 describe rules for modifying a tableau so that the conditions of the Littlewood-Richardson rule are preserved. These results are combined in Corollary 5.4, and the proof of Proposition 5.1 follows.

Notice that the total highest weights of the two products in Proposition 5.1 differ by a dominant weight. In particular Proposition 5.1 implies that, for dominant weights μ_1 and μ_2 , $\chi_{\mu_1+\lambda_1}\chi_{\mu_2+\lambda_d} \geq \chi_{\mu_1}\chi_{\mu_2}$. Now if ν is a dominant weight belonging to the interior of the Weyl chamber, and w is a dominant weight in $\Pi(\nu)$, then either $w \in F^j(\nu)$ for some j , or $w \in \Pi(\nu')$ where $\nu' = \nu - (\lambda_1 + \lambda_d)$. Thus in a product of irreducible characters whose total highest weight, ν , lies in the interior of the fundamental Weyl chamber, we may determine all irreducible characters with highest weights on the faces $F^j(\nu)$ by Proposition 4.6 and induction on the rank of \mathcal{G} , and all the remaining irreducible

characters by Proposition 5.1 and induction on v . This argument also works when the total highest weight lies on a boundary of the fundamental Weyl chamber, if the rank of \mathcal{G} is less than or equal to three. (In that case, the difference $v - v'$ is still a dominant weight, although it may be different from $\lambda_1 + \lambda_d$.)

Unfortunately, there are gaps if the rank of \mathcal{G} is greater than three and the total highest weight v lies on a boundary of the Weyl chamber. In particular, there are dominant weights w in $\Pi(v)$ which do not lie on any of the faces $F^j(v)$, and do not belong to any set $\Pi(v')$ for which $v' \prec v$ and the difference $v - v'$ is a dominant weight. To account for these gaps, we consider in Section 2 products in which the difference in total highest weights is not a dominant weight. The combinatorial arguments required here are more involved than those in the first section, and consequently we restrict our attention to products with total highest weights v of the form $n_1\lambda_1 + n_2\lambda_2$, or to specific cases in ranks four and five.

The results obtained in Section 2 are weaker than that of Proposition 5.1. For example, if $\mu_i := m_{i1}\lambda_1 + m_{i2}\lambda_2$ are dominant weights of A_d and $\chi_w \leq \chi_{\mu_0 - p\lambda_1} \chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_q + \lambda_2}$ for integers p and q satisfying $2q + p = d + 1$, then $\chi_w \leq \chi_{\mu_0} \cdots \chi_{\mu_q}$ if w lies on the face F^1 (Lemma 5.5). Similar results are proved for certain characters of A_5 in Lemmas 5.6, 5.7, and 5.8.

Lemma 5.9 applies to products of irreducible characters of A_d whose highest weights are sums of the fundamental weights λ_1 and λ_2 when d is even. We consider in particular the special case where the highest weights are all multiples of the fundamental weight λ_2 . In this case we obtain sufficient conditions for an irreducible character $\chi_w \leq \chi_{\mu_0 - \lambda_1 + \lambda_2} \chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_t + \lambda_2}$ to appear in $\chi_{\mu_0} \cdots \chi_{\mu_t}$. Analogous results hold for certain characters of A_4 and A_5 (Lemmas 5.10, 5.11, and 5.12). The results of Section 2 are used in proving saturation of certain characters of A_d when $d > 3$ (Proposition 6.7 and Theorem 6.8).

1. An inequality for products of irreducible characters in A_d

Let $\lambda = l_1\lambda_1 + l_2\lambda_2 + \cdots + l_d\lambda_d$ be a dominant weight. A *Young diagram* of shape λ consists of (at most) $d + 1$ rows of boxes such that, for $j = 1, 2, \dots, d$, the number of boxes in row j minus the number of boxes in row $j + 1$ equals l_j . The basic Young diagram of shape λ has $l_j + l_{j+1} + \cdots + l_d$ boxes in row j , for $j = 1, 2, \dots, d$. All other Young diagrams of shape λ can be obtained from the basic one by adding the same number of boxes to each of rows 1 to $d + 1$.

Suppose that $v = n_1\lambda_1 + \cdots + n_d\lambda_d$. If $n_j + n_{j+1} + \cdots + n_d \geq l_j + l_{j+1} + \cdots + l_d$ for $j = 1, 2, \dots, d$, then a *skew tableau* of shape $v \setminus \lambda$ is a Young diagram of shape v in which the $(n_j + n_{j+1} + \cdots + n_d) - (l_j + \cdots + l_d)$ boxes at the right end of row j have been filled with entries from a set S (here, the set S will consist of integers or pairs of integers) for $j = 1, 2, \dots, d$. For example, if $v = \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4$ and $\lambda = 2\lambda_1 + \lambda_2 + \lambda_3$ then Figure 5(A) below is a skew tableau of shape $v \setminus \lambda$.

| | | | | |
|---|---|---|---|---|
| | | | | 1 |
| | | 1 | 2 | |
| | 3 | 3 | | |
| 4 | | | | |

Figure 5(A)

Call the part of the skew tableau without entries the *base* of the tableau.

The following is the Littlewood-Richardson rule for computing the product of irreducible characters of A_d with highest weights $\lambda = l_1\lambda_1 + l_2\lambda_2 + \cdots + l_d\lambda_d$ and $\mu = m_1\lambda_1 + \cdots + m_d\lambda_d$ respectively.

Beginning with the basic Young diagram of shape λ , add $m_i + m_{i+1} + \cdots + m_d$ boxes with entry i for $i = 1, 2, \dots, d$ in such a way that the resulting skew tableau has at most $d + 1$ rows and satisfies the following two conditions.

- LR 1: The entries in the tableau increase weakly along rows and strictly down columns. When this holds, we say that the tableau is *column strict*.
- LR 2: Let $w = (w_1, w_2, \dots, w_k)$ be the sequence of entries in the tableau read from right to left and top to bottom. For each positive integer $j \leq k$ and each positive integer $i \leq d - 1$ the number of i 's in the sequence (w_1, w_2, \dots, w_j) is greater than or equal to the number of $i + 1$'s.

The multiplicity of an irreducible character χ_ν in the product $\chi_\lambda\chi_\mu$ is the number of distinct skew tableau of shape $\nu \setminus \lambda$ so obtained.

To compute the product of more than two irreducible characters, say $\chi_{\mu_0}\chi_{\mu_1} \cdots \chi_{\mu_t}$, we can extend this algorithm, by adding to the Young diagram of shape μ_0 boxes whose entries are pairs of integers. A box with entry (i, j) is to be interpreted as a box with entry j added at the i^{th} step $(\chi_{\mu_0}\chi_{\mu_1} \cdots \chi_{\mu_{i-1}} \times \chi_{\mu_i})$ of the product; more precisely, it represents a box with entry j added to the Young diagram whose shape is $\nu \setminus \mu_0$, where ν is the highest weight of an irreducible character appearing in the product $\chi_{\mu_0} \cdots \chi_{\mu_{i-1}}$.

The extended algorithm goes as follows. If $\mu_i = m_{i_1}\lambda_1 + m_{i_2}\lambda_2 + \cdots + m_{i_d}\lambda_d$, then we add for $i \in \{1, 2, \dots, t\}$ and $j \in \{1, 2, \dots, d\}$, $m_{i_j} + m_{i_{j+1}} + \cdots + m_{i_d}$ boxes with entry (i, j) to the basic Young diagram of shape μ_0 in such a way that the resulting skew tableau has at most $d + 1$ rows and satisfies the following two conditions.

- MLR 1: The entries in the tableau increase weakly from left to right and strictly from top to bottom in the lexicographic ordering on the pairs. (We still call such a tableau column strict.)
- MLR 2: For each $i \in \{1, 2, \dots, t\}$ the sequence $w_i = (w_{i_1}, w_{i_2}, \dots, w_{i_k})$ of second coordinates of entries (i, w_{i_n}) in the tableau read from right to left and top to bottom satisfies LR 2.

The multiplicity of the irreducible character χ_ν in the product $\chi_{\mu_0}\chi_{\mu_1} \cdots \chi_{\mu_t}$ will be the number of distinct skew tableau of shape $\nu \setminus \mu_0$ so obtained.

For example let $\lambda = \lambda_1 + 2\lambda_2$, $\mu = \lambda_1 + 3\lambda_2$, and $\nu = 3\lambda_2$ be dominant weights of A_2 . The tableaux in Figure 5(B), (a) and (b) below show respectively that $\chi_\mu \leq \chi_\lambda^2$ and $\chi_\nu \leq \chi_\mu\chi_\lambda$ —they satisfy the conditions LR 1 and LR 2—so $\chi_\nu \leq \chi_\lambda^3$. These two results are combined in the single tableau (c), which satisfies the conditions MLR 1 and MLR 2.

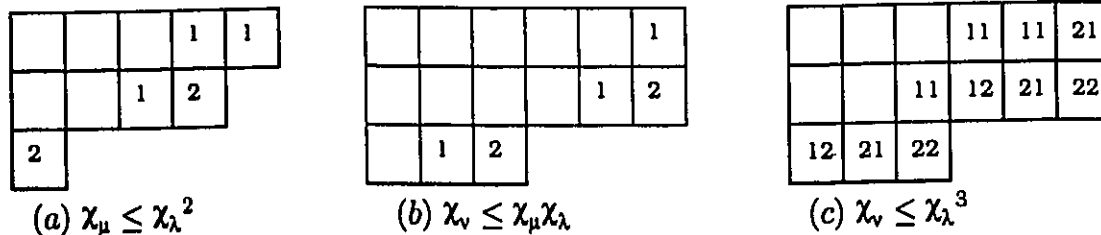


Figure 5(B)

The next two lemmas give a method for adding boxes to a tableau so that the conditions MLR1 and MLR2 are preserved.

Suppose that we are given a tableau T in which the entries increase weakly along rows, and we add to T a number of boxes (with or without entries). We add the boxes to T in such a way that the entries still increase weakly along rows; that is, when adding a box with entry (i, j) to row m of the tableau T , we put it as far to the left as possible, so that all entries in row m to the left of the added box are strictly less than (i, j) and those to the right are greater than or equal to (i, j) . We may consider blank boxes as boxes with entry $(0, 0)$, and these we add to the first column of the tableau.

Let S be a subset consisting of pairs (i, j) where either $i = j = 0$, or both i and j are strictly positive integers. Let T' be the tableau obtained from T by adding, for each $(i, j) \in S$, a box with entry (i, j) to T in such a way that the entries in the tableau still increase weakly along rows.

Lemma 5.2. Suppose that T satisfies MLR1.

- (i) If all the boxes are added to row one, then T' is column strict.
- (ii) If, for $m > 1$, we only add a box with entry (i, j) to row m of T after adding a box with entry $(i', j') < (i, j)$ to row $m - 1$, or if $i = j = 0$, after adding a box with entry $(0, 0)$ to row $m - 1$, then T' is column strict.

Proof:(i) Suppose that $|S| = 1$, and that T' is the tableau obtained after adding a box with entry (i, j) to row one, column n , of T . If $i = j = 0$, then $n = 1$; otherwise, for $k < n$ the entries in column k of the tableau have not changed, so they still increase strictly from top to bottom. For $k \geq n$, the entry in row one, column k of T' is less than or equal to the entry in row one, column k of T , (it is equal to the entry in row one, column $k - 1$ of T), so entries in column k still increase strictly. By induction on $|S|$, we can add any number of boxes to row one of T in this way, and T' will be column strict.

(ii) Suppose that we have added a box with entry (i', j') to row $m - 1$, column n' of T so that the resulting tableau is still column strict, and we then add a box with entry $(i, j) > (i', j')$ —or if $i' = j' = 0$ a box with entry $(0, 0)$ —to row m , column n . Then the entries in columns $k < n$ have not changed, so still increase strictly, and the entries in columns $k \geq n$ increase strictly from row 1 to $m - 1$ and from row m to $d + 1$. We therefore only need to check that, for $k \geq n$, the entry in row $m - 1$, column k is strictly less than that in row m , column k .

If $n \leq k \leq n'$, then the entry in row m column k is greater than or equal to (i, j) since $k \geq n$. The entry in row $m - 1$, column k of T' is less than or equal to (i', j') since $k \leq n'$, so it is strictly less than (i, j) . If $k \geq n$ and $k \geq n'$, then the entries in rows $m - 1$ and m column k of T' are equal to those in rows $m - 1$ and m , column k respectively of T , so they increase strictly. \ast

Example: Suppose that we are given a tableau T of shape $v \setminus \mu$ obtained using the Littlewood-Richardson algorithm for the product $\chi_\lambda \chi_\mu$. If $w_1 \geq w_2 \geq \dots \geq w_d$ is a sequence of integers, and we add w_1 blank boxes to row 1 of T , w_2 blank boxes to row 2 of T , and so on up to w_d blank boxes to row d of T , then the resulting tableau T' will still satisfy LR1 by Lemma 5.2. It will also satisfy LR2, since we haven't changed any of the entries in T . Consequently, $(\chi_v, \chi_\lambda \chi_\mu) \leq (\chi_{v+w}, \chi_\lambda \chi_{\mu+w})$ for any dominant weight $w = (w_1 - w_2)\lambda_1 + (w_2 - w_3)\lambda_2 + \dots + (w_{d-1} - w_d)\lambda_{d-1} + w_d \lambda_d$.

Let $\mu_0 := 2\lambda_1 + \lambda_3 + \lambda_4$, $\mu_1 := 3\lambda_1 + 2\lambda_2$, $v := 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4$, and $w := \lambda_1 + \lambda_2 + \lambda_4$. The tableaux in Figure 5(C) (a) and (c) below show that $\chi_v \leq \chi_{\mu_0} \chi_{\mu_1}$ and $\chi_{v+w} \leq \chi_{\mu_0+w} \chi_{\mu_1}$ respectively.

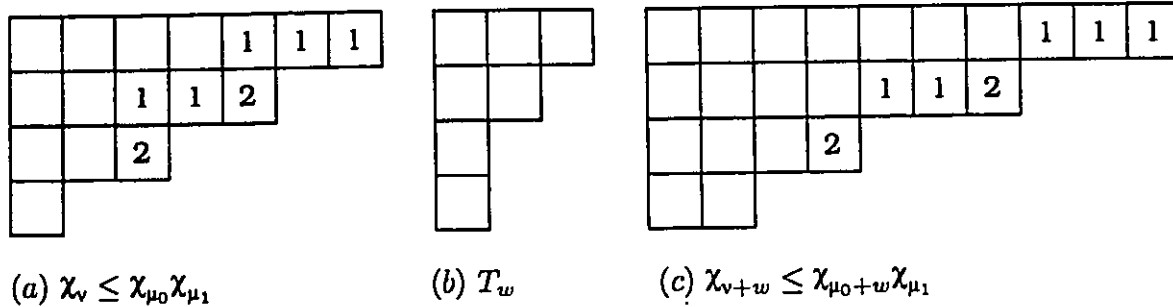


Figure 5(C)

The tableau T' obtained in the above example automatically satisfied MLR2, since we added only blank boxes to the tableau T . The next Lemma gives conditions for adding boxes with entries to the tableau T , so that T' satisfies MLR2.

Lemma 5.3. Suppose that T satisfies MLR2. If for each $i \geq 1$ and $j \geq 2$, we only add a box with entry (i, j) to row m of T after adding a box with entry $(i, j - 1)$ to row $m' < m$ of T , then T' still satisfies MLR2.

Proof: For each fixed i the sequence w_i consists of the second co-ordinates of the entries in T , read from right to left, top to bottom. By adding boxes according to the rules given above, we ensure that, for $j = 2, 3, \dots, d$, the number of $j - 1$'s in this sequence is increased by 1 before the number of j 's is, so that T' still satisfies MLR2. *

Example: Before giving the proof of Proposition 5.1, we will use Lemmas 5.2 and 5.3 to prove a special case of this result. Figure 5(D) illustrates the fact that $\chi_{\mu} \chi_{\lambda} \leq \chi_{\mu+\lambda} \chi_{\lambda+\lambda_2}$ for characters of A_2 with dominant weights $\mu := 2\lambda_1$ and $\lambda := 2\lambda_2$.

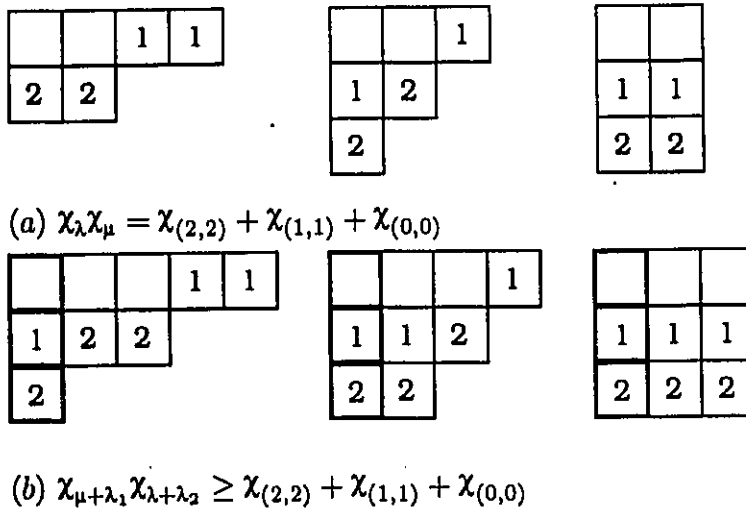


Figure 5(D)

In this example, we added to each tableau in (a) the same sequence of boxes in the same rows (i.e. a blank box to row one, a box with entry 1 to row two, and a box with entry 2

to row three) to obtain the corresponding tableau in (b). In fact Lemmas 5.2 and 5.3 together imply that, if we are given any tableau T_ν representing an irreducible character χ_ν in an arbitrary product $\chi_\mu\chi_\lambda$ of irreducible characters of A_2 , and we make these same additions, then the resulting tableau T'_ν represents χ_ν in the product $\chi_{\mu+\lambda_1}\chi_{\lambda+\lambda_2}$. In other words, we have shown that, for irreducible characters of A_2 , $\chi_\mu\chi_\lambda \leq \chi_{\mu+\lambda_1}\chi_{\lambda+\lambda_2}$. We can easily extend this example to show that $\chi_\lambda\chi_\mu \leq \chi_{\lambda+\lambda_d}\chi_{\mu+\lambda_1}$ for irreducible characters of A_d : add to any tableau representing an irreducible character in the product $\chi_\lambda\chi_\mu$ a blank box to row one, a box with entry 1 to row two, and so on, up to a box with entry d to row $d+1$ to obtain a tableau representing the same irreducible character in $\chi_{\lambda+\lambda_d}\chi_{\mu+\lambda_1}$.

More generally, the results of Lemmas 5.2 and 5.3 may be combined in the following Corollary.

Let A be the array whose rows are left justified and whose i^{th} row consists of the entries added to row i of T , arranged in (lexicographically) nondecreasing order. (In case the same number of boxes is added to each row of T , we can let A be a matrix.)

Corollary 5.4. Suppose that T satisfies MLR1 and MLR2. If the entries of A increase strictly along columns after the first nonzero entry, and if for each $i \geq 1$ and $j \geq 2$ the pair (i, j) lies in a lower row than the entry $(i, j-1)$, then T' satisfies both MLR1 and MLR2.

Proof of Proposition 5.1: Suppose that χ_ν is an irreducible character appearing in the product $\chi_{\mu_0}\chi_{\mu_1}\cdots\chi_{\mu_t}$. Then there is a tableau T_ν of shape $\nu \setminus \mu_0$ satisfying the conditions MLR 1 and MLR 2; T_ν has a base of shape μ_0 and for $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, d$, there are $m_{i_j} + m_{i_{j+1}} + \cdots + m_{i_d}$ boxes in T_ν with entry (i, j) . We say that T_ν represents the irreducible character χ_ν in the product $\chi_{\mu_0}\chi_{\mu_1}\cdots\chi_{\mu_t}$. We will construct from T_ν a tableau T'_ν representing χ_ν in the product $\chi_{\mu_0+\lambda_{i_0}}\chi_{\mu_1+\lambda_{i_1}}\cdots\chi_{\mu_t+\lambda_{i_t}}$. In order to achieve this we must add boxes to T_ν so that the new base has shape $\mu_0 + \lambda_{i_0}$ —that is, we must add an empty box to each of rows 1 to i_0 —and for $i = 1, 2, \dots, t$, there are $m_{i_j} + \cdots + m_{i_d} + 1$ boxes with entry (i, j) if $j \in \{1, 2, \dots, i_i\}$, and $m_{i_j} + \cdots + m_{i_d}$ boxes with entry (i, j) if $j \in \{i_{i+1}, \dots, d\}$. We must add the same number of boxes to each row of T_ν , and they must be added in such a way that T'_ν still satisfies MLR 1 and MLR 2.

Suppose that $\sum_{j=0}^t i_j = N(d+1)$. Then we must add N boxes to each of rows one to $d+1$ of T_ν . The $N(d+1)$ boxes must have entries in the set

$$S = \{(0, 0), \dots, (0, 0); (1, 1), (1, 2), \dots, (1, i_1); (2, 1), (2, 2), \dots, (2, i_2); \dots; (t_1), (t_2), \dots, (t, i_t)\}$$

and must be added to T_ν so that the $d+1 \times N$ matrix A satisfies the conditions of Corollary 5.4.

Begin filling A by putting the i_0 entries $(0,0)$ into rows 1 to i_0 of column one. These correspond to blank boxes, so this ensures that the base of T'_ν has shape $\mu_0 + \lambda_{i_0}$. If $i_0 + i_1 \leq d+1$, then fill the next i_1 rows of column one with entries $(1, 1), (1, 2), \dots, (1, i_0)$. Continue adding elements of S to column one until $k_j := i_0 + i_1 + i_2 + \cdots + i_j - (d+1) > 0$. Then fill the first k_j rows of column two with entries $(j, 1), (j, 2), \dots, (j, k_j)$ and the last $i_j - k_j$ rows of column one with the entries $(j, k_j + 1), \dots, (j, i_j)$. Since $k_j \leq i_j \leq d$, there will always be enough space in a single column for the entries $(j, 1), (j, 2), \dots, (j, k_j)$. We continue to fill the matrix A from top to bottom, left to right with the elements in the set S (in increasing order) ensuring that, for a given $j \in \{1, 2, \dots, t\}$ either all the entries (j, k) lie in a single column of A , or if there are only $i_j - k_j < i_j$ spaces remaining in the leftmost column m of A being filled, we put the pairs $(j, 1), (j, 2), \dots, (j, k_j)$ in

the first k_j rows of column $m + 1$, and the pairs $(j, k_j + 1), \dots, (i_j)$ into the last $i_j - k_j$ rows of column m . *

Examples of A in some special cases.

1. $\chi_\lambda^{d+1} \leq \chi_{\lambda+\lambda_1}^{d+1}$:
 A is the $d + 1 \times 1$ matrix

$$\begin{pmatrix} (0, 0) \\ (1, 1) \\ \vdots \\ (d, 1) \end{pmatrix}.$$

2. $\chi_\lambda^{d+1} \leq \chi_{\lambda+\lambda_2}^{d+1}$:
 If $d + 1 = 2n$, then A is the $2n \times 2$ matrix

$$\begin{pmatrix} (0, 0) & (n, 1) \\ (0, 0) & (n, 2) \\ \vdots & \vdots \\ (n-1, 1) & (2n, 1) \\ (n-1, 2) & (2n, 2) \end{pmatrix}.$$

If $d + 1 = 2n + 1$, then A is the $(2n + 1) \times 2$ matrix

$$\begin{pmatrix} (0, 0) & (n, 1) \\ (0, 0) & (n+1, 1) \\ (1, 1) & (n+1, 2) \\ \vdots & \vdots \\ (n-1, 1) & (2n-1, 2) \\ (n-1, 2) & (2n, 1) \\ (n, 2) & (2n, 2) \end{pmatrix}.$$

2. Restricted Inequalities

There are two basic types of comparisons considered in this section. We compare $\chi_{\mu_1} \cdots \chi_{\mu_n}$ with products of the form $\chi_{\mu_1 - k\lambda_i} \chi_{\mu_2 + \lambda_{i+1}} \cdots \chi_{\mu_n + \lambda_{i+1}}$ (Lemma 5.5 and Lemmas 5.6–5.8), and with products of the form $\chi_{\mu_1 - \lambda_i + \lambda_{i+1}} \chi_{\mu_2 + \lambda_{i+1}} \cdots \chi_{\mu_n + \lambda_{i+1}}$ (Lemma 5.9 and Lemmas 5.10–5.12). Lemma 5.5 and Lemma 5.9 hold for irreducible characters in any rank; the other results apply to special cases in rank 4 and 5. Before discussing these, we need to describe the tableaux corresponding to the faces $F^j(\lambda)$.

Beginning with a Young diagram T_λ of shape λ , we construct, for any simple root α_j , a diagram of shape $\lambda - \alpha_j$ by moving a box from row j to row $j + 1$. By repeating this process, we can construct a diagram T_w from T_λ for any dominant weight $w \prec \lambda$. A diagram of shape $w \in F^1(\lambda)$ constructed in this way will have the same number of boxes in row 1 as does T_λ , since the difference $\lambda - w$ is a sum of simple roots other than α_1 . If $w \in F^j(\lambda)$, then T_w is formed without moving any box in T_λ from row j to row $j + 1$. Thus the number of boxes in rows 1 to j of T_w equals the number of boxes in rows 1 to j of T_λ . Also, the number of boxes in row j of

row 3 of T'_w are those with entry $(i, 3)$. In any column then, the entry in row 2 is strictly less than that in row 3. It follows that T'_w satisfies MLR1 as well.

Now form the tableau T''_w by adding to T'_w the three boxes $(1,1)$, $(2,1)$, and $(3,1)$ to row 1, boxes $(1,2)$, $(2,2)$, and $(3,2)$ to row 2, and then boxes $(1,3)$, $(2,3)$, $(3,3)$, and $(4,1)$ respectively to rows 3 to 6 (one to each row). (See Figure 5(G))

By Corollary 5.4, T''_w satisfies MLR1 and MLR2; it therefore represents χ_w in the product $\chi_{\mu_0+2\lambda_2}\chi_{\mu_1+\lambda_3}\chi_{\mu_2+\lambda_3}\chi_{\mu_3+\lambda_3}\chi_{\mu_4+\lambda_1}$. *

In figure 5(G) below, $\mu_0 := \lambda_1 + 2\lambda_2 + 3\lambda_3$, $\mu_1 := \lambda_1 + \lambda_3$, $\mu_2 := 2\lambda_1 + \lambda_2 + \lambda_3$, $\mu_3 := \lambda_1$, and $\mu_4 := \lambda_1 + 2\lambda_2 + \lambda_3$.

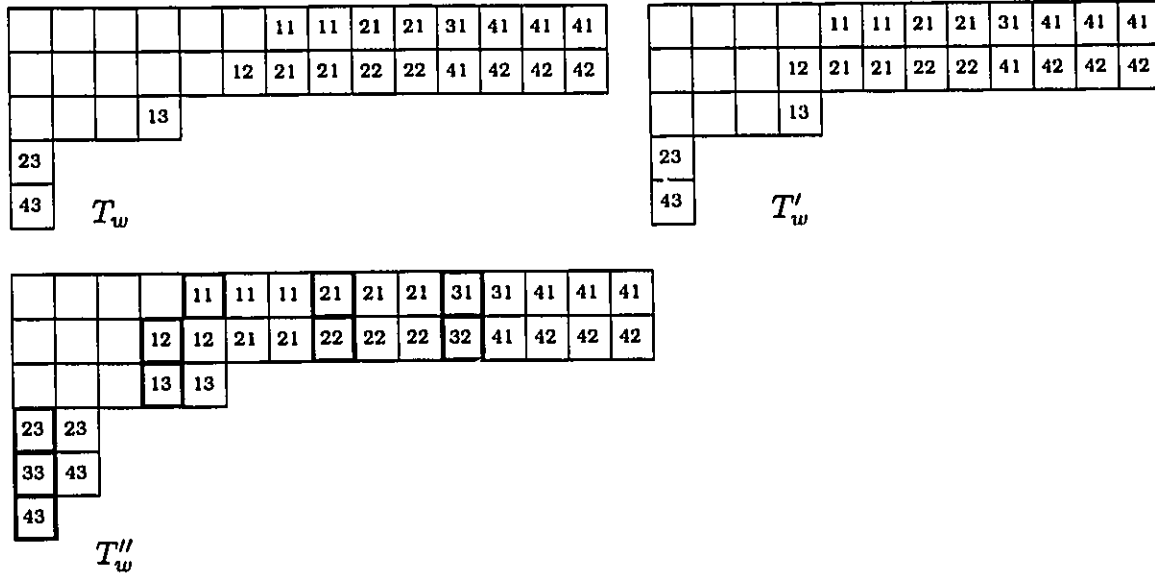


Figure 5(G)

For Lemma 5.7, suppose that $\mu_4 = 0$, and $m_{01} > 0$.

Lemma 5.7. If $\chi_w \leq \chi_{\mu_0}\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3}$ and $w \in F^1(\nu)$ then

$$\chi_w \leq \chi_{\mu_0-\lambda_1}\chi_{\mu_1+\lambda_2}\chi_{\mu_2+\lambda_2}\chi_{\mu_3+\lambda_3}.$$

Proof: Let T_w be a tableau representing χ_w in the product $\chi_{\mu_0} \cdots \chi_{\mu_3}$, and let T'_w be the tableau obtained by removing a single blank box from row 1 of T_w . Since $w \in F^1(\nu)$, the tableau T'_w satisfies both MLR1 and MLR2.

Construct the tableau T''_w by adding to T'_w boxes $(1,1)$, and $(2,1)$ to row 1, and the boxes $(1,2)$, $(2,2)$, $(3,1)$, $(3,2)$, and $(3,3)$ to rows 2 to 6 respectively. By Lemma 5.4, T''_w satisfies MLR1 and MLR2, and therefore represents χ_w in the product $\chi_{\mu_0-\lambda_1}\chi_{\mu_1+\lambda_2}\chi_{\mu_2+\lambda_2}\chi_{\mu_3+\lambda_3}$. *

In Lemma 5.8, we assume that $m_{i1} = 0$ for $i = 1, 2, 3, 4$ and $m_{02} \geq 2$.

Lemma 5.8. If $\chi_w \leq \chi_{\mu_0} \chi_{\mu_1} \cdots \chi_{\mu_4}$ and $w \in F^3(\nu)$ then

$$\chi_w \leq \chi_{\mu_0 - 2\lambda_3} \chi_{\mu_1 + \lambda_4} \chi_{\mu_2 + \lambda_4} \chi_{\mu_3 + \lambda_2} \chi_{\mu_4 + \lambda_2}.$$

Proof: Given a tableau representing χ_w in the product $\chi_{\mu_0} \cdots \chi_{\mu_4}$, let T'_w be the tableau formed by removing 2 blank boxes from each of rows 1 to 3 of T_w . Then T'_w satisfies MLR2, and since $w \in F^3(\nu)$, T'_w satisfies MLR1 as well. We construct a tableau T''_w of shape w by adding to T'_w boxes (1,1), (2,1), and (3,1) to row 1; boxes (1,2), (2,2), (3,2) to row 2; boxes (1,3), (2,3), (4,1) to row 3; and boxes (1,4), (2,4), and (4,2) to rows 4 to 6 respectively. Then T''_w satisfies MLR1 and MLR2 by Corollary 5.4, and represents χ_w in $\chi_{\mu_0 - 2\lambda_3} \chi_{\mu_1 + \lambda_4} \chi_{\mu_2 + \lambda_4} \chi_{\mu_3 + \lambda_2} \chi_{\mu_4 + \lambda_2}$. *

Suppose that $d = 2t$. For $i = 0, 1, \dots, t$ let $\mu_i = m_{i1}\lambda_1 + m_{i2}\lambda_2$ be dominant weights of A_d such that $m_{01} > 0$. Set $\nu = \mu_0 + \cdots + \mu_t$.

Lemma 5.9. Suppose that $\chi_w \leq \chi_{\mu_0} \chi_{\mu_1} \cdots \chi_{\mu_t}$.

(i) If $w \in F^1(\nu)$, then $\chi_w \leq \chi_{\mu_0 - \lambda_1 + \lambda_2} \chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_t + \lambda_2}$;

(ii) If $m_{01} = 1$, $m_{i1} = 0$ for $i \geq 1$, and $w \in F^d(\nu)$ then $\chi_w \leq \chi_{\mu_0 - \lambda_1 + \lambda_2} \chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_t + \lambda_2}$.

Proof: Let T_w be a tableau representing χ_w in the product $\chi_{\mu_0} \cdots \chi_{\mu_t}$ and form the tableau T'_w by adding to T_w a box with entry (1,1) to row 1, a blank box to row 2, and boxes with entries (1, 2), (2, 1), (2, 2) \dots , (k, 2) to rows 3 to d+1 respectively (one to each row). Then T'_w has shape w , and it satisfies MLR2 by Lemma 5.3. By Lemma 5.2, the entries in rows 2 to d+1 of T'_w increase strictly down columns.

(i) We will show that T'_w also satisfies MLR1, and so represents χ_w in the product given in (i). If $w \in F^1(\nu)$, all boxes with entry $(i, 1)$ in T_w belong to row 1, and the only entries in row 2 of T_w or T'_w are those of the form $(i, 2)$. In each column therefore, the entry in row 1 is strictly less than that in row 2, so MLR1 holds.

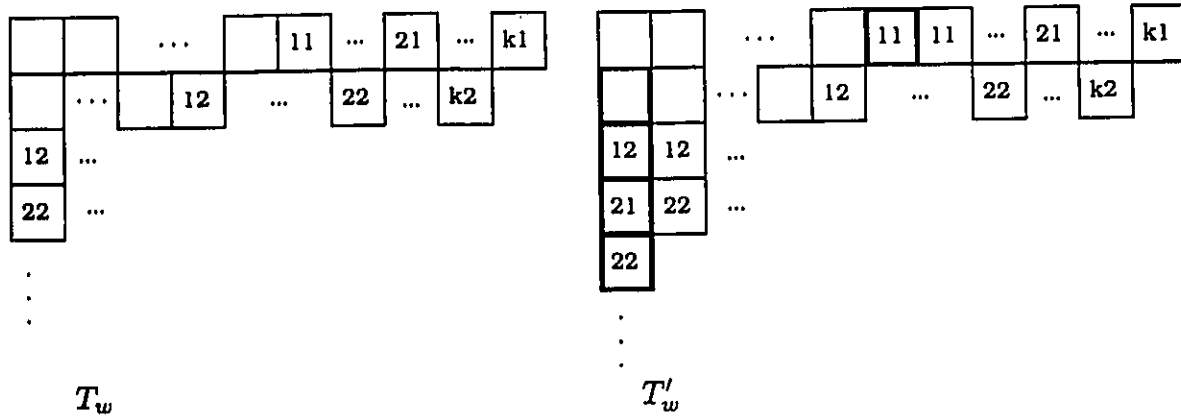


Figure 5(H)

(ii) If $m_{01} = 1$ then the base of T_w consists of $m_{02} + 1$ blank boxes in row 1 and m_{02} blank boxes in row 2; the base of T'_w consists of $m_{02} + 1$ blank boxes in rows 1 and 2. We will consider the entries in rows 1 and 2, column $k \geq m_{02} + 2$ of T'_w .

If $k > m_{02} + 2$, then the entries in rows 1 and 2, column k of T'_w equal the entries in rows 1 and 2, column $k - 1$ of T_w ; the entry in row 1 is therefore strictly less than that in row 2.

The entry in row 1 column $m_{02} + 2$ of T'_w equals (1,1); the entry in row 2 column $m_{02} + 2$ of T'_w equals that in row 2 column $m_{02} + 1$ of T_w . If this entry does not equal (1,1), then it is strictly greater than (1,1), and T'_w satisfies MLR1. (See Figure 5(I))

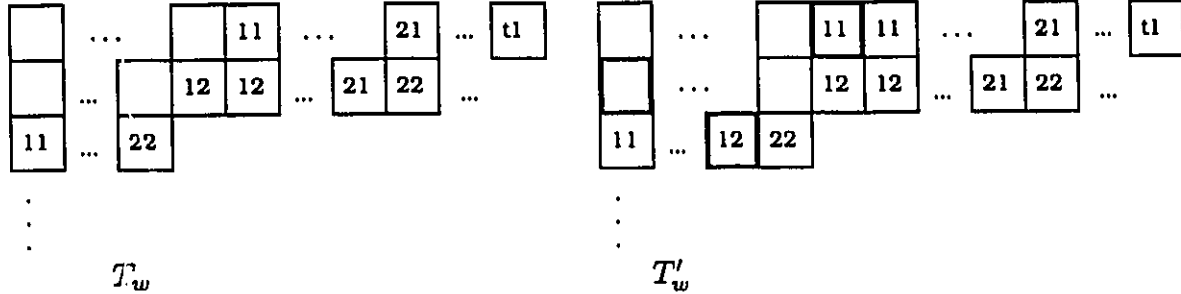


Figure 5(I)

If the entry in row 2 column $m_{02} + 2$ of T'_w equals (1,1) then we proceed as follows. (The reader may wish to refer to the example in Figure J, on the following page.) We note that there is at least one entry (1,2) in row 3 of T'_w , since we added one to row 3 of T_w in the first step; the leftmost entry (1,2) in row 3 of T'_w lies in column $l < m_{02} + 2$. (Since there is a (1,1) in row 2 column $m_{02} + 1$ of T_w , any entry (1,1) in row 3 of T_w or T'_w lies in a column strictly to the left of $m_{02} + 1$, so the leftmost entry (1,2) in row 3 lies in column $l \leq m_{02} + 1$.) Form the tableau T''_w by exchanging the entry (1,1) in row 2 column $m_{02} + 2$ of T'_w with the leftmost entry (1,2) in row 3. (See Figure J(c))

We observe first that the entries in columns $k \neq m_{02} + 2$ of T''_w increase strictly. Those in columns $k > m_{02} + 2$ have not changed; the only change in the entries in columns $k < m_{02} + 2$ is that the entry (1,2) in row 3 column l has been replaced by (1,1). The latter is strictly less than the entry in row 4, column l and lies below a blank box in row 2 of T''_w , so entries in column l still increase strictly. If there are no entries (1,2) in row 3 of T''_w , then the entry in row 3, column $m_{02} + 2$ is strictly greater than (1,2), so T''_w satisfies MLR1. The absence of entries (1,2) in row 3 of T''_w also implies that MLR2 is satisfied so T''_w represents χ_w in (ii).

If T''_w does not satisfy both MLR1 and MLR2, then we form $T_w^{(3)}$ as follows. Observe that there is at least one entry (2,1) in row 4 of T''_w , since we added one to T_w in the first step; the leftmost entry (2,1) in row 4 lies in column $p \leq l$ (since the leftmost (1,2) in row 3 of T''_w lies in column l). We form $T_w^{(3)}$ by exchanging the rightmost entry (1,2) in row 3 of T''_w (which lies in column $p' \geq l$, strictly to the right of column p) with the leftmost entry (2,1) in row 4 of T''_w . Then $T_w^{(3)}$ satisfies MLR2, and it will satisfy MLR1 unless there is a box with entry (2,1) in row 4, column p' . (Figure J(d))

If there is a box with entry (2,1) in row 4, column p' of $T_w^{(3)}$, we exchange this entry with the leftmost entry (2,2) in row 5. (Again there is at least one, and it lies to the left of the (2,1) in row 4.) Call the resulting tableau $T_w^{(4)}$. If this tableau does not satisfy both MLR1 and MLR2, then we continue this process until we obtain a tableau satisfying MLR1 and MLR2, or until we arrive at $T_w^{(d-1)}$, obtained by exchanging the rightmost entry $(t-1, 2)$ in row $d-1$ with the leftmost entry $(t, 1)$ in row d . Then $T_w^{(d-1)}$ satisfies MLR2. Since $w \in F^d(v)$, there are no boxes in row $d+1$ of T_w . The hypothesis $m_{i1} = 0$ for $i > 0$ implies that there are no boxes with entry $(i, 1)$ in row d

of T_w , so there are no boxes with entry $(i, 1)$ remaining in row d of $T_w^{(d-1)}$. The latter therefore satisfies MLR1 as well, and so represents χ_w in the product $\chi_{\mu_0 - \lambda_1 + \lambda_2} \chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_t + \lambda_2}$. *

Example:

Here $d = 6$ and we consider the product $\chi_{\mu_0} \chi_{\mu_1} \chi_{\mu_2} \chi_{\mu_3}$, where $\mu_0 := 4\lambda_2 + \lambda_1$, $\mu_1 := 5\lambda_2$, $\mu_2 := 5\lambda_2$, and $\mu_3 := 2\lambda_2$.

| | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | 11 | 12 | 21 | 22 | | | |
| 11 | 11 | 12 | 12 | 22 | 22 | | | | |
| 12 | 12 | 21 | 21 | | | | | | |
| 22 | 22 | 32 | | | | | | | |
| 32 | | | | | | | | | |
| | | | | | | | | | |

(a) T_w

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | | 11 | 12 | 21 | 22 | | | |
| 11 | 11 | 12 | 12 | 12 | 22 | 22 | | | | |
| 12 | 12 | 21 | 21 | 21 | | | | | | |
| 22 | 22 | 22 | 32 | | | | | | | |
| 31 | 32 | | | | | | | | | |
| 32 | | | | | | | | | | |

(b) T'_w

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | | 12 | 12 | 21 | 22 | | | |
| 11 | 11 | 11 | 12 | 12 | 22 | 22 | | | | |
| 12 | 12 | 21 | 21 | 21 | | | | | | |
| 22 | 22 | 22 | 32 | | | | | | | |
| 31 | 32 | | | | | | | | | |
| 32 | | | | | | | | | | |

(c) T''_w

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | | 12 | 12 | 21 | 22 | | | |
| 11 | 11 | 11 | 12 | 21 | 22 | 22 | | | | |
| 12 | 12 | 12 | 21 | 21 | | | | | | |
| 22 | 22 | 22 | 32 | | | | | | | |
| 31 | 32 | | | | | | | | | |
| 32 | | | | | | | | | | |

(d) T_w^3

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | | 12 | 12 | 21 | 22 | | | |
| 11 | 11 | 11 | 12 | 21 | 22 | 22 | | | | |
| 12 | 12 | 12 | 21 | 22 | | | | | | |
| 21 | 22 | 22 | 32 | | | | | | | |
| 31 | 32 | | | | | | | | | |
| 32 | | | | | | | | | | |

(e) T_w^4

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| | | | | 11 | 11 | 11 | 21 | 21 | 31 | 31 |
| | | | | 12 | 12 | 21 | 22 | | | |
| 11 | 11 | 11 | 12 | 21 | 22 | 22 | | | | |
| 12 | 12 | 12 | 21 | 22 | | | | | | |
| 21 | 22 | 31 | 32 | | | | | | | |
| 22 | 32 | | | | | | | | | |
| 32 | | | | | | | | | | |

(f) T_w^5

Figure 5(J)

In the remaining lemmas we consider characters of A_4 and A_5 ; the results (and proofs) are similar to those of Lemma 5.9.

For $i = 0, 1, 2$, let $\mu_i = m_{i1}\lambda_1 + m_{i2}\lambda_2 + m_{i3}\lambda_3$ be dominant weights of A_4 and suppose that $m_{02} > 0$. Define $\nu := \mu_0 + \mu_1 + \mu_2$.

Lemma 5.10. Suppose that $\chi_w \leq \chi_{\mu_0}\chi_{\mu_1}\chi_{\mu_2}$. Then $\chi_w \leq \chi_{\mu_0-\lambda_2+\lambda_3}\chi_{\mu_1+\lambda_3}\chi_{\mu_2+\lambda_1}$ provided any of the following conditions holds:

- (i) $w \in F^2(\nu)$;
- (ii) $m_{02} - 1 = m_{12} = m_{22} = 0$ and $w \in F^1(\nu)$;
- (iii) $m_{02} - 1 = m_{12} = m_{22} = 0$, $m_{i1} = 0$ for all i , and $w \in F^4(\nu)$.

Proof: Let T_w be a tableau representing χ_w in the product $\chi_{\mu_0}\chi_{\mu_1}\chi_{\mu_2}$. Construct the tableau T'_w from T_w by adding a box with entry (1,1) to row 1, a box with entry (1,2) to row 2, a blank box to row 3, and boxes with entries (1,3) and (2,1) to rows 4 and 5 respectively. Then T'_w satisfies MLR2 by Lemma 5.3, and the entries in rows 1 and 2 and in rows 3 to 5 increase strictly down columns, by Lemma 5.2.

- (i) If w lies on $F^2(\nu)$ then the only nonempty boxes in row 3 are those with entries $(i, 3)$ and all boxes with entries with entries $(i, 2)$ lie in row 2. In any column, the entry in row 2 of T'_w is therefore strictly less than that in row 3, so T'_w satisfies MLR1.
- (ii) The tableau T_w has $m_{03} + 1$ blank boxes in row 2 and m_{03} blank boxes in row 3. Since $w \in F^1(\nu)$ the only possible entries in row 3 column $m_{03} + 1$ of T_w are (1,2) or (1,3).

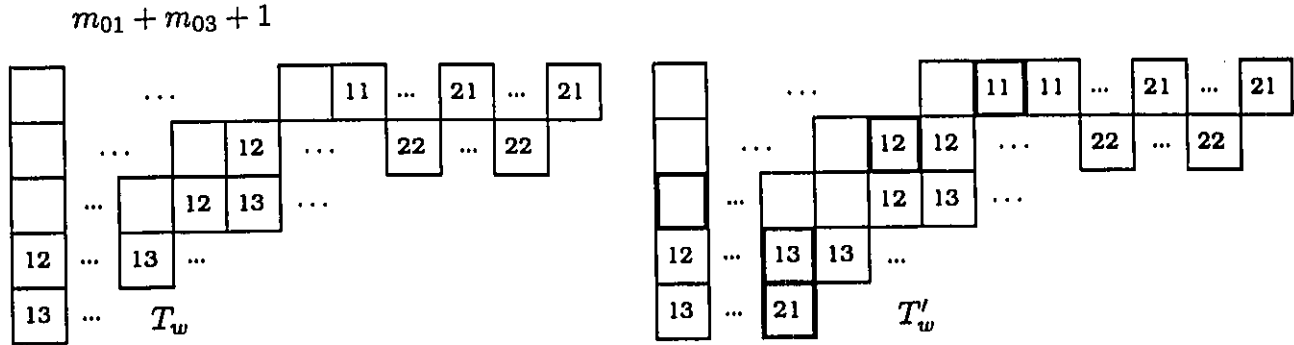


Figure 5(K)

In the latter case T'_w satisfies MLR1. In the former, we let T''_w be the tableau which results when we exchange the entry (1,2) in row 3, column $m_{03} + 2$ of T'_w with the leftmost entry (1,3) in row 4. (The latter lies in column $l \leq m_{03} + 1$ since the rightmost entry (1,2) in this row lies to the left of column $m_{03} + 1$.) If T''_w does not satisfy MLR1 and MLR2, then we form $T^{(3)}$ by exchanging the rightmost (1,3) in row 4 of T''_w with the leftmost entry (2,1) in row 5. Then $T^{(3)}$ satisfies MLR2; since $w \in F^1(\nu)$ there are no boxes with entry (2,1) remaining in row 5 of $T^{(3)}$, so MLR1 is satisfied as well.

(iii) In this case, the tableau T_w has $m_{03} + 1$ blank boxes in rows 1 and 2, and m_{03} blank boxes in row 3. Since $w \in F^4(\nu)$, there are no boxes in row 5 of T_w ; the condition MLR2 implies that there are no boxes with entry $(i, 1)$ below row 2, and no boxes with entry $(i, 2)$ below row 3. (See Figure 5(L).)

The tableau T'_w will satisfy MLR1 unless there is an entry (1,2) in row 3 column $m_{03} + 2$ of T'_w . If there is, then we construct T''_w and if necessary $T^{(3)}$ as in part (ii).*

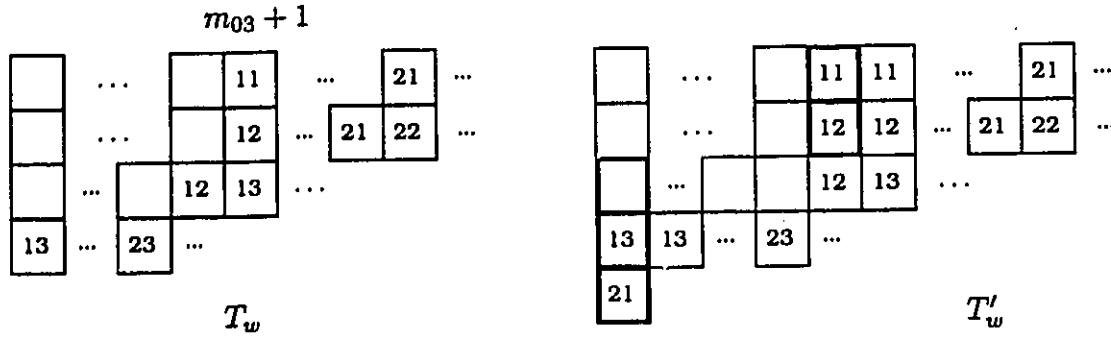


Figure 5(L)

Let $\mu_i = m_{i1}\lambda_1 + m_{i2}\lambda_2 + m_{i3}\lambda_3 + m_{i4}\lambda_4$ be dominant weights of A_5 for $i = 0, 1, 2$, and set $\nu := \mu_1 + \mu_2 + \mu_3$. Suppose for the next Lemma that $m_{02} > 0$.

Lemma 5.11. If $\chi_w \leq \chi_{\mu_0}\chi_{\mu_1}\chi_{\mu_2}$, then $\chi_w \leq \chi_{\mu_0-\lambda_2+\lambda_3}\chi_{\mu_1+\lambda_4}\chi_{\mu_2+\lambda_1}$ provided any of the following conditions holds:

- (i) $w \in F^2(\nu)$;
- (ii) $m_{02} - 1 = m_{12} = m_{22} = 0$ and $w \in F^1(\nu)$;
- (iii) $m_{02} - 1 = m_{12} = m_{22} = 0$, $m_{i1} = m_{i3} = 0$ for all i , and $w \in F^5(\nu)$.

Proof: If T_w is a tableau representing χ_w in the product $\chi_{\mu_0}\chi_{\mu_1}\chi_{\mu_2}$, let T'_w be the tableau formed by adding a box with entry (1,1) to row 1, a box with entry (1,2) to row 2, a blank box to row 3, and boxes with entries (1,3), (1,4), and (2,1) to rows 4,5, and 6 respectively. It follows from Lemma 5.3 that T'_w satisfies MLR2, and from Lemma 5.2 that the entries in rows 1 and 2, and in rows 3 to 6 increase strictly down columns.

(i) If $w \in F^2(\nu)$ then in T_w and T'_w the only entries in row 3 are those of the form $(i, 3)$, which implies that in any column, the entry in row 2 is strictly less than that in row 3. Thus T'_w satisfies MLR1, and represents χ_w in the product $\chi_{\mu_0-\lambda_2+\lambda_3}\chi_{\mu_1+\lambda_4}\chi_{\mu_2+\lambda_1}$.

(ii) The tableau T_w has $m_{03} + m_{04} + 1$ blank boxes in row 2 and $m_{03} + m_{04}$ blank boxes in row 3. Since $w \in F^1(\nu)$, the tableau T'_w will satisfy MLR1 unless the entry in row 3, column $m_{03} + m_{04} + 2$ equals (1,2).

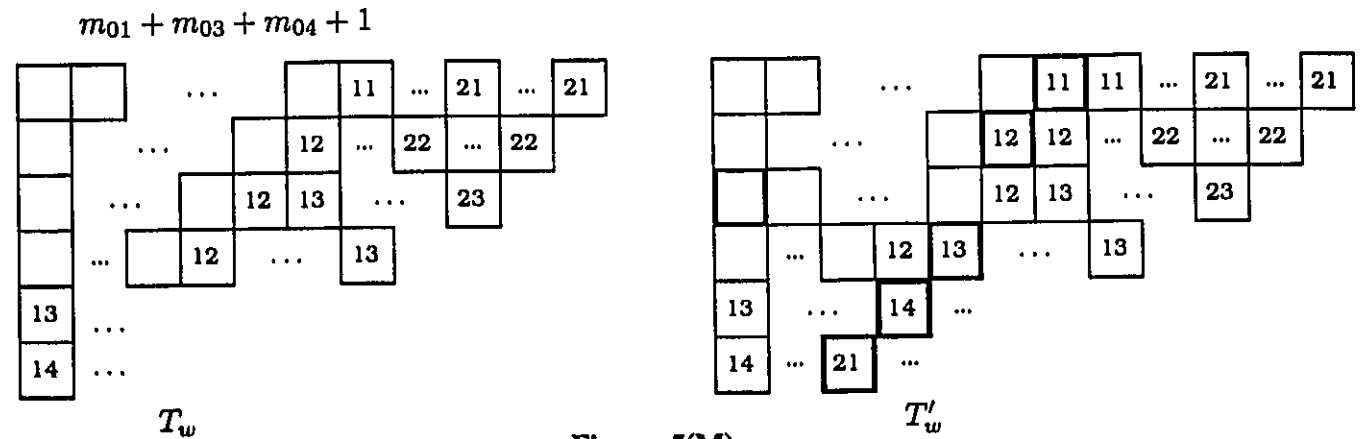


Figure 5(M)

If T'_w does not satisfy MLR1, let T''_w be the tableau obtained by exchanging the (1,2) in row 3 column $m_{03} + m_{04} + 2$ of T'_w with the leftmost entry (1,3) in row 4. (It is easy to check that

the latter lies in column $l < m_{03} + m_{04} + 2$.) If there are no entries (1,3) in row 4 of T_w'' , then it satisfies MLR1 and MLR2, and we are done.

If T_w'' does not satisfy both MLR1 and MLR2, form $T_w^{(3)}$ by exchanging the rightmost entry (1,3) in row 4 of T_w'' with the leftmost entry (1,4) in row 5. If one of MLR1 or MLR2 does not hold then there is at least one entry (1,4) in row 5 of $T_w^{(3)}$. Let $T_w^{(4)}$ be the tableau obtained by exchanging the rightmost entry (1,4) in row 5 of $T_w^{(3)}$ with the leftmost (2,1) in row 6. Then $T_w^{(4)}$ satisfies MLR2, and since $w \in F^1(\nu)$, there are no entries (2,1) in row 6 of $T_w^{(4)}$, so MLR1 is satisfied as well.

(iii) The tableau T_w has $m_{04} + 1$ blank boxes in rows 1 and 2, and m_{04} blank boxes in rows 3 and 4. Also, the number of entries (1, j) in T_w equals m_{14} for all j . Since $w \in F^5(\nu)$, this condition together with MLR2 implies that there are no boxes with entry (1,1) below row 2. The tableau T_w' will therefore satisfy MLR1 unless there is an entry (1,2) in row 3 column $m_{04} + 2$.

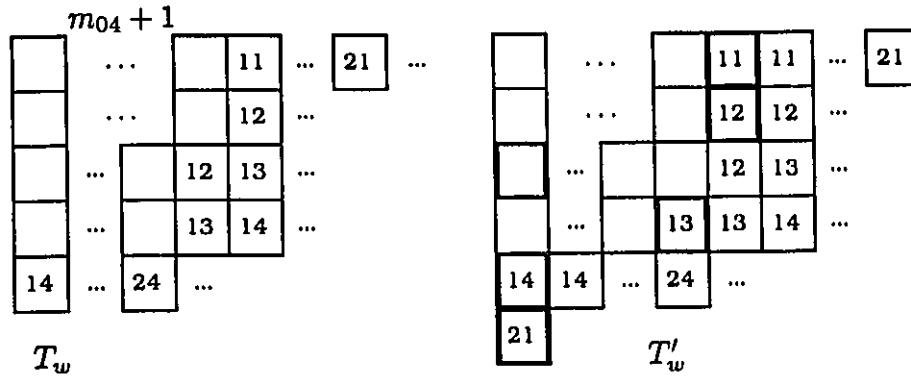


Figure 5(N)

If there is, then we proceed as in (ii). (In this case the condition $w \in F^5(\nu)$ implies that there are no boxes with entry (2,1) in row 6 of $T_w^{(3)}$.) *

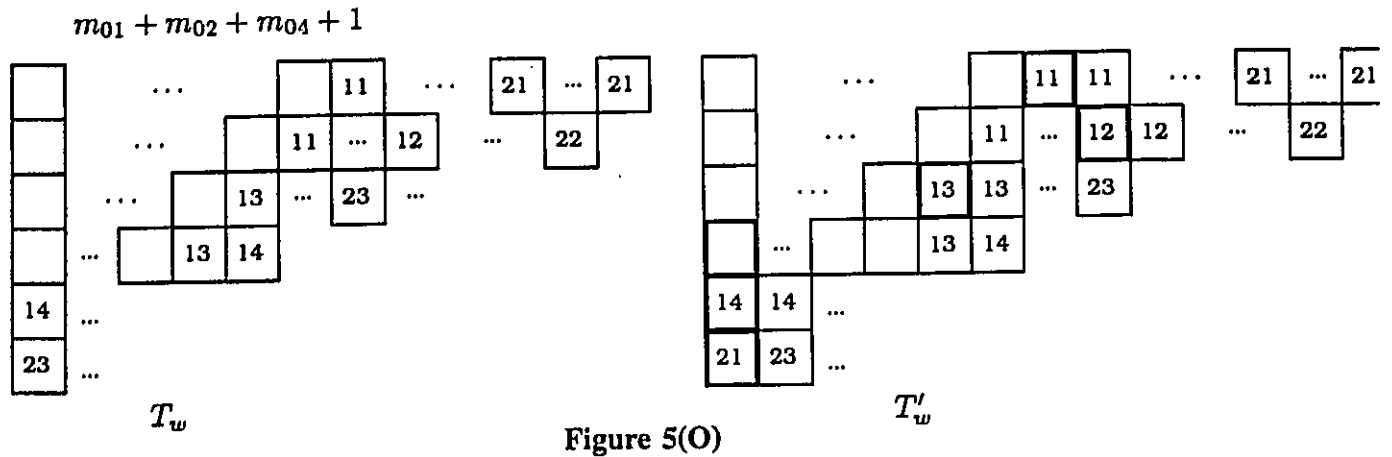
Finally, suppose that $m_{03} > 0$.

Lemma 5.12. If $\chi_w \leq \chi_{\mu_0} \chi_{\mu_1} \chi_{\mu_2}$ then $\chi_w \leq \chi_{\mu_0 - \lambda_3 + \lambda_4} \chi_{\mu_1 + \lambda_4} \chi_{\mu_2 + \lambda_1}$ provided any of the following conditions holds:

- (i) $w \in F^3(\nu)$;
- (ii) $m_{03} - 1 = m_{13} = m_{23} = 0$ and $w \in F^2(\nu)$;
- (iii) $m_{03} - 1 = m_{13} = m_{23} = 0$, $m_{i2} = 0$ for all i , and $w \in F^1(\nu) \cap F^5(\nu)$;
- (iv) $m_{03} - 1 = m_{13} = m_{23} = 0$, $m_{i1} = m_{i2} = 0$ for all i , and $w \in F^5(\nu)$.

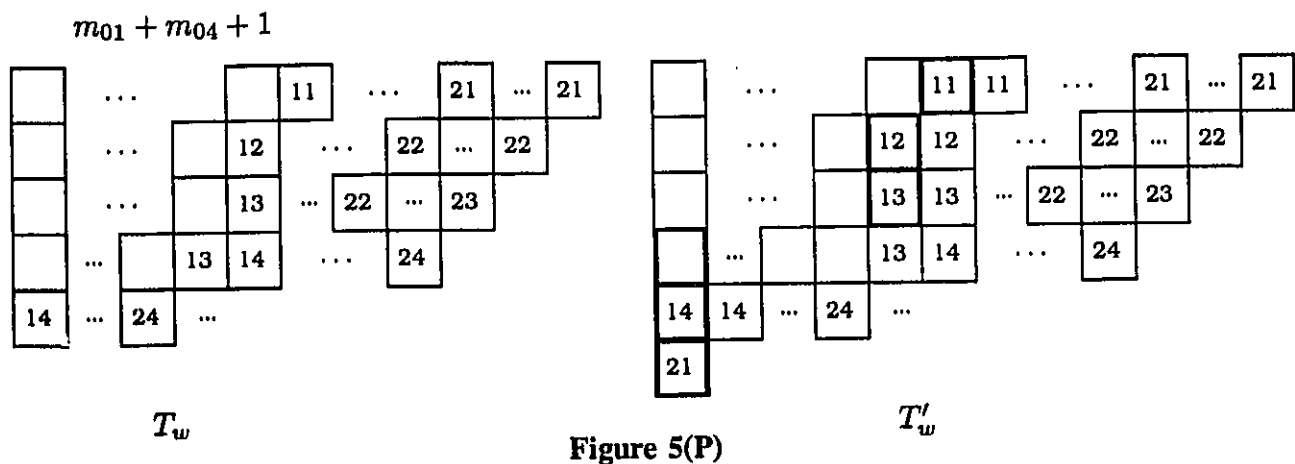
Proof: Given a tableau T_w representing χ_w in the product $\chi_{\mu_0} \chi_{\mu_1} \chi_{\mu_2}$, form T_w' by adding a box with entry (1,1) to row 1, a box with entry (1,2) to row 2, a box with entry (1,3) to row 3, a blank box to row 4, and boxes with entries (1,4) and (2,1) to rows 5 and 6 respectively. The tableau T_w' satisfies MLR2 and the entries in rows 1 to 3 and in rows 4 to 6 increase strictly along columns by Lemmas 5.3 and 5.2 respectively.

- (i) In row 4 of T_w , the only entries are those of the form $(i, 4)$, thus T_w' satisfies MLR1.
- (ii) In rows 3 and 4 of T_w there are respectively $m_{04} + 1$ and m_{04} blank boxes. Since $w \in F^2(\nu)$, all the entries $(i, 1)$ and $(i, 2)$ lie above row 3 of T_w . Thus T_w' will satisfy MLR1 unless there is a box with entry (1,3) in row 4, column $m_{04} + 2$. (See Figure 5(O).)



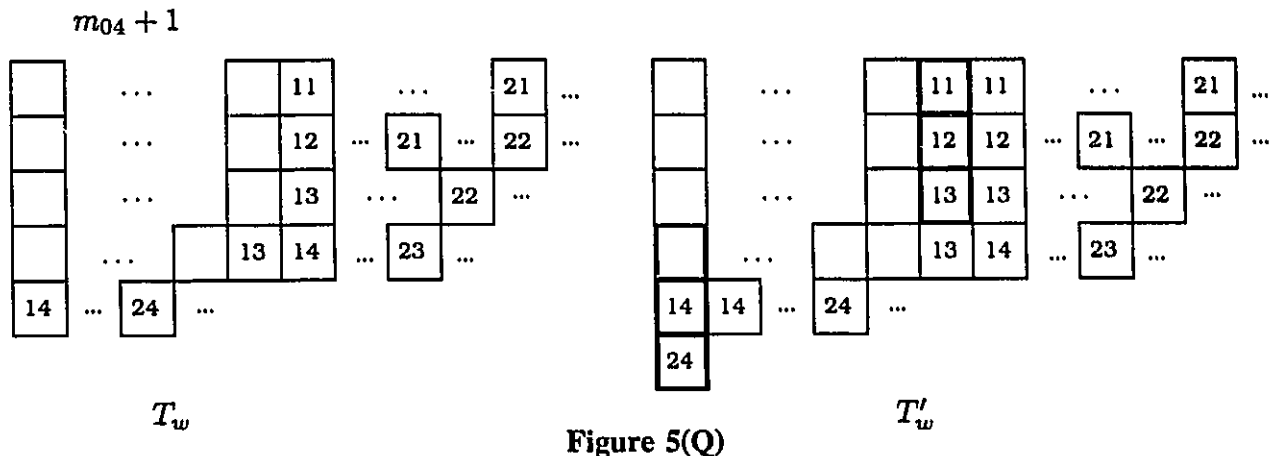
If T'_w does not satisfy MLR1, form the tableau T''_w by exchanging the (1,3) in row 4, column $m_{04} + 2$ with the leftmost (1,4) in row 5 of T'_w . (There is at least one such entry, by construction, and it lies in a column $l < m_{04} + 2$.) If T''_w does not satisfy MLR1 and MLR2, then there is at least one entry (1,4) in row 5; form $T_w^{(3)}$ by exchanging the rightmost (1,4) in row 5 of T''_w with the leftmost (2,1) in row 6. Then $T_w^{(3)}$ satisfies MLR2, and since there are no boxes (2,1) remaining in row 6, it satisfies MLR1 as well.

(iii) The tableau T_w has $m_{04} + 1$ blank boxes in rows 2 and 3, and m_{04} blank boxes in row 4, and all boxes with entry $(i, 1)$ belong to row 1. The conditions $m_{i2} = m_{i3} = 0$ for $i > 0$ and $w \in F^5(\nu)$ (and MLR2) imply that there are no boxes with entry $(i, 2)$ below row 3 of T_w . Thus T'_w will satisfy MLR1 unless the entry in row 4, column $m_{04} + 2$ equals (1,3).



If T'_w does not satisfy MLR1, then we proceed as in (ii), noting in this case that there are no entries (2,1) in row 6 of $T_w^{(3)}$ since $w \in F^1(\nu)$.

(iv) Here T_w has $m_{04} + 1$ blank boxes in rows 1, 2, and 3, and m_{04} blank boxes in row 4. The remaining conditions imply that there are no boxes with entry $(i, 1)$ below row 2, no boxes with entry $(i, 2)$ below row 3, and no boxes at all in row 6.



The tableau T'_w will satisfy MLR1 unless there is a box with entry (1,3) in row 4, column $m_{04} + 2$. If there is, then we argue as in (ii). *

Remark: Suppose that $\lambda = l_1\lambda_1 + \dots + l_n\lambda_n$ and $\mu = m_1\lambda_1 + \dots + m_k\lambda_k$ are dominant weights. We can consider them as weights of A_d for any $d \geq \max\{n, k\}$. Let $\mathcal{D}_{\lambda, \mu}$ be the set of tableaux (having at most $d + 1$ rows) obtained using the Littlewood-Richardson algorithm for the product $\chi_\lambda\chi_\mu$. Any tableau satisfying LR1 and LR2 has at most $n + k$ rows, hence the set of tableaux obtained is the same for all $d \geq n + k - 1$.

If $d \geq n + k$, then all the weights in $\mathcal{D}_{\lambda, \mu}$ lie on the face $F^d(\lambda + \mu)$ since all the boxes belong to rows 1 to $n + k$. It therefore follows from Proposition 4.6 that the decomposition of the product $\chi_\lambda\chi_\mu$ is "the same" for all algebras A_d with $d \geq n + k - 1$. More precisely, it is the same for $d \geq n + k$ once we identify the fundamental weights in all the algebras. (This was observed in [BBL; Corollary 7.2], where the authors also show that it holds for algebras of type B_d , C_d , and D_d .) If $d = n + k - 1$, the set $\mathcal{D}_{\lambda, \mu}$ differs from that for $d = n + k$ by a constant multiple of λ_d , and the corresponding multiplicities are equal (for A_d).

Chapter 6

A quantitative saturation result for characters of A_d

In this chapter, we use the results of Chapters 4 and 5 to show that if χ_λ is an irreducible character of A_d and $d \leq 5$ then χ_λ^n is saturated for $n \geq d + 1$ (Theorem 6.8). We also prove this result for certain irreducible characters of A_d when d is greater than 5. The strategy is to show the saturation of a more general form of product, which includes the products arising when Proposition 5.1 is applied to χ_λ^n . (For example, we consider products of the form $\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_d}^{n_2}$.) Using Proposition 4.6, we argue by induction on the rank of \mathcal{G} that saturation holds on the faces of the set of weights of the product. We then use Proposition 5.1 and induction on the total highest weight to show that most of the remaining irreducible characters appear. In case $d \leq 3$, or λ belongs to the interior of the Weyl chamber, we can account for all irreducible characters in this way. Otherwise, results from Section 2 of Chapter 5 are then used to account for the remaining irreducible characters.

One disadvantage of this approach is that we must first prove saturation for products of irreducible characters whose highest weights lie on a boundary of the Weyl chamber. There are also a number of minimal cases (where Proposition 5.1 does not apply) which have to be dealt with separately. As a consequence, the method becomes impractical when the rank of \mathcal{G} is greater than five.

Before we can apply the results of Chapter 5 to prove saturation, we need to describe the weights in $\Pi(\lambda)$ which do not lie on the dominant faces. When λ belongs to the interior of the Weyl chamber, a dominant weight $w \in \Pi(\lambda)$ either lies on a face $F^i(\lambda)$, or satisfies $w \prec \lambda - (\alpha_1 + \cdots + \alpha_d)$. If $\lambda = l_1\lambda_1 + \cdots + l_m\lambda_m$ lies on a boundary of the Weyl chamber, then we show in Lemma 6.1 that dominant weights w which do not lie on a face of $\Pi(\lambda)$ satisfy $w \prec \lambda - (\alpha_1 + \cdots + \alpha_{m-1} + (d+1-m)\alpha_m + \cdots + 2\alpha_{d-1} + \alpha_d)$. From Lemma 6.1 it follows that if $l_1 + 2l_2 + \cdots + ml_m \leq d$, then all dominant weights in $\Pi(\lambda)$ lie on the face $F^d(\lambda)$ (Corollary 6.3). This leads to a geometric interpretation of the ‘‘stability’’ of the set $\Pi(\lambda)$ [BBL; Theorems 4.1(i) and 8.5(i)] for irreducible characters of A_d .

In Section 2, we prove that χ_λ^m is saturated for all integers $m \geq d + 1$ if χ_λ is an irreducible character of A_d with highest weight of the form $\lambda = l_1\lambda_1 + l_2\lambda_2$ (Propositions 6.4 and 6.7) or $\lambda = l_1\lambda_1 + l_d\lambda_d$ (Proposition 6.5). We conclude the chapter with the proof of this result for irreducible characters of A_d when $d \leq 5$ (Theorem 6.8). We also show that if $\lambda = \lambda^*$ in any of the above cases, then χ_λ^d is saturated.

1. Dominant weights in $\Pi(\lambda)$

Lemma 6.1 and its corollaries describe the dominant weights in $\Pi(\nu)$ when ν lies on a boundary of the Weyl chamber. Let $\lambda = l_1\lambda_1 + l_2\lambda_2 + \cdots + l_m\lambda_m$ be a weight of A_d (not necessarily dominant).

Lemma 6.1. If w is a dominant weight and $w \prec \lambda$, then either $\langle \lambda - w, \lambda_d \rangle = 0$ or $w \prec \lambda - [(d+1-m)\alpha_m + \cdots + 2\alpha_{d-1} + \alpha_d]$.

Proof: By hypothesis, $\langle w, \alpha_j \rangle \geq 0$ for $j = 1, 2, \dots, d$ and there exist nonnegative integers

k_1, k_2, \dots, k_d such that $w = \lambda - k_1\alpha_1 - k_2\alpha_2 - \dots - k_d\alpha_d$. If $m = d$, then the statement is obvious. Otherwise, note that $\langle w, \alpha_d \rangle \geq 0$ implies that $k_{d-1} \geq 2k_d$, and for $i = 1, 2, \dots, d-m$, $\langle w, \alpha_{d+1-i} \rangle \geq 0$ implies that $k_{d-i} \geq (i+1)k_d$. Therefore, there are nonnegative integers k'_1, \dots, k'_{d-1} such that

$$w = \lambda - k'_1\alpha_1 - \dots - k'_{d-1}\alpha_{d-1} - k_d[(d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d].$$

If $\langle \lambda - w, \lambda_d \rangle \neq 0$ then $k_d \geq 1$ and the result follows. \ast

If λ is dominant, it follows that any dominant weight w in $\Pi(\lambda)$ either lies on a face $F^j(\lambda)$ or is less than $\lambda - \alpha_1 - \dots - \alpha_{m-1} - ((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d)$.

Corollary 6.2. If w is a dominant weight and $w \prec \lambda$, then there exist nonnegative integers k_1, k_2, \dots, k_d such that

$$w = \lambda - k_1\alpha_1 - \dots - k_m\alpha_m - k_{m+1}(2\alpha_m + \alpha_{m+1}) - \dots - k_d((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d).$$

Proof: We argue by induction on the rank of \mathcal{G} ; the assertion is obviously true if the rank of \mathcal{G} equals 1. It follows from the proof of Lemma 6.1 that there exist nonnegative integers n_1, n_2, \dots, n_d such that

$$w = \lambda - n_d((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d) - n_1\alpha_1 - \dots - n_{d-1}\alpha_{d-1}.$$

Let $\lambda' = \lambda - n_d((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d)$. Then $p_d w = p_d \lambda' - n_1\alpha_1 - \dots - n_{d-1}\alpha_{d-1}$, (where we recall that p_d is the projection onto the span of $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$). Since $p_d \lambda'$ is a sum involving only the first m fundamental weights of A_{d-1} and $p_d w$ is a dominant weight of A_{d-1} , it follows from the induction hypothesis that

$$\begin{aligned} p_d(\lambda' - w) &= k_1\alpha_1 + \dots + k_m\alpha_m + k_{m+1}(2\alpha_m + \alpha_{m+1}) + \dots \\ &\quad + k_{d-1}((d-m)\alpha_m + \dots + 2\alpha_{d-2} + \alpha_{d-1}). \end{aligned}$$

Hence $w = \lambda - p_d(\lambda' - w) - n_d((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d)$ has the required form. \ast

Potentially distracting remark: Corollary 6.2 implies that if $\langle \lambda, \alpha_j \rangle > 0$ for $1 \leq j \leq m$, then the edges (at λ) of the cone of dominant weights less than λ are rays along the vectors $\alpha_1, \dots, \alpha_m, (2\alpha_m + \alpha_{m+1}), \dots, (d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d$.

Corollary 6.3. If $l_1 + 2l_2 + \dots + ml_m < d+1$, then all dominant weights less than λ lie on $F^d(\lambda)$.

Proof: Write $w = \lambda - k_1\alpha_1 - \dots - k_d((d+1-m)\alpha_m + \dots + 2\alpha_{d-1} + \alpha_d)$ as in the previous lemma. Assume that w does not lie on $F^d(\lambda)$, so that $k_d \geq 1$. The conditions $\langle w, \alpha_{m-j} \rangle \geq 0$ for $j \in \{0, 1\}$ imply that

$$\begin{aligned} k_{m-1} &\geq 2k_m + 3k_{m+1} + \dots + (d+2-m)k_d - l_m, \quad \text{and} \\ k_{m-2} &\geq 2k_{m-1} - k_m - 2k_{m+1} - \dots - (d+1-m)k_d - l_{m-1} \end{aligned} \tag{6.1.1}$$

respectively, and for $j \in \{2, \dots, m-2\}$,

$$k_{m-j-1} \geq 2k_{m-j} - k_{m-j+1} - l_{m-j}. \quad (6.1.2)$$

Now $\langle w, \alpha_1 \rangle \geq 0$ if and only if $l_1 \geq 2k_1 - k_2$; by substituting the inequalities (6.1.1) and (6.1.2) successively into this last expression we obtain $l_1 \geq d+1 - (ml_m - \dots - 2l_2)$. This contradicts the hypothesis $(l_1 + \dots + ml_m < d+1)$, so w must lie on $F^d(\lambda)$.*

It follows from Corollary 6.3 and Proposition 4.1 that if $l_1 + 2l_2 + \dots + ml_m \leq d$, the set of dominant weights in $\Pi(\lambda)$ equals the set of dominant weights of the irreducible character of A_{d-1} whose highest weight is $p_d\lambda$ (up to translation by a multiple of λ_d). In other words, the sets $\{\lambda - w \mid w \in \Pi(\lambda) \cap \Lambda^+\}$ and $\{p_d\lambda - w \mid w \in \Pi(p_d\lambda) \cap \Lambda^+\}$ are equal, where $\Pi(p_d\lambda)$ is the set of weights of the irreducible character of A_{d-1} with highest weight $p_d\lambda$. Moreover, the multiplicities of the corresponding weights w are equal.

This may be interpreted as a ‘‘stability’’ result if we regard the weight λ as a dominant weight of any algebra A_r with $r \geq m$, and consider the set of dominant weights which are less than λ as the rank r increases. It implies [BBL; Theorems 4.1(i) and 8.5(i)] for $\mathcal{G} = A_d$, and actually gives a little more. In particular, [BBL; Theorems 4.1(i) and 8.5(i)] state that the sets of dominant weights in $\Pi(\lambda)$ and their multiplicities are equal for $r \geq d$ (once we identify the fundamental weights in all of the algebras A_r). Corollary 6.3 implies that the sets $\{\lambda - w \mid w \in \Pi(\lambda) \cap \Lambda^+\}$ and the multiplicities of the corresponding weights (w) are equal for all algebras A_r with $r \geq d-1$. Here we assume that the set of simple roots of A_r is $\{\alpha_1, \dots, \alpha_r\}$ and that of A_{r+1} is $\{\alpha_1, \dots, \alpha_r, \alpha_{r+1}\}$. The stability results for $\mathcal{G} = B_d, C_d$, and D_d given in [BBL; Theorems 4.1(ii) and 8.5(ii)] cannot be obtained by a result similar to Corollary 6.3; that is, the dominant weights do not all lie on a face $F^j(\lambda)$ when the highest weight is close enough to the origin. (If all the dominant weights did lie on a face, then Proposition 4.6 would imply that the weight multiplicities stabilize as well, and this is not the case.) For example, if $\mathcal{G} = B_d$, then

$$\begin{aligned} \lambda_1 &= \alpha_1 + \dots + \alpha_d, \\ \lambda_2 &= \alpha_1 + 2\alpha_2 + \dots + 2\alpha_d, \dots, \\ \lambda_{d-1} &= \alpha_1 + 2\alpha_2 + \dots + (d-1)\alpha_{d-1} + (d-1)\alpha_d, \text{ and} \\ 2\lambda_d &= \alpha_1 + 2\alpha_2 + \dots + d\alpha_d. \end{aligned}$$

Unless $\lambda = \lambda_d$ (the only minimal weight of B_d), at least one of the weights $\lambda - \lambda_i$ or $\lambda - 2\lambda_d$ is dominant and belongs to $\Pi(\lambda)$, but does not lie on a face $F^j(\lambda)$.

2. Saturation of χ_λ^{d+1}

Using the Clebsch-Gordon rule, it is not difficult to show that χ_λ^n is saturated for any irreducible character of A_1 when $n \geq 2$. Proposition 6.4 implies the saturation of a more general form of product, and applies to irreducible characters of A_d whose highest weights are multiples of λ_1 .

Proposition 6.4. Let $\lambda = l\lambda_1$ be a dominant weight of A_d . If n_1 and n_2 are nonnegative integers and $n_1 + n_2 \geq d+1$ then the product $\chi_\lambda^{n_1} \chi_{\lambda - \lambda_1}^{n_2}$ is saturated.

Proof: We argue by induction on the total highest weight $v := n_1\lambda + n_2(\lambda - \lambda_1)$ and on the rank of \mathcal{G} . If $\mathcal{G} = A_1$, and $n_2 \geq 1$, then either the product equals χ_{λ_1} (which is saturated since λ_1 is

minimal) or by Proposition 5.1

$$\chi_\lambda^{n_1} \chi_{\lambda-\lambda_1}^{n_2} \geq \begin{cases} \chi_\lambda^{n_1-2} \chi_{\lambda-\lambda_1}^{n_2+2} & \text{if } n_1 \geq 2, \\ \chi_{\lambda-\lambda_1}^{n_2+2n_1-2} \chi_{\lambda-2\lambda_1}^{2-n_1} & \text{otherwise.} \end{cases} \quad (6.2.1)$$

The products on the right of (6.2.1) are saturated by the induction hypothesis since they have total highest weight $\lambda - 2\lambda_1 = \lambda - \alpha_1$. If w is a dominant weight and $w \prec v$, then either $w = v$, in which case χ_w appears in the product, or $w \prec \lambda - \alpha_1$ and χ_w appears by (6.2.1).

Now suppose that the saturation result holds if the rank of \mathcal{G} is less than d . Then by Proposition 4.6, all irreducible characters with highest weights on the face $F^d(v)$ appear in $\chi_\lambda^{n_1} \chi_{\lambda-\lambda_1}^{n_2}$. If $\lambda = \lambda_1$ and $n_1 \leq d$ then $v = n_1 \lambda_1$; by Corollary 6.3 there are no dominant weights in $\Pi(v)$ other than those on $F^d(v)$ and the product is saturated. Otherwise it follows from Proposition 5.1 that

$$\chi_\lambda^{n_1} \chi_{\lambda-\lambda_1}^{n_2} \geq \begin{cases} \chi_\lambda^{n_1-(d+1)} \chi_{\lambda-\lambda_1}^{n_2+(d+1)} & \text{if } n_1 \geq d+1, \\ \chi_{\lambda-\lambda_1}^{n_2+2n_1-(d+1)} \chi_{\lambda-2\lambda_1}^{d+1-n_1} & \text{if } n_1 \leq d. \end{cases} \quad (6.2.2)$$

The products on the right hand side of (6.2.2) have total highest weight

$v - (d+1)\lambda_1 = v - (d\alpha_1 + \dots + 2\alpha_{d-1} + \alpha_d) \prec v$, so are saturated by our induction hypothesis.

If w is a dominant weight in $\Pi(v)$ and w does not belong to $F^d(v)$, then by Lemma 6.1, $w \prec v - (d+1)\lambda_1$. Thus w is a weight of the products on the right hand side of (6.2.2), so $\chi_w \leq \chi_\lambda^{n_1} \chi_{\lambda-\lambda_1}^{n_2}$. *

We have shown in particular that the product χ_λ^n is saturated for $n \geq d+1$ if $\lambda = l_1 \lambda_1$. The exponent $d+1$ is the smallest one which guarantees the saturation of products of this form. For example, for characters of A_2 we have

$$\chi_{l_1 \lambda_1}^2 = \chi_{2l_1 \lambda_1} + \chi_{2l_1 \lambda_1 - \alpha_1} + \dots + \chi_{2l_1 \lambda_1 - l_1 \alpha_1}.$$

These weights all lie on an edge of $\Pi(2l_1 \lambda_1)$, and therefore the product is only saturated if $l_1 = 1$. If $d \geq 2$, then the highest weights of the irreducible characters appearing in $\chi_{l_1 \lambda_1}^n$ all lie on the face $F^d(nl_1 \lambda_1)$ if $n \leq d$, (see the remark at the end of Chapter 5), so this product is not saturated in general.

If λ is the highest weight of an irreducible representation, then the highest weight of the contragredient representation is λ^* , where $\lambda_i^* = \lambda_{d+1-i}$ for weights of A_d . Since $(\chi_w, \chi_\lambda \chi_\mu) = (\chi_{w^*}, \chi_{\lambda^*} \chi_{\mu^*})$, the product χ_λ^n is saturated if and only if χ_λ^n is saturated. Therefore, a result analogous to Proposition 6.4 holds for irreducible characters of A_d with highest weights of the form $l_d \lambda_d$.

Proposition 6.5. Let $\lambda = l_1 \lambda_1 + l_d \lambda_d$ be a dominant weight of A_d , and let n be a nonnegative integer such that $n \geq d+1$. Then the product χ_λ^n is saturated. If $l_1 = l_d$ then $n \geq d$ is sufficient.

Proof: We prove the following more general result, arguing by induction as in Proposition 6.4.

Let n_0, n_1, n_2 , and n_3 be nonnegative integers such that

$$n_0 + n_1 + n_2 + n_3 \geq d+1, \text{ and if both } n_0 > 0 \text{ and } n_3 > 0 \text{ then } n_1 + n_2 > 0 \quad (*).$$

Then the product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_d}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_d)}^{n_3} \quad (6.2.3)$$

is saturated.

If either $l_1 = 0$ or $l_d = 0$ then saturation follows from Proposition 6.4. Otherwise, let v be the total highest weight of the product (6.2.3). It follows from Proposition 4.6 that all irreducible characters with highest weights on a face $F^j(v)$ appear in (6.2.3) if and only if the corresponding product of characters of \mathcal{G}^j is saturated. (By "corresponding", we mean the product of irreducible characters of \mathcal{G}^j whose highest weights equal the projections onto E_j of the highest weights of the irreducible factors in the product (6.2.3).) If $j = 1$ or $j = d$, then the product in $\mathcal{G}^j = A_{d-1}$ satisfies the hypotheses of Proposition 6.4; otherwise \mathcal{G}^j is a direct product $A_{j-1} \times A_{d+1-j}$ and in each of the components of \mathcal{G}^j the hypotheses of Proposition 6.4 are satisfied. The products are therefore saturated, and as a result any irreducible character whose highest weight lies on one of the faces $F^j(v)$ appears in (6.2.3).

If w is a dominant weight in $\Pi(v)$ which does not lie on one of the faces $F^j(v)$ then $w \prec v - (\alpha_1 + \cdots + \alpha_d) = v - (\lambda_1 + \lambda_d)$. It follows from Proposition 5.1 that

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_d}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_d)}^{n_3} \geq \begin{cases} \chi_\lambda^{n_0-2} \chi_{\lambda-\lambda_1}^{n_1+1} \chi_{\lambda-\lambda_d}^{n_2+1} \chi_{\lambda-(\lambda_1+\lambda_d)}^{n_3} & \text{if } n_0 \geq 2 \\ \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_d}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_d)}^{n_3+1} & \text{if } n_0 = 1 \\ \chi_{\lambda-\lambda_1}^{n_1-1} \chi_{\lambda-\lambda_d}^{n_2-1} \chi_{\lambda-(\lambda_1+\lambda_d)}^{n_3+2} & \text{if } n_0 = 0. \end{cases}$$

(Note that it suffices to consider the cases $n_0 \geq 2$, $n_0 = 1$ and either $n_1 \geq 1$ or $n_2 \geq 1$, or $n_0 = 0$ and both $n_1 \geq 1$ and $n_2 \geq 1$, because of condition (*).)

The products on the right hand side of the above expression have total highest weight $v - (\lambda_1 + \lambda_d) = v - (\alpha_1 + \cdots + \alpha_d) \prec v$, so they are saturated by our induction hypothesis. Thus χ_w appears in (6.2.3), and saturation follows.

If $l_1 = l_d$, and $n_0 + n_1 + n_2 + n_3 \geq d$, then any irreducible character whose highest weight lies on a face of $\Pi(v)$ appears in the product (6.2.3) by Proposition 4.6 and Proposition 6.4; we only require that $n_0 + n_1 + n_2 + n_3 \geq d$, since the rank of any simple factor of \mathcal{G}^j is at most $d-1$. For the remaining weights, we apply Proposition 5.1, reducing the product in (6.2.3) to either the trivial character, χ_{λ_1} or χ_{λ_d} all of which are saturated. *

Corollary 6.6. If χ_λ is an irreducible character of A_2 then χ_λ^n is saturated for $n \geq 3$. If $\lambda = \lambda^*$ then χ_λ^2 is saturated as well.

Proof: This follows immediately from the two previous Propositions. *

Proposition 6.7. If $\lambda = l_1\lambda_1 + l_2\lambda_2$ is a dominant weight of A_d , then χ_λ^n is saturated for all integers $n \geq d+1$.

Proof: We will prove the following: let n_0, n_1, n_2 , and n_3 be nonnegative integers satisfying condition (*). Then the product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_2}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_2)}^{n_3} \quad (6.2.4)$$

is saturated. The statement reduces to Proposition 6.5 if $l_2 = 0$ or to Corollary 6.6 if $d = 2$. Assume that $l_2 > 0$ and $d \geq 3$, and let $v = v_1\lambda_1 + v_2\lambda_2$ be the total highest weight of the product (6.2.4). If w is a dominant weight lying on a face $F^j(v)$, then χ_w appears in the product by Proposition 4.6 and induction on the rank d (and, for $j = 1, 2$, or 3 , by previous results).

Let $a := \lfloor \frac{d+1}{2} \rfloor$. We first consider the case when $v_2 \leq a$. If $v_1 < d+1-2v_2$, then by Corollary 6.3, there are no dominant weights in $\Pi(v)$ other than those on the face $F^d(v)$, so the product is saturated in this case. Otherwise it follows from Lemma 6.1 that if w is a dominant weight in $\Pi(v)$ which does not lie on $F^d(v)$ then $w = v - k_1\alpha_1 - k_2\alpha_2 - k_3((d-1)\alpha_2 + \cdots + \alpha_d)$ for some integers $k_1, k_2 \geq 0$ and $k_3 \geq 1$. Now $\langle w, \alpha_2 \rangle \geq 0$ implies that $k_1 \geq d - v_2$, hence

$$\begin{aligned} w &< v - ((d - v_2)\alpha_1 + (d - 1)\alpha_2 + \cdots + \alpha_d) \\ &= v - (d + 1 - 2v_2)\lambda_1 - v_2\lambda_2 \end{aligned}$$

Since $v_1 \geq d + 1 - 2v_2$, the weight $v' := v - (d + 1 - 2v_2)\lambda_1 - v_2\lambda_2$ is dominant, and we can use Proposition 5.1 and induction on the total highest weight to conclude that χ_w appears in (6.2.4).

Now suppose $v_2 > a$. If $w \in \Pi(v)$ is a dominant weight which does not lie on a face thereof then $w < v - \alpha_1 - (d - 1)\alpha_2 - \cdots - 2\alpha_{d-1} - \alpha_d$ by Lemma 6.1. We will consider the cases $2a = d + 1$ and $2a = d$ separately.

If $2a = d + 1$, then $a\lambda_2 = (a - 1)\alpha_1 + (d - 1)\alpha_2 + \cdots + 2\alpha_{d-1} + \alpha_d$. It follows from Proposition 5.1 that $\chi_{\mu_1 + \lambda_2} \cdots \chi_{\mu_a + \lambda_2} \geq \chi_{\mu_1} \cdots \chi_{\mu_a}$ for dominant weights μ_1, \dots, μ_a of A_d . The product (6.0.6) is therefore greater than a product which satisfies the hypothesis and has total highest weight equal to $v - a\lambda_2$. Thus by induction on the total highest weight, χ_w appears in (6.2.4) if $w < v - a\lambda_2$.

To account for the remaining weights, define for positive integers j the weights

$$\begin{aligned} v_j &:= v - (j\alpha_1 + (d - 1)\alpha_2 + \cdots + \alpha_d) \\ &= v + (d - 1 - 2j)\lambda_1 - (d - j)\lambda_2. \end{aligned}$$

If $w < v$ is a dominant weight which does not lie on a face $F^k(v)$ and does not belong to $\Pi(v - a\lambda_2)$, then w lies on $F^1(v_j)$ for some j in $\{1, 2, \dots, a - 2\}$. We now apply Lemma 5.5; we first give the details for the product χ_λ^n .

If

$$\chi_w \leq \chi_{\lambda + (d-1-2j)\lambda_1} \chi_{\lambda - \lambda_2}^{d-j} \chi_\lambda^{n-d-1+j}$$

and $w \in F^1(v_j)$ then $\chi_w \leq \chi_\lambda^n$ by Lemma 5.5. On the face $F^1(v_j)$ the product of irreducible characters of \mathcal{G}^1 corresponding to that on the right hand side of the above has irreducible components with highest weights of the form $p_1(l_2\lambda_2) = (l_2, 0, \dots, 0)$. This product is saturated by Proposition 6.4, so all irreducible characters with highest weights on $F^1(v_j)$ appear in χ_λ^n . In general, that is, for any product of the form (6.2.4), when we apply Lemma 5.5 we deduce that (6.2.4) exceeds a product whose total highest weight equals v_j and whose component highest weights, when projected onto E_1 (the span of $\{\alpha_2, \dots, \alpha_d\}$) satisfy the hypothesis of Proposition 6.4. Thus all irreducible characters with highest weights on $F^1(v_j)$ appear in the product with total highest weight v_j , and therefore by Lemma 5.5 they appear in the product (6.2.4).

If $2a = d$, then $\lambda_1 + a\lambda_2 = a\alpha_1 + (d - 1)\alpha_2 + \cdots + 2\alpha_{d-1} + \alpha_d$. If $w \in \Pi(v)$ is a dominant weight which does not lie on a face, and $v_1 \geq 1$, then either $w \in \Pi(v - \lambda_1 - a\lambda_2)$ or w belongs to $F^1(v_j)$ for some j in $\{1, 2, \dots, a - 1\}$. We argue as in the preceding paragraph that χ_w appears in the product.

If $v_1 = 0$, then (6.2.4) has the form $\chi_\lambda^{n_0} \chi_{\lambda - \lambda_2}^{n_2}$ with $n_0 + n_2 \geq d + 1$. If $v_2 < d + 1$, then $\lambda = \lambda_2$ and we claim that the product $\chi_{\lambda_2}^{n_0}$ is saturated. This follows from [Ha3; Lemma 2.2], which states that products of saturated characters are saturated, however we will prove it by induction on the

rank of \mathcal{G} . Assume that all irreducible characters with highest weights on a face $F^j(n_0\lambda_2)$ appear in the product $\chi_{\lambda_2}^{n_0}$. If $2n_0 \leq d$, then all dominant weights in $\Pi(n_0\lambda_2)$ lie on the face $F^d(n_0\lambda_2)$, and we are done.

Suppose that $\lfloor \frac{d+1}{2} \rfloor \leq n_0 < d+1$. If w is a dominant weight in $\Pi(n_0\lambda_2)$ which does not lie on a face, then by Corollary 6.2, there exist integers $k_i \geq 0$, with $k_1 > 0$ and $k_d > 0$ such that

$$w \prec n\lambda_2 - k_1\alpha_1 - k_2\alpha_2 - \cdots - k_d((d-1)\alpha_2 + \cdots + 2\alpha_{d-1} + \alpha_d).$$

Since w is dominant, we must have $\langle w, \alpha_2 \rangle = n_0 + k_1 - 2k_2 - \cdots - dk_d \geq 0$ so $k_1 \geq d - n_0$. Thus

$$\begin{aligned} w &\prec n_0\lambda_2 - (d - n_0)\alpha_1 - (d - 1)\alpha_2 - \cdots - 2\alpha_{d-1} - \alpha_d \\ &= n_0\lambda_2 - ((d + 1 - 2n_0)\lambda_1 + n_0\lambda_2) \\ &= (2n_0 - (d + 1))\lambda_1 \end{aligned}$$

The product $\chi_{\lambda_1}^{2n_0-(d+1)}$ is saturated by Proposition 6.4, so follows from Lemma B.1 that $\chi_w \leq \chi_{\lambda_2}^{n_0}$. Thus $\chi_{\lambda_2}^{n_0}$ is saturated.

If $v_2 \geq (d+1)$, then by Proposition 5.1, χ_w appears in the product (6.2.4) if $w \prec v - (d+1)\lambda_2$. Now $(d+1)\lambda_2 = (d-1)\alpha_1 + 2(d-1)\alpha_2 + \cdots + 4\alpha_{d-1} + 2\alpha_d$; weights w which do not lie on a face of $\Pi(v)$ or in $\Pi(v - (d+1)\lambda_2)$ lie either on $F^1(v_j)$ or $F^1(v_j - k\alpha_1 - (d-1)\alpha_2 - \cdots - \alpha_d)$ for some j, k in $\{1, 2, \dots, a-1\}$, or on $F^d(v_{a-1})$. We apply Lemma 5.5 or Lemma 5.9; we give the details in case $n_0 \geq d - a + 2$.

If $w \in F^1(v_j)$, then

$$\chi_w \leq \chi_{\lambda+(d-1-2j)\lambda_1} \chi_{\lambda-\lambda_2}^{n_2+(d-j)} \chi_{\lambda}^{n_0-(d-j+1)}$$

(since this product has total highest weight v_j and is saturated on the face $F^1(v_j)$ by Proposition 6.4). By Lemma 5.5, $\chi_w \leq \chi_{\lambda}^{n_0} \chi_{\lambda-\lambda_2}^{n_2}$. If $w \in F^1(v_j - k\alpha_1 - (d-1)\alpha_2 - \cdots - \alpha_d)$, then we apply Lemma 5.5 again to deduce that χ_w appears in the product as well. Finally, if $w \in F^d(v_{a-1})$ then

$$\chi_w \leq \chi_{\lambda+\lambda_1-\lambda_2} \chi_{\lambda-\lambda_2}^{n_2+(d-a)} \chi_{\lambda}^{n_0-(d-a-1)}$$

(the product on the right hand side has total highest weight v_{a-1} and is saturated on the face $F^d(v_{a-1})$ by induction on the rank d). Now Lemma 5.9 implies that $\chi_w \leq \chi_{\lambda}^{n_0} \chi_{\lambda-\lambda_2}^{n_2}$. *

The argument used in Proposition 6.7 becomes unwieldy when we try to apply it to arbitrary irreducible characters, mostly because of the huge number of special cases to check. For example, if $\lambda = l_1\lambda_1 + \cdots + l_m\lambda_m$ then we must consider the m congruence classes of $d+1 \pmod{m}$ separately and within each of these classes there are numerous boundary cases. We also require results (similar those in Section 2 of Chapter 5, for example) to account for irreducible characters whose highest weights lie "in between" the faces and a smaller set of weights corresponding to a saturated product. For example, if $m = 3$, then a weight $w \prec v$ which does not lie on a face of $\Pi(\lambda)$ satisfies $w \prec v - \alpha_1 - \alpha_2 - ((d-2)\alpha_3 + \cdots + \alpha_d)$ while the total weight of a smaller saturated product might have the form $v - k_1\alpha_1 - k_2\alpha_2 - ((d-2)\alpha_3 + \cdots + \alpha_d)$ where k_1 and k_2 are integers greater than 1; in general there will be $m-1$ variables in the description of such weights. Even for $m = 3$, the details are overwhelming.

If we restrict our attention to irreducible characters of A_d when $d \leq 5$, then the method is feasible and we can prove the following.

Theorem 6.8. If χ_λ is an irreducible character of A_d for $d \leq 5$ then χ_λ^n is saturated for all integers $n \geq d + 1$. If $\lambda = \lambda^*$ and $2 \leq d \leq 5$, then χ_λ^d is saturated as well.

Proof: We have seen that the theorem holds if $d = 1$ or $d = 2$ in Proposition 6.4 and Corollary 6.6 respectively. We will deal with the cases $d = 3, 4$ and 5 separately.

1. If $d = 3$, then the saturation of χ_λ^n for $n \geq 4$ follows from Proposition 6.5 if λ equals $l_1\lambda_1 + l_2\lambda_2$ or $l_2\lambda_2 + l_3\lambda_3$, and from Proposition 6.7 if λ equals $l_1\lambda_1 + l_3\lambda_3$. In the latter case, we have also shown that χ_λ^3 is saturated if $\lambda = \lambda^*$. It remains to show that the theorem holds if λ belongs to the interior of the fundamental Weyl chamber, and that χ_λ^3 is saturated when $\lambda = l_2\lambda_2$.

(i) We will show first that if λ belongs to the interior of the fundamental Weyl chamber, then any product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_3}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3} \quad (6.2.5)$$

is saturated whenever the integers n_0, n_1, n_2 , and n_3 satisfy condition (*). The demonstration is similar to that of Proposition 6.4. Let ν be the total highest of the product (6.2.5), and suppose that w is a dominant weight in $\Pi(\nu)$. Then either w lies on a face $F^j(\nu)$, in which case χ_w appears in (6.2.5) by Proposition 4.6 and Corollary 6.6, or $w \in \Pi(\nu - (\lambda_1 + \lambda_3))$, and the corresponding irreducible character appears in (6.2.5) by Proposition 5.1 and induction on the total highest weight.

(ii) Suppose $\lambda = \lambda^* = l_2\lambda_2$. We will show that if n_0 and n_1 are nonnegative integers and $n_0 + n_1 = 3$, then $\chi_\lambda^{n_0} \chi_{\lambda-\lambda_2}^{n_1}$ is saturated. Again a straightforward induction argument applies. If a dominant weight w of this product lies on a face $F^j(\nu)$ (where ν is the total highest weight of the product) then the corresponding irreducible character appears by the usual argument. Otherwise, $w \in \Pi(\nu - (\alpha_1 + 2\alpha_2 + \alpha_3)) = \Pi(\lambda - 2\lambda_2)$ and we apply Proposition 5.1 and induction, reducing the product to either the trivial character, or to χ_{λ_2} both of which are saturated.

(iii) If $\lambda = \lambda^*$ and λ lies in the interior of the Weyl chamber, then we can use the arguments of parts (i) and (ii) to show that any product of the form $\chi_\lambda^m \chi_{\lambda-\lambda_1}^n \chi_{\lambda-\lambda_3}^r$ or $\chi_{\lambda-\lambda_1}^n \chi_{\lambda-\lambda_3}^r \chi_{\lambda-(\lambda_1+\lambda_3)}^m$ is saturated if $m + 2n = 3$. (We apply Proposition 5.1 as in (i), and reduce the product to one satisfying the conditions in part (ii).)

2. If $d = 4$, then by Proposition 6.5 and Proposition 6.7, the assertion holds when λ equals $l_1\lambda_1 + l_2\lambda_2, l_3\lambda_3 + l_4\lambda_4$ or $l_1\lambda_1 + l_4\lambda_4$. We check the remaining cases.

(i) Suppose that $\lambda = l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3$ and both l_2 and l_3 are non-zero. We will show that the product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_2}^{n_1} \chi_{\lambda-\lambda_3}^{n_2} \chi_{\lambda-(\lambda_2+\lambda_3)}^{n_3} \quad (6.2.6)$$

is saturated whenever the integers n_0, n_1, n_2 , and n_3 satisfy condition (*).

If ν is the total highest weight of (6.2.6), and w is a dominant weight on $F^j(\nu)$ where $j = 1, 2$, or 3 , then χ_w appears in (6.2.6) by Proposition 4.6 and Proposition 6.5 or Corollary 6.6. If $j = 4$, then χ_w appears provided products of the form (6.2.6) are saturated for characters of A_3 ; this holds by Proposition 6.5 if $l_1 = 0$ or by an argument analogous to that in 1(i) if $l_1 \neq 0$. If $w \prec \nu - (\lambda_2 + \lambda_3) = \nu - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$, then χ_w appears in (6.2.6) by Proposition 5.1 and induction on ν . When $l_1 = 0$, or $l_1 \neq 0$ but $l_3 = 1$, there are no other dominant weights in $\Pi(\nu)$, and the product is saturated.

Otherwise, dominant weights which do not lie on the faces $F^j(\nu)$ or below $\nu - (\lambda_2 + \lambda_3)$ belong to $F^2(\nu - (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)) = F^2(\nu - \lambda_1 + \lambda_2 - 2\lambda_3)$ by Lemma 6.1. To show that the

corresponding irreducible characters appear in (6.2.6), we apply Lemma 5.10. We will demonstrate this when $n_0 \geq 3$ and $n_1 \geq 1$; the other cases are proved similarly. Lemma 5.10 says that if

$$\chi_w \leq \chi_{\lambda-\lambda_1} \chi_{\lambda}^{n_0-2} \chi_{\lambda-\lambda_2}^{n_1-1} \chi_{\lambda-\lambda_3}^{n_2+2} \chi_{\lambda-(\lambda_2+\lambda_3)}^{n_3} \quad (6.2.7)$$

and w lies on $F^2(\nu - \lambda_1 + \lambda_2 - 2\lambda_3)$ then χ_w appears in (6.2.6). On the face F^2 the product (6.2.7) is saturated, so all irreducible characters with highest weights on this face appear in (6.2.6), and we are done.

(ii) Suppose that $\lambda = l_1\lambda_1 + l_3\lambda_3$, where both $l_1 > 0$ and $l_3 > 0$. We will prove saturation of products

$$\chi_{\lambda}^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_3}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3} \quad (6.2.8)$$

where n_0, n_1, n_2 and n_3 satisfy condition (*). Let $\nu = v_1\lambda_1 + v_3\lambda_3$ be the total highest weight of (6.2.8). If w is a dominant weight on a face of $\Pi(\nu)$ then χ_{λ} appears in (6.2.8) by the usual argument. To account for the remaining irreducible characters, we consider the cases $v_1 = 1$ and $v_1 \geq 2$ separately.

When $v_1 = 1$, the product (6.2.8) has the form $\chi_{\lambda} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3}$, where $\lambda = \lambda_1 + l_3\lambda_3$. If $l_3 = 1$ and n_1 equals 1 or 2, then (6.2.8) would equal $\chi_{\lambda_1+\lambda_3} \chi_{\lambda_3}$ or $\chi_{\lambda_1+\lambda_3} \chi_{\lambda_3}^2$ respectively. In the first case, dominant weights in the product which do not lie on a face of $\Pi(\nu)$ must be less than $\nu - \lambda_1 + \lambda_2 - 2\lambda_3$ by Lemma 6.1; but $\nu - \lambda_1 + \lambda_2 - 2\lambda_3 = \lambda_2$ which is minimal, and it is easy to check that the corresponding irreducible character appears in the product. In the second case, it follows from Lemma 5.10 that if $\chi_w \leq \chi_{\lambda_2} \chi_{\lambda_1} \chi_{\lambda_3}$ and w lies on F^j where $j = 1, 2$ or 4 then χ_w appears in the product. Otherwise, $w \prec \nu - (3\lambda_3 + \lambda_1) = 0$ and we can check that the trivial character appears.

If $v_1 = 1$ and $v_3 \geq 3$, then again Lemma 5.10 implies that χ_w appears in the product (6.2.8) if w belongs to $F^j(\nu - \lambda_1 + \lambda_2 - 2\lambda_3)$ for $j = 1, 2$ or 4 .

Otherwise $w \prec \nu - (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) = \nu - (\lambda_1 + 3\lambda_3)$ and by Proposition 5.1

$$\chi_{\lambda} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3} \geq \begin{cases} \chi_{\lambda-\lambda_1}^{n_1-2} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3+2} & \text{if } n_1 \geq 3 \\ \chi_{\lambda-\lambda_1} \chi_{\lambda-(\lambda_1+\lambda_3)}^{n_3+(3-n_1)} \chi_{\lambda-(\lambda_1+2\lambda_3)} & \text{if } n_1 < 3. \end{cases}$$

By induction on the total highest weight, the products on the right hand side of the above expression are saturated, so χ_w appears in (6.2.8) whenever $w \prec \nu - (\lambda_1 + 3\lambda_3)$.

If $v_1 \geq 2$, then it follows from Proposition 5.1 and induction that all irreducible characters with highest weights w less than $\nu - (2\lambda_1 + \lambda_3) = \nu - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$ appear in (6.2.8). Dominant weights which do not lie on faces or below $\nu - (2\lambda_1 + \lambda_3)$ belong to $F^j(\nu - \lambda_1 + \lambda_2 - 2\lambda_3)$ for $j = 1$ or $j = 2$. By Lemma 5.10 all the corresponding irreducible characters appear.

(iii) Suppose that $\lambda = l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3 + l_4\lambda_4$ where both $l_1 > 0$ and $l_4 > 0$. Then the product

$$\chi_{\lambda}^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_4}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_4)}^{n_3} \quad (6.2.9)$$

is saturated whenever n_0, n_1, n_2 and n_3 satisfy condition (*). The usual induction argument applies here (as in Proposition 6.5 for example), since dominant weights in this product lie either on a face of the weight diagram or below $\nu - (\lambda_1 + \lambda_4)$.

(iv) If $\lambda = \lambda^*$ and $l_1 = 0$, then we argue as in 2(i) that any product $\chi_\lambda^{n_0} \chi_{\lambda-\lambda_2}^{n_1} \chi_{\lambda-\lambda_3}^{n_2}$ or $\chi_{\lambda-\lambda_2}^{n_1} \chi_{\lambda-\lambda_3}^{n_2} \chi_{\lambda-(\lambda_2+\lambda_3)}^{n_0}$ is saturated whenever $m + 2n = 4$. (In this case, we reduce the product to the trivial character using Proposition 5.1.) If $l_1 > 0$, then any product of the form (6.2.9) with $n_1 = n_2$ and either $n_0 = 0$ or $n_3 = 0$ is saturated if $n_0 + 2n_1 = 4$. (here we use Proposition 5.1 to reduce the product to one of the form $\chi_{\lambda'}^4$, where $\lambda' = l_2(\lambda_2 + \lambda_3)$.)

3. If $d = 5$, then the theorem holds if $\lambda = l_1\lambda_1 + l_2\lambda_2$, $\lambda = l_4\lambda_4 + l_5\lambda_5$, or $\lambda = l_1\lambda_1 + l_5\lambda_5$. There are several cases remaining which must be checked.

(i) Suppose that $\lambda = l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3$. We will show that any product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_3}^{n_1} \tag{6.2.10}$$

is saturated if $n_0 + n_1 \geq 6$.

Let $v = v_1\lambda_1 + v_2\lambda_2 + v_3\lambda_3$ be the total highest weight of (6.2.10). Any irreducible character whose highest weight lies on a face of $\Pi(v)$ appears in the product by Proposition 4.6 and saturation results for $d < 5$.

Suppose $v_3 \geq 2$. If $w \prec v - 2\lambda_3 = v - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5)$ then by Proposition 5.1 and induction (on the total highest weight) the corresponding irreducible character appears in (6.2.10). If $l_1 = v_1 = 0$ or if $v_3 = 2$, then there are no other dominant weights in $\Pi(v)$ and the product is saturated. Otherwise, it follows from Lemma 6.1 that dominant weights in $\Pi(v)$ which do not lie on a face lie below $v - (\alpha_1 + \alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) = v - \lambda_1 + 2\lambda_2 - 3\lambda_3$. If such a weight does not lie below $v - 2\lambda_3$, then it belongs to $F^2(v - \lambda_1 + 2\lambda_2 - 3\lambda_3)$. Now we apply Lemma 5.6: the product $\chi_w \leq \chi_{\lambda+2\lambda_2} \chi_{\lambda-\lambda_3}^{n_1+2} \chi_{\lambda-\lambda_1} \chi_\lambda^{n_0-2}$ is saturated on the face $F^2(v - \lambda_1 + 2\lambda_2 - 3\lambda_3)$, so all irreducible characters with highest weights on this face appear in its decomposition. By Lemma 5.6, these irreducible characters also appear in (6.2.10).

If $v_3 = l_3 = 1$, then the product (6.2.10) has the form $\chi_\lambda \chi_{\lambda-\lambda_3}^{n_1}$ where $\lambda = l_1\lambda_1 + l_2\lambda_2 + \lambda_3$ and $n_1 \geq 5$. If $l_1 = l_2 = 0$, then the product equals χ_{λ_3} (which is saturated). If $l_1 > 0$ and $l_2 = 0$, then a dominant weight $w \in \Pi(v)$ which does not lie on a face $F^j(v)$ is less than $v - (3\alpha_1 + 3\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) = v - (3\lambda_1 + \lambda_3)$. Proposition 5.1 and induction imply that the corresponding irreducible character appears in (6.2.10). If $l_1 > 0$ and $l_2 > 0$, then $w \prec v - (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) = v + \lambda_1 - 2\lambda_2 - \lambda_3$. If w lies on $F^1(v + \lambda_1 - 2\lambda_2 - \lambda_3)$, then by Lemma 5.7, χ_w appears in (6.2.10). Otherwise, $w \prec v - (2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) = v - (\lambda_1 + \lambda_2 + \lambda_3)$, and the corresponding irreducible characters appear in the product by Proposition 5.1 and Proposition 6.5.

In case $l_3 = 1, l_2 > 0$ but $l_1 = 0$, we consider products

$$\chi_{\lambda-\lambda_2} \chi_{\lambda-(\lambda_2+\lambda_3)}^{n_1} \chi_{\lambda-\lambda_3}^{n_2}$$

with total highest weight $v' = v'_2\lambda_2 + \lambda_3$. If $v'_2 = 1$, then there are no dominant weights in $\Pi(v')$ other than those on the faces, by Corollary 6.3, and the product is saturated. If $v'_2 = 2$, then dominant weights which do not lie on a face of $\Pi(v')$ are less than $v' + \lambda_1 - 2\lambda_2 - \lambda_3 = \lambda_1$; this weight is minimal and the corresponding irreducible character appears by Lemma 5.7. Finally, if $v'_2 \geq 3$, then the remaining dominant weights lie either on $F^1(v + \lambda_1 - 2\lambda_2 - \lambda_3)$, in which case the corresponding irreducible character appears in the product by Lemma 5.7, or are less than $v' - 3\lambda_2$ and Proposition 5.1 applies.

(ii) Suppose that $\lambda = l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3 + l_4\lambda_4$, and both $l_2 > 0$ and $l_4 > 0$. We will show that the product

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_2}^{n_1} \chi_{\lambda-\lambda_4}^{n_2} \chi_{\lambda-(\lambda_2+\lambda_4)}^{n_3} \quad (6.2.11)$$

is saturated if the exponents n_0, n_1, n_2 and n_3 satisfy condition (*). As usual, we let ν be the total highest weight of (6.2.11); if w is a dominant weight of (6.2.11) lying on a face of $\Pi(\nu)$, then χ_w appears in the product. (We note that on the face $F^5(\nu)$, the product has the form (6.2.11) for irreducible characters of A_4 ; saturation of such products follows from an argument analogous to that in 2(iii).) To account for the remaining weights, we consider several cases.

If $v_4 = 1$, then $w \prec \nu - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) = \nu - \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4$ when both $l_1 > 0$ and $l_3 > 0$; otherwise, $w \prec \nu - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) = \nu - (\lambda_2 + \lambda_4)$. In the latter case the irreducible character χ_w appears in (6.2.11) by Proposition 5.1 and induction. In the former case, (6.2.11) has the form $\chi_{\lambda-\lambda_2} \chi_{\lambda-\lambda_4} \chi_{\lambda-(\lambda_2+\lambda_4)}^{n_3}$. By Lemma 5.11, if $\chi_w \leq \chi_{\lambda-\lambda_1-\lambda_4} \chi_{\lambda-\lambda_3-\lambda_4} \chi_{\lambda-(\lambda_2+\lambda_4)}^{n_3}$ and w belongs to $F^2(\nu - \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)$ then χ_w appears in (6.2.11). The remaining dominant weights satisfy $w \prec \nu - (\lambda_2 + \lambda_4)$, and the corresponding irreducible characters appear by Proposition 5.1 and induction.

If $v_4 \geq 2$ and $v_1 > 0$, then dominant weights w which do not lie on the faces of $\Pi(\nu)$ are less than $\nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) = \nu - \lambda_1 + \lambda_3 - 2\lambda_4$. By Lemma 5.12(i), if $w \in F^3(\nu - \lambda_1 + \lambda_3 - 2\lambda_4)$ then χ_w appears in (6.2.11), and if $l_3 = 0$ and $w \in F^2(\nu - \lambda_1 + \lambda_3 - 2\lambda_4)$ then χ_w appears as well. If $l_3 > 0$ and $w \in F^2(\nu - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5))$ then χ_w appears by Lemma 5.11. The remaining dominant weights lie below $\nu - (\lambda_2 + \lambda_4)$ and so the corresponding irreducible characters appear in (6.2.11) by Proposition 5.1.

If $v_4 \geq 2$, $v_1 = l_1 = 0$, and $v_2 \geq 2$, then dominant weights which do not lie on the faces of $\Pi(\nu)$ and do not lie below $\nu - (\lambda_2 + \lambda_4)$ belong to $F^3(\nu - (\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5))$. We have $\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 = 2\lambda_2 - 2\lambda_3 + 2\lambda_4$; by Lemma 5.8 the irreducible characters on $F^3(\nu - (2\lambda_2 - 2\lambda_3 + 2\lambda_4))$ appear in (6.2.11).

If $v_4 \geq 2$, $v_1 = 0$ and $v_2 = 1$, then dominant weights in $\Pi(\nu)$ which do not lie on a face thereof are less than $\nu - (\lambda_2 + \lambda_4)$; the corresponding irreducible characters appear in the product by Proposition 5.1.

(iii) Suppose that $\lambda = l_1\lambda_1 + l_3\lambda_3 + l_4\lambda_4$ where the integers l_1, l_3 , and l_4 are all strictly greater than zero. We will show that any product of the form

$$\chi_\lambda^m \chi_{\lambda-\lambda_1}^n \chi_{\lambda-(\lambda_1+\lambda_4)}^n, \quad (6.2.12)$$

$$\chi_\lambda^n \chi_{\lambda-\lambda_4}^n \chi_{\lambda-(\lambda_1+\lambda_4)}^m, \quad \text{or} \quad (6.2.13)$$

$$\chi_\lambda^m \chi_{\lambda-\lambda_1}^2 \chi_{\lambda-\lambda_4} \quad (6.2.14)$$

is saturated if $m + 2n \geq 6$ in (6.2.12) and (6.2.13), or if $m \geq 3$ in (6.2.14).

Let $\nu = v_1\lambda_1 + v_3\lambda_3 + v_4\lambda_4$ be the total highest weight of the product; any irreducible character whose highest weight lies on a face of $\Pi(\nu)$ appears in the product, by Proposition 4.6 and Proposition 6.5.

If $v_1 \geq 2$, then we can use Proposition 5.1 and induction to argue that any irreducible character whose highest weight w is less than $\nu - (2\lambda_1 + \lambda_4) = \nu - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ appears in the product. When $v_4 \geq 2$, the remaining dominant weights in $\Pi(\nu)$ lie on the faces of $\Pi(\nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)) = \Pi(\nu - (\lambda_1 - \lambda_3 + 2\lambda_4))$. If w lies on $F^3(\nu - \lambda_1 + \lambda_3 - 2\lambda_4)$, then

χ_w appears in the product by Lemma 5.12. Otherwise, $w \in F^j(\nu - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5))$ where $j = 1$ or $j = 2$; the corresponding irreducible characters appear by Lemma 5.11. When $v_4 = 1$, dominant weights which are not less than $\nu - (2\lambda_1 + \lambda_4)$ (and do not lie on the faces of $\Pi(\nu)$) belong to $F^2(\nu - \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)$. The corresponding irreducible characters appear in the product by Lemma 5.11.

If $v_1 = 1$, then the product has the form

$$\chi_\lambda \chi_{\lambda - (\lambda_1 + \lambda_4)}^n \chi_{\lambda - \lambda_1}^n \quad (6.2.15)$$

where $\lambda = \lambda_1 + l_3\lambda_3 + l_4\lambda_4$, and $2n \geq 5$. (This implies $n \geq 3$.) By Proposition 5.1, any irreducible character χ_w satisfying

$$\chi_w \leq \chi_{\lambda - \lambda_1}^{n-2} \chi_{\lambda - (\lambda_3 + \lambda_1)} \chi_{\lambda - (\lambda_1 + \lambda_4)}^{n+2} \quad (6.2.16)$$

appears in (6.2.15). The product on the right hand side of (6.2.16) has the form $\chi_{\lambda'}^{n-2} \chi_{\lambda' - \lambda_3} \chi_{\lambda' - \lambda_4}^{n+2}$ where $\lambda' = l_3\lambda_3 + l_4\lambda_4$, and so is saturated by Proposition 6.5. Thus if $w < \nu - (\lambda_1 + \lambda_3 + 2\lambda_4) = \nu - (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 4\alpha_4 + 2\alpha_5)$, then χ_w appears in (6.2.15).

The remaining dominant weights w in $\Pi(\nu)$ satisfy $w < \nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$. If w lies on $F^3(\nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5))$ then χ_w appears in (6.2.15) by Lemma 5.12. If w belongs to $F^j(\nu - \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ for $j = 1$ or $j = 2$, then χ_w appears in the product by Lemma 5.11. The remaining dominant weights are less than $\nu - 2\lambda_3$. We now reduce the product (6.2.15) by $2\lambda_3$ using Proposition 5.1 until we arrive at a product of the form

$$\chi_{l_4\lambda_4}^n \chi_{(l_4-1)\lambda_4}^{n-1} \chi_{\lambda_1 + \lambda_4} \chi_{\lambda_3 + (l_4-1)\lambda_4}. \quad (6.2.17)$$

As before, weights w which do not lie on the faces of $\Pi(\nu)$ lie below $\nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$. It follows from Lemma 5.12 and Lemma 5.11 that if $w \in F^3(\nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5))$ or respectively $w \in F^j(\nu - (\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5))$ for $j = 1, 2$, or 5 , then χ_w appears in the product (6.2.17). The remaining dominant weights satisfy $w < \nu - (\lambda_1 + \lambda_3 + 2\lambda_4)$, and so appear in the (saturated) product $\chi_{l_4\lambda_4}^n \chi_{(l_4-1)\lambda_4}^{n-1}$; by Proposition 5.1, they appear in the product (6.2.17).

(iv) Suppose $\lambda = l_1\lambda_1 + l_4\lambda_4$ with both $l_1 > 0$ and $l_4 > 0$. We will show that any product

$$\chi_\lambda^{n_0} \chi_{\lambda - \lambda_1}^{n_1} \chi_{\lambda - \lambda_4}^{n_2} \chi_{\lambda - (\lambda_1 + \lambda_4)}^{n_3} \quad (6.2.18)$$

is saturated if the integers n_i satisfy condition (*).

Again, we let $\nu = v_1\lambda_1 + v_4\lambda_4$ be the total highest weight of (6.2.16), and note that if w is a dominant weight on a face of $\Pi(\nu)$ then χ_w appears in (6.2.18).

If $v_4 = 1$, and $v_1 = 1$ then by Corollary 6.3, there are no other dominant weights in $\Pi(\nu)$ and the product (equal to $\chi_{\lambda_1} \chi_{\lambda_4}$) is saturated.

If $v_4 = 1$ and $v_2 \geq 2$, then the remaining dominant weights in $\Pi(\nu)$ are less than $\nu - (2\lambda_1 + \lambda_4)$, and saturation of (6.2.18) follows from Proposition 5.1 and induction.

If $v_4 \geq 2$, then by Lemma 6.1, the remaining dominant weights in $\Pi(\nu)$ are less than $\nu' := \nu - (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) = \nu - (\lambda_1 - \lambda_3 + 2\lambda_4)$. It follows from Lemma 5.12(i),(ii) and saturation results for $d \leq 5$ that if w belongs to $F^2(\nu')$ or to $F^3(\nu')$, then χ_w appears in (6.2.18); this also holds if w lies on $F^1(\nu') \cap F^5(\nu')$ by Lemma 5.12(iii). If $w \in F^1(\nu')$ but w does not belong to $F^j(\nu')$ for $j = 2, 3$ or 5 , then $w < \nu - (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5) = \nu - 3\lambda_4$; by Proposition 5.1 the corresponding irreducible character appears in (6.2.18).

If w does not belong to $F^1(v')$, $F^2(v')$ or $F^3(v')$, and $v_1 \geq 2$ then $w \prec v - (2\lambda_1 + \lambda_4)$ and by Proposition 5.1 the irreducible character χ_w appears in (6.2.18).

If $v_1 = 1$, then Lemma 5.12(iv) also implies that if $w \in F^5(v')$ then χ_w appears in the product (6.2.18). If w does not lie on a face of $\Pi(v')$, then $w \prec v - 3\lambda_4$, and the corresponding irreducible character appears by the usual argument.

(v) Suppose that $\lambda = l_1\lambda_1 + l_2\lambda_2 + \cdots + l_5\lambda_5$ and both $l_1 > 0$ and $l_5 > 0$. We consider products

$$\chi_\lambda^{n_0} \chi_{\lambda-\lambda_1}^{n_1} \chi_{\lambda-\lambda_5}^{n_2} \chi_{\lambda-(\lambda_1+\lambda_5)}^{n_3} \quad (6.2.19)$$

with total highest weight v where n_0, n_1, n_2 and n_3 satisfy condition (*). Dominant weights in $\Pi(v)$ lie either on a face $F^j(v)$ or are less than $v - (\lambda_1 + \lambda_5)$, and the usual arguments imply that the product is saturated.

(vi) Finally, if $\lambda = \lambda^*$, then arguments similar to those in (i), (ii), and (v) can be used to show that χ_λ^5 is saturated as well. The details: In (i) we have $\lambda = l_3\lambda_3$, and we reduce any product of the form (6.2.10) with $n_0 + n_1 = 5$ (using Proposition 5.1) to the trivial character or to χ_{λ_3} . In (ii), we consider products of the form (6.2.11) with $n_1 = n_2$, and either $n_0 = 0$ and $2n_1 + n_3 = 5$ or $n_3 = 0$ and $2n_1 + n_0 = 5$. We reduce such a product to one of the form (6.2.10) with $\lambda = l_3\lambda_3$ and apply (i). In (v), we impose the same conditions on the exponents as in (ii), and reduce the product in (6.2.19) to one of the form (6.2.11). *

For irreducible characters of A_2 , the sets $\mathcal{D}_{\lambda,\mu}$, consisting of the highest weights of all irreducible characters appearing in $\chi_\lambda\chi_\mu$, are relatively convex with respect to the root lattice. This can be deduced from the description of these sets given in the next chapter (Proposition A.4). We would like to know whether Λ_r -convexity also holds for products of irreducible characters of A_d , or more generally, whenever the weights λ and μ satisfy the conditions for eventual saturation. It is difficult to get a handle on this question, if only because there is no natural way to formulate Λ_r -convexity algebraically. Also, we would expect that if the set $\mathcal{D}_{\lambda,\mu}$ were relatively convex, then the same would be true for the set of highest weights appearing in a multiple product $\chi_{\mu_1}\chi_{\mu_2}\cdots\chi_{\mu_t}$. If we could establish these properties, then we could extend the result of Theorem 6.8 to all d . (All irreducible characters with highest weights lying on the edges along $-\alpha_i, i \in I_\lambda$ would appear in the product χ_λ^M for $M \geq 2$ by Proposition A.6, and the trivial character appears in χ_λ^{d+1} .)

Appendix A

Products in A_d

The induction argument developed in Chapters 4 and 5 to prove saturation of certain powers of irreducible characters might also be used to obtain more specific information about the decomposition of products $\chi_\lambda \chi_\mu$. In particular, it may be used to describe the “geometry” of the set of irreducible characters appearing in the decomposition, since the argument is based on the geometry of the weight polytope. We use this approach here to study products of irreducible characters of A_d .

Let $\mathcal{D}_{\lambda,\mu}$ be the set of all weights ν such that χ_ν appears in the orthogonal decomposition of the product $\chi_\lambda \chi_\mu$. For any semisimple complex Lie algebra, the highest weight of an irreducible character appearing in the decomposition of a product $\chi_\lambda \chi_\mu$ must belong to the set $\mathcal{C}_{\lambda,\mu}$ defined by

$$\mathcal{C}_{\lambda,\mu} := (\Pi(\lambda) + \mu) \cap (\Pi(\mu) + \lambda) \cap \bar{\mathcal{C}};$$

this can be deduced from Steinberg’s formula for example (see [Hu1; Exercise 24.12]). If $\mathcal{G} = A_1$, then $\mathcal{D}_{\lambda,\mu} = \mathcal{C}_{\lambda,\mu}$; however, the two sets are not equal in general if the rank of \mathcal{G} is greater than 1.

Let w_0 be the longest element of W . It is shown in [PRV] that the dominant weight conjugate to $w_0\lambda + \mu$, which we denote $\{w_0\lambda + \mu\}$, belongs to $\mathcal{D}_{\lambda,\mu}$ and is less than any other weight in $\mathcal{D}_{\lambda,\mu}$. We will show in Section 1 that, when $\mathcal{G} = A_2$, $\mathcal{D}_{\lambda,\mu}$ equals the set of all weights in $\mathcal{C}_{\lambda,\mu}$ which are greater than $\{w_0\lambda + \mu\}$. We also show that this set consists of all weights in $\mathcal{C}_{\lambda,\mu}$ of the form $\lambda + \mu - k_0(\alpha_1 + \alpha_2) - k_j\alpha_j$ where $j \in \{1, 2\}$ and $\lambda + \mu - k_0(\alpha_1 + \alpha_2) \in \mathcal{C}_{\lambda,\mu}$ (Proposition A.4).

If the rank of \mathcal{G} is greater than 2, then it is not true in general that all the weights in $\mathcal{C}_{\lambda,\mu}$ which are greater than $\{w_0\lambda + \mu\}$ belong to $\mathcal{D}_{\lambda,\mu}$. For example, if $d = 3$ and $\lambda = 2\lambda_2$, then the weight $\lambda_2 + 2\lambda_3 = 4\lambda_2 - (\alpha_1 + 2\alpha_2)$ belongs to $\mathcal{C}_{\lambda,\lambda}$ and is greater than $\{w_0\lambda + \mu\} = 0$, but the corresponding irreducible character does not appear in the product χ_λ^2 . We make no attempt here to determine additional constraints on the weights in $\mathcal{D}_{\lambda,\mu}$. Instead we use Proposition 4.6 and Proposition 5.1 to describe certain “lines” of weights in $\mathcal{D}_{\lambda,\mu}$ in Section 2.

1. Products in A_2

Let λ and μ be dominant weight of A_2 . Recall the definition of the dominant faces given in Chapter 4:

$F^i(\lambda) := \{x \in \Pi(\lambda) \mid \langle \lambda_i, x \rangle = \langle \lambda_i, \lambda \rangle\}$. In case $\mathcal{G} = A_2$, weights on the face $F^i(\lambda)$ have the form $\lambda - k\alpha_{3-i}$ where k is a positive integer. For $i \in \{1, 2\}$, define faces of $\mathcal{C}_{\lambda,\mu}$:

$$F^i(\lambda, \mu) := (F^i(\lambda) + \mu) \cap (F^i(\mu) + \lambda) \cap \bar{\mathcal{C}};$$

weights on the face $F^i(\lambda, \mu)$ have the form $\lambda + \mu - k\alpha_{3-i}$. We first establish in Lemma A.2 that if w is a weight in $\mathcal{C}_{\lambda,\mu}$, then either w lies on a face $F^i(\lambda, \mu)$ for $i \in \{1, 2\}$, or w belongs to one of $\mathcal{C}_{\lambda-\lambda_1, \mu-\lambda_2}$ or $\mathcal{C}_{\lambda-\lambda_2, \mu-\lambda_1}$. In Lemma A.3, we describe products of irreducible characters with highest weights on the boundary of $\bar{\mathcal{C}}$. We then argue by induction on the total highest weight $\lambda + \mu$ to establish Proposition A.4 in all cases.

For $i \in \{1, 2\}$, the set $\Pi(\lambda - \lambda_i) + \mu - \lambda_{3-i}$ is contained in $\Pi(\lambda) + \mu$. By superimposing the two sets, we readily deduce that the difference is included in the union of faces of $\Pi(\lambda) + \mu$.

Lemma A.1. The difference $(\Pi(\lambda) + \mu) \setminus (\Pi(\lambda - \lambda_i) + \mu - \lambda_{3-i})$ is equal to the set of weights on

$$(F^i(\lambda) \cup \sigma_i F^i(\lambda) \cup F^{3-i}(\lambda)) + \mu.$$

Proof: It suffices to establish the result when $i = 1$, and this follows from Figure A(i). (In case $i = 2$, the weight diagrams are reflections of those in Figure A(i).) Observe in (b) that if λ is a multiple of the fundamental weight λ_1 , then the face $F^1(\lambda) + \mu$ consists of a single weight $\lambda + \mu$, so $F^1(\lambda) + \mu \subset F^2(\lambda) + \mu$, and similarly $\sigma_1 F^1(\lambda) + \mu \subset F^2(\lambda) + \mu$. Notice also that $w_0 \lambda + \mu$ equals $w_0(\lambda_i) + \mu - \lambda_{3-i}$, and is therefore a common vertex of $\Pi(\lambda) + \mu$ and $\Pi(\lambda - \lambda_i) + (\mu - \lambda_{3-i})$.*

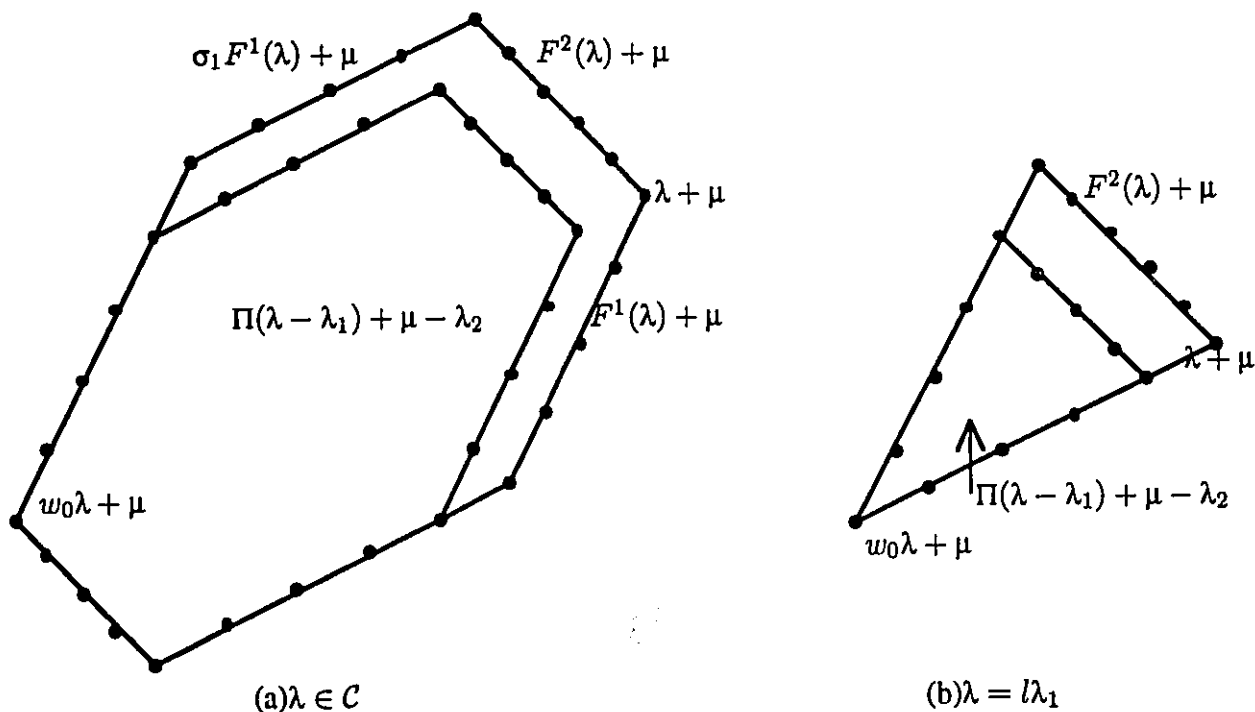


Figure A(i)

Lemma A.2. Suppose that $\mu \in C$.

- (i) If $\lambda = l\lambda_i$ for $i \in \{1, 2\}$, then any weight in $C_{\lambda, \mu}$ which does not lie on the face $F^{3-i}(\lambda, \mu)$ belongs to $C_{\lambda - \lambda_i, \mu - \lambda_{3-i}}$.
- (ii) If $\lambda \in C$, then any weight in $C_{\lambda, \mu}$ which does not lie on a face $F^i(\lambda, \mu)$ belongs to either $C_{\lambda - \lambda_1, \mu - \lambda_2}$ or to $C_{\lambda - \lambda_2, \mu - \lambda_1}$.

Proof: (i) To simplify the demonstration, we will assume that $i = 1$. (The proof is analogous if $i = 2$.) Suppose that w belongs to $C_{\lambda, \mu}$ and does not lie on $F^2(\lambda) + \mu$. Then w belongs to $\Pi(\lambda) + \mu$ but not to $F^2(\lambda) + \mu$; by Lemma A.1, w belongs to $\Pi(\lambda - \lambda_1) + \mu - \lambda_2$ (See Figure A(i)

(b). We claim that w also belongs to $\Pi(\mu - \lambda_2) + \lambda - \lambda_1$, and so to $\mathcal{C}_{\lambda-\lambda_1, \mu-\lambda_2}$. By assumption, $w \in \Pi(\mu) + \lambda$, and w does not lie on $F^2(\mu) + \lambda$. Thus either w belongs to $\Pi(\mu - \lambda_2) + \lambda - \lambda_1$, or w lies on one of the faces $F^1(\mu) + \lambda$, $\sigma_1 F^1(\mu) + \lambda$ by Lemma A.1. Since $\lambda = l\lambda_1$, we have $F^1(\mu) + \lambda \cap \Pi(\lambda) + \mu = \{\lambda + \mu\} \subset F^2(\lambda, \mu)$, and $\sigma_1 F^1(\mu) + \lambda \cap \Pi(\lambda) + \mu = \{\sigma_1 \mu + \lambda\} \subset F^2(\lambda, \mu)$. Therefore, $w \in \Pi(\mu - \lambda_2) + \lambda - \lambda_1$, and consequently $w \in \mathcal{C}_{\lambda-\lambda_1, \mu-\lambda_2}$.

(ii) Suppose that $w \in \mathcal{C}_{\lambda, \mu}$ does not lie on $F^i(\lambda, \mu)$ for $i \in \{1, 2\}$. Then by Lemma A.1,

$$w \in (\Pi(\lambda - \lambda_1) + \mu - \lambda_2) \cup (\Pi(\lambda - \lambda_2) + \mu - \lambda_1), \quad \text{and} \quad (7.0.2)$$

$$w \in (\Pi(\mu - \lambda_1) + \lambda - \lambda_2) \cup (\Pi(\mu - \lambda_2) + \lambda - \lambda_1) \quad (7.0.3).$$

If w does not belong to one of $\mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$, then for some i , $w \in \Pi(\lambda - \lambda_i) + \mu - \lambda_{3-i}$ but $w \notin \Pi(\lambda - \lambda_{3-i}) + \mu - \lambda_i$, and $w \in \Pi(\mu - \lambda_i) + \lambda - \lambda_{3-i}$ but $w \notin \Pi(\mu - \lambda_{3-i}) + \lambda - \lambda_i$. We will show that this is not possible. If it were, then $w \in (\sigma_{3-i} F^{3-i}(\lambda) + \mu) \cap (\sigma_{3-i} F^{3-i}(\mu) + \lambda) \cap \bar{\mathcal{C}}$ by Lemma A.1. This set is evidently empty unless $(\lambda, \alpha_{3-i}) = (\mu, \alpha_{3-i})$. On the other hand, since $w \in \sigma_{3-i} F^{3-i} + \mu$, $w = \sigma_{3-i} x' + \mu$ for some weight $x' = \lambda - k\alpha_i \neq \lambda$ on the face $F^{3-i}(\lambda)$. Therefore

$$\begin{aligned} (w, \alpha_{3-i}) &= (\sigma_2 x', \alpha_{3-i}) + (\mu, \alpha_{3-i}) \\ &= -(\lambda, \alpha_{3-i}) - k + (\mu, \alpha_{3-i}). \end{aligned}$$

Thus if $(\lambda, \alpha_2) = (\mu, \alpha_2)$, then $(w, \alpha_2) < 0$, so w does not belong to $\bar{\mathcal{C}}$. Consequently, w belongs to one of $\mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$. *

Lemma A.3. Suppose that m and n are integers such that $m \geq n > 0$. Then for $i = 1, 2$

$$(i) \quad \chi_{m\lambda_i} \chi_{n\lambda_i} = \sum_{k=0}^n \chi_{(m+n)\lambda_i - k\alpha_i}$$

$$(ii) \quad \chi_{m\lambda_i} \chi_{n\lambda_{3-i}} = \sum_{k=0}^n \chi_{m\lambda_i + n\lambda_{3-i} - k(\alpha_1 + \alpha_2)}.$$

Proof: (i) All irreducible characters with highest weights $\lambda + \mu - k\alpha_i$ for $k \in \{1, 2, \dots, n\}$ appear (with multiplicity 1) in the product by Proposition 4.6 and the Clebsch-Gordon formula (see Proposition A.6). No other irreducible characters appear, since the lowest weight in $\mathcal{D}_{\lambda, \mu}$ equals $\lambda + \mu - n\alpha_i$.

(ii) The only dominant weights in $(\Pi(\lambda) + \mu) \cap (\Pi(\mu) + \lambda)$ are those of the form $\lambda + \mu - k(\alpha_1 + \alpha_2)$, where $k \in \{1, 2, \dots, n\}$. All the corresponding irreducible characters appear by repeated application of Proposition 5.1. The multiplicity of any irreducible character equals 1 since this holds for the corresponding weight multiplicities.*

Proposition A.4. Suppose that $\mathcal{G} = A_2$

(i) $\mathcal{D}_{\lambda, \mu}$ consists of all $w \in \mathcal{C}_{\lambda, \mu}$ satisfying $w \succ \{w_0\lambda + \mu\}$.

(ii) $\mathcal{D}_{\lambda, \mu}$ consists of all weights $w \in \mathcal{C}_{\lambda, \mu}$ of the form $w = \lambda + \mu - k_0(\alpha_1 + \alpha_2) - k_i\alpha_i$ such that $\lambda + \mu - k_0(\alpha_1 + \alpha_2)$ belongs to $\mathcal{C}_{\lambda, \mu}$.

Proof: If both λ and μ lie on the boundary of $\bar{\mathcal{C}}$ then the proposition reduces to Lemma A.3.

(i) It suffices to show that all weights w satisfying the hypothesis belong to $\mathcal{D}_{\lambda,\mu}$. We argue by induction on the total highest weight, $\lambda + \mu$. Let $w \in \mathcal{C}_{\lambda,\mu}$ and suppose that $w \succ \{w_0\lambda + \mu\}$. Note that $\{w_0\lambda + \mu\} = \{w_0(\lambda - \lambda_i) + \mu - \lambda_{3-i}\}$, so $w \succ \{w_0(\lambda - \lambda_i) + \mu - \lambda_{3-i}\}$ for $i \in \{1, 2\}$.

Suppose first that $\lambda = l\lambda_1$ and $\mu \in \mathcal{C}$. If w lies on the face $F^{3-i}(\lambda)$, then χ_w appears by Proposition 4.6 and the Clebsch-Gordon formula (see Proposition A.6(i) in Section 2, for example). Otherwise, $w \in \mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$ by Lemma A.2(i). Since $w \succ \{w_0\lambda + \mu\} = \{w_0(\lambda - \lambda_i) + \mu - \lambda_{3-i}\}$, our induction hypothesis implies that χ_w appears in the product $\chi_{\lambda-\lambda_i}\chi_{\mu-\lambda_{3-i}}$. By Proposition 5.1, $\chi_{\lambda-\lambda_i}\chi_{\mu-\lambda_{3-i}} \leq \chi_\lambda\chi_\mu$, thus χ_w appears in $\chi_\lambda\chi_\mu$.

Now suppose that both λ and μ belong to \mathcal{C} . If w lies on one of the faces $F^i(\lambda, \mu)$, then χ_w appears in the product by Proposition A.6(i). Otherwise, Lemma A.2(ii) implies that w belongs to either $\mathcal{C}_{\lambda-\lambda_1, \mu-\lambda_2}$ or $\mathcal{C}_{\lambda-\lambda_2, \mu-\lambda_1}$. By our induction assumption, $\chi_w \leq \chi_{\lambda-\lambda_1}\chi_{\mu-\lambda_2}$ or $\chi_w \leq \chi_{\lambda-\lambda_2}\chi_{\mu-\lambda_1}$ respectively. The assertion now follows from Proposition 5.1.

(ii) We assume as an induction hypothesis that $\mathcal{D}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$ consists of all weights in $\mathcal{C}_{\lambda,\mu}$ of the form $\lambda + \mu - k_0(\alpha_1 + \alpha_2) - k_j\alpha_j$ where $j \in \{1, 2\}$, $k_0 \geq 1$ and $\lambda + \mu - k_0(\alpha_1 + \alpha_2) \in \mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$. Since $\chi_{\lambda-\lambda_i}\chi_{\mu-\lambda_{3-i}} \leq \chi_\lambda\chi_\mu$ (Proposition 5.1), all such weights belong to $\mathcal{D}_{\lambda,\mu}$ as well.

We first consider the case $\lambda = l\lambda_i$. Suppose that both $w := \lambda + \mu - k_0(\alpha_1 + \alpha_2) - k_j\alpha_j$ and $w' := \lambda + \mu - k_0(\alpha_1 + \alpha_2)$ belong to $\mathcal{C}_{\lambda,\mu}$. If $k_0 = 0$ then w lies on the face $F^{3-i}(\lambda, \mu)$, so $w \in \mathcal{D}_{\lambda,\mu}$. If $k_0 > 0$, then by Lemma A.1, w belongs to $\mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$, as does w' . Thus $w \in \mathcal{D}_{\lambda-\lambda_i, \mu-\lambda_{3-i}} \subset \mathcal{D}_{\lambda,\mu}$. If w belongs to $\mathcal{C}_{\lambda,\mu}$ but w' does not, then in particular, $k_0 > 0$. Thus w belongs to $\mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$, but w' does not belong to this set. By our induction assumption, $w \notin \mathcal{D}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$. It follows from part (i) that $w \not\succeq \{w_0(\lambda - \lambda_i) + \mu - \lambda_{3-i}\} = \{w_0\lambda + \mu\}$, hence the corresponding irreducible character does not appear in $\chi_\lambda\chi_\mu$.

Now suppose that $\lambda \in \mathcal{C}$ and let $w := \lambda + \mu - k_0(\alpha_1 + \alpha_2) - k_j\alpha_j$ and $w' := \lambda + \mu - k_0(\alpha_1 + \alpha_2)$. If w belongs to $\mathcal{C}_{\lambda,\mu}$ and $k_0 = 0$, then w lies on a face $F^i(\lambda, \mu)$, so $w \in \mathcal{D}_{\lambda,\mu}$. If $k_0 > 0$, then w belongs to either $\mathcal{C}_{\lambda-\lambda_1, \mu-\lambda_2}$ or to $\mathcal{C}_{\lambda-\lambda_2, \mu-\lambda_1}$ by Lemma A.2(ii). If $w' \in \mathcal{C}_{\lambda,\mu}$ then w' belongs to both of these sets. (Since w' has the form $\lambda + \mu - k(\alpha_1 + \alpha_2)$, it does not lie on any of the faces $F^i(\lambda) + \mu$ or $\sigma_i F^i(\lambda) + \mu$, so it belongs to both sets by Lemma A.1.) Thus w belongs to either $\mathcal{D}_{\lambda-\lambda_1, \mu-\lambda_2}$ or $\mathcal{D}_{\lambda-\lambda_2, \mu-\lambda_1}$ by our induction hypothesis. If w belongs to $\mathcal{C}_{\lambda,\mu}$ but w' does not, then w' does not belong to either of the sets $\mathcal{C}_{\lambda-\lambda_i, \mu-\lambda_{3-i}}$. We argue as in the preceding paragraph that $w \notin \mathcal{D}_{\lambda,\mu}$.*

Corollary A.5. The set $\mathcal{D}_{\lambda,\mu} - \lambda - \mu$ is relatively convex with respect to the root lattice.

*

The example in Figure A(ii) on the next page was computed using the software *Simple*, and shows $\mathcal{D}_{\lambda,\mu}$ when $\lambda = 15\lambda_1 + 18\lambda_2$ and $\mu = 12\lambda_1 + 21\lambda_2$. The multiplicities $(\chi_w, \chi_\lambda\chi_\mu)$ are indicated at each of the points w . Note the pattern: the multiplicities are constant on ‘‘shells’’, and increase as the shells approach the centre of $\mathcal{D}_{\lambda,\mu}$. This is similar to the pattern of weight multiplicities in irreducible characters of A_2 (see [AS], for example).

$A_2: (15, 18) \otimes (12, 21).$

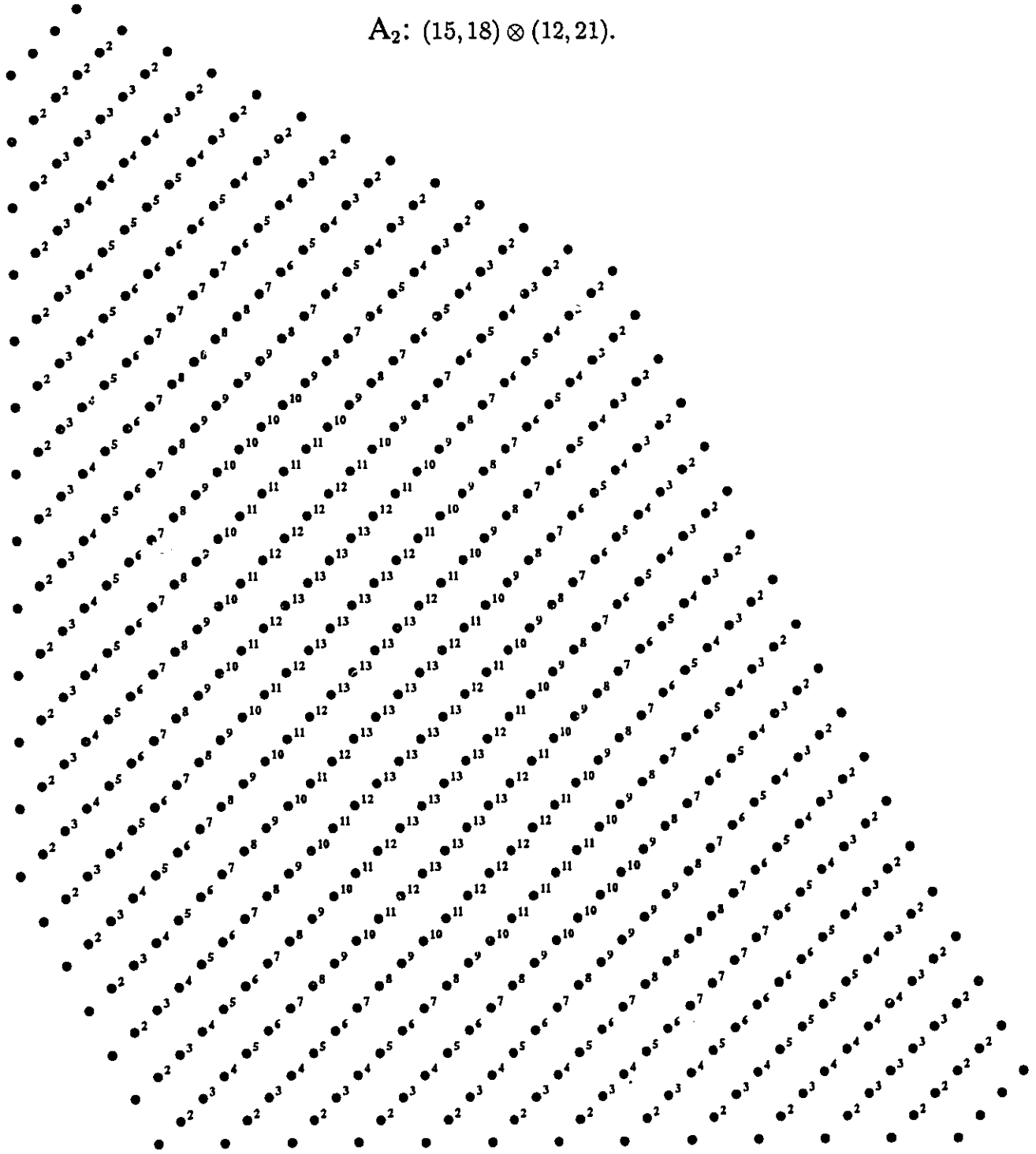


Figure A(ii)

2. Lines of weights in $\mathcal{D}_{\lambda,\mu}$

Let $\mathcal{D}_{M\lambda}$ be the set of highest weights of irreducible characters appearing in χ_λ^M . We show that the sets $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{M\lambda}$ include certain “lines” of weights. (If the set $\mathcal{D}_{\lambda,\mu}$ were relatively convex, some of these lines would describe edges of this set.)

It is well known that all weights of the form $\lambda + \mu - k_i\alpha_i$ in $\mathcal{C}_{\lambda,\mu}$ belong to $\mathcal{D}_{\lambda,\mu}$; we prove this result using Proposition 5.1 and Proposition 4.6, and then argue by induction on M that all dominant weights of the form $M\lambda - k_i\alpha_i$ belong to $\mathcal{D}_{M\lambda}$ (Proposition A.6). We can therefore account for all weights along those edges of $\text{cvx } \mathcal{D}_{\lambda,\mu}$ (or $\text{cvx } \mathcal{D}_{M\lambda}$) in the direction of the simple roots $\alpha_i, i \in I_\lambda \cap I_\mu$ ($i \in I_\lambda$ respectively).

If \mathcal{G} is type A_d , and β is any positive root, then all dominant weights of the form $M\lambda - k\beta$ belong to $\mathcal{D}_{M\lambda}$, and certain weights of the form $\lambda + \mu - k\beta$ belong to $\mathcal{D}_{\lambda,\mu}$ (Proposition A.8).

Define weights $\omega_j := \lambda_j + \lambda_{d+1-j}$ for $j = 1, 2, \dots, d' := \lfloor \frac{d+1}{2} \rfloor$. By repeated application of Proposition 5.1, we show that when $\mathcal{G} = A_d$, all dominant weights of the form $M\lambda - k_1\omega_1 - \dots - k_{d'}\omega_{d'}$ belong to $\mathcal{D}_{M\lambda}$, and certain weights of the form $\lambda + \mu - k_1\omega_1 - \dots - k_{d'}\omega_{d'}$ belong to $\mathcal{D}_{\lambda,\mu}$ (Proposition A.10). Using Proposition 4.6, we deduce corresponding results for weights on the faces of $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{M\lambda}$ when \mathcal{G} is one of B_d, C_d , or D_d (Corollary A.11).

For $i \in I$, let $N_i \equiv N_i(\lambda, \mu) := \min\{\langle \lambda, \alpha_i \rangle, \langle \mu, \alpha_i \rangle\}$.

Proposition A.6.

- (i) The irreducible character with highest weight $\lambda + \mu - k\alpha_i$ appears in the decomposition of $\chi_\lambda\chi_\mu$, necessarily with multiplicity equal to 1, if and only if $k \in \{0, 1, \dots, N_i\}$. The weight $\lambda + \mu - N_i\alpha_i$ is a vertex of $\text{cvx } \mathcal{D}_{\lambda,\mu}$.
- (ii) All irreducible characters with highest weights of the form $M\lambda - k\alpha_i, i \in I_\lambda$, appear in χ_λ^M , and the weight $M\lambda - \lfloor \frac{M}{2} \rfloor \alpha_i$ is a vertex of $\text{cvx } \mathcal{D}_{M\lambda}$.

Proof: (i) The weights $\lambda - k\alpha_i$ lie on the edge of $\text{cvx } \Pi(\lambda)$ for which the corresponding Lie algebra is type A_1 with simple root α_i . The results now follow from Proposition 4.6 and the Clebsch-Gordon formula for products of irreducible characters of A_1 .

(ii) We argue by induction on M . In case $M = 2$, $N_i = \langle \lambda, \alpha_i \rangle$, and $2\lambda - k\alpha_i$ is dominant if and only if $k \leq N_i$, so the result follows from (i).

Suppose that (ii) holds for χ_λ^n when $n < M$. The weight $M\lambda - k\alpha_i$ is dominant if and only if $k \leq \lfloor \frac{M}{2} \rfloor \langle \lambda, \alpha_i \rangle$. We will find nonnegative integers k_1 and k_2 such that $k = k_1 + k_2$, $k_1 \leq \lfloor \frac{M-1}{2} \rfloor \langle \lambda, \alpha_1 \rangle$, and $k_2 \leq \min\{\langle \lambda, \alpha_1 \rangle, (M-1)\langle \lambda, \alpha_1 \rangle - 2k_1\}$. It will then follow from (i) and our induction assumption that

$$\chi_{M\lambda - k\alpha_1} \leq \chi_{(M-1)\lambda - k_1\alpha_1} \chi_\lambda \leq \chi_\lambda^{(M-1)} \chi_\lambda.$$

If $k \leq \lfloor \frac{M-1}{2} \rfloor \langle \lambda, \alpha_1 \rangle$, then we may take $k_1 = k$, and $k_2 = 0$. Otherwise, $k = \lfloor \frac{M}{2} \rfloor \langle \lambda, \alpha_1 \rangle - x$ and $x < \langle \lambda, \alpha_1 \rangle$. In that case we let $k_1 := \lfloor \frac{M-2}{2} \rfloor \langle \lambda, \alpha_1 \rangle$, and $k_2 := k - k_1$. Then $\min\{\langle \lambda, \alpha_1 \rangle, (M-1)\langle \lambda, \alpha_1 \rangle - 2k_1\} = \langle \lambda, \alpha_1 \rangle$, and $0 \leq k_2 = \langle \lambda, \alpha_1 \rangle - x \leq \langle \lambda, \alpha_1 \rangle$. *

Suppose that $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}$ are distinct simple roots such that $\langle \alpha_{i_r}, \alpha_{i_s} \rangle = 0$ if $r \neq s$.

Corollary A.7.

- (i) If $w := \lambda + \mu - k_1\alpha_{i_1} - \dots - k_m\alpha_{i_m}$, then the irreducible character χ_w appears in $\chi_\lambda\chi_\mu$ (with multiplicity 1 necessarily) if and only if $k_j \in \{0, 1, \dots, N_{i_j}\}$ for $j = 1, 2, \dots, m$. The weight $\lambda + \mu - N_{i_1}\alpha_{i_1} - \dots - N_{i_m}\alpha_{i_m}$ is a vertex of cvx $\mathcal{D}_{\lambda,\mu}$. (Note that this weight is the dominant weight conjugate to $\lambda + \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_m}\mu$.)
- (ii) All irreducible characters with highest weights of the form $M\lambda - k_1\alpha_{i_1} - \dots - k_m\alpha_{i_m}$ appear in χ_λ^M , and the weights $M\lambda - [\frac{M}{2}] \sum_{j=1}^m \alpha_{i_j}$ are vertices of $\mathcal{D}_{\lambda,\mu}$.

Proof: (i) The weights $\lambda - k_1\alpha_{i_1} - \dots - k_m\alpha_{i_m}$ lie on a face of cvx $\Pi(\lambda)$ for which the corresponding Lie algebra is a direct sum of m algebras of type A_1 with simple roots $\alpha_{i_1}, \dots, \alpha_{i_m}$. The irreducible character with highest weight $\lambda + \mu - k_1\alpha_{i_1} - \dots - k_m\alpha_{i_m}$ will therefore appear in the product $\chi_\lambda\chi_\mu$ if and only if, for each $j \in \{1, 2, \dots, m\}$ the irreducible character $\chi_{\lambda+\mu-k_j\alpha_{i_j}}$ appears there-in, so the claim follows from (i).

Set $J = I \setminus \{i_1, i_2, \dots, i_m\}$. The set of weights in $\mathcal{D}_{\lambda,\mu}$ lying on the face F^J is therefore an n -box (an n -cube like object with different length edges) whose set of vertices equals $\left\{ \lambda + \mu - \sum_{j \in K} N_{i_j} \alpha_{i_j} \mid K \subset \{1, 2, \dots, m\} \right\}$. If we suppose that the roots are ordered so that $N_{i_j} = \langle \lambda, \alpha_{i_j} \rangle$ for $j \leq k$ and $N_{i_j} = \langle \mu, \alpha_{i_j} \rangle$ otherwise, then $\lambda + \mu - \sum_{j=1}^m N_{i_j} \alpha_{i_j} = \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k} \lambda + \sigma_{i_{k+1}} \dots \sigma_{i_m} \mu$. Since all the reflections σ_{i_j} commute, the latter is conjugate to $\sigma_{i_1} \dots \sigma_{i_m} \lambda + \mu$.

- (ii) This follows from Proposition A.6(ii) and part (i). *

Using Proposition 5.1, we show in the next proposition that if β is an arbitrary positive root of A_d , then some weights of the form $\lambda + \mu - k\beta$ belong to $\mathcal{D}_{\lambda,\mu}$. If $\lambda = \mu$ then we show that all dominant weights of this form belong. (Note, however, that the weights $\lambda + \mu - k\beta$ usually do not lie on a face of cvx $\Pi(\lambda)$.)

Let $N_{j,m} \equiv N_{j,m}(\lambda, \mu) := \min\{\langle \lambda, \alpha_j \rangle, \langle \mu, \alpha_m \rangle\} + \min\{\langle \mu, \alpha_j \rangle, \langle \lambda, \alpha_m \rangle\}$, and suppose that β is a positive non-simple root of A_d . If $\beta = \alpha_j + \alpha_{j+1} + \dots + \alpha_m$ for some $1 \leq j < m \leq d$, let $N_\beta := N_{j,m}$.

Proposition A.8.

- (i) If k is an integer and $k \leq N_\beta$ then the irreducible character with highest weight $\lambda + \mu - k\beta$ appears in the product $\chi_\lambda\chi_\mu$.
- (ii) All irreducible characters with highest weights of the form $M\lambda - k\beta$ occur in χ_λ^M .

Proof: (i) We first consider the case where $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_d$ and $k \leq N_{1,d}$. Then we may write $k = k_1 + k_2$ where $k_1 \leq \min\{\langle \lambda, \alpha_1 \rangle, \langle \mu, \alpha_d \rangle\}$ and $k_2 \leq \min\{\langle \mu, \alpha_1 \rangle, \langle \lambda, \alpha_d \rangle\}$. By repeated application of Proposition 5.1, we have

$$\chi_\lambda\chi_\mu \geq \chi_{\lambda-k_1\lambda_1-k_2\lambda_d}\chi_{\mu-k_1\lambda_d-k_2\lambda_1}. \tag{A1}$$

Since the total highest weight of the product on the right hand side of (A1) equals $\lambda + \mu - (k_1 + k_2)(\lambda_1 + \lambda_d) = \lambda + \mu - k\beta$, the corresponding irreducible character appears in both products of (A1), thus (i) holds when $\beta = \alpha_1 + \dots + \alpha_d$. Now we argue by induction on the rank d . If $d = 2$, then we are done. If $d > 2$ and $\beta = \alpha_j + \dots + \alpha_m$ with either $j > 1$ or $m < d$, then the weights $\lambda + \mu - k\beta$ lie on the (proper) face $F^1(\lambda)$ or $F^d(\lambda)$ respectively of cvx $\Pi(\lambda)$, where the corresponding Lie algebra has rank $d - 1$. By Proposition 4.6 and our induction hypothesis, the assertion holds.

(ii) The weight $M\lambda - k(\alpha_1 + \alpha_2 + \cdots + \alpha_d)$ is dominant if and only if $k \leq M \min\{\langle \lambda, \alpha_1 \rangle, \langle \lambda, \alpha_d \rangle\}$. If $M = 2$ then this bound equals $N_{1,d}$, and (ii) holds in this case.

Assume (ii) holds for integers less than M if $\beta = \alpha_1 + \cdots + \alpha_d$, and suppose that $k \leq M \min\{\langle \lambda, \alpha_1 \rangle, \langle \lambda, \alpha_d \rangle\}$. We will find nonnegative integers k_1 and k_2 such that $k = k_1 + k_2$,

$$k_1 \leq \min\{(M-1)\langle \lambda, \alpha_d \rangle, (M-1)\langle \lambda, \alpha_1 \rangle\},$$

and

$$k_2 \leq \min\{(M-1)\langle \lambda, \alpha_1 \rangle - k_1, \langle \lambda, \alpha_d \rangle\} + \min\{(M-1)\langle \lambda, \alpha_d \rangle - k_1, \langle \lambda, \alpha_1 \rangle\}.$$

It will follow from (i) and induction that $\chi_{M\lambda - k\beta} \leq \chi_{(M-1)\lambda - k_1\beta} \chi_{k_2\beta} \leq \chi_\lambda^{(M-1)} \chi_{k_2\beta}$.

Assume for simplicity that $\langle \lambda, \alpha_d \rangle \leq \langle \lambda, \alpha_1 \rangle$. (An analogous argument works if $\langle \lambda, \alpha_1 \rangle \leq \langle \lambda, \alpha_d \rangle$.) If $k \leq (M-1)\langle \lambda, \alpha_d \rangle$ then let $k_1 := k$. Otherwise, say $k = M\langle \lambda, \alpha_d \rangle - x$, and let $k_1 := (M-2)\langle \lambda, \alpha_d \rangle$ and $k_2 := k - k_1 = 2\langle \lambda, \alpha_d \rangle - x$. Then

$$k_2 \leq 2\langle \lambda, \alpha_d \rangle$$

$$\leq \min\{(M-1)\langle \lambda, \alpha_1 \rangle - k_1, \langle \lambda, \alpha_d \rangle\} + \min\{(M-1)\langle \lambda, \alpha_d \rangle - k_1, \langle \lambda, \alpha_1 \rangle\},$$

so (ii) holds for all M if $\beta = \alpha_1 + \cdots + \alpha_d$. If $d = 2$ then we are done. Otherwise, if $\beta = \alpha_j + \cdots + \alpha_m$ with either $j > 1$ or $m < d$, then the weights $M\lambda - k\beta$ lie on the face $F^1(\lambda)$ or $F^d(\lambda)$ respectively of $\text{cvx } \mathcal{D}_{\lambda, \mu}$. The result now follows by Proposition 4.6 and induction. \ast

Let β_1, \dots, β_t be positive non-simple roots such that for $r \neq s$, $\langle \beta_r, \beta_s \rangle = 0$.

Corollary A.9.

- (i) If k_j are integers and $0 \leq k_j \leq N_{\beta_j}$ for $j = 1, 2, \dots, t$, then the irreducible character with highest weight $\lambda + \mu - k_1\beta_1 - \cdots - k_t\beta_t$ appears in $\chi_\lambda \chi_\mu$.
- (ii) All irreducible characters with highest weights of the form $M\lambda - k_1\beta_1 - \cdots - k_t\beta_t$ appear in χ_λ^M .

Proof: Observe that the weights $\lambda - k_1\beta_1 - \cdots - k_t\beta_t$ lie on a face of $\text{cvx } \Pi(\lambda)$ for which the corresponding Lie algebra is a direct sum of t algebras whose ranks are less than d . The assertions now follow from Proposition A.8 and Proposition 4.6. \ast

Unfortunately, we can not always account for all the weights of the form $\lambda + \mu - k\beta$ in $\mathcal{D}_{\lambda, \mu}$ using Proposition A.8 (unless $\lambda = \mu$; see the remark after Proposition A.10). For example, if we consider the irreducible characters of A_3 with highest weights $\lambda = \lambda_1 + \lambda_3$ and $\mu = \lambda_2$, then $N_{1,3} = 0$, but the irreducible character with highest weight $\lambda + \mu - (\alpha_1 + \alpha_2 + \alpha_3) = \lambda_2$ appears in the product $\chi_\lambda \chi_\mu$. On the other hand, we may be able to account for all weights on a ‘‘root line’’ if $(I_\lambda \cup I_\mu) \cap \{d', d' + 1, \dots, d\} = \emptyset$ for some d' . This holds trivially if $d' \leq \lfloor \frac{d+1}{2} \rfloor$ since in this case, all weights in $\mathcal{D}_{\lambda, \mu}$ lie on the face $F^{d'}(\lambda + \mu)$ (as we observed at the end of Chapter 5), and $N_{j,m} = 0$ for all pairs (j, m) .

For $j = 1, 2, \dots, \lfloor \frac{d+1}{2} \rfloor$, define

$$\omega_j := \lambda_j + \lambda_{d+1-j}$$

$$= \alpha_1 + 2\alpha_2 + \cdots + j\alpha_j + \cdots + j\alpha_{d+1-j} + \cdots + 2\alpha_{d-1} + \alpha_d$$

and let $M_j := N_{j, \alpha_{d+1-j}} = \min\{\langle \lambda, \alpha_j \rangle, \langle \mu, \alpha_{d+1-j} \rangle\} + \min\{\langle \lambda, \alpha_{d+1-j} \rangle, \langle \mu, \alpha_j \rangle\}$. Set $d' = \lfloor \frac{d+1}{2} \rfloor$.

Proposition A.10.

- (i) If $k_1, k_2, \dots, k_{d'}$ are nonnegative integers with $k_j \leq M_j$, then the irreducible character with highest weight $\lambda + \mu - k_1\omega_1 - \dots - k_{d'}\omega_{d'}$ appears in the product $\chi_\lambda\chi_\mu$.
- (ii) All irreducible characters with highest weights of the form $M\lambda - k_1\omega_1 - \dots - k_{d'}\omega_{d'}$ appear in χ_λ^M .

Proof: Part (i) is proved by repeated application of Proposition 5.1 as in the proof of Proposition A.8(i).

- (ii) The weight $M\lambda - \sum_{j=1}^{d'} k_j(\lambda_j + \lambda_{d+1-j})$ is dominant if and only if

$$k_j \leq M \min\{\langle \lambda, \alpha_j \rangle, \langle \lambda, \alpha_{d+1-j} \rangle\}.$$

If $M = 2$ then the upper bound corresponds with M_j . Otherwise, an induction argument analogous to the one given in the proof of Proposition A.8(ii) works (with α_j and α_{d+1-j} replacing α_1 and α_d respectively). *

Remark: If $\lambda = \mu$, then the weights $2\lambda - N_{j,m}(\alpha_j + \dots + \alpha_m)$ are vertices of $\mathcal{D}_{\lambda,\mu}$, as are the weights $2\lambda - M_1\omega_1 - \dots - M_p\omega_p$, since weights of the form $2\lambda - k(\alpha_j + \dots + \alpha_m)$ or $2\lambda - k_1\omega_1 - \dots - k_p\omega_p$ are dominant if and only if, respectively $k \leq N_{j,m}$ or $k_i \leq M_i$.

Finally, we can use Proposition 4.6 to extend the results of Propositions A.8 and A.10 to certain faces of the sets $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{M\lambda}$ when \mathcal{G} is type B_d, C_d , and D_d .

Let $\omega'_j := \alpha_1 + 2\alpha_2 + \dots + j\alpha_j + \dots + j\alpha_{d-j} + \dots + 2\alpha_{d-2} + \alpha_{d-1}$ for $j = 1, 2, \dots, [\frac{d}{2}]$.

Corollary A.11. Let \mathcal{G} be a Lie algebra of type B_d, C_d , or D_d .

- (i) For any positive non-simple root of the form $\beta = \alpha_j + \alpha_{j+1} + \dots + \alpha_m$ with $1 \leq j < m \leq d-1$, the irreducible character with highest weight $\lambda + \mu - k\beta$ appears in $\chi_\lambda\chi_\mu$ if $k \leq N_{j,m}$.
- (ii) All dominant weights of the form $M\lambda - k\beta$ appear in χ_λ^M .
- (iii) If there are nonnegative integers k_j such that $k_j \leq N_{j,d-j}$, then the irreducible character with highest weight $\lambda + \mu - k_1\omega'_1 - \dots - k_{[\frac{d}{2}]} \omega'_{[\frac{d}{2}]}$ appears in $\chi_\lambda\chi_\mu$.
- (iv) All dominant weights of the form $M\lambda - k_1\omega'_1 - \dots - k_{[\frac{d}{2}]} \omega'_{[\frac{d}{2}]}$ appear in χ_λ^M .

Proof: The relative face $F^d(\lambda)$ of $\text{cvx } \Pi(\lambda)$ corresponds to the Lie algebra \mathcal{G}^d of type A_{d-1} with simple roots $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$. The claims now follow from Proposition 4.6 and Propositions A.8 and A.10. *

Remark: Parts (i) and (ii) also hold for roots of D_d of the form $\alpha_1 + \dots + \alpha_{d-2} + \alpha_d$, and parts (iii) and (iv) hold if we replace α_{d-1} with α_d in the definition of ω'_j .

Appendix B

A comparison lemma for powers of χ_{λ_2}

In order to show that, for characters of A_d , $\chi_{\lambda_2}^n$ is saturated when $n < d+1$ (Proposition 6.7), we require the following Lemma.

Lemma B.1. If $\lceil \frac{d+1}{2} \rceil \leq n < d+1$, then $\chi_{\lambda_1}^{2n-(d+1)} \leq \chi_{\lambda_2}^n$.

Proof: Let $y := 2n - (d+1)$. Suppose that $\chi_w \leq \chi_{\lambda_1}^y$, and let T_w be a tableau representing χ_w in the product. Then the base of T_w consists of a single blank box, and there is exactly one box with entry $(i, 1)$ for each $i \in \{1, 2, \dots, y-1\}$. We will construct a tableau T'_w representing χ_w in the product $\chi_{\lambda_2}^n$ as follows. Let $x := d+1 - n = n - y$. Replace each entry $(i, 1)$ in T_w with the entry $(x+i, 1)$. Now add a box with entry $(1, 1)$ to row 1, a blank box to row 2, and boxes with entries $(1, 2), (2, 1), (2, 2), \dots, (x, 1), (x, 2)$ to rows 3 to $2x+1$ respectively, and boxes with entries $(x+1, 2), \dots, (x+y-1, 2)$ to rows $2x+2$ to $d+1$ respectively. We claim that the resulting tableau T'_w represents χ_w in the product $\chi_{\lambda_2}^n$. Observe that T'_w has a base of shape λ_2 , and includes one box with entry $(i, 1)$ and one with entry $(i, 2)$ for each $i \in \{1, 2, \dots, n-1\}$. The condition MLR2 is satisfied, since for each $i \in \{1, 2, \dots, x\}$, the entry $(i, 1)$ lies above the entry $(i, 2)$ by construction, and for each $i \in \{1, 2, \dots, y-1\}$, the entry $(x+i, 1)$ lies in row $j \leq i+1$, while the entry $(x+i, 2)$ lies in row $2x+i+1$. It follows from Lemma 5.2 that the entries in rows 2 to $d+1$ of T'_w increase strictly down columns. The entries in rows 1 and 2, column k increase strictly when $k > 2$ (since these equal the entries in rows 1 and 2 column $k-1$ of T_w), so we simply have to check that entry in row 1, column 2 is strictly less than that in row 2, column 2. This is obvious, since the entry in row 1, column 2 equals $(1,1)$, which is strictly less than any other entry in T'_w . *

Appendix C

Normality of certain cones C_λ for E_d

We will show that the cones C_λ are normal when \mathcal{G} is type E_d and $\{1\} \subset I_\lambda \subset \{1, 6, 7, 8\}$. We identify the root lattice with \mathbf{Z}^d , and note that $\mathcal{L}(C_\lambda) = \mathbf{Z}^d$ by Lemma 1.5.

Suppose that x belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. Recall that if $x = (x_1, \dots, x_d)$ (with respect to the basis of simple roots) then $x_i \equiv f_i(x)$ and $(\tau x)_i \equiv \tau^{-1} f_i(x)$ are nonnegative integers for all $i \in I$ and $\tau \in W_\lambda$. If $\tau f_i(x) = 0$ for some τ and i , then x lies on the face τF^i of $\text{cvx } C_\lambda$. Now any face τF^i of $\text{cvx } C_\lambda$ is a convex cone generated by the elements of $\mathcal{E}_\lambda \cap \tau F^i$. A generator $\delta \in \mathcal{E}_\lambda$ lies on the face F^i if and only if $\delta_i = 0$. Let $\hat{\delta} := (\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_d)$. We deduce from the description of the sets \mathcal{E}_λ in Chapter 3 (or from Lemma 4.4) that $\{\hat{\delta} \mid \delta \in F^i\}$ is the set of generators $\mathcal{E}_{p_i\lambda}$ of the cone $C_{p_i\lambda}$ of \mathcal{G}^i (where $p_i\lambda = (l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_d)$ with respect to the basis of fundamental weights of \mathcal{G}^i). By the results in Chapter 3, these cones are normal when $i \leq 6$, thus if $x \in F^i \cap \mathbf{Z}^d$ then $x \in C_\lambda$. (In case $d = 7$ or $d = 8$ and $i > 6$, we will argue by induction that the corresponding cone is normal and $x \in C_\lambda$.) Since C_λ is invariant under W_λ , x belongs to C_λ whenever $\tau f_i(x) = 0$ for some τ and i . In order to show that C_λ is normal, we must therefore show that x belongs to C_λ whenever $\tau f_i(x)$ is a positive integer for all τ in W_λ and i in I_λ .

In case I_λ is one of $\{1, 6\}$, $\{1, 7\}$, or $\{1, 8\}$, we argue as follows. In each of these cases, there is a generator δ^0 of C_λ , and a proper subset I^0 of I such that $\tau f_i(\delta^0) \in \{0, 1, 2\}$ for all $\tau \in W_\lambda$, $i \in I$, and, if $i \notin I^0$, $\tau f_i(\delta^0) \in \{0, 1\}$. We show that if $x \in \text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $\tau f_i(x) = 1$ for some $i \in I^0$, then $x \in C_\lambda$. Then we observe that there is a nonnegative integer m such that $y := x - m\delta^0$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and either y lies on a face of $\text{cvx } C_\lambda$ or $\tau f_i(y) = 1$ for some $\tau \in W_\lambda$ and $i \in I^0$. Thus $y \in C_\lambda$ and consequently $x \in C_\lambda$. We deduce normality in all the remaining cases ($\{1, 6\} \subset I_\lambda$ or $\{1, 7\} \subset I_\lambda$) from Lemma 3.6(ii).

For convenience, we order the roots as follows: the simple roots corresponding to nodes in the graph Γ of \mathcal{G} along the straight line of length 5, 6, or 7 are labelled 1, 2, 3, 5, 6, ... —the “branch” node here is labelled by 3. A list of generators for the cones C_λ corresponding to the cases $I_\lambda = \{1, 6\}$, $I_\lambda = \{1, 7\}$ and $I_\lambda = \{1, 8\}$ is included at the end of this appendix.

Assume that $x \in \text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$. The first Lemma applies to all cases.

Lemma C.1. If $x_3 = 1$, then $x \in C_\lambda$.

Proof: Note that all generators δ of C_λ satisfy $\delta_3 \geq \delta_4$, so $x_3 \geq x_4$. By assumption, $x_4 > 0$, so $x_3 = x_4 = 1$, and x belongs to the convex hull of the cone C' generated by $\{\delta \in \mathcal{E}_\lambda \mid \delta_3 = \delta_4\}$. The latter consists of elements of the form e_J where $J \subset I$ is an interval, hence x belongs to C_λ by Lemma 3.4(i).*

Since $x \in C_\lambda$ if and only if $\tau x \in C_\lambda$ for all $\tau \in W_\lambda$, it follows from Lemma C.1 that $x \in C_\lambda$ if $(\tau x)_3 = 1$ for some $\tau \in W_\lambda$.

1. Cases $\{1, 6\} \subset I_\lambda \subset \{1, 6, 7, 8\}$

We begin with the case $I_\lambda = \{1, 6\}$. Suppose that $x \in \text{cvx } C_\lambda \cap \mathbf{Z}^d$, and $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$.

Lemma C.2. If $x_2 = 1$ then $x \in C_\lambda$.

Proof: By assumption, $x_1 \neq 0$, so $x_1 \geq x_2$. There is exactly one generator $\delta^* := (1, 0, 0, \dots, 0)$ of C_λ for which $\delta_1 > \delta_2$. Therefore, the element $x' := x - (x_1 - x_2)\delta^*$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and satisfies $x'_1 = x'_2 = 1$. We will show that x' belongs to C_λ .

We may write

$$x' = \sum_{\beta_1 = \beta_2} r_\beta \beta + \sum_{\gamma_2 > \gamma_1} r_\gamma (\gamma + \delta^*)$$

where the first sum runs over those generators β of C_λ with $\beta_1 = \beta_2$ and the second over generators γ with $\gamma_2 - \gamma_1 = 1$, and the coefficients r_β and r_γ are nonnegative. We verify below that each of the sums $\gamma + \delta^*$ can be written as a sum of generators β with $\beta_1 = \beta_2$. The generator $(\delta_1, \dots, \delta_6, a, b)$ below should be interpreted as $(\delta_1, \dots, \delta_6)$ if $d = 6$, as $(\delta_1, \dots, \delta_6, a)$ with $a \in \{0, 1\}$ if $d = 7$, and as $(\delta_1, \dots, \delta_6, a, b)$ with $(a, b) \in \{(0, 0), (1, 0), (1, 1)\}$ if $d = 8$.

$$\begin{aligned} (1, 0, \dots, 0) + (1, 2, 2, 1, 1, 0, 0, 0) &= (1, 1, 0, \dots, 0) && + (1, 1, 2, 1, 1, 0, 0, 0) \\ (1, 0, \dots, 0) + (0, 1, 1, 0, 1, 1, a, b) &= (1, 1, 0, \dots, 0) && + (0, 0, 1, 0, 1, 1, a, b) \\ (1, 0, \dots, 0) + (0, 1, 1, 1, 1, 1, a, b) &= (1, 1, 0, \dots, 0) && + (0, 0, 1, 1, 1, 1, a, b) \\ (1, 0, \dots, 0) + (0, 1, 2, 1, 1, 1, a, b) &= (1, 1, 1, 0, \dots, 0) && + (0, 0, 1, 1, 1, 1, a, b) \\ (1, 0, \dots, 0) + (0, 1, 2, 1, 2, 1, a, b) &= (1, 1, 1, 0, 1, 0, \dots, 0) && + (0, 0, 1, 1, 1, 1, a, b). \end{aligned}$$

Thus x' belongs to the convex hull of the cone C' generated by the set $\{\delta \in \mathcal{E}_\lambda \mid \delta_1 = \delta_2\}$. The cone C' is isomorphic to the cone $C_{\lambda'}$ of D_{d-1} with $I_{\lambda'} = \{d-5, d-1\}$. (To see this, map $(\delta_1, \delta_2, \dots, \delta_d)$ to $(\delta_d, \delta_{d-1}, \dots, \delta_5, \delta_3, \delta_4, \delta_2)$.) The cone C' is therefore normal by the results in Chapter 3. Since the group generated by C' is also generated by intervals, x' belongs to $\text{cvx } C' \cap \mathcal{L}(C')$ by Lemma 3.3, so $x' \in C'$ and $x \in C_\lambda$.*

Proposition C.3. If $I_\lambda = \{1, 6\}$, then C_λ is normal.

Proof: Suppose that $x \in \text{cvx } C_\lambda \cap \mathbf{Z}^d$. If x lies on a face of $\text{cvx } C_\lambda$, then $x \in C_\lambda$ (we argue by induction on the rank in case $d = 7$ or $d = 8$ and $i > 6$). Otherwise, $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$. Let $\delta^0 := (1, 2, 2, 1, 1, 0)$ if $d = 6$, $\delta^0 := (1, 2, 2, 1, 1, 0, 0)$ if $d = 7$, and $\delta^0 := (1, 2, 2, 1, 1, 0, 0, 0)$ if $d = 8$. Note that δ^0 is a generator of C_λ , $\tau f_i(\delta^0) \in \{0, 1, 2\}$, and, if $i \notin \{2, 3\}$, $\tau f_i(\delta^0) \in \{0, 1\}$. Let n be the minimum of the set

$$\{\tau f_i(x) \mid \tau \in W_\lambda, i \notin \{2, 3\}\} \cup \left\{ \left\lceil \frac{\tau f_i(x)}{2} \right\rceil \mid \tau \in W_\lambda, i \in \{2, 3\} \right\}.$$

Then $y := x - n\delta^0$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and either y lies on a face of $\text{cvx } C_\lambda$, or $\tau f_i(y) = 1$ for some $\tau \in W_\lambda, i \in \{2, 3\}$. In any case, y belongs to C_λ (we invoke Lemma C.1 or C.2 in the latter case), hence $x \in C_\lambda$.*

Corollary C.4. If $\{1, 6\} \subset I_\lambda$ then C_λ is normal.

Proof: We have to prove the assertion when $\{1, 6\}$ is a proper subset of I_λ . This follows from Lemma 3.5(ii). (If $\{1, 6, 7\} \subset I_\lambda$ then the generator $\delta = \alpha_7$ of C_λ satisfies the condition of Lemma 3.5(ii); if $\{1, 6, 8\} \subset I_\lambda$, then $\delta = \alpha_8$ works.)*

2. Cases $\{1, 7\} \subset I_\lambda \subset \{1, 7, 8\}$.

We consider the case $I_\lambda = \{1, 7\}$ first.

Suppose that $x \in \text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$.

Lemma C.5. If $x_6 = 1$, or $x_5 = 1$, then $x \in C_\lambda$.

Proof: We will consider the cases $d = 7$ and $d = 8$ separately.

In case $d = 7$, we argue as in Lemma C.2. Suppose that $x_6 = 1$. Since x does lie on a face of $\text{cvx } C_\lambda$, $x_7 \geq x_6$. There is a single generator $\delta^* := (0, 0, 0, 0, 0, 0, 1)$ of C_λ satisfying $\delta_7 > \delta_6$, so $x' := x - (x_7 - x_6)\delta^*$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $x'_6 = x'_7 = 1$. We may therefore write

$$x' = \sum_{\beta_6 = \beta_7} r_\beta \beta + \sum_{\gamma_6 > \gamma_7} r_\gamma (\gamma + \delta^*),$$

where the coefficients are nonnegative real numbers, the first sum runs over those generators β of C_λ with $\beta_6 = \beta_7$, and the second over generators with $\gamma_6 > \gamma_7$. (In case $\gamma_6 > \gamma_7$, the difference $\gamma_6 - \gamma_7$ equals 1.) We verify below that each of the sums $\gamma + \delta^*$ can be expressed as a sum of generators β with $\beta_6 = \beta_7$.

$$\begin{aligned} (0, 0, 0, 0, 0, 0, 1) + (1, 1, 1, 0, 1, 1, 0) &= (1, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 1, 1, 1, 1, 1, 0) &= (1, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 1, 2, 1, 1, 1, 0) &= (1, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 2, 2, 1, 1, 1, 0) &= (1, 1, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 1, 2, 1, 2, 1, 0) &= (1, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 2, 2, 1, 2, 1, 0) &= (1, 1, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 1, 2, 1, 0) &= (1, 1, 1, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 2, 2, 1, 0) &= (1, 1, 1, 1, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1) + (0, 1, 2, 1, 2, 2, 1) &= (0, 1, 1, 1, 1, 1, 1) + (0, 0, 1, 0, 1, 1, 1) \end{aligned}$$

Thus x' belongs to the convex hull of the cone C' generated by $\{\delta \in \mathcal{E}_\lambda \mid \delta_6 = \delta_7\}$. This cone is isomorphic to the cone $C_{\lambda'}$ of E_6 with $I_{\lambda'} = \{1, 6\}$ (map $(\delta_1, \dots, \delta_6, \delta_7)$ to $(\delta_1, \dots, \delta_6)$), so is normal by Proposition C.3. Since the group $\mathcal{L}(C')$ is generated by intervals, $x' \in \text{cvx } C' \cap \mathcal{L}(C') = C'$, so $x \in C_\lambda$.

If $x_5 = 1$, then $x_6 \geq x_5$. The generator $\delta' := (0, 0, 0, 0, 0, 1, 1)$ is the single generator of C_λ with $\delta_6 > \delta_5$, thus $x'' := x - (x_6 - x_5)\delta'$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $x''_6 = x''_5 = 1$. By the previous argument, $x'' \in C_\lambda$, thus $x \in C_\lambda$.

In case $d = 8$, there is a single generator $\delta^* := (0, 0, 0, 0, 0, 0, 1, 1)$ of C_λ with $\delta_8 > \delta_6$. If $x_6 = 1$, then $x' := x - (x_8 - x_6)\delta^*$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and satisfies $x_6 = x_8 = 1$. We will show that x may be expressed as a combination of generators β of C_λ with $\beta_6 = \beta_8$. Note first

that if $\delta \in \mathcal{E}_\lambda$ and $\delta_6 > \delta_8$, then $\delta_6 - \delta_8 = 1$, unless $\delta = \zeta := (0, 1, 2, 1, 2, 2, 1, 0)$. Thus we have an expression

$$x' = \sum_{\beta_6 = \beta_8} r_\beta \beta + \sum_{\gamma_6 - \gamma_8 = 1} r_\gamma (\gamma + \delta^*) + r_\zeta (\zeta + 2\delta^*),$$

where the coefficients are nonnegative real numbers.

We now verify that each of the sums $\gamma + \delta^*$ and $\zeta + 2\delta^*$ may be expressed as a sum of generators β with $\beta_6 = \beta_8$.

$$\begin{aligned} (0, 0, 0, 0, 0, 0, 1, 1) + (1, 1, 1, 0, 1, 1, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 1, 1, 1, 1, 1, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 1, 2, 1, 1, 1, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 2, 2, 1, 1, 1, 0, 0) &= (1, 1, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 1, 2, 1, 2, 1, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 2, 2, 1, 2, 1, 0, 0) &= (1, 1, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 2, 3, 1, 2, 1, 0, 0) &= (1, 1, 1, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (1, 2, 3, 2, 2, 1, 0, 0) &= (1, 1, 1, 1, 0, 0, 0, 0) + (0, 1, 2, 1, 2, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 0, 0, 0, 0, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 0, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 0, 0, 0, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 0, 1, 0, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 0, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 1, 1, 0, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 1, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 0, 1, 1, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 0, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 1, 1, 1, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 1, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 1, 2, 1, 1, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 1, 2, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 0, 1, 1) + (0, 1, 2, 1, 2, 1, 1, 0) &= (0, 0, 0, 0, 0, 0, 1, 0) + (0, 1, 2, 1, 2, 1, 1, 1), \end{aligned}$$

$$\begin{aligned} 2(0, 0, 0, 0, 0, 0, 1, 1) + (0, 1, 2, 1, 2, 2, 1, 0) &= \\ (0, 0, 0, 0, 0, 0, 1, 0) + (0, 1, 2, 1, 2, 1, 1, 1) + (0, 0, 0, 0, 0, 1, 1, 1). \end{aligned}$$

Thus x' belongs to the convex hull of the cone C' generated by $\{\delta \in \mathcal{E}_\lambda \mid \delta_6 = \delta_8\}$. Observe that if $\delta \in \mathcal{E}_\lambda$ and $\delta_6 = \delta_8$ then $\delta_6 = \delta_7 = \delta_8$. The cone C' is therefore isomorphic to the cone $C_{\lambda'}$ of E_6 with $I_{\lambda'} = \{1, 6\}$ (map $(x_1, \dots, x_6, x_7, x_8)$ to (x_1, \dots, x_6)), so C' is normal, and $x' \in C'$. Thus $x \in C_\lambda$.

Suppose that $x_5 = 1$. Replace x by $x - (x_8 - x_6)\delta^*$ if $x_8 > x_6$, so that $x_6 \leq x_8$. We may therefore express x as a combination, with nonnegative real coefficients, of generators $\delta \neq \delta^*$. (In an expression for x involving δ^* , we replace sums $\gamma + \delta^*$ (where $\gamma_6 > \gamma_8$), and $\zeta + 2\delta^*$ with sums of generators β satisfying $\beta_6 = \beta_8$, as above). By assumption, $x_8 \geq x_5 = 1$, and among the generators $\delta \neq \delta^*$ of C_λ , there is a single generator $\delta' := (0, 0, 0, 0, 0, 1, 1, 1)$ such

that $x_8 > x_5$. Thus $x' := x - (x_8 - x_5)\delta'$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$ and $x'_5 = x'_8$. We therefore have an expression

$$x' = \sum_{\beta_5=\beta_8} r_\beta + \sum_{\gamma_5-\gamma_8=1} r_\gamma(\gamma + \delta') + \sum_{\gamma_5-\gamma_8=2} r_\gamma(\gamma + 2\delta').$$

Our goal is to obtain an expression for x' in which δ' is the only generator involved (with non zero coefficient) for which $\delta_5 \neq \delta_6$. We note that if β is a generator of C_λ and $\beta_5 = \beta_8$, then $\beta_5 = \beta_6 = \beta_8$. We verify below that if γ is a generator of C_λ and $\gamma_5 - \gamma_8 = 1$, then either $\gamma_5 = \gamma_6$ or we may write $\gamma + \delta'$ as a sum of generators β with $\beta_5 = \beta_6 = \beta_8$. We also check that if $\gamma_5 - \gamma_8 = 2$, and $\gamma_5 > \gamma_6$, then $\gamma + \delta'$ equals a sum of generators $\beta + \gamma'$, where $\beta_5 = \beta_6 = \beta_8$ and $\gamma'_5 = \gamma'_6 = \gamma'_8 + 1$.

$$\begin{aligned} (0, 0, 0, 0, 0, 1, 1, 1) + (1, 1, 1, 0, 1, 0, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 1, 1, 1, 1, 0, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 1, 2, 1, 1, 0, 0, 0) &= (1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 2, 2, 1, 1, 0, 0, 0) &= (1, 1, 0, 0, 0, 0, 0, 0) + (0, 1, 2, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 1, 2, 1, 2, 1, 0, 0) &= (1, 1, 2, 1, 1, 1, 0, 0) + (0, 0, 0, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 2, 2, 1, 2, 1, 0, 0) &= (1, 2, 2, 1, 1, 1, 0, 0) + (0, 0, 0, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 2, 3, 1, 2, 1, 0, 0) &= (1, 2, 2, 1, 1, 1, 0, 0) + (0, 0, 1, 0, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (1, 2, 3, 2, 2, 1, 0, 0) &= (1, 2, 2, 1, 1, 1, 0, 0) + (0, 0, 1, 1, 1, 1, 1, 1) \\ (0, 0, 0, 0, 0, 1, 1, 1) + (0, 1, 2, 1, 2, 1, 1, 0) &= (0, 1, 1, 1, 1, 1, 1, 0) + (0, 0, 1, 0, 1, 1, 1, 1). \end{aligned}$$

Consequently,

$$x' = \sum_{\beta_5=\beta_6=\beta_8} r_\beta \beta + \sum_{\gamma_5=\gamma_6=\gamma_8+1} r_\gamma(\gamma + \delta') + r_\zeta(\zeta + 2\delta'),$$

where $\zeta = (0, 1, 2, 1, 2, 2, 1, 0)$, and the coefficients are all nonnegative.

Now if $x'_6 > x'_5$, then the difference $x_6 - x_5$ equals $2\zeta + \sum r_\gamma$, since δ' is the only generator in the above expression with $x_6 > x_5$, and $x'' := x' - (x_6 - x_5)\delta'$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. Since $x''_5 = x''_6 = 1$, $x'' \in C_\lambda$ so $x \in C_\lambda$. *

Proposition C.6. If $I_\lambda = \{1, 7\}$ then C_λ is normal.

Proof: Suppose that x belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. If x lies on a face of $\text{cvx } C_\lambda$, then $x \in C_\lambda$. (In case $d = 8$ and $x \in \sigma F^7(\lambda)$, we argue by induction on the rank.) Otherwise, $\tau f_i(x) > 0$ for all $i \in I$ and all $\tau \in W_\lambda$.

Let $\delta^0 := (0, 1, 2, 1, 2, 2, 1)$ if $d = 7$ and $\delta^0 = (0, 1, 2, 1, 2, 2, 1, 0)$ if $d = 8$. Then $\delta^0 \in \mathcal{E}_\lambda$, and for all $\tau \in W_\lambda$, $\tau f_i(\delta^0) \in \{0, 1, 2\}$, and, if $i \notin \{3, 5, 6\}$, $\tau f_i(\delta^0) \in \{0, 1\}$. Let n be the minimum of the set

$$\{\tau f_i(x) \mid \tau \in W_\lambda, i \in I \setminus \{3, 5, 6\}\} \cup \left\{ \left\lfloor \frac{\tau f_i(x)}{2} \right\rfloor \mid \tau \in W_\lambda, i \in \{3, 5, 6\} \right\}.$$

Since $\tau f_i(\delta^0) \in \{0, 1, 2\}$ for all $\tau \in W_\lambda$ and $i \in I$, and $\tau f_i(\delta^0) \in \{0, 1\}$ if $i \notin \{3, 5, 6\}$, the element $y := x - n\delta^0$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. Either y lies on a face of $\text{cvx } C_\lambda$ or $(\tau y)_i = 1$ for some $\tau \in W_\lambda$ and $i \in \{3, 5, 6\}$. In the former case, y belongs to C_λ ; in the latter, τy belongs to C_λ by Lemma C.5, so y does as well. Thus x belongs to C_λ .*

Corollary C.7. If $I_\lambda = \{1, 7, 8\}$, then C_λ is normal.

Proof: The assertion follows from Lemma 3.5(ii), with $\delta_0 = \alpha_8$.*

3. Finally, suppose that $d = 8$ and $I_\lambda = \{1, 8\}$. The proof that C_λ is normal is analogous to that for the case $I_\lambda = \{1, 7\}$ when $d = 7$.

Suppose that $x \in \text{cvx } C_\lambda$ and $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$.

Lemma C.8. If $x_5 = 1$, $x_6 = 1$, or $x_7 = 1$, then $x \in C_\lambda$.

Proof: Suppose that $x_7 = 1$. We have $x_8 \geq x_7$, and a single generator $\delta^* := (0, 0, 0, 0, 0, 0, 0, 1)$ of C_λ satisfying $\delta_8 > \delta_7$, so $x' := x - (x_8 - x_7)\delta^*$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. Since $x'_7 = x'_8$, we may write

$$x' = \sum_{\beta_7 = \beta_8} r_\beta + \sum_{\gamma_7 > \gamma_8} r_\gamma(\gamma + \delta^*).$$

(In case $\gamma_7 > \gamma_8$, the difference $\gamma_7 - \gamma_8$ equals 1.) We verify that each of the sums $\gamma + \delta^*$ can be written as a sum of generators β with $\beta_7 = \beta_8$.

$$\begin{aligned} (0, 0, 0, 0, 0, 0, 0, 1) + (1, 1, 1, 0, 1, 1, 1, 0) &= (0, 1, 1, 0, 1, 1, 1, 1) + (1, 0, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 1, 1, 1, 1, 1, 1, 0) &= (0, 1, 1, 1, 1, 1, 1, 1) + (1, 0, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 1, 2, 1, 1, 1, 1, 0) &= (0, 1, 2, 1, 1, 1, 1, 1) + (1, 0, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 2, 1, 1, 1, 1, 0) &= (0, 1, 2, 1, 1, 1, 1, 1) + (1, 1, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 1, 2, 1, 2, 1, 1, 0) &= (0, 1, 2, 1, 2, 1, 1, 1) + (1, 0, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 1, 2, 1, 2, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 0, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 2, 1, 2, 1, 1, 0) &= (0, 1, 2, 1, 2, 1, 1, 1) + (1, 1, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 1, 2, 1, 1, 0) &= (0, 1, 2, 1, 2, 1, 1, 1) + (1, 1, 1, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 2, 2, 1, 1, 0) &= (0, 1, 2, 1, 2, 1, 1, 1) + (1, 1, 1, 1, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 2, 1, 2, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 1, 2, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 1, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 1, 3, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 1, 0, 1, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 2, 2, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 1, 1, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 3, 2, 3, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 1, 1, 1, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 2, 4, 2, 3, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 1, 2, 1, 1, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (1, 3, 4, 2, 3, 2, 1, 0) &= (0, 1, 2, 1, 2, 2, 1, 1) + (1, 2, 2, 1, 1, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0, 0, 1) + (0, 1, 2, 1, 2, 2, 2, 1) &= (0, 1, 2, 1, 2, 2, 1, 1) + (0, 0, 0, 0, 0, 0, 1, 1). \end{aligned}$$

Thus x' belongs to the convex hull of the cone C' generated by $\{\delta \in \mathcal{E}_\lambda \mid \delta_7 = \delta_8\}$. The cone C' is isomorphic to the cone $C_{\lambda'}$ for E_7 with $I_{\lambda'} = \{1, 7\}$ (via the map $(\delta_1, \dots, \delta_7, \delta_8) \mapsto (\delta_1, \dots, \delta_7)$), so $x' \in C' \cap \mathcal{L}(C') = C'$, and therefore $x \in C$.

Suppose the $x_6 = 1$, so that $x_7 \geq x_6$. Since $\delta' := (0, 0, 0, 0, 0, 0, 1, 1)$ is the only generator of C_λ satisfying $\delta_7 > \delta_6$, the element $x' := x - (x_7 - x_6)\delta'$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. It follows from the previous argument that $x' \in C_\lambda$ since $x'_7 = x'_6 = 1$, thus $x \in C_\lambda$.

If $x_5 = 1$, then $x_6 \geq x_5$. Again, there is a single generator $\delta'' := (0, 0, 0, 0, 0, 1, 1, 1)$ of C_λ satisfying $\delta_6 > \delta_5$, so $x' := x - (x_6 - x_5)\delta''$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. Since $x'_6 = x'_5 = 1$, it follows from the preceding paragraph that $x' \in C_\lambda$, hence $x \in C_\lambda$. *

Proposition C.9. If $I_\lambda = \{1, 8\}$ then C_λ is normal.

Proof: Suppose that x belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$. If x lies on a face of $\text{cvx } C_\lambda$, then x belongs to C_λ . Otherwise, $\tau f_i(x) > 0$ for all $\tau \in W_\lambda$ and $i \in I$. Let $\delta^0 := (0, 1, 2, 1, 2, 2, 2, 1)$. Then $\tau f_i(\delta^0) \in \{0, 1, 2\}$, and $\tau f_i(\delta^0) \in \{0, 1\}$ if $i \notin \{3, 5, 6, 7\}$. If we let n equal the minimum of the set

$$\{\tau f_i(x) \mid \tau \in W_\lambda, i \notin \{3, 5, 6, 7\}\} \cup \left\{ \left\lfloor \frac{\tau f_i(x)}{2} \right\rfloor \mid \tau \in W_\lambda, i \in \{3, 5, 6, 7\} \right\},$$

then $y := x - n\delta^0$ belongs to $\text{cvx } C_\lambda \cap \mathbf{Z}^d$, and either y lies on a face of $\text{cvx } C_\lambda$, or $\tau f_i(y) = 1$ for some $\tau \in W_\lambda$ and $i \in \{3, 5, 6, 7\}$. In any case, it follows that $y \in C_\lambda$, hence $x \in C_\lambda$. *

Generators of C_λ

We list here the elements of the set \mathcal{E}_λ of generators of C_λ in case I_λ is one of $\{1, 6\}$, $\{1, 7\}$, or $\{1, 8\}$.

The generators listed below are for E_8 ; to obtain those for E_6 or E_7 , just take the first 6 (respectively 7) coordinates. The lists are each separated into 2 W_λ orbits.

$I_\lambda = \{1, 6\}$:

$$\begin{aligned} &(1, 0, 0, 0, 0, 0, 0, 0) \\ &(1, 1, 0, 0, 0, 0, 0, 0) \\ &(1, 1, 1, 0, 0, 0, 0, 0) \\ &(1, 1, 1, 0, 1, 0, 0, 0) \\ &(1, 1, 1, 1, 0, 0, 0, 0) \\ &(1, 1, 1, 1, 1, 0, 0, 0) \\ &(1, 1, 2, 1, 1, 0, 0, 0) \\ &(1, 2, 2, 1, 1, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} &(0, 0, 0, 0, 0, 1, 0, 0), & (0, 0, 0, 0, 0, 1, 1, 0), & (0, 0, 0, 0, 0, 1, 1, 1) \\ &(0, 0, 0, 0, 1, 1, 0, 0), & (0, 0, 0, 0, 1, 1, 1, 0), & (0, 0, 0, 0, 1, 1, 1, 1) \\ &(0, 0, 1, 0, 1, 1, 0, 0), & (0, 0, 1, 0, 1, 1, 1, 0), & (0, 0, 1, 0, 1, 1, 1, 1) \\ &(0, 1, 1, 0, 1, 1, 0, 0), & (0, 1, 1, 0, 1, 1, 1, 0), & (0, 1, 1, 0, 1, 1, 1, 1) \\ &(0, 0, 1, 1, 1, 1, 0, 0), & (0, 0, 1, 1, 1, 1, 1, 0), & (0, 0, 1, 1, 1, 1, 1, 1) \\ &(0, 1, 1, 1, 1, 1, 0, 0), & (0, 1, 1, 1, 1, 1, 1, 0), & (0, 1, 1, 1, 1, 1, 1, 1) \\ &(0, 1, 2, 1, 1, 1, 0, 0), & (0, 1, 2, 1, 1, 1, 1, 0), & (0, 1, 2, 1, 1, 1, 1, 1) \\ &(0, 1, 2, 1, 2, 1, 0, 0), & (0, 1, 2, 1, 2, 1, 1, 0), & (0, 1, 2, 1, 2, 1, 1, 1) \end{aligned}$$

$I_\lambda = \{1, 7\}$:

(1, 0, 0, 0, 0, 0, 0, 0)
(1, 1, 0, 0, 0, 0, 0, 0)
(1, 1, 1, 0, 0, 0, 0, 0)
(1, 1, 1, 0, 1, 0, 0, 0)
(1, 1, 1, 1, 0, 0, 0, 0)
(1, 1, 1, 1, 1, 0, 0, 0)
(1, 1, 2, 1, 1, 0, 0, 0)
(1, 2, 2, 1, 1, 0, 0, 0)
(1, 1, 1, 0, 1, 1, 0, 0)
(1, 1, 1, 1, 1, 1, 0, 0)
(1, 1, 2, 1, 1, 1, 0, 0)
(1, 2, 2, 1, 1, 1, 0, 0)
(1, 1, 2, 1, 2, 1, 0, 0)
(1, 2, 2, 1, 2, 1, 0, 0)
(1, 2, 3, 1, 2, 1, 0, 0)
(1, 2, 3, 2, 2, 1, 0, 0)

(0, 0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 0, 1, 1)
(0, 0, 0, 0, 0, 1, 1, 0), (0, 0, 0, 0, 0, 1, 1, 1)
(0, 0, 0, 0, 1, 1, 1, 0), (0, 0, 0, 0, 1, 1, 1, 1)
(0, 0, 1, 0, 1, 1, 1, 0), (0, 0, 1, 0, 1, 1, 1, 1)
(0, 1, 1, 0, 1, 1, 1, 0), (0, 1, 1, 0, 1, 1, 1, 1)
(0, 0, 1, 1, 1, 1, 1, 0), (0, 0, 1, 1, 1, 1, 1, 1)
(0, 1, 1, 1, 1, 1, 1, 0), (0, 1, 1, 1, 1, 1, 1, 1)
(0, 1, 2, 1, 1, 1, 1, 0), (0, 1, 2, 1, 1, 1, 1, 1)
(0, 1, 2, 1, 2, 1, 1, 0), (0, 1, 2, 1, 2, 1, 1, 1)
(0, 1, 2, 1, 2, 2, 1, 0), (0, 1, 2, 1, 2, 2, 1, 1)

$I_\lambda = \{1, 8\}$:

| | | |
|---------------------------|---------------------------|--------------------------|
| (1, 0, 0, 0, 0, 0, 0, 0), | (1, 1, 0, 0, 0, 0, 0, 0), | (1, 1, 1, 0, 0, 0, 0, 0) |
| (1, 1, 1, 0, 1, 0, 0, 0), | (1, 1, 1, 1, 0, 0, 0, 0), | (1, 1, 1, 1, 1, 0, 0, 0) |
| (1, 1, 2, 1, 1, 0, 0, 0), | (1, 2, 2, 1, 1, 0, 0, 0), | (1, 1, 1, 0, 1, 1, 0, 0) |
| (1, 1, 1, 1, 1, 1, 0, 0), | (1, 1, 2, 1, 1, 1, 0, 0), | (1, 2, 2, 1, 1, 1, 0, 0) |
| (1, 1, 2, 1, 2, 1, 0, 0), | (1, 2, 2, 1, 2, 1, 0, 0), | (1, 2, 3, 1, 2, 1, 0, 0) |
| (1, 2, 3, 2, 2, 1, 0, 0), | (1, 1, 1, 0, 1, 1, 1, 0), | (1, 1, 1, 1, 1, 1, 1, 0) |
| (1, 1, 2, 1, 1, 1, 1, 0), | (1, 2, 2, 1, 1, 1, 1, 0), | (1, 1, 2, 1, 2, 1, 1, 0) |
| (1, 1, 2, 1, 2, 2, 1, 0), | (1, 2, 2, 1, 2, 1, 1, 0), | (1, 2, 3, 1, 2, 1, 1, 0) |
| (1, 2, 3, 2, 2, 1, 1, 0), | (1, 2, 2, 1, 2, 2, 1, 0), | (1, 2, 3, 1, 2, 2, 1, 0) |
| (1, 2, 3, 1, 3, 2, 1, 0), | (1, 2, 3, 2, 2, 2, 1, 0), | (1, 2, 3, 2, 3, 2, 1, 0) |
| (1, 2, 4, 2, 3, 2, 1, 0), | (1, 3, 4, 2, 3, 2, 1, 0), | |
| | | |
| (0, 0, 0, 0, 0, 0, 0, 1), | (0, 0, 0, 0, 0, 0, 1, 1), | (0, 0, 0, 0, 0, 1, 1, 1) |
| (0, 0, 0, 0, 1, 1, 1, 1), | (0, 0, 1, 0, 1, 1, 1, 1), | (0, 1, 1, 0, 1, 1, 1, 1) |
| (0, 0, 1, 1, 1, 1, 1, 1), | (0, 1, 1, 1, 1, 1, 1, 1), | (0, 1, 2, 1, 1, 1, 1, 1) |
| (0, 1, 2, 1, 2, 1, 1, 1), | (0, 1, 2, 1, 2, 2, 1, 1), | (0, 1, 2, 1, 2, 2, 2, 1) |

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Index of Symbols

| | |
|---|---|
| $\mathcal{G} \dots 6$ $\chi_\lambda \dots 6$ $\Pi(\lambda) \dots 6$ $\text{cvx } A \dots 7$ $\mathcal{L}(A) \dots 7$ $C_v \dots 7$ $\alpha_i \dots 8$ $(,) \dots 8$ $\langle , \rangle \dots 8$ $\lambda_i \dots 8$ $I \dots 8$ $\Lambda_r \dots 8$ $\Lambda \dots 8$ $\Lambda^+ \dots 8$ $P_\lambda \dots 9$ $I_\lambda \dots 9$ $W \dots 9$ $W_\lambda \dots 9$ $C_\lambda \dots 9$ $f_{i,\dots} \dots 10$ $\bar{C} \dots 10, 12$ $\Gamma \dots 11$ $\mathcal{C} \dots 12$ $(\chi_w, \chi) \dots 12$ $V_\lambda \dots 13$ $x_i \dots 13$ $Q \dots 16$ $\text{Log } Q \dots 16$ $R_Q \dots 16$ $K \dots 16$ $\alpha_{cs} \dots 19, 26$ $\alpha_{cd} \dots 19$ $R_K \dots 23$ $e_J \dots 23$ $C_S \dots 23$ $H_S \dots 23$ $F^j \dots 25$ $F^J \dots 25$ $F^j(\lambda) \dots 30$ $G^j \dots 31$ | $\Delta_j \dots 31$ $E_j \dots 31$ $W_j \dots 31$ $p_j \dots 31$ $m(\lambda; \mu) \dots 31$ $m_j(\lambda; \mu) \dots 31$ $M(\lambda, \mu; \nu + \mu) \dots 32$ $M_j(\lambda, \mu; \nu + \mu) \dots 32$ $F^J(\lambda) \dots 33$ $\mathcal{G}^J \dots 33$ $p_J \dots 33$ $m_J(\lambda; \mu) \dots 33$ $M_J(\lambda, \mu; \nu + \mu) \dots 33$ $S_J \dots 34$ $\text{LR1, LR2} \dots 43$ $\text{MLR1, MLR2} \dots 43$ $T \dots 44$ $T_\nu \dots 46$ $\mathcal{D}_{\lambda, \mu} \dots 58, 71, 72$ $\lambda^* \dots 62$ $C_{\lambda, \mu} \dots 72$ $w_0 \dots 72$ $\{w_0\lambda + \mu\} \dots 72$ $F^i(\lambda, \mu) \dots 72$ $\mathcal{D}_{M\lambda} \dots 77$ $\omega_j \dots 77, 79$ $N_i \dots 77$ $N_{j,m} \dots 78$ $M_j \dots 79$ $\omega'_j \dots 80$ $\hat{\delta} \dots 82$ |
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