Higher Specht Polynomials for Representations of Iwahori-Hecke Algebras

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Abstract

In this thesis we construct a generalization of the higher Specht polynomials to the Hecke algebra $\mathcal{H}_q(S_n)$. These polynomials form a basis of the coinvariant algebra $\mathfrak{C}$ with respect to the action of S_n, and they will decompose $\mathfrak{C}$ into irreducible representations of the Hecke algebra. These irreducible representations are q-Specht modules $\mathcal{S}_\lambda^q$. In this construction, if we consider $q = 1$ then we obtain the original higher Specht polynomials for S_n.

We will also introduce a generalization of the divided difference and Demazure operators in the setting of the ring of Laurent polynomials $\mathfrak{L}$. We will construct a coinvariant algebra for the action of the hyperoctahedral group W_n on $\mathfrak{L}$. From these operators, we will be able to find a faithful representation of the Hecke algebra $\mathcal{H}_{q,p}(W_n)$ over $\mathfrak{L}$.

Dedication

To my parents, brothers, and friends, all of whom supported me as I pursued my passion.

Acknowledgement

I offer my gratitude to my supervisor, Dr. Hadi Salmasian, for his exceptional guidance and support during the course of my masters. It is thanks to his mentorship I were able to learn as much as I did. I also would like to thank Dr. Colin Ingalls and Dr. Alistair Savage for reading this thesis and providing great comments and questions.

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Chapter 0

Introduction

Let $\Bbbk$ be a field of characteristic 0 and let $\mathscr{P}$ be the polynomial ring with n variables over $\Bbbk$. Consider a finite subgroup $G \subset \mathrm{GL}_n(\Bbbk)$, where G acts naturally on the variables of $\mathscr{P}$. There are two natural questions pertaining to the theory of invariants: when does the invariant subring $\mathscr{P}^G$ obtain a polynomial ring structure, and what is the structure of $\mathscr{P}$ as a $\mathscr{P}^G$-module? Claude Chevalley expanded on a theorem by G.C. Shephard and J.A Todd (see [8, chap. 4.1]). This theorem states three equivalent conditions:

- G is generated by pseudo-reflections (i.e a pseudo-reflection group).
- $\mathscr{P}^G$ is isomorphic to a polynomial ring with $\dim V$ generators.
- $\mathscr{P}$ is a free $\mathscr{P}^G$-module of rank $|G|$.

This result is known as the Chevalley-Shephard-Todd theorem. In fact we can relate this result to representations of a pseudo-reflection group G in the following way: consider the ideal $m_G \subset \mathscr{P}$ generated by all invariants of G with no constant term. The quotient $\mathfrak{C} = \mathscr{P}/m_G$ is called the **coinvariant algebra** of G. A result known to Chevalley tells us that $\mathfrak{C} \cong \Bbbk[G]$ as G-modules (see [8, Theorem 4.1]). Furthermore, it has been proven that $\mathscr{P} \cong \mathscr{P}^G \otimes_{\Bbbk} \mathfrak{C}$ (see [8, Chapter 3.3.1]). This means that we can study the action of G on $\mathscr{P}$ by its action on the coinvariant algebra. By a classical result in representations of finite groups, if S is the set of all irreducible G-representations, then

$$\mathfrak{C} \cong \Bbbk[G] \cong \bigoplus_{s \in S} s^{\oplus \dim s}.$$

It is possible to find a basis of each copy of $s \in S$ in $\mathfrak{C}$. This would produce a basis of $\mathfrak{C}$ which respects the decomposition of $\mathfrak{C}$ into different irreducible G representations. Particularly for the case that $G = S_n$, it is well known that the irreducible representations of S_n are given by Specht modules S_λ, indexed by partitions $\lambda \vdash n$ (see [16, Chapter 4]). Therefore, in this case we obtain

$$\mathfrak{C} \cong \bigoplus_{\lambda \vdash n} S_\lambda^{\oplus \dim S_\lambda}$$

In their paper [4], S. Ariki, T. Terasoma, and H. Yamada, produced a basis of $\mathfrak{C}$, which generalizes a polynomial basis of the irreducible representations of S_n called the Specht polynomials. This new basis of $\mathfrak{C}$ was then called the **higher Specht polynomials**. In fact, the higher Specht polynomials have been generalized for a higher class of groups. Denote $\mathrm{ST}(\lambda)$ to be the set of standard tableaux of shape λ. The higher Specht polynomials are a set of polynomials indexed by two standard tableaux of the same shape $\{F_T^V \mid \lambda \vdash n \text{ and } T, V \in \mathrm{ST}(\lambda)\}$. If we fix $\lambda \vdash n$ and a standard tableau $V \in \mathrm{ST}(\lambda)$ then the set $\{F_T^V \mid T \in \mathrm{ST}(\lambda)\}$ becomes a basis for a submodule of $\mathfrak{C}$ which is isomorphic to S_λ.

In this thesis, we generalize this construction to the Hecke algebra of S_n, denoted by $\mathcal{H}_q(S_n)$. The Hecke algebra, in this thesis, is a finite dimensional algebra over a field $\Bbbk$ indexed by an element $q \in \Bbbk$. The relations of this algebra are dependent of q. If we choose $q = 1$ then the Hecke algebra is the group ring of S_n over $\mathcal{H}_q(S_n)$. Akihiko Gyoja constructed a generalization of the Specht modules in their paper [19]. These q-Specht modules, denoted by S_λ^q, are indexed by partitions of n, with the property that $S_\lambda^1 \cong S_\lambda$. Another important construction, is the action of $\mathcal{H}_q(S_n)$ on $\mathscr{P}$ given by Alain Lascoux (see [21]). By using Gyoja's description of S_λ^q and the action of $\mathcal{H}_q(S_n)$ on $\mathscr{P}$, we generalize the higher Specht polynomials to a Hecke algebra version. These polynomials will also be indexed by two standard tableaux of the same shape, $\{\mathfrak{F}_T^V \mid \lambda \vdash n \text{ and } T, V \in \mathrm{ST}(\lambda)\}$. Similar to the S_n version, if we fix $\lambda \vdash n$ and $V \in \mathrm{ST}(\lambda)$ then the set $\{\mathfrak{F}_T^V \mid T \in \mathrm{ST}(\lambda)\}$ is a basis of a submodule of $\mathfrak{C}$ which is isomorphic to $\mathcal{S}_\lambda^q$. We also obtain that the set of q-higher Specht polynomials is a basis of $\mathbb{C}$ over $\Bbbk$. Furthermore, in the $q = 1$ case, we obtain the original higher Specht polynomials. In other words, if $q = 1$ then

$$\mathfrak{F}_T^V = F_T^V$$

The second goal of this thesis is to construct an action of the Hecke algebra of the hyperoctahedral group $\mathcal{H}_{q,p}(W_n)$ on the ring of Laurent polynomials. In the classical case, a version of the higher Specht polynomials has been constructed for almost all the pseudo-reflection groups (see [23]). These constructions rely on the fact that there is a natural action of G on $\mathscr{P}$, and one may construct a coinvariant algebra of G. In this thesis, we will consider a different construction of this setting for the case that $G = W_n$. We will consider a natural action of W_n on $\mathfrak{L}$ and use Chevalley-Shephard Todd to construct the invariant subring $\mathfrak{L}^{W_n}$ and a version of the coinvariant algebra $\mathfrak{LC}$ as a quotient ring of $\mathfrak{L}$. In this setting, we will be able to generalize symmetrizing operators of $\mathscr{P}$ to $\mathfrak{L}$, and using these operators we will construct an action of $\mathcal{H}_{p,q}(W_n)$ on $\mathfrak{L}$.

Chapter 1

Preliminaries

In this chapter we aim at giving a concise summary of the necessary definitions and results concerning partitions, Young tableaux, and their application to the basic representation theory of the permutation group S_n and the hyperoctahedral group W_n. We give all these results in detail in order to provide some proof techniques that will appear in later chapters. In Section 1.1 we will cover partitions and multi-partitions of an integer and its associated Young diagram. We will also cover in this section the definition of a Young tableau and a standard Young tableau and some of their properties. Next, in Section 1.2 we will use the combinatorics of Young tableaux to obtain irreducible representations of the permutation group. We will finish the section with a description of the one-dimensional representations of the hyperoctahedral group in Section 1.3.

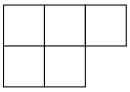
Section 1.1: Tableaux Combinatorics

We begin our discussion with the combinatorics of integer partitions. For a given natural number n we define a **partition** λ of n, denoted by $\lambda \vdash n$, to be a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\sum_i \lambda_i = n$ and $\lambda_i \geqslant \lambda_j$ if $i \geqslant j$. For a given partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n we denote the **length** of the partition by $l(\lambda) := k$. Given a partition λ of n its **Young diagram** is a set of tuples of integers defined in the following way:

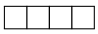
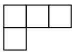
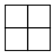
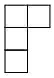
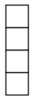
$$D(\lambda) = \{(r, c) \,|\, 1 \leqslant r \leqslant l(\lambda)\,,\, 1 \leqslant c \leqslant \lambda_r\}$$

Each element of $D(\lambda)$ is called a **cell** of the diagram. It is easy to see that for a given partition $\lambda \vdash n$ such that $\lambda = (\lambda_1, \ldots, \lambda_k)$ then the size of its young diagram as a set, denoted by $|D(\lambda)|$ is $\lambda_1 + \cdots + \lambda_k = n$. This gives a way to visualize partitions of an integer as arrangements of row and columns as indicated by Example 1.1 below.

Example 1.1: Consider the partition $\lambda = (3, 2)$ and its associated Young diagram below:

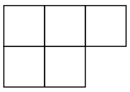
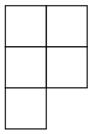
$D(\lambda) =$ 

If we are given a diagram $D = \{(r, c) \,|\, 1 \leqslant r \leqslant k\,,\, 1 \leqslant c \leqslant d_r\}$ such that $d_i \geqslant d_j$ for any $i \geqslant j$ then D produces a unique partition of $n = |D|$ given by $(d_1, \ldots, d_k)$. Informally, this means that there is a bijection between diagrams D with $|D| = n$ and partitions of n. For this reason, we will consider a partition $\lambda \vdash n$ and its Young diagram interchangeably. Therefore, λ may refer to the partition and its diagram, and we will be dropping the $D(\lambda)$ notation. For instance, we give the correspondence between partitions of 4 and diagrams of size 4 below:

$(4) \leftrightarrow$  $(3,1) \leftrightarrow$  $(2,2) \leftrightarrow$  $(2,1,1) \leftrightarrow$  $(1,1,1,1) \leftrightarrow$ 

A consequence of this is that operations on diagrams will give operations on partitions of n. An example of this is conjugation. Given a partition $\lambda \vdash n$ we define the **conjugate** partition, denoted by λ', as the partition associated with the diagram $D'(\lambda) = \{(c, r) \,|\, 1 \leqslant r \leqslant l(\lambda)\,,\, 1 \leqslant c \leqslant \lambda_i\}$. Geometrically, this is equivalent to transposing the diagram in the same way we transpose a matrix.

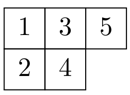
Example 1.2: Consider the partition $\lambda = (3, 2)$. We give the diagram corresponding to the partition and its conjugate below.

$\lambda = (3, 2) \longleftrightarrow D(\lambda) =$  $D'(\lambda) =$  $\longleftrightarrow \lambda' = (2, 2, 1)$

One important construction which uses the diagram of a partition is the **Young tableau**. Formally, given a partition λ we define a Young tableau T of shape λ to be an injective function from the set of tuples of the diagram of λ to the set $\{1, \ldots, n\}$:

$$T : D(\lambda) \to \{1, \ldots, n\}$$

Example 1.3: Consider a partition $\lambda = (3, 2) \vdash 5$. We can visualize any of those functions by filling up the diagram of λ with the corresponding value of T. Thus, a Young tableaux of shape λ is given below:

$T =$ 

Given a Young tableau T of shape λ, we can enumerate the individual rows and columns of T. Using the individual rows (or columns), we would be able to partition the set $\{1, \ldots, n\}$ into the set of numbers that appear in the same row (or column). To formally describe this process, we will first we need some definitions. Given the tableau T, the value of T at row r and column c will be written as

$T(r,c)$. With this notation, we will define the indexed row (or indexed column) sets as

$$\mathrm{Row}(r,T) = \{T(r,c)\,|\,1 \leqslant c \leqslant \lambda_r\} \text{ and } \mathrm{Col}(c,T) = \mathrm{R}(c,T').$$

We can consider the set of rows and columns of a tableau, denoted by $\mathrm{R}(T)$ and $\mathrm{C}(T)$ respectively.

Example 1.4: Given the same tableau T from Example 1.3 we give its row and column sets respectively:

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \implies \mathrm{Row}(T) = \{\mathrm{Row}(1,T), \mathrm{Row}(2,T)\} = \{\{1,3,5\},\{2,4\}\}$$

$$\mathrm{Col}(T) = \{\mathrm{Col}(1,T), \mathrm{Col}(2,T), \mathrm{Col}(3,T)\} = \{\{1,2\},\{3,4\},\{5\}\}$$

A tableau T is called **standard** if the values are written in increasing order in all rows and columns. In other words, if $r < r'$ then $T(r,c) < T(r',c)$. Likewise, if $c < c'$ then $T(r,c) < T(r,c')$. The tableau in Example 1.4 gives a standard tableau of shape $\lambda = (3,2)$.

Definition 1.5: For a given partition λ we denote T_λ to be the tableau constructed by numbering the cell from left to right starting from the top row. Similarly, denote $T^\lambda = (T_{\lambda'})'$.

Example 1.6: For $\lambda = (4,2,2,1) \vdash 9$ we have:

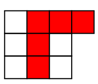
$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} 9 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \text{and } T^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 8 & 9 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

Let $\mathrm{ST}(\lambda)$ be the set of all standard tableaux of a given shape λ. Furthermore, let h_λ be the number of standard tableaux of shape λ, so that $h_\lambda = |\mathrm{ST}(\lambda)|$. In order to describe a formula for h_λ, we will discuss the hook-length of a cell. Let us consider any partition λ of n, and pick a cell (r,c) of its diagram. Consider the following set:

$$H(r,c) = \{(r',c) \in D(\lambda)\,|\,r' \geqslant r\} \bigcup \{(r,c') \in D(\lambda)\,|\,c' \geqslant c\}$$

The set $H(r,c)$ consists of all the cells directly to the right and directly downwards from the cell (r,c). We call this set a **hook** of the cell (r,c).

Example 1.7: As an example of this, consider the cell $(1,2)$ of the partition $\lambda = (4,3,3)$. Then we give the hook of this cell below:

$H(1,2) =$ 

For a given cell (r,c) of a Young diagram $D(\lambda)$ we define its **hook length** $l_\lambda(r,c) := |H(r,c)|$. This notation gives us the following formula for the number of standard tableaux.

Theorem 1.8: For a given partition λ of size n, the number of standard tableaux of this shape is given by:

$$h_\lambda = \frac{n!}{\prod_{(r,c)\in D(\lambda)} l_\lambda(r,c)}$$

A proof by Frame, Robinson and Thrall can be found in [15, Theorem 1].

Example 1.9: Using Theorem 1.8, we can compute the number of standard tableaux of shape $\lambda = (4,3,1) \vdash 8$. The entries of the diagram $D(\lambda)$ below indicate the hook lengths of the cells.

$$D(\lambda) = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} \quad \longrightarrow \quad h_\lambda = \frac{8!}{6\cdot 4\cdot 4\cdot 3\cdot 2} = 5\cdot 7\cdot 4$$

Corollary 1.10: $|\mathrm{ST}(\lambda)| = |\mathrm{ST}(\lambda')|$ for all partitions $\lambda \vdash n$.

***Proof*:** [16, Chapter 4.1] This result is well known, we give a proof based on the hook length formula.Consider any cell (c,r) in the conjugate diagram $D(\lambda')$. Its hook on the conjugate diagram will be defined by $H_{\lambda'}(c,r) = \{(c',r) \mid c' \geqslant c\} \bigcup \{(c,r') \mid r' \geqslant r\}$. Note that there is a bijection between the set $H_{\lambda'}(c,r)$ and the set $H_\lambda(r,c)$ given by switching the order of the tuples. Thus the hook length $h_{\lambda'}(c,r) = h_\lambda(r,c)$ giving us the equality:

$$\prod_{(c,r)\in D(\lambda')} h(c,r) = \prod_{(r,c)\in D(\lambda)} h(r,c) \qquad \blacksquare$$

Section 1.2: Representations of the Permutation Group

The permutation group S_n is the group of bijections from the set $\{1,\ldots,n\}$ to itself. The aim of this section is to classify all representations of S_n over a field $\Bbbk$ of characteristic 0. This classification will be done using the combinatorial tools we discussed so far. First we define a natural action of S_n on the set of Young tableaux of shape $\lambda \vdash n$. This action is defined by the following: If $\sigma \in S_n$ and T is a Young tableau of shape λ then

$$(\sigma \cdot T)(r,c) = \sigma\left(T(r,c)\right)$$

Definition 1.11: Given a tableau T of shape λ we define two subgroups of S_n. Consider all the permutations which permute elements only within the same row. These permutations form the **row symmetrizer**. Similarly, the **column symmetrizer** which are the permutations that permute elements within the same column. We denote these subgroups as $\mathrm{R}(T)$ and $\mathrm{C}(T)$ and they are formally given by:

$$\mathrm{R}(T) = \{\sigma \in S_n \,|\, \mathrm{Row}(r, T) = \mathrm{Row}(r, \sigma T)\}$$
$$\mathrm{C}(T) = \{\sigma \in S_n \,|\, \mathrm{Col}(c, T) = \mathrm{Col}(c, \sigma T)\}$$

Example 1.12: Consider a partition $\lambda = (3, 2) \vdash 5$. We give a tableau T of shape λ and $\mathrm{R}(T)$ and $\mathrm{C}(T)$ below. For notation purposes, given a set A we denote $\mathrm{Perm}(A)$ the group of permutations of elements of A. This way we can express the group of row and column symmetrizers of T in the following way

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array} \qquad \begin{array}{l}\mathrm{R}(T) = \mathrm{Perm}(\{1,2,3\}) \times \mathrm{Perm}(\{2,5\}) \\ \mathrm{C}(T) \ = \mathrm{Perm}(\{1,2\}) \times \mathrm{Perm}(\{3,5\})\end{array}$$

Theorem 1.13: For a given Young tableau T of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ if we have $\lambda' = (\nu_1, \ldots, \nu_l)$ then $\mathrm{R}(T) \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ and $\mathrm{C}(T) \cong S_{\nu_1} \times \cdots \times S_{\nu_l}$.

***Proof*:** [26, Given in Definition 2.3.1] If the sets of rows and columns of T are $\{r_1, \ldots, r_n\}$ and $\{c_1, \ldots, c_l\}$ respectively, then the size of each row $|r_i| = \lambda_i$, similarly $|c_i| = \nu_i$. Thus for each row r_i we have that $\mathrm{Perm}(r_i) \cong S_{\lambda_i}$ from which it follows that

$$\mathrm{R}(T) \cong \mathrm{Perm}(r_1) \times \cdots \times \mathrm{Perm}(r_k) \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$$

A similar argument can be given for the columns, as $\mathrm{Perm}(c_i) \cong S_{\nu_i}$ for each $1 \leqslant i \leqslant l$. ■

Suppose we are given a representation V of a group G, where V is a vector space over some field $\Bbbk$ of characteristic 0. Then V as a G-representation is equivalent to a module over its group ring $\Bbbk G$. Thus finding irreducible representation of G is equivalent to finding irreducible modules of $\Bbbk G$. In our case, we want to find the irreducible representations of $G = S_n$. These representations will first be defined in the form of left ideals of the group ring $\Bbbk S_n$. We will construct these ideals with a set of idempotent elements called Young symmetrizers. These elements are constructed with the following symmetrizing and anti-symmetrizing elements of $\Bbbk S_n$.

Definition 1.14: Fix a tableau T and consider the following elements of $\Bbbk[S_n]$.

$$r(T) = \sum_{\sigma \in \mathrm{R}(T)} \sigma \qquad \text{and} \qquad c(T) = \sum_{\sigma \in \mathrm{C}(T)} \mathrm{sgn}(\sigma)\sigma$$

By construction, given a Young tableau T and any permutation $p \in \mathrm{R}(T)$ then $pr(T) = r(T)$ and similarly for any $p \in \mathrm{C}(T)$ we have $pc(T) = \mathrm{sgn}(p)c(T)$. Therefore, we obtain the $\Bbbk\mathrm{R}(T)$-module (respectively the $\Bbbk\mathrm{C}(T)$-module) spanned by $r(T)$ and $c(T)$. We can see that they form two one-dimensional representation of $\mathrm{R}(T)$ and $\mathrm{C}(T)$.

Proposition 1.15: Consider a tableau T of shape $\lambda \vdash n$. Then $r(T)^2 = |\mathrm{R}(T)|r(T)$ and $c(T)^2 = |\mathrm{C}(T)|c(T)$.

***Proof*:** [16, Lemma 4.21] Since for any $p \in \mathrm{R}(T)$ we have that $pr(T) = r(T)$ then its easy to see the following computation.

$$r(T)^2 = \left(\sum_{p\in\mathrm{R}(T)} p\right) r(T) = |\mathrm{R}(T)|r(T)$$

Similarly, given any $p \in \mathrm{C}(T)$ we have that $pc(T) = \mathrm{sgn}(p)c(T)$ then

$$c(T)^2 = \left(\sum_{p\in\mathrm{C}(T)} \mathrm{sgn}(p)p\right) c(T) = \sum_{p\in\mathrm{C}(T)} \left(\mathrm{sgn}(p)^2\right) c(T) = |\mathrm{C}(T)|c(T). \qquad \blacksquare$$

Definition 1.16: Let T be a young tableau of shape λ. The **Young symmetrizer** of T is defined as the following element of $\Bbbk S_n$.

$$\varepsilon_T = r(T)c(T) = \sum_{\sigma\in\mathrm{R}(T)} \sum_{\pi\in\mathrm{C}(T)} \mathrm{sgn}(\pi)\sigma\pi$$

Definition 1.17: Consider a partition $\lambda \vdash n$. Then we define the following ideal of $\Bbbk S_n$ which we call the **Specht module** associated with λ denoted by $S_\lambda := \Bbbk S_n \varepsilon_{T^\lambda}$.

Remark 1.18: The choice of using T^λ above is arbitrary. One may use any Young tableaux of shape λ to define the Specht modules. However, this choice will make some proofs easier.

As left ideals of the group ring of S_n generated by ε_{T^λ}, it is clear that Specht modules form representations of S_n. The goal now is to show that for each $\lambda \vdash n$ the module S_λ is in fact an irreducible representation of S_n. Furthermore, we must show that the set $\{S_\lambda \mid \lambda \vdash n\}$ forms a full set of inequivalent irreducible representations of S_n. For this second statement is enough to recall the following well known result [16, Proposition 2.30].

Proposition 1.19: For a finite group G, the number of irreducible representations of G is the same as the number of conjugacy classes of G.

Proposition 1.20: Consider two tableaux T_1 and T_2 and assume there exists a transposition $\sigma = (a, b) \in S_n$ such that $\sigma \in \mathrm{C}(T_1)$ and $\sigma \in \mathrm{R}(T_2)$ then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$.

***Proof*:** [16, Lemma 4.21] We must show that $c(T_1)r(T_2) = 0$, note that $\sigma \in \mathrm{C}(T_1) \bigcap \mathrm{R}(T_2)$ then we can show this is true with the following computation:

$$c(T_1)r(T_2) = -c(T_1)\sigma r(T_2) = -c(T_2)r(T_2) \Longleftrightarrow c(T_1)r(T_2) = 0 \qquad \blacksquare$$

Lemma 1.21: The element ε_T is the unique element, up to scalar multiples, which for any element $a \in \mathrm{R}(T)$ and $b \in \mathrm{C}(T)$ satisfying $a\varepsilon_T = \varepsilon_T$ and $\varepsilon_T b = -\varepsilon_T$.

***Proof*:** [16, Lemma 4.21] Assume that there exists another element $f = \sum_{\sigma \in S_n} c_\sigma \sigma$ that satisfies this condition. We must show two things, that for any $\sigma = ab \in \mathrm{R}(T)\mathrm{C}(T)$ we have that $c_\sigma = k\mathrm{sgn}(b)$ for some constant $k \in \mathbb{C}$, and if $\sigma \notin \mathrm{R}(T)\mathrm{C}(T)$ then $c_\sigma = 0$. For our reference we can compute the coefficients of each term in afb in the following way

$$afb = \sum_\sigma c_\sigma a\sigma b = \sum_\sigma c_{a^{-1}\sigma b^{-1}} \sigma.$$

The first assertion is easier to show. If f has the property mentioned then for any $a \in \mathrm{R}(T)$ and $b \in \mathrm{C}(T)$ we have that $afb = \mathrm{sgn}(b)f$ then $c_{a^{-1}\sigma b^{-1}} = \mathrm{sgn}(b)c_\sigma$. This means that the coefficient for $ab \in \mathrm{R}(T)\mathrm{C}(T)$ in f is given by $c_{ab} = \mathrm{sgn}(b)c_e$. This shows immediately the first assertion with $k = c_e$.

Before we show the second part, we will show that if $\sigma \notin \mathrm{R}(T)\mathrm{C}(T)$ then there exists a transposition $t \in \mathrm{R}(T) \bigcap \mathrm{C}(\sigma T)$. Denote $U = \sigma T$, then we must show that there are two numbers belonging to a row of T and a column of U. Assume via contradiction that there are no such integers. Since there are no such pair of integers, then all the numbers appearing in the first row of T must appear in different columns of U. There exists permutation $g_1 \in \mathrm{R}(T)$ and $h_1 \in \mathrm{C}(U)$ such that $g_1 T$ and $h_1 U$ have the same first row. We can make this same argument for the second row of $g_1 T$ and $h_1 U$. This means that there must be permutations g_2 and h_2 such that $g_2 g_1 T$ and $h_2 h_1 U$ have the same first two rows. Repeat this process for all rows, then we obtain two permutation g and h such that $gT = hU$. This means however that $gT = h\sigma T$ which gives us that $g = h\sigma$. Note that $g \in \mathrm{R}(T)$ and $h \in \mathrm{C}(U) = \sigma \mathrm{C}(T)\sigma^{-1}$ thus write $h = \sigma h' \sigma^{-1}$ with $h' \in \mathrm{C}(T)$. We then get that $g = \sigma h'$ thus $\sigma = g(h')^{-1} \in \mathrm{R}(T)\mathrm{C}(T)$ a contradiction.

The argument above shows that for a given permutation $\sigma \notin \mathrm{R}(T)\mathrm{C}(T)$ there must exist a transposition $t \in \mathrm{R}(T) \bigcap \mathrm{C}(\sigma T)$. Consider such transposition t and define $g = \sigma^{-1} t \sigma \in \mathrm{C}(T)$ then $t\sigma g = t\sigma\sigma^{-1}t\sigma = \sigma$. We have now constructed a transposition $t \in \mathrm{R}(T)$ and a element $g \in \mathrm{C}(T)$ therefore $c_{t\sigma g} = \mathrm{sgn}(g)c_\sigma = -c_\sigma$. Furthermore, we have that $t\sigma g = \sigma$ so that $c_{t\sigma g} = c_\sigma$. Therefore $c_\sigma = c_{t\sigma g} = -c_\sigma$ meaning that $c_\sigma = 0$ as desired. ■

Corollary 1.22: For any $g \in \Bbbk S_n$ we have that $\varepsilon_T g \varepsilon_T = k\varepsilon_T$ for some $k \in \varepsilon_T$.

***Proof*:** [16, p. 53, Lemma 4.21] Note that for $a \in \mathrm{R}(T)$ and $b \in \mathrm{C}(T)$ then $a\varepsilon_T g \varepsilon_T b = \mathrm{sgn}(b)\varepsilon_T g \varepsilon_T$. However, from the last Lemma 1.21 we have that $\varepsilon_T g \varepsilon_T = k\varepsilon_T$ for some $k \in \mathbb{C}$. ■

Consider two partitions $\lambda_1 = (a_1, \ldots, a_k)$ and $\lambda_2 = (b_1, \ldots, b_j)$. We say that $\lambda_1 > \lambda_2$ if comparing $(a_1, \ldots, a_k, 0, \ldots) < (b_1, \ldots, b_j, 0, \ldots)$ with lexicographical ordering. Therefore, it is easy to see that it must be a total ordering on partitions of n. This ordering is called the **lexicographical ordering** on partitions.

Example 1.23: The lexicographical ordering for all the partitions of $n = 4$ is given by:

$$(4) < (3,1) < (2,2) < (2,1,1) < (1,1,1,1)$$

Lemma 1.24: Consider two tableaux T_1 and T_2 of shapes λ_1 and λ_2 respectively. If $\lambda_1 > \lambda_2$ then there exist two numbers belonging to a row of T_1 and a column of T_2.

***Proof*:** [29, Chapter 14.7, Lemma 1] We will proceed with an argument similar to the one in the proof of Lemma 1.21. If the diagram $D(\lambda_1)$ has more columns than the diagram $D(\lambda_2)$ then note that in the first row of T_1 there are more elements than the number of columns of T_2, thus there must be a column containing more than one element of the first row of T_1 by a pigeonhole argument. Thus, we can assume that $D(\lambda_1)$ and $D(\lambda_2)$ have the same number of columns. Let $\lambda_1 = (a_1, \ldots, a_l)$ and $\lambda_2 = (b_1, \ldots, b_k)$ then since $\lambda_1 > \lambda_2$ then let i be the minimal integer such that $a_i \neq b_i$ by definition of the lexicographic ordering it must be the case that $a_i > b_i$. In terms of the diagram, this means that row i is the first row that $D(\lambda_1)$ and $D(\lambda_2)$ does not have the same amount of cells. Furthermore, there are more cells in row i of $D(\lambda_1)$ then $D(\lambda_2)$. We proceed with the following inductive argument. If there exists a column in T_2 containing more than one number from the first row of T_1 then we are done. Otherwise, each element of the first row of T_1 is placed at different columns of T_2, thus there exists a permutation $\pi_1 \in \mathrm{C}(T_2)$ such that the first row of $T_2^{(1)} := \pi_1 T_2$ contains the same numbers as the first row of T_1. We repeat this process until we reach row i and $T_2^{(i)} := \pi_i \pi_{i-1} \cdots \pi_1 T_2$. However, note that the set of number written above row i in T_1 and $T_2^{(i)}$ are the same, and row i has more elements in T_1 then in $T_2^{(i)}$. Thus, as with the first case there must exist two numbers in row i of T_1 belonging to the same column of $T_2^{(i)}$. However, since each permutation $\pi_1, \cdots, \pi_i$ preserve the columns this shows that there are two numbers in a row of T_1 that appear in the same column of T_2. ■

Corollary 1.25: If T_1 and T_2 are of shapes λ_1 and λ_2 respectively with $\lambda_1 > \lambda_2$ then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$.

***Proof*:** [16, Lemma 4.23] We make use of the lemma 1.24, knowing that there exists two elements belonging to the same row of T_1 and a column of T_2. We use Lemma 1.21 to infer that $\varepsilon_{T_1}\varepsilon_{T_2} = 0$. ■

Corollary 1.26: Consider $g \in \Bbbk S_n$, and two tableaux T_1 and T_2 of shape λ_1 and λ_2 respectively. If $\lambda_1 > \lambda_2$ then $\varepsilon_{T_1} g \varepsilon_{T_2} = 0$.

***Proof*:** [16, Lemma 4.23] We only need to show this in the case that $g \in S_n$. Note that $\varepsilon_{T_2} = g^{-1}\varepsilon_{g \cdot T_2} g$. Thus $\varepsilon_{T_1} g \varepsilon_{T_2} = \varepsilon_{T_1}\varepsilon_{gT_2} g = 0$ from the last corollary. ■

Theorem 1.27: The set of Specht modules $\{S_\lambda \mid \lambda \vdash n\}$ forms a complete set of irreducible non-equivalent representations of S_n.

***Proof*:** [16, Lemma 4.25] First, we show that S_λ is irreducible for a given $\lambda \vdash n$. Note that by definition $S_\lambda = \Bbbk S_n \varepsilon_{T^\lambda}$ then $\varepsilon_{T^\lambda} S_\lambda \varepsilon_{T^\lambda} \subseteq \Bbbk \varepsilon_{T^\lambda}$ by corollary 1.22. Let us consider any possible sub-representation $V \subseteq S_\lambda$. Then we immediately have two possibilities, $\varepsilon_{T^\lambda} V = \Bbbk \varepsilon_{T^\lambda}$ or $\varepsilon_{T^\lambda} V = 0$. In the first case, it is easy to see that

$$S_\lambda = \Bbbk S_n \varepsilon_{T^\lambda} = \Bbbk S_n \varepsilon_{T^\lambda} V \subseteq V.$$

This means that $S_\lambda = V$ thus V is a trivial sub-representations of S_λ. On the other hand if $\Bbbk S_n \varepsilon_{T^\lambda} V = 0$ then $V = 0$. To see this consider $\Bbbk S_n$ as a vector space with basis $\{g \mid g \in S_n\}$ and consider a hermitian form $(-,-) : \Bbbk S_n^2 \mapsto \Bbbk S_n$ defined by $(g_i^*, g_j) = 1$ if $i = j$ or 0 otherwise, where g_i^* is the conjugate transpose of g_i. If we define the orthogonal space $V' = \{v \in \Bbbk S_n \mid (v, V) = 0\}$ then $\Bbbk S_n = V + V'$. This means that $1 = v_1 + v_2$ for some $v_1 \in V$ and $v_2 \in V'$. However, as $V \cdot V = 0$ then $1^2 = 1 = v_2 \in V'$ therefore $V' = \Bbbk S_n$ and $V = 0$. This argument shows that there are no non-trivial sub-representations of S_λ meaning that it must be an irreducible representation. Note that this shows that $\varepsilon_{T^\lambda}^2 \neq 0$.

What remains for us to show is that if λ_1 and λ_2 are two different partitions of n then $S_{\lambda_1} \ncong S_{\lambda_2}$. Note that since $\lambda_1 \neq \lambda_2$ then either $\lambda_1 < \lambda_2$ or $\lambda_2 < \lambda_1$, both cases will lead to an equivalent argument so we will assume the first holds. Because of this we have that $\varepsilon_{\lambda_1} x \varepsilon_{\lambda_2} = 0$ for any $x \in \Bbbk S_n$ by Corollary 1.26. This shows that $\varepsilon_{\lambda_1} S_{\lambda_2} = 0$, but by the previous argument we have that $\varepsilon_{\lambda_1} S_{\lambda_1} = \Bbbk \varepsilon_{\lambda_1}$. This means S_{λ_1} and S_{λ_2} are non-isomorphic as $\Bbbk S_n$ modules. Because the conjugacy classes of S_n are in one-to-one correspondence with partitions of n, then by Proposition 1.19 the set $\{S_\lambda \mid \lambda \vdash n\}$ is a full set of inequivalent irreducible modules of S_n, which is what we needed to show. ■

Proposition 1.28: For a given T of shape λ we have that $\varepsilon_T^2 = \frac{n!}{\dim S_\lambda}\varepsilon_T$

***Proof*:** [16, Lemma 4.26] Note that in the proof of Theorem 1.27 one can write $\Bbbk S_n = S_\lambda + S'_\lambda$ where S'_λ is the space orthogonal to S_λ. Define $M : \Bbbk S_n \to \Bbbk S_n$ to be the map given by $x \mapsto x\varepsilon_{T^\lambda}$. Clearly $M(S_\lambda) = S_\lambda$ and $M(S'_\lambda) = 0$ since M is a constant multiple of an idempotent. More specifically, for any $x \in S_\lambda$ we have that $Mx = c_\lambda x$ for some non-zero constant $c_\lambda \in \mathbb{C}$ such that $\varepsilon_{T^\lambda}^2 = c_\lambda \varepsilon_{T^\lambda}$. This means that choosing a basis for S_λ and S'_λ one writes M as a matrix with trace $c_\lambda \dim(S_\lambda)$. However, note that we can use the standard basis for $\Bbbk S_n$ using the group elements $\{g \mid g \in S_n\}$. Writing M as a matrix over this basis, we can compute its trace to be $n!$ as the coefficient of g in $g\varepsilon_{T^\lambda}$ is 1. Since both traces must be the same it shows that $n! = c_\lambda \dim(S_\lambda)$. ■

We will finish this discussion with a formula for the dimension of the Specht modules, as well as give an explicit basis for each S_λ. For this, we must discuss a formula for the characters of each Specht module. Consider the following polynomials. First, denote the power sums $p_m = x_1^m + \cdots + x_n^m$ and the Vandermonde determinant $\Delta = \prod_{i<j}(x_i - x_j)$. For any polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$ and positive integers $d_1, \ldots, d_n$ we denote $[P]_{(d_1,\ldots,d_n)}$ to be the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$. We can give an explicit formula for the characters in the following theorem, which can be found at [16, Frobenius formula 4.10].

Theorem 1.29: Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and define $L_i = \lambda_i + k - i$ for each $1 \leqslant i \leqslant k$. Denote C_ν to be the conjugacy class of S_n associated with the partition $\nu = (\nu, \ldots, \nu_l) \vdash n$ and for any $1 \leqslant i \leqslant n$ let $\nu(i)$ be the number of times i appears in the list ν. Then for any $g \in C_\nu$ then the character value of g in S_λ is equal to $\chi_\lambda(g) = [\Delta \prod_{i=1}^n (p_{n(i)})^i]_{(L_1,\ldots,L_k)}$

One could use this to deduce the formula for the dimension of S_λ by computing the character at the conjugacy class of the identity element. In other words letting $\nu = (n)$ we can compute the dimension of S_λ. Computing the dimension of S_λ this way yields the following result, whose proof can be found in [12, p.49-50].

Proposition 1.30: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $L_i = \lambda_i + k - i$. Then $\dim S_\lambda = \frac{n!}{L_1! \ldots L_k!} \prod_{i<j}(L_i - L_j)$.

Using this formula for the dimension of the Specht module, we may deduce a nicer formula which provides a good tool for finding a basis for the Specht module. See [16, Chapter 4.12] for a proof of this.

Theorem 1.31: The dimension of the Specht module is equal to the number of standard tableaux, so that $\dim S_\lambda = h_\lambda$

In the remainder of this section, we will construct a basis for S_λ using the standard tableaux. First we shall do an ordering of the standard tableaux called the last letter ordering [LL-ordering], this ordering was introduced in [4] by Ariki, Terasoma and Yamada.

Definition 1.32: For two standard tableaux T_1 and T_2 of the same shape, let $1 \leqslant x \leqslant n$ be the largest integer that is written in a different cell in T_1 and T_2. Let $T_1(r_1, c_1) = T_2(r_2, c_2) = x$. We say that $T_1 < T_2$ according **the last letter ordering**, if $r_1 < r_2$.

Example: If $\lambda = (3, 2) \vdash 4$ then below we present two standard tableaux T_1 and T_2. Clearly the greatest integer written in a different position in both tableaux is 5. Since 5 is written in T_1 in a row above T_2 then $T_1 < T_2$.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array} < T_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array}$$

Proposition 1.33: The last letter ordering on $\mathrm{ST}(\lambda)$ is a total ordering.

***Proof*:** The authors of [4] use this fact without proof, we offer a proof of this in here. Let $T_1, T_2 \in \mathrm{ST}(\lambda)$, and assume that n is written in different cells in both T_1 and T_2. Then n cannot be written in the same row in both T_1 and T_2, otherwise they would not be standard tableaux. Since n is written in different rows, then either $T_1 < T_2$ or $T_2 < T_1$.

Assume that $T_1, T_2 \in \mathrm{ST}(\lambda)$ and $1 \leqslant x \leqslant n$ is the largest integer written in different cells. Then let T_1^* and T_2^* be the tableau T_1 and T_2 with the cells containing $x+1, \ldots, n$ removed. By the argument above, x must be written in different rows of T_1^* and T_2^*. This implies that x must be written in a different row in both T_1 and T_2. Therefore, either $T_1 < T_2$ or $T_2 < T_1$. ■

Remark 1.34: One can also see that for a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ the maximum element in respect to the LL-ordering is T_λ. This can be seen in the following way; consider T to be a standard tableau with $1 \leqslant x \leqslant n$ being the largest element written in different cells in both T and T_λ. If $x = n$ then note that n is written in the last row of T_λ, and since the position must change in T we have that $T < T_\lambda$. In the case that $x < n$ then let T^* and $(T_\lambda)^*$ be the tableau with $x+1, \ldots, n$ removed, and use the previous case. By a similar argument, we can show that T^λ is the smallest element with respect to LL-ordering of the standard tableaux.

Proposition 1.35: If $T_1 < T_2$ for $T_1, T_2 \in \mathrm{ST}(\lambda)$ then there are two numbers which are shared between a row of T_1 and a column of T_2.

***Proof*:** [4, Lemma 4] Let x be the largest element that is written in different positions both of T_1 and T_2. Let T_1' and T_2' be the tableaux T_1 and T_2 with $x+1, \ldots, n$ removed. Note that since these elements removed were written in the same position, then T_1' and T_2' must have the same shape $\tau = (\tau_1, \ldots, \tau_k) \vdash x$. Let W_1 and W_2 be two tableaux constructed by removing x from T_1' and T_2' respectively, with shapes τ_1 and τ_2. Since $T_1' < T_2'$ then x is written in row p in T_1' and row q in T_2' with $p < q$. Thus $\tau_1 = (\tau_1, \ldots, \tau_p - 1, \ldots, \tau_k)$ and $\tau_2 = (\tau_1, \ldots, \tau_q - 1, \ldots, \tau_k)$ and by ordering of partitions we have that $\tau_1 > \tau_2$. Thus, using Lemma 1.24 we obtain the result we need. ■

Corollary 1.36: If $T_1 < T_2$ for $T_1, T_2 \in \mathrm{ST}(\lambda)$ then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$

***Proof*:** This corollary is immediate by using the Proposition 1.35 and Corollary 1.25. ■

Theorem 1.37: The set $\{\pi\varepsilon_{T^\lambda} \mid \pi T^\lambda = U \text{ for } U \in \mathrm{ST}(\lambda)\}$ is a basis for S_λ.

***Proof*:** [4, p.184] First, we order the set $\mathrm{ST}(\lambda) = (T_1, \ldots, T_k)$ such that $T_i < T_j$ if $i < j$. By a previous argument, T_1 is T^λ. Let π_i be the permutation which $\pi_i \cdot T_1 = T_i$ for $1 \leqslant i \leqslant n$. What we will show is that the set $\{\pi_i\varepsilon_{T_1} \mid 1 \leqslant i \leqslant k\}$ is a linearly independent set in $\Bbbk S_n \varepsilon_{T_1}$. Since $\dim S_\lambda$ is the number of standard tableaux of shape λ and $T_1 = T^\lambda$ this would show that the set $\{\pi_i \varepsilon_{T_1} \mid 1 \leqslant i \leqslant k\}$ is a basis for $S_\lambda = \Bbbk S_n \varepsilon_{T_1}$. Consider $c_1, \ldots, c_k$ such that the following equation holds

$$c_1\pi_1\varepsilon_{T_1} + \cdots + c_k\pi_k\varepsilon_{T_1} = \sum_{i=1}^{k} c_i\varepsilon_{T_i}\pi_i = 0.$$

We will show by an inductive process that $c_i = 0$ for all $1 \leqslant i \leqslant k$. Now using the fact that if $i < j$ then $\varepsilon_{T_i}\varepsilon_{T_j} = 0$. Furthermore, for any $1 \leqslant i \leqslant k$ is not hard to see that

$$\pi_i\varepsilon_{T_1}\pi^{-1} = \varepsilon_{T_i}\pi_i$$

Using this, We have apply ε_{T_i} to both sides of the equation. This creates the following system of equations:

$$\begin{aligned}
&c_1\frac{n!}{\dim S_\lambda}\varepsilon_{T_1}\pi_1 &&= 0\\
&c_1\varepsilon_{T_2}\varepsilon_{T_1}\pi_1 + c_2\frac{n!}{\dim S_\lambda}\varepsilon_{T_2}\pi_2 &&= 0\\
&\qquad\vdots\\
&c_1\varepsilon_{T_k}\varepsilon_{T_1}\pi_1 + c_2\varepsilon_{T_k}\varepsilon_{T_2} + \cdots + c_k\varepsilon_{T_k}\pi_k &&= 0
\end{aligned}$$

Since $\frac{n!}{\dim S_\lambda}\varepsilon_{T_i}\pi_{T_i} \neq 0$ because $\mathrm{R}(T) \cap \mathrm{C}(T) = \{1\}$, then the only solution to the system of equation above is $c_1 = \cdots = c_k = 0$. Since $\pi_i\varepsilon_{T_1} = \varepsilon_{T_i}\pi_i$ we clearly have that $\{\pi_i\varepsilon_{T_1} \mid 1 \leqslant i \leqslant k\}$ is a linearly independent set of size $\dim S_\lambda$, thus it must be a basis for S_λ. ■

Example 1.38: With this Theorem 1.37 we have a complete description of the irreducible representations of S_n together with a basis for each representation. We finish this discussion with some examples of Specht modules. Clearly the only two one dimensional representations of S_n are given by the trivial partition $\tau = (n)$ and its transpose τ'. The partition $(n) \vdash n$ gives the **trivial representation** and the partition $(1, \ldots, 1) \vdash n$ gives the one dimensional representation we call the **sign representation** we denote S_{sgn}. This representation is given by the sign map of each permutation $\pi \mapsto \mathrm{sgn}(\pi)$.

Section 1.3: Hyperoctahedral Group

Consider a permutation π of the set $\{\pm 1, \ldots, \pm n\}$ such that $\pi(-x) = -\pi(x)$ for all x. Such a permutation is called a **signed permutation**. The set of signed permutations forms a group which we call the **hyperoctahedral Group** which we denote by W_n. It is easy to see that any signed permutation is completely determined by what happens to the set $\{1, \ldots, n\}$ since $\pi(-m) = -\pi(m)$ for any $1 \leqslant m \leqslant n$. In fact, any injection from the set $\{1, \ldots, n\}$ to $\{\pm 1, \ldots, \pm n\}$ can be extended to a signed permutation. Thus, we can calculate the order of the hyperoctahedral group by counting all such injections, which gives $|W_n| = n!2^n$. We will make distinctions between two different types of elements in W_n. First we define a **positive permutation** to be an element of the form $\pi_1\pi_2$ where π_1 only permutes the positive elements and π_2 only permutes the negative elements. Consequently we define a **positive transposition** to be any element of the form $(a,b)(-a,-b)$ for $1 \leqslant a < b \leqslant n$. We call a **negative transposition** to be an element of the form $(a,-a)$ for some $1 \leqslant a \leqslant n$.

Consider a field $\Bbbk$ of characteristic 0, and consider the group of **signed permutation matrices** which we will denote $SP_n \subset GL_n(\Bbbk)$. These are permutation matrices where the entries are allowed to be ± 1. It is not hard to see that one can map signed permutations to signed permutation matrices.

Proposition 1.39: The signed permutation group is isomorphic to W_n.

***Proof*:** This is a well known result (see [25]) we give a concise proof here. Let $e_1, \ldots, e_n$ be a basis for V .Consider the map $\varphi : W_n \to SP_n \subseteq \mathrm{GL}(V)$ given in the following way, if $\pi \in W_n$ such that $\pi(i) = \pm j$ then $\varphi(\pi) = M$ such that $M(e_i) = \pm e_j$. Clearly this is a injective map and a homomorphism. One can count the number of signed permutation matrices to be $n!2^n = |W_n|$, meaning this homomorphism is a bijection, thus $SP_n \cong W_n$. ■

Consider any signed permutation matrix M. One could decompose $M = DP$ where D is a diagonal matrix with ± 1 in its diagonal and P is a permutation matrix. Note that under the map φ described in the proof above, any permutation matrix π can be written in W_n as $\pi_1\pi_2$ where π_1 permutes the positive elements in the set $\{1, \ldots, n\}$ and π_2 permute the set $\{-1, \ldots, -n\}$. Similarly, under the map above any diagonal matrix D can be written as a product of negative transpositions $(a_1, -a_1) \cdots (a_k, -a_k)$. This means that since we can always decompose a signed permutation matrix in this way, we can always decompose a signed permutation as disjoint positive and negative transpositions. Thus, we can decompose any signed permutation as a product of negative and positive transpositions, with the parity of negative transpositions and positive transpositions being unique for each signed permutation. We can summarize this in the following results (see [14, Section 2] and [1, Section 2] for the Proposition 1.40 and Theorem 1.41):

Proposition 1.40: The set of negative and positive transpositions generate W_n.

Theorem 1.41: Consider the set $\{w_0, \ldots, w_{n-1}\} \subset W_n$ where $t_0 = (1, -1)$ and if $i > 0$ we have then $w_i = (i, i+1)(-i, -i-1)$. Then $\{w_0, \ldots, w_n\}$ is a generating set of W_n

Proposition 1.42: We can give an embedding $W_n \to S_{2n}$ via a homomorphism φ defined below:

$$\varphi((a,b)(-a,-b)) = (a,b)(a+n, b+n) \text{ and } \varphi((a,-a)) = (a, a+n)$$

***Proof*:** This is almost trivial by the definition of W_n. Consider a bijection ψ from the set $\{1, \ldots, n, -1, \ldots, -n\}$ to the set $\{1, \ldots, 2n\}$ where

$$\psi(i) = \begin{cases} i & \text{if } 1 \leqslant i \leqslant n \\ -i + n & \text{otherwise} \end{cases}$$

Since W_n is defined as a subgroup of the permutations of the set $\{1, \ldots, n, -1, \ldots, -n\}$ then one can identify W_n as a subgroup of S_{2n} under this map. ■

Example 1.43: Recall that for any $\pi \in W_n$, then there exists integers k and l such that $\pi = n_1 \cdots n_k p_1 \cdots p_l$ where n_i are negative transpositions and p_i are positive transpositions. Decomposing all signed permutations into positive and negative transpositions gives us an insight on the one-dimension representations of W_n. We define 4 one-dimensional representations of W_n we will denote $I, I_{\text{sgn}}, I_p, I_n$. We define each of the representations by what they do to each element.

- $I(\pi) = 1$
- $I_p(\pi) = (-1)^l$
- $I_n(\pi) = (-1)^k$
- $I_{\text{sgn}}(\pi) = (-1)^{k+l}$

In fact, these are the only one-dimensional representations of W_n. We will show this in a more general result in the next chapter. We will re-introduce both permutation groups and hyperoctahedral groups as Coxeter groups and generalize them to Iwahori-Hecke algebras.

Chapter 2

Iwahori-Hecke Algebras

The purpose of this section is to define the Hecke algebras of type A and B as deformations of the permutation group and the hyperoctahedral group. We will study their representations and give a full description of the irreducible representations of the Hecke algebra of type A with some conditions on the base field. We start in Section 2.1, by recalling the definition and basic properties of Coxeter groups. We define reduced words and the Bruhat ordering on Coxeter systems. In Section 2.2 we will define the Hecke algebra over a general Coxeter system and study their structure and basic properties. We will talk about the specialization arguments and some basic theory about representations of the Hecke algebras. We will generalize the concept of the Young idempotent in the setting of a Hecke algebra of Type A in Section 2.3. This generalization will allow us to construct the irreducible representations of the Hecke algebra of type A. Most of this chapter is based on two sources, "Reflection Groups and Coxeter groups" by James Humphrey [20] and the last section is based on Akihiko Gyoja's paper "A q-Analogue of Young Symmetrizer" [19]. There will be several other sources, and we will introduce them as needed.

Section 2.1: Coxeter groups

In Chapter 1 we discussed the combinatorics and representations of the group S_n and W_n. These groups are in fact part of a larger class of Coxeter groups which have been extensively studied. In order to define the Hecke algebra deformation of S_n and W_n we must first define Coxeter groups and prove some of their properties which will be necessary for our constructions of the associated Hecke algebra. In this section, we will give some general constructions and theorems about Coxeter groups, as well as prove that S_n and W_n are indeed Coxeter groups.

Definition 2.1: Let G be a group and $S \subseteq G$ be a generating subset of G. We call the pair (G, S) a **Coxeter system** with rank $|S|$, when the group presentation is given with the relations:

$$\{(s_1 s_2)^{m_{s_1,s_2}} = 1 \,|\, \forall s_1, s_2 \in S \;\; m_{s_1,s_2} \in \mathbb{N}\},$$

where for each $s_1, s_2 \in S$ we have that $m_{s_1,s_2} = m_{s_2,s_1}$ and $m_{s_1,s_1} = 1$. Furthermore, if $s_1 \neq s_2$ then $m_{s_1,s_2} \geqslant 2$. We say that $m_{s_1,s_2} = \infty$ if there are no integers m such that $(s_1 s_2)^m = 1$.

Definition 2.2: A **Coxeter group** G is any group which possesses a set of generators $S \subseteq G$ that form a Coxeter system (G, S).

Example 2.3: Consider a group G with generators $\{g_1, \ldots, g_k\}$ with relations $g_i^2 = 1$ for $1 \leqslant i \leqslant n$. This group does indeed satisfy the Coxeter relations with $m_{g_i,g_j} = 1$ if and only if $i = j$ and ∞ otherwise. This group is called the **universal Coxeter group** of rank k. It is constructed so that all other Coxeter systems of rank k are obtained as quotients of this group.

Lemma 2.4: Recall that the permutation group S_n is generated by transpositions $s_i := (i, i+1)$ for $1 \leqslant i \leqslant n-1$. Consider the set of such transposition $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$. With this notation, the tuple $(S_n, \mathcal{S})$ is a Coxeter system.

***Proof*:** [6, Chapter 1 Ex. 5] Since the set $\mathcal{S}$ does generate S_n, we must show that each pair s_i, s_j satisfy a Coxeter relation, and there are no other relations. Clearly for each $1 \leqslant i \leqslant n-1$ we have that $s_i^2 = 1$ therefore $m_{s_i,s_i} = 1$. Next for $1 \leqslant i < i+1 < j \leqslant n-1$ we have the following:

$$(s_i s_j) = (i, i+1)(j, j+1) = (j, j+1)(i, i+1) = s_j s_i.$$

Since s_i and s_j commute, we have $s_i s_j s_i s_j = s_i^2 s_j^2 = 1$ and therefore $m_{s_i,s_j} = m_{s_j,s_i} = 2$. The remaining case is then $1 \leqslant i \leqslant n-2$ and $j = i+1$. In this case, it is easy to see that $s_i s_{i+1} = (i, i+1, i+2)$. Therefore, the order of $s_i s_{i+1}$ and $s_{i+1} s_i = (s_i s_{i+1})^{-1}$ is 3. Which shows that $m_{s_i,s_{i+1}} = m_{s_{i+1},s_i} = 3$. Therefore, $(S_m, \mathcal{S})$ is a Coxeter system.

It remains to show that the quotient of the free group generated by $n-1$ elements modulo the above Coxeter relations is canonically isomorphic to S_n. We will proceed in the following way. Consider G_n to be a group generated by $\{t_1, \ldots, t_{n-1}\}$, such that the t_i satisfy the relations described in the above argument. What we have shown so far is that there exists a surjective homomorphism $G_n \to S_n$ such that $t_i \mapsto s_i$. We now show that this homomorphism must be injective. Since this homomorphism is surjective, it would suffice to prove that $|G_n| \leqslant |S_n|$. We will do this inductively over n. Note that $|G_2| = 2 \leqslant |S_2|$, which will be our base case. Let G_k be the subgroup of G_n generated by $t_1, \ldots, t_{k-1}$ such that $G_k \subset G_{k+1}$. We compute $[G_{k+1} : G_k]$. Consider a left-coset $h = t_{i_1} \cdots t_{i_p} G_k$. We can perform two reductions on this coset:

1. If $i_p \neq k$ then $t_{i_p} \in G_k$ thus $h = t_{i_1} \cdots t_{i_{p-1}} G_k$.

2. If $i_p = k$ and $i_{p-1} \neq k-1$ then $t_{i_{p-1}} t_{i_p} = t_{i_p} t_{i_{p-1}}$ therefore $h = t_{i_1} \cdots t_{i_{p-2}} t_{i_p} G_k$.

This means that the only possible left cosets are of the following form $t_a t_{a+1} \cdots t_{k-1} t_k G_k$. Therefore, there are a maximum of $k+1$ left cosets of G_k in G_{k+1}. This implies that $[G_{k+1} : G_k] \leqslant (k-1)$. Consequently $|G_n| = |G_2||G_2 : G_1| \cdots |G_{n-2} : G_{n-1}| \leqslant n! = |S_n|$. This proves that the surjection $G_k \to S_k$ must be an injection. ■

We now have a Coxeter structure for the permutation group S_n using the transposition set $\mathcal{S}$. We call a Coxeter group G with Coxeter system (G, D) a **Coxeter group of type A** if there exists an isomorphism $\varphi : G \to S_n$ such that $\varphi(D) = \mathcal{S}$. Thus, we can abstractly talk about a Coxeter group of type A as a group G with generators $\{s_1, \ldots, s_{n-1}\}$ with the following relations:

- $s_i^2 = 1$ for $1 \leqslant i \leqslant n-1$.
- $(s_i s_j)^2 = 1$ for $1 \leqslant i, j \leqslant n-1$ and $|i-j| > 1$.
- $(s_i s_{i+1})^3 = 1$ for $1 \leqslant i < n-2$.

Corollary 2.5: Recall from Theorem 1.41 that the hyperoctahedral group W_n has a generating set $\mathcal{W} = \{w_0, \ldots, w_{n-1}\}$. These generators are defined as $w_i = (i, i+1)(-i, -i-1)$ for $1 \leqslant i \leqslant n-1$ and $w_0 = (1, -1)$. With this generating set, the tuple $(W_n, \mathcal{W})$ is a Coxeter system.

***Proof*:** [6, Chapter 8 Ex. 1] Recall Proposition 1.42 which shows that there is an embedding $\varphi : S_n \to W_n$ given by $\varphi(s_i) = w_i$. This means that the subgroup G of W_n generated by $\{w_1, \ldots, w_{n-1}\}$ is isomorphic to S_n. By Lemma 2.4, we have that $(G, \{w_1, \ldots, w_{n-1}\})$ is a Coxeter system. This means that to check that W_n satisfies Coxeter relations with generating set $\mathcal{W}$, we only need to check pairs containing w_0. First, we have that $w_0^2 = 1$, therefore $m_{w_0,w_0} = 2$. We now have the following:

$$(w_0 w_1)^4 = ((1,-1)(-1,-2)(1,2))^4 = ((1,2,-1,-2))^4 = 1$$

Therefore $m_{w_0,w_1} = 4$. A similar computation shows that $m_{w_1,w_0} = 4$ as well. Lastly, if $2 \leqslant i \leqslant n-1$ then $w_0 w_i = (1,-1)(-i,-i-1)(i,i+1) = (-i,-i-1)(i,i+1)(1,-1) = w_i w_0$. Thus $m_{0,i} = m_{i,0} = 2$. It follows that $\mathcal{W}$ does in fact have Coxeter relations. Similarly, as in Lemma 2.4, we show that a group G_n with relations $\{t_0, \ldots, t_{n-1}\}$ satisfying the above relations have at most $|W_n| = 2^n n!$ elements. Since the map $\varphi : t_i \to w_i$ gives a surjective map from G_n to W_n this would show that $G_n \cong W_n$. Consider the embedding $G_{n-1} \subseteq G_n$ where G_{n-1} is the subgroup generated by $\{t_0, \ldots, t_{n-2}\}$. We would like to show that the number of left cosets $[G_{n-1} : G_n] \leqslant 2n$. Via an induction argument, this would mean that $|G_n| \leqslant 2^n n!$. Let $t_{i_1} \cdots t_{i_p} G_{n-1}$ be a right coset of G_{n-1}. We will prove that $0 \leqslant p \leqslant 2n-1$ and that, without changing this right coset we can assume,

$$(i_1, \ldots, i_p) = \begin{cases} (n-p, \ldots, n-1) & \text{if } 1 \leqslant p \leqslant n \\ (p-n, p-n-1 \ldots, 0, 1, \ldots, n-1) & \text{if } n < p \leqslant 2n-1 \end{cases}$$

Let $1 \leqslant p \leqslant n$, the strategy is to show via induction that the sequence $t_{i_1}, \ldots, t_{i_p}$ must have the form $t_{n-p}, \ldots, t_{n-1}$. In the base case, where $p = 1$, if $i_1 \neq n-1$ we have that $t_{i_1} \in G_{n-1}$ therefore the $t_{i_1}G_{n-1} = G_{n-1}$. Therefore $i = n-1$ and $t_{i_1}G_{n-1} = t_{n-1}G_{n-1}$. Consider the case that $(i_2, \ldots, i_p) = (n-p+1, \ldots, n-1)$ and $i_1 \neq n-p$. Then we have three cases; either $i_1 = n-p+1$, or $i_1 = n-p+2$, or $|i_1 - n - p + 1| > 1$.

1. In the first case, we have that $i_1 = n-p+1 = i_2$ then $t_{i_1}t_{n-p+1}\cdots t_{n-p} = t_{n-p+1}t_{n-p+1}\cdots t_{n-p} = t_{n-p+2}\cdots t_{n-p+2}$ which results in a sequence of length less than p.

2. In the second case we have that $i_1 = n-p+2 = i_3$ and $i_2 = n-p+1$. Therefore, $t_{n-p+2}t_{n-p+1}t_{n-p+2} = t_{n-p+1}t_{n-p+2}t_{n-p+1}$ meaning that $t_{i_1}t_{i_2}t_{i_3} = t_{i_2}t_{i_3}t_{i_2}$. Using the fact that t_{i_2} commutes with t_{i_k} for $k \geqslant 4$, we see the following

$$t_{i_1}t_{i_2}t_{i_3}t_{i_4}\cdots t_{n-1}G_{n-1} = t_{i_2}t_{i_3}t_{i_2}t_{i_4}\cdots t_{n-1}G_{n-1} = t_{i_2}t_{i_3}t_{i_4}\cdots t_{n-1}t_{i_2}G_{n-1}.$$

   Which gives us $t_{i_1}\cdots t_{n-1}t_{i_2}G_{n-1} = t_{i_2}\cdots t_{n-1}G_{n-1}$. This gives us a sequence of length shorter sequence then length p.

3. If $|i_1 - n - p + 1| > 1$ then there are two options, either $i_1 < n-p$ or $i_1 > n-p+2$. In the first case, let $k \geqslant 2$, since $i_1 + 1 < i_k$, then t_{i_1} commutes with t_{i_k}. Therefore, we obtain the sequence, $t_{i_1}\cdots t_{i_p}G_{n-1} = t_{i_2}\cdots t_{i_p}t_{i_1}G_{n-1} = t_{i_2}\cdots t_{i_p}G_{n-1}$. In the second case then $i_1 = i_k$ for some $2 \leqslant k \leqslant p$. Then $t_{i_1}\cdots t_{i_p} = t_{i_2}\cdots t_{i_{k-2}}t_{i_1}t_{i_{k-1}}\cdots t_{i_p}$. Note that the sequence $t_{i_1}t_{i_{k-1}}t_{i_k}\cdots t_{i_p}G_{n-1}$ is reduced to the second case, since $i_1 + 1 = i_{k-1}$. This means we can reduce this sequence further.

Therefore, in either case the sequence obtained by multiplying t_{i_1} becomes shorter. This shows that if $1 \leqslant p \leqslant n$ then the only possible sequence which we may obtain is $t_{n-p}\cdots t_{n-1}$. A similar argument can be made to show the case that if $n < p \leqslant 2n-1$ then the sequence will have the form defined above. Since these sequences are the only possible sequences, we can produce without shortening the sequence produced then $[G_{n-1} : G_n] = 2n$. This means that $|G_n| = |G_1|\cdots|G_n| \leqslant 2n$. Which shows the assertion. ■

Just as we have given a presentation for the permutation group S_n, the proof of Theorem 2.5 gives us a presentation of the hyperoctahedral group. The generators of W_n are $w_0, \ldots w_{n-1}$ and their relations are the following:

- $w_i^2 = 1$ for $0 \leqslant i \leqslant n-1$.
- $(w_iw_j)^2 = 1$ for $|i-j| > 1$.
- $(w_iw_{i+1})^3 = 1$ for $1 \leqslant i < n-2$.
- $(w_0w_1)^4 = 1$

Given a Coxeter system (G, S) and isomorphism $\varphi : G \to W_n$ such that $\varphi(S) = \{w_0, \ldots, w_{n-1}\}$, we call (G, S) **a Coxeter system of type B**.

Subsection 2.1: One-Dimensional Representations of Coxeter Groups

Consider a Coxeter system (G, S). In this section we will briefly classify all possible homomorphisms $\varphi : G \to \Bbbk$ where $\Bbbk$ is a field of characteristic not 2. We will present an overview of the one-dimensional representations of G.

Remark 2.6: When not specified, s_i mean arbitrary elements of S and not the generators of the S_n. It will be specified when we use the notation s_i to mean the transposition $(i, i+1)$. This slight abuse of notation will spare us from using double subscripts like s_{i_1} and s_{i_2}.

Lemma 2.7: [16, Chapter 2] Let $s_1, s_2 \in S$ be conjugate elements in G and $\varphi : G \to \Bbbk$ be a representation. Then $\varphi(s_1) = \varphi(s_2)$.

Lemma 2.8: Let (G, S) be a Coxeter system and $s, t \in S$ such that $(st)^m = 1$ and m is an odd number. Then s and t are conjugates in S.

***Proof*:** Since m is odd we may write it of the form $m = 2a + 1$ for some $a \in \mathbb{N}$. Using this we may write $(st)^m$ in the following way

$$(st)^m = (c_1 \cdots c_{2m}) \text{ where } c_i = \begin{cases} s & \text{if } i \text{ is odd} \\ t & \text{if } i \text{ is even} \end{cases}$$

Note that we can split the sequence as $A = (c_1 \cdots c_m)$ and $B = (c_{m+1} \cdots c_{2m})$ so that $AB = (st)^m$. Since m is odd and $c_1 = s$ it is clear that $B = sAt$ so we have that $AsAt = 1$ which implies that $AsA = t$. However, note that $A^{-1} = A$ as $c_0 \cdots c_m c_0 \cdots c_m = (st \cdots s)(s \cdots ts) = 1$. Therefore $AsA^{-1} = t$. ∎

Corollary 2.9: Let (G, S) be a Coxeter system and $S = S_1 \cup \cdots \cup S_k$ where S_i are disjoint classes of conjugate elements of S. In other words, $s, t \in S_i$ if and only if $s = gtg^{-1}$ for some $g \in G$. Let $C = \{s_1, \ldots, s_k\}$ where s_i is a choice of representative of S_i. Then there is a one-to-one correspondence between the set of functions from C to $\{1, -1\}$ and one-dimensional representations of G.

***Proof*:** The backwards direction of this correspondence is easy to check. First, any homomorphism $\varphi : G \to \Bbbk^*$ defines a function from C to $\{1, -1\}$ by restriction of φ to C. Consider two homomorphisms φ and ψ such that $\varphi \neq \psi$. We must check that φ and ψ when restricted to C give different functions. Recall that if $s, t \in S_i$ then $\varphi(s) = \varphi(t)$ and $\psi(s) = \psi(t)$. This means that φ and ψ are completely determined by their values on C. Since φ and ψ are not the same homomorphism, there must be $a, b \in C$ such that $a \neq b$ and $\varphi(a) \neq \varphi(b)$. This shows what we needed.

To prove the forward direction, consider $f : C \to \{1, -1\}$. Define $\varphi_f : G \to \Bbbk^*$ such that if $x \in S_i$

then $\varphi_f(x) = f(s_i)$ and for any $g \in G$ such that $g = s_1 \cdots s_k$ then $\varphi_f(g) = \varphi_f(s_1) \cdots \varphi_f(s_k)$. We must show that this function is in fact well-defined and a homomorphism. What we must show is that this map preserves Coxeter relations. Let $s, t \in S$ and $0 \leqslant m \in \mathbb{Z}$ such that $(st)^m = 1$. Note that by our definition of φ_f we have that $\varphi_f((st)^m) = \varphi_f(s)^m \varphi_f(t)^m$. If m is even, we are done as $\varphi_f(s)^m \varphi_f(t)^m = (\pm 1)^m (\pm 1)^m = 1$. If m is odd, then by Lemma 2.8 we have that s and t are conjugates, so that $\varphi_f(s) = \varphi_f(t) = f(s)$ for some $s \in C$. This means that $\varphi_f(s)^m \varphi_f(t)^m = f(s)^{2m} = 1$. It is trivial to see that two functions $f, g : C \to \{1, -1\}$ are different then φ_f and φ_g are two different representations. ∎

Example 2.10: Let $G = S_n$ with $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$. Note that since all s_i are conjugate by $(i, j)(i+1, j+1)s_i(j+1, i+1)(i, j) = s_j$ then there is only one conjugate class over $\mathcal{S}$. This means that as we expect there are only two one-dimensional representations of G over $\Bbbk$. These representations are the following:

- The trivial representation given by $\iota : s_i \mapsto 1$ for all i.
- The sign representation given by $\varsigma : s_i \mapsto -1$ for all i.

Example 2.11: In the case that $G = W_n$ and $\mathcal{S} = \{w_0, \ldots, w_{n-1}\}$. Recall that $w_0 = (1, -1)$ and $w_i = (i, i+1)(-i, -i-1)$ for $1 \leqslant i \leqslant n-1$. Note that the group G generated by $\{w_1, \ldots, w_{n-1}\}$ is isomorphic to S_n given by $\varphi : G \to S_n$ where $\varphi(w_i) = s_i$ (here s_i are the transposition $(i, i+1)$). With this embedding of G onto S_n it is clear that for $1 \leqslant i, j \leqslant n$ the elements w_i and w_j are conjugate by the argument in Example 2.10. In Proposition 1.42 we showed that there is a injective homomorphism $\varphi : W_n \to S_{2n}$ given by $\varphi(w_0) = (1, n+1)$ and $\varphi(w_i) = (i, i+1)(n+i, n+i+1)$ for $1 \leqslant i \leqslant n-1$. Using φ we see that $\varphi(w_0)$ and $\varphi(w_i)$ cannot be conjugate as they have different cycle structure. This means that w_0 and w_i cannot be conjugate for any $1 \leqslant i \leqslant n-1$. Thus we split $\mathcal{S} = \{w_0\} \cup \{w_1, \cdots, w_{n-1}\}$ and we obtain 4 representations of W_n onto $\Bbbk$. These representations are given by the following (where $1 \leqslant i \leqslant n-1$):

1. $\iota(w_0) = 1$ and $\iota(w_i) = 1$.
2. $\varsigma_0(w_0) = -1$ and $\varsigma_0(w_i) = 1$.
3. $\varsigma_1(w_0) = 1$ and $\varsigma_1(w_i) = -1$.
4. $\varsigma(w_0) = -1$ and $\varsigma_1(w_i) = -1$

These representations correspond to the representations $\iota, I_p, I_n, I_{\text{sgn}}$ in Chapter 1.3. Thus, we did obtain all possible one-dimensional representations of W_n.

Note that in these examples there are two representations which are always possible in any Coxeter group G: The trivial representation and the representation which sends all $s \in S$ to -1. The trivial representation can be defined for any group G. However, using the structure of Coxeter groups, the second representation (i.e. the sign representation) just as we did for S_n and W_n can be defined in the following way:

Definition 2.12: Let (G, S) be a Coxeter system and $\Bbbk$ a field. We call the representation $\varsigma : G \to \Bbbk$ such that $\varsigma(s) = -1$ for all $s \in S$ the **sign representation** of (G, S).

Subsection 2.1: Reduced Words and Length

Definition 2.13: Consider a Coxeter system (G, S) with $g \in G$. We may write $g = s_1 \cdots s_k$ for some $s_1, \ldots, s_k \in S$. Such a decomposition of g is called a **word** of g. If k is the minimal integer such that this decomposition of g is possible over S then $s_1 \cdots s_k$ is called a **reduced word**.

Remark 2.14: For a given element $g \in G$ of a Coxeter group, the choice of a reduced word is not unique. For example, consider the group $G = S_3$ generated by $\mathcal{S} = \{s_1, s_2\}$ that the permutation $(1, 3) = s_1 s_2 s_1 = s_2 s_1 s_2$.

Definition 2.15: Let (G, S) be a Coxeter system and $g \in S$. Let k is the length of a reduced word, such that $s_1 \cdots s_k = g$. We say that the length of g, denoted by $l(g)$, is k. The identity element $1 \in G$ is defined to have length $l(1) := 0$.

Lemma 2.16: The following properties hold for a given Coxeter system (G, S) and element $g, h \in G$.

1. $l(g) = l(g^{-1})$.
2. $l(g) - l(h) \leqslant l(gh) \leqslant l(g) + l(h)$

***Proof*:** [20, Chapter 5.2] Consider a choice of reduced words $g = s_1 \cdots s_k$ and $h = d_1 \cdots d_j$, with the property $s_1, \ldots, s_k, d_1, \ldots, d_j \in S$. It is easy to see that $g^{-1} = s_k \cdots s_1$. If the word $s_1 \cdots s_k$ was in fact not reduced, then this means we could further reduce $s_k \cdots s_1$. This creates an upper bound on the length of $l(g^{-1}) \leqslant l(g)$. Repeating the same process for g^{-1} we obtain that $l(g) \leqslant l(g^{-1})$. This means that $l(g) = l(g^{-1})$. For the second assertion, note that $gh = s_1 \cdots s_k d_1 \cdots d_l$ means that $l(gh) \leqslant k + l = l(g) + l(h)$. Repeating this argument for $u = gh$ and $v = h^{-1}$ we obtain

$$l(g) = l(uv) \leqslant l(u) + l(v) = l(gh) + l(h^{-1}) = l(gh) + l(h)$$

This implies that $l(g) - l(h) \leqslant l(gh)$ as desired. ∎

Lemma 2.17: Let (G, S) be a Coxeter system and $\varsigma : G \to \Bbbk^*$ be the sign representation as discussed in Definition 2.12. Then for all $g \in G$ we have that

$$\varsigma(g) = (-1)^{l(g)}$$

***Proof*:** This is almost trivial to check. If $g = s_1 \cdots s_k$ where $k = l(g)$ then

$$\varsigma(g) = \varsigma(s_1) \cdots \varsigma(s_k) = (-1)^k.$$ ∎

Corollary 2.18: For any given $g \in G$ and $s \in S$ we have that $l(gs) = l(g) \pm 1$.

***Proof*:** [20, Prop. 5.3] Let $\varsigma : G \to \Bbbk^*$ be the sign representation. If $l(gs) = l(g)$ then by Prop 2.17 we have

$$(-1)^{l(g)} = \varsigma(gs) = \varsigma(w)\varsigma(s) = (-1)^{l(k)+1}$$

Which is a contradiction, therefore $l(gs) \neq l(g)$. Furthermore, by Lemma 2.16 we have the inequality $l(g) - l(s) \leqslant l(gs) \leqslant l(g) + l(s)$. Since $l(s) = 1$ then we have that $l(gs) = l(g) \pm 1$. ∎

One important question is that given $g = s_1 \cdots s_r$ such that $l(g) < r$, what is the relation between a representing word of length $l(g)$ and $s_1 \cdots s_r$. We may answer that with the **exchange condition** (Theorem 2.19). The importance of this condition is that it classifies all Coxeter groups. Often a stronger version of this theorem is proven (see [20, Theorem 5.8] and [6, Theorem 1.4.8]).

Theorem 2.19: Let $g \in G$ and $s \in S$ with $g = s_{i_1} \cdots s_{i_k}$ a reduced word for g. If $l(gs) < l(g)$ then there exists a unique integer $1 \leqslant j \leqslant k$ such that $gs = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k}$. (A similar statement holds if $l(sg) < l(g)$).

Corollary 2.20: For a word $s_1 \cdots s_k$, we denote $s_1 \cdots \overline{s_i} \cdots s_k$ to be the expression $s_1 \cdots s_{i-1} s_{i+1} \cdots s_k$. Given $g \in G$ and a decomposition $g = s_1 \cdots s_r$ where $r > l(g)$, there exists $m \in \mathbb{N}$ and a sequence of increasing integers $(i_1, \ldots, i_{2m})$ such that the reduced form of g is given by $s_1 \cdots \overline{s_{i_1}} \cdots \overline{s_{i_{2m}}} \cdots s_r$. (i.e. by removing an even number of the $s_i's$)

***Proof*:** [20, Corollary 5.8] If $g = s_1 \cdots s_r$ and $r > l(g)$ then consider i to be the minimum integer such that $w = s_1 \cdots s_{i-1}$ is an irreducible expression. This means that, by the exchange condition, there exists an integer $1 \leqslant j \leqslant i - 1$ such that $ws_i = s_1 \cdots \overline{s_j} \cdots s_{i-1}$. Thus $g = ws_i \cdots s_r = s_1 \cdots \overline{s_j} \cdots \overline{s_i} \cdots s_r$. This shows that if $r > l(g)$ then we may always find two integers $1 \leqslant i < j \leqslant r$ such that we may omit s_i and s_j from the expression given for g. Therefore, we may recursively omit two generators from the expression for g until we reach length $l(g)$. ∎

Corollary 2.20 is called the **deletion condition** on Coxeter groups. Together with the exchange condition, it also classifies Coxeter groups, in the sense that if a group G has generators S, with the condition that all elements of S are of order 2, then the following are equivalent (see [6, Theorem 1.5.1]):

- (G, S) is a Coxeter system.
- (G, S) satisfies the exchange condition.
- (G, S) satisfies the deletion condition.

Corollary 2.21: Let (G, S) be a Coxeter system with $g \in G$ and $s, t \in S$. If $l(sgt) = l(g)$ and $l(sg) = l(gt)$ then $sgt = g$.

***Proof*:** [20, Lemma 7.8] If $l(g) = k$ with $g = s_1 \cdots s_k$ then we have two possibilities for $l(sg)$:

1. If $l(sg) > l(g)$ then since $l(sg) = l(gt)$ we have that $l(sgt) = l(g) < l(sg)$. Since $l(sg) > l(g)$ then $ss_1 \cdots s_k = sg$ is a reduced word. Applying the exchange condition, we have that either $sgt = s_1 \cdots s_k = g$ or $sgt = ss_1 \cdots \overline{s_i} \cdots s_k$. In the second case let $h = s_1 \cdots \overline{s_i} \cdots s_k$. Then $sgt = sg'$. Note that this implies that $gt = g'$ thus $l(gt) = l(g') < l(g)$ a contradiction.

2. If $l(sg) < l(g)$ then we repeat the last case letting $g' = sg$ so that $l(g') < l(sg')$. This means that $s(g')t = g'$ equivalently $gt = s(sg)t = sg$. ■

Recall Lemma 2.4 which shows that S_n is in fact a Coxeter group. We may study combinatorics of S_n using its Coxeter structure. We will be making use of these connections when we generalize the results we described in Chapter 1 about the representations of S_n to its Hecke algebra in the next section. In the next few theorems, we consider $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$.

Definition 2.22: Consider a permutation $w \in S_n$. We define its inverse set as the following $\mathrm{Inv}(w) := \{1 \leqslant i < j \leqslant n \,|\, w(i) < w(j)\}$. The inversion number is defined as $\mathrm{Inv}(w) := |\mathrm{Inv}(w)|$.

Proposition 2.23: For any permutation $w \in S_n$ we have $\mathrm{Inv}(w) = \mathrm{Inv}(w^{-1})$.

***Proof*:** This is a well known result (see [19, Section 2]), we give a proof here. Consider $i < j$ and $w(j) < w(i)$. Let $a = w(i)$ and $b = w(j)$ then clearly $b < a$ and $w^{-1}(a) < w^{-1}(b)$. This shows that any inversion of w is an inversion of w^{-1}. This argument is symmetrical, and one can show that all inversions of w^{-1} are inversions of w. Thus, w and w^{-1} must have the same number of inversions. ■

Proposition 2.24: For a given $w \in S_n$ and $s \in \mathcal{S}$, where $s = (i, i+1)$, we have the following:

$$\mathrm{Inv}(ws) = \begin{cases} \mathrm{Inv}(w) + 1 & \text{if } w(i) < w(i+1) \\ \mathrm{Inv}(w) - 1 & \text{if } w(i) > w(i+1) \end{cases}$$

***Proof*:** [6, Chapter 1.5] Let $w \in S_n$ and $s \in \mathcal{S}$ such that $s = (i, i+1)$. Then $ws(i) = w(s(i)) = w(i+1)$ and $ws(i+1) = w(i)$. If it was the case that $w(i) < w(i+1)$ then $ws(i) > ws(i+1)$ thus $\mathrm{Inv}(ws) = \mathrm{Inv}(w) + 1$. Otherwise, if $w(i) > w(i+1)$ then $ws(i) < ws(i+1)$, meaning that $\mathrm{Inv}(ws) = \mathrm{Inv}(w) - 1$. ■

Corollary 2.25: For any $w \in S_n$ we have that $l(w) = \mathrm{Inv}(w)$ (see [6, Proposition 8.3.1])

***Proof*:** We give a simple proof in this thesis. Note that for any $w \in S_n$ we may write $w = s_1 \cdots s_k$ where $l(w) = k$. Therefore $\text{Inv}(w) = \text{Inv}(s_1 \cdots s_k)$. If we consider a sequence $w_i = s_1 \cdots s_i$ with $w_0 = 1$ then $\text{Inv}(w_0) = 0$. Using an induction argument, one can see that $\text{Inv}(w) \leqslant k = l(w)$. Thus, we must show that $l(w) \leqslant \text{Inv}(w)$ for any given $w \in S_n$. We will prove this via induction on $m = \text{Inv}(w)$. In the case that $m = 0$, there are no inversions, meaning that $w(1) < \cdots < w(n)$ and the only possibility for this is that $w(i) = i$. Therefore, if $m = 0$ then $w = 1$ and thus $l(w) = 0$. Assume that for any $w \in S_n$ such that $\text{Inv}(w) \leqslant k$ then $l(w) = \text{Inv}(w)$. Consider $\text{Inv}(w) = k + 1$, if there are no $1 \leqslant i \leqslant n - 1$ such that $\text{Inv}(ws_i) = \text{Inv}(w) - 1$ then Proposition 2.24 tells us that for all i we have $w(i) < w(i + 1)$. This means that $w = e$ which is cannot happen as we are assuming that $\text{Inv}(w) > 0$. Thus let $s \in S$ such that $\text{Inv}(ws) = \text{Inv}(w) - 1 = k$. Then $l(ws) \leqslant k$ meaning that $l(w) \leqslant k + 1$. Thus $l(w) \leqslant \text{Inv}(w) \leqslant l(w)$ for any $w \in S_n$ which proves our assertion. ■

Corollary 2.26: There exists a unique element $w \in S_n$ such that $l(w) > l(g)$ for any permutation $g \neq w$. This maximal element is given by $w = (n, n-1, \cdots, 1)$ and $l(w) = \frac{1}{2}(n-1)n$.

***Proof*:** We will use the fact that $l(w) = \text{Inv}(w)$ and maximize the number of inversions. Let $w = (n, n-1, \ldots, 1)$ where $w_m(i) = n-i+1$. For any given $i < j$ then $w(j) = n-i+1 < n-j+1 = w(j)$. This means that every pair $i < j$ gives an inversion. Specifically for any $i < i + 1$ we have that $w(i + 1) < w(i)$. Thus $\text{Inv}(ws_i) = \text{Inv}(w) - 1$. Therefore w has the maximal number of inversions with $l(w) = |\{(i, j)|1 \leqslant i < j \leqslant n\}| = n(n-1)/2$. ■

Subsection 2.1: Parabolic Subgroups

Definition 2.27: Let (G, S) be a Coxeter system and $I \subseteq S$. Consider the subgroup generated by I given by $G_I = \{w \in G \,|\, w = s_1 \cdots s_k \text{ and } s_i \in I\}$. We call G_I a **parabolic subgroup** of G

Lemma 2.28: Let (G, S) be a Coxeter system and $I \subset S$, then the following statements hold:

1. (G_I, I) is a Coxeter system.
2. Let l_I be the length function of the Coxeter system (G_I, I). Then for any $x \in G_I$ we have that $l_I(x) = l(x)$.

***Proof*:** [20, Chapter 5.5] By definition, we have that I is in fact a generating set for G_I. Therefore, if $s_1 \cdots s_n = 1$ where $s_i \in I$ then this relation must also hold for G with generating set S. Thus, all relations of (G_I, I) are relations in (G, S). Meaning that (G_I, I) must be a Coxeter system, as the only relations that may hold for its generating set are Coxeter relations. Now consider $x \in G_I$ so that $x = s_1 \cdots s_m$ where $s_i \in I$. If this word for x is not reduced, then by the deletion condition (Corollary 2.20) there exists $1 \leqslant a_1 < \cdots < a_k \leqslant m$ such that $s_1 \cdots s_m = s_{a_1} \cdots s_{a_k} \in G_I$. Therefore, there exists a reduced word for x in G that is given by generators in I. This means that $l_I(x) = l(x)$ for any $x \in G_I$. ■

Theorem 2.29: Let $I, J \subset S$ and $g \in G$. We define $G_I g G_J = \{h_1 g h_2 \mid h_1 \in G_I \text{ and } h_2 \in G_J\}$. There exists a unique element $w \in G_I g G_J$ such that $l(w) \leqslant l(w')$ for any $w' \in G_I g G_J$. Furthermore, for any $z \in G_I g G_J$ there exists $x \in G_I$ and $y \in G_J$ such that $z = xwy$ and $l(z) = l(x) + l(w) + l(y)$.

***Proof*:** [7, Chapter 4 Ex. 3] Let $w \in G_I g G_J$ be of minimal length over $G_I g G_J$. Then for any $s \in I$ and $t \in J$ we have that $l(w) < l(sw)$ and $l(w) < l(wt)$. Otherwise, we would contradict that w is of minimal length. Note that for any element $h \in G_I g G_J$ we have that $G_I h G_J = G_I g G_J$. To see this, note that $h = agb$ for $a \in G_I$ and $b \in G_J$. Then $g = a^{-1} h b^{-1} \in G_I h G_J$ which shows that the two double-cosets are equal. This means in particular that $G_I g G_J = G_I w G_J$. Consider a reduced word for $w = s_{i_1} \cdots s_{i_k}$. Let $x = s_{j_1} \cdots s_{j_r} \in G_I$ with $l(x) = r$. Then $xw = s_{j_1} \cdots s_{j_r} s_{i_1} \cdots s_{i_k}$, and our aim is to show that this is a reduced word for xw. Assume that $l(xw) < r + k$. Then there exists a maximal integer $i_1 \leqslant k \leqslant i_k$ such that $s_{j_1} \cdots s_{j_r} s_{i_1} \cdots s_{i_{m-1}}$ is reduced, however $s_{j_1} \cdots s_{j_r} s_{i_1} \cdots s_{i_m}$ is not. Therefore, by the deletion condition (Corollary 2.20) there exist two integers $p < q$ such that $xw = s_{j_1} \cdots \overline{s_p} \cdots \overline{s_q} \cdots s_{i_m}$. Now we have three cases:

1. If $j_1 \leqslant p < q \leqslant j_r$. Then $x = (xw)w^{-1} = s_{j_1} \cdots \overline{s_p} \cdots \overline{s_q} \cdots s_{j_r}$. This implies that $l(x) < r$ which is a contradiction.
2. If $i_1 \leqslant p < q \leqslant i_m$ then by a similar argument as above we have that $l(w) < k$.
3. If $j_1 \leqslant p \leqslant j_r$ and $i_1 \leqslant q \leqslant i_m$ then this implies that $w' = s_{i_1} \cdots \overline{s_q} \cdots s_{i_k} \in G_I g G_J$. However $l(w') < k = l(w)$. This contradicts the fact that w was of minimum length in $G_I g G_J$.

Therefore in each case we arrive at a contradiction. This means that for any $x \in G_I$ we have that $l(xw) = l(x) + l(w)$. A similar argument can be made for G_J so that for any $x \in G_I$ and $y \in G_J$ we have $l(xwy) = l(x) + l(w) + l(y)$. The last thing we must show is that w is in fact unique. Assume that $w' \in G_I g G_J$ is another minimal element. Then by an earlier argument $G_I w G_J = G_I w' G_J$ therefore there exists $x \in G_I$ and $y \in G_J$ such that $w' = xwy$ therefore $l(w') = l(x) + l(w) + l(y) = l(w)$. This means that $l(x) = l(y) = 0$ meaning that $x = y = 1$. ∎

In the case of the permutation group $G = S_n$ generated by $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ we may precisely describe all possible parabolic subgroups. Picking $I \subset \mathcal{S}$ we can write $I = I_1 \cup \cdots \cup I_k$ where $I_i = \{s_{i_1}, \ldots, s_{i_1 + m_i - 1}\}$ and for any $s_j \in I_p$ and $s_l \in I_q$, where $p \neq q$, we have that $|j - l| > 1$. Meaning that we write I as a union of consecutive transpositions. It is clear that way that G_{I_i} only permute elements $\{i_1, \ldots, i_1 + m_i - 1\} \subset \{1, \ldots n\}$. This means that for each i we have that $G_{I_i} \cong S_{m_i}$. Concluding that $S_I \cong S_{m_1} \times \cdots \times S_{m_k}$ where S_{m_i} permutes consecutive integers. In fact, any parabolic subgroup of S_n is of this form. We will discuss parabolic subgroups of S_n given by partitions of n. We may restate many of the definitions and theorems given in Section 1.2 in terms of parabolic subgroups. Consider a partition $\lambda \vdash n$ and recall the definition of the standard tableaux T_λ as stated in Definition 1.5. T_λ is defined to be the tableaux constructed by filling each row with consecutive numbers from 1 to n. Define the tableaux T^λ to be $T'_{\lambda'}$. In other words, the tableaux constructed by filling each column from left to right, and each row from top to bottom.

Example 2.30: If $\lambda = (4, 3, 2)$ then the tableaux T_λ and T^λ are shown below:

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array} \qquad T^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array}$$

For a given tableau $T \in \mathrm{ST}(\lambda)$, in Section 1.2 we defined two subgroups of S_n given T (see Definition 1.11). These subgroups are the column symmetrizers $\mathrm{C}(T)$ and the row symmetrizers $\mathrm{R}(T)$. The column symmetrizers are defined as permutations which only permute elements within the same column. Similarly, we define the row symmetrizers $\mathrm{R}(T)$ as the elements that permute within the same row. Note that if $T = T_\lambda$ then all the rows of T are consecutive increasing integers. This means that $\mathrm{R}(T_\lambda)$ is generated by $\mathrm{R}(T_\lambda) \cap \mathcal{S}$. In fact, $\mathrm{R}(T_\lambda)$ is a parabolic subgroup of S_n. One can make the same argument for $\mathrm{C}(T^\lambda)$ as it is generated by $\mathrm{C}(T^\lambda) \cap \mathcal{S}$.

Example 2.31: Take the partition $\lambda = (4, 3, 2) \vdash 9$ and the tableaux T_λ and T^λ as computed in Example 2.30. Note that S_9 is generated by $\mathcal{S} = \{s_1, \ldots, s_8\}$. The first row of T_λ is given by $\{1, 2, 3, 4\}$ so all permutations of this set belongs to $\mathrm{R}(T_\lambda)$. Permutations of $\{1, 2, 3, 4\}$ are in fact generated by $\{s_1, s_2, s_3\}$. Continuing along the rows, we may see the subset of $\mathcal{S}$ that generated $\mathrm{R}(T_\lambda)$.

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array} \qquad \Longrightarrow \qquad \mathrm{R}(T_\lambda) = \langle \{s_1, s_2, s_3\} \cup \{s_5, s_6\} \cup \{s_8\} \rangle$$

The same can be computed for T^λ and $\mathrm{C}(T^\lambda)$. The first column of T^λ is given by $\{1, 2, 3\}$. The group which permutes this column is generated by $\{s_1, s_2\}$. Continuing this way, we can compute $\mathcal{S} \cap C(T^\lambda)$.

$$T^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array} \qquad \Longrightarrow \qquad \mathrm{C}(T^\lambda) = \langle \{s_1, s_2\} \cup \{s_4, s_5\} \cup \{s_7\} \rangle$$

Definition 2.32: Given tableaux T and V of shape $\lambda \vdash n$, we define $\pi(T, V) \in S_n$ such that $\pi(T, V) \cdot T = V$.

It is clear that given a shape λ and $T \in \mathrm{ST}(\lambda)$ then only when $T = T_\lambda$ or $T = T^\lambda$ are the subgroups $\mathrm{R}(T)$ and $\mathrm{C}(T)$ parabolic. With this notation, it is easy to see that $\pi(T, V)^{-1} = \pi(V, T)$. Furthermore, for any tableaux T and V of the same shape, we have $\mathrm{R}(T) = \pi(V, T)\mathrm{R}(V)\pi(T, V)$ and $\mathrm{C}(T) =$

$\pi(V, T)\mathrm{C}(V)\pi(T, V)$. Using Corollary 2.25 we obtain the following:

Proposition 2.33: Let T be a standard tableau of shape λ. Then $l(\pi(T, T_\lambda)) + l(\pi(T, T^\lambda)) = l(\pi(T_\lambda, T^\lambda))$.

***Proof*:** [19, Proof of Lemma 2.3.1] Recall that for a tableau T with rows k rows and m columns, we define $T(r_i, c_j)$ to be the integer located at row i and column j. Let $\lambda = (\lambda_1 \cdots, \lambda_k)$ and $D(\lambda)$ to be its Young diagram. Then $D(\lambda)$ has exactly k rows and m columns. The first thing we will show is that $\mathrm{Inv}(\pi(T_\lambda, T^\lambda))$ is bijective with the set

$$A := \{((r, c), (r', c')) \in D(\lambda) \times D(\lambda) \,|\, 1 \leqslant r < r' \leqslant k \text{ and } 1 \leqslant c' < c \leqslant m\}.$$

For notation purpose, let $\pi = \pi(T_\lambda, T^\lambda)$. It is easy to see that T_λ can be used as an enumeration of the cells in $D(\lambda)$. Using this enumeration of the cells, we match $1 \leqslant i \leqslant n$ with a cell (r_i, c_i) where i is written in T_λ. We will give a condition on the cells of T such that an inversion happens. To start, consider $1 \leqslant i < j \leqslant n$ and let (r_i, c_i) and (r_j, c_j) the positions in which i and j are written in T_λ. By the construction of T_λ and the fact that $i < j$, we can deduce two cases for (r_i, c_i) and (r_j, c_j). Either $r_i = r_j$ and $c_i > c_j$ or $r_i < r_j$. One can easily see this as T_λ is constructed by enumerating each row with consecutive integers, from the top row to the bottom row. Note that since T^λ is standard, then if $r_i \leqslant r_j$ and $c_i \leqslant c_j$ then $T^\lambda(r_i, c_i) \geqslant T^\lambda(r_j, c_j)$ and $T_\lambda(r_i, c_j) \leqslant T_\lambda(r_j, c_j)$. Thus, the only possible case to look at is when $r_i < r_j$ and $c_j < c_i$. Since T_λ is constructed by enumerating the rows top to bottom, then $T_\lambda(r_i, c_i) < T_\lambda(r_j, c_j)$ meaning that $i < j$. In a similar argument, as T^λ is constructed by enumeration of the columns, this means that $T^\lambda(r_i, c_i) > T^\lambda(r_j, c_j)$ meaning that $\pi(i) > \pi(j)$. In other words, we have an inversion. This shows that there is a bijection between $\mathrm{Inv}(\pi)$ and A.

Now that we have described the possible pairs of cells of $D(\lambda)$ that an inversion occur in π we will prove our assertion. Let T be any standard tableau and $x = \pi(T_\lambda, T)$ and $y = \pi(T, T^\lambda)$. Similar to the above argument, we will use T_λ and T^λ to enumerate the positions of $D(\lambda)$. We will argue that if $i < j$ gives an inversion on $\pi = \pi(T_\lambda, T^\lambda)$, then either it must be an inversion in x or y. Consider $1 \leqslant i < j \leqslant n$ and let (r_i, c_i) and (r_j, c_j) be their respective positions in $D(\lambda)$. The argument above shows that $((r_i, c_i,), (r_j, c_j)) \in A$ therefore $r_i < r_j$ and $c_j < c_i$. This means that there are two cases. If $a = T(r_i, c_i) > T(r_j, c_j) = b$ then we have an inversion $x(i) > x(j)$. Otherwise, we have that $b = T(r_j, c_j) > T(r_i, c_i) = a$. By the positions of (r_i, c_i) and (r_j, c_j), we have that $T^\lambda(r_j, c_j) < T^\lambda(r_i, c_i)$. Thus, since $a < b$ and $y(b) < y(a)$ we obtain an inversion of y. This argument shows that for $1 \leqslant r_i < r_j \leqslant k$ and $1 \leqslant c_j < c_i \leqslant m$ either these positions represent a inversion in x or y. This shows that $|\mathrm{Inv}(x)| + |\mathrm{Inv}(y)| = |A|$. which means that $l(x) + l(y) = l(\pi(T_\lambda, T^\lambda))$. ∎

Lemma 2.34: Let $\lambda \vdash n$ and $\pi = \pi(T^\lambda, T_\lambda)$. For any given $g \in G$ we have that the following statements are logically equivalent:

1. $g\mathrm{R}(T_\lambda)g^{-1} \cap \pi\mathrm{C}(T^\lambda)\pi^{-1} = \{1\}$.

2. $g \in (\pi\mathrm{C}(T^\lambda)\pi^{-1})\mathrm{R}(T_\lambda)$.

***Proof*:** [19, Lemma 2.1.4] First, we show that (1) implies (2). Let T be the tableau obtained by $g(T_\lambda) = T$. This means that $g\mathrm{R}(T_\lambda)g^{-1} = \mathrm{R}(T)$. In the same manner, it is easy to see that $\pi\mathrm{C}(T^\lambda)\pi^{-1} = \mathrm{C}(T_\lambda)$. By [29, Lemma 14, 7] consider two tableaux V and W of shapes λ and μ respectively. If $\mu \leqslant \lambda$ lexicographically then if there are no transpositions shared between $\mathrm{R}(W)$ and $\mathrm{C}(V)$ then $\lambda = \mu$ and there exists $x \in \mathrm{R}(W)$ and $y \in \mathrm{R}(W)$ such that $xy(V) = W$. Assume that condition 1 holds, so that we obtain $\mathrm{R}(T) \cap \mathrm{C}(T_\lambda) = \{1\}$. This implies that there must be $a \in \mathrm{C}(T_\lambda)$ and $b \in \mathrm{R}(T)$ such that $ab(T_\lambda) = T$. This means that $x = ab \in \mathrm{C}(T_\lambda)\mathrm{R}(T_\lambda)$ as desired.

Next we consider the opposite direction. If $g \in \mathrm{C}(T_\lambda)\mathrm{R}(T_\lambda)$ then $g = cr$ for some $c \in \mathrm{C}(T_\lambda)$ and $r \in \mathrm{R}(T_\lambda)$. Assuming that (1) does not hold we have that there exist $a \in \mathrm{R}(T_\lambda)$ and $b \in \mathrm{C}(T_\lambda)$ where either a or b is not 1. Such that the following holds:

$$rar^{-1} = c^{-1}bc$$

Note that the right-hand side is an element of $\mathrm{R}(T_\lambda)$ and the left-hand side belongs to $\mathrm{C}(T_\lambda)$. By the definition of $\mathrm{R}(T_\lambda)$ and $\mathrm{C}(T_\lambda)$. This can only happen if $a = b = 1$. This contradicts our assumption, which proves our assertion. ■

Subsection 2.1: Bruhat Ordering

Definition 2.35: Consider a Coxeter system (G, S) and $g \in G$ with reduced word $g = s_1 \cdots s_k$. A **subword** of $s_1 \cdots s_k$ is any element of G consisting of a product $s_{i_1} \cdots s_{i_j}$ where $1 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant n$.

Let $w = s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$ be two reduced words of w. It has been shown that if $g = s_1 \cdots s_r$ is a subword of $s_{i_1} \cdots s_{i_k}$ then there exists a subword of $s_{j_1} \cdots s_{j_k}$ which is equal to g (see [7, Chapter 5, Theorem 10]). This means that the set of subwords of w in a Coxeter group is not dependent on the choice of reduced words. Using this fact we may order the elements of a Coxeter group depending on if they are subwords of another element.

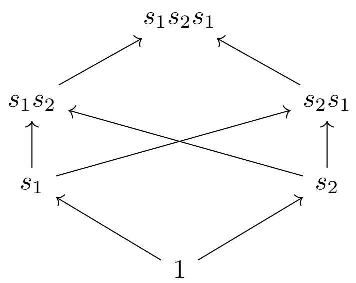
Definition 2.36: Given a Coxeter group G with Coxeter system (G, S), we say that $h \leqslant g$ if there exists a reduced words $g = s_1 \cdots s_k$ and $h = d_1 \cdots d_j$ so that $d_1 \cdots d_j$ is a subword of $s_1 \cdots s_k$. This ordering on the elements of G is called the **Bruhat** order on G.

Note that if $l(sw) = l(w) + 1$ and $w = s_1 \cdots s_n$ then $w < sw$ as $ss_1 \cdots s_n$ would be a reduced expression for sw. Similarly, if $l(ws) = l(w) + 1$ then $w < ws$. This is useful to describe the ordering of certain elements in a Coxeter group.

Proposition 2.37: Let (G, S) be a Coxeter system and $I, J \subset S$ and let $a \in G$. The unique element $w \in G_I a G_J$ of minimal length satisfies the following: For any $x \in G_I$ and $y \in G_J$ we have that $w \leqslant xw$ and $w \leqslant wy$. In addition, $xw \leqslant xwy$ and $wy \leqslant xwy$.

***Proof*:** This is a known result (see [19, Section 2]), we give a short proof. Theorem 2.29 states that there exists an element $w \in G_I a G_J$ such that $l(w)$ is minimal in this double coset. Furthermore, for any $g \in G_I a G_J$ there exist element x and y such that $g = xwy$ and $l(g) = l(x) + l(w) + l(y)$. Choose reduced words $w = s_1 \cdots s_k$, $x = a_1 \cdots a_p$ and $y = b_1 \cdots b_q$. Using the fact that if $l(sw) = l(w) + 1$ implies $w < sw$ we obtain that $w < xw$. Similarly, if $l(ws) = l(w) + 1$ we have that $w < wy$. We may use the same argument to show that $xw, wy \leqslant xwy$ by iterating this same argument on the reduced words of x and y. ∎

Example 2.38: The group S_3 is generated by $\{s_1, s_2\}$. The six elements of S_3 can be written as $\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$. Thus we show the Bruhat ordering on S_3 below:



Section 2.2: Hecke Algebra of a Coxeter Group

For this section, we let R be a commutative ring and (G, S) be a Coxeter system for a finite group G. Recall that RG is the group ring over R. We will begin by defining a deformation of RG. This algebra is unital, and is described as a deformation of the Coxeter relations of (G, S). We will describe a deformation of this algebra using a list of elements over R. Let M be a free R-module of rank $|G|$ with basis $\{m_w \mid w \in G\}$. First, we describe a general deformation of the group ring in the following proposition (see [20, Section 7.1-7.3] for a proof of this Proposition 2.39).

Proposition 2.39: For each element $s \in S$, we associate a choice of elements $q_s, p_s \in R$. We will further assume that given $s, t \in S$, if s and t are conjugate in G then $p_s = p_t$ and $q_s = q_t$. There exists a unique algebra structure with base module M, and basis $\{m_w \mid w \in G\}$, such that if $s \in S$ and $g \in G$ then:

$$m_s m_w = \begin{cases} m_{sw} & \text{if } l(sw) > l(w) \\ q_s m_w + p_s m_{sw} & \text{if } l(sw) < l(w) \end{cases} \qquad m_w m_s = \begin{cases} m_{ws} & \text{if } l(ws) > l(w) \\ q_s m_w + p_s m_{ws} & \text{if } l(ws) < l(w) \end{cases}$$

Definition 2.40: The algebra described in Proposition 2.39 is called the **generic algebra** with parameters $P = \{p_s \mid s \in S\}$ and $Q = \{q_s \mid s \in S\}$. We denote this algebra by $\mathcal{A}_{P,Q}(G)$.

Definition 2.41: Consider a subset $Q = \{q_s \mid s \in S\} \subset R$ such that $q_s = q_t$ if s and t are conjugate in G. Then a **Iwahori-Hecke algebra** (also referred to as a **Hecke algebra** in this thesis) with parameter Q is the generic algebra over G with parameters $P = \{(q_s - 1) \mid s \in S\}$ and $Q = \{q_s \mid s \in S\}$. We denote this algebra by $\mathcal{H}_Q(G, S)$. It has a basis $\{\mathfrak{h}(w) \mid w \in G\}$ with multiplication defined as the following:

$$\mathfrak{h}(s)\mathfrak{h}(w) = \begin{cases} \mathfrak{h}(sw) & \text{if } l(sw) > l(w) \\ (q_s - 1)\mathfrak{h}(sw) + q_s\mathfrak{h}(w) & \text{if } l(sw) < l(w) \end{cases}$$

The equation above is a special case of the first equation in Proposition 2.39. A similar equation for $\mathfrak{h}(w)\mathfrak{h}(s)$ as the second equation in Proposition 2.39 holds.

Remark 2.42: For a given Hecke algebra $\mathcal{H} = \mathcal{H}_Q(G, S)$, then $\{\mathfrak{h}(s) \mid s \in S\}$ is a generating set of $\mathcal{H}$. In addition to this, for two choices of reduced words $g = s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$ we have that $\mathfrak{h}(s_{i_1}) \cdots \mathfrak{h}(s_{i_k}) = \mathfrak{h}(s_{j_1}) \cdots \mathfrak{h}(s_{j_k})$ (see [20, Chapter 7.1]). Therefore, for the remainder of this thesis, we will consider $\mathfrak{h}(g)$ to be the product of the Hecke algebra generators corresponding to a choice of reduced word for g.

Remark 2.43: For notation purposes, if $Q = \{q_s \mid s \in S\}$ is the choice of parameters for a Hecke algebra $\mathcal{H}_Q(G, S)$, we will assume that it has the propriety that $q_s = q_t$ whenever $s, t \in S$ are conjugates.

Remark 2.44: If we choose $q_s = 1 \in R$ for all $s \in S$ then the Hecke algebra $\mathcal{H}_Q(G, S) \cong RG$. Note that since $\{\mathfrak{h}(s) \mid s \in S\}$ generate $\mathcal{H}_Q(G, S)$, we may give a presentation for the Hecke algebra using the formula given in Definition 2.41. Given $s, t \in S$ we have;

$$\mathfrak{h}(s)\mathfrak{h}(t) = \begin{cases} \mathfrak{h}(st) & \text{if } s \neq t \\ \mathfrak{h}(1) & \text{otherwise} \end{cases}$$

Therefore the map $\varphi : RG \to \mathcal{H}_Q(G, S)$ given by $\varphi : w \to \mathfrak{h}(w)$ is a algebra isomorphism.

Proposition 2.45: Let $Q = \{q_s \mid s \in S\} \subset R$ be the parameters of a Hecke algebra $\mathcal{H}_Q(G, S)$. If q_s are units in R for all $s \in S$ then $\mathfrak{h}(g)$ is invertible for any $g \in S_n$.

***Proof*:** Given $s \in S$ then since $\mathfrak{h}(s)^2 = (q_s - 1)\mathfrak{h}(s) + q_s$ we have that $\mathfrak{h}(s)(q_s^{-1}\mathfrak{h}(s) + (q_s^{-1} - 1)) = 1$. This means that $\mathfrak{h}(s)$ is invertible for every $s \in S$. Since $\{\mathfrak{h}(s) \mid s \in S\}$ generates $\mathcal{H}_Q(G, S)$ then all elements are invertible. ■

Proposition 2.46: Let $Q = \{q_s \mid s \in S\} \subset R$ where each q_s is invertible for all $s \in S$. If $s, t \in S$ are conjugate elements, then $\mathfrak{h}(s)$ and $\mathfrak{h}(t)$ are conjugate.

***Proof*:** Note that since all elements of Q are invertible then $\mathfrak{h}(g)$ is invertible for all $g \in G$. If s, t are conjugates, then $q_s = q_t$ and there exists $w \in G$ such that $sw = wt$. Since $l(sw) = l(wt)$ then we obtain two cases:

1. If $l(sw) = l(w) + 1$ then we have that $\mathfrak{h}(s)\mathfrak{h}(w) = \mathfrak{h}(sw) = \mathfrak{h}(wt) = \mathfrak{h}(w)\mathfrak{h}(t)$.

2. If $l(sw) = l(w) - 1$ then $\mathfrak{h}(s)\mathfrak{h}(w) = (q_s - 1)\mathfrak{h}(w) + q_s\mathfrak{h}(sw) = (q_t - 1)\mathfrak{h}(w) + q_t\mathfrak{h}(wt) = \mathfrak{h}(w)\mathfrak{h}(t)$. ■

Proposition 2.47: Let $g, g' \in G$. Then

$$\mathfrak{h}(g)\mathfrak{h}(g') = \sum_{a \leqslant g} \sum_{b \leqslant g'} c_{a,b}\mathfrak{h}(ab)$$

where the coefficients $c_{a,b} \in R$.

***Proof*:** [20, Proposition 7.4] Let $g' = s_1 \cdots s_k$ be a reduced word. Let $g'_0 = 1$ and $g'_i = g'_{i-1}s_i$. We will prove that $\mathfrak{h}(g)\mathfrak{h}(g') = \sum_{b \leqslant g'} c_b\mathfrak{h}(gb)$. To show this, we will use induction on $0 \leqslant i \leqslant k$ to show that

$$\mathfrak{h}(g)\mathfrak{h}(s_1)\cdots\mathfrak{h}(s_i) = \sum_{b \leqslant g'_i} d_{b,i}\mathfrak{h}(gb)$$

for some $d_{b,i} \in R$. Since $g'_i \leqslant g'$ for any $1 \leqslant i \leqslant k$ this would show our claim. In the base case that $i = 0$ the claim is trivial to check as $g_0 = 1$. Assume that the claim works for all $0 \leqslant j \leqslant i - 1$. By the induction hypothesis we obtain the following;

$$\mathfrak{h}(g)\mathfrak{h}(s_1)\cdots\mathfrak{h}(s_i) = \sum_{b \leqslant g'_{i-1}} d_{b,i-1}\mathfrak{h}(gb)\mathfrak{h}(s_i)$$

Since if $b \leqslant g'_{i-1}$ then $bs_i \leqslant g'_i$ by the definition of g'_i. With this argument, fixing $b \leqslant g'_{i-1}$ we obtain the following:

$$\mathfrak{h}(gb)\mathfrak{h}(s_i) = \begin{cases} \mathfrak{h}(gbs_i) & \text{if } l(gbs_i) = l(gb) + 1 \\ (q_{s_i} - 1)\mathfrak{h}(gbs_i) + q_s\mathfrak{h}(gb) & \text{if } l(gbs_i) = l(gb) - 1 \end{cases}$$

In either case, $\mathfrak{h}(gb)\mathfrak{h}(s_i)$ is described as a linear combination of elements of the form $\mathfrak{h}(ga)$ for $a \leqslant g'_i$. This proves that $\mathfrak{h}(g)\mathfrak{h}(g') = \sum_{b \leqslant g'} c_b\mathfrak{h}(gb)$ for some $c_b \in R$. Consider $g = t_1 \cdots t_l$ and define the sequence $g_0 = 1$ and $g_i = t_{l-i}g_{i-1}$. Fixing $b \leqslant g'$, using a similar proof as above, we can show that $\mathfrak{h}(g)\mathfrak{h}(b) = \sum_{a \leqslant g} d_{a,b}\mathfrak{h}(ab)$ using induction on $0 \leqslant i \leqslant l$. Therefore we obtain

$$\mathfrak{h}(g)\mathfrak{h}(g') = \sum_{b \leqslant g'} c_b\mathfrak{h}(gb) = \sum_{b \leqslant g'} \sum_{a \leqslant g} c_b d_{a,b}\mathfrak{h}(ab).$$

This shows what we needed ■

Definition 2.48: Let (G, S) be a Coxeter system, let $S = S_1 \cup \ldots \cup S_k$ where S_i are distinct conjugacy classes of elements in S. Consider a Hecke algebra $\mathcal{H}_Q(G, S)$ for a choice of parameters $Q \subset R$. We call $\mathcal{H}_Q(G, S)$ a **generic Hecke algebra** if $R = \mathbb{Z}[q_1, \ldots, q_k]$ and $q_s = q_i$ if $s \in S_i$.

Example 2.49: Let $G = S_n$ and recall that $s_i = (i, i+1)$ are generators of S_n which give a Coxeter group structure. Note that all s_i are in fact conjugate in S_n. This means that the generic Hecke algebra of S_n is defined on $R = \mathbb{Z}[q]$. Let $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ and $Q = \{q\}$, the generic Hecke algebra $\mathcal{H}_Q(S_n, \mathcal{S})$ is generated by $\mathfrak{h}_i := \mathfrak{h}(s_i)$. We can give a presentation of $\mathcal{H}_Q(S_n, \mathcal{S})$ using the generators $\{\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-1}\}$ in the following way:

1. $\mathfrak{h}_i^2 = (q-1)\mathfrak{h}_i + q$ for $1 \leqslant i \leqslant n-1$.
2. $\mathfrak{h}_i\mathfrak{h}_j = \mathfrak{h}_j\mathfrak{h}_i$ for $|i-j| > 1$.
3. $\mathfrak{h}_i\mathfrak{h}_{i+1}\mathfrak{h}_i = \mathfrak{h}_{i+1}\mathfrak{h}_i\mathfrak{h}_{i+1}$ for $1 \leqslant i \leqslant n-2$.

Example 2.50: Another example is the case that $G = W_n$. We know that the hyperoctahedral group is generated by $\mathcal{W} = \{w_0, \ldots, w_{n-1}\}$ with Coxeter relations given in Example 2.1. In order to compute the generators and relations of the generic Hecke algebra of W_n we must find which elements of S are conjugate to each other. Recall that W_n are permutations of $\{-n, \ldots, -1, 1, \ldots, n\}$ and the generators are $w_0 = (1, -1)$ and $w_i = (i, i+1)(-i, -i-1)$. Clearly, w_i and w_j are conjugate for $i, j > 0$ with $w_i = a w_j a^{-1}$ where $a = (i, j)(-i, -j)(i+1, j+1)(-i-1, -j-1)$. Note that w_0 has a different cycle structure than w_i for $i > 0$. Therefore, w_0 is not conjugate to w_i for $i > 0$. There are only two sets of conjugate elements in S. Therefore we only need two indeterminates to compute the generic Hecke algebra of W_n. Let $R = \mathbb{Z}[p, q]$ and $Q = \{p, q\}$, then using the relations given in Example 2.1 and the Definition 2.48 we give a presentation for $\mathcal{H}_Q(W_n, \mathcal{W})$. The generators are $\mathfrak{h}_i := \mathfrak{h}(w_i)$ for $0 \leqslant i \leqslant n-1$ with the following relations:

1. $\mathfrak{h}_0^2 = (p-1)\mathfrak{h}_0 + p$
2. $\mathfrak{h}_i^2 = (q-1)\mathfrak{h}_i + q$ for $1 \leqslant i \leqslant n-1$.
3. $\mathfrak{h}_i\mathfrak{h}_j = \mathfrak{h}_j\mathfrak{h}_i$ for $|i-j| > 1$.
4. $\mathfrak{h}_i\mathfrak{h}_{i+1}\mathfrak{h}_i = \mathfrak{h}_{i+1}\mathfrak{h}_i\mathfrak{h}_{i+1}$ for $1 \leqslant i \leqslant n-2$.
5. $(\mathfrak{h}_0\mathfrak{h}_1)^2 = (\mathfrak{h}_1\mathfrak{h}_0)^2$

Remark 2.51: In Example 2.49 we only needed one parameter for the Hecke algebra $\mathcal{H}_Q(S_n, \mathcal{S})$. This is true for any Hecke algebra using the Coxeter system $(S_n, \mathcal{S})$. In order to shorten the notation, given $q \in R$ then we will denote $\mathcal{H}_q(S_n)$ for $\mathcal{H}_Q(S_n, \mathcal{S})$. Similarly, in Example 2.50 we only needed two parameters. Let $p, q \in R$ then we denote $\mathcal{H}_{p,q}(W_n) = \mathcal{H}_Q(W_n, \mathcal{W})$, here p is the parameter associated with w_0 and q is the parameter associated with w_i for $i \neq 0$. Using this notation, if we fix $p = q = 1 \in R$ then we have that $\mathcal{H}_1(S_n) = RS_n$ and $\mathcal{H}_{1,1}(W_n) = RW_n$.

Subsection 2.2: Specialization of a Generic Hecke Algebra

Let $s_1, \ldots, s_k$ be representatives of the conjugates classes of S. Let $Q = \{q_s \mid s \in S\}$ be a set of indeterminates as in the definition of a generic Hecke algebra. Consider the ring $F = \mathbb{Z}[q_{s_1}, \ldots, q_{s_k}]$ and let R be a commutative ring. Consider a map $f : q_{s_i} \mapsto r_i$ which defines a ring homomorphism $\varphi_f : F \to R$. Note that R can be given an F-module structure via the map φ_f. Let $P = \varphi_f(Q) = \{r_1, \ldots, r_k\}$. We use this ring homomorphism to define an Iwahori-Hecke algebra given by extending the scalars of $\mathcal{H}_Q(G, S)$ to R using the map f.

$$\mathcal{H}_P(G, S) \cong R \otimes_F \mathcal{H}_Q(G, S).$$

One can show this isomorphism by giving the right-hand side an R-module structure. When a Hecke algebra is obtained from a generic Hecke algebra this way, we say it is obtained by a **specialization** of the indeterminates. It is easy to see that all Hecke algebras can be obtained by a specialization of $\mathcal{H}_Q(G, S)$ by considering a map from R to F, mapping the indeterminates Q to the parameters $P = \{r_s \mid s \in S\}$ of a Hecke algebra $\mathcal{H}_P(G, S)$.

Example 2.52: Let $G = S_n$. In Example 2.49 we computed the generic Hecke algebra of S_n with generators $\{\mathfrak{h}_i \mid 1 \leqslant i \leqslant n\}$. The first relation is given in that example is $\mathfrak{h}_i^2 = (q-1)\mathfrak{h}_i + q$. It is easy to see from these relations that if we specialize $q \to 1$ over any ring R, the first relation becomes $\mathfrak{h}_i^2 = 1$. This new relations match the Coxeter relations for S_n (see Section 2.1). Then the specialization $q \to 1$ of $\mathcal{H}(S_n)$ is isomorphic to RS_n. Another example of specialization is by setting $q \to 0 \in R$. The first relation then becomes $\mathfrak{h}_i^2 = \mathfrak{h}_i$. This algebra is known as the nil-Hecke algebra of S_n over R. One may define this algebra for any Coxeter group.

Definition 2.53: Consider $\mathcal{H}(G)$ the generic Hecke algebra for a Coxeter group G over $F = \mathbb{Z}[q_1, \ldots, q_n]$. Consider any ring R. The specialization $q_i \to 0 \in R$ is called the **nil-Hecke algebra** of G over R.

Remark 2.54: The nil-Hecke algebra $\mathcal{H}(G)$ has a basis over R given by $\{\mathfrak{h}(w) \mid w \in G\}$.

As before, let $S = S_1 \cup \cdots S_k$ where S_i are different conjugacy classes of S. Let $R = \Bbbk(q_1, \ldots, q_k)$ and $Q = \{q_1, \ldots, q_k\}$ so that we obtain the Hecke algebra $\mathcal{H}_Q(G, S)$. A different way we will want to consider specialization is to evaluate q_i to some element in $\Bbbk$. In this way, we have to define elements of $\mathcal{H}_Q(G, S)$ which are specializable.

Definition 2.55: Let $R = \Bbbk(q_1, \ldots, q_n)$ and $Q = \{q_1, \ldots, q_k\}$ be the choice of parameters for a Hecke algebra $\mathcal{H}_Q(G, S)$ consider $P = \{r_1, \ldots, r_k\} \in \Bbbk$, and an element $x \in \mathcal{H}_Q(G, S)$ so that $x = \sum_{g \in G} c_g \mathfrak{h}(g)$ for $c_g \in \Bbbk(q_1, \ldots, q_k)$. Let $c_g = a_g/b_g$ where $a_g, b_g \in \Bbbk[q_1, \ldots, q_k]$. We say that x is specializable at P if $b_g(r_1, \ldots, r_k) \neq 0$ for all $g \in G$.

For a given $g \in G$ we have that $\mathfrak{b}(g)$ is always specializable to any set $P = \{r_1, \ldots, r_k\}$. Furthermore, if $s \in S$ and $g \in G$ such that $s \in S_i$ then $\mathfrak{b}(s)\mathfrak{b}(g)$ is always specializable and their multiplication is given by

$$\mathfrak{b}(s)\mathfrak{b}(g) = \begin{cases} \mathfrak{b}(sg) & \text{if } l(sg) = l(g) + 1 \\ (r_i - 1)\mathfrak{b}(sg) + r_i\mathfrak{b}(g) & \text{if } l(sg) = l(g) - 1 \end{cases}$$

A similar equation can be produced for $\mathfrak{b}(g)\mathfrak{b}(s)$. These equations match the multiplication of the Hecke algebra $\mathcal{H}_P(G, S)$. Therefore, if we have specializable element $x \in \mathcal{H}_Q(G, S)$ and let y be the specialization of x at P then we may consider $y \in \mathcal{H}_P(G, S)$. This specialization argument will be very important to prove results about Hecke algebra modules, as we will see in the upcoming section.

Subsection 2.2: Hecke Algebra Modules

We will finish this section with some basic properties of representations of the Hecke algebras. Let (G, S) be a Coxeter system and assume that S can be split into r conjugacy classes. Let $R = \Bbbk(q_1, \ldots, q_r)$ with $Q = \{q_1, \ldots, q_r\}$ where $\Bbbk$ is a field of characteristic 0. We will classify all one-dimensional representations of $\mathcal{H}_Q(G, S)$.

Definition 2.56: Let M be a left $\mathcal{H}_Q(G, S)$-module where M is a finite-dimensional vector space over R. For each $s \in S$ consider the matrix obtained by the linear homomorphism $\varphi_s : \mathcal{H}_Q(G, S) \to \mathcal{H}_Q(G, S)$ defined by $\varphi_s(x) = \mathfrak{b}(s)x$. We say that M is integral if there exists a basis of M over R such that the matrix obtained by φ_s has entries in $\Bbbk[q_1, \ldots, q_r]$ for all $s \in S$.

Example 2.57: The Hecke algebra $\mathcal{H}_Q(G, S)$ is integral as a left $H_Q(G, S)$ module. Note that for each $s \in S$ and $w \in W$ we have that either $\mathfrak{b}(s)\mathfrak{b}(w) = \mathfrak{b}(sw)$ or $\mathfrak{b}(s)\mathfrak{b}(w) = (q_s - 1)\mathfrak{b}(w) + q_s\mathfrak{b}(ws)$. In either case $\mathfrak{b}(s)\mathfrak{b}(w)$ has coefficients in $\Bbbk[q_1, \ldots, q_r]$. Thus, the matrix obtained by the left multiplication is integral.

It is clear that if a module M is integral, then we may specialize $q_1, \ldots, q_r \to 1$ to obtain a representation of G. This specialization argument will be very important, as we will see later that for the Hecke algebras of Type A all modules are integral. In Corollary 2.9 we classified all one-dimensional representations of a Coxeter group. We can generalize that argument to classify all one-dimensional representations of $\mathcal{H}(G)$. We will show from this classification that all 1-dimensional representations of $\mathcal{H}(G)$ are integral.

Proposition 2.58: Let $S = S_1 \cup \cdots \cup S_r$ where S_i are distinct conjugacy classes of S. Consider a choice of representatives $C = \{s_1, \ldots, s_r\}$. There is a one-to-one correspondence between the one-dimensional representations of $\mathcal{H}_Q(G, S)$ and functions $\{F : C \to R \,|\, F(s_i) \in \{-1, q_i\}\}$. Given such a function F there exists a unique one-dimensional representation φ such that for all $s \in S_i$ we have that $\varphi(\mathfrak{h}(s)) = F(s_i)$.

***Proof*:** This is a well known result (see [3, Section 2]), we will give a proof here. Let $s \in S_i$ and consider a representation $\varphi : \mathcal{H}_Q(G,S) \to R = \Bbbk(q_1,\ldots,q_r)$. Consider $\varphi(\mathfrak{h}(s)) = a \in R$. Then note that $\varphi(\mathfrak{h}(s)^2) = a^2$ and since $\mathfrak{h}(s)^2 = (q_i-1)\mathfrak{h}(s) + q_i$ then $a^2 = (q_i-1)a + q_i$. This means that $(a-q_i)(a+1) = 0$ and therefore there are only two choices for $a \in \{q_i, -1\}$. Since $s \in S_i$ then s_i and s are conjugates, meaning that there exists $w \in G$ such that $\mathfrak{h}(w)\mathfrak{h}(s)\mathfrak{h}(w)^{-1} = \mathfrak{h}(s_i)$ therefore $\varphi(\mathfrak{h}(s_i)) = \varphi(\mathfrak{h}(s))$ this shows that all one-dimensional representation gives a function $F : C \to R$ given by $F(s_i) = \varphi(\mathfrak{h}(s_i)) \in \{1, q_i\}$.

Now consider a function $F : C \to R$ where $F(s_i) \in \{1, q_i\}$. Consider $s \in S_i$ and define a map given by $\varphi(\mathfrak{h}(s)) = F(s_i)$. We claim that φ can be extended to a homomorphism. Note that $\mathcal{H}_Q(G,S)$ is generated by $\{\mathfrak{h}(s) \mid s \in S\}$, thus it is enough to show that $\varphi(\mathfrak{h}(s)\mathfrak{h}(t)) = \varphi(\mathfrak{h}(s))\varphi(\mathfrak{h}(t))$ for all $s, t \in S$. Note that if $s \neq t$ then $\mathfrak{h}(s)\mathfrak{h}(t) = \mathfrak{h}(st)$, therefore $\varphi(\mathfrak{h}(s)\mathfrak{h}(t)) = \varphi(\mathfrak{h}(s))\varphi(\mathfrak{h}(t))$. Therefore, we only need to check the case that $s = t$. Assuming that $s \in S_i$ then $\mathfrak{h}(s)^2 = (q_i-1)\mathfrak{h}(s) + q_i$. Note that $\varphi(\mathfrak{h}(s)^2) = F(s_i)^2$ and $\varphi((q_i-1)\mathfrak{h}(s) + q_i) = (q_i-1)F(s_i) + q_i$. Since $F(s_i) \in \{-1, q_i\}$ then it is easy to see that $(q_i-1)F(s_i) + q_i = F(s_i)^2$. Therefore $\varphi(\mathfrak{h}(s)^2) = \varphi(\mathfrak{h}(s))\varphi(\mathfrak{h}(s))$. ■

Corollary 2.59: All one-dimensional representations of $\mathcal{H}_Q(G,S)$ are integral.

***Proof*:** Note that from Proposition 2.58 we have that if φ is a one-dimensional representation then $\varphi(s) \in \{-1, q_i\} \subseteq \Bbbk[q_1,\ldots,q_n]$. Therefore φ is integral. ■

Example 2.60: Let $G = S_n$ and $\mathcal{S} = \{s_1,\ldots,s_{n-1}\}$. Since all elements of $\mathcal{S}$ are conjugate to each other, the base field of $\mathcal{H}(S_n)$ is $R = \Bbbk(q)$. Therefore, we obtain only two one-dimensional representations of the Hecke algebra:

- $\iota(q) : \mathfrak{h}(s) \to q$ for all $s \in \mathcal{S}$ is called the **trivial** representation.
- $\varsigma(q) : \mathfrak{h}(s) \to -1$ for all $s \in \mathcal{S}$ is called the **sign** representation.

Note that if we let $q \to 1$ we obtain the original 1 dimensional representation of S_n as discussed in Example 2.10.

Example 2.61: Let $G = W_n$ with $\mathcal{S} = \{w_0,\ldots,w_{n-1}\}$. Then there are two conjugacy classes splitting $\mathcal{S}$. Thus the base field of $\mathcal{H}(G)$ is $\Bbbk(q,p)$ where for $\mathfrak{h}(w_0)^2 = (p-1)\mathfrak{h}(w_0) + p$ and $\mathfrak{h}(w_i)^2 = (q-1)\mathfrak{h}(w_i) + q$ for $i > 0$. Thus we obtain 4 one-dimensional representations:

- $\iota(q,p) : \mathfrak{h}(w_0) \mapsto p$ and $\iota_q : \mathfrak{h}(w_i) \mapsto q$ for $i > 0$.
- $\varsigma_0(q,p) : \mathfrak{h}(w_0) \mapsto -1$ and $\varsigma_0(p,q) : \mathfrak{h}(w_i) \mapsto q$ for $i > 0$.
- $\varsigma_1(q,p) : \mathfrak{h}(w_0) \mapsto p$ and $\varsigma_1(p,q) : \mathfrak{h}(w_i) \mapsto -1$ for $i > 0$.

- $\varsigma(q,p) : \mathfrak{b}(w_0) \mapsto -1$ and $\varsigma(p,q) : \mathfrak{b}(w_i) \mapsto -1$ for $i > 0$.

There are two fundamental questions involving the representations of Hecke algebras.

- The first question is, for which choice of a ring R and parameters $Q \subset R$ is the Hecke algebra $\mathcal{H}_Q(G)$ semisimple?
- What are the irreducible modules of $\mathcal{H}_Q(G)$?

A theorem due to Jacques Tits partially answers both questions. This theorem shows that if $R = \mathbb{C}$ then there exists a finite set $A \subset \mathbb{C}$ such that if $Q \subseteq \mathbb{C} - A$ then $\mathcal{H}_Q(G) \cong \mathbb{C}G$. (see [11, Theorem 68.17 and 68.21]). Later, George Lusztig proved [18, Theorem 3.1] the following theorem.

Theorem 2.62: Let $\Bbbk$ be a field of characteristic 0 and let $R = \Bbbk(x)$. Choose $q_s = x^2$ for all $s \in S$ so that $Q = \{x^2\}$. We then have $\mathcal{H} = \mathcal{H}_Q(G,S) \cong RG$.

This means that if we choose all parameters to be the same, and extend the scalars to include the square root of the parameter, then we have that the Hecke algebra is isomorphic to the group ring. In this case $\mathcal{H}_Q(G,S)$ is semisimple, and it has as many irreducible representations as G. In this thesis, we will only need the irreducible representations of $\mathcal{H}_q(S_n)$ and the one-dimension representations of $\mathcal{H}_{q,p}(W_n)$. We already defined the one-dimensional representations $\mathcal{H}_q(S_n)$ and $\mathcal{H}_{q,p}(W_n)$. For $G = S_n$, we have the following result due to Richard Dipper and Gordon James [13, Theorem 5.2, Corollary 5.3]

Theorem 2.63: Let $R = \Bbbk(q)$ where $\Bbbk$ is a field of characteristic 0. Consider $\mathcal{H} = \mathcal{H}_q(S_n)$. Then the following hold:

- $\mathcal{H}$ is semisimple.
- There is a one-to-one correspondence between partitions of n and irreducible representations of $\mathcal{H}$.
- Given a partition $\lambda \vdash n$ the irreducible representation associated with λ has dimension h_λ (here h_λ is the hook-length number for $D(\lambda)$ defined in Theorem 1.8).
- Consider a specialization $q \to a \in \mathbb{Q} \subseteq \Bbbk$ where $0 < a$. Then $\mathcal{H}_a(S_n) \cong \mathbb{Q}S_n$

In fact, Dipper and James in the same paper compute the irreducible representations of $\mathcal{H}_q(S_n)$ for a general ring R and choice of parameter $q \in R$. In the next section, we will compute the irreducible representations based on a q-generalization of the Specht module. Before we finish this section, we will introduce a special notation for the specialization of the Hecke algebras of S_n and W_n.

Remark 2.64: Let $R = \Bbbk(q)$ and consider the Hecke algebra $\mathcal{H}_q(S_n)$. Consider $x \in \mathcal{H}_q(S_n)$ such that x is specializable to y when evaluating q at $a \in \Bbbk$. We denote

$$\lim_{q \to a} x = y \in \mathcal{H}_a(S_n).$$

Remark 2.65: Let $R = \Bbbk(q, p)$ and let $x \in \mathcal{H}_{q,p}(W_n)$. If evaluating (q, p) at $(a, b) \in \Bbbk^2$ specializes x to y then we denote

$$\lim_{(q,p) \to (a,b)} x = y \in \mathcal{H}_{a,b}(W_n).$$

Section 2.3: Specht Modules for Hecke Algebras

In this section, we will present a generalization of the Young symmetrizer. Using this generalization, we will construct q-variant of the Specht modules. The results in this section are due to Akihiko Gyoja [19]. For this section $\Bbbk$ is a field of characteristic 0 and $R = \Bbbk(q)$ and we will consider the Hecke algebra $\mathcal{H} = \mathcal{H}_q(S_n)$. The following theorem is given in [19, Section 1]

Theorem 2.66: Let V be a finite-dimensional representation of $\mathcal{H}_q(S_n)$ over R. Then the following hold:

- V is integral.
- Let V' be the S_n representation obtained from specializing $q \to 1$. Then V is irreducible if and only if V is irreducible.

The above theorem will prove very useful, since for any representation of $\mathcal{H}_q(S_n)$ over $\Bbbk$ is specializable $q \to 1$. Furthermore, if the specialization $q \to 1$ gives a irreducible module over S_n then we have found an irreducible representation of $\mathcal{H}_q(S_n)$. Before we begin describing the irreducible $\mathcal{H}(S_n)$-modules, we will need one last result.

Lemma 2.67: Let $X = \{x_1, \ldots, x_r\}$ be a subset of $\mathcal{H}_q(S_n)$ such that the specialization $q \to 1$ of x_i is $y_i \in \Bbbk S_n$. So we obtain a set $Y = \{y_1, \ldots, y_r\} \subset \Bbbk S_n$. If Y is linearly independent over $\Bbbk$ then X is linearly independent over $\Bbbk(q)$.

***Proof*:** If Y is linearly independent, then we may assume that $y_i \neq 0$ for all i. Assume that X is linearly dependant. Then there exists $c_1, \ldots, c_r \in \Bbbk(q)$ such that $c_1 x_1 + \cdots + c_r x_r = 0$. Note that we may write $c_i = (q-1)^{d_i} c_i'$ so that $d \in \mathbb{Z}$ and $(q-1)$ does not appear as a factor in c_i'. Let $m = \max\{d_i \mid 1 \leqslant i \leqslant r\}$. We define $h_i \in \Bbbk(q)$ to be $h_i = \frac{c_i}{(q-1)^m}$. Therefore, by construction $h_1 x_1 + \cdots h_r x_r = 0$ and there are no factors of $(q-1)$ in the denominator. Furthermore, there exists $1 \leqslant j \leqslant r$ such that h_j has no factor of $(q-1)$ in both the denominator and nominator. Therefore, the specialization of $q \to 1$ gives $h_j' \notin \Bbbk^*$. This means that the specialization $q \to 1$ gives $h_1' y_1 + \cdots h_r' y_r = 0$. We know that since Y is linearly independent, then $h_i' = 0$ for all i. However, by construction we have that $h'_j \neq 0$. Thus we have a contradiction. ■

Subsection 2.3: q-Generalization of the Specht Module

Recall that for a given partition, $\lambda \vdash n$ we have that $\mathrm{R}(T_\lambda)$ and $\mathrm{C}(T^\lambda)$ are parabolic subgroups of S_n. Also recall that for two tableaux T and V of the same shape, we defined the permutation $\pi(T,V)$ to be the permutation such that $\pi(T,V) \cdot T = V$ (see Definition 2.32).

Definition 2.68: Let $I = \mathcal{S} \cap \mathrm{R}(T_\lambda)$ and define $\mathcal{H}_\lambda = \{\mathfrak{h}(s) \mid s \in I\}$. Similarly, let $J = \mathcal{S} \cap \mathrm{C}(T^\lambda)$ then $\mathcal{H}^\lambda = \{\mathfrak{h}(s) \mid s \in J\}$.

Definition 2.69: Given a partition $\lambda \vdash n$ we define the following elements of $\mathcal{H}_q(S_n)$.

$$c_\lambda^q = \sum_{c \in \mathrm{C}(T^\lambda)} (-q)^{-l(c)} \mathfrak{h}(c) \qquad r_\lambda^q = \sum_{r \in \mathrm{R}(T_\lambda)} \mathfrak{h}(r) \tag{2.1}$$

For simplicity, in this section we will consider $r_\lambda^q = r_{T_\lambda}^q$ and $c_\lambda^q = c_{T^\lambda}^q$. Furthermore, if we specialize $q \to a \neq 0$ then we will denote elements c_λ^q and r_λ^q under this specialization as c_λ^a and r_λ^a.

Lemma 2.70: We have that $c_\lambda^1 = c(T^\lambda)$ and $r_\lambda^1 = r(T_\lambda)$ (see Definition 1.14).

***Proof*:** [19, Section 2] Note that by the definition of $r_\lambda^q = \sum_{g \in \mathrm{R}(T_\lambda)} \mathfrak{h}(g)$ the coefficient of each $\mathfrak{h}(g)$ is $1 \in R$. Therefore, taking the specialization $q \to 1$ we obtain $r_\lambda^1 = \sum_{g \in \mathrm{R}(T_\lambda)} g$. Similarly, the coefficient of $\mathfrak{h}(g)$ is $(-q)^{-l(g)} = (-1)^{-l(g)} q^{-l(g)}$. It is easy to see that $(-1)^{l(g)} = \mathrm{sgn}(g)$ and setting $q = 1$ we have $(-1)^{-l(g)} q^{-l(g)} = \mathrm{sgn}(g)$. ■

Proposition 2.71: For any $g \in \mathrm{R}(T_\lambda)$ we have $\mathfrak{h}(g) r_\lambda^q = q^{l(g)} r_\lambda^q$ and for any $g \in \mathrm{C}(\lambda)$ we have $\mathfrak{h}(g) c_\lambda^q = (-1)^{l(g)} c_\lambda^q$.

***Proof*:** [19, Section 2] Note that for all $g \in S_n$ and $s \in S$ by Corollary 2.18 we have $l(sg) = l(g) \pm 1$. If $g = s_1 \cdots s_k$ where $l(g) = k$ then using the exchange condition (Theorem 2.19) we have that if $l(g) < l(sg)$ then $g < sg$ since otherwise we have that $g = s_{j_1} \cdots s_{j_k}$ where $s_{j_1} = s$. This means that for any given $s \in \mathcal{S} \cap \mathrm{R}(T_\lambda)$

$$r_\lambda^q = \sum_{\substack{x \in \mathrm{R}(T_\lambda) \\ x < sx}} (\mathfrak{h}(1) + \mathfrak{h}(s)) \mathfrak{h}(x).$$

Therefore we may decompose $r_\lambda^q = (1 + \mathfrak{h}(s))A$ for some $A \in \mathcal{H}_q(S_n)$.Since $\mathfrak{h}(s)(\mathfrak{h}(1) + \mathfrak{h}(s)) = q(\mathfrak{h}(1) + \mathfrak{h}(s))$ we have that $\mathfrak{h}(s) r_\lambda^q = \mathfrak{h}(s)(1 + \mathfrak{h}(s))A = q(1 + \mathfrak{h}(s))A = q r_\lambda^q$ for any $s \in \mathcal{S} \cap \mathrm{R}(T)$. In the same way we can show that we may write

$$c^q_\lambda = \sum_{\substack{x \in \mathrm{C}(T^\lambda) \\ x < sx}} (1 - q^{-1}\mathfrak{b}(s))\mathfrak{b}(x).$$

It is easy to see that $\mathfrak{b}(s)(1 - q^{-1}\mathfrak{b}(s)) = (-1)(1 - q^{-1}\mathfrak{b}(s))$. Therefore $\mathfrak{b}(s)c^q_\lambda = -c^q_\lambda$ for all $s \in \mathcal{S} \cap \mathrm{C}(T^\lambda)$. The proposition follows from these computations by an induction argument on the length $l(g)$. ■

Corollary 2.72: The left ideals $\mathcal{H}_\lambda r^q_\lambda$ and $\mathcal{H}^\lambda c^q_\lambda$ are one-dimensional representations of $\mathcal{H}_\lambda$ and $\mathcal{H}^\lambda$ respectively.

***Proof*:** [19, Section 2] This follows immediately from the Proposition 2.71 as the representation $\varphi : \mathcal{H}_\lambda \to \mathrm{End}(\mathcal{H}_\lambda r^q_\lambda)$ given by $\varphi(\mathfrak{b}(w)) = \mathfrak{b}(w)r^q_\lambda$ has the image $\varphi(\mathcal{H}) \subseteq Rr^q_\lambda$. A similar argument can be given for $\mathcal{H}^\lambda c^q_\lambda$. ■

Corollary 2.73: Let $Q_\lambda = \sum_{w \in \mathrm{R}(T_\lambda)} p^{l(w)}$ and $Q^\lambda = \sum_{w \in \mathrm{C}(T^\lambda)} (-p)^{-l(w)}$. Then $(r^q_\lambda)^2 = Q_\lambda r^q_\lambda$ and $(c^q_\lambda)^2 = Q^\lambda c^q_\lambda$.

***Proof*:** [19, Section 2] From Proposition 2.71 we have that if $g \in \mathrm{R}(T_\lambda)$ then $gr^q_\lambda = q^{l(g)}r^q_\lambda$. Then we can compute $(r^q_\lambda)^2$ in the following way

$$(r^q_\lambda)^2 = \sum_{g \in \mathrm{R}(T_\lambda)} \mathfrak{b}(g)r^q_\lambda = \sum_{g \in \mathrm{R}(T_\lambda)} q^{l(g)}r^q_\lambda = Q_\lambda r^q_\lambda.$$

A similar computation can be done for $(c^q_\lambda)^2$. ■

Theorem 2.74: Recall that $\mathrm{ST}(\lambda)$ is the set of standard tableau of shape λ, as defined in Section 1.1. Given standard tableau $T \in \mathrm{ST}(\lambda)$. We have that

$$\dim_k \hom_{\Bbbk S_n}(\Bbbk S_n c(T), \Bbbk S_n r(T)) = \dim_k \hom_{\Bbbk S_n}(\Bbbk S_n r(T), \Bbbk S_n c(T)) = 1.$$

***Proof*:** [19, Proposition 1.3] Consider $f \in \hom_{\Bbbk S_n}(\Bbbk S_n r(T), \Bbbk S_n c(T))$. Then f is determined by $f(r(T)) \in \Bbbk S_n c(T)$. Note that by Proposition 1.15 we have that $r(T)^2 = |\mathrm{R}(T)|r(T)$ and $c(T)^2 = |\mathrm{C}(T)|c(T)$. Furthermore, for any given $x \in \mathcal{H}_q(S_n)$ we have that $f(xr(T)) = xf(r(T))$, therefore we may write

$$f(r(T)) = f\left(\frac{r(T)^2}{|\mathrm{R}(T)|}\right)\frac{c(T)}{|\mathrm{C}(T)|} = \frac{r(T)f(r(T))c(T)}{|\mathrm{R}(T)||\mathrm{C}(T)|}.$$

Since $x \in \mathrm{R}(T)$ then $r(T)x = r(T)$ and similarly if $y \in \mathrm{C}(T)$ then $yc(T) = \mathrm{sgn}(y)c(T)$. If $g \in \mathrm{R}(T)\mathrm{C}(T)$ then $r(T)gc(T) = \mathrm{sgn}(g)r(T)c(T)$. Furthermore, if $g \notin \mathrm{R}(T)\mathrm{C}(T)$ then in Lemma 1.21 we showed that there exists a transposition $t \in \mathrm{R}(T) \cap \mathrm{C}(gT)$. This means that

$$r(T)gc(T) = r(T)c(gT)g^{-1} = r(T)tc(gT)g^{-1} = -r(T)c(gT)g^{-1} = -r(T)gc(T).$$

This shows that $r(T)gc(T) = 0$ if $g \notin \mathrm{R}(T)\mathrm{C}(T)$. Combining our results, we have then following:

$$r(T)gc(T) = \begin{cases} \mathrm{sgn}(g)r(T)c(T) & \text{if } g \in \mathrm{R}(T)\mathrm{C}(T) \\ 0 & \text{otherwise} \end{cases}$$

Therefore, since $f(r(T))$ can be expressed as a linear sum of terms with the form $r(T)gc(T)$, then there must exist $k \in \Bbbk$ such that is that $f(r(T)) = kc(T)r(T)$. Since $c(T) \in \mathrm{C}(T)$ and $r(T) \in \mathrm{R}(T)$ and $\mathrm{C}(T) \cap \mathrm{R}(T) = \{1\}$ then $c(T)r(T) \neq 0$. Thus showing $\dim_{\Bbbk} \hom_{\Bbbk S_n}(\Bbbk S_n r(T), \Bbbk S_n r(T)) = 1$. A similar proof can be made for $f \in \hom_{\Bbbk S_n}(\Bbbk S_n c(T), \Bbbk S_n r(T))$. ■

Consider $f \in \hom_{\Bbbk S_n}(\Bbbk S_n c(T), \Bbbk S_n r(T))$ and note that the image of f must lie in $\Bbbk S_n c(T)r(T)$. In fact, this image is exactly the Specht module where $\varepsilon(T) = c(T)r(T)$. We have constructed the q-generalizations of $c(T^\lambda)$ and $r(T_\lambda)$. We will see that the Young idempotent will be the image of a similar map in $\mathcal{H}$.

Proposition 2.75: Let $T \in \mathrm{ST}(\lambda)$ and consider $x = \mathfrak{h}(\pi(T_\lambda, T))$ and $y = \mathfrak{h}(\pi(T^\lambda, T))$. Let $0 \neq f \in \hom_{\mathcal{H}}(\mathcal{H}c^q_\lambda, \mathcal{H}r^q_\lambda)$ and $0 \neq g \in \hom_{\mathcal{H}}(\mathcal{H}r^q_\lambda, \mathcal{H}c^q_\lambda)$. Then for some choices $a, b \in R$ we have that

$$f(c_\lambda) = ac^q_\lambda y^{-1} x r^q_\lambda \text{ and } g(r_\lambda) = br^q_\lambda x^{-1} y c^q_\lambda$$

***Proof*:** [19, Section 2] We begin with a similar computation to the proof of Theorem 2.74. We have that f is determined by $f(c^q_\lambda)$. From Corollary 2.73 we have that

$$f(c^q_\lambda) = \frac{1}{Q_\lambda Q^\lambda} c^q_\lambda f(c^q_\lambda) r^q_\lambda.$$

Therefore, by expressing $f(c^q_\lambda)$ as a linear combination of elements of $\mathcal{H}_q(S_n)$, it follows that there exist $a_g \in R$ for all $g \in S_n$ such that

$$f(c^q_\lambda) = \sum_{g \in S_n} a_g c^q_\lambda \mathfrak{h}(g) r^q_\lambda. \tag{2.2}$$

The strategy is to show that $a_g = 0$ for almost all $g \in S_n$, with only one exception. This would show that there is only one choice for f. Define $I = \mathcal{S} \cap \mathrm{R}(T_\lambda)$ and $J = \mathcal{S} \cap \mathrm{C}(T^\lambda)$. By Proposition 2.71, if $s \in I$ and $t \in J$ then

$$c^q_\lambda gsr^q_\lambda = qc^q_\lambda gr^q_\lambda \text{ and } c^q_\lambda tgr^q_\lambda = c^q_\lambda gr^q_\lambda.$$

This means that if $g \in S_n$ has the property that $g \in \mathrm{C}(T^\lambda)w\mathrm{R}(T_\lambda)$ where w is minimal (see Theorem 2.29). For $g = xwy$ for $x \in C(T^\lambda)$ and $y \in R(T_\lambda)$ with lengths a and b respectively, then

$$c^q_\lambda gr^q_\lambda = q^a(-1)^b c^q_\lambda wr^q_\lambda.$$

Therefore if we define $A := \{g \in S_n \mid g < tg\,,\ g < gs\,,\ t \in J, s \in I\}$ then we can reduce Equation 2.2 to the following

$$f(c_\lambda^q) = \sum_{g \in A} a_g c_\lambda^q \mathfrak{b}(g) r_\lambda^q.$$

To see this by assuming $g \notin A$, then there exists $y \in \mathrm{C}(T^\lambda)$ and $x \in \mathrm{R}(T_\lambda)$ such that $g = xwy$ where $l(g) = l(x) + l(w) + l(y)$ by Theorem 2.29. Furthermore, for any $s \in I$ and $t \in J$ then $w \leqslant sw$ and $w \leqslant wt$ by Proposition 2.37. In other words, $w \in A$. Computing $c_\lambda^q \mathfrak{b}(g) r_\lambda^q = q^{l(x)}(-1)^{l(y)} c_\lambda^q \mathfrak{b}(w) r_\lambda^q$ the summation can be reduced to one only on the elements of A. What remains to show is that there is only one element in A. For this, consider $\sigma = \pi(T_\lambda, T^\lambda)$. Then since both T_λ and T^λ are standard, then for any $(i, i+1) = s \in I$ we have that $\pi(T_\lambda, T^\lambda)(i) < \pi(T_\lambda, T^\lambda)(i+1)$. Therefore $\sigma < \sigma s$ by Proposition 2.24 and Corollary 2.25. A similar argument shows that $\sigma^{-1} < \sigma^{-1} t$ for $t \in J$, thus $\sigma < t\sigma$. This shows that $\sigma \in A$. Note that any element $g \in S_n$ is of the form $\sigma\tau$ by picking $\tau = (\sigma^{-1})g$. This means that all elements of A may be obtained this way. Now consider $\sigma\tau \in A$ such that $\tau \neq 1$. Note that since $\sigma \in A$ we have that $\sigma\tau \notin \mathrm{C}(T^\lambda)\sigma\mathrm{R}(T_\lambda)$, since the only way for this to be true is to have $\tau \in \mathrm{R}(T_\lambda)$, which would contradict $\sigma\tau \in A$. By Lemma 2.34 we have that

$$\tau \mathrm{R}(T_\lambda) \tau^{-1} \cap \sigma^{-1} \mathrm{C}(T^\lambda) \sigma \neq \{1\}.$$

Therefore we may find $z_1 \in \mathrm{R}(T_\lambda)$ and $z_2 \in \mathrm{C}(T^\lambda)$ where $z_1, z_2 \neq 1$, since the intersection contains elements which are not identity. These elements satisfy

$$\tau z_1 \tau^{-1} = \sigma^{-1} z_2 \sigma \iff \sigma\tau z_1 = z_2 \sigma\tau.$$

Therefore note that we have the following:

$$\begin{aligned} c_\lambda^q \mathfrak{b}(\sigma\tau z_1) r_\lambda^q &= q^{l(z_1)} c_\lambda^q \mathfrak{b}(\sigma\tau) r_\lambda^q \\ c_\lambda^q \mathfrak{b}(z_2 \sigma\tau) r_\lambda^q &= (-1)^{l(w_2)} c_\lambda^q \mathfrak{b}(\sigma\tau) r_\lambda^q \end{aligned}$$

Therefore since $\sigma\tau z_1 = z_2\sigma\tau$ we have the following equation

$$q^{l(z_1)} c_\lambda^q \mathfrak{b}(\sigma\tau) r_\lambda^q = (-1)^{l(w_2)} c_\lambda^q \mathfrak{b}(\sigma\tau) r_\lambda^q$$

Since either $z_1 \neq 1$ or $z_2 \neq 2$ then the equation on top implies that $c_\lambda^q \mathfrak{b}(\sigma\tau) r_\lambda^q = 0$. This means that

$$f(c_\lambda^q) = c_\lambda^q \mathfrak{b}(\sigma) r_\lambda^q.$$

It is clear that $c_\lambda^q \mathfrak{b}(\sigma) r_\lambda^q \neq 0$ since the limit $q \to 1$ gives $c(T^\lambda)\sigma r(T_\lambda) = c(T^\lambda) r(T^\lambda) \sigma^{-1} \neq 0$. Furthermore note that $\pi(T^\lambda, T)\pi(T_\lambda, T) = \sigma$ and thus $c_\lambda^q y^{-1} x r_\lambda^q \neq 0$ by a specialization argument. This shows that $f = a c_\lambda^q y^{-1} x r_\lambda^q$. To show the same for the map g, one would simply repeat these steps. ■

Theorem 2.76: Let $\lambda \vdash n$ then

$$\dim_R \hom_{\mathcal{H}}(\mathcal{H}c_\lambda^q, \mathcal{H}r_\lambda^q) = \dim_R \hom_{\mathcal{H}}(\mathcal{H}r_\lambda^q, \mathcal{H}c_\lambda^q) = 1.$$

***Proof*:** This is a direct result from Proposition 2.75. In that proposition we showed that there are non-trivial elements in $M = \hom_{\mathcal{H}}(\mathcal{H}c_\lambda^q, \mathcal{H}r_\lambda^q)$. Furthermore, let $\sigma = \pi(T_\lambda, T^\lambda)$ and $a \in R$, then all maps $f \in M$ are of the following form:

$$f(x) = xac_\lambda^q \sigma r_\lambda^q$$

Thus if there are two different, non-trivial maps $f, g \in M$ then $f = ag$ for some $a \in R$. This shows that $\dim_R M = 1$. Let $N = \hom_{\mathcal{H}}(\mathcal{H}r_\lambda^q, \mathcal{H}c_\lambda^q)$, then by a similar argument one may show that $\dim_R N = 1$. ■

Definition 2.77: Let $T \in \mathrm{ST}(\lambda)$ then we define the q-**Young symmetrizer** to be

$$\varepsilon_T^q = \mathfrak{b}(\pi(T^\lambda, T))c_\lambda^q \mathfrak{b}(\pi(T^\lambda, T))^{-1}\mathfrak{b}(\pi(T_\lambda, T))r_\lambda^q \mathfrak{b}(\pi(T_\lambda, T))^{-1}$$

Lemma 2.78: The element ε_T^q is specializable at $q = 1$ and $\varepsilon_T^1 = \varepsilon_T$ (See Definition 1.16).

***Proof*:** This is a direct consequence of Lemma 2.70. Note that by setting $q \to 1$ we have that $c_\lambda^q \to c(T^\lambda)$ and $r_\lambda^q \to r(T_\lambda)$. Since all elements of the forms $\mathfrak{b}(w)$ specialize w, then we have that

$$\lim_{q\to 1} \varepsilon_T^q = \pi(T^\lambda, T)c(T^\lambda)(\pi(T, T^\lambda)\pi(T_\lambda, T)r(T_\lambda)\pi(T, T_\lambda) = c(T)r(T) = \varepsilon_T$$

Which shows exactly what we needed. ■

Definition 2.79: Let $\lambda \vdash n$ and define $\varepsilon_\lambda^q = \varepsilon_{T^\lambda}^q$. Then we define the q-**Specht module** of $\mathcal{H}$ to be the representation of $\mathcal{H}$ given by $S_\lambda^q := \mathcal{H}\varepsilon_\lambda^q$

Theorem 2.80: The Specht module S_λ^q is an irreducible $\mathcal{H}$-module. As an R-module, it has a basis given by

$$B = \{\mathfrak{b}(\pi(T^\lambda, T))\varepsilon_\lambda^q \mid T \in \mathrm{ST}(\lambda)\}.$$

Furthermore for different partitions $\lambda_1 \neq \lambda_2$ we have that $S_{\lambda_1}^q \ncong S_{\lambda_2}^q$.

***Proof*:** Note that since $\varepsilon_\lambda^q \to \varepsilon_{T^\lambda}$ as $q \to 1$ the module $S_\lambda^q \to S_\lambda$. Therefore, by Theorem 2.66 we have that S_λ^q is an irreducible representation of $\mathcal{H}$. If there exists $\lambda_1 \neq \lambda_2$ such that $S_{\lambda_1}^q \cong S_{\lambda_2}^q$ then let $\varphi : S_{\lambda_1}^q \to S_{\lambda_2}^q$ be such isomorphism. Then $\varphi(\varepsilon_{\lambda_1}^q) = x\varepsilon_{\lambda_2}^q$ for some $x \in \mathcal{H}$. Note if $0 \neq k \in R = \Bbbk(q)$ then $\varphi_k : x \mapsto k\varphi(x)$ is also an isomorphism. Thus let $x = \sum_{g\in S_n} a_g \mathfrak{b}(g)$ such that $a_g \in \Bbbk(q)$ then $a_g = (q-1)^{m_g} a_g'$ where $m_g \in \mathbb{Z}$ and a_g' has no factor of $(q-1)$ in both the denominator and numerator. Let $M = \min\{m_g \mid g \in S_n\}$ and define $b_g = \frac{a_g}{(q-1)^M}$. This guaranties that for all $g \in G$

there are no $(q-1)$ factors in the denominator. Furthermore, it guaranties that there exists one $g \in S_n$ such that there are no factors of $(q-1)$ in the nominator as well. Let $k = \frac{1}{(q-1)^M}$ then $kx = \sum_{g \in S_n} b_g \mathfrak{b}(g)$ is specializable by setting $q = 1$. Also, $k\varepsilon^q_{\lambda_1}$ is not 0 at the specialization, since there exists a $g \in G$ such that b_g has no $(q-1)$ factors. By this construction, the map φ_k defines a map $\varphi'_k : S_{\lambda_1} \to S_{\lambda_2}$ by a specialization $q \neq 1$. Furthermore φ'_k is non-trivial, which is a contradiction.

So far, we proved that $\mathcal{S}^q_\lambda$ is an irreducible representation of $\mathcal{H}_q(S_n)$. Furthermore, for different partitions $\lambda_1 \neq \lambda_2$ of n we proved that $S^q_{\lambda_1} \ncong S^q_{\lambda_2}$. It remains to show that B is a basis for S^q_λ. Note that, if we specialize $q \to 1$, then the elements of B gives us $B' = \{\pi(T, T_\lambda)\varepsilon_\lambda \,|\, T \in \mathrm{ST}(\lambda)\}$. By Theorem 1.37 we have that B' is in fact a basis for S_λ. Therefore by Lemma 2.67 we have that B is linearly independent as B' is. This means that for every $\lambda \vdash n$ we have that $\dim_R S^q_\lambda \geqslant \dim_{\Bbbk} S_\lambda$. It will then be enough to prove $\dim_R S^q_\lambda = \dim_{\Bbbk} S_\lambda$. To do this, we recall Theorem 2.63 which tells us that $\mathcal{H}$ is semisimple. This means that

$$\mathcal{H} \cong \bigoplus_{\lambda \vdash n} (S^q_\lambda)^{\dim(S^q_\lambda)}.$$

Since $\dim_k \Bbbk S_n = \dim_R \mathcal{H} = n!$ we obtain the following;

$$n! = \sum_{\lambda \vdash n} (\dim_k S_\lambda)^2 \leqslant \sum_{\lambda \vdash n} (\dim_R S^q_\lambda)^2 = n!$$

Which proves the equality as we needed. ■

Subsection 2.3: Structure of the Hecke Young Symmetrizer

So far, in Theorem 2.80 we have shown that $\mathcal{S}^q_\lambda = \mathcal{H}\varepsilon^q_\lambda$ is an irreducible representation of $\mathcal{H}$. In the same theorem, we also have constructed a basis of S^q_λ. We will finish this chapter by showing that ε^q_λ is an idempotent. We also recover some relations for the q-generalization of the Young symmetrizer. Namely, we determine for which T_1 and T_2 one has $\varepsilon^q_{T_1}\varepsilon^q_{T_2} = 0$. Lastly, we will give all the Specht modules for $\mathcal{H}_q(S_3)$.

Proposition 2.81: For each $\lambda \vdash n$ there exists $k_\lambda \in R$ such that $(\varepsilon^q_T)^2 = k_\lambda \varepsilon^q_T$.

***Proof*:** Consider $f \in M = \hom_{\mathcal{H}}(\mathcal{H}c^q_\lambda, \mathcal{H}r^q_\lambda)$ and $g \in N = \hom_{\mathcal{H}}(\mathcal{H}r^q_\lambda, \mathcal{H}c^q_\lambda)$ such that $f, g \neq 0$. By Theorem 2.76 we have that $\dim_R M = \dim_R N = 1$. This implies that there exists a k such that

$$f(g(f(c^q_\lambda))) = kf(c^q_\lambda).$$

We will use Proposition 2.75 to expand the above with $x = \mathfrak{b}(\pi(T_\lambda, T)$ and $y = \mathfrak{b}(\pi(T^\lambda, T))$. Then we obtain the following

$$f(g(f(c^q_\lambda))) = f(g(c^q_\lambda y^{-1} x r^q_\lambda)) = f(c^q_\lambda y^{-1} x r^q_\lambda x^{-1} y c^q_\lambda) = c^q_\lambda y^{-1} x r^q_\lambda x^{-1} y c^q_\lambda y^{-1} x r^q_\lambda$$

This means that we have the relation

$$c_\lambda^q y^{-1} x r_\lambda^q x^{-1} y c_\lambda^q y^{-1} x r_\lambda^q = k c_\lambda^q y^{-1} x r_\lambda^q.$$

Recall that $\varepsilon_T^q = y c_\lambda^q y^{-1} x r_\lambda^q x^{-1}$ which means that

$$(\varepsilon_T^q)^2 = y(c_\lambda^q y^{-1} x r_\lambda^q x^{-1} y c_\lambda^q y^{-1} x r_\lambda^q) x^{-1} = y(k c_\lambda^q y^{-1} x r_\lambda^q) x^{-1} = k\varepsilon_T^q.$$

Since the specialization at $q \to 1$ of $(\varepsilon_T^q)^2$ is $(\varepsilon_T)^2$ and we know by Proposition 1.28 that $(\varepsilon_T)^2 \neq 0$ meaning that $k \neq 0$. ■

Corollary 2.82: For $\lambda_1, \lambda_2 \vdash n$ such that $\lambda_1 \neq \lambda_2$ we have $\varepsilon_{\lambda_1}^q \varepsilon_{\lambda_2}^q = 0$.

***Proof*:** Note that $S_{\lambda_1}^q$ and $S_{\lambda_2}^q$ are two inequivalent irreducible representations of $\mathcal{H}$ and therefore we have that $S_{\lambda_1}^q \cap S_{\lambda_2}^q = \{0\}$. Furthermore by Proposition 1.28 the elements $\varepsilon_{\lambda_1}^q$ and $\varepsilon_{\lambda_2}^q$ are idempotents. This means that $A = \mathcal{H}\varepsilon_{\lambda_1}^q \varepsilon_{\lambda_2}^q$ is a submodule of $S_{\lambda_2}^q$. Therefore either $A = 0$ or $A = S_{\lambda_2}^q$. Since $S_{\lambda_1}^q$ and $S_{\lambda_2}^q$ are inequivalent the latter cannot be true, therefore $A = 0$ meaning that $\varepsilon_{\lambda_1}^q \varepsilon_{\lambda_2}^q = 0$. ■

Lemma 2.83: For a given $T_1, T_2 \in \mathrm{ST}(\lambda)$ if $T_1 \neq T_2$ and $l(\pi(T^\lambda, T_1)) \geqslant l(\pi(T^\lambda, T_2))$ then $\varepsilon_{T_1}^q \varepsilon_{T_2}^q = 0$.

***Proof*:** Let $x_1 = \mathfrak{h}(T_\lambda, T_1)$ and $y_1 = \mathfrak{h}(T_\lambda, T_1)$. Similarly define $x_2 = \mathfrak{h}(T^\lambda, T_2)$ and $y_2 = \mathfrak{h}(T^\lambda, T_2)$. Thus we may write

$$\varepsilon_{T_1}^q \varepsilon_{T_2}^q = x_1 c_\lambda^q x_1^{-1} y_1 r_\lambda^q y_1^{-1} x_2 c_\lambda^q x_2^{-1} y_2 r_\lambda^q y_2^{-1}.$$

Therefore it would be enough to show that $r_\lambda^q y_1^{-1} x_2 c_\lambda^q = 0$. First, we have proven in Proposition 2.33 that for any $T \in \mathrm{ST}(\lambda)$ we have $l(x_2) + l(y_2) = l(y_1) + l(x_1) = l(\pi(T_\lambda, T^\lambda))$. Furthermore, by Lemma 2.16 we have that $l(y_1^{-1} x_2) \leqslant l(y_1) + l(x_2)$ and our assumption of the lemma is that $l(y_1) \geqslant l(x_2)$ and hence

$$l(y_1^{-1} x_2) \leqslant l(y_1) + l(x_2) \leqslant l(\pi(T_\lambda, T^\lambda)).$$

Let $A = \{ab \,|\, a \leqslant \mathfrak{h}(T_\lambda, T_1) \text{ and } b \leqslant \mathfrak{h}(T^\lambda, T_2)\}$ then by Proposition 2.47 we have that

$$y_1^{-1} x_2 = \sum_{g \in A} a_g \mathfrak{h}(g).$$

Note that for all $y^{-1} x_2 \neq g \in A$ we have that $g < y_1^{-1} x_2$. Since $y_1^{-1} x_2 \neq \pi(T^\lambda, T_\lambda)$ then we have

$$A \cap H_\lambda \mathfrak{h}(\pi(T^\lambda, T_\lambda)\mathcal{H}^\lambda = \varnothing.$$

Therefore we for any $g \in A$ we use Lemma 2.34 and repeat the same proof for Proposition 2.75. Namely we assume $\mathfrak{h}(\pi(T_\lambda, T^\lambda))g \in A$ and we show that $r_\lambda^q \mathfrak{h}(\pi(T_\lambda, T^\lambda)) g c_\lambda^q = 0$. Which shows that for any $r_\lambda^q y_1^{-1} x_2 c_\lambda^q = 0$. ■

In Chapter 1 we ordered $\mathrm{ST}(\lambda)$ such that if $T_1 < T_2$ then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$. Lemma 2.83 shows that we may order $\mathrm{ST}(\lambda)$ similarly with $\varepsilon^q_{T_1}\varepsilon^q_{T_2} = 0$. We will finish this section by computing all the Specht modules for $\mathcal{H}_q(S_3)$.

Example 2.84: For the case for S_3 we have three partitions of 3. They are given by $\lambda_1 = (3)$, $\lambda_2 = (2,1)$, and $\lambda_3 = (1,1,1)$. We give T_λ and T^λ for all all of these partitions below:

$$T_{\lambda_1} = T^{\lambda_1} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad T_{\lambda_2} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T^{\lambda_2} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad T_{\lambda_3} = T^{\lambda_3} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

We will compute $S^q_{\lambda_1}$, $S^q_{\lambda_3}$, and $S^q_{\lambda_2}$ in this order. For each of these let $T = T_\lambda$, $x = \pi(T^\lambda, T)$, $s_1 = (12)$ and $s_2 = (23)$.

1. For $S^q_{\lambda_1}$, note that $\mathrm{R}(T_\lambda) = S_3$ and $\mathrm{C}(T^\lambda) = \{1\}$. Therefore $c^q_\lambda = 1$ and r^q_λ is given by $r^q_\lambda = \sum_{g\in S_3} \mathfrak{h}(g)$. In this case $x = 1$ as $T_{\lambda_1} = T^{\lambda_2}$ which gives us $\varepsilon^q_\lambda = r^q_\lambda$. Since $\dim_R S^q_{\lambda_1} = \dim_{\Bbbk} S_{\lambda_1} = 1$ then we have that $S^q_\lambda = R\varepsilon^q_\lambda = Rr^q_\lambda$. By Proposition 2.71 we have that for any $g \in S_3$ we have $\mathfrak{h}(g)r^q_\lambda = q^{l(g)}r^q_\lambda$. This gives us the **trivial representation** for $\mathcal{H}(S_3)$ as expected.

2. Similarly for $S^q_{\lambda_3}$ we have that $x = y = 1$ and $\mathrm{R}(T_\lambda) = \{1\}$. Therefore we have $c^q_\lambda = \sum_{g\in S_3}(-q)^{-l(g)}$ which gives $\varepsilon^q_\lambda = c^q_\lambda$. Since $S^q_{\lambda_3}$ has dimension 1 this gives us the **sign representation** for $\mathcal{H}(S_3)$ as $gc^q_\lambda = (-1)^{l(g)}c^q_\lambda$.

3. For the last representation we have two standard tableaux over λ_2. This means that $\dim_R S^q_{\lambda_2} = 2$. Here $x = \pi(T^\lambda, T_\lambda) = s_2$. Furthermore $\mathrm{C}(T^\lambda) = \mathrm{R}(T_\lambda) = \{1, s_1\}$. We compute $c^q_\lambda = 1 - q^{-1}\mathfrak{h}(s_1)$ and $r^q_\lambda = 1 + \mathfrak{h}(s_1)$ giving us

$$\varepsilon^q_\lambda = \mathfrak{h}(s_2)(1 - q^{-1}\mathfrak{h}(s_1))\mathfrak{h}(s_2)^{-1}(1 + \mathfrak{h}(s_1)) = 2 + (1 - q^{-1})\mathfrak{h}(s_1)$$

   Furthermore by Theorem 2.80 we have that a basis for S_{λ_2} is given by $B = \{\varepsilon^q_{lm_2}, \mathfrak{h}(s_2)\varepsilon^q_{\lambda_2}\}$. We give the matrix associated with $\mathfrak{h}(s_1)$ and $\mathfrak{h}(s_2)$ bellow.

$$\mathfrak{h}(s_1) = \begin{pmatrix} q^2 & q^3 - q^2 + q - 1 \\ -q & -q^2 + q - 1 \end{pmatrix} \qquad \mathfrak{h}(s_2) = \begin{pmatrix} 0 & q \\ 1 & q - 1 \end{pmatrix}$$

Chapter 3

symmetric polynomials and Symmetrizing Operators

We will dedicate this Chapter to studying the structure of a polynomial ring $\mathscr{P} = \Bbbk[x_1, \ldots, x_n]$ as a module over the ring of symmetric polynomials. As an application, we present a known polynomial representation of the Hecke algebras of type A (see [21]). Using the machinery obtained from these constructions, we will obtain a representation of the Hecke algebra of Type B over the ring of Laurent polynomials. Beginning with Section 3.1 we will define pseudo-reflection groups and their actions on $\mathscr{P}$. We state the Chevalley-Shephard-Todd Theorem on the structure of $\mathscr{P}$ as a module over $\mathscr{P}^G$, where G is the reflection group. We also define the coinvariant algebra $\mathscr{P}_G$ and study its structure as a G-module. In Sections 3.2 and 3.3 we study the special case $G = S_n$ where the subring of invariants is called the ring of symmetric polynomials, denoted by $\mathfrak{Sym}$. We give a basis for $\mathscr{P}$ as a $\mathfrak{Sym}$-module, and we discuss various symmetrizing operators such as the divided difference and Demazure operators. We will be generalizing all results obtained in the case where $G = S_n$ acting on $\mathscr{P}$, to the case where G is the Hyperoctahedral group W_n acting on the ring of Laurent polynomials. We will be generalizing the coinvariant algebra, and the symmetrizing operators to the Laurent polynomials. Using this material in Section 3.4, we construct a representation of the Hecke algebra $H_q(S_n)$ over $\mathscr{P}$. We also discuss a way to represent the Hecke algebra $H_{q,p}(W_n)$ over the Laurent polynomials.

Section 3.1: Reflection Groups

Throughout this Chapter we fix a field $\Bbbk$ of characteristic 0. Consider a vector space V over $\Bbbk$. Given a linear map $r \in \mathrm{GL}(V)$, we call r a **pseudo-reflection** if $\dim\ker(r - I) = \dim V - 1$ and $r^d = I$ for some $d \geqslant 2$. A pseudo-reflection R is called a **true reflection** if $r^2 = I$. If a subgroup of $\mathrm{GL}(V)$ is generated by pseudo-reflections we call it a **pseudo-reflection group**. If the subgroup is generated by true reflections then we call it a **reflection group**.

Example 3.1: Let $\{v_1, \ldots, v_n\}$ be a basis for V. Then one can consider S_n as a subgroup of $\mathrm{GL}(V)$ composed of permutations of the basis of V. Consider a transposition s_i which swaps v_i and v_{i+1}. We can show that s_i is in fact a reflection. First we compute what $s_i - I$ does to the basis elements:

$$(s_i - I)(v_j) = \begin{cases} 0 & \text{if } j \neq i, i+1 \\ v_{i+1} - v_i & \text{if } j = i \\ v_i - v_{i+1} & \text{if } j = i+1 \end{cases}$$

From the above computation $\ker(s_i - I) = \langle \{v_j \mid j \neq i, i+1\} \bigcup \{v_i + v_{i+1}\} \rangle$ thus $\dim \ker(s_i - I) = n - 1 = \dim V - 1$. Since S_n is in fact generated by transpositions, it follows that S_n is a reflection group.

Pseudo-reflection groups have been extensively studied, and they have important connections to the theory of polynomial invariants. The structure of the subring of invariants under a pseudo-reflection group is given by a really powerful theorem due to G. C Shephard and J. A. Todd in [28] which was further expanded by C. Chevalley in [9]. To state this theorem, we first introduce some notation. Let V be the vector space spanned by the variables $\{x_1, \ldots, x_n\}$ of P. For a given $v \in V$ and $p \in \mathscr{P}$ where $v = (v_1, \ldots, v_n)$, we have that $p(v) = p(v_1, \ldots, v_n)$. For a given matrix $g \in \mathrm{GL}(V)$ we can define the action of g on a polynomial via the formula $g \cdot p(v) = p(g^{-1}v)$. Thus, for a given subgroup $G \subseteq \mathrm{GL}(V)$ we define its invariant ring to be

$$\mathscr{P}^G = \{p \in \mathscr{P} \mid g \cdot p = p \, \forall g \in G\}.$$

Definition 3.2: Let M_G be the ideal generated by all the polynomials $f \in \mathscr{P}^G$ with no constant term. Then the **coinvariant algebra** of G over $\mathscr{P}$ is defined as $\mathscr{P}_G := \mathscr{P}/M_G$.

The following result, known as the **Chevalley-Shephard-Todd** Theorem (CST), shows the connection between invariant subrings and pseudo-reflection groups:

Theorem 3.3: If G is a finite subgroup of $\mathrm{GL}(V)$ then the following are equivalent:

1. G is a pseudo-reflection group.
2. $\mathscr{P}^G$ is isomorphic to a polynomial algebra with n variables.
3. $\mathscr{P}$ is a free module of finite rank over $\mathscr{P}^G$.
4. $\mathscr{P} \cong \mathscr{P}^G \otimes \mathscr{P}_G$

Now let G be a pseudo-reflection group. By Theorem 3.3 we may find algebraically independent homogeneous polynomials $f_1, \ldots, f_n \in \mathscr{P}^G$ such that $\mathscr{P}^G = \Bbbk[f_1, \ldots, f_n]$. Therefore, by the definition of the coinvariant algebra we can see that $\mathscr{P}_G = \mathscr{P}/(f_1, \ldots, f_n)$. Recall that for a commutative ring R and a group G we use RG to denote the group ring of G over R. Let $d_i := \deg f_i$.

Theorem 3.4: Let G be a finite pseudo-reflection group in $\mathrm{GL}(V)$. Then the following hold:

1. $|G| = d_1 \cdots d_n$

2. As G-representations we have $\mathscr{P} \cong \mathscr{P}^G \otimes_{\Bbbk} \Bbbk G$ and $\mathscr{P}_G \cong \Bbbk G$.

For a proof of the above theorems see [8, Chap. 4.1]. Given a pseudo-reflection group G over V we may ask two important questions: what invariant algebraically independent polynomials $f_1, \ldots, f_n \in \mathscr{P}^G$ do satisfy $\mathscr{P}^G = \Bbbk[f_1, \ldots, f_n]$, and what is a basis for $\mathscr{P}$ as a $\mathscr{P}^G$-module? In Section 1.2 we aim at answering these questions when G is the permutation group S_n as realized in Example 3.1.

Section 3.2: Ring of Symmetric Polynomials

We discussed in Example 3.1 that S_n is a reflection group acting on an n-dimensional vector space V. This defines an action of S_n on $\mathscr{P}$ in the following way: for a given permutation $\sigma \in S_n$ and a polynomial $p \in \mathscr{P}$ we define $\sigma \cdot p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. We denote the subring of polynomials which are invariant under this action of S_n by $\mathfrak{Sym} = \{p \in \mathscr{P} \mid \sigma p = p \, \forall \sigma \in S_n\}$ and we will call this the subring of **symmetric polynomials**. Thus, we can apply Chevallay-Shephard-Todd (Theorem 3.3) to obtain the following result:

Corollary 3.5: The following hold for $\mathscr{P}$ and $\mathfrak{Sym}$.

- There exist algebraically independent $f_1, \ldots, f_n \in \mathscr{P}$ such that $\mathfrak{Sym} = \Bbbk[f_1, \ldots, f_n]$.
- $\mathscr{P}$ is a free and finitely generated $\mathfrak{Sym}$ module.

There are multiple ways to choose $f_1, \ldots, f_n \in \mathscr{P}$ satisfying the properties given in Corollary 3.5. As well as to choose a basis of $\mathscr{P}$ over $\mathfrak{Sym}$. This study of symmetric polynomials and their relationship to the structure of the polynomial ring has been well studied, with considerable applications to and from the combinatorics of S_n and partitions. We will present in this section a study on the structure of $\mathfrak{Sym}$ and its relation to $\mathscr{P}$.

Subsection 3.2: Polynomial Ring Structure of $\mathfrak{Sym}$

We begin this section by studying a particular generating set of $\mathfrak{Sym}$ and its connection to combinatorics. Consider the polynomial ring $\Bbbk[t, x_1, \ldots, x_n]$ and the element $F = (t + x_1) \cdots (t + x_n)$. Then clearly F is invariant under permutations of the variables $\{x_1, \ldots, x_n\}$. We can expand F as a polynomial with variable t and coefficients in $\Bbbk[x_1, \ldots, x_n]$. For each $0 \leqslant i \leqslant n$ we denote e_i to be the coefficient of t^i such that:

$$F = \sum_{i=0}^{n} e_i(x_1, \ldots, x_n) t^{n-i}$$

Since F is invariant under permutations of the variables $\{x_1, \ldots, x_n\}$, the polynomials $e_i(x_1, \ldots, x_n)$ are symmetric for each $0 \leqslant i \leqslant n$. These symmetric polynomials are called the **elementary symmetric polynomials** and they serve the purpose for Corollary 3.5. Let $S(m, n)$ be the set of all subsets of $\{1, \ldots, n\}$ of cardinality m. By expanding F we can give an explicit formula for each elementary symmetric polynomial e_j for $1 \leqslant j \leqslant n$:

$$e_j(x_1, \ldots, x_n) = \sum_{A \in S(j,n)} \left(\prod_{i \in A} x_i \right). \tag{3.1}$$

It is known that the elementary symmetric polynomials do satisfy the conditions in Corollary 3.5 as a consequence of the **Fundamental Theorem of symmetric polynomials**. We give this theorem and an algorithmic proof due to Gauss, as is found in [10, Chapter 7.1, Theorem 3]. However, to show the uniqueness part of the proof, we will make a simpler and more conceptual argument than the proof found in [10].

Theorem 3.6: Given a symmetric polynomial $f \in \mathfrak{Sym}$ there exists a unique polynomial $F \in \mathscr{P}$ such that $f = F(e_1, \ldots, e_n)$.

***Proof*:** First, consider the lexicographical ordering on the monomials of $\mathscr{P} = \Bbbk[x_1, \ldots, x_n]$ where $x_i > x_j$ if $i < j$. Recall that the lexicographical order is a well ordering of the monomials of $\mathscr{P}$ where given two monomials $m_1 = x_1^{d_1} \cdots x_n^{d_n}$ and $m_2 = x_1^{l_1} \cdots x_n^{l_n}$ we have that $m_1 < m_2$ if $(d_1, \ldots, d_n) < (l_1, \ldots, l_n)$ according to the lexicographical ordering of $\mathbb{N}^n$. Using this, consider any symmetric polynomial $f \in \mathfrak{Sym}$, denote $LT(f)$ to be the leading term of f under this lexicographical ordering. Note that $LT(f) = cx_1^{d_1} \cdots x_n^{d_n}$ where $d_1 \geqslant \cdots \geqslant d_n$. Otherwise, there would be a permutation $\sigma \in S_n$ where $d_{\sigma(1)} \geqslant \cdots \geqslant d_{\sigma(k)}$ and since f is symmetric then $\sigma(LT(f))$ is a term in f with $\sigma(LT(f)) > LT(f)$ which is a contradiction. Now define a symmetric polynomial g given in the following way:

$$g = ce_1^{d_1 - d_2} \cdots e_{n-1}^{d_{n-1} - d_n} e_n^{d_n}$$

Then we have $LT(cg) = cLT(e_1)^{d_1 - d2} \cdots LT(e_n)^{d_n} = cx_1^{d_1} \cdots x_n^{d_n} = LT(f)$, since the leading term of polynomials is multiplicative, we have Next, consider a new polynomial $f_1 = f - g$. Since both f and g have the same leading term then $\deg(f_1) < \deg(f)$. Furthermore, both f and g are symmetric, so f_1 must also be symmetric. Continuing this process and constructing g_i for each f_i, we obtain a sequence of symmetric polynomials $(f, f_1, f_2, \ldots)$ and $(g, g_1, g_2, \ldots)$. Since $\deg(f_{i+1}) < \deg(f_i)$ it must be the case that this sequence stops at some value k so that $f_{k+1} = 0$ and we obtain $f = cg + c_1 g_1 + \cdots + c_k g_k$. By construction, each g_i is a polynomial in $e_1, \ldots, e_k$ and thus f must also be a polynomial in $e_1, \ldots, e_k$.

Now we show uniqueness. Consider another polynomial ring $\mathscr{P}' = \Bbbk[y_1, \ldots, y_n]$ and consider the map $\varphi : y_i \mapsto e_i$ which gives a homomorphism from $\mathscr{P}'$ to $\mathscr{P}$. The argument above shows that all symmetric polynomials lie in the image of this morphism, we show uniqueness by showing this map is in fact injective. We will prove this by contradiction. Consider two polynomials $g_1, g_2 \in \mathscr{P}'$ such that $\varphi(g_1) = \varphi(g_2)$ with $g := g_1 - g_2 \neq 0$ in $\mathscr{P}'$ however the image of $g \in \ker \varphi$. The polynomial g induces a map $\tilde{g} : \Bbbk^n \to \Bbbk$ where $\tilde{g}(e_1(t_1, \ldots, t_n), \ldots, e_n(t_1, \ldots, t_n)) = 0$ for all points $(t_1, \ldots, t_n) \in \Bbbk^n$. Since $g \neq 0$ then $\tilde{g} \neq 0$. We can give a geometric argument will lead to a contradiction. First let $f : \Bbbk^n \to \Bbbk^n$ be defined by $f(T) = (e_1(T), \ldots, e_n(T))$, where $e_i(T)$ is the evaluation of e_i at a point $T \in \Bbbk^n$. Let $\overline{k}$ be the algebraic closure of $\Bbbk$, what we will show is that the image $\mathrm{Im}(f) \subseteq \overline{\Bbbk}^n$ has the property that the Zariski closure $C(\mathrm{Im}(f)) = \overline{k}^n$. First, we consider an extension $f^* : \overline{\Bbbk}^n \to \overline{\Bbbk}^n$ of f where $f^*(T) = (e_1(T), \ldots, e_n(T))$ for all $T \in \overline{\Bbbk}^n$. Next we show that f^* is surjective. Consider any point $(a_0, \ldots, a_{n-1}) \in \overline{k}^n$ and consider the polynomial:

$$F = t^n + a_{n-1}t^{n-1} + \cdots + a_0$$

Since $\overline{\Bbbk}$ is algebraically closed, then F completely splits with $F = (t - r_1) \cdots (t - r_n)$. However, it is easy to see that $e_i(r_0, \ldots, r_{n-1}) = a_i$. This means that $f^*(r_1, \ldots r_n) = (a_0, \ldots, a_{n-1})$. This shows that $\mathrm{Im}(f^*) = \overline{k}^n$. Since g vanishes on the image of f, hence $\tilde{g}$ vanishes on all of $C(\mathrm{Im}(f)) = \overline{k}^n$. In other words $\tilde{g} = 0$, therefore $g = 0$, which is a contradiction. ■

Corollary 3.7: The set of elementary symmetric polynomials is an algebraically independent set, and it forms a generating set for $\mathfrak{Sym}$.

The result above gives a concrete description of the first part of Corollary 3.5. More specifically, it gives a concrete description of the polynomials $f_1, \ldots, f_n$. We can state that $\mathscr{P}^{S_n} = \mathfrak{Sym} = \Bbbk[e_1, \ldots, e_n]$. There are other known choices for a generating set of $\mathfrak{Sym}$, an example of which is the **power sum** symmetric polynomials. We define the j'th power sum as the following:

$$p_j = \sum_{i=1}^{n} x_i^j$$

Lemma 3.8: Define the $e_0 = 1$ as the 0'th elementary symmetric polynomial. For any $1 \leqslant j \leqslant n$ the following formula holds

$$e_j = \sum_{i=1}^{j} e_{j-i}p_i \tag{3.2}$$

The set of equations above relating both polynomials is known as **Newton's Identity**. This result is well known, (see [10, Chap. 7, Thm. 8]). These identities show that one may use $\{p_1, \ldots, p_n\}$ as a generating set for $\mathfrak{Sym}$. So far, we have discussed two generating sets for the ring of symmetric polynomials. We will next give a generalization of both elementary symmetric polynomials and power sums which will serve as a basis of $\mathfrak{Sym}$ over k.

Definition 3.9: Define a **word** of length n to be a sequence $a = (a_1, \ldots, a_n)$ of non-negative integers. We define the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$.

Definition 3.10: Given a word a let $O(a)$ be all the distinct permutations of a. In other words $O(a) = \{(a_{\pi(1)}, \ldots, a_{\pi(n)}) \mid \pi \in S_n\}$. We define the **monomial symmetric polynomials** indexed by a as the following:

$$\mathsf{m}_a = \sum_{b \in S(a)} x^b$$

Example 3.11: Given $a = (3, 3, 1)$ then $O(a) = \{(3, 3, 1), (3, 1, 3), (1, 3, 3)\}$ which is the orbit of a under the action of S_n. Thus the monomial symmetric polynomial indexed by a is:

$$\mathsf{m}_{(3,3,1)} = x_1^3 x_2^3 x_3 + x_1^3 x_2 x_3^3 + x_1 x_2^3 x_3^3$$

Proposition 3.12: The elementary symmetric polynomials and the power sums are monomial symmetric polynomials.

***Proof*:** Let $v_k = (1^k, 0^{n-k})$ consisting of k consecutive ones and $n - k$ consecutive zeroes, and let $O(v_k)$ be the orbit of v_k. Furthermore, consider the set $S(k, n) = \{s \subseteq \{1, \ldots, n\} \mid |s| = k\}$ as in Formula 3.1 for the elementary symmetric polynomials. We can show that $\mathsf{m}_{v_k} = e_k$ with the following computation:

$$\mathsf{m}_{v_k} = \sum_{a \in O(v_k)} x_1^{a_1} \cdots x_n^{a_n} = \sum_{s \in S(k,n)} x_{s_1} \cdots x_{s_k} = e_k$$

For the power sums, consider the partition $l_k = (k, 0^{n-1})$. We follow a similar computation as above:

$$\mathsf{m}_{l_k} = \sum_{i=1}^{n} x_i^k = p_k \qquad \blacksquare$$

Lemma 3.13: The monomial symmetric functions span $\mathfrak{Sym}$ over $\Bbbk$.

***Proof*:** Consider $f \in \mathfrak{Sym}$ and let $M(f)$ be the set of all monomials appearing in f. This means that $f = \sum_{m \in M(f)} c_m m$. Pick a monomial $m_1 \in M(f)$. Since f is symmetric, given $\pi \in S_n$ we have that $c_1(\pi(m_1))$ is also a term in f. If $m_1 = x^{A_1}$ for some word A_1, then the orbit of S_n on m_1 is by definition m_{A_1}. Define $f_1 = f$ and $g_1 = 0$, define $f_2 = f_1 - c_1 \mathsf{m}_{A_1}$ and $g_2 = g_1 + c_1 \mathsf{m}_{A_1}$. We may repeat the same process for f_2 and g_2 creating a $f_2 = f - c_1 \mathsf{m}_{A_1} - c_2 \mathsf{m}_{A_2}$ and $g_2 = c_1 \mathsf{m}_{A_1} + c_2 \mathsf{m}_{A_2}$. Repeating this process, we note that $|M(f_i)| > |M(f_{i+1})|$ if $f_i \neq 0$. Because of this, there must exist a minimal integer $j > 0$ such that $f_j = 0$. This way we have that

$$g_j = c_1 \mathsf{m}_{A_1} + \cdots + c_j \mathsf{m}_{A_j} \text{ and } f_{j-1} - c_j \mathsf{m}_{A_j} = f - g_j = 0$$

This means that $f = g_j$. However, as g_j is by construction a combination of monomial symmetric polynomials, then so is f. ■

We have now three families of symmetric polynomials; the elementary symmetric polynomials, the Power sum polynomials, and the monomial symmetric polynomials. Next we will construct a basis of $\mathscr{P}$ as a free $\mathfrak{Sym}$-module called the descent monomials. We will be making use of the monomial symmetric polynomials to show that the descent monomials are in fact a basis of $\mathscr{P}$.

Subsection 3.2: Basis of P over $\mathfrak{Sym}$

So far we have found two generating sets of $\mathfrak{Sym}$, which give specific instances of the polynomials $f_1, \ldots, f_n$ in Corollary 3.5. We use these to explicitly define the coinvariant algebra of S_n. Consider the ideal $\mathfrak{Sym}_+$ of $\mathscr{P}$ generated by all symmetric polynomials with no constant term. Using the elementary symmetric polynomials or the power sums we obtain $\mathfrak{Sym}_+ = (e_1, \ldots, e_n) = (p_1, \ldots, p_n)$, which implies that the coinvariant algebra for S_n is given by $\mathfrak{C} = \mathscr{P}/\mathfrak{Sym}_+$. By the Chevalley-Shephard-Todd (see Theorems 3.3 and 3.4) we have that $\mathfrak{C}$ is a finite dimensional vector space over $\Bbbk$ with $\dim_{\Bbbk}(\mathfrak{C}) = n!$. There is a well known monomial basis called the **descent monomial** basis for the coinvariant algebra. An elementary proof that this set of monomials are a basis is given by E. Allen in [2, Section 1].

Definition 3.14: For a permutation $\sigma \in S_n$ the **descent set** of σ is defined as

$$D(\sigma) = \{1 \leqslant i \leqslant n-1 \,|\, \sigma(i) > \sigma(i+1)\}.$$

We define the **descent monomial** associated with σ in the following way:

$$x_\sigma = \prod_{i \in D(\sigma)} \prod_{1 \leqslant j \leqslant i} x_{\sigma(j)}$$

We will later prove that the set $\{x_\sigma \,|\, \sigma \in S_n\}$ does indeed form a basis for $\mathscr{P}$ over $\mathfrak{Sym}$ and consequently a basis for $\mathfrak{C}$ over $\Bbbk$. If we assume this is true for now, we can explicitly compute the cases for S_2 and S_3.

Example 3.15: For S_2 it is clear that there are only two permutations $\{e, \sigma\}$, whose descents are $D(e) = \varnothing$ and $D(\sigma) = \{1\}$. We conclude that the following is the descent basis for $\Bbbk[x_1, x_2]$ over its ring of symmetric polynomials $\Bbbk[x_1 + x_2, x_1x_2]$.

$$x_e = 1 \qquad x_\sigma = x_2$$

Example 3.16: When considering S_3 we have 6 permutations. Thus similar to the calculation above, we compute the descent set and basis elements below:

In order to prove that the descent monomials do indeed form a basis we will need to define an

permutation	descents	x_σ
$(1,2,3)$	$\varnothing$	1
$(2,1,3)$	$\{1\}$	x_2
$(1,3,2)$	$\{2\}$	x_1x_3
$(3,2,1)$	$\{1,2\}$	$x_2x_3^2$
$(2,3,1)$	$\{2\}$	x_2x_3
$(3,1,2)$	$\{1\}$	x_3

operation on words of length n called the **charge**, which we will use to relate any given monomial to a descent monomial.

Definition 3.17: For a given word $a = (a_1, \ldots, a_n)$ let $b_1 \leqslant \cdots \leqslant b_n$ be a sorted list of all integers in a. Let $p(j)$ be the integer where $a_{p(j)} = b_j$ such that if $b_j = b_{j+1}$ then $p(j) > p(j+1)$. This means that if there are repeated values $b_j = b_{j+1} = \cdots = b_{j+k}$ then $p(j)$ is the rightmost position in a such that the value b_j occurs. With this enumeration, define the **co-charge** $J(a) = (J_1, \ldots, J_n)$ recursively in the following fashion.

1. $J_{p(1)} = 0$.
2. if $p(i+1) < p(i)$ then $J_{p(i+1)} = J_{p(i)}$, otherwise $J_{p(i+1)} = J_{p(i)} + 1$.

Example 3.18: Consider $a = (5, 2, 1, 3, 1, 2)$. The sorted list of integers in a is given by $b = (1, 1, 2, 2, 3, 5)$. Note that 1 and 2 appear repeatedly and we account for this in our enumeration p of a. The first instance of the number is the rightmost position that it is written in, making the enumeration $p = (5, 3, 6, 2, 4, 1)$. So that $a_{p(1)} = a_5 = 1 = b_1$ where this is the 5 is the position of the first instance of 1 on a, reading from right to left. Therefore, $J(a)$ is the list $(J_1, \ldots, J_6) = (2, 1, 0, 2, 0, 1)$.

Definition 3.19: Given a word $a = (a_1, \ldots, a_n)$ and cocharge $J(a) = (J_1, \ldots, J_n)$, we define the **charge** $I(a) := a - J(a) = (a_1 - J_1, \ldots, a_n - J_n)$.

The pair of the charge and co-charge splits a word a into two components $a \mapsto (I(a), J(a))$. This map is in fact injective, since if $a \neq b$ then the pairs $(I(a), J(a))$ and $(I(b), J(b))$ have the property that $I(a) + J(a) = a$ and $I(b) + J(b) = b$ (here the addition is entry-wise addition between the two words of same length).Thus the pairs are not equal. This splitting of a pair will be useful to relate monomials to descent monomials and symmetric polynomials. Before we show this, we must first define an ordering on the words of a fixed length. Given a word $a = (a_1, \ldots, a_n)$, define $\bar{a}$ to be a permutation of a such that $\bar{a}_i \geqslant \bar{a}_{i+1}$. Given a and b, we say that $a \prec b$ if:

1. $\bar{a} < \bar{b}$ (here $<$ refers to the lexicographical ordering)
2. or $\bar{a} = \bar{b}$ and $a < b$.

Proposition 3.20: The ordering "$\prec$" is a total ordering on the set of words of a fixed length.

***Proof*:** The proof of the above is almost trivial as the lexicographical ordering is a total ordering, and we would only fall into three cases $\bar{a} = \bar{b}$, $\bar{a} < \bar{b}$ or $\bar{b} < \bar{a}$. In either case we would be able to compare a and b using the $\prec$ ordering. ■

Proposition 3.21: For any given word a, the monomial $x^{J(a)}$ is a descent monomial.

***Proof*:** [2, Th 2.1] Let $J(a) = (J_1, \ldots, J_n)$. Since the co-charge is built by adding 1 consecutively, starting from 0, and the addition only depends on the position of the next integer it is easy to see that $0 \leqslant J_i \leqslant n-1$ for any i. Let $k = \max\{J_1, \ldots, J_n\}$ and define $B(j) = \{i \mid J_i = j\}$. Consider $B(j) = \{l_1, \ldots, l_k\}$ where $l_i < l_{i+1}$ then define $\bar{B}(j) = (l_1, \ldots, l_k)$. Now for each $1 \leqslant j \leqslant k$ we have constructed the ordered list $\bar{B}(j)$ of elements in $B(j)$. Define $B = \tilde{B}(k) \dot\cup \cdots \dot\cup \tilde{B}(0)$, where $\tilde{B}(i) \dot\cup \bar{B}(i+1)$ means joining both lists via concatenation in this particular order. Note that B is a re-ordering of the positions in $J(a)$. Let π be the permutation such that $(J_{\pi(1)}, \ldots, J_{\pi(n)}) = B$. Our goal is to show that $x^{J(a)} = x_\pi$.

Let $\tilde{B}(i) = (l_1, \ldots, l_p)$ and $\tilde{B}(i-1) = (w_1, \ldots, w_q)$. Note that w_1 corresponds to the left-most position such that $J(a)_{w_1} = i-1$, and l_p corresponds to the right-most position where $J(a)_{l_p} = i$. By the construction of the co-charge it must be that $w_1 < l_p$. Define $\bar{b}(i) = \max B(i)$ and let $1 \leqslant p_i \leqslant n$ be the integer such that $\pi(p_i) = \bar{b}(i)$. Then by the above argument the descent set $D(\pi) = \{p_i \mid k \geqslant i \geqslant 1\}$. This guarantees that if $l \in B(i)$ then the exponent of x_l in x_π is i. We can see this from the definition of the descent monomials that

$$x_\pi = \prod_{b \in D(\pi)} (x_{\pi(1)} \cdots x_{\pi(b)}) = \prod_{1 \leqslant i \leqslant k} (x_{\pi(1)} \cdots x_{\pi(p_i)}).$$

Let $p_{i-1} < j \leqslant p_i$ then the number of times $\pi(j)$ appears is exactly i times in x_π. Since if $l \in B(i)$ then there exists $p_{i-1} < j \leqslant p_i$ where $\pi(j) = l$, which shows that we needed. It is easy to check that the same applies for $x^{J(a)}$, since $l \in B(i)$ then the exponent of x_l must be i. Therefore, since the exponents match for each variable in both $x^{J(a)}$ and x_π then $x^{J(a)} = x_\pi$. ■

Theorem 3.22: For any word a we have the following:

$$\mathsf{m}_{I(a)} x^{J(a)} = x^a + \sum_{b \prec a} c_b x^b$$

***Proof*:** [2, Proposition 2.1] Denote $\text{fix}(a)$ to be the stabilizer of a under the action of S_n. We will further set $C(a)$ to be a set of right coset representatives of $S_n/\text{fix}(a)$. Using $C(a)$ we describe the orbit of a by the action of S_n as $O(a) = \{\pi(a) \mid \pi \in C(a)\}$. Set $J = J(a)$ and $I = I(b) = a - J$. Then clearly $\mathsf{m}_I = \sum_{\pi \in C(I)} x^{\pi(I)}$. Using this description of m_I we may write $\mathsf{m}_I x^J$ in the following way;

$$\mathsf{m}_I x^J = \sum_{\pi \in C(I)} x^{\pi(I)} x^J = \sum_{\pi \in C(I)} x^{\pi(I)+J}$$

Therefore, to prove our statement, it suffices to show that if $\pi \in C(I)$ is not the identity then $\pi(I) + J \prec a$. We may then assume that π is not in fix(I). There are two cases to consider: either $\pi \in \sigma\text{fix}(J)$ for some $\sigma \in \text{fix}(I)$ or not. If we assume that $\pi \in \sigma\text{fix}(J)$ then $\pi = \sigma\nu$ where $\nu \in \text{fix}(J)$. Then we may decompose $\pi(I) + J = \sigma(I + J) = \sigma(a)$. Therefore, $\overline{\pi(I) + J} = \overline{a}$ as $\pi(I) + J$ and a are a permutation of each other. By the definition of the co-charge, if $i < j$ and $J_i = J_j$, then $a_i \geqslant a_j$ and $I_i \geqslant I_j$. Since σ does not fix I there must exist elements a_i and a_j described as above but $a_i < a_j$ such that $\pi(a_i) = a_j$. This means that $\pi(I_i) = I_j$ and thus $(\pi(J) + I)_i < (a)_i$ and this means that $(\pi(J) + I) < a$ lexicographically proving that $(\pi(J) + I) \prec a$.

The second case gives $\pi \notin \sigma\text{fix}(I)$ for any $\sigma \in \text{fix}(J)$. This means that π must permute entries in a which have different values in the co-charge J. Consider $a_k = \max\{a_i \,|\, \pi(J_i) \neq J_i\}$. In other words, a_k is the largest integer occurring in a where a_k and $\pi(a_k)$ have different corresponding co-charge values. We let a_i and a_j be such that $\pi(a_k) = a_i$ and $\pi(a_j) = a_k$. Since a_k is the largest integer for which a_k and $\pi(a_k)$ have different co-charge values, it is clear that $a_k > a_i, a_j$ and $I_k > I_i, I_j$. With this we obtain the following:

$$(\pi(I) + J)_k = (a_j - J_j + J_k) < a_k - J_k + J_k = a_k$$

Consequently it is easy to see that on the cycle containing a_k all elements are less then a_k. This means that $\overline{\pi(I) + J} < \overline{a}$ lexicographically. Therefore $(\pi(I) + J) \prec J$ as desired. ■

Corollary 3.23: The set $\{\mathsf{m}_\lambda x_\sigma \,|\, l(\lambda) = n \text{ and } \sigma \in s_n\}$ spans $\mathscr{P}$ as a vector space over $\Bbbk$.

***Proof*:** This is a known result (see [2, Section 2]) we give a proof here. Consider an enumeration $\{a_1, a_2, a_3, \ldots\}$ of words of length n where if $i < j$ then $a_i \prec a_j$. Using this, we consider an ordering of the monomial of $\mathscr{P}$, given by $x^{a_i} < x^{a_j}$ if and only if $i < j$. Clearly $a_1 = (0, 0, \ldots, 0)$. Using this enumeration we can show this via induction on index of $\{a_1, a_2, \ldots\}$ given above. For our base case, $x^{a_1} = 1 = \mathsf{m}_{a_1} x_e$ where e is the identity permutation. Thus, let us assume that the claim works for all monomials up to x^{a_k}. Note that we may write $x^{a_{k+1}}$ in the following way using Theorem 3.22.

$$x^{a_{k+1}} = \mathsf{m}_{I(a_{k+1})} x^{J(a_{k-1})} - \sum_{i=1}^{k} c_i x^{a_i}$$

From Proposition 3.21 and the induction hypothesis, we have that the right-hand side is a linear combination of elements of the form $\mathsf{m}_\lambda x_\sigma$ for $\sigma \in S_n$ and λ a word of length n. ■

Corollary 3.24: The set $\{e_1^{a_1} \cdots e_n^{a_n} x_\sigma \,|\, \sigma \in S_n \text{ and } a_1, \ldots, a_n \geqslant 0\}$ is a spanning set for $\mathscr{P}$ over $\Bbbk$.

***Proof*:** This is trivial from Corollary 3.24 using fact that $\{e_1^{a_1}\cdots e_n^{a_n}\,|\,a_1,\ldots,a_n \geqslant 0\}$ and $\{\mathsf{m}_\lambda\,|\,l(\lambda) = n\}$ are spanning sets for $\mathfrak{Sym}$ over $\mathbb{k}$. ■

Definition 3.25: Let R be a graded ring over $\mathbb{k}$ where $R = \oplus_{i=0}^{\infty} R_i$ and $\dim_{\mathbb{k}}(R_i) = d_i$. Then the Hilbert series of R is the following power series $h(R) = \sum_{i=0}^{\infty} d_i t^i$.

To show why the above result is important, we present a theorem presented by E. Allen in [2], the proof of which is given by A. Garcia in [17, Proposition 1.2].

Theorem 3.26: Let R be a graded ring over $\mathbb{k}$ and fix homogeneous elements $f_1,\ldots,f_n \in R$ with $\deg(f_i) > 0$ and define $H := R/(f_1,\ldots,f_n)$.Assume that there are elements $b_1,\ldots,b_k \in R$ that satisfy the following:

$$h(R) = \left(\sum_{1=1}^{k} t^{\deg(b_i)}\right) / \prod_{i=1}^{n}\left(1 - t^{\deg f_i}\right)$$

Then the following statements are equivalent:

1. $\{f_1^{a_1}\cdots f_n^{a_n} b_i\,|\,1 \leqslant i \leqslant k \text{ and } a_1,\ldots a_i \geqslant 0\}$ spans R over $\mathbb{k}$.

2. $f_1,\ldots,f_n$ are algebraically independent over $\mathbb{k}$, and R is a free module over $\mathbb{k}[f_1,\ldots,f_n]$ with basis $\{b_1,\ldots,b_k\}$.

3. $b_1,\ldots,b_k$ is a basis of H over $\mathbb{k}$.

If we can show that the elementary symmetric polynomials $\{e_1,\ldots,e_n\}$ and the descent monomials $\{x_\sigma\,|\,\sigma \in S_n\}$ satisfy the condition of the Hilbert series in Theorem 3.26 then using Corollary 3.24 we can deduce that the descent monomials are a basis for $\mathscr{P}$ over $\mathfrak{Sym}$. We will present one last result before continuing with this proof. this is a classical result by P. A. MacMahon (see [27, Formula 1.2]).

Proposition 3.27: For a permutation $\sigma \in S_n$ we define $d(\sigma) = \sum_{i \in D(\sigma)} i$. The following equation holds

$$\sum_{\pi \in S_n} t^{d(\pi)} = \prod_{i=1}^{n-1}(1 + t + \cdots t^i).$$

Theorem 3.28: The set $\{x_\pi\,|\,\pi \in S_n\}$ is a basis for $\mathscr{P}$ as a module over $\mathfrak{Sym}$ and for $\mathfrak{C}$ as a vector space over $\mathbb{k}$.

***Proof*:** (See [2, Section 2]). First we show that it satisfies the conditions for the Hilbert series as stated in Theorem 3.26. It is clear that for an elementary symmetric polynomial e_j the degree $\deg(e_j) = j$. Let $\sigma \in S_n$ with descent set $D(\sigma) = \{i\,|\,\sigma(i) > \sigma(i+1)\}$ then

$$\deg(x_\sigma) = \deg\left(\prod_{i\in D(\sigma)} (x_{\sigma(1)} \cdots x_{\sigma(i)})\right) = \sum_{i\in D(\sigma)} i = d(\sigma).$$

Now using Proposition 3.27 we show the Hilbert series condition:

$$\begin{aligned}\left(\sum_\sigma t^{\deg(x_\sigma)}\right) / \prod_{i=1}^{n} (1 - t^{\deg e_i}) &= \left(\sum_\sigma t^{d(\sigma)}\right) / \prod_{i=1}^{n} (1 - t^n) \\ &= \left(\prod_{i=1}^{n}(1 + t + \cdots + t^i)\right) / \left((1-t)^n \prod_{i=1}^{n}(1 + t + \cdots + t^i)\right) \\ &= \frac{1}{(1-t)^n}\end{aligned}$$

The Hilbert series of $\mathscr{P}$ is known to be $1/(1-t)^n$. Therefore, the above computation shows that using the elementary symmetric polynomials and the descent monomials, we satisfy the conditions for Theorem 3.26. With Corollary 3.24 satisfying one of the equivalent conditions we obtain the result we needed. ■

Subsection 3.2: W_n Action on Laurent Polynomials

So far, by making use of the action of S_n on $\mathscr{P}$ and the Chevalley-Shephard-Todd Theorem (Theorems 3.3 and 3.4), we have proven strong results involving $\mathscr{P}$ as a $\mathfrak{Sym}$-module. We may summarize these results in the following points:

1. $\mathfrak{Sym} = \Bbbk[e_1, \ldots, e_n]$, where e_i is the i'th elementary symmetric polynomial.
2. $\mathscr{P}$ is a free module of finite rank over $\mathfrak{Sym}$.
3. We constructed a basis $\{x_\sigma \mid \sigma \in S_n\}$ for $\mathscr{P}$ over $\mathfrak{Sym}$ (and $\mathfrak{C}$ over $\Bbbk$).

We would like to generalize some of these results to the action of the Hyperoctahedral group W_n on the ring of Laurent polynomials $\mathfrak{L} = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. In this section we define an action of W_n on $\mathfrak{L}$ and define its invariant subring $\mathfrak{LSym} := \{f \in \mathfrak{L} \mid w \cdot f = f \text{ for } w \in W_n\}$ which we will call the **symmetric Laurent polynomials**. We claim that the relation between $\mathfrak{L}$ and $\mathfrak{LSym}$ is similar to $\mathscr{P}$ and $\mathfrak{Sym}$, where $\mathfrak{L}$ is a free module of finite rank over $\mathfrak{LSym}$. In this section, we will describe a generating set of $\mathfrak{LSym}$, using a generalization of the elementary symmetric polynomials. We will also construct a basis of $\mathfrak{L}$ over $\mathfrak{LSym}$ using the descent monomials. Furthermore, we will be able to construct a version of the coinvariant algebra $\mathfrak{LC}$ for the W_n action over $\mathfrak{L}$. Before we start with this, we must discuss a generalization of the earlier results in this Chapter. This will help us build the results for the Laurent polynomials.

Definition 3.29: Let S_n act on $[1, n] = \{1, \ldots, n\}$ and partition $[1, n]$ into k non-empty, pairwise non-intersecting sets $I_1, \ldots I_k$. We define S_{I_j} as the set of permutations that only permute elements of I_j. We call the subgroup $G = S_{I_1} \times \cdots S_{I_k}$ a **Young subgroup**.

Proposition 3.30: Let $I_1, \ldots, I_k$ be a partitioning of the set $[1, n]$ with $n_i = |I_i|$. Then the Young subgroup $S_{I_1} \times \cdots \times S_{I_k} \cong S_{n_1} \times \cdots \times S_{n_k}$

The proof for the above claim is trivial, as each component is defined as all possible permutations of the set I_j. Clearly any Young subgroup G is a reflection group and thus $\mathscr{P}^G$ is isomorphic to a polynomial ring with $\mathscr{P}$ a free module of finite rank over $\mathscr{P}^G$. We will briefly mention how to find a generating set of $\mathscr{P}^G$ and a basis for $\mathscr{P}$ over $\mathscr{P}^G$.

Definition 3.31: Let $X \subseteq \{x_1, \ldots, x_n\}$ be a subset of the variables. We define $\mathfrak{Sym}(X)$ as the polynomials in $\mathscr{P}$ which are invariant under the permutation of elements in X.

Remark 3.32: We may drop the set notation when referring to $\mathfrak{Sym}(X)$. So that if we consider $\{x_1, x_3\} \subset \{x_1, x_2, x_3\}$ then $\mathfrak{Sym}(x_1, x_3) = \mathfrak{Sym}(\{x_1, x_3\})$.

Example 3.33: Consider $X = \{x_1, x_3\} \subset \{x_1, x_2, x_3\}$. Then $\mathfrak{Sym}(X) = \mathfrak{Sym}(x_1, x_3)$ are polynomials which are symmetric in the x_1 and x_3 variables. Clearly if we let $\varphi : \Bbbk[y_1, y_2] \to \Bbbk[x_1, x_2, x_3]$ with $\varphi(y_1) = x_1$ and $\varphi(y_2) = x_3$ then $\mathfrak{Sym}(y_1, y_2) \cong \Bbbk[y_1 + y_2, y_1 y_2]$ using the elementary symmetric polynomials over y_1, y_2. Using φ we can uncover that $\mathfrak{Sym}(x_1, x_3) = \Bbbk[x_1 + x_3, x_1 x_3, x_2]$. One can easily generalize this example into actions of any Young subgroup.

Lemma 3.34: Let $I_1, \ldots I_k$ be a partition of $\{1, \ldots, n\}$ and G be the Young subgroup associated with it. Let $d_i = |I_i|$ and $X_i = \{x_j \,|\, j \in I_i\}$ so that S_{d_i} can be identified with the subgroup of S_n which only permutes variables in X_i. Then we have the isomorphism $\mathscr{P}^G \cong \Bbbk[X_1]^{S_{d_1}} \otimes \cdots \otimes \Bbbk[X_k]^{S_{d_k}}$. Furthermore, define $e_j^{(d_i)}$ to be the j'th elementary symmetric polynomial on the ring $\Bbbk[y_1, \ldots, y_{d_i}]$. We write $e_j^{(d_i)}(X_i)$ to be the evaluation of the polynomial $e_j^{(d_i)}$ at the variables from X_i. Then $\Bbbk[X_i]^{S_{d_i}}$ is generated by $\{e_j^{(d_i)} \,|\, 1 \leqslant j \leqslant d_i\}$.

***Proof*:** Part of this result is well known (see [4, Section 1]). We give a proof for this here. It is well known that there exist an isomorphism $\varphi : \Bbbk[X_1] \otimes \cdots \otimes \Bbbk[X_k] \to \mathscr{P}$ defined by $\varphi(f_1 \otimes \cdots \otimes f_k) = f_1 \cdots f_k$. It is clear that S_{d_i} can be identified with the subgroup of G which only permutes the variables in X_i and acts like the identity on all other variables. Therefore, one can see that $\varphi(\Bbbk[X_1]^{S_{d_1}} \otimes \cdots \otimes \Bbbk[X_k]^{S_{d_k}}) \subseteq \mathscr{P}^G$. To see it in the other direction, pick any polynomial $f \in \mathscr{P}^G$ and consider a monomial x^a which appears in f. Note that we may factor this monomial in the following way $x^a = m_1 \cdots m_k$ where each m_i only contain the variables in X_i. Using this factorization is easy to see that the orbit of x^a over the action of G gives that the following term is a part of f:

$$m_G = \prod_{i=1}^{k} \left(\sum_{\sigma \in S_{d_i}} \sigma(m_i) \right)$$

One can see that $m_G = f_1 \cdots f_k$ where $f_i \in \Bbbk[X_i]^{S_{d_i}}$. This shows the first statement. The second statement is an immediate consequence of Theorem 3.6, since the $\Bbbk[X_i]^{S_{d_i}}$ is generated by elementary symmetric polynomials using the variables in X_i. ■

We have constructed an instance of a generating set for the subring invariant under the action of any given Young subgroup $G \subseteq S_n$. The next question is to find a basis for $\mathscr{P}$ as a module over $\mathscr{P}^G$. This is surprisingly easy to find. First, consider the following well known result [5, remark after Proposition 2.12].

Lemma 3.35: Let $R_1, \ldots, R_k$ be a family of $\Bbbk$-algebras with $S_i \subseteq R_i$ such that R_i is a free module of rank d_i over S_i. Let B_i be a basis for R_i over S_i. Let $R = R_1 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} R_k$ and $S = S_1 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} S_k$, then R is a free module over S with basis $\{a_1 \otimes \cdots \otimes a_k \mid a_i \in B_i\}$

Corollary 3.36: Consider a Young subgroup G with X_i and d_i defined as in Lemma 3.34. Then $\Bbbk[X_i]$ is a free module of finite rank over $\Bbbk[X_i]^{S_{d_i}}$ with the descent monomial as basis $M_i = \{\mathsf{m}^i_\sigma \mid \sigma \in S_{d_i}\}$. Then the set $M = \{m_1 \cdots m_k \mid m_i \in M_i\}$ is a basis for $\mathscr{P}^G$

Lemma 3.34 and Corollary 3.36 provides a generalization of our results about $\mathscr{P}$ as a $\mathfrak{Sym}$-module. We may summarize these results in the following way; partition $\{x_1, \cdots, x_n\}$ into the disjoint sets $X_1, \ldots, X_k$, where $|X_i| = n_i$. Consider the action of $G \cong S_{n_1} \times \cdots \times S_{n_k} \subseteq S_n$ on $\mathscr{P}$ where S_{n_i} permutes the variables X_i. For $1 \leqslant i \leqslant k$ we define $e^{(i)}_j$ to be the j'th elementary symmetric polynomial on the subring $\Bbbk[X_i]$. Similarly for $1 \leqslant i \leqslant j$ we consider $M_i = \{\mathsf{m}^i_\sigma \mid \sigma \in \mathcal{S}_{d_i}\}$ as a basis of $\Bbbk[X_i]$ as a $\mathfrak{Sym}(X_i)$-module. Then we have the following:

1. $\mathscr{P}^G = \Bbbk[e^{(1)}_1, \cdots, e^{(1)}_{n_1}, \cdots, e^{(k)}_1, \cdots, e^{(k)}_{n_k}]$.
2. $\{m_1 \cdots m_k \,| m_i \in M_i\}$ is a basis of $\mathscr{P}$ as a $\mathscr{P}^G$-module.

Example 3.37: Let $R = \Bbbk[x_1, y_1, \ldots, x_n, y_n]$ and let $G = S_2 \times \cdots \times S_2$ act on R where the i'th copy of S_2 permutes x_i and y_i. Then by Lemma 3.34 we have that $R^G = \Bbbk[x_1 + y_1, x_1y_1, \ldots, x_n + y_n, x_ny_n]$. Furthermore, $\Bbbk[x_1, y_1]$ is a free and finite rank module over $\Bbbk[x_1, y_1]^{S_2}$ with basis $\{1, x_1\}$. Consequently by Corollary 3.36 we have that $\{\prod_{s \in S} x_s \mid S \subseteq \{1, \ldots, n\}\}$ is a basis for R as a R^G-module.

Definition 3.38: The ring of Laurent polynomials is defined as the following quotient:

$$\mathfrak{L} := \Bbbk[x_1, y_1, \ldots, x_n, y_n]/(x_1y_1 - 1, \ldots, x_ny_n - 1) = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm n}].$$

Remark 3.39: Recall that the Hyperoctahedral group W_n is the subgroup of permutations on the set $\{\pm 1, \ldots, \pm n\}$ such that if $\pi \in W_n$ then $\pi(-k) = -\pi(k)$.

Proposition 3.40: For a given $\pi \in W_n$ and x_i a generator of $\mathfrak{L}$ we define

$$\pi(x_i) = \begin{cases} x_{\pi(i)} & \text{if } \pi(i) > 0 \\ x^{-1}_{-\pi(i)} & \text{otherwise.} \end{cases}$$

This map extends to an action of W_n on $\mathfrak{L}$

***Proof*:** By Proposition 1.42 there exists a subgroup $G \subseteq S_{2n}$, such that $G \cong W_n$. Using the action of S_{2n} over R we immediately obtain an action of W_n on $R = \Bbbk[x_1, y_1, \ldots, x_n, y_n]$. For a given $w \in W_n$ and $1 \leqslant i \leqslant n$ We may describe this action in the following way; if $w(i) > 0$ then $w(x_i) = x_{w(i)}$ and $w(y_i) = y_{w(i)}$. If $w(i) < 0$ then $w(x_i) = y_{|w(i)|}$ and $w(y_i) = x_{|w(i)|}$. Let $I = (x_1y_1 - 1, \ldots, x_ny_n - 1)$, it is easy to see that $W_n(I) \subseteq I$. This means that the action of W_n on R extends to $R/I \cong \mathfrak{L}$. This action of W_n on $\mathfrak{L}$ is equivalent to the action described. ■

Our goal is to find a normal subgroup $N \lhd G \cong W_n$, such that N is a Young subgroup of S_{2n}. If we find such a group then we may use Lemma 3.34 and Corollary 3.36 to find a generating set for R^N and a basis for R over R^N. These results will extend to the quotient $(R/I) \cong \mathfrak{L}$ as a module over $(R/I)^N \cong \mathfrak{L}^N$. We will then be able to show that $(R/I)^N$ is isomorphic to a polynomial algebra. This implies that there exist elements $f_1, \ldots, f_k \in \mathfrak{L}^N$ such that $\mathfrak{L}^N = \Bbbk[f_1, \ldots, f_k]$. Lastly, we will show that the quotient group G/N acts on $\mathfrak{L}^N$ as a reflection group. Combining these results, we will be able to generalize our results from $\mathscr{P}$ as a $\mathfrak{Sym}$-module to $\mathfrak{L}$ as a $\mathfrak{L}^{W_n}$-module. For the following results, unless cited, the author has not found in literature and the proofs are original.

Proposition 3.41: Let N be the subgroup of W_n generated by the negative transpositions $(i, -i)$ (see Section 1.3). Let $f_i = x_i + x_i^{-1}$ then have that $\mathfrak{L}^N = \Bbbk[f_1, \ldots, f_n]$ and f_i are algebraically independent. Furthermore, $\mathfrak{L}$ is a free module of finite rank over $\mathfrak{L}^N$ with basis given by the set $\{\prod_{s \in S} x_s \mid S \subseteq \{1, \ldots, n\}\}$

***Proof*:** Since we may identify W_n as a subgroup of S_{2n} then N is identified as the subgroup consisting of permutations of the form $(k, k + n + 1)$ for $1 \leqslant k \leqslant n$. Consider the action of N on the ring $R = \Bbbk[x_1, y_1 \ldots, x_n, y_n]$. From Example 3.37 we see that $R^N = \Bbbk[x_1 + y_1, x_1y_1, \ldots, x_n + y_n, x_ny_n]$. Let $I = (x_1y_1 - 1, \ldots, x_ny_n - 1) \subset R$, so that $\mathfrak{L} \cong R/I$. For any commutative ring H, if $h_1, \ldots, h_n$ generates H, then for any ideal J of H the elements $h_1 + I, \ldots, h_n + I$ generate H/J. Thus, $\{x_1+y_1+I, x_1y_1+I, \ldots, x_n+y_n, x_ny_n+I\}$ must generate R/I. Consider the isomorphism $\varphi : R/I \to \mathfrak{L}$. Then $\varphi(x_i + y_i + I) = x_i + x_i^{-1}$ and $\varphi(x_iy_i + I) = 1$. Set $f_i = x_i + x_i^{-1}$. This argument show that $\{f_i \mid 1 \leqslant i \leqslant n\}$ generates $\mathfrak{L}^N$.

Since $\mathfrak{L}$ has no zero divisors. Pick any polynomial $f \in \Bbbk[x_1, \ldots, x_n]$ and consider its evaluation at $g = f(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}) \in \mathfrak{L}$. Let d be the degree of f in $\Bbbk[x_1, \ldots, x_n]$ then if we assume $f \neq 0$ then the highest degree component of g is d. This means that $g \neq 0$. However, this implies that there are no polynomials with roots $x_i + x_i^{-1}$ in $\mathscr{P}$. Therefore, the elements $x_i + x_i^{-1}$ are algebraically independent.

Note that the set $\{1, x_i\}$ is a basis of $\Bbbk[x_i, y_i]$ as a $\Bbbk[x_i, y_i]^{S_2}$-module. This means $\{1, x_i\}$ spans $\Bbbk[x_i^{\pm 1}]$ as a $\Bbbk[x_i^{\pm 1}]^{S_2}$-module. We must show that $1, x_1$ are linearly independent over $\Bbbk[x_i^{\pm 1}]^{S_2}$. By the previous arguments in this proof, we have that $\Bbbk[x_i^{\pm}]^{S_2} = \Bbbk[x_i + x_i^{-1}]$. Let $f, g \in \Bbbk[x_i + x_i^{-1}]$ be polynomials such that $fx_i + g = 0$. In this case, we have that $fx_i = -g \in \Bbbk[x_i^{\pm}]^{S_2}$ meaning that $fx_i = fx_i^{-1}$. Since $x_i + x_i^{-1} \neq 0$ and $(x_i + x_i^{-1})f = 0$ then $f = 0$ and $g = 0$. Thus, $fx_i + g = 0$ if and only if $f = g = 0$. This means that x_1 and 1 are linearly independent over $\Bbbk[x_i + x_i^{-1}]$. Using Lemma 3.35 and the fact that $\mathfrak{L} \cong \otimes_{i=1}^{n} \Bbbk[x_i^{\pm 1}]$ we have that $\{\prod_{s \in S} x_s \mid S \subseteq \{1, \ldots, n\}\}$ is a basis for $\mathfrak{L}$ over $\mathfrak{L}^N$. ■

These next two results, Lemma 3.42 and Lemma 3.43, can be found in literature under Galois theory over commutative rings. These two results can be found in [30, Theorem 3.1].

Lemma 3.42: Let R be a commutative ring and consider groups $N \subseteq G \subseteq \mathrm{Aut}(R)$ where N is a normal subgroup of G. Then there is a natural action of G/N on the invariant ring R^N, and we have equality $R^G = (R^N)^{G/N}$.

***Proof*:** Consider $f \in R^N$ and $g_1, g_2 \in G$ such that $g_1 N = g_2 N$. There exist $n_1, n_2 \in N$ we have that $g_1 n_1 = g_2 n_2$. Thus $g_1 \cdot f = g_1 n_1 \cdot f = g_2 n_2 \cdot f = g_2 \cdot f$. This means that the action of G/N over R^N defined by $gN \cdot f = g \cdot f$ is well-defined, as it is not dependent on the coset representative. With this action, it is clear that $R^G \subseteq (R^N)^{G/N}$. Now pick $f \in (R^N)^{G/N}$ and $g \in G$. Since N is normal there is a coset representative h and $n \in N$ such that $g = hn$. By our assumptions, f is invariant under n and h, since it is invariant under N and G/N. Therefore $g \cdot f = (hn) \cdot f = f$. ■

Lemma 3.43: Let R be a commutative ring, G be a finite group subgroup $\mathrm{Aut}(R)$ and $N \subseteq G$ a normal subgroup. Assume that the following holds:

- R is a free R^N-module with basis $\{b_1, \ldots, b_k\}$.
- R^N is a free $(R^N)^{G/N}$-module with basis $\{c_1, \ldots, c_k\}$.

Then $\{b_i c_j \mid 1 \leqslant i \leqslant k \, 1 \leqslant j \leqslant l\}$ is a basis of R over R^G.

***Proof*:** Let $f \in R$. Then there exists $f_1, \ldots, f_k \in R^N$ such that $f = f_1 b_1 + \cdots f_k b_k$. Furthermore, for each i there exists $g_1^i, \ldots, g_l^i \in (R^N)^{G/N} = R^G$ such that $f_i = g_1^i c_1 + \cdots + g_l^i c_l$. This means that we have the following equation:

$$f = \sum_{i=1}^{k} \sum_{j=1}^{l} g_j^i c_j b_i.$$

Thus, we have shown that the set $\{b_i c_j \mid 1 \leqslant i \leqslant k \, 1 \leqslant j \leqslant l\}$ spans R as a R^G-module. We must now show it is a basis. Assume that there exists $g_i^j \in R^G$ such that $\sum_i \sum_j g_i^j c_j b_i = 0$. Let $p_i = \sum_{j=1}^n g_j^i c_j$. Since the elements $b_1, \ldots, b_n$ is a basis of R over R^N and $\sum_i p_i b_i = 0$ then $p_i = \sum_k g_i^j c_j = 0$ for all $1 \leqslant i \leqslant k$. Since $c_1, \ldots, c_l$ are a basis of R^N, then by a similar argument we have that $g_i^j = 0$ for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. This means that the set $\{b_i c_j \mid 1 \leqslant i \leqslant k \, 1 \leqslant j \leqslant l\}$ is linearly independent. ■

Theorem 3.44: Define $e_i^* = e_i(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1})$ where e_i is the i'th elementary symmetric polynomial on n-variables. Then $\mathfrak{L}^{W_n} = \Bbbk[e_1^*, \ldots, e_n^*]$. Furthermore, given a permutation $\pi \in S_n$ let $b_\pi = x_\pi(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1})$ so that b_π is the descent monomial associated with π evaluated at $x_i \mapsto x_i + x_i^{-1}$. Then the set $\{b_\pi \prod_{s \in S} x_s \mid \pi \in S_n \text{ and } S \subset \{1, \ldots n\}\}$ is a basis of $\mathfrak{L}$ over $\mathfrak{L}^{W_n}$.

***Proof*:** The author has not found this result in literature, we give a proof of this result here. Note that N as defined in Proposition 3.41 is a normal subgroup of W_n generated by $(i, -i)$. Furthermore, by Theorem 1.39 we $S_n \cong W_n/N$. From Proposition 3.41 we have that

$$\left\{ \prod_{s \in S} x_s \mid S \subseteq \{1, \ldots, n\} \right\}$$

is a basis of $\mathfrak{L}$ over $\mathfrak{L}^N$. Furthermore, we have proven that $\mathfrak{L}^N = \Bbbk[f_1, \ldots, f_n]$ where $f_i = x_i + x_i^{-1}$. Note that as in Lemma 3.43 and Lemma 3.42 there is a natural action of W_n/N over $\mathfrak{L}^N$. Let $\varphi : S_n \to W_n$ given by $\varphi(s_i) = (i, i+1)(-i, -i-1)$. It is clear that for each $\pi \in S_n$ we have a different coset $\varphi(\pi)N$. Therefore, we compute the action of W_n/N over $\Bbbk[f_1, \ldots, f_n]$. Let $\varphi(\pi)N \in W_n/N$ we have

$$\varphi(\pi)N f_i = \varphi(\pi)(x_i + x_i^{-1}) = x_{\pi(i)} + x_{\pi(i)}^{-1} = f_{\pi(i)}.$$

The action of W_n/S_2^n on $\mathfrak{L}^N$ is then equivalent to the action of S_n permuting the polynomials $f_1, \ldots, f_n$. Therefore W_n/N acts as a reflection group on $\mathfrak{L}^N$. We apply Corollary 3.5 to $\mathfrak{L}^N$ as an $(\mathfrak{L}^N)^{W_n/N}$-module and deduce two results:

- The set $\{e_i^* \mid 1 \leqslant i \leqslant n\}$ where $e_i^* = e_i(f_1, \ldots, f_n)$ is a generating set for $(\mathfrak{L}^N)^{S_n}$. Meaning that $(\mathfrak{L}^N)^{S_n} = \Bbbk[e_1^*, \ldots, e_n^*]$.
- The set $\{b_\pi \mid \pi \in S_n\}$ is a basis for $\mathfrak{L}^N$ as a $(\mathfrak{L}^N)^{S_n}$-module.

Using Lemma 3.42 we have that $(\mathfrak{L}^{S_2^n})^{S_n} = \mathfrak{L}^{W_n}$ meaning that $\mathfrak{L}^{W_n} = \Bbbk[e_1^*, \ldots, e_n^*]$. Furthermore, since $\{\prod_{s \in S} x_s \mid S \subseteq \{1, \ldots, n\}\}$ is a basis of $\mathfrak{L}$ as a $\mathfrak{L}^N$-module, by Lemma 3.43 we have that the set $\{b_\pi \prod_{s \in S} x_s \mid \pi \in S_n \text{ and } S \subset \{1, \ldots n\}\}$ is an basis of $\mathfrak{L}$ over $\mathfrak{L}^{W_n}$. ■

Definition 3.45: The polynomial e_i^* as defined in Theorem 3.44 above is called the i'th **Laurent elementary symmetric polynomial**.

Definition 3.46: We will call the invariant subring $\mathfrak{L}^{W_n}$ the **symmetric Laurent polynomials** and we will denote $\mathfrak{LSym} = \mathfrak{L}^{W_n}$.

Note that $\mathfrak{L}^{W_n} = \Bbbk[e_1^*, \ldots, e_n^*]$ which is isomorphic to a polynomial ring. Therefore, we may generalize the coinvariant algebra in the following way:

Definition 3.47: The **Laurent coinvariant algebra** is defined as $\mathfrak{LC} = \mathfrak{L}/(e_1^*, \ldots, e_n^*)$.

Theorem 3.44 gives an $\mathfrak{L}$-version of the results we obtained for the S_n action on $\mathscr{P}$ case. It follows that the ring of symmetric Laurent polynomials $\mathfrak{LSym} = \mathfrak{L}^{W_n}$ is isomorphic to a polynomial ring. We have a generating set of $\mathfrak{LSym}$ which are the Laurent elementary symmetric sums. Furthermore, we have that $\mathfrak{L}$ is a free module of finite rank over $\mathfrak{LSym}$. Before we end this section, we will compute the case in which $n = 2$.

Example 3.48: For W_2 acting on $\mathfrak{L} = \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}]$ the Laurent elementary symmetric polynomials are given by $e_1^* = (x_1 + x_1^{-1}) + (x_2 + x_2^{-1})$ and $e_2^* = (x_1 + x_1^{-1})(x_2 + x_2^{-1})$. The ring of symmetric Laurent polynomials is then $\mathfrak{LSym} = \Bbbk[e_1^*, e_2^*]$. The subsets of $\{1, 2\}$ are given by $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. These correspond to the basis elements of $\mathfrak{L}$ over $\mathfrak{L}^{S_2 \times S_2}$ which are $\{1, x_1, x_2, x_1x_2\}$. A basis for $\mathfrak{L}^{S_2^n}$ over $\mathfrak{L}^{W_n}$ comes from the descent monomials b_π where $\pi \in S_2$. Let $S_2 = \{e, \sigma\}$. Then $b_e = 1$ and $b_\sigma = f_2 = (x_2 + x_2^{-1})$ from Example 3.15. Thus using Theorem 3.44 a basis for $\mathfrak{L}$ over $\mathfrak{LSym}$ can be computed as shown below.

subset	permutation	descent monomials b_π	subset basis	basis element
$\varnothing$	e	1	1	1
$\varnothing$	σ	$x_2 + x_2^{-1}$	1	$(x_2 + x_2^{-1})$
$\{1\}$	e	1	x_1	x_1
$\{1\}$	σ	$x_2 + x_2^{-1}$	x_1	$(x_2 + x_2^{-1})x_1$
$\{2\}$	e	1	x_2	x_2
$\{2\}$	σ	$x_2 + x_2^{-1}$	x_2	$(x_2 + x_2^{-1})x_2$
$\{1, 2\}$	e	1	x_1x_2	x_1x_2
$\{1, 2\}$	σ	$x_2 + x_2^{-1}$	x_1x_2	$(x_2 + x_2^{-1})x_1x_2$

Section 3.3: Symmetrizing Operators on the Polynomial Ring

We will exploit the structure of $\mathscr{P}$ as a $\mathfrak{Sym}$ module to define certain operators which will allow us to construct polynomial representations of the Hecke algebra. Most of the material introduced in this section is due to A. Lascoux in [22]. We start with the divided difference operators on $\mathscr{P}$ and we verify some relations that they satisfy.

Proposition 3.49: Recall that s_i is the transposition $(i, i + 1)$. For any polynomial $f \in \mathscr{P}$ we have that $x_i - x_{i+1}$ divides $f - s_i \cdot f$.

***Proof*:** Let $j = i+1$ and consider $m = x_i^a x_j^b$. Then $m - s_i \cdot m = x_i^a x_j^b - x_i^b x_j^a$. Let $c = \min\{a, b\}$ then we have two cases, either $a \leqslant b$ or $b \leqslant a$. In the first case, we have that $m = (x_i x_j)^c (x_j^{b-c} - x_i^{b-c})$. Note that for any variables x, y and integer k, we have that

$$x^k - y^k = (x - y) \sum_{i=0}^{k} (x^{k-i} - y^{i-1-1}).$$

This means that $(x_i - x_j)$ divides $m - s_i \cdot m$. Thus, for any given monomial $\partial_i(m)$ is well-defined. Note that since the action of S_n is linear over $\Bbbk$ then $(x_i - x_{i+1})$ must divide $f - s_i \cdot f$ for any polynomial $f \in \mathscr{P}$. ■

Definition 3.50: The **divided difference** operator $\partial_i \in \hom_{\Bbbk}(\mathscr{P}, \mathscr{P})$ is defined in the following way:

$$\partial_i : f \mapsto \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

Proposition 3.51: Recall that for a subset $X \subseteq \{x_1, \ldots, x_n\}$ the ring $\mathfrak{Sym}(X)$ is the subring of $\mathscr{P}$ consisting of polynomials which are symmetric over the variables in X (see Definition 3.31). For a given polynomial $f \in \mathfrak{Sym}(x_i, x_{i+1})$ and $g \in \mathscr{P}$ we have that $\partial_i(fg) = f\partial_i(g)$.

***Proof*:** This is easy to show from the formula given for ∂_i. Given $f \in \mathscr{P}$ such that $f = s_i \cdot f$ and $g \in \mathscr{P}$ we compute the following:

$$\partial_i(fg) = \frac{(fg) - s_i \cdot (fg)}{x_i - x_{i+1}} = \frac{fg - f(s_i \cdot g)}{x_i - x_{i+1}} = f\partial_i(g)$$ ■

Corollary 3.52: The divided difference operator ∂_i is $\mathfrak{Sym}$-linear.

***Proof*:** This is an immediate consequence of Proposition 3.51 and the fact that $\mathfrak{Sym} \subseteq \mathfrak{Sym}(X)$ for any $X \subseteq \{x_1, \ldots, x_n\}$. ■

Proposition 3.53: The divided difference operator satisfies the following relations:

1. $\partial_i^2 = 0$
2. $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| > 1$
3. $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

***Proof*:** We can check each relation individually. Consider $f = x_i^a x_{i+1}^b m$ where m is any monomial which does not contain the variables x_i and x_{i+1}. Then from the Proposition 3.51, it is easy to see that If $|i - j| > 1$ then $\partial_i(f) = m\partial_i(x_i^a x_{i+1}^b)$. Consequently

$$\partial_j \partial_i (x_i^a x_{i+1}^b m) = \partial_j(m) \partial_i(x_i^a x_{i+1}^b) = \partial_i \partial_j (x_i^a x_{i+1}^b m))$$

By Corollary 3.52, to check the first equation, we only need to show that it is true for the generators of $\mathscr{P}$ as a $\mathfrak{Sym}(x_i, x_{i+1})$-module. We may use the descent monomials computed in Example 3.15 in order to check this. The descent monomials are $\{1, x_i\}$. When we apply ∂_i^2 to both we obtain the following:

$$\partial_i^2(1) = \partial_i \frac{1-1}{x_i - x_{i+1}} = 0 \text{ and } \partial_i^2(x_i) = \partial_i \frac{x_i - x_{i+1}}{x_i - x_{i+1}} = 0.$$

This shows the first equation is true. For the last equation, since we are using both ∂_i and ∂_{i+1}, we may use a similar strategy. However, since ∂_i is $\mathfrak{Sym}(x_i, x_{i+1})$-linear and ∂_{i+1} is $\mathfrak{Sym}(x_{i+1}, x_{i+2})$-linear, we will need to check if $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$ on $\mathscr{P}$ as a $\mathfrak{Sym}(x_i, x_{i+1}, x_{i+2})$ module. Repeating the technique above we only need to show this for the basis of $\mathscr{P}$ over $\mathfrak{Sym}(x_i, x_{i+1}, x_{i+2})$. Using the descent monomials computed in Example 3.16 one can show this equation to be true. The descent monomial basis is $B_i = \{1, x_{i+1}, x_{i+2}, x_i x_{i+2}, x_{i+1}x_{i+2}, x_{i+1}x_{i+2}^2\}$ and we compare the operators below:

Basis	$\partial_i\partial_{i+1}\partial_i$	$\partial_{i+1}\partial_i\partial_{i+1}$
1	0	0
x_{i+1}	0	0
x_{i+2}	0	0
$x_i x_{i+2}$	0	0
$x_{i+1}x_{i+2}$	0	0
$x_{i+1}x_{i+2}^2$	-1	-1

By Corollary 3.36 the set B_i is a basis for $\mathscr{P}$ as a $\mathfrak{Sym}(x_i, x_{i+1}, x_{i+2})$-module. Furthermore, ∂_i and ∂_{i+1} is $\mathfrak{Sym}(x_i, x_{i+1}, x_{i+2})$-linear. The computation above shows that $\partial_i\partial_{i+1}\partial_i \cdot b = \partial_{i+1}\partial_i\partial_{i+1} \cdot b$ for all $b \in B_i$. Therefore, these operators are the same, proving the third relation. ■

Proposition 3.54: Let $X_i : \mathscr{P} \to \mathscr{P}$ be the operator $X_i \in \hom_{\Bbbk}(\mathscr{P}, \mathscr{P})$ defined by $X_i : f \mapsto x_i f$. The following relations between X_i and ∂_i hold:

1. $\partial_i X_i - X_{i+1}\partial_i = 1$
2. $X_i \partial_i - \partial_i X_{i+1} = 1$
3. $\partial_i X_j - X_j \partial_i = 0$ for $|i - j| > 1$

***Proof*:** It is easy to see that equation 3 holds since the variable $x_j \in \mathfrak{Sym}(x_i, x_{i+1})$. Equation 1 can be checked with the following computation below:

$$
\begin{aligned}
(\partial_i X_i - X_{i+1}\partial_i)\cdot f &= \frac{x_i f - x_{i+1}s_i\cdot f}{x_i - x_{i+1}} - \frac{x_{i+1}f - x_{i+1}s_i\cdot f}{x_i - x_{i+1}} \\
&= \frac{x_i f - x_{i+1}s_i\cdot f - x_{i+1}f - x_{i+1}s_i\cdot f}{x_i -_{i+1}} \\
&= \frac{x_i f - x_{i+1}f}{x_i - x_{i+1}} = 1\cdot f
\end{aligned}
$$

Equation 2 can be checked with a very similar computation as above. ■

Lemma 3.55: Given $1 \leqslant i < n$ we have the following:

1. $\partial_i X_i - X_i\partial_i = s_i$
2. $s_i\partial_i = -\partial_i s_i$
3. $(\partial_i X_i)^2 = \partial_i X_i$

***Proof*:** For the first relation described, we can do the following computation:

$$
(\partial_i x_i - x_i\partial_i)(f) = \frac{x_i f - x_{i+1}s_i\cdot f}{x_i - x_{i+1}} - \frac{x_i f - x_i s_i\cdot f}{x_i - x_{i+1}} = s_i\cdot f\frac{x_i - x_{i+1}}{x_i - x_{i+1}} = s_i\cdot f
$$

The second relation described can be seen easily as s_i is just a transposition of $\{x_i, x_{i+1}\}$. Thus, we prove this relation with the following computation:

$$
s_i\partial_i(f) = \frac{s_i\cdot f - f}{x_{i+1} - x_i} = -\frac{s_i\cdot f - f}{x_i - x_{i+1}} = -\partial_i s_i
$$

Lastly, for the third relation we use the relations found for ∂_i in the following way:

$$
(\partial_i X_i\partial_i X_i) = \partial_i(1 + \partial_i X_{i+1})X_i = \partial_i X_i + \partial_i^2 X_{i+1}X_i = \partial_i X_i
$$

Which shows exactly what we needed. ■

Definition 3.56: The **isobaric divided difference** also known as the **Demazure operator** is the operator in $\hom_{\Bbbk}(\mathscr{P}, \mathscr{P})$ defined by

$$
\delta_i := \partial_i X_i.
$$

Proposition 3.57: The operators δ_i satisfy the following relations:

1. $\delta_i^2 = \delta_i$ for all i.
2. $\delta_i\delta_j = \delta_j\delta_i$ if $|i - j| > 1$
3. $\delta_i\delta_{i+1}\delta_i = \delta_i\delta_{i+1}\delta_i$

***Proof*:** These relations can be computed directly from Propositions 3.53 and 3.54, and Lemma 3.55. Note that, as $\delta_i = \partial_i X_i$ then equation 3 from Lemma 3.55 shows the first relation. For the second relation, recall that the maps ∂_i and X_i are endomorphisms of $\Bbbk[x_i, x_{i+1}]$, therefore $X_j\partial_i = \partial_i X_j$ if $|i-j| > 1$. Thus we obtain the following:

$$\delta_i\delta_j = \partial_i X_i \partial_j X_j = \partial_j X_j \partial_i X_i = \delta_j \delta_i$$

The third and final we repeat the technique used in Proposition 3.53. Note that both δ_i and δ_{i+1} are $\mathfrak{Sym}$-linear endomorphisms of $\Bbbk[x_i, x_{i+1}, x_{i+2}]$, as they are a composition of δ_i and X_i. Furthermore, from Corollary 3.5 we have that $\Bbbk[x_i, x_{i+1}, x_{i+2}]$ is a free module of finite rank module over the ring of symmetric polynomials over $\{x_i, x_{i+1}, x_{i+2}\}$. A basis for this module called was given in Example 3.16:

$$\{1, x_{i+1}, x_{i+2}, x_i x_{i+2}, x_{i+1}x_{i+2}, x_{i+1}x_{i+2}^3\}$$

Thus we compute the maps $\delta_i\delta_{i+1}\delta_i$ and $\delta_{i+1}\delta_i\delta_{i+1}$ on the basis above, and show that they do in fact match.

Basis	$\delta_i\delta_{i+1}\delta_i$	$\delta_{i+1}\delta_i\delta_{i+1}$
1	1	1
x_{i+1}	0	0
x_{i+2}	0	0
$x_i x_{i+2}$	0	0
$x_{i+1}x_{i+2}$	0	0
$x_{i+1}x_{i+2}^2$	0	0

Since both columns match above, it shows that both operators $\delta_i\delta_{i+1}\delta_i$ and $\delta_{i+1}\delta_i\delta_{i+1}$ are the same. More surprisingly, both operators zero all non-symmetric polynomials, which would be hard to show if we were to do a direct computation. This shows that equation 3 does in fact hold. ■

Remark 3.58: The relations given in Proposition 3.57 match the relations for the nil-Hecke $\mathcal{H}_0(S_n)$ algebra for S_n (see Definition 2.53 in Chapter 2). Therefore, the algebra generated by the Demazure operators δ_i is isomorphic to $\mathcal{H}_0(S_n)$. This gives us a polynomial representation of the nil-Hecke algebra.

Theorem 3.59: Let $f \in \mathscr{P}$. Then the following are equivalent:

1. f is a symmetric polynomial.
2. f is invariant under δ_i (meaning that $\delta_i \cdot f = f$) for $1 \leqslant i \leqslant n-1$.
3. $f \in \ker(\partial_i)$ for all $1 \leqslant i \leqslant n-1$

***Proof*:** First, we show that (1) implies (3), for that let $f \in \mathscr{P}$ be a symmetric polynomial. Then $s_i \cdot f = f$ for all $1 \leqslant i \leqslant n-1$ implying that $\partial_i \cdot f = 0$ by the definition of ∂_i.

Next, we show that (3) implies (2). Assume that $\partial_i f = 0$ for all $1 \leqslant i \leqslant n-1$. Then using Proposition 3.54, specifically relation 1, we see that $\delta_i f = \partial_i X_i f = (1 - X_{i+1}\partial_i)f = 1f$. Thus applying $\delta_i f$ is the same as multiplying by 1, meaning f is invariant.

Lastly, we show that (2) implies (1). We can easily see this from the following

$$\frac{x_i f - x_{i+1} s_i \cdot f}{x_i - x_{i+1}} = f \Longleftrightarrow x_i f - s_i \cdot (x_i f) = (x_i - x_{i+1}) f \Longleftrightarrow s_i \cdot f = f$$

This shows that if $\delta_i \cdot f = f$ then $s_i \cdot f = f$, since this is true for all $1 \leqslant i \leqslant n-1$ then f must be invariant. ■

Subsection 3.3: Symmetrizing Operators of $\mathfrak{L}$

We have defined three operators s_i, ∂_i and σ_i which are $\mathfrak{Sym}(x_i, x_{i+1})$-linear. Each operator gives an equivalent condition for a polynomial f to be symmetric in the variables x_i and x_{i+1} in Proposition 3.59. We will generalize these operators to $\mathfrak{L}\mathfrak{Sym}$-linear operators over $\mathfrak{L}$. The author has not found this material in literature, therefore these results and their proofs presented are original. First, recall that W_n is generated by $\{w_0, \ldots, w_{n-1}\}$ where $w_0 = (1, -1)$ and $w_i = (i, i+1)(-i, -i-1)$. The action of w_n on $\mathfrak{L}$ as discussed in Section 3.2.3 can be explicitly written in the following way:

$$w_i(x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} x_1^{-a_1} \cdots x_n^{a_n} & \text{if } i = 0 \\ x_1^{a_1} \cdots x_i^{a_{i+1}} x_{i+1}^{a_i} \cdots x_n^{a_n} & \text{otherwise} \end{cases} \tag{3.3}$$

Definition 3.60: We define the **divided difference** over $\mathfrak{L}$ to be the following $\Bbbk$-linear operator.

$$\partial_i(f) = \begin{cases} (f - w_0 \cdot f) / \left(x_1^{-1} - x_1\right) & \text{if } i = 0 \\ (f - w_i \cdot f) / (x_i - x_{i+1}) & \text{otherwise} \end{cases}$$

Proposition 3.61: The operator $\partial_i \in \hom_k(\mathfrak{L}, \mathfrak{L})$ is well-defined.

***Proof*:** In the case that $i > 0$ then the action of w_i is the same as permuting the variables x_i and x_{i+1}. Therefore, a similar computation can be made as in Proposition 3.49. What we must show is that ∂_0 is in fact well-defined over the monomials and $\Bbbk$-linear. Consider $m = x_1^a m'$ where m' is a monomial which does not contain x_1. Then we may compute the following:

$$m - w_0 \cdot m = (x_1^a - x_1^{-a})m'.$$

Thus $x_1^{-1} - x_1$ divides $m - w_0 \cdot m$. Since division by a constant is $\Bbbk$-linear and so is the action by w_0 then we have that ∂_0 is well-defined. ■

In the previous section, given a subset of the variables $X \subset \{x_1, \ldots, x_n\}$ we defined $\mathfrak{Sym}(X)$ to be the polynomials which are symmetric under the permutations of the variables in X. We may generalize this to the ring of symmetric Laurent polynomials.

Definition 3.62: Given $X = \{x_{i_1}, \cdots, x_{i_k}\}$ we define $\mathfrak{LSym}(X)$ to be the polynomials in $\mathfrak{L}$ which are invariants under the elements of W_n which only permutes elements of the set $\{i_1, \ldots, i_k, -i_1, \ldots, -i_k\}$.

Proposition 3.63: The operator ∂_i is $\mathfrak{LSym}(x_i, x_{i+1})$-linear for $1 \leqslant i \leqslant n-1$. The operator ∂_0 is $\mathfrak{LSym}(x_1)$ linear.

***Proof*:** Let $1 \leqslant i \leqslant n-1$ and consider the polynomials $f \in \mathfrak{L}$ and $g \in \mathfrak{LSym}(x_i, x_{i+1})$ then $w_i(g) = g$. Therefore $gf - w_i(gf) = g(f - w_i f)$ which shows that ∂_i is $\mathfrak{LSym}(x_i, x_{i+1})$-linear. For the second statement, $w_0 = (1, -1)$ so for any polynomial $g \in \mathfrak{LSym}(x_1)$ we have that $w_0(g) = g$. Thus we repeat the same argument as $fg - w_0(fg) = g(f - w_0 f)$. ■

Definition 3.64: Let $X_i : \mathfrak{L} \to \mathfrak{L}$ be the linear operator over $\mathfrak{L}$ defined by $X_i : f \mapsto x_i f$. The **Laurent isobaric divided difference** or **Laurent Demazure operator** is defined to be the following linear operator over $\mathfrak{L}$:

$$\delta_i = \begin{cases} \partial_i X_i & \text{if } i > 0 \\ \partial_0 X_1^{-1} & \text{otherwise} \end{cases}$$

Proposition 3.65: The Laurent Demazure operators δ_i are $\mathfrak{LSym}(x_i, x_{i+1})$-linear for $i > 0$. Likewise, the operator δ_0 is $\mathfrak{LSym}(x_1)$-linear.

***Proof*:** This follows immediately from Proposition 3.63. Since ∂_i and X_i are $\mathfrak{LSym}(x_i, x_{i+1})$-linear for $i > 0$, then $\delta_i = \partial_i X_i$ must be $\mathfrak{LSym}(x_i, x_{i+1})$-linear. The same argument applies for δ_0. ■

Proposition 3.66: The operators δ_i satisfy the following relations for $0 \leqslant i, j \leqslant n-1$:

1. $\delta_i^2 = \delta_i$.
2. $\delta_i \delta_j = \delta_j \delta_i$ if $|i - j| > 1$
3. $\delta_i \delta_{i+1} \delta_i = \delta_i \delta_{i+1} \delta_i$ if $i > 0$.
4. $(\delta_0 \delta_1)^2 = (\delta_1 \delta_0)^2$

***Proof*:** For a given polynomial $f \in \mathfrak{L}$ there exists an integer $d \geqslant 0$ such that $(x_1 \cdots x_n)^d f \in \mathscr{k}[x_1, \ldots, x_n]$. Denote $M_d = (x_1 \cdots x_n)^d$. For $i > 0$ we have that ∂_i and δ_i act on polynomials $f \in \mathscr{k}[x_1, \ldots, x_n]$ equivalently to their counterpart in $\mathscr{P}$, since the action of w_i is to transpose the variables (x_i, x_{i+1}). Furthermore, $\partial_i(M_d f) = M_d \partial_i(f)$ and $\delta_i(M_d f) = M_d \delta_i(f)$. This means that all relations involving only ∂_i where $i > 0$ can be immediately checked from Proposition 3.53. For example, if $i > 0$ then for any $f \in \mathfrak{L}$, let d be an integer such that $M_d f \in \Bbbk[x_1, \ldots, x_n]$ then

$$\delta_i^2(f) = M_d^{-1}\delta_i^2(M_d f) = \delta_i(f).$$

Thus, we only need to check the relations involving the case that $i = 0$. To check the first relation, it suffices to check if it holds over $\Bbbk[x_1^{\pm 1}]$, since δ_0 only acts nontrivially on x_1. Note that, from Theorem 3.44 we have that $\Bbbk[x_1^{\pm 1}]$ is a free module of finite rank module over $\mathfrak{LSym}(x_1)$ with basis $\{1, x_1\}$. Thus, it suffices to check if the equation $\delta_0^2 = \delta_0$ holds for 1 and x_1. Note that

$$\delta_0(1) = \frac{x_1^{-1} - x_1}{x_1^{-1} - x_1} = 1 \text{ and } \delta_0(x_1) = \frac{x_1^{-1}x_1 - x_1 x_1^{-1}}{x_1^{-1} - x_1} = 0.$$

In both cases we can see that $\delta_0^2(1) = \delta_0(1) = 1$ and $\delta_1^2(x_1) = \delta_1(0) = 0$. This proves the first relation. To prove relation 2, let $i > 1$ and let $m = x_1^a m'$ where $m' \in \mathfrak{L}$ is a monomial not containing x_1. Then since δ_0 is $\mathfrak{LSym}(x_1)$-linear and δ_i is $\mathfrak{LSym}(x_i, x_{i+1})$-linear where. Since $x_1 \in \mathfrak{LSym}(x_i, x_{i+1})$ and $\{x_i, x_{i+1}\} \in \mathfrak{LSym}(x_1)$ then:

$$\delta_0\delta_i(m) = \delta_0(x_1^a)\delta_i(m') = \delta_i(m')\delta_0(x_1^a) = \delta_i\delta_0(m)$$

Relation 4 involves the operators δ_0 and δ_1. Both operators are $\mathfrak{LSym}(x_1, x_2, x_3)$-Linear. Thus, we only need to check this equation using the basis obtained in Example 3.48. We apply the operators to $(\delta_0\delta_1)^2$ and $(\delta_1\delta_0)^2$ and see that they do in fact match.

Basis	$(\delta_0\delta_1)^2$	$(\delta_1\delta_0)^2$
1	1	1
x_1	0	0
x_2	0	0
x_1x_2	0	0
$x_2 + x_2^{-1}$	$x_1 + x_2 + x_1^{-1} + x_2^{-1}$	$x_1 + x_2 + x_1^{-1} + x_2^{-1}$
$x_1(x_2 + x_2^{-1})$	0	0
$x_2(x_2 + x_2^{-1})$	1	1
$x_1x_2(x_2 + x_2^{-1})$	0	0

This shows that $(\delta_0\delta_1)^2 = (\delta_1\delta_0)^2$ as desired. ■

Theorem 3.67: Let $f \in \mathfrak{L}$, then the following are equivalent:

1. $f \in \mathfrak{LSym}$.
2. f is invariant under δ_i for all $0 \leqslant i \leqslant n - 1$.
3. $f \in \ker \partial_i$ for all $0 \leqslant i \leqslant n - 1$

***Proof*:** We may prove this with two if and only if proofs. First we will show that $f \in \mathfrak{LSym}$ if and only if $f \in \ker \partial_i$ for $0 \leqslant i \leqslant n-1$. Recall that the numerator of $\partial_i \cdot f$ is $f - w_i \cdot f$. Thus $f \in \ker(\partial_i)$ if and only if $f - w_i \cdot f = 0$. With this we have the following

$$f \in \mathfrak{LSym} \Leftrightarrow f = w_i \cdot f \Leftrightarrow f - w_i \cdot f = 0 \Leftrightarrow \partial_i \cdot f = 0.$$

This shows that conditions 1 and 3 are equivalent. Now we show that (3) and (2) are equivalent. We will do this in two cases, the first case is that $i > 0$. In this case we have the following:

$$\begin{aligned}\delta_i \cdot f = f &\Leftrightarrow x_i f - x_{i+1} w_i \cdot f = f(x_i - x_{i+1}) \\ &\Leftrightarrow x_{i+1}(f - w_i(f)) = 0 \\ &\Leftrightarrow f - w_i(f) = 0\end{aligned}$$

Thus $\delta_i(f) = f$ if and only if $\partial_i(f) = 0$. With a similar computation, we may show this same case for δ_0 and ∂_0.

$$\begin{aligned}\delta_0(f) = f &\Leftrightarrow x_1^{-1} f - x_1 w_0(f) = f(x_1^{-1} - x_1) \\ &\Leftrightarrow x_1(f - w_i(f)) = 0 \\ &\Leftrightarrow \partial_0(f) = 0\end{aligned}$$

Which shows that $\delta_i(f) = f$ if and only if $f \in \ker \partial_0$. ■

Section 3.4: Representations of the Hecke Algebras

For this section, we consider $\Bbbk = \mathbb{F}(q)$ where $\mathbb{F}$ is a field of characteristic 0 and q is an indeterminate, and $\mathscr{P} = \Bbbk[x_1, \ldots, x_n]$. Recall from Example 2.49 that the Hecke algebra $\mathcal{H}_q(S_n)$ is the algebra over $\Bbbk$ with generators $\{\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-1}\}$ and the following relations:

1. $\mathfrak{h}_i^2 = (q-1)\mathfrak{h}_i + q$ for all $1 \leqslant i \leqslant n-1$.

2. $\mathfrak{h}_i \mathfrak{h}_j = \mathfrak{h}_j \mathfrak{h}_i$ for $|j - i| > 1$.

3. $\mathfrak{h}_i \mathfrak{h}_{i+1} \mathfrak{h}_i = \mathfrak{h}_{i+1} \mathfrak{h}_i \mathfrak{h}_{i_1}$ for $1 \leqslant i \leqslant n-2$.

Our goal is to describe a polynomial representation $\Phi : \mathcal{H}_q(S_n) \to \hom_{\Bbbk}(\mathscr{P}, \mathscr{P})$. This representation is described by A. Lascoux in [21] and it will make use of the operators δ_i and ∂_i described in Section 3.3. We will describe some properties of this representation, and prove that it is a faithful representation of $\mathcal{H}_q(S_n)$.

Definition 3.68: Define the operator $\zeta_i : \mathscr{P} \to \mathscr{P}$ given by

$$\zeta_i := \delta_i(q-1) + s_i$$

Lemma 3.69: The operator ζ_i is $\mathfrak{Sym}$-linear.

***Proof*:** From Proposition 3.59 we know that the operators δ_i and s_i are $\mathfrak{Sym}$-linear. Since ζ_i is the linear sum of two $\mathfrak{Sym}$-linear operator, it must too be $\mathfrak{Sym}$-linear. ■

Proposition 3.70: The operators ζ_i satisfy the same relations as the Hecke algebra $\mathcal{H}_q(S_n)$. Meaning that the following holds true.

1. $\zeta_i^2 = (q-1)\zeta_i + q$ for all $1 \leqslant i \leqslant n-1$
2. $\zeta_i\zeta_j = \zeta_j\zeta_i$ if $|i-j| > 1$
3. $\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}$

***Proof*:** We will begin with the first relation, for which we can give a direct computation below:

$$\begin{aligned}
\zeta_i^2 &= \delta_i^2(q-1)^2 + (q-1)(\delta_i s_i + s_i\delta_i) + s_i^2 \\
&= (q-1)^2\delta_i + (q-1)(s_i(\partial_i X_i - \partial_i X_{i+1})) + 1 \\
&= (q-1)^2\delta_i + (q-1)(s_i(\partial_i X_i - X_i\partial_i + 1)) + 1 \\
&= (q-1)^2\delta_i + (q-1)(s_i(s_i + 1)) + 1 \\
&= (q-1)(\delta_i(q-1) + s_i) + q \\
&= (q-1)\zeta_i + q
\end{aligned}$$

For the second relation, we recall from Proposition 3.57 that $\delta_i\delta_j = \delta_j\delta_i$ for $|i-j| > 1$. Furthermore, it is known that $s_i s_j = s_j s_i$ for $|i-j| > 1$. Therefore, we may compute the following:

$$\begin{aligned}
\zeta_i\zeta_j &= (q-1)^2\delta_i\delta_j + (q-1)(\delta_i s_j + s_i\delta_j) + s_i s_j \\
&= (q-1)^2\delta_j\delta_i + (q-1)(\delta_j s_i + s_j\delta_i) + s_j s_i \\
&= \zeta_j\zeta_i
\end{aligned}$$

To show the third and final relation, we utilize the same strategy in the proof of Proposition 3.57. We know that ζ_i and ζ_{i+1} are $\mathfrak{Sym}$-linear maps on $\Bbbk[x_i, x_{i+1}, x_{i+2}]$. We then compute $\zeta_i\zeta_{i+1}\zeta_i$ and $\zeta_{i+1}\zeta_i\zeta_{i+1}$ on the basis of $\Bbbk[x_1, x_{i+1}, x_{i+2}]$ over $\mathfrak{Sym}(x_i, x_{i+1}, x_{i+2})$. We obtain the following result:

Basis	$\delta_i\delta_{i+1}\delta_i$	$\delta_{i+1}\delta_i\delta_{i+1}$
1	q^3	q^3
x_{i+1}	$(q^2-q)x_i + q^2x_{i+1}$	$(q^2-q)x + q^2x_{i+1}$
x_{i+2}	qx_i	qx_i
x_ix_{i+2}	$(q^2-q)x_ix_{i+1} + q^2x_ix_{i+2}$	$(q^2-q)x_ix_{i+1} + q^2x_ix_{i+2}$
$x_{i+1}x_{i+2}$	qx_ix_{i+1}	qx_ix_{i+1}
$x_{i+1}x_{i+2}^2$	$x_i^2x_{i+1} + (q-q^2)x_ix_{i+1}x_{i+3}$	$x_i^2x_{i+1} + (q-q^2)x_ix_{i+1}x_{i+3}$

Since both columns are the same, then the maps $\zeta_i\zeta_{i+1}\zeta_i$ and $\zeta_i\zeta_{i+1}\zeta_i$ are the same. ■

The following corollaries follow immedately from Proposition 3.70.

Corollary 3.71: There exists a representation $\Phi_q : \mathcal{H}_q(S_n) \to \hom_k(\mathscr{P}, \mathscr{P})$ given by $\Phi_q(\mathfrak{h}_i) \mapsto \zeta_i$.

Corollary 3.72: The representation Φ_q is integral (see Definition 2.56).

Remark 3.73: The fact that Φ_q is integral means that if we pick any $a \in \mathbb{F}$ we obtain a representation Φ_a given by evaluating $q \mapsto a$. Particularly, by definition of ζ_i we have that Φ_1 gives the representation of S_n over $\mathscr{P}$ by permutation of the variables.

Theorem 3.74: The map Φ_q is a faithful representation of the Hecke algebra $\mathcal{H}_q(S_n)$.

***Proof*:** We proceed via contradiction, assuming a non-zero element $x \in \mathcal{H}_q(S_n)$ is in the kernel of Φ. Recall that $\mathcal{H}_q(S_n)$ has a basis of $\{\mathfrak{h}(\sigma) \,|\, |\sigma \in S_n\}$ where $\mathfrak{h}(\sigma) = \mathfrak{h}(s_{i_1}) \cdots \mathfrak{h}(s_{i_k})$ where $s_{i_1} \cdots s_{i_k} = \sigma$ is a minimal word. Thus, we express x as a linear combination of this basis. For notation purpuses, given a reduced word $s_{i_1} \cdots s_{i_k} = \sigma$ we deinte $\zeta(\sigma) = \zeta_{i_1} \cdots \zeta_{i_k}$.

$$\Phi_q(x) = \Phi_q \left(\sum_{\sigma \in S_n} c_\sigma \mathfrak{h}(\sigma) \right) = \sum_{\sigma \in S_n} c_\sigma \zeta(\sigma) = 0$$

We have in fact obtained a relation between the ζ operators above. However, since the specialization of $q \to 1$ gives $\zeta_i \to s_i$ then clearly $\zeta(\sigma) \to \sigma$ under this specialization. By evaluating $q = 1$ we obtain the following

$$\sum_{\sigma \in S_n} c_\sigma \sigma = 0$$

This can only happen as $c_\sigma = 0$ for all $\sigma \in S_n$. This is a contradiction, as this shows $x = 0$. ■

Proposition 3.75: The representation Φ_q extends to a representation of the Hecke algebra on the coinvariant algebra $\mathfrak{C}$. Specifically $I = (e_1, \ldots, e_n)$ so that $\mathfrak{C} = \mathscr{P}/I$. Furthermore, there is an action of $H_q(S_n)$ on $\mathfrak{C}$, defined in the following way: Given $x \in \mathcal{H}_q(S_n)$ and $f + I \in \mathfrak{C}$:

$$\Phi_q(x)(f + I) = \Phi_q(x)(f) + I.$$

***Proof*:** Since ζ_i is $\mathfrak{Sym}$-linear we immediately obtain that $\varphi(I) \subseteq I$. This shows that the representation of $\mathcal{H}_q(S_n)$ given above is well-defined. To see this consider $a, b \in \mathscr{P}$ such that $a + I = b + I$. This means that for some $x, y \in I$ we have that $a + x = b + y$. We compute then

$$\zeta_i(a) + \zeta_i(x) = \zeta_i(b) + \zeta_i(y).$$

Therefore $\zeta_i(a + x)$ and $\zeta_i(b + y)$ maps to the same element in the quotient $\mathscr{P}/I$. ■

So far, we have obtained one example of a faithful representation of the Hecke $\mathcal{H}_q(S_n)$ over $\mathscr{P}$. This representation gives an action of $\mathcal{H}_q(S_n)$ on the coinvariant algebra $\mathfrak{C}$. A. Lascoux makes use of this representation in [21] to build a sub-representation of $\mathscr{P}$ (or $\mathfrak{C}$) isomorphic to the q-Specht module S^q_λ. We will further generalize these q-Specht modules to decompose $\mathfrak{C}$ into irreducible modules of the Hecke algebra $\mathcal{H}_q(S_n)$. These will be a generalization of the higher Specht polynomials, originaly constructed in [4].

Subsection 3.4: Representations of the Hecke Algebra of Type B

In the previous section, we built a representation of the Hecke algebra $\mathcal{H}_q(S_n)$ using the divided difference and Demazure operators. In Section 3.3 we generalized these operators to the ring of Laurent polynomial $\mathfrak{L}$. The goal of this section is to use the generalization of the operators ∂_i, δ_i, and w_i to build a representation of the Hecke algebra $\mathcal{H}_{p,q}(W_n)$. The results of this section were not found in literature by the author. These results are proven by the author in this section. For this section, we will consider $\Bbbk = \mathbb{F}(p,q)$. Recall that the Hecke algebra $\mathcal{H}_{p,q}(W_n)$ has generators $\{\mathfrak{b}_0, \ldots, \mathfrak{b}_{n-1}\}$ and the following relations:

1. $\mathfrak{b}_i^2 = (q-1)\mathfrak{b}_i + q$ for $i > 0$.
2. $\mathfrak{b}_0^2 = (p-1)\mathfrak{b}_o + p$.
3. $\mathfrak{b}_i\mathfrak{b}_j = \mathfrak{b}_j\mathfrak{b}_i$ for $|i-j| > 1$
4. $\mathfrak{b}_i\mathfrak{b}_{i+1}\mathfrak{b}_i = \mathfrak{b}_{i+1}\mathfrak{b}_i\mathfrak{b}_{i+1}$ for $i > 0$.
5. $(\mathfrak{b}_0\mathfrak{b}_1)^2 = (\mathfrak{b}_1\mathfrak{b}_0)^2$

Definition 3.76: Define the operator $\xi_i \in \hom_{\Bbbk}(\mathfrak{L}, \mathfrak{L})$ in the following way:

$$\xi_i := \begin{cases} \delta_i(q-1) + w_i & \text{if } i > 0 \\ \delta_0(p-1) + w_0 & \text{otherwise} \end{cases}$$

Lemma 3.77: The operators ζ_i are $\mathfrak{LSym}$-Linear.

***Proof*:** This is a direct consequence of the fact that δ_i and w_i are $\mathfrak{LSym}$-linear for any $0 \leqslant i \leqslant n-1$.

Theorem 3.78: The map given by $\Psi_{q,p} : \mathfrak{b}_i \to \xi_i$ for $0 \leqslant i \leqslant n-1$ induces a representation of $\mathcal{H}_{q,p}(W_n)$ into $\hom_{\Bbbk}(\mathfrak{L}, \mathfrak{L})$.

***Proof*:** All we must show is that the operators ξ_i have the same relations as the Hecke algebra generators $\mathfrak{b}_0, \ldots, \mathfrak{b}_{n-1}$ as shown in 3.4.1. Note that if $i > 0$ and $f \in \Bbbk[x_1, \ldots, x_n] \subseteq \mathfrak{L}$, the operators ∂_i, and δ_i act the same as their counterparts in $\mathscr{P}$. This is due to the fact that $w_i(x_j) = x_{w_i(j)}$ if $i > 0$. Therefore, for $f \in \Bbbk[x_1, \ldots, x_n]$ the operator $\xi_i(f) = \zeta_i(f)$ for $i > 0$. Furthermore, using the same technique as in proof of Proposition 3.66 we consider $M_d = (x_1 \cdots x_n)^d$. One can see that $\xi_i(f) = M_d^{-1}\xi_i(M_d f)$. This means we can prove relations 1 and 4 in the following way. Let $f \in \mathfrak{L}$ and let d be a positive integer such that $M_d f \in \Bbbk[x_1, \ldots, x_n]$. Then the following hold:

$$\xi_i^2(f) = M_d^{-1}\zeta_i^2(M_d f) = M_d^{-1}[(q-1)\zeta_i(M_d f) + qM_d f] = (q-1)\xi_i(f) + q$$

This shows that $\xi_i^2 = (q-1)\xi + q$. With a similar argument, we show relation 4 in the following way:

$$\xi_i\xi_{i+1}\xi_i(f) = M_d^{-1}\zeta_i\zeta_{i+1}\zeta_i(M_d f) = \zeta_{i+1}\zeta_i\zeta_{i+1}(M_d f) = \xi_{i+1}\xi_i\xi_{i+1}(f)$$

Which shows that $\xi_i\xi_{i+1}\xi_i = \xi_{i+1}\xi_i\xi_{i+1}$. Now we proceed with the relations involving ζ_0. Using the fact that δ_0 and w_0 are $\mathfrak{LSym}(x_1)$-linear by Proposition 3.65, then we may use the free basis computed in the proof for Proposition 3.66 of $\Bbbk[x_1^{\pm 1}]$ over $\mathfrak{LSym}(x_1)$. We must check if the claim is true on $\{1, x_1\}$.

$$\xi_0^2(1) = \xi_0\left((p-1)+1\right) = p^2 = (p-1)\xi_0(1) + p \tag{3.4}$$

$$\xi_0^2(x_1) = \xi_0(x_1^{-1}) = (p-1)y_1 + px_1 = (p-1)\xi_0(x_1) + p(x_1) \tag{3.5}$$

This shows that the basis elements satisfy relation 2, and ξ_0^2 and $(p-1)\xi_0 + p$ are the same on the basis $\{1, x_1\}$. For relation 3 consider $i > 1$ and $m = x_1^a m'$ with m' a monomial that does not contain x_1. From Proposition 3.65 and the fact that $w_0(x_i) = x_i$ for $i > 1$ then

$$w_0 w_i(m) = w_0(x_1)w_i(m') = w_i w_o(m) \text{ and } \delta_0\delta_i(m) = \delta_0(x_1^a)\delta_i(m') = \delta_i\delta_0(m).$$

Then relation 3 must hold:

$$\xi_0\xi_i(m) = ((p-1)\delta_0(x_1^a) + w_0)((q-1)\delta_i(m') + w_i) = \xi_i\xi_0(m)$$

Lastly, the relation $(\xi_0\xi_1)^2 = (\xi_1\xi_0)^2$ can be checked with the basis for $\mathfrak{L}$ over $\mathfrak{LSym}(x_1, x_2)$ computed in Example 3.48. We compare $(\xi_0\xi_1)^2$ and $(\xi_1\xi_0)^2$ over the basis

$$\left\{1,\ x_1,\ x_2,\ x_1x_2,\ x_2 + x_2^{-1},\ x_1x_2 + x_1x_2^{-1},\ x_2^2 + 1,\ x_1x_2^2 + x_1\right\}$$

We present the result of applying both $(\xi_0\xi_1)^2$ and $(\xi_1\xi_0)^2$ in the same column.

Basis	$(\xi_0\xi_1)^2$ or $(\xi_1\xi_0)^2$
1	q^2p^2
x_1	$qpx_1^{-1} + (qp-p)x_2^{-1}$
x_2	px_2^{-1}
x_1x_2	$qx_1^{-1}x_2^{-1} - q^2p + qp$
$x_2 + x_2^{-1}$	$(q^2p^2 - qp^2)x_1 + q^2p^2x_2 + (q^2p^2 - qp^2)x_1^{-1} + (q^2p^2 - qp^2 - qp + p^2 + p)x_2^{-1}$
$x_1x_2 + x_1x_2^{-1}$	$q^2p\,x_2x_1^{-1} + \left(q^2p - q\,p\right)x_1x_2^{-1} + \left(q^2p - q^2 + q\right)x_1^{-1}x_2^{-1}$
$x_2^2 + 1$	$(-q\,p + p)\,x_1x_2^{-1} + (-q\,p + p)\,x_1^{-1}x_2^{-1} + p\,x_2^{-2} + q^2p^2 + q^2p - q\,p^2 - q\,p + p$
$x_1x_2^2 + x_1$	$x_1^{-1}x_{-2}^2 + (-q\,p + p)\,x_1 + q\,x_1^{-1}$

One could check that applying $(\xi_0\xi_1)^2 - (\xi_1\xi_0)^2$ on the matrix, all entries are 0. Meaning that both operators are the same. ■

Chapter 4

Higher Specht Polynomial for the Hecke Algebra

So far we have seen that the polynomial ring $\mathscr{P}$ is a free module of finite rank over the ring of symmetric functions $\mathfrak{Sym}$ via the Chevalley-Shephard Todd Theorem (see Theorems 3.3 and 3.4). In Chapter 3 we have constructed a descent monomial basis of $\mathscr{P}$ over $\mathfrak{Sym}$ and of the coinvariant algebra $\mathfrak{C} = \mathscr{P}/\mathfrak{Sym}_+$ (see Definition 3.14). In Section 4.1 we will construct a basis of $\mathfrak{C}$, called the higher Specht polynomials, which respects the decomposition of $\mathfrak{C}$ into the different irreducible S_n-modules. This basis was constructed by S. Ariki, T. Terasoma, and H.F. Yamada in [4]. We will give their constructions and a brief version of their proof in Section 4.1. We will then generalize the higher Specht polynomials in Section 4.2 using the Hecke algebra $\mathcal{H}_q(S_n)$. Using the action of $\mathcal{H}_q(S_n)$ we will see that these q-deformations of the higher Specht polynomials form a basis of $\mathfrak{C}$. Furthermore, they decompose $\mathfrak{C}$ into the irreducible $\mathcal{H}_q(S_n)$ modules. In fact, Alain Lascoux described a class of submodules of $\mathscr{P}$ (and consequently submodules of $\mathfrak{C}$) which is isomorphic to irreducible representations of $\mathcal{H}_q(S_n)$ in [21]. The decomposition of $\mathfrak{C}$ obtained by the q-version of the higher Specht polynomials will contain the modules described by Lascoux in [21] as a special case. The construction of the q-Higher Specht polynomials have not been found in literature by the author. Their construction, and proof that they indeed generalize the original higher Specht polynomial is original work by the author.

Section 4.1: Higher Specht Polynomials

Let $\Bbbk$ be a field of characteristic 0. Recall from the Chevalley-Shephard-Todd Theorem (i.e. Theorems 3.3 and 3.4) that as an S_n-module we have $\mathfrak{C} \cong \Bbbk S_n$. Furthermore, recall that the regular representation $\Bbbk G$ contains each irreducible module V of G exactly $\dim(V)$ times [16, Cor. 2.18]. In Theorem 1.27, it was shown that the set of Specht modules $\{S_\lambda \mid \lambda \vdash n\}$ was a full set of irreducible, and nonequivalent representations of S_n. Thus, we have

$$\Bbbk S_n \cong \bigoplus_{\lambda \vdash n} S_\lambda^{\oplus \dim S_\lambda}.$$

Since $\mathfrak{C} \cong \Bbbk S_n$, this means that it is possible to find a basis of $\mathfrak{C}$ which decomposes it into different copies of the Specht module S_λ. A sub-representation of $\mathscr{P}$ isomorphic to S_λ, spanned by polynomials called Specht polynomials, is well known (see [16, Chapter 4]). However, Ariki et al. in their paper [4] were able to generalize this to a basis of $\mathfrak{C}$, called the higher Specht polynomials, which respects the decomposition mentioned above. In this section we introduce the higher Specht polynomials as originally defined by S. Ariki, T. Terasoma, and H. Yamada in [4]. In order to do this, we will first generalize the concepts of charge and cocharge (see Definitions 3.17 and 3.19) to Young tableaux. Then we will define alternating polynomials and a bilinear form over $\mathfrak{C}$. This bilinear form will then be used to show that the higher Specht polynomials are a basis of $\mathfrak{C}$.

Subsection 4.1: Tableaux Words and the Charge of a Tableau

Recall that for a sequence of integers $a = (a_0, \dots, a_n)$ we defined the charge $I(a)$ and cocharge $J(a)$ (see Definitions 3.17 and 3.19). These were defined by a procedure done by comparing the positions of the sequence. The charge was fundamental in our proof that the descent monomials where in fact a basis for $\mathscr{P}$ over $\mathfrak{Sym}$, and for $\mathfrak{C}$ over $\Bbbk$. Recall that a Young tableau T of shape $\lambda \vdash n$ is a function $T : D(\lambda) \to \{1, \dots, n\}$, where $D(\lambda)$ is the young diagram of λ. We will need to define the charge and cocharge of a Young tableau in order to define the higher Specht polynomials.

Definition 4.1: For the tableau T let C_j be a list of numbers appearing in the j'th column of T from left to right. Thus $C_j = (c_j^1, \cdots, c_j^{n_j})$ with the entries of the column written from bottom to top. Let k be the number of columns of T, we define the **tableau-word** of T to be the following list

$$w(T) = (c_1^1 - 1, \dots, c_1^{n_1} - 1, \dots, c_k^{n_k} - 1).$$

In other words $w(T)$ lists the entries from each column (subtracted by 1), reading from left to right, and from the lowest cell in the column to the highest cell.

Example 4.2: Consider T of shape $\lambda = (3, 2, 1)$ as shown below. According to Definition 4.1, the columns read $(6, 3, 1)$, $(5, 2)$, and (4). These are read from number written in the last row in that column, to the first row. Therefore, the word $w(T)$ is given by $(5, 2, 0, 4, 2, 3)$.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \Longrightarrow \qquad w(T) = (5, 2, 0, 4, 1, 3)$$

Remark 4.3: The construction $T \mapsto w(T)$ is reversible if we know the shape of T. This means that if we have a word $w = (w_0, \ldots, w_{n-1})$ with $\{w_0, \ldots, w_{n-1}\} = \{0, \ldots, n-1\}$, then we may construct a unique tableau $\mathrm{Tab}(w, \lambda)$ given a shape λ by following the reverse process.

Definition 4.4: Consider a tableau T of shape λ. We define the **charge** of T denoted by $I(T)$ to be the tableau $\mathrm{Tab}(I(w(T)), \lambda)$.

Definition 4.5: Define the **co-charge** of T to be $J(T) := \mathrm{Tab}(J(w(T)), \lambda)$.

Example 4.6: Following the same tableau as in Example 4.2 we compute $I(T)$ and $J(T)$ below:

$$I(w(T)) = (3, 1, 0, 2, 0, 1) \qquad J(w(T)) = (2, 1, 0, 2, 1, 2)$$

$I(T) =$

0	0	1
1	2	
3		

$J(T) =$

0	1	2
1	2	
2		

Recall that a tableau T is said to be standard if its rows and columns are written in increasing order. The set of standard tableaux of shape λ is denoted by $\mathrm{ST}(\lambda)$. We will next discuss some properties of the charge and co-charge of standard tableaux.

Proposition 4.7: Let T be a Young tableau of shape λ and (r, c) be the cell in the r'th row and c'th column of $D(\lambda)$. Then $J(T)(r, c) + I(T)(r, c) = T(a, b) - 1$.

***Proof*:** [4, Lemma 1] Let $a = w(T)$ with $a = (a_0, \ldots, a_{n-1})$. Using the fact the the set of entries appearing a Young tableau is $\{1, \ldots, n\}$ we see that $\{a_0, \ldots, a_{n-1}\} = \{0, \ldots, n-1\}$. Consider $a_i = k$ then by the definition of the cocharge we have that $J(a)_i = k - I(a)_i$. Therefore $I(a)_i + J(a)_i = k$. Recall that for a given tableaux T and cell (r, c), we denote $T(r, c)$ to be the entrie of T appearing in that cell. Therefore, consider the cell (r, c) such that $T(r, c) = k + 1$ then $I(T)(r, c) + J(T)(r, c) = k$ which shows what we needed. ■

Proposition 4.8: Let $a = (a_0, \cdots, a_{n-1})$ such that $\{a_0, \ldots, a_{n-1}\} = \{0, \ldots, n-1\}$. The co-charge $J(a)$ can be built by following the reverse process as the charge. In other words, let $p(i)$ be a number such that $a_{p(i)} = i$ for $1 \leqslant i \leqslant n$. Then the co-charge $J = (J_1, \ldots, J_n)$ is built recursively in the following way:

1. $J_{p(1)} = 0$.
2. If $p(i) < p(i+1)$ then $J_{p(i)} = J_{p(i+1)} + 1$ and otherwise $J_{p(i)} = J_{p(i+1)}$.

***Proof*:** [4, Lemma 1] Consider $I = I(a)$ and let J be the word $(J_1, \ldots, J_n)$ defined by the construction above. Let $1 \leqslant k, l \leqslant n$ be two integers such that $a_l = a_k + 1$. If $k < l$ then $I_l = I_k$ and $J_l = J_k + 1$, otherwise $I_l = I_k + 1$ and $J_l = J_k$. In either case, we have that $I_l + J_l = I_k + J_k + 1$. With this formula, it is easy to show that $I_i + J_i = a_i$ inductively on the position $p(i)$ such that $a_{p(i)} = i$. Note that this matches with the original definition of $J(a)$ as $J(a)_i = a_i - I(a)_i$ ■

Lemma 4.9: For a given standard tableau T and $1 \leqslant i < i+1 \leqslant n$, let (r_1, c_1) and (r_2, c_2) be the cells in which i and $i+1$ are written in T respectively. Then either $r_2 \leqslant r_1$ and $c_1 < c_2$ or $r_1 < r_2$ and $c_2 \leqslant c_1$.

***Proof*:** Let T_1 be the tableau T with the cells containing entries in $\{i+2, \ldots, n\}$ removed. Let (r_1, c_1) and (r_2, c_2) be the cells of T_1 with entries i and $i+1$ respectively. If i and $i+1$ belong to the same row (or the same column) then the claim is trivial. We may then assume that i and $i+1$ does not belong to the same row or column. If we assume that the claim above is false, then we obtain two cases: Either $r_1 < r_2$ and $c_1 < c_2$ or $r_2 \leqslant r_1$ and $c_2 \leqslant c_1$.

For the first case, assume that $r_1 < r_2$ and $c_1 < c_2$. Therefore, there either exists a cell (r_1, c_2) or (r_2, c_1) in T_1. In either case, this implies that there exists a number written directly to the right, or below i which is not $i+1$. This contradicts the fact that T_1 is standard, since $i+1$ is the largest entry in T. In a similar argument, one can show that if $r_2 \leqslant r_1$ and $c_2 \leqslant c_1$ then there exists a cell, either directly to the right or below (r_2, c_2) which contains an entry less than $i+1$. This contradicts the fact that T is standard. ■

Corollary 4.10: For any given standard tableau T we have that $I(T) = J(T')'$

***Proof*:** [4, Lemma 1] Assume that T has size n. Let $1 \leqslant i < n-1$, we must show that the order in which i and $i+1$ appear in $w(T)$ and $w(T')$ are reversed. This would show that $w(T')$ is the reverse of the word for $w(T)$ which would show that $I(w(T)) = J(w(T'))$ using Proposition 4.8. Fix $1 \leqslant i < i+1 \leqslant n-1$, and let (r_1, c_1) and (r_2, c_2) be the cells in which i and $i+1$ is written in T respectively. Since T is standard, then we have that $r_1 \leqslant r_2$ and $c_1 \leqslant r_2$. Consider integers $p_1(i)$ and $p_1(i+1)$ so that $w(T)_{p_1(i)} = i$ and $w(T)_{p_1(i+1)} = i+1$. Likewise, let $p_2(i)$ and $p_2(i+1)$ be integers satisfying $w(T')_{p_2(i)} = i$ and $w(T')_{p_2(i+1)} = i+1$. Our goal is to show that the order of $p_1(i)$ and $p_1(i+1)$ is the opposite of $p_2(i)$ and $p_2(i+1)$. By Lemma 4.9 there are two cases to consider; either $r_2 \leqslant r_1$ and $c_1 < c_2$ or $r_1 < r_2$ and $c_2 \leqslant c_1$.

If $r_1 < r_2$ and $c_2 \leqslant c_1$ then $i+1$ is written in a column before i or directly below i in T. This implies that $p_1(i) < p_2(i+1)$ by the construction of the tableaux word. Similarly $p_2(i+1) < p_2(i)$, since $i+1$ is written in a column to the right of i. This shows that the orders of $p_1(i)$ and $p_2(i+1)$ is the opposite of $p_2(i)$ and $p_2(i+1)$. Note that the case in which $r_1 < r_2$ and $c_2 \leqslant c_1$ can be addressed by lettering $U = T'$ and applying the first case to U. Therefore, the order in which i and $i+1$ appear in $w(T)$ is the opposite to the order in which they appear in $W(T')$. ■

Recall the definition of the last letter ordering on standard tableaux (1.32). For standard tableaux T_1 and T_2 of shape λ, let m be the maximum integer that is written in different places for T_1 and T_2. Let $T_1(r_1, c_1) = T_2(r_2, c_2) = m$. We say $T_1 < T_2$ if $r_1 < r_2$. (See Definition 1.32).

Proposition 4.11: Let T_1 and T_2 be standard tableaux of shape λ such that $T_1 < T_2$. Then we have $T_2' < T_1'$.

***Proof*:** Since $T_1 < T_2$, there exists an integer m such that for all $i > m$ we have that the positions of i in T_1 and T_2 are the same, and m is written in different positions in T_1 and T_2. If $m < n$ then consider V_1 and V_2 to be the tableaux T_1 and T_2 with the cells containing $m+1, \cdots, n$ removed. Let $V_1(r_1, c_1) = m$ and $V_2(r_2, c_2) = m$. Since $T_1 < T_2$ then $r_1 < r_2$, furthermore since both V_1 and V_2 are standard, this means that $c_2 < c_1$. Therefore, since $T_1'(c_1, r_1) = m$ and $T_2'(c_2, r_2) = m$ and $c_2 < c_1$ then $T_2' < T_1'$. ■

Subsection 4.1: Alternating Polynomials

One more important concept to introduce before the higher Specht polynomials is the alternating polynomials. We will use the structure of alternating polynomials to construct a bilinear form over the coinvariant algebra.

Definition 4.12: We call a polynomial $p \in \mathscr{P}$ an **alternating polynomial** if $\sigma \cdot p = \text{sgn}(p)p$ for all $\sigma \in S_n$.

Definition 4.13: The **Vandermonde determinant** $\Delta \in \mathscr{P}$ is given by

$$\Delta = \prod_{i<j}(x_i - x_j).$$

Theorem 4.14: If $p \in \mathscr{P}$ is an alternating polynomial then there exists $\sigma \in \mathfrak{Sym}$ such that $p = \sigma\Delta$.

***Proof*:** This is a well known result (see [24, Section 6.2 and 6.3]), we give a quick proof. It is easy to see that Δ is in fact an alternating polynomial. Note that any transposition of the form $t = (i, i+1)$ will permute the factors of the form $t : (x_k - x_i) \leftrightarrow (x_j - x_{i+1})$ for $j < i$. Similarly, $t : (x_i - x_j) \leftrightarrow (x_{i+1} - x_j)$ for $i + 1 < j$. Since the factor $t : (x_i - x_{i-1}) = (-1)(x_i - x_{i+1})$ then $t\Delta = -\Delta$. We may conclude by an induction argument that for any $\sigma \in S_n$ we have that $\sigma\Delta = (-1)^{l(\sigma)}\Delta = \text{sgn}(\sigma)\Delta$.

Now consider any f such that for all $\sigma \in S_n$ we have that $\sigma \cdot f = \text{sgn}(\sigma) \cdot f$. Then for a transposition of the form $t = (i, j)$ we have that $t \cdot f = -f$. Thus, if we evaluate f at $x_i = x_j$ we get that $f = t \cdot f = -f$. This implies that $f = 0$. Meaning that $(x_i - x_j)$ must be a factor of f. Since we can do this for any pair (i, j) for $i < j$ we obtain that $f = g\Delta$ for some $g \in \mathscr{P}$. It remains to show that

$g \in \mathfrak{Sym}$. To do this, consider the action of S_n on rational functions. Let $\sigma \in S_n$

$$\sigma \cdot \frac{g\Delta}{\Delta} = \frac{(-1)g\Delta}{(-1)\Delta} = \frac{g\Delta}{\Delta}.$$

The above shows that for any $\sigma \in S_n$ we have $\sigma \cdot g = g$, meaning that $g \in \mathfrak{Sym}$. ■

Theorem 4.14 completely classifies all possible alternating polynomials. It implies that if f is alternating then $\deg(f) \geqslant \deg(\Delta)$. This is an extremely useful classification. We will see that we may obtain all alternating polynomials using an operator from the group algebra

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma \in \Bbbk[S_n].$$

It is quite clear that if we apply this operator to any polynomial, then we obtain an alternating polynomial. The following result shows how one can build Δ from this operator. Then we will use this operator to construct a bilinear form for $\mathfrak{C}$.

Definition 4.15: The **Vandermonde matrix** υ is the following $n \times n$ matrix with entries in $\mathscr{P}$.

$$\upsilon = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$$

Proposition 4.16: The following hold for the Vandermonde determinant:

1. $\det(\upsilon) = \Delta$.
2. $\deg(\Delta) = n(n-1)/2$.
3. $\Delta = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma(x_1^{n-1} \cdots x_{n-1}^1)$

***Proof*:** This is a well known result (see [4, Section 2]), we give a quick proof here. For the second assertion is quick to check, as $n(n-1)/2 = |\{(i,j) \mid 1 \leqslant i < j \leqslant n\}|$, proving that $\deg(\Delta) = n(n-1)/2$. For the first assertion, note that $\det(\upsilon) \in \mathscr{P}$. For a given transposition t we have that

$$t \cdot \det(\upsilon) = \det(t \cdot \upsilon) = -\det(\upsilon).$$

This means that $\det(\upsilon)$ is an alternating polynomial, and by Theorem 4.14 we have $\det(\upsilon) = g\Delta$ for some $g \in \mathfrak{Sym}$. Furthermore, we can show that $\det(\upsilon)$ is a homogeneous polynomial.

$$\det(\upsilon)(\lambda x_1, \ldots, \lambda x_n) = \det\left(\begin{bmatrix} 1 & \lambda x_1 & \cdots & \lambda^{n-1}x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda x_n & \cdots & \lambda^{n-1}x_n^{n-1} \end{bmatrix}\right) = \lambda^{\frac{n(n-1)}{2}} \det(\upsilon).$$

This proves that $\det(\upsilon)$ is a homogeneous polynomial of degree $n(n-1)/2$. Note that $\deg\det(\upsilon) = \deg(\Delta)$ therefore $\det(\upsilon) = \Delta$. To show the final assertion, we use the Leibniz formula for the determinant. Let $E_{i,j}$ be the entry of υ^T in row i and column j. Note that $E_{i,j} = x_j^{i-1}$

$$\Delta = \det(\upsilon^T) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) E_{1,\pi(1)} \cdots E_{n,\pi(n)} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) x^0_{\pi(1)} \cdots x^{n-1}_{\pi(n)} \qquad \blacksquare$$

Proposition 4.17: Consider the ring homomorphism $N : \mathscr{P} \to \Bbbk$ given by $N(x_i) = 0$ for all $1 \leqslant i \leqslant n$. This way, N is the same as evaluating all variables at 0. Consider $f, g \in \mathfrak{C}$. Denote $\bar{f}$ and $\bar{g}$ to be a lifting of f and g to $\mathscr{P}$. The map $\langle -, - \rangle : \mathfrak{C} \times \mathfrak{C} \to \Bbbk$, defined below, is well-defined and a bilinear form:

$$\langle f, g \rangle = N\left(\frac{1}{\Delta} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma(\bar{f}\bar{g})\right).$$

***Proof*:** This result is used in [4], we give it a quick proof here. Denote the operator $\theta = \sum_\sigma \operatorname{sgn}(\sigma)\sigma : \mathscr{P} \to \mathscr{P}$. Then it is easy to see that $\theta(f)$ is alternating, therefore Δ divides $\theta(f)$. Furthermore, it is easy to see that if $f, g \in \mathscr{P}$ and $a, b \in \mathfrak{Sym}$ we obtain

$$\theta(af + bg) = a\theta(f) + b\theta(g).$$

Therefore, the operator θ is $\mathfrak{Sym}$-linear. Note that N is definitely $\Bbbk$-linear and division by Δ is also $\Bbbk$-linear, then if we fix $c, d \in \Bbbk$ and $f, g, h \in \mathscr{P}$ we obtain

$$N\left(\frac{\theta((cf + dg)h)}{\Delta}\right) = cN\left(\frac{\theta(fh)}{\Delta}\right) + dN\left(\frac{\theta(gh)}{\Delta}\right).$$

This proves that $\langle -, - \rangle$ is $\Bbbk$-bilinear. It remains to show that the value of $\langle -, - \rangle$ does not depend on the choice of lifting. Let $f \in \mathfrak{C}$ and F_1 and F_2 be two choices of lifting of f. Consider the ideal generated by the elementary symmetric functions $I = (e_1, \ldots, e_n) \subseteq \mathscr{P}$ so that $\mathfrak{C} = \mathscr{P}/I$. There exists polynomials $\{b_1, \ldots, b_{n!}\} \subset \mathscr{P}$ such that $\{b_i + I \,|\, 1 \leqslant i \leqslant n!\}$ is a basis of $\mathfrak{C}$ (see Theorem 3.5). This means that there exists $h_1, h_2 \in I$ such that

$$F_1 = \sum_i c_i b_i + h_1 \quad F_2 = \sum_i c_i b_i + h_2$$

Note that since $h_1, h_2 \in I$ then neither h_1 or h_2 have a constant term. Note that for any $h \in I$ there exists polynomials $h_1, \ldots, h_n \in \mathscr{P}$ such that $h = h_1 e_1 + \cdots + h_n e_n$. Therefore, since N is equivalent to evaluating all variables to 0, and θ is $\mathfrak{Sym}$-linear, we have

$$N(\theta(h)) = N(e_1)N(\theta(h_1)) + \cdots + N(e_n)N(\theta(h_n)) = 0.$$

Thus for any choice $p \in \mathscr{P}$ we have that $N(\theta(h_1 p)) = N(\theta(h_2 p)) = 0$. Therefore we obtain the following

$$N(\theta(F_1 p)) = N\left(\sum_i c_i b_i\right) + N(\theta(h_1 p)) = N\left(\sum_i c_i b_i\right) + N(\theta(h_1 p)) = N(\theta(F_2 p)).$$

This shows that $\langle f, g \rangle$ does not depend on the choice of lifting for f. A similar argument can be made for two different liftings of g. ■

Proposition 4.18: For any $\sigma \in S_n$ we have that $\langle \sigma f, g\rangle = \text{sgn}(\sigma)\langle f, \sigma^{-1}g\rangle$

***Proof*:** [4, Lemma 2] Let A and B be liftings of f and g respectively. Then we have the following:

$$\begin{aligned}\sum_{\pi\in S_n}\text{sgn}(\pi)\pi((\sigma A)B) &= \sum_{\pi\in S_n}\text{sgn}(\pi)\pi(\sigma\sigma^{-1}(\sigma A)B)\\ &= \text{sgn}(\sigma)\sum_{\rho\in S_n}\text{sgn}(\rho)\rho(A(\sigma^{-1}B)) \qquad \text{where } \rho = \pi\sigma\end{aligned}$$

The computation above proves the assertion. ■

Definition 4.19: For a sequence of non-negative integers $a = (a_1, \ldots, a_n)$ we denote $\bar{a}$ the sequence obtained by ordering a in a weakly increasing way. We also denote $|a| = a_1 + \cdots + a_n$. Furthermore, for two sequences a and b of equal length, we denote $a + b$ to be the componentwise sum of both sequences.

Lemma 4.20: Let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be two words such that $\langle x_w^a, x_w^b\rangle \neq 0$ then the following holds:

1. $|a| + |b| = \frac{n(n-1)}{2}$ and $\{a_0 + b_0, \ldots, a_{n-1} + b_{n-1}\} = \{0, \ldots, n-1\}$.
2. If $\overline{a} + \overline{b} = (0, \ldots, n-1)$ then there exists a unique $\pi \in S_n$ such that $\pi(a + b) = (0, \ldots, n-1)$ with $\pi(a) = \overline{a}$ and $\pi(b) = \overline{b}$

***Proof*:** [4, Lemma 3] Let $h = x_1^{a_0+b_0}\cdots x_n^{a_{n-1}+b_{n-1}}$ and let $\theta = \sum_\sigma \text{sgn}(\sigma)\sigma$. It is clear that $D(h)$ is an alternating polynomial of degree $\deg(\theta(h)) = \deg(h) = |a| + |b|$. If $|a| + |b| < n(n-1)/2$ then $\theta(h)$ is an alternating polynomial of degree $\deg(\theta(h)) < \deg(\Delta)$ therefore $\theta(h) = 0$ by Theorem 4.14 and Proposition 4.16. Otherwise, if $|a| + |b| > n(n-1)/2$ then there exists homogeneous $0 \neq g \in \mathfrak{Sym}$ such that $\theta(h) = g\Delta$. This means that $\frac{1}{\Delta}\theta(h) = g$. However, since $g \neq 0$, setting all variables $x_1 = x_2 = \cdots = x_n = 0$ would make $g = 0$. This means that $\langle x_w^a, x_w^b\rangle = 0$. If it is the fact that $|a| + |b| = 0$ then by Proposition 4.16 we have that $h = \pi(x_1^{n-1}\cdots x_n^0)$ meaning that $\{a_0 + b_0, \ldots, a_{n-1} + b_{n-1}\} = \{0, \ldots, n-1\}$.

To prove the second part, if $\overline{a} + \overline{b} = (0, \ldots, n-1)$. Then by the first part we also have that $\{a_0 + b_0, \ldots, a_{n-1} + b_{n-1}\} = \{0, \ldots, n-1\}$. This means that there exists a permutation $\pi \in S_n$ such that $a_{\pi(i)} + b_{\pi(i)} = i$. Since $\overline{a}_i + \overline{b}_i = i$ this means that $\pi a = \overline{a}$ and $\pi b = \overline{b}$ as desired. ■

Subsection 4.1: Higher Specht Polynomials

We now introduce the higher Specht polynomials and show that they are indeed a basis of the coinvariant algebra. We will first define the Specht polynomials, then use the bilinear form (see

Definition 4.17) to show that the Specht polynomials are indeed a basis. We will also generalize alternating polynomials in order to describe the structure of the higher Specht polynomials.

Definition 4.21: Fix two tableaux standard T and V of shape $\lambda \vdash n$. Let $w(T) = (a_1, \ldots, a_n)$ then let $\alpha = (a_1 + 1, \ldots, a_n + 1)$ and $b = w(V)$. Then the **charge monomial** of T and V is defined as

$$x_T^V = x_{a_1}^{b_1} \cdots x_{a_n}^{b_n}$$

Example 4.22: Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $V = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ then $w(T) = (2,0,1)$ and $w(V) = (1,0,2)$. Computing $\alpha = (3,1,2)$ we give the charge monomial is given by

$$x_T^V = x_3^2 x_1^0 x_2^1 = x_2 x_3^2$$

Recall the definition of the Young symmetrizer of a tableau T (see Definition 1.16). If $\mathrm{C}(T)$ and $\mathrm{R}(T)$ are the column and row symmetrizers (respectively) then we define

$$\varepsilon_T = \left(\sum_{c \in \mathrm{C}(T)} \mathrm{sgn}(c) c \right) \left(\sum_{r \in \mathrm{R}(T)} r \right).$$

Furthermore, recall Definition 1.5. Given $\lambda \vdash n$ the tableau T^λ is constructed by enumerating each cell from left to right, for each consecutive row. Similarly, $T^\lambda = (T_\lambda)'$ is constructed by enumerating each cell, from top to bottom, for each consecutive column. We have seen in Definition 1.17 and Theorem 1.27 that the irreducible representations of S_n are the Specht modules $S_\lambda = \mathbb{C}[S_n]\varepsilon_{T^\lambda}$.

Definition 4.23: Consider a partition $\lambda \vdash n$ and let $T, W \in \mathrm{ST}(\lambda)$. We define the **higher Specht polynomial** indexed by T and V to be

$$F_T^V = \varepsilon_T \cdot x_T^{I(V)}.$$

Example 4.24: Following the same tableaux in Example 4.22, let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $V = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Then we know that $x_T^V = x_2 x_3$. Furthermore $\varepsilon_T = (1 - (13))(1 + (12)) = 1 - (13) + (12) - (123)$. Therefore

$$F_T^V = \varepsilon_T \cdot x_2 x_3 = x_2(x_3 - x_1).$$

Lemma 4.25: For any $V \in \mathrm{ST}(\lambda)$ the subspace of $\mathscr{P}$ spanned by $\{F_T^V \mid T \in \mathrm{ST}(\lambda)\}$ is isomorphic to the Specht module S_λ as a representation of S_n.

***Proof*:** [4, Lemma 5] For $M \in \mathrm{ST}(\lambda)$ let π_T be the permutation such that $\pi_T(T_\lambda) = M$. In Theorem 1.37 it was proven that $\{\pi_M \varepsilon_{T_\lambda} \mid T \in \mathrm{ST}(\lambda)\}$ is a basis for S_n. Therefore, let $\sigma \in S_n$ and $T \in \mathrm{ST}(\lambda)$ then there exists $c_M \in \Bbbk$ for every $M \in \mathrm{ST}(\lambda)$ such that

$$\sigma \pi_T \varepsilon_{T_\lambda} = \sum_{M \in \mathrm{ST}(\lambda)} c_M \pi_M \varepsilon_{T_\lambda} = \sum_{M \in \mathrm{ST}(\lambda)} c_M \varepsilon_M \pi_M.$$

Considering that $F_T^V = \varepsilon_T x_T^V = \pi_T \varepsilon_{T_\lambda} x_{T^\lambda}^V$ then we compute

$$\sigma F_T^V = \sigma \pi_T \varepsilon_{T^\lambda} x_{T_\lambda}^V = \sum_{M \in \mathrm{ST}(\lambda)} c_M F_M^V.$$

This shows that the space spanned by $\{F_T^V \mid T \in \mathrm{ST}(\lambda)\}$ is isomorphic to a sub-representation of S_λ. Since this space is non-zero, this means that it must be isomorphic to S_λ since the Specht modules are irreducible. ■

So far, we have that proven that the higher Specht polynomials respects the irreducible S_n-module decomposition of $\mathfrak{C}$. We will now show that they do indeed form a basis of $\mathfrak{C}$ by making use of the bilinear form defined in Definition 4.17. We will show there is an ordering of the higher Specht polynomials, in which the Gramian of the bilinear form computed at the Specht polynomials forms an upper triangular matrix with nonzero diagonal. This would show that the Gramian is invertible, thus the higher Specht polynomials are a basis of $\mathfrak{C}$.

Proposition 4.26: For any tableau T and $f, g \in \mathfrak{C}$ we have $\langle \varepsilon_T f, g \rangle = \langle f, \varepsilon_{T'} g \rangle$

***Proof*:** [4, Lemma 2] Note that $\mathrm{R}(T') = \mathrm{C}(T)$ and $\mathrm{C}(T') = \mathrm{R}(T)$. We show this statement with the following computation:

$$\begin{aligned}
\langle \varepsilon_T f, g \rangle &= \sum_{c \in C(T)\, r \in R(T)} \mathrm{sgn}(c) \langle crf, g \rangle \\
&= \sum_{c \in C(T)\, r \in R(T)} \mathrm{sgn}(c)\mathrm{sgn}(rc) \langle f, r^{-1}c^{-1}g \rangle && \text{by Proposition 4.18} \\
&= \sum_{c \in \mathrm{R}(T')\, r \in \mathrm{C}(T')} \mathrm{sgn}(r) \langle f, rcg \rangle \\
&= \langle f, \varepsilon_{T'} g \rangle
\end{aligned}$$ ■

Corollary 4.27: [4, Lemma 4] Consider standard tableaux $T_1, T_2, V_1, V_2 \in \mathrm{ST}(\lambda)$. If $T_1 > T_2$ in the last letter ordering, then $\langle F_{T_1}^{S_1}, F_{T_2'}^{S_2'} \rangle = 0$

***Proof*:** This is an immediate consequence of Proposition 4.26 and Proposition 1.36 together with Corollary 1.36. Using these results, we know that if $T_1 < T_2$ then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$. Furthermore, since $T_1 > T_2$ implies $T_1' < T_2'$ then we obtain $\langle \varepsilon_{T_1} f, \varepsilon_{T_2'} g \rangle = \langle f, \varepsilon_{T_1'} \varepsilon_{T_2} g \rangle = 0$. ■

For the next few results, we will fix some notation. Consider two sequences of length n, a and b, of non-negative integers. We say $a < b$ if there exists m such that for all $m + 1 \leqslant j \leqslant n$ the sequences match $a_j = b_j$ and $a_m < b_m$. We will call this ordering of sequences the **reverse-lexicographical ordering**. Recall that for T of shape λ the Young idempotent given $\varepsilon_T^2 = \frac{n!}{\dim S_\lambda} \varepsilon_T$ (see Proposition 1.28). This can be used to prove the following lemma.

Lemma 4.28: Consider $T, V_1, V_2 \in \mathrm{ST}(\lambda)$ and let $C = \mathrm{C}(T)$ and $R = \mathrm{R}(T)$. Then

$$\langle F_T^{V_1}, F_{T'}^{V_2'}\rangle = \frac{n!}{\dim S_\lambda} \sum_{c\in C\, r\in R} \langle r \cdot x_T^{I(V_1)}, c \cdot x_T^{J(V_2)}\rangle.$$

***Proof*:** [4, Section 2] Recall from Corollary 4.10 that $I(V_2) = J(V_2')'$ we have

$$\begin{aligned}\langle F_T^{V_1}, F_{T'}^{V_2'}\rangle &= \langle \varepsilon_T x_T^{I(V_1)}, \varepsilon_{T'} x_{T'}^{I(T')}\rangle = \langle \varepsilon_{T'}\varepsilon_T x_T^{I(V_1)}, x_T^{J(V_2)}\rangle \\ &= \frac{n!}{\dim S_\lambda} \sum_{c\in C\, r\in R} \mathrm{sgn}(c)\langle cr \cdot x_T^{I(V_1)}, x_T^{J(V_2)}\rangle \\ &= \frac{n!}{\dim S_\lambda} \sum_{c\in C\, r\in R} \langle r \cdot x_T^{I(V_1)}, c \cdot x_T^{J(V_2)}\rangle.\end{aligned}$$

■

Remark 4.29: For the next results, we will fix some notation given in [4]. For a given sequence of non-negative integers $a = (a_1, \ldots, a_n)$, we say $\overline{a}$ is a reordering of the word a in a weakly increasing order.

Proposition 4.30: Consider $T, V_1, V_2 \in \mathrm{ST}(\lambda)$. If $\overline{I(w(V_1))} < \overline{I(w(V_2))}$ then $\langle F_T^{V_1}, F_{T'}^{V_2'}\rangle = 0$

***Proof*:** [4, Theorem 1] Denote $C = \mathrm{C}(T)$ and $R = \mathrm{R}(T)$ then from Lemma 4.28 we obtain

$$\langle F_T^{V_1}, F_{T'}^{V_2'}\rangle = \frac{n!}{\dim S_\lambda} \sum_{c\in C\, r\in R} \langle r \cdot x_T^{I(V_1)}, c \cdot x_T^{J(V_2)}\rangle$$

Furthermore, since $\overline{I(w(V_1))} < \overline{I(w(V_2))}$ this means that $\overline{I(w(V_1))} + \overline{J(w(V_2))} < (0, \ldots, n-1)$. We can see this as $\overline{I(w(V_2))} + \overline{J(w(V_2))} = (0, \ldots, n-1)$. Particularly this means that if $a = w(r \cdot I(V_1))$ and $b = w(c \cdot J(V_2))$ for $r \in R$ and $c \in C$ we have that $|a| + |b| < |(0, \ldots, n-1)| = n(n-1)/2$. Therefore, we obtain by Lemma 4.20

$$\langle r \cdot x_T^{I(V_1)}, c \cdot x_T^{J(V_2)}\rangle = 0.$$

■

Proposition 4.31: Let $T, V_1, V_2 \in \mathrm{ST}(\lambda)$ with $\overline{I(w(V_1))} = \overline{I(w(V_2))}$ and $V_1 < V_2$ (according to the last letter ordering). Then $\langle F_T^{V_1}, F_{T'}^{V_2'}\rangle = 0$.

***Proof*:** [4, Proposition 1] We will prove this via contradiction. From Lemma 4.28 we have

$$\langle F_T^{V_1}, F_{T'}T^{V_2'}\rangle = \frac{n!}{\dim S_\lambda}\sum_{c\in C\, r\in R}\operatorname{sgn}(c)\langle r\cdot x_T^{I(V_1)}, c\cdot x_T^{J(V_2)}\rangle$$

Before we continue the proof, notice that T assigns each cell of $D(\lambda)$ a unique number from 1 to n. This means that for any particular tableau M of shape λ, and permutation $\pi \in S_n$ we can define an action of π on M by permuting the cells corresponding to the numbers of T. This means that if $\pi(i) = j$ and (r_1, c_1) and (r_2, c_2) are the cells corresponding to i and j in T respectively, then $\pi \cdot M(r_1, c_1) = M(r_2, c_2)$. We will use this particular action of S_n on tableaux of shape λ using T for this proof. Particularly if $r \in \mathrm{R}(T)$ and $c \in \mathrm{C}(T)$ then

$$\langle r\cdot x_T^{I(V_1)}, c\cdot x_T^{J(V_2)}\rangle = \langle x_T^{r\cdot I(V_1)}, x_T^{c\cdot J(V_2)}\rangle.$$

Let us assume that for some $r \in \mathrm{R}(T)$ and $c \in \mathrm{C}(T)$ we have that $\langle x_T^{r\cdot I(V_1)}, x_T^{c\cdot J(V_2)}\rangle \neq 0$. Let $a = w(r \cdot I(V_1))$ and $b = w(c \cdot J(V_2))$. By Proposition 4.7, Lemma 4.20 and our assumption that $I(V_1) = I(V_2)$ we have that $\overline{a} + \overline{b} = (0, 1, \ldots, n-1)$ and $\{a_0 + b_0, \ldots, a_{n-1} + b_{n-1}\} = \{0, \ldots, n-1\}$. Furthermore, there exists a permutation σ such that $\sigma a = \overline{a}$ and $\sigma b = \overline{b}$. Since $V_1 < V_2$ there exists $1 \leqslant m \leqslant n$ such that if $k > m$ then the number k is written in the same cell in both V_1 and V_2. Furthermore, m is written in a row above in V_1 then V_2. Assume that $m = n$, meaning that n is written in a row ρ_1 in V_1 and ρ_2 in V_2 such that $\rho_1 < \rho_2$. By Lemma 4.20, since $\langle x_T^{r\cdot I(V_1)}, x_T^{r\cdot J(V_2)}\rangle \neq 0$ then $r \cdot a + c \cdot b$ is a permutation of $(0, \ldots, n-1)$ and $\overline{a} + \overline{b} = (0, \ldots, n-1)$. This means that there should exist $1 \leqslant k \leqslant n$ such that $(r \cdot a)_k + (c \cdot b)_k = n - 1$. However, since n is written in a row ρ_i in V_1 and row ρ_j in V_2 with $i > j$ then there are no positions l in which $(r \cdot a)_l + (c \cdot b) = n - 1$ which contradicts our assumptions.

The above argument addresses the case that $m = n$. However, we must show that this holds for $m < n$. To do this let $P_{m+1}, \ldots, P_n$ be the cells in V_1 and V_2 such that $V_1(P_k) = V_2(P_k) = k$ for $m < k \leqslant n$. We will argue that $(r \cdot I(V_1))(P_k) = (c \cdot J(V_2))(P_K)$ for $m < k \leqslant n$. By our assumption that $\langle x_T^{r\cdot a}, x_T^{c\cdot b}\rangle \neq 0$ and that $\overline{a} + \overline{b} = (0, 1, \ldots, n-1)$ we have that the cell P_n is the only cell such that $I(V_1)(P_n) + J(V_2)(P_n) = n - 1$. This means that in order for the bilinear form to not be zero, P_n must be the only position that satisfies $(r \cdot I(V_1))(P_n) + (c \cdot I(V_2))(P_n) = n - 1$. However, this means that $(r \cdot I(V_1))(P_n) = (c \cdot J(V_2))(P_n)$. Next, let $V_1^{(1)}$ and $V_2^{(2)}$ be the tableaux V_1 and V_2 with cell P_n removed. We repeat the same argument until we obtain $V_1^{(m)}$ and $V_2^{(m)}$. Note that m is the greatest integer that is written in different positions in $V_1^{(m)}$ and $V_2^{(m)}$. Therefore, by the argument in the first paragraph, we have a contradiction with Lemma 4.27. ■

Proposition 4.32: For $T, V \in \mathrm{ST}(\lambda)$ the bilinear form $\langle F_T^V, F_{T'}^{V'}\rangle \neq 0$.

***Proof*:** [4, Propostion 1] We begin again by writing the bilinear form as in Lemma 4.28. Let $R = \mathrm{R}(T)$ and $C = \mathrm{C}(T)$. Then

$$\langle F_T^V, F_{T'}^{V'} \rangle = \frac{n!}{\dim S_\lambda} \sum_{\pi \in C} \sum_{\rho \in R} \langle \rho \cdot x_T^{I(V)}, \pi \cdot x_T^{J(V)} \rangle$$

By the argument in Proposition 4.31 we saw that there exists an action of S_n on the diagram $D(\lambda)$, by using the index of each cell given by T. Assume that for given $\rho \in R$ and $\pi \in C$ we have that $\rho \cdot I(V_1) \neq I(V_1)$ and $\pi \cdot J(V_2) \neq J(V_2)$. Let $a = w(\rho \cdot I(V))$ and $b = w(\pi \cdot J(V))$. For all integers $1 \leqslant i \leqslant n$, let $(r_i, c_i) \in D(\lambda)$ be the cell that i is written in V. With this notation, define m to be the largest integer such that the cell (r_m, c_m) is not fixed by ρ or π. Since ρ does not fix $I(V)$ and π does not fix $J(V)$, then for any $1 \leqslant i \leqslant m$ we have that

$$(\rho \cdot I(V))\,(r_i, c_i) + (\rho \cdot I(V))\,(r_i, c_i) < m - 1.$$

This means that there are no integers i such that $a_i + b_i = n - 1$. This contradicts 4.20 as $\{a_1 + b_1, \cdots, a_n + b_n\} = \{0, \cdots, n-1\}$. Therefore $\langle \rho \cdot x_T^{I(V_1)}, \pi \cdot x_T^{J(V_2)} \rangle = 0$ unless $\rho \cdot I(V_1) = I(V_1)$ and $\pi \cdot J(V_2) = J(V_2)$. Consider the following subgroups of S_n;

$$A = \{\rho \in R \,|\, r \cdot I(V_1) = I(V_1)\} \text{ and } B = \{\pi \in C \,|\, c \cdot J(V_2) = J(V_2)\}.$$

Let $M = \langle x_T^{I(V_1)}, x_T^{J(V_2)} \rangle$. Then we obtain

$$\frac{n!}{\dim S_\lambda} \sum_{c \in C} \sum_{r \in R} \langle r \cdot x_T^{I(V_1)}, c \cdot x_T^{J(V_2)} \rangle = \frac{n!}{\dim S_\lambda} \sum_{c \in B} \sum_{r \in A} M = \frac{n!}{\dim S_\lambda} |A||B| M \qquad \blacksquare$$

Theorem 4.33: The set $\{F_T^V \,|\, T, V \in \mathrm{ST}(\lambda)\}$ is a basis for $\mathfrak{C}$.

***Proof*:** [4, Theorem 1] First, let Λ be the ordered list containing all partitions of n. Assume that there are k partitions of n so that we may list $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$, where each Λ_i is a partition of n. Now we order the set of higher Specht polynomials. For a given $F_1 = F_{T_1}^{V_1}$ and $F_2 = F_{T_2}^{V_2}$ we say that $F_1 < F_2$ if any of the following applies:

1. If $T_1, V_1 \in \mathrm{ST}(\Lambda_i)$ and $T_2, V_2 \in \mathrm{ST}(\Lambda_j)$ with $i < j$.
2. If $T_1, T_2, V_1, V_2 \in \mathrm{ST}(\Lambda_i)$ and $T_1 > T_2$ by the last letter ordering.
3. If $T_1 = T_2$ and $\overline{I(w(V_1))} < \overline{I(w(V_2))}$ by the reverse lexicographical ordering
4. If $T_1 = T_2$ and $\overline{I(w(V_1))} = \overline{I(w(V_2))}$ with $V_1 < V_2$ by the last letter ordering.

Using this we see that if $F_{T_1}^{V_1} < F_{T_2}^{V_2}$ then by Propositions 4.27, 4.30, and 4.31 then $\langle F_{T_1}^{V_1}, F_{T_2'}^{V_2'} \rangle = 0$. Consider the Gramian matrix

$$M = \begin{bmatrix} \langle F_{T_1}^{V_1}, F_{T_1'}^{V_1'} \rangle & \cdots & \langle F_{T_n}^{V_n}, F_{T_1'}^{V_1'} \rangle \\ \vdots & \ddots & \vdots \\ \langle F_{T_1}^{V_1}, F_{T_n'}^{V_n'} \rangle & \cdots & \langle F_{T_n}^{V_n}, F_{T_n'}^{V_n'} \rangle \end{bmatrix}$$

Note that M is upper triangular, with non-zero elements of $\Bbbk$ in the diagonal entries. This means that M is invertable, which shows that the set $\{F_T^V \mid T, V \in \mathrm{ST}(\lambda) \text{ for } \lambda \vdash n\}$ is a basis of $\mathfrak{C}$. ■

Subsection 4.1: Structure of Higher Specht Polynomials

We will finish this section by defining a generalization of the alternating polynomials. Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. It is natural to consider $S_{\lambda_1} \times \cdots \times S_{\lambda_k} \subseteq S_n$ by identifying the subgroups S_{λ_i} as groups which permute a subset of $\{1, \ldots, n\}$ of size λ_i. Thus, we can arrange the elements that each subgroup permutes in the columns of a tableau of shape λ'. This motivates the following definitions:

Definition 4.34: Let T be a tableau of shape λ and $\mathrm{C}(T)$ be the subgroups of column permutations of T. Then we define the T-symmetric functions as $\mathfrak{Sym}_T := \{f \in \mathscr{P} \mid \sigma f = f \text{ for all } \sigma \in \mathrm{C}(T)\}$.

It is easy to see that $\mathrm{C}(T)$ is a reflection group acting on $\mathscr{P}$. By the Chevalley-Shephard-Todd theorem (see Theorems 3.3 and 3.4) we have that $\mathscr{P}$ is free-module of finite rank over $\mathscr{P}^{\mathrm{C}(T)}$.

Definition 4.35: For a tableau T of shape λ, we say that $f \in \mathscr{P}$ is T-**alternating** if $\sigma f = \mathrm{sgn}(\sigma) f$ for all $\sigma \in \mathrm{C}(T)$.

We may classify all T-alternating polynomials by modifying the definition of the Vandermonde determinant in the following way:

Definition 4.36: Let T be a tableau of shape λ. Let $C_1, \ldots, C_k$ be the set of numbers appearing in each column of T. We define the T-**Vandermonde determinant** to be

$$\Delta_T := \prod_{i=1}^{k} \prod_{\substack{j,l \in C_i \\ j<l}} (x_j - x_l).$$

Example 4.37: Consider the standard tableau $T = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$ of shape $\lambda = (2,2)$. Here we have $\{a,b,c,d\} = \{1,2,3,4\}$. The columns of T are $C_1 = \{a,c\}$ and $C_2 = \{b,d\}$. Therefore, we may compute the T-vandermonde determinant as

$$\Delta_T = (x_a - x_c)(x_b - x_d).$$

Proposition 4.38: If f is T-alternating, then there exists $g \in \mathfrak{Sym}_T$ such that $f = g\Delta_T$.

***Proof*:** The fact that Δ_T is T-alternating is immediate from Theorem 4.14. Let $C_1, \ldots, C_k$ be the set of numbers appearing in the columns of T. Then $\mathrm{C}(T) \cong S_{C_1} \times \cdots \times S_{C_k}$. If $\sigma \in S_{C_i}$ then σ leaves all components of δ_T with elements not in C_i invariant. By Theorem 4.14 we have that

$$\sigma \prod_{j<l \in C_i} (x_i - x_j) = \mathrm{sgn}(\sigma) \prod_{j<l \in C_i} (x_i - x_j)$$

This shows that $\sigma\Delta_T = \mathrm{sgn}(\sigma)\Delta_T$. Therefore, Δ_T is in fact T-alternating. Now, consider f a T-alternating polynomial. By evaluating f at $x_i = x_j$ for i and j in the same column of T, that $f|_{x_i=x_j} = 0$. This means that if i and j are in the same column, then $(x_i - x_j)$ divides f. This means that if $f = \Delta_T g$. To show that $g \in \mathfrak{Sym}_T$ is identical to the proof in Theorem 4.14. ∎

Lemma 4.39: For a given, $T, V \in \mathrm{ST}(\lambda)$ there exists $g \in \mathfrak{Sym}_T$ such that $F_T^V = g\Delta_T$.

***Proof*:** This is almost trivial to check from the definition of the higher Specht polynomials.

$$F_T^V = \varepsilon_T x_T^V = \left(\sum_{c \in \mathrm{C}(T)} \mathrm{sgn}(c)c\right)\left(\sum_{r \in \mathrm{R}(T)} r\right) \cdot x_T^V$$

It is clear that for any $\sigma \in \mathrm{C}(T)$ that $\sigma\varepsilon_T = \mathrm{sgn}(c)\varepsilon_T$. Therefore, $\varepsilon_T \cdot x_T^V$ is an T-alternating polynomial. By Proposition 4.38 we have that there must exist $g \in \mathfrak{Sym}_T$ such that $F_T^V = g\Delta_T$. ∎

Example 4.40: We compute all higher Specht polynomials for S_3. First, we compute all standard tableaux of size 3 and compute their charge and Young symmetrizer. Here we denote T_r and T_a to be the standard tableaux corresponding to the trivial and alternating representations of S_n respectively. In addition to this, we consider T_1 and T_2 to be the two standard tableaux of shape $(2,1)$. We give the following list of tableaux, their charge, and their corresponding Young symmetrizers.

	Young Tableaux T	**Tableaux Charge $I(T)$**	**Young Symmetrizer ε_T**
T_r	$\boxed{1}\boxed{2}\boxed{3}$	$\boxed{0}\boxed{0}\boxed{0}$	$e + (12) + (13) + (23) + (123) + (132)$
T_1	$\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array}$	$\begin{array}{l}\boxed{0}\boxed{0}\\ \boxed{1}\end{array}$	$(e - (13))(e + (12))$
T_2	$\begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array}$	$\begin{array}{l}\boxed{0}\boxed{1}\\ \boxed{1}\end{array}$	$(e - (12))(e + (13))$
T_a	$\begin{array}{l}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{array}$	$\begin{array}{l}\boxed{0}\\ \boxed{1}\\ \boxed{2}\end{array}$	$e - (12) - (13) - (23) + (123) + (132)$

There is only one copy of the trivial representation and the signed representation of S_3 as sub-representations of $\mathfrak{C}$. The higher Specht polynomials that give a basis for each in $\mathfrak{C}$ are the following:

$$F_{T_r}^{T_r} = \varepsilon_{T_r}(1) = 6 \qquad F_{T_a}^{T_a} = \varepsilon_{T_a}(x_2x_3^2) = (-1)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

There are two copies of $S_{(2,1)}$ in $\mathfrak{C}$. These are given by the following higher Specht polynomial basis:

- $F_{T_1}^{T_1} = \varepsilon_{T_1}(x_3) = 2(x_3 - x_1)$
- $F_{T_1}^{T_2} = \varepsilon_{T_1}(x_2x_3) = x_2(x_3 - x_1)$
- $F_{T_2}^{T_1} = \varepsilon_{T_2}(x_2) = 2(x_2 - x_1)$
- $F_{T_2}^{T_2} = \varepsilon_{T_2}(x_2x_3) = x_3(x_2 - x_1)$

This means that spaces spanned by $\{2(x_3 - x_1), 2(x_2 - x_3)\}$ and $\{x_2(x_3 - x_1), x_3(x_2 - x_1)\}$ are isomorphic to $S_{(2,1)}$ as a S_3-representation. Note that for all higher Specht polynomials we computed were of the form $F_T^V = g\Delta_T$ for some $g \in \mathfrak{Sym}_T$. This is in fact a true statement for all higher Specht polynomials.

Section 4.2: q-Deformation of Higher Specht Polynomial

For this section, we consider $\Bbbk = \mathbb{F}(q)$ for an indeterminate q. Like the previous section, consider $\mathscr{P} = \Bbbk[x_1, \ldots, x_n]$ and $\mathfrak{Sym}$ be the subring of $\mathscr{P}$ of symmetric polynomials. Consider the Hecke algebra of type A over $\mathbb{F}$ given by $H = H_q(S_n)$. The generators of H are given by $\{\mathfrak{h}_i \mid 1 \leqslant i \leqslant n-1\}$ with the following presentation (see Example 2.49)

1. $\mathfrak{h}_i^2 = (q-1)\mathfrak{h}_i + q$ for $1 \leqslant i \leqslant n-1$.
2. $\mathfrak{h}_i\mathfrak{h}_j = \mathfrak{h}_j\mathfrak{h}_i$ for $|i - j| > 1$.
3. $\mathfrak{h}_i\mathfrak{h}_{i+1}\mathfrak{h}_i = \mathfrak{h}_{i+1}\mathfrak{h}_i\mathfrak{h}_{i+1}$ for $1 \leqslant i \leqslant n-2$.

These represent q-deformation of the transpositions $s_1, \ldots, s_{n-1}$ which generate S_n. In Chapter 3 we constructed a faithful representation of $\mathcal{H}$ over $\mathscr{P}$ (subsequently over $\mathfrak{C}$, see Proposition 3.70 and Corollary 3.74). This representation was given by the operators ζ_i

$$\mathfrak{h}_i \mapsto \zeta_i := \delta_i(q-1) + s_i \tag{4.1}$$

This representation is a 1-parameter version of the 2-parameter presentation given by Alain Lascoux in [22, Section 2]. In Chapter 2 we built a q-deformation of the Young symmetrizers which gave us the Specht modules. Recall that for a partition λ we defined two elements of $\mathcal{H}_q(S_n)$ given by (see Definition 2.69)

$$c_\lambda^q = \sum_{c \in \mathrm{C}(T^\lambda)} (-q)^{-l(c)}\tau(c) \qquad r_\lambda^q = \sum_{r \in \mathrm{R}(T_\lambda)} \tau(r) \tag{4.2}$$

For tableaux T and V of the same shape, we define $\pi(T, V) \in S_n$ to be the permutation such that $\pi(T, V) \cdot T = V$ (see Definition 2.32). Using this notation, we defined the Young symmetrizer as the following element of $\mathcal{H}_q(S_n)$ (see Definition 2.77):

$$\varepsilon_T^q := \mathfrak{h}(\pi(T^\lambda, T))c_\lambda^q\mathfrak{h}(\pi(T^\lambda, T))^{-1}\mathfrak{h}(\pi(T_\lambda, T))r_\lambda^q\mathfrak{h}(\pi(T_\lambda, T))^{-1}$$

The q-deformation of the Young symmetrizers were central to our generalization of the Specht modules to $\mathcal{H}_q(S_n)$. These modules are given by

$$S_\lambda^q := \mathcal{H}_q(S_n)\varepsilon_{T^\lambda}^q.$$

To keep the notation simple, denote $\varepsilon_\lambda^q = \varepsilon_{T_\lambda}^q$ and $\mathfrak{b}(T,V) = \mathfrak{b}(\pi(T,V))$. We proved that S_λ^q are irreducible modules of $\mathcal{H}_q(S_n)$ (see Theorem 2.80) with the following basis

$$\{\mathfrak{b}(T^\lambda, T)\varepsilon_\lambda^q \mid T \in \mathrm{ST}(\lambda)\}$$

Definition 4.41: For a given $T, V \in \mathrm{ST}(\lambda)$ we define the q-**higher Specht polynomials** as the following polynomial:

$$\mathfrak{F}_T^V = \mathfrak{b}(T^\lambda, T)\varepsilon_\lambda^q \cdot x_{T^\lambda}^V$$

Proposition 4.42: Specializing $q = 1$ we obtain that $\mathfrak{F}_T^V = F_T^V$

***Proof*:** It is easy to see from Lemma 2.78 that as $q \to 1$ we have that $\varepsilon_\lambda^q \to \varepsilon_\lambda$ and $\mathfrak{b}(T^\lambda, T) \to \pi(T^\lambda, T)$. This means that specializing $q \to 1$ we obtain

$$\mathfrak{F}_T^V \to \pi(T^\lambda, T)\varepsilon_{T^\lambda} \cdot x_{T^\lambda}^T = e_T x_T^V = F_T^V. \qquad \blacksquare$$

Theorem 4.43: The set $\{\mathfrak{F}_T^V \mid T, V \in \mathrm{ST}(\lambda) \text{ for all } \lambda \vdash n\}$ is a basis of $\mathfrak{C}$ as a $\mathbb{F}$-vector space.

***Proof*:** Note that by the Chevalley-Shephard-Todd Theorem 3.5 we have that $\mathfrak{C}$ is a vector space of dimension $n!$. Furthermore by specializing $q \to 1$ we have that

$$\{\mathfrak{F}_T^V \mid T, V \in \mathrm{ST}(\lambda) \text{ for all } \lambda \vdash n\} \to \{F_T^V \mid T, V \in \mathrm{ST}(\lambda) \text{ for all } \lambda \vdash n\}.$$

From Theorem 4.33 we know that the higher Specht polynomials are a basis of over $\mathfrak{C}$. Therefore, by Lemma 2.67 the set $\{\mathfrak{F}_T^V \mid T, V \in \mathrm{ST}(\lambda) \text{ for all } \lambda \vdash n\}$ must be linearly independent of size $n! = \dim \mathfrak{C}$. Therefore, they are a basis of $\mathfrak{C}$. ■

Theorem 4.44: Let $V \in \mathrm{ST}(\lambda)$. Then the space spanned by $\{\mathfrak{F}_T^V \mid T \in \mathrm{ST}(\lambda)\}$ is isomorphic to S_λ^q as an $\mathcal{H}_q(S_n)$ representation.

***Proof*:** Recall that Theorem 2.80 proves that $S_\lambda^q = \mathcal{H}\varepsilon_\lambda^q$ has a basis given by

$$\{\mathfrak{b}(T^\lambda, T)\varepsilon_\lambda^q \mid T \in \mathrm{ST}(\lambda).\}$$

Therefore for any $h \in \mathcal{H}_q(S_n)$ and $T \in \mathrm{ST}(\lambda)$, there exists $c_U \in \mathbb{F}$ for all $U \in \mathrm{ST}(\lambda)$ such that

$$h\mathfrak{F}_T^V = h\mathfrak{b}(T_\lambda, T)\varepsilon_\lambda^q \cdot x_{T_\lambda}^V = \sum_{U \in \mathrm{ST}(\lambda)} c_U \mathfrak{F}_U^V.$$

This shows that $\{\mathfrak{F}_T^V \mid T \in \mathrm{ST}(\lambda)\}$ spans a non-trivial submodule of S_λ^q. This implies that the space it spans is isomorphic to S_λ^q as the Specht modules are irreducible. ■

Both Theorems 4.44 and 4.43 imply that we have indeed generalized the higher Specht polynomials to $\mathcal{H}_q(S_n)$. We now want to obtain a good description of the polynomial $\mathfrak{F}_T^V$ in order to compute them. In Lemma 4.39 we related the Specht polynomials with T-alternating polynomials. In order to obtain a similar description, we must generalize alternating polynomials to the Hecke algebra $\mathcal{H}_q(S_n)$ action on $\mathscr{P}$. Recall that the trivial representation φ and sign representation ψ of $\mathcal{H}_q(S_n)$ are defined in the following way (see Example 2.60);

$$\varphi(\mathfrak{h}_i) = q \qquad \psi(\mathfrak{h}_i) = -1$$

Definition 4.45: A polynomial $f \in \mathscr{P}$ is q-alternating if $\zeta_i f = -f$ for all $1 \leqslant i \leqslant n-1$.

Definition 4.46: The q-Vandermonde determinant is defined as the following polynomial

$$\Delta_q = \prod_{i<j}(x_i - qx_j).$$

Proposition 4.47: Assume that f is q-alternating in $\mathbb{F}[x_1, \ldots, x_n]$. Let $s_i = (i, i+1)$. Then $s_i(f)$ is given by:

$$s_i(f) = f\frac{qx_i - x_{i+1}}{qx_{i+1} - x_i}.$$

***Proof*:** If we assume $\zeta_i(f) = -f$ then we obtain that

$$\frac{x_i f - x_{i+1}s_i(f)}{x_i - x_{i+1}}(q-1) + s_i(f) = -f \Longleftrightarrow [qx_i - x_{i+1}]f = s_i(f)[qx_{i+1} - x_i] \qquad \blacksquare$$

Corollary 4.48: If f is a q-alternating polynomial and $\sigma = s_{k-1}\cdots s_i$ for some $1 \leqslant i < k \leqslant n$ then we have the following formula for $\sigma(f)$.

$$\sigma(f) = f\frac{(qx_i - x_k)(qx_{i+1} - x_k)\cdots(qx_{k-1} - x_k)}{(qx_k - x_{k-1})(qx_k - x_{k-2})\cdots(qx_k - x_i)}$$

***Proof*:** We shall prove this inductively over k. The base case is $k = i+1$ which is done in Proposition 4.47. Thus we assume it works for $k-1$, consider $\sigma' = s_{k-2}\cdots s_i$ thus $\sigma = s_{k-1}\sigma'$. Then via induction we know that

$$\sigma'(f) = f\frac{(qx_i - x_{k-1})(qx_{i+1} - x_{k-1})\cdots(qx_{k-2} - x_{k-1})}{(qx_{k-1} - x_{k-2})(qx_{k-1} - x_{k-3})\cdots(qx_{k-1} - x_i)}$$

We compute a formula for $\sigma(f)$ by applying s_{k-1} to this equation.

$$\begin{aligned}\sigma(f) &= s_{k-1}(f)s_{k-1}\left(\frac{(qx_i - x_{k-1})(qx_{i+1} - x_{k-1})\cdots(qx_{k-2} - x_{k-1})}{(qx_{k-1} - x_{k-2})(qx_{k-1} - x_{k-3})\cdots(qx_{k-1} - x_i)}\right)\\ &= f\left(\frac{qx_{k-1} - x_k}{qx_k - x_{k-1}}\right)\left(\frac{(qx_i - x_k)(qx_{i+1} - x_k)\cdots(qx_{k-2} - x_k)}{(qx_k - x_{k-2})(qx_k - x_{k-3})\cdots(qx_k - x_i)}\right)\end{aligned} \qquad \blacksquare$$

Corollary 4.49: If f is q-alternating, then for $1 \leqslant i < k \leqslant n$ we have that $(x_i - qx_{i+1})$ divides f.

***Proof*:** In Corollary 4.48, we consider $\sigma = \sigma = s_{k-1} \cdots s_i$ which gives us

$$\sigma(f) = f\frac{(qx_i - x_k)(qx_{i+1} - x_k)\cdots(qx_{k-1} - x_k)}{(qx_k - x_{k-1})(qx_k - x_{k-2})\cdots(qx_k - x_i)}.$$

Note however that $\sigma(f)$ is a polynomial, thus all polynomials of the form $(qx_k - x_j)$ in the denominator must divide the nominator. More specifically, this means that $(qx_k - x_i)$ must divide the nominator. However, note that $(qx_k - x_i)$ does not divide any polynomial of the form $(x_j - qx_k)$ therefore $(qx_k - x_i)$ must divide f. ∎

Theorem 4.50: For a given polynomial f, we have that f is q-alternating if and only if f is of the form $f = \Delta^q g$ for a symmetric polynomial g.

***Proof*:** In direction is trivial; if $f = \Delta^q g$ where g is symmetric, then $\tau_i(f) = g\tau_i(\Delta^q) = -g\Delta^q = -f$ and thus f is q-alternating. On the other hand, if f is alternating, then $(x_i - qx_j)$ divides f for all $1 \leqslant i < j \leqslant n$, meaning that Δ^q divides f. Since this has to be true for all q-alternating polynomials, then $f = \Delta^q g$ for some polynomial g. It remains to show that g is symmetric. Consider another q-alternating polynomial $p \neq 0$, and consider the ratio f/p. Note that if we apply s_i to this ratio, we can use Coronary 4.49 in the following way:

$$s_i(f/p) = f\frac{qx_i - x_{i+1}}{qx_{i+1} - x_i}/p\frac{qx_i - x_{i+1}}{qx_{i+1} - x_i} = f/p$$

The ratio f/p is a symmetric function. In fact the ratio of any two q-alternating function is a symmetric function. Thus of we consider $p = \Delta^q$, since $f = \Delta^q g$ then the ratio $f/p = g$ this shows that g is in fact symmetric. ∎

Definition 4.51: Let $\lambda \vdash n$. Then we define $f \in \mathscr{P}$ to be **λ-symmetric** if for all $\sigma \in \mathrm{C}(T^\lambda)$ we have $\sigma f = f$. We will denote the subring of λ-symmetric polynomials by $\mathfrak{Sym}_\lambda$

Let $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ be the transpositions generating S_n. By the way that T^λ is defined, it is clear that $\mathrm{C}(T^\lambda)$ is generated by a subset of $\mathcal{S}$. Since the operators ζ_i are $\mathfrak{Sym}$-linear, they must also be $\mathfrak{Sym}_\lambda$-linear.

Definition 4.52: Let $\lambda \vdash n$ then $f \in \mathscr{P}$ is **(q, λ)-alternating** if for all $s_i \in \mathrm{C}(T^\lambda)$, the corresponding Hecke algebra element satisfies $\mathfrak{h}_i \cdot f = \zeta_i(f) = -f$

Definition 4.53: Let $C_1, \ldots C_k$ be the set of numbers appearing in the columns of T^λ. We define the (q,λ)**-Vandermonde polynomial** to be

$$\Delta_\lambda^q = \prod_{i=1}^{k} \prod_{\substack{j,l\in C_i \\ j<l}} (x_j - qx_l).$$

Proposition 4.54: If $f \in \mathscr{P}$ is (q,λ)-alternating, then there exists a polynomial $g \in \mathfrak{Sym}_\lambda$ such that $f = g\Delta_\lambda^q$.

***Proof*:** This is almost immediate by applying Theorem 4.50 and restricting the group to $\mathrm{C}(T^\lambda)$. Since $\mathrm{C}(T^\lambda)$ is parabolic, then we just compute Δ_λ^q on each set of variables appearing in the columns of T^λ. ■

Lemma 4.55: Given $V \in \mathrm{ST}(\lambda)$ there exists $g \in \mathfrak{Sym}_\lambda$ such that $\mathfrak{F}_{T^\lambda}^V = g\Delta_\lambda^q$.

***Proof*:** Because $\mathfrak{h}(T^\lambda, T^\lambda) = 1$, then $\varepsilon_{T^\lambda} = c_\lambda^q \mathfrak{h}(T_\lambda, T^\lambda) r_\lambda^q \mathfrak{h}(T_\lambda, T^\lambda)^{-1}$. Recall that for any permutation $p \in \mathrm{C}(T^\lambda)$ we have $\mathfrak{h}(p)c_\lambda^q = (-1)^{l(p)} c_\lambda^q$. This gives the following

$$p\mathfrak{F}_{T^\lambda}^V = pc_\lambda^q \mathfrak{h}(T_\lambda, T^\lambda) r_\lambda^q \mathfrak{h}(T_\lambda, T^\lambda)^{-1} x_{T^\lambda}^V = (-1)^{l(p)} \varepsilon_{T^\lambda} x_{T^\lambda}^V.$$

This shows that $F_{T^\lambda}^V$ is a (q,λ)-alternating polynomial. Therefore, there must exist $g \in \mathfrak{Sym}_\lambda$ such that $F_{T^\lambda}^V = g\Delta_\lambda^q$. ■

Example 4.56: Given $\mathcal{H} = \mathcal{H}_q(S_3)$ we compute all possible higher Specht polynomials. From the description of the higher Specht polynomials, for a given $V \in \mathrm{ST}(\lambda)$ all higher Specht polynomials $\mathfrak{F}_T^V$ are given as the following

$$\mathfrak{F}_T^V = \mathfrak{h}(T^\lambda, T)\mathfrak{F}_{T^\lambda}^V.$$

Therefore for each partition, we only need to compute ε_λ^q and $\mathfrak{F}_{T^\lambda}^V$. Then for each tableau T we compute $\mathfrak{h}(T^\lambda, T)$ and let it act on $\mathfrak{F}_{T^\lambda}^V$. Since $\mathcal{H}_q(S_3)$ has two one-dimensional representations, we compute their higher Specht polynomials first. The partition giving the trivial representation is $\lambda_t = (3)$ and the partition giving the signed representation is $\lambda_a = (1,1,1)$.

$$\varepsilon_{\lambda_t} = 1 + \mathfrak{h}_1 + \mathfrak{h}_2 + \mathfrak{h}_1\mathfrak{h}_2 + \mathfrak{h}_2\mathfrak{h}_1 + \mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_1$$
$$\varepsilon_{\lambda_a} = 1 - \frac{1}{q}[\mathfrak{h}_1 + \mathfrak{h}_2] + \frac{1}{q^2}[\mathfrak{h}_1\mathfrak{h}_2 + \mathfrak{h}_2\mathfrak{h}_1] - \frac{1}{q^3}\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_1$$

Let T_t and T_a be the only standard tableau of shape λ_t and λ_a respectively. Then we compute the higher Specht polynomials as follows:

$$\mathfrak{F}_{T_t}^{T_t} = \varepsilon_{\lambda_t}(1) = 1 + 2q + 2q^2 + q^3 \qquad \mathfrak{F}_{T_a}^{T_a} = \frac{-1}{q^3}(x_1 - qx_2)(x_2 - qx_3)(x_2 - qx_3)$$

Next let $\lambda = (2,1)$ and consider $T_1 = \begin{smallmatrix}1&2\\3&\end{smallmatrix}$ and $T_2 = \begin{smallmatrix}1&3\\2&\end{smallmatrix}$. It is easy to see that $T_1 = T_\lambda$ and $T_2 = T^\lambda$.

Then $\pi(T^{\lambda}, T_{\lambda}) = (2,3)$ then $\mathfrak{h}(T^{\lambda}, T_{\lambda}) = \mathfrak{h}(s_2) = \mathfrak{h}_2$

$$\varepsilon_{T^{\lambda}} = \varepsilon_{T_2} = \left(1 - \frac{1}{q}\mathfrak{h}_1\right)\mathfrak{h}_2\left(1 + \mathfrak{h}_2\right)\mathfrak{h}_2^{-1}.$$

Using this we compute the polynomials $\mathfrak{F}_{T^{\lambda}}^{T_1}$ and $\mathfrak{F}_{T^{\lambda}}^{T_2}$. They will generate the different copies of S_{λ}^{q}.

$$\mathfrak{F}_{T^{\lambda}}^{T_1} = \varepsilon_{T^{\lambda}} x_{T^{\lambda}}^{T_1} = \frac{1-q}{q}(x_1 - qx_2) \qquad \mathfrak{F}_{T^{\lambda}}^{T_2} = \varepsilon_{T^{\lambda}} x_{T^{\lambda}}^{T_1} = \frac{-1}{q}x_3(x_1 - qx_2)$$

Note that for $V \in \mathrm{ST}(\lambda)$ a basis for the Specht module S_{λ}^{q} in $\mathfrak{C}$ is given by $\{\mathfrak{F}_{T^{\lambda}}^{V}, \mathfrak{h}(T_2, T_1)\mathfrak{F}_{T^{\lambda}}^{V}\}$. We may use them to compute the remainder of the higher Specht polynomials.

- $\mathfrak{F}_{T_2}^{T_1} = \frac{1-q}{q}(x_1 - qx_2)$
- $\mathfrak{F}_{T_2}^{T_2} = \frac{-1}{q}x_3(x_1 - qx_2)$
- $\mathfrak{F}_{T_1}^{T_1} = (1-q)(x_1 + (1-q)x_2 - qx_3)$
- $\mathfrak{F}_{T_1}^{T_2} = \frac{-1}{q}x_2(x_1 - qx_3)$

Note that not always does the higher Specht polynomials factor into q-Vandermonde determinants. In the $q = 1$ case, this is always true from Lemma 4.39. This is a property that has been lost in this generalization of the higher Specht polynomials to $\mathcal{H}_q(S_n)$. As we see from the computed example, $\mathcal{F}_{T_1}^{T_1}$ does not have a Δ_{λ}^{q} factor. However, as we see from the computations above, all higher Specht polynomials are obtainable by applying an appropriate Hecke algebra element to the polynomial $\mathfrak{F}_{T^{\lambda}}^{V}$ which does indeed always factor.

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