Polynomial-Like Behaviour of the Faithful Dimension of p-Groups

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Abstract

The *faithful dimension* of a finite group G over $\mathbb{C}$, denoted by $m_{\text{faithful}}(G)$, is defined to be the smallest integer m such that G can be embedded in $GL_m(\mathbb{C})$. We are interested in computing the faithful dimension of nilpotent p-groups of the form $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} R)$, where $\mathfrak{f}_{n,c}$ is the free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators, and R is a finite truncated valuation ring. In the special case of R being a finite field with a sufficiently large characteristic, we obtain a sharp result for the faithful dimension associated with free nilpotent $\mathbb{Z}$-Lie algebras of class $c = 4$. For a general finite truncated valuation ring R, we obtain asymptotically sharp upper and lower bounds. Our lower bound improves previously known results. Additionally, when $R = \mathbb{F}_q$ and $c = 5$ we compute an upper bound for the faithful dimension of magnitude n^5q^4, and a lower bound of magnitude q^2.

Dedications

To my family, husband and son Abdulaziz.

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Chapter 1

Introduction

Character theory is an essential component of finite group representations, with applications in various areas of mathematics such as number theory, combinatorics, analysis, and quantum computation. In the case of nilpotent p-groups, characters of irreducible representations can be obtained using *Kirillov's orbit method* [Kir62]. Originally, Kirillov's approach was developed for connected and simply connected Lie groups. Since then, it has been extended to different classes of nilpotent Lie groups, including p-groups [BS08, Kaz77, How77]. The main ingredient of Kirillov's method is the concept of coadjoint orbits, which arises from the action of the group on the dual of its Lie algebra. In the case of nilpotent p-groups, the associated Lie algebras are obtained by the *Lazard correspondence* [Khu98].

In this thesis we will be working with nilpotent p-groups of the form $G := \exp(\mathfrak{g} \otimes_{\mathbb{Z}} R)$, where $\mathfrak{g}$ is a finitely generated nilpotent $\mathbb{Z}$-Lie algebra, and R is a finite local ring with a nilpotent and principal maximal ideal. This ring is referred to as a *finite truncated valuation ring* [BMKRS23, McL73]. In particular, we are interested in determining the *faithful dimension of* G over a field K, denoted by $m_{\mathrm{faithful},K}(G)$, which is defined to be the smallest integer n such that G embeds into $GL_n(K)$. The faithful dimension of p-groups is closely related to the theory of essential dimension of algebraic groups [Rei21]. In fact, Karpenko and Merkurjev [KM08] proved that the essential dimension of a p-group G over a field K containing a primitive p^{th} root of unity is equal to $m_{\text{faithful},K}(G)$.

Using Kirillov's orbit method, Bardestani et al. [BMKRS23] expressed the faithful dimension of G over $\mathbb{C}$ as a minimization problem that depends on the concept of the *commutator matrix*, which is a skew-symmetric matrix of linear forms associated to the nilpotent $\mathbb{Z}$-Lie algebra $\mathfrak{g}$ [OV15]. In the case where R is a finite field of order p^f, there exists a partition $\{\mathscr{P}_1, \ldots, \mathscr{P}_r\}$ of sufficiently large prime numbers into Frobenius sets, as defined in Definition 3.1.4, and a partition $\{\mathscr{F}_1, \ldots, \mathscr{F}_s\}$ of natural numbers into sets that belong to the Boolean algebra generated by arithmetic progressions, and polynomials $g_{ij}(T)$ with non-negative integer coefficients for all $1 \leq i \leq r$ and $1 \leq j \leq s$, such that if the pair (p, f) belongs to some $\mathscr{P}_i \times \mathscr{F}_j$, then the faithful dimension is given by $m_{\text{faithful},\mathbb{C}}(G) = f g_{ij}(p^f)$. This polynomiality property has not yet been extended to any truncated valuation ring R, but it has been proven in the special case of pattern groups [BMKRS23, Thm. 1.5].

The minimization theorem presented in [BMKRS23] will be our main tool in this thesis for the computations of the faithful dimension of various classes of nilpotent p-groups. We consider first the family of groups given by $\exp(\mathfrak{g}_{\prec} \otimes_{\mathbb{Z}} R)$, where $\mathfrak{g}_{\prec}$ is a pattern Lie algebra associated to the partial order $\prec$ [BMKS19], and we provide an explicit formula for the faithful dimension of groups associated to nilradicals of parabolic subalgebras of the general linear Lie algebra $\mathfrak{gl}(n)$. Our main result concerns groups of the form $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} R)$, where $\mathfrak{f}_{n,c}$ is a free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators. The faithful dimension for nilpotency classes $c = 2$ and $c = 3$ has been computed when $R = \mathbb{F}_{p^f}$ by Bardestani et al. in [BMKS19]. Furthermore, for the case when $f = 1$ and $c = 4$, the faithful dimension was determined by Tielker [Tie18]. We systematically approach the problem of computing the minimal faithful dimension of $\mathfrak{f}_{n,c}$ by providing an explicit description of the commutator matrix in terms of Hall basis elements [Ser92]. We use programming tools to efficiently handle the tedious computations involved. For nilpotency class $c = 4$, we are able to extend the work of Tielker [Tie18] to any finite field $R = \mathbb{F}_{p^f}$. This is established in Theorem 6.7.1. To this end, we develop algorithms to describe the commutator matrix that can be adapted to higher nilpotency classes. Our next main result, stated in Theorem 6.8.12, proves explicit upper and lower bounds when R is a general finite truncated valuation ring and $c = 4$. This result improves the lower bound obtained in [BMKRS23].

Finally, in Theorems 8.1.3 and 8.1.5 we establish concrete upper and lower bounds for the faithful dimension for nilpotency class $c = 5$ and $R = \mathbb{F}_{p^f}$. To the best of our knowledge this is the first time nilpotency class $c = 5$ has been studied in a quantitative way.

The work initiated in this thesis opens the door towards further interesting directions to explore in the future. A natural next step would be to improve the bounds for the faithful dimension of $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ for nilpotency class $c = 5$, or even better, establish a precise result for the faithful dimension given that an explicit description of the commutator matrix has been obtained in this case. As observed in nilpotency class $c = 5$, the computations involved grow enormously compared to the case of $c = 4$. Hence, for a general c, new approaches are needed to bypass these issues. Another problem to consider would be to study the faithful dimension of $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} R)$ over a general finite truncated valuation ring R. A final direction we suggest would be to look into the faithful dimension associated with other finitely generated nilpotent $\mathbb{Z}$-Lie algebras.

Chapter 2

Background

Our starting point is a review of an approach to the representation theory of finite p-groups that is inspired by techniques from Lie theory and symplectic geometry. Given a finite nilpotent $\mathbb{Z}$-Lie algebra $\mathfrak{g}$ with a cardinality that is a prime power p^n, where p is strictly larger than the nilpotency of $\mathfrak{g}$, we first associate a p-group $\exp(\mathfrak{g})$ to $\mathfrak{g}$. Next, we provide a brief overview of Kirillov's orbit method. Finally, we introduce the *commutator matrix* of a $\mathbb{Z}$-Lie algebra. This tool is of great importance in this thesis and we will heavily rely on it in the proof of our results.

2.1 Very Basic Lie Theory

We begin this section with the following basic definition from group theory.

Definition 2.1.1. [DF04, Ch. 6] The *upper central series* of a group G is the following chain of subgroups

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

where $Z_0(G) = 1$, $Z_1(G) = Z(G)$ and $Z_{i+1}(G)$ is defined inductively by

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

A group G is *nilpotent* if $Z_c(G) = G$ for some $c \in \mathbb{N}$. The smallest c where $Z_c(G) = G$ is the called the *nilpotence class* of G.

It will turn out later in this section that there is a strong connection between finite p-groups, which are automatically nilpotent, and finite nilpotent $\mathbb{Z}$-Lie algebras with cardinality a power of p. Therefore, for the sake of completeness, we provide the definition of a Lie algebra over a general commutative unital ring.

Let $\mathbb{K}$ be a commutative ring with a unit element. Recall that a $\mathbb{K}$-*algebra* is a $\mathbb{K}$-module with a $\mathbb{K}$-bilinear binary operation.

Definition 2.1.2. [Ser92, Ch.1] A $\mathbb{K}$-*Lie algebra* is a $\mathbb{K}$-algebra $\mathfrak{g}$ with the binary operation expressed by a *bracket* (or *commutator*),

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y], \end{aligned}$$

that satisfies the following axioms:

i. $[x, x] = 0$ for all $x \in \mathfrak{g}$.

ii. $[x, [y, z]] + [y, [z, x]] + [x, [y, z]] = 0$ for $x, y, z \in \mathfrak{g}$. This is known as the *Jacobi identity*.

Remark 2.1.3. Throughout this thesis $\mathfrak{g}$ will denote a $\mathbb{Z}$-Lie algebra. In the literature, Lie algebras over $\mathbb{Z}$ are sometimes called *Lie rings*.

Definition 2.1.4. [BMKS19, Sec. 2] The *lower central series* of $\mathfrak{g}$ is defined inductively as $\mathfrak{g}^1 = \mathfrak{g}$, and $\mathfrak{g}^{\ell+1} = [\mathfrak{g}, \mathfrak{g}^\ell]$ for $\ell \geq 1$. The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}^\ell = 0$ for some $\ell \in \mathbb{N}$. Moreover, $\mathfrak{g}$ is called *c-step nilpotent* or *nilpotent of class c* if c is the smallest integer such that $\mathfrak{g}^{c+1} = 0$.

Remark 2.1.5. The commutator subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{g}' := \mathfrak{g}^2$.

Definition 2.1.6. The *upper central series* of the $\mathbb{Z}$-Lie algebra $\mathfrak{g}$ is given by

$$Z_0(\mathfrak{g}) \leq Z_1(\mathfrak{g}) \leq Z_2(\mathfrak{g}) \leq \dots$$

where $Z_0(\mathfrak{g}) = 0$, $Z_1(\mathfrak{g}) = Z(\mathfrak{g}) := \{x \in \mathfrak{g} : [x, y] = 0 \ \forall y \in \mathfrak{g}\}$ and $Z_{i+1}(\mathfrak{g})$ is defined inductively by

$$Z_{i+1}(\mathfrak{g})/Z_i(\mathfrak{g}) = Z(\mathfrak{g}/Z_i(\mathfrak{g})).$$

Lemma 2.1.7. *If $\mathfrak{g}$ is nilpotent of class c, then both the upper and lower central series have equal lengths. In particular, $\mathfrak{g}^{c+1-i} \leq Z_i(\mathfrak{g})$ for $i = 0, 1, \dots, c$.*

Proof: Note that $x \in Z_k(\mathfrak{g})$ if and only if $[x, y] \in Z_{k-1}(\mathfrak{g})$ for all $y \in \mathfrak{g}$. By this fact and induction on i we obatin the result. ∎

Proposition 2.1.8. *$\mathfrak{g}$ is nilpotent of class c if and only if $Z_c(\mathfrak{g}) = \mathfrak{g}$.*

Proof: Suppose that $\mathfrak{g}$ is nilpotent of class c. If we set $i = c$ in Lemma 2.1.7, we get that $\mathfrak{g} = Z_c(\mathfrak{g})$. Now, suppose that $Z_c(\mathfrak{g}) = \mathfrak{g}$. From Definition 2.1.6 one can show that if $x \in Z_c(\mathfrak{g})$, then

$$[y_1, [y_2, [\dots, [y_{c-1}, [x, y_c]] \dots]] = 0,$$

for any $y_1, y_2, \dots, y_c \in \mathfrak{g}$. Since $Z_c(\mathfrak{g}) = \mathfrak{g}$, we have $\mathfrak{g}^{c+1} = 0$. ∎

2.2 The Formal BCH Product

In this section we construct a group operation called the *Baker-Campbell-Hausdorff* formula (BCH) on a free nilpotent Lie algebra, which will be used in the Lazard correspondence between finite p-groups and a well-defined class of finite Lie rings. In the construction of the BCH formula we use the concept of a free Lie algebra. We remark that in this thesis we need to work with free Lie algebras from two different viewpoints: as a (Lie)-subalgebra of a free *associative* algebra, and as a quotient of a free *nonassociative* algebra. We recall the first viewpoint in the rest of this section. The second viewpoint will be deferred to Chapter 4.

This section follows [Khu98, Sec. 5, Sec. 9]. Let $\mathcal{A}$ be a free non-unital associative $\mathbb{Q}$-algebra on infinitely many generators $x_1, x_2, \ldots$. Hence, the basis of $\mathcal{A}$ consist of all monomials $x_{i_1}x_{i_2}\ldots x_{i_k}$ of degrees $k \in \mathbb{N}$, and the multiplication is juxtaposition:

$$(x_{i_1}x_{i_2}\ldots x_{i_k})(x_{j_1}x_{j_2}\ldots x_{j_\ell}) = x_{i_1}x_{i_2}\ldots x_{i_k}x_{j_1}x_{j_2}\ldots x_{j_\ell}.$$

Also the $\mathbb{Q}$-algebra $\mathcal{A}$ is a graded algebra where $\mathcal{A} = \bigoplus_{n\geq 1}\mathcal{A}_n$, and $\mathcal{A}_n$ is the $\mathbb{Q}$-linear span of the monomials of degree n. The bracket $[x, y] = xy - yx$ defines a $\mathbb{Q}$-Lie algebra structure on the additive group of $\mathcal{A}$. This $\mathbb{Q}$-Lie algebra will be denoted by $\mathcal{A}^{(-)}$. Let $\mathcal{L}$ be the $\mathbb{Z}$-Lie algebra generated by the x_i's in $\mathcal{A}^{(-)}$, i.e. the additive group of $\mathcal{L}$ is generated by the x_i's and their commutators of arbitrary length. The following result is a standard construction of free Lie algebras.

Theorem 2.2.1. [Khu98, Thm. 5.39 (b)] *$\mathcal{L}$ is a free $\mathbb{Z}$-Lie algebra on the free generators $x_1, x_2, x_3, \ldots$.*

Let $\mathbb{Q}\mathcal{L} = \{r\ell : r \in \mathbb{Q}, \ell \in \mathcal{L}\}$ be the free $\mathbb{Q}$-Lie algebra generated by the x_i's. Note that $\mathbb{Q}\mathcal{L} \neq \mathcal{A}^{(-)}$ since for example $x_1x_2 \in \mathcal{A}^{(-)}$ but $x_1x_2 \notin \mathcal{L}$.

Let $\mathcal{A}^{c+1} := \bigoplus_{i \geq c+1} \mathcal{A}_i$. Then $A := \mathcal{A}/\mathcal{A}^{c+1}$ is a free nilpotent associative $\mathbb{Q}$-algebra of nilpotency class c with free generators $x_1, x_2, \ldots$. The algebra A has a natural basis consisting of all monomials $x_{i_1} x_{i_2} \ldots x_{i_k}$ where $k \leq c$.

Lemma 2.2.2. [Khu98] *$\mathcal{L}^{c+1} = \mathcal{L} \cap \mathcal{A}^{c+1}$, where $\mathcal{L}^{c+1} = [\mathcal{L}, \mathcal{L}^c]$.*

Proof: Let $w_1, w_2 \in \mathcal{A}$ such that $w_1 \in \mathcal{A}^{c_1}, w_2 \in \mathcal{A}^{c_2}$, for some $c_1, c_2 \in \mathbb{N}$. Since $\mathcal{A}$ is graded, the commutator $[w_1, w_2] \in \mathcal{A}^{c_1+c_2}$. Also, note that $\mathcal{L}^k \subseteq \mathcal{A}^k$, for all $k \geq 2$. This can be shown by induction. First, note that for any generators $x_i, x_j \in \mathcal{L}$, $[x_i, x_j] = x_i x_j - x_j x_i \in \mathcal{A}^2$. Hence, $\mathcal{L}^2 \subseteq \mathcal{A}^2$. Now, suppose that $x_i \in \mathcal{L}, y \in \mathcal{L}^k$. By the induction hypothesis, $y \in \mathcal{A}^k$. So, $[x_i, y] \in \mathcal{A}^{k+1}$. As a result, $\mathcal{L}^{k+1} \subseteq \mathcal{A}^{k+1}$.

From the above observation, it is clear that $\mathcal{L}^{c+1} \subseteq \mathcal{L} \cap \mathcal{A}^{c+1}$. To prove the other inclusion, let $z \in \mathcal{L} \cap \mathcal{A}^{c+1}$. Since $z \in \mathcal{L}$, by [Khu98, Lemma 5.6] we can write $z = \sum_i r_i [u_i, v_i]$, where $r_i \in \mathbb{Z}$, u_i is one of the generators x_j and $v_i \in \mathcal{L}^{t_i}$ for some $t_i \geq 1$. Since $[\mathcal{L}, \mathcal{L}^{t_i}] = \mathcal{L}^{t_i+1} \subseteq \mathcal{A}^{t_i+1}$ for all i and $z \in \mathcal{A}^{c+1}$, we must have $t_i = c$ for all i. As a result, $z \in \mathcal{L}^{c+1}$. Hence, $\mathcal{L} \cap \mathcal{A}^{c+1} \subseteq \mathcal{L}^{c+1}$. ∎

Let $L := \mathcal{L}/\mathcal{L}^{c+1} = \mathcal{L}/(\mathcal{L} \cap \mathcal{A}^{c+1})$.

Proposition 2.2.3. *L is a free nilpotent $\mathbb{Z}$-Lie algebra of class c with free generators x_i, where $i = 1, 2, 3, \ldots$. Also, $\mathbb{Q}L = \{r\ell : r \in \mathbb{Q}, \ell \in L\}$ is a free nilpotent $\mathbb{Q}$-Lie algebra.*

Proof: This follows from universality of $\mathcal{L}$, which is what Theorem 2.2.1 asserts. ∎

Now we will construct a free nilpotent group within A. We start by adjoining an outer unity 1 to A by forming the associative $\mathbb{Q}$-algebra $\tilde{A} = \mathbb{Q} \oplus A$, with multiplication given by:

$$(r, a)(r', a') = (rr', aa' + ra' + r'a).$$

Let $1+a := (1,a)$, and consider the group $1+A = \{1+a : a \in A\} \subseteq \tilde{A}$, where the identity element is 1, and $1 - a + a^2 - a^3 + \cdots + (-1)^c a^c$ is the inverse of $1+a$. We define the exponential and logarithm maps as follows:

$$\begin{aligned} \exp : A \to 1+A, \qquad & a \mapsto 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots + \frac{a^c}{c!} = e^a, \\ \log : 1+A \to A, \quad & 1+a \mapsto a - \frac{a^2}{2} + \frac{a^3}{3} - \cdots + (-1)^{c-1}\frac{a^c}{c} = \log(1+a) \end{aligned}$$

By direct calculations we can show that $\log(e^a) = a$ and $e^{\log(1+a)} = 1+a$. Since these maps are mutual inverses, we have a bijection between A and $1+A$. Hence every element of $1+A$ is expressible as e^a for some $a \in A$, with its inverse being e^{-a}.

For $x, y \in A$, let $H(x,y) \in A$ be defined by $e^{H(x,y)} = e^x e^y$, in other words $H(x,y) = \log(e^x e^y)$.

Theorem 2.2.4. [Khu98, Thm. 9.11] *Let $x, y \in \{x_1, x_2, \dots\}$, then $H(x,y) \in \mathbb{Q}L$. In particular,*

$$H(x,y) = \sum_{n=1}^{c} H_n(x,y),$$

where each $H_n(x,y)$ is a linear combination of brackets of length n in x and y.

More specifically, for generators x and y, we have

$$H(x,y) = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{(a_1,b_1),\dots,(a_n,b_n) \\ a_j+b_j \geq 1}} \frac{\left(\sum_{1\leq i\leq n} a_i + b_i\right)^{-1}}{a_1!b_1!\dots a_n!b_n!}[x^{\mathbf{a}}y^{\mathbf{b}}], \tag{2.2.1}$$

where $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are n-tuples of non-negative integers, and the left-aligned bracket above is defined in the following manner:

$$[x^{\mathbf{a}}y^{\mathbf{b}}] := [\underbrace{x, [x, \dots, [x,}_{a_1} \underbrace{[y, [y, \dots, [y,}_{b_1} \dots, \underbrace{[x, [x, \dots, [x,}_{a_n} \underbrace{[y, [y, \dots, [y, y}_{b_n}]] \dots]. \tag{2.2.2}$$

Observe that if $b_n > 1$, or if $b_n = 0$ and $a_n > 1$, the corresponding term in Equation

(2.2.1) vanishes. Also, note that any term which is a bracket of length at least 2 has at least one x and one y. The expression in Equation (2.2.1) is called the *Baker-Campbell-Hausdorff* formula (BCH).

Remark 2.2.5. Since A is nilpotent of class c, any bracket of the form in Equation (2.2.2) with length at least $c+1$ becomes zero. As a result, the sum in the Baker-Campbell-Hausdorff formula $H(x,y)$ in Equation (2.2.1) is finite.

Corollary 2.2.6. *For any $a, b \in A$ we have:*

i. $H(a, H(b,c)) = H(H(a,b),c)$.

ii. $H(a,b) = \sum_{n=1}^{c} H_n(a,b) \in A$.

Proof: It is straightforward to check part *i.* as follows:

$$\begin{aligned} H(a, H(b,c)) = \log\left(e^a e^{H(b,c)}\right) &= \log\left(e^a e^b e^c\right) \\ &= \log\left(e^{H(a,b)} e^c\right) = H(H(a,b),c). \end{aligned}$$

From functoriality of exponential and logarithm, it follows that for any algebra morphism $\phi : A \to A$ we have

$$H(\phi(x), \phi(y)) = \phi(H(x,y)) = \sum_{n=1}^{c} H_n(\phi(x), \phi(y)),$$

where x, y as in Theorem 2.2.4. To prove part *ii.*, we pick $\phi(x) = a$ and $\phi(y) = b$. ∎

Remark 2.2.7. Note that in Equation (2.2.1) no prime divisor of the denominators exceeds the nilpotency class c. More specifically, if we set

$$\mathbb{Z}_{\leq c} := \left\{ \frac{m}{n} \in \mathbb{Q} : (m,n) = 1 \text{ , every prime divisor } q \text{ of } n \text{ satisfies } q \leq c \right\},$$

then $H(x,y) \in \mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} L$ for any generators x, y.

We define an action of $1+A$ on A as follows. Let $g \in 1+A$, and $b \in A$. Recall that each element of $1+A$ can be expressed as e^a for some $a \in A$, so we can write $g = e^a$. We set

$$g \cdot b := e^a b e^{-a}.$$

It is straightforward to check that $\exp(g \cdot b) = e^a e^b e^{-a}$. Also, note that this action can be expressed in terms of the BCH formula as follows:

$$g \cdot b = e^a b e^{-a} = \log\big(\exp\big(e^a b e^{-a}\big)\big) = \log\big(e^a e^b e^{-a}\big) = H(H(a,b), -a). \tag{2.2.3}$$

Thus, $g \cdot b \in \mathbb{Q}L$ when a, b are generators. In this case the bracket expression in Equation (2.2.3) can be simplified to the following form.

Proposition 2.2.8. *Let a, b be generators of A and set $g = e^a \in 1+A$. Then the following hold in $\mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} L$:*

$$g \cdot b = \sum_{m=0}^{c-1} \frac{1}{m!} ad_a^m(b),$$

where $ad_a^m(b) = [\underbrace{a, [a, \ldots, [a}_{m}, b]] \ldots]$.

Proof: For any generator $a \in A$, we define the operator of left multiplication by a as $L_a(x) := ax$, for $x \in A$. Similarly, the operator of right multiplication by a is given by $R_a(x) := xa$, for $x \in A$. It is clear that $L_a R_a = R_a L_a$, and $L_a^c = R_a^c = 0$. Also, note that $\mathrm{ad}_a(x) = [a, x] = (L_a + R_{-a})(x)$. Then we have

$$\begin{aligned} g \cdot b = e^a b e^{-a} = \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!n!} a^m b (-a)^n &= \sum_{m \geq 0} \sum_{n \geq 0} \frac{1}{m!n!} L_a^m R_{-a}^n (b) \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} L_a^{k-n} R_{-a}^n (b) \end{aligned}$$

$$= \sum_{k\geq 0} \frac{1}{k!}(L_a + R_{-a})^k(b) = \sum_{k\geq 0} \frac{1}{k!}\mathrm{ad}_a^k(b).$$

Since $\mathbb{Q}L$ is nilpotent of class c, we obtain $g \cdot b = \sum_{k=0}^{c-1} \frac{1}{k!}\mathrm{ad}_a^k(b)$. ∎

2.3 Lazard Correspondence

Suppose that $\mathfrak{g}$ is a finite $\mathbb{Z}$-Lie algebra with cardinality a power of p. Moreover, assume that $\mathfrak{g}$ is nilpotent of class $c < p$. We set

$$\exp(\mathfrak{g}) := (\mathfrak{g}, *),$$

to be the finite p-group obtained by equipping $\mathfrak{g}$ with the group operation induced by the BCH formula in Equation (2.2.1). In particular, for $x, y \in \mathfrak{g}$,

$$x * y = H(x, y) = \sum_{k=1}^{c} H_k(x, y),$$

where $H_k(x, y)$ is the linear combination of brackets of length k from Equation (2.2.1). It is straightforward to check that the identity element is 0 and $x^{-1} = -x$ for any $x \in \mathfrak{g}$. The associativity of this group operation follows from Corollary 2.2.6. The key point is that the BCH formula is well-defined in $\mathfrak{g}$ because its denominators are invertible mod p, as noted in Remark 2.2.7. For nilpotency class $c = 2$ and $p \geq 3$, the group operation $*$ is given by

$$x * y = x + y + \frac{1}{2}[x, y].$$

Observe that $\frac{1}{2}[x, y] = k[x, y] \in \mathfrak{g}$ for some value of k. Indeed, since $(2, p) = 1$ for $p \geq 3$, there exists a unique value of k such that $2k \equiv 1 \pmod{p^d}$, where p^d is divisible by the order of $[x, y]$ in the additive structure of $\mathfrak{g}$. Another example of the BCH formula is for nilpotency class $c = 3$ and $p \geq 5$, as shown below:

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]].$$

Corollary 2.3.1. *For $x, y \in \mathfrak{g}$, $x * y * x^{-1} = \sum_{m=0}^{c-1} \frac{1}{m!} ad_x^m(y)$.*

Proof: First, note that

$$x * y * x^{-1} = H(x * y, x^{-1}) = H(H(x, y), x^{-1}) = H(H(x, y), -x). \tag{2.3.1}$$

From Equation (2.2.3), we see that this corollary is equivalent to Proposition 2.2.8 but in the setting of the finite nilpotent $\mathbb{Z}$-Lie algebra $\mathfrak{g}$. By the universal property of the free nilpotent $\mathbb{Z}$-Lie algebra L on the set of generators $X = \{x_1, x_2, \dots\}$ [Khu98, Sec. 1.3], the identity in Proposition 2.2.8 holds in the finite nilpotent $\mathbb{Z}$-Lie algebra $\mathfrak{g}$. More specifically, given any map $\phi : X \to \mathfrak{g}$, there exists a unique homomorphism $\psi : L \to \mathfrak{g}$ extending ϕ. We can extend the scalars in both L and $\mathfrak{g}$ to include the ring $\mathbb{Z}_{\leq c}$, and obtain a homomorphism extending ψ given by:

$$\begin{aligned} \chi : \mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} L &\to \mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} \mathfrak{g} \\ \tfrac{m}{n} \otimes x &\mapsto \tfrac{m}{n} \otimes \psi(x). \end{aligned}$$

Since $\mathfrak{g}$ is finite with cardinality a power of p, we can show that $\mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} \mathfrak{g} \cong \mathfrak{g}$ with the isomorphism being the map $\mathfrak{g} \to \mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} \mathfrak{g}$, which sends $x \mapsto 1 \otimes x$. Surjectivity of the latter map can be checked as follows: Let $\frac{m}{n} \otimes x \in \mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} \mathfrak{g}$. Since $(n, p) = 1$, then there exists a value of $k \in \mathbb{Z}$ such that $nk \equiv 1 \pmod{|\mathfrak{g}|}$. As a result,

$$\frac{m}{n} \otimes x = \frac{m}{n} \otimes nkx = 1 \otimes mkx.$$

Hence, the map is surjective, and as a result $\mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} \mathfrak{g} \cong \mathfrak{g}$.

To be precise, let $x, y \in \mathfrak{g}$ such that $\phi(x_1) = x$, $\phi(x_2) = y$ and $\phi(x_i) = 0$ for any $i \neq 1, 2$. First note that from Corollary 2.2.6 we have

$$\chi\left(H(H(x_1, x_2), -x_1)\right) = H\left(H\left(\chi(x_1), \chi(x_2)\right), -\chi(x_1)\right) = H(H(x, y), -x).$$

Also, from Equation (2.2.3) and Proposition 2.2.8, the following equation holds in $\mathbb{Z}_{\leq c} \otimes_{\mathbb{Z}} L$:

$$H(H(x_1, x_2), -x_1) = \sum_{m=0}^{c-1} \frac{1}{m!} \mathrm{ad}_{x_1}^m(x_2).$$

By applying χ on both sides of the above equation we obtain the following equation in $\mathfrak{g}$:

$$H(H(x, y), -x) = \sum_{m=0}^{c-1} \frac{1}{m!} \mathrm{ad}_x^m(y).$$

By Equation (2.3.1), the proof of the statement is complete. ∎

Proposition 2.3.2. *The group* $\exp(\mathfrak{g})$ *is a p-group of nilpotency class c.*

Proof: The group $\exp(\mathfrak{g})$ is a p-group because its underlying set, $\mathfrak{g}$, has cardinality a power of p. Consequently, $\exp(\mathfrak{g})$ is nilpotent, as every finite p-group is a nilpotent group.

Let $G := \exp(\mathfrak{g})$. To show that the nilpotency class of G is equal to the nilpotency class of $\mathfrak{g}$, we will prove that $Z_i(\mathfrak{g}) = Z_i(G)$ as sets, for all $i \in \{1, \ldots, c\}$. This will be proven by induction on i.

We start by showing that $Z(\mathfrak{g}) = Z(G)$ as sets. Let $x \in Z(\mathfrak{g})$. So, $[x, y] = 0$ for any $y \in \mathfrak{g}$. We want to show that $x \in Z(G)$, i.e. $x * y = y * x$ for all $y \in \mathfrak{g}$. From Corollary 2.3.1, and the fact that $x \in Z(\mathfrak{g})$, we obtain

$$x * y * x^{-1} = \sum_{m=0}^{c-1} \frac{1}{m!} \mathrm{ad}_x^m(y) = y.$$

This shows that $Z(\mathfrak{g}) \subseteq Z(G)$. Now, suppose that $x \in Z(G)$. Our goal is to show that $x \in Z(\mathfrak{g})$. Note that since $\mathfrak{g}$ is nilpotent of class c, any left aligned bracket of length $c+1$ vanishes. Hence, $[x, w] = 0$ for any $w \in \mathfrak{g}^c$. To show that $x \in Z(\mathfrak{g})$, we need to show that $[x, w] = 0$ for any $w \in \mathfrak{g}$. Let us assume that $[x, w] = 0$ holds for all $w \in \mathfrak{g}^k$, where $k \leq c$ is fixed. We will prove that the same holds when k is replaced by $k-1$. Suppose that $y \in \mathfrak{g}^{k-1}$. Using Corollary 2.3.1, and the fact that $x * y * x^{-1} = y$ for $x \in Z(G)$, we obtain

$$\begin{aligned} 0 &= (x * y * x^{-1}) - y \\ &= \sum_{m=1}^{c-1} \frac{1}{m!} \operatorname{ad}_x^m(y) \\ &= [x, y] + \frac{1}{2}[x, [x, y]] + \frac{1}{6}[x, [x, [x, y]]] + \dots \end{aligned} \tag{2.3.2}$$

Note that since $y \in \mathfrak{g}^{k-1}$, the bracket $[x, y] \in \mathfrak{g}^k$. From the induction hypothesis we obtain that $[x, [x, y]] = 0$. Thus, $\operatorname{ad}_x^m(y) = 0$ for all $m \geq 2$. From (2.3.2), we get that $[x, y] = 0$ for any $y \in \mathfrak{g}^{k-1}$. As a result, $[x, y] = 0$ for all $y \in \mathfrak{g}$, i.e. $x \in Z(\mathfrak{g})$. Since $Z(G) \subseteq Z(\mathfrak{g})$, we conclude that $Z(G) = Z(\mathfrak{g})$.

Now, suppose that $Z_k(\mathfrak{g}) = Z_k(G)$ where $k \geq 1$ is fixed. We claim that $Z_{k+1}(\mathfrak{g}) = Z_{k+1}(G)$. First, note that

$$x * Z_k(G) = x + Z_k(\mathfrak{g}), \tag{2.3.3}$$

where $x * Z_k(G) \in G/Z_k(G)$, $x + Z_k(\mathfrak{g}) \in G/Z_k(\mathfrak{g})$ and $x \in \mathfrak{g}$. To prove this, suppose that $z \in Z_k(\mathfrak{g}) = Z_k(G)$. Then,

$$x * z = H(x, z) = x + z + H'(x, z),$$

where $H'(x, z)$ is a linear combination of left-aligned brackets as in Equation (2.2.2).

Since each bracket in $H'(x,z)$ has at least one z and since $z \in Z_k(\mathfrak{g})$, we have

$$H'(x,z) \in Z_{k-1}(\mathfrak{g}) \leq Z_k(\mathfrak{g}).$$

Since $z + H'(x,z) \in Z_k(\mathfrak{g})$, we get $x * z \in x + Z_k(\mathfrak{g})$. Thus, $x * Z_k(G) \subseteq x + Z_k(\mathfrak{g})$. Since $|G| = |\mathfrak{g}|$ and $|Z_k(\mathfrak{g})| = |Z_k(G)|$, we must have $|x * Z_k(G)| = |x + Z_k(\mathfrak{g})|$. As a result, $x * Z_k(G) = x + Z_k(\mathfrak{g})$.

Now, suppose that $x \in Z_{k+1}(\mathfrak{g})$ and $y \in \mathfrak{g}$. From Corollary 2.3.1,

$$\begin{aligned} x * y * x^{-1} &= \sum_{m=0}^{c-1} \frac{1}{m!} \mathrm{ad}_x^m(y) = y + [x,y] + \frac{1}{2}[x,[x,y]] + \frac{1}{6}[x,[x,[x,y]]] + \dots \\ &= y + K(x,y). \end{aligned}$$

Note that $K(x,y) \in Z_k(\mathfrak{g})$ since $x \in Z_{k+1}(\mathfrak{g})$. From Equation (2.3.3), there exists $K' \in Z_k(G)$ such that $y + K(x,y) = y * K'$. Hence,

$$x * y * x^{-1} * Z_k(G) = y * K' * Z_k(G) = y * Z_k(G).$$

So, $x \in Z_{k+1}(G)$. Thus, $Z_{k+1}(\mathfrak{g}) \subseteq Z_{k+1}(G)$.

The next step is to demonstrate that $Z_{k+1}(G) \subseteq Z_{k+1}(\mathfrak{g})$. Let $x \in Z_{k+1}(G)$. To show that $x \in Z_{k+1}(\mathfrak{g})$, we need to show that $[x,w] \in Z_k(\mathfrak{g})$ for all $w \in \mathfrak{g}$. Note that if $w \in \mathfrak{g}^{c-k}$, then $[x,w] \in Z_k(\mathfrak{g})$ since $\mathfrak{g}$ has nilpotency class c and

$$[y_1,[y_2,[\dots,[y_k,[x,w]]\dots]] = 0,$$

for any $y_1, \dots, y_k \in \mathfrak{g}$. Let us assume that $[x,w] \in Z_k(\mathfrak{g})$ holds for all $w \in \mathfrak{g}^s$, where $s \leq c-k$ is fixed. We will prove that the same holds when s is replaced by $s-1$. Suppose that $y \in \mathfrak{g}^{s-1}$. Using Corollary 2.3.1, and the fact that $x \in Z_{k+1}(G)$, we obtain

$$x * y * x^{-1} * Z_k(G) = y * Z_k(G).$$

Hence,

$$\left(\sum_{m=0}^{c-1} \frac{1}{m!} \operatorname{ad}_x^m(y)\right) + Z_k(\mathfrak{g}) = y + Z_k(\mathfrak{g}).$$

So, we have

$$\left(\sum_{m=1}^{c-1} \frac{1}{m!} \operatorname{ad}_x^m(y)\right) - y = [x, y] + \frac{1}{2}[x, [x, y]] + \frac{1}{6}[x, [x, [x, y]]] + \cdots \in Z_k(\mathfrak{g}).$$

Note that since $y \in \mathfrak{g}^{s-1}$, the bracket $[x, y] \in \mathfrak{g}^s$. From the induction hypothesis we obtain that $[x, [x, y]] \in Z_k(\mathfrak{g})$. Thus, $\operatorname{ad}_x^m(y) \in Z_k(\mathfrak{g})$ for all $m \geq 2$. So, we get that $[x, y] \in Z_k(\mathfrak{g})$ for any $y \in \mathfrak{g}^{s-1}$. As a result, $[x, y] \in Z_k(\mathfrak{g})$ for all $y \in \mathfrak{g}$, i.e. $x \in Z_{k+1}(\mathfrak{g})$.

∎

Finally, after defining the group $\exp(\mathfrak{g})$ and discussing some of its properties, we conclude this section by presenting the *Lazard correspondence*, which states that any nilpotent p-group is isomorphic to a group of the form $\exp(\mathfrak{g})$ for some finite nilpotent $\mathbb{Z}$-Lie algebra $\mathfrak{g}$.

Theorem 2.3.3. [Khu98, Ch.10] *Suppose that G is a p-group of nilpotency class $c < p$. Then there exists a unique (up to isomorphism) finite $\mathbb{Z}$-Lie algebra $\mathfrak{g}$ of nilpotency class c such that $G \cong \exp(\mathfrak{g})$.*

2.4 Kirillov's Orbit Method

The Kirillov's orbit method for classifying representations was initially introduced for connected and simply connected nilpotent Lie groups [Kir62]. However, from the works of Kazhdan [Kaz77], Howe [How77], etc. it turns out that an analogous idea is a powerful tool in finite group theory. For more details, we refer the reader to [BS08]. Let G be a p-group of nilpotency class $c < p$, and let $\mathfrak{g}$ be the $\mathbb{Z}$-Lie algebra associated to G by the Lazard correspondence. Consider the action of G on $\widehat{\mathfrak{g}} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{g}, \mathbb{C}^*)$, defined as

$$\theta^x(y) := \theta\left(x * y * x^{-1}\right) = \theta\left(\sum_{n=0}^{c} \frac{\mathrm{ad}_x^n(y)}{n!}\right),$$

where $x, y \in \mathfrak{g}$ and $\theta \in \widehat{\mathfrak{g}}$.

Theorem 2.4.1. [BS08, Thm. 2.6] *There exists a bijection between G-orbits $\Theta \subseteq \widehat{\mathfrak{g}}$ and irreducible representations ρ_Θ of G such that Kirillov's character formula holds:*

$$\chi_\Theta(x) := \chi_{\rho_\Theta}(x) = |\Theta|^{-1/2} \sum_{\theta \in \Theta} \theta(x), \quad x \in G.$$

Here, $\chi_\Theta(x) = \mathrm{tr}(\rho_\theta(x))$. We will not explain the details of this correspondence but we will demonstrate it with some examples.

Example 2.4.2. Suppose that G is an abelian p-group, and $\theta \in \widehat{\mathfrak{g}}$. Then the G-orbit of θ is $\Theta = \{\theta\}$. In this case G is just $\mathfrak{g}$ as an abelian group. The representation of G associated to Θ is just θ. As a result the character of the associated irreducible representation ρ_Θ of G is $\chi_{\rho_\Theta} = \theta$.

Example 2.4.3. Let

$$G = \mathrm{Heis}_3(\mathbb{F}_p) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\},$$

be the 2-step nilpotent Heisenberg Group, and $\theta \in \widehat{\mathfrak{g}}$. Then, the elements of the G-orbit of θ are given by

$$\theta^x(y) = \theta(y)\theta([x, y]),$$

for $x, y \in \mathfrak{g}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}.$$

Consider the following cases:

- If $\theta|_{Z(\mathfrak{g})} = 1$, then the G-orbit of θ is $\Theta = \{\theta\}$, since $[\mathfrak{g}, \mathfrak{g}] = Z(\mathfrak{g})$. As in Example 2.4.2, the character of irreducible representation ρ_Θ of G is $\chi_{\rho_\Theta} = \theta$.

- If $\theta|_{Z(\mathfrak{g})} \neq 1$, then by linear algebra computations the G-orbit of θ is given by $\Theta = \{\theta' : \theta'|_{Z(\mathfrak{g})} = \theta|_{Z(\mathfrak{g})}\}$. Let

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{F}_p \right\} \leq \mathfrak{g},$$

  and $\gamma = \theta|_{\mathfrak{h}}$. Let

$$H = \exp(\mathfrak{h}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{F}_p \right\} \leq G.$$

  Note that H is an abelian subgroup. As in Example 2.4.2, γ corresponds to a one dimensional representation of H (which is still γ if we identify $\mathfrak{h}$ and H). The representation of G corresponding to the G-orbit of θ is the induced representation of G from γ on H. This is a standard example, for more detail see [BMKS16].

2.5 Commutator Matrix

This section introduces the concept of the commutator matrix, an essential tool in this thesis. Let $\mathfrak{g}$ be a $\mathbb{Z}$-Lie algebra which is free as a $\mathbb{Z}$-module. The commutator matrix was first introduced by Grunewald and Segal [GS84]. It has proved to be an effective and versatile tool in representation theory [OV15, SV14]. For convenience we further assume the following:

i. $\mathfrak{g}/Z(\mathfrak{g})$ is a free $\mathbb{Z}$-module with basis represented by $v_1, \ldots, v_n \in \mathfrak{g}$.

ii. $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is a free $\mathbb{Z}$-module with basis $w_1, \ldots, w_{l_1} \in \mathfrak{g}$.

iii. $[\mathfrak{g}, \mathfrak{g}]/Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is a free $\mathbb{Z}$-module with basis represented by $w_{l_1+1}, \ldots, w_m \in \mathfrak{g}$, after passing to the quotient.

Note that the elements $w_1, \ldots, w_m$ form a $\mathbb{Z}$-module basis of $[\mathfrak{g}, \mathfrak{g}]$, and thus for any pair of representatives v_i, v_j of the basis elements of $\mathfrak{g}/Z(\mathfrak{g})$ we have:

$$[v_i, v_j] = \sum_{k=1}^{m} \lambda_{ij}^k w_k, \tag{2.5.1}$$

where $\lambda_{ij} \in \mathbb{Z}$.

Remark 2.5.1. The freeness conditions on the $\mathbb{Z}$-Lie algebra $\mathfrak{g}$ can be relaxed so that $\mathfrak{g}$ as an abelian group is any finitely generated abelian group. This is done in [BMKS19, Sec. 5]. However, the Lie algebra examples studied in this thesis satisfy the above freeness assumptions.

In the following definition, let n and m be as in i. and iii. above, and let λ_{ij} as in Equation (2.5.1).

Definition 2.5.2. [OV15, Def. 2.1] The *Commutator Matrix* of the $\mathbb{Z}$-Lie algebra $\mathfrak{g}$

is a matrix whose entries are linear forms in the parameters $T_1, \ldots, T_m$ given by

$$F_{\mathfrak{g}}(T_1, \ldots, T_m) := \left(\sum_{k=1}^{m} \lambda_{ij}^{k} T_k \right) \in M_{n \times n}(\mathbb{Z}[T_1, \ldots, T_m]),$$

where $\lambda_{ij} \in \mathbb{Z}$, for all $1 \leq i, j \leq n$.

The following example is taken from [BMKS19, Ex. 2.2]. We will use it throughout this thesis as a toy example to demonstrate constructions and various phenomena that occur in a more general setting.

Example 2.5.3. Let $\mathfrak{g}$ be a $\mathbb{Z}$-Lie algebra spanned as a free $\mathbb{Z}$-module by $\{v_1, \ldots, v_6\}$ such that :

$$[v_1, v_2] = [v_3, v_4] = v_5, \;\; [v_1, v_4] = [v_2, v_3] = v_6, \;\; [v_i, v_j] = 0 \text{ for all other } i < j.$$

Note that the commutator of any $u = \sum_{i=1}^{6} a_i v_i$ and $w = \sum_{i=1}^{6} b_i v_i$ in $\mathfrak{g}$ is given by:

$$[u, w] = (a_1 b_2 - a_2 b_1 + a_3 b_4 - a_4 b_3)\, v_5 + (a_1 b_4 - a_4 b_1 + a_2 b_3 - a_3 b_2)\, v_6, \tag{2.5.2}$$

where $a_i, b_j \in \mathbb{Z}$. From Equation (2.5.2), its clear that the axioms of the $\mathbb{Z}$-Lie algebra holds and that $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}_{\mathbb{Z}}\{v_5, v_6\}$. We claim that $Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. It is clear that $[\mathfrak{g}, \mathfrak{g}] \leq Z(\mathfrak{g})$ as $v_5, v_6 \in Z(\mathfrak{g})$. Suppose that $u = \sum_{i=1}^{6} a_i v_i \in Z(\mathfrak{g})$. If we choose $w = v_1, v_2$ in Equation (2.5.2), we obtain the following:

$$[u, v_1] = -a_2 v_5 - a_4 v_6 = 0$$
$$[u, v_2] = a_1 v_5 - a_3 v_6 = 0.$$

As a result, $a_1 = a_2 = a_3 = a_4 = 0$. Thus, $u = a_5 v_5 + a_6 v_6 \in [\mathfrak{g}, \mathfrak{g}]$. Since $Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}_{\mathbb{Z}}\{v_5, v_6\}$, we have $n = 4$ and $m = 2$. As a result the commutator matrix is given by

$$F_{\mathfrak{g}}(T_1, T_2) = \begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1 & 0 & T_2 & 0 \\ 0 & -T_2 & 0 & T_1 \\ -T_2 & 0 & -T_1 & 0 \end{pmatrix}.$$

Later on, we will see that the faithful dimension is closely related to the rank of such matrices.

Chapter 3

Faithful Dimension of Finite Groups

In this chapter, we first introduce the concept of the *faithful dimension* of finite groups over a field. Then we review the existing qualitative results regarding the polynomiality of the faithful dimension [BMKRS23], and a combinatorial formula for the faithful dimension of *Pattern Groups* [BMKS19]. The latter class of groups includes the Heisenberg group and the group of unitriangular matrices. Finally, we conclude this chapter by giving an explicit formula for the faithful dimension of the nilradicals of parabolics in $\mathrm{GL}_n(R)$, where R is a finite truncated valuation ring.

3.1 Qualitative Results on The Faithful Dimension

Let G be a finite group and let V be a finite dimensional vector space over a field K. Recall that a *representation of* G in V is a homomorphism from the group G into $GL(V)$. Here $GL(V)$ denotes the group of invertible linear maps on V. Moreover, the dimension of V is called the *dimension* of the representation.

Definition 3.1.1. [BMKS19, Sec. 1] The *faithful dimension of* G *over* K, denoted by $m_{\mathrm{faithful},K}(G)$, is the smallest possible dimension of a faithful representation of G.

In this thesis we will mainly focus on the faithful dimension of groups over the complex numbers $K = \mathbb{C}$. So, for simplicity we write

$$m_{\text{faithful}}(G) := m_{\text{faithful},\mathbb{C}}(G).$$

Our main interest in this thesis is computing the faithful dimension of groups associated by the Lazard correspondence to finitely generated $\mathbb{Z}$-Lie algebras of nilpotentcy class c. From now on, we set the following notation

$$\begin{aligned}\mathfrak{g}_q &:= \mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{F}_q,\\ \mathscr{G}_q &:= \exp(\mathfrak{g}_q) = \exp(\mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{F}_q).\end{aligned}$$

where $q := p^f$, $p > c$, and $\mathfrak{g}$ is a $\mathbb{Z}$-Lie algebra of nilpotentcy class c that is finitely generated as an abelian group.

Example 3.1.2. Let $\mathfrak{g}$ be the $\mathbb{Z}$-Lie algebra in Example 2.5.3, which was spanned as a free $\mathbb{Z}$-module by $\{v_1, \dots, v_6\}$. In Example 4.3.3 from Chapter 4, we will show that for $f = 1$ and p an odd prime we have

$$m_{\text{faithful}}(\mathscr{G}_p) = m_{\text{faithful}}(\exp(\mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{F}_p)) = \begin{cases} 2p & \text{if } p \equiv 1 (\text{mod } 4) \\ 2p^2 & \text{if } p \equiv 3 (\text{mod } 4). \end{cases}$$

Note that in the example above, the set of prime numbers

$$\mathbb{P} = \{2, 3, 5, \dots\}$$

larger than 2 is partitioned in a way that the faithful dimension is expressed as a polynomial in p. This partition of primes arises from *Frobenius sets*, which are defined below.

Definition 3.1.3. Let $g(T) \in \mathbb{Z}[T]$. We define an *elementary Frobenius set* to be

$$\mathcal{V}_g := \{p \in \mathbb{P} : g(T) \equiv 0 \pmod p \text{ has a solution}\}.$$

Definition 3.1.4. A *Frobenius set* is a finite union of sets of the form

$$\mathcal{V}_{g_1} \cap \cdots \cap \mathcal{V}_{g_k} \cap \mathcal{V}^c_{g_{k+1}} \cap \cdots \cap \mathcal{V}^c_{g_l},$$

for some $g_1(T), \dots, g_\ell(T) \in \mathbb{Z}[T]$.

Example 3.1.5. The sets of primes from Example 3.1.2 are expressed as Frobenius sets as follows:

$$\mathscr{P}_1 = \{p \in \mathbb{P} : p \geq 3,\ p \equiv 1 \pmod 4\} = \mathcal{V}_g \cap \mathcal{V}^c_{g'},$$
$$\mathscr{P}_2 = \{p \in \mathbb{P} : p \geq 3,\ p \equiv 3 \pmod 4\} = \mathcal{V}^c_g,$$

where $g(T) = T^2 + 1, g'(T) = 2$. This is because $T^2 + 1 \equiv 0 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 4$, for $p > 2$ [IR90, Ch.5].

Now we can present the polynomiality result of [BMKRS23].

Theorem 3.1.6. [BMKRS23, Thm. 1.1] *Let $\mathfrak{g}$ be a nilpotent $\mathbb{Z}$-Lie algebra which is finitely generated as an abelian group. Then there exist:*

i. a constant $M = M(\mathfrak{g})$ only depending on $\mathfrak{g}$,

ii. a partition $\{\mathscr{P}_1, \dots, \mathscr{P}_r\}$ of the set of prime numbers larger than M into Frobenius sets,

iii. a partition $\{\mathscr{F}_1, \dots, \mathscr{F}_s\}$ of natural numbers into arithmetic progressions, and

iv. polynomials $g_{ij}(T)$, for $1 \leq i \leq r$ and $1 \leq j \leq s$, with non-negative integer coefficients, *such that for all* $q := p^f$ *where* $(p, f) \in \mathscr{P}_i \times \mathscr{F}_j$ *for some* $1 \leq i \leq r$ *and* $1 \leq j \leq s$, *we have*

$$m_{faithful}(\mathscr{G}_q) = f g_{ij}(q).$$

The computation of the sets $\mathscr{P}_i$ and $\mathscr{F}_j$ is related to questions in number theory. For a concrete example of the computations involved in Theorem 3.1.6 see Example 4.3.3.

3.2 A Combinatorial Formula in the Case of Pattern Groups

Let $([n], \prec)$ be a partially ordered set, where $[n] := \{1, \ldots, n\}$. As presented in [BMKS19], we define the *pattern Lie algebra* associated to the partial order $\prec$ to be

$$\mathfrak{g}_{\prec} := \mathrm{Span}_{\mathbb{Z}}\{e_{ij} : i \prec j\} \subseteq \mathfrak{gl}(n, \mathbb{Z})$$

where e_{ij} is the $n \times n$ matrix whose unique nonzero entry is 1 in the (i, j) position, and the Lie bracket is induced from $\mathfrak{gl}(n, \mathbb{Z})$. In particular, for all $i \prec j$ and $k \prec \ell$ we have

$$[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{\ell i} e_{kj},$$

where $\delta_{jk}, \delta_{\ell i}$ are the *Kronecker delta functions.*

Example 3.2.1. The *Heisenberg Lie algebra* $\mathfrak{h}$ is defined to be the pattern Lie algebra associated to the following partial order on $[k+2]$:

$$1 \prec 2, 3, \ldots, k+1 \prec k+2.$$

In particular, the Heisenberg Lie algebra $\mathfrak{h}$ is given by:

$$\mathfrak{h} = \mathfrak{g}_{\prec} = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & \dots & a_k & c \\ 0 & 0 & 0 & \dots & 0 & b_1 \\ 0 & 0 & 0 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_k \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} : a_1, a_2, \dots, a_k, c, b_1, b_2, \dots, b_k \in \mathbb{Z} \right\}.$$

Example 3.2.2. The *Lie algebra of strictly upper triangular matrices* $\mathfrak{u}_k$ is the pattern Lie algebra associated to the following partial order on $[k]$:

$$1 \prec 2 \prec 3 \prec \cdots \prec k.$$

Specifically, the Lie algebra of strictly upper triangular matrices $\mathfrak{u}_k$ is given by

$$\mathfrak{u}_k = \mathfrak{g}_{\prec} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1k-1} & a_{1k} \\ 0 & 0 & a_{23} & \dots & a_{2k-1} & a_{2k} \\ 0 & 0 & 0 & \dots & a_{3k-1} & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{k-1k} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} : a_{ij} \in \mathbb{Z} \right\}.$$

Definition 3.2.3. Recall that $([n], \prec)$ is a poset.

i. For $i \prec j$, define $\alpha(i,j) := \#\{k \in [n] : i \prec k \prec j\}$.

ii. The *length of* $([n], \prec)$, denoted by $\lambda_{\prec}$, is the maximum value r such that a chain

$$i_0 \prec \cdots \prec i_r$$

exists in $([n], \prec)$.

iii. An ordered pair (i, j) is called an *extreme pair* if:

- i is a minimal element in $([n], \prec)$.

- j is a maximal element in $([n], \prec)$.
- $i \prec j$.

The set of extreme pairs will be denoted by I_{ex}.

Note that the pattern Lie algebra $\mathfrak{g}_\prec$ associated to the partial order $\prec$ is nilpotent, since it is a subalgebra of $\mathfrak{u}_n$ (the algebra of strictly upper triangular matrices).

Proposition 3.2.4. *The nilpotency class of $\mathfrak{g}_\prec$ is equal to $\lambda_\prec$.*

Proof: For the proof, refer to [BMKS19, Lemma 4.1]. ∎

Example 3.2.5. From Example 3.2.1, the Heisenberg Lie algebra $\mathfrak{h}$ has nilpotency class 2 since $\lambda_\prec = 2$. Also, from Example 3.2.2, the Lie algebra of strictly upper triangular matrices $\mathfrak{u}_k$ has nilpotency class $k-1$ since $\lambda_\prec = k-1$.

We are now going to introduce the p-groups $\mathscr{G}_R := \exp(\mathfrak{g}_\prec \otimes_{\mathbb{Z}} R)$, where R is a *finite truncated valuation ring.* These groups naturally extend the family $\mathscr{G}_q = \exp(\mathfrak{g}_\prec \otimes_{\mathbb{Z}} \mathbb{F}_q)$. The following definitions follows [BMKRS23].

Definition 3.2.6. A *finite truncated valuation ring* R is a finite local ring with a nilpotent and principal maximal ideal $\mathfrak{m}$.

Remark 3.2.7. It is an elementary statement to verify that in any ring as defined in Definition 3.2.6, every principal ideal is a power of the maximal ideal.

Definition 3.2.8. The *associated parameters* of the finite truncated valuation ring R are given by a quadruple of integers (d, p, e, f) defined as follows:

- d is the smallest positive integer satisfying $\mathfrak{m}^d = 0$,

- $p = \mathrm{char}(R/\mathfrak{m})$,

- $e \geq 1$ satisfies $pR = \mathfrak{m}^e$,

- $f \geq 1$ satisfies $R/\mathfrak{m} \cong \mathbb{F}_{p^f}$.

Example 3.2.9. The finite field $\mathbb{F}_{p^f}$ is an example of a finite truncated valuation ring, where the maximal ideal $\mathfrak{m} = (0)$, and the quadruple is given by $(d, p, e, f) = (1, p, 1, f)$. Another example of a finite truncated valuation ring is $R = \mathbb{Z}/p^n\mathbb{Z}$, where $\mathfrak{m} = p\mathbb{Z}/p^n\mathbb{Z}$ and the quadruple is given by $(d, p, e, f) = (n, p, 1, 1)$.

The faithful dimension of the pattern groups $\mathscr{G}_R$ can be computed using the following result of [BMKRS23], which depends on the associated parameters of R, and the set of extreme pairs of the partial order.

Theorem 3.2.10. [BMKRS23, Thm. 1.5] *Let $\mathfrak{g}_\prec$ be the pattern Lie algebra associated to the partial order $\prec$, and let R be a finite truncated valuation ring with associated parameters (d, p, e, f) such that*

$$p > \max\{\alpha(i, j) : i \prec j\} + 1.$$

Then

$$m_{\text{faithful}}(\mathscr{G}_R) = m_{\text{faithful}}(\exp(\mathfrak{g}_\prec \otimes_{\mathbb{Z}} R)) = \sum_{\ell=0}^{e-1} \sum_{(i,j)\in I_{\text{ex}}} f\, p^{f(d-\ell)\alpha(i,j)}.$$

Example 3.2.11. The *Heisenberg Group* is given by

$$\mathrm{Heis}_{2k+1}(R) = \exp(\mathfrak{h} \otimes_{\mathbb{Z}} R) = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_k & c \\ 0 & 1 & 0 & \dots & 0 & b_1 \\ 0 & 0 & 1 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_k \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} : a_i, c, b_j \in R \right\},$$

where $\mathfrak{h}$ is the Heisenberg Lie algebra. From Example 3.2.1, we assume that

$$p > \max\{\alpha(i, j) : i \prec j\} + 1 = k + 1.$$

Since $I_{ex} = \{(1, k+2)\}$ and $\alpha(1, k+2) = k$, from Theorem 3.2.10 it follows that

$$m_{\text{faithful}}(\text{Heis}_{2k+1}(R)) = m_{\text{faithful}}(\exp(\mathfrak{h} \otimes_{\mathbb{Z}} R)) = \sum_{\ell=0}^{e-1} f\, p^{f(d-\ell)k}.$$

Example 3.2.12. The group of *Unitriangular Matrices* is given by

$$U_k(R) = \exp(\mathfrak{u}_k \otimes_{\mathbb{Z}} R) = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1k-1} & a_{1k} \\ 0 & 1 & a_{23} & \dots & a_{2k-1} & a_{2k} \\ 0 & 0 & 1 & \dots & a_{3k-1} & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{k-1k} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} : a_{ij} \in R \right\},$$

where $\mathfrak{u}_k$ is the Lie algebra of strictly upper triangular matrices. From Example 3.2.2, we assume that

$$p > \max\{\alpha(i,j) : i \prec j\} + 1 = (k-2) + 1 = k - 1.$$

Since $I_{ex} = \{(1, k)\}$ and $\alpha(1, k) = k - 2$, by Theorem 3.2.10 we get that

$$m_{\text{faithful}}(U_k(R)) = m_{\text{faithful}}(\exp(\mathfrak{u}_k \otimes_{\mathbb{Z}} R)) = \sum_{\ell=0}^{e-1} f\, p^{f(d-\ell)(k-2)}.$$

3.3 Nilradicals of Parabolics

In this section we extend the explicit result on the faithful dimension of the p-groups associated to the Heisenberg and the unitriangular algebras to a general class of nilpotent Lie algebras that in technical terms can be characterized as nilradicals of parabolic subalgebras of the general linear Lie algebra $\mathfrak{gl}(n)$. Let $n \in \mathbb{N}$, and let

$d_1, d_2, \ldots, d_r \in \mathbb{N}$, such that $\sum_{i=1}^{r} d_i = n$. Define a partial order $\prec$ on $[n]$ as follows:

$$\underbrace{1, \ldots, d_1}_{d_1} \prec \underbrace{d_1+1, \ldots, d_1+d_2}_{d_2} \prec \cdots \prec \underbrace{\sum_{i=1}^{r-1} d_i + 1, \ldots, \sum_{i=1}^{r} d_i}_{d_r}. \tag{3.3.1}$$

Note that $\lambda_{\prec} = r-1$. The pattern Lie algebra associated to this partial order is given by

$$\mathfrak{n} = \mathfrak{g}_{\prec} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} & \ldots & A_{1r-1} & A_{1r} \\ 0 & 0 & A_{23} & \ldots & A_{2r-1} & A_{2r} \\ 0 & 0 & 0 & \ldots & A_{3r-1} & A_{3r} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & A_{r-1r} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} : A_{ij} \in M_{d_i \times d_j}(\mathbb{Z}) \right\}.$$

Whereas, the pattern group associated to this partial order is given by

$$\mathcal{N}(R) = \exp(\mathfrak{n} \otimes_{\mathbb{Z}} R) = \left\{ \begin{pmatrix} I_1 & A_{12} & A_{13} & \ldots & A_{1r-1} & A_{1r} \\ 0 & I_2 & A_{23} & \ldots & A_{2r-1} & A_{2r} \\ 0 & 0 & I_3 & \ldots & A_{3r-1} & A_{3r} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_{r-1} & A_{r-1r} \\ 0 & 0 & 0 & \ldots & 0 & I_r \end{pmatrix} : A_{ij} \in M_{d_i \times d_j}(R) \right\}.$$

Note that this partial order generalizes both Heisenberg group and unitriangular matrices as follows:

- For $r = n$ and $d_1 = d_2 = \cdots = d_n = 1$, we have $\mathcal{N}(R) = U_n(R)$.
- For $r = 3$, $d_1 = d_3 = 1$ and $d_2 = n-2$, we have $\mathcal{N}(R) = \mathrm{Heis}_{2n-3}(R)$.

Using Theorem 3.2.10, we can find the faithful dimension for $\mathcal{N}(R)$.

Proposition 3.3.1. *Let $p > \max\{\alpha(i,j) : i \prec j\} + 1$. Then,*

$$m_{\mathit{faithful}}(\mathcal{N}(R)) = d_1 d_r \sum_{\ell=0}^{e-1} f\, p^{f(d-\ell)(n-(d_1+d_r))}.$$

Proof: From the generalized partial order in Equation (3.3.1), we get the following set of extreme pairs:

$$I_{ex} = \left\{ (i,j) : i \in \{1, \dots, d_1\}, j \in \{\sum_{i=1}^{r-1} d_i + 1, \dots, \sum_{i=1}^{r} d_i\} \right\},$$

Since

$$\alpha(i,j) = d_2 + d_3 + \cdots + d_{r-1} = n - (d_1 + d_r),$$

for any pair $(i,j) \in I_{ex}$, we can compute the faithful dimension using Theorem 3.2.10 as follows

$$m_{\text{faithful}}\left(\mathcal{N}(R)\right) = d_1 d_r \sum_{\ell=0}^{e-1} f\ p^{f(d-\ell)(n-(d_1+d_r))}.$$

∎

Corollary 3.3.2. $m_{faithful}\left(\mathcal{N}(\mathbb{F}_p)\right) = m_{faithful}(\exp(\mathfrak{n} \otimes_{\mathbb{Z}} \mathbb{F}_p)) = d_1 d_r\ p^{n-(d_1+d_r)}$.

Example 3.3.3. Recall that for $\mathcal{N}(\mathbb{F}_p) = U_n(\mathbb{F}_p)$, we have $r = n$ and $d_1 = d_2 = \cdots = d_n = 1$. Using Corollary 3.3.2, we get $m_{\text{faithful}}\left(\mathcal{N}(\mathbb{F}_p)\right) = p^{n-2}$, which is consistent with our result in Example 3.2.12.

Chapter 4

Free Nilpotent Lie Algebras

This chapter presents the main tool for computing the faithful dimension for a more general class of groups associated to nilpotent $\mathbb{Z}$-Lie algebras. More specifically, we are interested in free nilpotent $\mathbb{Z}$-Lie algebras. We start by describing the Hall basis of these free nilpotent $\mathbb{Z}$-Lie algebras. Then we explain how the problem of computing the faithful dimension can be reduced to a rank minimization problem for a matrix of linear forms.

4.1 Hall Sets

In this section we define Hall sets which will be used later to describe a basis for the free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators. The presentation of this section follows [Ser92, Chapter 4].

Definition 4.1.1. [Ser92, Def. 1.1] A set M is called a *magma* if it is equipped with a map

$$M \times M \to M \ , \ (x, y) \mapsto xy.$$

Definition 4.1.2. Let X be a set. The *free magma on X* is defined to be the disjoint union

$$M(X) := \coprod_{n=1}^{\infty} X_n,$$

where the family of sets $\{X_n : n \geq 1\}$ is defined as follows :

i. $X_1 = X$.

ii. For $n \geq 1$, $X_n = \coprod_{n=p+q} X_p \times X_q$.

Note that the binary operation on the free magma $M(X)$ is defined to be

$$\begin{aligned} M(X) \times M(X) &\to M(X) \\ (x_p, x_q) &\mapsto x_p x_q := (x_p, x_q) \in X_p \times X_q \subset X_{p+q}, \end{aligned}$$

where $x_p \in X_p, x_q \in X_q$, for any $p, q \geq 1$.

Definition 4.1.3. Let $w \in M(X)$. The *length* $\ell(w)$ is the unique n such that $w \in X_n$.

Definition 4.1.4. [Ser92, Def. 5.1] A *Hall set* relative to X is a subset $\mathcal{H}$ of $M(X)$ that is equipped with a total order and has the following properties:

i. If $u, v \in \mathcal{H}$ and $\ell(u) < \ell(v)$, then $u < v$.

ii. For $u \in M(X)$:

a. If $\ell(u) = 1$, then $u \in \mathcal{H}$, i.e. $X \subset \mathcal{H}$.

b. If $\ell(u) = 2$, then $u \in \mathcal{H}$ if and only if $u = vw$ with $v, w \in \mathcal{H}$ and $v < w$.

c. If $\ell(u) \geq 3$, then $u \in \mathcal{H}$ if and only if the following conditions hold:

ci. $u = vw$ with $v, w \in \mathcal{H}$ and $v < w$.

cii. $w = w'w''$ with $w', w'' \in \mathcal{H}$, $w' < w''$, and $v \geq w'$.

Remark 4.1.5. In [Ser92, Def. 5.1], these Hall sets are called *P. Hall families* .

Notation 4.1.6. The subset of a Hall set $\mathcal{H}$ containing all elements of length i will be denoted by $\mathcal{H}^i := \mathcal{H} \cap X_i$. Note that one needs to make a choice of ordering on each $\mathcal{H}^i$.

From [Ser92, Lemma 5.2] it follows that a Hall set $\mathcal{H}$ exists (even though not unique) for any set X. This is demonstrated in the following example.

Example 4.1.7. Let $X = \{x, y\}$ such that $x \neq y$. To find $\mathcal{H}$, we need to find its subsets $\mathcal{H}_i$, which can be done successively using Definition 4.1.4 and assigning a total order to each subset (a lexicographical order for example). This process is explained in the proof of [Ser92, Lemma 5.2]. As an example we can get the following subsets of $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H}^1 &= \mathcal{H} \cap X_1 = \{x, y\}, && \text{(total order given by } \ x < y) \\
\mathcal{H}^2 &= \mathcal{H} \cap X_2 = \mathcal{H} \cap \{xy, yx\} = \{xy\}, && \\
\mathcal{H}^3 &= \mathcal{H} \cap X_3 = \mathcal{H} \cap \{x(xy), x(yx), y(xy), y(yx), (xy)x, (xy)y, (yx)x, (yx)y\} && \\
&= \{x(xy), y(xy)\}, && \text{(total order given by } \ x(xy) < y(xy)\ )
\end{aligned}$$

and so on.

4.2 Hall Bases of Free Nilpotent Lie Algebras

From now on let X be a set of n elements, and suppose that the binary operation on $M(X)$ is denoted by brackets.

Definition 4.2.1. [Ser92, Def. 2.1] The *free $\mathbb{Z}$-algebra on X* (or simply the *free algebra on X*) is the $\mathbb{Z}$-algebra of the free magma $M(X)$. It is denoted by $A(X)$.

This means that $A(X)$ is a free $\mathbb{Z}$-module with basis $M(X)$. So, an element $\alpha \in A(X)$ is a finite sum $\alpha = \sum_{m \in M(X)} c_m m$, where $c_m \in \mathbb{Z}$. Note that $A(X)$ is a free nonassociative algebra.

Definition 4.2.2. [Ser92, Def. 3.1] The *free $\mathbb{Z}$-Lie algebra on n generators* is defined to be

$$\mathfrak{f}_n := A(X)/I,$$

where I is the ideal generated by elements of the following forms:

i. $[u, u]$.

ii. $J(u, v, w) = [u, [v, w]] + [v, [w, u]] + [w, [u, v]]$.

where $u, v, w \in A(X)$.

Definition 4.2.3. [BMKS19, Sec. 7] The *free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators* is defined to be

$$\mathfrak{f}_{n,c} = \mathfrak{f}_n/\mathfrak{f}_n^{c+1},$$

where $\mathfrak{f}_n^{c+1} := [\mathfrak{f}_n, \mathfrak{f}_n^c]$, for all $c \geq 1$, and $f_n^1 = f_n$.

Remark 4.2.4. The definition of the free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators in Definition 4.2.3 agrees with the previous definition in Proposition 2.2.3 by considering a finite set of generators. In other words, Proposition 2.2.3 and Definition 4.2.3 provide two different ways of realizing the same algebra.

Remark 4.2.5. The image of $\mathcal{H}^k$ under the natural projection $\mathfrak{f}_n^k \to \mathfrak{f}_n^k/\mathfrak{f}_n^{k+1}$ forms a $\mathbb{Z}$-basis of $\mathfrak{f}_n^k/\mathfrak{f}_n^{k+1}$ [Ser92, Thm. 5.3]. Furthermore, we have Witt's Formula:

$$r_n(k) := |\mathcal{H}^k| = \mathrm{rk}_{\mathbb{Z}}(\mathfrak{f}_n^k/\mathfrak{f}_n^{k+1}) = \frac{1}{k}\sum_{d|k} \mu(d) n^{k/d}, \tag{4.2.1}$$

where μ is the *Möbius function* [Ser92, Thm. 4.2].

Proposition 4.2.6. [BMKS19, Prop. 7.2] *For $n \geq 2$ and $c \geq 2$,*

i. The image of $\cup_{i=1}^c \mathcal{H}^i$ under the natural projection $\mathfrak{f}_n \to \mathfrak{f}_{n,c}$ is a basis of $\mathfrak{f}_{n,c}$.

ii. The image of $\cup_{i=2}^{c}\mathcal{H}^i$ *under the natural projection* $\mathfrak{f}_n \to \mathfrak{f}_{n,c}$ *is a basis of* $\mathfrak{f}'_{n,c}$.

iii. The image of $\mathcal{H}^c$ *under the natural projection* $\mathfrak{f}_n \to \mathfrak{f}_{n,c}$ *is a basis of* $Z(\mathfrak{f}_{n,c})$.

From Definition 2.5.2, the commutator matrix of $\mathfrak{f}_{n,c}$ is an $m_1 \times m_1$ skew-symmetric matrix in m variables given by

$$F_{\mathfrak{f}_{n,c}}(T) = \begin{pmatrix} F_{11}(T^{(2)}) & F_{12}(T^{(3)}) & \dots & F_{1(c-1)}(T^{(c)}) \\ F_{21}(T^{(3)}) & F_{22}(T^{(4)}) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ F_{(c-1)1}(T^{(c)}) & 0 & \dots & 0 \end{pmatrix} \in M_{m_1\times m_1}(\mathbb{Z}(T)), \qquad (4.2.2)$$

where:

- $m = \mathrm{rk}_{\mathbb{Z}}(\mathfrak{f}'_{n,c}) = \sum_{k=2}^{c} r_n(k)$,

- $m_1 = \mathrm{rk}_{\mathbb{Z}}(\mathfrak{f}_{n,c}/Z(\mathfrak{f}_{n,c})) = \sum_{k=1}^{c-1} r_n(k)$,

- $T = (T^{(k)})_{2\le k\le c}$, where $T^{(k)} = (T_1^{(k)}, \dots, T_{r_n(k)}^{(k)})$,

and for $1 \le i, j \le c-1$ we have:

- $F_{ij}(T^{(i+j)})$ is the zero matrix if $i+j > c$,

- $F_{ij}(T^{(i+j)}) = -F_{ji}(T^{(i+j)})$.

We can construct a simplified form of the commutator matrix by substituting $F_{ij}(T^{(i+j)}) = 0$ whenever $i+j \neq c$ as follows:

Definition 4.2.7. The *reduced commutator matrix* of $\mathfrak{f}_{n,c}$ is

$$F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(T^{(c)}) = \begin{pmatrix} 0 & 0 & \dots & 0 & F_{1(c-1)}(T^{(c)}) \\ 0 & 0 & \dots & F_{2(c-2)}(T^{(c)}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & F_{(c-2)2}(T^{(c)}) & \dots & 0 & 0 \\ F_{(c-1)1}(T^{(c)}) & 0 & \dots & 0 & 0 \end{pmatrix}.$$

4.3 Rank Minimization Problem

Let $\mathfrak{g}$ be $\mathbb{Z}$-Lie algebra which is nilpotent of class c and finitely generated as an abelian group. The following key result of Bardestani, Mallahi-Karai and Salmasian [BMKS19] expresses the faithful dimension of $\mathscr{G}_q = \exp(\mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ as a solution to a rank minimization problem.

Theorem 4.3.1. [BMKS19, Thm. 5.3] *For sufficiently large p, we have*

$$m_{faithful}(\mathscr{G}_q) = \min \left\{ \sum_{\ell=1}^{\ell_1} f q^{\frac{rk_{\mathbb{F}_q}(F_{\mathfrak{g}}(x_{\ell 1}, \ldots, x_{\ell m}))}{2}} : \begin{pmatrix} x_{11} & \ldots & x_{1\ell_1} \\ \vdots & & \vdots \\ x_{\ell_1 1} & \ldots & x_{\ell_1 \ell_1} \end{pmatrix} \in GL_{\ell_1}(\mathbb{F}_q) \right\} + f\ell_2,$$

where $m := \mathrm{rk}_{\mathbb{Z}}([\mathfrak{g}, \mathfrak{g}])$, $\ell_1 := \mathrm{rk}_{\mathbb{Z}}([\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g}))$, and $\ell_2 := \mathrm{rk}_{\mathbb{Z}}(Z(\mathfrak{g})/[\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g}))$.

Remark 4.3.2. The rank of a $\mathbb{Z}$-module $\mathfrak{h}$, denoted by $\mathrm{rk}_{\mathbb{Z}}(\mathfrak{h})$, is the maximum number of $\mathbb{Z}$-linearly independent elements of $\mathfrak{h}$.

Example 4.3.3. Let $\mathfrak{g}$ be a $\mathbb{Z}$-Lie algebra in Example 2.5.3. Recall that it is spanned as a free $\mathbb{Z}$-module by $\{v_1, \ldots, v_6\}$, such that :

$$[v_1, v_2] = [v_3, v_4] = v_5, \ \ [v_1, v_4] = [v_2, v_3] = v_6, \ \ [v_i, v_j] = 0 \text{ for all } i < j.$$

Since

$$\begin{aligned} m &= \mathrm{rk}_{\mathbb{Z}}([\mathfrak{g}, \mathfrak{g}]) = 2, \\ \ell_1 &= \mathrm{rk}_{\mathbb{Z}}([\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g})) = 2, \\ \ell_2 &= \mathrm{rk}_{\mathbb{Z}}(Z(\mathfrak{g})/[\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g})) = 0, \end{aligned}$$

we get the following rank minimization problem from Theorem 4.3.1:

$$m_{\text{faithful}}(\mathscr{G}_q) = \min \left\{ f q^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(x_{11}, x_{12}))}{2}} + f q^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(x_{21}, x_{22}))}{2}} : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in GL_2(\mathbb{F}_q) \right\}.$$

Recall from Example 2.5.3, the commutator matrix was given by

$$F_{\mathfrak{g}}(T_1,T_2) = \begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1 & 0 & T_2 & 0 \\ 0 & -T_2 & 0 & T_1 \\ -T_2 & 0 & -T_1 & 0 \end{pmatrix}.$$

Note that $\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(T_1,T_2))$ is even since $F_{\mathfrak{g}}(T_1,T_2)$ is skew symmetric, and

$$\det(F_{\mathfrak{g}}(T_1,T_2)) = (T_1^2+T_2^2)^2.$$

It follows that

$$m_{\text{faithful}}(\mathscr{G}_q) = \begin{cases} fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(\alpha,1))}{2}} + fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(-\alpha,1))}{2}} = 2fq & \text{if } p \equiv 1 (\mathrm{mod}\ 4) \\ fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(\alpha,1))}{2}} + fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(-\alpha,1))}{2}} = 2fq & \text{if } p \equiv 3 (\mathrm{mod}\ 4),\ f \text{ even} \\ fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(a_1,b_1))}{2}} + fq^{\frac{\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(a_2,b_2))}{2}} = 2fq^2 & \text{if } p \equiv 3 (\mathrm{mod}\ 4),\ f \text{ odd} \end{cases}$$

where

- $\alpha \in \mathbb{F}_q$ and $\alpha^2 = -1$. Here we have $\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(\alpha,1)) = \mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(-\alpha,1)) = 2$
- $(a_i,b_i) \neq (0,0) \in \mathbb{F}_q$. Here we have

$$\det(F_{\mathfrak{g}}(a_i,b_i)) = (a_i^2+b_i^2)^2 \neq 0 \implies \mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{g}}(a_i,b_i)) = 4$$

As in Theorem 3.1.6, we have

$$m_{\text{faithful}}(\mathscr{G}_q) = \begin{cases} fg_{11}(q) = fg_{12}(q) & \text{if } (p,f) \in (\mathscr{P}_1 \times \mathscr{F}_1) \cup (\mathscr{P}_1 \times \mathscr{F}_2) \\ fg_{22}(q) & \text{if } (p,f) \in \mathscr{P}_2 \times \mathscr{F}_2 \\ fg_{21}(q) & \text{if } (p,f) \in \mathscr{P}_2 \times \mathscr{F}_1 \end{cases}$$

where $g_{11}(T) = g_{12}(T) = g_{22}(T) = 2T$, $g_{11}(T) = 2T^2$, and $\mathscr{P}_1, \mathscr{P}_2$ are the Frobenius sets from Example 3.1.5, and the arithmetic progressions $\mathscr{F}_1, \mathscr{F}_2$ are given by

$$\mathscr{F}_1 = \{f \in \mathbb{N} : f \equiv 1 (\mathrm{mod}\ 2)\},$$

$$\mathscr{F}_2 = \{f \in \mathbb{N} : f \equiv 0 (\bmod 2)\}.$$

Now let $\mathfrak{g} = \mathfrak{f}_{n,c}$, the free nilpotent $\mathbb{Z}$-Lie algebra of class c on n generators, as in Definition 4.2.3. Using the concept of the reduced commutator matrix in Definition 4.2.7, the rank minimization problem in Theorem 4.3.1 can be reduced as follows:

Proposition 4.3.4. [BMKS19, Prop. 7.4] *Let $F = F^{red}_{\mathfrak{f}_{n,c}}$ be the reduced commutator matrix of $\mathfrak{f}_{n,c}$. Then, the faithful dimension of $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ is given by*

$$\min\left\{\sum_{\ell=1}^{r_n(c)} f q^{\frac{1}{2} rk_{\mathbb{F}_q}\left(F(a_{\ell 1},\ldots,a_{\ell r_n(c)})\right)} : \begin{pmatrix} a_{11} & \ldots & a_{1r_n(c)} \\ \vdots & & \vdots \\ a_{r_n(c)1} & \ldots & a_{r_n(c)r_n(c)} \end{pmatrix} \in GL_{r_n(c)}(\mathbb{F}_q)\right\},$$

where $r_n(c)$ is as in Equation (4.2.1).

Proof: To compute the rank using Theorem 4.3.1, we need to find $r_n(c)$ vectors of the form

$$a_\ell = (a_\ell^{(k)})_{2\le k\le c} = (a_{\ell 1}^{(2)}, \ldots, a_{\ell r_n(2)}^{(2)}, a_{\ell 1}^{(3)}, \ldots, a_{\ell r_n(3)}^{(3)}, \ldots, a_{\ell 1}^{(c)}, \ldots, a_{\ell r_n(c)}^{(c)}),$$

where $\ell = 1, \ldots, r_n(c)$, such that $\sum_{\ell=1}^{r_n(c)} f q^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{f}_{n,c}(a_\ell)})}$ is minimized and

$$\begin{pmatrix} a_{11}^{(c)} & \ldots & a_{1r_n(c)}^{(c)} \\ \vdots & & \vdots \\ a_{r_n(c)1}^{(c)} & \ldots & a_{r_n(c)r_n(c)}^{(c)} \end{pmatrix} \in GL_{r_n(c)}(\mathbb{F}_q). \tag{4.3.1}$$

Note that

$$\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{f}_{n,c}}(a_\ell)) \ge \sum_{i+j=c} \mathrm{rk}_{\mathbb{F}_q}(F_{ij}(a_\ell^{(c)})). \tag{4.3.2}$$

Suppose that the vectors a_ℓ, $\ell = 1, \ldots, r_n(c)$, are the solution to the rank minimization problem. We can assume that $a_{\ell i}^{(k)} = 0$ for all $2 \le k < c$ and $i = 1, \ldots, r_n(k)$. This is true since the condition in Equation (4.3.1) still holds, and the rank does not increase as shown in Equation (4.3.2). So, by Equation (4.3.2), we get same minimum value for both the reduced and the nonreduced cases. Thus, it suffices to use the reduced commutator matrix for the rank computation. ∎

For nilpotency classes $c = 2, 3$, the faithful dimension was computed in Theorem 2.13 of [BMKS19], as follows

Theorem 4.3.5. [BMKS19, Thm 2.13] *Let $n \ge 2$ and let $\mathbb{F}_q$ be the finite field with $q = p^f$ elements. Then*

i. $m_{faithful}(\exp(\mathfrak{f}_{n,2} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = \frac{n^2-n}{2} fq = r_n(2) fq$ *for* $p \ge 3$.

ii. $m_{faithful}(\exp(\mathfrak{f}_{n,3} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = \frac{n^3-n}{3} fq = r_n(3) fq$ *for* $p \ge 5$.

Chapter 5

Free Lie Algebras of Nilpotency Class 4

The goal of this chapter is to use the machinery of the Hall basis to compute the reduced commutator matrix of $\mathfrak{f}_{n,4}$, as defined in Definition 4.2.7. For an explicit description of the reduced commutator matrix, we divide this chapter into two parts. In the first part, namely Sections 5.1, 5.2 and 5.3, we describe the entries of the commutator matrix as linear combinations of variables indexed by Hall basis elements. In the second part, Section 5.4, we determine the number of occurrences and the exact entries of each Hall basis indexed variable in the commutator matrix.

5.1 Entries of Reduced Commutator Matrix and Hall Basis

Let $X = \{x_1, \ldots, x_n\}$. From Proposition 4.2.6(i) we see that the natural projection of $\mathcal{H}^1 \cup \mathcal{H}^2 \cup \mathcal{H}^3 \cup \mathcal{H}^4$ is the basis of $\mathfrak{f}_{n,4}$. These Hall sets are described as follows:

$$\mathcal{H}^1 = \{x_1, \ldots, x_n\}.$$

$$\begin{aligned}
\mathcal{H}^2 &= \{x_{ij} := [x_i, x_j] : i < j\}.\\
\mathcal{H}^3 &= \{x_{ijk} := [x_i, [x_j, x_k]] : j \le i, \quad j < k\}.\\
\mathcal{H}^4 &= \{x_{ijk\ell} := [x_i, [x_j, [x_k, x_\ell]]] : k \le j \le i, \quad k < \ell\}\\
&\quad \cup \{x_{ij,i\ell} := [\,[x_i, x_j], [x_i, x_\ell]\,] : i < j < \ell\}\\
&\quad \cup \{x_{ij,k\ell} := [\,[x_i, x_j], [x_k, x_\ell]\,] : i < j, \quad i < k < \ell\}\,.
\end{aligned}$$

Recall that Hall sets must be equipped with a total order. We use the natural order of indices for $\mathcal{H}^1$ and the lexicographical order of the indices for $\mathcal{H}^i$, for $i = 2, 3$. The total order on $\mathcal{H}^4$ is defined as follows: First, $\mathcal{H}^4$ is the union of the three sets given above, and we order their elements linearly. In other words, each element of the first set is less than each element of the second set, and so on. Then, within each of the three sets, we order elements according to the lexicographic order on the indices.

For simplicity, we will use the sets R_1, R_2, R_3 and R_4 to parametrize the sets $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3$ and $\mathcal{H}^4$, respectively, by their indices as follows,

$$\begin{aligned}
R_1 &= \{1, \ldots, n\}.\\
R_2 &= \{ij : i < j\}.\\
R_3 &= \{ijk : j \le i, \quad j < k\}.\\
R_4 &= \{ijk\ell : k \le j \le i, \quad k < \ell\} \cup \; \{(ij)(i\ell) : i < j < \ell\}\\
&\quad \cup \; \{(ij)(k\ell) : i < j, \quad i < k < \ell\},
\end{aligned}$$

where all the parameters $i, j, k, \ell \in \{1, \ldots, n\}$. For example ijk stands for the bracket $x_{ijk} = [x_i, [x_j, x_k]]$, and $(ij)(k\ell)$ stands for the bracket $x_{ij,k\ell} = [[x_i, x_j], [x_k, x_\ell]]$. From Proposition 4.2.6(ii) we see that the natural projection of $\mathcal{H}^2 \cup \mathcal{H}^3 \cup \mathcal{H}^4$ is the basis of $[\mathfrak{f}_{n,4}, \mathfrak{f}_{n,4}]$. Hence the entries of the commutator matrix $F_{\mathfrak{f}_{n,4}}(T^{(4)})$ have the variables

$T = (T^{(k)})_{2 \leq k \leq 4}$, where

$$\begin{aligned} T^{(2)} &= (T_1^{(2)}, \ldots, T_{r_n(2)}^{(2)}) = (T_s : s \in R_2), \\ T^{(3)} &= (T_1^{(3)}, \ldots, T_{r_n(3)}^{(3)}) = (T_s : s \in R_3), \\ T^{(4)} &= (T_1^{(4)}, \ldots, T_{r_n(4)}^{(4)}) = (T_s : s \in R_4). \end{aligned}$$

where $r_n(2), r_n(3)$ and $r_n(4)$ are as defined in Equation (4.2.1). As a result, we get the following reduced commutator matrix $F(T^{(4)}) := F^{\text{red}}_{\mathfrak{f}_{n,4}}(T^{(4)})$

$$F(T^{(4)}) = \begin{array}{c c} & \begin{array}{ccc} \mathcal{H}^1 & \mathcal{H}^2 & \mathcal{H}^3 \end{array} \\ \begin{array}{c} \mathcal{H}^1 \\ \mathcal{H}^2 \\ \mathcal{H}^3 \end{array} & \begin{pmatrix} 0 & 0 & F_{13}(T^{(4)}) \\ 0 & F_{22}(T^{(4)}) & 0 \\ -F_{13}^{tr}(T^{(4)}) & 0 & 0 \end{pmatrix} \end{array} \tag{5.1.1}$$

Our goal is to explicitly describe the reduced commutator matrix $F(T^{(4)})$, which can be done as follows:

1. We write the entries of $F(T^{(4)})$ as a linear combination of variables T_i indexed by $i \in R_4$. This is done in this section.

2. For each variable T_i, where $i \in R_4$, we find how many times it appears in $F(T^{(4)})$ and in which entries. This is done in Section 5.4.

Let us start with the entries of $F_{13}(T^{(4)})$. From Equation (2.5.1) and Definition 2.5.2 we know that the entries of $F_{13}(T^{(4)})$ correspond to brackets of the form

$$[x_i, x_{jk\ell}] = [x_i, [x_j, [x_k, x_\ell]]],$$

where $x_i \in \mathcal{H}^1$ and $x_{jk\ell} := [x_j, [x_k, x_\ell]] \in \mathcal{H}^3$, such that $j \geq k$ and $k < \ell$. As in Section 2.2, we call brackets of this form *left-aligned* brackets. We want to write the bracket

$[x_i, x_{jk\ell}]$ as a linear combination of elements in $\mathcal{H}^4$. Since $i \in R_1$ and $jk\ell \in R_3$, we will start with partitioning R_3 into the following subsets

$$\alpha = \{jk\ell \in R_3 : k < \ell < j\},$$
$$\beta = \{jk\ell \in R_3 : k < j < \ell\},$$
$$\gamma = \{jk\ell \in R_3 : k = j, \quad k < \ell\} = \{jj\ell \in R_3 : j < \ell\},$$
$$\delta = \{jk\ell \in R_3 : k < j, \quad \ell = j\} = \{jkj \in R_3 : k < j\}.$$

Then, we consider all possibilities for the position of i with respect to the order of $jk\ell \in R_3$. This is demonstrated in Figure 5.1.

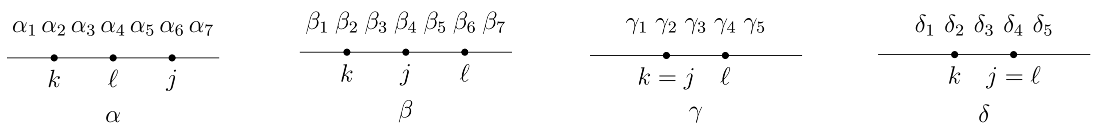

Figure 5.1: Possible positions of i with respect to j, k, ℓ.

For example, in Figure 5.1, the $\alpha_1, \ldots, \alpha_7$ represent the 7 possibilities of the position of i with respect to $k < \ell < j$, which is equivalent to $jk\ell \in \alpha$. The β_r's, γ_r's and δ_r's are defined analogously. Now, for each of the $7+7+5+5 = 24$ resulting cases we apply the Jacobi identity and the Hall basis conditions to express each bracket as a linear combination of elements of $\mathcal{H}^4$ (or equivalently, R_4). The description of these cases, and the basis expansion of their elements can be obtained by tedious calculations but it is also amenable to computer-assisted symbolic computation. Indeed we have done both. First we did the tedious calculation by hand (they are presented in Section 5.2). Then wrote a Python code based on the flowchart in Figure 5.2, which we will discuss further at the end of Section 5.2.

Proposition 5.1.1. *If $i \in R_1, jkl \in R_3$, then*

$$ijkl = \begin{cases} ijk\ell & ijk\ell \in \alpha_6 \cup \alpha_7 \cup \beta_{4,\dots,7} \cup \gamma_{2,\dots,5} \cup \delta_4 \cup \delta_5 \\ jki\ell - j\ell ik + (ij)(k\ell) & ijk\ell \in \alpha_1, \delta_1 \\ jki\ell - \ell jik + (ik)(j\ell) + (ij)(k\ell) & ijk\ell \in \beta_1, \gamma_1 \\ jik\ell + (ij)(k\ell) & ijk\ell \in \beta_2 \\ jik\ell - (k\ell)(ij) & o.w. \end{cases}$$

where $\beta_{4,\dots,7} = \cup_{m=4}^{7} \beta_m$ and $\gamma_{2,\dots,5} = \cup_{m=2}^{5} \gamma_m$.

Proposition 5.1.1 has the following corollary:

Corollary 5.1.2. *If $i \in R_1, jkl \in R_3$, then*

$$F_{13}(T^{(4)})_{i,jk\ell} = \begin{cases} T_{ijk\ell} & ijk\ell \in \alpha_6 \cup \alpha_7 \cup \beta_{4,\dots,7} \cup \gamma_{2,\dots,5} \cup \delta_4 \cup \delta_5 \\ T_{jki\ell} - T_{j\ell ik} + T_{ij,k\ell} & ijk\ell \in \alpha_1, \delta_1 \\ T_{jki\ell} - T_{\ell jik} + T_{ik,j\ell} + T_{ij,k\ell} & ijk\ell \in \beta_1, \gamma_1 \\ T_{jik\ell} + T_{ij,k\ell} & ijk\ell \in \beta_2 \\ T_{jik\ell} - T_{k\ell,ij} & o.w. \end{cases}$$

where $F_{13}(T^{(4)})_{i,jk\ell}$ is the entry in row $i \in R_1$ and column $jk\ell \in R_3$.

5.2 Proof of Proposition 5.1.1

The proof of Proposition 5.1.1 is a lengthy case-by-case argument. Note that in each case, we apply the Jacobi identity on a commutator $[x_i, [x_j, [x_k, x_\ell]]]$, where $i \in R_1$ and $jk\ell \in R_3$. For simplicity, we suppress the x's in commutators by writing

$$[i, [j, [k, \ell]]] := [x_i, [x_j, [x_k, x_\ell]]].$$

It is straightforward to verify that the last line of each case is a linear combination of elements of R_4.

1. Let $\alpha_1 := \{ijk\ell : i < k < \ell < j\}$. Then

$$\begin{aligned}[i,[j,[k,\ell]]] &= -[j,[[k,\ell],i]] - [[k,\ell],[i,j]] \\ &= [j,[i,[k,\ell]]] + [[i,j],[k,\ell]] \\ &= [j,-[k,[\ell,i]] - [\ell,[i,k]]] + [[i,j],[k,\ell]] \\ &= [j,[k,[i,\ell]]] - [j,[\ell,[i,k]]] + [[i,j],[[k,\ell]]]\end{aligned}$$

   So, $ijk\ell = jki\ell - j\ell ik + (ij)(k\ell)$.

2. Let $\alpha_2 := \{ijk\ell : k = i < \ell < j\}$. Then

$$\begin{aligned}[i,[j,[i,\ell]]] &= -[j,[[i,\ell],i]] - [[i,\ell],[i,j]] \\ &= [j,[i,[i,\ell]]] - [[i,\ell],[i,j]]\end{aligned}$$

   So, $iji\ell = jii\ell - (i\ell)(ij)$.

3. Let $\alpha_3 := \{ijk\ell : k < i < \ell < j\}$. Then

$$\begin{aligned}[i,[j,[k,\ell]]] &= -[j,[[k,\ell],i]] - [[k,\ell],[i,j]] \\ &= [j,[i,[k,\ell]]] - [[k,\ell],[i,j]]\end{aligned}$$

   So, $ijk\ell = jik\ell - (k\ell)(ij)$.

4. Let $\alpha_4 := \{ijki : k < i < j\}$. Then

$$\begin{aligned}[i,[j,[k,i]]] &= -[j,[[k,i],i]] - [[k,i],[i,j]] \\ &= [j,[i,[k,i]]] - [[k,i],[i,j]]\end{aligned}$$

So, $ijki = jiki - (ki)(ij)$.

5. Let $\alpha_5 := \{ijk\ell : k < \ell < i < j\}$. Then

$$\begin{aligned}[i,[j,[k,\ell]]] &= -[j,[[k,\ell],i]] - [[k,\ell],[i,j]] \\ &= [j,[i,[k,\ell]]] - [[k,\ell],[i,j]]\end{aligned}$$

So, $ijk\ell = jik\ell - (k\ell)(ij)$.

6. Let $\alpha_6 := \{iik\ell : k < \ell < i\}$. Then $iik\ell \in R_4$.

7. Let $\alpha_7 := \{ijk\ell : k < \ell < j < i\}$. Then $ijk\ell \in R_4$.

8. Let $\beta_1 := \{ijk\ell : i < k < j < \ell\}$. Then

$$\begin{aligned}[i,[j,[k,\ell]]] &= -[j,[[k,\ell],i]] - [[k,\ell],[i,j]] \\ &= [j,[i,[k,\ell]]] + [[i,j],[k,\ell]] \\ &= [j,-[k,[\ell,i]] - [\ell,[i,k]]] + [[i,j],[k,\ell]] \\ &= [j,[k,[i,\ell]]] - [j,[\ell,[i,k]]] + [[i,j],[k,\ell]] \\ &= [j,[k,[i,\ell]]] - (-[\ell,[[i,k],j]] - [[i,k],[j,\ell]]) + [[i,j],[k,\ell]] \\ &= [j,[k,[i,\ell]]] - [\ell,[j,[i,k]]] + [[i,k],[j,\ell]] + [[i,j],[k,\ell]]\end{aligned}$$

So, $ijk\ell = jki\ell - \ell jik + (ik)(j\ell) + (ij)(k\ell)$.

9. Let $\beta_2 := \{iji\ell : i < j < \ell\}$. Then

$$\begin{aligned}[i,[j,[i,\ell]]] &= -[j,[[i,\ell],i]] - [[i,\ell],[i,j]] \\ &= [j,[i,[i,\ell]]] + [[i,j],[i,\ell]]\end{aligned}$$

So, $iji\ell = jii\ell + (ij)(i\ell)$.

10. Let $\beta_3 := \{ijk\ell : k < i < j < \ell\}$. Then

$$\begin{aligned}[i,[j,[k,\ell]]] &= -[j,[[k,\ell],i]] - [[k,\ell],[i,j]] \\ &= [j,[i,[k,\ell]]] - [[k,\ell],[i,j]]\end{aligned}$$

So, $ijk\ell = jik\ell - (k\ell)(ij)$.

11. Let $\beta_4 := \{iik\ell : k < i < \ell\}$. Then $iik\ell \in R_4$.

12. Let $\beta_5 := \{ijk\ell : k < j < i < \ell\}$. Then $ijk\ell \in R_4$.

13. Let $\beta_6 := \{ijki : k < j < i\}$. Then $ijki\ell \in R_4$.

14. Let $\beta_7 := \{ijk\ell : k < j < \ell < i\}$. Then $ijk\ell \in R_4$.

15. Let $\gamma_1 := \{ijj\ell : i < j < \ell\}$. Then

$$\begin{aligned}[i,[j,[j,\ell]]] &= -[j,[[j,\ell],i]] - [[j,\ell],[i,j]] \\ &= [j,[i,[j,\ell]]] + [[i,j],[j,\ell]] \\ &= [j,-[j,[\ell,i]] - [\ell,[i,j]]] + [[i,j],[j,\ell]] \\ &= [j,[j,[i,\ell]]] - [j,[\ell,[i,j]]] + [[i,j],[j,\ell]] \\ &= [j,[j,[i,\ell]]] - (-[\ell,[[i,j],j]] - [[i,j],[j,\ell]]) + [[i,j],[j,\ell]] \\ &= [j,[j,[i,\ell]]] - ([\ell,[j,[i,j]]] - [[i,j],[j,\ell]]) + [[i,j],[j,\ell]] \\ &= [j,[j,[i,\ell]]] - [\ell,[j,[i,j]]] + [[i,j],[j,\ell]] + [[i,j],[j,\ell]] \\ &= [j,[j,[i,\ell]]] - [\ell,[j,[i,j]]] + 2[[i,j],[j,\ell]]\end{aligned}$$

So, $ijj\ell = jji\ell - \ell jij + 2(ij)(j\ell)$.

16. Let $\gamma_2 := \{iii\ell : i < \ell\}$. Then $iii\ell \in R_4$.

17. Let $\gamma_3 := \{ijj\ell : j < i < \ell\}$. Then $ijj\ell \in R_4$.

18. Let $\gamma_4 := \{ijji : j < i\}$. Then $ijji \in R_4$.

19. Let $\gamma_5 := \{ijj\ell : j < \ell < i\}$. Then $ijj\ell \in R_4$.

20. Let $\delta_1 := \{ijkj : i < k < j\}$. Then

$$\begin{aligned}[i,[j,[k,j]]] &= -[j,[[k,j],i]] - [[k,j],[i,j]] \\ &= [j,[i,[k,j]]] + [[i,j],[k,j]] \\ &= [j,-[k,[j,i]] - [j,[i,k]]] + [[i,j],[k,j]] \\ &= [j,[k,[i,j]] - [j,[j,[i,k]]] + [[i,j],[k,j]]\end{aligned}$$

So, $ijkj = jkij - jjik + (ij)(kj)$.

21. Let $\delta_2 := \{ijij : i < j\}$. Then

$$\begin{aligned}[i,[j,[i,j]]] &= -[j,[[i,j],i]] - [[i,j],[i,j]] \\ &= [j,[i,[i,j]]]\end{aligned}$$

So, $ijij = jiij$.

22. Let $\delta_3 := \{ijkj : k < i < j\}$. Then

$$\begin{aligned}[i,[j,[k,j]]] &= -[j,[[k,j],i]] - [[k,j],[i,j]] \\ &= [j,[i,[k,j]]] - [[k,j],[i,j]]\end{aligned}$$

So, $ijkj = jikj - (kj)(ij)$.

23. Let $\delta_4 := \{iiki : k < i\}$. Then $iiki \in R_4$.

24. Let $\delta_5 := \{ijkj : k < j < i\}$. Then $ijkj \in R_4$.

Let us now give an algorithmic flavour to the above proof. In Figure 5.2 we give a flowchart version of the strategies used throughout the computations in this section. We describe the different ways we can apply the Jacobi identity as follows:

$$\begin{array}{ll} \mathrm{I} & abcd = bacd - (cd)(ab), \\ \mathrm{II} & abcd = acbd - adbc. \end{array}$$

The Python code for this flowchart appears in Appendix A.1.

5.3 Computing $F_{22}(T^{(4)})$

Let us move on to the computation of the entries of $F_{22}(T^{(4)})$. From Equation (2.5.1) and Definition 2.5.2 we know that the entries of $F_{22}(T^{(4)})$ correspond to brackets of the following form

$$[x_{ij}, x_{k\ell}] = [[x_i, x_j], [x_k, x_\ell]],$$

where $x_{ij}, x_{k\ell} \in \mathcal{H}^2$. We call brackets of this form *balanced* brackets. We want to write $[x_{ij}, x_{k\ell}]$ as a linear combination of elements in $\mathcal{H}^4$. From Definition 4.1.4 and the lexicographic order $\mathcal{H}^2$ is equipped with, we see that if $i \leq k$, then $[[x_i, x_j], [x_k, x_\ell]] \in \mathcal{H}^4$. Otherwise, if $i > k$, then

$$[[x_i, x_j], [x_k, x_\ell]] = -[[x_k, x_\ell], [x_i, x_j]],$$

and $[[x_k, x_\ell], [x_i, x_j]] \in \mathcal{H}^4$. As a result, we obtain the following proposition, where we continue using the notation of Proposition 5.1.1.

Proposition 5.3.1. *If $ij \in R_2, k\ell \in R_2$, then*

$$(ij)(k\ell) = \begin{cases} (ij)(k\ell) & (ij)(k\ell) \in \mu \cup \nu_{1,\dots,5} \\ -(ij)(k\ell) & o.w. \end{cases}$$

where $\mu = \{(ij)(i\ell) : i < j < \ell\}$, *and*

$$\nu_{1,\ldots,5} = \cup_{m=1}^{5} \nu_m = \{(ij)(k\ell) : i < j, i < k < \ell\}.$$

The ν_m*'s are described in Figure 5.3 below.*

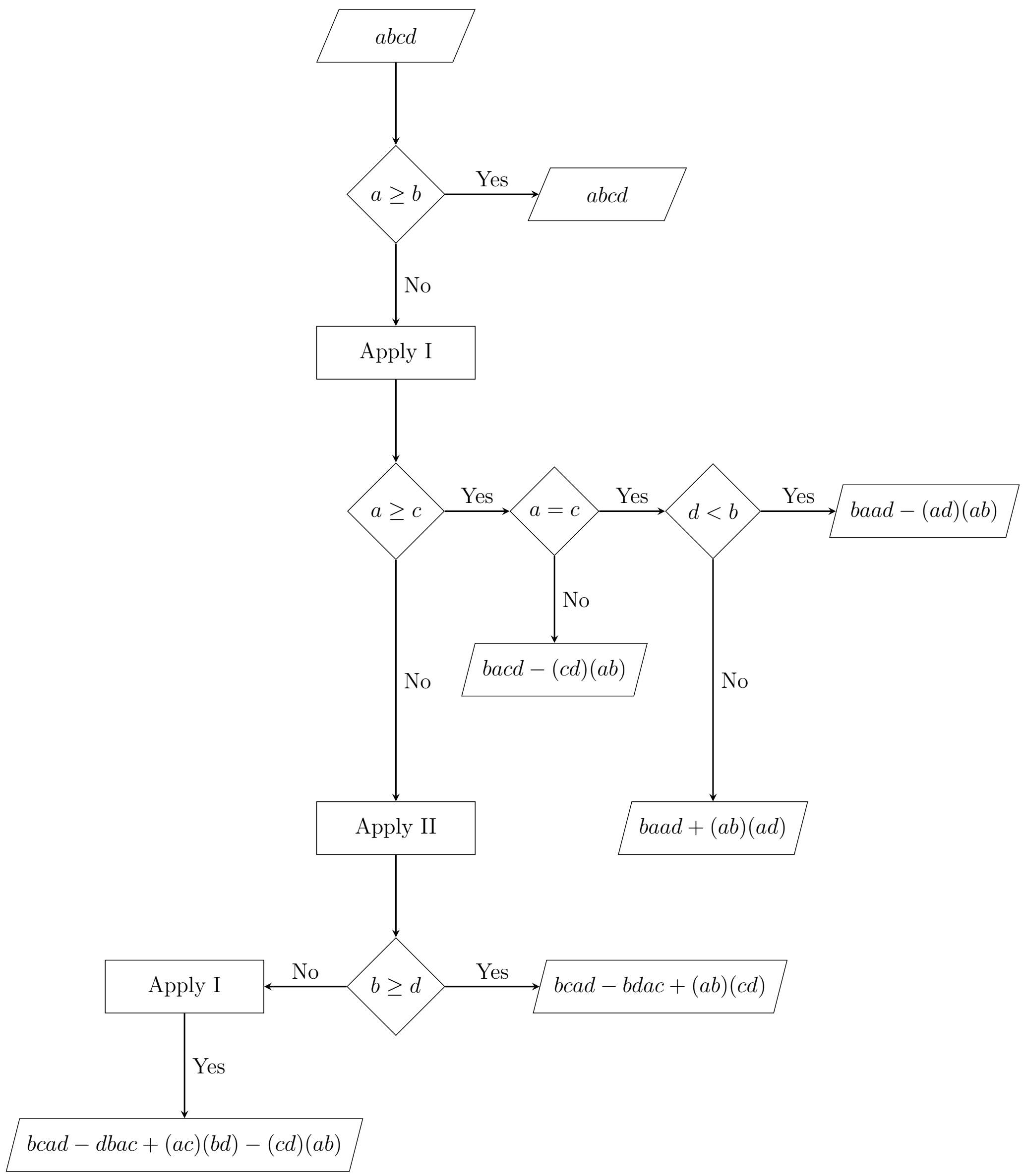

Figure 5.2: Hall basis expansion algorithms.

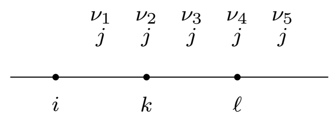

Figure 5.3: Possible positions of j with respect to i, k, ℓ.

Corollary 5.3.2. *If $ij \in R_2, k\ell \in R_2$, then*

$$F_{22}(T^{(4)})_{ij,k\ell} = \begin{cases} T_{ij,k\ell} & (ij)(k\ell) \in \mu \cup \nu_{1,\ldots,5} \\ -T_{k\ell,ij} & o.w. \end{cases}$$

where $F_{22}(T^{(4)})_{ij,k\ell}$ is the entry in row $ij \in R_2$ and column $k\ell \in R_2$.

5.4 Explicit Description of Reduced Commutator matrix $F(T^{(4)})$

Recall from Section 5.1 that for each $t \in R_4$, we want to find how many times T_t appears in $F(T^{(4)})$ and in which entries. Note that from Figure 5.1 we have

$$R_4 = (\alpha_6 \cup \alpha_7) \cup (\cup_{m=4}^{7} \beta_m) \cup (\cup_{m=2}^{5} \gamma_m) \cup (\delta_4 \cup \delta_5) \cup \mu \cup (\cup_{m=1}^{5} \nu_m). \tag{5.4.1}$$

Depending on which part of the above partition t belongs to in R_4, the word t falls into one of the following types: The *First Type* is when $t = abcd$, and the *Second Type* is when $t = (ab)(cd)$.

Remark 5.4.1. By Corollary 5.1.2 and Corollary 5.3.2, when t is of the first type, T_t can only appear in $F_{13}(T^{(4)})$ entries, whereas in the second type, T_t can appear in both $F_{13}(T^{(4)})$ and $F_{22}(T^{(4)})$ entries.

To find all the occurrences of T_t in $F(T^{(4)})$, we consider the type of t as follows:

- *First Type*: We need to find all possible $i \in R_1$ and $jk\ell \in R_3$ such that $t = abcd$ is part of the expression of $ijk\ell$ in terms of elements from R_4.

- *Second Type*: As in the first type, we need to find all possible $i \in R_1$ and $jk\ell \in R_3$ such that $t = (ab)(cd)$ is part of the basis expansion of $ijk\ell$. As mentioned in Remark 5.4.1, we also need to find all $ij \in R_2$ and $k\ell \in R_2$ such that $t = (ab)(cd)$ is in the basis expansion of $(ij)(k\ell)$. Proposition 5.3.1 shows that the only possibilities are $\{ij,\ k\ell\} \in \{ab,\ cd\}$, since $(ab)(cd) = t$ and $(cd)(ab) = -(ab)(cd) = -t$.

Therefore, we need to analyze each basis element $t \in R_4$. We consider the cases in the order that the parts appears in the partition in Equation (5.4.1) as follows:

1. If $t \in \alpha_6$, then $t = bbcd$ with $c < d < b$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

   - $i, j, k, \ell \in \{b, c, d\}$, with only b allowed to be repeated twice.
   - Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
     - If $k = c$, then we can take $\ell \in \{d, b\}$ and $j \in \{c, d, b\}$. Since only b can be repeated twice, we must have $(\ell, j) \in \{(d, b), (b, d), (b, b)\}$.
     - If $k = d$, then we can take $\ell = b$ and $j \in \{d, b\}$. Since only b can be repeated twice, we only have $\ell = j = b$.

   So, we have the following possibilities

   $$ijk\ell \in \{bbcd,\ bdcb,\ dbcb,\ cbdb\}.$$

   Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

   (a) $ijk\ell = bbcd = t \in \alpha_6$.

   (b) $ijk\ell = bdcb \in \beta_6$. Since $\beta_6 \subset R_4$ and $bdcb \neq t$, this case does not lead into an occurrence of T_t.

(c) $ijk\ell = dbcb \in \delta_3$. Then, $dbcb = bdcb - (cb)(db)$. Since t is not in the basis expansion of $dbcb$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = cbdb \in \delta_1$. Then, $cbdb = bdcb - bbcd + (cb)(db) = bdcb - t + (cb)(db)$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$; once with coefficient $+1$ and once with coefficient -1, as shown below:

	row	*column*	*entry*
1	b	bcd	$T_{bbcd} = T_t$
2	c	bdb	$T_{bdcb} - T_t + T_{(cb)(db)}$

As a result, T_t appears 4 times in $F(T^{(4)})$: two times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

2. If $t \in \alpha_7$, then $t = abcd$ with $c < d < b < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c, d\}$, with no repetition allowed.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = c$, then we can take $\ell \in \{d, b, a\}$ and $j \in \{c, d, b, a\}$. Since no repetition is allowed, then we must have

$$(\ell, j) \in \{(d, b), (d, a), (b, d), (b, a), (a, d), (a, b)\}.$$

  - If $k = d$, then we can take $\ell = \{b, a\}$ and $j \in \{d, b, a\}$. Since no repetition is allowed, we must have $(\ell, j) \in \{(b, a), (a, b)\}$.

So, we have the following possibilities

$$ijk\ell \in \{abcd,\ bacd,\ adcb,\ dacb,\ bdca,\ dbca,\ cadb,\ cbda\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = abcd = t \in \alpha_7$.

(b) $ijk\ell = bacd \in \alpha_5$. Then, $bacd = abcd - (cd)(ba) = t - (cd)(ba)$.

(c) $ijk\ell = adcb \in \beta_7$. Since $\beta_7 \subset R_4$ and $adcb \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = dacb \in \alpha_3$. Then, $dacb = adcb - (cb)(da)$. Since t is not in the basis expansion of $dacb$, this case does not lead into an occurrence of T_t.

(e) $ijk\ell = bdca \in \beta_5$. Since $\beta_5 \subset R_4$ and $bdca \neq t$, this case does not lead into an occurrence of T_t.

(f) $ijk\ell = dbca \in \beta_3$. Then, $dbca = bdca - (ca)(db)$. Since t is not in the basis expansion of $dbca$, this case does not lead into an occurrence of T_t.

(g) $ijk\ell = cadb \in \alpha_1$. Then, $cadb = adcb - abcd + (ca)(db) = adcb - t + (ca)(db)$.

(h) $ijk\ell = cbda \in \beta_1$. Then,

$$cbda = bdca - abcd + (cd)(ba) + (cb)(da) = bdca - t + (cd)(ba) + (cb)(da).$$

So, T_t appears 4 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bcd	$T_{abcd} = T_t$
2	b	acd	$T_t - T_{(cd)(ba)}$
3	c	adb	$T_{adcb} - T_t + T_{(ca)(db)}$
4	c	bda	$T_{bdca} - T_t + T_{(cd)(ba)} + T_{(cb)(da)}$

As a result, T_t appears 8 times in $F(T^{(4)})$: 4 times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

3. If $t \in \beta_4$, then $t = aacd$ with $c < a < d$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

   - $i, j, k, \ell \in \{a, c, d\}$, with only a allowed to be repeated twice.
   - Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
     - If $k = c$, then we can take $\ell \in \{a, d\}$ and $j \in \{c, a, d\}$. Since only a can be repeated twice, we must take $(\ell, j) \in \{(a, a), (a, d), (d, a)\}$.
     - If $k = a$, then we can take $\ell = d$ and $j \in \{a, d\}$. Since only a can be repeated twice, we only have $(\ell, j) = (d, a)$.

   So, we have the following possibilities

$$ijk\ell \in \{daca,\ adca,\ aacd,\ caad\}.$$

   Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

   (a) $ijk\ell = daca \in \delta_5$. Since $\delta_5 \subset R_4$ and $daca \neq t$, this case does not lead into an occurrence of T_t.

   (b) $ijk\ell = adca \in \alpha_4$. Then, $adca = daca - (ca)(ad)$. Since t is not in the basis expansion of $adca$, this case does not lead into an occurrence of T_t.

   (c) $ijk\ell = aacd = t \in \beta_4$.

   (d) $ijk\ell = caad \in \gamma_1$. Then, $caad = aacd - daca + 2(ca)(ad) = t - daca + 2(ca)(ad)$.

   So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	acd	$T_{aacd} = T_t$
2	c	aad	$T_t - T_{daca} + 2T_{(ca)(ad)}$

As a result, T_t appears 4 times in $F(T^{(4)})$: two times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

4. If $t \in \beta_5$, then $t = abcd$ with $c < b < a < d$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

   - $i, j, k, \ell \in \{a, b, c, d\}$, with no repetition allowed.
   - Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
     - If $k = c$, then we can take $\ell \in \{b, a, d\}$ and $j \in \{c, b, a, d\}$. Since no repetition is allowed, we must have $(\ell, j) \in \{(b, a), (b, d), (a, b), (a, d), (d, b), (d, a)\}$.
     - If $k = b$, then we can take $\ell \in \{a, d\}$ and $j \in \{b, a, d\}$. Since no repetition is allowed, we only have $(\ell, j) \in \{(a, d), (d, a)\}$.

   So, we have the following possibilities

   $$ijk\ell \in \{dacb,\ adcb,\ dbca,\ bdca,\ abcd,\ bacd,\ cdba,\ cabd\}.$$

   Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

   (a) $ijk\ell = dacb \in \alpha_7$. Since $\alpha_7 \subset R_4$ and $dacb \neq t$, this case does not lead into an occurrence of T_t.

   (b) $ijk\ell = adcb \in \alpha_5$. Then, $adcb = dacb - (cb)(ad)$. Since t is not in the basis expansion of $adcb$, this case does not lead into an occurrence of T_t.

   (c) $ijk\ell = dbca \in \beta_7$. Since $\beta_7 \subset R_4$ and $dbca \neq t$, this case does not lead into an occurrence of T_t.

   (d) $ijk\ell = bdca \in \alpha_3$. Then, $bdca = dbca - (ca)(bd)$. Since t is not in the basis expansion of $bdca$, this case does not lead into an occurrence of T_t.

   (e) $ijk\ell = abcd = t \in \beta_5$.

(f) $ijk\ell = bacd \in \beta_3$. Then, $bacd = abcd - (cd)(ba) = t - (cd)(ba)$.

(g) $ijk\ell = cdba \in \alpha_1$. Then, $cdba = dbca - dacb + (cd)(ba)$. Since t is not in the basis expansion of $cdba$, this case does not lead into an occurrence of T_t

(h) $ijk\ell = cabd \in \beta_1$. Then,

$$cabd = abcd - dacb + (cb)(ad) + (ca)(bd) = t - dacb + (cb)(ad) + (ca)(bd).$$

So, T_t appears 3 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bcd	$T_{abcd} = T_t$
2	b	acd	$T_t - T_{(cd)(ba)}$
3	c	abd	$T_t - T_{dacb} + T_{(cb)(ad)} + T_{(ca)(bd)}$

As a result, T_t appears 6 times in $F(T^{(4)})$: three times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

5. If $t \in \beta_6$, then $t = abca$ with $c < b < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c\}$, with only a allowed to be repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = c$, then we can take $\ell \in \{b, a\}$ and $j \in \{c, b, a\}$. Since only a ca be repeated twice, we must have $(\ell, j) \in \{(b, a), (a, b), (a, a)\}$.
  - If $k = b$, then we can take $\ell = a$ and $j \in \{b, a\}$. Since only a ca be repeated twice, we only have $j = \ell = a$.

So, we have the following possibilities

$$ijk\ell \in \{aacb,\ abca,\ baca,\ caba\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = aacb \in \alpha_6$. Since $\alpha_6 \subset R_4$ and $aacb \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = abca = t \in \beta_6$.

(c) $ijk\ell = baca \in \delta_3$. Then, $baca = abca - (ca)(ba) = t - (ca)(ba)$.

(d) $ijk\ell = caba \in \delta_1$. Then, $caba = abca - aacb + (ca)(ba) = t - aacb + (ca)(ba)$.

So, T_t appears 3 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bca	$T_{abca} = T_t$
2	b	aca	$T_t - T_{(ca)(ba)}$
3	c	aba	$T_t - T_{aacb} + T_{(ca)(ba)}$

As a result, T_t appears 6 times in $F(T^{(4)})$: three times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

6. If $t \in \beta_7$, then $t = abcd$ with $c < b < d < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c, d\}$, with no repetition allowed.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = c$, then we can take $\ell \in \{b, d, a\}$ and $j \in \{c, b, d, a\}$. Since no repetition allowed, we must have $(\ell, j) \in \{(b, d), (b, a), (d, b), (d, a), (a, b), (a, d)\}$.
  - If $k = b$, then we can take $\ell \in \{d, a\}$ and $j \in \{b, d, a\}$. Since no repetition allowed, we only have $(\ell, j) \in \{(d, a), (a, d)\}$.

So, we have the following possibilities

$$ijk\ell \in \{adcb,\ dacb,\ abcd,\ bacd,\ dbca,\ bdca,\ cabd,\ cdba\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = adcb \in \alpha_7$. Since $\alpha_7 \subset R_4$ and $adcb \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = dacb \in \alpha_5$. Then, $dacb = adcb - (cb)(da)$. Since t is not in the basis expansion of $dacb$, this case does not lead into an occurrence of T_t.

(c) $ijk\ell = abcd = t \in \beta_7$.

(d) $ijk\ell = bacd \in \alpha_3$. Then, $bacd = abcd - (cd)(ba) = t - (cd)(ba)$.

(e) $ijk\ell = dbca \in \beta_5$. Since $\beta_5 \subset R_4$ and $dbca \neq t$, this case does not lead into an occurrence of T_t.

(f) $ijk\ell = bdca \in \beta_3$. Then, $bdca = dbca - (ca)(bd)$. Since t is not in the basis expansion of $bdca$, this case does not lead into an occurrence of T_t.

(g) $ijk\ell = cabd \in \alpha_1$. Then, $cabd = abcd - adcb + (ca)(bd) = t - adcb + (ca)(bd)$.

(h) $ijk\ell = cdba \in \beta_1$. Then, $cdba = dbca - adcb + (cb)(da) + (cd)(ba)$. Since t is not in the basis expansion of $cdba$, this case does not lead into an occurrence of T_t.

So, T_t appears 3 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bcd	$T_{abcd} = T_t$
2	b	acd	$T_t - T_{(cd)(ba)}$
3	c	abd	$T_t - T_{adcb} + T_{(ca)(bd)}$

As a result, T_t appears 6 times in $F(T^{(4)})$: three times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

7. If $t \in \gamma_2$, then $t = aaad$ with $a < d$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

   - $i, j, k, \ell \in \{a, d\}$, with only a allowed to be repeated three times.
   - Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. This implies that we must take $k = a$, $\ell = d$ and $j \in \{a, d\}$. Since only a is allowed to be repeated three times, we can only take $j = a$.

   So, the only possibility is $ijk\ell = aaad = t$. Thus, T_t appears only once in $F_{13}(T^{(4)})$ in the following entry:

	row	*column*	*entry*
1	a	aad	$T_{aaad} = T_t$

   As a result, T_t appears 2 times in $F(T^{(4)})$: once in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

8. If $t \in \gamma_3$, then $t = abbc$ with $b < a < c$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

   - $i, j, k, \ell \in \{a, b, c\}$, with only b allowed to be repeated twice.
   - Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
     - If $k = b$, then we can take $\ell \in \{a, c\}$ and $j \in \{a, b, c\}$. Since only b can be repeated twice, we must have $(\ell, j) \in \{(a, b), (a, c), (c, a), (c, b)\}$.
     - If $k = a$, then we can take $\ell = c$ and $j \in \{a, c\}$. Since only b can be repeated twice, there is no possible choices for ℓ or j.

So, we have the following possibilities

$$ijk\ell \in \{cbba,\ bcba,\ babc,\ abbc\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = cbba \in \gamma_5$. Since $\gamma_5 \subset R_4$ and $cbba \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = bcba \in \alpha_2$. Then, $bcba = cbba - (ba)(bc)$. Since t is not in the basis expansion of $bcba$, this case does not lead into an occurrence of T_t.

(c) $ijk\ell = babc \in \beta_2$. Then, $babc = abbc + (ba)(bc) = t + (ba)(bc)$.

(d) $ijk\ell = abbc = t \in \gamma_3$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	b	abc	$T_t + T_{(ba)(bc)}$.
2	a	bbc	$T_{abbc} = T_t$.

As a result, T_t appears 4 times in $F(T^{(4)})$: twice in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

9. If $t \in \gamma_4$, then $t = abba$ with $b < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b\}$, with both a and b allowed to be repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. This implies that we must take $k = b$, $\ell = a$ and $j \in \{b, a\}$. Since both a and b are allowed to be repeated twice, we must have $(\ell, j) \in \{(a, b), (a, a)\}$.

So, we have the following possibilities

$$ijk\ell \in \{abba,\ baba\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = abba = t \in \gamma_4$.

(b) $ijk\ell = baba \in \delta_2$. Then, $baba = abba = t$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bba	$T_{abba} = T_t$.
2	b	aba	$T_{abba} = T_t$.

As a result, T_t appears 4 times in $F(T^{(4)})$: twice in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

10. If $t \in \gamma_5$, then $t = abbc$ with $b < c < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c\}$, with only b allowed to be repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = b$, then we can take $\ell \in \{c, a\}$ and $j \in \{b, c, a\}$. Since b is allowed to be repeated twice, we must have $(\ell, j) \in \{(c, b), (c, a), (a, b), (a, c)\}$.
  - If $k = c$, then we can take $\ell = a$ and $j \in \{c, a\}$. Since only b is allowed to be repeated twice, there are no possible choices for (ℓ, j).

So, we have the following possibilities

$$ijk\ell \in \{abbc,\ babc,\ cbba,\ bcba\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = abbc = t \in \gamma_5$.

(b) $ijk\ell = babc \in \alpha_2$. Then, $babc = abbc - (bc)(ba) = t - (bc)(ba)$

(c) $ijk\ell = cbba \in \gamma_3$. Since $\gamma_3 \subset R_4$ and $cbba \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = bcba \in \beta_2$. Then, $bcba = cbba + (bc)(ba)$. Since t is not in the basis expansion of $bcba$, this case does not lead into an occurrence of T_t.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	bbc	$T_{abbc} = T_t$.
2	b	abc	$T_t - T_{(bc)(ba)}$.

As a result, T_t appears 4 times in $F(T^{(4)})$: twice in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

11. If $t \in \delta_4$, then $t = bbcb$ with $c < b$.Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{b, c\}$, with only b allowed to be repeated three times.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. This implies that we must take $k = c$, $\ell = b$ and $j \in \{b, c\}$. Since only b is allowed to be repeated three times, we must have $(\ell, j) = (b, b)$.

So, the only possibility is $ijk\ell = bbcb = t$. Thus, T_t appears only once in $F_{13}(T^{(4)})$ in the following entry:

	row	*column*	*entry*
1	b	bcb	$T_{bbcb} = T_t$.

As a result, T_t appears 2 times in $F(T^{(4)})$: once in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

12. If $t \in \delta_5$, then $t = abcb$ with $c < b < a$. Thus, t belongs to the *First Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c\}$, with only both b allowed to be repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = c$, then we can take $\ell \in \{b, a\}$ and $j \in \{a, b, c\}$. Since b is allowed to be repeated twice, we must have $(\ell, j) \in \{(b, a), (b, b), (a, b)\}$.
  - If $k = b$, then we can take $\ell = a$ and $j \in \{a, b\}$. Since b is allowed to be repeated twice, there is only one possible choice for $(\ell, j) = (a, b)$.

So, we have the following possibilities

$$ijk\ell \in \{bacb,\ abcb,\ bbca,\ cbba\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = bacb \in \alpha_4$. Then, $bacb = abcb - (cb)(ba) = t - (cb)(ba)$.

(b) $ijk\ell = abcb = t \in \delta_5$.

(c) $ijk\ell = bbca \in \beta_4$. Since $\beta_4 \subset R_4$ and $bbca \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = cbba \in \gamma_1$. Then, $cbba = bbca - abcb + 2(cb)(ba) = bbca - t + 2(cb)(ba)$.

So, T_t appears 3 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	b	acb	$T_{abcb} = T_t - T_{(cb)(ba)}$.
2	a	bcb	$T_{abcb} = T_t$.
3	c	bba	$T_{bbca} - T_t + 2T_{(cb)(ba)}$.

As a result, T_t appears 6 times in $F(T^{(4)})$: three times in each of the blocks $F_{13}(T^{(4)})$ and $-F_{13}^{tr}(T^{(4)})$.

13. If $t \in \nu_1$, then $t = (ab)(ad)$ with $a < b < d$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, d\}$, with only both a allowed to be repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{b, d\}$ and $j \in \{a, b, d\}$. Since a is allowed to be repeated twice, we must have $(\ell, j) \in \{(b, a), (b, d), (d, a), (d, b)\}$.
  - If $k = b$, then we can take $\ell = d$ and $j \in \{b, d\}$. Since a is allowed to be repeated twice, there is no possible choice for (ℓ, j).

So, we have the following possibilities

$$ijk\ell \in \{daab,\ adab,\ baad,\ abad\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = daab \in \gamma_5$. Since $\gamma_5 \subset R_4$ and $daab \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = adab \in \alpha_2$. Then, $adab = daab - (ab)(ad) = daab - t$.

(c) $ijk\ell = baad \in \gamma_3$. Since $\gamma_3 \subset R_4$ and $baad \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = abad \in \beta_2$. Then, $abad = baad + (ab)(ad) = baad + t$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	a	dab	$T_{daab} - T_t$.
2	a	bad	$T_{baad} + T_t$.

Since $t = (ab)(ad)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	*column*	*entry*
1	ab	ad	$T_{(ab)(ad)} = T_t$.
2	ad	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

14. If $t \in \nu_2$, then $t = (ab)(cd)$ with $a < b < c < d$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c, d\}$, with no repetition allowed.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{b, c, d\}$ and $j \in \{a, b, c, d\}$. Since no repetition is allowed, we must have

$$(\ell, j) \in \{(b, c), (b, d), (c, b), (c, d), (d, b), (d, c)\}.$$

  - If $k = b$, then we can take $\ell = \{c, d\}$ and $j \in \{b, c, d\}$. Since no repetition is allowed, we must have $(\ell, j) \in \{(c, d), (d, c)\}$.

So, we have the following possibilities

$$ijk\ell \in \{dcab,\ cdab,\ dbac,\ bdac,\ cbad,\ bcad,\ adbc,\ acbd\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = dcab \in \alpha_7$. Since $\alpha_7 \subset R_4$ and $dcab \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = cdab \in \alpha_5$. Then, $cdab = dcab - (ab)(cd) = dcab - t$.

(c) $ijk\ell = dbac \in \beta_7$. Since $\beta_7 \subset R_4$ and $dbac \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = bdac \in \alpha_3$. Then, $bdac = dbac - (ac)(bd)$. Since t is not in the basis expansion of $bdac$, this case does not lead into an occurrence of T_t.

(e) $ijk\ell = cbad \in \beta_5$. Since $\beta_5 \subset R_4$ and $cbad \neq t$, this case does not lead into an occurrence of T_t.

(f) $ijk\ell = bcad \in \beta_3$. Then, $bcad = cbad - (ad)(bc)$. Since t is not in the basis expansion of $bcad$, this case does not lead into an occurrence of T_t.

(g) $ijk\ell = adbc \in \alpha_1$. Then, $adbc = dbac - dcab + (ad)(bc)$. Since t is not in the basis expansion of $adbc$, this case does not lead into an occurrence of T_t.

(h) $ijk\ell = acbd \in \beta_1$. Then,

$$acbd = cbad - dcab + (ab)(cd) + (ac)(bd) = cbad - dcab + t + (ac)(bd).$$

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	c	dab	$T_{dcab} - T_t$.
2	a	cbd	$T_{cbad} - T_{dcab} + T_t + T_{(ac)(bd)}$.

Since $t = (ab)(cd)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	*column*	*entry*
1	ab	cd	$T_{(ab)(cd)} = T_t$.
2	cd	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

15. If $t \in \nu_3$, then $t = (ab)(bd)$ with $a < b < d$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, d\}$, with only b repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{b, d\}$ and $j \in \{a, b, d\}$. Since only b is repeated twice, we must have $(\ell, j) \in \{(b, b), (b, d), (d, b)\}$.
  - If $k = b$, then we can take $\ell = \{d\}$ and $j \in \{b, d\}$. Since only b is repeated twice, we must have $(\ell, j) = (d, b)$.

So, we have the following possibilities

$$ijk\ell \in \{dbab,\ bdab,\ bbad,\ abbd\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = dbab \in \delta_5$. Since $\delta_5 \subset R_4$ and $dbab \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = bdab \in \alpha_4$. Then,$bdab = dbab - (ab)(bd) = dbab - t$.

(c) $ijk\ell = bbad \in \beta_4$. Since $\beta_4 \subset R_4$ and $bbad \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = abbd \in \gamma_1$. Then, $abbd = bbad - dbab + 2(ab)(bd) = bbad - dbab + 2t$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	b	dab	$T_{dbab} - T_t$.
2	a	bbd	$T_{bbad} - T_{dbab} + 2T_t$.

Since $t = (ab)(bd)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	*column*	*entry*
1	ab	bd	$T_{(ab)(bd)} = T_t$.
2	bd	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

16. If $t \in \nu_4$, then $t = (ab)(cd)$ with $a < c < b < d$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c, d\}$, with No repetition allowed.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{b, c, d\}$ and $j \in \{a, b, c, d\}$. Since no repetition is allowed, we must have

$$(\ell, j) \in \{(b, c), (b, d), (c, b), (c, d), (d, b), (d, c)\}.$$

- If $k = c$, then we can take $\ell = \{b, d\}$ and $j \in \{c, b, d\}$. Since no repetition is allowed, we must have $(\ell, j) \in \{(b, d), (d, b)\}$.

So, we have the following possibilities

$$ijk\ell \in \{dcab,\ cdab,\ dbac,\ bdac,\ cbad,\ bcad,\ adcb,\ abcd\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

- $ijk\ell = dcab \in \beta_7$. Since $\beta_7 \subset R_4$ and $dcab \neq t$, this case does not lead into an occurrence of T_t.
- $ijk\ell = cdab \in \alpha_3$. Then, $cdab = dcab - (ab)(cd) = dcab - t$.
- $ijk\ell = dbac \in \alpha_7$. Since $\alpha_7 \subset R_4$ and $dbac \neq t$, this case does not lead into an occurrence of T_t.
- $ijk\ell = bdac \in \alpha_5$. Then, $bdac = dbac - (ac)(bd)$. Since t is not in the basis expansion of $bdac$, this case does not lead into an occurrence of T_t.
- $ijk\ell = cbad \in \beta_3$. Then, $cbad = bcad - (ad)(cb)$. Since t is not in the basis expansion of $cbad$, this case does not lead into an occurrence of T_t.
- $ijk\ell = bcad \in \beta_5$. Since $\beta_5 \subset R_4$ and $bcad \neq t$, this case does not lead into an occurrence of T_t.
- $ijk\ell = adcb \in \alpha_1$. Then, $adcb = dcab - dbac + (ad)(cb)$. Since t is not in the basis expansion of $adcb$, this case does not lead into an occurrence of T_t.
- $ijk\ell = abcd \in \beta_1$. Then,

$$abcd = bcad - dbac + (ac)(bd) + (ab)(cd) = bcad - dbac + (ac)(bd) + t.$$

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	column	entry
1	c	dab	$T_{dcab} - T_t$.
2	a	bcd	$T_{bcad} - T_{dbac} + T_{(ac)(bd)} + T_t$.

Since $t = (ab)(cd)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	column	entry
1	ab	cd	$T_{(ab)(cd)} = T_t$.
2	cd	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

17. If $t \in \nu_5$, then $t = (ab)(cb)$ with $a < c < b$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c\}$, with only b repeated twice.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{c, b\}$ and $j \in \{a, b, c\}$. Since only b repeated twice, we must have $(\ell, j) \in \{(c, b), (b, b), (b, c)\}$.
  - If $k = c$, then we can take $\ell = \{b\}$ and $j \in \{c, b\}$. Since only b repeated twice, we must have $\ell = j = b$.

So, we have the following possibilities

$$ijk\ell \in \{bbac,\ cbab,\ bcab,\ abcb\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = bbac \in \alpha_6$. Since $\alpha_6 \subset R_4$ and $bbac \neq t$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = cbab \in \delta_3$. Then, $cbab = bcab - (ab)(cb) = bcab - t$.

(c) $ijk\ell = bcab \in \beta_6$. Since $\beta_6 \subset R_4$ and $bcab \neq t$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = abcb \in \delta_1$. Then, $abcb = bcab - bbac + (ab)(cb) = bcab - bbac + t$.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	*column*	*entry*
1	c	bab	$T_{bcab} - T_t$.
2	a	bcb	$T_{bcab} - T_{bbac} + T_t$.

Since $t = (ab)(cb)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	*column*	*entry*
1	ab	cb	$T_{(ab)(cb)} = T_t$.
2	cb	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

18. If $t \in \nu_6$, then $t = (ab)(cd)$ with $a < c < d < b$. Thus, t belongs to the *Second Type*, and the choices for $i \in R_1$ and $jk\ell \in R_3$ are as follows:

- $i, j, k, \ell \in \{a, b, c, d\}$, with No repetition allowed.
- Since $jk\ell \in R^3$, then we must have $k < \ell$ and $k \leq j$. So,
  - If $k = a$, then we can take $\ell \in \{c, d, b\}$ and $j \in \{a, b, c, d\}$. Since no repetition is allowed, we must have

$$(\ell, j) \in \{(c, b), (c, d), (d, b), (d, c), (b, c), (b, d)\}.$$

– If $k = c$, then we can take $\ell = \{d, b\}$ and $j \in \{c, d, b\}$. Since no repetition is allowed, we must have $(\ell, j) \in \{(d, b), (b, d)\}$.

So, we have the following possibilities

$$ijk\ell \in \{dbac,\ bdac,\ cbad,\ bcad,\ dcab,\ cdab,\ abcd,\ adcb\}.$$

Now, using Proposition 5.1.1, we write $ijk\ell$ in each of the above cases as a sum of Hall basis elements:

(a) $ijk\ell = dbac \in \alpha_5$. Then, $dbac = bdac - (ac)(db)$. Since t is not in the basis expansion of $dbac$, this case does not lead into an occurrence of T_t.

(b) $ijk\ell = bdac \in \alpha_7$. Since $\alpha_7 \subset R_4$ and $bdac \neq t$, this case does not lead into an occurrence of T_t.

(c) $ijk\ell = cbad \in \alpha_3$. Then, $cbad = bcad - (ad)(cb)$. Since t is not in the basis expansion of $cbad$, this case does not lead into an occurrence of T_t.

(d) $ijk\ell = bcad \in \beta_7$. Since $\beta_7 \subset R_4$ and $bcad \neq t$, this case does not lead into an occurrence of T_t.

(e) $ijk\ell = dcab \in \beta_5$. Since $\beta_5 \subset R_4$ and $dcab \neq t$, this case does not lead into an occurrence of T_t.

(f) $ijk\ell = cdab \in \beta_3$. Then, $cdab = dcab - (ab)(cd) = dcab - t$.

(g) $ijk\ell = abcd \in \alpha_1$. Then, $abcd = bcad - bdac + (ab)(cd) = bcad - bdac + t$.

(h) $ijk\ell = adcb \in \beta_1$. Then, $adcb = dcab - bdac + (ac)(db) + (ad)(cb)$. Since t is not in the basis expansion of $adcb$, this case does not lead into an occurrence of T_t.

So, T_t appears 2 times in $F_{13}(T^{(4)})$ in the following entries:

	row	column	entry
1	c	dab	$T_{dcab} - T_t$.
2	a	bcd	$T_{bcad} - T_{bdac} + T_t$.

Since $t = (ab)(cd)$ falls into the *Second Type*, T_t also appears in the following entries of $F_{22}(T^{(4)})$:

	row	column	entry
1	ab	cd	$T_{(ab)(cd)} = T_t$.
2	cd	ab	$-T_t$.

As a result, T_t appears 6 times in $F(T^{(4)})$: twice in $F_{22}(T^{(4)})$, twice in $F_{13}(T^{(4)})$ and twice in $-F_{13}^{tr}(T^{(4)})$.

We summarize the results of the previous 18 cases in Table 5.1 and Table 5.2, where the cases are ordered with respect to the number of occurrences.

R_4	$abcd$	*Occurrences in* $F_{13}(T^{(4)})$
γ_2	$a = b = c < d$	1
δ_4	$c < a = b = d$	1
γ_4	$b = c < a = d$	2
α_6	$c < d < a = b$	2
β_4	$c < a = b < d$	2
γ_3	$b = c < a < d$	2
γ_5	$b = c < d < a$	2
β_6	$c < b < a = d$	3
δ_5	$c < b = d < a$	3
β_5	$c < b < a < d$	3
β_7	$c < b < d < a$	3
α_7	$a < b < c < d$	4

Table 5.1: Occurrences of left-aligned basis elements in $F(T^{(4)})$.

R_4	$(ab)(cd)$	*Occurrences in* $F_{13}(T^{(4)})$	*Occurrences in* $F_{22}(T^{(4)})$
ν_1	$a = c < b < d$	2	2
ν_2	$a < b < c < d$	2	2
ν_3	$a < b = c < d$	2	2
ν_4	$a < c < b < d$	2	2
ν_5	$a < c < b = d$	2	2
ν_6	$a < c < d < b$	2	2

Table 5.2: Occurrences of balanced basis elements in $F(T^{(4)})$.

We can also obtain these occurrences using a Python code based on the flowchart in Figure 5.2. The details are presented in Appendix A.2.

Chapter 6

Faithful Dimension of Nilpotency Class 4

Our goal in this chapter is to compute the faithful dimension of $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ using the rank minimization problem presented in Proposition 4.3.4. Using the results from Chapter 5, we assign a vector to each basis element of R_5 such that these vectors are linearly independent and produce minimal rank. This process is explained for each type of basis element in Sections 6.2, 6.3, 6.4, 6.5 and 6.6.

6.1 Preliminaries

The partition of R_4 in Equation (5.4.1) induces an equivalence relation, where we declare two words of R_4 to be in the same equivalence class if they can be converted to each other by an order preserving labeling of the letters. For example, suppose that $r < s$ and $u < v$, then the words $rrrs$ and $uuuv$ are in the same equivalence class γ_2, by the relabeling $r \mapsto s$ and $u \mapsto v$. Throughout this chapter we set $w < x < y < z$, where $w, x, y, z \in \{1, \dots, n\}$. We represent a general element of each equivalence class by a word in w, x, y and z that meets the following conditions:

- If the words in the equivalence class contain only two letters, the representative must contain the letters w, x.
- If the words in the equivalence class contain only three letters, the representative must contain the letters w, x, y.

The representatives of the equivalence classes of R_4 are given in the last column of Table 6.1 and Table 6.2.

Recall that Proposition 4.3.4 provides a tool for computing the faithful dimension of $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)$. Therefore, our goal is to find

$$r_n(4) = |R_4| = \frac{1}{4}(n^4 - n^2)$$

linearly independent vectors that minimize the ranks in a suitable sense. More precisely, we need to find a set of vectors $\{a_\ell : \ell \in R_4\}$ which is linearly independent and the sum

$$\sum_{i=1}^{r_n(4)} f q^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,4}}(a_i)\right)}$$

is minimized. Our next goal is to prove that this can be achieved by the following greedy algorithm: having chosen vectors $a_1, \ldots, a_r$, we choose a_{r+1} such that $\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,4}}(a_{r+1})\right)$ is smallest possible, subject to the condition that $\{a_1, \ldots, a_{r+1}\}$ is linearly independent.

Definition 6.1.1. [KV06, Def. 13.3.] A *Matroid* is a set system $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a collection of subsets of E, such that

i. $\phi \in \mathcal{F}$.

ii. If $X \subseteq Y \in \mathcal{F}$, then $X \in \mathcal{F}$.

iii. If $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there is an $x \in X - Y$ with $Y \cup \{x\} \in \mathcal{F}$.

Lemma 6.1.2. *Let* $E = \{(a(1), \ldots, a(r_n(4))) : a(i) \in \mathbb{F}_q\}$*, and* $\mathcal{F} = \{F \subset E :$ F *linearly independent*$\}$*. Then,* $(E, \mathcal{F})$ *is a Matroid.*

Proof: Properties i. and ii. from Definition 6.1.1 are clear. To show property iii., suppose that there is no $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$, i.e. $Y \cup \{x\}$ is linearly dependent. Thus, X is in the span of Y. Since $|X| > |Y|$, then X is linearly dependent, which is a contradiction. ∎

Proposition 6.1.3. *The greedy algorithm produces the optimal solution to the rank minimization problem.*

Proof: Let $\overline{F}(a) = fq^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a)\right)}$ and suppose that:

- the set of vectors $\{a_1, \ldots, a_{r_n(4)}\}$ is the optimal solution to the rank minimization problem and without loss of generality we can assume that

$$\overline{F}(a_1) \leq \cdots \leq \overline{F}(a_{r_n(4)}).$$

- the set of vectors $\{b_1, \ldots, b_{r_n(4)}\}$ is the greedy algorithm output.

We claim that

$$\sum_{i=1}^{r_n(4)} \overline{F}(a_i) = \sum_{i=1}^{r_n(4)} \overline{F}(b_i).$$

It suffices to prove the stronger statement that

$$\sum_{i=1}^{k} \overline{F}(b_i) \leq \sum_{i=1}^{k} \overline{F}(a_i).$$

for $1 \leq k \leq r_n(4)$. We prove this by induction on k. Suppose that

$$\sum_{i=1}^{k} \overline{F}(b_i) \leq \sum_{i=1}^{k} \overline{F}(a_i),$$

for some $1 \leq k \leq r_n(c) - 1$. If

$$\sum_{i=1}^{k+1} \overline{F}(b_i) > \sum_{i=1}^{k+1} \overline{F}(a_i),$$

then

$$\overline{F}(b_{k+1}) > \overline{F}(a_{k+1}) \geq \cdots \geq \overline{F}(a_1).$$

Note that $|\{b_1, \ldots, b_k\}| < |\{a_1, \ldots, a_k, a_{k+1}\}|$. So, by Lemma 6.1.2 there exists $a \in \{a_1, \ldots, a_k, a_{k+1}\}$ such that the set $\{b_1, \ldots, b_k, a\}$ is linearly independent. In this case the greedy algorithm could have chosen the vector a instead of b_{k+1}. Hence,

$$\overline{F}(b_{k+1}) = \overline{F}(a) > \overline{F}(a),$$

which is a contradiction. So, we must have

$$\sum_{i=1}^{k+1} \overline{F}(b_i) \leq \sum_{i=1}^{k+1} \overline{F}(a_i),$$

which completes the inductive argument. ∎

Our approach to the rank minimization problem using the greedy algorithm will be as follows: first we choose an order on the equivalence classes of R_4. Then, we sequentially assign a vector to each element of the equivalence class of R_4 that minimizes the rank of $F(T^{(4)})$ subject to the condition that it is linearly independent from previously chosen vectors. The order we chose on R_4 depends on the type of the representative word (left-aligned or balanced) and number of distinct letters in it. The order on R_4 is as follows:

i. Left-aligned two-letter equivalence classes: γ_2, δ_4, γ_4.

ii. Left-aligned three-letter equivalence classes: α_6, β_4,γ_3, γ_5, β_6, δ_5.

iii. Left-aligned four-letter equivalence classes: β_5, β_7, α_7.

iv. Balanced three-letter equivalence classes: ν_1, ν_3, ν_5.

v. Balanced four-letter equivalence classes: ν_2, ν_4, ν_6.

R_4	$abcd$	$w < x < y < z$
α_6	$c < d < a = b$	$yywx$
α_7	$a < b < c < d$	$zywx$
β_4	$c < a = b < d$	$xxwy$
β_5	$c < b < a < d$	$yxwz$
β_6	$c < b < a = d$	$yxwy$
β_7	$c < b < d < a$	$zxwy$
γ_2	$a = b = c < d$	$wwwx$
γ_3	$b = c < a < d$	$xwwy$
γ_4	$b = c < a = d$	$xwwx$
γ_5	$b = c < d < a$	$ywwx$
δ_4	$c < a = b = d$	$xxwx$
δ_5	$c < b = d < a$	$yxwx$

Table 6.1: Representatives of left-aligned equivalence classes of R_4.

R_4	$(ab)(cd)$	$w < x < y < z$
ν_1	$a = c < b < d$	$(wx)(wy)$
ν_2	$a < b < c < d$	$(wx)(yz)$
ν_3	$a < b = c < d$	$(wx)(xy)$
ν_4	$a < c < b < d$	$(wy)(xz)$
ν_5	$a < c < b = d$	$(wy)(xy)$
ν_6	$a < c < d < b$	$(wz)(xy)$

Table 6.2: Representatives of balanced equivalence classes of R_4.

We will refer to the following lemma often in the upcoming sections to prove linear independence.

Lemma 6.1.4. *Let $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_s\}$ be sets of vectors in F^N, where F is a field and $N \in \mathbb{N}$, such that*

- *The set $\{u_1, \ldots, u_r\}$ is linearly independent.*
- *For each $1 \leq k \leq s$, the vector $v_k = (v_k(i) : 1 \leq i \leq N)$ has a position i_k such that :*
  - $v_k(i_k) \neq 0$.
  - $v_{k'}(i_k) = 0$ *for* $1 \leq k' \leq s,\ k' \neq k$.
  - $u_{k'}(i_k) = 0$ *for* $1 \leq k' \leq r$.

Then, the set $\{u_1, \ldots, u_r, v_1, \ldots, v_s\}$ is linearly independent.

Proof: Suppose that

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = 0, \tag{6.1.1}$$

where $a_i, b_j \in F$. Let $1 \leq k_0 \leq s$, and i_{k_0} be the position of v_{k_0} where $v_{k_0}(i_{k_0}) \neq 0$. From Equation (6.1.1), we get $b_{k_0} = 0$. Thus, $b_k = 0$ for all $1 \leq k \leq s$. Since

$\{u_1, \ldots, u_r\}$ is linearly independent, we get that $a_i = 0$ for all $1 \leq i \leq r$. As a result, the set $\{u_1, \ldots, u_r, v_1, \ldots, v_s\}$ is linearly independent. ∎

Let $a = (a(t) : t \in R_4) \in \mathbb{F}_q^{r_n(4)}$. From the reduced commutator matrix $F(T^{(4)})$ in Equation (5.1.1), the rank $F(T^{(4)})$ evaluated at a is given by the following formula:

$$rk(F(a)) = 2\, rk(F_{13}(a)) + rk(F_{22}(a)). \tag{6.1.2}$$

Lemma 6.1.5. *If $a \in \mathbb{F}_q^{r_n(4)}$ has a nonzero entry at a left-aligned postion, then $rk(F(a)) \geq 2$.*

Proof: Suppose that the aforementioned nonzero entry of $a = (a(t) : t \in R_4)$ is $a(t_0)$ such that t_0 is left-aligned. From Corollary 5.1.2 there exists an entry in $F_{13}(T^{(4)})$ consisting solely of the monomial T_{t_0}. Hence, $rk(F_{13}(a)) \geq 1$. However, T_0 does not appear in $F_{22}(T^{(4)})$ as shown in Corollary 5.3.2. So, $rk(F_{22}(a)) = 0$. As a result,

$$rk(F(a)) = 2\, rk(F_{13}(a)) \geq 2,$$

which completes the proof. ∎

Lemma 6.1.6. *If $a \in \mathbb{F}_q^{r_n(4)}$ has a nonzero entry at a balanced position, then we have $rk(F(a)) \geq 4$.*

Proof: Suppose that the nonzero entry of $a = (a(t) : t \in R_4)$ is $a(t_0)$. such that t_0 is balanced. From Table 5.2, we see that T_{t_0} appears in both $F_{13}(T^{(4)})$ and $F_{22}(T^{(4)})$. In $F_{13}(T^{(4)})$, T_{t_0} appears in an entry of the form $T_t - T_{t_0}$, for some left-aligned $t \in R_4$. If the value of this entry in $F_{13}(a)$ is zero, then the value of the entry T_t in $F_{13}(a)$ is nonzero. So, in both cases $F_{13}(a)$ has a nonzero entry that grantees that $rk(F_{13}(a)) \geq 1$. Since $F_{22}(T^{(4)})$ is skew-symmetric, the $rk(F_{22}(a))$ is even. From Corollary 5.3.2, there exists an entry in $F_{22}(T^{(4)})$ consisting solely of the monomial T_{t_0}. Since $F_{22}(a)$ has a nonzero entry, $rk(F_{22}(a)) \geq 2$. As a result,

$$rk(F(a)) = 2\, rk(F_{13}(a)) + rk(F_{22}(a)) \geq 4,$$

Which completes the proof. ∎

We conclude this section with Table 6.3, which provides labels for the 4-letter words in the subscripts of the T's in the commutator matrix $F(T^{(4)})$. We introduced this labeling for better readability, as we represent words (e.g. $wwwx$, $xxwx$, ...) using capital letters (e.g. A, B, ...). Note that in Table 6.3 we are labeling some words of R_4 not equivalence classes, for example $A = wwwx$ and $J = xxxy$ are in the equivalence class γ_2.

6.2 Left-Aligned Two-Letter Words

In this section we consider left-aligned words with only two letters w and x such that $w < x$. According to Table 6.1, these words are: $wwwx \in \gamma_2$, $xwwx \in \gamma_4$ and $xxwx \in \delta_4$. From Table 5.1, the number of occurrences of each type is given below:

Type	*Word*	*Occurrences in* $F_{13}(T^{(4)})$
γ_2	$wwwx$	1
δ_4	$xxwx$	1
γ_4	$xwwx$	2

We analyze each case in the above table as follows:

1. Let $\ell = wwwx \in \gamma_2$. From Case 7 in Section 5.4, T_ℓ appears once in $F_{13}(T^{(4)})$ in the following position:

$$F_{13}(T^{(4)}) = \begin{array}{cc} & wwx \\ w & \begin{pmatrix} & \vdots & \\ \dots & T_\ell & \dots \\ & \vdots & \end{pmatrix} \end{array}.$$

$A = wwwx$	$B = xxwx$	$C = xwwx$	$D = yywx$
$E = ywwx$	$F = yyxy$	$G = xxwy$	$H = wwwy$
$I = xwwy$	$J = xxxy$	$K = yywy$	$L = ywwy$
$M = yxwy$	$N = yxxy$	$O = yxwx$	$P_0 = wwwz$
$P_1 = xwwz$	$P_2 = ywwz$	$P_3 = xxwz$	$P_4 = yxwz$
$P_5 = xxxz$	$P_6 = yxxz$	$P_7 = yywz$	$P_8 = yyxz$
$P_9 = yyyz$	$Q_0 = zwwy$	$Q_1 = zxwy$	$Q_2 = zxxy$
$Q_3 = zzwy$	$Q_4 = zzxy$	$Q_5 = zzyz$	$R_0 = zwwx$
$R_1 = zwwz$	$R_2 = zxwx$	$R_3 = zxwz$	$R_4 = zxxz$
$R_5 = zywx$	$R_6 = zywz$	$R_7 = zyxy$	$R_8 = zyxz$
$R_9 = zyyz$	$S_0 = zzwx$	$S_1 = zzwz$	$S_2 = zzxz$
$T_0 = zywy$	$V_1 = (wx)(wy)$	$V_2 = (wx)(yz)$	$V_3 = (wx)(xy)$
$V_4 = (wy)(xz)$	$V_5 = (wy)(xy)$	$V_6 = (wz)(xy)$	

Table 6.3: Table of labels.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell \\ 0 & i \neq \ell, \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

We visually represent the vector a_ℓ constructed above by highlighting the nonzero entries as follows:

$$\begin{array}{cccccccccccccc} & & & & & & & A & & & & & & \\ \gamma_2 & (& 0 & , & \dots & , & 0 & , & 1 & , & 0 & , & \dots & , & 0 &) \end{array}$$

where $A = wwwx$ as labeled in Table 6.3.

Next we prove that the set of vectors $\{a_\ell : \ell \in \gamma_2\}$ is linearly independent. Let $b = \sum_{\ell \in \gamma_2} c_\ell a_\ell = (b(i) : i \in R_4)$. Then, for $i \in R_4$ we have

$$b(i) = \begin{cases} c_i & i \in \gamma_2 \\ 0 & i \notin \gamma_2, \end{cases}$$

If $b = 0$, then $b(i) = 0$ for all $i \in R_4$. As a result, $c_i = 0$ for all $i \in \gamma_2$.

2. Let $\ell = xxwx \in \delta_4$. From Case 11 in Section 5.4, T_ℓ appears once in $F_{13}(T^{(4)})$ in the following position:

$$F_{13}(T^{(4)}) = \begin{array}{c} \\ x \end{array} \begin{array}{c} xwx \\ \begin{pmatrix} & \vdots & \\ \dots & T_\ell & \dots \\ & \vdots & \end{pmatrix} \end{array}.$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell \\ 0 & i \neq \ell, \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

We visually represent the vector a_ℓ constructed above by highlighting its nonzero entries and comparing it to the vectors produced so far as follows:

$$\begin{array}{cc} & \begin{array}{cccc} & A & B & \end{array} \\ \gamma_2 & (\cdots 1 \; 0 \cdots) \\ \delta_4 & (\cdots 0 \; 1 \cdots) \end{array}$$

where the nonzero positions are labeled by A and $\ell = B$ as in Table 6.3.

Next we prove that the current set of vectors associated with γ_2 and δ_4 is linearly independent. Let u be a vector associated to an element of δ_4. Then, for some position $\ell_0 = x_0x_0w_0x_0 \in \delta_4$, where $w_0 < x_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ 0 & i \neq \ell_0. \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated to δ_4 when $v \neq u$, or for any vector v associated to γ_2. This will be demonstrated as follows:

(a) Let v be a vector associated with δ_4 such that $v \neq u$. We claim that $v(x_0x_0w_0x_0) = 0$. Otherwise, $x_0x_0w_0x_0 = x_1x_1w_1x_1$ for some $w_1 < x_1$. So must have $v = u$, which is a contradiction.

(b) Let v be a vector associated with γ_2. We claim that $v(x_0x_0w_0x_0) = 0$. Otherwise, we must have $x_0x_0w_0x_0 = w_2w_2w_2x_2$, for some $w_2 < x_2$. So, we must have $x_0 = w_2 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

By Lemma 6.1.4, the vectors associated with $\gamma_2 \cup \delta_4$ are linearly independent

3. Let $\ell = xwwx \in \gamma_4$. From Case 9 in Section 5.4, T_ℓ appears twice in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c c} & \begin{array}{cc} wwx & xwx \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} T_{wwwx} & T_\ell \\ T_\ell & T_{xxwx} \end{pmatrix} \end{array},$$

where both T_{wwwx} and T_{xxwx} occur once in $F_{13}(T^{(4)})$, as seen in Section 5.4. If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwx,\ \ xxwx,\ \ \ell = xwwx\} \\ 0 & e.w., \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $A, B, \ell = C$ as in Table 6.3, is as follows:

$$\begin{array}{ll} & \quad\ \ A\ B\ C \\ \gamma_2 & (\cdots 1\ 0\ 0 \cdots) \\ \delta_4 & (\cdots 0\ 1\ 0 \cdots) \\ \gamma_4 & (\cdots 1\ 1\ 1 \cdots) \end{array}$$

Remark 6.2.1. Note that the vectors associated with elements of γ_2, δ_4 and γ_4 have nonzero values only in positions of words of consisting of two letters, as seen in the color-coded representation of vectors.

Next we prove that the current set of vectors associated to γ_2, δ_4 and γ_4 is linearly independent. Let u be a vector associated to an element of γ_4. Then, for some position $\ell_0 = x_0w_0w_0x_0 \in \gamma_4$, where $w_0 < x_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0x_0,\ \ x_0x_0w_0x_0,\ \ \ell_0\} \\ 0 & \text{elsewhere} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with γ_4 when $v \neq u$, or for any vector v associated to γ_2, δ_4 or γ_4. This will be demonstrated as follows:

(a) Let v be a vector associated with γ_4 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_1w_1w_1x_1,\ \ x_1x_1w_1x_1,\ \ x_1w_1w_1x_1\},$$

for some $w_1 < x_1$. So we have the following possibilities:

i. If $x_0w_0w_0x_0 = w_1w_1w_1x_1$, then $x_0 = w_1 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

ii. If $x_0w_0w_0x_0 = x_1x_1w_1x_1$, then $x_0 = x_1 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iii. If $x_0w_0w_0x_0 = x_1w_1w_1x_1$, then $x_0 = x_1$ and $w_0 = w_1$. So, $v = u$, which is a contradiction.

(b) Let v be a vector associated with γ_2. We claim that $v(\ell_0) = 0$. Otherwise, $x_0w_0w_0x_0 = w_2w_2w_2x_2$ for some $w_2 < x_2$. So, we must have $x_0 = w_2 = w_0$, which contradict $w_0 < x_0$.

(c) Let v be a vector associated with δ_4. We claim that $v(\ell_0) = 0$. Otherwise, $x_0w_0w_0x_0 = x_3x_3w_3x_3$ for some $w_3 < x_3$. So, we must have $x_0 = x_3 = w_0$, which contradict $w_0 < x_0$.

By Lemma 6.1.4, the vectors associated with $\gamma_2 \cup \delta_4 \cup \gamma_4$ are linearly independent.

6.3 Left-Aligned Three-Letter Words

In this section we consider left-aligned words with only three letters w, x and y such that $w < x < y$. The following table lists these three-letter words and the number of their occurrences, as shown in Table 5.1.

Type	*Word*	*Occurrences in* $F_{13}(T^{(4)})$
α_6	$yywx$	2
β_4	$xxwy$	2
γ_3	$xwwy$	2
γ_5	$ywwx$	2
β_6	$yxwy$	3
δ_5	$yxwx$	3

We analyze each case in the above table as follows:

1. Let $\ell = yywx \in \alpha_6$. From Case 1 in Section 5.4, T_ℓ appears twice in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{cc} & \begin{array}{ccc} wwx & ywx & yxy \end{array} \\ \begin{array}{c} w \\ y \end{array} & \begin{pmatrix} T_{wwwx} & T_{ywwx} - T_{(wx)(wy)} & T_{yxwy} - T_\ell + T_{(wy)(xy)} \\ T_{ywwx} & T_\ell & T_{yyxy} \end{pmatrix} \end{array}.$$

Remark 6.3.1. Based on Section 5.4, we construct such submatrices of $F_{13}(T^{(4)})$ in a way that guarantees the consideration of all the occurrences of T_X, where X is left-aligned.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{ywwx,\ wwwx,\ \ell = yywx\} \\ -1 & i = yyxy \\ 0 & \text{elsewhere} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$. Note that since p is assumed to be sufficiently large, we have $1 \neq -1$.

Remark 6.3.2. In our selection of vectors associated with left-aligned words, the positions of balanced words have zero values. This means that vectors with at least one nonzero balanced position will not produce a minimal rank, as, in this case, $rk(F_{22}(a_\ell)) \geq 2$ instead of zero.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $A, \ell = D, E, F$ as in Table 6.3, is as follows:

$$\begin{array}{cccccccccl}
 & & A & B & C & D & E & F & & \\
\gamma_2 & (\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 & \cdots) \\
\delta_4 & (\cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 & \cdots) \\
\gamma_4 & (\cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 & \cdots) \\
\alpha_6 & (\cdots & 1 & 0 & 0 & 1 & 1 & -1 & 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 & \cdots)
\end{array}$$

Next we prove that the current set of vectors associated with $\gamma_2, \delta_4, \gamma_4$ and α_6 is linearly independent. Let u be a vector associated to an element of α_6. Then, for some position $\ell_0 = y_0y_0w_0x_0 \in \alpha_6$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{y_0w_0w_0x_0,\ w_0w_0w_0x,\ \ell_0\} \\ -1 & i = y_0y_0x_0y_0 \\ 0 & \text{elsewhere} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with α_6 when $v \neq u$, or for any vector v associated to γ_2, δ_4 or γ_4. This will be demonstrated as follows:

(a) Let v be a vector associated with α_6 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_1w_1w_1x_1,\ w_1w_1w_1x_1,\ y_1y_1w_1x_1,\ y_1y_1x_1y_1\},$$

for some $w_1 < x_1 < y_1$. So, we have the following possibilities:

i. If $y_0y_0w_0x_0 = y_1w_1w_1x_1$, then $y_1 = y_0 = w_1$, but $w_1 < y_1$. So we have a contradiction.

ii. If $y_0y_0w_0x_0 = w_1w_1w_1x_1$, then $y_0 = w_1 = w_0$, but $w_0 < y_0$. So we have a contradiction.

iii. If $y_0y_0w_0x_0 = y_1y_1w_1x_1$. So, we must have $v = u$, which is a contradiction.

iv. If $y_0y_0w_0x_0 = y_1y_1x_1y_1$, then $y_0 = y_1 = x_0$, but $x_0 < y_0$. So we have a contradiction.

(b) Let v be a vector associated with γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

By Lemma 6.1.4, the vectors associated with $\gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6$ are linearly independent.

2. Let $\ell = xxwy \in \beta_4$. From Case 3 in Section 5.4, T_ℓ appears twice in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c c} & \begin{array}{ccc} wwy & xwy & xxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} T_{wwwy} & T_{xwwy} + T_{(wx)(wy)} & T_\ell - T_{yxwx} + 2T_{(wx)(xy)} \\ T_{xwwy} & T_\ell & T_{xxxy} \end{pmatrix} \end{array}.$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwy,\ xwwy,\ xxxy,\ \ell = xxwy\} \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $H, I, J, \ell = G$ as in Table 6.3, is as follows:

$$
\begin{array}{lccccccccccl}
 & A & B & C & D & E & F & G & H & I & J & \\
\gamma_2 & (\cdots 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_4 & (\cdots 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_4 & (\cdots 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\alpha_6 & (\cdots 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_4 & (\cdots 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots)
\end{array}
$$

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6$ and β_4 is linearly independent. Let u be a vector associated to an element of β_4. Then, for some position $\ell_0 = x_0x_0w_0y_0 \in \beta_4$, where $w_0 < x_0 < y_0$, we have

$$
u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0y_0,\ \ x_0w_0w_0y_0,\ \ x_0x_0x_0y_0,\ \ \ell_0\} \\ 0 & \text{elsewhere.} \end{cases}
$$

We will show that $v(\ell_0) = 0$ for any vector v associated to β_4 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4$ or α_6. This will be demonstrated as follows:

(a) Let v be a vector associated with β_4 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise, we must have

$$
\ell_0 \in \{w_1w_1w_1y_1,\ \ x_1w_1w_1y_1,\ \ x_1x_1x_1y_1,\ \ x_1x_1w_1y_1\}.
$$

for some $w_1 < x_1 < y_1$. So, we have the following possibilities:

i. If $x_0x_0w_0y_0 = w_1w_1w_1y_1$, then $x_0 = w_1 = w_0$, but $w_0 < x_0$. So we have a contradiction.

ii. If $x_0x_0w_0y_0 = x_1w_1w_1y_1$, then $x_1 = x_0 = w_1$, but $w_1 < x_1$. So we have a contradiction.

iii. If $x_0x_0w_0y_0 = x_1x_1x_1y_1$, then $x_0 = x_1 = w_0$, but $w_0 < x_0$. So we have a contradiction.

iv. If $x_0x_0w_0y_0 = x_1x_1w_1y_1$, then $x_0 = x_1$ and $w_0 = w_1$ and $y_0 = y_1$, but $u \neq v$. So we have a contradiction.

(b) Let v be a vector associated to α_6. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_2w_2w_2x_2,\ \ w_2w_2w_2x_2,\ \ y_2y_2x_2y_2,\ \ y_2y_2w_2x_2\},$$

for some $w_2 < x_2 < y_2$. So, we have the following possibilities:

i. If $x_0x_0w_0y_0 = y_2w_2w_2x_2$, then $y_2 = x_0 = w_2$, but $w_2 < y_2$. So we have a contradiction.

ii. If $x_0x_0w_0y_0 = w_2w_2w_2x_2$, then $x_0 = w_2 = w_0$, but $w_0 < x_0$. So we have a contradiction.

iii. If $x_0x_0w_0y_0 = y_2y_2x_2y_2$, then $x_0 = y_2 = y_0$, but $x_0 < y_0$. So we have a contradiction.

iv. If $x_0x_0w_0y_0 = y_2y_2w_2x_2$, then $x_0 = y_2$ and $w_0 = w_2$ and $y_0 = x_2$. Since $x_2 < y_2$, we must have $y_0 < x_0$, but $x_0 < y_0$. So we have a contradiction.

(c) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

By Lemma 6.1.4, the vectors associated with elements of $\gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6 \cup \beta_4$ are linearly independent.

3. Let $\ell = xwwy \in \gamma_3$. From Case 8 in Section 5.4, T_ℓ appears twice in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c c} & \begin{array}{ccc} wwy & xwy & xxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} T_{wwwy} & T_\ell + T_{(wx)(wy)} & T_{xxwy} - T_{yxwx} + 2T_{(wx)(xy)} \\ T_\ell & T_{xxwy} & T_{xxxy} \end{pmatrix} \end{array}.$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{\ell = xwwy,\ \ xxxy\} \\ -1 & i \in \{xxwy,\ \ wwwy\} \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $G, H, \ell = I, J$ as in Table 6.3, is as follows:

$$\begin{array}{ll}
 & \qquad A\ B\ C\ D\ E\ F\ G\ H\ I\ J \\
\gamma_2 & (\cdots 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_4 & (\cdots 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_4 & (\cdots 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\alpha_6 & (\cdots 1\ 0\ 0\ 1\ 1\ {-1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_4 & (\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_3 & (\cdots 0\ 0\ 0\ 0\ 0\ 0\ {-1}\ {-1}\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots)
\end{array}$$

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4$ and γ_3 is linearly independent. In this case the argument is not an immediate consequence of Lemma 6.1.4. Rather, there is another twist. Let

$$b = \sum_{i \in \mathcal{S}_1} r_i v_i + \sum_{j \in \gamma_3} s_j u_j, \tag{6.3.1}$$

where $\mathcal{S}_1 = \gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6 \cup \beta_4$, v_i are the vectors associated with $\mathcal{S}_1$ and u_j are the vectors associated with γ_3. We claim that $r_i = s_j = 0$ for all i, j, when $b = 0$.

Let u be a vector associated to an element of γ_3. Then, for some position $\ell_0 =$

$x_0w_0w_0y_0$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{\ell_0,\ \ x_0x_0x_0y_0\} \\ -1 & i \in \{x_0x_0w_0y_0,\ \ w_0w_0w_0y_0\} \\ 0 & \text{elsewhere.} \end{cases}$$

We will start by showing that $v(\ell_0) = 0$ for any vector v associated with γ_3 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6$. This will be demonstrated as follows:

(a) Let v be a vector associated with γ_3 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{x_1w_1w_1y_1,\ \ x_1x_1x_1y_1,\ \ x_1x_1w_1y_1,\ \ w_1w_1w_1y_1\},$$

for some $w_1 < x_1 < y_1$. So, we have the following possibilities:

i. If $x_0w_0w_0y_0 = x_1w_1w_1y_1$, then $u = v$, which is a contradiction.

ii. If $x_0w_0w_0y_0 = x_1x_1x_1y_1$, then $x_0 = x_1 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iii. If $x_0w_0w_0y_0 = x_1x_1w_1y_1$, then $x_0 = x_1 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iv. If $x_0w_0w_0y_0 = w_1w_1w_1y_1$, then $x_0 = w_1 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

(b) Let v be a vector associated with α_6. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_2w_2w_2x_2,\ \ w_2w_2w_2x_2,\ \ y_2y_2x_2y_2,\ \ y_2y_2w_2x_2\},$$

for some $w_2 < x_2 < y_2$. We have the following possibilities:

i. If $x_0w_0w_0y_0 = y_2w_2w_2x_2$, then $x_0 = y_2$ and $w_0 = w_2$ and $y_0 = x_2$. Since $x_2 < y_2$, we must have $y_0 < x_0$, which is a contradiction.

ii. If $x_0w_0w_0y_0 = w_2w_2w_2x_2$, then $x_0 = w_2 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iii. If $x_0w_0w_0y_0 = y_2y_2x_2y_2$, then $x_0 = y_2 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iv. If $x_0w_0w_0y_0 = y_2y_2w_2x_2$, then $x_0 = y_2 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

(c) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

Up to now, we can use Lemma 6.1.4 to infer that all the vectors excluding the ones associated to β_4, i.e. $\gamma_2, \delta_4, \gamma_4, \alpha_6$ and γ_3, are linearly independent. Next , we prove that linear independence holds after including the β_4. Let v be a vector associated with β_4. If $v(\ell_0) \neq 0$, we must have

$$\ell_0 \in \{w_3w_3w_3y_3,\ \ x_3w_3w_3y_3,\ \ x_3x_3x_3y_3,\ \ x_3x_3w_3y_3\},$$

where $w_3 < x_3 < y_3$. So, we have the following possibilities:

i. If $x_0w_0w_0y_0 = w_3w_3w_3y_3$, then $x_0 = w_3 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

ii. If $x_0w_0w_0y_0 = x_3w_3w_3y_3$, then $x_0 = x_3$, $w_0 = w_3$ and $y_0 = y_3$

iii. If $x_0w_0w_0y_0 = x_3x_3x_3y_3$, then $x_0 = x_3 = w_0$, but $w_0 < x_0$. So, we have a contradiction.

iv. If $x_0w_0w_0y_0 = x_3x_3w_3y_3$, then $x_0 = x_3 = w_0$ and $w_0 < x_0$. So, we have a contradiction.

So the only case where $v(\ell_0) \neq 0$ is when v is the vectors associated with the word $\ell_1 = x_0x_0w_0y_0 \in \beta_4$, given by

$$v(i) = \begin{cases} 1 & i \in \{w_0w_0w_0y_0,\ \ell_0,\ x_0x_0x_0y_0,\ \ell_1\} \\ 0 & \text{elsewhere.} \end{cases}$$

From the analysis above, we showed that the only vectors with nonzero entries at the position $\ell_0 = x_0w_0w_0y_0$ are u and v. Moreover, these vectors are also the only vectors with nonzero entries at the position $\ell_1 = x_0x_0w_0y_0$, since $v'(\ell_1) = 0$ for any vector v' associated with $\gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6$. This can be visualized by the color-coded representation of the nonzero entries, where the positions $\ell_0 = x_0w_0w_0y_0$ and $\ell_1 = x_0x_0w_0y_0$ correspond to I and G, respectively. From Equation (6.3.1), we have

$$b(k) = \sum_{i\in\mathcal{S}_1} r_iv_i(k) + \sum_{j\in\gamma_3} s_ju_j(k) = 0.$$

where $k \in R_4$. In particular,

- if $k = \ell_0$, then $b(\ell_0) = r\,v(\ell_0) + s\,u(\ell_0) = r + s = 0.$
- if $k = \ell_1$, then
$$b(\ell_1) = r\,v(\ell_1) + s\,u(\ell_1) = r - s = 0.$$

As a result, $s = r = 0$. Hence, $s_j = 0$ for all $j \in \gamma_3$ in Equation (6.3.1).

Since the vectors associated with $\mathcal{S}_1 = \gamma_2\cup\delta_4\cup\gamma_4\cup\alpha_6\cup\beta_4$ are linearly independent, we must have $r_i = 0$ for all $i \in \mathcal{S}_1$ in Equation (6.3.1).

4. Let $\ell = ywwx \in \gamma_5$. From Case 10 in Section 5.4, T_ℓ appears twice in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c} \\ w \\ y \end{array}\!\!\begin{array}{c} \begin{array}{ccc} wwx & ywx & yxy \end{array} \\ \begin{pmatrix} T_{wwwx} & T_\ell - T_{(wx)(wy)} & T_{yxwy} - T_{yywx} + T_{(wy)(xy)} \\ T_\ell & T_{yywx} & T_{yyxy} \end{pmatrix} \end{array}.$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{yywx,\ wwwx,\ yyxy\} \\ -1 & i = \ell = ywwx \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $A, D, \ell = E, F$ as in Table 6.3, is as follows:

$$\begin{array}{*{46}{c}}
 & & A & B & C & D & E & F & G & H & I & J \\
\gamma_2 & (\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\delta_4 & (\cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\gamma_4 & (\cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\alpha_6 & (\cdots & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\beta_4 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\gamma_3 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\gamma_5 & (\cdots & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots)
\end{array}$$

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3$ and γ_5 is linearly independent. Again, the proof is not immediate from Lemma 6.1.4 and therefore we give the details. Let

$$b = \sum_{i \in \mathcal{S}_2} r_i v_i + \sum_{j \in \gamma_5} s_j u_j, \tag{6.3.2}$$

where $\mathcal{S}_2 = \gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3$, v_i are the vectors associated with $\mathcal{S}_2$ and u_j are the vectors associated with γ_5. We claim that $r_i = s_j = 0$ for all i, j, when $b = 0$.

Let u be a vector associated to an element of γ_5. Then, for some position $\ell_0 = y_0w_0w_0x_0$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{y_0y_0w_0x_0,\ \ w_0w_0w_0x_0,\ \ y_0y_0x_0y_0\} \\ -1 & i = \ell_0 \\ 0 & \text{elsewhere} \end{cases}$$

We will start by showing that $v(\ell_0) = 0$ for any vector v associated with γ_5 when $v \neq u$, or for any vector v associated to $\mathcal{S}_2 - \alpha_6$. This will be demonstrated as follows:

(a) Let v be a vector associated with γ_5 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_1y_1w_1x_1,\ \ w_1w_1w_1x_1,\ \ y_1y_1x_1y_1,\ \ y_1w_1w_1x_1\}.$$

for some $w_1 < x_1 < y_1$. So, have the following possibilities:

i. If $y_0w_0w_0x_0 = y_1y_1w_1x_1$, then $y_0 = y_1 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

ii. If $y_0w_0w_0x_0 = w_1w_1w_1x_1$, then $y_0 = w_1 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iii. If $y_0w_0w_0x_0 = y_1y_1x_1y_1$, then $y_0 = y_1 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iv. If $y_0w_0w_0x_0 = y_1w_1w_1x_1$, then $y_0 = y_1$ and $w_0 = w_1$ and $x_0 = x_1$. So, we must have $u = v$, but this is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with β_4. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_2w_2w_2y_2,\ \ x_2w_2w_2y_2,\ \ x_2x_2x_2y_2,\ \ x_2x_2w_2y_2\}.$$

for some $w_2 < x_2 < y_2$. So, we have the following possibilities:

i. If $y_0w_0w_0x_0 = w_2w_2w_2y_2$, then $y_0 = w_2 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

ii. If $y_0w_0w_0x_0 = x_2w_2w_2y_2$, then $y_0 = x_2$ and $w_0 = w_2$ and $x_0 = y_2$. Since $x_2 < y_2$, then we must have $y_0 < x_0$, which is a contradiction.

iii. If $y_0w_0w_0x_0 = x_2x_2x_2y_2$, then $y_0 = x_2 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iv. If $y_0w_0w_0x_0 = x_2x_2w_2y_2$, then $y_0 = x_2 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

(d) Let v be a vector associated with γ_3, We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{x_3w_3w_3y_3,\ \ x_3x_3x_3y_3,\ \ x_3x_3w_3y_3,\ \ w_3w_3w_3y_3\}.$$

for some $w_3 < x_3 < y_3$. So, we have the following possibilities:

i. If $y_0w_0w_0x_0 = x_3w_3w_3y_3$, then $y_0 = x_3$ and $w_0 = w_3$ and $x_0 = y_3$. Since $x_3 < y_3$, then we must have $y_0 < x_0$, which is a contradiction.

ii. If $y_0w_0w_0x_0 = x_3x_3x_3y_3$, then $y_0 = x_3 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iii. If $y_0w_0w_0x_0 = x_3x_3w_3y_3$, then $y_0 = x_3 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iv. If $y_0w_0w_0x_0 = w_3w_3w_3y_3$, then $y_0 = w_3 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

Finally, let v be a vector associated with α_6. If $v(\ell_0) \neq 0$, we must have

$$\ell_0 \in \{y_4w_4w_4x_4,\ \ w_4w_4w_4x_4,\ \ y_4y_4x_4y_4,\ \ y_4y_4w_4x_4\}.$$

for some $w_4 < x_4 < y_4$. So, we have the following possibilities:

i. If $y_0w_0w_0x_0 = y_4w_4w_4x_4$, then $y_0 = y_4$ and $w_0 = w_4$ and $x_0 = x_4$.

ii. If $y_0w_0w_0x_0 = w_4w_4w_4x_4$, then $y_0 = w_4 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iii. If $y_0w_0w_0x_0 = y_4y_4x_4y_4$, then $y_0 = y_4 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iv. If $y_0w_0w_0x_0 = y_4y_4w_4x_4$, then $y_0 = y_4 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

So the only case where $v(\ell_0) \neq 0$ is when v is one of the vectors associated with the word $\ell_1 = y_0y_0w_0x_0 \in \alpha_6$, given by

$$v(i) = \begin{cases} 1 & i \in \{\ell_0,\ \ w_0w_0w_0x_0,\ \ \ell_1\} \\ -1 & i = y_0y_0x_0y_0 \\ 0 & \text{elsewhere.} \end{cases}$$

From the analysis above, we showed that the only vectors with nonzero entries at the position $\ell_0 = y_0w_0w_0x_0$ are u and v. Moreover, we claim that the vectors u and v are also the only vectors with nonzero entries at the position $\ell_1 = y_0y_0w_0x_0$. This can be visualized by the color-coded representation of the nonzero entries, where the positions $\ell_0 = y_0w_0w_0x_0$ and $\ell_1 = y_0y_0w_0x_0$ correspond to E and D, respectively. Indeed, we will show that $v'(\ell_1) = 0$ for any vector v' associated with γ_5 when $v' \neq u$, or for any vector v' associated to $\mathcal{S}_2 - \alpha_6$. This will be demonstrated as follows:

(a) Let v' be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v'(\ell_1) = 0$.

(b) Let v' be a vector associated with γ_5 such that $v' \neq u$. We claim that $v'(\ell_1) = 0$. Otherwise,

$$\ell_1 \in \{y_1y_1w_1x_1,\ w_1w_1w_1x_1,\ y_1y_1x_1y_1,\ y_1w_1w_1x_1\}.$$

for some $w_1 < x_1 < y_1$. So, have the following possibilities:

i. If $y_0y_0w_0x_0 = y_1y_1w_1x_1$, then $u = v'$, which is a contradiction.

ii. If $y_0y_0w_0x_0 = w_1w_1w_1x_1$, then $y_0 = w_1 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iii. If $y_0y_0w_0x_0 = y_1y_1x_1y_1$, then $y_0 = y_1 = x_0$, but $x_0 < y_0$. So, we have a contradiction.

iv. If $y_0y_0w_0x_0 = y_1w_1w_1x_1$, then $y_0 = w_1 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

(c) Let v' be a vector associated with β_4. We claim that $v'(\ell_1) = 0$. Otherwise,

$$\ell_1 \in \{w_2w_2w_2y_2,\ x_2w_2w_2y_2,\ x_2x_2x_2y_2,\ x_2x_2w_2y_2\}.$$

for some $w_2 < x_2 < y_2$. So, we have the following possibilities:

i. If $y_0y_0w_0x_0 = w_2w_2w_2y_2$, then $y_0 = w_2 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

ii. If $y_0y_0w_0x_0 = x_2w_2w_2y_2$, then $x_2 = y_0 = w_2$ but $w_2 < x_2$. So, we have a contradiction.

iii. If $y_0y_0w_0x_0 = x_2x_2x_2y_2$, then $y_0 = x_2 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iv. If $y_0y_0w_0x_0 = x_2x_2w_2y_2$, then $y_0 = x_2$ and $w_0 = w_2$ and $x_0 = y_2$. Since $x_2 < y_2$, we must have $y_0 < x_0$, which is a contradiction.

(d) Let v' be a vector associated with γ_3, We claim that $v'(\ell_1) = 0$. Otherwise,

$$\ell_1 \in \{x_3w_3w_3y_3,\ \ x_3x_3x_3y_3,\ \ x_3x_3w_3y_3,\ \ w_3w_3w_3y_3\}.$$

for some $w_3 < x_3 < y_3$. So, we have the following possibilities:

i. If $y_0y_0w_0x_0 = x_3w_3w_3y_3$, then $x_3 = y_0 = w_3$ but $w_3 < x_3$. So, we have a contradiction.

ii. If $y_0y_0w_0x_0 = x_3x_3x_3y_3$, then $y_0 = x_3 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

iii. If $y_0y_0w_0x_0 = x_3x_3w_3y_3$, then $y_0 = x_3$ and $w_0 = w_3$ and $x_0 = y_3$. Since $x_3 < y_3$, we must have $y_0 < x_0$, which is a contradiction.

iv. If $y_0y_0w_0x_0 = w_3w_3w_3y_3$, then $y_0 = w_3 = w_0$, but $w_0 < y_0$. So, we have a contradiction.

From Equation (6.3.2), we have

$$b(k) = \sum_{i\in\mathcal{S}_2} r_iv_i(k) + \sum_{j\in\gamma_5} s_ju_j(k) = 0.$$

where $k \in R_4$. In particular,

- if $k = \ell_0$, then $b(\ell_0) = r\,v(\ell_0) + s\,u(\ell_0) = r - s = 0$.
- if $k = \ell_1$, then $b(\ell_1) = r\,v(\ell_1) + s\,u(\ell_1) = r + s = 0$.

As a result, $s = r = 0$. Hence, $s_j = 0$ for all $j \in \gamma_5$ in Equation (6.3.2).

Since the vectors associated with $\mathcal{S}_2 = \gamma_2 \cup \delta_4 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3$ are linearly independent, we must have $r_i = 0$ for all $i \in \mathcal{S}_2$ in Equation (6.3.2).

5. Let $\ell = yxwy \in \beta_6$. From Case 5 in Section 5.4, T_ℓ appears 3 times in $F_{13}(T^{(4)})$ in the following positions:

$$\begin{array}{cc} & \begin{array}{ccccc} wwy & xwy & xxy & ywy & yxy \end{array} \\ \begin{array}{c} w \\ x \\ y \end{array} & \begin{pmatrix} T_{wwwy} & T_{xwwy} + T_{(wx)(wy)} & T_{xxwy} - T_{yxwx} + 2T_{(wx)(xy)} & T_{ywwy} & T_\ell - T_{yywx} + T_{(wy)(xy)} \\ T_{xwwy} & T_{xxwy} & T_{xxxy} & T_\ell - T_{(wy)(xy)} & T_{yxxy} \\ T_{ywwy} & T_\ell & T_{yxxy} & T_{yywy} & T_{yyxy} \end{pmatrix} \end{array}$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwy,\ xwwy,\ xxwy,\ ywwy,\ \ell = yxwy, \\ & \qquad xxxy,\ yxxy,\ yywy,\ yyxy\} \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $H, I, G, L, \ell = M, J, N, F, K$ as in Table 6.3, is as follows:

$$\begin{array}{ll} & \quad\;\; A\,B\,C\,D\,E\,F\,G\,H\,I\,J\,K\,L\,M\,N \\ \gamma_2 & (\cdots 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \delta_4 & (\cdots 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \gamma_4 & (\cdots 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \alpha_6 & (\cdots 1\ 0\ 0\ 1\ 1\ {-1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \beta_4 & (\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \gamma_3 & (\cdots 0\ 0\ 0\ 0\ 0\ 0\ {-1}\ {-1}\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \gamma_5 & (\cdots 1\ 0\ 0\ 1\ {-1}\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \\ \beta_6 & (\cdots 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots) \end{array}$$

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5$ and β_6 is linearly independent. Let u be a vector associated to an element of β_6. Then, for some position $\ell_0 = y_0x_0w_0y_0 \in \beta_4$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0y_0,\ x_0w_0w_0y_0,\ x_0x_0w_0y_0,\ y_0w_0w_0y_0,\ \ell_0, \\ & \qquad x_0x_0x_0y_0,\ y_0x_0x_0y_0,\ y_0y_0w_0y_0,\ y_0y_0x_0y_0\} \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with β_6 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3$ or γ_5. This will be demonstrated as follows:

(a) Let v be a vector associated with β_6 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_1w_1w_1y_1,\ x_1w_1w_1y_1,\ x_1x_1w_1y_1,\ y_1w_1w_1y_1,\ y_1x_1w_1y_1, \\ x_1x_1x_1y_1,\ y_1x_1x_1y_1,\ y_1y_1w_1y_1,\ y_1y_1x_1y_1\}$$

for some $w_1 < x_1 < y_1$. Note that each word in

$$\{w_1w_1w_1y_1,\ x_1w_1w_1y_1,\ x_1x_1w_1y_1,\ y_1w_1w_1y_1,\ x_1x_1x_1y_1, \\ y_1x_1x_1y_1,\ y_1y_1w_1y_1,\ y_1y_1x_1y_1\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = y_0x_0w_0x_0$ does not have any letter in two consecutive positions. The last possibility is when $\ell_0 = y_0x_0w_0y_0 = y_1x_1w_1y_1$. So, we must have $y_0 = y_1$, $x_0 = x_1$ and $w_0 = w_1$. As a result, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with α_6. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_2 w_2 w_2 x_2,\ \ w_2 w_2 w_2 x_2,\ \ y_2 y_2 x_2 y_2,\ \ y_2 y_2 w_2 x_2\}.$$

for some $w_2 < x_2 < y_2$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(d) Let v be a vector associated with β_4. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_3 w_3 w_3 y_3,\ \ x_3 w_3 w_3 y_3,\ \ x_3 x_3 x_3 y_3,\ \ x_3 x_3 w_3 y_3\}.$$

for some $w_3 < x_3 < y_3$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(e) Let v be a vector associated with γ_3. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{x_4 w_4 w_4 y_4,\ \ x_4 x_4 x_4 y_4,\ \ x_4 x_4 w_4 y_4,\ \ w_4 w_4 w_4 y_4\}.$$

for some $w_4 < x_4 < y_4$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(f) Let v be a vector associated with γ_5. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 = y_0 x_0 w_0 y_0 \in \{y_5 y_5 w_5 x_5,\ \ w_5 w_5 w_5 x_5,\ \ y_5 y_5 x_5 y_5,\ \ y_5 w_5 w_5 x_5\}.$$

for some $w_5 < x_5 < y_5$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction. We have the following possibilities:

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6$ are linearly independent.

6. Let $\ell = yxwx \in \delta_5$. From Case 12 in Section 5.4, T_ℓ appears 3 times in $F_{13}(T^{(4)})$ in the following positions:

$$\begin{array}{cc}
 & \begin{array}{ccccc} wwx & xwx & xxy & ywx & yxy \end{array} \\
\begin{array}{c} w \\ x \\ y \end{array} & \begin{pmatrix} T_{wwwx} & T_{xwwx} & T_{xxwy} - T_\ell + 2T_{(wx)(xy)} & T_{ywwx} - T_{(xw)(wy)} & T_{yxwy} - T_{yywx} + T_{(wy)(xy)} \\ T_{xwwx} & T_{xxwx} & T_{xxxy} & T_\ell - T_{(wx)(xy)} & T_{yxxy} \\ T_{ywwx} & T_\ell & T_{yxxy} & T_{yywx} & T_{yyxy} \end{pmatrix}.
\end{array}$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwx,\ xxwx,\ \ell = yxwx,\ xxxy,\ yxxy,\ yywx,\ yyxy\} \\ -1 & i \in \{xwwx,\ ywwx\} \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $A, B, \ell = O, J, N, D, F, C, E$ as in Table 6.3, is as follows:

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	
γ_2	$(\cdots$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
δ_4	$(\cdots$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
γ_4	$(\cdots$	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
α_6	$(\cdots$	1	0	0	1	1	-1	0	0	0	0	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
β_4	$(\cdots$	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
γ_3	$(\cdots$	0	0	0	0	0	0	-1	-1	1	1	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
γ_5	$(\cdots$	1	0	0	1	-1	1	0	0	0	0	0	0	0	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
β_6	$(\cdots$	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$
δ_5	$(\cdots$	1	1	-1	1	-1	1	0	0	0	1	0	0	0	1	1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $0\cdots)$

Remark 6.3.3. Note that the vectors associated with elements of $\alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ and δ_5 have nonzero values only in positions of words consisting at most three

letters, as seen in the color-coded representation of vectors.

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ and δ_5 is linearly independent. Let u be a vector associated to an element of δ_5. Then, for some position $\ell_0 = y_0x_0w_0x_0 \in \delta_5$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0x_0,\ x_0x_0w_0x_0,\ \ell_0,\ x_0x_0x_0y_0, \\ & \qquad y_0x_0x_0y_0,\ y_0y_0w_0x_0,\ y_0y_0x_0y_0\} \\ -1 & i \in \{x_0w_0w_0x_0,\ y_0w_0w_0x_0\} \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with δ_5 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5$ or β_6. This will be demonstrated as follows:

(a) Let v be a vector associated with δ_5 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_1w_1w_1x_1,\ x_1x_1w_1x_1,\ y_1x_1w_1x_1,\ x_1x_1x_1y_1,\ y_1x_1x_1y_1, \\ y_1y_1w_1x_1,\ y_1y_1x_1y_1,\ x_1w_1w_1x_1,\ y_1w_1w_1x_1\}$$

for some $w_1 < x_1 < y_1$. Note that each word in

$$\{w_1w_1w_1x_1,\ x_1x_1w_1x_1,\ x_1x_1x_1y_1,\ y_1x_1x_1y_1, \\ y_1y_1w_1x_1,\ y_1y_1x_1y_1,\ x_1w_1w_1x_1,\ y_1w_1w_1x_1\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = y_0x_0w_0x_0$ don't have any letter in two consecutive positions. The last possibility is when $\ell_0 = y_0x_0w_0x_0 = y_1x_1w_1x_1$. So, we must have $y_0 = y_1$, $x_0 = x_1$ and $w_0 = w_1$. As a result, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with α_6. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_2w_2w_2x_2,\ \ w_2w_2w_2x_2,\ \ y_2y_2x_2y_2,\ \ y_2y_2w_2x_2\},$$

for some $w_2 < x_2 < y_2$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(d) Let v be a vector associated with β_4. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_3w_3w_3y_3,\ \ x_3w_3w_3y_3,\ \ x_3x_3x_3y_3,\ \ x_3x_3w_3y_3\},$$

for some $w_3 < x_3 < y_3$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(e) Let v be a vector associated with γ_3. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{x_4w_4w_4y_4,\ \ x_4x_4x_4y_4,\ \ x_4x_4w_4y_4,\ \ w_4w_4w_4y_4\},$$

for some $w_4 < x_4 < y_4$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(f) Let v be a vector associated with γ_5. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{y_5y_5w_5x_5,\ \ w_5w_5w_5x_5,\ \ y_5y_5x_5y_5,\ \ y_5w_5w_5x_5\},$$

for some $w_5 < x_5 < y_5$. Since each possible word from the set above has a letter in two consecutive positions and ℓ_0 does not have any letter in two consecutive positions, we get a contradiction.

(g) Let v be a vector associated with β_6. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_6w_6w_6y_6,\ x_6w_6w_6y_6,\ x_6x_6w_6y_6,\ y_6w_6w_6y_6,\ y_6x_6w_6y_6,\\ x_6x_6x_6y_6,\ y_6x_6x_6y_6,\ y_6y_6w_6y_6,\ y_6y_6x_6y_6\},$$

for some $w_6 < x_6 < y_6$. Note that each word in

$$\{w_6w_6w_6y_6,\ x_6w_6w_6y_6,\ x_6x_6w_6y_6,\ y_6w_6w_6y_6,\ x_6x_6x_6y_6,\\ y_6x_6x_6y_6,\ y_6y_6w_6y_6,\ y_6y_6x_6y_6\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = y_0x_0w_0x_0$ don't have any letter in two consecutive positions. The last possibility is when $\ell_0 = y_0x_0w_0x_0 = y_6x_6w_6y_6$. Then we must have $y_0 = y_6 = x_0$, but $x_0 < y_0$. So we have a contradiction.

In conclusion, by Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5$ are linearly independent.

6.4 Left-Aligned Four-Letter Words

In this section we consider left-aligned words with four letters w, x, y and z such that $w < x < y < z$. The following table lists these four-letter words and the number of their occurrences, as shown in Table 5.1.

Type	*Word*	*Occurrences in* $F_{13}(T^{(4)})$
β_5	$yxwz$	3
β_7	$zxwy$	3
α_7	$zywx$	4

We analyze each case in the above table as follows:

1. Let $\ell = yxwz \in \beta_5$. From Case 4 in Section 5.4, T_ℓ appears 3 times in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c} \\ w \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} wwz & xwz & xxz & ywz & yxz & yyz \end{array} \\ \begin{pmatrix} T_{wwwz} & T_{xwwz} & T_{xxwz} - T_{zxwx} & T_{ywwz} & T_\ell & T_{yywz} - T_{zywy} \\ T_{xwwz} & T_{xxwz} & T_{xxxz} & T_\ell & T_{yxxz} & T_{yyxz} - T_{zyxy} \\ T_{ywwz} & T_\ell & T_{yxxz} & T_{yywz} & T_{yyxz} & T_{yyyz} \end{pmatrix} \end{array}$$

Remark 6.4.1. Note that the variables T_X where $X \in \mu \cup \nu_{1,\dots,5}$, i.e. variables indexed by balanced words, will be omitted in the commutator matrix $F_{13}(T^{(4)})$ for better visibility. This will not affect the rank computations as the vectors we will assign to left-aligned words will have zero entries in balanced words positions.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwz,\ xwwz,\ ywwz,\ xxwz,\ \ell = yxwz,\ xxxz,\ yxxz, \\ & \quad yywz,\ yyxz,\ yyyz\} \\ 0 & \text{elsewhere,} \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $P_0, P_1, P_2, P_3, \ell = P_4, P_5, P_6, P_7, P_8, P_9$ as in Table 6.3, is as follows:

$$
\begin{array}{cccccccccccccccccccccccccccc}
 & & A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & \\
\gamma_2 & (\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_4 & (\cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_4 & (\cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\alpha_6 & (\cdots & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_4 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_3 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_5 & (\cdots & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_6 & (\cdots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_5 & (\cdots & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_5 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots)
\end{array}
$$

Next we prove that the current set of vectors associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6, \delta_5$ and β_5 is linearly independent. Let u be a vector associated to an element of β_5. Then, for some position $\ell_0 = y_0x_0w_0z_0 \in \beta_5$, where $w_0 < x_0 < y_0 < z_0$, we have

$$
u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0z_0,\ x_0w_0w_0z_0,\ y_0w_0w_0z_0,\ x_0x_0w_0z_0,\ \ell_0, \\ & \quad x_0x_0x_0z_0,\ y_0x_0x_0z_0,\ y_0y_0w_0z_0,\ y_0y_0x_0z_0,\ y_0y_0y_0z_0\} \\ 0 & \text{elsewhere.} \end{cases}
$$

We will show that $v(\ell_0) = 0$ for any vector v associated with β_5 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ or δ_5. This will be demonstrated as follows:

(a) Let v be a vector associated with β_5 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$
\begin{aligned}
\ell_0 \in \{&w_1w_1w_1z_1,\ x_1w_1w_1z_1,\ y_1w_1w_1z_1,\ x_1x_1w_1z_1,\ y_1x_1w_1z_1,\ x_1x_1x_1z_1, \\
&y_1x_1x_1z_1,\ y_1y_1w_1z_1,\ y_1y_1x_1z_1,\ y_1y_1y_1z_1\}
\end{aligned}
$$

for some $w_1 < x_1 < y_1 < z_1$. Note that each word in

$$\{w_1w_1w_1z_1,\ x_1w_1w_1z_1,\ y_1w_1w_1z_1,\ x_1x_1w_1z_1,\ x_1x_1x_1z_1,\\ y_1x_1x_1z_1,\ y_1y_1w_1z_1,\ y_1y_1x_1z_1,\ y_1y_1y_1z_1\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = y_0x_0w_0z_0$ does not have any letter in two consecutive positions. The last possibility is when $\ell_0 = y_0x_0w_0z_0 = y_1x_1w_1z_1$. So, we must have $y_0 = y_1$, $x_0 = x_1$, $w_0 = w_1$ and $z_0 = z_1$. As a result, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with $\alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ or δ_5. As observed in Remark 6.3.3, $v(\ell_0) = 0$.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5$ are linearly independent.

2. Let $\ell = zxwy \in \beta_7$. From Case 6 in Section 5.4, T_ℓ appears 3 times in $F_{13}(T^{(4)})$ in the following positions:

$$F_{13}(T^{(4)}) = \begin{array}{c} \\ w \\ x \\ z \end{array} \begin{array}{c} \begin{array}{cccccc} wwy & xwy & xxy & zwy & zxy & zyz \end{array} \\ \begin{pmatrix} T_{wwwy} & T_{xwwy} & T_{xxwy} - T_{yxwx} & T_{zwwy} & T_\ell & T_{zywz} - T_{zzwy} \\ T_{xwwy} & T_{xxwy} & T_{xxxy} & T_\ell & T_{zxxy} & T_{zyxz} - T_{zzxy} \\ T_{zwwy} & T_\ell & T_{zxxy} & T_{zzwy} & T_{zzxy} & T_{zzyz} \end{pmatrix} \end{array}$$

As in Remark 6.4.1, the variables indexed by balanced words in $F_{13}(T^{(4)})$ are omitted. If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{wwwy,\ xwwy,\ zwwy,\ xxwy,\ \ell = zxwy,\ xxxy, \\ & zxxy,\ zzwy,\ zzxy\} \\ -1 & i = zzyz \\ 0 & \text{elsewhere}, \end{cases}$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$.

The color-coded representation of the nonzero entries of a_ℓ, labeled by $H, I, Q_0, G, \ell = Q_1, J, Q_2, Q_3, Q_4, Q_5$ as in Table 6.3, is as follows:

$$\begin{array}{cccccccccccccccccccccccccccccccccc}
 & & A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & Q_0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & \\
\gamma_2 & (\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_4 & (\cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_4 & (\cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\alpha_6 & (\cdots & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_4 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_3 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\gamma_5 & (\cdots & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_6 & (\cdots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\delta_5 & (\cdots & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_5 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots) \\
\beta_7 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots)
\end{array}$$

Next we prove that the current set of vectors associated to γ_2, δ_4, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5 and β_7 is linearly independent. Let u be a vector associated to an element of β_7. Then, for some position $\ell_0 = z_0x_0w_0y_0 \in \beta_7$, where $w_0 < x_0 < y_0 < z_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{w_0w_0w_0y_0,\ x_0w_0w_0y_0,\ z_0w_0w_0y_0,\ x_0x_0w_0y_0,\ \ell_0, \\ & \quad x_0x_0x_0y_0,\ z_0x_0x_0y_0,\ z_0z_0w_0y_0,\ z_0z_0x_0y_0\} \\ -1 & i = z_0z_0y_0z_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with β_7 when $v \neq u$, or for any vector v associated to $\gamma_2, \delta_4, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6, \delta_5$ or β_5. This will be demonstrated as follows:

(a) Let v be a vector associated with β_7 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_1w_1w_1y_1,\ x_1w_1w_1y_1,\ z_1w_1w_1y_1,\ x_1x_1w_1y_1,\ z_1x_1w_1y_1,\ x_1x_1x_1y_1, \\ z_1x_1x_1y_1,\ z_1z_1w_1y_1,\ z_1z_1x_1y_1,\ z_1z_1y_1z_1\}$$

for some $w_1 < x_1 < y_1 < z_1$. Note that each word in

$$\{w_1w_1w_1y_1,\ x_1w_1w_1y_1,\ z_1w_1w_1y_1,\ x_1x_1w_1y_1,\ x_1x_1x_1y_1, \\ z_1x_1x_1y_1,\ z_1z_1w_1y_1,\ z_1z_1x_1y_1,\ z_1z_1y_1z_1\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = z_0x_0w_0y_0$ don't have any letter in two consecutive positions. The last possibility is when $\ell_0 = z_0x_0w_0y_0 = z_1x_1w_1y_1$. So, we must have $y_0 = y_1$, $x_0 = x_1$, $w_0 = w_1$ and $z_0 = z_1$. As a result, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with $\alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ or δ_5. As observed in Remark 6.3.3, $v(\ell_0) = 0$.

(d) Let v be a vector associated with β_5. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_2w_2w_2z_2,\ x_2w_2w_2z_2,\ y_2w_2w_2z_2,\ x_2x_2w_2z_2,\ y_2x_2w_2z_2,\\ x_2x_2x_2z_2,\ y_2x_2x_2z_2,\ y_2y_2w_2z_2,\ y_2y_2x_2z_2,\ y_2y_2y_2z_2\}.$$

for some $w_2 < x_2 < y_2 < z_2$. Note that each word in

$$\{w_2w_2w_2z_2,\ x_2w_2w_2z_2,\ y_2w_2w_2z_2,\ x_2x_2w_2z_2,\ x_2x_2x_2z_2,\\ y_2x_2x_2z_2,\ y_2y_2w_2z_2,\ y_2y_2x_2z_2,\ y_2y_2y_2z_2\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = z_0x_0w_0y_0$ don't have any letter in two consecutive positions. The last possibility is when $\ell_0 = z_0x_0w_0y_0 = y_2x_2w_2z_2$. So, we must have $z_0 = y_2$ and $x_0 = x_2$ and $w_0 = w_2$ and $y_0 = z_2$. Since $y_2 < z_2$, we get that $z_0 < y_0$, which is a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7$ are linearly independent.

3. Let $\ell = zywx \in \alpha_7$. From Case 2 in Section 5.4, T_ℓ appears 4 times in $F_{13}(T^{(4)})$ as shown in Figure 6.1, where the indices are labeled as in Table 6.3. As in Remark 6.4.1, the variables indexed by balanced words in $F_{13}(T^{(4)})$ are omitted.

$$
\begin{array}{c}
\\ w \\ x \\ y \\ z
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
wwx & wwz & xwx & xwz & xxy & xxz & ywx & ywz & yxy & yxz & yyz & zwx & zwz & zxy & zxz & zyz
\end{array}\\
\begin{pmatrix}
T_A & T_{P_0} & T_C & T_{P_1} & -T_O & T_{P_3}-T_{R_2} & T_E & T_{P_2} & -T_D & T_{P_4}-T_\ell & T_{P_7} & T_{R_0} & T_{R_1} & -T_\ell & T_{R_3}-T_{S_0} & T_{R_6} \\
T_C & T_{P_1} & T_B & T_{P_3} & T_J & T_{P_5} & T_O & T_{P_4} & T_N & T_{P_6} & T_{P_8}-T_{R_7} & T_{R_2} & T_{R_3} & T_{Q_2} & T_{R_4} & T_{R_8}-T_{Q_4} \\
T_E & T_{P_2} & T_O & T_{P_4} & T_N & T_{P_6} & T_D & T_{P_7} & T_F & T_{P_8} & T_{P_9} & T_\ell & T_{R_6} & T_{R_7} & T_{R_8} & T_{R_9} \\
T_{R_0} & T_{R_1} & T_{R_2} & T_{R_3} & T_{Q_2} & T_{R_4} & T_\ell & T_{R_6} & T_{R_7} & T_{R_8} & T_{R_9} & T_{S_0} & T_{S_1} & T_{Q_4} & T_{S_2} & T_{Q_5}
\end{pmatrix}
\end{array}
$$

Figure 6.1: Occurrences of T_ℓ in $F_{13}(T^4)$, $\ell = zywx \in \alpha_7$.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$
a_\ell(i) = \begin{cases}
2 & i \in \{P_4,\ P_3,\ R_3,\ P_1,\ R_6,\ R_1,\ S_1,\ R_9,\ Q_5,\ P_7,\ P_9,\ P_2,\ P_0\} \\
1 & i \in \{\ell = R_5,\ P_8,\ R_2,\ P_6,\ P_5,\ R_4,\ R_0,\ R_8,\ S_0,\ S_2,\ D,\ E,\ O, \\
 & \qquad A,\ C,\ B\} \\
-1 & i \in \{R_7,\ Q_2,\ Q_4,\ N,\ F,\ J\} \\
0 & \text{elsewhere,}
\end{cases}
$$

we get $rk(F_{13}(a_\ell)) = 1$. As a result, $rk(F(a_\ell)) = 2$. The rank computations for this case were checked through the use of a code.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	S_0	S_1	S_2
γ_2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
δ_4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
γ_4	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
α_6	1	0	0	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
β_4	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
γ_3	0	0	0	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
γ_5	1	0	0	1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
β_6	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
δ_5	1	1	−1	1	−1	1	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
β_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
β_7	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
α_7	1	1	1	1	1	−1	0	0	0	−1	0	0	0	−1	1	2	2	2	2	2	1	1	2	1	2	0	0	−1	0	−1	2	1	2	1	2	1	1	2	−1	1	2	1	2	1

Figure 6.2: Color-coded representation of a_ℓ, $\ell = zywx \in \alpha_7$

Next we prove that the current set of vectors associated to δ_4, γ_2,γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 and α_7 is linearly independent. Let u be a vector associated to an element of α_7. Then, for some position $\ell_0 = z_0y_0w_0x_0 \in \alpha_7$, where $w_0 < x_0 < y_0 < z_0$, we have

$$u(i) = \begin{cases} 2 & i \in \{y_0x_0w_0z_0,\ x_0x_0w_0z_0,\ z_0x_0w_0z_0,\ x_0w_0w_0z_0,\ z_0y_0w_0z_0, \\ & \quad z_0w_0w_0z_0,\ z_0z_0w_0z_0,\ z_0y_0y_0z_0,\ z_0z_0y_0z_0,\ y_0y_0w_0z_0, \\ & \quad y_0y_0y_0z_0,\ y_0w_0w_0z_0,\ w_0w_0w_0z_0\} \\ 1 & i \in \{\ell_0,\ y_0y_0x_0z_0,\ z_0x_0w_0x_0,\ y_0x_0x_0z_0,\ x_0x_0x_0z_0, \\ & \quad z_0x_0x_0z_0,\ z_0w_0w_0x_0,\ z_0y_0x_0z_0,\ z_0z_0w_0x_0,\ z_0z_0x_0z_0,\ y_0y_0w_0x_0, \\ & \quad y_0w_0w_0x_0,\ y_0x_0w_0x_0,\ w_0w_0w_0x_0,\ x_0w_0w_0x_0,\ x_0x_0w_0x_0\} \\ -1 & i \in \{z_0y_0x_0y_0,\ z_0x_0x_0y_0,\ z_0z_0x_0y_0,\ y_0x_0x_0y_0, \\ & \quad y_0y_0x_0y_0,\ x_0x_0x_0y_0\} \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with α_7 when $v \neq u$, or for any vector v associated to $\delta_4, \gamma_2, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6, \delta_5, \beta_5$ or β_7. This will be demonstrated as follows:

(a) Let v be a vector associated with α_7 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\begin{aligned} \ell_0 \in \{\ & y_1x_1w_1z_1,\ x_1x_1w_1z_1,\ z_1x_1w_1z_1,\ x_1w_1w_1z_1,\ z_1y_1w_1z_1, \\ & z_1w_1w_1z_1,\ z_1z_1w_1z_1,\ z_1y_1y_1z_1,\ z_1z_1y_1z_1,\ y_1y_1w_1z_1, \\ & y_1y_1y_1z_1,\ y_1w_1w_1z_1,\ w_1w_1w_1z_1,\ z_1y_1w_1x_1,\ y_1y_1x_1z_1, \\ & z_1x_1w_1x_1,\ y_1x_1x_1z_1,\ x_1x_1x_1z_1,\ z_1x_1x_1z_1,\ z_1w_1w_1x_1, \\ & z_1y_1x_1z_1,\ z_1z_1w_1x_1,\ z_1z_1x_1z_1,\ y_1y_1w_1x_1,\ y_1w_1w_1x_1, \\ & y_1x_1w_1x_1,\ w_1w_1w_1x_1,\ x_1w_1w_1x_1,\ x_1x_1w_1x_1,\ z_1y_1x_1y_1, \end{aligned}$$

$$z_1x_1x_1y_1,\ z_1z_1x_1y_1,\ y_1x_1x_1y_1,\ y_1y_1x_1y_1,\ x_1x_1x_1y_1\ \}$$

for some $w_1 < x_1 < y_1 < z_1$. Note that each word in

$$\{x_1x_1w_1z_1\ x_1w_1w_1z_1,\ z_1w_1w_1z_1,\ z_1z_1w_1z_1,\ z_1y_1y_1z_1,\ z_1z_1y_1z_1,\ y_1y_1w_1z_1,$$
$$y_1y_1y_1z_1,\ y_1w_1w_1z_1,\ w_1w_1w_1z_1,\ y_1y_1x_1z_1,\ y_1x_1x_1z_1,\ x_1x_1x_1z_1,\ z_1x_1x_1z_1,$$
$$z_1w_1w_1x_1,\ z_1z_1w_1x_1,\ z_1z_1x_1z_1,\ y_1y_1w_1x_1,\ y_1w_1w_1x_1,\ w_1w_1w_1x_1,\ x_1w_1w_1x_1,$$
$$x_1x_1w_1x_1,\ z_1x_1x_1y_1,\ z_1z_1x_1y_1,\ y_1x_1x_1y_1,\ y_1y_1x_1y_1,\ x_1x_1x_1y_1\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = z_0y_0w_0x_0$ does not have any letter in two consecutive positions. We check the rest of the possibilities as follows:

i. If $\ell_0 = z_0y_0w_0x_0 = y_1x_1w_1z_1$, then $z_0 = y_1$ and $x_0 = z_1$. Since $y_1 < z_1$, then we must have $z_0 < x_0$, which is a contradiction.

ii. If $\ell_0 = z_0y_0w_0x_0 = z_1x_1w_1z_1$, then $z_0 = z_1 = x_0$, but $x_0 < z_0$. So, we get a contradiction.

iii. If $\ell_0 = z_0y_0w_0x_0 = z_1y_1w_1z_1$, then $z_0 = z_1 = x_0$, but $x_0 < z_0$. So, we get a contradiction.

iv. If $\ell_0 = z_0y_0w_0x_0 = z_1y_1w_1x_1$, then $z_0 = z_1$, $y_0 = y_1$, $w_0 = w_1$ and $x_0 = x_1$. So, $u = v$, which is a contradiction.

v. If $\ell_0 = z_0y_0w_0x_0 = z_1x_1w_1x_1$, then $y_0 = x_1 = x_0$, but $x_0 < y_0$. So, we get a contradiction.

vi. If $\ell_0 = z_0y_0w_0x_0 = z_1y_1x_1z_1$, then $z_0 = z_1 = x_0$, but $x_0 < z_0$. So, we get a contradiction.

vii. If $\ell_0 = z_0y_0w_0x_0 = y_1x_1w_1x_1$, then $y_0 = x_1 = x_0$, but $x_0 < y_0$. So, we get a contradiction.

viii. If $\ell_0 = z_0y_0w_0x_0 = z_1y_1x_1y_1$, then $y_0 = y_1 = x_0$, but $x_0 < y_0$. So, we get a contradiction.

(b) Let v be a vector associated with δ_4, γ_2 or γ_4. As observed in Remark 6.2.1, $v(\ell_0) = 0$.

(c) Let v be a vector associated with $\alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6$ or δ_5. As observed in Remark 6.3.3, $v(\ell_0) = 0$.

(d) Let v be a vector associated with β_5. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_2w_2w_2z_2,\ x_2w_2w_2z_2,\ y_2w_2w_2z_2,\ x_2x_2w_2z_2,\ y_2x_2w_2z_2,\\ x_2x_2x_2z_2,\ y_2x_2x_2z_2,\ y_2y_2w_2z_2,\ y_2y_2x_2z_2,\ y_2y_2y_2z_2\},$$

for some $w_2 < x_2 < y_2 < z_2$. Note that each word in

$$\{w_2w_2w_2z_2,\ x_2w_2w_2z_2,\ y_2w_2w_2z_2,\ x_2x_2w_2z_2,\ x_2x_2x_2z_2,\\ y_2x_2x_2z_2,\ y_2y_2w_2z_2,\ y_2y_2x_2z_2,\ y_2y_2y_2z_2\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = z_0y_0w_0x_0$ does not have any letter in two consecutive positions. The last possibility is when $\ell_0 = z_0y_0w_0x_0 = y_2x_2w_2z_2$. So, we must have $z_0 = y_2$ and $y_0 = x_2$ and $w_0 = w_2$ and $x_0 = z_2$. Since $y_2 < z_2$, we get that $z_0 < x_0$, which is a contradiction.

(e) Let v be a vector associated with β_7. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{w_3w_3w_3y_3,\ x_3w_3w_3y_3,\ z_3w_3w_3y_3,\ x_3x_3w_3y_3,\ z_3x_3w_3y_3,\\ x_3x_3x_3y_3,\ z_3x_3x_3y_3,\ z_3z_3w_3y_3,\ z_3z_3x_3y_3,\ z_3z_3y_3z_3\}$$

for some $w_3 < x_3 < y_3 < z_3$. Note that each word in

$$\{w_3w_3w_3y_3,\ \ x_3w_3w_3y_3,\ \ z_3w_3w_3y_3,\ \ x_3x_3w_3y_3,\ \ x_3x_3x_3y_3,\\ z_3x_3x_3y_3,\ \ z_3z_3w_3y_3,\ \ z_3z_3x_3y_3,\ \ z_3z_3y_3z_3\}$$

has a letter in two consecutive positions, which leads to a contradiction since $\ell_0 = z_0y_0w_0x_0$ don't have any letter in two consecutive positions. The last case is when $\ell_0 = z_0y_0w_0x_0 = z_3x_3w_3y_3$. So, we must have $z_0 = z_3$ and $y_0 = x_3$ and $w_0 = w_3$ and $x_0 = y_3$. Since $x_3 < y_3$, we get that $y_0 < x_0$, which is a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7$ are linearly independent.

6.5 Balanced Three-Letter Words

In this section we consider balanced words with only three letters w, x and y such that $w < x < y$. The following table lists these three-letter words and the number of their occurrences, as shown in Table 5.2.

Type	*Word*	*Appearances in* $F_{22}(T^{(4)})$	*Appearances in* $F_{13}(T^{(4)})$
ν_1	$(wx)(wy)$	2	2
ν_3	$(wx)(xy)$	2	2
ν_5	$(wy)(xy)$	2	2

Before analyzing the cases in the above table, note that, at this point, the greedy algorithm has exhausted all possible linearly independent vectors a that produce a minimal rank of $rk(F(a)) = 2$. This is because the algorithm has constructed N linearly independent vectors lying in an N-dimensional subspace of $\mathbb{F}_q^{r_4(c)}$, where N is the number of left-aligned words. To ensure linear independence, the greedy algorithm is forced to choose vectors a with nonzero balanced entries. Thus, the minimal rank will be $rk(F(a)) \geq 4$, as shown in Lemma 6.1.6. With this in mind, we proceeded with the case by case analysis as follows:

1. Let $\ell = (wx)(wy) \in \nu_1$. From Case 13 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$ in the following positions:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} w & x & z & xw & wy & wwx & wwz & xwz & zwx \end{array} \\
\begin{array}{c} w \\ x \\ z \\ xw \\ wy \\ wwx \\ wwz \\ xwz \\ zwx \end{array} &
\left(\begin{array}{ccc|cc|cccc}
 & & & & & & & T_{xwwz}+T_\ell & T_{zwwx}-T_\ell \\
 & & & & & & T_{xwwz} & & \\
 & & & & & T_{zwwx} & & & \\ \hline
 & & & 0 & T_\ell & & & & \\
 & & & -T_\ell & 0 & & & & \\ \hline
 & & -T_{zwwx} & & & & & & \\
 & -T_{xwwz} & & & & & & & \\
-T_{xwwz}-T_\ell & & & & & & & & \\
-T_{zwwx}+T_\ell & & & & & & & &
\end{array}\right)
\end{array}$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell = (wx)(wy) \\ 0 & \text{elsewhere,} \end{cases}$$

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\, rk(F_{13}(a_\ell)) = 2 + 2 \times 1 = 4.$$

The color-coded representation of the nonzero entries of a_ℓ is given below:

$$\begin{array}{cc} & \begin{array}{ccccccccc} & & & & V_1 & & & & \end{array} \\ \nu_1 & \left(\begin{array}{ccccccccc} 0 & , & \cdots & , & 1 & , & \cdots & , & 0 \end{array} \right) \end{array}$$

where $V_1 = (wx)(wy)$ as labeled in Table 6.3.

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7 and ν_1 is linearly independent. Let u be a vector associated to an element of ν_1. Then, for some position $\ell_0 = (w_0x_0)(w_0y_0) \in \nu_1$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_1 when $v \neq u$, or for any vector v associated to $\delta_4, \gamma_2, \gamma_4, \alpha_6, \beta_4, \gamma_3, \gamma_5, \beta_6, \delta_5, \beta_5, \beta_7$ or α_7. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_1 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_1x_1)(w_1y_1)$ for some $w_1 < x_1 < y_1$. Then, $w_0 = w_1$, $x_0 = x_1$ and $y_0 = y_1$. So, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1$ are linearly independent.

2. Let $\ell = (wx)(xy) \in \nu_3$. From Case 15 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$ in the following positions:

$$\begin{array}{cc}
 & \begin{array}{ccc|cc|cccc} w & x & y & wx & xy & xwx & xwy & xxy & ywx \end{array} \\
\begin{array}{c} w \\ x \\ y \\ wx \\ xy \\ xwx \\ xwy \\ xxy \\ ywx \end{array} &
\left(\begin{array}{ccc|cc|cccc}
 & & & & & & & T_{xxwy}-T_{yxwx}+2T_\ell & \\
 & & & & & & T_{xxwy} & & T_{yxwx}-T_\ell \\
 & & & & & T_{yxwx} & & & \\
\hline
 & & & 0 & -T_\ell & & & & \\
 & & & T_\ell & 0 & & & & \\
\hline
 & & -T_{yxwx} & & & & & & \\
 & -T_{xxwy} & & & & & & & \\
-T_{xxwy}+T_{yxwx}-2T_\ell & & & & & & & & \\
 & -T_{yxwx}+T_\ell & & & & & & &
\end{array}\right)
\end{array}$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell = (wx)(xy) \\ -2 & i = xxwy \\ 0 & \text{elsewhere,} \end{cases}$$

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\, rk(F_{13}(a_\ell)) = 2 + 2 \times 1 = 4.$$

The color-coded representation of the nonzero entries of a_ℓ, labeled by $G, \ell = V_3$ as in Table 6.3, is as follows:

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	S_0	S_1	S_2	V_1	V_2	V_3	V_4	V_5	V_6	
ν_1	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	$\cdots)$
ν_3	$(\cdots$	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	$\cdots)$

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1 and ν_3 is linearly independent. Let u be a vector associated to an element of ν_3. Then, for some position $\ell_0 = (w_0x_0)(x_0y_0) \in \nu_3$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ -2 & i = x_0x_0w_0y_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_3 when $v \neq u$, or for any vector v associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7 or ν_1. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_3 such that $v \neq u$. We claim that $v(\ell_0) = 0$.

Otherwise,

$$\ell_0 \in \{(w_1x_1)(x_1y_1), x_1x_1w_1y_1\}.$$

for some $w_1 < x_1 < y_1$. Since ℓ_0 is a balanced element, then we must have $\ell_0 = (w_1x_1)(x_1y_1)$. So, $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

(c) Let v be a vector associated with ν_1. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0x_0)(x_0y_0) = (w_2x_2)(w_2y_2)$ for some $w_2 < x_2 < y_2$. Then, $w_0 = w_2 = x_0$, but $w_0 < x_0$. So, we get a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1 \cup \nu_3$ are linearly independent.

3. Let $\ell = (wy)(xy) \in \nu_5$. From Case 17 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$ in the following positions:

$$\begin{array}{c} \\ w \\ x \\ y \\ wy \\ xy \\ xwy \\ ywx \\ ywy \\ yxy \end{array}
\begin{array}{c}
\begin{array}{ccc|cc|cccc} w & x & y & wy & xy & xwy & ywx & ywy & yxy \end{array} \\
\left(\begin{array}{ccc|cc|cccc}
 & & & & & & & & T_{yxwy}-T_{yywx}+T_\ell \\
 & & & & & & & T_{yxwy}-T_\ell & \\
 & & & & & T_{yxwy} & T_{yywx} & & \\
\hline
 & & & 0 & T_\ell & & & & \\
 & & & -T_\ell & 0 & & & & \\
\hline
 & & -T_{yxwy} & & & & & & \\
 & & -T_{yywx} & & & & & & \\
 & -T_{yxwy}+T_\ell & & & & & & & \\
-T_{yxwy}+T_{yywx}-T_\ell & & & & & & & &
\end{array}\right)
\end{array}$$

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i \in \{\ell = (wy)(xy), yxwy\} \\ 2 & i = yywx \\ 0 & \text{elsewhere,} \end{cases}$$

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\,rk(F_{13}(a_\ell)) = 2 + 2 \times 1 = 4.$$

The color-coded representation of the nonzero entries of a_ℓ, labeled by $D, M, \ell = V_5$ as in Table 6.3, is as follows:

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	S_0	S_1	S_2	V_1	V_2	V_3	V_4	V_5	V_6	
ν_1	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	$\cdots)$
ν_3	$(\cdots$	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	$\cdots)$
ν_5	$(\cdots$	0	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$\cdots)$

Remark 6.5.1. Note that the vectors associated with elements of ν_1, ν_3 and ν_5 have nonzero values only in positions of words of consisting of at most three letters, as seen in the color-coded representation of vectors.

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1, ν_3 and ν_5 is linearly independent. Let u be a vector associated to an element of ν_5. Then, for some position $\ell_0 = (w_0y_0)(x_0y_0)$, where $w_0 < x_0 < y_0$, we have

$$u(i) = \begin{cases} 1 & i \in \{\ell_0 = (w_0y_0)(x_0y_0), y_0x_0w_0y_0\} \\ 2 & i = y_0y_0w_0x_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_5 when $v \neq u$, or for any vector v associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1 or ν_3. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_5 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{(w_1y_1)(x_1y_1),\ \ y_1x_1w_1y_1,\ \ y_1y_1w_1x_1\},$$

for some $w_1 < x_1 < y_1$. Since ℓ_0 is a balanced element, we must have $\ell_0 = (w_1y_1)(x_1y_1)$. So, we must have $u = v$, which is ac contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

(c) Let v be a vector associated with ν_1. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0y_0)(x_0y_0) = (w_2x_2)(w_2y_2)$ for some $w_2 < x_2 < y_2$. Then, $w_0 = w_2 = x_0$, but $w_0 < x_0$. So, we get a contradiction.

(d) Let v be a vector associated with ν_3. We claim that $v(\ell_0) = 0$. Otherwise,

$$\ell_0 \in \{(w_3x_3)(x_3y_3),\ \ x_3x_3w_3y_3\},$$

for some $w_3 < x_3 < y_3$. Since ℓ_0 is a balanced element, we must have $\ell_0 = (w_0y_0)(x_0y_0) = (w_3x_3)(x_3y_3)$. Then, $y_0 = x_3 = x_0$, but $x_0 < y_0$. So, we get a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1 \cup \nu_3 \cup \nu_5$ are linearly independent.

6.6 Balanced Four-Letter Words

In this section we consider balanced words with four letters w, x, y and z such that $w < x < y < z$. The following table lists these four-letter words and the number of their occurrences, as shown in Table 5.2.

Type	*Word*	*Appearances in* $F_{22}(T^{(4)})$	*Appearances in* $F_{13}(T^{(4)})$
ν_2	$(wx)(yz)$	2	2
ν_4	$(wy)(xz)$	2	2
ν_6	$(wz)(xy)$	2	2

From now on, the greedy algorithm is forced to choose vectors with nonzero entries in balanced four-letter positions to ensure linear independence. Recall that in the proof of Lemma 6.1.6, we showed that if a is a vector with a nonzero entry in a balanced position, then the rank of $F_{13}(a)$ is at least one. In Subsections 6.6.1, 6.6.2 and 6.6.3, we will prove the following results:

Proposition 6.6.1. *If $a(\ell) \neq 0$ for some $\ell \in \nu_2$, then $rk(F_{13}(a)) \geq 2$.*

Proposition 6.6.2. *If $a(\ell) \neq 0$ for some $\ell \in \nu_4$, then $rk(F_{13}(a)) \geq 2$.*

Proposition 6.6.3. *If $a(\ell) \neq 0$ for some $\ell \in \nu_6$, then $rk(F_{13}(a)) \geq 2$.*

Hence, all the vectors a chosen by the greedy algorithm from now on satisfy

$$rk(F(a)) = 2rk(F_{13}(a)) + rk(F_{22}(a)) = 2 \times 2 + 2 \geq 6.$$

Consequently, it is enough to complete our list of vectors into a basis with vectors a such that $rk(F(a)) = 6$, as shown in the following case by case analysis:

1. Let $\ell = (wx)(yz) \in \nu_2$. From Case 14 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$. It appears two times in $F_{22}(T^4)$ in the following positions:

$$F_{22}(T^{(4)}) = \begin{array}{cc} & \begin{array}{cc} wx & yz \end{array} \\ \begin{array}{c} wx \\ yz \end{array} & \begin{pmatrix} 0 & T_\ell \\ -T_\ell & 0 \end{pmatrix} \end{array},$$

and two times in $F_{13}(T^4)$ as shown in Figure 6.3. Note that in Figure 6.3 the T's in the variables are suppressed for simplicity (e.g. A instead of T_A, etc.), and the indices are labeled as in Table 6.3.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell = (wx)(yz) \\ 0 & \text{elsewhere.} \end{cases}$$

$$
\begin{array}{c}
\\ w \\ x \\ y \\ z
\end{array}
\begin{array}{cccccccccccccccc}
wwx & wwz & xwx & xwz & xxy & xxz & ywx & ywz & yxy & yxz & yyz & zwx & zwz & zxy & zxz & zyz
\end{array}
$$

$$
\begin{array}{c}
w \\ x \\ y \\ z
\end{array}
\begin{pmatrix}
A & P_0 & C & P_1 & -O & P_3-R_2 & E & P_2 & -D & P_4-R_5+\ell & P_7 & R_0 & R_1 & -R_5 & R_3-S_0 & R_6 \\
C & P_1 & B & P_3 & J & P_5 & O & P_4 & N & P_6 & P_8-R_7 & R_2 & R_3 & Q_2 & R_4 & R_8-Q_4 \\
E & P_2 & O & P_4 & N & P_6 & D & P_7 & F & P_8 & P_9 & R_5-\ell & R_6 & R_7 & R_8 & R_9 \\
R_0 & R_1 & R_2 & R_3 & Q_2 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & S_0 & S_1 & Q_4 & S_2 & Q_5
\end{pmatrix}
$$

Figure 6.3: Occurrences of T_ℓ in $F_{13}(T^4)$, $\ell = (wx)(yz) \in \nu_2$.

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\, rk(F_{13}(a_\ell)) = 2 + 2 \times 2 = 6.$$

The color-coded representation of the nonzero entry of a_ℓ, labeled by $\ell = V_2$ as in Table 6.3, is as follows:

$$
\begin{array}{c|c|cccccccccccccccccccccccccccccccccccccccccccccccccc|c}
 & & A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & Q_0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & R_0 & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & S_0 & S_1 & S_2 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & \\
\nu_1 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots) \\
\nu_3 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots) \\
\nu_5 & (\cdots & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots) \\
\nu_2 & (\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots)
\end{array}
$$

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1, ν_3, ν_5 and ν_2 is linearly independent. Let u be a vector associated to an element of ν_2. Then, for some position $\ell_0 = (w_0x_0)(y_0z_0)$, where $w_0 < x_0 < y_0 < z_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_2 when $v \neq u$, or for any vector v associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1 , ν_3, or ν_5. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_2 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0x_0)(y_0z_0) = (w_1x_1)(y_1z_1)$ for some $w_1 < x_1 < y_1 < z_1$.

So, we must have $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

(c) Let v be a vector associated with ν_1, ν_3 or ν_5. As observed in Remark 6.5.1, $v(\ell_0) = 0$.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1 \cup \nu_3 \cup \nu_5 \cup \nu_2$ are linearly independent.

2. Let $\ell = (wy)(xz) \in \nu_4$. From Case 16 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$. It appears two times in $F_{22}(T^4)$ in the following positions:

$$F_{22}(T^{(4)}) = \begin{array}{c} \\ wy \\ xz \end{array} \begin{array}{c} \begin{array}{cc} wy & xz \end{array} \\ \begin{pmatrix} 0 & T_\ell \\ -T_\ell & 0 \end{pmatrix} \end{array}$$

and two times in $F_{13}(T^4)$ as shown in Figure 6.4. Note that in Figure 6.4 the T's in the variables are suppressed for simplicity, and the indices are labeled as in Table 6.3.

$$\begin{array}{c|cccccccccccccccc}
 & wwx & wwy & xwx & xwy & xxy & xxz & ywx & ywy & yxy & yxz & yyz & zwx & zwy & zxy & zxz & zyz \\
\hline
w & A & H & C & I & G-O & -R_2 & E & L & M-D & -R_5+\ell & -T_0 & R_0 & Q_0 & Q_1-R_5 & -S_0 & -Q_3 \\
x & C & I & B & G & J & P_5 & O & M & N & P_6 & P_8-R_7 & R_2 & Q_1-\ell & Q_2 & R_4 & R_8-Q_4 \\
y & E & L & O & M & N & P_6 & D & K & F & P_8 & P_9 & R_5 & T_0 & R_7 & R_8 & R_9 \\
z & R_0 & Q_0 & R_2 & Q_1 & Q_2 & R_4 & R_5 & T_0 & R_7 & R_8 & R_9 & S_0 & Q_3 & Q_4 & S_2 & Q_5
\end{array}$$

Figure 6.4: Occurrences of T_ℓ in $F_{13}(T^4)$, $\ell = (wy)(xz) \in \nu_4$.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell = (wy)(xz) \\ 0 & e.w. \end{cases}$$

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\, rk(F_{13}(a_\ell)) = 2 + 2 \times 2 = 6.$$

The color-coded representation of the nonzero entry of a_ℓ, labeled by $\ell = V_4$ as in Table 6.3, is as follows:

$$\begin{array}{l*{52}{c}}
 & & A&B&C&D&E&F&G&H&I&J&K&L&M&N&O&P_0&P_1&P_2&P_3&P_4&P_5&P_6&P_7&P_8&P_9&Q_0&Q_1&Q_2&Q_3&Q_4&Q_5&R_0&R_1&R_2&R_3&R_4&R_5&R_6&R_7&R_8&R_9&S_0&S_1&S_2&V_1&V_2&V_3&V_4&V_5&V_6 & \\
\nu_1 & (\cdots & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 & \cdots) \\
\nu_3 & (\cdots & 0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 & \cdots) \\
\nu_5 & (\cdots & 0&0&0&2&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 & \cdots) \\
\nu_2 & (\cdots & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 & \cdots) \\
\nu_4 & (\cdots & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 & \cdots)
\end{array}$$

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1, ν_3, ν_5, ν_2 and ν_4 is linearly independent. Let u be a vector associated to an element of ν_4. Then, for some position $\ell_0 = (w_0y_0)(x_0z_0)$, where $w_0 < x_0 < y_0 < z_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_4 when $v \neq u$, or for any vector v associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1 , ν_3, ν_5 or ν_2. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_4 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0y_0)(x_0z_0) = (w_1y_1)(x_1z_1)$ for some $w_1 < x_1 < y_1 < z_1$. So, we must have $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

(c) Let v be a vector associated with ν_1, ν_3 or ν_5. As observed in Remark 6.5.1, $v(\ell_0) = 0$.

(d) Let v be a vector associated with ν_2. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0y_0)(x_0z_0) = (w_2x_2)(y_2z_2)$ for some $w_2 < x_2 < y_2 < z_2$. So, we must have $y_0 = x_2$ and $x_0 = y_2$. Since $x_2 < y_2$, we get $y_0 < x_0$ which is a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1 \cup \nu_3 \cup \nu_5 \cup \nu_2 \cup \nu_4$ are linearly independent.

3. Let $\ell = (wz)(xy) \in \nu_6$. From Case 18 in Section 5.4, T_ℓ appears 6 times in $F(T^{(4)})$. It appears two times in $F_{22}(T^4)$ in the following positions:

$$F_{22}(T^{(4)}) = \begin{array}{c c} & \begin{array}{cc} wz & xy \end{array} \\ \begin{array}{c} wz \\ xy \end{array} & \begin{pmatrix} 0 & T_\ell \\ -T_\ell & 0 \end{pmatrix} \end{array}$$

and two times in $F_{13}(T^4)$ as shown in Figure 6.5. Note that in Figure 6.5 the T's in the variables are suppressed for simplicity, and the indices are labeled as in Table 6.3.

$$\begin{array}{c c} & \begin{array}{cccccccccccccccc} wwy & wwz & xwy & xwz & xxy & xxz & ywy & ywz & yxy & yxz & yyz & zwy & zwz & zxy & zxz & zyz \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} H & P_0 & I & P_1 & G & P_3 & L & P_2 & M & P_4 & P_7-T_0 & Q_0 & R_1 & Q_1+\ell & R_3 & R_6-Q_3 \\ I & P_1 & G & P_3 & J & P_5 & M & P_4-\ell & N & P_6 & P_8-R_7 & Q_1 & R_3 & Q_2 & R_4 & R_8-Q_4 \\ L & P_2 & M & P_4 & N & P_6 & K & P_7 & F & P_8 & P_9 & T_0 & R_6 & R_7 & R_8 & R_9 \\ Q_0 & R_1 & Q_1 & R_3 & Q_2 & R_4 & T_0 & R_6 & R_7 & R_8 & R_9 & Q_3 & S_1 & Q_4 & S_2 & Q_5 \end{pmatrix} \end{array}$$

Figure 6.5: Occurrences of T_ℓ in $F_{13}(T^4)$, $\ell = (wz)(xy) \in \nu_6$.

If we choose the vector $a_\ell = (a_\ell(i) : i \in R_4)$, such that

$$a_\ell(i) = \begin{cases} 1 & i = \ell = (wz)(xy) \\ 0 & \text{elsewhere.} \end{cases}$$

we get

$$rk(F(a_\ell)) = rk(F_{22}(a_\ell)) + 2\, rk(F_{13}(a_\ell)) = 2 + 2 \times 2 = 6.$$

The color-coded representation of the nonzero entry of a_ℓ, labeled by $\ell = V_6$ as in Table 6.3, is as follows:

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	S_0	S_1	S_2	V_1	V_2	V_3	V_4	V_5	V_6	
ν_1	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	$\cdots)$
ν_3	$(\cdots$	0	0	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	$\cdots)$
ν_5	$(\cdots$	0	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$\cdots)$
ν_2	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$\cdots)$
ν_4	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	$\cdots)$
ν_6	$(\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	$\cdots)$

Next we prove that the current set of vectors associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1, ν_3, ν_5, ν_2, ν_4 and ν_6 is linearly independent. Let u be a vector associated to an element of ν_6. Then, for some position $\ell_0 = (w_0 z_0)(x_0 y_0)$, where $w_0 < x_0 < y_0 < z_0$, we have

$$u(i) = \begin{cases} 1 & i = \ell_0 \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $v(\ell_0) = 0$ for any vector v associated with ν_6 when $v \neq u$, or for any vector v associated to δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7, α_7, ν_1 , ν_3, ν_5 ,ν_2 or ν_4. This will be demonstrated as follows:

(a) Let v be a vector associated with ν_6 such that $v \neq u$. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0 z_0)(x_0 y_0) = (w_1 z_1)(x_1 y_1)$ for some $w_1 < x_1 < y_1 < z_1$. So, we must have $u = v$, which is a contradiction.

(b) Let v be a vector associated with δ_4, γ_2, γ_4, α_6, β_4, γ_3, γ_5, β_6, δ_5, β_5, β_7 or α_7. As observed in Remark 6.3.2, $v(\ell_0) = 0$.

(c) Let v be a vector associated with ν_1, ν_3 or ν_5. As observed in Remark 6.5.1, $v(\ell_0) = 0$.

(d) Let v be a vector associated with ν_2. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0z_0)(x_0y_0) = (w_2x_2)(y_2z_2)$ for some $w_2 < x_2 < y_2 < z_2$. So, we must have $z_0 = x_2$ and $x_0 = y_2$. Since $x_2 < y_2$, we get $z_0 < x_0$ which is a contradiction.

(e) Let v be a vector associated with ν_4. We claim that $v(\ell_0) = 0$. Otherwise, $\ell_0 = (w_0z_0)(x_0y_0) = (w_3y_3)(x_3z_3)$ for some $w_3 < x_3 < y_3 < z_3$. So, we must have $z_0 = y_3$ and $y_0 = z_3$. Since $y_3 < z_3$, we get $z_0 < y_0$ which is a contradiction.

By Lemma 6.1.4, the vectors associated with $\delta_4 \cup \gamma_2 \cup \gamma_4 \cup \alpha_6 \cup \beta_4 \cup \gamma_3 \cup \gamma_5 \cup \beta_6 \cup \delta_5 \cup \beta_5 \cup \beta_7 \cup \alpha_7 \cup \nu_1 \cup \nu_3 \cup \nu_5 \cup \nu_2 \cup \nu_4 \cup \nu_6$ are linearly independent.

Before proving Propositions 6.6.1, 6.6.2 and 6.6.3 in the following subsections, let us note the following simple facts.

Lemma 6.6.4. *The determinant of any 2×2 submatrix of a rank one matrix is zero.*

Proof: Since the submatrix has rank at most one, it is not invertible. ∎

Lemma 6.6.5. *If a rank one matrix has a zero entry, then either the row or the column containing it has all zero entries.*

Proof: Suppose that some of the entries of the row and the column containing the zero entry are nonzero. Then, there exists a 2×2 upper (or lower) triangular submatrix with nonzero diagonal (or anti-diagonal), contradicting Lemma 6.6.4. ∎

6.6.1 Proof of Proposition 6.6.1

Suppose that $rk(F_{13}(a)) = 1$. For simplicity let ℓ denote $a(\ell)$. So, $\ell \neq 0$. Consider the following submatrices from Figure 6.3:

$$\begin{array}{c|cc} & wwx & ywx \\ \hline y & E & D \\ z & R_0 & R_5 \end{array}\;\text{i.e.}\; \begin{array}{c} \\ y \\ z \end{array}\begin{array}{c} \begin{matrix} wwx & ywx \end{matrix} \\ \begin{pmatrix} E & D \\ R_0 & R_5 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ w \\ y \end{array}\begin{array}{c} \begin{matrix} ywx & zwx \end{matrix} \\ \begin{pmatrix} E & R_0 \\ D & R_5-\ell \end{pmatrix} \end{array}.$$

From Lemma 6.6.4, we get:

$$ER_5 = DR_0.$$
$$E(R_5-\ell) = DR_0$$

As a result,

$$ER_5 = E(R_5-\ell).$$

Note that we must have $E=0$. Otherwise, $R_5-\ell = R_5$, which implies that $\ell = 0$. Since we are assuming that $\ell \neq 0$, we get a contradiction. Since $E = 0$ and appears two times in $F_{13}(a_\ell)$, from Lemma 6.6.5 we have the following possibilities:

i. If the first row of Figure 6.3 is zero, then $R_5 = 0$, $P_7 = 0$ and $R_0 = 0$. So, $R_5 - \ell = -\ell \neq 0$. From submatrix

$$\begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{matrix} ywz & zwx \end{matrix} \\ \begin{pmatrix} P_4 & R_2 \\ P_7 & R_5-\ell \end{pmatrix} \end{array},$$

we get $P_4(R_5-\ell) = R_2P_7$. Since $P_7 = 0$ and $R_5 - \ell = -\ell \neq 0$, we get that $P_4 = 0$. Also, from submatrix

$$\begin{array}{c} \\ w \\ y \end{array}\begin{array}{c} \begin{matrix} yxz & zwx \end{matrix} \\ \begin{pmatrix} P_4 - R_5 + \ell & R_0 \\ P_8 & R_5-\ell \end{pmatrix} \end{array},$$

we get $(P_4 - R_5 + \ell)(R_5 - \ell) = R_0 P_8$. Since $R_0 = 0$, then

$$(P_4 - R_5 + \ell)(R_5 - \ell) = (0 - 0 + \ell)(0 - \ell) = -\ell^2 = 0,$$

which implies that $\ell = 0$. Since $\ell \neq 0$, we get a contradiction.

ii. If the third row Figure 6.3 is zero, then $R_5 - \ell = 0$. So, $R_5 = \ell \neq 0$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} ywx & zxy \end{array} \\ \begin{array}{c} w \\ z \end{array} & \begin{pmatrix} E & -R_5 \\ R_5 & Q_4 \end{pmatrix} \end{array},$$

we get $EQ_4 = -R_5^2$. Since $E = 0$, we get $R_5 = 0$, which is a contradiction.

iii. If both columns containing E in Figure 6.3 are zero, then $R_5 = 0$ and $R_0 = 0$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} yyz & zwx \end{array} \\ \begin{array}{c} w \\ y \end{array} & \begin{pmatrix} P_7 & R_0 \\ P_9 & R_5 - \ell \end{pmatrix} \end{array},$$

we get $P_7(R_5 - \ell) = R_0 P_9$. Since $R_0 = 0$ and $R_5 - \ell = 0 - \ell = -\ell \neq 0$, then $P_7 = 0$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} ywz & zwx \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} P_4 & R_2 \\ P_7 & R_5 - \ell \end{pmatrix} \end{array},$$

we get $P_4(R_5 - \ell) = R_2 P_7$. Since $P_7 = 0$ and $R_5 - \ell = 0 - \ell = -\ell \neq 0$, then $P_4 = 0$. From submatrix

$$\begin{array}{c|cc} & yxz & zwx \\ \hline w & P_4 - R_5 + \ell & R_0 \\ y & P_8 & R_5 - \ell \end{array},$$

we get $(P_4 - R_5 + \ell)(R_5 - \ell) = R_0 P_8$. Since $R_0 = 0$, then

$$(P_4 - R_5 + \ell)(R_5 - \ell) = (0 - 0 + \ell)(0 - \ell) = -\ell^2 = 0,$$

which implies that $\ell = 0$. Since $\ell \neq 0$, we get a contradiction.

As a result, $rk(F_{13}(a)) \neq 1$ when $a(\ell) \neq 0$.

6.6.2 Proof of Proposition 6.6.2

Suppose that $rk(F_{13}(a)) = 1$. For simplicity let ℓ denote $a(\ell)$. So, $\ell \neq 0$. Consider the following submatrices from Figure 6.4:

$$\begin{array}{c|cc} & wwy & zwy \\ \hline x & I & Q_1 - \ell \\ z & Q_0 & Q_3 \end{array}, \quad \begin{array}{c|cc} & xwy & zwy \\ \hline w & I & Q_0 \\ z & Q_1 & Q_3 \end{array}.$$

From Lemma 6.6.4, we get:

$$IQ_3 = (Q_1 - \ell)Q_0.$$
$$IQ_3 = Q_1 Q_0$$

As a result

$$(Q_1 - \ell)Q_0 = Q_1 Q_0.$$

Note that we must have $Q_0 = 0$. Otherwise, $Q_1 - \ell = Q_1$, which implies that $\ell = 0$. Since we are assuming that $\ell \neq 0$, we get a contradiction. Since $Q_0 = 0$ and appears two times in $F_{13}(T^4)$, we have the following possibilities from Lemma 6.6.5:

i. If the fourth row in Figure 6.4 is zero, then $Q_1 = 0$. Since Q_0 appears in the first row also, then the first row must also have zero entries, as the column of Q_0 contains the entry $Q_1 - \ell = \ell \neq 0$. As a result, $-R_5 + \ell = 0$ and $Q_1 - R_5 = 0$, since they are entries of the first row. So, $\ell = R_5 = Q_1 = 0$, which is a contradiction.

ii. If the first row in Figure 6.4 is zero, then $-R_5 + \ell = 0$ and $S_0 = 0$. So, $R_5 = \ell \neq 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} ywx & zwx \end{array} \\ \begin{array}{c} y \\ z \end{array} & \begin{pmatrix} D & R_5 \\ R_5 & S_0 \end{pmatrix}, \end{array}$$

we get $DS_0 = R_5^2$. Since $S_0 = 0$, we get $R_5 = \ell = 0$, which is a contradiction.

iii. If both columns containing Q_0 in Figure 6.4 are zero, then $I = 0$ and $Q_1 - \ell = 0$. As a result, $Q_1 = \ell \neq 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} xwy & yxz \end{array} \\ \begin{array}{c} w \\ z \end{array} & \begin{pmatrix} I & -R_5 + \ell \\ Q_1 & R_8 \end{pmatrix}, \end{array}$$

we get $IR_8 = (-R_5 + \ell)Q_1$. Since $I = 0$ and $Q_1 \neq 0$, we get $-R_5 + \ell = 0$. So, $R_5 = \ell \neq 0$. Since T_0 is an entry of a the second column containing Q_0, then $T_0 = 0$. Since T_0 appears also in the last row, then the column containing it must be zero, since the row contains the nonzero entry Q_1. Since M is an entry in this column we must have $M = 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} xwy & zwx \end{array} \\ \begin{array}{c} y \\ z \end{array} & \begin{pmatrix} M & R_5 \\ Q_1 & S_0 \end{pmatrix}, \end{array}$$

we get $MS_0 = R_5Q_1$. Since $M = 0$ and $Q_1 \neq 0$, we get $R_5 = \ell = 0$, which is a contradiction.

Thus, $rk(F_{13}(a)) \neq 1$.

6.6.3 Proof of Proposition 6.6.3

Suppose that $rk(F_{13}(a)) = 1$. For simplicity let ℓ denote $a(\ell)$. So, $\ell \neq 0$. Consider the following submatrices from Figure 6.5:

$$\begin{array}{cc} & \begin{array}{cc} yxz & zxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} P_4 & Q_1+\ell \\ P_6 & Q_2 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} xwy & yxz \end{array} \\ \begin{array}{c} x \\ z \end{array} & \begin{pmatrix} G & P_6 \\ Q_1 & R_8 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} xxy & yxz \end{array} \\ \begin{array}{c} w \\ z \end{array} & \begin{pmatrix} G & P_4 \\ Q_2 & R_8 \end{pmatrix} \end{array}.$$

From Lemma 6.6.4, we get:

$$\begin{aligned} P_4Q_2 &= P_6(Q_1+\ell), \\ GR_8 &= P_6Q_1, \\ GR_8 &= P_4Q_2. \end{aligned}$$

As a result

$$P_6(Q_1+\ell) = P_4Q_2 = GR_8 = P_6Q_1.$$

Note that we must have $P_6 = 0$. Otherwise, $Q_1 + \ell = Q_1$, which implies that $\ell = 0$. Since we are assuming that $\ell \neq 0$, we get a contradiction. Since $P_6 = 0$ and appears two times in $F_{13}(T^4)$, we have the following possibilities from Lemma 6.6.5:

i. If the third row in Figure 6.5 is zero, then $P_4 = 0$, $R_6 = 0$ and $P_2 = 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} xwy & ywz \end{array} \\ \begin{array}{c} x \\ z \end{array} & \begin{pmatrix} G & P_4-\ell \\ Q_1 & R_6 \end{pmatrix} \end{array}$$

we get $GR_6 = Q_1(P_4-\ell)$. Since $R_6 = 0$ and $P_4 - \ell = -\ell \neq 0$, then $Q_1 = 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} ywz & zxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} P_2 & Q_1+\ell \\ P_4-\ell & Q_2 \end{pmatrix} \end{array}$$

we get $P_2Q_2 = (P_4-\ell)(Q_1+\ell)$. Since $P_2 = 0$, then $(P_4-\ell)(Q_1+\ell) = (0-\ell)(0+\ell) = -\ell^2 = 0$, which is a contradiction since $\ell \neq 0$.

ii. If the second row Figure 6.5 is zero, then $P_1 = 0$ and $P_4 - \ell = 0$. So, $P_4 = \ell \neq 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} xwz & yxz \end{array} \\ \begin{array}{c} w \\ y \end{array} & \begin{pmatrix} P_1 & P_4 \\ P_4 & P_8 \end{pmatrix} \end{array}$$

we get $P_1P_8 = P_4^2$. Since $P_1 = 0$, then $P_4 = \ell = 0$, which is a contradiction.

iii. If both columns containing P_6 Figure 6.5 are zero, then $R_4 = 0$ and $P_4 = 0$. From submatrix

$$\begin{array}{c c} & \begin{array}{cc} ywz & zxz \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} P_2 & R_3 \\ P_4-\ell & R_4 \end{pmatrix} \end{array}$$

we get $P_2R_4 = R_3(P_4 - \ell)$. Since $R_4 = 0$ and $P_4 - \ell = 0 - \ell = -\ell \neq 0$, then $R_3 = 0$. From Lemma 6.6.5, we see that either the second row or the column of R_3 is zero. Since $P_4 - \ell = -\ell \neq 0$, then the the column containing R_3 is zero. Since R_6 is in this column, we have $R_6 = 0$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} xxy & ywz \end{array} \\ \begin{array}{c} x \\ z \end{array} & \begin{pmatrix} J & P_4 - \ell \\ Q_2 & R_6 \end{pmatrix} \end{array}$$

we get $JR_6 = Q_2(P_4 - \ell)$. Since $R_6 = 0$ and $P_4 - \ell = -\ell \neq 0$, then $Q_2 = 0$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} zwy & zxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} Q_0 & Q_1 + \ell \\ Q_1 & Q_2 \end{pmatrix} \end{array}$$

we get $Q_0Q_2 = Q_1(Q_1 + \ell)$. Since $Q_2 = 0$, then either $Q_1 = 0$ or $Q_1 + \ell = 0$. We consider each case as follows:

- If $Q_1 = 0$, then from submatrix

$$\begin{array}{cc} & \begin{array}{cc} ywz & zxy \end{array} \\ \begin{array}{c} w \\ x \end{array} & \begin{pmatrix} P_2 & Q_1 + \ell \\ P_4 - \ell & Q_2 \end{pmatrix} \end{array}$$

  we get $P_2Q_2 = (Q_1 + \ell)(P_4 - \ell)$. Since $Q_2 = 0$, $Q_1 = 0$ and $P_4 = 0$, we get $(Q_1 + \ell)(P_4 - \ell) = -\ell^2 = 0$. So, $\ell = 0$, which is a contradiction.
- If $Q_1 + \ell = 0$, then $Q_1 = -\ell$. From submatrix

$$\begin{array}{cc} & \begin{array}{cc} xwy & ywz \end{array} \\ \begin{array}{c} x \\ z \end{array} & \begin{pmatrix} G & P_4 - \ell \\ Q_1 & R_6 \end{pmatrix} \end{array}$$

we get $GR_6 = Q_1(P_4 - \ell)$. Since $R_6 = 0$, $P_4 = 0$ and $Q_1 = -\ell$, we get $Q_1(P_4 - \ell) = \ell^2 = 0$. So, $\ell = 0$, which is a contradiction

Thus, $rk(F_{13}(a)) \neq 1$.

6.7 The Faithful Dimension of $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)$

We have successfully found $r_n(4) = |R_4|$ linearly independent vectors that yield minimal rank. The minimal ranks for each type are summarized in Table 6.4 and Table 6.5. Now, we can compute the faithful dimension of $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ using Proposition 4.3.4.

Type in R_4	$w < x < y < z$	$rk(F)$
α_6	$yywx$	2
α_7	$zywx$	2
β_4	$xxwy$	2
β_5	$yxwz$	2
β_6	$yxwy$	2
β_7	$zxwy$	2
γ_2	$wwwx$	2
γ_3	$xwwy$	2
γ_4	$xwwx$	2
γ_5	$ywwx$	2
δ_4	$xxwx$	2
δ_5	$yxwx$	2

Table 6.4: Rank summary for left-aligned equivalence classes of R_4.

Type in R_4	$w < x < y < z$	$rk(F)$
ν_1	$(wx)(wy)$	4
ν_3	$(wx)(xy)$	4
ν_5	$(wy)(xy)$	4
ν_2	$(wx)(yz)$	6
ν_4	$(wy)(xz)$	6
ν_6	$(wz)(xy)$	6

Table 6.5: Rank summary for balanced equivalence classes of R_4.

Theorem 6.7.1. *For $n \geq 2$ and p sufficiently large,*

$$m_{faithful}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = \frac{1}{8} f \left((n^4 + 2n^3 - n^2 - 2n)q + 4(n^3 - 3n^2 + 2n)q^2 \right.$$
$$\left. + (n^4 - 6n^3 + 11n^2 - 6n)q^3 \right)$$

Proof: Let $\{a_\ell : \ell \in R_4\}$ be the set of $r_n(4)$ linearly independent vectors associated with the elements of R_4 obatined in the previous sections. From Proposition 4.3.4,

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = \sum_{\ell \in R_4} f q^{\frac{1}{2}\text{rk}_{\mathbb{F}_q}(F(a_\ell))}.$$

We can calculate this sum using the results from Table 6.4 and Table 6.5, which can be summarized in terms of ranks and number of letters as follows:

$rk(F)$	*Two-Letter Types*	*Three-Letter Types*	*Four-Letter Types*
2	$\gamma_2, \gamma_4, \delta_4$	$\alpha_6, \beta_4, \beta_6, \gamma_3, \gamma_5, \delta_5$	$\alpha_7, \beta_5, \beta_7$
4		ν_1, ν_3, ν_5	
6			ν_2, ν_4, ν_6

Table 6.6: Rank summary for R_4.

Now, we need to find the number of elements in each type. The size of each type depends on the number of letters in its representative. Let

$$X_2 := \binom{n}{2} = \frac{n(n-1)}{2}$$

$$X_3 := \binom{n}{3} = \frac{n(n-1)(n-2)}{3}$$

$$X_4 := \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

Thus, for representatives with 2, 3, or 4 letters, the number of words in each corresponding type is X_2, X_3, and X_4, respectively. So, from Table 6.6, we see that there are $3X_2 + 6X_3 + 3X_4$ vectors with rank 2, $3X_3$ vectors of rank 4, and $3X_4$ vectors of rank 6. As a result,

$$\begin{aligned} m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)) &= \sum_{\ell \in R_4} f q^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}(F(a_\ell))} \\ &= (3X_4 + 6X_3 + 3X_2)\, fq + 3X_3 fq^2 + 3X_4 fq^3 \\ &= \frac{1}{8}n(n-1)(n+1)(n+2)fq + \frac{1}{2}n(n-1)(n-2)fq^2 \\ &\quad + \frac{1}{8}n(n-1)(n-2)(n-3)fq^3 \\ &= \frac{1}{8}f\left((n^4 + 2n^3 - n^2 - 2n)q + 4(n^3 - 3n^2 + 2n)q^2 \right. \\ &\quad \left. +(n^4 - 6n^3 + 11n^2 - 6n)q^3\right), \end{aligned}$$

which completes the proof of the theorem. ∎

6.8 The Faithful Dimension of $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)$

In this section we are interested in p-groups of the form $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)$, where R is a finite truncated valuation ring as in Definition 3.2.6, with the associated parameters (d, p, e, f) given in Definition 3.2.8.

Proposition 6.8.1. [McL73] *$R \cong \mathcal{O}/\mathfrak{p}^d$, where $\mathcal{O}$ is the valuation ring of a local field and $\mathfrak{p}$ is the prime ideal of $\mathcal{O}$.*

So, we can assume that $R = \mathcal{O}/\mathfrak{p}^d$. In the notation of Definition 3.2.6, $\mathfrak{m} = \mathfrak{p}/\mathfrak{p}^d$, and the associated parameters (d, p, e, f) are given as follows:

- $p = \mathrm{char}(\mathcal{O}/\mathfrak{p})$, and $\#\mathcal{O}/\mathfrak{p} = p^f$ for some $f \geq 1$.

- $p\mathcal{O}/\mathfrak{p}^d = \mathfrak{p}^e/\mathfrak{p}^d$ for some $e \geq 1$.

The rank minimization problem in Theorem 4.3.1 was extended to finite truncated valuation rings in [BMKRS23, Prop. 3.2]. Let $\mathfrak{g}$ be a nilpotent $\mathbb{Z}$-Lie algebra which is finitely generated as an abelian group, with m, ℓ_1, ℓ_2 as in Theorem 4.3.1.

Define the map

$$\begin{aligned} \mathrm{proj}: \qquad \left(\mathcal{O}/\mathfrak{p}^d\right)^{\oplus m} &\to \left(\mathcal{O}/\mathfrak{p}^e\right)^{\oplus \ell_1} \\ (a(1), \ldots, a(m)) &\mapsto (\overline{a(1)}, \ldots, \overline{a(\ell_1)}), \end{aligned}$$

where $a \mapsto \overline{a}$ is the natural projection $\mathcal{O}/\mathfrak{p}^d \to \mathcal{O}/\mathfrak{p}^e$.

Let $\mathcal{O}(d, m, e, \ell_1)$ denote the set of $fe\ell_1$-tuples $(a_1, \ldots, a_{fe\ell_1})$ such that $a_i \in (\mathcal{O}/\mathfrak{p}^d)^{\oplus m}$ for all $1 \leq i \leq fe\ell_1$, and the projections $\{\mathrm{proj}(a_i)\}_{i=1}^{fe\ell_1}$ form a basis of the $\mathbb{F}_p$-vector space $(\mathcal{O}/\mathfrak{p}^e)^{\oplus \ell_1}$.

Proposition 6.8.2. [BMKRS23, Prop. 3.2] *Then there exists a constant $M > 0$*

such that for $p > M$ the faithful dimension of $\exp(\mathfrak{g} \otimes_{\mathbb{Z}} R)$ *is given by*

$$\min \left\{ \sum_{i=1}^{fe\ell_1} \left(\frac{p^{fd(rk(\mathfrak{g})-\ell_1-\ell_2)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F_{\mathfrak{g}}(a_i))} \right)^{\frac{1}{2}} : (a_1, \ldots, a_{fe\ell_1}) \in \mathcal{O}(d, m, e, \ell_1) \right\} + f\ell_2 e,$$

where $F_{\mathfrak{g}}$ is the commutator matrix of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{f}_{n,c}$. As in Proposition 4.3.4, using the reduced commutator matrix, defined in Definition 4.2.7, the minimization problem in Proposition 6.8.2 can be reduced to the following:

Proposition 6.8.3. *Let $F = F^{red}_{\mathfrak{f}_{n,c}}$ be the reduced commutator matrix of $\mathfrak{f}_{n,c}$. Then, the faithful dimension of* $\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} R)$ *is given by*

$$\min \left\{ \sum_{i=1}^{fe\ell_1} \left(\frac{p^{fd(rk(\mathfrak{f}_{n,c})-\ell_1)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F(a_i))} \right)^{\frac{1}{2}} : (a_1, \ldots, a_{fe\ell_1}) \in \mathcal{O}(d, \ell_1, e, \ell_1) \right\},$$

where $rk(\mathfrak{f}_{n,c}) = \sum_{k=1}^{c} r_n(k)$ and $\ell_1 = r_n(c)$.

Proof: The proof follows that of Proposition 4.3.4, with the following observation in mind. Recall that from Proposition 4.2.6, we have $\ell_1 = r_n(c)$, $\ell_2 = 0$ and $m = \sum_{k=2}^{c} r_n(k)$. Thus, for any $a_i \in (\mathcal{O}/\mathfrak{p}^d)^{\oplus m}$ where $1 \leq i \leq fe\ell_1$, we can express the entries of a_i as follows:

$$a_i = (a_i^{(k)})_{2 \leq k \leq c} = (a_{i1}^{(2)}, \ldots, a_{ir_n(2)}^{(2)}, a_{i1}^{(3)}, \ldots, a_{ir_n(3)}^{(3)}, \ldots, a_{i1}^{(c)}, \ldots, a_{ir_n(c)}^{(c)}).$$

As in Equation (4.3.2), we have

$$\mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{f}_{n,c}}(a_i)) \geq \mathrm{rk}_{\mathbb{F}_q}(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a_i)).$$

where $\mathrm{rk}_{\mathbb{F}_q}(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a_i)) = \sum_{i+j=c} \mathrm{rk}_{\mathbb{F}_q}(F_{ij}(a_i^{(c)}))$. From the rank-nullity theorem, if we set $C = \sum_{k=1}^{c-1} r_n(k)$, then

$$C - \mathrm{rk}_{\mathbb{F}_q}(F_{\mathfrak{f}_{n,c}}(a_i)) \leq C - \mathrm{rk}_{\mathbb{F}_q}(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a_i)),$$

i.e.

$$\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F_{\mathfrak{f}_{n,c}}(a_i)) \leq \# \ker_{\mathcal{O}/\mathfrak{p}^d}(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a_i)).$$

As a result,

$$\left(\frac{p^{fd(\mathrm{rk}(\mathfrak{g})-\ell_1)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F_{\mathfrak{f}_{n,c}}(a_i))} \right)^{\frac{1}{2}} \geq \left(\frac{p^{fd(\mathrm{rk}(\mathfrak{g})-\ell_1)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F^{\mathrm{red}}_{\mathfrak{f}_{n,c}}(a_i))} \right)^{\frac{1}{2}}.$$

The rest of the argument is similar to that of Proposition 4.3.4 ∎

As mentioned earlier, we are mainly interested in p-groups of the form $\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)$, where $R = \mathcal{O}/\mathfrak{p}^d$. Suppose that the vectors $a_i \in (\mathcal{O}/\mathfrak{p}^d)^{\oplus r_n(4)}$, where $1 \leq i \leq fe\, r_n(4)$, are the optimal solution to the minimization problem in Proposition 6.8.3. So,

$$m_{\mathrm{faithful}}(\exp(\mathfrak{f}_{n,c} \otimes_{\mathbb{Z}} R)) = \sum_{i=1}^{fe\, r_n(4)} \left(\frac{p^{fd(\mathrm{rk}(\mathfrak{f}_{n,c})-\ell_1)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F(a_i))} \right)^{\frac{1}{2}}, \tag{6.8.1}$$

and the projections $\{\mathrm{proj}(a_i)\}_{i=1}^{fe\ell_1}$ form a basis of the $\mathbb{F}_p$-vector space $(\mathcal{O}/\mathfrak{p}^e)^{\oplus r_n(4)}$.

Lemma 6.8.4. [BMKRS23, Lemma 4.5] *Let V be a finite dimensional vector space over an arbitrary field and let S be a basis for $V^{\oplus k}$ where $k \geq 1$. Then S can be partitioned into subsets $S_1, \dots, S_k$ such that each of the sets $\{\pi_\ell(w) : w \in S_\ell\}$ is a basis of V for $1 \leq \ell \leq k$, where $\pi_\ell(w(1), \dots, w(k)) := w(\ell)$.*

As a consequence of Lemma 6.8.4, the set of vectors

$$\mathcal{S} := \{a_i \in (\mathcal{O}/\mathfrak{p}^d)^{\oplus r_n(4)} : 1 \leq i \leq ef\, r_n(4)\}$$

can be partitioned into $r_n(4)$ subsets S_ℓ, where $\ell \in R_4$, such that $|S_\ell| = ef$ and the set

$$\{\pi_\ell(\mathrm{proj}(a)) = \overline{a(\ell)} : a \in S_\ell\}, \tag{6.8.2}$$

forms an $\mathbb{F}_p$-basis for $\mathcal{O}/\mathfrak{p}^e$. Furthermore, for each $\ell \in R_4$, the projection of the elements of the set in Equation (6.8.2) to $\mathcal{O}/\mathfrak{p}$ forms a spanning set since $\mathcal{O}/\mathfrak{p}$ is a quotient of $\mathcal{O}/\mathfrak{p}^e$. Hence, there exits a subset $T_\ell \subset S_\ell$ for each $\ell \in R_4$ such that the projections of $\cup_{\ell\in R_4} T_\ell$ forms an $\mathbb{F}_p$-basis for $(\mathcal{O}/\mathfrak{p})^{\oplus r_n(4)}$.

Define the following subsets of $\mathcal{S}$:

$$\mathcal{T} := \cup_{\ell\in R_4} T_\ell, \qquad \mathcal{S}_{\mathrm{I}} := \cup_{\ell\in R_{4,\mathrm{I}}} S_\ell - T_\ell, \qquad \mathcal{S}_{\mathrm{II}} := \cup_{\ell\in R_{4,\mathrm{II}}} S_\ell - T_\ell,$$

where

$$R_{4,\mathrm{I}} = \{\ell \in R_4 : \ell \text{ left-aligned}\},$$
$$R_{4,\mathrm{II}} = \{\ell \in R_4 : \ell \text{ balanced 3-letter}\}.$$

In the the following we use basics of valuations of local fields, for more details we refer the reader to [Ser79].

Remark 6.8.5. Note that if $a \in S_\ell - T_\ell \subset \mathcal{S}_{\mathrm{I}} \cup \mathcal{S}_{\mathrm{II}}$, then $\nu(a(\ell)) \geq 1$ since we are excluding the vectors of T_ℓ which forms a basis for the units $\mathcal{O}/\mathfrak{p}$.

Proposition 6.8.6. *Let*

$$K(a) = \frac{p^{fd(rk(\mathfrak{f}_{n,c})-r_n(4))}}{\#\ker_{\mathcal{O}/\mathfrak{p}^d}(F(a))},$$

for $a \in (\mathcal{O}/\mathfrak{p}^d)^{\oplus r_n(4)}$. Then,

i. For $a \in S_\ell - T_\ell \subset \mathcal{S}_I$, if $1 \leq \nu(a(\ell)) \leq e-1$, then $K(a) \geq p^{2f(d-\nu(a(\ell)))}$.

ii. For $a \in S_\ell - T_\ell \subset \mathcal{S}_{II}$, if $1 \leq \nu(a(\ell)) \leq e-1$, then $K(a) \geq p^{4f(d-\nu(a(\ell)))}$.

Proof: To obtain a lower bound for $K(a)$, we aim to find an upper bound for the size of the kernel of the reduced commutator matrix $F_{\mathfrak{f}_{n,c}}(a)$, given by

$$\ker_{\mathcal{O}/\mathfrak{p}^d}(F(a)) = \{x \in \left(\mathcal{O}/\mathfrak{p}^d\right)^{\oplus(\mathrm{rk}(\mathfrak{f}_{n,c})-r_n(4))} : F(a)x^T = 0\}.$$

Let $x = (x(i) : 1 \leq i \leq \mathrm{rk}(\mathfrak{f}_{n,c}) - r_n(4))$. We prove the required upper bound for each case as follows:

i. Let $a \in S_\ell - T_\ell \subset \mathcal{S}_{\mathrm{I}}$. Since $1 \leq \nu(a(\ell)) \leq e-1 \leq d-1$, we know that $a(\ell) \neq 0$ in $\mathcal{O}/\mathfrak{p}^d$. As in the proof of Lemma 6.1.5, there exists an entry of $F_{13}(a)$ consisting solely of $a(\ell)$. Then, from the vector equation that defines the kernel we have

$$a(\ell)x(i_1) = \sum_{\substack{i\in R_3\\ i\neq i_1}} -s_i x(i) \tag{6.8.3}$$

$$-a(\ell)x(i_2) = \sum_{\substack{i\in R_1\\ i\neq i_2}} -t_i x(i), \tag{6.8.4}$$

where $s_i, t_i \in \mathcal{O}/\mathfrak{p}^d$. Note that the first equation is obtained from the submatrix $F_{13}(a)$, whereas the second equation is obtained from the submatrix $-F_{13}^{\mathrm{tr}}(a)$. Let $f : \mathcal{O}/\mathfrak{p}^d \to \mathcal{O}/\mathfrak{p}^d$, such that $x \mapsto a(\ell)x$. We claim that $\#\ker f \leq p^{f\nu(a(\ell))}$. To see this, let $x \in \ker f$, then

$$\nu(a(\ell)x) = \nu(a(\ell)) + \nu(x) \geq d.$$

So, we must have $\nu(x) \geq d-\nu(a(\ell))$, i.e. $x \in \mathfrak{p}^{d-\nu(a(\ell))}/\mathfrak{p}^d$. Since $\#\mathfrak{p}^{d-\nu(a(\ell))}/\mathfrak{p}^d = p^{f\nu(a(\ell))}$, we proved our claim. Note that for any choice for the variables $x(i)$, where $i \neq R_3$ and $i \neq i_1$, in the right hand side of Equation (6.8.3), there are at most $p^{f\nu(a(\ell))}$ choices for $x(i_1)$ such that Equation (6.8.3) holds. Similarly, since $\nu(-a(\ell)) = \nu(a(\ell))$, we can show that for any choice of the variables $x(i)$, where $i \neq R_1$ and $i \neq i_2$, in the right hand side of Equation (6.8.4), there are at most $p^{f\nu(a(\ell))}$ choices for $x(i_2)$ such that Equation (6.8.4) is satisfied. As a result, we obtain the following upper bound

$$\#\ker_{\mathcal{O}/\mathfrak{p}^d}(F(a)) \leq p^{fd(\mathrm{rk}(\mathfrak{f}_{n,c})-r_n(4)-2)}p^{2f\nu(a(\ell))}.$$

Consequently, $K(a) \geq p^{2f(d-\nu(a_\ell))}$.

ii. Let $a \in S_\ell - T_\ell \subset \mathcal{S}_{\mathrm{II}}$. Since $1 \leq \nu(a(\ell)) \leq e-1 \leq d-1$, we know that $a(\ell) \neq 0$ in $\mathcal{O}/\mathfrak{p}^d$. From Table 5.2, we see that T_ℓ appears twice in both $F_{13}(T^{(4)})$ and $F_{22}(T^{(4)})$. In particular, $a(\ell)$ appears twice in the submatrix $F_{22}(a)$ as entries of the form $a(\ell)$ and $-a(\ell)$, and it appears in the submatrix $F_{13}(a)$ in an entry of the form $r + a(\ell)$, where $r \in \mathcal{O}/\mathfrak{p}^d$ and there exits an entry of $F_{13}(a)$ that consists solely of r. If $r \neq -a(\ell)$, then $r + a(\ell) \neq 0$ and we consider the following 4 equations that define the kernel:

$$\begin{aligned}
(r + a(\ell))\, x(i_1) &= \sum_{\substack{i \in R_3 \\ i \neq i_1}} -b_i x(i), \\
a(\ell)x(i_2) &= \sum_{\substack{i \in R_2 \\ i \neq i_2, i_3}} -s_i x(i), \\
-a(\ell)x(i_3) &= \sum_{\substack{i \in R_2 \\ i \neq i_2, i_3}} -t_i x(i), \\
-(r + a(\ell))\, x(i_4) &= \sum_{\substack{i \in R_1 \\ i \neq i_4}} -u_i x(i),
\end{aligned}$$

where $b_i, s_i, t_i, u_i \in \mathcal{O}/\mathfrak{p}^d$. Note that $s_{i_3} = t_{i_2} = 0$ since $F_{22}(a)$ is skew-symmtric. Following the argument in proof of part i., we obtain the upper bound

$$\#\ker_{\mathcal{O}/\mathfrak{p}^d}(F(a)) \leq p^{fd(\mathrm{rk}(\mathfrak{f}_{n,c})-r_n(4)-4)}p^{4f\nu(a(\ell))}. \tag{6.8.5}$$

On the other hand, if $r = -a(\ell)$, then we can instead consider the equation that involves the position where r shows up solely, and by the same reasoning we obtain the upper bound in Equation (6.8.5). As a result, $K(a) \geq p^{4f(d-\nu(a_\ell))}$.

∎

So far, we were able to find lower bounds for $K(a)$ when $a \in \mathcal{S}_{\mathrm{I}} \cup \mathcal{S}_{\mathrm{II}}$. Our next step is find a lower bound when $a \in \mathcal{T} = \cup_{\ell \in R_4} T_\ell$. To achieve that, we need the following lemmas.

Lemma 6.8.7. *Let $g(x) = \sum_{j=0}^{k} g_j x^j \in \mathbb{Z}[x]$ be a polynomial with nonzero integer coefficients, and let $y_1, \ldots, y_{fm}$ nonzero integers such that $\sum_{i=1}^{fm} p^{fy_i} \geq fg(p^f)$. Then, for $p \geq \max\{m, \sum_{j=0}^{k} g_j\}$, we have $\sum_{i=1}^{fm} p^{dfy_i} \geq fg(p^{fd})$.*

Proof: The argument is identical to the inequalities used for the lower bound of [BMKRS23, Thm. 1.3] and therefore we do not repeat it here. ∎

Lemma 6.8.8. [BMKS19, Lemma 4.6] *Let V be an m-dimensional $\mathbb{F}_q$-vector space. Suppose that $B = \{v_1, \ldots, v_{fm}\}$ is a basis of V viewed as a vector space over the subfield $\mathbb{F}_p$. Then there exists a partition $B_1, \ldots, B_f$ of B into f sets of size m such that each B_i is a basis of V as an $\mathbb{F}_q$-vector space.*

Proposition 6.8.9. *Let*

$$g(x) = \frac{1}{8}\left((n^4 + 2n^3 - n^2 - 2n)x + 4(n^3 - 3n^2 + 2n)x^2 \right.$$
$$\left. +(n^4 - 6n^3 + 11n^2 - 6n)x^3\right).$$

Then, $\sum_{a \in \mathcal{T}} (K(a))^{\frac{1}{2}} \geq fg(p^{fd})$.

Proof: Recall that the projections of the vectors of $\mathcal{T}$ form an $\mathbb{F}_p$-basis of $(\mathcal{O}/\mathfrak{p})^{\oplus r_n(4)}$. From Lemma 6.8.8, there exists a subset $\mathcal{T}_0 \subseteq \mathcal{T}$ of size $|\mathcal{T}_0| = r_n(4)$ that forms an $\mathbb{F}_q$-basis of $(\mathcal{O}/\mathfrak{p})^{\oplus r_n(4)}$. Let

$$x_{i,d} := (K(a_i))^{\frac{1}{2}} = \left(\frac{p^{fd(\mathrm{rk}(\mathfrak{f}_{n,4}) - r_n(4))}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F(a_i))} \right)^{\frac{1}{2}}.$$

Note that $x_{i,d} \geq x_{i,1}^d$, and $x_{i,d} = p^{fy_{i,d}}$ for a non-negative integer $y_{i,d}$. By Proposition 4.3.4,

$$\sum_{i=1}^{r_n(4)} p^{fy_{i,1}} = \sum_{i=1}^{r_n(4)} x_{i,1} \geq fg(p^f),$$

where

$$g(x) = \frac{1}{8} \big((n^4 + 2n^3 - n^2 - 2n)x + 4(n^3 - 3n^2 + 2n)x^2 \\ + (n^4 - 6n^3 + 11n^2 - 6n)x^3 \big).$$

By Lemma 6.8.7,

$$\sum_{i=1}^{r_n(4)} x_{i,d} \geq \sum_{i=1}^{r_n(4)} x_{i,1}^d = \sum_{i=1}^{r_n(4)} p^{dfy_{i,1}} \geq fg(p^{fd}).$$

As a result,

$$\sum_{a \in \mathcal{T}} (K(a))^{\frac{1}{2}} \geq \sum_{i=1}^{r_n(4)} x_{i,d} \geq fg(p^{fd}),$$

which completes the proof. ∎

Remark 6.8.10. The polynomial $g(x)$ in Proposition 6.8.9 is the polynomial of the faithful dimension in Theorem 6.7.1. In particular, $m_{\mathrm{faithful}}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = fg(q)$.

Lemma 6.8.11. [BMKRS23, Lemma 4.6] *Let $a_0, \ldots, a_{m-1}$ be non-negative real numbers with $\sum_{k=0}^{m-1} a_k = m$. Assume that for any $0 \leq k \leq m-1$ we have $a_k + \cdots + a_{m-1} \leq m - k$. Then for any decreasing sequence $x_0 \geq \cdots \geq x_{m-1}$,* *we have* $\sum_{k=0}^{m-1} a_k x_k \geq \sum_{k=0}^{m-1} x_k$.

Theorem 6.8.12. *Let R be a finite truncated valuation ring with associated parameters (d, p, e, f). Then,*

$$\sum_{i=0}^{e-1} g(p^{f(d-i)}) \geq m_{faithful}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)) \\ \geq \frac{1}{8} f(n^4 - 6n^3 + 11n^2 - 6n)p^{3fd} + \sum_{i=0}^{e-1} g_1(p^{f(d-i)}),$$

where $g_1(x) = \frac{1}{8} f(n^4 + 2n^3 - n^2 - 2n)x + \frac{1}{2} f(n^3 - 3n^2 + 2n)x^2$, *and* $g(x) = g_1(x) + \frac{1}{8} f(n^4 - 6n^3 + 11n^2 - 6n)x^3$.

Proof: As a consequence of Propositions 6.8.6 and 6.8.9, we can write Equation (6.8.1) as follows

$$\begin{aligned} m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)) &= \sum_{i=1}^{fe\, r_n(4)} \left(\frac{p^{fd(\mathrm{rk}(\mathfrak{f}_{n,c}) - \ell_1)}}{\# \ker_{\mathcal{O}/\mathfrak{p}^d}(F(a_i))} \right)^{\frac{1}{2}} \\ &\geq \sum_{a \in \mathcal{S}_{\mathrm{I}}} (K(a))^{\frac{1}{2}} + \sum_{a \in \mathcal{S}_{\mathrm{II}}} (K(a))^{\frac{1}{2}} + \sum_{a \in \mathcal{T}} (K(a))^{\frac{1}{2}} \\ &\geq \sum_{\ell \in R_{4,\mathrm{I}}} \sum_{a \in S_\ell - T_\ell} p^{f(d - \nu(a(\ell)))} + \sum_{\ell \in R_{4,\mathrm{II}}} \sum_{a \in S_\ell - T_\ell} p^{2f(d - \nu(a(\ell)))} \\ &\quad + fg(p^{fd}), \end{aligned}$$

where

$$g(x) = \frac{1}{8} \left((n^4 + 2n^3 - n^2 - 2n)x + 4(n^3 - 3n^2 + 2n)x^2 \right. \\ \left. + (n^4 - 6n^3 + 11n^2 - 6n)x^3 \right).$$

We can simplify the sums in the above expression using Lemma 6.8.11. Let

$$a_k := \frac{1}{f}\#\{w \in \mathcal{O}/\mathfrak{p}^e : \ \nu(w) = k+1\},$$

for $0 \le k \le e-2$. Note that $\sum_{k=1}^{e-2} a_k = e-1$, and $a_k + \cdots + a_{e-2} \le (e-1)-k$. Then,

$$\begin{aligned}
\sum_{a\in S_\ell} p^{f(d-\nu(a(\ell)))} &= fa_1p^{f(d-1)} + fa_2p^{f(d-2)} + \cdots + fa_{e-2}p^{f(d-(e-2))} \\
&\ge fp^{f(d-1)} + fp^{f(d-2)} + \cdots + fp^{f(d-(e-2))} \\
&= \sum_{i=1}^{e-1} fp^{f(d-i)}.
\end{aligned}$$

Similarly, we can show that

$$\sum_{a\in S_\ell - T_\ell} p^{2f(d-\nu(a(\ell)))} = \sum_{i=1}^{e-1} fp^{2f(d-i)}.$$

As a result,

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4}\otimes_{\mathbb{Z}} R)) \ge \sum_{\ell\in R_{4,\mathrm{I}}}\sum_{i=1}^{e-1} fp^{f(d-i)} + \sum_{\ell\in R_{4,\mathrm{II}}}\sum_{i=1}^{e-1} fp^{2f(d-i)} + fg(p^{fd}).$$

In Section 6.7, we found that

$$|R_{4,\mathrm{I}}| = \frac{1}{8}(n^4+2n^3-n^2-2n), \qquad |R_{4,\mathrm{II}}| = \frac{1}{2}(n^3-3n^2+2n).$$

As a result, the lower bound is given by

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4}\otimes_{\mathbb{Z}} R)) \ge \frac{1}{8}f(n^4-6n^3+11n^2-6n)p^{3fd} + \sum_{i=0}^{e-1} g_1(p^{f(d-i)}),$$

where $g_1(x) = \frac{1}{8}f(n^4 + 2n^3 - n^2 - 2n)x + \frac{1}{2}f(n^3 - 3n^2 + 2n)x^2$. An upper bound can also be obtained by [BMKRS23, Thm. 1.3]. Note that

$$g(x) = g_1(x) + \frac{1}{8}f(n^4 - 6n^3 + 11n^2 - 6n)x^3,$$

then

$$\begin{aligned}\sum_{i=0}^{e-1} g(p^{f(d-i)}) &\geq m_{\text{faithful}}(\exp(\mathfrak{f}_{n,4} \otimes_{\mathbb{Z}} R)) \\ &\geq \frac{1}{8}f(n^4 - 6n^3 + 11n^2 - 6n)p^{3fd} + \sum_{i=0}^{e-1} g_1(p^{f(d-i)}).\end{aligned}$$

which completes the proof. ∎

Chapter 7

Free Lie Algebras of Nilpotency Class 5

In this chapter, we present a comprehensive description of the reduced commutator matrix of $\mathfrak{f}_{n,5}$, as defined in Definition 4.2.7. As in Chapter 5, we first start in Section 7.1 with describing the entries of the commutator matrix as linear combinations of variables indexed by Hall basis elements. Then, in Section 7.2, we use computer-assisted symbolic computations to find the number of occurrences and the exact positions of each Hall basis indexed variable in the commutator matrix

7.1 Entries of Reduced Commutator Matrix and Hall Basis

Let $X = \{x_1, \ldots, x_n\}$. From Proposition 4.2.6(i) we see that the natural projection of $\mathcal{H}^1 \cup \mathcal{H}^2 \cup \mathcal{H}^3 \cup \mathcal{H}^4 \cup \mathcal{H}^5$ is the basis of $\mathfrak{f}_{n,5}$. These Hall sets are described as follows:

$$\mathcal{H}^1 = \{x_1, \ldots, x_n\}.$$

$$\mathcal{H}^2 = \{x_{ij} := [x_i, x_j] : i < j\}.$$

$$\begin{aligned}
\mathcal{H}^3 &= \{x_{ijk} := [x_i,[x_j,x_k]] : j \le i,\ \ j<k\}.\\
\mathcal{H}^4 &= \{x_{ijk\ell} := [x_i,[x_j,[x_k,x_\ell]]] : k \le j \le i,\ \ k<\ell\}\\
&\quad \cup \{x_{ij,i\ell} := [\,[x_i,x_j],[x_i,x_\ell]\,] : i<j<\ell\}\\
&\quad \cup \{x_{ij,k\ell} := [\,[x_i,x_j],[x_k,x_\ell]\,] : i<j,\ \ i<k<\ell\}\\
\mathcal{H}^5 &= \{x_{ijk\ell m} := [x_i,[x_j,[x_k,[x_\ell,x_m]]]] : \ell \le k \le j \le i,\ \ \ell<m\}\\
&\quad \cup \{x_{ij,k\ell m} := [\,[x_i,x_j],[x_k,[x_\ell,x_m]]\,] : i<j,\ \ \ell<m,\ \ \ell \le k\}\,.
\end{aligned}$$

The total order on $\mathcal{H}^i$, for $i=1,2,3,4$ is given in Section 5.1. Whereas the total order on $\mathcal{H}^5$ is defined as follows: First, $\mathcal{H}^5$ is the union of the two sets given above, and we order their elements linearly. In other words, each element of the first set is less than each element of the second set. Then, within each of the two sets, we order elements according to the lexicographic order on the indices. As in nilpotency class 4, we will use the sets R_1,R_2,R_3,R_4 and R_5 to parametrize the sets $\mathcal{H}^1,\mathcal{H}^2,\mathcal{H}^3,\mathcal{H}^4$ and $\mathcal{H}^5$, respectively, by their indices as follows:

$$\begin{aligned}
R_1 &= \{1,\dots,n\}.\\
R_2 &= \{ij : i<j\}.\\
R_3 &= \{ijk : j \le i,\ \ j<k\}.\\
R_4 &= \{ijk\ell : k \le j \le i,\ \ k<\ell\} \cup \{(ij)(i\ell) : i<j<\ell\}\\
&\quad \cup \{(ij)(k\ell) : i<j,\ \ i<k<\ell\}\\
R_5 &= \{ijk\ell m : \ell \le k \le j \le i,\ \ \ell<m\} \cup \{(ij)(k\ell m) : i<j,\ \ \ell<m,\ \ \ell \le k\}\,.
\end{aligned}$$

From Proposition 4.2.6(ii) we see that the natural projection of $\mathcal{H}^2\cup\mathcal{H}^3\cup\mathcal{H}^4\cup\mathcal{H}^5$ is the basis of $[\mathfrak{f}_{n,5},\mathfrak{f}_{n,5}]$. Hence the entries of the commutator matrix $F_{\mathfrak{f}_{n,5}}(T^{(5)})$ have the variables $T=(T^{(k)})_{2\le k\le 5}$, where

$$T^{(2)} = (T^{(2)}_1,\dots,T^{(2)}_{r_n(2)}) = (T_s : s \in R_2),$$

$$T^{(3)} = (T_1^{(3)}, \ldots, T_{r_n(3)}^{(3)}) = (T_s : s \in R_3),$$
$$T^{(4)} = (T_1^{(4)}, \ldots, T_{r_n(4)}^{(4)}) = (T_s : s \in R_4).$$
$$T^{(5)} = (T_1^{(5)}, \ldots, T_{r_n(5)}^{(5)}) = (T_s : s \in R_5).$$

As a result, we get the following reduced commutator matrix $F^{\mathrm{red}}_{\mathfrak{f}_{n,5}}(T^{(5)})$, which will be denoted by $F(T^{(5)})$:

$$F(T^{(5)}) = \begin{array}{c c} & \begin{array}{cccc} \mathcal{H}^1 & \mathcal{H}^2 & \mathcal{H}^3 & \mathcal{H}^4 \end{array} \\ \begin{array}{c} \mathcal{H}^1 \\ \mathcal{H}^2 \\ \mathcal{H}^3 \\ \mathcal{H}^4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & F_{14}(T^{(5)}) \\ 0 & 0 & F_{23}(T^{(5)}) & 0 \\ 0 & -F_{23}^{tr}(T^{(5)}) & 0 & 0 \\ -F_{14}^{tr}(T^{(5)}) & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{7.1.1}$$

where the nonzero entries of $F(T^{(5)})$ are linear combinations of variables from

$$T^{(5)} = (T_1^{(5)}, \ldots, T_{r_n(5)}^{(5)}) = (T_i : i \in R_5).$$

These linear combinations are given in Definition 2.5.2. The scalars in these entries are obtained from brackets of elements from $\mathcal{H}^1 \cup \mathcal{H}^2 \cup \mathcal{H}^3 \cup \mathcal{H}^4$, as shown in Equation (2.5.1).

Proposition 7.1.1. *The entries of the submatrix $F_{23}(T^{(5)})$ are monomials of the form $T_{(ij)(k\ell m)}$, where $(ij)(k\ell m) \in R_5$.*

Proof: For $u \in \mathcal{H}^2$ and $v \in \mathcal{H}^3$ the bracket $[u, v]$ is an element of $\mathcal{H}^5$. Hence, all of the entries of the submatrix $F_{23}(T^{(5)})$ are of the form $T_{(ij)(k\ell m)}$, where $(ij)(k\ell m)$ is a basis elements from R_5. ∎

For the submatrix $F_{14}(T^{(5)})$, the basis expansion of the bracket $[u, v]$, where $u \in \mathcal{H}^1$ and $v \in \mathcal{H}^4$, depends on the type of v. There are two types of brackets to consider:

i. Left-aligned brackets, which are brackets of the form $[a, [b, [c, d]]]$.

ii. Balanced brackets, which are brackets of the form $[[a, b], [c, d]]$.

If $v \in \mathcal{H}^4$ is left-aligned, then the algorithm described in Figure 7.1 produces the basis expansion of the bracket $[u, v]$. In this algorithm, we use the following notations to express different types of brackets:

$$\begin{aligned}
1abcde &:= [a, [b, [c, [d, e]]]],\\
2abcde &:= [a, [[b, c], [d, e]]],\\
3abcde &:= [[a, b], [c, [d, e]]].
\end{aligned}$$

Moreover, the numerals I, II and III in Figure 7.1 refer to the way the Jacobi Identity in Definition 4.2.2(ii) is applied on a given bracket, which is described below:

$$\begin{array}{lll}
\text{I} & 1abcde & = -1abdec - 1abecd\\
\text{II} & 1abcde & = 1acbde + 2abcde\\
\text{III} & 1abcde & = 1bacde + 3abcde
\end{array}$$

On the other hand, if $v \in \mathcal{H}^4$ is balanced, then the algorithm described in Figure 7.2 produces the basis expansion of the bracket $[u, v]$.

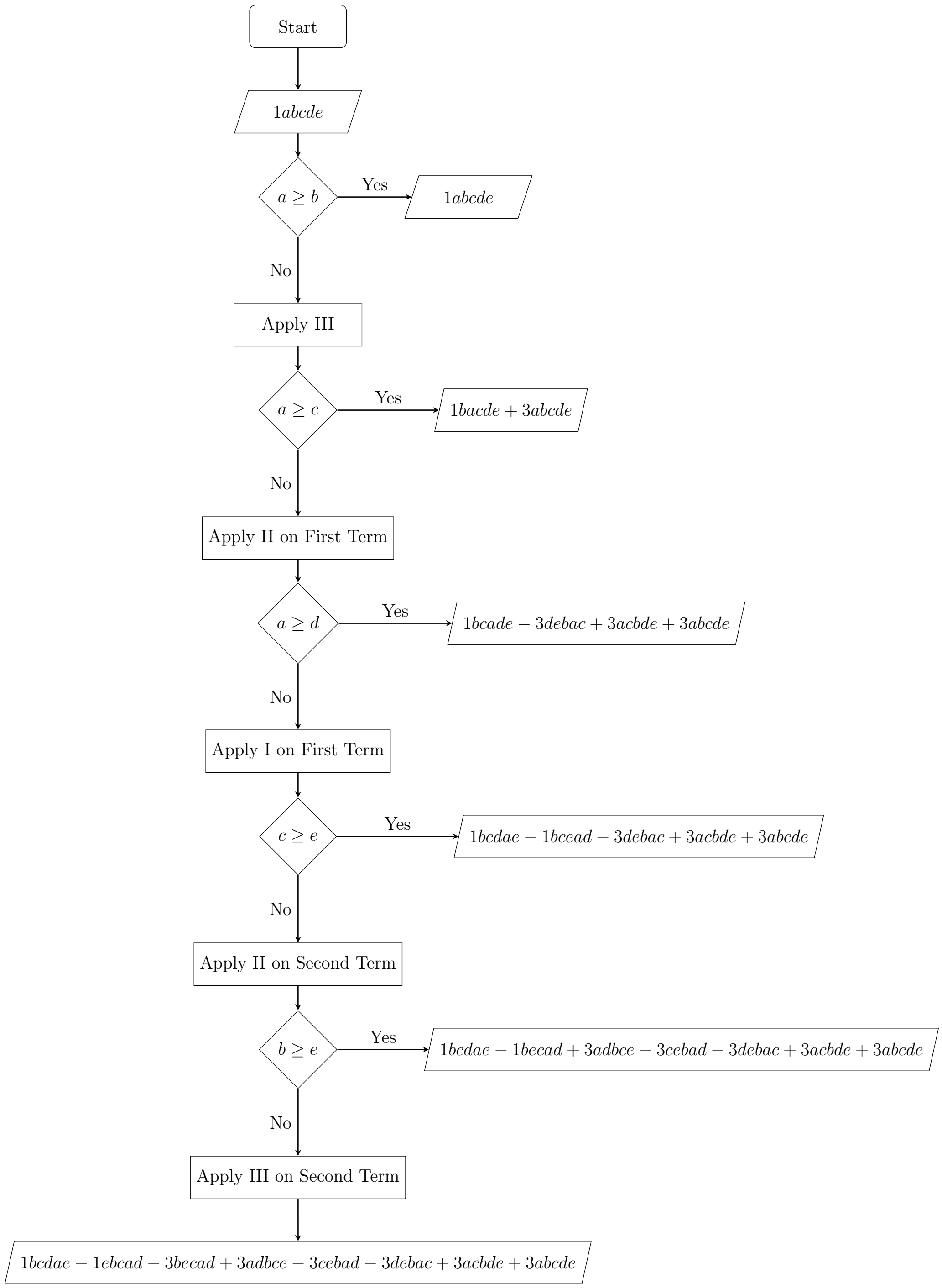

Figure 7.1: Hall basis algorithm for words of type $1abcde$.

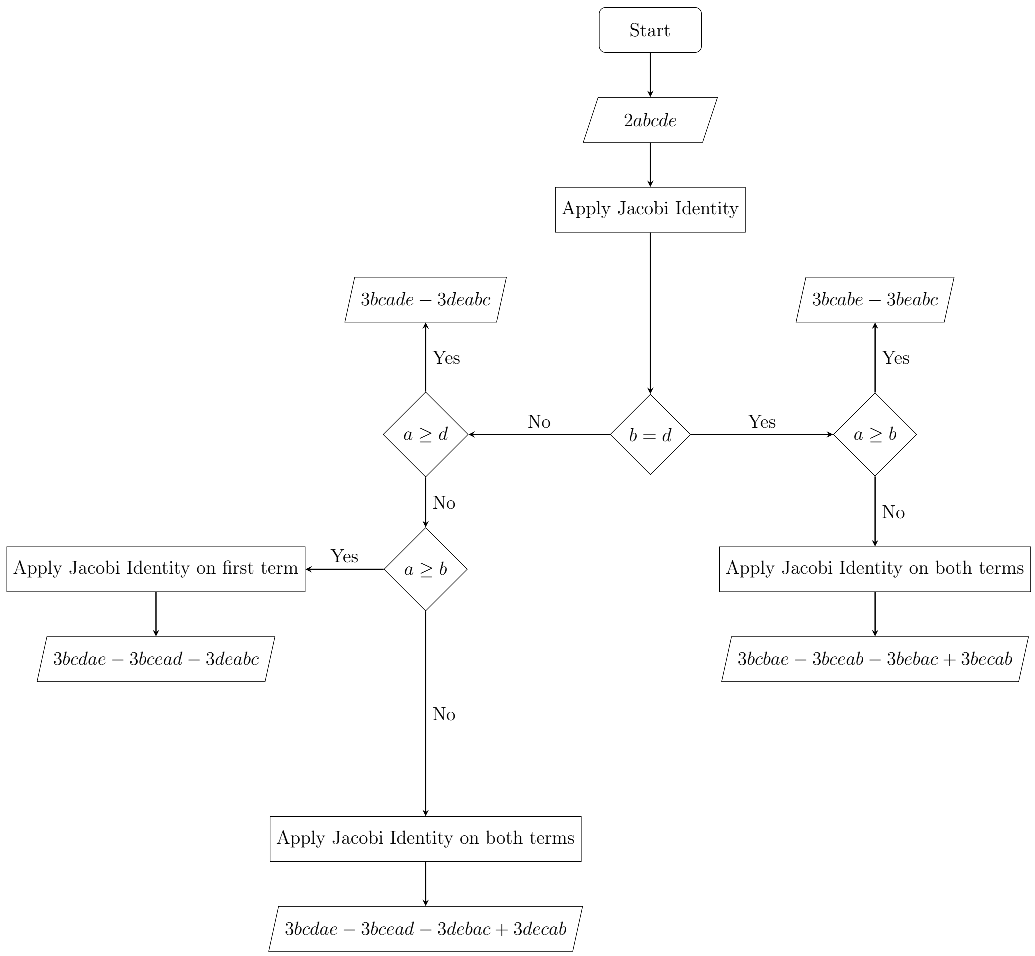

Figure 7.2: Hall basis algorithm for words of type $2abcde$.

Note that, in Figure 7.2 we apply the Jacobi identity depending on the type of the word as follows:

$$2abcde = 3bcade - 3deabc$$

$$3abcde = 3abdce - 3abecd.$$

7.2 Explicit Description of Reduced Commutator matrix $F(T^{(5)})$

Let $a, b, c, d, e \in \{1, \ldots, n\}$, such that $a < b < c < d < e$. The set of basis elements R_5, which is defined as

$$R_5 = \{1ijk\ell m : \ell \leq k \leq j \leq i,\ \ \ell < m\} \cup \{3ijk\ell m : i < j,\ \ \ell < m,\ \ \ell \leq k\}, \quad (7.2.1)$$

can be partitioned into different equivalence classes, where we declare two words to be in the same equivalence class if they can be converted to each other by an order preserving labeling of the letters. For example, the words $1aaaab$ and $1ccccd$ are in the same equivalence class by the relabeling $a \mapsto c$ and $b \mapsto d$. Consequently, the representative of each equivalence class will be a word that meets the following conditions:

- If the words in the equivalence class contain only two letters, the representative must contain the letters a, b.
- If the words in the equivalence class contain only three letters, the representative must contain the letters a, b, c.
- If the words in the equivalence class contain only four letters, the representative must contain the letters a, b, c, d.

Let x be the representative of an equivalence class. Its equivalence class will be denoted by $[x]$, and the elements of $[x]$ are said to be of type x. For example, $[1aaaab]$ denotes the equivalence class containing $1ccccd$, and $1ccccd$ is of type $1aaaab$.

We are able to show that there are 108 types of basis elements in R_5. This is obtained using the code in Appendix B.1. Let Q be the set of types of basis elements in R_5. Thus, we have $|Q| = 108$. Since Q is lexicographically ordered, each type can be identified by its position in Q, as seen in Listing 7.1. As a result, R_5 can be described as follows:

$$R_5 = \cup_{i=0}^{107}[Q[i]], \tag{7.2.2}$$

where $Q[i]$ denotes the i-th element of the lexicographically ordered set Q. We can divide Q into subsets with respect to the number of different letters in each word. Let $B2, B3, B4$ and $B5$ be sets of basis element types with 2,3,4 and 5 letters, respectively. Then, we get the following sets:

$$B2 = \{1aaaab, 1baaab, 1bbaab, 1bbbab, 3abaab, 3abbab\} \tag{7.2.3}$$

$$\begin{aligned} B3 = \{&1baaac, 1bbaac, 1bbbac, 1caaab, 1cbaab, 1cbaac, 1cbbab, 1cbbac, 1ccaab, 1ccbab,\\ &1ccbac, 1cccab, 3abaac, 3abbac, 3abbbc, 3abcab, 3abcac, 3abcbc, 3acaab, 3acbab,\\ &3acbac, 3acbbc, 3accab, 3accbc, 3bcaab, 3bcaac, 3bcbab, 3bcbac, 3bccab, 3bccac\} \end{aligned} \tag{7.2.4}$$

$$\begin{aligned} B4 = \{&1cbaad, 1cbbad, 1ccbad, 1dbaac, 1dbbac, 1dcaab, 1dcbab, 1dcbac, 1dcbad, 1dccab,\\ &1ddbac, 1ddcab, 3abcad, 3abcbd, 3abccd, 3abdac, 3abdbc, 3abdcd, 3acbad, 3acbbd,\\ &3accbd, 3acdab, 3acdbc, 3acdbd, 3adbac, 3adbbc, 3adcab, 3adcbc, 3adcbd, 3addbc,\\ &3bcaad, 3bcbad, 3bccad, 3bcdab, 3bcdac, 3bcdad, 3bdaac, 3bdbac, 3bdcab, 3bdcac,\\ &3bdcad, 3bddac, 3cdaab, 3cdbab, 3cdbac, 3cdbad, 3cdcab, 3cddab\} \end{aligned} \tag{7.2.5}$$

$$\begin{aligned} B5 = \{&1dcbae, 1ecbad, 1edbac, 1edcab, 3abdce, 3abecd, 3acdbe, 3acebd, 3adcbe, 3adebc,\\ &3aecbd, 3aedbc, 3bcdae, 3bcead, 3bdcae, 3bdeac, 3becad, 3bedac, 3cdbae, 3cdeab,\\ &3cebad, 3cedab, 3debac, 3decab\} \end{aligned} \tag{7.2.6}$$

```
Q[0] = 1aaaab
Q[1] = 1baaab
Q[2] = 1baaac
Q[3] = 1bbaab
Q[4] = 1bbaac
Q[5] = 1bbbab
Q[6] = 1bbbac
Q[7] = 1caaab
Q[8] = 1cbaab
Q[9] = 1cbaac
Q[10] = 1cbaad
Q[11] = 1cbbab
Q[12] = 1cbbac
Q[13] = 1cbbad
Q[14] = 1ccaab
Q[15] = 1ccbab
Q[16] = 1ccbac
Q[17] = 1ccbad
Q[18] = 1cccab
Q[19] = 1dbaac
Q[20] = 1dbbac
Q[21] = 1dcaab
Q[22] = 1dcbab
Q[23] = 1dcbac
Q[24] = 1dcbad
Q[25] = 1dcbae
Q[26] = 1dccab
```

```
Q[27] = 1ddbac
Q[28] = 1ddcab
Q[29] = 1ecbad
Q[30] = 1edbac
Q[31] = 1edcab
Q[32] = 3abaab
Q[33] = 3abaac
Q[34] = 3abbab
Q[35] = 3abbac
Q[36] = 3abbbc
Q[37] = 3abcab
Q[38] = 3abcac
Q[39] = 3abcad
Q[40] = 3abcbc
Q[41] = 3abcbd
Q[42] = 3abccd
Q[43] = 3abdac
Q[44] = 3abdbc
Q[45] = 3abdcd
Q[46] = 3abdce
Q[47] = 3abecd
Q[48] = 3acaab
Q[49] = 3acbab
Q[50] = 3acbac
Q[51] = 3acbad
Q[52] = 3acbbc
Q[53] = 3acbbd
```

```
Q[54] = 3accab
Q[55] = 3accbc
Q[56] = 3accbd
Q[57] = 3acdab
Q[58] = 3acdbc
Q[59] = 3acdbd
Q[60] = 3acdbe
Q[61] = 3acebd
Q[62] = 3adbac
Q[63] = 3adbbc
Q[64] = 3adcab
Q[65] = 3adcbc
Q[66] = 3adcbd
Q[67] = 3adcbe
Q[68] = 3addbc
Q[69] = 3adebc
Q[70] = 3aecbd
Q[71] = 3aedbc
Q[72] = 3bcaab
Q[73] = 3bcaac
Q[74] = 3bcaad
Q[75] = 3bcbab
Q[76] = 3bcbac
Q[77] = 3bcbad
Q[78] = 3bccab
Q[79] = 3bccac
Q[80] = 3bccad
```

```
Q[81] = 3bcdab
Q[82] = 3bcdac
Q[83] = 3bcdad
Q[84] = 3bcdae
Q[85] = 3bcead
Q[86] = 3bdaac
Q[87] = 3bdbac
Q[88] = 3bdcab
Q[89] = 3bdcac
Q[90] = 3bdcad
Q[91] = 3bdcae
Q[92] = 3bddac
Q[93] = 3bdeac
Q[94] = 3becad
Q[95] = 3bedac
Q[96] = 3cdaab
Q[97] = 3cdbab
Q[98] = 3cdbac
Q[99] = 3cdbad
Q[100] = 3cdbae
Q[101] = 3cdcab
Q[102] = 3cddab
Q[103] = 3cdeab
Q[104] = 3cebad
Q[105] = 3cedab
Q[106] = 3debac
Q[107] = 3decab
```

Listing 7.1: Basis Element Types in R_5

Our goal in this section is to find the number of times T_t appears in $F(T^{(5)})$ for each $t \in R_5$, and in which positions. This is achieved using the Python code in

Appendix B.2. The output of this code is presented in Listings 7.2, 7.3, 7.4 and 7.5 in Subsections 7.2.1,7.2.2,7.2.3 and 7.2.4, respectively, where types of basis elements are divided by the number of different letters. Note that in the listings below if the basis element $t \in R_5$ appears in a basis expansion of certain bracket, then the position of the variable T_t will depend on the type of the bracket as follows:

	row	*column*
$1abcde$	a	$bcde$
$2abcde$	a	$(bc)(de)$
$3abcde$	ab	cde

Table 7.1: Positions in commutator matrix.

7.2.1 Two-Letter Words Occurrences

From the set $B2$ in Equation (7.2.3), there are 6 types of basis element words in R_5 with only two letters, 4 of them are left-aligned and 2 are balanced. Listing 7.2 shows the number of times they appear in $F(T^{(5)})$ and in which entries.

Listing 7.2: Python Code Output for $B2$

```
1) 1aaaab:Q[0] appears  1 times in:
1. 1aaaab = 1aaaab

2) 1baaab:Q[1] appears  2 times in:
1. 1abaab = 1baaab+3abaab
2. 1baaab = 1baaab

3) 1bbaab:Q[3] appears  2 times in:
1. 1abbab = 1bbaab-3abbab+3abbab+3abbab
2. 1bbaab = 1bbaab

4) 1bbbab:Q[5] appears  1 times in:
1. 1bbbab = 1bbbab
```

```

5) 3abaab:Q[32] appears  2 times in:
1. 1abaab = 1baaab+3abaab
2. 3abaab = 3abaab

6) 3abbab:Q[34] appears  2 times in:
1. 1abbab = 1bbaab-3abbab+3abbab+3abbab
2. 3abbab = 3abbab
```

7.2.2 Three-Letter Words Occurrences

From the set $B3$ in Equation (7.2.4), there are 30 types of basis element words in R_5 with only three letters, 12 of them are left-aligned and 18 are balanced. Listing 7.3 shows the number of times they appear in $F(T^{(5)})$ and in which entries.

Listing 7.3: Python Code Output $B3$

```
1) 1baaac:Q[2]  appears  2 times in:
1. 1abaac = 1baaac+3abaac
2. 1baaac = 1baaac

2) 1bbaac:Q[4]  appears  2 times in:
1. 1abbac = 1bbaac-3acbab+3abbac+3abbac
2. 1bbaac = 1bbaac

3) 1bbbac:Q[6]  appears  2 times in:
1. 1abbbc = 1bbbac-1cbbab-3bcbab+3abbbc-3bcbab-3bcbab+3abbbc+3abbbc
2. 1bbbac = 1bbbac

4) 1caaab:Q[7]  appears  2 times in:
1. 1acaab = 1caaab+3acaab
2. 1caaab = 1caaab

5) 1cbaab:Q[8]  appears  3 times in:
```

```
1. 1acbab = 1cbaab-3abcab+3abcab+3acbab
2. 1bcaab = 1cbaab+3bcaab
3. 1cbaab = 1cbaab

6) 1cbaac:Q[9]  appears  3 times in:
1. 1acbac = 1cbaac-3accab+3abcac+3acbac
2. 1bcaac = 1cbaac+3bcaac
3. 1cbaac = 1cbaac

7) 1cbbab:Q[11]  appears  3 times in:
1. 1abbbc = 1bbbac-1cbbab-3bcbab+3abbbc-3bcbab-3bcbab+3abbbc+3abbbc
2. 1bcbab = 1cbbab+3bcbab
3. 1cbbab = 1cbbab

8) 1cbbac:Q[12]  appears  3 times in:
1. 1acbbc = 1cbbac-1ccbab+3abcbc-3bccab-3bccab+3abcbc+3acbbc
2. 1bcbac = 1cbbac+3bcbac
3. 1cbbac = 1cbbac

9) 1ccaab:Q[14]  appears  2 times in:
1. 1accab = 1ccaab-3abcac+3accab+3accab
2. 1ccaab = 1ccaab

10) 1ccbab:Q[15]  appears  3 times in:
1. 1acbbc = 1cbbac-1ccbab+3abcbc-3bccab-3bccab+3abcbc+3acbbc
2. 1bccab = 1ccbab-3abcbc+3bccab+3bccab
3. 1ccbab = 1ccbab

11) 1ccbac:Q[16]  appears  3 times in:
1. 1accbc = 1ccbac-1cccab-3bccac+3accbc+3accbc
2. 1bccac = 1ccbac-3accbc+3bccac+3bccac
3. 1ccbac = 1ccbac
```

```
12) 1cccab:Q[18]  appears  2 times in:
1. 1accbc = 1ccbac-1cccab-3bccac+3accbc+3accbc
2. 1cccab = 1cccab

13) 3abaac:Q[33]  appears  3 times in:
1. 1abaac = 1baaac+3abaac
2. 2aabac = 3abaac-3acaab
3. 3abaac = 3abaac

14) 3abbac:Q[35]  appears  4 times in:
1. 1abbac = 1bbaac-3acbab+3abbac+3abbac
2. 2aabbc = 3abbac-3abcab-3bcaab
3. 2babac = 3abbac-3acbab
4. 3abbac = 3abbac

15) 3abbbc:Q[36]  appears  3 times in:
1. 1abbbc = 1bbbac-1cbbab-3bcbab+3abbbc-3bcbab-3bcbab+3abbbc+3abbbc
2. 2babbc = 3abbbc-3bcbab
3. 3abbbc = 3abbbc

16) 3abcab:Q[37]  appears  2 times in:
1. 2aabbc = 3abbac-3abcab-3bcaab
2. 3abcab = 3abcab

17) 3abcac:Q[38]  appears  4 times in:
1. 1acbac = 1cbaac-3accab+3abcac+3acbac
2. 1accab = 1ccaab-3abcac+3accab+3accab
3. 2cabac = 3abcac-3accab
4. 3abcac = 3abcac

18) 3abcbc:Q[40]  appears  4 times in:
1. 1acbbc = 1cbbac-1ccbab+3abcbc-3bccab-3bccab+3abcbc+3acbbc
2. 1bccab = 1ccbab-3abcbc+3bccab+3bccab
```

```
3. 2cabbc = 3abcbc-3bccab
4. 3abcbc = 3abcbc

19) 3acaab:Q[48]  appears  3 times in:
1. 1acaab = 1caaab+3acaab
2. 2aabac = 3abaac-3acaab
3. 3acaab = 3acaab

20) 3acbab:Q[49]  appears  4 times in:
1. 1abbac = 1bbaac-3acbab+3abbac+3abbac
2. 1acbab = 1cbaab-3abcab+3abcab+3acbab
3. 2babac = 3abbac-3acbab
4. 3acbab = 3acbab

21) 3acbac:Q[50]  appears  3 times in:
1. 1acbac = 1cbaac-3accab+3abcac+3acbac
2. 2aacbc = 3acbac-3accab-3bcaac
3. 3acbac = 3acbac

22) 3acbbc:Q[52]  appears  3 times in:
1. 1acbbc = 1cbbac-1ccbab+3abcbc-3bccab-3bccab+3abcbc+3acbbc
2. 2bacbc = 3acbbc-3bcbac
3. 3acbbc = 3acbbc

23) 3accab:Q[54]  appears  5 times in:
1. 1acbac = 1cbaac-3accab+3abcac+3acbac
2. 1accab = 1ccaab-3abcac+3accab+3accab
3. 2aacbc = 3acbac-3accab-3bcaac
4. 2cabac = 3abcac-3accab
5. 3accab = 3accab

24) 3accbc:Q[55]  appears  4 times in:
1. 1accbc = 1ccbac-1cccab-3bccac+3accbc+3accbc
```

```
2. 1bccac = 1ccbac-3accbc+3bccac+3bccac
3. 2cacbc = 3accbc-3bccac
4. 3accbc = 3accbc

25) 3bcaab:Q[72]  appears  3 times in:
1. 1bcaab = 1cbaab+3bcaab
2. 2aabbc = 3abbac-3abcab-3bcaab
3. 3bcaab = 3bcaab

26) 3bcaac:Q[73]  appears  3 times in:
1. 1bcaac = 1cbaac+3bcaac
2. 2aacbc = 3acbac-3accab-3bcaac
3. 3bcaac = 3bcaac

27) 3bcbab:Q[75]  appears  4 times in:
1. 1abbbc = 1bbbac-1cbbab-3bcbab+3abbbc-3bcbab-3bcbab+3abbbc+3abbbc
2. 1bcbab = 1cbbab+3bcbab
3. 2babbc = 3abbbc-3bcbab
4. 3bcbab = 3bcbab

28) 3bcbac:Q[76]  appears  3 times in:
1. 1bcbac = 1cbbac+3bcbac
2. 2bacbc = 3acbbc-3bcbac
3. 3bcbac = 3bcbac

29) 3bccab:Q[78]  appears  4 times in:
1. 1acbbc = 1cbbac-1ccbab+3abcbc-3bccab-3bccab+3abcbc+3acbbc
2. 1bccab = 1ccbab-3abcbc+3bccab+3bccab
3. 2cabbc = 3abcbc-3bccab
4. 3bccab = 3bccab

30) 3bccac:Q[79]  appears  4 times in:
1. 1accbc = 1ccbac-1cccab-3bccac+3accbc+3accbc
```

```
2. 1bccac = 1ccbac-3accbc+3bccac+3bccac
3. 2cacbc = 3accbc-3bccac
4. 3bccac = 3bccac
```

7.2.3 Four-Letter Words Occurrences

From the set $B4$ in Equation (7.2.5), there are 48 types of basis element words in R_5 with only four letters, 12 of them are left-aligned and 36 are balanced. Listing 7.4 shows the number of times they appear in $F(T^{(5)})$ and in which entries.

Listing 7.4: Python Code Output $B4$

```
1) 1cbaad:Q[10]  appears  3 times in:
1. 1acbad = 1cbaad-3adcab+3abcad+3acbad
2. 1bcaad = 1cbaad+3bcaad
3. 1cbaad = 1cbaad

2) 1cbbad:Q[13]  appears  3 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 1bcbad = 1cbbad+3bcbad
3. 1cbbad = 1cbbad

3) 1ccbad:Q[17]  appears  3 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 1bccad = 1ccbad-3adcbc+3bccad+3bccad
3. 1ccbad = 1ccbad

4) 1dbaac:Q[19]  appears  3 times in:
1. 1adbac = 1dbaac-3acdab+3abdac+3adbac
2. 1bdaac = 1dbaac+3bdaac
3. 1dbaac = 1dbaac

5) 1dbbac:Q[20]  appears  3 times in:
1. 1adbbc = 1dbbac-1dcbab+3abdbc-3bcdab-3bcdab+3abdbc+3adbbc
```

```
2. 1bdbac = 1dbbac+3bdbac
3. 1dbbac = 1dbbac

6) 1dcaab:Q[21]  appears  3 times in:
1. 1adcab = 1dcaab-3abdac+3acdab+3adcab
2. 1cdaab = 1dcaab+3cdaab
3. 1dcaab = 1dcaab

7) 1dcbab:Q[22]  appears  5 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 1adbbc = 1dbbac-1dcbab+3abdbc-3bcdab-3bcdab+3abdbc+3adbbc
3. 1bdcab = 1dcbab-3abdbc+3bcdab+3bdcab
4. 1cdbab = 1dcbab+3cdbab
5. 1dcbab = 1dcbab

8) 1dcbac:Q[23]  appears  4 times in:
1. 1adcbc = 1dcbac-1dccab-3bcdac+3acdbc+3adcbc
2. 1bdcac = 1dcbac-3acdbc+3bcdac+3bdcac
3. 1cdbac = 1dcbac+3cdbac
4. 1dcbac = 1dcbac

9) 1dcbad:Q[24]  appears  4 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1bdcad = 1dcbad-3addbc+3bcdad+3bdcad
3. 1cdbad = 1dcbad+3cdbad
4. 1dcbad = 1dcbad

10) 1dccab:Q[26]  appears  4 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 1adcbc = 1dcbac-1dccab-3bcdac+3acdbc+3adcbc
3. 1cdcab = 1dccab+3cdcab
4. 1dccab = 1dccab
```

```
11) 1ddbac:Q[27]  appears  3 times in:
1. 1addbc = 1ddbac-1ddcab-3bcdad+3addbc+3addbc
2. 1bddac = 1ddbac-3acdbd+3bddac+3bddac
3. 1ddbac = 1ddbac

12) 1ddcab:Q[28]  appears  4 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1addbc = 1ddbac-1ddcab-3bcdad+3addbc+3addbc
3. 1cddab = 1ddcab-3abdcd+3cddab+3cddab
4. 1ddcab = 1ddcab

13) 3abcad:Q[39]  appears  4 times in:
1. 1acbad = 1cbaad-3adcab+3abcad+3acbad
2. 2aabcd = 3abcad-3abdac-3cdaab
3. 2cabad = 3abcad-3adcab
4. 3abcad = 3abcad

14) 3abcbd:Q[41]  appears  4 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 2babcd = 3abcbd-3abdbc-3cdbab
3. 2cabbd = 3abcbd-3bdcab
4. 3abcbd = 3abcbd

15) 3abccd:Q[42]  appears  3 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 2cabcd = 3abccd-3cdcab
3. 3abccd = 3abccd

16) 3abdac:Q[43]  appears  5 times in:
1. 1adbac = 1dbaac-3acdab+3abdac+3adbac
2. 1adcab = 1dcaab-3abdac+3acdab+3adcab
3. 2aabcd = 3abcad-3abdac-3cdaab
4. 2dabac = 3abdac-3acdab
```

```
5. 3abdac = 3abdac

17) 3abdbc:Q[44]  appears  5 times in:
1. 1adbbc = 1dbbac-1dcbab+3abdbc-3bcdab-3bcdab+3abdbc+3adbbc
2. 1bdcab = 1dcbab-3abdbc+3bcdab+3bdcab
3. 2babcd = 3abcbd-3abdbc-3cdbab
4. 2dabbc = 3abdbc-3bcdab
5. 3abdbc = 3abdbc

18) 3abdcd:Q[45]  appears  4 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1cddab = 1ddcab-3abdcd+3cddab+3cddab
3. 2dabcd = 3abdcd-3cddab
4. 3abdcd = 3abdcd

19) 3acbad:Q[51]  appears  4 times in:
1. 1acbad = 1cbaad-3adcab+3abcad+3acbad
2. 2aacbd = 3acbad-3acdab-3bdaac
3. 2bacad = 3acbad-3adbac
4. 3acbad = 3acbad

20) 3acbbd:Q[53]  appears  3 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 2bacbd = 3acbbd-3bdbac
3. 3acbbd = 3acbbd

21) 3accbd:Q[56]  appears  4 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 2baccd = 3accbd-3acdbc-3cdbac
3. 2cacbd = 3accbd-3bdcac
4. 3accbd = 3accbd

22) 3acdab:Q[57]  appears  5 times in:
```

```
1. 1adbac = 1dbaac-3acdab+3abdac+3adbac
2. 1adcab = 1dcaab-3abdac+3acdab+3adcab
3. 2aacbd = 3acbad-3acdab-3bdaac
4. 2dabac = 3abdac-3acdab
5. 3acdab = 3acdab

23) 3acdbc:Q[58]  appears  5 times in:
1. 1adcbc = 1dcbac-1dccab-3bcdac+3acdbc+3adcbc
2. 1bdcac = 1dcbac-3acdbc+3bcdac+3bdcac
3. 2baccd = 3accbd-3acdbc-3cdbac
4. 2dacbc = 3acdbc-3bcdac
5. 3acdbc = 3acdbc

24) 3acdbd:Q[59]  appears  4 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1bddac = 1ddbac-3acdbd+3bddac+3bddac
3. 2dacbd = 3acdbd-3bddac
4. 3acdbd = 3acdbd

25) 3adbac:Q[62]  appears  4 times in:
1. 1adbac = 1dbaac-3acdab+3abdac+3adbac
2. 2aadbc = 3adbac-3adcab-3bcaad
3. 2bacad = 3acbad-3adbac
4. 3adbac = 3adbac

26) 3adbbc:Q[63]  appears  3 times in:
1. 1adbbc = 1dbbac-1dcbab+3abdbc-3bcdab-3bcdab+3abdbc+3adbbc
2. 2badbc = 3adbbc-3bcbad
3. 3adbbc = 3adbbc

27) 3adcab:Q[64]  appears  5 times in:
1. 1acbad = 1cbaad-3adcab+3abcad+3acbad
2. 1adcab = 1dcaab-3abdac+3acdab+3adcab
```

```
3. 2aadbc = 3adbac-3adcab-3bcaad
4. 2cabad = 3abcad-3adcab
5. 3adcab = 3adcab

28) 3adcbc:Q[65]  appears  4 times in:
1. 1adcbc = 1dcbac-1dccab-3bcdac+3acdbc+3adcbc
2. 1bccad = 1ccbad-3adcbc+3bccad+3bccad
3. 2cadbc = 3adcbc-3bccad
4. 3adcbc = 3adcbc

29) 3adcbd:Q[66]  appears  4 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 2badcd = 3adcbd-3addbc-3cdbad
3. 2cadbd = 3adcbd-3bdcad
4. 3adcbd = 3adcbd

30) 3addbc:Q[68]  appears  5 times in:
1. 1addbc = 1ddbac-1ddcab-3bcdad+3addbc+3addbc
2. 1bdcad = 1dcbad-3addbc+3bcdad+3bdcad
3. 2badcd = 3adcbd-3addbc-3cdbad
4. 2dadbc = 3addbc-3bcdad
5. 3addbc = 3addbc

31) 3bcaad:Q[74]  appears  3 times in:
1. 1bcaad = 1cbaad+3bcaad
2. 2aadbc = 3adbac-3adcab-3bcaad
3. 3bcaad = 3bcaad

32) 3bcbad:Q[77]  appears  4 times in:
1. 1bcbad = 1cbbad+3bcbad
2. 2abcbd = 3bcbad-3bcdab-3bdbac+3bdcab
3. 2badbc = 3adbbc-3bcbad
4. 3bcbad = 3bcbad
```

```
33) 3bccad:Q[80]  appears  4 times in:
1. 1bccad = 1ccbad-3adcbc+3bccad+3bccad
2. 2abccd = 3bccad-3bcdac-3cdbac+3cdcab
3. 2cadbc = 3adcbc-3bccad
4. 3bccad = 3bccad

34) 3bcdab:Q[81]  appears  5 times in:
1. 1adbbc = 1dbbac-1dcbab+3abdbc-3bcdab-3bcdab+3abdbc+3adbbc
2. 1bdcab = 1dcbab-3abdbc+3bcdab+3bdcab
3. 2abcbd = 3bcbad-3bcdab-3bdbac+3bdcab
4. 2dabbc = 3abdbc-3bcdab
5. 3bcdab = 3bcdab

35) 3bcdac:Q[82]  appears  5 times in:
1. 1adcbc = 1dcbac-1dccab-3bcdac+3acdbc+3adcbc
2. 1bdcac = 1dcbac-3acdbc+3bcdac+3bdcac
3. 2abccd = 3bccad-3bcdac-3cdbac+3cdcab
4. 2dacbc = 3acdbc-3bcdac
5. 3bcdac = 3bcdac

36) 3bcdad:Q[83]  appears  4 times in:
1. 1addbc = 1ddbac-1ddcab-3bcdad+3addbc+3addbc
2. 1bdcad = 1dcbad-3addbc+3bcdad+3bdcad
3. 2dadbc = 3addbc-3bcdad
4. 3bcdad = 3bcdad

37) 3bdaac:Q[86]  appears  3 times in:
1. 1bdaac = 1dbaac+3bdaac
2. 2aacbd = 3acbad-3acdab-3bdaac
3. 3bdaac = 3bdaac

38) 3bdbac:Q[87]  appears  4 times in:
```

```
1. 1bdbac = 1dbbac+3bdbac
2. 2abcbd = 3bcbad-3bcdab-3bdbac+3bdcab
3. 2bacbd = 3acbbd-3bdbac
4. 3bdbac = 3bdbac

39) 3bdcab:Q[88]  appears  5 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 1bdcab = 1dcbab-3abdbc+3bcdab+3bdcab
3. 2abcbd = 3bcbad-3bcdab-3bdbac+3bdcab
4. 2cabbd = 3abcbd-3bdcab
5. 3bdcab = 3bdcab

40) 3bdcac:Q[89]  appears  4 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 1bdcac = 1dcbac-3acdbc+3bcdac+3bdcac
3. 2cacbd = 3accbd-3bdcac
4. 3bdcac = 3bdcac

41) 3bdcad:Q[90]  appears  4 times in:
1. 1bdcad = 1dcbad-3addbc+3bcdad+3bdcad
2. 2abdcd = 3bdcad-3bddac-3cdbad+3cddab
3. 2cadbd = 3adcbd-3bdcad
4. 3bdcad = 3bdcad

42) 3bddac:Q[92]  appears  5 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1bddac = 1ddbac-3acdbd+3bddac+3bddac
3. 2abdcd = 3bdcad-3bddac-3cdbad+3cddab
4. 2dacbd = 3acdbd-3bddac
5. 3bddac = 3bddac

43) 3cdaab:Q[96]  appears  3 times in:
1. 1cdaab = 1dcaab+3cdaab
```

```
2. 2aabcd = 3abcad-3abdac-3cdaab
3. 3cdaab = 3cdaab

44) 3cdbab:Q[97]  appears  4 times in:
1. 1acbbd = 1cbbad-1dcbab-3cdbab+3abcbd-3bdcab-3bdcab+3abcbd+3acbbd
2. 1cdbab = 1dcbab+3cdbab
3. 2babcd = 3abcbd-3abdbc-3cdbab
4. 3cdbab = 3cdbab

45) 3cdbac:Q[98]  appears  4 times in:
1. 1cdbac = 1dcbac+3cdbac
2. 2abccd = 3bccad-3bcdac-3cdbac+3cdcab
3. 2baccd = 3accbd-3acdbc-3cdbac
4. 3cdbac = 3cdbac

46) 3cdbad:Q[99]  appears  4 times in:
1. 1cdbad = 1dcbad+3cdbad
2. 2abdcd = 3bdcad-3bddac-3cdbad+3cddab
3. 2badcd = 3adcbd-3addbc-3cdbad
4. 3cdbad = 3cdbad

47) 3cdcab:Q[101]  appears  5 times in:
1. 1accbd = 1ccbad-1dccab-3cdcab+3abccd-3cdcab-3bdcac+3accbd+3accbd
2. 1cdcab = 1dccab+3cdcab
3. 2abccd = 3bccad-3bcdac-3cdbac+3cdcab
4. 2cabcd = 3abccd-3cdcab
5. 3cdcab = 3cdcab

48) 3cddab:Q[102]  appears  5 times in:
1. 1adcbd = 1dcbad-1ddcab+3abdcd-3cddab-3bddac+3acdbd+3adcbd
2. 1cddab = 1ddcab-3abdcd+3cddab+3cddab
3. 2abdcd = 3bdcad-3bddac-3cdbad+3cddab
4. 2dabcd = 3abdcd-3cddab
```

```
5. 3cddab = 3cddab
```

7.2.4 Five-Letter Words Occurrences

From the set $B5$ in Equation (7.2.6), there are 24 types of basis element words in R_5 with five letters, 4 of them are left-aligned and 20 are balanced. Listing 7.5 shows the number of times they appear in $F(T^{(5)})$ and in which entries.

Listing 7.5: Python Code Output $B5$

```
1) 1dcbae:Q[25]  appears  4 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 1bdcae = 1dcbae-3aedbc+3bcdae+3bdcae
3. 1cdbae = 1dcbae+3cdbae
4. 1dcbae = 1dcbae

2) 1ecbad:Q[29]  appears  4 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 1becad = 1ecbad-3adebc+3bcead+3becad
3. 1cebad = 1ecbad+3cebad
4. 1ecbad = 1ecbad

3) 1edbac:Q[30]  appears  4 times in:
1. 1aedbc = 1edbac-1edcab-3bcead+3adebc+3aedbc
2. 1bedac = 1edbac-3acebd+3bdeac+3bedac
3. 1debac = 1edbac+3debac
4. 1edbac = 1edbac

4) 1edcab:Q[31]  appears  6 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
3. 1aedbc = 1edbac-1edcab-3bcead+3adebc+3aedbc
4. 1cedab = 1edcab-3abecd+3cdeab+3cedab
5. 1decab = 1edcab+3decab
```

```
6. 1edcab = 1edcab

5) 3abdce:Q[46]  appears  4 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 2cabde = 3abdce-3abecd-3decab
3. 2dabce = 3abdce-3cedab
4. 3abdce = 3abdce

6) 3abecd:Q[47]  appears  5 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 1cedab = 1edcab-3abecd+3cdeab+3cedab
3. 2cabde = 3abdce-3abecd-3decab
4. 2eabcd = 3abecd-3cdeab
5. 3abecd = 3abecd

7) 3acdbe:Q[60]  appears  4 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 2bacde = 3acdbe-3acebd-3debac
3. 2dacbe = 3acdbe-3bedac
4. 3acdbe = 3acdbe

8) 3acebd:Q[61]  appears  5 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 1bedac = 1edbac-3acebd+3bdeac+3bedac
3. 2bacde = 3acdbe-3acebd-3debac
4. 2eacbd = 3acebd-3bdeac
5. 3acebd = 3acebd

9) 3adcbe:Q[67]  appears  4 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 2badce = 3adcbe-3adebc-3cebad
3. 2cadbe = 3adcbe-3becad
4. 3adcbe = 3adcbe
```

```

10) 3adebc:Q[69]  appears  5 times in:
1. 1aedbc = 1edbac-1edcab-3bcead+3adebc+3aedbc
2. 1becad = 1ecbad-3adebc+3bcead+3becad
3. 2badce = 3adcbe-3adebc-3cebad
4. 2eadbc = 3adebc-3bcead
5. 3adebc = 3adebc

11) 3aecbd:Q[70]  appears  4 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 2baecd = 3aecbd-3aedbc-3cdbae
3. 2caebd = 3aecbd-3bdcae
4. 3aecbd = 3aecbd

12) 3aedbc:Q[71]  appears  5 times in:
1. 1aedbc = 1edbac-1edcab-3bcead+3adebc+3aedbc
2. 1bdcae = 1dcbae-3aedbc+3bcdae+3bdcae
3. 2baecd = 3aecbd-3aedbc-3cdbae
4. 2daebc = 3aedbc-3bcdae
5. 3aedbc = 3aedbc

13) 3bcdae:Q[84]  appears  4 times in:
1. 1bdcae = 1dcbae-3aedbc+3bcdae+3bdcae
2. 2abcde = 3bcdae-3bcead-3debac+3decab
3. 2daebc = 3aedbc-3bcdae
4. 3bcdae = 3bcdae

14) 3bcead:Q[85]  appears  5 times in:
1. 1aedbc = 1edbac-1edcab-3bcead+3adebc+3aedbc
2. 1becad = 1ecbad-3adebc+3bcead+3becad
3. 2abcde = 3bcdae-3bcead-3debac+3decab
4. 2eadbc = 3adebc-3bcead
5. 3bcead = 3bcead
```

```
15) 3bdcae:Q[91]  appears  4 times in:
1. 1bdcae = 1dcbae-3aedbc+3bcdae+3bdcae
2. 2abdce = 3bdcae-3bdeac-3cebad+3cedab
3. 2caebd = 3aecbd-3bdcae
4. 3bdcae = 3bdcae

16) 3bdeac:Q[93]  appears  5 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 1bedac = 1edbac-3acebd+3bdeac+3bedac
3. 2abdce = 3bdcae-3bdeac-3cebad+3cedab
4. 2eacbd = 3acebd-3bdeac
5. 3bdeac = 3bdeac

17) 3becad:Q[94]  appears  4 times in:
1. 1becad = 1ecbad-3adebc+3bcead+3becad
2. 2abecd = 3becad-3bedac-3cdbae+3cdeab
3. 2cadbe = 3adcbe-3becad
4. 3becad = 3becad

18) 3bedac:Q[95]  appears  5 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 1bedac = 1edbac-3acebd+3bdeac+3bedac
3. 2abecd = 3becad-3bedac-3cdbae+3cdeab
4. 2dacbe = 3acdbe-3bedac
5. 3bedac = 3bedac

19) 3cdbae:Q[100]  appears  4 times in:
1. 1cdbae = 1dcbae+3cdbae
2. 2abecd = 3becad-3bedac-3cdbae+3cdeab
3. 2baecd = 3aecbd-3aedbc-3cdbae
4. 3cdbae = 3cdbae
```

```
20) 3cdeab:Q[103]  appears  5 times in:
1. 1aecbd = 1ecbad-1edcab+3abecd-3cdeab-3bdeac+3acebd+3aecbd
2. 1cedab = 1edcab-3abecd+3cdeab+3cedab
3. 2abecd = 3becad-3bedac-3cdbae+3cdeab
4. 2eabcd = 3abecd-3cdeab
5. 3cdeab = 3cdeab

21) 3cebad:Q[104]  appears  4 times in:
1. 1cebad = 1ecbad+3cebad
2. 2abdce = 3bdcae-3bdeac-3cebad+3cedab
3. 2badce = 3adcbe-3adebc-3cebad
4. 3cebad = 3cebad

22) 3cedab:Q[105]  appears  5 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 1cedab = 1edcab-3abecd+3cdeab+3cedab
3. 2abdce = 3bdcae-3bdeac-3cebad+3cedab
4. 2dabce = 3abdce-3cedab
5. 3cedab = 3cedab

23) 3debac:Q[106]  appears  4 times in:
1. 1debac = 1edbac+3debac
2. 2abcde = 3bcdae-3bcead-3debac+3decab
3. 2bacde = 3acdbe-3acebd-3debac
4. 3debac = 3debac

24) 3decab:Q[107]  appears  5 times in:
1. 1adcbe = 1dcbae-1edcab-3decab+3abdce-3cedab-3bedac+3acdbe+3adcbe
2. 1decab = 1edcab+3decab
3. 2abcde = 3bcdae-3bcead-3debac+3decab
4. 2cabde = 3abdce-3abecd-3decab
5. 3decab = 3decab
```

Chapter 8

Faithful Dimension of Nilpotency Class 5

In this chapter, we provide upper and lower bounds on the faithful dimension of $\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)$ using the rank minimization problem presented in Proposition 4.3.4 and the results from Chapter 5.

8.1 Bounds for The Faithful Dimension of $\mathfrak{f}_{n,5}$

In the computation of the faithful dimension of $\mathfrak{f}_{n,4}$ the greedy algorithm provided an optimal result as seen in Chapter 6. However, for the case of $\mathfrak{f}_{n,5}$ using the greedy algorithm appears to be a substantially more difficult task because of the larger size of the commutator matrix and the presence of more vectors to analyze.

Proposition 4.3.4 provides a method for computing an upper bound for the faithful dimension of $\mathfrak{f}_{n,5}$ using the standard basis vectors e_i, where for each $i \in R_5$ the only nonzero coordinate is in the i^{th} position, that is

$$e_i = (0, \ldots, 0, \underset{\substack{\downarrow \\ i^{\text{th}} \text{ position}}}{1}, 0, \ldots, 0).$$

An upper bound is obtained by computing the rank of the commutator matrix evaluated at the vectors e_i, where $i \in R_5$.

Lemma 8.1.1. *For $i \in R_5$,*

$$rk(F(e_i)) = \begin{cases} 2\, rk(F_{14}(e_i)) & \text{if } i = 1abcde, \\ 2\, rk(F_{14}(e_i)) + 2 & \text{if } i = 3abcde. \end{cases}$$

Proof: From the commutator matrix in Equation (7.1.1), we have

$$rk(F(e_i)) = 2\, rk(F_{14}(e_i)) + 2\, rk(F_{23}(e_i)).$$

As a consequence of Proposition 7.1.1, $rk(F_{23}(e_{1abcde})) = 0$ for $1abcde \in R_5$ since T_{1abcde} is not an entry of $F_{23}(T^{(5)})$. Hence, $rk(F(e_{1abcde})) = 2\, rk(F_{14}(e_{1abcde}))$. Also, from Proposition 7.1.1 we have $rk(F_{23}(e_{3abcde})) = 1$ for $3abcde \in R_5$, since T_{3abcde} appears only once in $F_{23}(T^{(5)})$, specifically in row $ab \in R_2$ and column $cde \in R_3$. As a result, $rk(F(e_{3abcde})) = 2\, rk(F_{14}(e_{3abcde})) + 2$. ∎

To compute the ranks of the commutator matrices $F_{14}(e_i)$ for all $i \in R_5$, it is enough to compute the ranks of $F_{14}(e_{Q[i]})$, for $0 \leq i \leq 107$, since $R_5 = \cup_{i=0}^{107}[Q[i]]$. Since the vectors $e_{Q[i]}$ for $0 \leq i \leq 107$ are nonzero only in the position $Q[i]$, we are only interested in the entries of $F_{14}(T^{(5)})$ containing the variable $T_{Q[i]}$. These entries are explicitly described in Listings 7.2, 7.3,7.4 and 7.5. The following is an example of the rank computation process.

Example 8.1.2. Consider the four letter word $Q[44] = 3abdbc$. From line 91, in Listing 7.4, the variable T_{3abdbc} appears 4 times in $F(T^{(5)})$. More specifically, the variable T_{3abdbc} appears only one time in $F_{23}(T^{(5)})$, which follows from Proposition 7.1.1, and 3 times in $F_{14}(T^{(5)})$ in the following positions

$$F_{14}(T^{(5)}) = \begin{array}{c c} & \begin{array}{cccc} dbbc & dcab & (ab)(bc) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} A & & & \\ & B & & C \\ & & D & \end{pmatrix} \end{array},$$

where

$$A = T_{1dbbac} - T_{1dcbab} + 2T_{3abdbc} - 2T_{3bcdab} + T_{3adbbc},$$

$$B = T_{1dcbab} - T_{3abdbc} + T_{3bcdab} + T_{3bdcab},$$

$$C = T_{3abcbd} - T_{3abdbc} - T_{3cdbab},$$

$$D = T_{3abdbc} - T_{3bcdab}.$$

Since we are interested in the vector $e_{Q[44]} = e_{3abdbc}$, which has the value 1 in the position $3abdbc$ and 0 elsewhere, we get the following matrix.

$$F_{14}(e_{Q[44]}) = F_{14}(e_{3abdbc}) = \begin{array}{c c} & \begin{array}{cccc} dbbc & dcab & (ab)(bc) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} 2 & & & \\ & -1 & & -1 \\ & & 1 & \end{pmatrix} \end{array}.$$

From Lemma 8.1.1,

$$rk(F(e_{3abdbc})) = 2\,rk(F_{14}(e_{3abdbc})) + 2 = 2 \times 3 + 2 = 8.$$

In a similar fashion we can continue the rank computation of the matrices $F_{14}(e_{Q[i]})$, for $0 \leq i \leq 107$. The computation results are presented in Tables 8.2, 8.3, 8.4 and 8.5 in Section 8.2. These calculations are summarized in Table 8.1, where the entry in row $i \in \{1,2,3,4\}$ and column B_j, for $j \in \{2,3,4,5\}$, is the number of words in B_j with rank $2i$.

$\frac{1}{2}rk(F(e_i))$	B_2	B_3	B_4	B_5
1	2	0	0	0
2	4	11	0	0
3	0	15	24	0
4	0	4	24	24

Table 8.1: Ranks summary for R_5.

From Proposition 4.3.4, we can find an upper bound for the faithful dimension using the standard basis vectors e_i, where $i \in R_5$.

Theorem 8.1.3. *For $n \geq 2$ and p sufficiently large, we have*

$$m_{faithful}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)) \leq \; f\, n(n-1)\left[q + \frac{1}{6}(11n-10)q^2 + \frac{1}{2}(n-2)(2n-1)q^3 + \frac{1}{15}(n-2)(3n^2-6n+1)q^4\right].$$

Proof: From Proposition 4.3.4,

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)) \leq \sum_{\ell=1}^{r_n(5)} f q^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,5}}(e_{Q[\ell]})\right)}, \tag{8.1.1}$$

where $e_{Q[\ell]}$ is the standard basis vectors for $1 \leq \ell \leq r_n(5)$. Using Table 8.1, we get

$$\sum_{\ell=1}^{r_n(5)} f q^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,5}}(e_{Q[\ell]})\right)} = f \times 2X_2 \times q + f(4X_2 + 11X_3)q^2 + f(15X_3 + 24X_4)q^3 + f(4X_3 + 24X_4 + 24X_5)q^4, \tag{8.1.2}$$

where $X_i = \binom{n}{i} = \frac{n!}{i!(n-i)!}$, for $i = 2, 3, 4, 5$. Each term in Equation (8.1.2) can be simplified as follows:

$$2X_2 = n(n-1)$$

$$4X_2 + 11X_3 = 2n(n-1) + \frac{11}{6}n(n-1)(n-2) = \frac{1}{6}n(n-1)(11n-10)$$

$$\begin{aligned}15X_3 + 24X_4 &= \frac{5}{2}n(n-1)(n-2) + n(n-1)(n-2)(n-3)\\ &= \frac{1}{2}n(n-1)(n-2)(2n-1)\end{aligned}$$

$$\begin{aligned}4X_3 + 24X_4 + 24X_5 &= \frac{2}{3}n(n-1)(n-2) + n(n-1)(n-2)(n-3)\\ &+ \frac{24}{120}n(n-1)(n-2)(n-3)(n-4)\\ &= \frac{1}{15}n(n-1)(n-2)\left[10 + 15(n-3) + 3(n-3)(n-4)\right]\\ &= \frac{1}{15}n(n-1)(n-2)(3n^2-6n+1)\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\ell=1}^{r_n(5)} fq^{\frac{1}{2}\mathrm{rk}_{\mathbb{F}_q}\left(F^{\mathrm{red}}_{\mathfrak{f}_{n,5}}(e_{Q[\ell]})\right)} &= fn(n-1)q + \frac{1}{6}fn(n-1)(11n-10)q^2\\ &+ \frac{1}{2}fn(n-1)(n-2)(2n-1)q^3\\ &+ \frac{1}{15}fn(n-1)(n-2)(3n^2-6n+1)q^4\\ &= fn(n-1)\left[q + \frac{1}{6}(11n-10)q^2 + \frac{1}{2}(n-2)(2n-1)q^3\right.\\ &\left.+\frac{1}{15}(n-2)(3n^2-6n+1)q^4\right].\end{aligned}$$

As a result of (8.1.1), we obtain an upper bound for the faithful dimension. ∎

We conclude this section with the following lower bound result. This result improves the lower bound obtained in [BMKS19], which was given by

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)) \geq r_n(5) f q.$$

Lemma 8.1.4. *Let $v_1, \ldots v_r \in F^r$, where F is a field. If the vectors $v_1, \ldots v_r$ are linearly independent, then for all $i \in \{1, \ldots, r\}$ there exits a vector v_j such that $v_j(i) \neq 0$.*

Proof: Suppose that there exists $i \in \{1, \ldots, r\}$ such that $v_j(i) = 0$ for all $j \in \{1, \ldots, r\}$. So, these r vectors can be imbedded in F^{r-1}. Thus, they must be linearly dependent, which is a contradiction. ∎

Theorem 8.1.5. *For p sufficiently large, we have*

$$m_{\textit{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)) \geq f q^2.$$

Proof: Suppose that the vectors $v_1, \ldots, v_{r_n(5)} \in \mathbb{F}_q^{r_n(5)}$ are the optimal solution to the rank minimization in Proposition 4.3.4. By Lemma 8.1.4, for any $i \in R_5$ there exists $j \in \{1, \ldots, r_n(5)\}$ such that $v_j(i) \neq 0$. Let $i = 3abacc = Q[33]$ and v be the vector where $v(i) \neq 0$. From Listing 7.3, we see that T_i appears 3 times in $F(T^{(5)})$, twice in $F_{14}(T^{(5)})$ and once in $F_{23}(T^{(5)})$. In particular, the occurrences of T_i can be summarized in the following table:

row	*column*	*entry*
a	$baac$	$T_{1baaac} + T_i$
a	$(ab)(ac)$	$T_i - T_{3acaab}$
ab	acc	T_i

Since $v(i) \neq 0$ and the entries of $F_{23}(T^{(5)})$ are monomials, we have $rk(F_{23}(v)) \geq 1$. Furthermore, if $v(1baaac) \neq -v(i)$, then $F_{14}(v)$ has a nonzero entry in row a and column $baac$. Otherwise, we must have $v(1baaac) = -v(i) \neq 0$, which means that $F_{14}(v)$ has a nonzero entry since $1baaac$ is a basis element. So, in both cases we have $rk(F_{14}(v)) \geq 1$. As a result,

$$rk(F(v)) = 2F_{14}(v) + 2F_{23}(v) \geq 4.$$

From Proposition 4.3.4,

$$m_{\text{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q)) = \sum_{i=1}^{r_n(5)} fq^{\frac{1}{2}\text{rk}_{\mathbb{F}_q}(F(v_i))} \geq fq^{\frac{1}{2}\text{rk}_{\mathbb{F}_q}(F(v))} = fq^2,$$

which completes the proof. ∎

Remark 8.1.6. The bounds in Theorem 8.1.3 and Theorem 8.1.5 provide a large interval for the exact value of $m_{\text{faithful}}(\exp(\mathfrak{f}_{n,5} \otimes_{\mathbb{Z}} \mathbb{F}_q))$. It is an interesting problem to prove a sharp asymptotic formula for this faithful dimension.

8.2 Rank Computations

Listings 7.2, 7.3, 7.4 and 7.5 provide the exact entries of $F_{14}(T^{(5)})$ containing the variable $T_{Q[i]}$ for each $0 \leq i \leq 107$. A case by case computation is presented in the following tables.

Table 8.2: Ranks of $B2$ words.

1)	$Q[0] = 1aaaab$	$\begin{array}{cc} & aaab \\ a & \begin{pmatrix} 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1aaaab})) = 1$ $rk(F(e_{1aaaab})) = 2$

Continued on next page

Table 8.2: ranks of $B2$ words (cont'd)

2)	$Q[1] = 1baaab$	$\begin{array}{cc} & \begin{matrix} aaab & baab \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1baaab})) = 2$ $rk(F(e_{1baaab})) = 4$
3)	$Q[3] = 1bbaab$	$\begin{array}{cc} & \begin{matrix} baab & bbab \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbaab})) = 2$ $rk(F(e_{1bbaab})) = 4$
4)	$Q[5] = 1bbbab$	$\begin{array}{cc} & bbab \\ b & \begin{pmatrix} 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbbab})) = 1$ $rk(F(e_{1bbbab})) = 2$
5)	$Q[32] = 3abaab$	$\begin{array}{cc} & baab \\ a & \begin{pmatrix} 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbbab})) = 1$ $rk(F(e_{1bbbab})) = 4$
6)	$Q[34] = 3abbab$	$\begin{array}{cc} & bbab \\ a & \begin{pmatrix} 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbbab})) = 1$ $rk(F(e_{1bbbab})) = 4$

Table 8.3: Ranks of $B3$ words.

1)	$Q[2] = 1baaac$	$\begin{array}{cc} & \begin{matrix} aaac & baac \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1baaac})) = 2$ $rk(F(e_{1baaac})) = 4$
2)	$Q[4] = 1bbaac$	$\begin{array}{cc} & \begin{matrix} baac & bbac \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbaac})) = 2$ $rk(F(e_{1bbaac})) = 4$

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Table 8.3: ranks of $B3$ words (cont'd)

3)	$Q[6] = 1bbbac$	$\begin{array}{cc} & \begin{array}{cc} bbac & bbbc \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1bbbac})) = 2$ $rk(F(e_{1bbbac})) = 4$
4)	$Q[7] = 1caaab$	$\begin{array}{cc} & \begin{array}{cc} aaab & caab \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1caaab})) = 2$ $rk(F(e_{1caaab})) = 4$
5)	$Q[8] = 1cbaab$	$\begin{array}{cc} & \begin{array}{ccc} baab & caab & cbab \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbaab})) = 3$ $rk(F(e_{1cbaab})) = 6$
6)	$Q[9] = 1cbaac$	$\begin{array}{cc} & \begin{array}{ccc} baac & caac & cbac \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbaac})) = 3$ $rk(F(e_{1cbaac})) = 6$
7)	$Q[11] = 1cbbab$	$\begin{array}{cc} & \begin{array}{ccc} bbab & bbbc & cbab \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & -1 & \\ & & 1 \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbbab})) = 3$ $rk(F(e_{1cbbab})) = 6$
8)	$Q[12] = 1cbbac$	$\begin{array}{cc} & \begin{array}{ccc} bbac & cbac & cbbc \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbbac})) = 3$ $rk(F(e_{1cbbac})) = 6$

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Table 8.3: ranks of $B3$ words (cont'd)

9)	$Q[14] = 1ccaab$	$\begin{array}{cc} & \begin{array}{cc} caab & ccab \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ccaab})) = 2$ $rk(F(e_{1ccaab})) = 4$
10)	$Q[15] = 1ccbab$	$\begin{array}{cc} & \begin{array}{ccc} cbab & cbbc & ccab \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & -1 & \\ & & 1 \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ccbab})) = 3$ $rk(F(e_{1ccbab})) = 6$
11)	$Q[16] = 1ccbac$	$\begin{array}{cc} & \begin{array}{ccc} cbac & ccac & ccbc \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ccbac})) = 3$ $rk(F(e_{1ccbac})) = 6$
12)	$Q[18] = 1cccab$	$\begin{array}{cc} & \begin{array}{cc} ccab & ccbc \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cccab})) = 2$ $rk(F(e_{1cccab})) = 4$
13)	$Q[33] = 3abaac$	$\begin{array}{cc} & \begin{array}{cc} baac & (ab)(ac) \end{array} \\ a & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abaac})) = 1$ $rk(F(e_{3abaac})) = 4$
14)	$Q[35] = 3abbac$	$\begin{array}{cc} & \begin{array}{ccc} bbac & (ab)(ac) & (ab)(bc) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 2 & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abbac})) = 2$ $rk(F(e_{3abbac})) = 6$
15)	$Q[36] = 3abbbc$	$\begin{array}{cc} & \begin{array}{cc} bbbc & (ab)(bc) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 3 & \\ & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abbbc})) = 2$ $rk(F(e_{3abbbc})) = 6$

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Table 8.3: ranks of $B3$ words (cont'd)

16)	$Q[37] = 3abcab$	$\begin{array}{cc} & \begin{array}{c} (ab)(bc) \end{array} \\ \begin{array}{c} a \end{array} & \begin{pmatrix} -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abcab})) = 1$ $rk(F(e_{3abcab})) = 4$
17)	$Q[38] = 3abcac$	$\begin{array}{cc} & \begin{array}{ccc} cbac & ccab & (ab)(ac) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} 1 & -1 & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abcac})) = 2$ $rk(F(e_{3abcac})) = 6$
18)	$Q[40] = 3abcbc$	$\begin{array}{cc} & \begin{array}{ccc} cbbc & ccab & (ab)(bc) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 2 & & \\ & -1 & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abcbc})) = 3$ $rk(F(e_{3abcbc})) = 8$
19)	$Q[48] = 3acaab$	$\begin{array}{cc} & \begin{array}{cc} caab & (ab)(ac) \end{array} \\ \begin{array}{c} a \end{array} & \begin{pmatrix} 1 & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acaab})) = 1$ $rk(F(e_{3acaab})) = 4$
20)	$Q[49] = 3acbab$	$\begin{array}{cc} & \begin{array}{ccc} bbac & cbab & (ab)(ac) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} -1 & 1 & \\ & & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acbab})) = 2$ $rk(F(e_{3acbab})) = 6$
21)	$Q[50] = 3acbac$	$\begin{array}{cc} & \begin{array}{cc} cbac & (ac)(bc) \end{array} \\ \begin{array}{c} a \end{array} & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acbac})) = 1$ $rk(F(e_{3acbac})) = 4$
22)	$Q[52] = 3acbbc$	$\begin{array}{cc} & \begin{array}{cc} cbbc & (ac)(bc) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acbbc})) = 2$ $rk(F(e_{3acbbc})) = 6$
23)	$Q[54] = 3accab$	$\begin{array}{cc} & \begin{array}{cccc} cbac & ccab & (ac)(bc) & (ab)(ac) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} -1 & 2 & -1 & \\ & & & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3accab})) = 2$ $rk(F(e_{3accab})) = 6$

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Table 8.3: ranks of $B3$ words (cont'd)

24)	$Q[55] = 3accbc$	$\begin{array}{cc} & \begin{matrix} ccac & ccbc & (ac)(bc) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} & 2 & \\ -1 & & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3accbc})) = 3$ $rk(F(e_{3accbc})) = 8$
25)	$Q[72] = 3bcaab$	$\begin{array}{cc} & \begin{matrix} caab & (ab)(bc) \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcaab})) = 2$ $rk(F(e_{3bcaab})) = 6$
26)	$Q[73] = 3bcaac$	$\begin{array}{cc} & \begin{matrix} caac & (ac)(bc) \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcaac})) = 2$ $rk(F(e_{3bcaac})) = 6$
27)	$Q[75] = 3bcbab$	$\begin{array}{cc} & \begin{matrix} bbbc & cbab & (ab)(bc) \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} -3 & & \\ & 1 & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcbab})) = 2$ $rk(F(e_{3bcbab})) = 6$
28)	$Q[76] = 3bcbac$	$\begin{array}{cc} & \begin{matrix} cbac & (ac)(bc) \end{matrix} \\ b & \begin{pmatrix} 1 & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcbac})) = 1$ $rk(F(e_{3bcbac})) = 4$
29)	$Q[78] = 3bccab$	$\begin{array}{cc} & \begin{matrix} cbbc & ccab & (ab)(bc) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} -2 & & \\ & 2 & \\ & & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bccab})) = 3$ $rk(F(e_{3bccab})) = 8$
30)	$Q[79] = 3bccac$	$\begin{array}{cc} & \begin{matrix} ccac & ccbc & (ac)(bc) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} & -1 & \\ 2 & & \\ & & -1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bccac})) = 3$ $rk(F(e_{3bccac})) = 8$

Table 8.4: Ranks of $B4$ words.

1)	$Q[10] = 1cbaad$	$\begin{array}{c c} & \begin{array}{ccc} baad & caad & cbad \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbaad})) = 3$ $rk(F(e_{1cbaad})) = 6$
2)	$Q[13] = 1cbbad$	$\begin{array}{c c} & \begin{array}{ccc} bbad & cbad & cbbd \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1cbbad})) = 3$ $rk(F(e_{1cbbad})) = 6$
3)	$Q[17] = 1ccbad$	$\begin{array}{c c} & \begin{array}{ccc} cbad & ccad & ccbd \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ccbad})) = 3$ $rk(F(e_{1ccbad})) = 6$
4)	$Q[19] = 1dbaac$	$\begin{array}{c c} & \begin{array}{ccc} baac & daac & dbac \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dbaac})) = 3$ $rk(F(e_{1dbaac})) = 6$
5)	$Q[20] = 1dbbac$	$\begin{array}{c c} & \begin{array}{ccc} bbac & dbac & dbbc \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dbbac})) = 3$ $rk(F(e_{1dbbac})) = 6$

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Table 8.4: ranks of $B4$ words (cont'd)

6)	$Q[21] = 1dcaab$	$\begin{array}{cc} & \begin{matrix} caab & daab & dcab \end{matrix} \\ \begin{matrix} a \\ c \\ d \end{matrix} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dcaab})) = 3$ $rk(F(e_{1dcaab})) = 6$
7)	$Q[22] = 1dcbab$	$\begin{array}{cc} & \begin{matrix} cbab & cbbd & dbab & dbbc & dcab \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} & -1 & & -1 & \\ & & & & 1 \\ & & 1 & & \\ 1 & & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dcbab})) = 4$ $rk(F(e_{1dcbab})) = 8$
8)	$Q[23] = 1dcbac$	$\begin{array}{cc} & \begin{matrix} cbac & dbac & dcac & dcbc \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dcbac})) = 4$ $rk(F(e_{1dcbac})) = 8$
9)	$Q[24] = 1dcbad$	$\begin{array}{cc} & \begin{matrix} cbad & dbad & dcad & dcbd \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dcbad})) = 4$ $rk(F(e_{1dcbad})) = 8$
10)	$Q[26] = 1dccab$	$\begin{array}{cc} & \begin{matrix} ccab & ccbd & dcab & dcbc \end{matrix} \\ \begin{matrix} a \\ c \\ d \end{matrix} & \begin{pmatrix} & -1 & & -1 \\ & & 1 & \\ 1 & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1dccab})) = 3$ $rk(F(e_{1dccab})) = 6$

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Table 8.4: ranks of $B4$ words (cont'd)

11)	$Q[27] = 1ddbac$	$\begin{array}{cc} & \begin{array}{ccc} dbac & ddac & ddbc \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ddbac})) = 3$ $rk(F(e_{1ddbac})) = 6$
12)	$Q[28] = 1ddcab$	$\begin{array}{cc} & \begin{array}{cccc} dcab & dcbd & ddab & ddbc \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} & -1 & & -1 \\ & & 1 & \\ 1 & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1ddcab})) = 3$ $rk(F(e_{1ddcab})) = 6$
13)	$Q[39] = 3abcad$	$\begin{array}{cc} & \begin{array}{ccc} cbad & (ab)(ad) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abcad})) = 2$ $rk(F(e_{3abcad})) = 6$
14)	$Q[41] = 3abcbd$	$\begin{array}{cc} & \begin{array}{ccc} cbbd & (ab)(bd) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abcbd})) = 3$ $rk(F(e_{3abcbd})) = 8$
15)	$Q[42] = 3abccd$	$\begin{array}{cc} & \begin{array}{cc} ccbd & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abccd})) = 2$ $rk(F(e_{3abccd})) = 6$
16)	$Q[43] = 3abdac$	$\begin{array}{cc} & \begin{array}{cccc} dbac & dcab & (ab)(ac) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ d \end{array} & \begin{pmatrix} 1 & -1 & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abdac})) = 2$ $rk(F(e_{3abdac})) = 6$

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Table 8.4: ranks of $B4$ words (cont'd)

17)	$Q[44] = 3abdbc$	$\begin{array}{cc} & \begin{array}{cccc} dbbc & dcab & (ab)(bc) & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} 2 & & & \\ & -1 & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abdbc})) = 3$ $rk(F(e_{3abdbc})) = 8$
18)	$Q[45] = 3abdcd$	$\begin{array}{cc} & \begin{array}{ccc} dcbd & ddab & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abdcd})) = 3$ $rk(F(e_{3abdcd})) = 8$
19)	$Q[51] = 3acbad$	$\begin{array}{cc} & \begin{array}{ccc} cbad & (ac)(ad) & (ac)(bd) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acbad})) = 2$ $rk(F(e_{3acbad})) = 6$
20)	$Q[53] = 3acbbd$	$\begin{array}{cc} & \begin{array}{cc} cbbd & (ac)(bd) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acbbd})) = 2$ $rk(F(e_{3acbbd})) = 6$
21)	$Q[56] = 3accbd$	$\begin{array}{cc} & \begin{array}{ccc} ccbd & (ac)(bd) & (ac)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3accbd})) = 3$ $rk(F(e_{3accbd})) = 8$
22)	$Q[57] = 3acdab$	$\begin{array}{cc} & \begin{array}{cccc} dbac & dcab & (ab)(ac) & (ac)(bd) \end{array} \\ \begin{array}{c} a \\ d \end{array} & \begin{pmatrix} -1 & 1 & & -1 \\ & & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acdab})) = 2$ $rk(F(e_{3acdab})) = 6$

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Table 8.4: ranks of $B4$ words (cont'd)

23)	$Q[58] = 3acdbc$	$\begin{array}{cc} & \begin{array}{cccc} dcac & dcbc & (ac)(bc) & (ac)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} & 1 & & \\ -1 & & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acdbc})) = 3$ $rk(F(e_{3acdbc})) = 8$
24)	$Q[59] = 3acdbd$	$\begin{array}{cc} & \begin{array}{ccc} dcbd & ddac & (ac)(bd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acdbd})) = 3$ $rk(F(e_{3acdbd})) = 8$
25)	$Q[62] = 3adbac$	$\begin{array}{cc} & \begin{array}{ccc} dbac & (ac)(ad) & (ad)(bc) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & & 1 \\ & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adbac})) = 2$ $rk(F(e_{3adbac})) = 6$
26)	$Q[63] = 3adbbc$	$\begin{array}{cc} & \begin{array}{cc} dbbc & (ad)(bc) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adbbc})) = 2$ $rk(F(e_{3adbbc})) = 6$
27)	$Q[64] = 3adcab$	$\begin{array}{cc} & \begin{array}{cccc} cbad & dcab & (ab)(ad) & (ad)(bc) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} -1 & 1 & & -1 \\ & & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adcab})) = 2$ $rk(F(e_{3adcab})) = 6$
28)	$Q[65] = 3adcbc$	$\begin{array}{cc} & \begin{array}{ccc} ccad & dcbc & (ad)(bc) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adcbc})) = 3$ $rk(F(e_{3adcbc})) = 8$

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Table 8.4: ranks of $B4$ words (cont'd)

29)	$Q[66] = 3adcbd$	$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} dcbd & (ad)(bd) & (ad)(cd) \end{array} \\ \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adcbd})) = 3$ $rk(F(e_{3adcbd})) = 8$
30)	$Q[68] = 3addbc$	$\begin{array}{c} \\ a \\ b \\ d \end{array} \begin{array}{c} \begin{array}{cccc} dcad & ddbc & (ad)(bc) & (ad)(cd) \end{array} \\ \begin{pmatrix} & 2 & & \\ -1 & & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3addbc})) = 3$ $rk(F(e_{3addbc})) = 8$
31)	$Q[74] = 3bcaad$	$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} caad & (ad)(bc) \end{array} \\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcaad})) = 2$ $rk(F(e_{3bcaad})) = 6$
32)	$Q[77] = 3bcbad$	$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} cbad & (ad)(bc) & (bc)(bd) \end{array} \\ \begin{pmatrix} & & 1 \\ 1 & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcbad})) = 2$ $rk(F(e_{3bcbad})) = 6$
33)	$Q[80] = 3bccad$	$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} ccad & (ad)(bc) & (bc)(cd) \end{array} \\ \begin{pmatrix} & & 1 \\ 2 & & \\ & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bccad})) = 3$ $rk(F(e_{3bccad})) = 8$
34)	$Q[81] = 3bcdab$	$\begin{array}{c} \\ a \\ b \\ d \end{array} \begin{array}{c} \begin{array}{cccc} dbbc & dcab & (ab)(bc) & (bc)(bd) \end{array} \\ \begin{pmatrix} -2 & & & -1 \\ & 1 & & \\ & & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcdab})) = 3$ $rk(F(e_{3bcdab})) = 8$

Continued on next page

Table 8.4: ranks of $B4$ words (cont'd)

35) $Q[82] = 3bcdac$

$$\begin{array}{cc} & \begin{matrix} dcac & dcbc & (ac)(bc) & (bc)(cd) \end{matrix} \\ \begin{matrix} a \\ b \\ d \end{matrix} & \begin{pmatrix} & -1 & & -1 \\ 1 & & & \\ & & -1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bcdac})) = 3$

$rk(F(e_{3bcdac})) = 8$

36) $Q[83] = 3bcdad$

$$\begin{array}{cc} & \begin{matrix} dcad & ddbc & (ad)(bc) \end{matrix} \\ \begin{matrix} a \\ b \\ d \end{matrix} & \begin{pmatrix} & -1 & \\ 1 & & \\ & & -1 \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bcdad})) = 3$

$rk(F(e_{3bcdad})) = 8$

37) $Q[86] = 3bdaac$

$$\begin{array}{cc} & \begin{matrix} daac & (ac)(bd) \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bdaac})) = 2$

$rk(F(e_{3bdaac})) = 6$

38) $Q[87] = 3bdbac$

$$\begin{array}{cc} & \begin{matrix} dbac & (ac)(bd) & (bc)(bd) \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{pmatrix} & & -1 \\ 1 & -1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bdbac})) = 2$

$rk(F(e_{3bdbac})) = 6$

39) $Q[88] = 3bdcab$

$$\begin{array}{cc} & \begin{matrix} cbbd & dcab & (ab)(bd) & (bc)(bd) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} -2 & & & 1 \\ & 1 & & \\ & & -1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bdcab})) = 3$

$rk(F(e_{3bdcab})) = 8$

40) $Q[89] = 3bdcac$

$$\begin{array}{cc} & \begin{matrix} ccbd & dcac & (ac)(bd) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3bdcac})) = 3$

$rk(F(e_{3bdcac})) = 8$

Continued on next page

Table 8.4: ranks of $B4$ words (cont'd)

41)	$Q[90] = 3bdcad$	$\begin{array}{cc} & \begin{array}{ccc} dcad & (ad)(bd) & (bd)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & 1 \\ 1 & & \\ & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bdcad})) = 3$ $rk(F(e_{3bdcad})) = 8$
42)	$Q[92] = 3bddac$	$\begin{array}{cc} & \begin{array}{cccc} dcbd & ddac & (ac)(bd) & (bd)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} -1 & & & -1 \\ & 2 & & \\ & & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bddac})) = 3$ $rk(F(e_{3bddac})) = 8$
43)	$Q[96] = 3cdaab$	$\begin{array}{cc} & \begin{array}{cc} daab & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3cdaab})) = 2$ $rk(F(e_{3cdaab})) = 6$
44)	$Q[97] = 3cdbab$	$\begin{array}{cc} & \begin{array}{ccc} cbbd & dbab & (ab)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} -1 & & \\ & & -1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3cdbab})) = 3$ $rk(F(e_{3cdbab})) = 8$
45)	$Q[98] = 3cdbac$	$\begin{array}{cc} & \begin{array}{ccc} dbac & (ac)(cd) & (bc)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3cdbac})) = 3$ $rk(F(e_{3cdbac})) = 8$
46)	$Q[99] = 3cdbad$	$\begin{array}{cc} & \begin{array}{ccc} dbad & (ad)(cd) & (bd)(cd) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3cdbad})) = 3$ $rk(F(e_{3cdbad})) = 8$

Continued on next page

Table 8.4: ranks of $B4$ words (cont'd)

47) $Q[101] = 3cdcab$

$$\begin{array}{c} \\ a \\ c \end{array} \begin{array}{c} \begin{array}{cccc} ccbd & dcab & (ab)(cd) & (bc)(cd) \end{array} \\ \begin{pmatrix} -2 & & & 1 \\ & 1 & -1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3cdcab})) = 2$

$rk(F(e_{3cdcab})) = 6$

48) $Q[102] = 3cddab$

$$\begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} dcbd & ddab & (ab)(cd) & (bd)(cd) \end{array} \\ \begin{pmatrix} -1 & & & 1 \\ & 2 & & \\ & & -1 & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{3cddab})) = 3$

$rk(F(e_{3cddab})) = 8$

Table 8.5: Ranks of $B5$ words.

1) $Q[25] = 1dcbae$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} cbae & dbae & dcae & dcbe \end{array} \\ \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{1dcbae})) = 4$

$rk(F(e_{1dcbae})) = 8$

2) $Q[29] = 1ecbad$

$$\begin{array}{c} \\ a \\ b \\ c \\ e \end{array} \begin{array}{c} \begin{array}{cccc} cbad & ebad & ecad & ecbd \end{array} \\ \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \end{array}$$

$rk(F_{14}(e_{1ecbad})) = 4$

$rk(F(e_{1ecbad})) = 8$

Continued on next page

Table 8.5: ranks of $B5$ words (cont'd)

3)	$Q[30] = 1edbac$	$\begin{array}{cc} & \begin{matrix} dbac & ebac & edac & edbc \end{matrix} \\ \begin{matrix} a \\ b \\ d \\ e \end{matrix} & \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1edbac})) = 4$ $rk(F(e_{1edbac})) = 8$
4)	$Q[31] = 1edcab$	$\begin{array}{cc} & \begin{matrix} dcab & dcbe & ecab & ecbd & edab & edbc \end{matrix} \\ \begin{matrix} a \\ c \\ d \\ e \end{matrix} & \begin{pmatrix} & -1 & & -1 & & -1 \\ & & & & 1 & \\ & & 1 & & & \\ 1 & & & & & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{1edcab})) = 4$ $rk(F(e_{1edcab})) = 8$
5)	$Q[46] = 3abdce$	$\begin{array}{cc} & \begin{matrix} dcbe & (ab)(ce) & (ab)(de) \end{matrix} \\ \begin{matrix} a \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abdce})) = 3$ $rk(F(e_{3abdce})) = 8$
6)	$Q[47] = 3abecd$	$\begin{array}{cc} & \begin{matrix} ecbd & edab & (ab)(cd) & (ab)(de) \end{matrix} \\ \begin{matrix} a \\ c \\ e \end{matrix} & \begin{pmatrix} 1 & & & \\ & -1 & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3abecd})) = 3$ $rk(F(e_{3abecd})) = 8$
7)	$Q[60] = 3acdbe$	$\begin{array}{cc} & \begin{matrix} dcbe & (ac)(be) & (ac)(de) \end{matrix} \\ \begin{matrix} a \\ b \\ d \end{matrix} & \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acdbe})) = 3$ $rk(F(e_{3acdbe})) = 8$

Continued on next page

Table 8.5: ranks of $B5$ words (cont'd)

8)	$Q[61] = 3acebd$	$\begin{array}{cc} & \begin{matrix} ecbd & edac & (ac)(bd) & (ac)(de) \end{matrix} \\ \begin{matrix} a \\ b \\ e \end{matrix} & \begin{pmatrix} 1 & & & \\ & -1 & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3acebd})) = 3$ $rk(F(e_{3acebd})) = 8$
9)	$Q[67] = 3adcbe$	$\begin{array}{cc} & \begin{matrix} dcbe & (ad)(be) & (ad)(ce) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adcbe})) = 3$ $rk(F(e_{3adcbe})) = 8$
10)	$Q[69] = 3adebc$	$\begin{array}{cc} & \begin{matrix} ecad & edbc & (ad)(bc) & (ad)(ce) \end{matrix} \\ \begin{matrix} a \\ b \\ e \end{matrix} & \begin{pmatrix} & 1 & & \\ -1 & & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3adebc})) = 3$ $rk(F(e_{3adebc})) = 8$
11)	$Q[70] = 3aecbd$	$\begin{array}{cc} & \begin{matrix} ecbd & (ae)(bd) & (ae)(cd) \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3aecbd})) = 3$ $rk(F(e_{3aecbd})) = 8$
12)	$Q[71] = 3aedbc$	$\begin{array}{cc} & \begin{matrix} dcae & edbc & (ae)(bc) & (ae)(cd) \end{matrix} \\ \begin{matrix} a \\ b \\ d \end{matrix} & \begin{pmatrix} & 1 & & \\ -1 & & & -1 \\ & & 1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3aedbc})) = 3$ $rk(F(e_{3aedbc})) = 8$
13)	$Q[84] = 3bcdae$	$\begin{array}{cc} & \begin{matrix} dcae & (ae)(bc) & (bc)(de) \end{matrix} \\ \begin{matrix} a \\ b \\ d \end{matrix} & \begin{pmatrix} & & 1 \\ 1 & & \\ & -1 & \end{pmatrix} \end{array}$	$rk(F_{14}(e_{3bcdae})) = 3$ $rk(F(e_{3bcdae})) = 8$

Continued on next page

Table 8.5: ranks of $B5$ words (cont'd)

14)	$Q[85] = 3bcead$	$\begin{array}{c} \\ a \\ b \\ e \end{array} \begin{array}{c} \begin{array}{cccc} ecad & edbc & (ad)(bc) & (bc)(de) \end{array} \\ \left(\begin{array}{cccc} & -1 & & -1 \\ 1 & & & \\ & & -1 & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3bcead})) = 3$ $rk(F(e_{3bcead})) = 8$
15)	$Q[91] = 3bdcae$	$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} dcae & (ae)(bd) & (bd)(ce) \end{array} \\ \left(\begin{array}{ccc} & & 1 \\ 1 & & \\ & -1 & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3bdcae})) = 3$ $rk(F(e_{3bdcae})) = 8$
16)	$Q[93] = 3bdeac$	$\begin{array}{c} \\ a \\ b \\ e \end{array} \begin{array}{c} \begin{array}{cccc} ecbd & edac & (ac)(bd) & (bd)(ce) \end{array} \\ \left(\begin{array}{cccc} -1 & & & -1 \\ & 1 & & \\ & & -1 & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3bdeac})) = 3$ $rk(F(e_{3bdeac})) = 8$
17)	$Q[94] = 3becad$	$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} ecad & (ad)(be) & (be)(cd) \end{array} \\ \left(\begin{array}{ccc} & & 1 \\ 1 & & \\ & -1 & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3becad})) = 3$ $rk(F(e_{3becad})) = 8$
18)	$Q[95] = 3bedac$	$\begin{array}{c} \\ a \\ b \\ d \end{array} \begin{array}{c} \begin{array}{cccc} dcbe & edac & (ac)(be) & (be)(cd) \end{array} \\ \left(\begin{array}{cccc} -1 & & & -1 \\ & 1 & & \\ & & -1 & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3bedac})) = 3$ $rk(F(e_{3bedac})) = 8$
19)	$Q[100] = 3cdbae$	$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} dbae & (ae)(cd) & (be)(cd) \end{array} \\ \left(\begin{array}{ccc} & & -1 \\ & -1 & \\ 1 & & \end{array}\right) \end{array}$	$rk(F_{14}(e_{3cdbae})) = 3$ $rk(F(e_{3cdbae})) = 8$

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Table 8.5: ranks of $B5$ words (cont'd)

20) $Q[103] = 3cdeab$

$$\begin{array}{cc} & \begin{array}{cccc} ecbd & edab & (ab)(cd) & (be)(cd) \end{array} \\ \begin{array}{c} a \\ c \\ e \end{array} & \begin{pmatrix} -1 & & & 1 \\ & 1 & & \\ & & -1 & \end{pmatrix} \end{array}$$

$$rk(F_{14}(e_{3cdeab})) = 3 \qquad rk(F(e_{3cdeab})) = 8$$

21) $Q[104] = 3cebad$

$$\begin{array}{cc} & \begin{array}{ccc} ebad & (ad)(ce) & (bd)(ce) \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{array}$$

$$rk(F_{14}(e_{3cebad})) = 3 \qquad rk(F(e_{3cebad})) = 8$$

22) $Q[105] = 3cedab$

$$\begin{array}{cc} & \begin{array}{cccc} dcbe & edab & (ab)(ce) & (bd)(ce) \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} -1 & & & 1 \\ & 1 & & \\ & & -1 & \end{pmatrix} \end{array}$$

$$rk(F_{14}(e_{3cedab})) = 3 \qquad rk(F(e_{3cedab})) = 8$$

23) $Q[106] = 3debac$

$$\begin{array}{cc} & \begin{array}{ccc} ebac & (ac)(de) & (bc)(de) \end{array} \\ \begin{array}{c} a \\ b \\ d \end{array} & \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{array}$$

$$rk(F_{14}(e_{3debac})) = 3 \qquad rk(F(e_{3debac})) = 8$$

24) $Q[107] = 3decab$

$$\begin{array}{cc} & \begin{array}{cccc} dcbe & ecab & (ab)(de) & (bc)(de) \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} -1 & & & 1 \\ & & -1 & \\ & 1 & & \end{pmatrix} \end{array}$$

$$rk(F_{14}(e_{3decab})) = 3 \qquad rk(F(e_{3decab})) = 8$$

Appendix A

Python Codes for Nilpotency Class 4

A.1 Implementation of Basis Expansion Algorithm

The following Python code is created as another approach for finding the basis expansion for left-aligned words in Section 5.2. The list `L` consist of $w, x, y, z \in \{1, \ldots, n\}$ where $w < x < y < z$.

```
L=['w','x','y','z']
```

The list `W` represents all the 24 possible four-letter left-aligned word types as in Figure 5.1.

```
W=[]
for i1 in L:
    for i2 in L:
        for i3 in L:
            for i4 in L:
                if i2 >= i3 and i3<i4:
                    st ='t'
                    IL=[]
                    c1= L.index(i1)
                    c2 = L.index(i2)
                    c3 = L.index(i3)
```

```
                    c4 = L.index(i4)
                    IL = [c1,c2,c3,c4]
                    IL.sort()
                    for j in range(3):
                        if IL[j] < IL[j+1]-1:
                            st= 'f'
                    if IL[0] != 0:
                            st = 'f'
                    if st == 't':
                        i_t = i1 + i2 + i3 + i4
                        W = W + [i_t]

print("length of W =",len(W))
```

Output:

```
length of W = 24
```

The elements of the list W corresponds to the types of Figure 5.1 as follows:

$$
\begin{array}{cccccc}
W[0] = \gamma_2 & W[1] = \delta_2 & W[2] = \beta_2 & W[3] = \gamma_1 & W[4] = \alpha_2 & W[5] = \delta_1 \\
W[6] = \beta_1 & W[7] = \alpha_1 & W[8] = \gamma_4 & W[9] = \gamma_3 & W[10] = \delta_4 & W[11] = \beta_4 \\
W[12] = \alpha_4 & W[13] = \delta_3 & W[14] = \beta_3 & W[15] = \alpha_3 & W[16] = \gamma_5 & W[17] = \delta_5 \\
W[18] = \beta_6 & W[19] = \beta_5 & W[20] = \alpha_6 & W[21] = \alpha_5 & W[22] = \beta_7 & W[23] = \alpha_7
\end{array}
$$

The algorithm presented in Figure 5.2 is implemented to obtain the basis expansion for each element of the list `W`. The resulting list is denoted as `H`.

```
def inner(a,b,c,d):
    return([a,c,b,d],'-', [a,d,b,c])

def Convert(string):
    list1=[]
    list1[:0]=string
    return list1
```

```python

H=[]
for w in W:
    u1 = Convert(w)
    if u1[0] >= u1[1]:
        H=H+[u1[0] + u1[1]+u1[2] + u1[3] ]
    else:
        if u1[2] + u1[3] != u1[0] + u1[1] and u1[1] + u1[0] != u1[2] + u1[3] :
            u2 = u1[1] + u1[0] + u1[2] + u1[3] + '-' + '(' +u1[2] + u1[3]+')' + '(' +u1[0] + u1[1] + ')'
        else:
            u2 = u1[1] + u1[0] + u1[2] + u1[3]
        k1 = u1[1] + u1[0] + u1[2] + u1[3]
        if k1[1] == k1[2]:
            if k1[3] < k1[0]:
                if  k1[1]  + k1[3] != k1[1] + k1[0] and  k1[3]  + k1[1] != k1[1] + k1[0]:
                    H = H + [ k1[0]+ k1[1]+ k1[1]+ k1[3] + '-' + '(' + k1[1]  + k1[3] +')' + '('+ k1[1] + k1[0]+')']
                else:
                    H = H + [k1[0] + k1[1] + k1[1] + k1[3] ]
            else:
                if k1[1] + k1[3] != k1[1] + k1[0] and k1[3] + k1[1] != k1[1] + k1[0]:
                    H = H +[k1[0]+ k1[1]+ k1[1]+ k1[3]+ '+'+ '(' + k1[1] + k1[0] + ')' + '(' + k1[1] +  k1[3] + ')']
                else:
                    H = H + [k1[0] + k1[1] + k1[1] + k1[3] ]
        elif k1[1] > k1[2]:
            H= H +[u2]
        elif k1[1] < k1[2]:
            u3 = inner(k1[0], k1[1], k1[2], k1[3])
```

```
                if k1[0] >= k1[3]:
                    if k1[1] + k1[0] != k1[2] + k1[3] and k1[0] + k1
    [1] != k1[2] + k1[3]:
                        H= H + [k1[0]+ k1[2] + k1[1] + k1[3] + '-'+ k1
    [0] + k1[3] + k1[1] + k1[2] +'+' +'(' + k1[1] + k1[0] +')' + '('
    + k1[2] + k1[3] +')']
                    else:
                        H = H + [ k1[0] + k1[2] + k1[1] + k1[3] + '-' +
     k1[0] + k1[3] + k1[1] + k1[2] ]
                else:
                    u5 = u3[2]
                    u4 = u5[1] + u5[0] + u5[2] + u5[3] + '-' +'(' + u5
    [2] + u5[3] +')' +'(' +u5[0] + u5[1] +')'
                    is1 = " "
                    is2 = " "
                    if u5[2] + u5[3] != u5[0] + u5[1] and  u5[2] + u5
    [3] != u5[1] + u5[0]:
                        is1 =  '(' + u5[2] + u5[3] +')' +'(' +u5[0] +
    u5[1] +')'
                    if k1[1] + k1[0] != k1[2] + k1[3]  and k1[1] + k1
    [0] != k1[3] + k1[2]:
                        is2 = '(' + k1[1] + k1[0] +')' + '(' + k1[2] +
    k1[3] +')'

                    H = H + [k1[0] + k1[2] + k1[1] + k1[3] + '-' + u5
    [1] + u5[0] + u5[2] + u5[3] + '+' + is1 + '+' + is2 ]

for i in range(24):
    print(f'W[{i}]:', W[i], "=", H[i])
```

Output:

```
W[0]: wwwx = wwwx
W[1]: wxwx = xwwx
```

```
W[2]: wxwy = xwwy+(wx)(wy)
W[3]: wxxy = xxwy-yxwx+(wx)(xy)+(wx)(xy)
W[4]: wywx = ywwx-(wx)(wy)
W[5]: wyxy = yxwy-yywx+(wy)(xy)
W[6]: wyxz = yxwz-zywx+(wx)(yz)+(wy)(xz)
W[7]: wzxy = zxwy-zywx+(wz)(xy)
W[8]: xwwx = xwwx
W[9]: xwwy = xwwy
W[10]: xxwx = xxwx
W[11]: xxwy = xxwy
W[12]: xywx = yxwx-(wx)(xy)
W[13]: xywy = yxwy-(wy)(xy)
W[14]: xywz = yxwz-(wz)(xy)
W[15]: xzwy = zxwy-(wy)(xz)
W[16]: ywwx = ywwx
W[17]: yxwx = yxwx
W[18]: yxwy = yxwy
W[19]: yxwz = yxwz
W[20]: yywx = yywx
W[21]: yzwx = zywx-(wx)(yz)
W[22]: zxwy = zxwy
W[23]: zywx = zywx
```

A.2 Hall Basis Occurrences

The list Q in the code below represents left-aligned Hall basis words.

```
Q = [W[20], W[23], W[11], W[19], W[18], W[22], W[0], W[9], W[8], W
    [16], W[10], W[17]]
```

```
Q1 = [20, 23, 11, 19, 18, 22, 0, 9,  8, 16, 10, 17]
print( "length of Q =", len(Q))
```

Output:

```
length of Q = 12
```

The following code finds the occurrences of each variable T_t in $F_{13}(T^{(4)})$, where t is a left-aligned Hall basis element. Note that if the basis element t appears in a basis expansion of a word $abcd$, then the variable T_t will be in row a column bcd.

```
for j in range(12):
    T= 0
    M=[]
    N=[]
    for k in range(24):
        if Q[j] in H[k]:
            T = T + 1
            M= M + [W[k]]
            N= N + [H[k]]
    print(f'W[{Q1[j]}]: {Q[j]} appears ', T, 'times in :', M, "=", N
    )

```

Output:

```
W[20]: yywx appears  2 times in basis expansion of:
 ['wyxy', 'yywx']
 = ['yxwy-yywx+(wy)(xy)', 'yywx']
W[23]: zywx appears  4 times in basis expansion of:
 ['wyxz', 'wzxy', 'yzwx', 'zywx']
 = ['yxwz-zywx+(wx)(yz)+(wy)(xz)', 'zxwy-zywx+(wz)(xy)',
   'zywx-(wx)(yz)', 'zywx']
W[11]: xxwy appears  2 times in basis expansion of:
```

```
 ['wxxy', 'xxwy']
 = ['xxwy-yxwx+(wx)(xy)+(wx)(xy)', 'xxwy']
W[19]: yxwz appears  3 times in basis expansion of:
 ['wyxz', 'xywz', 'yxwz']
 = ['yxwz-zywx+(wx)(yz)+(wy)(xz)', 'yxwz-(wz)(xy)', 'yxwz']
W[18]: yxwy appears  3 times in basis expansion of:
 ['wyxy', 'xywy', 'yxwy']
 = ['yxwy-yywx+(wy)(xy)', 'yxwy-(wy)(xy)', 'yxwy']
W[22]: zxwy appears  3 times in basis expansion of:
 ['wzxy', 'xzwy', 'zxwy']
 = ['zxwy-zywx+(wz)(xy)', 'zxwy-(wy)(xz)', 'zxwy']
W[0]: wwwx appears  1 times in basis expansion of:
 ['wwwx'] = ['wwwx']
W[9]: xwwy appears  2 times in basis expansion of:
 ['wxwy', 'xwwy']
 = ['xwwy+(wx)(wy)', 'xwwy']
W[8]: xwwx appears  2 times in basis expansion of:
 ['wxwx', 'xwwx']
 = ['xwwx', 'xwwx']
W[16]: ywwx appears  2 times in basis expansion of:
 ['wywx', 'ywwx']
 = ['ywwx-(wx)(wy)', 'ywwx']
W[10]: xxwx appears  1 times in basis expansion of:
 ['xxwx'] = ['xxwx']
W[17]: yxwx appears  3 times in basis expansion of:
 ['wxxy', 'xywx', 'yxwx']
 = ['xxwy-yxwx+(wx)(xy)+(wx)(xy)', 'yxwx-(wx)(xy)', 'yxwx']
```

Appendix B

Python Codes for Nilpotency Class 5

B.1 Hall Basis Types

The following Python code is created to find all types of basis elements in R_5. The list `L` consists of $a, b, c, d, e \in \{1, \dots, n\}$ where $a < b < c < d < e$.

```
L=['a','b','c','d','e']
```

The list `W1` contains all possible words of the form $1rstuv$, where $r \in R_1$ and $stuv \in R_4$ is left-aligned.

```
W1=[]
for i1 in L:
    for i2 in L:
        for i3 in L:
            for i4 in L:
                for i5 in L:
                    if i2 >= i3 >= i4  and i4<i5:
                        st ='t'
                        IL=[]
                        c1= L.index(i1)
                        c2 = L.index(i2)
                        c3 = L.index(i3)
```

```
                        c4 = L.index(i4)
                        c5 = L.index(i5)
                        IL = [c1,c2,c3,c4,c5]
                        IL.sort()
                        for j in range(4):
                            if IL[j] < IL[j+1]-1:
                                st= 'f'
                        if IL[0] != 0:
                                st = 'f'
                        if st == 't':
                            i_t ='1'+ i1 + i2 + i3 + i4 + i5
                            W1 = W1 + [i_t]
```

The list W2 contains all possible words of the form $2rstuv$, where $r \in R_1$ and $(st)(uv) \in R_4$ is balanced.

```
W2=[]
for i1 in L:
   for i2 in L:
      for i3 in L:
          for i4 in L:
              for i5 in L:
                 b1= bool(i2<i3)
                 b2= bool(i4<i5)
                 b3= bool(i2==i4)
                 b4= bool(i3<i5)
                 b5= bool(i2<i4)
                 if b1 and b2 and ((b3 and b4) or b5):
                    st ='t'
                    IL=[]
                    c1= L.index(i1)
                    c2 = L.index(i2)
                    c3 = L.index(i3)
                    c4 = L.index(i4)
```

```
                    c5 = L.index(i5)
                    IL = [c1,c2,c3,c4,c5]
                    IL.sort()
                    for j in range(4):
                        if IL[j] < IL[j+1]-1:
                            st= 'f'
                    if IL[0] != 0:
                            st = 'f'
                    if st == 't':
                        i_t ='2'+ i1 + i2 + i3 + i4 + i5
                        W2 = W2 + [i_t]
```

The list W3 contains all possible words of the form $3rstuv$, where $rs \in R_2$ and $tuv \in R_3$.

```
W3=[]
for i1 in L:
   for i2 in L:
      for i3 in L:
          for i4 in L:
              for i5 in L:
                 b1= bool(i1<i2)
                 b2= bool(i3>=i4)
                 b3= bool(i4<i5)
                 if b1 and b2 and b3:
                    st ='t'
                    IL=[]
                    c1= L.index(i1)
                    c2 = L.index(i2)
                    c3 = L.index(i3)
                    c4 = L.index(i4)
                    c5 = L.index(i5)
                    IL = [c1,c2,c3,c4,c5]
                    IL.sort()
```

```
                    for j in range(4):
                        if IL[j] < IL[j+1]-1:
                            st= 'f'
                    if IL[0] != 0:
                            st = 'f'
                    if st == 't':
                        i_t ='3'+ i1 + i2 + i3 + i4 + i5
                        W3 = W3 + [i_t]

W = W1 + W2 + W3
```

The list Q1 contains all Hall basis types of the form $1rstuv$.

```
Q1=[]
for i1 in L:
   for i2 in L:
      for i3 in L:
          for i4 in L:
              for i5 in L:
                 if i1 >= i2 >= i3 >= i4  and i4<i5:
                    st ='t'
                    IL=[]
                    c1= L.index(i1)
                    c2 = L.index(i2)
                    c3 = L.index(i3)
                    c4 = L.index(i4)
                    c5 = L.index(i5)
                    IL = [c1,c2,c3,c4,c5]
                    IL.sort()
                    for j in range(4):
                        if IL[j] < IL[j+1]-1:
                            st= 'f'
                    if IL[0] != 0:
                            st = 'f'
```

```
                    if st == 't':
                        i_t ='1'+ i1 + i2 + i3 + i4 + i5
                        Q1 = Q1 + [i_t]
```

The list `Q` contains all Hall basis types in R_5.

```
Q= Q1 + W3
print('Length of Q =',len(Q))
```

Output:

```
Length of Q = 108
```

```
for i in range(len(Q)):
    print(f'Q[{i}] = {Q[i]}')
```

The output is presented in Listing 7.1.

B.2 Hall Basis Occurrences

The algorithms presented in Figure 7.1 and Figure 7.2 are implemented to obtain the basis expansion for each element of the list `W`. The resulting list is denoted as `H`.

```
H=[]
for u in W:
    if u[0] == '1':
        w0=u[1]
        w1=u[2]
        w2=u[3]
        w3=u[4]
        w4=u[5]
        if w0 >= w1:
            H = H + ['1' + w0 + w1 + w2 + w3 + w4]
        else:
            if w0 >= w2:
                H = H + [
```

```
                '1'+ w1 + w0 + w2 + w3 + w4 + '+' + '3' + w0 + w1
 + w2 + w3 + w4 ]
            else:
                if w0 >= w3:
                    H = H + [
                        '1'+ w1 + w2 + w0 + w3 + w4  + '-' + '3' + w3
 + w4  + w1 + w0 + w2 + '+' + '3' + w0 + w2  + w1 + w3 + w4 + '+'
 + '3' + w0 + w1  + w2 + w3 + w4]
                else:
                    if w2 >= w4:
                        H = H + ['1'+ w1 + w2 + w3 + w0 + w4 + '-' +
'1' + w1 + w2 + w4 + w0 + w3+ '-' + '3' + w3 + w4  + w1 + w0 + w2
                            + '+' + '3' + w0 + w2  + w1 + w3 + w4
                            + '+' + '3' + w0 + w1  + w2 + w3 + w4]
                    else:
                        if w1 >= w4:
                            H = H + [
                                '1'+ w1 + w2 + w3 + w0 + w4
                                + '-' +'1' + w1 + w4 + w2 + w0 + w3
                                + '+' + '3' + w0 + w3  + w1 + w2 +
w4
                                + '-' + '3' + w2 + w4 + w1 + w0 + w3
                                + '-' + '3' + w3 + w4  + w1 + w0 +
w2
                                + '+' + '3' + w0 + w2 + w1 + w3 + w4
                                + '+' + '3' + w0 + w1 + w2 + w3 + w4
]
                        else:
                            H = H + [
                                '1' + w1 + w2 + w3 + w0 + w4
                                + '-' + '1' + w4 + w1 + w2 + w0 + w3
                                + '-' + '3' + w1 + w4  + w2 + w0 +
w3
```

```
                             + '+' + '3' + w0 + w3  + w1 + w2 + w4
                             + '-' + '3' + w2 + w4  + w1 + w0 + w3
                             + '-' + '3' + w3 + w4  + w1 + w0 + w2
                             + '+' + '3' + w0 + w2  + w1 + w3 + w4
                             + '+' + '3' + w0 + w1  + w2 + w3 + w4]
 if u[0]=='2':
     if u[2] == u[4]:
         if u[1] >= u[2]:
             H = H + ['3' + u[2] + u[3]+ u[1]+u[2]+u[5] + '-' + '3' + u[2] + u[5]+ u[1]+u[2]+u[3]]
         else:
             H = H + ['3' + u[2] + u[3] + u[2] + u[1] + u[5] + '-' + '3' + u[2] + u[3] + u[5] + u[1] + u[2]
                     + '-' + '3' + u[2] + u[5] + u[2] + u[1] + u[3] + '+' + '3' + u[2] + u[5] + u[3] + u[1] + u[2] ]
     if u[2] < u[4]:
         if u[1] >= u[4]:
             H = H + ['3' + u[2] + u[3] + u[1] + u[4] + u[5] + '-' + '3' + u[4] + u[5] + u[1] + u[2] + u[3]]
         else:
             if u[1] >= u[2]:
                 H = H + ['3' + u[2] + u[3] + u[4] + u[1] + u[5] + '-' + '3' + u[2] + u[3] + u[5] + u[1] + u[4]
                          +'-' + '3' + u[4] + u[5] + u[1] + u[2] + u[3]]
             else:
                 H = H + ['3' + u[2] + u[3] + u[4] + u[1] + u[5] + '-' + '3' + u[2] + u[3] + u[5] + u[1] + u[4]
```

```
                                    + '-' + '3' + u[4] + u[5] + u[2] + u[1] + u[3]
                                    + '+' + '3' + u[4] + u[5] + u[3] + u[1] + u[2]]

    if u[0]== '3':
        H = H+[u[0]+u[1]+u[2]+u[3]+u[4]+u[5]]
```

The lists `B2`, `B3`, `B4` and `B5` contain elements of `Q` with 2,3, 4 and 5 letters, respectively.

```
B2=[]
B3=[]
B4=[]
B5=[]
for m in Q:
    t0 = L.index(m[1])
    t1 = L.index(m[2])
    t2 = L.index(m[3])
    t3 = L.index(m[4])
    t4 = L.index(m[5])
    e = [t0, t1, t2, t3, t4]
    e = list(dict.fromkeys(e))
    if len(e) == 2:
        B2= B2 + [m]
    if len(e) == 3 :
        B3 = B3 + [m]
    if len(e) == 4 :
        B4 = B4 + [m]
    if len(e) == 5 :
        B5 = B5 + [m]

T2=[]
S2=[]
for i in B2:
```

```
    if i[0] == '1' :
        T2 = T2 + [i[0]]
    if i[0] == '3' :
        S2 = S2 + [i[0]]

print('Length of B2 =', len(B2))
print(f'Left aligned words in B2: {len(T2)}')
print(f'Balanced words in B2 :{len(S2)}')
```

Output:

```
Length of B2 = 6
Left aligned words in B2: 4
Balanced words in B2 :2
```

```
for b in B2:
    j = Q.index(b)
    X=0
    M=[]
    N=[]
    for k in range(len(H)):
        if b in H[k]:
            X1 = [i for i in range(len(H[k])) if H[k].startswith(b,
i)]
            X2=[]
            if X1[0] == 0:
                X3 = [H[k][X1[j]-1] for j in range(1,len(X1))]
                X2 = X2 + ['+'] + X3
            else:
                X2 = [H[k][X1[j]-1] for j in range(len(X1))]

            PX2 =0
            NX2 =0
            for i in range(len(X2)):
```

```
                if X2[i] == '+':
                    PX2 = PX2 +1
                elif X2[i] == '-':
                    NX2 = NX2 + 1

            if PX2 != NX2:
                X = X + 1
                M = M + [W[k]]
                N = N + [H[k]]
            else:
                X=X
                M=M
                N=N
    print("")
    print(f'{B2.index(b)+1}) {b}:Q[{j}] appears ',X, f'times in:')
    for i in range(X):
        print(f'{i+1}. {M[i]} = {N[i]}')
```

The output is presented in Listing 7.2.

```
T3=[]
S3=[]
for i in B3:
    if i[0] == '1' :
        T3 = T3 + [i[0]]
    if i[0] == '3' :
        S3 = S3 + [i[0]]

print('Length of B3 =', len(B3))
print(f'Left aligned words in B3: {len(T3)}')
print(f'Balanced words in B3 :{len(S3)}')
```

Output:

```
Length of B3 = 30
```

```
Left aligned words in B3: 12

Balanced words in B3 :18
```

```
for b in B3:
    j=Q.index(b)
    X=0
    M=[]
    N=[]
    for k in range(len(H)):
        if b in H[k]:
            X1 = [i for i in range(len(H[k])) if H[k].startswith(b,
  i)]
            X2 = []
            if X1[0] == 0:
                X3 = [H[k][X1[j] - 1] for j in range(1, len(X1))]
                X2 = X2 + ['+'] + X3
            else:
                X2 = [H[k][X1[j] - 1] for j in range(len(X1))]

            PX2 = 0
            NX2 = 0
            for i in range(len(X2)):
                if X2[i] == '+':
                    PX2 = PX2 + 1
                elif X2[i] == '-':
                    NX2 = NX2 + 1

            if PX2 != NX2:
                X = X + 1
                M = M + [W[k]]
                N = N + [H[k]]
            else:
                X = X
```

```
                M = M
                N = N
    print("")
    print(f'{B3.index(b)+1}) {b}:Q[{j}]  appears ',X, f'times in:')
    for i in range(X):
        print(f'{i+1}. {M[i]} = {N[i]}')
```

The output is presented in Listing 7.3.

```
T4=[]
S4=[]
for i in B4:
    if i[0] == '1' :
        T4 = T4 + [i[0]]
    if i[0] == '3' :
        S4 = S4 + [i[0]]

print('Length of B4 =', len(B4))
print(f'Left aligned words in B4: {len(T4)}')
print(f'Balanced words in B4 :{len(S4)}')
```

Output:

```
Length of B4 = 48
Left aligned words in B4: 12
Balanced words in B4 :36
```

```
for b in B4:
    j = Q.index(b)
    X=0
    M=[]
    N=[]
    for k in range(len(H)):
        if b in H[k]:
            X1 = [i for i in range(len(H[k])) if H[k].startswith(b,
   i)]
```

```
            X2 = []
            if X1[0] == 0:
                X3 = [H[k][X1[j] - 1] for j in range(1, len(X1))]
                X2 = X2 + ['+'] + X3
            else:
                X2 = [H[k][X1[j] - 1] for j in range(len(X1))]

            PX2 = 0
            NX2 = 0
            for i in range(len(X2)):
                if X2[i] == '+':
                    PX2 = PX2 + 1
                elif X2[i] == '-':
                    NX2 = NX2 + 1

            if PX2 != NX2:
                X = X + 1
                M = M + [W[k]]
                N = N + [H[k]]
            else:
                X = X
                M = M
                N = N
    print("")
    print(f'{B4.index(b)+1}) {b}:Q[{j}]  appears ',X, f'times in:')
    for i in range(X):
        print(f'{i+1}. {M[i]} = {N[i]}')
```

The output is presented in Listing 7.4.

```
T5=[]
S5=[]
for i in B5:
    if i[0] == '1' :
```

```
        T5 = T5 + [i[0]]
    if i[0] == '3' :
        S5 = S5 + [i[0]]

print('Length of B5 =', len(B5))
print(f'Left aligned words in B5: {len(T5)}')
print(f'Balanced words in B5 :{len(S5)}')
```

Output:

```
Length of B5 = 24
Left aligned words in B5: 4
Balanced words in B5 :20
```

```
for b in B5:
    j = Q.index(b)
    X=0
    M=[]
    N=[]
    for k in range(len(H)):
        if b in H[k]:
            X1 = [i for i in range(len(H[k])) if H[k].startswith(b,
i)]
            X2 = []
            if X1[0] == 0:
                X3 = [H[k][X1[j] - 1] for j in range(1, len(X1))]
                X2 = X2 + ['+'] + X3
            else:
                X2 = [H[k][X1[j] - 1] for j in range(len(X1))]

            PX2 = 0
            NX2 = 0
            for i in range(len(X2)):
                if X2[i] == '+':
```

```
                    PX2 = PX2 + 1
                elif X2[i] == '-':
                    NX2 = NX2 + 1

            if PX2 != NX2:
                X = X + 1
                M = M + [W[k]]
                N = N + [H[k]]
            else:
                X = X
                M = M
                N = N
    print("")
    print(f'{B5.index(b)+1}) {b}:Q[{j}]  appears ',X, f'times in:')
    for i in range(X):
        print(f'{i+1}. {M[i]} = {N[i]}')
```

The output is presented in Listing 7.5.

Bibliography

[BMKRS23] M. Bardestani, K. Mallahi-Karai, D. Rumiantsau, and H. Salmasian. Polynomiality of the faithful dimension for nilpotent groups over finite truncated valuation rings. *Trans. Amer. Math. Soc.*, 376(12):8795–8823, 2023.

[BMKS16] M. Bardestani, K. Mallahi-Karai, and H. Salmasian. Minimal dimension of faithful representations for p-groups. *J. Group Theory*, 19(4):589–608, 2016.

[BMKS19] M. Bardestani, K. Mallahi-Karai, and H. Salmasian. Kirillov's orbit method and polynomiality of the faithful dimension of p-groups. *Compos. Math.*, 155(8):1618–1654, 2019.

[BS08] M. Boyarchenko and M. Sabitova. The orbit method for profinite groups and a p-adic analogue of Brown's theorem. *Israel J. Math.*, 165:67–91, 2008.

[DF04] D. S. Dummit and R. M. Foote. *Abstract algebra.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[GS84] F. Grunewald and D. Segal. Reflections on the classification of torsion-free nilpotent groups. In *Group theory*, pages 121–158. Academic Press, London, 1984.

[How77] R. E. Howe. Kirillov theory for compact p-adic groups. *Pacific J. Math.*, 73(2):365–381, 1977.

[IR90] K. Ireland and M. Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[Kaz77] D. Kazhdan. Proof of Springer's hypothesis. *Israel J. Math.*, 28(4):272–286, 1977.

[Khu98] E. I. Khukhro. *p-automorphisms of finite p-groups*, volume 246 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[Kir62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. *Uspehi Mat. Nauk*, 17(4(106)):57–110, 1962.

[KM08] N. A. Karpenko and A. S. Merkurjev. Essential dimension of finite p-groups. *Invent. Math.*, 172(3):491–508, 2008.

[KV06] B. Korte and J. Vygen. *Combinatorial optimization*, volume 21 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, third edition, 2006. Theory and algorithms.

[McL73] K. R. McLean. Commutative artinian principal ideal rings. *Proc. London Math. Soc. (3)*, 26:249–272, 1973.

[OV15] E. A. O'Brien and C. Voll. Enumerating classes and characters of p-groups. *Trans. Amer. Math. Soc.*, 367(11):7775–7796, 2015.

[Rei21] Z. Reichstein. From Hilbert's 13th problem to essential dimension and back. *Eur. Math. Soc. Mag.*, (122):4–15, 2021.

[Ser79] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser92] J.-P. Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 1992. 1964 lectures given at Harvard University.

[SV14] A. Stasinski and C. Voll. Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B. *Amer. J. Math.*, 136(2):501–550, 2014.

[Tie18] E. Tielker. Topics in the representation theory of finite p-groups. Master's thesis, Bielefeld University, 2018.