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A probabilistic approach to detecting quantum entanglement

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Abstract

A key concept in quantum information theory is the notion of separable states; states for which there is no quantum entanglement. Determining if a given state is entangled has been shown to be NP-hard, and so no deterministic and efficient algorithm will likely be created for this purpose. Current tests are either computationally intensive , return false positives or worsen as the dimension of the problem grows larger.

These large dimensions arise naturally in quantum computing. It is therefore necessary to develop a reliable test that is naturally scalable to higher dimensions if one wishes to create efficient quantum computers. Using random matrix theory, a new algorithm has been developed that, with high probability, can determine if a quantum state is separable or entangled. The purpose of this research is to implement this algorithm, checking its efficiency through numerical tests, as well as testing the limitations of the theory behind it.

Mathematical preliminary

In quantum mechanics, quantum states with n degrees of freedom are represented as $n \times n$ density matrices; positive-semi definite matrices with unit trace. A quantum state, represented as a density matrix ρ, is said to be separable if it can be written as the convex combination of product states:

$$\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k$$

where $\sum_k p_k = 1$, ρ_1^i and ρ_2^i are density matrices of rank one. States that can't be decomposed in such a way are said to be entangled. The main idea behind detecting separable and entangled states arises through positive maps. A linear map $\Phi\colon \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is said to be positive if, for all positive-semi definite matrices A in $\mathbb{C}^{n\times n}$, $\Phi(A)$ in $\mathbb{C}^{m\times m}$ is also positive-semi definite.

It is rather surprising that the tensor product of two positive maps isn't necessarily positive. Consider the image of the positive matrix ϱ under the partial transposition map $\mathrm{Id}_2 \otimes \mathrm{T}_2$

$$(\mathrm{Id}_2 \otimes \mathrm{T}_2)\varrho = (\mathrm{Id}_2 \otimes \mathrm{T}_2)\begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Because it has a negative eigenvalue, it isn't a positive-semi definite matrix. Now consider the image of this map under any separable state ρ in $\mathbb{C}^{4\times 4}$

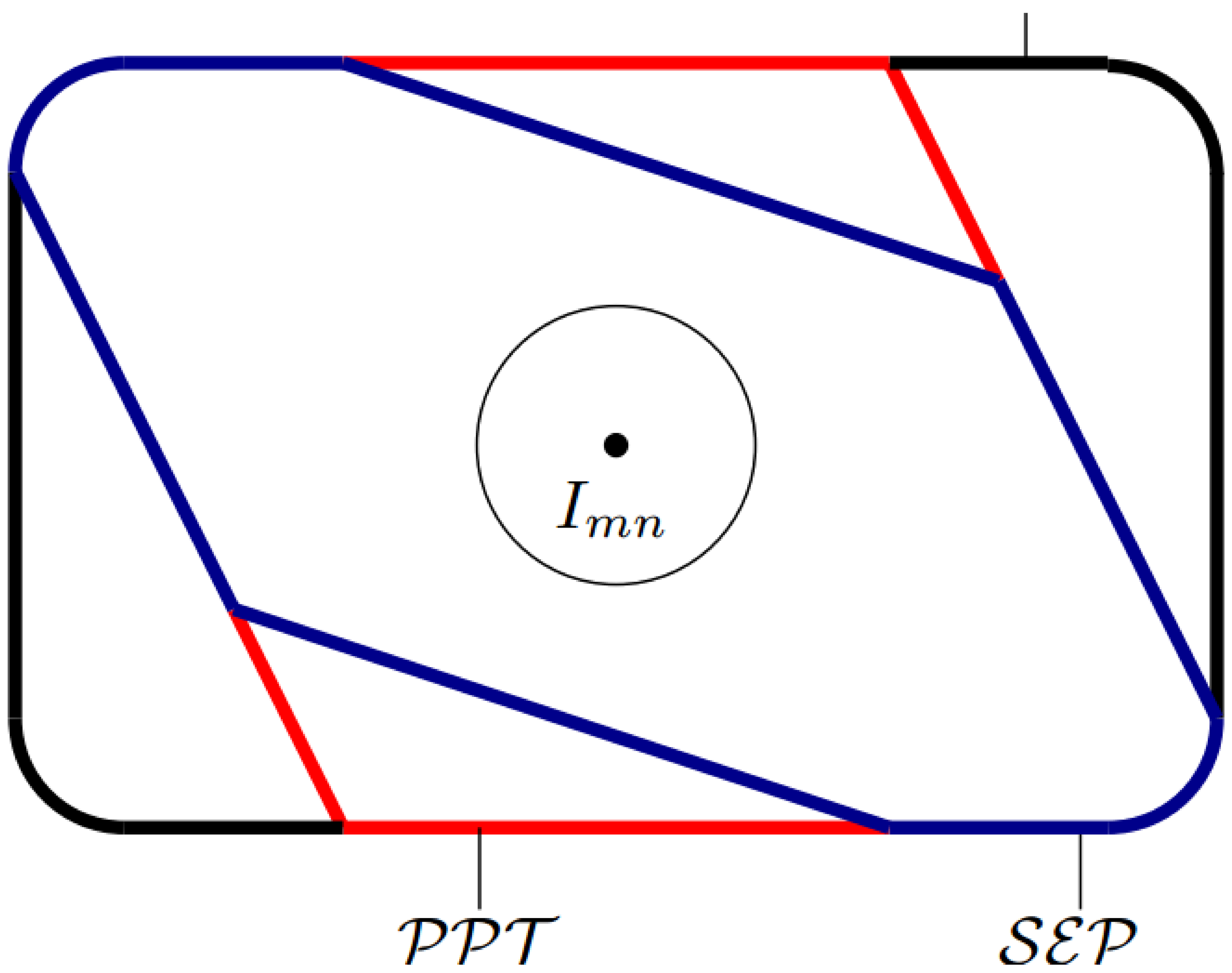
$$(\mathrm{Id}_2 \otimes \mathrm{T}_2)\rho = (\mathrm{Id}_2 \otimes \mathrm{T}_2)\sum_k p_k \rho_1^k \otimes \rho_2^k = \sum_k p_k \rho_1^k \otimes \mathrm{T}_2(\rho_2^k)$$

Because T_2 is a positive map, the resulting matrix will always be positive-semi definite. Therefore, the matrix ϱ is entangled because the image under $\mathrm{Id}_2 \otimes \mathrm{T}_2$ is not positive. A matrix whose image under the partial transposition is still positive is said to be PPT. This gives rise to a necessary , but not sufficient, criteria for separability. Maps such as the partial transposition are called entanglement witnesses.

Creating positive maps

Although the partial transpose is an example for which the tensor product fails to be positive, there are positive maps Φ for which $\mathrm{Id}_k \otimes \Phi$ is positive for all $k \in \mathbb{N}$. These maps, called completely positive maps, are useless as entanglement witnesses. It is therefor necessary to create positive but not completely positive maps, the starting point being Choi's theorem.

Choi's theorem on completely positive maps gives a necessary and sufficient condition for a map Φ to be completely positive. Every map $\Phi\colon \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is associated with a matrix $C_\Phi \in \mathbb{C}^{nm\times nm}$, and if C_Φ is positive, then Φ is completely positive. Furthermore, the map $\Phi \to C_\Phi$ sends Hermitian preserving maps to Hermitian matrices. Therefore, the first step towards creating positive maps is using a Hermitian Choi matrix which, by the spectral theorem, has an eigendecomposition $C_\Phi = UDU^*$ with U unitary.



A simplified geometric representation of PPT and separable states.

If one wishes to systematically generate positive maps with this method, it is most useful to take a probabilistic approach. There are many ways to create a random positive map, and many different approaches were tested throughout the research.

Part of purpose of this research is to find a better test for entanglement than the PPT criterion. Thus, it is necessary to test the randomly created positive maps on entangled PPT states. The definition of separable states allow a simple geometric argument when deciding which states to test. Simply put, the entangled states that will most likely be detected by the entanglement witnesses are those just on the edge of PPT states.

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Results

The algorithm to create and test positive maps was implemented in the computational software program *Mathematica* which was chosen for its ease of use. There are many ways of creating a random positive map, and many were tested throughout the project.

However, no example of an entanglement witness was found. Either the matrix had no negative eigenvalues, or, when it did, the compression test also returned negative eigenvalues.

Conclusion

The test introduced was presumed to be detect some PPT states that are non-separable, however the attempts we conducted during this project showed that this task is more arduous than expected. We made substantial progress in understanding states that are PPT but not separable, we don't know a general description of such states, and a more systematic investigation should be undertaken .

There is also a possibility that our family of tests do not detect PPT non separable states. This result, while disappointing from a quantum information theoretic point of view, would have a deep and unexpected meaning in free probability theory. Either way, our investigations confirm that elucidating the problem is an arduous task, and whatever the solution to this problem, it would shed important light on the relation between free probability and quantum information theory.

References

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