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LOCALLY NILPOTENT DERIVATIONS AND THEIR RINGS OF CONSTANTS

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Abstract

Given a UFD R containing $\mathbb{Q}$, we study R-elementary derivations of $B = R[Y_1, \dots, Y_m]$, i.e., R-derivations satisfying $D(Y_i) \in R$ for all i; in the particular case of $m = 3$, we will show that if R is a polynomial ring in n variables over a field k (of characteristic zero), and $a_1, a_3, a_3 \in R$ are three monomials, then the kernel of the derivation $\sum_{i=1}^{3} a_i\partial/\partial Y_i$ of B is generated over R by at most three linear elements in the Y_i's. This gives a partial answer to a question of A. van den Essen ([27]) about the existence of elementary derivations in dimension six whose kernels are not finitely generated. A set of generators is given for the kernel of R-elementary fixed point free derivations of B. Also, some interesting examples of elementary derivations in dimensions six and seven are provided as well as a criterion for a derivation of $R^{[2]}$ (i.e., a polynomial ring in two variables over R) to be R-elementary.

Given a field k of characteristic zero, it is well-known that the kernel of any linear derivation of $k[X_1, \dots, X_n]$ (that is, a k-derivation which maps each X_i to a linear form in $X_1, \dots, X_n$) is a finitely generated k-algebra (see [28]). All known proofs of this result are non-constructive in the sense that we do not know a generating set for the kernel. Nowicki conjectured in [25] that the kernel of the derivation $d = \sum_{i=1}^{n} X_i\partial/\partial Y_i$ of $k[X_1, \dots, X_n, Y_1, \dots, Y_n]$ is generated over k by the elements $u_{ij} = X_iY_j - X_jY_i$ for $1 \leq i < j \leq n$. Using the theory of Groebner bases, we prove this conjecture in the more general case of the derivation $D = \sum_{i=1}^{n} X_i^{t_i}\partial/\partial Y_i$ where each t_i is a nonnegative integer. Note that in the case of the derivation D, the finite generation of the kernel is no longer evident. Note also that the generators of $\ker D$ are linear in the Y_i's over $k[X_1, \dots, X_n]$; we will show that this is not always the

case for elementary derivations by giving an example of an elementary derivation in dimension seven whose kernel is finitely generated but cannot be generated by linear forms.

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I would like to express my sincere thanks to my supervisor, Professor Daniel Daigle for introducing me to this field, for his guidance, and for his never ending patience and valuable suggestions during the course of the development of this thesis.

I also would like to thank the department of mathematics and statistics at the University of Ottawa for providing me with financial support during the years of my graduate studies.

Dedication

I dedicate this work to the one whose love and support made my life much easier, to my wife Antoinette.

Introduction

Throughout this thesis, all rings are commutative and have an identity element. If A is a ring, the notation $B = A^{[n]}$ means that B is A-isomorphic to a polynomial ring in n variables over A. The symbol k denotes a field of characteristic zero.

One of the problems related to the study of $\mathbf{A}_k^n$, the affine space of dimension n over k, is the description of the algebraic structure of its group $\mathrm{GA}_n(k)$ of automorphisms. This problem, often referred to as the *Automorphism Problem*, is equivalent to studying the k-algebra automorphisms of $k^{[n]}$.

Very little is known about the structure of $\mathrm{GA}_n(k)$ for $n > 2$.

One way to approach the Automorphism Problem is to study how various algebraic groups act on $\mathbf{A}_k^n$. In particular, consider the algebraic actions of $\mathbf{G}_a$ on $\mathbf{A}_k^n$, where $\mathbf{G}_a$ is the additive group of k. Each such action $f : \mathbf{G}_a \times \mathbf{A}_k^n \to \mathbf{A}_k^n$ determines a subgroup $\{f_t \mid t \in k\}$ of $\mathrm{GA}_n(k)$ isomorphic to $(k, +)$. It is a well-known fact (see Chapter 1) that algebraic $\mathbf{G}_a$-actions on $\mathbf{A}_k^n$ are equivalent to locally nilpotent derivations of $k^{[n]}$ (see definition 1.1.1 below). In other words, locally nilpotent derivations are the algebraic interpretation of $\mathbf{G}_a$-actions.

Given a locally nilpotent derivation $D : B \to B$ of an integral domain B containing k, an interesting object of study is the k-algebra $\ker D = \{x \in B \mid Dx = 0\}$, called the *kernel* or the *ring of constants* of D. In fact it is known (see [3]) that the study of locally nilpotent derivations essentially reduces to that of their kernels.

An important question to ask, when studying a derivation of $k^{[n]}$, is whether or not its kernel is finitely generated as a k-algebra. This question was answered positively by Nagata and Nowicki ([23]) in the case $n < 4$. In higher dimension, many examples of locally nilpotent derivations having non finitely generated kernels have been found, and each of these examples gives (by letting $L = \text{qt}(\ker D)$) a counterexample to the Fourteenth's Problem of Hilbert, which can be stated as follows:

> If L is a subfield of $k(X_1, \dots, X_n)$ containing k, is $L \cap k[X_1, \dots, X_n]$ a finitely generated k-algebra?

The first counterexample to Hilbert's problem was given by Nagata ([13]) in 1958, with $n = 32$. It is now known that, for each $n > 4$, there exists a locally nilpotent derivation of $k^{[n]}$ whose kernel is not finitely generated. This leaves Hilbert's problem open only in dimension four. For more details, refer to section 1.4.

Of particular importance for this thesis is a counterexample found by Roberts in 1990 ([12]) in dimension seven, and which was used by Deveney and Finston ([11]) to show that, for any $t \geq 2$, the kernel of the derivation

$$D = X_1^{t+1}\frac{\partial}{\partial Y_1} + X_2^{t+1}\frac{\partial}{\partial Y_2} + X_3^{t+1}\frac{\partial}{\partial Y_3} + (X_1X_2X_3)^t\frac{\partial}{\partial Y_4} \qquad (1)$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ is not finitely generated as a k-algebra. This Robert-Deveney-Finston example suggests that we study a special type of derivation of polynomial rings, called *elementary derivations* (see below). In fact, most of this thesis is devoted to the study of elementary derivations and of their kernels.

Let D be a k-derivation of the polynomial ring $k[Z_1, \dots, Z_N]$ ($N \geq 2$). We say that D is *elementary* if for some partition

$$\{Z_1, \dots, Z_N\} = \{X_1, \dots, X_n\} \cup \{Y_1, \dots, Y_m\}$$

D is of the form

$$D = a_1(X_1, \dots, X_n)\frac{\partial}{\partial Y_1} + \cdots + a_m(X_1, \dots, X_n)\frac{\partial}{\partial Y_m}$$

where each a_i is in $k[X_1, \dots, X_n]$. Elementary derivations are easily seen to be locally nilpotent. We say that D is a *monomial* derivation if it is of the form

$$D = M_1\frac{\partial}{\partial Z_1} + \cdots + M_N\frac{\partial}{\partial Z_N}$$

where each M_i is a monomial in $Z_1, \dots, Z_N$.

Note that (1) is an elementary monomial derivation of $k^{[7]}$ (i.e., a derivation which is both elementary and monomial). We will show in Chapter 2 that every elementary monomial derivation of $k^{[6]}$ has finitely generated kernel. Moreover, a corollary of the main result in section 2.3 states that the kernel of any elementary monomial derivation of $k[X_1, \ldots, X_n, Y_1, Y_2, Y_3]$ is generated over $k[X_1, \ldots, X_n]$ by at most three linear elements in the Y_i's (this holds for any n).

Elementary derivations of polynomial rings were studied in [13], where it was shown (in particular) that the kernel of any elementary derivation of the polynomial ring $B = k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ is a finitely generated k-algebra whenever $n+m \leq 5$. It was also shown in [27] that if $n \geq 3$ and $m \geq 4$, then the kernel of any derivation of type (1) is not a finitely generated k-algebra. There remains the case where $m = 3$ and $n \geq 3$, about which little is known (see Question 4.3 in [27]). In Chapter 2, we will prove some results for the case $m = 3$ in the more general case where the subalgebra $k[X_1, \ldots, X_n]$ of B is replaced by a UFD.

Once we know that the kernel of a given derivation $D : B \to B$ is finitely generated over k, one may ask for a set of generators of the kernel. Indeed, one of the standard techniques for investigating a $\mathbf{G}_a$-action is to study the scheme $\operatorname{Spec} A$ and the morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$, where $A = \ker D$. If we know a set $\{f_1, \ldots, f_r\}$ such that $A = k[f_1, \ldots, f_r]$, then we readily have a closed immersion $\operatorname{Spec} A \to \mathbf{A}_k^r$, which is very helpful for investigating these geometric objects.

The problem of finding a set of generators for the kernel seems to be hard even for some derivations that look (at least at the first glance) easy to deal with. For instance, a well-known result of R. Weitzenbock ([30]) states that any locally nilpotent *linear* derivation, i.e., a locally nilpotent derivation of the form

$$\sum_{i=1}^{n}(a_{i1}X_1 + \cdots + a_{in}X_n)\frac{\partial}{\partial X_i} \;:\; k[X_1, \ldots, X_n] \longrightarrow k[X_1, \ldots, X_n]$$

($a_{ij} \in k$ for all i, j), has a finitely generated kernel. In his proof (as well as in all the others that appeared later), Weitzenbock gave no set of generators for the kernel. It is interesting to note that even for the simplest linear derivation

$$d_n = \sum_{i=1}^{n} X_i \frac{\partial}{\partial Y_i} \tag{2}$$

of $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, the problem of finding a set of generators for the kernel doesn't seem to be easy. In fact, for the derivation (2), Nowicki ([25]) found explicitly a set of generators for the kernel in the case $n \leq 4$, and gave a finite set of elements of $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ that he conjectured to be a set of generators for $\ker d_n$ for arbitrary n (see Conjecture 6.9.10 in [25]). The same set was claimed by H. Kojima and M. Miyanishi ([20]) to be a set of generators for $\ker d_n$; but, as we will show in Chapter 3, their proof fails at one point. In Chapter 3, we will prove Nowicki's conjecture in the more general case of the derivation

$$\delta_n = \sum_{i=1}^{n} X_i^{t_i} \frac{\partial}{\partial Y_i}$$

of $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ where $t_1, \ldots, t_n$ are nonnegative integers. Our main tool for this will be the theory of Groebner bases.

Another interesting linear derivation is Δ_n defined on $k[X_1, \ldots, X_n]$ by $\Delta_n(X_1) = 0$ and $\Delta_n(X_i) = X_{i-1}$ for $i \geq 2$. When $n \leq 5$, sets of generators for $\ker \Delta_n$ can be found in [25]. For arbitrary n, [15] proposed a finite set of generators for $\ker \Delta_n$ but it was shown later (see [29]) that this wasn't a generating set.

This thesis is divided into three chapters. Chapter 1 presents some known facts about locally nilpotent derivations, facts that will be used in the rest of the work. Section 1, however, which explains the equivalence between locally nilpotent derivations of $k^{[n]}$ and $\mathbf{G}_a$-actions on $\mathbf{A}_k^n$, will not be needed in later parts of the thesis. The last section explains the connection between kernels of derivations of $k^{[n]}$ and the fourteenth problem of Hilbert and gives a brief description of the recent progress concerning this problem.

Given a UFD R containing $\mathbb{Q}$, we study in Chapter 2 the class of R-elementary derivations of $B = R[Y_1, \ldots, Y_m]$, i.e, R-derivations of the form

$$\sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \quad : \quad R[Y_1, \ldots, Y_m] \longrightarrow R[Y_1, \ldots, Y_m] \qquad \text{where } a_1, \ldots, a_m \in R. \tag{3}$$

This study was inspired by a question of van den Essen (see question **4.3** in [27]), asking whether there exist $k[X_1, X_2, X_3]$-elementary derivations of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$

whose kernels are not finitely generated over k. The derivation (3) is said to be *fixed point free* if 1 belongs to the ideal of R generated by the elements $a_1, \dots, a_m$. We say that (3) is a *standard* derivation if its kernel is generated over R by *standard linear constants* L_{ij}, that we define as follows: given $i, j \in \{1, \dots, m\}$, define $L_{ij} = \frac{a_i}{g_{ij}}Y_j - \frac{a_j}{g_{ij}}Y_i$ where:

$$g_{ij} = \begin{cases} \gcd(a_i, a_j) & \text{if } a_i \neq 0 \text{ or } a_j \neq 0 \\ 1 & \text{if } a_i = 0 = a_j. \end{cases}$$

We start in section 1 by proving a (well-known) sufficient condition for finite generation of the kernel of a locally nilpotent derivation. Section 2 shows that if (3) is fixed point free, then it is standard. Note that it was shown in [27] that, if $R \cong k^{[n]}$, then the kernel of any R-elementary fixed point free derivation of $R^{[m]}$ $(m \geq 1)$ is a polynomial ring in $n+m-1$ variables over k. In Section 3, we are mostly concerned with the case $m = 3$. In this case we will give (under certain assumptions) a necessary and sufficient condition on the elements a_1, a_2, a_3 of R so that the derivation (3) is standard. As a corollary, it is shown that if $R = k[X_1, \dots, X_n]$, then every R-elementary monomial derivation of $R^{[3]}$ is standard; this gives a partial answer to van den Essen's question. Section 3 also investigates some situations where one can show the finite generation of the kernel of R-elementary derivations. The case of elementary derivations of $k^{[6]}$ is studied separately in section six where it is shown that if $R = k^{[2]}$ then the kernel of any monomial R-elementary derivation of $R^{[4]}$ is a polynomial ring over R. In section five, a criterion for a derivation of $R^{[2]}$ to be R-elementary is provided in the form of necessary and sufficient condition. Some interesting examples of elementary derivations of $k^{[6]}$ are studied in section 4, giving further partial answers to van den Essen's question. We finish Chapter 2 with some results concerning monomial elementary derivations in dimension seven (see (1) above).

Chapter 3 is mainly devoted to the solution of a generalization of Nowicki's conjecture. As mentioned earlier, our main tool for that is the theory of Groebner bases. A self-contained introduction to this theory is included in section 1, together with some of its interesting applications. Elimination theory (Theorem 3.1.28) is of crucial importance for the proof of the conjecture. In section 2 we develop some techniques that are helpful, in some cases, for determining the primeness of certain ideals of $A^{[n]}$, where A is a domain but not necessarely a UFD; in particular, we give a necessary and sufficient condition for the primeness of the principal ideal $(a_1X_1 + \cdots + a_nX_n)$ of $A[X_1, \ldots, X_n]$, where $a_1, \ldots, a_n \in A$. With the help of those techniques, we then examine the proof of Nowicki's Conjecture which is given in ([20]) and show that one step of that proof is incorrect. It is because we were unable to fix that mistake that we wrote our own proof, which we give in section 3. The generalization of Nowicki's Conjecture is stated as Theorem 1, which is the main result of Chapter 3. The proof of that theorem is not very long, but one step in the proof requires Proposition 3.3.4, whose proof fills more than 50 pages.

Regarding the nature of the proof of Proposition 3.3.4, some comments are in order. The proposition asserts that, *for every positive integer n*, a certain finite set G_n is a Groebner basis of a certain ideal of $k^{[n(n+5)/2]}$. For a given value of n, deciding whether G_n is a Groebner basis is done by carrying out, for each pair $(f, g) \in G_n \times G_n$, a finite computation whose outcome is either YES or NO; in other words, we have to evaluate a certain function $\mathcal{F}_n : G_n \times G_n \to \{\text{YES, NO}\}$ at each element (f, g) of its domain (then G_n is a Groebner basis if and only if $\mathcal{F}_n(f, g) = \text{YES}$ for all (f, g)). So the proof of Proposition 3.3.4 by brute force would require the evaluation the function $\mathcal{F} : \mathcal{G} \to \{\text{YES, NO}\}$ at each element of its domain, where $\mathcal{G} = \cup_n (G_n \times G_n)$ and $\mathcal{F} = \cup_n \mathcal{F}_n$. However, this cannot be done because the domain of $\mathcal{F}$ is an infinite set. In essence, our proof of Proposition 3.3.4 consists in finding a finite partition $P_1 \cup \cdots \cup P_N$ of dom$(\mathcal{F})$, proving that $\mathcal{F}$ is constant on each P_i and, for each i, verifying that $\mathcal{F}(f, g) = \text{YES}$ for some $(f, g) \in P_i$. The number N of sets in the partition is initially quite large but, after many simplifications, the number of pairs (f, g) at which $\mathcal{F}$ has to be evaluated is brought down to 106. All computations are displayed in the proof of the proposition.

The thesis closes with an interesting example of a derivation of the polynomial ring $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ which is monomial, $k[X_1, X_2, X_3]$-elementary and whose kernel is finitely generated over k but cannot be generated over $k[X_1, X_2, X_3]$ by linear forms (i.e., polynomials of the form $\sum_i a_i Y_i$ with $a_i \in k[X_1, X_2, X_3]$). This shows that Question 2.7.6, proposed at the end of Chapter two, is not true in dimensions higher than six. Again, we use the theory of Groebner bases to show the finite generation of the kernel.

Contents

Chapter 1

PRELIMINARIES

In this chapter we study some basic facts and properties of locally nilpotent derivations that will be needed in the coming chapters. In the first section we introduce the relation between locally nilpotent derivations on a domain B (containing k) and the algebraic actions of $\mathbf{G}_a$ (see definition 1.1.7 below) on the affine scheme $X = \text{Spec}(B)$. From a geometrical point of view, this relation is one of the main motivations for studying locally nilpotent derivations on B. In the second section, we study basic properties of locally nilpotent derivations of integral domains in general. Although these basic properties are well known, we include the proof of each of them for the sake of completeness. The important case of polynomial rings will be studied in section three where we present the most recent progress in this case. Finally, a brief history of Hilbert's fourteenth problem and its connection to locally nilpotent derivations of polynomial rings is given in section four.

This chapter does not contain new results. Most of the content of sections 1 and 2 is "folklore"; for this type of material, we included the proofs but (in many cases) could not find references. The treatment of 1.1.10 is a suggestion of my supervisor. The results given in sections 3 and 4 are properly referenced.

1.1 $\mathbf{G}_a$-actions on affine spaces

In this section, and unless mentioned otherwise, B always denotes a k-domain, i.e., an integral domain containing the field k.

Definition 1.1.1. Let B be any ring.
(a) A *derivation* of B is a map $D : B \longrightarrow B$ satisfying:

1. $D(x+y) = D(x) + D(y)$
2. $D(xy) = xD(y) + yD(x)$

for all $x, y \in B$.
(b) A derivation D of B is called *locally nilpotent* if for every x in B there exists $n \geq 0$ (depending on x) such that $D^n(x) = 0$.
(c) If A is any subring of B, then the derivation D of B is called an A-*derivation* if $D(a) = 0$ for all a in A.
(d) We denote by $\text{LND}(B)$ the set of locally nilpotent derivations of B.

Example 1.1.2. If $B = A[X_i;\ i \in I]$ is a polynomial ring over a ring A with variables indexed by an arbitrary set I, then for each $i \in I$, the usual partial derivative with respect to X_i, denoted by $\frac{\partial}{\partial X_i}$, is an A-derivation of B which is locally nilpotent.

Proposition 1.1.3 (The Leibnitz' identity). *Let B be an integral domain of characteristic zero and $D : B \rightarrow B$ a derivation. Then, for all $n \in \mathbb{N}$ and $x, y \in B$,*

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^i(x) D^{n-i}(y).$$

Proof
By induction on n. ∎

Proposition 1.1.4. *Let B be a subring of a $\mathbb{Q}$-algebra C, let $D : B \rightarrow B$ be a locally nilpotent derivation and let $\gamma \in C$. Then*

$$\begin{aligned} \exp(\gamma D) : B &\longrightarrow C \\ b &\longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b)\, \gamma^n \end{aligned}$$

is a homomorphism of A-algebras, where $A = \ker(D)$.

Remark 1.1.5. We are not claiming that the homomorphism $\exp(\gamma D)$ coincides with the exponential of the derivation γD. However, the two concepts are the same whenever γD is locally nilpotent.

Proof

It is clear that $\exp(\gamma D)$ preserves addition and restricts to the identity map on A, so it's enough to verify that

$$\left(\sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)\,\gamma^i\right)\left(\sum_{j\in\mathbb{N}} \frac{1}{j!} D^j(y)\,\gamma^j\right) = \sum_{n\in\mathbb{N}} \frac{1}{n!} D^n(xy)\,\gamma^n \tag{4}$$

holds for all $x, y \in B$. Now the coefficient of γ^n in the left hand side of (4) is

$$\sum_{i+j=n} \frac{1}{i!\,j!} D^i(x)D^j(y) = \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!\,j!} D^i(x)D^j(y),$$

and this is equal to $\frac{1}{n!} D^n(xy)$ by the Leibnitz' identity. ∎

1.1.6. We recall some of the properties of the contravariant functor Spec, which goes from the category of k-algebras to that of affine schemes over k.

1. If A is a k-algebra then $\operatorname{Spec} A$ is the set of prime ideals of A, endowed with extra structure so as to make it an affine scheme over k. (The "extra structure" consists of a topology, a sheaf of rings and the morphism $\operatorname{Spec}(\nu) : \operatorname{Spec} A \to \operatorname{Spec} k$, where $\nu : k \to A$ is the inclusion map.)

2. Every affine scheme over k is isomorphic to $\operatorname{Spec} A$ for some k-algebra A.

3. If $A = k^{[n]}$, then $\operatorname{Spec} A$ is denoted $\mathbf{A}^n_k$ (or simply $\mathbf{A}^n$) and is called the *n-dimensional affine space over k*. When k is algebraically closed, $\mathbf{A}^n_k$ may be identified with the algebraic variety k^n.

4. If A, B are k-algebras then Spec gives a bijection from the set $\operatorname{Hom}_k(A, B)$ of homomorphisms of k-algebras to the set $\operatorname{Hom}_k(\operatorname{Spec} B, \operatorname{Spec} A)$ of morphisms of schemes over k. If $\phi : A \to B$ is a homomorphism of k-algebras then, as a set map, $f = \operatorname{Spec}(\phi) : \operatorname{Spec} B \to \operatorname{Spec} A$ is defined by $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of B.

5. If A, B are k-algebras then the product $\text{Spec}(A) \times \text{Spec}(B)$ (in the category of affine schemes over k) is isomorphic to $\text{Spec}(A \otimes_k B)$. For example, $\mathbf{A}^m \times \mathbf{A}^n \cong \mathbf{A}^{m+n}$. Another important example is $\mathbf{A}^1 \times \text{Spec}\, B \cong \text{Spec}\,(B^{[1]})$.

Definition 1.1.7. If k is algebraically closed, the symbol G_a (or $G_a(k)$) denotes the group $(k, +)$ viewed as an algebraic group. For arbitrary k, one defines $G_a = \mathbf{A}^1$ as an algebraic variety and the group operation is the morphism

$$G_a \xleftarrow{\mu} G_a \times G_a$$

which corresponds to the k-algebra homomorphism

$$\begin{array}{ccccc} k[T] & \longrightarrow & k[T] \otimes_k k[T] & \xrightarrow{\cong} & k[X,Y] \\ T & \longmapsto & T \otimes 1 + 1 \otimes T & \longmapsto & X + Y \end{array}.$$

Definition 1.1.8. Let B be a k-algebra and $X = \text{Spec}\, B$. An *algebraic action of G_a on X* (or simply a G_a-action on X) is a morphism

$$f : G_a \times X \to X$$

(of schemes over k) satisfying the following two conditions:

1. The composition $X \xrightarrow{\epsilon} G_a \times X \xrightarrow{f} X$ is 1_X, where ϵ is defined by:

$$\begin{array}{ccc} G_a \times X & \xleftarrow{\epsilon} & X \\ B[T] & \xrightarrow{\text{ev}_0} & B \\ T & \longmapsto & 0. \end{array}$$

2. The diagram:

$$\begin{array}{ccc} G_a \times G_a \times X & \xrightarrow{1_{G_a} \times f} & G_a \times X \\ {\scriptstyle \mu \times 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ G_a \times X & \xrightarrow{f} & X \end{array}$$

   is commutative.

Remark 1.1.9. If k is algebraically closed, G_a can be identified with the additive group $(k, +)$ of k and note that a G_a-action on X is a morphism $f : k \times X \to X$ which satisfies:

1. $f(0, x) = x$ for all $x \in X$

2. $f(a+b, x) = f(a, f(b, x))$ for all $a, b \in k$ and $x \in X$.

1.1.10. Let B be a k-domain. We show that the concept of a G_a-action on $\operatorname{Spec} B$ is equivalent to that of a locally nilpotent derivation $B \to B$. Since $G_a \times \operatorname{Spec} B \cong \operatorname{Spec} B[T]$, there is a bijection between the set of morphisms

$$\operatorname{Spec} B \longleftarrow G_a \times \operatorname{Spec} B$$

and the set of k-algebra homomorphisms

$$B \longrightarrow B[T].$$

Fix a morphism $f : G_a \times \operatorname{Spec} B \to \operatorname{Spec} B$ and let $\phi : B \to B[T]$ be the corresponding k-homomorphism. Then f satisfies the conditions (1) and (2) of 1.1.8 if and only if ϕ satisfies the following conditions (1′) and (2′):

(1′) The composition

$$B \xrightarrow{\phi} B[T] \xrightarrow{\text{ev}_0} B$$

is the identity map of B. In other words, for each $b \in B$ the constant term of $\phi(b) \in B[T]$ is b itself.

(2′) Given indeterminates T_1, T_2 over B, each element $g(T) = \sum_i b_i T^i$ ($b_i \in B$) of the image of ϕ satisfies

$$g(T_1 + T_2) = g^{(\phi_{T_1})}(T_2),$$

where $g^{(\phi_{T_1})}(T_2) = \sum_i \phi_{T_1}(b_i) T_2^i$ and where $\phi_{T_1} : B \longrightarrow B[T_1]$ is the k-homomorphism determined by f. In other words, ϕ_{T_1} is the composition

$$\begin{array}{ccccc} B & \xrightarrow{\phi} & B[T] & \xrightarrow{\cong} & B[T_1] \\ & & T & \longmapsto & T_1. \end{array}$$

Hence, $\phi \mapsto \operatorname{Spec}(\phi)$ is a bijection from

$$\Sigma \stackrel{\text{def}}{=} \text{set of } k\text{-algebra homomorphisms } B \xrightarrow{\phi} B[T] \text{ satisfying (1′) and (2′)}$$

to the set of G_a-actions on Spec B. We now proceed to define bijections $\Sigma \to \mathrm{LND}(B)$ and $\mathrm{LND}(B) \to \Sigma$ which are inverses of each other. Given $\phi \in \Sigma$, let $D : B \to B$ be the composition $B \xrightarrow{\phi} B[T] \xrightarrow{d/dT} B[T] \xrightarrow{\mathrm{ev}_0} B$, where d/dT is the usual T-derivative. Since the two maps ev_0 and d/dT are additive, D is also additive. Also, if $a, b \in B$, then

$$
\begin{aligned}
D(ab) &= \mathrm{ev}_0\left[d/dT(\phi(ab))\right] \\
&= \mathrm{ev}_0\left[d/dT(\phi(a)\phi(b))\right] \\
&= \mathrm{ev}_0\left[\phi(a)d/dT(\phi(b)) + \phi(b)d/dT(\phi(a))\right] \\
&= \mathrm{ev}_0(\phi(a))\mathrm{ev}_0\left[d/dT(\phi(b))\right] + \mathrm{ev}_0(\phi(b))\mathrm{ev}_0\left[d/dT(\phi(a))\right] \\
&= aD(b) + bD(a) \qquad \text{by condition } (1') \text{ above.}
\end{aligned}
$$

This proves that D is indeed a derivation of B. Next, we prove that D is locally nilpotent. For this, we will use condition $(2')$. Let T, T_1, T_2 be three indeterminates over B, then for any $b \in B$, $\phi_T(b)(T_1 + T_2) = \phi_T(b)^{(\phi_{T_1})}(T_2)$ and so if we write

$$\phi_T(b) = \sum_{i=0}^{t} b_i T^i$$

then we have

$$\sum_{i=0}^{t} b_i (T_1 + T_2)^i = \sum_{j=0}^{t} \phi_{T_1}(b_j) T_2{}^j. \tag{5}$$

Now by comparing the coefficients of $T_2{}^l$ in both sides of equation (5) we get that

$$\textstyle\sum_{i=l}^{t} b_i \begin{pmatrix} i \\ l \end{pmatrix} T_1{}^{i-l} \quad = \quad \phi_{T_1}(b_l) \tag{6}$$

for any l. Comparing the coefficients of T_1 in both sides of (6) we get that

$$D(b_l) = (l+1)b_{l+1} \tag{7}$$

for any l. Taking $l = 0$ gives that $D(b) = b_1$ (since $b_0 = b$ by $(1')$). Consequently,

$$D^i(b) = i! b_i \tag{8}$$

for any i. Since $b_i = 0$ for any $i > t$, then $D^i(b) = 0$ for any $i > t$, and D is locally nilpotent. So $D \in \text{LND}(B)$.

Conversely, let $D \in \text{LND}(B)$ and consider

$$\begin{aligned} \phi = \exp(TD) : B &\longrightarrow B[T] \\ b &\longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b)\, T^n, \end{aligned}$$

which is a homomorphism of k-algebras, by Proposition 1.1.4. Obviously, it satisfies condition (1′). For condition (2′), let $b \in B$ and let T_1, T_2 be two variables over B, then if $g = \sum_{i=0}^{s} \frac{D^i(b)}{i!} T^i$ ($b \in B$) is in the image of $\exp(TD)$ with $s = \max\{r \geq 0 \mid D^r(b) \neq 0\}$ we have that

$$\begin{aligned} g(T_1 + T_2) &= \sum_{i=0}^{s} \frac{D^i(b)}{i!} (T_1 + T_2)^i \\ &= \sum_{i=0}^{s} \frac{D^i(b)}{i!} \left(\sum_{l=0}^{i} \binom{i}{l} T_1^l T_2^{i-l} \right) \\ &= \sum_{i=0}^{s} \sum_{l=0}^{i} \frac{D^i(b)}{l!(i-l)!} T_1^l T_2^{i-l} \end{aligned}$$

On the other hand,

$$\begin{aligned} g^{(\exp(T_1 D))}(T_2) &= \sum_{i=0}^{s} \exp(T_1 D) \left(\frac{D^i(b)}{i!} \right) T_2^i \\ &= \sum_{i=0}^{s} \frac{1}{i!} \left(\sum_{j \geq 0} \frac{1}{j!} D^{i+j}(b) T_1^j \right) T_2^i \\ &= \sum_{i=0}^{s} \sum_{j \geq 0} \frac{D^{i+j}(b)}{i! j!} T_1^j T_2^i. \end{aligned}$$

Let $l = i + j$, then the above calculation shows that

$$g^{(\exp(T_1 D))}(T_2) = \sum\nolimits_{l=0}^{s} \sum\nolimits_{j=0}^{l} \frac{D^l(b)}{j!(l-j)!} T_1^j T_2^{l-j} = g(T_1 + T_2).$$

So, $\exp(TD) \in \Sigma$.

It remains to show that the two maps $\Sigma \xrightarrow{\alpha} \text{LND}(B)$ and $\text{LND}(B) \xrightarrow{\beta} \Sigma$ defined above are inverses of each other. Fix $b \in B$. If $\phi \in \Sigma$, write $\phi(b) = \sum_{j\geq 0} b_j T^j$, where $b_j \in B$ for all j with $b_0 = b$ by condition (1′). Let $D = \alpha(\phi)$, then $\beta(D)(b) = \sum_{j\geq 0} \frac{D^j(b)}{j!} T^j$. But $D^j(b) = j! b_j$ (by (8)) for any j and hence $\beta(D)(b) = \sum_{j\geq 0} b_j T^j = \phi(b)$. This shows that $\beta \circ \alpha(\phi) = \phi$. Conversely, if D is a locally nilpotent derivation on B, then by definition of $\exp(TD)$, we have that $d/dT(\exp(TD)(b))|_{T=0} = D(b)$. In other words, $\alpha \circ \beta(D) = D$.

To summarize, we have shown that the following are well-defined bijections:

$$\begin{array}{lllll} \text{LND}(B) & \longrightarrow & \Sigma & \longrightarrow & \text{set of } \mathbf{G}_a\text{-actions on Spec}(B) \\ D & \mapsto & \exp(TD) & & \\ & & \phi & \mapsto & \text{Spec}(\phi) \end{array}$$

where Σ is the set of k-homomorphisms $B \to B[T]$ satisfying (1′) and (2′).

1.2 Properties of locally nilpotent derivations

In this section, B denotes a domain of characteristic zero, $D : B \longrightarrow B$ is a nonzero derivation of B, and $A = \ker D$.

Note that A is a subring of B, so in particular $D(v) = 0$ for every $v \in \mathbb{Z}$.

Definition 1.2.1. An element b of B is called a *local slice* of the derivation D if $D(b) \in \ker(D)\backslash\{0\}$. It is called a *slice* if $D(b) = 1$.

Remark 1.2.2. If D is a nonzero locally nilpotent derivation, then it has a local slice. Indeed, choose $b_0 \in B$ such that $D(b_0) \neq 0$, and let $\mathcal{F} = \{m \in \mathbb{N} \mid D^m(b_0) = 0\}$. Let $n = \inf \mathcal{F}$ (since D is locally nilpotent, we know that $\mathcal{F}$ is not empty), so $D^n(b_0) = 0$, and $1 \leq n \leq m$ whenever $D^m(b_0) = 0$. Note that $n \geq 2$ and let $b = D^{n-2}(b_0) \in B$, then $D(b) = D^{n-1}(b_0) \neq 0$ and $D^2(b) = D^n(b_0) = 0$, so b is a local slice for D.

Proposition 1.2.3. *Let R be a domain, $E = R[Y_1, \dots, Y_n] \cong R^{[n]}$ a polynomial ring in n variables over R. If $D : E \longrightarrow E$ is an R-derivation of E satisfying $D^{n_i}(Y_i) = 0$ for some $n_i \geq 1$, then D is a locally nilpotent derivation of E.*

Proof

Using the Leibnitz' identity, it is easy to prove that the set C of elements of E killed by some power of D is an R-subalgebra of E. Since C contains all the Y_i's, it is equal to E. ∎

Definition 1.2.4. A subring C of B is called *factorially closed* if for all $x, y \in B$, we have

$$xy \in C\backslash\{0\} \Rightarrow x, y \in C.$$

Definition 1.2.5. A *degree function* on a domain B is a map

$$\nu : B \to \mathbf{Z} \cup \{-\infty\}$$

satisfying the following properties:

1. $\nu(b) = -\infty$ if and only if $b = 0$;
2. $\nu(x + y) \leq \max(\nu(x), \nu(y))$ for every $x, y \in B$;
3. $\nu(xy) = \nu(x) + \nu(y)$ for every $x, y \in B$.

If $D : B \to B$ is a locally nilpotent derivation, one can associate to D the map

$$\nu_D : B \longrightarrow \mathbf{N} \cup \{-\infty\}$$

defined by the formula

$$\nu_D(x) = \begin{cases} \max\{n \geq 0 \mid D^n(x) \neq 0\} & \text{if } x \neq 0 \\ -\infty & \text{if } x = 0 \end{cases}$$

Lemma 1.2.6. *If $D : B \to B$ is a locally nilpotent derivation of B, then the map $\nu_D : B \to \mathbf{N} \cup \{-\infty\}$ is a degree function on B satisfying condition*
4. $\nu_D(D(x)) = \nu_D(x) - 1$ *if* $D(x) \neq 0$.

Proof

We prove first that the three conditions of definition 1.2.5 are satisfied for ν_D. Condition 1 is clear from the definition of ν_D. For conditions 2 and 3, let T be an indeterminate over B, $K = \operatorname{Frac} B$ and consider the homomorphism

$$\begin{array}{rccc} \exp(TD): & B & \longrightarrow & K[T] \\ & b & \mapsto & \sum_{j\geq 0} \frac{D^j(b)}{j!}T^j \end{array}$$

defined in section 1. It is clear that ν_D is the composition

$$B \overset{\exp(TD)}{\longrightarrow} K[T] \overset{\deg}{\longrightarrow} \mathbf{N} \cup \{-\infty\}$$

where deg is the usual T-degree on $B[T]$. Since $\exp(TD)$ is a homomorphism (Proposition 1.1.4), conditions 2 and 3 follow. For condition 4, let $x \in B\backslash\{0\}$ such that $n = \nu_D(x) > 0$, then $D^{n-1}(D(x)) = D^n(x) \neq 0$ and if $p > n-1$, then $D^p(D(x)) = D^{p+1}(x) = 0$ since $p+1 > n$. Hence $\nu_D(D(x)) = n-1 = \nu_D(x) - 1$. ∎

It turns out that the degree function associated to a locally nilpotent derivation D is very useful to prove some of the properties of D, as we will see in the following Propositions.

Proposition 1.2.7. *If $D: B \to B$ is locally nilpotent, then its kernel is a factorially closed subring of B.*

Proof

If $x, y \in B\backslash\{0\}$ with $D(xy) = 0$, then $\nu_D(xy) = 0$. Lemma 1.2.6 gives that $\nu_D(x) + \nu_D(y) = 0$ and consequently $\nu_D(x) = \nu_D(y) = 0$, so $x, y \in \ker D$. ∎

Corollary 1.2.8. *If B is a k-domain and $D: B \to B$ is locally nilpotent, then $B^* \subseteq \ker D$ and consequently D is a k-derivation.*

Proposition 1.2.9. *Let $D: B \to B$ be a derivation of B, $A = \ker D$ and S a multiplicatively closed subset of $B\backslash\{0\}$. Then:*

1. *D can be extended to a derivation $S^{-1}D: S^{-1}B \to S^{-1}B$.*

2. *$S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$.*

3. *If $S \subseteq A$, then $\ker S^{-1}D = S^{-1}A$ and $S^{-1}A \cap B = A$.*

Proof

1. If $b \in B$ and $s \in S$, define $S^{-1}D\left(\frac{b}{s}\right)$ by the formula

$$S^{-1}D\left(\frac{b}{s}\right) = \frac{sD(b) - bD(s)}{s^2} \quad \text{(the quotient rule of derivation).}$$

It is easy to see that $S^{-1}D$ is indeed a derivation on $S^{-1}B$ that extends D.

2. If $S^{-1}D$ is locally nilpotent, then clearly D is locally nilpotent. Moreover, if $s \in S$, then $S^{-1}D(\frac{1}{s}s) = S^{-1}D(1) = 0$, and so $S^{-1}D(\frac{1}{s}) = 0 = S^{-1}D(s)$ by Proposition 1.2.7. This shows that $s \in B \cap \ker(S^{-1}D) = A$, and hence $S \subseteq A$. For the converse, assume that D is locally nilpotent and $S \subseteq A$ and let $b \in B$, $s \in S$. We claim that $(S^{-1}D)^n\left(\frac{b}{s}\right) = \frac{1}{s}D^n(b)$ for all $n \in \mathbb{N}$. Indeed, the claim is obviously true if $n = 0$. Assume $n \geq 1$, then

$$\begin{aligned}
(S^{-1}D)^n\left(\frac{b}{s}\right) &= S^{-1}D\left[(S^{-1}D)^{n-1}\left(\frac{b}{s}\right)\right] \\
&= S^{-1}D\left(\frac{1}{s}D^{n-1}(b)\right) \quad \text{(by the induction hypothesis)} \\
&= S^{-1}D\left(\frac{1}{s}\right)D^{n-1}(b) + \frac{1}{s}D^n(b) \\
&= \frac{1}{s}D^n(b) \quad \text{since } S^{-1}D\left(\frac{1}{s}\right) = 0 \quad (S \subseteq A).
\end{aligned}$$

This proves the claim. Choose $n \geq 0$ such that $D^n(b) = 0$ (D is locally nilpotent), then $(S^{-1}D)^n\left(\frac{b}{s}\right) = \frac{1}{s}D^n(b) = 0$ and so $S^{-1}D$ is locally nilpotent.

3. Assume that $S \subseteq A$, and let $\frac{b}{s} \in \ker S^{-1}D$, then

$$S^{-1}D\left(\frac{b}{s}\right) = 0 = \frac{sD(b) - bD(s)}{s^2} = \frac{D(b)}{s}.$$

Hence $D(b) = 0$ (B is a domain) and $b \in A$. So $\ker S^{-1}D \subseteq S^{-1}A$. Conversely, if $\frac{b}{s} \in S^{-1}A$, then $b \in \ker D$, and hence $D(b) = 0$, so $S^{-1}D\left(\frac{b}{s}\right) = \frac{D(b)}{s} = 0$, and $\frac{b}{s} \in \ker S^{-1}D$. Thus, $\ker S^{-1}D = S^{-1}A$. In particular,

$$S^{-1}A \cap B = \ker S^{-1}D \cap B = A.$$

∎

Proposition 1.2.10. ***(Proposition 2.1,** [32]) Assume that $\mathbb{Q} \subseteq B$ and let $D : B \to B$ be a locally nilpotent derivation. If $b \in B$ is a slice of D, then*

1. $\ker D$ *is the image of the homomorphism* $\exp(-bD) : B \to B$.
2. $B = A[b] = A^{[1]}$ *where* $A = \ker D$.

Proof

Write $\zeta = \exp(-bD)$.

(1) If $x \in \ker D$, then $\zeta(x) = x$ and $x \in \operatorname{im}\zeta$. Conversely, suppose that $y = \zeta(x)$ for some $x \in B$, then

$$y = \sum_{j=0}^{n} \frac{1}{j!}(-b)^j D^j(x) \tag{9}$$

where $n = \nu_D(x)$. If $n = 0$, then $y = x \in \ker D$. If $n \geq 1$, applying D to (9) gives that

$$D(y) = D(x) - D(x) - bD^2(x) + bD^2(x) + \cdots + (-1)^{n-1}\frac{b^{n-1}}{(n-1)!}D^n(x) + (-1)^n\frac{b^{n-1}}{(n-1)!}D^n(x) + (-1)^n\frac{b^n}{n!}D^{n+1}(x) = 0.$$

So $y \in \ker D$.

(2) If $B \neq A[b]$, then one can choose $x \in B \backslash A[b]$ such that $\nu_D(x)$ is minimal. Since $\nu_D(D^i x) < \nu_D(x)$ for all $i \geq 1$, the minimality of $\nu_D(x)$ implies that the element

$$\sum_{i=1}^{\infty} \frac{1}{i!}(-b)^i D^i x$$

of B is in $A[b]$. Part (1), together with the fact that

$$\zeta(x) = x + \sum_{i=1}^{\infty} \frac{1}{i!}(-b)^i D^i x$$

imply that $x \in A[b]$. This is a contradiction. So $B = A[b]$ and since A is algebraically closed in B (easily verified), $B = A[b] = A^{[1]}$. ∎

Corollary 1.2.11. *Assume that $\mathbb{Q} \subseteq B$. Let $D : B \to B$ be a locally nilpotent derivation, and $b \in B$ a local slice for D. Put $S = \{(D(b))^n; n \geq 0\}$, then $S^{-1}B = (S^{-1}A)[b] = (S^{-1}A)^{[1]}$.*

Proof

Clearly, S is a multiplicatively closed subset of $A = \ker D$, hence D can be extended to a locally nilpotent derivation $S^{-1}D$ on $S^{-1}B$ with kernel equal to $S^{-1}A$ (Proposition 1.2.9). On the other hand, it is clear that $\frac{b}{D(b)}$ is a slice for $S^{-1}D$, hence Proposition 1.2.10 implies that $S^{-1}B = (S^{-1}A)[b/D(b)] = (S^{-1}A)^{[1]}$. ∎

Corollary 1.2.12. *If $D : B \to B$ is a locally nilpotent derivation, $A = \ker D$ and $S = A \backslash \{0\}$, then $S^{-1}B = (\mathrm{Frac}\, A)^{[1]}$ and $(\mathrm{Frac}\, A) \cap B = A$.*

Proof

Note first that S is a multiplicatively closed subset of $B\backslash\{0\}$. Choose a local slice b of D, and let $T = \{(D(b))^n; n \geq 0\}$, then $T \subseteq S$. Now, $S^{-1}B = (TS)^{-1}B = S^{-1}((T^{-1}A)^{[1]})$ (Corollary 1.2.11) $= (S^{-1}(T^{-1}A))^{[1]} = ((TS)^{-1}A)^{[1]} = (S^{-1}A)^{[1]} = (\mathrm{Frac}\, A)^{[1]}$. Also, by propostion 1.2.9, we have that $S^{-1}A \cap B = A$ and hence $(\mathrm{Frac}\, A) \cap B = A$. ∎

Another corollary of proposition 1.2.10 is the following

Lemma 1.2.13. *Let $R \supseteq \mathbb{Q}$ be an integral domain, $B = R^{[m]}$ and $D : B \to B$ a locally nilpotent R-derivation. If D admits a slice which is a variable of B over R, then there exists a coordinate system $(Z_1, \ldots, Z_m)$ of B over R such that $D = \partial/\partial Z_m$. In particular, $\ker D = R^{[m-1]}$.*

Proof

Let $A = \ker D$. If s is any slice of D then $B = A[s] = A^{[1]}$ (Proposition 1.2.10), so $B/sB \cong A$. If s is also a variable of B over R then $B = R[s_1, \ldots, s_{m-1}, s]$, so $B/sB \cong R^{[m-1]}$. It follows that $A \cong R^{[m-1]}$, i.e., we may choose $Z_1, \ldots, Z_{m-1}$ such that $A = R[Z_1, \ldots, Z_{m-1}]$. Let $Z_m = s$, then $B = A[Z_m]$, so we are done. ∎

Let $b \in B$, and define the map $bD : B \longrightarrow B$ by $(bD)(x) = bD(x)$ for all $x \in B$. It is easy to see that bD is a derivation of B whenever D is one. A less obvious fact is the following.

Proposition 1.2.14. *Let $D : B \to B$ be a derivation of B. For $b \in B \backslash \{0\}$, the derivation bD of B is locally nilpotent if and only if D is locally nilpotent and $b \in A = \ker D$.*

Proof

Assume that D is locally nilpotent, and that $b \in A$. Using an easy induction argument (and the fact that $b \in A$), one can see that $(bD)^n(x) = b^n D^n(x)$ for all $x \in B$ and all $n > 0$. This proves in particular that bD is locally nilpotent. For the converse, assume that bD is locally nilpotent; we need to show that D is locally nilpotent and that $b \in A$. We may assume that $D \neq 0$. Then (Remark 1.2.2) bD has a local slice $s \in B$, i.e., $0 \neq (bD)(s) \in \ker(bD)$, so $bDs \in \ker(bD) \backslash \{0\}$ and so $b \in \ker(bD) = \ker D = A$ since $\ker(bD)$ is factorially closed in B. If $x \in B$ then (for some $n \in \mathbb{N}$) $(bD)^n(x) = 0$, but $b \in A$ implies that $(bD)^n(x) = b^n D^n(x)$, so $D^n(x) = 0$. Hence, D is locally nilpotent. ∎

Proposition 1.2.15. *Let $D : B \to B$ be a derivation, $f \in B[T] \cong B^{[1]}$. There exists a unique derivation D_1 of $B[T]$ extending D and satisfying $D_1(T) = f$. Moreover, D_1 is locally nilpotent if and only if D is locally nilpotent and $f \in B$.*

Proof

The proof requires some work. Let $D_1 : B[T] \longrightarrow B[T]$ be the map defined by

$$D_1\left(\sum_{i=0}^{n} b_i T^i\right) = \sum_{i=0}^{n}[D(b_i)T^i + ib_i f T^{i-1}].$$

It is straightforward to check that D_1 is indeed a derivation of $B[T]$ extending D, and that D_1 is the unique derivation of $B[T]$ sending T to f.

Now, assume that D is locally nilpotent, and $f \in B$. Then the set

$$\{x \in B[T] | D_1^n(x) = 0 \text{ for some } n \in \mathbb{N}\}$$

is a subring of $B[T]$ containing B and T; hence equal to $B[T]$. So D_1 is locally nilpotent.

Conversely, assume that D_1 is locally nilpotent, then D is also locally nilpotent since

D_1 is an extension of D. It remains to prove that $\deg f = 0$. Let $g = \sum_{i=0}^{n} b_i T^i$ be an element of $B[T]$, then

$$D_1(g) = g^{(D)}(T) + fg'(T) \tag{10}$$

where

$$g^{(D)}(T) = D(b_0) + D(b_1)T + \cdots + D(b_n)T^n.$$

If $\deg f > 1$, then $\deg fg'(T) > \deg g$ and since $\deg g^{(D)}(T) \leq \deg g$, equation (10) gives that $\deg D_1(g) = \deg fg'(T) > \deg g$, and this contradicts the assumption that D_1 is locally nilpotent. If now we assume that $\deg f = 1$, then equation (10) becomes

$$D_1(g) = g^{(D)}(T) + g'(T)(aT + b)$$

for some $a, b \in B$ with $a \neq 0$. So the coefficient of T^n in $D_1(g)$ is $D(b_n) + nab_n$. We prove now that this coefficient is not zero. If $D(b_n) = -nab_n$, then we can apply the map ν_D to both sides of this equality and get that

$$\nu_D(b_n) - 1 = \nu_D(a) + \nu_D(b_n).$$

This means that $\nu_D(a) = -1$, which is not possible. So $D(b_n) + nab_n \neq 0$ and $\deg D_1(g) = \deg g$, this also contradicts the assumption that D_1 is locally nilpotent. We conclude that $\deg f(T) = 0$, and $f \in B$. ∎

A special type of derivation we will often look at called an *irreducible derivation*.

Definition 1.2.16. A derivation $D : B \longrightarrow B$ of a domain B is called *irreducible* if the only principal ideal of B containing $D(B)$ is B itself.

Proposition 1.2.17. *Let B be a UFD, $D : B \longrightarrow B$ a nonzero derivation of B. Then:*

1. *$D = \alpha D_0$ for some $\alpha \in B$ and some irreducible derivation D_0 of B. Moreover, α and D_0 are unique up to multiplication by units of B.*

2. *D is locally nilpotent if and only if D_0 is locally nilpotent and $\alpha \in \ker D$.*

Proof
Part (2) is clear from Proposition 1.2.14. Part (1) is clear if D is irreducible, so we may assume that $(DB) \subset \alpha_1 B$ for some $\alpha_1 \in B \backslash B^*$. In other words, $D = \alpha_1 D_1$ for some derivation D_1 of B. If D_1 is irreducible, we are done. If not, we can find $\alpha_2 \in B \backslash B^*$, and a derivation D_2 of B such that $D = \alpha_1 \alpha_2 D_2$, so on $\cdots$. If D_n is not irreducible for all n, then for $b \in B$ with $Db \neq 0$, we can construct an infinite sequence of principal ideals

$$(Db) \subset (D_1 b) \subset \ldots \subset (D_n b) \subset \ldots$$

of B. This is not possible since B is a UFD.
For the uniqueness (up to a multiple by a unit) of α and D_0, assume that $D = \alpha_1 D_1 = \alpha_2 D_2$ for some $\alpha_1, \alpha_2 \in B$ and irreducible derivations D_1, D_2, and let $d = \gcd(\alpha_1, \alpha_2)$. Then $(\alpha_1/d)D_1 = (\alpha_2/d)D_2$ and $D(B) \subseteq (\alpha_1)$, $D(B) \subseteq (\alpha_2)$. Now, since α_1/d and α_2/d are relatively prime,

$$D_1(B) \subseteq (\alpha_2/d) \text{ and } D_2(B) \subseteq (\alpha_1/d).$$

But D_1 and D_2 are irreducible, so (α_1/d) and (α_2/d) are units. Hence α_1 and α_2 are associates. This finishes the proof. ∎

1.3 Case of polynomial rings

One case of special interest for us is the case of derivations of polynomial rings. In this case, we have, among other attractive properties, a nice expression of the derivations. We start with a general setting.

For any ring B, recall that $\mathrm{Der}\,(B)$ and $\mathrm{LND}(B)$ are respectively the sets of derivations and locally nilpotent derivations on B. It is clear that if d_1, d_2 are two derivations of B and $b \in B$, then $d_1 + d_2$, bd_1 are also derivations on B. This gives $\mathrm{Der}\,(B)$ the structure of B-module. Also, if d_1, d_2 are elements of $\mathrm{Der}\,(B)$, then $[d_1, d_2] := d_1 d_2 - d_2 d_1$ is also a derivation on B. This means that $\mathrm{Der}\,(B)$ is also a Lie algebra. If A is a subalgebra of B, then $\mathrm{Der}_A\,(B)$ will denote the submodule of

Der (B) consisting of all A-derivations of B. If $B = A[X_1, \dots, X_n]$ is a polynomial ring over A, we will show that $\mathrm{Der}_A(B)$ is a free B-module. This result will be often used in the coming chapters.

Proposition 1.3.1. *Let $B = A[X_1, \dots, X_n]$ be a polynomial ring over a ring A.*

1. *$\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}$ is a basis of the B-module $Der_A(B)$.*
2. *If $f \in B$ and $d \in Der_A(B)$, then $d(f) = \sum_{i=1}^{n} d(X_i)\frac{\partial f}{\partial X_i}$.*

Proof

(1) Let $d \in \mathrm{Der}_A(B)$, and let $a_i = d(X_i)$ for each $i \in \{1, \dots, n\}$. Put $D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial X_i}$, then it is clear that D is an A-derivation and $d(X_i) = D(X_i) = a_i$. Hence $d = D$, and the set $\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\}$ generates $\mathrm{Der}_A(B)$ over B. For the uniqueness of the coefficients, assume that $d \in \mathrm{Der}_A(B)$ is such that $d = \sum_{i=1}^{n} a_i \frac{\partial}{\partial X_i} = \sum_{i=1}^{n} b_i \frac{\partial}{\partial X_i}$ for some $a_i, b_i \in B$, then evaluting both sums at X_i gives that $a_i = b_i$ for all i.
(2) Clear from the proof of (1). ∎

In the case of polynomial rings one can also classify locally nilpotent derivations according to their ranks. First a definition.

Definition 1.3.2. Let B be a ring and R a subring of B such that $B \cong R^{[m]}$. A *coordinate system of B over R* is an ordered m-tuple $(Y_1, \dots, Y_m)$ of elements of B satisfying $B = R[Y_1, \dots, Y_m]$.

Note that if $B = k^{[m]}$ (where k is a field) then we may simply speak of a "coordinate system of B", without mentioning k, since k is uniquely determined by B ($k = B^* \cup \{0\}$).

Now we define the notion of "rank":

Definition 1.3.3. *([18])* If $B = k^{[n]}$ is a polynominal ring in n variables over k and D is a locally nilpotent derivation of B, we define the *rank* of D to be the least integer $r \geq 0$ for which there exists a coordinate system $(X_1, ..., X_n)$ of B such that $k[X_1, ..., X_{n-r}] \subseteq \ker D$.

If $B = k^{[n]}$, then locally nilpotent derivations of B of low ranks are easy to understand. For instance, it is clear that if $D : B \to B$ is of rank zero, then it is the zero derivation. Also, it was shown in [18] that locally nilpotent derivations of rank one of $B = k^{[n]}$ can be characterized by the two conditions

$$\ker D = k^{[n-1]} \text{ and } B = (\ker D)^{[1]}.$$

Also, the locally nilpotent derivations of polynomial rings of low dimensions are understood. For instance, any nonzero locally nilpotent derivation of $k[T]$ is of the form $a(\mathrm{d}/\mathrm{d}T)$ for some $a \in k$. As for locally nilpotent derivations of $k^{[2]}$, the following theorem of Rentschler ([26]) provides a full description of them.

Theorem 1.3.4. *(**Rentschler**) If L is a field of characteristic zero and D is a nonzero, locally nilpotent derivation of $L[X,Y] = L^{[2]}$ then there exists P, Q such that $L[X,Y] = L[P,Q]$ and $\ker D = L[P]$. Moreover, there exists $\alpha \in L[P]$ such that*

$$Dh = \alpha \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix} \quad \textit{for all } h \in L[X,Y].$$

Remark 1.3.5. With the notation of Theorem 1.3.4, we have $D = \alpha\frac{\partial}{\partial Q}$.

D. Daigle and G. Freudenburg gave the following nice generalization of Rentschler's Theorem for locally nilpotent derivations of higher dimensions.

Theorem 1.3.6. *(**Theorem 2.4**, [7]) Let R be a UFD containing the rationals, let $B = R[X,Y] = R^{[2]}$ and let $K = \operatorname{Frac} R$. For an R-derivation $D \neq 0$ of B, the following are equivalent:*

1. *D is locally nilpotent*
2. *$D = \alpha\Delta_P$, for some $P \in B$ which is a variable of $K[X,Y]$ and satisfies $gcd_B(P_X, P_Y) = 1$, and for some $\alpha \in R[P]\backslash\{0\}$.*

Moreover, if the above conditions are satisfied then $\ker D = R[P]$.

Here, Δ_P is the R-derivation of $B = R[X, Y]$ defined by the formula

$$\Delta_P(h) = \begin{vmatrix} P_X & P_Y \\ h_X & h_Y \end{vmatrix}$$

for all $h \in B$.

As a consequence of the above Theorem, one can get a full description of rank two locally nilpotent derivations of $k^{[n]}$. First some notation:
Given a coordinate system $\gamma = (X_1, \dots, X_n)$ of $k^{[n]}$, and an element $P \in k[X_1, \dots, X_n]$, define a derivation Δ_P^γ of $k^{[n]}$ by

$$\Delta_P^\gamma = -P_{X_n}\frac{\partial}{\partial X_{n-1}} + P_{X_{n-1}}\frac{\partial}{\partial X_n}.$$

Corollary 1.3.7. *(Corollary 3.2, [7]) For a k-derivation $D \neq 0$ of $R_n = k^{[n]}$, the following are equivalent:*

1. *D is locally nilpotent and $\operatorname{rank} D \leq 2$;*
2. *$D = \alpha\Delta_P^\gamma$ for some γ, P and α satisfying*
   - *$\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ is a coordinate system of R_n,*
   - *$P \in R_n$ is a variable of $k(X_1, \dots, X_{n-2})[Y, Z]$ satisfying $gcd_{R_n}(P_Y, P_Z) = 1$.*
   - *α is a nonzero element of $k[X_1, \dots, X_{n-2}, P]$.*

Moreover, if the above two conditions are satisfied then the following hold:

1. *$\ker D = k[X_1, \dots, X_{n-2}, P]$;*
2. *Δ_P^γ is irreducible;*
3. *$\Delta_P^\gamma(R_n)$ contains a nonzero element of $k[X_1, \dots, X_{n-2}]$.*

Another class of derivations special to polynomial rings are the triangular derivations.

Definition 1.3.8. Let $B = k^{[n]}$ and $D : B \to B$ a derivation of B.

1. D is called *triangular* with respect to the coordinate system $(X_1, \dots, X_n)$ of B if D can be written as

$$D = a_1 \frac{\partial}{\partial X_1} + \cdots + a_n \frac{\partial}{\partial X_n}$$

with $a_1 \in k$, and $a_i \in k[X_1, \dots, X_{i-1}]$ for $2 \le i \le n$.

2. D is called *triangulable* if there exists a coordinate system of B with respect to which D is triangular.

In the next section we will see that most counterexamples to Hilbert's fourteenth problem found till now are quotient fields of kernels of triangular derivations. Note that one can see easily that triangular derivations are in particular locally nilpotent. The converse is not true, as was shown in [1], where H. Bass gave the first example of a non-triangular locally nilpotent derivation of $k^{[3]}$. In [8], Daigle gave a criterion for triangulability of derivations of $k[X, Y, Z]$ in the form of a necessary and sufficient condition.

1.4 Derivations and Hilbert's fourteenth problem

Derivations of polynomial rings and the fourteenth problem of Hilbert are closely related as we will see in this section. This gives another interesting aspect of derivations. Recall that one form of the fourteenth problem of Hilbert can be stated as follows:

(H14) If L is a subfield of $k(X_1, ..., X_n)$ (the quotient field of $k^{[n]}$), is $L \cap k[X_1, ..., X_n]$ a finitely generated k-algebra?

By a counterexample to (H14) in dimension n, we mean a subfield L of $k(X_1, \dots, X_n)$ such that $L \cap k[X_1, ..., X_n]$ is not a finitely generated k-algebra. We also consider the following important special case

(AC-H14) If L is a subfield of $k(X_1, ..., X_n)$ and $L \cap k[X_1, ..., X_n]$ is algebraically closed in $k[X_1, ..., X_n]$, is $L \cap k[X_1, ..., X_n]$ a finitely generated k-algebra?

Relationship

Derivations of polynomial rings and Hilbert's fourteenth problem are related in the following way: if one has a derivation of $k^{[n]}$ whose kernel is not a finitely generated k-algebra, then one obtains a counterexample to (AC-H14) Hilbert's fourteenth problem by taking the quotient field of $\ker D$. Conversely, Nowicki showed the following.

Theorem 1.4.1. *(Theorem 5.4, [24]) Let B be a domain which is a finitely generated k-algebra. For a subalgebra A of B, the following are equivalent:*

1. *There exists a k-derivation D of B such that $A = \ker D$*
2. *A is algebraically closed in B.*

This implies that every counterexample to (AC-H14) can be realised as the quotient field of the kernel of a derivation. A similar result was also proved by Derksen ([9]).

Progress

There has been substantial progress in Hilbert's fourteenth problem in the last decade. In what follows, we list the important stages of this problem.

1. In 1958, Nagata ([13]) solved the problem by finding the first counterexample to (H-14), and it was in dimension 32.
2. In 1993, Derksen ([21]) proved that Nagata's example can be realized as the kernel of a derivation of $k^{[32]}$.
3. Another counterexample to Hilbert's fourteenth was found by Roberts in 1990 ([12]) in dimension seven, and it was used by Deveney and Finston ([11]) to show that the kernel of the derivation
$$D = X_1^{t+1}\frac{\partial}{\partial Y_1} + X_2^{t+1}\frac{\partial}{\partial Y_2} + X_3^{t+1}\frac{\partial}{\partial Y_3} + (X_1X_2X_3)^t\frac{\partial}{\partial Y_4} \qquad (11)$$
of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ is not a finitely generated k-algebra for any $t \geq 2$.

4. In 1998, a counterexample in dimension six was constructed by Freudenburg ([17]) as the field of fractions of the kernel of the derivation

$$d = X_1^3\frac{\partial}{\partial Y_1} + X_2^3Y_1\frac{\partial}{\partial Y_2} + X_2^3Y_2\frac{\partial}{\partial Y_3} + X_1^2X_2^2\frac{\partial}{\partial Y_4} \tag{12}$$

of $k[X_1, X_2, Y_1, Y_2, Y_3, Y_4]$.

5. In the same year, Daigle and Freudenburg ([6]) constructed a counterexample in dimension five as the field of fractions of the kernel of the derivation

$$T = X_1^3\frac{\partial}{\partial X_2} + X_2\frac{\partial}{\partial X_3} + X_3\frac{\partial}{\partial X_4} + X_1^2\frac{\partial}{\partial X_5} \tag{13}$$

of $k[X_1, X_2, X_3, X_4, X_5]$.

For $n < 4$, Nagata and Nowicki ([23]) showed the following important fact.

Theorem 1.4.2. *(Theorem 7.1.3, [25]) If D is a k-derivation of $k[X_1, \dots, X_n]$, where $n \leq 3$, then the ring $\ker D$ is finitely generated over k.*

This leaves Hilbert's problem open only in dimension four. Note that the derivations (11), (12), and (13) above are all triangular and monomial in the sense of the following definition.

Definition 1.4.3. A derivation $D = \sum_{i=1}^{n} a_i\partial/\partial X_i$ of $B = k[X_1, \dots, X_n]$ is called *monomial* if a_i is a monomial of B for all i.

Attempts to answer Hilbert's fourteenth problem in dimension 4 resulted in the following important results.

1. Maubach ([22]) proved the following result in dimension 4:

   Theorem 1.4.4. *Let $A = k[X_1, X_2, X_3, X_4]$. Then the kernel of every monomial triangular derivation $D : A \to A$ is generated by at most four elements.*

2. In an attempt to prove that the triangular hypothesis for a derivation in dimension four is enough for the finite generation of its kernel, a work of Daigle and Freudenburg ([5]) resulted in the following interesting Theorem

Theorem 1.4.5. *Given any integer $n \geq 3$, there exists a triangular derivation Δ of the polynomial ring $k[X_1, X_2, X_3, X_4]$ whose kernel cannot be generated by fewer than n elements.*

3. Finally, Daigle and Freudenburg were able to prove that the kernel of every triangular derivation of $k^{[4]}$ is a finitely generated k-algebra ([4]).

A consequence of the main result in chapter 2 will state that we cannot construct a counterexample to Hilbert's fourteenth problem in dimension six which is of type (11) above.

Chapter 2

ELEMENTARY DERIVATIONS OVER A UFD

Definition 2.0.1. Let B be a ring and R a subring of B such that $B = R^{[m]}$. A derivation $D : B \to B$ is R-elementary if $D(R) = \{0\}$ and if there exists a coordinate system $(Y_1, \ldots, Y_m)$ of B over R (see definition 1.3.2) such that $DY_i \in R$ for all i.

In this case we have:

$$D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \qquad \text{(where } a_i \in R\text{)}.$$

Definition 2.0.2. Let $B = k^{[N]}$: A derivation $D : B \to B$ is elementary if, for some integers $m, n \geq 0$ such that $m + n = N$, there exists a coordinate system $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ of B satisfying:

$$k[X_1, \ldots, X_n] \subseteq \ker D \quad \text{and} \quad \forall\, i \;\; DY_i \in k[X_1, \ldots, X_n].$$

In this case, D is $k[X_1, \ldots, X_n]$-elementary:

$$D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \qquad \text{(where } a_i \in k[X_1, \ldots, X_n]\text{)}.$$

An immediate consequence of the above definition is that all elementary derivations are locally nilpotent.

Here are some facts that make elementary derivations interesting to study:

- In [12], A. van den Essen and E. Hubbers showed that a large class of polynomial automorphisms of $k^{[n]}$ can be written as a finite product of automorphisms of the form $\exp(D)$ where D is an elementary derivation of $k^{[n]}$.
- It was shown in [26] that after a change of variables, any locally nilpotent derivation of $k[X, Y]$ is an elementary derivation.
- In [12], P. Roberts showed that the kernel of the following elementary derivation
$$D = X_1^3\frac{\partial}{\partial Y_1} + X_2^3\frac{\partial}{\partial Y_2} + X_3^3\frac{\partial}{\partial Y_3} + (X_1X_2X_3)^2\frac{\partial}{\partial Y_4} \tag{14}$$
of $k^{[7]}$ is not a finitely generated k-algebra.

Given a ring R, and an R-elementary derivation $D = a_1\partial/\partial Y_1 + \cdots + a_n\partial/\partial Y_n$ of $B = R[Y_1, \ldots, Y_n]$ ($a_i \in R$ for all i), one would like to describe $\ker D$. For a general ring R and an arbitrary n, the problem of finding $\ker D$ seems to be hard; however, if R is a field or if R is a UFD containing $\mathbb{Q}$ and $n = 2$, then one can prove that $\ker D$ is finitely generated over R and even give a set of generators (see Corollary 2.2.5 and Proposition 2.5.5).

Let $T = k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ ($m, n \geq 1$), and let D be an elementary derivation, $D = \sum_{i=1}^{m} a_i(X)\frac{\partial}{\partial Y_i} : T \to T$. In [13], A. van den Essen and T. Janssen proved that $\ker D$ is a finitely generated k-algebra in each of the following cases

1. D is a fixed point free derivation of T (see definition 2.2.1)
2. $n \leq 2$, m arbitrary
3. $m \leq 2$, n arbitrary
4. $n + m \leq 5$.

This means that if $n + m \leq 6$, then the only possibility for an elementary derivation of T to have a kernel which is not finitely generated over k is when $n = m = 3$. On the other hand, it was shown in [13], using a generalization of the derivation (14) above, that if $n \geq 3$ and $m \geq 4$, then there exists an elementary derivation of T whose kernel is not finitely generated. The case $m = n = 3$ was the subject of the following question raised in [13] (Question 4.3):

Question 2.0.3. Does there exist an elementary derivation D of T with $m = n = 3$ whose kernel is not a finitely generated k-algebra?

In this chapter, we investigate R-elementary derivations of $R[Y_1, \ldots, Y_m]$ where R is a UFD. Although we are mostly concerned with the case $m = 3$, some concepts and questions can be stated in general:

Definition 2.0.4. Let $R \supseteq \mathbb{Q}$ be a UFD, $B = R[Y_1, \ldots, Y_m] = R^{[m]}$ and consider an R-elementary derivation

$$D = \sum_{i=1}^{m} a_i \partial_i \quad : \quad B \longrightarrow B$$

where $a_i \in R$ and $\partial_i = \partial/\partial Y_i$ for all i.

1. Any element of $\ker(D)$ of the form

   $$r_1 Y_1 + \cdots + r_m Y_m \qquad \text{(where } r_i \in R\text{)}$$

   is said to be a linear constant of D.

2. Given $i, j \in \{1, \ldots, m\}$, define $L_{ij} = \frac{a_i}{g_{ij}} Y_j - \frac{a_j}{g_{ij}} Y_i$ where:

   $$g_{ij} = \begin{cases} \gcd(a_i, a_j) & \text{if } a_i \neq 0 \text{ or } a_j \neq 0 \\ 1 & \text{if } a_i = 0 = a_j \end{cases}$$

   It is clear that $L_{ij} \in \ker(D)$, $L_{ii} = 0$ and $L_{ji} = -L_{ij}$ (for all i, j). We call the elements L_{ij} the standard linear constants of D.

3. If $\ker(D)$ is generated as an R-algebra by the standard linear constants, we say that D is a standard R-elementary derivation.

Consider R-elementary derivations $D : B \to B$, where R and $B = R^{[m]}$ are as in the above definition. Then this chapter seeks *extra hypotheses* that would imply that D satisfies one or the other of the following conditions:

(i) *D is standard*

(ii) $\ker(D)$ *is generated by linear constants*

(iii) $\ker(D)$ *is finitely generated as an R-algebra.*

[Note that (i) $\implies$ (ii) is trivial and that (ii) $\implies$ (iii) is easy to see if we assume that R is noetherian. However, (iii) $\not\Longrightarrow$ (ii) (Theorem 3.4.1) and (ii) $\not\Longrightarrow$ (i) (example 2.4.2).]

In the first section we prove a sufficient condition for finite generation of the kernel of an elementary derivation. This result will be our main tool for the proof of the majority of the results of this chapter. In section 2 we study elementary fixed point free derivations of $R^{[m]}$ (R is a UFD). In section 3, we look at some cases where the kernel of an R-elementary derivation of $R^{[3]}$ is a finitely generated R-algebra. In particular we show that any monomial R-elementary derivation of $R[Y_1, Y_2, Y_3]$ is standard in the case where $R = k[X_1, \ldots, X_n]$, $n \geq 1$. Section 4 studies some specific examples of elementary derivations of $k^{[6]}$ to which none of the results of section 3 can be applied. A criterion for elementariness of irreducible derivations of $R[X, Y]$ (R is a UFD containing $\mathbb{Q}$) in the form of necessary and sufficient condition will be given in section 5. In section 6 we show that monomial elementary derivations of $k^{[6]}$ are standard; in particular, it will be shown that if $R = k[U, V] \cong k^{[2]}$, then the kernel of every R-elementary monomial derivation of $R^{[m]}$ ($m \geq 1$) is a polynomial ring in $m - 1$ variables over R. Finally, we investigate in Section 7 the case of elementary monomial derivations in dimension seven (see derivation (14) above).

2.1 A condition for finite generation and some basic facts

We begin this section by describing the algorithm presented by A. van den Essen in [14] to compute the kernel of a locally nilpotent derivation of polynomial rings over k. Note, however, that the algorithm will not be used in this thesis.

Let $B := k[X_1, \dots, X_n]$ be a polynomial ring in n variables over k, let D be a nonzero locally nilpotent derivation of B and put $A := \ker D$. Fix a local slice s of D and let $d := D(s)$. Consider the n elements

$$b_i := \sum_{j \geq 0} \frac{1}{j!} \left(-\frac{s}{d}\right)^j D^j(X_i)$$

of the localization B_d of B at d. Since D is locally nilpotent, this sum is finite, so $b_i \in B_d$ and we can choose $n_i \in \mathbb{N}$ such that $f_i := d^{n_i} b_i \in B$ for all i. Let $R_0 := k[f_1, \dots, f_n, d]$, then a straightforward calculation shows that $R_0 \subseteq A$. Also it was shown in [14] that

$$R_0 \subseteq A \subseteq R_0[d^{-1}] \tag{15}$$

as subrings of B_d. Next, we define R_m $(m \geq 1)$ inductively as the k-subalgebra of B generated by the elements $h \in B$ satisfying $dh \in R_{m-1}$. Then each R_m is a finitely generated k-algebra and can be computed inductively as follows. Assume that $R_{m-1} = k[f_1, \dots, f_r]$ for some $f_i \in B$ and $r \geq 1$, and consider the ideal

$$I := \{P \in k[T_1, \dots, T_r] \cong k^{[r]} |\ P(f_1, \dots, f_r) \equiv 0 \bmod (d)\}$$

of $k[T_1, \dots, T_r]$. Since $k^{[r]}$ is a noetherian ring, I is generated by a finite number of elements $P_1, \dots, P_s$ of $k[T_1, \dots, T_r]$. For each $i \in \{1, \dots, s\}$, let $g_i \in B$ be such that $P_i(f_1, \dots, f_r) = dg_i$. It was shown in [14] that

$$R_m = k[f_1, \dots, f_r, g_1, \dots, g_s] \tag{16}$$

and that

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq \ker D \quad \text{and} \quad \ker D = \textstyle\bigcup_{i \geq 0} R_i. \tag{17}$$

The essence of the algorithm is the following Proposition:

Proposition 2.1.1. *With the above notation,* $\ker D$ *is a finitely generated* k*-algebra if and only if there exists* $n_0 \in \mathbb{N}$ *such that* $R_{n_0} = R_{n_0+1}$*. In this case* $\ker D = R_{n_0}$.

Proof

Assume that $\ker D$ is finitely generated as a k-algebra by $g_1, \ldots, g_r, r \geq 1$, and choose $n_0 \in \mathbf{N}$ such that $g_i \in R_{n_0}$ for all $i \in \{1, \ldots, r\}$ (this is possible since $\ker D = \cup_{i \geq 0} R_i$), then $\ker D = k[g_1, \ldots, g_r] \subseteq R_{n_0}$ and hence $\ker D = R_{n_0}$ and $R_n = R_{n_0}$ for all $n \geq n_0$. Conversely, assume that $R_{n_0} = R_{n_0+1}$ for some $n_0 \in \mathbf{N}$. If $R_{n_0} \neq \ker D$, then we can choose $x \in \ker D \backslash R_{n_0}$ such that $dx \in R_{n_0}$ (this is because $(\ker D)_d = (R_{n_0})_d$). Then $x \in R_{n_0+1}$ (by the definition of R_{n_0+1}) and this contradicts our hypothesis that $R_{n_0} = R_{n_0+1}$. ∎

As we mentioned before, the above algorithm will not be used in the text. What we will use instead is the following:

Proposition 2.1.2. *Let $E \subseteq A_0 \subseteq A \subseteq C$ be integral domains, where E is a UFD. Suppose that some element d of $E \backslash \{0\}$ satisfies:*

- $(A_0)_d = A_d$
- $pC \cap A_0 = pA_0$ *for each prime divisor p of d, (in E)*

then $A_0 = A$.

Proof

The assumption $pC \cap A_0 = pA_0$ implies (by an easy induction argument) that if q is a finite product of prime factors of d, then $qC \cap A_0 = qA_0$. In particular, $d^nC \cap A_0 = d^nA_0$ for all $n \geq 0$. Now if $y \in A$, then $d^ny \in A_0$ for some $n \geq 0$, so $d^ny \in d^nC \cap A_0 = d^nA_0$ and $y \in A_0$. ∎

Corollary 2.1.3. *Let R be a UFD, B an integral domain containing R, and $D : B \to B$ a locally nilpotent R-derivation. Assume that $s \in B$ satisfies $D(s) \in R \setminus \{0\}$. Let $d = D(s)$ and let A be an R-subalgebra of $\ker(D)$ satisfying $A_d = (\ker D)_d$. If $A \cap pB = pA$ holds for each prime factor p of d in R, then $\ker D = A$.*

Proposition 2.1.4. *Let $a_1, \ldots, a_m$, $m \geq 1$, be elements of a UFD R containing $\mathbb{Q}$, and let A be the kernel of the corresponding R-elementary derivation $D = a_1\partial_1 + \cdots +$*

$a_m\partial_m$ of $B := R[Y_1, \ldots, Y_m]$. Fix $i \in \{1, \ldots, m\}$ such that $a_i \neq 0$ and consider the R-algebra A_i generated by the $m-1$ elements

$$L_{ij} := \tfrac{a_i}{g_{ij}}Y_j - \tfrac{a_j}{g_{ij}}Y_i, \quad j \in \{1, \ldots, m\}\backslash\{i\}$$

where $g_{ij} = gcd(a_i, a_j)$. Then $A_{a_i} = (A_i)_{a_i}$.

Proof

For the proof, we may clearly assume that $i = 1$ (so $a_1 \neq 0$). Note that if $i \neq j$, then $D(L_{ij}) = 0$ and hence $A_1 := R[L_{1j} : j > 1] \subseteq A$. Let $S := \{a_1^n; n \geq 0\}$, then S is a multiplicatively closed subset of $R \subseteq A$, and hence D induces a locally nilpotent derivation $S^{-1}D : S^{-1}B \longrightarrow S^{-1}B$ (defined by the quotient rule of differentiation) satisfying $A_{a_1} = S^{-1}A = \ker S^{-1}D$. On the other hand, $\frac{Y_1}{a_1}$ is a slice for $S^{-1}D$ (i.e., $S^{-1}D\left(\frac{Y_1}{a_1}\right) = 1$) so (Proposition 1.2.10) $\ker S^{-1}D$ is equal to im ζ where ζ is the homomorphism

$$\begin{array}{ccc} S^{-1}B & \longrightarrow & S^{-1}B \\ c & \mapsto & \sum_{j\geq 0} \frac{1}{j!}\left(-\frac{Y_1}{a_1}\right)^j (S^{-1}D)^j(c) \end{array}$$

We have $\zeta(Y_1) = 0$ and, for $i > 1$, $\zeta(Y_i) = Y_i - \frac{a_i}{a_1}Y_1 = \frac{g_{1i}}{a_1}L_{1i}$, where $\frac{g_{1i}}{a_1}$ is a unit of R_{a_1}. So, $A_{a_1} = \text{im}\,\zeta = R_{a_1}[L_{12}, \ldots, L_{1m}] = (A_1)_{a_1}$, which proves the proposition. ∎

We end this section with a proposition that allows us, for the purpose of proving that the kernel of a certain derivation of $k^{[n]}$ is a finitely generated k-algebra, to assume that k is an algebraically closed field.

Let $k \subseteq k'$ be a field extension, B a k-algebra, and let D be a k-derivation of B. Let $B' = B\otimes_k k'$ and consider the k'-derivation $D' = D \otimes 1$ of B' (D' is the map defined by $D'(x \otimes a) = D(x) \otimes a$ for all $x \in B$ and $a \in k'$). Let $A = \ker D$ and $A' = \ker D'$, then we have the following

Proposition 2.1.5. *(**Proposition 4.1**, [13]) A is a finitely generated k-algebra if and only if A' is a finitely generated k'-algebra.*

Proof
We start the proof by claiming that $A\otimes_k k'$ and $\ker D'$ are isomorphic as k'-algebras. Indeed, since $-\otimes_k k'$ is an exact functor, the exact sequence

$$0 \to A \overset{i}{\hookrightarrow} B \overset{D}{\to} B$$

of k-modules induces the exact sequence

$$0 \to A\otimes_k k' \overset{i\otimes 1}{\to} B\otimes_k k' \overset{D'}{\to} B\otimes_k k'$$

of k'-modules. Consequently $\ker D' = \mathrm{Im}(i \otimes 1) \cong A\otimes_k k'$ and the claim is proved. Next, assume that $\{f_1, \dots, f_r\}$ generates A over k, then clearly $\{f_1 \otimes 1, \dots, f_r \otimes 1\}$ generates A' over k'. Conversely, assume that $A' \cong A\otimes_k k'$ is finitely generated over k', then one can choose a finite subset F of A such that $\{f \otimes 1,\ f \in F\}$ generates A' over k' (this is because $\{a \otimes 1 \mid a \in A\}$ generates $A\otimes_k k'$ over k'). Since $k[F]$ is a k-submodule of A, we have the following exact sequence of k-modules

$$0 \to k[F] \overset{i}{\hookrightarrow} A \overset{\pi}{\to} A/k[F] \to 0$$

(π is the natural epimorphism) which yields the following exact sequence of k'-modules

$$0 \to k[F]\otimes_k k' \overset{i\otimes 1}{\to} A\otimes_k k' \overset{\pi\otimes 1}{\to} (A/k[F]) \otimes_k k' \to 0.$$

Thus, $\pi\otimes 1$ is onto and $(A/k[F])\otimes_k k' \cong (A\otimes_k k')/\mathrm{Im}(i\otimes 1) \cong (A\otimes_k k')/(k[F]\otimes_k k') = 0$. We conclude that $A = k[F]$ is a finitely generated k-algebra. ∎

Remark 2.1.6. We also see that if F is any subset of A then the condition $A = k[F]$ is equivalent to $A' = k'[F]$.

2.2 Elementary fixed point free derivations over a UFD

Let B be an integral domain containing $\mathbb{Q}$, and let $D : B \longrightarrow B$ be a locally nilpotent derivation.

Then (see Chapter 1) there is an associated $\mathbf{G}_a$-action, $\alpha : \mathbf{G}_a \times \operatorname{Spec} B \to \operatorname{Spec} B$, and it turns out that the set of fixed points of α is the closed subset $V(I)$ of $\operatorname{Spec} B$, where I denotes the ideal (DB) of B generated by DB (the image of B). In particular, α is fixed point free if and only if $(DB) = B$. This motivates:

Definition 2.2.1. The locally nilpotent derivation $D : B \to B$ is called *fixed point free* if the ideal of B generated by the image of D is equal to B.

Obviously, if D admits a slice (i.e., if there exists $s \in B$ such that $Ds = 1$) then D is fixed point free. It is well-known that the converse is not true in general, but that it does hold for elementary derivations:

Lemma 2.2.2. *Let $R \supseteq \mathbf{Q}$ be a domain, $B = R[Y_1, \dots, Y_m] = R^{[m]}$, and $D : B \to B$ an R-elementary derivation.*

1. *If D is fixed point free, then it admits a slice.*
2. *If D is fixed point free and $R = k^{[n]}$, then there exists a coordinate system $(Z_1, \dots, Z_m)$ of B over R such that $D = \partial/\partial Z_m$.*

Proof

Write $D = \sum_{i=1}^m a_i \partial_i$ where $a_i \in R$ and $\partial_i = \partial/\partial Y_i$. If D is fixed point free then $B = (DB) = (a_1, \dots, a_m)B$, so $(a_1, \dots, a_m)R = R$, and $\sum_{i=1}^m a_i r_i = 1$ for some $r_1, \dots, r_m \in R$. Then $s = \sum_{i=1}^m r_i Y_i$ is a slice of D. If $R = k^{[n]}$ then, by the Quillen-Suslin theorem, $(r_1 \ \dots \ r_m)$ is the first row of a matrix $U \in \mathrm{Gl}_m(R)$ and it follows that s is a variable of B over R. The result follows from Lemma 1.2.13. ∎

Remark 2.2.3. The use of Quillen-Suslin theorem in the above lemma appeared in the proof of Theorem 2.1 of [13].

Proposition 2.2.4. *Let $\mathbf{Q} \subseteq R \subseteq B$ be domains and let $D : B \to B$ be a locally nilpotent R-derivation with a slice s. Then*

$$\begin{aligned} \zeta : \ B &\longrightarrow B \\ x &\mapsto \sum_{i \geq 0} \frac{1}{i!}(-s)^i D^i(x) \end{aligned}$$

is a homomorphism of R-algebras with image equal to ker D*. In particular, if* B *is finitely generated over* R*,* $B = R[e_1, \dots, e_m]$*, then* ker D *is finitely generated over* R *and*

$$\ker D = R[\zeta(e_1), \dots, \zeta(e_m)].$$

Proof

The result follows from Proposition 1.2.10. ∎

As a consequence we have the following:

Corollary 2.2.5. *Let* $R \supseteq \mathbb{Q}$ *be a UFD,* $B = R[Y_1, \dots, Y_m] = R^{[m]}$ *and* $D : B \to B$ *an R-elementary derivation. If* $DY_i \in R^*$ *for some* i*, then* ker D *is generated by* $m-1$ *standard linear constants.*

Proof

We may assume that $DY_1 \in R^*$. Define $s = (DY_1)^{-1}Y_1$, then s is a slice of D and consequently the map $B \xrightarrow{\xi} B$, $\xi(x) = \sum_{j\geq 0} \frac{1}{j!}(-s)^j D^j(x)$, is a homomorphism of R-algebras with image equal to $\ker D$. Thus $\ker D = R[\xi(Y_1), \dots, \xi(Y_m)]$ and we are done because $\xi(Y_j) = Y_j - (DY_j)s = L_{1,j}$ for each j. ∎

We prove now the main result of this section.

Theorem 2.2.6. *Let* $R \supseteq \mathbb{Q}$ *be a UFD,* $B = R[Y_1, \dots, Y_m] = R^{[m]}$ *and* $D : B \to B$ *an R-elementary derivation. If* D *is fixed point free, then it is standard.*

Proof

Write $D = \sum_{i=1}^m a_i \partial_i$, where $a_i \in R$ and $\partial_i = \partial/\partial Y_i$. Since D is fixed point free, there exist $r_1, \dots, r_m \in R$ such that $\sum_{i=1}^m r_i a_i = 1$. Hence, $s = \sum_{i=1}^m r_i Y_i$ is a slice of D and consequently the map $B \xrightarrow{\xi} B$, $\xi(x) = \sum_{j\geq 0} \frac{1}{j!}(-s)^j D^j(x)$, is a homomorphism of R-algebras with image equal to $\ker D$. Therefore $\ker D = R[\xi(Y_1), \dots, \xi(Y_m)]$, where each $\xi(Y_i) = Y_i - a_i s$ is a linear constant. We obtain:

$$\ker D \text{ is generated as an } R\text{-algebra by } m \text{ linear constants.} \tag{18}$$

So it suffices to show that each linear constant is a linear combination (over R) of the standard linear constants. In other words, we have to show that the R-module $T(D)$ is trivial, where:

$$\begin{aligned} \text{LC}(D) &= \text{ set of linear constants of } D \text{ (an } R\text{-submodule of } \ker D), \\ \text{SLC}(D) &= R\text{-submodule of } \text{LC}(D) \text{ generated by the standard linear constants,} \\ T(D) &= \text{LC}(D)/\,\text{SLC}(D). \end{aligned}$$

Let $\mathfrak{m}$ be a maximal ideal of R and consider the derivation $D_\mathfrak{m} : B_\mathfrak{m} \to B_\mathfrak{m}$ obtained by localization at the set $R \setminus \mathfrak{m}$. Now $R_\mathfrak{m}$ is a UFD, $B_\mathfrak{m} = R_\mathfrak{m}[Y_1, \ldots, Y_m] = R_\mathfrak{m}^{[m]}$ and $D_\mathfrak{m} = \sum_{i=1}^{m} a_i \partial_i$ is an $R_\mathfrak{m}$-elementary derivation. Since D is fixed point free, we have $(a_1, \ldots, a_m)R \not\subseteq \mathfrak{m}$ so, for some i, a_i is a unit of $R_\mathfrak{m}$. By the previous result, $D_\mathfrak{m}$ is standard, so $T(D_\mathfrak{m}) = 0$. It is immediate that $\text{LC}(D_\mathfrak{m}) = \text{LC}(D)_\mathfrak{m}$ and $\text{SLC}(D_\mathfrak{m}) = \text{SLC}(D)_\mathfrak{m}$, so $T(D_\mathfrak{m}) = T(D)_\mathfrak{m}$ and we have shown:

$$T(D)_\mathfrak{m} = 0 \quad \text{for all maximal ideals } \mathfrak{m} \text{ of } R.$$

We conclude that $T(D) = 0$ and the result follows. ∎

Corollary 2.2.7. *If R is a PID containing $\mathbb{Q}$, then any nonzero R-elementary derivation of $R[Y] = R[Y_1, \cdots, Y_n] \cong R^{[n]}$ is standard.*

Proof

Let $D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial Y_i}$ be such a derivation of $R[Y]$ ($a_i \in R$ for all i). For the purpose of proving that $\ker D$ is generated over R by standard linear elements, we may clearly assume that D is irreducible, and hence we have that $(a_1, \ldots, a_n)R = R$ (R is a PID). This implies that D is fixed point free, and the Corollary can be now deduced from Theorem 2.2.6. ∎

In particular, the kernel of any $k[X]$-elementary derivation of $k[X, Y_1, \cdots, Y_{n-1}] = k^{[n]}$ is a finitely generated $k[X]$-algebra.

Remark 2.2.8. The results of Theorem 2.2.6 and Corollary 2.2.7 remain true if B is replaced by any polynomial ring over R (not necessarily of finite type over R).

Remark 2.2.9. In section 2.4, we will show that the kernel of the $k[X_1, X_2, X_3]$-elementary derivation

$$D = (X_1^2 - X_2X_3)\frac{\partial}{\partial Y_1} + (X_2^2 - X_1X_3)\frac{\partial}{\partial Y_2} + (X_3^2 - X_1X_2)\frac{\partial}{\partial Y_3}$$

of $B = k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is generated over $k[X_1, X_2, X_3]$ by the two elements

$$f = X_3Y_1 + X_1Y_2 + X_2Y_3, \quad g = X_2Y_1 + X_3Y_2 + X_1Y_3.$$

We claim that D is not standard in this case. To see this, it is enough to notice that f is homogeneous of degree 2 in the X_i's and the Y_j's while each standard linear constant is homogeneous of degree 3. In other words, $f \in \ker D \backslash R[L_1, L_2, L_3]$ where L_1, L_2, L_3 are the standard linear constants of D. This shows that the condition "fixed point free" of Theorem 2.2.6 is not superfluous (clearly, D is not fixed point free).

2.3 Main Theorems

Let $n, m \geq 1$, $k[X, Y] := k[X_1, \dots, X_n, Y_1, \dots, Y_m] \cong k^{[n+m]}$. A corollary of the main result of this section (Theorem 2.3.3) will state that for n arbitrary and $m = 3$, any monomial $k[X_1, \dots, X_n]$-elementary derivation of $k[X, Y]$ is either zero or standard. This will give a negative answer to Question 2.0.3 in the particular case of monomial derivations.

Elementary derivations will be studied in the more general case where we replace $k[X_1, \dots, X_n]$ by a unique factorization domain which is a finitely generated k-algebra.

In what follows we introduce some notation and prove some facts which will be used in the proof of the main result in this chapter. Let R be a UFD which is a finitely generated k-algebra, and $B = R[Y_1, Y_2, Y_3] \cong R^{[3]}$. For $i \in \{1, 2, 3\}$, let ∂_i denote the partial derivative with respect to Y_i. If a_1, a_2, a_3 are relatively prime elements of R, define $g_i := \gcd(a_j, a_k)$ for $i = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$, and fix three elements of B

$$L_1 = \tfrac{a_3}{g_1}Y_2 - \tfrac{a_2}{g_1}Y_3, \quad L_2 = -\tfrac{a_3}{g_2}Y_1 + \tfrac{a_1}{g_2}Y_3, \quad L_3 = \tfrac{a_2}{g_3}Y_1 - \tfrac{a_1}{g_3}Y_2,$$

with the understanding that $g_i = 1$ and $L_i = 0$ when $a_i = a_k = 0$. In other words, L_1, L_2, L_3 are the standard linear constants. Then we have the following easy lemma.

Lemma 2.3.1. (1) *g_1, g_2, g_3 are pairwise relatively prime in R.*
(2) *If $\{i, j, k\} = \{1, 2, 3\}$, then $g_i g_j$ is a divisor of a_k in R.*
(3) *Write $a_k = \alpha_k g_i g_j$ for $\{i, j, k\} = \{1, 2, 3\}$, then $\alpha_1, \alpha_2, \alpha_3$ are pairwise relatively prime in R.*
(4) *$L_i \in \ker D$ for all $i \in \{1, 2, 3\}$ where D is the elementary derivation $a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ of B.*

Proof
(1) Let $i \neq j \in \{1, 2, 3\}$ and let $d = \gcd(g_i, g_j)$, then d is a common divisor of a_1, a_2, a_3 and so $d = 1$ since the a_i's are assumed to be relatively prime.
(2) If $\{i, j, k\} = \{1, 2, 3\}$, write

$$a_k = e_k g_i = b_k g_j \tag{19}$$

for some $e_k, b_k \in R$. Since g_i, g_j are relatively prime by part (1), equation (19) implies that $e_k = \alpha_k g_j$ for some $\alpha_k \in R$. Hence $a_k = e_k g_i = \alpha_k g_i g_j$.
(3) For $\{i, j, k\} = \{1, 2, 3\}$, write $a_i = \beta_i g_k$, $a_j = \beta_j g_k$ for some relatively prime elements β_i and β_j of R. By part (2), $a_i = \alpha_i g_j g_k$ and $a_j = \alpha_j g_i g_k$ and so $\beta_i = \alpha_i g_j$ and $\beta_j = \alpha_j g_i$. If now d is a common divisor of α_i and α_j, then d is also a common divisor of β_i and β_j and hence $d \in R^*$. ∎

As a corollary of Proposition 2.1.4, we also have:

Corollary 2.3.2. *With the above notation, $(\ker D)_{a_i} = (R[L_1, L_2, L_3])_{a_i}$ for any $i \in \{1, 2, 3\}$.*

The main result of this section gives a necessary and a sufficient condition for D to be standard in the case where one of the a_i is such that R/pR is a UFD for every prime divisor p of a_i. Namely we have the following

Theorem 2.3.3. *Let R be a UFD which is a finitely generated k-algebra, and let $B = R[Y_1, Y_2, Y_3] \cong R^{[3]}$. Let $D = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ be an irreducible R-elementary derivation of B (i.e. $gcd(a_1, a_2, a_3) = 1$) and let g_i, α_i, L_i $(i = 1, 2, 3)$ be as above. Assume that $a_3 \neq 0$ and that for every prime divisor p of a_3, the ring $\overline{R} := R/pR$ is a UFD. Then $\ker D = R[L_1, L_2, L_3]$ if and only if $gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$ in $\overline{R}$ for every prime divisor p of a_3.*

Remark 2.3.4. With the notation of Theorem 2.3.3, if only one of the a_i's, say a_3, is zero, then theorem 2.4 of [7] implies that the kernel of the derivation $a_1\partial_1 + a_2\partial_2$ of $R[Y_1, Y_2]$ is $R[a_1Y_2 - a_2Y_1] = R[L_3]$, and so $\ker D = R[L_3, Y_3] = R[L_1, L_2, L_3]$ (since in this case $L_1 = -Y_3$ and $L_2 = Y_3$). If two of the a_i's are equal to zero, say $a_1 = a_2 = 0$, then clearly $\ker D = R[Y_1, Y_2] = R[L_1, L_2, L_3]$ (since $L_1 = Y_2, L_2 = -Y_1$ and $L_3 = 0$ in this case). So if one at least of the a_i's is zero, $\ker D = R[L_1, L_2, L_3]$.

For the proof of Theorem 2.3.3, we need the following lemma.

Lemma 2.3.5. *With the notation of Theorem 2.3.3, if $p \in R$ is a prime element such that $\overline{R} := R/pR$ is a UFD, then $pB \cap R[L_1, L_2, L_3] = pR[L_1, L_2, L_3]$ if and only if the elements $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ of $\overline{R}$ are relatively prime.*

Proof

If one of the a_i's is zero, then Remark 2.3.4 implies that $\ker D = R[L_1, L_2, L_3]$ and hence $R[L_1, L_2, L_3]$ is a factorially closed subring of B; then the equality $pB \cap R[L_1, L_2, L_3] = pR[L_1, L_2, L_3]$ is true. On the other hand, it is easy to see that one of the α_i's is invertible in this case, and so $\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$. Thus, we may, and will assume that $a_i \neq 0$ for all i.

Let $R_0 = R[L_1, L_2, L_3] \subseteq \ker D$. Assume first that $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ are relatively prime in $\overline{R}$, we need to prove that $pB \cap R_0 \subseteq pR_0$ (the other inclusion being clear). To see this, consider the ring homomorphism

$$\phi : \overline{R}[S, T, U] \longrightarrow \overline{R}[\overline{L_1}, \overline{L_2}, \overline{L_3}]$$

sending S, T, U to $\overline{L_1}, \overline{L_2}, \overline{L_3}$ respectively (here $\overline{R}[S, T, U] \cong \overline{R}^{[3]}$), and let $\wp$ be the kernel of ϕ.

Claim $\wp = \overline{q}\overline{R}[S,T,U]$ where $\overline{q} = \overline{\alpha_1}S + \overline{\alpha_2}T + \overline{\alpha_3}U$.
Indeed, modulo p we have that $\overline{L_1} = \overline{\alpha_3 g_2}Y_2 - \overline{\alpha_2 g_3}Y_3$, $\overline{L_2} = -\overline{\alpha_3 g_1}Y_1 + \overline{\alpha_1 g_3}Y_3$, $\overline{L_3} = \overline{\alpha_2 g_1}Y_1 - \overline{\alpha_1 g_2}Y_2$ and since $\overline{\alpha_1 L_1} + \overline{\alpha_2 L_2} + \overline{\alpha_3 L_3} = 0$ in $\overline{R}$ (in fact $\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 = 0$ in R), then $\operatorname{trdeg}_{\overline{R}} \overline{R}\,[\overline{L_1}, \overline{L_2}, \overline{L_3}] \leq 2$. Next we prove that $\operatorname{trdeg}_{\overline{R}} \overline{R}\,[\overline{L_1}, \overline{L_2}, \overline{L_3}] \geq 2$. Since we assumed that $a_i \neq 0$ for all i, the coefficients $\frac{a_2}{g_1}$ and $\frac{a_3}{g_1}$ of L_1 are relatively prime in R. So, $\overline{L_1} \neq 0$ and, similarly, $\overline{L_2} \neq 0$ and $\overline{L_3} \neq 0$. This means that every row of the following matrix (with entries in the field of quotients of $\overline{R}$)

$$Q = \begin{pmatrix} 0 & \overline{\alpha_3 g_2} & -\overline{\alpha_2 g_3} \\ -\overline{\alpha_3 g_1} & 0 & \overline{\alpha_1 g_3} \\ \overline{\alpha_2 g_1} & -\overline{\alpha_1 g_2} & 0 \end{pmatrix}$$

is nonzero. Now any matrix

$$Q' = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$

with entries in a field and with no zero row has rank at least 2. Hence, rank $Q = 2$ and consequently

$$\operatorname{trdeg}_{\overline{R}} \overline{R}\,[\overline{L_1}, \overline{L_2}, \overline{L_3}] = 2.$$

Thus, the height of $\wp$ is one, and $\wp$ is a principal ideal of $\overline{R}[S,T,U]$, since $\overline{R}$ is a UFD. Consider the element $\overline{q} = \overline{\alpha_1}S + \overline{\alpha_2}T + \overline{\alpha_3}U \in \overline{R}[S,T,U]$, then clearly $\phi(\overline{q}) = 0$, and since $\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$ by assumption, $\overline{q}$ is irreducible. Thus $\wp = \overline{q}\overline{R}[S,T,U]$ as claimed.

From the claim, it follows easily that the kernel of the homomorphism

$$\psi : R[S,T,U] \longrightarrow R[L_1, L_2, L_3] \longrightarrow \overline{R}[\overline{L_1}, \overline{L_2}, \overline{L_3}]$$

is the ideal $(\alpha_1 S + \alpha_2 T + \alpha_3 U, p)$ of $R[S,T,U]$. Now we prove the inclusion $pB \cap R_0 \subseteq pR_0$. Let $x \in pB \cap R_0$ and choose $\Phi \in R[S,T,U]$ and $b \in B$ such that $x = pb = \Phi(L_1, L_2, L_3)$, then clearly $\Phi \in \ker(\psi)$ and hence we can write $\Phi =$

$(\alpha_1 S + \alpha_2 T + \alpha_3 U)\Phi_1 + p\Phi_2$ for some $\Phi_1, \Phi_2 \in R[S,T,U]$. This shows that $x = p\Phi_2(L_1, L_2, L_3) \in pR_0$.

Next, we prove the other direction. Assume that $pB \cap R_0 = pR_0$, we show that $\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$ in $\overline{R}$. Let $g \in R$ be such that $\overline{g} = \gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3})$ in $\overline{R}$ and write $\overline{\alpha_i} = \overline{\beta_i}\overline{g}$ for some $\beta_i \in R$, and $\gcd(\overline{\beta_1}, \overline{\beta_2}, \overline{\beta_3}) = 1$ in $\overline{R}$. Also, choose $\zeta_i \in R$ such that

$$\alpha_i = \beta_i g + \zeta_i p \tag{20}$$

for $i \in \{1,2,3\}$. Now since $\overline{\alpha_1 L_1} + \overline{\alpha_2 L_2} + \overline{\alpha_3 L_3} = 0$, then either $\overline{g} = 0$ or $\overline{\beta_1 L_1} + \overline{\beta_2 L_2} + \overline{\beta_3 L_3} = 0$. If ever $\overline{g} = 0$, then $g = rp$ for some $r \in R$, and equation (20) implies that $p|\alpha_i$ for all i, which gives a contradiction to the fact that the α_i's are relatively prime (Lemma 2.3.1). We deduce that $\beta_1 L_1 + \beta_2 L_2 + \beta_3 L_3 \in pB \cap R_0 = pR_0$. Choose $\Phi \in R[S,T,U]$ such that $\beta_1 L_1 + \beta_2 L_2 + \beta_3 L_3 = p\Phi(L_1, L_2, L_3)$ and write $\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n$ where Φ_i is the homogeneous component of Φ of degree i. Since each L_j is homogeneous of degree 1, then each $\Phi_i(L_1, L_2, L_3)$ is also homogeneoous of degree i, and this means that $\Phi_i(L_1, L_2, L_3) = 0$ for all $i \neq 1$. Thus, $\beta_1 L_1 + \beta_2 L_2 + \beta_3 L_3 = p\Phi_1(L_1, L_2, L_3) = p(\gamma_1 L_1 + \gamma_2 L_2 + \gamma_3 L_3)$ where $\gamma_i \in R$ for all i and this gives the equation

$$(\beta_1 - p\gamma_1)L_1 + (\beta_2 - p\gamma_2)L_2 + (\beta_3 - p\gamma_3)L_3 = 0.$$

Let $\lambda_i := \beta_i - p\gamma_i$ for all $i \in \{1,2,3\}$, then we have the equations

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0 \tag{21}$$

$$\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 = 0. \tag{22}$$

Let K be the field of fractions of R, then clearly L_i, L_j are linearly independent over K as vectors of the K-vector space

$$V = \{P \in K[Y_1, Y_2, Y_3] | P \text{ is homogeneous of degree one}\} \cup \{0\}$$

Also, since $\overline{\beta_i} \neq 0$ for at least one $i \in \{1,2,3\}$ (otherwise, the β_i's would not be relatively prime), we deduce that $\lambda_i \neq 0$ for at least one i. Assume that $\lambda_1 \neq 0$, then from equations (21) and (22) above we can deduce that

$$\frac{\alpha_2}{\alpha_1}L_2 + \frac{\alpha_3}{\alpha_1}L_3 = \frac{\lambda_2}{\lambda_1}L_2 + \frac{\lambda_3}{\lambda_1}L_3$$

as elements of V. This gives the two equations

$$\alpha_i\lambda_1 = \alpha_1\lambda_i,\ i = 2,3 \tag{23}$$

Now since α_1, α_i are relatively prime for $i = 2,3$ (Lemma 2.3.1), then equation (23) shows that α_i divides λ_i for $i = 1,2,3$, and equations (23) imply that $\lambda_i = \mu\alpha_i$ for $i = 1,2,3$, where $\mu = \frac{\lambda_1}{\alpha_1} \in R$. Hence $\beta_i - p\gamma_i = \mu\alpha_i = \mu(\beta_i g + \zeta_i p)$ for all $i = 1,2,3$. In other words, $\overline{\beta_i}(1 - \overline{\mu g}) = 0$ for all $i \in \{1,2,3\}$. Choose i such that $\overline{\beta_i} \neq 0$, then $\overline{\mu g} = 1$ and hence $\overline{g} \in \overline{R}^*$. This shows that $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ are relatively prime in $\overline{R}$ and the Lemma is proved. ■

The main Theorem of this section can now be deduced easily from the above Lemma.

Proof of Theorem 2.3.3 If $\ker D = R[L_1, L_2, L_3]$, then in particular, $R[L_1, L_2, L_3]$ is factorially closed in B (as the kernel of a locally nilpotent derivation of B). Let p be a prime divisor of a_3 and let $x \in pB \cap R[L_1, L_2, L_3]$, and write $x = pb$ for some $b \in B$, then $b \in R[L_1, L_2, L_3]$ since the latter is factorially closed, and so $pB \cap R[L_1, L_2, L_3] = pR[L_1, L_2, L_3]$. Then Lemma 2.3.5 gives that $\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$. Conversely, assume that for each prime divisor p of a_3, the elements $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ of $\overline{R} := R/pR$ are relatively prime, then by Lemma 2.3.5, $pB \cap R[L_1, L_2, L_3] = pR[L_1, L_2, L_3]$ for all prime divisors p of a_3. By Corollary 2.3.2 and Proposition 2.1.2, we deduce that $\ker D = R[L_1, L_2, L_3]$. ■

Example 2.3.6. Consider the derivation

$$D = (X_1^a + X_2^b)\frac{\partial}{\partial Y_1} + (X_1^c + X_3^d)\frac{\partial}{\partial Y_2} + X_1^{t_1}X_2^{t_2}X_3^{t_3}\frac{\partial}{\partial Y_3}$$

of $C = k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$, where $t_1, t_2, t_3, a, b, c, d \in \mathbb{N}$. It is clear that the ring

$$k[X_1, X_2, X_3]/pk[X_1, X_2, X_3]$$

is a UFD for every prime divisor p of $a_3 = X_1^{t_1}X_2^{t_2}X_3^{t_3}$, and that a_1, a_2, a_3 are pairwise relatively prime in $k[X_1, X_2, X_3]$. Moreover, for any $i \in \{1,2,3\}$, taking $X_i = 0$ in the expressions of a_1 and a_2 (that is taking a_1, a_2 modulo X_i) clearly yields relatively prime elements in $k[X_j, X_k]$ where $\{1,2,3\} = \{i,j,k\}$. Theorem 2.3.3 implies that $\ker D$ is generated over $k[X_1, X_2, X_3]$ by L_1, L_2, L_3.

An important consequence of Theorem 2.3.3 is the following.

Corollary 2.3.7. *If $R = k[X_1, \dots, X_n]$ is a polynomial ring in n variables ($n \geq 1$) over k, then every nonzero R-elementary monomial derivation of $R[Y_1, Y_2, Y_3]$ is standard.*

Proof

Let $D = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ be an elementary monomial derivation of $B = R[Y_1, Y_2, Y_3]$. We may assume that D is irreducible and (by Remark 2.3.4) that $a_i \neq 0$ for all i. Let α_i, L_i be as above. For any $i \in \{1, \dots, n\}$, we can choose $j \neq k$ such that α_j, α_k are not divisible by X_i (since the α_i's are pairwise relatively prime). This means that

$$\alpha_j, \alpha_k \in k[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n] \cong k[X_1, \dots, X_n]/(X_i)$$

and therefore the α_i's are relatively prime modulo X_i for any i. The result now follows now from Theorem 2.3.3. ∎

Example 2.3.8. The kernel of the derivation

$$X_1^a \frac{\partial}{\partial Y_1} + X_2^b \frac{\partial}{\partial Y_2} + (X_1X_2X_3)^c \frac{\partial}{\partial Y_3}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is a finitely generated k-algebra for any nonnegative integers a, b, c (see derivation (14) above).

Remark 2.3.9. The above corollary can be generalized as follows: let $R = k[X_i, i \in I]$ be an arbitrary polynomial ring over k, and let $B = R[Y_1, Y_2, Y_3] \cong R^{[3]}$. Let D be the k-derivation of B sending X_i to zero for all $i \in I$, and Y_1, Y_2 and Y_3 to monomials a_1, a_2 and a_3 respectively of R. Choose a finite subset I_0 of I such that a_1, a_2, a_3 are elements of $R_0 = k[X_i; i \in I_0] \subseteq R$ and let D_0 be the restriction of D to $R_0[Y_1, Y_2, Y_3]$. Clearly, $\ker D = (\ker D_0)[X_i; i \in I \backslash I_0]$, and by Corollary 2.3.7, $\ker D_0 = R_0[L_1, L_2, L_3]$; so $\ker D = R[L_1, L_2, L_3]$.

Corollary 2.3.10. *Let $R = k[X_1, \dots, X_n]$ be a polynomial ring in n variables ($n \geq 1$) over k, and let a_1, a_2, a_3 be relatively prime elements of R satisfying the following properties:*

- *a_3 is a polynomial in only one variable, say X_1, over k*
- *If b is any root of a_3 in $\overline{k}$ (the algebraic closure of k), then the elements*

$$\alpha_i(b, X_2, \dots, X_n),\ i = 1, 2, 3$$

of $\overline{k}[X_2, \dots, X_n]$ are relatively prime.

Then the kernel of the derivation $D = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ of $R[Y_1, Y_2, Y_3]$ is equal to $R[L_1, L_2, L_3]$.

Proof

By Proposition 2.1.5, we may assume that $\overline{k} = k$. In this case

$$a_3 = \prod_{s=1}^{r}(X_1 - b_s)^{t_s},\ t_s \geq 1,$$

where $\{b_1, \dots, b_r\}$ is the set of distinct roots of a_3 in $\overline{k}$. Clearly, $R/(X_1 - b_s)$ is a UFD for each s and the image of α_k in the quotient ring $R/(X_1 - b_s)$ is nothing but $\alpha_k(b_s, X_2, \dots, X_n)$. The Corollary can now be deduced from Theorem 2.3.3. ∎

Example 2.3.11. The derivation

$$D = (X_1^2X_2X_3 + 2X_1X_3^3)\frac{\partial}{\partial Y_1} + (-3X_1^4X_3 + 5X_2^3X_3 + 2)\frac{\partial}{\partial Y_2} + (X_1^2 - 3X_1 + 2)\frac{\partial}{\partial Y_3}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is standard. Indeed, it is clear that a_1, $a_2\, a_3$ are pairwise relatively prime in this case (so $\alpha_j = a_j$ for $j = 1, 2, 3$), and $a_3 = (X_1 - 1)(X_1 - 2)$. On the other hand, we have that

$$a_1(1, X_2, X_3) = X_2X_3 + 2X_3^3, \quad a_2(1, X_2, X_3) = -3X_3 + 5X_2^3X_3 + 2$$
$$a_1(2, X_2, X_3) = 4X_2X_3 + 4X_3^3, \quad a_2(2, X_2, X_3) = -48X_3 + 5X_2^3X_3 + 2,$$

and hence $\gcd(a_1(i, X_2, X_3), a_2(i, X_2, X_3)) = 1$ for $i = 1, 2$, and the result can be deduced from Corollary 2.3.10.

With the notations of Theorem 2.3.3, if $a_3 = p$ for some prime element p of R satisfying R/pR is a UFD, then the assumption "$\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$" can be dropped and $\ker D$ will remain finitely generated over R by some linear constants. But the fact that D is "standard" is no longer evident. More precisely, we have the following

Proposition 2.3.12. *Let R be a UFD which is a finitely generated k-algebra, and let $B = R[Y_1, Y_2, Y_3] \cong R^{[3]}$. If $p \in R$ is a prime element satisfying R/pR is a UFD, then for any $a_1, a_2 \in R$, the kernel of the R-elementary derivation $D = a_1\partial_1 + a_2\partial_2 + p\partial_3$ of B is generated as an R-algebra by at most three linear constants.*

Proof

For the proof, we may assume that D is irreducible. This implies that either the two elements a_1, a_2 are in $R \backslash pR$ or exactly one of them is in the ideal pR. We will treat each of these two cases separately.

Case 1. $a_1, a_2 \notin pR$. In this case, let $d \in R$ be such that $\overline{d} = \gcd(\overline{a_1}, \overline{a_2})$ in $\overline{R} = R/pR$, and write

$$a_i = \beta_i d + \beta_i' p, \quad i = 1, 2$$

for some $\beta_i, \beta_i' \in R$ with $\gcd(\overline{\beta_1}, \overline{\beta_2}) = 1$ in $\overline{R}$. Consider the elements

$$f = pY_2 - a_2Y_3, \quad g = -pY_1 + a_1Y_3, \quad h = -\beta_2Y_1 + \beta_1Y_2 - (\beta_1\beta_2' - \beta_2\beta_1')Y_3$$

of B. Clearly, $f, g \in \ker D$ and

$$\begin{aligned} D(h) &= -\beta_2a_1 + \beta_1a_2 - (\beta_1\beta_2' - \beta_2\beta_1')p \\ &= -\beta_2(\beta_1d + \beta_1'p) + \beta_1(\beta_2d + \beta_2'p) - (\beta_1\beta_2' - \beta_2\beta_1')p \\ &= 0. \end{aligned}$$

So $R_0 := R[f, g, h] \subseteq \ker D$. By Proposition 2.1.4, we also have that $(R_0)_p = (\ker D)_p$. Consider the homomorphism of $\overline{R}$-algebras

$$\phi : \overline{R}[S, T, U] \cong \overline{R}^{[3]} \longrightarrow \overline{R}[\overline{f}, \overline{g}, \overline{h}]$$

sending S, T, U to $\overline{f}, \overline{g}, \overline{h}$ respectively. Since $\overline{f} = -\overline{a_2}Y_3$, $\overline{g} = \overline{a_1}Y_3$, and $\overline{h} = -\overline{\beta_2}Y_1 + \overline{\beta_1}Y_2 - (\overline{\beta_1\beta_2'} - \overline{\beta_2\beta_1'})Y_3$, the ring $\overline{R_0}$ has transcendence degree two over $\overline{R}$, and so the height of the kernel $\wp$ of ϕ is one. Let $q = \overline{\beta_1}S + \overline{\beta_2}T \in \overline{R}[S, T, U]$, then $q \in \wp$ since $\phi(\overline{d}q) = \phi(\overline{a_1}S + \overline{a_2}T) = 0$ and $\overline{d} \neq 0$. Also q is an irreducible element of $\overline{R}[S, T, U]$ since $\gcd(\overline{\beta_1}, \overline{\beta_2}) = 1$ in $\overline{R}$. We deduce that $\wp = q\overline{R}[S, T, U]$.

Now we prove that $pR_0 = pB \cap R_0$. Let $x \in pB \cap R_0$, and write $x = \Phi(f, g, h) = pb$ for some $\Phi \in R[S, T, U]$ and $b \in B$. The image $\overline{\Phi}$ of Φ in $\overline{R}[S, T, U]$ is in $\wp$ and hence $\Phi = (\beta_1 S + \beta_2 T)\Psi + \Omega p$ for some $\Psi, \Omega \in R[S, T, U]$. This shows that

$$\begin{aligned} x &= \Phi(f, g, h) \\ &= (\beta_1 f + \beta_2 g)\Psi(f, g, h) + \Omega(f, g, h)p \\ &= \Psi(f, g, h)ph + \Omega(g, f, h)p \ \text{(since } \beta_1 f + \beta_2 g = ph) \\ &= (\Psi(f, g, h)h + \Omega(f, g, h))\, p \in pR_0 \end{aligned}$$

and we are done by Proposition 2.1.2.

Case 2. $a_1 \in pR$ Write $a_1 = \beta_1 p^r$ for some $\beta_1 \in R \backslash pR$ and $r \geq 1$. Note that in this case $g_1 = \gcd(a_2, a_3) = 1$, $g_2 = \gcd(a_1, a_3) = p$ and so $\alpha_3 = \frac{a_3}{g_1 g_2} = \frac{p}{p} = 1$. Thus, $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ are relatively prime in $\overline{R}$. Theorem 2.3.3 shows now that $\ker D = R[L_1, L_2, L_3]$. ∎

We continue in what follows to investigate some particular situations where the kernel of an R-elementary derivation of B is finitely generated over R. For the next result, we need the following

Lemma 2.3.13. *Let A be any domain, $a \in A^*$ and $b \in A$, then the ring $A[X]/(aX + b)$ is a domain, where $A[X]$ is the polynomial ring in one variable over A.*

Proof

Consider the homomorphism of A-algebras

$$\begin{aligned} f : A[X] &\longrightarrow A \\ X &\mapsto -a^{-1}b \end{aligned}$$

Clearly $aX + b \in \ker f$ and hence the ideal $(aX + b)$ of $A[X]$ generated by $aX + b$ is contained in $\ker f$. Conversely, let $F \in A[X]$ satisfying $f(F) = 0$, and write $F = G(aX + b) + r$ for some $G \in A[X]$, $r \in A$ (division algorithm in the ring $A[X]$), then $r = f(F) = 0$, and hence $F \in (aX + b)$, so $\ker f = (aX + b)$ and consequently $A[X]/(aX + b)$ is a domain. ∎

Theorem 2.3.14. *Let R be a UFD which is a finitely generated k-algebra, and let $B = R[Y_1, Y_2, Y_3] \cong R^{[3]}$. Let a_1, a_2, a_3 be three relatively prime elements of R and* *assume that there exist $b_1, b_2, b_3 \in R$ such that the element $a := a_1b_1 + a_2b_2 + a_3b_3$ of R is nonzero and is either a unit or for every prime divisor p of a, the following are true:*

1. *R/pR is a UFD, and*

2. *$\gcd(\overline{a_1}, \overline{a_2}) = 1$ in $\overline{R} = R/pR$ and $\overline{b_3} \in \overline{R}^*$.*

Then the kernel of the R-elementary derivation $D = \sum_{i=1}^{3} a_i \frac{\partial}{\partial Y_i}$ of B is generated over R by at most four linear constants.

Proof

If a or one of the a_i's is a unit, then the derivation D is fixed point free, and the result can be deduced from Theorem 2.2.6. So we may assume that a, a_i are not units of R for $i = 1, 2, 3$. Let $s := b_1Y_1 + b_2Y_2 + b_3Y_3 \in B$, then $D(s) = a_1b_1 + a_2b_2 + a_3b_3 = a \in \ker D\backslash\{0\}$, and so s is a local slice for D. Let $f_i = -aY_i + a_is$, $i = 1, 2, 3$ and $g = a_2Y_1 - a_1Y_2$ and let R_0 be the subalgebra $R[f_i, g : i = 1, 2, 3]$ of B. Since $D(f_i) = -aa_i + a_ia = 0$ and $D(g) = 0$, we have that $R_0 \subseteq \ker D$. Also, $(R_0)_a = (\ker D)_a$. Indeed, one can show that $(R[f_1, f_2, f_3])_a = (\ker D)_a$ using a technique similar to the one used in the proof of Proposition 2.1.4. Thus, it is enough (Proposition 2.1.2) to show that $pB \cap R_0 \subseteq pR_0$ for any prime divisor p of a. So, let $x \in pB \cap R_0$, and write $x = \Phi(f_1, f_2, f_3, g)$ for some $\Phi \in R[T_1, T_2, T_3, T_4] \cong R^{[4]}$. Let ϕ be the homomorphism of $\overline{R}$-algebras

$$\phi : \overline{R}[T_1, T_2, T_3, T_4] \longrightarrow \overline{R}[\overline{f_1}, \overline{f_2}, \overline{f_3}, \overline{g}]$$

sending T_i to $\overline{f_i}$, $i \in \{1, 2, 3\}$, respectively, and T_4 to $\overline{g}$. Then the image $\overline{\Phi}$ of Φ in $\overline{R}[T_1, T_2, T_3, T_4]$ belongs to $\wp = \ker \phi$. Consider the two elements $q = \overline{b_1}T_1 + \overline{b_2}T_2 + \overline{b_3}T_3$ and $r = \overline{a_2}T_1 - \overline{a_1}T_2$ of $\overline{R}[T_1, T_2, T_3, T_4]$. We will show that $\wp$ is generated by q and r over $\overline{R}$. First note that it is easy to verify that $b_1f_1 + b_2f_2 + b_3f_3 = 0$, and hence $\phi(q) = 0$; also it is easily seen that $\phi(r) = 0$. Hence the ideal (q, r) of $\overline{R}[T_1, T_2, T_3, T_4]$ is contained in $\wp$. Moreover, since $\overline{a_1}$ and $\overline{a_2}$ are relatively prime in $\overline{R}$, r is a prime element of $\overline{R}[T_1, T_2, T_3, T_4]$. We claim that the following sequence of ideals

$$0 \subset (r) \subset (r, q) \subseteq \wp \tag{24}$$

is a prime chain in $\overline{R}[T_1, T_2, T_3, T_4]$. Indeed, note that $r \neq 0$ since $\overline{a_1}, \overline{a_2} \neq 0$ (otherwise, $\gcd(\overline{a_1}, \overline{a_2}) \neq 1$), and if q belongs to the ideal generated by r, then we would be able to choose $\psi \in R[T_1, T_2, T_3, T_4]$ satisfying

$$\overline{b_1}T_1 + \overline{b_2}T_2 + \overline{b_3}T_3 = (\overline{a_2}T_1 - \overline{a_1}T_2)\overline{\psi}. \tag{25}$$

Sending T_1, T_2 to zero in (25) will give that $\overline{b_3} = 0$ which contradicts the assumption $\overline{b_3} \in \overline{R}^*$. This shows that the first two inclusions in (24) are indeed strict. It remains to show that the ideal (r, q) of $\overline{R}[T_1, T_2, T_3, T_4]$ is prime. To see this, let A be the quotient ring $\overline{R}[T_1, T_2]/(r) = \overline{R}[t_1, t_2]$ (where t_1, t_2 are the images of T_1, T_2, respectively, in A), then A is a domain since r is prime, and we have

$$\begin{aligned} \overline{R}[T_1, T_2, T_3, T_4]/(q, r) &\cong A[T_3, T_4]/\left(\overline{b_3}T_3 + \overline{b_1}t_1 + \overline{b_2}t_2\right) \\ &= \left(A[T_3]/(\overline{b_3}T_3 + \overline{b_1}t_1 + \overline{b_2}t_2)\right)[T_4] \end{aligned}$$

Since $\overline{b_3} \in \overline{R}^* \subseteq A^*$, Lemma 2.3.13 gives that $E := A[T_3]/(\overline{b_3}T_3 + \overline{b_1}t_1 + \overline{b_2}t_2)$ is a domain, and hence $\overline{R}[T_1, T_2, T_3, T_4]/(q, r) \cong E[T_4]$ is also a domain. This proves the claim. Now, the transcendence degree of $\overline{R}[\overline{f_1}, \overline{f_2}, \overline{f_3}, \overline{g}]$ over $\overline{R}$ is two since $\overline{f_i} = \overline{a_i}\,\overline{s}$ for $i = 1, 2, 3$ and $\overline{g} = \overline{a_2}Y_1 - \overline{a_1}Y_2$ and since at least one of $\overline{a_1}, \overline{a_2}$ is not zero (otherwise their gcd is not one), and $\overline{b_3} \neq 0$; thus the height of $\wp$ is two and sequence (24) implies that $\wp = (r, q)$.

To finish the proof of the theorem, find $\alpha, \beta, \gamma \in R[T_1, T_2, T_3, T_4]$ such that

$$\Phi = (b_1T_1 + b_2T_2 + b_3T_3)\alpha + (a_2T_1 - a_1T_2)\beta + p\gamma.$$

Then $x = \Phi(f_1, f_2, f_3, g) = -a\beta(f_1, f_2, f_3, g)g + p\gamma(f_1, f_2, f_3, g) \in pR_0$ (since $\sum_{i=1}^{3} b_i f_i = 0$ and $a_2 f_1 - a_1 f_2 = -ag$). This finishes the proof. ∎

Remark 2.3.15. The condition "$\overline{b_3} \in (R/pR)^*$ for every prime divisor p of a" is equivalent to the condition "b_3 is a unit modulo a". Indeed, let $a = p_1^{t_1} \cdots p_r^{t_r}$ be the prime decomposition of a, and let $\overline{R} = R/aR$ and $R(i) = R/p_iR$ for $i = 1, \dots, r$. If $x \in R$, denote by $\overline{x}, \overline{x}^i$ the images of x in $\overline{R}$ and $R(i)$ respectively. If $\overline{x} \in \overline{R}^*$, then we can choose $\alpha, \beta \in R$ such that $\alpha x = 1 + \beta a$, and so for each i, $\alpha x = 1 + \beta_i p_i$ for some $\beta_i \in R$; this means that $\overline{x}^i \in R(i)^*$ for all $i = 1, \ldots, r$. Conversely, if $\overline{x}^i \in R(i)^*$ for all i, then for each i, we can choose $\alpha_i, \beta_i \in R$ satisfying $\alpha_i x = 1 + \beta_i p_i$ and so $(\alpha_i x - 1)^{t_i} = \gamma_i p_i^{t_i}$ for some $\gamma_i \in R$. Thus $\prod_{i=1}^{r}(\alpha_i x - 1)^{t_i} = \gamma a$ for some $\gamma \in R$. Now $\prod_{i=1}^{r}(\alpha_i x - 1)^{t_i} = \delta x + (-1)^r$ for some $\delta \in R$. Hence $1 \in (x, a)$ and $\overline{x} \in \overline{R}^*$.

If in the above theorem we assume that a is a prime element of R such that R/aR is a UFD, then we can (as in Proposition 2.3.12) drop the assumption "$\gcd(a_1, a_2) = 1$" and we will have the following.

Proposition 2.3.16. *With the notation and assumptions of Theorem* 2.3.14, *if $a = p$ is a prime element of R such that R/pR is a UFD, and $\overline{b_3}$ is a unit in R/pR, then* ker D *is generated by at most four linear constants.*

Proof

Let $d \in R$ be such that $\overline{d} = \gcd(\overline{a_1}, \overline{a_2})$, and write

$$a_i = \beta_i d + \beta_i' p, \quad i = 1, 2$$

for some $\beta_i, \beta_i' \in R$, for $i = 1, 2$, with $\gcd(\overline{\beta_1}, \overline{\beta_2}) = 1$ in $\overline{R}$. Consider the elements

$$g = -\beta_2 Y_1 + \beta_1 Y_2 + (\beta_1 \beta_2' - b_2 \beta_2')\, s, \quad f_i = -pY_i + a_i s, \ i = 1, 2, 3$$

where $s = b_1 Y_1 + b_2 Y_2 + b_3 Y_3$. An argument similar to the one in Proposition 2.3.12 allows us to prove that $\ker D = R[f_1, f_2, f_3, g]$. ∎

Proposition 2.3.17. *Let R be a UFD containing $\mathbb{Q}$, $B = R[Y_1, Y_2, Y_3]$, and $a_1, a_2, a_3 \in R$ be three relatively prime elements of R and $D = \sum_{i=1}^{3} a_i \partial/\partial Y_i$. If a_3 is in the ideal of R generated by a_1 and a_2, then* $\ker D = R[Y_3 - \mu_1 Y_1 - \mu_2 Y_2, L_3] = R^{[2]}$ *for some μ_1, μ_2 in R.*

Proof

Choose $\mu_1, \mu_2 \in R$ such that $a_3 = \mu_1 a_1 + \mu_2 a_2$, and let

$$Y_3' = Y_3 - \mu_1 Y_1 - \mu_2 Y_2.$$

Let $R' = R[Y_3']$, then $R' \subseteq \ker D$ is a UFD containing $\mathbb{Q}$, $B = R'[Y_1, Y_2]$ and D is an R'-elementary derivation of B. We conclude that $\ker D = R'[L_{1,2}]$. ∎

For the next result, we need the following known Lemma.

Lemma 2.3.18. *Let A be any ring, $B = A[X_1, \dots, X_n] \cong A^{[n]}$ $(n \geq 1)$, and let f be a nonzero element of B. Then f is a zero divisor of B if and only if $af = 0$ for some $a \in A \backslash \{0\}$.*

Proof

If $af = 0$ for some $a \in A\backslash\{0\}$, then f is a zero divisor of B. Conversely, if f is a zero divisor of B, we prove that $af = 0$ for some $a \in A\backslash\{0\}$ by induction on the number of variables. For $n = 1$, write $f = \sum_{i=0}^{m} a_i X_1^i$ $(a_i \in A,\ m \geq 0)$, and choose $g = \sum_{i=0}^{r} b_i X_1^i \in A[X_1]\backslash\{0\}$ $(b_i \in A,\ r \geq 0)$ of minimal degree r such that $fg = 0$. If $b_0 = 0$, then $g = g_1 X_1$ for some $g_1 \in A[X_1]\backslash\{0\}$ and so $(fg_1)X_1 = 0$. This implies that $fg_1 = 0$ since X_1 is not a zero divisor of $A[X_1]$; this contradicts the minimality of r. So $b_0 \neq 0$. Next we claim that $a_i g = 0$ for every $i = 0, \dots, m$. Indeed, since $fg = 0$ we have that $a_m b_r = 0$ and hence $a_m g$ has degree strictly less than r. But $(a_m g)f = 0$ and so $a_m g = 0$ by the minimality of r. Assume now that $a_{m-j} g = 0$ for some $0 \leq j \leq i-1$, and let $f_i = a_0 + a_1 X_1 + \dots + a_{m-i} X_1^{m-i}$, then $f = f_i + a_{m-i+1} X_1^{m-i+1} + \dots + a_m X_1^m$ and $fg = f_i g = 0$. This means that $a_{m-i} b_r = 0$, and $a_{m-i} g$ has degree strictly less than r. The minimality of r gives once again that $a_{m-i} g = 0$ and the claim is proved. As a consequence of the claim, we get that $b_0 a_i = 0$ for all $i = 0, \dots, m$ (since $a_i g = 0$ for all i) and hence $b_0 f = 0$. Since $b_0 \neq 0$, the proof in the one variable case is complete. Next, assume that $m \geq 2$, and let $A' = A[X_1, \dots, X_{m-1}]$, then $B = A'[X_m]$. Choose $g \in B\backslash\{0\}$ such that $fg = 0$, then by the one variable case, we have that $hf = 0$ where $h = g(X_1, X_2, \dots, X_{m-1}, 0)$. Write $f = \sum_{i=0}^{r} f_i X_m^i$ where $f_i \in A'$, then $f_i h = 0$ for all i. By the induction hypothesis $af_i = 0$ for all i where $a = h(0, \dots, 0) \in A\backslash\{0\}$. We conclude that $af = 0$. ∎

Corollary 2.3.19. *Let A be a ring, $f \in A[X_1, \dots, X_m]\backslash\{0\}$. If f is a zero divisor of $A[X_1, \dots, X_m]$ then every nonzero coefficient of f is a zero divisor of A.*

Definition 2.3.20. *([21])* A sequence of elements $a_1, \dots, a_n$ of a ring A is called *regular* if the following conditions are satisfied:

1. $(a_1, \dots, a_n)A \neq A$

2. a_1 is not a zero divisor of A, and for $i \geq 2$, a_i is not a zero divisor in

$$A/(a_1, \dots, a_{i-1})A.$$

Proposition 2.3.21. *Let R be a UFD containing $\mathbb{Q}$, $B = R[Y_1, Y_2, Y_3]$, and let a_1, a_2, a_3 be three pairwise relatively prime elements of R such that for every prime divisor p of a_3, the sequence $(\overline{a_1}, \overline{a_2})$ of $\overline{R} = R/pR$ is regular. Then the derivation $D = \sum_{i=1}^{3} a_i \frac{\partial}{\partial Y_i}$ of B is standard.*

Proof

Consider the standard linear constants

$$L_1 = a_1Y_2 - a_2Y_1, \quad L_2 = a_1Y_3 - a_3Y_1, \quad L_3 = a_2Y_3 - a_3Y_2$$

and let $R_0 = R[L_1, L_2, L_3]$ (clearly, $(\ker D)_{a_3} = (R_0)_{a_3}$). Let p be a prime divisor of a_3, $\overline{R} = R/pR$ and $\overline{k}$ be the field of quotients of $\overline{R}$. We need to prove that $pB \cap R_0 \subseteq pR_0$. For this, consider the commutative diagram

$$\begin{array}{ccc} \overline{k}[T_1, T_2, T_3] & \xrightarrow{\overline{\phi}} & \overline{k}[\overline{L_1}, \overline{L_2}, \overline{L_3}] \\ \uparrow & & \uparrow \\ \overline{R}[T_1, T_2, T_3] & \xrightarrow{\phi} & \overline{R}[\overline{L_1}, \overline{L_2}, \overline{L_3}] \end{array}$$

where $\phi(T_i) = \overline{L_i}$ for $i = 1, 2, 3$ and $\overline{\phi}$ is the extension of ϕ to $\overline{k}[T_1, T_2, T_3]$. Since $\overline{a_2}\overline{L_2} - \overline{a_1}\overline{L_3} = 0$, it is clear that $\ker \overline{\phi}$ is generated by $\overline{a_2}T_2 - \overline{a_1}T_3$. Let $F \in \ker \phi$, then

$$F = (\overline{a_2}T_2 - \overline{a_1}T_3)G \tag{26}$$

for some $G \in \overline{k}[T_1, T_2, T_3]$. On the other hand, the polynomial $\overline{a_2}T_2 - \overline{a_1}T_3$ is monic as an element of the polynomial ring $\overline{R}_{\overline{a_1}}[T_1, T_2][T_3]$ (both $\overline{a_1}$ and $\overline{a_2}$ are nonzero in $\overline{R}$), and so the division algorithm in this ring allows us to write

$$F = (\overline{a_2}T_2 - \overline{a_1}T_3)G_1 + r \tag{27}$$

for some $G_1 \in \overline{R}_{\overline{a_1}}[T_1, T_2, T_3]$ and $r \in \overline{R}_{\overline{a_1}}[T_1, T_2]$. Comparing the two equations, (26) and (27), gives that

$$\overline{a_1}^t F = (\overline{a_2}T_2 - \overline{a_1}T_3)\rho \tag{28}$$

for some $t \geq 0$ and $\rho \in \overline{R}[T_1, T_2, T_3]$. Now $(\overline{a_1}^t, \overline{a_2})$ is a regular sequence of $\overline{R}$, so a_2 is not a zero divisor in the ring $\overline{R}/(\overline{a_1}^t\overline{R})$, then $\overline{a_2}T_2 - \overline{a_1}T_3$ is not a zero divisor in the ring $(\overline{R}/\overline{a_1}^t\overline{R})[T_1, T_2, T_3]$, by Corollary 2.3.19. Thus, taking equation (28) modulo $\overline{a_1}^t$ gives that $\rho \equiv 0 \pmod{\overline{a_1}^t}$ and, hence, $\overline{a_1}^t$ divides ρ in $\overline{R}[T_1, T_2, T_3]$. This implies that $F \in (\overline{a_2}T_2 - \overline{a_1}T_3)$ and $\ker \phi$ is the principal ideal generated by $\overline{a_2}T_2 - \overline{a_1}T_3$.

Let $x = \Phi(L_1, L_2, L_3) \in pB \cap R_0$ ($\Phi \in R[T_1, T_2, T_3]$), then $\Phi = \alpha(a_2T_2 - a_1T_3) + \beta p$ for some $\alpha, \beta \in R[T_1, T_2, T_3]$, and

$$x = \alpha(L_1, L_2, L_3)a_3L_1 + \beta(L_1, L_2, L_3)p \in pR_0$$

and so $pB \cap R_0 \subseteq pR_0$. ∎

2.4 Some examples

In this section, we will prove the finite generation of the kernels of some elementary derivations of $k^{[6]}$ to which none of the results of the previous sections can be applied. We will assume in this section that k is an algebraically closed field (see Proposition 2.1.5).

Proposition 2.4.1. *Let D be the elementary derivation*

$$(X_1^2 - X_2^3 + X_2)^{t_1}\frac{\partial}{\partial Y_1} + a_2\frac{\partial}{\partial Y_2} + (X_3^2 - X_1X_2)^{t_2}\frac{\partial}{\partial Y_3}, \ (t_1, t_2 \geq 0)$$

of $B = k[X_1, X_2, X_3, Y_1, Y_2, Y_3] \cong k^{[6]}$ with $a_2 \in R = k[X_1, X_2.X_3]$ satisfying:

1. *$(X_1^2 - X_2^3 + X_2)$ and $(X_3^2 - X_1X_2)$ do not divide a_2 in R*
2. *$X_1^2 - X_2^3 + X_2$ does not divide a_2 in $R/(X_3^2 - X_1X_2)R$.*

Then D is standard.

Remark 2.4.2. Note that $p := X_1^2 - X_2^3 + X_2$ and $q := X_3^2 - X_1X_2$ are both prime elements of $R = k[X_1, X_2, X_3]$ but R/pR and R/qR are not UFD's. Indeed, it is straightforward to prove that p and q are prime. The fact that R/pR is not a UFD is the subject of Exercise 6.2 in the first chapter of [19]. To see that R/qR is not a UFD, it is enough find a prime ideal of height one which is not principal in this ring. Indeed, let x_1, x_2, x_3 be the images of X_1, X_2, X_3 respectively in $k[X_1, X_2, X_3]/(p)$, and let $\wp$ be the prime ideal (x_1, x_3) of $R/(q)$, then $\wp$ has height one (because of the relation $x_3^2 = x_1x_2$), but it is not difficult to see that $\wp$ is not principal.

Proof of Proposition 2.4.1. Let $a_1 = p^{t_1}$, $a_3 = q^{t_2}$ where p and q are as in Remark 2.4.2. By Theorem 2.2.6, we may assume that $t_1, t_2 \geq 1$. Let A be the coordinate ring of the plane curve (C) defined by p, i.e., $A = k[X_1, X_2]/(X_1^2 - X_2^3 + X_2)$. We claim that A is an integrally closed domain. To see this, it is enough to prove that (C) is a nonsingular curve of the affine plane $\mathbf{A}^2$ over k. For this purpose, note that the solution of the system

$$\frac{\partial p}{\partial X_1} = 2X_1 = 0$$
$$\frac{\partial p}{\partial X_2} = -3X_2^2 + 1 = 0$$

consists of the two points $(0, \frac{\sqrt{3}}{3})$ and $(0, -\frac{\sqrt{3}}{3})$. Since these points are not on the curve, we deduce that (C) has no singular points and the claim is proved. Next, let $r := x_1x_2 \in A$, K be an algebraically closed field containing A, and let $x = \sqrt{r} \in K$. We claim that $x \notin A$. Indeed, assume $x \in A$. Then we can choose $\Phi \in k[X_1, X_2]$ such that $x = \Phi(x_1, x_2)$. So $r = (\Phi(x_1, x_2))^2$ and consequently

$$\Phi^2 = X_1X_2 + (X_1^2 - X_2^3 + X_2)\Psi$$

for some $\Psi \in k[X_1, X_2]$. The division algorithm in $k[X_1, X_2]$ allows us to find $\Phi_1, \Phi_2 \in k[X_1, X_2]$ with $\deg_{X_1}\Phi_1 \leq 1$ and such that $\Phi = \Phi_1 + p\Phi_2$. This means that

$$\Phi_1^2 = \Phi^2 - 2\Phi\Phi_2 p + p^2\Phi_2^2 = X_1X_2 + p\Gamma \tag{29}$$

for some $\Gamma \in k[X_1, X_2]$. Since $\deg_{X_1}\Phi_1 \leq 1$, we can find $\alpha, \beta \in k[X_2]$ such that $\Phi_1 = \alpha + \beta X_1$. Then equation (29) gives that

$$\alpha^2 + \beta^2(X_2^3 - X_2) = \Upsilon \tag{30}$$

where $\Upsilon = X_1(X_2 - 2\alpha\beta) + p(\Gamma - \beta^2) \in k[X_1, X_2]$. If $\Upsilon \neq 0$, then its degree in X_1 is clearly greater than or equal to one (since $X_2 - 2\alpha\beta \in k[X_2]$), and hence equation

(30) is not possible since $\alpha^2 + \beta^2(X_2^3 - X_2)$ is an element of $k[X_2]$. Thus $\Upsilon = 0$, and in particular

$$X_2 = 2\alpha\beta; \quad \text{(in particular } \alpha, \beta \neq 0) \tag{31}$$

and $\Gamma = \beta^2$. But the fact that $\Upsilon = \alpha^2 + \beta^2(X_2^3 - X_2) = 0$ implies that $\alpha = \beta = 0$ since $\deg_{X_2}\alpha^2$ is even and $\deg_{X_2}\beta^2(X_3 - X_2)$ is odd; this is a contradiction to (31). We conclude that $x \notin A$.

Let $\xi = X_3^2 - r \in A[X_3]$. We will show next that ξ is a prime element of $A[X_3]$. To see this, consider the homomorphism of A-algebras:

$$\begin{array}{rcl} \phi : A[X_3] & \longrightarrow & A[x] \\ X_3 & \mapsto & x. \end{array}$$

Clearly, $\xi \in \ker\phi$, and conversely, if $F \in \ker\phi$, write $F = \gamma(X_3^2 - r) + aX_3 + b$ for some $\gamma \in A[X_3]$ and $a, b \in A$, then $\phi(F) = ax + b = 0$. If $a \neq 0$, then $x = -\frac{b}{a} \in \mathrm{qt}(A)$, and consequently $x \in A$ since A is integrally closed and x is integral over A; this is a contradiction. Thus $a = b = 0$, and $F = \gamma(X_3^2 - r)$. This shows that $\ker\phi$ is generated by ξ in $A[X_3]$ and so ξ is a prime element of $A[X_3]$.

As a consequence, the image p' of p in $R' = R/qR$ is a prime element of R' since

$$\begin{array}{rcl} R'/p'R' & \cong & (k[X_1, X_2, X_3]/(p))\,/(X_3^2 - r) \\ & \cong & A[X_3]/(X_3^2 - r) \end{array}$$

and $A[X_3]/(X_3^2 - r)$ is a domain.

Using Proposition 2.1.2, we can finish the proof if we show that $qB \cap R_0 \subseteq qR_0$ where $B = R[Y_1, Y_2, Y_3]$, $R_0 = R[L_1, L_2, L_3]$ and L_1, L_2, L_3 are the standard linear constants as above. So let $x \in qB \cap R_0$, and write $x = \Phi(L_1, L_2, L_3)$ for some $\Phi \in R[T_1, T_2, T_3] \cong R^{[3]}$, then the image Φ' of Φ in $R'[T_1, T_2, T_3]$ is in the kernel $\wp$ of the homomorphism

$$\begin{array}{rcl} R'[T_1, T_2, T_3] & \xrightarrow{\phi} & R'[L_1', L_2', L_3'] \\ T_i & \mapsto & L_i' \end{array}$$

sending T_i to $L_i' = L_i + qB$, $i = 1,2,3$. If k' is the field of quotients of R', then ϕ can be extended to a homomorphism $\phi' : k'[T_1, T_2, T_3] \to k'[L_1', L_2', L_3']$ such that the diagram

$$\begin{array}{ccc} k'[T_1,T_2,T_3] & \xrightarrow{\phi'} & k'[L_1',L_2',L_3'] \\ \uparrow & & \uparrow \\ R'[T_1,T_2,T_3] & \xrightarrow{\phi} & R'[L_1',L_2',L_3'] \end{array} \tag{32}$$

is commutative. Since a_1' and a_2' are nonzero elements of R', it is clear that the transcendence degree of $k'[L_1', L_2', L_3']$ over k' is two, and hence the height of $\wp' = \ker \phi'$ is one. The element $u = a_1'T_1 + a_2'T_2$ of $k'[T_1, T_2, T_3]$ is a generator of $\wp'$ since it is prime and $\phi(u) = 0$. Let $F \in R'[T_1, T_2, T_3]$ be an element of $\wp$, then F is also in $\wp'$ and so we can write

$$F = Gu \tag{33}$$

for some $G \in k'[T_1, T_2, T_3]$. In the ring $R'_{a_1'}[T_1, T_2, T_3]$, u is monic in T_1, and so the division algorithm in this ring allows us to write

$$F = Hu + r \tag{34}$$

for some $H \in R'_{a_1'}[T_1, T_2, T_3]$ and some $r \in R'_{a_1'}[T_2, T_3]$. Comparing equations (33) and (34), we can see that $G = H \in R'_{a_1'}[T_1, T_2, T_3]$ and $r = 0$. Thus, there exists $t > 0$ such that

$$p'^{t} F = (p'^{t_1} T_1 + a_2'T_2)g \tag{35}$$

for some $g \in R'[T_1, T_2, T_3]$. Since p' is a prime element of R', then equation (35) shows that p' divides either $p'^{t_1}T_1 + a_2'T_2$ or g. By assumption, p' is not a divisor of a_2' in $R'/p'R'$, and so p' cannot divide $p'^{t_1}T_1 + a_2'T_2$ in $R'[T_1, T_2, T_3]$; consequently $F \in uR'[T_1, T_2, T_3]$, and $\wp$ is generated by u. Write $\Phi = (a_1T_1 + a_2T_2)\Psi + q\Upsilon$ for some $\Psi, \Upsilon \in R[T_1, T_2, T_3]$, then $x = -a_3L_3\Psi(L_1, L_2, L_3) + q\Upsilon(L_1, L_2, L_3) \in qR_0$. ∎

Example 2.4.3. The derivation

$$D = (X_1^2 - X_2^3 + X_2)^{t_1}\frac{\partial}{\partial Y_1} + (X_2^2 - X_1^3 + X_1)^{t_2}\frac{\partial}{\partial Y_2} + (X_3^2 - X_1X_2)^{t_3}\frac{\partial}{\partial Y_3}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is standard for all nonnegative integers t_1, t_2, t_3. Indeed, it is not difficult to prove that the two conditions of Proposition 2.4.1 are satisfied for $a_2 = (X_2^2 - X_1^3 + X_1)^{t_2}$. Note that in the above example, $D = \sum_{i=1}^{3} a_i \frac{\partial}{\partial Y_i}$ with $k[X_1, X_2, X_3]/(a_i)$ is not a UFD for all $i = 1, 2, 3$.

So far, all the kernels of elementary derivations of $B = R[Y_1, Y_2, Y_3]$ that we have seen are generated by three elements over R. The following proposition gives an example of an elementary derivation of B whose kernel is generated by only two linear elements over R.

Proposition 2.4.4. *The kernel of the elementary derivation*

$$D = (X_1^2 - X_2X_3)\frac{\partial}{\partial Y_1} + (X_2^2 - X_1X_3)\frac{\partial}{\partial Y_2} + (X_3^2 - X_1X_2)\frac{\partial}{\partial Y_3}$$

of $B = k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is a polynomial ring in two variables over $k[X_1, X_2, X_3]$.

Remark 2.4.5. The derivation D of proposition 2.4.4 is not standard but its kernel is generated by two linear constants.

Proof

Let $a_1 = X_1^2 - X_2X_3$, $a_2 = X_2^2 - X_1X_3$, $a_3 = X_3^2 - X_1X_2$, and let $R = k[X_1, X_2, X_3]$. Then a_1, a_2, a_3 are pairwise relatively prime elements of R and R/a_iR is not a UFD for $i \in \{1, 2, 3\}$. Consider the two elements of B

$$f = X_3Y_1 + X_1Y_2 + X_2Y_3, \quad g = X_2Y_1 + X_3Y_2 + X_1Y_3$$

and the usual standard linear constants

$$\begin{aligned} L_1 &= a_3Y_2 - a_2Y_3 &&= X_3^2Y_2 - X_1X_2Y_2 - X_2^2Y_3 + X_1X_3Y_3 \\ L_2 &= -a_3Y_1 + a_1Y_3 &&= -X_3^2Y_1 + X_1X_2Y_1 + X_1^2Y_3 - X_2X_3Y_3 \\ L_3 &= a_2Y_1 - a_1Y_2 &&= X_2^2Y_1 - X_1X_3Y_1 - X_1^2Y_2 + X_2X_3Y_2. \end{aligned}$$

It is immediate that $D(f) = D(g) = 0$ and that the following relations are true

$$L_1 = -X_2f + X_3g, \quad L_2 = -X_2f + X_1g, \quad L_3 = -X_1f + X_2g.$$

Let $R_0 := R[f, g]$, then $R[L_1, L_2, L_3] \subseteq R_0$ and since $(R[L_1, L_2, L_3])_{a_3} = (\ker D)_{a_3}$ (Proposition 2.1.4), then $(R_0)_{a_3} = (\ker D)_{a_3}$. We will show that $\ker D = R[f, g]$; so, it is enough (Proposition 2.1.2) to show that $a_3B \cap R_0 \subseteq a_3R_0$. Let $\overline{R} = R/a_3R$ and consider the ring homomorphism

$$\phi : \overline{R}[T_1, T_2] \longrightarrow \overline{R}[\overline{f}, \overline{g}]$$

sending T_1 to $\overline{f}$ and T_2 to $\overline{g}$. We claim that ϕ is an isomorphism. Indeed, since the elements $\overline{f}$ and $\overline{g}$ are not algebraic over $\overline{R}$, the transcendence degree of $\overline{R}[\overline{f}, \overline{g}]$ over $\overline{R}$ is either one or two. If it is one, then $\overline{f}, \overline{g}$ are linearly dependent over $K := \mathrm{qt}(\overline{R})$ and so there exists an $\overline{\alpha} \in \overline{R}$ such that $x_3 = \overline{\alpha}x_2$, $x_1 = \overline{\alpha}x_3$, $x_2 = \overline{\alpha}x_1$; in particular, $x_2^2 = x_1x_3$ in $\overline{R}$ and so

$$X_2^2 = X_1X_3 + (X_3^2 - X_1X_2)\Upsilon$$

for some $\Upsilon \in R$. This is absurd. Thus, $\mathrm{trdeg}_{\overline{R}}\overline{R}[\overline{f}, \overline{g}] = 2$, and so the height of $\ker\phi$ is zero. This shows that ϕ is injective, and hence an isomorphism. To finish the proof, consider an element $x = \Phi(f, g) = a_3b$ of $a_3B \cap R_0$ ($\Phi \in R[T_1, T_2]$ and $b \in B$), then the image $\overline{\Phi}$ of Φ in $\overline{R}[T_1, T_2]$ is in the kernel of ϕ, and consequently it is zero, so $\Phi = a_3h$ for some $h \in R[T_1, T_2]$, and hence $x = \Phi(f, g) \in a_3R_0$ as desired. ∎

2.5 A criterion for elementariness in $R[X, Y]$

Given a UFD R containing $\mathbb{Q}$, we try in this section, to understand under what conditions a locally nilpotent R-derivation of $B = R^{[2]}$ is R-elementary. Recall that a derivation D of $R^{[n]}$ is called R-elementary if there exists a coordinate system $\gamma = (X_1, \ldots, X_n)$ of $R^{[n]}$ such that $D(X_i) \in R$ for all i. Since R-elementary derivations of $R^{[2]}$ are now fully understood, such a criterion would help to find the kernel of some derivations of $R^{[2]}$. We begin with a Proposition from [7].

Proposition 2.5.1. ***(Proposition 2.1,** [7]) If $D \neq 0$ is a locally nilpotent R-derivation of $R[X, Y]$ then $\ker D$ is a polynomial ring in one variable over R.*

Lemma 2.5.2. *Let E be a domain, and $L = E[T] = E^{[1]}$, a polynomial ring in one variable over E. If $E[f] = E[g]$ for some $f, g \in L$ transcendental over E, then $f = ag + b$ for some $a \in E^*$ and $b \in E$.*

Proof

Write $f = \phi(g)$ for some $\phi \in E[T]$. Since f, g are both transcendental over E, we have that $\phi = aT + b$ for some $a, b \in E$. Similarly, $g = \psi(f)$ for some $\psi = \alpha T + \beta \in E[T]$. So $f = a(\alpha f + \beta) + b$, and $a\alpha = 1$ (since f is transcendental over E). ∎

Lemma 2.5.3. *Let E be a domain, and $V = E[X_1, \ldots, X_n]$ be a polynomial ring in n variables over E. If $\gamma = (F_1, \cdots, F_n)$ is a coordinate system of V, then the determinant of the matrix*

$$A = \left(\frac{\partial F_i}{\partial X_j}\right)_{1 \leq i,j \leq n}$$

is a unit of E.

Proof

Fix $i \in \{1, \ldots, n\}$. Since $E[X_1, \ldots, X_n] = E[F_1, \ldots, F_n]$, we can find an element G_i of $E[T_1, \ldots, T_n] \cong E^{[n]}$ such that $X_i = G_i(F_1, \ldots, F_n)$. The chain rule of differentiation gives then the two equations

$$\begin{aligned} 1 &= \sum_{k=1}^{n} \frac{\partial G_i}{\partial F_k} \frac{\partial F_k}{\partial X_i} \\ 0 &= \sum_{k=1}^{n} \frac{\partial G_j}{\partial F_k} \frac{\partial F_k}{\partial X_i} \quad \text{for any } j \neq i. \end{aligned}$$

This means that the matrix

$$\left(\frac{\partial G_i}{\partial F_j}\right)_{1 \leq i,j \leq n}$$

is the inverse of A, and the Lemma follows. ∎

Given an R-elementary derivation D if B, the following Lemma gives us all the R-elementary derivations of B of the form bD, $b \in B$.

Lemma 2.5.4. *Let R be a UFD, $B = R^{[2]}$ and let D be an R-derivation of B. For any $b \in B \backslash \{0\}$, we have*

$$bD \text{ is } R\text{-elementary} \Leftrightarrow D \text{ is } R\text{-elementary and } b \in R.$$

Proof

If D is R-elementary and $b \in R$, then we can choose a coordinate system $\gamma = (X_1, X_2)$ of B such that $D(X_1), D(X_2)$ are in R. Hence $bD(X_1)$ and $bD(X_2)$ are also in R and bD is then R-elementary. Conversely, if $bD(U_1)$, $bD(U_2) \in R$ for some coordinate system (U_1, U_2) of B, then b, $D(U_i)$ are elements of R for $i = 1, 2$. In particular, D is R-elementary. ∎

Proposition 2.5.5. *Let R be a UFD containing $\mathbb{Q}$ and which is a finitely generated k-algebra, $B = R^{[2]}$ be a polynomial ring in two variables over R. An irreducible R-derivation D of B is R-elementary if and only if there exists a coordinate system $\gamma = (U, V)$ of B with respect to which* $\ker D$ *is generated over R by an element of the form $F = aU + bV$ for some relatively prime elements a, b of R.*

Proof

Assume first that D is R-elementary and let (U, V) be a coordinate system of B such that $D(U) = a$ and $D(V) = b$ for some relatively prime elements a, b of R. Let $F = bU - aV$, then $R[F] \subseteq \ker D$ and for any prime divisor p of b, the equality $pB \cap R[F] = pR[F]$ is easily verified. This shows that $\ker D = R[F]$. Conversely, suppose that $\gamma = (U, V)$ is a coordinate system of B such that $\ker D$ is generated over R by an element of the form $F = aU + bV$ where a, b are relatively prime elements of R. Since D is an R-derivation, we have that

$$aD(U) = -bD(V). \tag{36}$$

If $D(U)$ and $D(V)$ have a common divisor $f \in B$, then the ideal (DB) of B would be included in the ideal fB of B, and so f is a unit by the irreducibility of D. So $D(U)$ and $D(V)$ are relatively prime in B, and equation (36) implies that $D(U)$ divides b and $D(V)$ divides a and hence $D(U), D(V)$ are both in R. Thus D is R-elementary. ∎

Lemma 2.5.6. *Let E be a UFD, $C = E[X, Y] = E^{[2]}$. Let $F = aX + bY \in C$ where $a, b \in E$. The following are equivalent:*

1. *F is a variable of C over E;*

2. *the ideal $(a,b)E$ of E generated by a, b is equal to E.*

Proof

If F is a variable, we can choose $G \in C$ such that $C = E[F, G]$, and the determinant of the matrix

$$A = \begin{pmatrix} a & b \\ G_X & G_Y \end{pmatrix}$$

is an element of E^* by Lemma 2.5.3. This means that we can find $\alpha, \beta \in E$ with $\alpha a + \beta b = 1$, and hence $(a,b)E = E$. Conversely, assume that $(a,b)E = E$ and choose $\alpha, \beta \in E$ with

$$\alpha a + \beta b = 1.$$

Let

$$A = \begin{pmatrix} a & b \\ -\beta & \alpha \end{pmatrix} \in \mathcal{M}(2 \times 2, R)$$

then A is an invertible matrix with inverse

$$B = \begin{pmatrix} \alpha & -b \\ \beta & a \end{pmatrix}.$$

Let $G = -\beta X + \alpha Y \in C$, then

$$\begin{pmatrix} F \\ G \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$$

and so

$$\begin{cases} X = \alpha F - bG \\ Y = \beta F + aG. \end{cases}$$

Thus (F, G) is a coordinate system and F is a variable. ∎

Proposition 2.5.7. *Let R be a PID containing $\mathbb{Q}$, $B = R^{[2]}$, and let $D \neq 0$ be an irreducible locally nilpotent R-derivation of B. Then the following are equivalent:*

1. *D is R-elementary;*

2. *every generator of* $\ker D$ *is a variable of B over R;*

3. *D is surjective;*

4. *D has a slice.*

Proof

We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). If (1) is true, one can choose a coordinate system (U, V) of B over R such that $\ker D$ is generated over R by an element of the form $F = aU + bV$ for some $a, b \in R$ relatively prime (Proposition 2.5.5). Since R is a PID, $(a, b)R = R$ and so F is a variable of B over R by Lemma 2.5.6. Now if F' is another generator of $\ker D$, then we have that

$$F' = \alpha F + \beta$$

for some $\alpha \in R^*$ and $\beta \in R$ (Lemma 2.5.2) , and so F' is also a variable of B over R. This proves (2).

Next assume (2) is satisfied, then Theorem 1.3.6 and the fact that D is irreducible imply that there exists a coordinate system (F, G) of B over R such that $D = F_Y\partial/\partial X - F_X\partial/\partial Y$ and $\ker D = R[F]$. Write $D = f\partial/\partial F + g\partial/\partial G$ for some $f, g \in B$, then $D(F) = f = 0$, and

$$g = D(G) = \begin{vmatrix} F_X & F_Y \\ G_X & G_Y \end{vmatrix} \in R^*,$$

since (F, G) is a coordinate system of B. Thus $D = g\partial/\partial G$ and the latter is clearly surjective. This proves condition (3).

Condition (3) implies condition (4) trivially. So assume that (4) is true, we need to prove that D is R-elementary. Let $G \in B$ such that $D(G) = 1$, then for any generator F of $\ker D$, we have that $B = R[F][G]$ and hence (F, G) is a coordinate system of B with respect to which D is elementary, and (1) is proved. ∎

Remark 2.5.8. Conditions (3) and (4) of the above proposition remain equivalent if we replace the assumption that R is a PID containing $\mathbb{Q}$ by the weaker assumption that R is a domain containing $\mathbb{Q}$. Indeed, we only need to prove (4) $\Rightarrow$ (3). If $G \in B$ is such that $D(G) = 1$, then $B = (\ker D)[G] = R[F][G]$ for some $F \in B$. If $\phi \in B$, then $D(\phi) = \frac{\partial\phi}{\partial F}D(F) + \frac{\partial\phi}{\partial G}D(G) = \frac{\partial\phi}{\partial G}$. So $D = \frac{\partial}{\partial G}$, which is clearly surjective.

Corollary 2.5.9. *Let $B = k[X,Y,Z] = k^{[3]}$, $D : B \to B$ be a rank two locally nilpotent k-derivation of B such that $D(X) = 0$. Then D is $k[X]$-elementary if and only if its kernel is generated by a variable of B over $k[X]$.*

Proof

Follows immediately from Proposition 2.5.7. ∎

Example 2.5.10. Consider the k-derivation D of $k[X,Y,Z]$ defined by $D(X) = 0, D(Y) = X, D(Z) = Y$. We claim that there exists no coordinate system $\gamma = (X,U,V)$ of B such that D is $k[X]$-elementary with respect to γ. Indeed, note first that D is triangular and hence locally nilpotent. Also, its kernel is clearly generated over $k[X]$ by the element

$$F = XZ - \frac{1}{2}Y^2.$$

If F were a variable of $k[X,Y,Z]$ over $k[X]$, then we would be able to choose $G \in k[X,Y,Z]$ such that $k[X][F,G] = k[X][F,G]$, and hence

$$k^* \ni \begin{vmatrix} \frac{\partial F}{\partial Y} & \frac{\partial F}{\partial Z} \\ \frac{\partial G}{\partial Y} & \frac{\partial G}{\partial Z} \end{vmatrix} = \begin{vmatrix} -Y & X \\ \frac{\partial G}{\partial Y} & \frac{\partial G}{\partial Z} \end{vmatrix} = -Y\frac{\partial G}{\partial Y} - X\frac{\partial G}{\partial Z},$$

which is clearly not true. So F is not a variable of $k[X,Y,Z]$ over $k[X]$, and the derivation D is not $k[X]$-elementary.

2.6 Elementary monomial derivations in dimension six

If $B = R[Y_1, \dots, Y_n] \cong R^{[m]}$, denote by ∂_i the partial derivative $\frac{\partial}{\partial Y_i}$ of B.

Recall that Robert's counterexample to Hilbert's fourteenth problem in dimension seven can be realized as the field of fractions of the kernel of the elementary monomial derivation

$$D = X_1^3\partial_1 + X_2^3\partial_2 + X_3^3\partial_3 + X_1^2X_2^2X_3^2\partial_4$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ (although he did not present it as such).
The purpose of this section is to show that we cannot construct a counterexample to Hilbert's fourteenth problem in dimension six which is of Robert's type. More generally, we will prove the following.

Theorem 2.6.1. *Let $R = k[X_1, \dots, X_n] \cong k^{[n]}$, $B = R[Y_1, \dots, Y_m] \cong R^{[m]}$ and $0 \neq D : B \longrightarrow B$ an R-elementary monomial derivation. If $n \leq 2$ or $m \leq 3$, then D is standard.*

Note the following two corollories to Theorem 2.6.1.

Corollary 2.6.2. *Every monomial elementary derivation of $k^{[6]}$ is either zero or standard.*

Proof

$n + m = 6$ implies that either $n \leq 2$ or $m \leq 3$. ∎

Corollary 2.6.3. *With the notation of the above Theorem, let $D : B \longrightarrow B$ be an R-elementary monomial derivation of $k^{[7]}$ whose kernel is not finitely generated as an R-algebra. Then $n = 3$ and $m = 4$.* ∎

With the notations of Theorem 2.6.1, it is known ([13]) that if $n \leq 2$ (m arbitrary) or $m \leq 2$ (n arbitrary), then ker D is a finitely generated k-algebra. If in addition we assume that D is monomial, then we show that it is standard. For $n = 2$, we will show (Theorem 2.6.6) that the kernel of any R-elementary monomial derivation of $R^{[m]}$ is in fact a polynomial ring in $m-1$ variables over R. As for the case $n = m = 3$, even the fact that the kernel is a finitely generated k-algebra seems to be a new result. Note also that for the proof of the Theorem 2.6.1 we may clearly assume that the derivation D is irreducible. Moreover, if $a_i = \alpha_i X_1^{e_1} X_2^{e_2} \dots X_n^{e_n}$ ($e_i \in \mathbb{N}$ and $\alpha_i \in k$ for all i), then we may assume that $\alpha_i = 1$. We start with a proposition.

Proposition 2.6.4. *Let R be a UFD containing $\mathbb{Q}$, $a_1, \ldots, a_m \in R$, and D the R-elementary derivation $a_1\partial_1 + \cdots + a_m\partial_m$ of $B := R[Y_1, \ldots, Y_m]$. If $a_i \in R^*$ for some i, then* ker D *is generated by $m-1$ standard linear constants (in fact* $\ker D$ *is a polynomial ring in $m-1$ variables over R).*

Proof
We may clearly assume that $a_1 = 1$. In this case consider the elements

$$f_1 = a_2Y_1 - Y_2, f_2 = a_3Y_1 - Y_3, \ldots, f_{m-1} = a_mY_1 - Y_m$$

of B. Clearly $A' := R[f_1, \ldots, f_{m-1}] \subseteq C := \ker D$, and since $Y_j = a_jY_1 - f_{j-1}$ for all $j \geq 2$, we can easily see that $B = A'[Y_1]$. Since $A' \subseteq C \subset B$ and C is algebraically closed in B, it follows that $A' = C$. ∎

Corollary 2.6.5. *Theorem* 2.6.1 *is true if $n = 1$ or if $a_i = 1$ for some i.*

Proof
If $n = 1$, then $D = X_1^{e_1}\partial_1 + \ldots + X_1^{e_n}\partial_n$, and by the irreducibility of D we may assume that $e_1 = 0$ and we are done by the above Proposition. ∎

By Corollary 2.3.7, Theorem 2.6.1 is true in the case $m = 3$. We study next the case $n = 2$. In this case, we will prove a more general result. Namely we have the following.

Theorem 2.6.6. *If $R = k[U, V] = k^{[2]}$ and $a_1, \ldots, a_m$, $m \geq 1$ are monomials in U, V (not all zero), then the kernel of the elementary derivation $D = \sum_{i=1}^{m} a_i\partial_i$ $(D \neq 0)$ of $B = R[Y_1, \ldots, Y_m]$ is generated by $m-1$ standard linear constants and is a polynomial ring in $m-1$ variables over R.*

The proof uses Lemma 2.6.7 (see below) which holds in the following more general situation: R is a UFD, $B = R[Y_1, \ldots, Y_m]$ and $D : B \longrightarrow B$ an R-elementary derivation of B, i.e., of the form

$$D = \sum_{i=1}^{m} a_i\partial_i$$

where $a_i \in R$ and ∂_i means the partial derivative with respect to Y_i. Recall the definition of the standard linear constants of D: for each pair $(i, j) \in \mathbb{N}^2$ with $1 \le i < j \le m$, define

$$L_{i,j} = \begin{cases} \left(\frac{a_j}{\gcd(a_i,a_j)}\right) Y_i - \left(\frac{a_i}{\gcd(a_i,a_j)}\right) Y_j & \text{if } a_i \neq 0 \text{ or } a_j \neq 0 \\ 0 & \text{if } a_i = a_j = 0 \end{cases}$$

For any integer $k \ge 0$, we say that "D has the property $P(k)$" if $D = 0$ or ker D can be generated (as R-algebra) by k-elements of the $L_{i,j}$'s. With this terminology, we have the following.

Lemma 2.6.7. *Suppose that for some $i \in \{1, \dots, m\}$, we have*

1. *The restriction D_i of D to $B_i = R[Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_m]$ has the property $P(k)$;*

2. *$a_j \mid a_i$ for some $j \neq i$.*

Then D has the property $P(k + 1)$.

Proof

If $D = 0$, there is nothing to prove, so we may assume $D \neq 0$ and consequently, we may choose $j \neq i$ such that $a_j \neq 0$ and $a_j \mid a_i$. The element $L = Y_i - \frac{a_i}{a_j} Y_j$ belongs to ker D and clearly $B = B_i[L]$, so ker $D =$ (ker D_i)$[L]$ and D has the property $P(k + 1)$. ∎

Proof of Theorem 2.6.6. We proceed by induction on m, the case $m = 1$ being obvious. By the induction hypothesis and Lemma 2.6.7, we may assume that a_i does not divide a_j whenever $i \neq j$; in particular $a_i \neq 0$ for all i, so, multiplying each a_i by a unit if necessary, we may assume that $a_i = U^{u_i} V^{v_i}$ for all i. We may also relabel the Y_i's in such a way that

$$u_1 > \dots > u_m \quad \text{and} \quad v_1 < \dots < v_m.$$

Note that $L_{i,j} = V^{v_j - v_i} Y_i - U^{u_i - u_j} Y_j$, where L_{ij} is as above. Let $A = R[L_{i,i+1}, 1 \le i \le m - 1]$, we will show that ker $D = A$.

A simple calculation shows that

$$L_{i,j} = U^{u_i - u_{i+1}} L_{i+1,j} + V^{v_j - v_{i+1}} L_{i,i+1}$$

whenever $j - i > 1$. In particular, $L_{i,j}$ is in the R-module generated by $L_{i+1,j}$ and $L_{i,i+1}$, and hence is in R-module generated by the set $\{L_{i,i+1} \mid 1 \leq i \leq m-1\}$. This shows that $L_{i,j} \in A$ for all i, j satisfying $i < j$. Also, it follows from Proposition 2.1.4 that $A_U = (\ker)_U$ (by the irreducibility of D, we can find $i \in \{1, \ldots, m\}$ such that $a_i = U^{\mu}$ for some $\mu \geq 1$), and consequently, it suffices (by Theorem 2.3.3) to prove that

$$A \cap UB = UA. \tag{37}$$

Clearly, $UA \subseteq A \cap UB$. Conversely, let $x \in A \cap UB$ and write $x = \Phi(L_{1,2}, \ldots, L_{m-1,m})$ where $\Phi \in R[T_1, \ldots, T_{m-1}]$ is a polynomial in $m-1$ variables. Let $\overline{R} = R/UR$, $\overline{B} = B/UB = \overline{R}[Y_1, \ldots, Y_m]$ and let us take the images via $B \longrightarrow \overline{B}$. Since $x \mapsto 0$ and $L_{i,i+1} \mapsto V^{v_{i+1} - v_i} Y_i$, this gives

$$\overline{\Phi}(V^{v_2 - v_1} Y_1, \ldots, V^{v_m - v_{m-1}} Y_{m-1}) = 0$$

and consequently $\overline{\Phi} = 0$. This means that each coefficient of Φ is divisible by U, and so $x \in UA$ and (37) is proved. ∎

In particular, Theorem 2.6.6 shows that D has the property $P(m-1)$, and this means that if $n = 2$, then D is standard. It remains now to consider the cases $m = 2$ and $m = 1$. In the first case, the derivation has the form $D = a_1 \partial_1 + a_2 \partial_2$ where $a_i \in k[X_1, ..., X_n]$ and in this case, Theorem 4.1 of [13] shows that the kernel of D is $k[X_1, \ldots, X_n, a_2 Y_1 - a_1 Y_2]$. In the case $m = 1$, it is easy to verify that the kernel is simply $k[X_1, \ldots, X_n]$. This finishes the proof of Theorem 2.6.1.

We finish this tour of elementary derivations in dimension six with a self-contained proof of the fact that every linear (see definition below) and elementary derivation of $k^{[6]}$ has a finitely generated kernel over k. Note that a well known result of Weitzenbock ([31]) stated that every linear and locally nilpotent derivation of $k^{[n]}$ has a finitely generated kernel. A modern proofs of this fact is due to Seshadri ([28]). All the known proof of this fact are non-constructive in the sense that it is not easy to describe the ring of constants of a given linear elementary derivation.

Definition 2.6.8. A derivation D of $k[X_1, \ldots, X_n]$ is called *linear* if it is of the form

$$D = \sum_{i=1}^{n} (a_{i1}X_1 + \cdots + a_{in}X_n) \frac{\partial}{\partial X_i}$$

where $a_{ij} \in k$ for all i and all j.

Locally nilpotent linear derivations of $k^{[n]}$ satisfy the following nice property.

Proposition 2.6.9. *(Lemma 3, [18]) If D is a linear locally nilpotent derivation of $k[X_1, \ldots, X_n] \cong k^{[n]}$, then the rank of D (see Definition 1.3.3) is equal to the rank of D as a vector space map on $V = < X_1, \ldots, X_n >$.*

Proposition 2.6.10. *Let $\mathcal{A} = (a_{ij})_{i,j=1}^{3}$ be a 3×3 matrix with entries in k, then the elementary derivation*

$$D = \sum_{i=1}^{3} (a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3) \frac{\partial}{\partial Y_i}$$

of $E = k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is either zero or standard.

Proof

By Proposition 2.6.9 above we have that the rank of D is equal to the rank of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \end{pmatrix}$$

which is clearly equal to rank $(\mathcal{A})$, hence $\operatorname{rank} D \leq 3$. If $\operatorname{rank} D \leq 2$, then we are done by Corollary 1.3.7. If $\operatorname{rank} D = 3$, then $\mathcal{A} \in GL_3(k)$, and so (a_1, a_2, a_3) is a coordinate system of $R = k[X_1, X_2, X_3]$, where

$$a_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3 \quad (i = 1, 2, 3).$$

So we may regard D as an elementary monomial derivation: Each DY_i is a monomial in $R = k[a_1, a_2, a_3]$. So D is standard. ∎

2.7 Elementary monomial derivations in dimension seven

Let $C = k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4] \cong k^{[7]}$. Recall that the kernel of the elementary derivation

$$X_1^{t+1}\frac{\partial}{\partial Y_1} + X_2^{t+1}\frac{\partial}{\partial Y_2} + X_3^{t+1}\frac{\partial}{\partial Y_3} + (X_1X_2X_3)^t\frac{\partial}{\partial Y_4} \tag{38}$$

of C is not a finitely generated k-algebra for any value of $t \geq 2$. It is then natural to ask the following

Question 2.7.1. Do there exist other elementary monomial derivations of C whose kernels are not finitely generated over k?

The purpose of this section is to investigate a little further the family of monomial elementary derivations of C. Although this type of derivation looks easy to handle at the first glance, in fact, even in some simple cases, the question of finite generation of the kernel seems to be very hard to answer. We will look at some cases of elementary monomial derivations of C where we can prove the finite generation of the kernel. First some notation.

Let R be a UFD containing $\mathbb{Q}$, and let a_1, a_2, a_3, a_4 be four nonzero elements of R. Let $B := R[Y_1, Y_2, Y_3, Y_4]$ be a polynomial ring in four variables over R. For

$1 \leq i < j \leq 4$, write g_{ij} for the gcd of a_i and a_j and recall that the standard linear constants L_{ij} of $D = \sum_{i=1}^{4} a_i \frac{\partial}{\partial Y_i}$ are defined by:

$$L_{ij} := \frac{a_j}{g_{ij}} Y_i - \frac{a_i}{g_{ij}} Y_j. \tag{39}$$

Proposition 2.7.2. *Let R be a UFD containing $\mathbb{Q}$ which is a finitely generated k-algebra, and $B = R[Y_1, Y_2, Y_3, Y_4]$ a polynomial ring in four variables over R. Let a_1, a_2, a_3, a_4 be four nonzero relatively prime elements of R satisfying $a_4 \in (a_1, a_2, a_3)R$ and the following conditions for every prime divisor p of a_1:*

- *The ring $\overline{R} = R/pR$ is a UFD;*
- $\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1$ *where*

$$\alpha_1 = \frac{a_1}{g_{12}g_{13}}, \quad \alpha_2 = \frac{a_2}{g_{12}g_{23}}, \quad \alpha_3 = \frac{a_3}{g_{13}g_{23}}.$$

Then the kernel of the R-elementary derivation

$$D = a_1 \frac{\partial}{\partial Y_1} + a_2 \frac{\partial}{\partial Y_2} + a_3 \frac{\partial}{\partial Y_3} + a_4 \frac{\partial}{\partial Y_4}$$

of B is equal to $R[L_{12}, L_{13}, L_{23}, L]$ for some linear constant L of D.

Proof

Write $a_4 = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$ for some elements μ_1, μ_2, μ_3 of R. Let

$$L = Y_4 - \mu_1 Y_1 - \mu_2 Y_2 - \mu_3 Y_3$$

and $R' = R[L]$. Clearly $L \in \ker D$ and $B = R'[Y_1, Y_2, Y_3] = R'^{[3]}$ and D is R'-elementary, so the result follows from Theorem 2.3.3. ∎

Corollary 2.7.3. *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring in n variables ($n \geq 1$) over k, and let a_1, a_2, a_3, a_4 be four monomials of R satisfying $a_i \mid a_j$ for some $i \neq j$. Then D is standard.*

Proof

We may clearly assume that D is irreducible and that $a_i \neq 0$ for all i. Assume $a_1 \mid a_4$. By the above Proposition, one has only to show that

$$\gcd(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) = 1 \tag{40}$$

for every prime divisor of a_1. Since the α_i's are pairwise relatively prime, for each $i \in \{1, \dots, n\}$ we can choose distinct $j, k \in \{1, 2, 3\}$ such that a_j, a_k are not divisible by X_i. In this case $\alpha_j, \alpha_k \in k[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n] \cong k[X_1, \cdots, X_n]/(X_i)$, and relation (40) follows. ∎

Example 2.7.4. The kernel of the derivation

$$X_1^{t_1}\frac{\partial}{\partial Y_1} + X_2^{t_2}\frac{\partial}{\partial Y_2} + X_3^{t_3}\frac{\partial}{\partial Y_3} + (X_1X_2X_3)^t\frac{\partial}{\partial Y_4}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ is finitely generated over k whenever $t_i \leq t$ for some $i \in \{1, 2, 3\}$ (see derivation (38) above).

The obvious question now is what can be said about the kernel of the elementary monomial derivation $d = \sum_{i=1}^4 a_i\partial/\partial Y_i$ of B if a_i does not divide a_j for all $i \neq j$ (clearly, derivation (38) is of this type). The following Proposition answers this question in some particular cases.

Proposition 2.7.5. *The kernel of the derivation*

$$D = X_1^{t_1}\frac{\partial}{\partial Y_1} + X_2^{t_2}\frac{\partial}{\partial Y_2} + X_1^aX_2^b\frac{\partial}{\partial Y_3} + X_2^cX_3^d\frac{\partial}{\partial Y_4}$$

is generated by at most six linear elements in the Y_i's for all nonnegative integers t_1, t_2, a, b, c, d with $b \leq c$.

Proof

Let $R = k[X_1, X_2, X_3]$ and $B = R[Y_1, Y_2, Y_3, Y_4]$. By Proposition 2.1.2 above we may assume that $t_1 > a > 0$ and $0 < b \leq c < t_2 > b$. In this case, the standard kernel elements of D are

$$L_{12} = X_1^{t_1}Y_2 - X_2^{t_2}Y_1, \quad L_{13} = X_1^{t_1-a}Y_3 - X_2^bY_1, \quad L_{14} = X_1^{t_1}Y_4 - X_2^cX_3^dY_1$$
$$L_{23} = X_2^{t_2-b}Y_3 - X_1^aY_2, \quad L_{24} = X_2^{t_2-c}Y_4 - X_3^dY_2, \quad L_{34} = X_1^aY_4 - X_2^{c-b}X_3^dY_3.$$

Let $R[T_{ij} : 1 \le i < j \le 4]$ be a polynomial ring in 6 variables over R, $\overline{R} = R/X_1R$, and let $\overline{L_{ij}}$ be the image of L_{ij} in $\overline{R}[Y_1, Y_2, Y_3, Y_4]$. Then we have:

$$\begin{array}{lll} \overline{L_{12}} = -X_2^{t_2}Y_1, & \overline{L_{13}} = -X_2^bY_1, & \overline{L_{14}} = -X_2^cX_3^dY_1 \\ \overline{L_{23}} = X_2^{t_2-b}Y_3, & \overline{L_{24}} = X_2^{t_2-c}Y_4 - X_3^dY_2, & \overline{L_{34}} = -X_2^{c-b}X_3^dY_3 \end{array}$$

and hence the transcendence degree of $\overline{R}[\overline{L_{ij}} : 1 \le i < j \le 4]$ over $\overline{R}$ is three. Consider the following homomorphism of $\overline{R}$-algebras:

$$\phi : \overline{R}[T_{ij} : 1 \le i < j \le 4] \to \overline{R}[\overline{L_{ij}} : 1 \le i < j \le 4]$$

sending T_{ij} to $\overline{L_{ij}}$. It is clear that the elements

$$X_3^dT_{23} + X_2^{t_2-c}T_{34},\ T_{12} - X_2^{t_2-b}T_{13},\ T_{14} - X_2^{c-b}X_3^dT_{13}$$

of $R[T_{ij} : 1 \le i \le j \le 4]$ are in $\wp = \ker \phi$. Since $X_3^dT_{23} + X_2^{t_2-c}T_{34}$ is clearly prime, the ring $S := R[T_{ij} : 1 \le i < j \le 4]/(X_3^dT_{23} + X_2^{t_2-c}T_{34})$ is an integral domain. Now

$$\begin{aligned} R[T_{ij} : 1 \le i < j \le 4]/(X_3^dT_{23} + X_2^{t_2-c}T_{34}, T_{12} - X_2^{t_2-b}T_{13}, T_{14} - X_2^{c-b}X_3^dT_{13}) &\cong \\ S[T_{12}, T_{13}, T_{14}, T_{24}]/(T_{12} - x_2^{t_2-b}T_{13}, T_{14} - x_3^cT_{13}) & \end{aligned}$$

(x_i is the image of X_i in S) and the latter ring is clearly an integral domain. This shows that the ideal

$$(X_3^dT_{23} + X_2^{t_2-c}T_{34}, T_{12} - X_2^{t_2-b}T_{13}, T_{14} - X_2^{c-b}X_3^dT_{13})$$

of $R[T_{ij} : 1 \le i < j \le 4]$ is prime of height 3. On the other hand, ht$\wp$ is also three since the transcendence degree of $\overline{R}[\overline{L_{ij}} : 1 \le i < j \le 4]$ over R is three and the dimension of $R[T_{ij} : 1 \le i \le j \le 4]$ is six. This proves that

$$\wp = (X_3^dT_{23} + X_2^{t_2-c}T_{34}, T_{12} - X_2^{t_2-b}T_{13}, T_{14} - X_2^{c-b}X_3^dT_{13})$$

and the relation $X_1B \cap R[L_{ij} : 1 \le i < j \le 4] = X_1R[L_{ij} : 1 \le i < j \le 4]$ follows from the following relations:

$$\begin{aligned} X_3^dL_{23} + X_2^{t_2-c}L_{34} &= X_1^aL_{24} \\ L_{12} - X_2^{t_2-b}L_{13} &= -X_1^{t_1-a}L_{23} \\ L_{14} - X_2^{c-b}X_3^dL_{13} &= X_1^{t_1-a}L_{34}. \end{aligned}$$

We conclude, by Proposition, 2.1.2 that $\ker D = R[L_{ij} : 1 \leq i < j \leq 4]$. ∎

Note that the elementary derivations encountered in this chapter satisfy the following condition.

Question 2.7.6. If the kernel of an R-elementary derivation D of $B = R[Y_1, Y_2, Y_3]$ is a finitely generated R-algebra, is $\ker D$ generated by linear constants?

Remark 2.7.7. Theorem 3.4.1 will show that the derivation

$$X_1^2 \frac{\partial}{\partial Y_1} + X_2^2 \frac{\partial}{\partial Y_2} + X_3^2 \frac{\partial}{\partial Y_3} + X_2 X_3 \frac{\partial}{\partial Y_4}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ has a finitely generated kernel which cannot be generated by linear elements in the Y_i's (see derivation 38 above).

Chapter 3

On a Conjecture of Nowicki

The following is the main result of this chapter:

Theorem 1. *Let n and $t_1, \dots, t_n$ be positive integers, and consider the derivation*

$$D = \sum_{i=1}^{n} X_i^{t_i} \frac{\partial}{\partial Y_i}$$

of the polynomial ring $k[X_1, \dots, X_n, Y_1, \dots, Y_n]$. Then

$$\ker D = k[X_1, \dots, X_n, L_{ij} : 1 \leq i < j \leq n], \tag{41}$$

where $L_{ij} = X_i^{t_i} Y_j - X_j^{t_j} Y_i$.

The special case where all the t_i's are equal to 1 was considered by Nowicki in [25]. Since, in that case, D is a linear derivation (that is, a derivation which maps each X_i and Y_i to a linear form in $X_1, \dots, X_n, Y_1, \dots, Y_n$), it is known (see [28]) that $\ker D$ was a finitely generated k-algebra, but no set of generators was found for arbitrary n. Nowicki conjectured (41) in that case ($t_i = 1$ for all i), basing his conjecture on his computation of the cases $n = 2, 3, 4$.

It was argued in [20] that (41) holds in the case where all the t_i's are equal to each other and are greater than or equal to 3. However, we will show in Section 3.2 that the proof presented in [20] contains an error. Note that in the case where the t_i's are not all 1 (i.e, D is not linear), it is no longer evident that $\ker D$ is finitely generated as a k-algebra. However, we will show in Section 3 that we can restrict ourself to the linear case.

In the first section, we present an overview of the theory of Groebner bases and some of its applications. The elimination theory (Theorem 3.1.28) will play a crucial role in the proof of Nowicki's Conjecture. In the second section, we look at the argument of H. Kojima and M. Miyanishi presented in [20] and we develop a technique which, in some cases, can help decide whether a certain ideal is prime. In Section 3 we present the solution to Nowicki's Conjecture, i.e., the proof of Theorem 1. In Section 4 we give an example of a $k[X_1, X_2, X_3]$-elementary derivation of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ whose kernel is finitely generated over k but cannot be generated by linear elements in the Y_i's.

Remark 3.0.8. After finishing the proof of Nowicki's Conjecture using the theory of Groebner bases, it was brought to our attention by Harm Derksen that one can give a shorter proof of this Conjecture by extending the $\mathbf{G}_a$-action to an SL_2-action. As far as we know, the short proof has not been published.

3.1 Groebner bases

Groebner bases were first introduced in 1965 by Bruno Buchberger who was a student of Wolfgang Groebner. The basic idea behind this theory can be described as a generalization of the division algorithm for polynomials in one variable. In particular, one can use the Groebner basis of the ideal of $k[X_1, \ldots, X_n]$ generated by the elements $f_1, \ldots, f_s$ to define the remainder of the division of an element $f \in k[X_1, \cdots, X_n]$ by the f_i's. In the multivariables case, we have to define, of course, the appropriate concept of division. Another important concept of Groebner bases is the elimination theory that we will also look at in this section. It is important to bear in mind that the notion of Groebner bases depends on the choice of variables and on the choice of a "monomial ordering" on $k[X_1, \ldots, X_n]$, so for a given ideal, different monomial orderings may give different Groebner bases.

In this section, the letter k stands for an arbitrary field (not necessarily of characteristic zero), and $k[\underline{X}]$ will be an abreviation for the polynomial ring in n variables $k[X_1, \ldots, X_n]$ over k for $n \geq 1$.

Consider the following problem:
Ideal Membership Problem Given $f_1, \dots, f_s \in k[\underline{X}]$, let I be the ideal of $k[\underline{X}]$ generated by the f_i's. For $g \in k[\underline{X}]$, decide if $g \in I$.

In the case of a polynomial ring in one variable $k[X_1]$, the solution to the above problem can be given easily: let $f \in k[\underline{X}]$ be the greatest common divisor of the f_i's, then $I = (f)$ since $k[X_1]$ is a PID in this case. Now use the division algorithm in $k[X_1]$ to find unique $q, r \in k[\underline{X}]$ such that $g = qf + r$ where r is either zero or $\deg_{X_1} r < \deg_{X_1} f$. So $g \in I$ if and only if $r = 0$.

If we properly define the notions of division algorithm and remainder in $k[\underline{X}]$, then Groebner bases will be the analog of greatest common divisor for the elements f_i in the sense that if $\{f_1, \dots, f_s\}$ is a Groebner basis for the ideal $I = (f_1, \dots, f_s)$, then an element g of $k[\underline{X}]$ is in I if and only if the remainder of the division of g by the f_i's is zero.

Before we can state the division algorithm of $k[\underline{X}]$, we need to define the notion of monomial ordering in $k[\underline{X}]$.

Monomial ordering in $k[\underline{X}]$

Let $R \cong k^{[n]}$, i.e, R is k-isomorphic to a polynomial algebra in n variables over k.

Recall that a *coordinate system* of R means an ordered n-tuple $\underline{X} = (X_1, \dots, X_n) \in R^n$ satisfying

$$R = k[X_1, \dots, X_n].$$

We write $R = k[\underline{X}]$ to indicate that the coordinate system $\underline{X} = (X_1, \dots, X_n)$ has been chosen. Note that this choice includes, in particular, the ordering of the variables ($\underline{X}$ is an *ordered* n-tuple). In that situation, we make the following definitions.

1. Given $\underline{\alpha} \in \mathbb{Z}^n_{\geq 0}$, $X^{\underline{\alpha}} := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

2. The elements of $\mathcal{M} = \mathcal{M}_{\underline{X}} = \{X^{\underline{\alpha}} \mid \underline{\alpha} \in \mathbf{Z}^n_{\geq 0}\}$ are called *monomials*. Note that $\underline{\alpha} \mapsto X^{\underline{\alpha}}$ is an isomorphism from the additive monoid $\mathbf{Z}^n_{\geq 0}$ to the multiplicative monoid $\mathcal{M}$.

3. A *monomial ordering* on $k[\underline{X}]$ is a relation $>$ on $\mathcal{M}$ satisfying

   - If $M, N \in \mathcal{M}$ with $M < N$, then for all $P \in \mathcal{M}$ we have $MP < NP$.
   - $>$ is a well ordering of $\mathcal{M}$.

In view of the isomorphism $\mathbf{Z}^n_{\geq 0} \cong \mathcal{M}$, we may also say that a monomial ordering on $k[\underline{X}]$ is a well ordering of $\mathbf{Z}^n_{\geq 0}$ satisfying

$$\underline{\alpha} < \underline{\beta} \Rightarrow \underline{\alpha} + \underline{\gamma} < \underline{\beta} + \underline{\gamma}$$

for all $\underline{\alpha}, \underline{\beta}, \underline{\gamma} \in \mathbf{Z}^n_{\geq 0}$.

In what follows we give two examples of monomial orderings on $k[\underline{X}]$.
(1) Lexicographic ordering Given $\underline{\alpha}$, $\underline{\beta}$ in $\mathbf{Z}^n_{\geq 0}$, we say that $\underline{\alpha}$ is greater than $\underline{\beta}$ in the Lexicographic order, and we denote $\underline{\alpha} >_{\text{lex}} \underline{\beta}$, if the first nonzero entry from the left in $\underline{\alpha} - \underline{\beta}$ is positive.

Proposition 3.1.1. *The lexicographic ordering on $\mathbf{Z}^n_{\geq 0}$ is a monomial ordering.*

Proof
The first part of the definition of a monomial ordering is clearly verified. For the second part, assume that $>_{\text{lex}}$ were not a well ordering on $\mathbf{Z}^n_{\geq 0}$, then we can find a nonempty subset S of $\mathbf{Z}^n_{\geq 0}$ which has no smallest element. If $\underline{\alpha}_1$ is a fixed element of S, then one can find $\underline{\alpha}_2 \in S$ with $\underline{\alpha}_1 >_{\text{lex}} \underline{\alpha}_2$. Also, since $\underline{\alpha}_2$ is not the smallest element of S, we can find $\underline{\alpha}_3 \in S$ with $\underline{\alpha}_2 >_{\text{lex}} \underline{\alpha}_3$. Continuing this way, we obtain a strictly decreasing infinite sequence

$$\underline{\alpha}_1 >_{\text{lex}} \underline{\alpha}_2 >_{\text{lex}} \underline{\alpha}_3 >_{\text{lex}} \cdots \tag{42}$$

of elements of S. Write $\underline{\alpha}_i = (\alpha_{1i}, \alpha_{2i}, \ldots, \alpha_{ni})$ for all $i \geq 1$. Using the definition of $>_{\text{lex}}$, we have that the sequence $(\alpha_{1i})_i$ of $\mathbf{Z}^n_{\geq 0}$ is nonincreasing; hence there exists k

such that $\alpha_{1i} = \alpha_{1k}$ for all $i \geq k$. Now consider the sequence

$$\underline{\alpha}_k >_{\text{lex}} \underline{\alpha}_{k+1} >_{\text{lex}} \underline{\alpha}_{k+2} >_{\text{lex}} \cdots. \tag{43}$$

Using the same argument as before, one sees that the sequence of the second entries of elements in (43) will eventually stabilize. Continuing the same way, we will be able to find s such that $\underline{\alpha}_i = \underline{\alpha}_s$ for all $i \geq s$, contradicting the fact that (42) is a strictly decreasing sequence. ∎

Remark 3.1.2. Because, say, $k[X, Y, Z] = k[Y, X, Z]$, the expression "Lexicographic ordering on $k[X, Y, Z]$" is ambiguous. A convenient way to specify that the chosen coordinate system is (X, Y, Z) (not (Y, X, Z) or some other permutation) is to write

"lexicographic ordering on $k[X, Y, Z]$ with $X > Y > Z$".

Example 3.1.3. In $k[X, Y, Z]$ with $X > Y > Z$, we have

- $X^2YZ >_{\text{lex}} XY^3Z^5$
- $X >_{\text{lex}} Y^aZ^b$ for any $a, b \in \mathbb{N}$.

Given $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$, define *the total degree* $|\ \underline{\alpha}\ |$ of $\underline{\alpha}$ to be $\sum_{i=1}^n \alpha_i$. One can show that the following is a monomial ordering on $\mathbf{Z}^n_{\geq 0}$.
(2) Graded Lexicographic ordering Given $\underline{\alpha}$, $\underline{\beta}$ in $\mathbf{Z}^n_{\geq 0}$, then

$$\underline{\alpha} >_{\text{grlex}} \underline{\beta} \Leftrightarrow \begin{cases} |\ \underline{\alpha}\ | > |\ \underline{\beta}\ | \\ \text{or} \\ |\ \underline{\alpha}\ | = |\ \underline{\beta}\ |, \text{ and } \underline{\alpha} >_{\text{lex}} \underline{\beta} \end{cases}$$

Example 3.1.4. In $k[X, Y, Z]$ with $X > Y > Z$, we have

- $XY^2Z^3 >_{\text{grlex}} X^2YZ^2$
- $X^2Z^3 >_{\text{grlex}} Y^3Z^2$.

Remark 3.1.5. One can easily verify the following two facts:

1. The total degree is not a monomial ordering on $k[\underline{X}]$ since one could have that $|\underline{\alpha}| = |\underline{\beta}|$ even if $X^{\underline{\alpha}}$ and $X^{\underline{\beta}}$ are two different monomials.

2. For the lexicographic and the graded lexicographic orderings defined above, $X^{\underline{\alpha}} \geq 1$ for any $\underline{\alpha} \in \mathbf{Z}^n_{\geq 0}$.

From now on, we will assume that a monomial ordering $>$ on $\mathbf{Z}^n_{\geq 0}$ is fixed, and all the notions and results that will follow in this section will be relative to this ordering.

Definition 3.1.6. Let $f = \sum_{\underline{\alpha}} a_{\underline{\alpha}} X^{\underline{\alpha}}$ be a nonzero element of $k[\underline{X}]$.

1. The *multidegree of* f (denoted $\mathrm{mdeg}(f)$) is

$$\mathrm{mdeg}(f) = \max\{\underline{\alpha} \in \mathbf{Z}^n_{\geq 0}; a_{\underline{\alpha}} \neq 0\}.$$

2. The *leading coefficient of* f (denoted $\mathrm{LC}(f)$) is

$$\mathrm{LC}(f) = a_{\mathrm{mdeg}(f)} \in k^*.$$

3. The *leading monomial of* f (denoted $\mathrm{LM}(f)$) is

$$\mathrm{LM}(f) = X^{\mathrm{mdeg}(f)}.$$

4. The *leading term of* f (denoted $\mathrm{LT}(f)$) is

$$\mathrm{LT}(f) = \mathrm{LC}(f)\,\mathrm{LM}(f).$$

As in the one variable case, one can prove easily the following lemma.

Lemma 3.1.7. *Let* $f, g \in k[\underline{X}]\backslash\{0\}$, *then we have the following*

1. $\mathrm{mdeg}(fg) = \mathrm{mdeg}(f) + \mathrm{mdeg}(g)$

2. If $f + g \neq 0$, *then* $\mathrm{mdeg}(f+g) \leq \max\{\mathrm{mdeg}(f), \mathrm{mdeg}(g)\}$, *and the equality holds if* $\mathrm{mdeg}(f) \neq \mathrm{mdeg}(g)$.

We now return to the ideal membership problem, this time for $k[X_1, \ldots, X_n]$, $n \geq 2$. Recall that in the one variable case, a division algorithm was used to determine if an element of $k[\underline{X}]$ is in the ideal generated by $f_1, \ldots, f_s$ or not. A less efficient version of this algorithm exists in the multivariable case. Namely, we have the following theorem.

Theorem 3.1.8. *(Division Algorithm, Theorem 3, page 63 of [2]) Let $F = (f_1, \ldots, f_s)$ be an s-tuple of elements of $k[\underline{X}]$, and let $f \in k[\underline{X}]$. Then there exist elements $a_1, \ldots, a_n, r$ of $k[\underline{X}]$ with the following properties*

1. $f = a_1 f_1 + \cdots + a_n f_n + r$
2. *either $r = 0$ or no term of r is divisible by any of* $\mathrm{LT}(f_i)$
3. $\mathrm{mdeg}(a_i f_i) \leq \mathrm{mdeg}(f)$ *whenever* $a_i f_i \neq 0$.

The most remarkable difference between the above theorem and the division algorithm in the one variable case is that the remainder is no longer unique in Theorem 3.1.8. To see this, consider the example

Example 3.1.9. Consider the three elements $f_1 = X_2^2 - 1, f_2 = X_1X_2 + 1$ and $f = X_1X_2^2 - X_1$ of $k[X_1, X_2]$ equipped with the lex order with $X_1 > X_2$. On one hand, $f = X_1 f_1$ (so the remainder is zero), and on he other hand $f = X_2 f_2 + (-X_1 - X_2)$ (so the remainder is $(-X_1 - X_2)$). This example shows also that the fact that the remainder in the division algorithm of f by $f_1, \ldots f_n$ is zero, is not a necessary condition for f to belong to the ideal generated by the f_i's (it is clearly sufficient though).

To solve the ideal membership problem, one hopes to find "nice generators" for the ideal $I = \langle f_1, \ldots, f_n \rangle$, in the sense that every $f \in k[\underline{X}]$ has a unique remainder upon division by the f_i's. These "nice generators" will be provided by the elements of a Groebner basis for I.

Theorem 3.1.10. *Let $G = \{g_1, \ldots, g_r\}$ be a finite subset of $k[\underline{X}]$ and suppose that I is an ideal of $k[\underline{X}]$ containing G. The following are equivalent:*

1. *The ideal* $\langle \mathrm{LT}(f); f \in I\backslash\{0\}\rangle$ *of* $k[\underline{X}]$, *generated by the leading terms of nonzero elements of* I, *is equal to* $\langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_r)\rangle$.

2. *If* $f \in I\backslash\{0\}$, *then there exists* $i \in \{1, \dots, r\}$ *such that* $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$.

3. *For every* $f \in I\backslash\{0\}$, *there exist* $a_1, \dots, a_r \in k[\underline{X}]$ *with the following properties*

   (a) $f = \sum_{i=1}^{r} a_i g_i$

   (b) $\mathrm{mdeg}(a_i g_i) \leq \mathrm{mdeg}(f)$ *whenever* $a_i g_i \neq 0$.

Proof

(1) $\Rightarrow$ (2). Let $f \in I\backslash\{0\}$, then by (1), $\mathrm{LT}(f) = \sum_{i=1}^{r} b_i \mathrm{LT}(g_i)$ for some b_i's in $k[\underline{X}]$. This can only happen when $\mathrm{LT}(f) = h\,\mathrm{LT}(g_i)$ for some $h \in k[\underline{X}]$ and some $i \in \{1, \dots, r\}$. Hence $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$.

(2) $\Rightarrow$ (3). If (3) is not true, we can choose $f \in I\backslash\{0\}$ of least multidegree such that f does not have the specified representation in (3). By (2), there exist $i \in \{1, \dots, r\}$ and a monomial $c \in k[\underline{X}]$ such that $\mathrm{LT}(f) = c\,\mathrm{LT}(g_i)$. Write $f = \mathrm{LT}(f) + f_0$ and $g_i = \mathrm{LT}(g_i) + g_{i0}$ where $f_0, g_{i0} \in k[\underline{X}]$ with $\mathrm{mdeg}(f_0) < \mathrm{mdeg}(f)$ and $\mathrm{mdeg}(g_{i0}) < \mathrm{mdeg}(g_i)$. Let $h = f - cg_i = f_0 - cg_{i0}$, then $\mathrm{mdeg}(h) < \mathrm{mdeg}(f)$ (since $h = f - cg_i$ and f and cg_i have the same leading terms). But $h \in I$, so the minimality of the multidegree of f implies that either $h = 0$ or

$$h = \sum_{j=1}^{r} a_j g_j \text{ with } \mathrm{mdeg}(h) \geq \mathrm{mdeg}(a_j g_j) \text{ whenever } a_j g_j \neq 0. \tag{44}$$

If $h = 0$, then $f = cg_i$ and we get a contradiction to our choice of f. Hence h must satisfy (44), and then

$$f = \left(\sum_{j \neq i} a_j g_j\right) + (c + a_i) g_i \tag{45}$$

where $\mathrm{mdeg}(f) > \mathrm{mdeg}(h) \geq \mathrm{mdeg}(a_j g_j)$ whenever $a_j g_j \neq 0$ and $i \neq j$. Moreover, we have that

$$\begin{aligned}
\mathrm{mdeg}\,((c+a_i)g_i) &= \mathrm{mdeg}(cg_i + a_ig_i)\\
&\leq \max\{\mathrm{mdeg}(cg_i), \mathrm{mdeg}(a_ig_i)\}\\
&= \max\{\mathrm{mdeg}(f), \mathrm{mdeg}(a_ig_i)\}\\
&\leq \max\{\mathrm{mdeg}(f), \mathrm{mdeg}(h)\}\\
&= \mathrm{mdeg}(f).
\end{aligned}$$

Relation (45) then gives a contradiction to our choice of f. So (3) is true.
(3) $\Rightarrow$ (1). Let $f \in I\backslash\{0\}$. By (3), $f = \sum_{i=1}^r a_ig_i$ for some a_i's in $k[\underline{X}]$ with $\mathrm{mdeg}(f) \geq \mathrm{mdeg}(a_ig_i)$ whenever $a_ig_i \neq 0$. Choose $i \in \{1, \ldots, r\}$ such that $\mathrm{mdeg}(f) = \mathrm{mdeg}(a_ig_i)$, then $\mathrm{LT}(f) = \mu\,\mathrm{LT}(a_ig_i) = \mu\,\mathrm{LT}(a_i)\,\mathrm{LT}(g_i)$ for some $\mu \in k^*$. So (1) is true. ∎

Definition 3.1.11. For a nonzero ideal I of $k[\underline{X}]$, a subset $G = \{g_1, \ldots, g_r\}$ of I is called a *Groebner basis* for I if one (hence all) of the three statements of theorem 3.1.10 is true for G.

Remark 3.1.12. (1) By property (3) of Theorem 3.1.10, we can see that a Groebner basis for an ideal I is in particular a set of generators for I.
(2) Given any nonzero ideal I of $k[\underline{X}]$, consider the ideal J of $k[\underline{X}]$ generated by the leading terms of nonzero elements of I. Since $k[\underline{X}]$ is noetherian, we can find $g_1, \ldots, g_r$ in I such that $J = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_r)\rangle$. This simply means (by condition (1) of Theorem 3.1.10) that $\{g_1, \ldots, g_r\}$ is a Groebner basis for I. Using the convention that the empty set is a Groebner basis for $\{0\}$, this shows that every ideal of $k[\underline{X}]$ has a Groebner basis.
(3) A Groebner basis for an ideal depends on the choice of a monomial ordering on $k[\underline{X}]$. Hence, different monomial orderings may give different Groebner bases for I.

Definition 3.1.13. Let $G = \{g_1, \ldots, g_r\} \subset k[\underline{X}]$, and let $f \in k[\underline{X}]\backslash\{0\}$. We say that f is *in standard form relative to* G if $f = \sum_{i=1}^r a_ig_i$ for some $a_1, \ldots, a_r \in k[\underline{X}]$ with $\mathrm{mdeg}(f) \geq \mathrm{mdeg}(a_ig_i)$ whenever $a_ig_i \neq 0$.

Theorem 3.1.10 asserts then that $G = \{g_1, \ldots, g_r\}$ is a Groebner basis for $I = \langle G\rangle$ if and only if every element of I is either zero or in standard form relative to G.

Example 3.1.14. In Example 3.1.9, $\{f_1, f_2\}$ is not a Groebner basis for $I = \langle f_1, f_2\rangle$ for the lex ordering with $X_1 > X_2$. Indeed, let $f = X_1 + X_2 \in I$ and suppose that

$$X_1 + X_2 = a_1(X_2^2 - 1) + a_2(X_1X_2 + 1) \tag{46}$$

for some $a_1, a_2 \in k[X_1, X_2]$ with $\text{mdeg}(a_1 f_1) \leq \text{mdeg}(f)$ and $\text{mdeg}(a_2 f_2) \leq \text{mdeg}(f)$. Then (46) implies that $a_2 = 0$ since $\text{mdeg}(f_2) > \text{mdeg}(f)$ and so $X_1 + X_2 = a_1(X_2^2 - 1)$ which is clearly not possible. Thus, f cannot be written in standard form relative to $\{f_1, f_2\}$.

We now can prove the uniqueness of the remainder in Theorem 3.1.8 for Groebner bases.

Proposition 3.1.15. *Let $G = \{g_1, \ldots, g_s\}$ be a Groebner basis for $I = \langle g_1, \ldots, g_s\rangle$, and let $f \in k[\underline{X}]\backslash\{0\}$. If r_1, r_2 are two remainders of f upon division by G, then $r_1 = r_2$.*

Proof

Write $f = h_1 + r_1 = h_2 + r_2$ where $h_1, h_2 \in I$ and $r_1, r_2 \in k[\underline{X}]$ such that either $r_i = 0$ or no term of r_i is divisible by any of $\text{LT}(g_1), \ldots, \text{LT}(g_s)$, for $i = 1, 2$. Assume that $r_1 \neq r_2$, then $r_1 - r_2 = h_2 - h_1 \in I\backslash\{0\}$, and so $\text{LT}(r_1 - r_2)$ is divisible by $\text{LT}(g_i)$ for some i, since G is a Groebner basis for I. But $\text{LT}(r_1 - r_2)$ is a term of either r_1 or r_2. This is a contradiction. ∎

Definition 3.1.16. Let G be a Groebner basis for a nonzero ideal I of $k[\underline{X}]$. If $f \in k[\underline{X}]\backslash\{0\}$, then *the remainder of f upon division by G* (denoted by $\overline{f}^G$) is the unique r of the above proposition.

Remark 3.1.17. With the notation of Proposition 3.1.15, if $f = a_1g_1 + \cdots + a_sg_s + r$, where r is the remainder of f upon division by G, r is unique but the a_i's are not.

Now the solution of the ideal membership problem becomes easy using the notion of the remainder defined above. Namely let I be a nonzero ideal of $k[\underline{X}]$, and $f \in k[\underline{X}]\backslash\{0\}$. Compute a Groebner basis G for I with respect to the fixed monomial ordering on $k[\underline{X}]$. Then $f \in I$ if and only if $\overline{f}^G = 0$. (There are algorithms for computing a Groebner basis, but we will not discuss this aspect.)

Buchberger's criterion

The question that we will answer next is the following:
Question: How can we decide if $\{g_i, \dots, g_r\}$ is a Groebner basis for the ideal $I = \langle g_1, \dots, g_r \rangle$?

We already have an answer to this question given by Theorem 3.1.10: G is a Groebner basis for $I = \langle G \rangle$ iff every nonzero element of I is in standard form relative to G. The problem is that this criterion is not usually practical. Buchberger proved a similar criterion that consists of doing the same thing but only for a finite number of elements of I.

Definition 3.1.18. (The S-polynomials) Let $f, g \in k[\underline{X}] \setminus \{0\}$.

1. If $\mathrm{mdeg}(f) = \underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\mathrm{mdeg}(g) = \underline{\beta} = (\beta_1, \dots, \beta_n)$, let $\gamma_i = \max(\alpha_i, \beta_i)$ for $i \in \{1, \dots, n\}$ and $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$. We call $X^{\underline{\gamma}}$ *the least common multiple* of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$.

2. The S-polynomial of f and g is defined as follows

$$S(f,g) = \frac{X^{\underline{\gamma}}}{\mathrm{LT}(f)} f - \frac{X^{\underline{\gamma}}}{\mathrm{LT}(g)} g.$$

Example 3.1.19. In $k[X,Y,Z]$, let $f = 2X^2YZ - 3Y^2$ and $g = 3Y^3Z + Z^2$. Using the lex order on $k[X,Y,Z]$ with $X > Y > Z$, we have

$$\begin{aligned} S(f,g) &= \frac{X^2Y^3Z}{2X^2YZ}(2X^2YZ - 3Y^2) - \frac{X^2Y^3Z}{3Y^3Z}(3Y^3Z + Z^2) \\ &= -\frac{3}{2}Y^4 - \frac{1}{3}X^2Z^2. \end{aligned}$$

Remark 3.1.20. (1) It is clear (see the above example) that an S-polynomial is designed to produce cancellation of leading terms, so it is not difficult to check that

$$\mathrm{mdeg}(S(f,g)) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g)))$$

(2) $S(f,g) = -S(g,f)$
(3) $S(f,f) = 0$

(4) if h is the greatest common divisor of $\mathrm{LT}(f)$ and $\mathrm{LT}(g)$, it is easy to check that

$$S(f,g) = \frac{\mathrm{LT}(g)}{h}f - \frac{\mathrm{LT}(f)}{h}g.$$

Lemma 3.1.21. *Let $f, g \in k[\underline{X}]\backslash\{0\}$, and let a, b be two nonzero monomials of $k[\underline{X}]$. If $\mathrm{LT}(af) = \mathrm{LT}(bg)$, then $af - bg = \lambda\,\mathrm{GCD}(a,b)S(f,g)$ for some $\lambda \in k^*$.*

Proof

Let $c = \mathrm{GCD}(a,b)$, and $h = \mathrm{GCD}(\mathrm{LT}(f), \mathrm{LT}(g))$. Since $a\,\mathrm{LT}(f) = b\,\mathrm{LT}(g)$, we have that

$$\left(\frac{a}{c}\right)\left(\frac{\mathrm{LT}(f)}{h}\right) = \left(\frac{b}{c}\right)\left(\frac{\mathrm{LT}(g)}{h}\right). \tag{47}$$

Since $\mathrm{GCD}\left(\frac{a}{c}, \frac{b}{c}\right) = \mathrm{GCD}\left(\frac{\mathrm{LT}(f)}{h}, \frac{\mathrm{LT}(g)}{h}\right) = 1$, equation (47) implies that there exists $\lambda \in k^*$ with

$$\begin{aligned} \frac{a}{c} &= \lambda\left(\frac{\mathrm{LT}(g)}{h}\right) \\ \frac{b}{c} &= \lambda\left(\frac{\mathrm{LT}(f)}{h}\right). \end{aligned}$$

So,

$$\begin{aligned} af - bg &= \lambda c\left(\frac{\mathrm{LT}(g)}{h}f - \frac{\mathrm{LT}(f)}{h}g\right) \\ &= \lambda c S(f,g). \end{aligned}$$

∎

Definition 3.1.22. Let $G = \{g_1, \ldots, g_r\}$ be a finite subset of $k[\underline{X}]$, and let $f \in k[\underline{X}]\backslash\{0\}$. For a monomial $t \in k[\underline{X}]\backslash\{0\}$, we say that f has a *t-representation with respect to G* if $f = \sum_{i=1}^{r} a_i g_i$ for some $a_1, \ldots, a_r \in k[\underline{X}]$ with $\mathrm{mdeg}(a_i g_i) \leq \mathrm{mdeg}(t)$ whenever $a_i g_i \neq 0$.

In particular, if f is in standard form relative to G, then f has an $\mathrm{LM}(f)$-representation with respect to G.

We can now give Buchberger's criterion for a set of generators to be a Groebner basis. This criterion, together with its two improvements, will be our main tool to detect Groebner bases for some ideals in the coming sections.

Theorem 3.1.23. *Let $G = \{g_1, \dots, g_r\}$ be a finite subset of $k[\underline{X}]$ such that $0 \notin G$. Then G is a Groebner basis for $I = \langle G \rangle$ if and only if for every $1 \leq i < j \leq r$, either $S(g_i, g_j) = 0$ or there exists a monomial $t_{ij} \in k[\underline{X}]\backslash\{0\}$ such that $\mathrm{mdeg}(t_{ij}) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j)))$ and $S(g_i, g_j)$ has a t_{ij}-representation with respect to G.*

Proof

If G is a Groebner basis, then for every i, j with $1 \leq i < j \leq r$, either $S(g_i, g_j) = 0$ or it is in standard form relative to G (Theorem 3.1.10). In the second case, $S(g_i, g_j)$ has a $\mathrm{LM}(S(g_i, g_j))$-representation with respect to G, and we know that $\mathrm{mdeg}(S(g_i, g_j)) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j)))$.

Conversely, assume that for every $1 \leq i < j \leq r$, either $S(g_i, g_j) = 0$ or it has a t_{ij}-representation with respect to G for some monomial t_{ij} with

$$\mathrm{mdeg}(t_{ij}) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))).$$

If G is not a Groebner basis for I, then we can choose $f \in I\backslash\{0\}$ which is not in standard form relative to G (theorem 3.1.10). Choose a representation

$$f = \sum_{i=1}^{r} a_i g_i \tag{48}$$

of f ($a_i \in k[\underline{X}]$) satisfying the following conditions:

1. $M := \max\{\mathrm{LM}(a_i g_i); 1 \leq i \leq r\}$ has the least multidegree $\underline{\alpha}$ among all representations of f of the form (48) (of course, the "maximum" considered in the definition of M is with respect to the monomial ordering fixed earlier).

2. f has the fewest possible i such that $\mathrm{mdeg}(a_i g_i) = \underline{\alpha}$: since f is not in standard form relative to G, we have that $\underline{\alpha} > \mathrm{mdeg}(f)$ and so at least two terms in the representation (48) of f must have multidegree $\underline{\alpha}$.

We will get a contradiction by finding another representation of f having fewer terms with multidegree $\underline{\alpha}$.

Rearrange the g_i's if necessary to assume that

$$\mathrm{mdeg}(a_1 g_1) = \mathrm{mdeg}(a_2 g_2) = \underline{\alpha}. \tag{49}$$

Let $m_i = \mathrm{LT}(a_i)$ for $i = 1, 2$, then $\mathrm{mdeg}(m_1 g_1) = \mathrm{mdeg}(m_2 g_2) = \underline{\alpha}$ and so $\mathrm{LT}(m_1 g_1) = \lambda\,\mathrm{LT}(m_2 g_2) = \mathrm{LT}(\lambda m_2 g_2)$ for some $\lambda \in k^*$. By Lemma 3.1.21, there exists $\mu \in k^*$ such that $m_1 g_1 - \lambda m_2 g_2 = \mu m_3 S(g_1, g_2)$ where $m_3 = \mathrm{GCD}\left(\frac{m_1}{LC(a_1)}, \frac{m_2}{LC(a_2)}\right)$. Now by our assumption, $S(g_1, g_2) = \sum_{i=1}^{r} d_i g_i$ for some $d_1, \ldots, d_r \in k[\underline{X}]$ and $\mathrm{mdeg}(d_i g_i) \leq \mathrm{mdeg}(t_{12})$ for some monomial t_{12} satisfying $\mathrm{mdeg}(t_{12}) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$.

Let $t = m_3 t_{12}$, $c_i = \mu m_3 d_i$ for $i \in \{1, \ldots, r\}$, then

$$m_1 g_1 - \lambda m_2 g_2 = \sum_{i=1}^{r} c_i g_i$$

with

$$\begin{aligned} \mathrm{mdeg}(c_i g_i) &\leq \mathrm{mdeg}(t) \\ &< \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(m_3 g_1), \mathrm{LM}(m_3 g_2))) \\ &\leq \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(m_1 g_1), \mathrm{LM}(\lambda m_2 g_2))) \\ &= \underline{\alpha}. \end{aligned}$$

So $\mathrm{mdeg}(c_i g_i) < \underline{\alpha}$ for all i. Now we have the following representation of f:

$$f = (a_1 - m_1 + c_1)g_1 + (a_2 + \lambda m_2 + c_2)g_2 + (a_3 + c_3)g_3 + \cdots + (a_r + c_r)g_r \tag{50}$$

with

$$\begin{aligned} \mathrm{mdeg}(a_1 - m_1 + c_1)g_1 &= \mathrm{mdeg}((a_1 - m_1)g_1 + c_1 g_1) \\ &\leq \max\{\mathrm{mdeg}((a_1 - m_1)g_1), \mathrm{mdeg}(c_1 g_1)\} \\ &< \max\{\mathrm{mdeg}(m_1 g_1), \mathrm{mdeg}(c_1 g_1)\} \\ &= \underline{\alpha} \end{aligned}$$

and

$$\begin{aligned}\text{mdeg}(a_2 + \lambda m_2 + c_2)g_2 &\leq \max\{\text{mdeg}(a_2g_2), \text{mdeg}(c_2g_2), \text{mdeg}(\lambda m_2g_2)\} \\ &= \underline{\alpha}.\end{aligned}$$

Also, for $i \in \{3, \dots, r\}$, we have

$$\text{mdeg}(a_i + c_i)g_i \leq \max\{\text{mdeg}(a_ig_i), \text{mdeg}(c_ig_i)\} < \underline{\alpha}$$

whenever $\text{mdeg}(a_ig_i) < \underline{\alpha}$. This shows that (50) is a representation of f with fewer terms having multidegree equal to $\underline{\alpha}$. This contradicts the assumptions on f and finishes the proof. ∎

Corollary 3.1.24. *$G = \{g_1, \dots, g_r\}$ is a Groebner basis for $I = \langle G \rangle$ if and only if $S(g_i, g_j)$ is in standard form relative to G whenever i, j satisfy $1 \leq i < j \leq r$.*

The proof follows immediately from the above theorem.

Improvements on Buchberger's criterion

Although Corollary 3.1.24 has a finite number of cases to treat, it is still not very efficient in many cases. In what follows we will discuss two improvements of Theorem 3.1.24 that can speed up greatly our calculations.

Proposition 3.1.25. *(**Buchberger's First Criterion**) Let G be a finite subset of $k[\underline{X}]$. Suppose that for some elements $f, g \in G$, we have $\text{GCD}(\text{LM}(f), \text{LM}(g)) = 1$, then $S(f, g)$ is in standard form relative to G.*

Proof

Let $l := \text{LT}(f)$ and $m := \text{LT}(g)$ and write $f = l + f_0$, $g = m + g_0$ where $\text{mdeg}(f_0) < \text{mdeg}(f)$ and $\text{mdeg}(g_0) < \text{mdeg}(g)$. If $a = LC(f)$ and $b = LC(g)$, it is easy to check that $S(f', g') = S(f, g)$ where $f' = f/a$ and $g' = g/b$. So we may assume that $LC(f) = LC(g) = 1$. In this case,

$$S(f, g) = mf - lg = (g - g_0)f - (f - f_0)g = f_0g - g_0f. \tag{51}$$

If $\mathrm{LM}(f_0 g) = \mathrm{LM}(g_0 f)$, then $\mathrm{LM}(f_0)\,\mathrm{LM}(g) = \mathrm{LM}(g_0)\,\mathrm{LM}(f)$ and so $\mathrm{LM}(f)$ must divide $\mathrm{LM}(f_0)$ since it is relatively prime with $\mathrm{LM}(g)$. But this is not possible since $\mathrm{mdeg}(f_0) < \mathrm{mdeg}(f)$. Thus the leading monomials of $f_0 g$ and $g_0 f$ are distinct, and consequently,

$$\mathrm{mdeg}(S(f,g)) = \max\{\mathrm{mdeg}(f_0 g), \mathrm{mdeg}(g_0 f)\}. \tag{52}$$

Relations (51) and (52) show that $S(f,g)$ is in standard form relative to $\{f,g\} \subseteq G$. ■

Proposition 3.1.26. ***(Buchberger's Second Criterion)*** *Let $G = \{f_1, \dots, f_r\} \subset k[\underline{X}]$, and let $g_1, p, g_2 \in k[\underline{X}]$ with the following properties:*

1. $\mathrm{LM}(p)$ *divides* $\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$
2. $S(g_i, p)$ *has a t_i-representation with respect to G for some monomial t_i satisfying*

$$\mathrm{mdeg}(t_i) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(p))) \text{ for } i = 1, 2.$$

Then $S(g_1, g_2)$ has a t-representation with respect to G for some monomial t with

$$\mathrm{mdeg}(t) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))).$$

Proof

By condition (2), we can choose $a_1, \dots, a_r, b_1, \dots, b_r \in k[\underline{X}]$ such that

$$S(g_1, p) = \textstyle\sum_{i=1}^{r} a_i f_i \quad \text{and} \quad S(g_2, p) = \sum_{i=1}^{r} b_i f_i$$

with

$$\mathrm{mdeg}(a_i f_i) \leq \mathrm{mdeg}(t_1) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(p)))$$

and

$$\mathrm{mdeg}(b_i f_i) \leq \mathrm{mdeg}(t_2) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_2), \mathrm{LM}(p)))$$

for some monomials t_1, t_2 of $k[\underline{X}]$. Since $\mathrm{LM}(p)$ divides $\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, it is not difficult to check that $\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ is divisible by both $\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(p))$ and $\mathrm{LCM}(\mathrm{LM}(g_2), \mathrm{LM}(p))$, and so we can find monomials s_1, s_2 such that

$$\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2)) = s_1 \,\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(p)) = s_2 \,\mathrm{LCM}(\mathrm{LM}(g_2), \mathrm{LM}(p)). \quad (53)$$

Now, using the definition of the S-polynomials, it is easy to check that

$$s_1 S(g_1, p) + s_2 S(p, g_2) = S(g_1, g_2).$$

Thus,

$$S(g_1, g_2) = \sum_{i=1}^{r} (s_1 a_i + s_2 b_i) f_i. \quad (54)$$

Let $t = \max\{s_1 \,\mathrm{LM}(a_i f_i), s_2 \,\mathrm{LM}(b_i f_i), i = 1, \dots, r\}$, then clearly t is a monomial of $k[\underline{X}]$ with $\mathrm{mdeg}((s_1 a_i + s_2 b_i) f_i) \leq \mathrm{mdeg}(t)$ for all i and

$$\begin{aligned}
\mathrm{mdeg}(t) &= \max\{\mathrm{mdeg}(s_1 a_i f_i), \mathrm{mdeg}(s_2 b_i f_i)\} \\
&\leq \max\{\mathrm{mdeg}(s_1 t_1), \mathrm{mdeg}(s_2 t_2)\} \\
&< \max\{\mathrm{mdeg}(s_1 \,\mathrm{LCM}(\mathrm{LM}(p), \mathrm{LM}(g_1))), \\
&\quad (s_2 \,\mathrm{LCM}(\mathrm{LM}(p), \mathrm{LM}(g_2)))\} \\
&= \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))).
\end{aligned}$$

Equation (54) is then a t-representation of $S(g_1, g_2)$ with respect to G. ∎

Given a set of generators $G = \{g_1, \dots, g_r\}$ for an ideal I of $k[\underline{X}]$, the two criteria above suggest the following strategy for deciding whether or not G is a Groebner basis for I : denote by $\mathcal{N}$ the set of all pairs (i, j) of integers satisfying $1 \leq i < j \leq n$. A given pair in $\mathcal{N}$ is marked "treated" if $S(g_i, g_j)$ has a t-representation with respect to G for some monomial t with $\mathrm{mdeg}(t) < \mathrm{mdeg}(\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j)))$. Theorem 3.1.23 says that G is a Groebner basis for I if and only if every pair in $\mathcal{N}$ is marked "treated". For a given pair $(i, j) \in \mathcal{N}$, if $\mathrm{LM}(g_i)$ and $\mathrm{LM}(g_j)$ are relatively prime, $S(g_i, g_j)$ is "treated" by the first criterion. If not, we look for $k \in \{1, \dots, n\} \backslash \{i, j\}$ such that $\mathrm{LM}(g_k)$ divides $\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))$ and $S(g_i, g_k)$, $S(g_j, g_k)$ are both marked "treated". If such k exists, then $S(g_i, g_j)$ is "treated" by the second criterion, and if not we carry on with our calculation of $S(g_i, g_j)$.

Elimination theory

Among the wide range of applications of the Groebner basis theory is its effective contribution to solve the ideal membership problem we discussed earlier. Another important one we will look at is its application to elimination theory. In fact this application is the main reason for us to study this theory.

Given an ideal I of $k[\underline{X}]$ and a subset $\underline{Y}$ of $\underline{X}$, we want to find a set of generators for the ideal $J = I \cap k[\underline{Y}]$. Actually we will find a Groebner basis for J.

Definition 3.1.27. Let I be a nonzero ideal of $k[X_1, \ldots, X_n]$, $n \geq 2$. If $k \in \{1, \ldots, n-1\}$, the *k^{th}-elimination ideal of I* is the ideal

$$I_k = I \cap k[X_{k+1}, \ldots, X_n]$$

of $k[X_{k+1}, \ldots, X_n]$.

Theorem 3.1.28. ***(Elimination Theorem)*** *Let I be a nonzero ideal of $k[X_1, \ldots, X_n]$, $n \geq 2$, and let G be a Groebner basis for I with respect to a monomial ordering $>$ on $k[X_1, \ldots, X_n]$ satisfying the following property for some $k \in \{1, \ldots, n-1\}$:*

Any nonconstant monomial in $k[X_1, \ldots, X_k]$ has a strictly larger multidegree than any monomial in $k[X_{k+1}, \ldots, X_n]$. (55)

Then $G_k = G \cap k[X_{k+1}, \ldots, X_n]$ is a Groebner basis for the k^{th}-elimination ideal I_k of I with respect to the monomial ordering on $k[X_{k+1}, \ldots, X_n]$ induced by $>$. In particular, G_k generates I_k.

Proof

Write $G = \{g_1, \ldots, g_r\}$ and rearrange the g_i's if necessary to assume that $G_k = \{g_1, \ldots, g_s\}$ for some $s \leq r$. By Corollary 3.1.24, it suffices to show that every nonzero element of I_k is in standard form relative to G_k. Let $f \in I_k \backslash \{0\}$, then there exits $a_1, \ldots, a_r \in k[\underline{X}]$ such that

$$f = a_1 g_1 + \cdots + a_r g_r \tag{56}$$

and

$$\text{mdeg}(f) \geq \text{mdeg}(a_i g_i) \tag{57}$$

whenever $a_i g_i \neq 0$ (G is a Groebner basis for I). By property (55), and since $f \in k[X_{k+1}, \dots, X_n]$, any polynomial of $k[\underline{X}]$ containing any of $X_1, \dots, X_k$ has multidegree strictly greater than the multidegree of f. So, in equation (56), $a_i = 0$ for all $i \in \{s+1, \dots, n\}$. For the same reason, either $a_i = 0$ or $a_i \in k[X_{k+1}, \dots, X_n]$ for all $i \in \{1, \dots, s\}$. This shows that G_k is a Groebner basis for I_k with respect to the monomial ordering induced by $>$ on $k[X_{k+1}, \dots, X_n]$. ∎

Remark 3.1.29. If we use the lex ordering on $k[X_1, \dots, X_n]$ with $X_1 > \dots > X_n$, then property (55) of the above theorem is true for any $k \in \{1, \dots, n-1\}$.

Remark 3.1.30. If in Theorem 3.1.28, G is any set of generators for I (not necessarily a Groebner basis), then the result of the theorem is not true. To see this, consider the ideal $I = \langle X + Y, X \rangle$ of $k[X, Y]$. With respect to the lex ordering on $k[X, Y]$ with $X > Y$, $\{X + Y, X\}$ is not a Groebner basis for I since Y cannot be written in standard form relative to $\{X + Y, X\}$. If Theorem 3.1.28 were true in this case, $\{X + Y, X\} \cap k[Y] = \emptyset$ would be a Groebner basis for the ideal $I \cap k[Y]$ of $k[Y]$ and hence the latter would be the zero ideal. But note that $I = \langle X, Y \rangle$, so $I \cap k[Y] = \langle Y \rangle$.

3.2 On the derivation $\sum_{i=1}^{n} X_i^{t+1} \partial/\partial Y_i$

In [20], after giving a sufficient condition for the G_a-invariant subring to be finitely generated over k, H. Kojima and M. Miyanishi propose a proof of the following

Theorem 3.2.1. *(**Theorem 1.2**, [20]) Let $m \geq 2$, let $A = k[X_1, \dots, X_m, Y_1, \dots, Y_m]$ be a polynomial ring in $2m$ variables and let $\Delta = \sum_{i=1}^{m} X_i^{t+1} \partial/\partial Y_i$ be a locally nilpotent k-derivation of A, where $t \geq 2$. Then the invariant subring $A_0 := \Delta^{-1}(0)$ is given as*

$$\begin{aligned} A_0 &= k[X_1, \dots, X_m, X_i^{t+1} Y_j - X_j^{t+1} Y_i : 1 \leq i < j \leq m] \\ &\cong \frac{k[X_1, \dots, X_m, U_{ij} : 1 \leq i < j < k \leq m]}{(X_i^{t+1} U_{jk} - X_j^{t+1} U_{ik} + X_k^{t+1} U_{ij} : 1 \leq i < j < k \leq m)}. \end{aligned}$$

Here, in the second presentation of the ring A_0, we adjoin variables U_{ij} to the polynomial ring $k[X_1, \dots, X_m]$ for all possible pairs (i, j) with $1 \le i < j \le m$ and consider the residue ring modulo the ideal generated by the elements

$$X_i^{t+1}U_{jk} - X_j^{t+1}U_{ik} + X_k^{t+1}U_{ij}$$

for all possible triplets (i, j, k) with $1 \le i < j < k \le m$.

In this section, we will show that the proof of the above theorem, as it is given in [20], fails at one stage. In our argument we use $m = 4$ for simplicity. Note that the fact that $A_0 = k[X_1, \dots, X_m, X_i^{t+1}Y_j - X_j^{t+1}Y_i : 1 \le i < j \le m]$ is a special case of Theorem 1. Before discussing the proof given in [20], we will establish some preliminary results.

Lemma 3.2.2. *Given positive integers t_1, t_2, t_3 ,the ideal*

$$I = (X_1^{t_1}U_2 - X_2^{t_2}U_1, X_1^{t_1}U_3 - X_3^{t_3}U_1, X_2^{t_2}U_3 - X_3^{t_3}U_2)$$

of $k[X_1, X_2, X_3, U_1, U_2, U_3]$ is prime.

Proof

Consider the $\mathbb{N}^4$-gradings on the polynomials rings

$$k[X_1, X_2, X_3, U_1, U_2, U_3] \text{ and } k[X_1, X_2, X_3, W]$$

defined by

$$\deg X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \deg X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \deg X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \deg U_1 = \begin{pmatrix} t_1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\deg U_2 = \begin{pmatrix} 0 \\ t_2 \\ 0 \\ 1 \end{pmatrix} \quad \deg U_3 = \begin{pmatrix} 0 \\ 0 \\ t_3 \\ 1 \end{pmatrix} \quad \deg W = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(where each element of k^* is homogeneous of degree zero). Then the homomorphism $\phi : k[X_1, X_2, X_3, U_1, U_2, U_3] \to k[X_1, X_2, X_3, W]$ sending X_i to X_i and U_i to $X_i^{t_i}W$ $(i = 1, 2, 3)$ is homogeneous of degree zero with respect to the above gradings. Also, it is clear that the ideal I is contained in the kernel P of ϕ. To prove that P is contained in I , it suffices to show that every homogeneous element of P is contained in I. So let

$$F = \sum \mu_i X_1^{a_i} X_2^{b_i} X_3^{c_i} U_1^{r_i} U_2^{s_i} U_3^{u_i}$$

$(\mu_i \in k)$ be a homogeneous element of P of degree (d_1, d_2, d_3, d_4), then

$$0 = \sum \mu_i X_1^{a_i + r_i t_1} X_2^{b_i + s_i t_2} X_3^{c_i + u_i t_3} W^{r_i + s_i + u_i}. \tag{58}$$

Since F is homogeneous, we have that

$$\begin{aligned} a_i + r_i t_1 &= d_1 \\ b_i + s_i t_2 &= d_2 \\ c_i + u_i t_3 &= d_3 \\ r_i + s_i + u_i &= d_4 \end{aligned}$$

and so relation (58) becomes

$$0 = \sum_i \mu_i X_1^{d_1} X_2^{d_2} X_3^{d_3} W^{d_4}$$

which is equivalent to $\sum_i \mu_i = 0$. Hence,

$$F = \sum_i \mu_i \left(X_1^{a_i} X_2^{b_i} X_3^{c_i} U_1^{r_i} U_2^{s_i} U_3^{u_i} - X_1^{a} X_2^{b} X_3^{c} U_1^{r} U_2^{s} U_3^{u} \right) \tag{59}$$

for some fixed monomial $X_1^a X_2^b X_3^c U_1^r U_2^s U_3^u$ of degree (d_1, d_2, d_3, d_4).
Let $M_1 - M_2$ be a difference of the two monomials appearing in a term of the expression of F given in (59) (in particular the degrees of M_1 and M_2 are equal to the degree of F), then it is not difficult to see that

$$\frac{M_1}{M_2} = \left(\frac{X_1^{t_1} U_2}{X_2^{t_2} U_1} \right)^x \left(\frac{X_1^{t_1} U_3}{X_3^{t_3} U_1} \right)^y \tag{60}$$

where $x = s_i - s$ and $y = u_i - u$. Two cases are then to be considered.

1. $\underline{x \geq 0,\ y \geq 0}$. In this case:

$$M_1 = X_1^{t_1(x+y)} U_2^x U_3^y, \quad M_2 = X_2^{t_2 x} X_3^{t_3 x} U_1^{x+y}$$

and so

$$\begin{aligned} M_1 - M_2 &= \begin{vmatrix} (X_1^{t_1}U_2)^x & (X_3^{t_3}U_1)^y \\ (X_2^{t_2}U_1)^x & (X_1^{t_1}U_3)^y \end{vmatrix} \\ &= \begin{vmatrix} (X_1^{t_1}U_2)^x - (X_2^{t_2}U_1)^x & (X_3^{t_3}U_1)^y - (X_1^{t_1}U_3)^y \\ (X_2^{t_2}U_1)^x & (X_1^{t_1}U_3)^y \end{vmatrix}. \end{aligned}$$

So $M_1 - M_2 \in I$ in this case, because the entries of the first row belong to I.

2. $\underline{x \geq 0,\ y \leq 0}$. In this case, we consider two subcases:
(*i*) $\underline{x \leq -y}$. In this case:

$$M_1 = X_3^{-t_3 y} U_1^{-x-y} U_2^x, \quad M_2 = X_1^{-t_1(y+x)} X_2^{t_2 x} U_3^{-y} .$$

So

$$\begin{aligned} M_1 - M_2 &= \begin{vmatrix} (X_3^{t_3}U_1)^{-x-y} & (X_2^{t_2}U_3)^x \\ (X_1^{t_1}U_3)^{-y-x} & (X_3^{t_3}U_2)^x \end{vmatrix} \\ &= \begin{vmatrix} (X_3^{t_3}U_1)^{-y-x} - (X_1^{t_1}U_3)^{-y-x} & (X_2^{t_2}U_3)^x - (X_3^{t_3}U_2)^x \\ (X_1^{t_1}U_3)^{-y-x} & (X_3^{t_3}U_2)^x \end{vmatrix} \in I. \end{aligned}$$

(*ii*) $\underline{x \geq -y}$. In this case

$$M_1 = X_1^{t_1(x+y)} X_3^{-t_3 y} U_2^x, \quad M_2 = X_2^{t_2 x} U_1^{x+y} U_3^{-y}$$

and so

$$\begin{aligned} M_1 - M_2 &= \begin{vmatrix} (X_1^{t_1}U_2)^{x+y} & (X_2^{t_2}U_3)^{-y} \\ (X_2^{t_2}U_1)^{x+y} & (X_3^{t_3}U_2)^{-y} \end{vmatrix} \\ &= \begin{vmatrix} (X_1^{t_1}U_2)^{x+y} - (X_2^{t_2}U_1)^{x+y} & (X_2^{t_2}U_3)^{-y} - (X_3^{t_3}U_2)^{-y} \\ (X_2^{t_2}U_1)^{x+y} & (X_3^{t_3}U_2)^{-y} \end{vmatrix} \in I. \end{aligned}$$

So $M_1 - M_2 \in I$ in this case.

The cases ($x \leq 0$, $y \leq 0$) and ($x \leq 0$, $y \geq 0$) are reduced to the above two cases by considering $\frac{M_2}{M_1}$ instead of $\frac{M_1}{M_2}$. This proves the lemma. ∎

Given elements $a_1, \dots, a_n$ of a domain R, we give next a criterion for the primeness of the element $a_1X_1 + \dots + a_nX_n$ of $R[X_1, \dots, X_n] \cong R^{[n]}$. First a definition.

Definition 3.2.3. Given a domain R, we say that the elements $a_1, \cdots, a_n$ of R are *strongly relatively prime* if for every $\alpha, \beta \in R$, the condition ($\beta \mid a_i\alpha$ for all i) implies that $\beta \mid \alpha$.

Remark 3.2.4. 1. If $a_1, \cdots, a_n$ are strongly relatively prime and if $\beta \in R$ is a common divisor of $a_1, \cdots, a_n$, then β is a unit of R (take $\alpha = 1$ in the definition); in particular, at least one a_i is nonzero.
2. If R is a UFD, then "strongly relatively prime" is equivalent to "relatively prime".

Lemma 3.2.5. *For elements $a_1, \cdots, a_n$ of a domain R, the following are equivalent.*

1. *$a_1, \cdots, a_n$ are strongly relatively prime.*
2. *For all positive integers $e_1, \cdots, e_n$, the elements $a_1^{e_1}, \cdots, a_n^{e_n}$ are strongly relatively prime.*
3. *For some positive integers $e_1, \cdots, e_n$, the elements $a_1^{e_1}, \cdots, a_n^{e_n}$ are strongly relatively prime.*

Proof

Let P be the set of positive integers and let

$$E = \{(e_1, \cdots, e_n) \in P^n \mid a_1^{e_1}, \cdots, a_n^{e_n} \text{ are strongly relatively prime in } R\}.$$

Suppose (1) holds; then $(1, \cdots, 1) \in E$. We will show that if $(e_1, \cdots, e_n) \in E$, then for each $j = 1, \cdots, n$,

$$(e_1, \cdots, e_{j-1}, e_j + 1, e_{j+1}, \cdots, e_n) \in E. \tag{61}$$

Consider $\alpha, \beta \in R$ such that $\beta \mid a_j^{e_j+1}\alpha$ and for all $i \neq j$, $\beta \mid a_i^{e_i}\alpha$. Then for all i, $\beta \mid a_i^{e_i}a_j\alpha$, so the fact that $(e_1, \cdots, e_n) \in E$ implies that $\beta \mid a_j\alpha$. In particular,

$\beta \mid a_j^{e_j}\alpha$, so for all i, $\beta \mid a_i^{e_i}\alpha$ and hence $\beta \mid \alpha$. This shows that relation (61) holds for all j and it follows that $E = P^n$. Hence (1) $\Rightarrow$ (2).
(2) $\Rightarrow$ (3) is trivial, so we show that (3) $\Rightarrow$ (1). Assume that $(e_1, \cdots, e_n) \in E$ and consider $\alpha, \beta \in R$ such that $\beta \mid a_i\alpha$ for all i. Then clearly $\beta \mid a_i^{e_i}\alpha$ for all i and so $\beta \mid \alpha$ since $a_1^{e_1}, \dots, a_n^{e_n}$ are strongly relatively prime. So, $a_1, \dots, a_n$ are strongly relatively prime and the proof is complete. ∎

Lemma 3.2.6. *Let R be an integral domain and $K = \text{Frac}(R)$. If $f, g \in R[X]$ satisfy $f \mid g$ in $K[X]$ and if the leading coefficient of f is a unit of R then $f \mid g$ in $R[X]$.*

Proof

Since the leading coefficient of f is a unit of R, we may use the division algorithm in $R[X]$ to find $q, r \in R[X]$ satisfying $g = fq + r$ and $\deg(r) < \deg(f)$. Interpreting $g = fq + r$ in $K[X]$ gives $r = 0$, since $f \mid g$ in $K[X]$. ∎

Proposition 3.2.7. *Let R be a domain, $a_1, \cdots, a_n \in R$ and $L = a_1X_1 + \cdots + a_nX_n \in S = R[X_1, \cdots, X_n]$ $(n \geq 1)$. Then the following are equivalent:*

1. *L is a prime element of S.*

2. *$a_1, \dots, a_n$ are strongly relatively prime in R.*

Proof

Let us also consider the condition

3. *$a_1, \dots, a_n$ are strongly relatively prime in S.*

We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Suppose that L is a prime element of S and consider $\alpha, \beta \in R$ satisfying

$$\forall i\, (\beta \mid a_i\alpha); \tag{62}$$

we show that $\beta \mid \alpha$. Note that at least one of the a_i's is nonzero; so if $\beta = 0$ then condition (62) implies that $\alpha = 0$, so $\beta \mid \alpha$ is clear. Suppose now that $\beta \neq 0$. Then condition (62) implies that αL is divisible by β in S, so $\alpha L = \beta L'$ for some

$L' \in S$. Since L is prime and does not divide β (because $\beta \neq 0$), it must divide L' and $\frac{\alpha}{\beta} = \frac{L'}{L} \in S$. It follows that $\frac{\alpha}{\beta} \in R$ and that $\beta \mid \alpha$. Hence $a_1, \dots, a_n$ are strongly relatively prime in R.

Next, suppose that $a_1, \dots, a_n$ are strongly relatively prime in R and consider $f, g \in S$ such that

$$\forall i \, (g \mid a_i f); \tag{63}$$

we show that $g \mid f$. Again, we note that at least one a_i is nonzero and that, consequently, $g \mid f$ is clear if $g = 0$. Assume that $g \neq 0$. Note that $g \mid f$ in $K[X_1, \dots, X_n]$ where $K = \text{Frac(R)}$, so $gh = f$ for some $h \in K[X_1, \dots, X_n]$. For each i we have $a_i h = \frac{a_i f}{g} \in S$, so each coefficient c of h satisfies $a_i c \in R$, for all i. Write $c = \frac{\alpha}{\beta}$ with $\alpha, \beta \in R$ and $\beta \neq 0$; then $\beta \mid a_i \alpha$ for all i and consequently $\beta \mid \alpha$ in R. This means that each coefficient c of h belongs to R, i.e, $h \in S$ and $g \mid f$ in S. Thus $a_1, \dots, a_n$ are strongly relatively prime in S.

Finally, suppose that $a_1, \dots, a_n$ are strongly relatively prime in S and consider $f, g \in S$ such that $L \mid fg$ in S, so $L \mid f$ or $L \mid g$ in $K[X_1, \dots, X_n]$; we may assume that $L \mid f$ in $K[X_1, \dots, X_n]$. Then Lemma 3.2.6 implies that, for each $i = 1, \dots, n$, $L \mid f$ in $R_{a_i}[X_1, \dots, X_n]$ and consequently $L \mid a_i^{e_i} f$ in S for some positive integer e_i. Since $a_1^{e_1}, \dots, a_n^{e_n}$ are strongly relatively prime in S by lemma 3.2.5, it follows that $L \mid f$ in S. Hence L is a prime element of S and the proof is complete. ∎

We describe roughly now the proof of Theorem 3.2.1 given in [20].

Let $D = k[X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3]$, then with the notation of theorem 3.2.1, $A = D[Y_4]$, and

$$\Delta = \delta + X_4^{t+1} \frac{\partial}{\partial Y_4}$$

where

$$\delta = \sum_{i=1}^{3} X_i^{t+1} \frac{\partial}{\partial Y_i}.$$

Using an induction hypothesis (or a result from Chapter 2), we have that the kernel D_0 of δ has the form described in the theorem, i.e,

$$D_0 = k[X_1, X_2, X_3, X_4, X_i^{t+1}Y_j - X_j^{t+1}Y_i : 1 \leq i < j \leq 3]$$

and the isomorphism

$$D_0 \cong k[X_1, X_2, X_3, X_4, U_{12}, U_{13}, U_{23}]/(X_1^{t+1}U_{23} - X_2^{t+1}U_{13} + X_3^{t+1}U_{12}) \qquad (64)$$

follows easily. For the passage from $m = 3$ to $m = 4$, the authors need to know that the following quotient ring:

$$B := D_0[U_1, U_2, U_3]/(X_i^{t+1}U_j - X_j^{t+1}U_i + X_4^{t+1}\overline{U_{ij}} : 1 \leq i < j \leq 3)$$

is isomorphic to

$$A' := D_0[X_i^{t+1}Y_4 - X_4^{t+1}Y_i : 1 \leq i \leq 3]$$

via the natural D_0-homomorphism $\phi : B \to A'$ sending U_i to $X_i^{t+1}Y_4 - X_4^{t+1}Y_i$ $(1 \leq i < 3)$. Clearly ϕ is onto, and to show that it is injective, the authors argue first that B is an integral domain and then use the relation

$$\text{height}(\ker\phi) + \dim(B/\ker\phi) = \dim B$$

together with the fact that $\dim A' = \dim B$ to deduce that $\ker\phi = 0$. So, the only detail that remains to be checked is the fact that B is indeed a domain. To do this, the authors argue that the image x_4 of X_4 in B is a nonzero divisor of B and that the quotient ring B/x_4B is a domain. It is this latter assertion that fails in the proof. To be more precise, we will show that the ring B/x_4B is not a domain.

Note that B/x_4B in this case is the quotient of the polynomial ring in nine variables

$$C = k[X_1, X_2, X_3, U_{12}, U_{13}, U_{23}, U_1, U_2, U_3]$$

by the ideal

$$(X_1^{t+1}U_2 - X_2^{t+1}U_1, X_1^{t+1}U_3 - X_3^{t+1}U_1, X_2^{t+1}U_3 - X_3^{t+1}U_2,$$
$$X_1^{t+1}U_{23} - X_2^{t+1}U_{13} + X_3^{t+1}U_{12}).$$

Now, let Q be the quotient ring of $k[X_1, X_2, X_3, U_1, U_2, U_3]$ by the ideal

$$(X_1^{t+1}U_2 - X_2^{t+1}U_1, X_1^{t+1}U_3 - X_3^{t+1}U_1, X_2^{t+1}U_3 - X_3^{t+1}U_2)$$

then Q is an integral domain by Lemma 3.2.2 and

$$B/x_4B \cong Q[U_{12}, U_{13}, U_{23}]/(x_1^{t+1}U_{23} - x_2^{t+1}U_{13} + x_3^{t+1}U_{12})$$

where x_i is the image of X_i in Q. Let $s = t+1$, we will use the criterion of Proposition 3.2.7 to prove that the quotient ring

$$Q[U_{12}, U_{13}, U_{23}]/(x_1^sU_{23} - x_2^sU_{13} + x_3^sU_{12})$$

is not a domain.

In Q, we have that

$$x_1^s u_2 = x_2^s u_1$$
$$x_1^s u_3 = x_3^s u_1$$
$$x_2^s u_3 = x_3^s u_2.$$

Let $\alpha = u_1$ and $\beta = x_1^s$, then

$$\forall i,\ (\beta \mid x_i^s \alpha).$$

Now, x_1^s is not a divisor of u_1 in Q. Indeed, if x_1^s divides u_1 in Q, then one has a relation of the form

$$U_1 = X_1^s\Phi + (X_1^sU_2 - X_2^sU_1)\Phi_1 + (X_1^sU_3 - X_3^sU_1)\Phi_2 + (X_2^sU_3 - X_3^sU_2)\Phi_3$$

for some $\Phi, \Phi_i \in k[X_1, X_2, X_3, U_1, U_2, U_3]$. But this would mean that U_1 belongs to the ideal of $k[X_1, X_2, X_3, U_1, U_2, U_3]$ generated by X_1, X_2, X_3; this is absurd. Hence, the elements x_1^s, x_2^s, x_3^s of Q are not strongly relatively prime, and by Proposition 3.2.7, the element $x_1^sU_{23} - x_2^sU_{13} + x_3^sU_{12}$ of $Q[U_{12}, U_{13}, U_{23}]$ is not prime. We conclude that the ring B/x_4B is not a domain.

Remark 3.2.8. Of course, one may find an easier way to prove that the ring B/x_4B is not a domain, but we think that both the technique used in lemma 3.2.2 and the criterion of proposition 3.2.7 are worth mentioning since they can be useful in some cases to answer the hard question of the primeness of an ideal.

Remark 3.2.9. With the above notation, we still do not know if the quotient ring

$$B := D_0[U_1, U_2, U_3]/(X_i^{t+1}U_j - X_j^{t+1}U_i + X_4^{t+1}\overline{U_{ij}} : 1 \le i < j \le 3)$$

is a domain or not.

3.3 The solution of Nowicki's conjecture

Let $k[X, Y] := k[X_1, \dots, X_n, Y_1, \dots, Y_n]$ be a polynomial ring in $2n$ variables over k. Recall that we are interested in finding a set of generators of $\ker D$ where D is the derivation $\sum_{i=1}^n X_i^{t_i}\partial/\partial Y_i$ of $k[X, Y]$ ($t_1, \dots, t_n \in \mathbb{N}$). We begin by showing that it is enough to treat the case where all the t_i's are equal to 1.

Proposition 3.3.1. *Let R be a ring, R' a subring of R such that R is a free R'-module. Then every polynomial ring $R[Y_1, \dots, Y_t]$ over R is a free $R'[Y_1, \dots, Y_t]$-module. Moreover, if $\mathcal{B}$ is a basis of R over R', then $\mathcal{B}$ is also a basis of $R[Y_1, \dots, Y_t]$ over $R'[Y_1, \dots, Y_t]$.*

Proof

Clearly, it is enough to assume that $t = 1$. Let $f = \sum a_i Y_1^i \in R[Y_1]$ ($a_i \in R$). Since each a_j can be written uniquely as $\sum \alpha_i b_i$ with $\alpha_i \in R'$ and $b_i \in \mathcal{B}$, then f can be written uniquely as a finite sum

$$f = \sum f_i(Y_1) b_i$$

where $f_i(Y_1) \in R'[Y_1]$ and $b_i \in \mathcal{B}$ for all i. This shows that $\mathcal{B}$ is a basis of $R[Y_1]$ over $R'[Y_1]$. ∎

With assumptions and notations as in Proposition 3.3.1, let $D = \sum_{i=1}^t a_i\partial/\partial Y_i$ be an R-elementary derivation of $R[Y_1, \dots, Y_t]$ such that $a_i \in R'$ for all i and let D' be the restriction of D to $R'[Y_1, \dots, Y_t]$. We have the following.

Lemma 3.3.2. *If $\mathcal{B}$ is a basis of R over R', then* $\ker D$ *is a free* $\ker D'$*-module with basis $\mathcal{B}$. In particular, if $G \subset R'[Y_1, \ldots, Y_t]$ generates* $\ker D'$ *as an R'-algebra, then G generates* $\ker D$ *as an R-algebra.*

Proof

Let $f \in R[Y_1, \ldots, Y_t]$ and write $f = \sum f_i b_i$ where $f_i \in R'[Y_1, \ldots, Y_t]$ and $b_i \in \mathcal{B}$. Since $b_i \in R$, we have that $D(f) = \sum D(f_i) b_i = \sum D'(f_i) b_i$. Therefore,

$$f \in \ker D \Leftrightarrow \forall i, D' f_i = 0 \Leftrightarrow \forall i, f_i \in \ker D'.$$

∎

In our case, let $R = k[X_1, \ldots, X_n]$ and $R' = k[X_1^{t_1}, \ldots, X_n^{t_n}]$. Then R is clearly a free R'-module, and $R' \cong k^{[n]}$. Now let $Z_i = X_i^{t_i}$, then the restriction D' of D to $R'[Y_1, \ldots, Y_n]$ is the derivation $\sum_{i=1}^n Z_i \partial / \partial Y_i$. If $\{Z_i Y_j - Y_i Z_j : 1 \leq i < j \leq n\}$ generates $\ker D'$ over R', then Lemma 3.3.2 implies that the same set generates $\ker D$ over R. Thus the proof of Theorem 1 reduces to that of:

Theorem 3.3.3. *Let n be a positive integer and consider the derivation*

$$D = \sum_{i=1}^{n} X_i \frac{\partial}{\partial Y_i}$$

of the polynomial ring $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. Then

$$\ker D = k[X_1, \ldots, X_n, L_{ij};\ 1 \leq i < j \leq n]$$

where $L_{ij} = X_i Y_j - X_j Y_i$.

Before proving Theorem 3.3.3, we have to establish Proposition 3.3.4 below.

Let n be an integer greater than or equal to 2 and $k[X, Y, T]$ the following polynomial ring over k:

$$k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, T_1, \ldots, T_n, T_{ij} : 1 \leq i, j \leq n].$$

Let $<$ denote the lexicographic ordering on $k[X, Y, T]$ with

$$X_1 > \ldots > X_n > Y_1 > \ldots > Y_n > T_1 > \ldots > T_n > T_{ij}$$

for all i, j with $1 \leq i, j \leq n$ and

$$T_{ij} > T_{kl} \Longleftrightarrow \begin{cases} i < k \\ \text{or} \\ i = k \text{ and } j < l. \end{cases}$$

For $1 \leq i < j \leq n$, we consider the element

$$L_{ij} := X_i Y_j - X_j Y_i$$

of $k[X, Y, T]$. Let I be the ideal of $k[X, Y, T]$ generated by the elements

$$X_1, T_i - X_i, T_{jk} - L_{jk}, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n.$$

In what follows we will give a Groebner basis for the ideal I with respect to the monomial ordering on $k[X, Y, T]$ defined above. Namely, we have the following

Proposition 3.3.4. *With respect to the monomial ordering $<$ on $k[X, Y, T]$ defined above, a Groebner basis for the ideal I is given by the union of the following nine families of elements of $k[X, Y, T]$:*

$\mathcal{F}_1 = \{X_1, T_1\} \cup \{-T_i + X_i;\ 2 \leq i \leq n\}$

$\mathcal{F}_2 = \{T_{ij} + Y_i T_j - Y_j T_i;\ 1 \leq i < j \leq n\}$

$\mathcal{F}_3 = \{Y_i T_{jk} - Y_j T_{ik} + Y_k T_{ij};\ 1 \leq i < j < k \leq n\}$

$\mathcal{F}_4 = \{T_{ij} T_{kl} - T_{ik} T_{jl} + T_{il} T_{jk};\ 1 \leq i < j < k < l \leq n\}$

$\mathcal{F}_5 = \{T_i T_{1j} - T_j T_{1i};\ 2 \leq i < j \leq n\}$

$\mathcal{F}_6 = \{T_i T_{jk} - T_j T_{ik} + T_k T_{ij};\ 1 \leq i < j < k \leq n\}$

$$\mathcal{F}_7 = \left\{ \begin{vmatrix} T_{rj} & T_{rp} & T_{rq} \\ T_{sj} & T_{sp} & T_{sq} \\ T_{kj} & T_{kp} & T_{kq} \end{vmatrix};\ 1 \leq r < s < k \leq j < p < q \leq n \right\}$$

$$\mathcal{F}_8 = \left\{ \begin{vmatrix} T_r & T_s & T_u \\ T_{ir} & T_{is} & T_{iu} \\ T_{jr} & T_{js} & T_{ju} \end{vmatrix} ; \; 2 \le i < j \le r < s < u \le n \right\}$$

$$\mathcal{F}_9 = \left\{ \begin{vmatrix} Y_r & Y_s & Y_u \\ T_{ir} & T_{is} & T_{iu} \\ T_{jr} & T_{js} & T_{ju} \end{vmatrix} ; \; 1 \le i < j \le r < s < u \le n \right\}.$$

Remark 3.3.5. The element $T_{\alpha\beta}$ should be interpreted as zero whenever $\alpha = \beta$.

Remark 3.3.6. For families $\mathcal{F}_7, \mathcal{F}_8, \mathcal{F}_9$, the lexicographic order considered above is "diagonal" in the sense that the leading monomial of each element $f = \det(A)$ $(A \in \mathcal{M}_{3\times 3}(k[Y,T])$ of these families is simply the product of the entries of the main diagonal in the matrix A.

Proof of Proposition 3.3.4

First we prove that the ideal I can be generated by the elements of the families $\mathcal{F}_i$, $i \in \{1, \dots, 9\}$.

Lemma 3.3.7. *With the above notation, I is generated (as an ideal of $k[X,Y,T]$) by $G := \cup_{i=1}^{9} \mathcal{F}_i$.*

Proof

Let I_1 be the ideal of $k[X,Y,T]$ generated by G.

$\underline{I \subseteq I_1}$: Clearly, $T_i - X_i \in \mathcal{F}_1 \subseteq I_1$ for all $i \in \{1, \dots, n\}$. For $1 \le i < j \le n$, we have

$$\begin{aligned} T_{ij} - L_{ij} &= T_{ij} - X_iY_j + X_jY_i \\ &= (T_{ij} + Y_iT_j - Y_jT_i) - Y_j(-T_i + X_i) + Y_i(-T_j + X_j) \\ &\in I_1. \end{aligned}$$

$\underline{I_1 \subseteq I}$: This can be shown using the following identities:

1. $-T_i + X_i = -(T_i - X_i)$; $2 \le i \le n$, $T_1 = (T_1 - X_1) + X_1$

2. $T_{ij} + Y_iT_j - Y_jT_i = (T_{ij} - L_{ij}) + Y_i(T_i - X_i) - Y_j(T_j - X_j);\ 1 \le i < j \le n$

3. $Y_iT_{jk} - Y_jT_{ik} + Y_kT_{ij} = Y_i(T_{jk} - L_{jk}) - Y_j(T_{ik} - L_{ik}) + Y_k(T_{ij} - L_{ij});$
$1 \le i < j < k \le n$

4. $T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk} = T_{kl}(T_{ij} - L_{ij}) - T_{jl}(T_{ik} - L_{ik}) + T_{jk}(T_{il} - L_{il}) + L_{ij}(T_{kl} - L_{kl}) - L_{ik}(T_{jl} - L_{jl}) - L_{il}(T_{jk} - L_{jk});\ 1 \le i < j < k < l \le n$

5. $T_iT_{1j} - T_jT_{1i} = T_{1j}(T_i - X_i) - T_{1i}(T_j - X_j) + X_i(T_{1j} - L_{1j}) - X_j(T_{1i} - L_{1i}) + X_1L_{ij};\ 2 \le i < j \le n$

6. $T_iT_{jk} - T_jT_{ik} + T_kT_{ij} = T_k(T_{ij} - L_{ij}) - T_j(T_{ik} - L_{ik}) + T_i(T_{jk} - L_{jk}) + L_{jk}(T_i - X_i) - L_{ik}(T_j - X_j) + L_{ij}(T_k - X_k);\ 1 \le i < j < k \le n$

7. $$\begin{vmatrix} T_{rj} & T_{rp} & T_{rq} \\ T_{sj} & T_{sp} & T_{sq} \\ T_{kj} & T_{kp} & T_{kq} \end{vmatrix} =$$

$$- T_{kj} \underbrace{(T_{rs}T_{pq} - T_{rp}T_{sq} + T_{rq}T_{sp})}_{\in I \text{ by } (4)} + T_{kp} \underbrace{(T_{rs}T_{jq} - T_{rj}T_{sq} + T_{rq}T_{sj})}_{\in I \text{ by } (4)}$$
$$- T_{kq} \underbrace{(T_{rs}T_{jp} - T_{rj}T_{sp} + T_{rp}T_{sj})}_{\in I \text{ by } (4)} + T_{rs} \underbrace{(T_{kj}T_{pq} - T_{kp}T_{jq} + T_{kq}T_{jp})}_{\in I \text{ by } (4)};$$
$1 \le r < s < k \le j < p < q \le n$

8. $$\begin{vmatrix} T_r & T_s & T_u \\ T_{ir} & T_{is} & T_{iu} \\ T_{jr} & T_{js} & T_{ju} \end{vmatrix} =$$

$$- T_{jr} \underbrace{(T_iT_{su} - T_sT_{iu} + T_uT_{is})}_{\in I \text{ by } (5)} + T_{js} \underbrace{(T_iT_{ru} - T_rT_{iu} + T_uT_{ir})}_{\in I \text{ by } (5)}$$
$$- T_{ju} \underbrace{(T_iT_{rs} - T_rT_{is} + T_sT_{ir})}_{\in I \text{ by } (5)} - T_i \underbrace{(T_{jr}T_{su} - T_{js}T_{ru} + T_{ju}T_{rs})}_{\in I \text{ by } (4)};$$
$2 \le i < j \le r < s < u \le n.$

9. $\begin{vmatrix} Y_r & Y_s & Y_u \\ T_{ir} & T_{is} & T_{iu} \\ T_{jr} & T_{js} & T_{ju} \end{vmatrix} =$

$$- T_{jr} \underbrace{(Y_i T_{su} - Y_s T_{iu} + Y_u T_{is})}_{\in I \text{ by } (3)} + T_{js} \underbrace{(Y_i T_{ru} - Y_r T_{iu} + Y_u T_{ir})}_{\in I \text{ by } (3)}$$
$$- T_{ju} \underbrace{(Y_i T_{rs} - Y_r T_{is} + Y_s T_{ir})}_{\in I \text{ by } (3)} + Y_i \underbrace{(T_{jr} T_{su} - T_{js} T_{ru} + T_{ju} T_{rs})}_{\in I \text{ by } (4)};$$

$1 \leq i < j < r < s < u \leq n.$

This shows that the set G generates the ideal I. ∎

Next we show that G is indeed a Groebner basis for I with respect to the lexicographic ordering considered above. This part of the proof fills the next 51 pages. Our main tool for that objective will be Corollary 3.1.24. We will proceed as follows: given $i, j \in \{1, \dots, 9\}$, we take two distinct elements $f_i \in \mathcal{F}_i$ and $f_j \in \mathcal{F}_j$ and we prove that $S(f_i, f_j)$ is in standard form relative to G. This process will be denoted by case "$(\mathcal{F}_i, \mathcal{F}_j)$". Buchberger's first and second criteria will be used frequently to eliminate many unnecessary computations of the form $S(f_i, f_j)$. The following table shows which cases "$(\mathcal{F}_i, \mathcal{F}_j)$" can be eliminated, either by using Buchberger's first criterion or some other techniques.

Family	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_8$	$\mathcal{F}_9$
$\mathcal{F}_1$	B1	B1	B1	B1	B1	?	B1	B1	B1
$\mathcal{F}_2$	↔	?	?	B1	?	?	B1	?	?
$\mathcal{F}_3$	↔	↔	?	?	B1	?	?	?	?
$\mathcal{F}_4$	↔	↔	↔	?	?	(3,4)	?	(4,9)	?
$\mathcal{F}_5$	↔	↔	↔	↔	?	?	?	?	?
$\mathcal{F}_6$	↔	↔	↔	↔	↔	(3,3)	(3,7)	(3,9)	?
$\mathcal{F}_7$	↔	↔	↔	↔	↔	↔	?	(7,9)	?
$\mathcal{F}_8$	↔	↔	↔	↔	↔	↔	↔	(9,9)	?
$\mathcal{F}_9$	↔	↔	↔	↔	↔	↔	↔	↔	?

The following notation have been used in the above table :

- If a box contains the symbol "B1", then the corresponding case can be disregarded by Buchberger's first criterion.
- If a box $(\mathcal{F}_i, \mathcal{F}_j)$ contains "$\leftrightarrow$" then this means that the corresponding case can be deduced from the case $(\mathcal{F}_j, \mathcal{F}_i)$ using the relation $S(f,g) = -S(g,f)$.
- If a box $(\mathcal{F}_i, \mathcal{F}_j)$ contains a pair "(i', j')" then this means that case $(\mathcal{F}_i, \mathcal{F}_j)$ can be deduced from the case $(\mathcal{F}_{i'}, \mathcal{F}_{j'})$. In each case, the reduction is straightforward and is left to the reader.
- The symbol "?" in a box $(\mathcal{F}_i, \mathcal{F}_j)$ means that we have to do the case $(\mathcal{F}_i, \mathcal{F}_j)$.

For simplicity we will adopt the following notation in our calculations:
If $1 \le r < s < k \le j < p < q \le n$, let

$$|T_{rj}T_{sp}T_{kq}| := \begin{vmatrix} T_{rj} & T_{rp} & T_{rq} \\ T_{sj} & T_{sp} & T_{sq} \\ T_{kj} & T_{kp} & T_{kq} \end{vmatrix}, \ |T_jT_{sp}T_{kq}| := \begin{vmatrix} T_j & T_p & T_q \\ T_{sj} & T_{sp} & T_{sq} \\ T_{kj} & T_{kp} & T_{kq} \end{vmatrix}, \ |Y_jT_{sp}T_{kq}| := \begin{vmatrix} Y_j & Y_p & Y_q \\ T_{sj} & T_{sp} & T_{sq} \\ T_{kj} & T_{kp} & T_{kq} \end{vmatrix}.$$

Also we will in some cases indicate the leading monomial of an element f by underlining it. Another helpful remark is the following:

Remark 3.3.8. Given an element f of $k[X,Y,T]$, we denote by TERMS(f) the set of all monomials appearing in f. The greatest element of TERMS$(f)\backslash\{\text{LM}(f)\}$ (with respect to the above monomial ordering on $k[X,Y,T]$) is called the "trailing element of f" and denoted by TR(f). So, if $f,g \in k[X,Y,T]$, their S-polynomial has the form $S(f,g) = \Delta f - \Lambda g$, and if $\Delta\text{TR}(f) \neq \Lambda\text{TR}(g)$, then the leading monomial of $S(f,g)$ is the maximum of $\Delta\text{TR}(f)$ and $\Lambda\text{TR}(g)$.

Case $(\mathcal{F}_1, \mathcal{F}_6)$

It is clear that the leading monomials of an element f of $\mathcal{F}_1$ and an element g of $\mathcal{F}_6$ are always relatively prime except in the case where $f = T_1$ and $g = T_1T_{jk} - T_jT_{1k} + T_kT_{1j}$ for some j,k satisfying $2 \le j < k \le n$. In this case,

$$S(f,g) = \underline{T_jT_{1k}} - T_kT_{1j} \in \mathcal{F}_5$$

is in standard form relative to G.

Case $(\mathcal{F}_2, \mathcal{F}_2)$

Let

$$f = T_{ij} + \underline{Y_iT_j} - Y_jT_i, \qquad g = T_{kl} + \underline{Y_kT_l} - Y_lT_k$$
$$1 \le i < j \le n \qquad\qquad 1 \le k < l \le n$$

be two distinct elements of $\mathcal{F}_2$. By Buchberger's first criterion, two cases need to be considered:

(1) $\underline{i = k}$ In this case, we may clearly assume that $1 \le i = k < j < l \le n$ and so

$$\begin{aligned} S(f,g) &= T_l(T_{ij} + Y_iT_j - Y_jT_i) - T_j(T_{il} + Y_iT_l - Y_lT_i) \\ &= T_lT_{ij} - \underline{Y_jT_iT_l} - T_jT_{il} + Y_lT_iT_j \\ &= (T_iT_{jl} - T_jT_{il} + T_lT_{ij}) - T_i(T_{jl} + Y_jT_l - Y_lT_j). \end{aligned}$$

(2) $\underline{j = l}$ In this case we may assume $1 \le i < k < j = l \le n$ and so

$$\begin{aligned} S(f,g) &= Y_k(T_{ij} + Y_iT_j - Y_jT_i) - Y_i(T_{kj} + Y_kT_j - Y_jT_k) \\ &= Y_kT_{ij} - Y_kY_jT_i - Y_iT_{kj} + \underline{Y_iY_jT_k} \\ &= Y_j(T_{ik} + Y_iT_k - Y_kT_i) - (Y_iT_{kj} + Y_jT_{ik} - Y_kT_{ij}). \end{aligned}$$

This proves that $S(f,g)$ is in standard form relative to G in this case.

Case $(\mathcal{F}_2, \mathcal{F}_3)$

Let

$$f = T_{rs} + \underline{Y_rT_s} - Y_sT_r \in \mathcal{F}_2, \qquad g = \underline{Y_iT_{jk}} - Y_jT_{ik} + Y_kT_{ij} \in \mathcal{F}_3$$
$$1 \le r < s \le n \qquad\qquad 1 \le i < j < k \le n$$

We need to show that $S(f,g)$ is in standard form relative to G. For this we use Buchberger's first criterion to restrict to the case where $r = i$. In this case, $S(f,g) = T_{jk}f - T_sg$ and the leading monomial of $S(f,g)$ is either $Y_sT_rT_{jk}$ (if $s \le j$) or $Y_jT_sT_{ik}$ (if $j < s$). Consider the following possibilities:

$$(i1)\ 1 \le r = i < s \le j < k \le n$$
$$(i2)\ 1 \le r = i < j < s \le k \le n$$
$$(i3)\ 1 \le r = i < j < k < s \le n.$$

The leading monomial of $S(f,g)$ is $Y_sT_rT_{jk}$ in case ($i1$) and $Y_jT_sT_{ik}$ in cases ($i2$) et ($i3$). In case ($i1$), one can easily verify that

$$\begin{aligned} S(f,g) &= T_{ij}(T_{sk}+Y_sT_k-Y_kT_s)-T_{ik}(T_{sj}+Y_sT_j-Y_jT_s) \\ &+ (T_{is}T_{jk}-T_{ij}T_{sk}+T_{ik}T_{sj})-Y_s(T_iT_{jk}-T_jT_{ik}+T_kT_{ij}) \end{aligned} \tag{65}$$

which proves that $S(f,g)$ is in standard form relative to G in this case. One can verify that $S(f,g)$ is also in standard form relative to G in cases ($i2$) and ($i3$).

Case $(\mathcal{F}_2, \mathcal{F}_5)$

Let

$$f = T_{ij}+Y_iT_j-Y_jT_i \in \mathcal{F}_2, \qquad g = T_jT_{1r}-T_rT_{1j} \in \mathcal{F}_5$$
$$1 \le i < j \le n, \qquad 1 \le j < r \le n.$$

So, $1 \le i < j < r \le n$, and $S(f,g) = \underline{Y_iT_rT_{1j}} - Y_jT_i + T_{1r}T_{ij}$. One can also verify that

$$\begin{aligned} S(f,g) &= (T_{1i}T_{jr}-T_{1j}T_{ir}+T_{1r}T_{ij})+T_{1j}(T_{ir}+Y_iT_r-Y_rT_i) \\ &- T_{1i}(T_{jr}+Y_jT_r-Y_rT_j)-Y_j(T_iT_{1r}-T_rT_{1i}) \\ &+ Y_r(T_iT_{1j}-T_jT_{1i}). \end{aligned}$$

This proves that $S(f,g)$ is in standard form relative to G in this case.

Case $(\mathcal{F}_2, \mathcal{F}_6)$

Let

$$f = T_{rs}+Y_rT_s-Y_sT_r \in \mathcal{F}_2, \qquad g = T_iT_{jk}-T_jT_{ik}+T_kT_{ij} \in \mathcal{F}_6$$
$$1 \le r < s \le n, \qquad 1 \le i < j < k \le n.$$

The leading monomials of f and g are relatively prime except in the case

$$1 \le r < s = i < j < k \le n.$$

In this case, the leading monomial of $S(f,g)$ is $Y_rT_jT_{sk}$, and one can verify that

$$\begin{aligned} S(f,g) &= (T_{rs}T_{jk}-T_{rj}T_{sk}+T_{rk}T_{sj})+T_{sk}(T_{rj}+Y_rT_j-Y_jT_r) \\ &- T_{sj}(T_{rk}+Y_rT_k-Y_kT_r)-T_r(Y_sT_{jk}-Y_jT_{sk}+Y_kT_{sj}). \end{aligned}$$

Case $(\mathcal{F}_2, \mathcal{F}_8)$

Let

$$f = T_{ij} + \underline{Y_i T_j} - Y_j T_i \in \mathcal{F}_2, \qquad g = |T_r T_{as} T_{bu}| \in \mathcal{F}_8$$
$$1 \leq i < j \leq n \qquad 2 \leq a < b \leq r < s < u \leq n.$$

By Buchberger's first criterion, it is enough to consider the case where $j = r$. This leaves us with the following possibilities:

$$(i1)\ 1 \leq i \leq a < b \leq r = j < s < u \leq n$$
$$(i2)\ 2 \leq a < i \leq b \leq r = j < s < u \leq n$$
$$(i3)\ 2 \leq a < b < i < r = j < s < u \leq n.$$

In all three cases, the leading monomial of $S(f,g)$ is $Y_i T_r T_{au} T_{bs}$. For $(i1)$, we have that $S(f,g)$ is equal to

$$\begin{aligned}
& T_{au}T_{bs}(T_{ir} + Y_iT_r - Y_rT_i) + (T_{ar}T_{bu} - T_{au}T_{br})(T_{is} + Y_iT_s - Y_sT_i) \\
- \;& (T_{ar}T_{bs} - T_{as}T_{br})(T_{iu} + Y_iT_u - Y_uT_i) + (T_{ia}T_{bu} - T_{ib}T_{au})(T_{rs} + Y_rT_s - Y_sT_r) \\
- \;& (T_{ia}T_{bs} - T_{ib}T_{as})(T_{ru} + Y_rT_u - Y_uT_r) + (T_{ia}T_{br} - T_{ib}T_{ar})(T_{su} + Y_sT_u - Y_uT_s) \\
- \;& T_{su}(T_{ia}T_{br} - T_{ib}T_{ar} + T_{ir}T_{ab}) + T_{ru}(T_{ia}T_{bs} - T_{ib}T_{as} + T_{is}T_{ab}) \\
- \;& T_{rs}(T_{ia}T_{bu} - T_{ib}T_{au} + T_{iu}T_{ab}) + T_{ab}(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) \\
+ \;& (Y_sT_{bu} - Y_uT_{bs})(T_iT_{ar} - T_aT_{ir} + T_rT_{ia}) - (Y_rT_{bu} - Y_uT_{br})(T_iT_{as} - T_aT_{is} + T_sT_{ia}) \\
+ \;& (Y_rT_{bs} - Y_sT_{br})(T_iT_{au} - T_aT_{iu} + T_uT_{ia}) - (Y_sT_{iu} - Y_uT_{is})(T_aT_{br} - T_bT_{ar} + T_rT_{ab}) \\
+ \;& (Y_rT_{iu} - Y_uT_{ir})(T_aT_{bs} - T_bT_{as} + T_sT_{ab}) - (Y_rT_{is} - Y_sT_{ir})(T_aT_{bu} - T_bT_{au} + T_uT_{ab}) \\
+ \;& |T_{ir}T_{as}T_{bu}| - Y_u\,|T_bT_{ir}T_{as}| + Y_s\,|T_bT_{ir}T_{au}| - Y_r\,|T_bT_{is}T_{au}|.
\end{aligned}$$

One can also show that $S(f,g)$ is in standard form relative to G in cases $(i2)$ and $(i3)$.

Case $(\mathcal{F}_2, \mathcal{F}_9)$

Let

$$f = T_{ak} + \underline{Y_a T_k} - Y_k T_a \in \mathcal{F}_2, \qquad g = |Y_r T_{is} T_{ju}| \in \mathcal{F}_8$$
$$1 \leq a < k \leq n \qquad 1 \leq i < j \leq r < s < u \leq n.$$

The leading monomials of f and g are relatively prime except when $a = r$, in which case $S(f,g) = T_{is}T_{ju}f - T_{k}g$ with leading monomial equal to $Y_rT_kT_{iu}T_{js}$. One has the following possibilities:

$$(i1)\, 1 \le i < j < r = a < k \le s < u \le n$$
$$(i2)\, 1 \le i < j < r = a < s < k \le u \le n$$
$$(i3)\, 1 \le i < j < r = a < s < u < k \le n.$$

For $(i1)$, one can verify that

$$\begin{aligned} S(f,g) &= T_{iu}T_{js}(T_{rk} + Y_rT_k - Y_kT_r) - (T_{ir}T_{ju} - T_{iu}T_{jr})(T_{ks} + Y_kT_s - Y_sT_k) \\ &+ (T_{ir}T_{js} - T_{is}T_{jr})(T_{ku} + Y_kT_u - Y_uT_k) + T_{ju}(T_{ir}T_{ks} - T_{ik}T_{rs} + T_{is}T_{rk}) \\ &- T_{js}(T_{ir}T_{ku} - T_{ik}T_{ru} + T_{iu}T_{rk}) - T_{jr}(T_{ik}T_{su} - T_{is}T_{ku} + T_{iu}T_{ks}) \\ &+ T_{ik}(T_{jr}T_{su} - T_{js}T_{ru} + T_{ju}T_{rs}) - Y_k\,|T_rT_{is}T_{ju}|\,. \end{aligned} \tag{66}$$

This proves that $S(f,g)$ is in standard form relative to G in this case. Using (66) and homomorphisms of type $\phi_{\rho\theta}$, one can also show that $S(f,g)$ is in standard form relative to G in the other two cases $(i2)$ and $(i3)$.

Case $(\mathcal{F}_3, \mathcal{F}_3)$

Let

$$f = \underline{Y_iT_{jk}} - Y_jT_{ik} + Y_kT_{ij}, \qquad g = \underline{Y_rT_{su}} - Y_sT_{ru} + Y_uT_{rs}$$
$$1 \le i < j < k \le n, \qquad 1 \le r < s < u \le n$$

be two distinct elements of $\mathcal{F}_3$. The leading monomials of f and g are relatively prime if and only if exactly one of the two following cases occurs

$$(1)\, i = r \qquad (2)\, (j,k) = (s,u)$$

(1) $\underline{i = r}$ In this case $S(f,g) = T_{su}f - T_{jk}g$ and we have the following six possibilities

$$(i1)\, j < k \le s < u \quad (i2)\, j \le s < k \le u \quad (i3)\, j < s < u < k$$
$$(i4)\, s < u \le j < k \quad (i5)\, s \le j < u \le k \quad (i6)\, s < j < k < u.$$

Using the relation $S(f,g) = -S(g,f)$, one can see that if $S(f,g)$ is in standard form relative to G in the cases $(i1),(i2),(i3)$, then it is also in the cases $(i4),(i5),(i6)$.

Hence one can restrict to the first three cases.
In case ($i1$), the leading monomial of $S(f,g)$ is $Y_jT_{ik}T_{su}$ and one can verify that $S(f,g)$ has the following expression

$$-T_{ik}(Y_jT_{su} - Y_sT_{ju} + Y_uT_{js}) + Y_k(T_{ij}T_{su} - T_{is}T_{ju} + T_{iu}T_{js}) + |Y_kT_{is}T_{ju}| \qquad (67)$$

which proves that $S(f,g)$ is in standard form in this case. For cases ($i2$) and ($i3$), one can get a standard form of $S(f,g)$ by interchanging columns in the determinant of (67), and hence multiplying the determinant by -1.
(2) $\underline{(j,k) = (s,u)}$ In this case we may clearly assume that $1 \le i < r < j = s < k = u \le n$ and hence $S(f,g) = Y_rf - Y_ig$ and the leading monomial of $S(f,g)$ is clearly $Y_iY_jT_{rk}$. On the other hand,

$$S(f,g) = Y_j(Y_iT_{rk} - Y_rT_{ik} + Y_kT_{ir}) - Y_k(Y_iT_{rj} - Y_rT_{ij} + Y_jT_{ir}).$$

This proves that $S(f,g)$ is in standard form relative to G in this case.

Case $(\mathcal{F}_3, \mathcal{F}_4)$

Let

$$f = \underline{Y_pT_{qr}} - Y_qT_{pr} + Y_rT_{pq} \in \mathcal{F}_3, \qquad g = \underline{T_{ij}T_{kl}} - T_{ik}T_{jl} + T_{il}T_{jk} \in \mathcal{F}_4$$
$$1 \le p < q < r \le n, \qquad\qquad 1 \le i < j < k < l \le n$$

The leading monomials of f and g are relatively prime unless one of the following two cases is true:

$$(1)\ (q,r) = (i,j) \qquad (2)\ (q,r) = (k,l)$$

(1) $\underline{(q,r) = (i,j)}$ In this case we have

$$1 \le p < i = q < j = r < k < l \le n,$$

$S(f,g) = T_{kl}f - Y_pg$ and the leading monomial of $S(f,g)$ is $Y_pT_{ik}T_{jl}$. One can verify that

$$\begin{aligned} S(f,g) &= T_{jl}(Y_pT_{ik} - Y_iT_{pk} + Y_kT_{pi}) - T_{il}(Y_pT_{jk} - Y_jT_{pk} + Y_kT_{pj}) \\ &+ T_{pl}(Y_iT_{jk} - Y_jT_{ik} + Y_kT_{ij}) - Y_i(T_{pj}T_{kl} - T_{pk}T_{jl} + T_{pl}T_{jk}) \\ &+ Y_j(T_{pi}T_{kl} - T_{pk}T_{il} + T_{pl}T_{ik}) - Y_k(T_{pi}T_{jl} - T_{il}T_{pj} + T_{pl}T_{ij}) \end{aligned}$$

(2) $\underline{(q,r)=(k,l)}$ In this case, $S(f,g) = T_{ij}f - Y_pg$ and its leading monomial is $Y_pT_{ik}T_{jl}$. Now according to the position of p relative to $i < j$, we have the following possibilities

$$(i1)\ 1 \le p \le i < j < k = q < l = r \le n$$
$$(i2)\ 1 \le i < p \le j < k = q < l = r \le n$$
$$(i3)\ 1 \le i < j < p < k = q < l = r \le n$$

For (i), one has:

$$S(f,g) \;=\; -T_{il}(Y_pT_{jk} - Y_jT_{pk} + Y_kT_{pj}) + T_{ik}(Y_pT_{jl} - Y_jT_{pl} + Y_lT_{pj}) - |Y_jT_{pk}T_{il}| \qquad (68)$$

Applying ϕ_{pi} to (68) one can also see that $S(f,g)$ is in standard form relative to G in case $(i2)$. For $(i3)$, $S(f,g)$ has the following expression

$$Y_l(T_{ij}T_{pk} - T_{ip}T_{jk} + T_{ik}T_{jp}) - Y_k(T_{ij}T_{pl} - T_{ip}T_{jl} + T_{il}T_{jp}) + |Y_pT_{ik}T_{jl}|$$

Case $(\mathcal{F}_3, \mathcal{F}_6)$

Let

$$f = Y_iT_{jk} - Y_jT_{ik} + Y_kT_{ij} \in \mathcal{F}_3, \qquad g = T_rT_{su} - T_sT_{ru} + T_uT_{rs} \in \mathcal{F}_6$$
$$1 \le i < j < k \le n, \qquad 1 \le r < s < u \le n.$$

By Buchberger's first criterion, we may assume that $(j,k) = (s,u)$. Also, by the case $(\mathcal{F}_3, \mathcal{F}_3)$, we may assume that $i = r$. In this case, the leading monomial of $S(f,g)$ is $Y_iT_jT_{ik}$, and one can check that

$$S(f,g) \;=\; T_{ik}(T_{ij} + Y_iT_j - Y_jT_i) - T_{ij}(T_{ik} + Y_iT_k - Y_kT_i).$$

Case $(\mathcal{F}_3, \mathcal{F}_7)$

Let

$$f = \underline{Y_aT_{bc}} - Y_bT_{ac} + Y_cT_{ab} \in \mathcal{F}_3, \qquad g = |T_{rj}T_{sp}T_{kq}| \in \mathcal{F}_7$$
$$1 \le a < b < c \le n \qquad 1 \le r < s < k \le j < p < q \le n.$$

By Buchberger's first criterion, one has to look only at the following three cases

$$(1)\ (b,c) = (r,j) \quad (2)\ (b,c) = (s,p) \quad (3)\ (b,c) = (k,q)$$

(1) $\underline{(b,c)=(r,j)}$ In this case

$$1 \le a < r = b < s < k \le j = c < p < q \le n,$$

$S(f,g) = T_{sp}T_{kq}f - Y_a g$ and the leading monomial of $S(f,g)$ is $Y_aT_{rj}T_{sq}T_{kp}$. On the other hand, one can verify that in this case $S(f,g)$ is equal to the following expression:

$$\begin{aligned}
& T_{sq}T_{kp}(Y_aT_{rj} - Y_rT_{aj} + Y_jT_{ar}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(Y_aT_{rp} - Y_rT_{ap} + Y_pT_{ar}) \\
- \; & (T_{sj}T_{kp} - T_{sp}T_{kj})(Y_aT_{rq} - Y_rT_{aq} + Y_qT_{ar}) + (T_{ap}T_{kq} - T_{aq}T_{kp})(Y_rT_{sj} - Y_sT_{rj} + Y_jT_{rs}) \\
- \; & (T_{aj}T_{kq} - T_{aq}T_{kj})(Y_rT_{sp} - Y_sT_{rp} + Y_pT_{rs}) + (T_{aj}T_{kp} - T_{ap}T_{kj})(Y_rT_{sq} - Y_sT_{rq} + Y_qT_{rs}) \\
- \; & (T_{ap}T_{rq} - T_{aq}T_{rp})(Y_sT_{kj} - Y_kT_{sj} + Y_jT_{sk}) + (T_{aj}T_{rq} - T_{aq}T_{rj})(Y_sT_{kp} - Y_kT_{sp} + Y_pT_{sk}) \\
- \; & (T_{aj}T_{rp} - T_{ap}T_{rj})(Y_sT_{kq} - Y_kT_{sq} + Y_qT_{sk}) - Y_k\,|T_{aj}T_{rp}T_{sq}| + T_{sk}\,|Y_jT_{ap}T_{rq}| \\
- \; & T_{rs}\,|Y_jT_{ap}T_{kq}| + T_{ar}\,|Y_jT_{sp}T_{kq}|\,. \qquad (69)
\end{aligned}$$

This clearly shows that $S(f,g)$ is in standard form relative to G in this case.
(2) $\underline{(b,c)=(s,p)}$ In this case, $S(f,g) = T_{rj}T_{kq}f - Y_a g$ and its leading monomial is always $Y_aT_{rj}T_{sq}T_{kp}$. Now according to the position of a relative to r one has to look at the following two cases:

$$(i1)\, 1 \le r \le a < s = b < k \le j < p = c < q \le n$$
$$(i2)\, 1 \le a < r < s = b < k \le j < p = c < q \le n$$

For the case $(i1)$, we have that $S(f,g)$ is equal to

$$\begin{aligned}
& (T_{rp}T_{kq} - T_{rq}T_{kp})(Y_aT_{sj} - Y_sT_{aj} + Y_jT_{as}) + T_{rq}T_{kj}(Y_aT_{sp} - Y_sT_{ap} + Y_pT_{as}) \\
+ \; & (T_{rj}T_{kp} - T_{rp}T_{kj})(Y_aT_{sq} - Y_sT_{aq} + Y_qT_{as}) - (T_{rp}T_{aq} - T_{rq}T_{ap})(Y_sT_{kj} - Y_kT_{sj} + Y_jT_{sk}) \\
+ \; & (T_{rj}T_{aq} - T_{rq}T_{aj})(Y_sT_{kp} - Y_kT_{sp} + Y_pT_{sk}) - (T_{rj}T_{ap} - T_{rp}T_{aj})(Y_sT_{kq} - Y_kT_{sq} + Y_qT_{sk}) \\
- \; & Y_k\,|T_{rj}T_{ap}T_{sq}| + T_{sk}\,|Y_jT_{rp}T_{aq}| - T_{as}\,|Y_jT_{rp}T_{kq}| \qquad (70)
\end{aligned}$$

In the case $r = a$, we replacing a by r in (70) and we obtain the following expression of $S(f,g)$:

$$\begin{aligned}
S(f,g) \;=\; & (T_{rp}T_{kq} - T_{rq}T_{kp})(Y_rT_{sj} - Y_sT_{rj} + Y_jT_{rs}) + T_{rq}T_{kj}(Y_rT_{sp} - Y_sT_{rp} + Y_pT_{rs}) \\
+ \; & (T_{rj}T_{kp} - T_{rp}T_{kj})(Y_rT_{sq} - Y_sT_{rq} + Y_qT_{rs}) - T_{as}\,|Y_jT_{rp}T_{kq}| \qquad (71)
\end{aligned}$$

For the case $(i2)$, one can verify that $S(f,g)$ can be written as

$$\begin{aligned}
& T_{sq}T_{kp}(Y_aT_{rj} - Y_rT_{aj} + Y_jT_{ar}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(Y_aT_{rk} - Y_rT_{ak} + Y_kT_{ar}) \\
- & (T_{sj}T_{kp} - T_{sp}T_{kj})(Y_aT_{rq} - Y_rT_{aq} + Y_qT_{ar}) + (T_{ap}T_{kq} - T_{aq}T_{kp})(Y_rT_{sj} - Y_sT_{rj} + Y_jT_{rs}) \\
+ & T_{aq}T_{kj}(Y_rT_{sp} - Y_sT_{rp} + Y_pT_{rs}) + (T_{aj}T_{kp} - T_{ap}T_{kj})(Y_rT_{sq} - Y_sT_{rq} + Y_qT_{rs}) \\
- & (T_{ap}T_{rq} - T_{aq}T_{rp})(Y_sT_{kj} - Y_kT_{sj} + Y_jT_{sk}) + (T_{aj}T_{rq} - T_{aq}T_{rj})(Y_sT_{kp} - Y_kT_{sp} + Y_pT_{sk}) \\
- & (Y_pT_{kq} - Y_qT_{kp})(T_{ar}T_{sj} - T_{as}T_{rj} + T_{aj}T_{rs}) - Y_qT_{kj}(T_{ar}T_{sp} - T_{as}T_{rp} + T_{ap}T_{rs}) \\
- & (Y_jT_{kp} - Y_pT_{kj})(T_{ar}T_{sq} - T_{as}T_{rq} + T_{aq}T_{rs}) + (Y_pT_{rq} - Y_qT_{rp})(T_{as}T_{kj} - T_{ak}T_{sj} + T_{aj}T_{sk}) \\
- & (Y_jT_{rq} - Y_qT_{rj})(T_{as}T_{kp} - T_{ak}T_{sp} + T_{ap}T_{sk}) - 2(Y_pT_{aq} - Y_qT_{ap})(T_{rs}T_{kj} - T_{rk}T_{sj} + T_{rj}T_{sk}) \\
+ & 2(Y_jT_{aq} - Y_qT_{aj})(T_{rs}T_{kp} - T_{rk}T_{sp} + T_{rp}T_{sk}) - (Y_jT_{ap} - Y_pT_{aj})(T_{rs}T_{kq} - T_{rk}T_{sq} + T_{rq}T_{sk}) \\
+ & T_{sp}\left|Y_kT_{aj}T_{rq}\right| - T_{sj}\left|Y_kT_{ap}T_{rq}\right| + 3T_{sk}\left|Y_jT_{ap}T_{rq}\right| - T_{rk}\left|Y_jT_{ap}T_{sq}\right| \qquad (72)
\end{aligned}$$

(3) $\underline{(b,c) = (k,q)}$ In this case, $S(f,g) = T_{rj}T_{sp}f - Y_ag$ with leading monomial equal to $Y_aT_{rj}T_{sq}T_{kp}$. We consider the following three possibilities:

$$\begin{aligned}
&(i1)\, 1 \le a \le r < s < b = k \le j < p < q \le n \\
&(i2)\, 1 \le r \le a \le s < b = k \le j < p < q \le n \\
&(i3)\, 1 \le r < s < a < b = k \le j < p < q \le n
\end{aligned}$$

For $(i1)$, one can verify that

$$\begin{aligned}
S(f,g) &= -(T_{rp}T_{sq} - T_{rq}T_{sp})(Y_aT_{kj} - Y_kT_{aj} + Y_jT_{ak}) \\
&+ (T_{rj}T_{sq} - T_{rq}T_{sj})(Y_aT_{kp} - Y_kT_{ap} + Y_pT_{ak}) \\
&+ T_{rp}T_{sj}(Y_aT_{kq} - Y_kT_{aq} + Y_qT_{ak}) + T_{ak}\left|Y_jT_{rp}T_{sq}\right| - Y_k\left|T_{aj}T_{rp}T_{sq}\right| \quad (73)
\end{aligned}$$

$S(f,g)$ can be proven to be in standard form relative to G in cases $(i2)$ and $(i3)$ using (73) and the following remark.

Remark 3.3.9. In (73), $S(f,g) = \sum_{i=1}^{5} a_ih_i$ with $a_i \in k[Y,T]$, $h_1,h_2,h_3 \in \mathcal{F}_3$, $h_4 \in \mathcal{F}_9, h_6 \in \mathcal{F}_7$ with the following properties:

- The terms a_ih_i are uneffected by any permutation of the set $\{a,r,s\}$ for $i = 1,\ldots,5$.

- $Y_k\,|T_{\sigma j}T_{\lambda p}T_{\rho q}|$ is always smaller than the leading monomial of $S(f,g)$ in any of the cases $(i1), (i2), (i3)$ for $\{\sigma, \lambda, \rho\} = \{a, r, s\}$.

So to obtain an expression of $S(f,g)$ in standard form relative to G in the cases $(i2)$ and $(i3)$, we simply interchange rows in the last determinant of the right hand side of (73) and multiply the resulting determinant by a suitable sign.

Case $(\mathcal{F}_3, \mathcal{F}_8)$

Let

$$f = \underline{Y_aT_{bc}} - Y_bT_{ac} + Y_cT_{ab} \in \mathcal{F}_3, \qquad g = |T_rT_{is}T_{ju}| \in \mathcal{F}_8$$
$$1 \le a < b < c \le n \qquad\qquad 2 \le i < j \le r < s < u \le n.$$

The leading monomials of f and g are relatively prime except in the following two cases

$$(1)\ (b,c) = (i,s) \qquad\qquad (2)\ (b,c) = (j,u).$$

(1) $\underline{(b,c) = (i,s)}$ In this case, the leading monomial of $S(f,g)$ is $Y_aT_rT_{iu}T_{js}$, and one has the unique possibility

$$1 \le a < i = b < j \le r < s = c < u \le n$$

in which case, one can verify that $S(f,g)$ is equal to

$$\begin{aligned}
& T_{iu}T_{js}(T_{ar} + Y_aT_r - Y_rT_a) + (T_{ir}T_{ju} - T_{iu}T_{jr})(T_{as} + Y_aT_s - Y_sT_a) \\
- \;& (T_{ir}T_{js} - T_{is}T_{jr})(T_{au} + Y_aT_u - Y_uT_a) - T_{as}T_{ju}(T_{ir} + Y_iT_r - Y_rT_i) \\
- \;& T_{au}T_{ij}(T_{rs} + Y_rT_s - Y_sT_r) - (T_{ai}T_{js} - T_{as}T_{ij})(T_{ru} + Y_rT_u - Y_uT_r) \\
+ \;& (T_{ai}T_{jr} - T_{ar}T_{ij})(T_{su} + Y_sT_u - Y_uT_s) - T_{su}(T_{ai}T_{jr} - T_{aj}T_{ir} + T_{ar}T_{ij}) \\
+ \;& T_{ru}(T_{ai}T_{js} - T_{aj}T_{is} + T_{as}T_{ij}) + T_{is}(T_{aj}T_{ru} - T_{ar}T_{ju} + T_{au}T_{jr}) \\
- \;& T_{ir}(T_{aj}T_{su} - T_{as}T_{ju} + T_{au}T_{js}) + 2T_{ij}(T_{ar}T_{su} - T_{as}T_{ru} + T_{au}T_{rs}) \\
- \;& T_{au}(T_{ij}T_{rs} - T_{ir}T_{js} + T_{is}T_{jr}) + (Y_sT_{ju} - Y_uT_{js})(T_aT_{ir} - T_iT_{ar} + T_rT_{ai}) \\
+ \;& Y_uT_{jr}(T_aT_{is} - T_iT_{as} + T_sT_{ai}) + (Y_rT_{js} - Y_sT_{jr})(T_aT_{iu} - T_iT_{au} + T_uT_{ai}) \\
- \;& (Y_sT_{au} - Y_uT_{as})(T_iT_{jr} - T_jT_{ir} + T_rT_{ij}) + (Y_rT_{au} - Y_uT_{ar})(T_iT_{js} - T_jT_{is} + T_sT_{ij}) \\
- \;& (Y_rT_{as} - Y_sT_{ar})(T_iT_{ju} - T_jT_{iu} + T_uT_{ij}) - T_j\,|Y_rT_{as}T_{iu}| + |T_{ar}T_{is}T_{ju}|\,.
\end{aligned}$$

This proves that $S(f,g)$ is in standard form relative to G in this case.
(2) $\underline{(b,c)=(j,u)}$ Here, the leading monomial of $S(f,g)$ is always $Y_aT_rT_{iu}T_{js}$, and one has the following two possibilities:

$$(i1)\, 1 \le a \le i < b = j \le r < s < u = c \le n$$
$$(i2)\, 2 \le i < a < b = j \le r < s < u = c \le n.$$

In case $(i1)$, one has the following expression of $S(f,g)$:

$$\begin{aligned}
& T_{iu}T_{js}(T_{ar}+Y_aT_r-Y_rT_a)+(T_{ir}T_{ju}-T_{iu}T_{jr})(T_{as}+Y_aT_s-Y_sT_a) \\
- & (T_{ir}T_{js}-T_{is}T_{jr})(T_{au}+Y_aT_u-Y_uT_a)-T_{au}T_{is}(T_{jr}+Y_jT_r-Y_rT_j) \\
- & T_{au}T_{ij}(T_{rs}+Y_rT_s-Y_sT_r)-T_{ai}T_{js}(T_{ru}+Y_rT_u-Y_uT_r) \\
+ & (T_{ai}T_{jr}-T_{aj}T_{ir})(T_{su}+Y_sT_u-Y_uT_s)-T_{su}(T_{ai}T_{jr}-T_{aj}T_{ir}+T_{ar}T_{ij}) \\
+ & T_{ru}(T_{ai}T_{js}-T_{aj}T_{is}+T_{as}T_{ij})-Y_sT_r(T_{ai}T_{ju}-T_{aj}T_{iu}+T_{au}T_{ij}) \\
+ & T_{is}(T_{aj}T_{ru}-T_{ar}T_{ju}+T_{au}T_{jr})+T_{ij}(T_{ar}T_{su}-T_{as}T_{ru}+T_{au}T_{rs}) \\
+ & (Y_sT_{ju}-Y_uT_{js})(T_aT_{ir}-T_iT_{ar}+T_rT_{ai})+Y_uT_{jr}(T_aT_{is}-T_iT_{as}+T_sT_{ai}) \\
+ & (Y_rT_{js}-Y_sT_{jr})(T_aT_{iu}-T_iT_{au}+T_uT_{ai})-(Y_sT_{au}-Y_uT_{as})(T_iT_{jr}-T_jT_{ir}+T_rT_{ij}) \\
+ & (Y_rT_{au}-Y_uT_{ar})(T_iT_{js}-T_jT_{is}+T_sT_{ij})+Y_sT_{ar}(T_iT_{ju}-T_jT_{iu}+T_uT_{ij}) \\
- & Y_u\,|T_jT_{ar}T_{is}|+Y_s\,|T_jT_{ar}T_{iu}|+|T_{ar}T_{is}T_{ju}|\,. \qquad (74)
\end{aligned}$$

This shows that $S(f,g)$ is in standard form relative to G in this case. Replacing T_{ai} by $-T_{ia}$ and interchanging rows a and i in the determinants of (74), one can also show that $S(f,g)$ is in standard form relative to G in case $(i2)$. This finishes case $(\mathcal{F}_3,\mathcal{F}_8)$.

Case $(\mathcal{F}_4,\mathcal{F}_4)$

Let

$$f=\underline{T_{ab}T_{cd}}-T_{ac}T_{bd}+T_{ad}T_{bc}, \qquad g=\underline{T_{ij}T_{kl}}-T_{ik}T_{jl}+T_{il}T_{jk}$$
$$1\le a<b<c<d\le n \qquad\qquad 1\le i<j<k<l\le n$$

be two distinct elements of $\mathcal{F}_4$. We want to show that $S(f,g)$ is in standard form relative to G. By Buchberger's first criterion, one has to consider the following possibilities

$$(1)\,(a,b)=(i,j) \qquad (2)\,(a,b)=(k,l)$$
$$(3)\,(c,d)=(i,j) \qquad (4)\,(c,d)=(k,l)$$

(1) $(a,b)=(i,j)$ In this case $S(f,g) = T_{kl}f - T_{cd}g$. Using the relation $S(f,g) = -S(g,f)$, we can restrict ourselves to the following possibilities:

$$(i1)\, 1 \le a = i \le b = j < c < d \le k < l \le n$$
$$(i2)\, 1 \le a = i < b = j < c \le k \le d \le l \le n$$
$$(i3)\, 1 \le a = i < b = j < c \le k < l \le d \le n$$

For $(i1)$, the leading monomial of $S(f,g)$ is $T_{ic}T_{jd}T_{kl}$ and we have

$$\begin{aligned} S(f,g) &= -T_{jd}(T_{ic}T_{kl} - T_{ik}T_{cl} + T_{il}T_{ck}) + T_{jc}(T_{id}T_{kl} - T_{ik}T_{dl} + T_{il}T_{dk}) \\ &- T_{il}(T_{jc}T_{dk} - T_{jd}T_{ck} + T_{jk}T_{cd}) + T_{ik}(T_{jc}T_{dl} - T_{jd}T_{cl} + T_{jl}T_{cd}) \end{aligned}$$

For $(i2)$, the leading monomial of $S(f,g)$ is also $T_{ic}T_{jd}T_{kl}$ and we have

$$\begin{aligned} S(f,g) &= -T_{jd}(T_{ic}T_{kl} - T_{ik}T_{cl} + T_{il}T_{ck}) + T_{id}(T_{jc}T_{kl} - T_{jk}T_{cl} + T_{jl}T_{ck}) \\ &- |T_{ik}T_{jd}T_{cl}| \end{aligned}$$

For $(i3)$, the leading monomial of $S(f,g)$ is either $T_{ic}T_{jd}T_{kl}$ (if $c < k$) or $T_{ik}T_{jl}T_{cd}$ (if $c = k$). In both cases we have

$$\begin{aligned} S(f,g) &= -T_{jd}(T_{ic}T_{kl} - T_{ik}T_{cl} + T_{il}T_{ck}) + T_{id}(T_{jc}T_{kl} - T_{jk}T_{cl} + T_{jl}T_{ck}) \\ &+ |T_{ik}T_{jl}T_{cd}| \end{aligned}$$

(2) $(a,b)=(k,l)$ In this case we have the only possibility

$$1 \le i < j < a = k < b = l < c < d \le n.$$

Also, $S(f,g) = T_{ij}f - T_{cd}g$ and its leading monomial is $T_{ij}T_{kc}T_{ld}$. One can check that

$$\begin{aligned} S(f,g) &= -T_{ld}(T_{ij}T_{kc} - T_{ik}T_{jc} + T_{ic}T_{jk}) + T_{lc}(T_{ij}T_{kd} - T_{ik}T_{jd} + T_{id}T_{jk}) \\ &+ T_{jd}(T_{ik}T_{lc} - T_{il}T_{kc} + T_{ic}T_{kl}) - T_{jc}(T_{ik}T_{ld} - T_{il}T_{kd} + T_{id}T_{kl}) \\ &+ T_{jl}(T_{ik}T_{cd} - T_{ic}T_{kd} + T_{id}T_{kc}) - T_{jk}(T_{il}T_{cd} - T_{ic}T_{ld} + T_{id}T_{lc}) \\ &- |T_{il}T_{jc}T_{kd}| \end{aligned}$$

(3) $(c,d)=(i,j)$ One can use case (2) above and relation $S(\rho,\theta)=-S(\theta,\rho)$ to show that $S(f,g)$ is in standard form relative to G in this case.
(4) $(c,d)=(k,l)$ In this case $S(f,g)=T_{ij}f-T_{ab}g$ and as in case (1), we can restrict ourselves to the following possibilities

$$\begin{aligned}&(i1)\, 1\le a<b\le i<j<c=k<d=l\le n\\&(i2)\, 1\le a\le i\le b\le j<c=k<d=l\le n\\&(i3)\, 1\le a\le i<j\le b<c=k<d=l\le n\end{aligned}$$

If $a=i$, then we may clearly assume that $b<j$, and so the leading monomial of $S(f,g)$ in this case is $T_{ab}T_{ik}T_{jl}$, and one can check that

$$S(f,g) = -T_{il}(T_{ib}T_{jk}-T_{ij}T_{bk}+T_{ik}T_{bj})+T_{ik}(T_{ib}T_{jl}-T_{ij}T_{bl}+T_{il}T_{bj})$$

which proves that $S(f,g)$ is in standard form relative to G in this case. Thus, for the three cases $(i1),(i2),(i3)$ above, we may assume that $a<i$ and obtain that the leading monomial of $S(f,g)$ is $T_{ab}T_{ik}T_{jl}$ in all of the three cases.
For $(i1)$, we have

$$\begin{aligned}S(f,g) &= T_{jl}(T_{ab}T_{ik}-T_{ai}T_{bk}+T_{ak}T_{bi})-T_{jk}(T_{ab}T_{il}-T_{ai}T_{bl}+T_{al}T_{bi})\\&- T_{bl}(T_{ai}T_{jk}-T_{aj}T_{ik}+T_{ak}T_{ij})+T_{bk}(T_{ai}T_{jl}-T_{aj}T_{il}+T_{al}T_{ij})\\&+ T_{al}(T_{bi}T_{jk}-T_{bj}T_{ik}+T_{bk}T_{ij})-T_{ak}(T_{bi}T_{jl}-T_{bj}T_{il}+T_{bl}T_{ij})\\&+ |T_{aj}T_{bk}T_{il}|\,.\end{aligned}$$

For $(i2)$, we have

$$\begin{aligned}S(f,g) &= -T_{il}(T_{ab}T_{jk}-T_{aj}T_{bk}+T_{ak}T_{bj})+T_{ik}(T_{ab}T_{jl}-T_{aj}T_{bl}+T_{al}T_{bj})\\&+ |T_{aj}T_{ik}T_{bl}|\,.\end{aligned}$$

And finally for $(i3)$, we have

$$\begin{aligned}S(f,g) &= T_{al}(T_{ij}T_{bk}-T_{ib}T_{jk}+T_{ik}T_{jb})-T_{ak}(T_{ij}T_{bl}-T_{ib}T_{jl}+T_{il}T_{jb})\\&+ |T_{ab}T_{ik}T_{jl}|\,.\end{aligned}$$

This shows that $S(f,g)$ is in standard form relative to G whenever $f,g\in\mathcal{F}_4$.

Case $(\mathcal{F}_4, \mathcal{F}_5)$

Given an element f of $\mathcal{F}_4$ and an element g of $\mathcal{F}_5$, we can (for the purpose of proving that $S(f,g)$ is in standard form relative to G) assume that they have the following forms

$$f = T_{1j}T_{kl} - T_{1k}T_{jl} + T_{1l}T_{jk}, \qquad g = T_iT_{1j} - T_jT_{1i}$$
$$2 \le j < k < l \le n, \qquad 2 \le i < j \le n.$$

In this case $2 \le i < j < k < l \le n$, and so

$$\begin{aligned} S(f,g) &= \underline{T_iT_{1k}T_{jl}} - T_iT_{1l}T_{jk} - T_jT_{1i}T_{kl} \\ &= T_{jl}(T_iT_{1k} - T_kT_{1i}) - T_{jk}(T_iT_{1l} - T_lT_{1i}) \\ &- T_{1i}(T_jT_{kl} - T_kT_{jl} + T_lT_{jk}) \end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G in this case.

Case $(\mathcal{F}_4, \mathcal{F}_7)$

Let

$$f = \underline{T_{ab}T_{cd}} - T_{ac}T_{bd} + T_{ad}T_{bc} \in \mathcal{F}_4, \qquad g = |T_{rj}T_{sp}T_{kq}| \in \mathcal{F}_7$$
$$1 \le a < b < c < d \le n, \qquad 1 \le r < s < k \le j < p < q \le n$$

As usual our goal is to show that $S(f,g)$ is standard form relative to G. By Buchberger's first criterion, one has to look at the following possibilities:

$$(1)\,(a,b) = (r,j) \quad (2)\,(a,b) = (s,p) \quad (3)\,(a,b) = (k,q)$$
$$(4)\,(c,d) = (r,j) \quad (5)\,(c,d) = (s,p) \quad (6)\,(c,d) = (k,q)$$

(1) $\underline{(a,b) = (r,j)}$ In this case $S(f,g) = T_{sp}T_{kq}f - T_{cd}g$ is of degree four since $(c,d) \neq (s,p)$, $(c,d) \neq (k,q)$ and the leading monomial of $S(f,g)$ is $T_{rj}T_{sq}T_{kp}T_{cd}$.

According to the position of c, d relative to p, q, we have the following six possibilities

$$(i1)\ 1 \le r = a < s < k \le j = b < c < d \le p < q \le n$$
$$(i2)\ 1 \le r = a < s < k \le j = b < c \le p \le d \le q \le n$$
$$(i3)\ 1 \le r = a < s < k \le j = b < c \le p < q \le d \le n$$
$$(i4)\ 1 \le r = a < s < k \le j = b < p < q \le c < d \le n$$
$$(i5)\ 1 \le r = a < s < k \le j = b < p \le c \le q \le d \le n$$
$$(i6)\ 1 \le r = a < s < k \le j = b < p \le c < d \le q \le n.$$

For $(i1)$, one can check that

$$\begin{aligned} S(f,g) &= T_{sq}T_{kp}(T_{rj}T_{cd} - T_{rc}T_{jd} + T_{rd}T_{jc}) + (T_{rp}T_{kq} - T_{rq}T_{kp})(T_{sj}T_{cd} - T_{sc}T_{jd} + T_{sd}T_{jc}) \\ &- (T_{rp}T_{sq} - T_{rq}T_{sp})(T_{kj}T_{cd} - T_{kc}T_{jd} + T_{kd}T_{jc}) - T_{jd}\,|T_{rc}T_{sp}T_{kq}| + T_{jc}\,|T_{rd}T_{sp}T_{kq}| \end{aligned} \tag{75}$$

This proves that $S(f,g)$ is in standard form relative to G in case $(i1)$.

Remark 3.3.10. Equation (75) above will remain true in the cases $(i2), \ldots, (i6)$ provided that we interchange columns properly in the two determinants appearing in the RHS of the equation and multiplying by a suitable sign while keeping the other terms in the RHS of (75) unchanged. On the other hand, notice that the leading monomials of the last two determinats appearing in (75) have the form $T_{r\sigma}T_{s\rho}T_{k\theta}T_{j\wp}$ for some $\sigma \in \{c, d, p, q\}$ and so interchanging columns in the determinant will not change the fact the the corresponding term is less than or equal to $S(f,g)$. This shows that $S(f,g)$ is in standard form relative to G in each of the cases $(i1), \ldots, (i6)$.

(2) $\underline{(a,b) = (s,p)}$ In this case $S(f,g) = T_{rj}T_{kq}f - T_{cd}g$ is also of degree four and its leading monomial is either $T_{rj}T_{sq}T_{kp}T_{cd}$ (if $c \ge q$) or $T_{rj}T_{sc}T_{kq}T_{pd}$ (if $c < q$). We have the following possibilities

$$(i1)\ 1 \le r < s = a < k \le j < p = b < q \le c < d \le n$$
$$(i2)\ 1 \le r < s = a < k \le j < p = b < c < q \le d \le n$$
$$(i3)\ 1 \le r < s = a < k \le j < p = b < c < d < q \le n$$

For $(i1)$, one can check that $S(f,g)$ is equal to

$$\begin{aligned}
& T_{sd}T_{kq}(T_{rj}T_{pc} - T_{rp}T_{jc} + T_{rc}T_{jp}) - T_{sc}T_{kq}(T_{rj}T_{pd} - T_{rp}T_{jd} + T_{rd}T_{jp}) \\
+ \; & T_{sq}T_{kp}(T_{rj}T_{cd} - T_{rc}T_{jd} + T_{rd}T_{jc}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(T_{rp}T_{cd} - T_{rc}T_{pd} + T_{rd}T_{pc}) \\
- \; & (T_{sj}T_{kp} - T_{sp}T_{kj})(T_{rq}T_{cd} - T_{rc}T_{qd} + T_{rd}T_{qc}) - T_{rd}T_{kq}(T_{sj}T_{pc} - T_{sp}T_{jc} + T_{sc}T_{jp}) \\
+ \; & T_{rc}T_{kq}(T_{sj}T_{pd} - T_{sp}T_{jd} + T_{sd}T_{jp}) + T_{rd}T_{kp}(T_{sj}T_{qc} - T_{sq}T_{jc} + T_{sc}T_{jq}) \\
- \; & T_{rc}T_{kp}(T_{sj}T_{qd} - T_{sq}T_{jd} + T_{sd}T_{jq}) - T_{rd}T_{kj}(T_{sp}T_{qc} - T_{sq}T_{pc} + T_{sc}T_{pq}) \\
+ \; & T_{rc}T_{kj}(T_{sp}T_{qd} - T_{sq}T_{pd} + T_{sd}T_{pq}) - (T_{rc}T_{sd} - T_{rd}T_{sc})(T_{kj}T_{pq} - T_{kp}T_{jq} + T_{kq}T_{jp}) \\
- \; & T_{kq}\left|T_{rp}T_{sc}T_{jd}\right|
\end{aligned}$$

For $(i2)$, $S(f,g)$ is equal to

$$\begin{aligned}
& T_{sd}T_{kq}(T_{rj}T_{pc} - T_{rp}T_{jc} + T_{rc}T_{jp}) - T_{sc}T_{kq}(T_{rj}T_{pd} - T_{rp}T_{jd} + T_{rd}T_{jp}) \\
+ \; & T_{sq}T_{kp}(T_{rj}T_{cd} - T_{rc}T_{jd} + T_{rd}T_{jc}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(T_{rp}T_{cd} - T_{rc}T_{pd} + T_{rd}T_{pc}) \\
- \; & T_{rd}T_{kq}(T_{sj}T_{pc} - T_{sp}T_{jc} + T_{sc}T_{jp}) + T_{rc}T_{kq}(T_{sj}T_{pd} - T_{sp}T_{jd} + T_{sd}T_{jp}) \\
- \; & T_{rq}T_{kp}(T_{sj}T_{cd} - T_{sc}T_{jd} + T_{sd}T_{jc}) - (T_{rc}T_{sq} - T_{rq}T_{sc})(T_{kj}T_{pd} - T_{kp}T_{jd} + T_{kd}T_{jp}) \\
- \; & (T_{rq}T_{sd} - T_{rd}T_{sq})(T_{kj}T_{pc} - T_{kp}T_{jc} + T_{kc}T_{jp}) + T_{rq}T_{kj}(T_{sp}T_{cd} - T_{sc}T_{pd} + T_{sd}T_{pc}) \\
- \; & T_{jd}\left|T_{rp}T_{sc}T_{kq}\right| + T_{jp}\left|T_{rc}T_{sq}T_{kd}\right| - T_{kc}\left|T_{rp}T_{sq}T_{jd}\right| + T_{kp}\left|T_{rc}T_{sq}T_{jd}\right| \\
- \; & T_{sd}\left|T_{rp}T_{kc}T_{jq}\right| + T_{rd}\left|T_{sp}T_{kc}T_{jq}\right|
\end{aligned}$$

For $(i3)$, we have the following expression of $S(f,g)$

$$\begin{aligned}
& T_{sd}T_{kq}(T_{rj}T_{pc} - T_{rp}T_{jc} + T_{rc}T_{jp}) - T_{sc}T_{kq}(T_{rj}T_{pd} - T_{rp}T_{jd} + T_{rd}T_{jp}) \\
+ \; & T_{sq}T_{kp}(T_{rj}T_{cd} - T_{rc}T_{jd} + T_{rd}T_{jc}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(T_{rp}T_{cd} - T_{rc}T_{pd} + T_{rd}T_{pc}) \\
- \; & T_{rd}T_{kq}(T_{sj}T_{pc} - T_{sp}T_{jc} + T_{sc}T_{jp}) + T_{rc}T_{kq}(T_{sj}T_{pd} - T_{sp}T_{jd} + T_{sd}T_{jp}) \\
- \; & T_{rq}T_{kp}(T_{sj}T_{cd} - T_{sc}T_{jd} + T_{sd}T_{jc}) - (T_{rc}T_{sq} - T_{rq}T_{sc})(T_{kj}T_{pd} - T_{kp}T_{jd} + T_{kd}T_{jp}) \\
+ \; & (T_{rd}T_{sq} - T_{rq}T_{sd})(T_{kj}T_{pc} - T_{kp}T_{jc} + T_{kc}T_{jp}) + T_{rq}T_{kj}(T_{sp}T_{cd} - T_{sc}T_{pd} + T_{sd}T_{pc}) \\
- \; & T_{jd}\left|T_{rp}T_{sc}T_{kq}\right| + T_{jc}\left|T_{rp}T_{sd}T_{kq}\right| - 2T_{jp}\left|T_{rc}T_{sd}T_{kq}\right| - T_{sq}\left|T_{rp}T_{kc}T_{jd}\right| \\
+ \; & T_{rq}\left|T_{sp}T_{kc}T_{jd}\right|
\end{aligned}$$

This proves that $S(f,g)$ is in standard form relative to G in case (2).
(3) $\underline{(a,b)=(k,q)}$ In this case

$$1 \leq r < s < k = a \leq j < p < q = b < c < d \leq n,$$

$S(f,g) = T_{rj}T_{sp}f - T_{cd}g$ and its leading monomial is $T_{rj}T_{sp}T_{kc}T_{qd}$. One can check that $S(f,g)$ is equal to

$$\begin{aligned}
& T_{sp}T_{kd}(T_{rj}T_{qc} - T_{rq}T_{jc} + T_{rc}T_{jq}) - T_{sp}T_{kc}(T_{rj}T_{qd} - T_{rq}T_{jd} + T_{rd}T_{jq}) \\
+ \ & T_{sq}T_{kp}(T_{rj}T_{cd} - T_{rc}T_{jd} + T_{rd}T_{jc}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(T_{rp}T_{cd} - T_{rc}T_{pd} + T_{rd}T_{pc}) \\
- \ & (T_{sj}T_{kp} - T_{sp}T_{kj})(T_{rq}T_{cd} - T_{rc}T_{qd} + T_{rd}T_{qc}) - T_{rd}T_{kq}(T_{sj}T_{pc} - T_{sp}T_{jc} + T_{sc}T_{jp}) \\
+ \ & T_{rc}T_{kq}(T_{sj}T_{pd} - T_{sp}T_{jd} + T_{sd}T_{jp}) + T_{rd}T_{kp}(T_{sj}T_{qc} - T_{sq}T_{jc} + T_{sc}T_{jq}) \\
- \ & T_{rc}T_{kp}(T_{sj}T_{qd} - T_{sq}T_{jd} + T_{sd}T_{jq}) - T_{rd}T_{kj}(T_{sp}T_{qc} - T_{sq}T_{pc} + T_{sc}T_{pq}) \\
+ \ & T_{rc}T_{kj}(T_{sp}T_{qd} - T_{sq}T_{pd} + T_{sd}T_{pq}) - (T_{rc}T_{sd} - T_{rd}T_{sc})(T_{kj}T_{pq} - T_{kp}T_{jq} + T_{kq}T_{jp}) \\
- \ & T_{sp}\left|T_{rq}T_{kc}T_{jd}\right|
\end{aligned}$$

(4) $\underline{(c,d)=(r,j)}$ In this case we also have a unique possibility

$$1 \leq a < b < r = c < s < k \leq j = d < p < q \leq n,$$

in which case $S(f,g) = T_{sp}T_{kq}f - T_{ab}g$ and its leading monomial is $T_{ab}T_{rj}T_{sq}T_{kp}$. Here one can check that $S(f,g)$ is equal to the following expression

$$\begin{aligned}
& T_{sq}T_{kp}(T_{ab}T_{rj} - T_{ar}T_{bj} + T_{aj}T_{br}) + (T_{sj}T_{kq} - T_{sq}T_{kj})(T_{ab}T_{rp} - T_{ar}T_{bp} + T_{ap}T_{br}) \\
- \ & (T_{sj}T_{kp} - T_{sp}T_{kj})(T_{ab}T_{rq} - T_{ar}T_{bq} + T_{aq}T_{br}) + (T_{bp}T_{kq} - T_{bq}T_{kp})(T_{ar}T_{sj} - T_{as}T_{rj} + T_{aj}T_{rs}) \\
- \ & (T_{bj}T_{kq} - T_{bq}T_{kj})(T_{ar}T_{sp} - T_{as}T_{rp} + T_{ap}T_{rs}) + (T_{bj}T_{kp} - T_{bp}T_{kj})(T_{ar}T_{sq} - T_{as}T_{rq} + T_{aq}T_{rs}) \\
- \ & (T_{bp}T_{rq} - T_{bq}T_{rp})(T_{as}T_{kj} - T_{ak}T_{sj} + T_{aj}T_{sk}) + (T_{bj}T_{rq} - T_{bq}T_{rj})(T_{as}T_{kp} - T_{ak}T_{sp} + T_{ap}T_{sk}) \\
- \ & (T_{bj}T_{rp} - T_{bp}T_{rj})(T_{as}T_{kq} - T_{ak}T_{sq} + T_{aq}T_{sk}) - (T_{ap}T_{kq} - T_{aq}T_{kp})(T_{br}T_{sj} - T_{bs}T_{rj} + T_{bj}T_{rs}) \\
+ \ & (T_{aj}T_{kq} - T_{aq}T_{kj})(T_{br}T_{sp} - T_{bs}T_{rp} + T_{bp}T_{rs}) - (T_{aj}T_{kp} - T_{ap}T_{kj})(T_{br}T_{sq} - T_{bs}T_{rq} + T_{bq}T_{rs}) \\
+ \ & (T_{ap}T_{rq} - T_{aq}T_{rp})(T_{bs}T_{kj} - T_{bk}T_{sj} + T_{bj}T_{sk}) - (T_{aj}T_{rq} - T_{aq}T_{rj})(T_{bs}T_{kp} - T_{bk}T_{sp} + T_{bp}T_{sk}) \\
+ \ & (T_{aj}T_{rp} - T_{ap}T_{rj})(T_{bs}T_{kq} - T_{bk}T_{sq} + T_{bq}T_{sk}) - 2(T_{ap}T_{bq} - T_{aq}T_{bp})(T_{rs}T_{kj} - T_{rk}T_{sj} + T_{rj}T_{sk}) \\
+ \ & 2(T_{aj}T_{bq} - T_{aq}T_{bj})(T_{rs}T_{kp} - T_{rk}T_{sp} + T_{rp}T_{sk}) - 2(T_{aj}T_{bp} - T_{ap}T_{bj})(T_{rs}T_{kq} - T_{rk}T_{sq} + T_{rq}T_{sk}) \\
- \ & T_{sq}\left|T_{ak}T_{bj}T_{rp}\right| + T_{sp}\left|T_{ak}T_{bj}T_{rq}\right| - T_{sj}\left|T_{ak}T_{bp}T_{rq}\right| + 4T_{sk}\left|T_{aj}T_{bp}T_{rq}\right| - T_{rk}\left|T_{aj}T_{bp}T_{sq}\right| \qquad (76)
\end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G.

(5) $\underline{(c,d)=(s,p)}$ In this case $S(f,g) = T_{rj}T_{kq}f - T_{ab}g$ is always of degree four and its leading monomial is $T_{rj}T_{ab}T_{sq}T_{kp}$. Three cases are to be considered

$$(i1)\, 1 \le r \le a < b < s = c < k \le j < p = d < q \le n$$
$$(i2)\, 1 \le a \le r \le b < s = c < k \le j < p = d < q \le n$$
$$(i3)\, 1 \le a < b \le r < s = c < k \le j < p = d < q \le n$$

For $(i1)$, one has the following expression of $S(f,g)$

$$\begin{aligned} & (T_{rp}T_{kq} - T_{rq}T_{kp})(T_{ab}T_{sj} - T_{as}T_{bj} + T_{aj}T_{bs}) + T_{rq}T_{kj}(T_{ab}T_{sp} - T_{as}T_{bp} + T_{ap}T_{bs}) \\ + & (T_{rj}T_{kp} - T_{rp}T_{kj})(T_{ab}T_{sq} - T_{as}T_{bq} + T_{aq}T_{bs}) - (T_{rp}T_{bq} - T_{rq}T_{bp})(T_{as}T_{kj} - T_{ak}T_{sj} + T_{aj}T_{sk}) \\ + & (T_{rj}T_{bq} - T_{rq}T_{bj})(T_{as}T_{kp} - T_{ak}T_{sp} + T_{ap}T_{sk}) - (T_{rj}T_{bp} - T_{rp}T_{bj})(T_{as}T_{kq} - T_{ak}T_{sq} + T_{aq}T_{sk}) \\ + & (T_{rp}T_{aq} - T_{rq}T_{ap})(T_{bs}T_{kj} - T_{bk}T_{sj} + T_{bj}T_{sk}) - (T_{rj}T_{aq} - T_{rq}T_{aj})(T_{bs}T_{kp} - T_{bk}T_{sp} + T_{bp}T_{sk}) \\ + & (T_{rj}T_{ap} - T_{rp}T_{aj})(T_{bs}T_{kq} - T_{bk}T_{sq} + T_{bq}T_{sk}) - 2T_{sk}\,|T_{rj}T_{ap}T_{bq}| + T_{bk}\,|T_{rj}T_{ap}T_{sq}| \\ - & T_{ak}\,|T_{rj}T_{bp}T_{sq}| \end{aligned} \tag{77}$$

which proves that $S(f,g)$ is in standard form relative to G. The cases $(i2)$ and $(i3)$ follow from (77) and the following remark.

Remark 3.3.11. The equality in (77) remains true in cases $(i2)$ and $(i3)$ provided that we interchange suitable rows (if necessary) in the three determinants appearing in the RHS. A quick check shows that this would also preserves the fact that the leading monomial of each of theses determinants remains less than $T_{rj}T_{ab}T_{sq}T_{kp}$. This shows in particular that $S(f,g)$ is also in standard form relative to G in cases $(i2)$ and $(i3)$.

(6) $\underline{(c,d)=(k,q)}$ In this case $S(f,g) = T_{rj}T_{sp}f - T_{ab}g$ is of degree four and its leading monomial is either $T_{rj}T_{sp}T_{ak}T_{bq}$ (if $s < a$) or $T_{rj}T_{ab}T_{sq}T_{kp}$ (if $a \le s$). We treat first the case $s = a$ where one has the following possibility:

$$1 \le r < s = a < b < k = c \le j < p < q = d \le n$$

and one can check that in this case

$$\begin{aligned} S(f,g) &= -(T_{rp}T_{sq} - T_{rq}T_{sp})(T_{sb}T_{kj} - T_{sk}T_{bj} + T_{sj}T_{bk}) + T_{rp}T_{sj}(T_{sb}T_{kq} - T_{sk}T_{bq} + T_{sq}T_{bk}) \\ &+ (T_{rj}T_{sq} - T_{rq}T_{sj})(T_{sb}T_{kp} - T_{sk}T_{bp} + T_{sp}T_{bk}) - T_{sk}\,|T_{rj}T_{sp}T_{bq}|. \end{aligned} \tag{78}$$

This shows that $S(f,g)$ is in standard form relative to G in this case. Assume next that $s \neq a$ and consider the following six possibilities

$$(i1)\, 1 \leq r < s < a < b < k = c \leq j < p < q = d \leq n$$
$$(i2)\, 1 \leq r \leq a < s \leq b < k = c \leq j < p < q = d \leq n$$
$$(i3)\, 1 \leq r \leq a < b \leq s < k = c \leq j < p < q = d \leq n$$
$$(i4)\, 1 \leq a \leq r < s \leq b < k = c \leq j < p < q = d \leq n$$
$$(i5)\, 1 \leq a \leq r \leq b \leq s < k = c \leq j < p < q = d \leq n$$
$$(i6)\, 1 \leq a < b \leq r < s < k = c \leq j < p < q = d \leq n$$

For $(i1)$, one can ckeck that

$$\begin{aligned} S(f,g) &= -(T_{rp}T_{sq} - T_{rq}T_{sp})(T_{ab}T_{kj} - T_{ak}T_{bj} + T_{aj}T_{bk}) + T_{rp}T_{sj}(T_{ab}T_{kq} - T_{ak}T_{bq} + T_{aq}T_{bk}) \\ &+ (T_{rj}T_{sq} - T_{rq}T_{sj})(T_{ab}T_{kp} - T_{ak}T_{bp} + T_{ap}T_{bk}) + T_{bk}\,|T_{rj}T_{sp}T_{aq}| \\ &- T_{ak}\,|T_{rj}T_{sp}T_{bq}| \end{aligned} \tag{79}$$

Which proves that $S(f,g)$ is in standard form relative to G in this case. Now using a remark similar to the one used in previous cases, one can deduce from relation (79) that $S(f,g)$ is also in standard form relative to G in all cases $(i2),\ldots,(i6)$. This finishes the case $(\mathcal{F}_4,\mathcal{F}_7)$.

Case $(\mathcal{F}_5,\mathcal{F}_5)$

Let

$$f = T_iT_{1j} - T_jT_{1i} \in \mathcal{F}_5, \qquad g = T_rT_{1s} - T_sT_{1r} \in \mathcal{F}_5$$
$$2 \leq i < j \leq n, \qquad 2 \leq r < s \leq n$$

be two distinct elements of $\mathcal{F}_5$. By Buchberger's first criterion, one has to consider the following two possibilities

$$(1)\, i = r \quad (2)\, j = s.$$

In the first case, we may assume that $2 \leq i = r < j < s \leq n$, and so

$$S(f,g) = \underline{-T_jT_{1i}T_{1s}} + T_sT_{1i}T_{1j} = -T_{1i}(T_jT_{1s} - T_sT_{1j})$$

is in standard form relative to G.

In the second case, we may assume that $2 \leq i < r < j = s \leq n$, and so

$$S(f,g) = \underline{-T_rT_jT_{1i}} + T_iT_jT_{1r} = T_j(T_iT_{1r} - T_rT_{1i})$$

is also in standard form relative to G. This finishes the case $(\mathcal{F}_5, \mathcal{F}_5)$.

Case $(\mathcal{F}_5, \mathcal{F}_6)$

Let

$$f = T_iT_{jk} - T_jT_{ik} + T_kT_{ij} \in \mathcal{F}_6, \qquad g = T_rT_{1s} - T_sT_{1r} \in \mathcal{F}_5$$
$$1 \leq i < j < k \leq n, \qquad 2 \leq r < s \leq n.$$

The leading monomials of f and g are not relatively prime only if $i = r$, which leads to the following three possibilities:

$$(i1)\, 2 \leq i = r < s \leq j < k \leq n$$
$$(i2)\, 2 \leq i = r < j < s \leq k \leq n$$
$$(i3)\, 2 \leq i = r < j < k < s \leq n$$

In all the above cases, we have

$$S(f,g) = -T_jT_{1s}T_{ik} + T_kT_{1s}T_{ij} + T_sT_{1i}T_{jk}.$$

For $(i1)$, the leading monomial of $S(f,g)$ is $T_sT_{1i}T_{jk}$, and one can verify that

$$\begin{aligned} S(f,g) &= T_s(T_{1i}T_{jk} - T_{1j}T_{ik} + T_{1k}T_{ij}) - T_{ij}(T_sT_{1k} - T_kT_{1s}) \\ &+ T_{ik}(T_sT_{1j} - T_jT_{1s}) \end{aligned}$$

For $(i2)$, the leading monomial of $S(f,g)$ is $T_jT_{1s}T_{ik}$, and one can check that

$$\begin{aligned} S(f,g) &= -T_{ik}(T_jT_{1s} - T_sT_{1j}) - T_{ij}(T_sT_{1k} - T_kT_{1s}) \\ &+ T_s(T_{1i}T_{jk} - T_{1j}T_{ik} + T_{1k}T_{ij}). \end{aligned}$$

For $(i3)$, the leading monomial of $S(f,g)$ is also $T_jT_{1s}T_{ik}$, and here we have:

$$\begin{aligned} S(f,g) &= -T_{ik}(T_jT_{1s} - T_sT_{1j}) + T_{ij}(T_kT_{1s} - T_sT_{1k}) \\ &+ T_s(T_{1i}T_{jk} - T_{1j}T_{ik} + T_{1k}T_{ij}). \end{aligned}$$

This finishes the case $(\mathcal{F}_5, \mathcal{F}_6)$.

Case $(\mathcal{F}_5, \mathcal{F}_7)$

Let

$$f = |T_{ad}T_{be}T_{cf}| \in \mathcal{F}_7, \qquad g = T_iT_{1j} - T_jT_{1i} \in \mathcal{F}_5$$
$$1 \leq a < b < c \leq d < e < f \leq n, \qquad 2 \leq i < j \leq n.$$

To prove that $S(f,g)$ is in standard form relative to G, it is enough (Buchberger's first criterion) to suppose that $a = 1$ and $j = d$, which leaves us with the three possibilities

$$(i1)\, 2 \leq i \leq b < c \leq j = d < e < f \leq n$$
$$(i2)\, 2 \leq b \leq i < c \leq j = d < e < f \leq n$$
$$(i3)\, 2 \leq b < c < i < j = d < e < f \leq n.$$

For all the above three cases, the leading monomial of $S(f,g)$ is $T_iT_{1j}T_{bf}T_{ce}$, and one has

$$\begin{aligned} S(f,g) &= -T_{bf}T_{ce}(T_iT_{1j} - T_jT_{1i}) + (T_{bf}T_{cj} - T_{bj}T_{cf})(T_iT_{1e} - T_eT_{1i}) \\ &+ (T_{bj}T_{ce} - T_{be}T_{cj})(T_iT_{1f} - T_fT_{1i}) + T_{1i}\,|T_jT_{be}T_{cf}| \end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G in all the above three cases. This finishes the case $(\mathcal{F}_5, \mathcal{F}_7)$.

Case $(\mathcal{F}_5, \mathcal{F}_8)$

Let

$$f = |T_cT_{ad}T_{be}| \in \mathcal{F}_8, \qquad g = T_iT_{1j} - T_jT_{1i} \in \mathcal{F}_5$$
$$2 \leq a < b \leq c < d < e \leq n, \qquad 2 \leq i < j \leq n.$$

The leading monomials of f and g are relatively prime except in the case where $c = i$. In this case, we have the following possibilities

$$(i1)\, 2 \leq a < b \leq c = i < j \leq d < e \leq n$$
$$(i2)\, 2 \leq a < b \leq c = i < d < j \leq e \leq n$$
$$(i3)\, 2 \leq a < b \leq c = i < d < e < j \leq n.$$

In all cases, the leading monomial of $S(f,g)$ is $T_cT_{1j}T_{ae}T_{bd}$. For $(i1)$, one can show that

$$\begin{aligned} S(f,g) &= -T_{ae}T_{bd}(T_cT_{1j} - T_jT_{1c}) + (T_{aj}T_{bc} - T_{ac}T_{bj})(T_dT_{1e} - T_eT_{1d}) \\ &+ T_{1c}\,|T_jT_{ad}T_{be}| - T_e\,|T_{1c}T_{aj}T_{bd}| + T_d\,|T_{1c}T_{aj}T_{be}|\,. \end{aligned}$$

For $(i2)$,

$$\begin{aligned} S(f,g) &= -T_{ae}T_{bd}(T_cT_{1j} - T_jT_{1c}) + (T_{ae}T_{bc} - T_{ac}T_{be})(T_dT_{1j} - T_jT_{1d}) \\ &+ (T_{ad}T_{bc} - T_{ac}T_{bd})(T_jT_{1e} - T_eT_{1j}) + T_j\,|T_{1c}T_{ad}T_{be}|\,. \end{aligned}$$

For $(i3)$,

$$\begin{aligned} S(f,g) &= -T_{ae}T_{bd}(T_cT_{1j} - T_jT_{1c}) + (T_{ae}T_{bc} - T_{ac}T_{be})(T_dT_{1j} - T_jT_{1d}) \\ &+ (T_{ac}T_{bd} - T_{ad}T_{bc})(T_eT_{1j} - T_jT_{1e}) + T_j\,|T_{1c}T_{ad}T_{be}|\,. \end{aligned}$$

This finishes the case $(\mathcal{F}_5, \mathcal{F}_8)$.

Case $(\mathcal{F}_5, \mathcal{F}_9)$

Let

$$\begin{aligned} &f = |Y_cT_{ad}T_{be}| \in \mathcal{F}_9, && g = T_iT_{1j} - T_jT_{1i} \in \mathcal{F}_5 \\ &1 \le a < b \le c < d < e \le n, && 2 \le i < j \le n. \end{aligned}$$

The leading monomials of f and g are relatively prime except in the case where $a = 1$ and $j = d$. In this case, we have the following possibilities

$(i1)\, 2 \le b \le c \le i < j = d < e \le n$

$(i2)\, 2 \le b \le i < c < j = d < e \le n$

$(i3)\, 2 \le i < b \le c < j = d < e \le n.$

The leading monomial of $S(f,g)$ is $Y_cT_iT_{1e}T_{bd}$. For $(i1)$, one has:

$$\begin{aligned} S(f,g) &= (Y_dT_{be} - Y_eT_{bd})(T_cT_{1i} - T_iT_{1c}) - Y_eT_{bc}(T_iT_{1d} - T_dT_{1i}) \\ &+ (Y_dT_{bc} - Y_cT_{bd})(T_iT_{1e} - T_eT_{1i}) + T_{1i}T_{be}(T_{cd} + Y_cT_d - Y_dT_c) \\ &- T_{1i}T_{bd}(T_{ce} + Y_cT_e - Y_eT_c) + T_{1i}T_{bc}(T_{de} + Y_dT_e - Y_eT_d) \\ &- T_{bc}(T_{1i}T_{de} - T_{1d}T_{ie} + T_{1e}T_{id}) + T_{1e}(T_{bc}T_{id} - T_{bi}T_{cd} + T_{bd}T_{ci}) \\ &- T_{1d}(T_{bc}T_{ie} - T_{bi}T_{ce} + T_{be}T_{ci}) + |T_{1i}T_{bd}T_{ce}|\,. \end{aligned}$$

For $(i2)$, one can check that

$$\begin{aligned}
S(f,g) &= (Y_eT_{bd} - Y_dT_{be})(T_iT_{1c} - T_cT_{1i}) - Y_eT_{bc}(T_iT_{1d} - T_dT_{1i}) \\
&+ (Y_dT_{bc} - Y_cT_{bd})(T_iT_{1e} - T_eT_{1i}) + T_{1i}T_{be}(T_{cd} + Y_cT_d - Y_dT_c) \\
&- T_{1i}T_{bd}(T_{ce} + Y_cT_e - Y_eT_c) + T_{1i}T_{bc}(T_{de} + Y_dT_e - Y_eT_d) \\
&+ T_{de}(T_{1b}T_{ic} - T_{1i}T_{bc} + T_{1c}T_{bi}) - T_{ce}(T_{1b}T_{id} - T_{1i}T_{bd} + T_{1d}T_{bi}) \\
&+ T_{cd}(T_{1b}T_{ie} - T_{1i}T_{be} + T_{1e}T_{bi}) - T_{ie}(T_{1b}T_{cd} - T_{1c}T_{bd} + T_{1d}T_{bc}) \\
&+ T_{id}(T_{1b}T_{ce} - T_{1c}T_{be} + T_{1e}T_{bc}) - T_{ic}(T_{1b}T_{de} - T_{1d}T_{be} + T_{1e}T_{bd}) \\
&- T_{bi}(T_{1c}T_{de} - T_{1d}T_{ce} + T_{1e}T_{cd}) - |T_{1c}T_{bd}T_{ie}| \,.
\end{aligned}$$

For $(i3)$,

$$\begin{aligned}
S(f,g) &= (Y_eT_{bd} - Y_dT_{be})(T_iT_{1c} - T_cT_{1i}) - Y_eT_{bc}(T_iT_{1d} - T_dT_{1i}) \\
&+ (Y_dT_{bc} - Y_cT_{bd})(T_iT_{1e} - T_eT_{1i}) + T_{1i}T_{be}(T_{cd} + Y_cT_d - Y_dT_c) \\
&- T_{1i}T_{bd}(T_{ce} + Y_cT_e - Y_eT_c) + T_{1i}T_{bc}(T_{de} + Y_dT_e - Y_eT_d) \\
&- T_{de}(T_{1i}T_{bc} - T_{1b}T_{ic} + T_{1c}T_{ib}) + T_{ce}(T_{1i}T_{bd} - T_{1b}T_{id} + T_{1d}T_{ib}) \\
&- T_{cd}(T_{1i}T_{be} - T_{1b}T_{ie} + T_{1e}T_{ib}) - T_{ie}(T_{1b}T_{cd} - T_{1c}T_{bd} + T_{1d}T_{bc}) \\
&+ T_{id}(T_{1b}T_{ce} - T_{1c}T_{be} + T_{1e}T_{bc}) - T_{ic}(T_{1b}T_{de} - T_{1d}T_{be} + T_{1e}T_{bd}) \\
&+ T_{ib}(T_{1c}T_{de} - T_{1d}T_{ce} + T_{1e}T_{cd}) + |T_{1c}T_{id}T_{be}| \,.
\end{aligned}$$

This finishes the case $(\mathcal{F}_5, \mathcal{F}_9)$.

Case $(\mathcal{F}_6, \mathcal{F}_9)$

Let

$$f = T_aT_{bc} - T_bT_{ac} + T_cT_{ab} \in \mathcal{F}_6, \qquad g = |Y_rT_{is}T_{ju}| \in \mathcal{F}_9$$
$$1 \le a < b < c \le n, \qquad 1 \le i < j \le r < s < u \le n.$$

The leading monomials of f and g are relatively prime except in the following two cases:

$$(1)\ (b,c) = (i,s) \quad (2)\ (b,c) = (j,u)$$

(1) $(b,c) = (i,s)$ In this case, the leading monomial of $S(f,g) = Y_rT_{ju}f - T_ag$ is $Y_rT_aT_{iu}T_{js}$, and one has the unique possibility

$$1 \leq a < i = b < j \leq r < s = c < u \leq n.$$

In this case, one can verify that $S(f,g)$ is equal to the following expression

$$\begin{aligned}
& (T_{ai}T_{ju} - T_{au}T_{ij})(T_{rs} + Y_rT_s - Y_sT_r) - (T_{ai}T_{js} - T_{as}T_{ij})(T_{ru} + Y_rT_u - Y_uT_r) \\
+ & (T_{ai}T_{jr} - T_{ar}T_{ij})(T_{su} + Y_sT_u - Y_uT_s) - T_{su}(T_{ai}T_{jr} - T_{aj}T_{ir} + T_{ar}T_{ij}) \\
+ & T_{ru}(T_{ai}T_{js} - T_{aj}T_{is} + T_{as}T_{ij}) - T_{rs}(T_{ai}T_{ju} - T_{aj}T_{iu} + T_{au}T_{ij}) \\
- & T_{iu}(T_{aj}T_{rs} - T_{ar}T_{js} + T_{as}T_{jr}) + T_{is}(T_{aj}T_{ru} - T_{ar}T_{ju} + T_{au}T_{jr}) \\
- & T_{ir}(T_{aj}T_{su} - T_{as}T_{ju} + T_{au}T_{js}) + 2T_{ij}(T_{ar}T_{su} - T_{as}T_{ru} + T_{au}T_{rs}) \\
+ & (Y_sT_{ju} - Y_uT_{js})(T_aT_{ir} - T_iT_{ar} + T_rT_{ai}) + Y_uT_{jr}(T_aT_{is} - T_iT_{as} + T_sT_{ai}) \\
+ & (Y_rT_{js} - Y_sT_{jr})(T_aT_{iu} - T_iT_{au} + T_uT_{ai}) - (Y_sT_{au} - Y_uT_{as})(T_iT_{jr} - T_jT_{ir} + T_rT_{ij}) \\
+ & (Y_rT_{au} - Y_uT_{ar})(T_iT_{js} - T_jT_{is} + T_sT_{ij}) - (Y_rT_{as} - Y_sT_{ar})(T_iT_{ju} - T_jT_{iu} \\
+ & T_uT_{ij}) - T_j\,|Y_rT_{as}T_{iu}| + |T_{ar}T_{is}T_{ju}|\,.
\end{aligned}$$

This shows that $S(f,g)$ is in standard form relative to G in this case.

(2) $(b,c) = (j,u)$ In this case, the leading monomial of $S(f,g)$ is always $Y_rT_aT_{iu}T_{js}$ and one has the two possibilities

$$\begin{aligned}
&(i1)\, 1 \leq i < a < j = b \leq r < s < u = c \leq n \\
&(i2)\, 1 \leq a \leq i < j = b \leq r < s < u = c \leq n.
\end{aligned}$$

For $(i1)$, one can verify that $S(f,g)$ is equal to the following expression

$$\begin{aligned}
& -T_{iu}T_{aj}(T_{rs} + Y_rT_s - Y_sT_r) + T_{is}T_{aj}(T_{ru} + Y_rT_u - Y_uT_r) \\
- & T_{ir}T_{aj}(T_{su} + Y_sT_u - Y_uT_s) + T_{aj}(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) \\
- & (Y_sT_{iu} - Y_uT_{is})(T_aT_{jr} - T_jT_{ar} + T_rT_{aj}) + (Y_rT_{iu} - Y_uT_{ir})(T_aT_{js} - T_jT_{as} + T_sT_{aj}) \\
+ & Y_sT_{ir}(T_aT_{ju} - T_jT_{au} + T_uT_{aj}) - T_j\,|Y_rT_{is}T_{au}| \qquad (80)
\end{aligned}$$

which shows that $S(f,g)$ is in standard form relative to G in this case. Interchanging the last two columns in the determinant appearing in (80) and multiplying it by a negative sign allow us to confirm that $S(f,g)$ is also in standard form relative to G in case $(i2)$. This finishes case $(\mathcal{F}_6, \mathcal{F}_9)$.

Case $(\mathcal{F}_7, \mathcal{F}_7)$

Let

$$f = |T_{rj}T_{sp}T_{kq}|, \qquad g = |T_{ai}T_{bd}T_{ce}|$$
$$1 \le r < s < k \le j < p < q \le n, \qquad 1 \le a < b < c \le i < d < e \le n$$

be two distinct elements of $\mathcal{F}_7$. We need to show that $S(f,g)$ is in standard form relative to G. Using Buchberger's first criterion we can assume that the leading monomials of f and g are not relatively prime and this can only happen if one of the following cases occurs:

$$\begin{array}{lll} (1)\,(r,j)=(a,i) & (2)\,(r,j)=(b,d) & (3)\,(r,j)=(c,e) \\ (4)\,(s,p)=(a,i) & (5)\,(s,p)=(b,d) & (6)\,(s,p)=(c,e) \\ (7)\,(k,q)=(a,i) & (8)\,(k,q)=(b,d) & (9)\,(k,q)=(c,e) \end{array}$$

Using the relation $S(f,g) = -S(g,f)$, we can see that the cases (4), (7) and (8) above can be obtained from cases (2), (3) and(6) respectively. So we look at the following cases:

$$\begin{array}{lll} (1)\,(r,j)=(a,i) & (2)\,(r,j)=(b,d) & (3)\,(r,j)=(c,e) \\ (4)\,(s,p)=(b,d) & (5)\,(s,p)=(c,e) & (6)\,(k,q)=(c,e) \end{array}$$

In each of the above cases, the degree of $S(f,g)$ is either four or five depending on whether the degree of the greatest common divisor of LM(f) and LM(g) is one or two. We will treat these two possibilities seperately in each of the above six cases.
(1) $(r,j) = (a,i)$ In this case the degree of $S(f,g)$ is four when exactly one of the following possibilities occurs:

$$\begin{array}{ll} A.\,(b,d)=(s,p) & B.\,(b,d)=(k,q) \\ C.\,(c,e)=(s,p) & D.\,(c,e)=(k,q). \end{array}$$

In case A, $S(f,g) = T_{ce}f - T_{kq}g$ and we may use the relation $S(f,g) = -S(g,f)$ to assume that $k \le c$, so we are left with the two possibilities:

$$A1.\, 1 \le r = a < s = b < k \le c \le j = i < p = d < q \le e \le n$$
$$A2.\, 1 \le r = a < s = b < k \le c \le j = i < p = d < e < q \le n.$$

For $A1$, the leading monomial of $S(f,g)$ is $T_{rj}T_{sq}T_{kp}T_{ce}$, and one can check that

$$S(f,g) = -T_{kp}|T_{rj}T_{sq}T_{ce}| + T_{kj}|T_{rp}T_{sq}T_{ce}| - T_{se}|T_{rj}T_{kp}T_{cq}| + T_{re}|T_{sj}T_{kp}T_{cq}|. \tag{81}$$

Case $A2$ can be treated similarly.

In case B, we have the only possibility

$$1 \le r = a < s < k = b < c \le j = i < p < q = d < e \le n,$$

$S(f,g) = T_{ce}f - T_{sp}g$ and its leading monomial is equal to $T_{rj}T_{sp}T_{ke}T_{cq}$. Also we have

$$\begin{aligned} S(f,g) &= T_{cq}|T_{rj}T_{sp}T_{ke}| - T_{kp}|T_{rj}T_{sq}T_{ce}| + T_{kj}|T_{rp}T_{sq}T_{ce}| \\ &- T_{sj}|T_{rp}T_{kq}T_{ce}| + T_{re}|T_{sj}T_{kp}T_{cq}| - T_{rq}|T_{sj}T_{kp}T_{ce}|. \end{aligned}$$

In case C, we have also a unique possibility

$$1 \le r = a < b < s = c < k \le j = i < d < p = e < q \le n,$$

$S(f,g) = T_{bd}f - T_{kq}g$ with leading monomial equal to $T_{rj}T_{bd}T_{sq}T_{kp}$, and one can check that

$$\begin{aligned} S(f,g) &= -T_{kp}|T_{rj}T_{bd}T_{sq}| + T_{sd}|T_{rj}T_{bp}T_{kq}| - T_{sj}|T_{rd}T_{bp}T_{kq}| \\ &+ T_{bj}|T_{rd}T_{sp}T_{kq}| - T_{rq}|T_{bj}T_{sd}T_{kp}| + T_{rp}|T_{bj}T_{sd}T_{kq}|. \end{aligned}$$

For case D, $S(f,g) = T_{bd}f - T_{sp}g$ with leading monomial equal to $T_{rj}T_{sp}T_{be}T_{kd}$. Also, we may restrict to the following two possibilities:

$D1.\ 1 \le r = a < s \le b < k = c < j = i < p \le d < q = e \le n$

$D2.\ 1 \le r = a < s \le b < k = c < j = i < d < p < q = e \le n$

In case $D1$, we have

$$S(f,g) = T_{kd}|T_{rj}T_{sp}T_{bq}| + T_{sq}|T_{rj}T_{bp}T_{kd}| - T_{sj}|T_{rp}T_{bd}T_{kq}| + T_{rq}|T_{sj}T_{bp}T_{kd}|.$$

Case $D2$ is treated similarly.

This shows that $S(f,g)$ is in standard form relative to G in the case where it has degree four. So we may assume next that $S(f,g)$ is of degree five. We use the relation $S(f,g) = -S(g,f)$ to limit ourselves to the following cases:

$(i1)$ $1 \le a = r < s < k \le b < c \le j = i < p < q \le d < e \le n$

$(i2)$ $1 \le a = r < s < k \le b < c \le j = i < p \le d < q \le e \le n$

$(i3)$ $1 \le a = r < s < k \le b < c \le j = i < p < d < e < q \le n$

$(i4)$ $1 \le a = r < s < k \le b < c \le j = i < d < e \le p < q \le n$

$(i5)$ $1 \le a = r < s < k \le b < c \le j = i < d \le p < e \le q \le n$

$(i6)$ $1 \le a = r < s < k \le b < c \le j = i < d < p < q < e \le n$

$(i7)$ $1 \le a = r < s \le b < k \le c \le j = i < p < q \le d < e \le n$

$(i8)$ $1 \le a = r < s \le b < k \le c \le j = i < p \le d < q \le e \le n$

$(i9)$ $1 \le a = r < s \le b < k \le c \le j = i < p < d < e < q \le n$

$(i10)$ $1 \le a = r < s \le b < k \le c \le j = i < d < e \le p < q \le n$

$(i11)$ $1 \le a = r < s \le b < k \le c \le j = i < d \le p < e \le q \le n$

$(i12)$ $1 \le a = r < s \le b < k \le c \le j = i < d < p < q < e \le n$

$(i13)$ $1 \le a = r < s < b < c < k \le j = i < p < q \le d < e \le n$

$(i14)$ $1 \le a = r < s < b < c < k \le j = i < p \le d < q \le e \le n$

$(i15)$ $1 \le a = r < s < b < c < k \le j = i < p < d < e < q \le n$

$(i16)$ $1 \le a = r < s < b < c < k \le j = i < d < e \le p < q \le n$

$(i17)$ $1 \le a = r < s < b < c < k \le j = i < d \le p < e \le q \le n$

$(i18)$ $1 \le a = r < s < b < c < k \le j = i < d < p < q < e \le n.$

For $(i1)$, $S(f,g) = T_{bd}T_{ce}f - T_{sp}T_{kq}g$ and its leading monomial is $T_{rj}T_{sp}T_{kq}T_{be}T_{cd}$. One can check that

$$\begin{aligned} S(f,g) &= T_{be}T_{cd}\,|T_{rj}T_{sp}T_{kq}| + (T_{bj}T_{cp} - T_{bp}T_{cj})\,|T_{rq}T_{sd}T_{ke}| - T_{sq}T_{ce}\,|T_{rj}T_{kp}T_{bd}| \\ &+ T_{sq}T_{cd}\,|T_{rj}T_{kp}T_{be}| - T_{sq}T_{bp}\,|T_{rj}T_{kd}T_{ce}| + T_{sq}T_{bj}\,|T_{rp}T_{kd}T_{ce}| \\ &- T_{sj}T_{kq}\,|T_{rp}T_{bd}T_{ce}| + T_{rq}T_{ce}\,|T_{sj}T_{kp}T_{bd}| - T_{rq}T_{cd}\,|T_{sj}T_{kp}T_{be}| \\ &+ T_{rq}T_{bp}\,|T_{sj}T_{kd}T_{ce}| - T_{rq}T_{bj}\,|T_{sp}T_{kd}T_{ce}| + T_{re}T_{kq}\,|T_{sj}T_{bp}T_{cd}| \\ &- T_{rd}T_{kq}\,|T_{sj}T_{bp}T_{ce}|\,. \end{aligned} \tag{82}$$

This shows that $S(f,g)$ is in standard form relative to G in this case.

A closer look at expression (82) reveals the following observations:

1. $S(f,g) = \sum T_{ux} T_{vy} D$ where $u, v \in \{r, s, k, b, c\}$, $x, y \in \{j, p, q, d, e\}$ and $D \in \mathcal{F}_7$.

2. The leading term of each term in the RHS of (82) has the form $T_{r\theta} T_{s\kappa} T_{u\rho} T_{v\xi} T_{w\mu}$, where $\theta, \kappa, \rho, \xi, \mu \in \{j, p, q, d, e\}$ and $u, v, w \in \{k, b, c\}$ with $\theta \geq j$

3. Except for the first determinant, if the first entry of D is T_{rj} then the leading coefficient of the corresponding term has one variable of the form $T_{sq} (< T_{sp})$.

So, to prove that $S(f,g)$ is in standard form relative to G in each of the above cases $(i2), \ldots, (i18)$, we proceed as follows:

1. The equality in (82) will remain true provided that we interchange columns and rows while multiplying with a suitable sign in the determinants of the RHS.

2. if $\hbar$ is an arbitrary term in the RHS sum, then the above observations will guarantee that the relation $\hbar \leq \mathrm{LM}(S(f,g))$ will remain true.

This finishes case (1).

(2) $(r,j) = (b,d)$ As in the previous case, we treat the cases $\deg(S(f,g)) = 4$, and $\deg(S(f,g)) = 5$ separately. The first case can occur in either of the following two cases

$$A. (c,e) = (s,p) \qquad B. (c,e) = (k,q).$$

In case A, $S(f,g) = T_{ai} f - T_{kq} g$. Since $T_{ai} T_{kq}$ is always a divisor of the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$, Buchberger's second criterion allows us to assume that $i \geq k$ because otherwise $T_{ai} T_{kq}$ would be the leading monomial of an element of $\mathcal{F}_4$ and the case $(\mathcal{F}_4, \mathcal{F}_7)$ has been treated already. This leaves us with the following possibility

$$1 \leq a < r = b < s = c < k \leq i < j = d < p = e < q \leq n$$

in which case the leading monomial of $S(f,g)$ is $T_{ai}T_{rj}T_{sq}T_{kp}$, and we have

$$\begin{aligned} S(f,g) &= -T_{kp}\left|T_{ai}T_{rj}T_{sq}\right| + T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| - T_{si}\left|T_{aj}T_{rp}T_{kq}\right| \\ &+ T_{ri}\left|T_{aj}T_{sp}T_{kq}\right|. \end{aligned}$$

In case B, we also have the unique possibility:

$$1 \leq a < r = b < s < k = c \leq i < j = d < p < q = e \leq n,$$

in which case $S(f,g) = T_{ai}f - T_{sp}g$ with leading monomial equal to $T_{ai}T_{rj}T_{sq}T_{kp}$. One can check that

$$\begin{aligned} S(f,g) &= -T_{kp}\left|T_{ai}T_{rj}T_{sq}\right| + T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| - T_{sj}\left|T_{ai}T_{rp}T_{kq}\right| \\ &- T_{rq}\left|T_{ai}T_{sj}T_{kp}\right| + T_{ri}\left|T_{aj}T_{sp}T_{kq}\right| + T_{aq}\left|T_{ri}T_{sj}T_{kp}\right| \end{aligned}$$

and this proves what we want in this case. So, we may now safely assume that $S(f,g)$ is of degree five. Hence $S(f,g) = T_{ai}T_{ce}f - T_{sp}T_{kq}g$ with leading monomial equal to $T_{ai}T_{rj}T_{sq}T_{ce}T_{kp}$. By Buchberger's second criterion, we may also eliminate all possibilities where $i < k$ since in this case $T_{ai}T_{kq}$ is the leading monomial of an element of $\mathcal{F}_4$ and the case $(\mathcal{F}_4, \mathcal{F}_7)$ has been treated above. This leaves us with the following possibilities

($i1$) $1 \leq a < r = b < s < k \leq c \leq i < j = d < p < q \leq e \leq n$
($i2$) $1 \leq a < r = b < s < k \leq c \leq i < j = d < p \leq e < q \leq n$
($i3$) $1 \leq a < r = b < s < k \leq c \leq i < j = d < e < p < q \leq n$
($i4$) $1 \leq a < r = b < s \leq c \leq k \leq i < j = d < p < q \leq e \leq n$
($i5$) $1 \leq a < r = b < s \leq c \leq k \leq i < j = d < p \leq e < q \leq n$
($i6$) $1 \leq a < r = b < s \leq c \leq k \leq i < j = d < e < p < q \leq n$
($i7$) $1 \leq a < r = b < c \leq s < i = k \leq j = d < p < q \leq e \leq n$
($i8$) $1 \leq a < r = b < c \leq s < i = k \leq j = d < p \leq e < q \leq n$
($i9$) $1 \leq a < r = b < c \leq s < i = k \leq j = d < e < p < q \leq n$
($i10$) $1 \leq a < r = b < c < s < k < i < j = d < p < q \leq e \leq n$
($i11$) $1 \leq a < r = b < c < s < k < i < j = d < p \leq e < q \leq n$
($i12$) $1 \leq a < r = b < c < s < k < i < j = d < e < p < q \leq n$

For $(i1)$, one can check that

$$\begin{aligned} S(f,g) &= -T_{ce}T_{kp}\left|T_{ai}T_{rj}T_{sq}\right| - T_{kq}T_{cj}\left|T_{ai}T_{rp}T_{se}\right| + T_{kj}T_{ce}\left|T_{ai}T_{rp}T_{sq}\right| \\ &+ T_{kq}T_{ci}\left|T_{aj}T_{rp}T_{se}\right| - (T_{sj}T_{ce} - T_{se}T_{cj})\left|T_{ai}T_{rp}T_{kq}\right| - T_{se}T_{ci}\left|T_{aj}T_{rp}T_{kq}\right| \\ &- (T_{si}T_{cj} - T_{sj}T_{ci})\left|T_{ap}T_{rq}T_{ke}\right| - T_{rq}T_{ce}\left|T_{ai}T_{sj}T_{kp}\right| - T_{rq}T_{cj}\left|T_{ai}T_{sp}T_{ke}\right| \\ &+ T_{rq}T_{ci}\left|T_{aj}T_{sp}T_{ke}\right| + T_{ri}T_{ce}\left|T_{aj}T_{sp}T_{kq}\right| - T_{rq}T_{sp}\left|T_{ai}T_{kj}T_{ce}\right| \\ &+ T_{aq}T_{ce}\left|T_{ri}T_{sj}T_{kp}\right| + T_{aq}T_{cj}\left|T_{ri}T_{sp}T_{ke}\right| - T_{aq}T_{ci}\left|T_{rj}T_{sp}T_{ke}\right| \\ &+ T_{aq}T_{sp}\left|T_{ri}T_{kj}T_{ce}\right| \qquad (83) \end{aligned}$$

Now, if $(k,q)=(c,e)$, $S(f,g) = T_{ai}f - T_{sp}g$ with leading monomial equal to $T_{ai}T_{rj}T_{sq}T_{kp}$. One can check that

$$\begin{aligned} S(f,g) &= -T_{kp}\left|T_{ai}T_{rj}T_{sq}\right| + T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| - T_{sj}\left|T_{ai}T_{rp}T_{kq}\right| \\ &- T_{rq}\left|T_{ai}T_{sj}T_{kp}\right| + T_{ri}\left|T_{aj}T_{sp}T_{kq}\right| + T_{aq}\left|T_{ri}T_{sj}T_{kp}\right|. \qquad (84) \end{aligned}$$

This shows that $S(f,g)$ is in standard form relative to G. Now some observations about relation (84) similar to those above allow us to draw the same conclusion for the cases $(i2), \ldots, (i12)$. This finishes case (2).

(3) $\underline{(r,j)=(c,e)}$ In this case $S(f,g) = T_{ai}T_{bd}f - T_{sp}T_{kq}g$ is always of degree five since $(a,i) \neq (s,p)$, $(a,i) \neq (k,q)$ and $(b,d) \neq (s,p)$, $(b,d) \neq (k,q)$. As above, Buchberger's second criterion allows us to assume that $k \leq i$, and hence we can restrict to the unique case

$$1 \leq a < b < r = c < s < k \leq i < d < j = e < p < q \leq n.$$

The leading monomial of $S(f,g)$ is $T_{ai}T_{bd}T_{rj}T_{sq}T_{kp}$ and the following proves that $S(f,g)$ is in standard form relative to G in this case:

$$\begin{aligned} S(f,g) &= -T_{sq}T_{kp}\left|T_{ai}T_{bd}T_{rj}\right| - (T_{sj}T_{kq} - T_{sq}T_{kj})\left|T_{ai}T_{bd}T_{rp}\right| + (T_{sj}T_{kp} - T_{sp}T_{kj})\left|T_{ai}T_{bd}T_{rq}\right| \\ &+ T_{rd}T_{kq}\left|T_{ai}T_{bj}T_{sp}\right| - T_{rd}T_{kp}\left|T_{ai}T_{bj}T_{sq}\right| + T_{rd}T_{kj}\left|T_{ai}T_{bp}T_{sq}\right| \\ &- T_{ri}T_{kq}\left|T_{ad}T_{bj}T_{sp}\right| + T_{ri}T_{kp}\left|T_{ad}T_{bj}T_{sq}\right| - T_{ri}T_{kj}\left|T_{ad}T_{bp}T_{sq}\right| \\ &+ (T_{ri}T_{sd} - T_{rd}T_{si})\left|T_{aj}T_{bp}T_{kq}\right| + T_{bi}T_{kq}\left|T_{ad}T_{rj}T_{sp}\right| - T_{bi}T_{kp}\left|T_{ad}T_{rj}T_{sq}\right| \\ &+ T_{bi}T_{kj}\left|T_{ad}T_{rp}T_{sq}\right| - T_{bi}T_{sd}\left|T_{aj}T_{rp}T_{kq}\right| + T_{bi}T_{rd}\left|T_{aj}T_{sp}T_{kq}\right| \end{aligned}$$

(4) $\underline{(s,p)=(b,d)}$ Using the relation $S(g,f)=-S(f,g)$, one may assume that $a \leq r$. Here $S(f,g)$ is of degree four if one of the two cases occurs:

$$A.\,(a,i)=(r,j) \qquad B.\,(c,e)=(k,q).$$

In case A, we can assume that $k \leq c$ and hence restrict to the following two subcases

$$A1.\,1 \leq r=a<s=b<k \leq c \leq j=i<p=d<q \leq e \leq n$$
$$A2.\,1 \leq r=a<s=b<k \leq c \leq j=i<p=d<e<q \leq n.$$

In case $A1$, $S(f,g)=T_{ce}f-T_{kq}g$ with leading monomial equal to $T_{rj}T_{sq}T_{kp}T_{ce}$ and one can verify that

$$\begin{aligned} S(f,g) &= -T_{kp}\left|T_{rj}T_{sq}T_{ce}\right| + T_{kj}\left|T_{rp}T_{sq}T_{ce}\right| - T_{se}\left|T_{rj}T_{kp}T_{cq}\right| \\ &+ T_{re}\left|T_{sj}T_{kp}T_{cq}\right|. \end{aligned} \tag{85}$$

In case $A2$, the leading monomial of $S(f,g)$ is $T_{rj}T_{se}T_{kq}T_{cp}$, and one can deduce from (85) that

$$\begin{aligned} S(f,g) &= T_{kp}\left|T_{rj}T_{se}T_{cq}\right| - T_{kj}\left|T_{rp}T_{se}T_{cq}\right| - T_{se}\left|T_{rj}T_{kp}T_{cq}\right| \\ &+ T_{re}\left|T_{sj}T_{kp}T_{cq}\right|. \end{aligned}$$

In case B, $S(f,g)=T_{ai}f-T_{rj}g$ and one can restrict to the following possibilities:

$$B1.\,1 \leq a \leq r<s=b<k=c \leq j \leq i<p=d<q=e \leq n$$
$$B2.\,1 \leq a \leq r<s=b<k=c \leq i<j<p=d<q=e \leq n.$$

In both cases, the leading monomial of $S(f,g)$ is $T_{ai}T_{rj}T_{sq}T_{kp}$. For $B1$, we have

$$\begin{aligned} S(f,g) &= -T_{ki}\left|T_{rj}T_{ap}T_{se}\right| + T_{si}\left|T_{rj}T_{ap}T_{ke}\right| - T_{re}\left|T_{aj}T_{si}T_{kp}\right| \\ &+ T_{rp}\left|T_{aj}T_{si}T_{ke}\right|. \end{aligned} \tag{86}$$

This proves that $S(f,g)$ is in standard form in this case. Case $B2$ is easily deduced from (86).

Next we assume that $S(f,g)$ is of degree five. Hence $S(f,g) = T_{ai}T_{ce}f - T_{rj}T_{kq}g$, and we use the relation $S(g,f) = -S(f,g)$ together with Buchberger's second criterion to restrict to the following possibilities:

$(i1) 1 \le a \le r < s = b < k \le j \le c \le i < p = d < q \le e \le n$
$(i2) 1 \le a \le r < s = b < k \le j \le c \le i < p = d < e < q \le n$
$(i3) 1 \le a \le r < s = b < k \le c \le j \le i < p = d < q \le e \le n$
$(i4) 1 \le a \le r < s = b < k \le c \le j \le i < p = d < e < q \le n$
$(i5) 1 \le a \le r < s = b < k \le c \le i \le j < p = d < q \le e \le n$
$(i6) 1 \le a \le r < s = b < k \le c \le i \le j < p = d < e < q \le n$
$(i7) 1 \le a \le r < s = b < c \le i = k \le j < p = d < q \le e \le n$
$(i8) 1 \le a \le r < s = b < c \le i = k \le j < p = d < e < q \le n$

For $(i1)$, the leading monomial of $S(f,g)$ is $T_{ai}T_{rj}T_{sq}T_{ce}T_{kp}$ and we have

$$\begin{aligned} S(f,g) &= T_{kj}T_{ce}\left|T_{ai}T_{rp}T_{sq}\right| - T_{sj}T_{ce}\left|T_{ai}T_{rp}T_{kq}\right| + (T_{sj}T_{ki} - T_{si}T_{kj})\left|T_{ap}T_{rq}T_{ce}\right| \\ &- T_{rj}T_{kp}\left|T_{ai}T_{sq}T_{ce}\right| + T_{ri}T_{kj}\left|T_{ap}T_{sq}T_{ce}\right| - T_{rj}T_{se}\left|T_{ai}T_{kp}T_{cq}\right| \\ &+ (T_{rj}T_{si} - T_{ri}T_{sj})\left|T_{ap}T_{kq}T_{ce}\right| + (T_{ap}T_{cq} - T_{aq}T_{cp})\left|T_{rj}T_{si}T_{ke}\right| + T_{ae}T_{ci}\left|T_{rj}T_{sp}T_{kq}\right| \\ &- T_{ae}T_{kj}\left|T_{ri}T_{sp}T_{cq}\right| + T_{ae}T_{sj}\left|T_{ri}T_{kp}T_{cq}\right|. \end{aligned} \tag{87}$$

This proves that $S(f,g)$ is in standard form relative to G in this case. In all the remaining cases $(i2),\dots,(i8)$, one can use (87) to show that $S(f,g)$ is in standard form in these cases too.

(5) $\underline{(s,p) = (c,e)}$ In this case $S(f,g)$ is of degree four if one of the following two cases occurs:

$$A.\,(a,i) = (r,j) \qquad B.\,(b,d) = (r,j).$$

In case A, we have

$$1 \le r = a < b < s = c < k \le j = i < d < p = e < q \le n,$$

$S(f,g) = T_{bd}f - T_{kq}g$ with leading monomial equal to $T_{rj}T_{bd}T_{sq}T_{kp}$. Also one can check that

$$\begin{aligned} S(f,g) &= -T_{kp}\left|T_{rj}T_{bd}T_{sq}\right| + T_{sd}\left|T_{rj}T_{bp}T_{kq}\right| - T_{sj}\left|T_{rd}T_{bp}T_{kq}\right| \\ &+ T_{bj}\left|T_{rd}T_{sp}T_{kq}\right| - T_{rq}\left|T_{bj}T_{sd}T_{kp}\right| + T_{rp}\left|T_{bj}T_{sd}T_{kq}\right|. \end{aligned}$$

In case B, we use Buchberger's second criterion to assume that $k \leq i$ and restrict to the possibility

$$1 \leq a < r = b < s = c < k \leq i < j = d < p = e < q \leq n.$$

Here $S(f,g) = T_{ai}f - T_{kq}g$ with leading monomial equal to $T_{ai}T_{rj}T_{sq}T_{kp}$, and

$$\begin{aligned} S(f,g) &= -T_{kp}\,|T_{ai}T_{rj}T_{sq}| + T_{kj}\,|T_{ai}T_{rp}T_{sq}| - T_{si}\,|T_{aj}T_{rp}T_{kq}| \\ &+ T_{ri}\,|T_{aj}T_{sp}T_{kq}|\,. \end{aligned}$$

This proves that if $S(f,g)$ is of degree four, then it is in standard form relative to G. Assume next that $S(f,g) = T_{ai}T_{bd}f - T_{rj}T_{kq}g$ is of degree five, and use Buchberger's second criterion to restrict to the following possibilities:

$(i1) 1 \leq r \leq a < b < c = s \leq k \leq j \leq i < d < p = e < q \leq n$

$(i2) 1 \leq r \leq a < b < c = s \leq k \leq i \leq j \leq d < p = e < q \leq n$

$(i3) 1 \leq r \leq a < b < c = s \leq k \leq i < d \leq j < p = e < q \leq n$

$(i4) 1 \leq a \leq r \leq b < c = s \leq k \leq j \leq i < d < p = e < q \leq n$

$(i5) 1 \leq a \leq r \leq b < c = s \leq k \leq i \leq j \leq d < p = e < q \leq n$

$(i6) 1 \leq a \leq r \leq b < c = s \leq k \leq i < d \leq j < p = e < q \leq n$

$(i7) 1 \leq a < b \leq r < c = s \leq k \leq j \leq i < d < p = e < q \leq n$

$(i8) 1 \leq a < b \leq r < c = s \leq k \leq i \leq j \leq d < p = e < q \leq n$

$(i9) 1 \leq a < b \leq r < c = s \leq k \leq i < d \leq j < p = e < q \leq n.$

For all the above nine cases, one can check that $S(f,g)$ has a leading monomial equal to $T_{rj}T_{ai}T_{bd}T_{sq}T_{kp}$.

For $(i1)$, we have the following expression of $S(f,g)$:

$$\begin{aligned} S(f,g) &= -(T_{bi}T_{sd} - T_{bd}T_{si})\,|T_{rj}T_{ap}T_{kq}| + (T_{ai}T_{sd} - T_{ad}T_{si})\,|T_{rj}T_{bp}T_{kq}| \\ &+ T_{ad}T_{bi}\,|T_{rj}T_{sp}T_{kq}| - (T_{rp}T_{kq} - T_{rq}T_{kp})\,|T_{aj}T_{bi}T_{sd}| \\ &- T_{rq}T_{kj}\,|T_{ai}T_{bd}T_{sp}| - (T_{rj}T_{kp} - T_{rp}T_{kj})\,|T_{ai}T_{bd}T_{sq}| \qquad (88) \end{aligned}$$

For $(i2)$,

$$\begin{aligned} S(f,g) &= -T_{si}T_{kq}\,|T_{rj}T_{ad}T_{bp}| + T_{bi}T_{kq}\,|T_{rj}T_{ad}T_{sp}| - (T_{bi}T_{sj} - T_{bj}T_{si})\,|T_{rd}T_{ap}T_{kq}| \\ &- T_{ai}T_{kp}\,|T_{rj}T_{bd}T_{sq}| + T_{ai}T_{sd}\,|T_{rj}T_{bp}T_{kq}| - T_{aj}T_{si}\,|T_{rd}T_{bp}T_{kq}| \end{aligned}$$

$$\begin{aligned}
&+ \; T_{aj}T_{bi}\left|T_{rd}T_{sp}T_{kq}\right| + T_{rp}T_{kq}\left|T_{ai}T_{bj}T_{sd}\right| - T_{rd}T_{kp}\left|T_{ai}T_{bj}T_{sq}\right| \\
&- \; T_{rq}T_{kj}\left|T_{ai}T_{bd}T_{sp}\right| + T_{rp}T_{kj}\left|T_{ai}T_{bd}T_{sq}\right| + T_{rq}T_{si}\left|T_{aj}T_{bd}T_{kp}\right| \\
&- \; T_{rp}T_{si}\left|T_{aj}T_{bd}T_{kq}\right| - T_{rq}T_{bi}\left|T_{aj}T_{sd}T_{kp}\right| + T_{rp}T_{bi}\left|T_{aj}T_{sd}T_{kq}\right|.
\end{aligned}$$

One can easily see that $(i3)$ can be deduced from (88) using elementary operations. For $(i4)$, we have that

$$\begin{aligned}
S(f,g) \;=\; & T_{bd}T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| - T_{bd}T_{sj}\left|T_{ai}T_{rp}T_{kq}\right| - T_{rj}T_{kp}\left|T_{ai}T_{bd}T_{sq}\right| \\
+ \; & T_{rj}T_{sd}\left|T_{ai}T_{bp}T_{kq}\right| - T_{rj}T_{si}\left|T_{ad}T_{bp}T_{kq}\right| + T_{rj}T_{bi}\left|T_{ad}T_{sp}T_{kq}\right| \\
+ \; & (T_{ap}T_{kq} - T_{aq}T_{kp})\left|T_{rj}T_{bi}T_{sd}\right| + T_{aq}T_{kd}\left|T_{rj}T_{bi}T_{sp}\right| - T_{ap}T_{kd}\left|T_{rj}T_{bi}T_{sq}\right| \\
- \; & T_{aq}T_{kj}\left|T_{ri}T_{bd}T_{sp}\right| + T_{ap}T_{kj}\left|T_{ri}T_{bd}T_{sq}\right| - T_{aq}T_{si}\left|T_{rj}T_{bd}T_{kp}\right| \\
+ \; & T_{ap}T_{si}\left|T_{rj}T_{bd}T_{kq}\right| + 2T_{aq}T_{sj}\left|T_{ri}T_{bd}T_{kp}\right| - 2T_{ap}T_{sj}\left|T_{ri}T_{bd}T_{kq}\right| \\
+ \; & (T_{ap}T_{bq} - T_{aq}T_{bp})\left|T_{rj}T_{si}T_{kd}\right| - T_{aq}T_{bj}\left|T_{ri}T_{sd}T_{kp}\right| + T_{ap}T_{bj}\left|T_{ri}T_{sd}T_{kq}\right| \\
- \; & (T_{ap}T_{rq} - T_{aq}T_{rp})\left|T_{bj}T_{si}T_{kd}\right| - T_{aq}T_{rd}\left|T_{bj}T_{si}T_{kp}\right| + T_{ap}T_{rd}\left|T_{bj}T_{si}T_{kq}\right|
\end{aligned} \tag{89}$$

For $(i5)$,

$$\begin{aligned}
S(f,g) \;=\; & -T_{sq}T_{kp}\left|T_{ai}T_{rj}T_{bd}\right| + T_{sd}T_{kq}\left|T_{ai}T_{rj}T_{bp}\right| - T_{bj}T_{kp}\left|T_{ai}T_{rd}T_{sq}\right| \\
+ \; & T_{bd}T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| + T_{bi}T_{kp}\left|T_{aj}T_{rd}T_{sq}\right| + (T_{bj}T_{sd} - T_{bd}T_{sj})\left|T_{ai}T_{rp}T_{kq}\right| \\
- \; & T_{bi}T_{sd}\left|T_{aj}T_{rp}T_{kq}\right| - T_{ri}T_{kp}\left|T_{aj}T_{bd}T_{sq}\right| + T_{ri}T_{sd}\left|T_{aj}T_{bp}T_{kq}\right| \\
- \; & T_{rj}T_{si}\left|T_{ad}T_{bp}T_{kq}\right| + T_{rj}T_{bi}\left|T_{ad}T_{sp}T_{kq}\right| + T_{ap}T_{kq}\left|T_{ri}T_{bj}T_{sd}\right| \\
- \; & T_{ad}T_{kp}\left|T_{ri}T_{bj}T_{sq}\right| - T_{aq}T_{kj}\left|T_{ri}T_{bd}T_{sp}\right| + T_{ap}T_{kj}\left|T_{ri}T_{bd}T_{sq}\right| \\
+ \; & T_{aq}T_{si}\left|T_{rj}T_{bd}T_{kp}\right| - T_{ap}T_{si}\left|T_{rj}T_{bd}T_{kq}\right| - T_{aq}T_{bi}\left|T_{rj}T_{sd}T_{kp}\right| \\
+ \; & T_{ap}T_{bi}\left|T_{rj}T_{sd}T_{kq}\right| - (T_{ap}T_{rq} - T_{aq}T_{rp})\left|T_{bi}T_{sj}T_{kd}\right|
\end{aligned} \tag{90}$$

For $(i6)$, we have

$$\begin{aligned}
S(f,g) \;=\; & T_{sd}T_{kq}\left|T_{ai}T_{rj}T_{bp}\right| - T_{si}T_{kq}\left|T_{ad}T_{rj}T_{bp}\right| - T_{bd}T_{kp}\left|T_{ai}T_{rj}T_{sq}\right| \\
+ \; & T_{bd}T_{kj}\left|T_{ai}T_{rp}T_{sq}\right| + T_{bi}T_{kq}\left|T_{ad}T_{rj}T_{sp}\right| - (T_{bd}T_{sj} - T_{bj}T_{sd})\left|T_{ai}T_{rp}T_{kq}\right| \\
+ \; & (T_{bi}T_{sj} - T_{bj}T_{si})\left|T_{ad}T_{rp}T_{kq}\right| - (2T_{bi}T_{sd} - T_{bd}T_{si})\left|T_{aj}T_{rp}T_{kq}\right| - T_{rq}T_{kp}\left|T_{ai}T_{bd}T_{sj}\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \ (T_{ri}T_{sd} - T_{rd}T_{si})\,|T_{aj}T_{bp}T_{kq}| + T_{rd}T_{bi}\,|T_{aj}T_{sp}T_{kq}| - (T_{ap}T_{kq} - 2T_{aq}T_{kp})\,|T_{ri}T_{bd}T_{sj}| \\
&- \ T_{aq}T_{kj}\,|T_{ri}T_{bd}T_{sp}| - (T_{aj}T_{kp} - T_{ap}T_{kj})\,|T_{ri}T_{bd}T_{sq}| \\
&+ \ (T_{ap}T_{rq} - T_{aq}T_{rp})\,|T_{bi}T_{sd}T_{kj}|
\end{aligned}
$$

For ($i7$),

$$
\begin{aligned}
S(f,g) \ = \ & -(T_{sj}T_{kq} - T_{sq}T_{kj})\,|T_{ai}T_{bd}T_{rp}| + (T_{sj}T_{kp} - T_{sp}T_{kj})\,|T_{ai}T_{bd}T_{rq}| - T_{rj}T_{kp}\,|T_{ai}T_{bd}T_{sq}| \\
& + \ T_{rd}T_{kj}\,|T_{ai}T_{bp}T_{sq}| - T_{ri}T_{kj}\,|T_{ad}T_{bp}T_{sq}| + (T_{rj}T_{sd} - T_{rd}T_{sj})\,|T_{ai}T_{bp}T_{kq}| \\
& - \ (T_{rj}T_{si} - T_{ri}T_{sj})\,|T_{ad}T_{bp}T_{kq}| + T_{bi}T_{kj}\,|T_{ad}T_{rp}T_{sq}| - T_{bi}T_{sj}\,|T_{ad}T_{rp}T_{kq}| \\
& + \ T_{bi}T_{rj}\,|T_{ad}T_{sp}T_{kq}| - T_{aq}T_{kj}\,|T_{bi}T_{rd}T_{sp}| + T_{ap}T_{kj}\,|T_{bi}T_{rd}T_{sq}| \\
& + \ T_{aq}T_{sj}\,|T_{bi}T_{rd}T_{kp}| - T_{ap}T_{sj}\,|T_{bi}T_{rd}T_{kq}| - T_{aq}T_{rj}\,|T_{bi}T_{sd}T_{kp}| \\
& + \ T_{ap}T_{rj}\,|T_{bi}T_{sd}T_{kq}| - T_{aq}T_{bd}\,|T_{rj}T_{si}T_{kp}| + T_{ap}T_{bd}\,|T_{rj}T_{si}T_{kq}|\,.
\end{aligned}
$$

For ($i8$),

$$
\begin{aligned}
S(f,g) \ = \ & -(T_{sj}T_{kq} - T_{sq}T_{kj})\,|T_{ai}T_{bd}T_{rp}| + (T_{sj}T_{kp} - T_{sp}T_{kj})\,|T_{ai}T_{bd}T_{rq}| - T_{rj}T_{kp}\,|T_{ai}T_{bd}T_{sq}| \\
& + \ T_{rd}T_{kj}\,|T_{ai}T_{bp}T_{sq}| - T_{ri}T_{kj}\,|T_{ad}T_{bp}T_{sq}| + (T_{rj}T_{sd} - T_{rd}T_{sj})\,|T_{ai}T_{bp}T_{kq}| \\
& + \ (T_{ri}T_{sj} - T_{rj}T_{si})\,|T_{ad}T_{bp}T_{kq}| + T_{bi}T_{kj}\,|T_{ad}T_{rp}T_{sq}| - T_{bi}T_{sj}\,|T_{ad}T_{rp}T_{kq}| \\
& + \ T_{bi}T_{rj}\,|T_{ad}T_{sp}T_{kq}| + (T_{ap}T_{kq} - T_{aq}T_{kp})\,|T_{bi}T_{rj}T_{sd}| + T_{aq}T_{kd}\,|T_{bi}T_{rj}T_{sp}| \\
& - \ T_{ap}T_{kd}\,|T_{bi}T_{rj}T_{sq}| - T_{aq}T_{kj}\,|T_{bi}T_{rd}T_{sp}| + T_{ap}T_{kj}\,|T_{bi}T_{rd}T_{sq}| \\
& + \ T_{aq}T_{si}\,|T_{bj}T_{rd}T_{kp}| - T_{ap}T_{si}\,|T_{bj}T_{rd}T_{kq}| - T_{aq}T_{ri}\,|T_{bj}T_{sd}T_{kp}| \\
& + \ T_{ap}T_{ri}\,|T_{bj}T_{sd}T_{kq}| - (T_{ap}T_{bq} - T_{aq}T_{bp})\,|T_{ri}T_{sj}T_{kd}|\,.
\end{aligned}
$$

For ($i9$),

$$
\begin{aligned}
S(f,g) \ = \ & -T_{sq}T_{kp}\,|T_{ai}T_{bd}T_{rj}| - (T_{sj}T_{kq} - T_{sq}T_{kj})\,|T_{ai}T_{bd}T_{rp}| + (T_{sj}T_{kp} - T_{sp}T_{kj})\,|T_{ai}T_{bd}T_{rq}| \\
& - \ T_{rd}T_{kp}\,|T_{ai}T_{bj}T_{sq}| + T_{rd}T_{kj}\,|T_{ai}T_{bp}T_{sq}| + T_{ri}T_{kp}\,|T_{ad}T_{bj}T_{sq}| \\
& - \ T_{ri}T_{kj}\,|T_{ad}T_{bp}T_{sq}| - (T_{rd}T_{sj} - T_{rj}T_{sd})\,|T_{ai}T_{bp}T_{kq}| + (T_{ri}T_{sj} - T_{rj}T_{si})\,|T_{ad}T_{bp}T_{kq}| \\
& - \ T_{bq}T_{kp}\,|T_{ai}T_{rd}T_{sj}| + T_{bi}T_{kq}\,|T_{ad}T_{rj}T_{sp}| - T_{bi}T_{kp}\,|T_{ad}T_{rj}T_{sq}| \\
& + \ T_{bi}T_{kj}\,|T_{ad}T_{rp}T_{sq}| - T_{bi}T_{sd}\,|T_{aj}T_{rp}T_{kq}| + T_{bi}T_{rd}\,|T_{aj}T_{sp}T_{kq}| \\
& + \ T_{aq}T_{kp}\,|T_{bi}T_{rd}T_{sj}| + (T_{ap}T_{bq} - T_{aq}T_{bp})\,|T_{ri}T_{sd}T_{kj}|\,.
\end{aligned}
$$

In all the above cases, $S(f,g)$ is in standard form relative to G.

(6) $\underline{(k,q)=(c,e)}$ In this case the degree of $S(f,g)$ is as usual either 4 or 5. It is 4 when exactly one of the following possibilities occurs:

$$A.\,(r,j)=(a,i) \quad B.\,(s,p)=(a,i)$$
$$C.\,(r,j)=(b,d) \quad D.\,(s,p)=(b,d).$$

Using the relation $S(f,g)=-S(g,f)$, we do not need to consider case C.

$\underline{A.\,(a,i)=(r,j)}$ Again we use relation $S(f,g)=-S(g,f)$ to restrict to the two following cases

$$A1.\,1\le r=a<s\le b<k=c\le j=i<p\le d<q=e\le n$$
$$A2.\,1\le r=a<b\le s<k=c\le j=i<d\le p<q=e\le n.$$

In case $A1$, $S(f,g)=T_{bd}f-T_{sp}g$ with leading monomial equal to $T_{rj}T_{sp}T_{bq}T_{kd}$, and one can ckeck that

$$\begin{aligned} S(f,g) &= T_{kd}\,|T_{rj}T_{sp}T_{bq}| + T_{sq}\,|T_{rj}T_{bp}T_{kd}| - T_{sj}\,|T_{rp}T_{bd}T_{kq}| \\ &- T_{rd}\,|T_{sj}T_{bp}T_{kq}| \end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G in this case. Case $A2$ is similar.

$\underline{B.\,(a,i)=(s,p)}$ In this case,

$$1\le r<s=a<b<k=c\le j<p=i<d<q=e\le n,$$

$S(f,g)=T_{bd}f-T_{rj}g$, its leading monomial is $T_{rj}T_{sp}T_{bq}T_{kd}$ and one can verify that

$$\begin{aligned} S(f,g) &= T_{kd}\,|T_{rj}T_{sp}T_{bq}| - T_{kp}\,|T_{rj}T_{sd}T_{bq}| + T_{bp}\,|T_{rj}T_{sd}T_{kq}| \\ &+ T_{sq}\,|T_{rj}T_{bp}T_{kd}| - T_{sj}\,|T_{rp}T_{bd}T_{kq}| - T_{rq}\,|T_{sj}T_{bp}T_{kd}|\,. \end{aligned}$$

$\underline{D.\,(b,d)=(s,p)}$ We may restrict to the following two possibilities:

$$B1.\,1\le r\le a<s=b<k=c\le j\le i<p=d<q=e\le n$$
$$B2.\,1\le r\le a<s=b<k=c\le i\le j<p=d<q=e\le n.$$

In both cases $S(f,g) = T_{ai}f - T_{rj}g$, its leading monomial is $T_{rj}T_{ap}T_{si}T_{kq}$. In case $B1$ we have

$$\begin{aligned} S(f,g) &= -T_{ki}\,|T_{rj}T_{ap}T_{sq}| + T_{si}\,|T_{rj}T_{ap}T_{kq}| - T_{rq}\,|T_{aj}T_{si}T_{kp}| \\ &+ T_{rp}\,|T_{aj}T_{si}T_{kq}| \end{aligned} \tag{91}$$

Case $B2$ follows easily from (91). This proves that if $S(f,g)$ is of degree 4, then it is in standard form relative to G in this case. Hence we may now safely assume that $S(f,g) = T_{ai}T_{bd}f - T_{rj}T_{sp}g$ is of degree five.

Using the relation $S(f,g) = -S(g,f)$, one can restrict to the following possibilities:

$(i1) 1 \le r < s \le a < b < k = c \le j < p \le i < d < q = e \le n$

$(i2) 1 \le r < s \le a < b < k = c \le j \le i \le p \le d < q = e \le n$

$(i3) 1 \le r < s \le a < b < k = c \le j \le i < d < p < q = e \le n$

$(i4) 1 \le r < s \le a < b < k = c \le i < d \le j < p < q = e \le n$

$(i5) 1 \le r < s \le a < b < k = c \le i \le j \le d \le p < q = e \le n$

$(i6) 1 \le r < s \le a < b < k = c \le i \le j < p \le d < q = e \le n$

$(i7) 1 \le r \le a \le s \le b < k = c \le j < p \le i < d < q = e \le n$

$(i8) 1 \le r \le a \le s \le b < k = c \le j \le i \le p \le d < q = e \le n$

$(i9) 1 \le r \le a \le s \le b < k = c \le j \le i < d < p < q = e \le n$

$(i10) 1 \le r \le a \le s \le b < k = c \le i < d \le j < p < q = e \le n$

$(i11) 1 \le r \le a \le s \le b < k = c \le i \le j \le d \le p < q = e \le n$

$(i12) 1 \le r \le a \le s \le b < k = c \le i \le j < p \le d < q = e \le n$

$(i13) 1 \le r \le a < b \le s < k = c \le j < p \le i < d < q = e \le n$

$(i14) 1 \le r \le a < b \le s < k = c \le j \le i \le p \le d < q = e \le n$

$(i15) 1 \le r \le a < b \le s < k = c \le j \le i < d < p < q = e \le n$

$(i16) 1 \le r \le a < b \le s < k = c \le i < d \le j < p < q = e \le n$

$(i17) 1 \le r \le a < b \le s < k = c \le i \le j \le d \le p < q = e \le n$

$(i18) 1 \le r \le a < b \le s < k = c \le i \le j < p \le d < q = e \le n$

For the case $(i1)$ (as well as $(i2), \ldots, (i6)$), the leading monomial of $S(f,g)$ is equal to $T_{rj}T_{sp}T_{ai}T_{bq}T_{kd}$ and one can verify that in this case $S(f,g)$ has the following expresssion

$$-(T_{bi}T_{kd} - T_{bd}T_{ki})\,|T_{rj}T_{sp}T_{aq}| + (T_{ai}T_{kd} - T_{ad}T_{ki})\,|T_{rj}T_{sp}T_{bq}| + T_{ad}T_{bi}\,|T_{rj}T_{sp}T_{kq}|$$

$$- \quad T_{rp}T_{sj}\left|T_{ai}T_{bd}T_{kq}\right| + (T_{rp}T_{sq} - T_{rq}T_{sp})\left|T_{aj}T_{bi}T_{kd}\right| - (T_{rj}T_{sq} - T_{rq}T_{sj})\left|T_{ap}T_{bi}T_{kd}\right| \tag{92}$$

and this shows that $S(f,g)$ is in standard form relative to G in this case. The following remark allows us to deduce that $S(f,g)$ is in standard form relative to G in cases $(i2),\dots,(i6)$ without doing any computation.

Remark 3.3.12. A closer look at equation (92) reveals the following observations:

- $S(f,g)$ has the form $\sum_{m=1}^{6} u_m h_m$ where $u_m \in k[T_{ij} : 1 \le i < j \le n]$ is homogeneous of degree two, and $h_m \in \mathcal{F}_7$
- For each m, the leading monomial of $u_m h_m$ has the form $T_{r\lambda}T_{s\mu}T_{a\kappa}T_{b\theta}T_{k\delta}$ with $\lambda,\mu,\kappa,\theta,\delta$ are elements of $\{j,p,i,d,q\}$
- In each of the cases $(i2),\dots,(i6)$, the equality in (92) will remain true provided that we interchance, some columns in h_5 and h_6, if necessary, and choose a suitable sign. A for the relation

$$\text{mdeg}(u_m h_m) \le \text{mdeg}(S(f,g)), \tag{93}$$

  it is certainly true for $m \in \{1,2,3,4\}$ and since the leading monomial of u_5h_5 is a multiple of T_{rp}, (93) is also true for u_5h_5. For u_6h_6, the leading monomial is $T_{rj}T_{sq}T_{ap}T_{bi}T_{kd}$ which is always less than the leading monomial of $S(f,g)$ for $s \le a$. This proves that (93) is true for all $m = 1,\dots,6$ and $S(f,g)$ is in standard form relative to G for $(i1),\dots,(i6)$.

For $(i7),\dots,(i12)$, the leading monomial of $S(f,g)$ is $T_{rj}T_{ai}T_{sp}T_{bq}T_{kd}$. For $(i7)$, one has

$$\begin{aligned} S(f,g) &= -T_{bd}T_{kp}\left|T_{rj}T_{ai}T_{sq}\right| + T_{bd}T_{kj}\left|T_{rp}T_{ai}T_{sq}\right| - (T_{bj}T_{kp} - T_{bp}T_{kj})\left|T_{ri}T_{ad}T_{sq}\right| \\ &+ T_{sp}T_{kd}\left|T_{rj}T_{ai}T_{bq}\right| - T_{sp}T_{ki}\left|T_{rj}T_{ad}T_{bq}\right| - T_{sj}T_{kq}\left|T_{rp}T_{ai}T_{bd}\right| \\ &+ T_{sp}T_{bi}\left|T_{rj}T_{ad}T_{kq}\right| - T_{sj}T_{bi}\left|T_{rp}T_{ad}T_{kq}\right| + T_{sj}T_{bp}\left|T_{ri}T_{ad}T_{kq}\right| \end{aligned}$$

$$
\begin{aligned}
&+ \quad T_{aq}T_{kd}\,|T_{rj}T_{sp}T_{bi}| - T_{aq}T_{kp}\,|T_{rj}T_{si}T_{bd}| + T_{aq}T_{kj}\,|T_{rp}T_{si}T_{bd}| \\
&+ \quad T_{aq}T_{bp}\,|T_{rj}T_{si}T_{kd}| - T_{aq}T_{bj}\,|T_{rp}T_{si}T_{kd}| + T_{aq}T_{sd}\,|T_{rj}T_{bp}T_{ki}| \\
&- \quad T_{ap}T_{sj}\,|T_{ri}T_{bd}T_{kq}| + (T_{rd}T_{kq} - 2T_{rq}T_{kd})\,|T_{aj}T_{sp}T_{bi}| + T_{rq}T_{ki}\,|T_{aj}T_{sp}T_{bd}| \\
&+ \quad (T_{ri}T_{kd} - T_{rd}T_{ki})\,|T_{aj}T_{sp}T_{bq}| + T_{rq}T_{kp}\,|T_{aj}T_{si}T_{bd}| - T_{rq}T_{kj}\,|T_{ap}T_{si}T_{bd}| \\
&- \quad 2T_{rq}T_{bp}\,|T_{aj}T_{si}T_{kd}| + T_{rd}T_{bp}\,|T_{aj}T_{si}T_{kq}| + 2T_{rq}T_{bj}\,|T_{ap}T_{si}T_{kd}| \\
&- \quad T_{rd}T_{bj}\,|T_{ap}T_{si}T_{kq}| - T_{rq}T_{sd}\,|T_{aj}T_{bp}T_{ki}| + T_{ri}T_{sq}\,|T_{aj}T_{bp}T_{kd}| \\
&- \quad 2(T_{rd}T_{aq} - T_{rq}T_{ad})\,|T_{sj}T_{bp}T_{ki}| - T_{rq}T_{ai}\,|T_{sj}T_{bp}T_{kd}|
\end{aligned}
$$

For $(i8)$, one can check that

$$
\begin{aligned}
S(f,g) \;=\;& (T_{bi}T_{kd} - T_{bd}T_{ki})\,|T_{rj}T_{ap}T_{sq}| + (T_{ai}T_{kd} - T_{ad}T_{ki})\,|T_{rj}T_{sp}T_{bq}| + T_{ad}T_{bi}\,|T_{rj}T_{sp}T_{kq}| \\
&+ \quad T_{rq}T_{kd}\,|T_{aj}T_{si}T_{bp}| - T_{rp}T_{kd}\,|T_{aj}T_{si}T_{bq}| + T_{rq}T_{ki}\,|T_{aj}T_{sp}T_{bd}| \\
&+ \quad T_{rp}T_{ki}\,|T_{aj}T_{sd}T_{bq}| - T_{rq}T_{bd}\,|T_{aj}T_{si}T_{kp}| + T_{rp}T_{bd}\,|T_{aj}T_{si}T_{kq}| \\
&- \quad T_{rq}T_{bi}\,|T_{aj}T_{sp}T_{kd}| - T_{rp}T_{bi}\,|T_{aj}T_{sd}T_{kq}| + T_{rj}T_{sq}\,|T_{ai}T_{bp}T_{kd}| \\
&+ \quad (T_{rp}T_{aq} - T_{rq}T_{ap})\,|T_{sj}T_{bi}T_{kd}| - T_{rq}T_{aj}\,|T_{si}T_{bp}T_{kd}| - T_{rp}T_{aj}\,|T_{si}T_{bd}T_{kq}|
\end{aligned}
$$

For cases $(i9), \ldots, (i12)$, we always have that $i \leq p$ and hence these cases follow easily from $(i8)$ by an argument similar to the one in remark 3.3.12.
For the cases $(i13), \ldots, (i18)$, the leading monomial of $S(f,g)$ is $T_{rj}T_{ai}T_{bd}T_{sq}T_{kp}$. For $(i13)$,

$$
\begin{aligned}
S(f,g) \;=\;& -T_{sq}T_{kp}\,|T_{rj}T_{ai}T_{bd}| + T_{sp}T_{kd}\,|T_{rj}T_{ai}T_{bq}| - T_{sp}T_{ki}\,|T_{rj}T_{ad}T_{bq}| \\
&- \quad (T_{sj}T_{kq} - T_{sq}T_{kj})\,|T_{rp}T_{ai}T_{bd}| - (T_{sj}T_{kp} - T_{sp}T_{kj}\,|T_{ri}T_{ad}T_{bq}| \\
&- \quad T_{bi}T_{kp}\,|T_{rj}T_{ad}T_{sq}| + T_{bi}T_{kj}\,|T_{rp}T_{ad}T_{sq}| + (T_{bj}T_{kp} - T_{bp}T_{kj})\,|T_{ri}T_{ad}T_{sq}| \\
&+ \quad T_{bi}T_{sp}\,|T_{rj}T_{ad}T_{kq}| - T_{bi}T_{sj}\,|T_{rp}T_{ad}T_{kq}| + T_{bp}T_{sj}\,|T_{ri}T_{ad}T_{kq}| \\
&- \quad T_{aq}T_{kp}\,|T_{rj}T_{bi}T_{sd}| + T_{aq}T_{kj}\,|T_{rp}T_{bi}T_{sd}| - (T_{aj}T_{kp} - T_{ap}T_{kj})\,|T_{ri}T_{bd}T_{sq}| \\
&+ \quad T_{aq}T_{sp}\,|T_{rj}T_{bi}T_{kd}| - T_{aq}T_{sj}\,|T_{rp}T_{bi}T_{kd}| - T_{ap}T_{sj}\,|T_{ri}T_{bd}T_{kq}| \\
&+ \quad T_{aq}T_{bd}\,|T_{rj}T_{sp}T_{ki}| + 2T_{rq}T_{kp}\,|T_{aj}T_{bi}T_{sd}| - T_{rd}T_{kp}\,|T_{aj}T_{bi}T_{sq}| \\
&- \quad 2T_{rq}T_{kj}\,|T_{ap}T_{bi}T_{sd}| + T_{rd}T_{kj}\,|T_{ap}T_{bi}T_{sq}| - 2T_{rq}T_{sp}\,|T_{aj}T_{bi}T_{kd}|
\end{aligned}
$$

$$
\begin{aligned}
&+\; T_{rd}T_{sp}\,|T_{aj}T_{bi}T_{kq}| + 2T_{rq}T_{sj}\,|T_{ap}T_{bi}T_{kd}| - T_{rd}T_{sj}\,|T_{ap}T_{bi}T_{kq}| \\
&-\; T_{rq}T_{bd}\,|T_{aj}T_{sp}T_{ki}| + T_{ri}T_{bq}\,|T_{aj}T_{sp}T_{kd}| - (T_{rd}T_{aq} - T_{rq}T_{ad})\,|T_{bj}T_{sp}T_{ki}| \\
&-\; T_{rq}T_{ai}\,|T_{bj}T_{sp}T_{kd}| + T_{rd}T_{ai}\,|T_{bj}T_{sp}T_{kq}|\,.
\end{aligned}
$$

This shows that $S(f,g)$ is in standard form relative to G in this case. As for $(i14)$, $S(f,g)$ is also in standard form relative to G since it is equal to the following expression

$$
\begin{aligned}
&(T_{si}T_{kd} - T_{sd}T_{ki})\,|T_{rj}T_{ap}T_{bq}| + T_{ad}T_{bi}\,|T_{rj}T_{sp}T_{kq}| - (T_{rp}T_{kq} - T_{rq}T_{kp})\,|T_{aj}T_{bi}T_{sd}| \\
+\;& T_{rq}T_{kj}\,|T_{ai}T_{bp}T_{sd}| - (T_{rj}T_{kp} - T_{rp}T_{kj})\,|T_{ai}T_{bd}T_{sq}| - T_{rq}T_{sd}\,|T_{aj}T_{bi}T_{kp}| \\
+\;& T_{rp}T_{sd}\,|T_{aj}T_{bi}T_{kq}| - T_{rq}T_{si}\,|T_{aj}T_{bp}T_{kd}| - T_{rp}T_{si}\,|T_{aj}T_{bd}T_{kq}| \\
+\;& T_{rj}T_{bq}\,|T_{ai}T_{sp}T_{kd}| - T_{rj}T_{aq}\,|T_{bi}T_{sp}T_{kd}|\,. \qquad (94)
\end{aligned}
$$

$S(f,g)$ can be proven to be in standard form relative to G in the cases $(i14),\ldots,(i18)$ by interchanging columns in (94) and multiplying by a suitable sign.

This proves that $S(f,g)$ is in standard form relative to G in case (6) and finishes the proof of the case $(\mathcal{F}_7,\mathcal{F}_7)$.

Case $(\mathcal{F}_8,\mathcal{F}_9)$

Let

$$
\begin{aligned}
&f = |T_cT_{ad}T_{be}| \in \mathcal{F}_8, && g = |Y_rT_{is}T_{ju}| \in \mathcal{F}_9 \\
&2 \le a < b \le c < d < e \le n, && 1 \le i < j \le r < s < u \le n.
\end{aligned}
$$

The leading monomials of f and g are not relatively prime only if one of the following cases is true:

$$(1)\,(a,d) = (i,s) \quad (2)\,(a,d) = (j,u) \quad (3)\,(b,e) = (i,s) \quad (4)\,(b,e) = (j,u).$$

In each of the above four cases, the degree of $S(f,g)$ is either 4 or 5. We will treat these two possibilities separately for each case.

<u>(1) $(a,d) = (i,s)$</u> In this case, the degree of $S(f,g)$ is four if $(j,u) = (b,e)$ in which case $\mathrm{LM}(S(f,g)) = Y_rT_sT_{ic}T_{ju}$. If $r < c$, then Y_rT_c would be the leading monomial of an element of $\mathcal{F}_2$, Buchberger's second criterion shows that $S(f,g)$ is in standard

form relative to G in this case. So we may assume that $r \geq c$, and this leaves us with the unique possibility:

$$2 \leq i = a < j = b \leq c \leq r < s = d < u = e \leq n.$$

In this case, $S(f,g)$ is equal to:

$$\begin{aligned}
&- (T_{ic}T_{ju} - T_{iu}T_{jc})(T_{rs} + Y_rT_s - Y_sT_r) + (T_{ic}T_{js} - T_{is}T_{jc})(T_{ru} + Y_rT_u - Y_uT_r) \\
&- (T_{ic}T_{jr} - T_{ir}T_{jc})(T_{su} + Y_sT_u - Y_uT_s) + T_{ju}(T_{ic}T_{rs} - T_{ir}T_{cs} + T_{is}T_{cr}) \\
&- T_{js}(T_{ic}T_{ru} - T_{ir}T_{cu} + T_{iu}T_{cr}) + T_{jr}(T_{ic}T_{su} - T_{is}T_{cu} + T_{iu}T_{cs}) \\
&- T_{jc}(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) - Y_u\left|T_cT_{ir}T_{js}\right| + Y_s\left|T_cT_{ir}T_{ju}\right| \\
&- \left|T_{ir}T_{js}T_{cu}\right|
\end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G in this case. So, we may safely assume that the degree of $S(f,g)$ is five. In this case, $S(f,g) = Y_rT_{ju}f - T_cT_{be}g$ with leading monomial equal to either $Y_rT_cT_{ie}T_{ju}T_{bd}$ or $Y_rT_cT_{iu}T_{js}T_{be}$. By Buchberger's second criterion, we may assume that $r \geq c$ and $c \geq j$. This leaves us with the following possibilities:

$$\begin{aligned}
&(i1)\, 2 \leq a = i < j \leq b \leq c \leq r < d = s < e < u \leq n \\
&(i2)\, 2 \leq a = i < j \leq b \leq c \leq r < d = s < u \leq e \leq n \\
&(i3)\, 2 \leq a = i < b \leq j \leq c \leq r < d = s < u < e \leq n \\
&(i4)\, 2 \leq a = i < b \leq j \leq c \leq r < d = s < e \leq u \leq n.
\end{aligned}$$

For $(i1)$, the leading monomial of $S(f,g)$ is $Y_rT_cT_{ie}T_{ju}T_{bd}$, and one can verify that $S(f,g)$ has the following expression

$$\begin{aligned}
&- (T_{ic}T_{ju}T_{be} - T_{ie}T_{ju}T_{bc})(T_{rs} + Y_rT_s - Y_sT_r) + (T_{ic}T_{ju}T_{bs} - T_{is}T_{ju}T_{bc})(T_{re} + Y_rT_e - Y_eT_r) \\
&- (T_{iu}T_{jc}T_{br} - T_{iu}T_{jr}T_{bc})(T_{se} + Y_sT_e - Y_eT_s) - (T_{ic}T_{jr}T_{be} - T_{ir}T_{jc}T_{be})(T_{su} + Y_sT_u - Y_uT_s) \\
&- (T_{ir}T_{jc}T_{bs} - T_{ic}T_{jr}T_{bs})(T_{eu} + Y_eT_u - Y_uT_e) + T_{ju}T_{be}(T_{ic}T_{rs} - T_{ir}T_{cs} + T_{is}T_{cr}) \\
&- T_{ju}T_{bs}(T_{ic}T_{re} - T_{ir}T_{ce} + T_{ie}T_{cr}) + T_{jr}T_{be}(T_{ic}T_{su} - T_{is}T_{cu} + T_{iu}T_{cs}) \\
&- T_{jr}T_{bs}(T_{ic}T_{eu} - T_{ie}T_{cu} + T_{iu}T_{ce}) - T_{jc}T_{be}(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) \\
&+ T_{jc}T_{bs}(T_{ir}T_{eu} - T_{ie}T_{ru} + T_{iu}T_{re}) + T_{iu}T_{be}(T_{jc}T_{rs} - T_{jr}T_{cs} + T_{js}T_{cr})
\end{aligned}$$

$$
\begin{aligned}
- & \ T_{iu}T_{bs}(T_{jc}T_{re} - T_{jr}T_{ce} + T_{je}T_{cr}) - (T_{is}T_{be} - T_{ie}T_{bs})(T_{jc}T_{ru} - T_{jr}T_{cu} + T_{ju}T_{cr}) \\
+ & \ T_{iu}T_{br}(T_{jc}T_{se} - T_{js}T_{ce} + T_{je}T_{cs}) - T_{iu}T_{bc}(T_{jr}T_{se} - T_{js}T_{re} + T_{je}T_{rs}) \\
- & \ (T_{ie}T_{ju} - T_{iu}T_{je})(T_{bc}T_{rs} - T_{br}T_{cs} + T_{bs}T_{cr}) + (T_{is}T_{ju} - T_{iu}T_{js})(T_{bc}T_{re} - T_{br}T_{ce} + T_{be}T_{cr}) \\
- & \ T_cT_{bs}\left|Y_rT_{ie}T_{ju}\right| + T_rT_{bc}\left|Y_sT_{ie}T_{ju}\right| + T_cT_{iu}\left|Y_rT_{js}T_{be}\right| - Y_uT_{be}\left|T_cT_{ir}T_{js}\right| \\
+ & \ Y_uT_{bs}\left|T_cT_{ir}T_{je}\right| + (Y_sT_{be} - Y_eT_{bs})\left|T_cT_{ir}T_{ju}\right| - Y_uT_{bc}\left|T_rT_{is}T_{je}\right| + Y_uT_{jc}\left|T_rT_{is}T_{be}\right| \\
- & \ Y_eT_{iu}\left|T_cT_{jr}T_{bs}\right| + Y_sT_{iu}\left|T_cT_{jr}T_{be}\right| - Y_uT_{ic}\left|T_rT_{js}T_{be}\right| + Y_uT_e\left|T_{ic}T_{jr}T_{bs}\right| \\
- & \ Y_uT_s\left|T_{ic}T_{jr}T_{be}\right| - T_{ju}\left|T_{ir}T_{bs}T_{ce}\right|. \qquad (95)
\end{aligned}
$$

This shows that $S(f,g)$ is in standard form relative to G in this case. In case $(i2)$, the leading monomial of $S(f,g)$ is $Y_rT_cT_{iu}T_{js}T_{be}$. Note that relation (95) remains true if one replaces the elements $T_{eu}+Y_eT_u-Y_uT_e$, $T_{ic}T_{eu}-T_{ie}T_{cu}+T_{iu}T_{ce}$, $T_{ir}T_{eu}-T_{ie}T_{ru}+T_{iu}T_{re}$, $|Y_rT_{ie}T_{ju}|$ and $|Y_sT_{ie}T_{ju}|$ with $-(T_{ue}+Y_uT_e-Y_eT_u)$, $-(T_{ic}T_{ue}-T_{iu}T_{ce}+T_{ie}T_{cu})$, $-(T_{ir}T_{ue}-T_{iu}T_{re}+T_{ie}T_{ru})$, $-|Y_rT_{iu}T_{je}|$ and $-|Y_sT_{iu}T_{je}|$ respectively. Also, one can check that these changes give a standard form of $S(f,g)$ relative to G in this case.
For $(i3)$, the leading monomial of $S(f,g)$ is $Y_rT_cT_{iu}T_{js}T_{be}$, and one can check that $S(f,g)$ is equal to the following expression in this case:

$$
\begin{aligned}
- & \ (T_{ic}T_{be}T_{ju} - T_{ie}T_{bc}T_{ju})(T_{rs} + Y_rT_s - Y_sT_r) + (T_{ic}T_{bs}T_{ju} - T_{is}T_{bc}T_{ju})(T_{re} + Y_rT_e - Y_eT_r) \\
- & \ (T_{ic}T_{br}T_{ju} - T_{ir}T_{bc}T_{ju})(T_{se} + Y_sT_e - Y_eT_s) + T_{be}T_{ju}(T_{ic}T_{rs} - T_{ir}T_{cs} + T_{is}T_{cr}) \\
- & \ T_{bs}T_{ju}(T_{ic}T_{re} - T_{ir}T_{ce} + T_{ie}T_{cr}) + T_{br}T_{ju}(T_{ic}T_{se} - T_{is}T_{ce} + T_{ie}T_{cs}) \\
- & \ T_{bc}T_{ju}(T_{ir}T_{se} - T_{is}T_{re} + T_{ie}T_{rs}) + T_cT_{js}\left|Y_rT_{iu}T_{be}\right| - T_cT_{jr}\left|Y_sT_{iu}T_{be}\right| \\
- & \ T_cT_{ie}\left|Y_rT_{bs}T_{ju}\right| - Y_eT_{js}\left|T_cT_{ir}T_{bu}\right| + Y_sT_{ju}\left|T_cT_{ir}T_{be}\right| + Y_eT_{jr}\left|T_cT_{is}T_{bu}\right| \\
- & \ Y_eT_{bc}\left|T_rT_{is}T_{ju}\right| + Y_eT_{ic}\left|T_rT_{bs}T_{ju}\right| - T_{ce}\left|T_{ir}T_{bs}T_{ju}\right| - T_{js}\left|T_{ir}T_{bu}T_{ce}\right| \\
+ & \ T_{jr}\left|T_{is}T_{bu}T_{ce}\right| - T_{be}\left|T_{ir}T_{js}T_{cu}\right| + T_{ie}\left|T_{br}T_{js}T_{cu}\right|. \qquad (96)
\end{aligned}
$$

This proves that $S(f,g)$ is in standard form relative to G in this case. Case $(i4)$ can be obtained from (96) as in case $(i2)$. This finishes case (1).
<u>(2) $(a,d)=(j,u)$</u> In this case $S(f,g) = Y_rT_{is}f - T_cT_{be}g$ is always of degree five, and its leading monomial is equal to $Y_rT_cT_{is}T_{je}T_{bu}$. By Buchberger's second criterion, we may assume that $r \geq c$, and hence we have the unique possibility

$$1 \leq i < j = a < b \leq c \leq r < s < u = d < e \leq n.$$

One can check that $S(f,g)$ is equal to

$$
\begin{aligned}
- &\quad T_{is}(T_{jc}T_{be} - T_{je}T_{bc})(T_{ru} + Y_rT_u - Y_uT_r) + T_{is}(T_{jc}T_{bu} - T_{ju}T_{bc})(T_{re} + Y_rT_e - Y_eT_r) \\
- &\quad (2T_{is}(T_{jc}T_{br} - 2T_{jr}T_{bc}) + T_{ir}(T_{js}T_{bc} - T_{jc}T_{bs}) + T_{ic}(T_{jr}T_{bs} - T_{js}T_{br}))(T_{ue} + Y_uT_e - Y_eT_u) \\
+ &\quad (T_{jr}T_{bs} - T_{js}T_{br})(T_{ic}T_{ue} - T_{iu}T_{ce} + T_{ie}T_{cu}) - (T_{jc}T_{bs} - T_{js}T_{bc})(T_{ir}T_{ue} - T_{iu}T_{re} + T_{ie}T_{ru}) \\
+ &\quad 2(T_{jc}T_{br} - T_{jr}T_{bc})(T_{is}T_{ue} - T_{iu}T_{se} + T_{ie}T_{su}) + (T_{is}T_{be} + T_{ie}T_{bs})(T_{jc}T_{ru} - T_{jr}T_{cu} + T_{ju}T_{cr}) \\
- &\quad (T_{is}T_{bu} + T_{iu}T_{bs})(T_{jc}T_{re} - T_{jr}T_{ce} + T_{je}T_{cr}) - 2T_{ie}T_{br}(T_{jc}T_{su} - T_{js}T_{cu} + T_{ju}T_{cs}) \\
+ &\quad 2T_{iu}T_{br}(T_{jc}T_{se} - T_{js}T_{ce} + T_{je}T_{cs}) + 2T_{ie}T_{bc}(T_{jr}T_{su} - T_{js}T_{ru} + T_{ju}T_{rs}) \\
- &\quad 2T_{iu}T_{bc}(T_{jr}T_{se} - T_{js}T_{re} + T_{je}T_{rs}) + 2(T_{iu}T_{je} - T_{ie}T_{ju})(T_{bc}T_{rs} - T_{br}T_{cs} + T_{bs}T_{cr}) \\
- &\quad (T_{is}T_{je} - T_{ie}T_{js})(T_{bc}T_{ru} - T_{br}T_{cu} + T_{bu}T_{cr}) + (T_{is}T_{ju} - T_{iu}T_{js})(T_{bc}T_{re} - T_{br}T_{ce} + T_{be}T_{cr}) \\
- &\quad T_cT_{bu}\,|Y_rT_{is}T_{je}| + T_cT_{js}\,|Y_rT_{iu}T_{be}| - T_cT_{jr}\,|Y_sT_{iu}T_{be}| + T_cT_{ir}\,|Y_sT_{ju}T_{be}| \\
+ &\quad (Y_uT_{be} - Y_eT_{bu})\,|T_cT_{ir}T_{js}| + Y_eT_{bs}\,|T_cT_{ir}T_{ju}| - Y_uT_{bs}\,|T_cT_{ir}T_{je}| \\
- &\quad Y_eT_{bc}\,|T_rT_{is}T_{ju}| + Y_uT_{bc}\,|T_rT_{is}T_{je}| - Y_eT_{jr}\,|T_cT_{is}T_{bu}| + Y_uT_{jr}\,|T_cT_{is}T_{be}| \\
+ &\quad 2Y_eT_{jc}\,|T_rT_{is}T_{bu}| - 2Y_uT_{jc}\,|T_rT_{is}T_{be}| + (Y_uT_{ie} - Y_eT_{iu})\,|T_cT_{jr}T_{bs}| \\
- &\quad Y_eT_{ic}\,|T_rT_{js}T_{bu}| + Y_uT_{ic}\,|T_rT_{js}T_{be}| - Y_eT_s\,|T_{ic}T_{jr}T_{bu}| + Y_uT_s\,|T_{ic}T_{jr}T_{be}| \\
- &\quad 2T_{cr}\,|T_{is}T_{ju}T_{be}| + T_{br}\,|T_{is}T_{ju}T_{ce}| - T_{jr}\,|T_{is}T_{bu}T_{ce}| + T_{ie}\,|T_{jr}T_{bs}T_{cu}| \\
- &\quad T_{iu}\,|T_{jr}T_{bs}T_{ce}|\,.
\end{aligned}
$$

This shows that $S(f,g)$ is in standard form relative to G in this case.

<u>(3) $(b,e)=(i,s)$</u> In this case $S(f,g) = Y_rT_{ju}f - T_cT_{ad}g$ is also always of degree five. Its leading monomial is $Y_rT_cT_{ad}T_{iu}T_{js}$. By Buchberger's second criterion, we may assume that $r \geq c$ and $d \geq j$. This leaves us with the following possibilities:

$$
\begin{aligned}
&(i1)\, 2 \leq a < i = b < j \leq c \leq r \leq d < s = e < u \leq n \\
&(i2)\, 2 \leq a < i = b < j \leq c \leq d \leq r < s = e < u \leq n \\
&(i3)\, 2 \leq a < i = b < c \leq j \leq d \leq r < s = e < u \leq n \\
&(i4)\, 2 \leq a < i = b < c \leq j \leq r \leq d < s = e < u \leq n.
\end{aligned}
$$

For $(i1)$, $S(f,g)$ is equal to

$$
\begin{aligned}
&\quad T_{ju}(T_{ac}T_{id} - T_{ad}T_{ic})(T_{rs} + Y_rT_s - Y_sT_r) - T_{js}(T_{ac}T_{id} - T_{ad}T_{ic})(T_{ru} + Y_rT_u - Y_uT_r) \\
+ &\quad T_{jr}(T_{ac}T_{id} - T_{ad}T_{ic})(T_{su} + Y_sT_u - Y_uT_s) - T_{id}T_{ju}(T_{ac}T_{rs} - T_{ar}T_{cs} + T_{as}T_{cr})
\end{aligned}
$$

$$
\begin{aligned}
&+ T_{id}T_{js}(T_{ac}T_{ru} - T_{ar}T_{cu} + T_{au}T_{cr}) - T_{id}T_{jr}(T_{ac}T_{su} - T_{as}T_{cu} + T_{au}T_{cs}) \\
&- T_{ic}T_{ju}(T_{ar}T_{ds} - T_{ad}T_{rs} + T_{as}T_{rd}) + T_{ic}T_{js}(T_{ar}T_{du} - T_{ad}T_{ru} + T_{au}T_{rd}) \\
&+ T_{ic}T_{jr}(T_{ad}T_{su} - T_{as}T_{du} + T_{au}T_{ds}) + (T_{as}T_{ju} - T_{au}T_{js})(T_{ic}T_{rd} - T_{ir}T_{cd} + T_{id}T_{cr}) \\
&+ (T_{ar}T_{ju} - T_{au}T_{jr})(T_{ic}T_{ds} - T_{id}T_{cs} + T_{is}T_{cd}) - (T_{ar}T_{js} - T_{as}T_{jr})(T_{ic}T_{du} - T_{id}T_{cu} + T_{iu}T_{cd}) \\
&- (T_cT_{id} - T_dT_{ic})\,|Y_rT_{as}T_{ju}| - T_dT_{ac}\,|Y_rT_{is}T_{ju}| + Y_uT_{jr}\,|T_cT_{ad}T_{is}| \\
&+ (Y_rT_{js} - Y_sT_{jr})\,|T_cT_{ad}T_{iu}| - T_{cd}\,|T_{ar}T_{is}T_{ju}| - (Y_sT_{ju} - Y_uT_{js})\,|T_cT_{ar}T_{id}|. \qquad (97)
\end{aligned}
$$

This proves that $S(f,g)$ is in standard form relative to G in this case. Using elementary operations on relation (97) shows that $S(f,g)$ is also in standard form relative to G in case $(i2)$. For $(i3)$, $S(f,g)$ is equal to

$$
\begin{aligned}
&(T_{ac}T_{id}T_{ju} - T_{ad}T_{ic}T_{ju} - T_{ad}T_{iu}T_{cj})(T_{rs} + Y_rT_s - Y_sT_r) \\
&- (T_{aj}T_{id}T_{cs} - T_{ad}T_{ij}T_{cs} - T_{as}T_{id}T_{cj})(T_{ru} + Y_rT_u - Y_uT_r) \\
&- (T_{ar}T_{id}T_{cj} + T_{ad}T_{ij}T_{cr} - T_{aj}T_{id}T_{cr})(T_{su} + Y_sT_u - Y_uT_s) \\
&- (Y_rT_dT_{is} - Y_sT_rT_{id} + T_{id}T_{rs})(T_{ac}T_{ju} - T_{aj}T_{cu} + T_{au}T_{cj}) - T_{id}T_{cu}(T_{aj}T_{rs} - T_{ar}T_{js} + T_{as}T_{jr}) \\
&+ T_{id}T_{cs}(T_{aj}T_{ru} - T_{ar}T_{ju} + T_{au}T_{jr}) - T_{id}T_{cr}(T_{aj}T_{su} - T_{as}T_{ju} + T_{au}T_{js}) \\
&+ (T_{ic}T_{ju} + T_{iu}T_{cj})(T_{ad}T_{rs} - T_{ar}T_{ds} + T_{as}T_{dr}) - T_{ij}T_{cs}(T_{ad}T_{ru} - T_{ar}T_{du} + T_{au}T_{dr}) \\
&+ T_{ij}T_{cr}(T_{ad}T_{su} - T_{as}T_{du} + T_{au}T_{ds}) + T_{id}T_{cj}(T_{ar}T_{su} - T_{as}T_{ru} + T_{au}T_{rs}) \\
&+ (Y_rT_dT_{as} - Y_sT_rT_{ad} + T_{ar}T_{ds} - T_{as}T_{dr})(T_{ic}T_{ju} - T_{ij}T_{cu} + T_{iu}T_{cj}) \\
&- (T_{as}T_{cu} - T_{au}T_{cs})(T_{ij}T_{dr} - T_{id}T_{jr} + T_{ir}T_{jd}) + (T_{ar}T_{cu} - T_{au}T_{cr})(T_{ij}T_{ds} - T_{id}T_{js} + T_{is}T_{jd}) \\
&- (T_{ar}T_{cs} - T_{as}T_{cr})(T_{ij}T_{du} - T_{id}T_{ju} + T_{iu}T_{jd}) - T_{ad}(Y_sT_{iu} - Y_uT_{is})(T_cT_{jr} - T_jT_{cr} + T_rT_{cj}) \\
&+ T_{ad}(Y_rT_{iu} - Y_uT_{ir})(T_cT_{js} - T_jT_{cs} + T_sT_{cj}) \\
&- (Y_rT_{as}T_{id} - Y_sT_{ad}T_{ir})(T_cT_{ju} - T_jT_{cu} + T_uT_{cj}) \\
&- T_dT_{cj}\,|Y_rT_{as}T_{iu}| - (T_jT_{id} - T_dT_{ij})\,|Y_rT_{as}T_{cu}| - T_dT_{aj}\,|Y_rT_{is}T_{cu}| \\
&+ (Y_sT_{cu} - Y_uT_{cs})\,|T_jT_{ad}T_{ir}| + Y_uT_{cr}\,|T_jT_{ad}T_{is}| + (Y_rT_{cs} - Y_sT_{cr})\,|T_jT_{ad}T_{iu}| \\
&+ Y_uT_{cj}\,|T_dT_{ar}T_{is}| - Y_sT_{cj}\,|T_dT_{ar}T_{iu}| - T_{jd}\,|T_{ar}T_{is}T_{cu}|. \qquad (98)
\end{aligned}
$$

This shows that $S(f,g)$ is in standard form relative to G in this case. Using expression (98) with the roles of r and d interchanged, one can see that $S(f,g)$ is also in standard form relative to G in case $(i4)$. This finishes case (3).

(4) $(b,e) = (j,u)$ In this case, the degree of $S(f,g)$ is either four or five. It is four if $(i,s) = (a,d)$ in which case $\mathrm{LM}(S(f,g)) = Y_rT_sT_{ic}T_{ju}$. One has the following two possibilities:

$$(i1)\, 2 \le i = a < j = b \le c < r < s = d < u = e \le n$$
$$(i2)\, 2 \le i = a < j = b \le r < c < s = d < u = e \le n.$$

In case $(i1)$, one has

$$\begin{aligned} S(f,g) &= -(T_{ic}T_{ju} - T_{iu}T_{jc})(T_{rs} + Y_rT_s - Y_sT_r) + (T_{ic}T_{js} - T_{is}T_{jc})(T_{ru} + Y_rT_u - Y_uT_r) \\ &- (T_{ic}T_{jr} - T_{ir}T_{jc})(T_{su} + Y_sT_u - Y_uT_s) + T_{ju}(T_{ic}T_{rs} - T_{ir}T_{cs} + T_{is}T_{cr}) \\ &- T_{js}(T_{ic}T_{ru} - T_{ir}T_{cu} + T_{iu}T_{cr}) + T_{jr}(T_{ic}T_{su} - T_{is}T_{cu} + T_{iu}T_{cs}) \\ &- T_{jc}(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) - Y_u\left|T_cT_{ir}T_{js}\right| + Y_s\left|T_cT_{ir}T_{ju}\right| \\ &- \left|T_{ir}T_{js}T_{cu}\right|. \end{aligned} \tag{99}$$

This shows that $S(f,g)$ is in standard form relative to G in this case. Similarly, one can also show that $S(f,g)$ is also in standard form relative to G in case $(i2)$. So, we may safely assume that the degree of $S(f,g)$ is five. In this case, $S(f,g) = Y_rT_{is}f - T_cT_{ad}g$, and by Buchberger's second criterion, we may assume that $c \le r$ (since Y_rT_c is always a common divisor of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$). This leaves us with the following possibilities:

$$(i1)\, 1 \le i < a < b = j \le c < d \le r < s < u = e \le n$$
$$(i2)\, 1 \le i < a < b = j \le c \le r < d < s < u = e \le n$$
$$(i3)\, 1 \le i < a < b = j \le c \le r < s \le d < u = e \le n$$
$$(i4)\, 2 \le a \le i < b = j \le c < d \le r < s < u = e \le n$$
$$(i5)\, 2 \le a \le i < b = j \le c \le r < d \le s < u = e \le n$$
$$(i6)\, 2 \le a \le i < b = j \le c \le r < s \le d < u = e \le n.$$

In cases $(i1), (i2), (i3)$, the leading monomial of $S(f,g)$ is $Y_rT_cT_{is}T_{au}T_{jd}$, and the following expression of $S(f,g)$ shows that it is in standard form relative to G in case $(i1)$:

$$\begin{aligned} &- T_{iu}(T_{ac}T_{jd} - T_{ad}T_{jc})(T_{rs} + Y_rT_s - Y_sT_r) + T_{is}(T_{ac}T_{jd} - T_{ad}T_{jc})(T_{ru} + Y_rT_u - Y_uT_r) \\ &+ (T_{ic}(T_{ar}T_{jd} - T_{ad}T_{jr}) + T_{id}(T_{ac}T_{jr} - T_{ar}T_{jc}) - 2T_{ir}(T_{ac}T_{jd} - T_{ad}T_{jc}))(T_{su} + Y_sT_u - Y_uT_s) \end{aligned}$$

$$
\begin{aligned}
&+ \ (T_{ad}T_{jr} - T_{ar}T_{jd})(T_{ic}T_{su} - T_{is}T_{cu} + T_{iu}T_{cs}) - (T_{ac}T_{jr} - T_{ar}T_{jc})(T_{id}T_{su} - T_{is}T_{du} + T_{iu}T_{ds}) \\
&+ \ 2(T_{ac}T_{jd} - T_{ad}T_{jc})(T_{ir}T_{su} - T_{is}T_{ru} + T_{iu}T_{rs}) + T_{iu}T_{jr}(T_{ac}T_{ds} - T_{ad}T_{cs} + T_{as}T_{cd}) \\
&- \ T_{is}T_{jr}(T_{ac}T_{du} - T_{ad}T_{cu} + T_{au}T_{cd}) - T_{iu}T_{jd}(T_{ac}T_{rs} - T_{ar}T_{cs} + T_{as}T_{cr}) \\
&+ \ T_{is}T_{jd}(T_{ac}T_{ru} - T_{ar}T_{cu} + T_{au}T_{cr}) + T_{iu}T_{jc}(T_{ad}T_{rs} - T_{ar}T_{ds} + T_{as}T_{dr}) \\
&- \ T_{is}T_{jc}(T_{ad}T_{ru} - T_{ar}T_{du} + T_{au}T_{dr}) + (T_{is}T_{au} - T_{iu}T_{as})(T_{jc}T_{dr} - T_{jd}T_{cr} + T_{jr}T_{cd}) \\
&- \ (T_cT_{jd} - T_dT_{jc})\left|Y_rT_{is}T_{au}\right| - T_dT_{ac}\left|Y_rT_{is}T_{ju}\right| + Y_uT_{jd}\left|T_cT_{ir}T_{as}\right| \\
&- \ Y_sT_{jd}\left|T_cT_{ir}T_{au}\right| - Y_uT_{jc}\left|T_dT_{ir}T_{as}\right| + Y_sT_{jc}\left|T_dT_{ir}T_{au}\right| - Y_uT_{ad}\left|T_cT_{ir}T_{js}\right| \\
&+ \ Y_sT_{ad}\left|T_cT_{ir}T_{ju}\right| + Y_uT_{ac}\left|T_dT_{ir}T_{js}\right| - Y_sT_{ac}\left|T_dT_{ir}T_{ju}\right| + Y_rT_{iu}\left|T_cT_{ad}T_{js}\right| \\
&- \ Y_uT_r\left|T_{ic}T_{ad}T_{js}\right| + Y_sT_r\left|T_{ic}T_{ad}T_{ju}\right|. \qquad (100)
\end{aligned}
$$

Using (100), one can also show that $S(f,g)$ is in standard form relative to G in cases ($i2$) and ($i3$). For ($i4$), the leading monomial of $S(f,g)$ is $Y_rT_cT_{ad}T_{iu}T_{js}$, and one can show that in this case $S(f,g)$ is equal to the following expression:

$$
\begin{aligned}
&-(T_{au}T_{ic}T_{jd} - T_{au}T_{id}T_{jc})(T_{rs} + Y_rT_s - Y_sT_r) - T_{ar}(T_{ic}T_{jd} - T_{id}T_{jc})(T_{su} + Y_sT_u - Y_uT_s) \\
&+ \ (T_{ac}(T_{is}T_{jd} - T_{id}T_{js}) - T_{ad}(T_{is}T_{jc} - T_{ic}T_{js}))(T_{ru} + Y_rT_u - Y_uT_r) \\
&+ \ (T_{id}T_{js} - T_{is}T_{jd})(T_{ac}T_{ru} - T_{ar}T_{cu} + T_{au}T_{cr}) - (T_{ic}T_{js} - T_{is}T_{jc})(T_{ad}T_{ru} - T_{ar}T_{du} + T_{au}T_{dr}) \\
&+ \ (T_{ic}T_{jd} - T_{id}T_{jc})(T_{ar}T_{su} - T_{as}T_{ru} + T_{au}T_{rs}) + T_{au}T_{js}(T_{ic}T_{dr} - T_{id}T_{cr} + T_{ir}T_{cd}) \\
&- \ T_{ar}T_{js}(T_{ic}T_{du} - T_{id}T_{cu} + T_{iu}T_{cd}) + T_{as}T_{jd}(T_{ic}T_{ru} - T_{ir}T_{cu} + T_{iu}T_{cr}) \\
&- \ T_{as}T_{jc}(T_{id}T_{ru} - T_{ir}T_{du} + T_{iu}T_{dr}) + (T_{as}T_{iu} - T_{au}T_{is})(T_{jc}T_{dr} - T_{jd}T_{cr} + T_{jr}T_{cd}) \\
&+ \ (T_{ar}T_{is} - T_{as}T_{ir})(T_{jc}T_{du} - T_{jd}T_{cu} + T_{ju}T_{cd}) - T_dT_{ac}\left|Y_rT_{is}T_{ju}\right| \\
&+ \ (Y_sT_{ju} - Y_uT_{js})\left|T_cT_{ad}T_{ir}\right| + Y_uT_{jr}\left|T_cT_{ad}T_{is}\right| + (Y_rT_{js} - Y_sT_{jr})\left|T_cT_{ad}T_{iu}\right| \\
&- \ Y_uT_{id}\left|T_cT_{ar}T_{js}\right| + Y_sT_{id}\left|T_cT_{ar}T_{ju}\right| + Y_uT_{ic}\left|T_dT_{ar}T_{js}\right| - Y_sT_{ic}\left|T_dT_{ar}T_{ju}\right| \\
&+ \ Y_rT_{au}\left|T_cT_{id}T_{js}\right| - Y_uT_r\left|T_{ac}T_{id}T_{js}\right| - T_{cd}\left|T_{ar}T_{is}T_{ju}\right|. \qquad (101)
\end{aligned}
$$

This proves that $S(f,g)$ is in standard form relative to G in this case. For ($i5$), the leading monomial of $S(f,g)$ is also $Y_rT_cT_{ad}T_{iu}T_{js}$ and one can use (101) to show that $S(f,g)$ is also in standard form relative to G in this case.

For ($i6$), the leading monomial of $S(f,g)$ is either $Y_rT_cT_{ad}T_{iu}T_{js}$ in the case where $a < i$ or $T_rT_cT_{is}T_{iu}T_{jd}$ in the case where $a = i$. The first case ($a < i$) can be treated as in the cases ($i4$) and ($i5$). In the case $a = i$, $S(f,g)$ is equal to the following expression

$$\begin{aligned}
&-T_{is}(T_{ic}T_{ju} - T_{iu}T_{jc})(T_{rd} + Y_rT_d - Y_dT_r) + T_{is}(T_{ic}T_{jd} - T_{id}T_{jc})(T_{ru} + Y_rT_u - Y_uT_r)\\
&- \ T_{is}(T_{ic}T_{jr} - T_{ir}T_{jc})(T_{du} + Y_dT_u - Y_uT_d) + T_{is}T_{ju}(T_{ic}T_{rd} - T_{ir}T_{cd} + T_{id}T_{cr})\\
&- \ T_{is}T_{jd}(T_{ic}T_{ru} - T_{ir}T_{cu} + T_{iu}T_{cr}) + T_{is}T_{jr}(T_{ic}T_{du} - T_{id}T_{cu} + T_{iu}T_{cd})\\
&- \ T_{is}T_{jc}(T_{ir}T_{du} - T_{id}T_{ru} + T_{iu}T_{rd}) - T_cT_{iu}\,|Y_rT_{is}T_{jd}| + T_cT_{ir}\,|Y_sT_{id}T_{ju}|\\
&- \ Y_uT_{is}\,|T_cT_{ir}T_{jd}| + Y_dT_{is}\,|T_cT_{ir}T_{ju}| - T_{is}\,|T_{ir}T_{jd}T_{cu}|
\end{aligned}$$

which proves that $S(f,g)$ is in standard form relative to G in this case. This finishes case $(\mathcal{F}_8, \mathcal{F}_9)$.

Preliminaries for cases $(\mathcal{F}_3, \mathcal{F}_9)$, $(\mathcal{F}_4, \mathcal{F}_9)$, $(\mathcal{F}_7, \mathcal{F}_9)$ and $(\mathcal{F}_9, \mathcal{F}_9)$

In what follows we develop some techniques that will allow us to use the above computations to finish the proof of the cases $(\mathcal{F}_3, \mathcal{F}_9)$, $(\mathcal{F}_4, \mathcal{F}_9)$, $(\mathcal{F}_7, \mathcal{F}_9)$ and $(\mathcal{F}_9, \mathcal{F}_9)$ without computing the S-polynomials. We start with two remarks.

Remark 3.3.13. A closer look at cases "$(\mathcal{F}_4, \mathcal{F}_7)$" and "$(\mathcal{F}_7, \mathcal{F}_7)$" reveals the following important observations:

- With respect to the monomial ordering induced by $<$ on $k[T_{ij} : 1 \le i < j \le n]$, $\mathcal{F}_7$ is a Groebner basis for the ideal it generates in the ring $k[T_{ij} : 1 \le i < j \le n]$. In other words, if $f, g \in \mathcal{F}_7$, then $S(f,g) = \sum_{i=1}^{m} \phi_i h_i$ for some $\phi_i \in k[T_{ij} : 1 \le i < j \le n]$ and $h_i \in \mathcal{F}_7$ and $\phi_i h_i \le S(f,g)$ for all i.

- If $f \in \mathcal{F}_4$, $g \in \mathcal{F}_7$, then $S(f,g) = \sum_{i=1}^{x} \psi_i p_i$ for some $\psi_i \in k[T_{ij} : 1 \le i < j \le n]$ and $p_i \in \mathcal{F}_4 \cup \mathcal{F}_7$ with, of course, $\psi_i p_i \le S(f,g)$ for all i.

Remark 3.3.14. Let $(J, <)$ be a finite totally ordered set (with at least two elements), let $m = \min J$ and $M = \max J$. Let $\mathcal{R}(J)$ be the polynomial ring

$$\begin{aligned}
&k[X_m, \dots, X_M, Y_m, \dots, Y_M, T_m, \dots, T_M, T_{ij} : m \le i < j \le M]\\
&\qquad = k[\{X_i\}_{i\in J} \cup \{Y_i\}_{i\in J} \cup \{T_i\}_{i\in J} \cup \{T_{ij}\}_{i,j\in J,\, i<j}].
\end{aligned}$$

For each $i \in \{1, \ldots, 9\}$, define a family $\mathcal{F}_i(J) \subset \mathcal{R}(J)$ by replacing, in the definition of $\mathcal{F}_i$ given in the statement of Proposition 3.3.4, each occurence of "1" by "m" and each occurence of "n" by "M". Then the above computations show the following statement: Given $f \in \mathcal{F}_i(J)$ and $g \in \mathcal{F}_j(J)$, where $i, j \in \{1, \ldots, 7\}$, $S(f, g)$ is in standard form relative to $G(J) := \cup_{i=1}^{9} \mathcal{F}_i(J)$.

Now, let $J = \{1, 2, \ldots, n\}$, $J' = \{0, 1, \ldots, n\}$ and consider the two homomorphisms of k-algebras:

$$\phi : \mathcal{R}(J) \to \mathcal{R}(J'), \quad \psi : \mathcal{R}(J') \to \mathcal{R}(J)$$

where ϕ sends X_i to X_i, Y_i to T_{0i}, T_i to T_i and T_{kl} to T_{kl} for $i \geq 1$ and $1 \leq k < l \leq n$. As for ψ, it sends X_0, Y_0, T_0 to 0, X_i to X_i, Y_i to Y_i, T_i to T_i, T_{0i} to Y_i for $i \geq 1$, and T_{kl} to T_{kl} for $1 \leq k < l \leq n$. Clearly $\psi \circ \phi = \mathrm{id}_{\mathcal{R}(J)}$ and we have the following easy Lemma:

Lemma 3.3.15. *Let $\alpha, \beta \in k[T_{ij}, Y_k : 1 \leq i < j \leq n, 1 \leq k \leq n]$, $\lambda, \mu \in k[T_{ij} : 0 \leq i < j \leq n]$ be four nonzero monomials, then*

1. $\phi\,(LCM(\alpha, \beta)) = LCM(\phi(\alpha), \phi(\beta))$
2. *if $\alpha \leq \beta$, then $\phi(\alpha) \leq \phi(\beta)$*
3. *if $\lambda \leq \mu$, then $\psi(\lambda) \leq \psi(\mu)$.*

Proof

(1) is straightforward. (2) and (3) follow from the fact that $Y_i \leq Y_j$ if and only if $T_{0i} \leq T_{0j}$. ∎

As a Corollary, we have

Lemma 3.3.16. *Let $f, g \in \mathcal{F}_3(J) \cup \mathcal{F}_4(J) \cup \mathcal{F}_7(J) \cup \mathcal{F}_9(J)$ then*

$$\phi\,(S(f, g)) = S\,(\phi(f), \phi(g))\,.$$

Proof

This is a direct consequence of the definition of $S(f, g)$ and the above Lemma. ∎

Using the properties of ϕ and ψ and the above calculations, one no longer needs to carry out the computations of $S(f,g)$ in the case where f is an element of

$$\mathcal{F}_3(J) \cup \mathcal{F}_4(J) \cup \mathcal{F}_7(J) \cup \mathcal{F}_9(J)$$

and $g \in \mathcal{F}_9(J)$. The following two paragraphs explain why.

Cases $(\mathcal{F}_3, \mathcal{F}_9)$ and $(\mathcal{F}_4, \mathcal{F}_9)$

Let $f \in \mathcal{F}_3(J) \cup \mathcal{F}_4(J)$, $g \in \mathcal{F}_9(J)$, we want to show that $S(f,g)$ is in standard form relative to G. We have $\phi(f) \in \mathcal{F}_4(J')$ and $\phi(g) \in \mathcal{F}_7(J')$ so, by Remark 3.3.13 above, we have

$$S\left(\phi(f), \phi(g)\right) = \sum_{i=1}^{u} \Delta_i \rho_i \tag{102}$$

where $\Delta_i \in k[T_{ij} : 0 \leq i < j \leq n]$ and $\rho_i \in \mathcal{F}_4(J') \cup \mathcal{F}_7(J')$ and

$$\Delta_i \rho_i \leq S((\phi(f), \phi(g)),\ i = 1, \ldots, u. \tag{103}$$

Applying ψ to relation (102), we get (by Lemma 3.3.15):

$$S(f,g) = \sum_{i=1}^{u} \psi(\Delta_i)\psi(\rho_i) \tag{104}$$

with $\psi(\Delta_i) \in k[Y_1, \ldots, Y_n, T_{ij} : 1 \leq i < j \leq n]$ and $\psi(\rho_i) \in \mathcal{F}_3(J) \cup \mathcal{F}_4(J) \cup \mathcal{F}_7(J) \cup \mathcal{F}_9(J)$. Moreover, Lemma 3.3.15 applied to relation (103) gives that

$$\psi(\Delta_i)\psi(\rho_i) \leq S(f,g),\ i = 1, \ldots, u. \tag{105}$$

Now relations (104) and (105) show that $S(f,g)$ is in standard form relative to G.

Cases $(\mathcal{F}_7, \mathcal{F}_9)$ and $(\mathcal{F}_9, \mathcal{F}_9)$

Next, let $f \in \mathcal{F}_7(J) \cup \mathcal{F}_9(J)$, $g \in \mathcal{F}_9(J)$, we want to show that $S(f,g)$ is in standard form relative to G. Since both $\phi(f), \phi(g)$ are elements of $\mathcal{F}_7(J')$, Remark 3.3.13 gives that

$$S(\phi(f), \phi(g)) = \sum_{i=1}^{m} \nabla_i h_i \tag{106}$$

for some $\nabla_i \in k[T_{ij} : 0 \le i < j \le n]$ and $h_i \in \mathcal{F}_7(J')$ satisfying

$$\nabla_i h_i \le S((\phi(f), \phi(g)),\ i = 1, \dots, m. \tag{107}$$

Applying ψ to equation (106) gives that

$$S(f,g) = \sum_{i=1}^{m} \psi(\nabla_i)\psi(h_i) \tag{108}$$

where $\psi(\nabla_i)$ is now an element of $k[Y_i, T_{kl} : 1 \le k < l \le n]$ and $\psi(h_i)$ is in $\mathcal{F}_7(J) \cup \mathcal{F}_9(J)$. Now part (3) of Lemma 3.3.15 applied to relation (107) gives

$$\psi(\nabla_i)\psi(h_i) \le S(f,g),\ i = 1, \dots, m. \tag{109}$$

Relations (108) and (109) show that $S(f,g)$ is in standard form relative to G.

This finishes the proof of Proposition 3.3.4.

Remark 3.3.17. Most computer algebra systems give a *reduced* Groebner basis for an ideal I in the sense that if G is the Groebner basis, then for all $g \in G$, no monomial of g lies in $\langle \mathrm{LT}(G - \{g\}) \rangle$. Clearly, the Groebner basis given in Proposition 3.3.4 is not reduced, and so it is different from a Groebner basis given by a computer algebra system for a fixed value of the integer n.

Proof of Theorem 3.3.3

In this proof, the letters X, Y, T and L will stand repectively for the sets $\{X_i, 1 \le i \le n\}$, $\{Y_i, 1 \le i \le n\}$, $\{T_i, T_{jk}, 1 \le i \le n, 1 \le j < k \le n\}$ and $\{L_{ij}, 1 \le i < j \le n\}$, where $L_{ij} = X_i Y_j - X_j Y_i$.

Let $A_0 = k[X, L]$, then $A_0 \subseteq \ker D$ and Proposition 2.1.4 implies that $(A_0)_{X_i} = (\ker D)_{X_i}$ for each $i \in \{1, \dots, n\}$. By Proposition 2.1.2, it is enough to show that $X_1 k[X, Y] \cap A_0 = X_1 A_0$ (since this would mean that $\ker D = A_0$). We only prove that $X_1 k[X, Y] \cap A_0 \subseteq X_1 A_0$, the other inclusion being clear. So let $x \in X_1 k[X, Y] \cap A_0$ and choose $z \in k[X, Y]$, $\Phi \in k[T]$ such that $x = \Phi(X, L) = X_1 z$. This means that Φ is in the kernel of the homomorphism

$$\theta : k[T] \to k[X, L] \hookrightarrow k[X, Y] \to k[X, Y]/(X_1)$$

sending T_i to $\overline{X_i}$ and T_{jk} to $\overline{L_{jk}}$ for $1 \leq i \leq n$ and $1 \leq i < j \leq n$, where $\overline{w}$ means $w + X_1 k[X, Y]$ for $w \in k[X, Y]$. Also, consider the homomorphism

$$\kappa : k[X, Y, T] \xrightarrow{\sigma} k[X, Y] \xrightarrow{\pi} k[X, Y]/(X_1)$$

where π is the canonical epimorphism and σ is the homomorphism sending X_i to X_i, Y_i to Y_i, T_i to X_i and T_{jk} to L_{jk} for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. It is clear that θ is the restriction of κ to $k[T]$ and hence

$$\ker \theta = \ker \kappa \cap k[T]. \tag{110}$$

We claim that $\ker \kappa$ is the ideal I (considered above) of $k[X, Y, T]$ generated by the elements

$$X_1, T_i - X_i, T_{jk} - L_{jk},\ 1 \leq i \leq n,\ 1 \leq j < k \leq n.$$

Indeed, let $N = \frac{n(n+5)}{2}$, and let $\Gamma = (\gamma_1, \dots, \gamma_N)$ be the N-tuple

$$(X_1, \dots, X_n, Y_1, \dots, Y_n, T_1 - X_1, \dots, T_n - X_n, T_{12} - L_{12}, \dots, T_{n,n-1} - L_{n,n-1}),$$

then Γ is clearly a coordinate system of $k[X, Y, T]$ (that is $k[X, Y, T] = k[\gamma_1, \dots, \gamma_N]$). Let $\lambda : k[\gamma_1, \dots, \gamma_N] \to k[\gamma_2, \dots, \gamma_{2n}]$ be the homomorphism of k-algebras defined by the following commutative diagram

$$\begin{array}{lcl} k[X, Y, T] & \xrightarrow{\kappa} & k[X, Y]/(X_1) \cong k[\gamma_1, \dots, \gamma_{2n}]/(\gamma_1) \\ \downarrow \cong & & \downarrow \cong \\ k[\Gamma] & \xrightarrow{\lambda} & k[\gamma_2, \dots, \gamma_{2n}]. \end{array}$$

So

$$\lambda(\gamma_i) = \begin{cases} \gamma_i & \text{if } 2 \leq i \leq 2n \\ 0 & \text{if } i = 1 \text{ or } i > 2n \end{cases}$$

This means that $\ker \lambda = \ker \kappa = \langle \gamma_1, \gamma_{2n+1}, \gamma_{2n+2}, \cdots, \gamma_N \rangle = I$, and the claim is proved.

Since $G = \cup_{i=1}^{9} \mathcal{F}_i$ is a Groebner basis for the ideal I, the elimination theory (Theorem 3.1.28) together with (110) imply in particular that the set

$$\mathcal{H} := \{T_1\} \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$$

generates $\ker \theta$ as an ideal of $k[T]$ and hence

$$\Phi = \sum (\xi_i h_i(T)) + T_1 \rho(T) \tag{111}$$

for $\xi_i, \rho \in k[T]$ and $h_i \in \cup_{i=4}^{8} \mathcal{F}_i$. On the other hand, one can easily verify the following identities

$L_{ij}L_{kl} - L_{ik}L_{jl} + L_{il}L_{jk} = 0,\ 1 \leq i < j < k < l \leq n$

$X_iL_{1j} - X_jL_{1i} = X_1L_{ij} \in X_1A_0,\ 2 \leq i < j \leq n$

$X_iL_{jk} - X_jL_{ik} + X_kL_{jk} = 0,\ 1 \leq i < j < k \leq n$

$$\begin{vmatrix} L_{rj} & L_{rp} & L_{rq} \\ L_{sj} & L_{sp} & L_{sq} \\ L_{kj} & L_{kp} & L_{kq} \end{vmatrix} = 0,\ 1 \leq r < s < k \leq j < p < q \leq n$$

$$\begin{vmatrix} X_j & X_p & X_q \\ L_{sj} & L_{sp} & L_{sq} \\ L_{kj} & L_{kp} & L_{kq} \end{vmatrix} = 0,\ 2 \leq s < k \leq j < p < q \leq n.$$

This means that $x = \Phi(X, L) \in X_1A_0$, and the theorem is proved. ∎

3.4 A "nonlinear" example

Let D be a $k[X_1, \dots, X_r]$-elementary monomial derivation of $K[X_1, \dots, X_r, Y_1, \dots, Y_s]$ and assume that $\ker D$ is a finitely generated k-algebra. One can ask the following
Question Can we find a set of generators for $\ker D$ consisting of linear elements in the Y_i's?

All the results and examples we have seen so far answer this question positively. In this section we will prove that this is not always the case. Namely, we will prove the following

Theorem 3.4.1. *The kernel of the derivation*

$$D = X_1^2\frac{\partial}{\partial Y_1} + X_2^2\frac{\partial}{\partial Y_2} + X_3^2\frac{\partial}{\partial Y_3} + X_2X_3\frac{\partial}{\partial Y_4}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4] \cong k^{[7]}$ *is a finitely generated k-algebra which cannot be generated over* $k[X_1, X_2, X_3]$ *by linear forms in the* Y_i*'s.*

As in the previous section, we will use the theory of Groebner bases to prove the finite generation of $\ker D$.

Consider the following elements of $\ker D$

$$L_{12} = X_1^2Y_2 - X_2^2Y_1 \qquad L_{13} = X_1^2Y_3 - X_3^2Y_1$$
$$L_{14} = X_1^2Y_4 - X_2X_3Y_1 \qquad L_{24} = X_2Y_4 - X_3Y_2$$
$$L_{34} = X_3Y_4 - X_2Y_3$$
$$f = X_1^2Y_4^2 - X_1^2Y_2Y_3 + X_3^2Y_1Y_2 + X_2^2Y_1Y_3 - 2X_2X_3Y_1Y_4.$$

We will prove that $\ker D = k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}, f]$. For this, let $k[X, Y, T]$ denote the polynomial ring

$$k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4, T_1, T_2, T_3, T_4, T_{12}, T_{13}, T_{14}, T_{24}, T_{34}]$$

in 16 variables and let I be the ideal of $k[X, Y, T]$ generated by the elements

$$T_1 - X_1, T_2 - X_2, T_3 - X_3, T_4 - f, T_{12} - L_{12}, T_{13} - L_{13},$$
$$T_{14} - L_{14}, T_{24} - L_{24}, T_{34} - L_{34}, X_1.$$

Lemma 3.4.2. *A Groebner basis for I with respect to the lexicographic order on $k[X, Y, T]$ with*

$$X_1 > X_2 > X_3 > Y_1 > \ldots > Y_4 > T_1 > \ldots > T_4 > T_{12} > T_{13} > T_{14} > T_{24} > T_{34}$$

is given by the elements

$g_1 = -T_2 + X_2$

$g_2 = -T_3 + X_3$

$g_3 = X_1$

$g_4 = Y_1T_2^2 + T_{12}$

$g_5 = Y_1T_3^2 + T_{13}$

$g_6 = Y_1T_2T_3 + T_{14}$

$g_7 = T_1$

$g_8 = -Y_4T_2 + T_{24} + T_3Y_2$

$g_9 = Y_3T_2 - Y_4T_3 + T_{34}$

$g_{10} = Y_2T_{13} + Y_3T_{12} - 2Y_4T_{14} + T_4$

$g_{11} = -T_3T_{12} + T_{14}T_2$

$g_{12} = T_2T_{13} - T_3T_{14}$

$g_{13} = T_4 + Y_1T_3T_{24} + Y_3T_{12} - Y_4T_{14}$

$g_{14} = -Y_2T_{14} + Y_1T_2T_{24} + Y_4T_{12}$

$g_{15} = Y_1T_2T_{34} - Y_3T_{12} + Y_4T_{14}$

$g_{16} = -Y_3T_{14} + Y_1T_3T_{34} + Y_4T_{13}$

$g_{17} = T_3Y_3T_{12} - T_3Y_4T_{14} + T_{14}T_{34}$

$g_{18} = Y_3T_{12}T_{34} + Y_3T_{14}T_{24} - Y_4T_{13}T_{24} - Y_4T_{14}T_{34} + T_4T_{34}$

$g_{19} = -T_{14}^2 + T_{12}T_{13}$

$g_{20} = -T_{14}T_{34} + T_3T_4 - T_{13}T_{24}$

$g_{21} = T_2T_4 - T_{14}T_{24} - T_{12}T_{34}$

$g_{22} = -T_{13}Y_4T_3 + T_{13}T_{34} + Y_3T_3T_{14}$

$g_{23} = Y_1T_{24}^2 - Y_2Y_3T_{12} - Y_2T_4 + Y_4^2T_{12}$

$g_{24} = Y_1T_{24}T_{34} + Y_3Y_2T_{14} + Y_4T_4 - Y_4^2T_{14}$

$g_{25} = T_{14}^2Y_2 - 2Y_4T_{14}T_{12} + T_4T_{12} + Y_3T_{12}^2$

$g_{26} = Y_1T_{34}^2 + Y_3^2T_{12} - 2Y_3Y_4T_{14} + Y_4^2T_{13}$

$g_{27} = T_{34}Y_2T_{14} - T_{34}Y_4T_{12} - T_{24}Y_3T_{12} + T_{24}Y_4T_{14}$
$g_{28} = T_{13}Y_3T_{14}T_{24} + Y_3T_{34}T_{14}^2 - Y_4T_{13}^2T_{24} - T_{13}Y_4T_{14}T_{34} + T_{13}T_4T_{34}.$

Proof

We start the proof by showing that $\{g_1, \dots, g_{28}\}$ is a set of generators for the ideal I. For this we use the following notation:

$$[h_1, \dots, h_{10}]$$

to denote the element

$$\begin{aligned} z &= h_1(T_1 - X_1) + h_2(T_2 - X_2) + h_3(T_3 - X_3) \\ &+ h_4(T_4 - f) + h_5(T_{12} - L_{12}) + h_6(T_{13} - L_{13}) \\ &+ h_7(T_{14} - L_{14}) + h_8(T_{24} - L_{24}) + h_9(T_{34} - L_{34}) + h_{10}X_1 \end{aligned}$$

of I ($h_i \in k[X, Y, T]$ for all i).

The fact that the g_i's are in I is given by the following identities:

$g_1 = [0, -1, 0, 0, 0, 0, 0, 0, 0, 0]$

$g_2 = [0, 0, -1, 0, 0, 0, 0, 0, 0, 0]$

$g_3 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1]$

$g_4 = [-X_1Y_2, X_2Y_1 + Y_1T_2, 0, 0, 1, 0, 0, 0, 0, Y_2T_1]$

$g_5 = [-X_1Y_3, 0, X_3Y_1 + Y_1T_3, 0, 0, 1, 0, 0, 0, Y_3T_1]$

$g_6 = [-X_1Y_4, X_3Y_1, Y_1T_2, 0, 0, 0, 1, 0, 0, Y_4T_1]$

$g_7 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$

$g_8 = [0, -Y_4, Y_2, 0, 0, 0, 0, 1, 0, 0]$

$g_9 = [0, Y_3, -Y_4, 0, 0, 0, 0, 0, 1, 0]$

$g_{10} = [X_1Y_4^2 - X_1Y_3Y_2, 0, 0, 1, Y_3, Y_2, -2Y_4, 0, 0, Y_3Y_2T_1 - Y_4^2T_1]$

$g_{11} = [T_3Y_2X_1 - T_2X_1Y_4, -Y_1T_3X_2 - Y_1T_2T_3 + T_2X_3Y_1, Y_1T_2^2, 0, -T_3, 0, T_2, 0, 0, -T_3Y_2T_1 + T_2Y_4T_1]$

$g_{12} = [-Y_3T_2X_1 + T_3X_1Y_4, -T_3X_3Y_1, T_2X_3Y_1, 0, 0, T_2, -T_3, 0, 0, T_2Y_3T_1 - T_3Y_4T_1]$

$g_{13} = [0, Y_4X_3Y_1 - Y_1T_3Y_4, -X_3Y_1Y_2 + Y_4Y_1T_2, 1, Y_3, 0, -Y_4, Y_1T_3, 0, 0]$

$g_{14} = [0, Y_4Y_1X_2 - X_3Y_1Y_2, 0, 0, Y_4, 0, -Y_2, Y_1T_2, 0, 0]$

$g_{15} = [-X_1Y_4^2 + X_1Y_3Y_2, -Y_3X_2Y_1 + Y_4X_3Y_1, 0, 0, -Y_3, 0, Y_4, 0, Y_1T_2, -Y_3Y_2T_1 + Y_4^2T_1]$

$g_{16} = [0, -Y_3X_3Y_1 + Y_1T_3Y_3, Y_4X_3Y_1 - Y_3Y_1T_2, 0, 0, Y_4, -Y_3, 0, Y_1T_3, 0]$

$g_{17} = [-T_3X_1Y_3Y_2 + Y_4Y_3T_2X_1, Y_1T_3Y_3T_2 + Y_3T_{14} - Y_3T_2X_3Y_1 + Y_3Y_1T_3X_2, -Y_3Y_1T_2^2 - Y_4T_{14}, 0, T_3Y_3, 0, -Y_3T_2, 0, T_{14}, T_3T_1Y_3Y_2 - Y_4T_2Y_3T_1]$

$g_{18} = [0, Y_3^2T_{12} + 2Y_4(-Y_3T_{14} - Y_3Y_1T_3X_2 - Y_1T_3Y_3T_2 + Y_3T_2X_3Y_1 + Y_4T_{13} + Y_4T_3X_3Y_1) - Y_4^2T_{13} + Y_3T_4 - 2Y_3Y_1T_2X_3Y_4 + 2Y_3Y_4T_3X_2Y_1 + 2Y_3Y_4Y_1T_2T_3 - Y_3T_3Y_2X_3Y_1 - Y_4^2T_3X_3Y_1, 2Y_4(Y_4T_{14} + Y_3Y_1T_2^2 - Y_2T_{13} - Y_1T_2X_3Y_4) - Y_4^2T_{14} - Y_4T_{12}Y_3 - Y_4T_4 + Y_3Y_2T_{14} + Y_3Y_1T_2X_3Y_2 - 2Y_3Y_1T_2^2Y_4 + Y_1T_2X_3Y_4^2 + Y_4Y_2T_{13}, -Y_3T_2 + Y_4T_3, -Y_3^2T_2 + Y_4T_3Y_3, Y_4(-Y_4T_2 + T_3Y_2), 2Y_4(Y_3T_2 - Y_4T_3) - Y_3T_3Y_2 + Y_4^2T_3, -Y_4T_{13} + Y_3T_{14}, Y_3T_{12} + T_4 - Y_4T_{14}, 0]$

$g_{19} = [-Y_3T_{12}X_1 + T_{14}X_1Y_4 + Y_1T_3^2Y_2X_1 - Y_1T_3T_2X_1Y_4, -T_{14}X_3Y_1 - Y_1^2T_3^2X_2 - Y_1^2T_3^2T_2 + Y_1^2T_3T_2X_3, T_{12}Y_1X_3 + T_{12}Y_1T_3 - T_{14}Y_1T_2 + Y_1^2T_3T_2^2, 0, -Y_1T_3^2, T_{12}, -T_{14} + Y_1T_2T_3, 0, 0, T_{12}Y_3T_1 - T_{14}Y_4T_1 - Y_1T_3^2Y_2T_1 + Y_1T_3T_2Y_4T_1]$

$g_{20} = [0, -Y_3T_{14} - Y_3Y_1T_3X_2 - Y_1T_3Y_3T_2 + Y_3T_2X_3Y_1 + Y_4T_{13} + Y_4T_3X_3Y_1, Y_4T_{14} + Y_3Y_1T_2^2 - Y_2T_{13} - Y_1T_2X_3Y_4, T_3, 0, -Y_4T_2 + T_3Y_2, Y_3T_2 - Y_4T_3, -T_{13}, -T_{14}, 0]$

$g_{21} = [0, Y_1T_2X_3Y_4 - Y_4T_3X_2Y_1 - Y_4Y_1T_2T_3 + T_3Y_2X_3Y_1 + Y_4T_{14} - Y_3T_{12}, -Y_1T_2X_3Y_2 + Y_1T_2^2Y_4 - Y_2T_{14} + Y_4T_{12}, T_2, Y_3T_2 - Y_4T_3, 0, -Y_4T_2 + T_3Y_2, -T_{14}, -T_{12}, 0]$

$g_{22} = [Y_3X_1(Y_3T_2 - Y_4T_3), T_{13}Y_3 + Y_3T_3X_3Y_1, -Y_4T_{13} - Y_3T_2X_3Y_1, 0, 0, -Y_3T_2, T_3Y_3, 0, T_{13}, -Y_3T_1(Y_3T_2 - Y_4T_3)]$

$g_{23} = [0, -Y_1T_{24}Y_4 - 2Y_2Y_4X_3Y_1 + Y_2Y_1T_3Y_4 + Y_4^2Y_1X_2, Y_1T_{24}Y_2 + X_3Y_1Y_2^2 - Y_2Y_4Y_1T_2, -Y_2, -Y_2Y_3 + Y_4^2, 0, 0, Y_1T_{24} - Y_1Y_2T_3 + Y_4Y_1T_2, 0, 0]$

$g_{24} = [0, Y_1T_{24}Y_3 + Y_4^2X_3Y_1 - Y_1T_3Y_4^2 - Y_3Y_4Y_1X_2 + Y_3X_3Y_1Y_2, -Y_1T_{24}Y_4 - Y_2Y_4X_3Y_1 + Y_4^2Y_1T_2, Y_4, 0, 0, Y_2Y_3 - Y_4^2, Y_1T_3Y_4 - Y_3Y_1T_2, Y_1T_{24}, 0]$

$g_{25} = [Y_1T_{24}X_1(-Y_4T_2+T_3Y_2), -Y_1^2T_{24}T_3X_2 - Y_1^2T_{24}T_2T_3 + Y_1^2T_{24}T_2X_3 + Y_4T_{12}Y_1X_3 - Y_4T_{12}Y_1T_3 - T_{14}Y_4Y_1X_2 + T_{14}X_3Y_1Y_2, Y_1^2T_{24}T_2^2 - T_{12}X_3Y_1Y_2 + T_{12}Y_4Y_1T_2, T_{12}, -Y_1T_3T_{24} + Y_3T_{12} - Y_4T_{14}, 0, Y_1T_2T_{24} - Y_4T_{12} + Y_2T_{14}, T_{12}Y_1T_3 - T_{14}Y_1T_2, 0, -Y_1T_{24}T_1(-Y_4T_2+T_3Y_2)]$

$g_{26} = [-Y_3X_1(Y_2Y_3 - Y_4^2), Y_1T_{34}Y_3 + Y_3^2X_2Y_1 - 2Y_3Y_4X_3Y_1 + Y_4Y_1T_3Y_3, -Y_1T_{34}Y_4 + Y_4^2X_3Y_1 - Y_4Y_3Y_1T_2, 0, Y_3^2, Y_4^2, -2Y_4Y_3, 0, Y_1T_{34} - Y_3Y_1T_2 + Y_1T_3Y_4, Y_3T_1(Y_2Y_3 - Y_4^2)]$

$g_{27} = [T_{24}X_1(Y_2Y_3 - Y_4^2), -Y_1T_{34}X_2Y_4 + Y_1T_{34}X_3Y_2 - T_{24}Y_3X_2Y_1 + T_{24}Y_4X_3Y_1, 0, 0, - T_{34}Y_4 - T_{24}Y_3, 0, T_{34}Y_2 + T_{24}Y_4, -Y_1T_2T_{34}, Y_1T_2T_{24}, -T_{24}T_1(Y_2Y_3 - Y_4^2)]$

$g_{28} = [-Y_3T_{34}X_1(-Y_3T_{12} + Y_4T_{14} + Y_1Y_2T_3^2 - Y_4Y_1T_2T_3), T_{13}Y_3T_4 - T_{13}Y_3T_3Y_2X_3Y_1 + T_{13}Y_3^2T_{12} + Y_4^2T_{13}^2 + T_{13}Y_4^2T_3X_3Y_1 + Y_3T_{34}T_{14}X_3Y_1 + Y_3T_{34}Y_1^2T_3^2X_2 + Y_3T_{34}Y_1^2T_3^2T_2 - Y_3T_{34}Y_1^2T_3T_2X_3 - 2Y_4T_{13}Y_3T_{14}, -Y_4T_{13}T_4 + Y_4^2T_{13}T_{14} + T_{13}Y_3Y_1T_2X_3Y_2 + T_{13}Y_3Y_2T_{14} - Y_4T_{13}Y_3T_{12} - Y_4Y_2T_{13}^2 - T_{13}Y_1T_2X_3Y_4^2 - Y_3T_{34}T_{12}Y_1X_3 - Y_1T_{34}T_3Y_3T_{12} + Y_3T_{34}T_{14}Y_1T_2 - Y_3T_{34}Y_1^2T_3T_2^2, -Y_3T_2T_{13} + T_{13}Y_4T_3, -T_{13}Y_3^2T_2 + T_{13}Y_3Y_4T_3 + Y_3T_{34}Y_1T_3^2, -Y_4^2T_{13}T_2 + Y_4T_{13}T_3Y_2 - Y_3T_{12}T_{34}, -T_3T_{13}Y_2Y_3 - Y_4^2T_{13}T_3 + Y_3T_{34}T_{14} - Y_3T_{34}Y_1T_2T_3 + 2Y_4T_{13}Y_3T_2, T_{13}Y_3T_{14} - Y_4T_{13}^2, T_{13}T_4 - Y_4T_{13}T_{14} + T_{13}Y_3T_{12}, Y_3T_{34}T_1(-Y_3T_{12} + Y_4T_{14} + Y_1Y_2T_3^2 - Y_4Y_1T_2T_3)]$.

The following identities prove now that that $I \subseteq < g_1, \dots, g_{28} >$:

$T_1 - X_1 = -g_3 + g_7$

$T_2 - X_2 = -g_1$

$T_3 - X_3 = -g_2$

$T_4 - f = (-Y_3X_2Y_1 + 2Y_4X_3Y_1 - Y_3Y_1T_2)g_1 + (-X_3Y_1Y_2 - Y_1Y_2T_3 + 2Y_4Y_1T_2)g_2 + (X_1Y_3Y_2 - X_1Y_4^2)g_3 - Y_3g_4 - Y_2g_5 + 2Y_4g_6 + g_{10}$

$T_{12} - L_{12} = (X_2Y_1 + Y_1T_2)g_1 - X_1Y_2g_3 + g_4$

$T_{13} - L_{13} = (X_3Y_1 + Y_1T_3)g_2 - X_1Y_3g_3 + g_5$

$T_{14} - L_{14} = X_3Y_1g_1 + Y_1T_2g_2 - X_1Y_4g_3 + g_6$

$T_{24} - L_{24} = -Y_4g_1 + Y_2g_2 + g_8$

$T_{34} - L_{34} = Y_3g_1 - Y_4g_2 + g_9$.

This shows that $\{g_1, \dots, g_{28}\}$ is a set of generators for the ideal I. Next we prove that it is in fact a Groebner basis for I with respect to the monomial ordering on $k[X, Y, T]$ defined above. Let Γ be the set of all pairs (i, j) such that $1 \leq i < j \leq 28$

and let $S_{i,j}$ be the S-polynomial of g_i and g_j. We need to prove that $S_{i,j}$ is in standard form relative to $\{g_k, 1 \leq k \leq 28\}$ for all (i,j) in Γ. To do this we use Buchberger's first criterion to eliminate the cases where $\mathrm{LM}(g_i)$, $\mathrm{LM}(g_j)$ are relatively prime. Also we use Buchberger's second criterion to avoid computing $S(g_i, g_j)$ in the case where there exists l such that $\mathrm{LM}(g_l)$ divides $\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))$ and $S_{l,i}$, $S_{l,j}$ are both in standard form relative to $\{g_1, \ldots, g_{28}\}$. In the remaining cases, $S_{i,j}$ is computed and shown to be in standard form relative to $\{g_k, 1 \leq k \leq 28\}$ as indicated in the following identities:

$S_{4,5} = T_3^2T_{12} - T_2^2T_{13} = -T_3g_{11} - T_2g_{12}$

$S_{4,6} = T_3T_{12} - T_2T_{14} = -g_{11}$

$S_{4,9} = Y_1Y_4T_2T_3 - Y_1T_2T_{34} + Y_3T_{12} = Y_4g_6 - g_{15}$

$S_{4,11} = Y_1T_2T_3T_{12} + T_{12}T_{14} = T_{12}g_6$

$S_{4,12} = Y_1T_2T_3T_{14} + T_{12}T_{13} = T_{14}g_6 + g_{19}$

$S_{4,13} = -Y_3T_2^2T_{12} + Y_4T_2^2T_{14} - T_2^2T_4 + T_3T_{12}T_{24} = -T_2T_{12}g_9 + (Y_4T_2 - T_{24})g_{11} - T_2g_{21}$

$S_{4,14} = Y_2T_2T_{14} - Y_4T_2T_{12} + T_{12}T_{24} = T_{12}g_8 + Y_2g_{11}$

$S_{4,15} = Y_3T_2T_{12} - Y_4T_2T_{14} + T_{12}T_{34} = T_{12}g_9 - Y_4g_{11}$

$S_{4,16} = Y_3T_2^2T_{14} - Y_4T_2^2T_{13} + T_3T_{12}T_{34} = T_2T_{14}g_9 - T_{34}g_{11} - Y_4T_2g_{12}$

$S_{4,21} = Y_1T_2T_{14}T_{24} + Y_1T_2T_{12}T_{34} + T_4T_{12} = Y_1T_{24}g_{11} + T_{12}g_{13} + T_{12}g_{15}$

$S_{4,23} = Y_2Y_3T_2^2T_{12} + Y_2T_2^2T_4 - Y_4^2T_2^2T_{12} + T_{12}T_{24}^2 = (Y_4T_2T_{12} + T_{12}T_{24})g_8 + Y_2T_2T_{12}g_9 + Y_2T_{24}g_{11} + Y_2T_2g_{21}$

$S_{4,24} = -Y_2Y_3T_2^2T_{14} + Y_4^2T_2^2T_{14} - Y_4T_2^2T_4 + T_{12}T_{24}T_{34} = (-Y_4T_2T_{14} + T_{12}T_{34})g_8 - Y_2T_2T_{14}g_9 + Y_2T_{34}g_{11} - Y_4T_2g_{21}$

$S_{4,26} = -Y_3^2T_2^2T_{12} + 2Y_3Y_4T_2^2T_{14} - Y_4^2T_2^2T_{13} + T_{12}T_{34}^2 = (-Y_3T_2T_{12} + 2Y_4T_2T_{14} - Y_4T_3T_{12} + T_{12}T_{34})g_9 + (Y_4^2T_3 - 2Y_4T_{34})g_{11} - Y_4^2T_2g_{12}$

$S_{5,6} = T_2T_{13} - T_3T_{14} = g_{12}$

$S_{5,8} = Y_1Y_4T_2T_3 - Y_1T_3T_{24} + Y_2T_{13} = Y_4g_6 + g_{10} - g_{13}$

$S_{5,13} = -Y_3T_3T_{12} + Y_4T_3T_{14} - T_3T_4 + T_{13}T_{24} = -g_{17} - g_{20}$

$S_{5,14} = Y_2T_3^2T_{14} - Y_4T_3^2T_{12} + T_2T_{13}T_{24} = T_3T_{14}g_8 + Y_4T_3g_{11} + T_{24}g_{12}$

$S_{5,15} = Y_3T_3^2T_{12} - Y_4T_3^2T_{14} + T_2T_{13}T_{34} = T_{34}g_{12} + T_3g_{17}$

$S_{5,16} = Y_3T_3T_{14} - Y_4T_3T_{13} + T_{13}T_{34} = g_{22}$
$S_{5,17} = -Y_1T_3T_{14}T_{34} + Y_3T_{12}T_{13} + Y_1Y_4T_3^2T_{14} = Y_4T_{14}g_5 - T_{14}g_{16} + Y_3g_{19}$
$S_{5,20} = Y_1T_3T_{13}T_{24} + Y_1T_3T_{14}T_{34} + T_4T_{13} = T_{13}g_{13} + T_{14}g_{16} - Y_3g_{19}$
$S_{5,22} = Y_1Y_4T_3^2T_{13} - Y_1T_3T_{13}T_{34} + Y_3T_{13}T_{14}$
$S_{5,23} = Y_2Y_3T_3^2T_{12} + Y_2T_4T_3^2 - Y_4^2T_3^2T_{12} + T_{13}T_{24}^2 = (Y_3T_3T_{12})g_8 + Y_4T_3T_{12}g_9 + Y_4T_{34}g_{11} + Y_4T_{24}g_{12} - T_{24}g_{17} + (Y_4T_2 - T_{24})g_{20}$
$S_{5,24} = -Y_2Y_3T_3^2T_{14} + Y_4^2T_3^2T_{14} - Y_4T_3^2T_4 + T_{13}T_{24}T_{34} = -Y_3T_3T_{14}g_8 - Y_4T_3T_{14}g_9 - Y_4T_3g_{20} + T_{24}g_{22}$
$S_{5,26} = -Y_3^2T_3^2T_{12} + 2Y_3Y_4T_3^2T_{14} - Y_4^2T_3^2T_{13} + T_{13}T_{34}^2 = -Y_3T_3g_{17} + (Y_4T_3 + T_{34})g_{22}$

$S_{6,8} = Y_1Y_4T_2^2 - Y_1T_2T_{24} + Y_2T_{14} = Y_4g_4 - g_{14}$
$S_{6,9} = Y_1TY_4T_3^2 - Y_1T_3T_{34} + Y_3T_{14} = Y_4g_5 - g_{16}$
$S_{6,11} = Y_1T_3^2T_{12} + T_{14}^2 = T_{12}g_5 - g_{19}$
$S_{6,12} = Y_1T_3^2T_{14} + T_{13}T_{14} = T_{14}g_5$
$S_{6,13} = -Y_3T_2T_{12} + Y_4T_2T_{14} - T_2T_4 + T_{14}T_{24} = -T_{12}g_9 + Y_4g_{11} - g_{21}$
$S_{6,14} = Y_2T_3T_{14} - Y_4T_3T_{12} + T_{14}T_{24} = T_{14}g_8 + Y_4g_{11}$
$S_{6,15} = Y_3T_3T_{12} - Y_4T_3T_{14} + T_{14}T_{34} = g_{17}$
$S_{6,16} = Y_3T_2T_{14} - Y_4T_2T_{13} + T_{14}T_{34} = T_{14}g_9 - Y_4g_{12}$
$S_{6,17} = Y_1Y_4T_2T_3T_{14} - Y_1T_2T_{14}T_{34} + Y_3T_{12}T_{14} = Y_4T_{14}g_6 - Y_1T_{34}g_{11} - T_{12}g_{16} + Y_4g_{19}$
$S_{6,20} = Y_1T_2T_{13}T_{24} + Y_1T_2T_{14}T_{34} + T_4T_{14} = Y_1T_{34}g_{11} + Y_1T_{24}g_{12} + T_{14}g_{13} + T_{12}g_{16} - Y_4g_{19}$
$S_{6,21} = Y_1T_3T_{12}T_{34} + Y_1T_3T_{14}T_{24} + T_4T_{14} = T_{14}g_{13} + T_{12}g_{16} - Y_4g_{19}$
$S_{6,22} = Y_1Y_4T_2T_3T_{13} - Y_1T_2T_{13}T_{34} + Y_3T_{14}^2 = Y_4T_{13}g_6 - Y_1T_{34}g_{12} - T_{14}g_{16}$
$S_{6,23} = Y_2Y_3T_2T_3T_{12} + Y_2T_2T_3T_4 - Y_4^2T_2T_3T_{12} + T_{14}T_{24}^2 = (Y_3T_2T_{12} + T_2T_4)g_8 + (Y_4T_2T_{12} - T_{12}T_{24})g_9 + Y_4T_{24}g_{11} + (Y_4T_2 - T_{24})g_{21}$
$S_{6,24} = Y_2Y_3T_2T_3T_{14} + Y_4^2T_2T_3T_{14} - Y_4T_2T_3T_4 + T_{14}T_{24}T_{34} = -Y_3T_2T_{14}g_8 + (-Y_4T_2T_{14} + T_{14}T_{24})g_9 - Y_4T_{24}g_{12} - Y_4T_2g_{20}$
$S_{6,26} = -Y_3^2T_2T_3T_{12} + 2Y_3Y_4T_2T_3T_{14} - Y_4^2T_2T_3T_{13} + T_{14}T_{34}^2 = (-Y_3T_3T_{12} + 2Y_4T_3T_{14})g_9 - Y_4^2T_3g_{12} + (T_{34} - Y_4T_3)g_{17}$

$S_{8,10} = -Y_3T_3T_{12} - Y_4T_2T_{13} + 2Y_4T_3T_{14} - T_3T_4 + T_{13}T_{24} = -Y_4g_{12} - g_{17} - g_{20}$
$S_{8,13} = -Y_1Y_4T_2T_{24} + Y_1T_{24}^2 - Y_2Y_3T_{12} + Y_2Y_4T_{14} - Y_2T_4 = -Y_4g_{14} + g_{23}$
$S_{8,16} = -Y_1Y_4T_2T_{34} + Y_1T_{24}T_{34} + Y_2Y_3T_{14} - Y_2Y_4T_{13} = -Y_4g_{10} - Y_4g_{15} + g_{24}$
$S_{8,17} = Y_2Y_4T_3T_{14} - Y_2T_{14}T_{34} - Y_3Y_4T_2T_{12} + Y_3T_{12}T_{24} = Y_4T_{14}g_8 - Y_4T_{12}g_9 + Y_4^2g_{11} - g_{27}$
$S_{8,20} = Y_2T_{13}T_{24} + Y_2T_{14}T_{34} - Y_4T_2T_4 + T_4T_{24} = T_{24}g_{10} - Y_4g_{21} + g_{27}$
$S_{8,22} = Y_2Y_4T_3T_{13} - Y_2T_{13}T_{34} - Y_3Y_4T_2T_{14} + Y_3T_{14}T_{24} = Y_4T_{13}g_8 - Y_4T_{14}g_9 - T_{34}g_{10} + Y_4^2g_{12} + g_{18}$
$S_{8,25} = -Y_3T_3T_{12}^2 - Y_4T_2T_{14}^2 + 2Y_4T_3T_{12}T_{14} - T_3T_4T_{12} + T_{14}^2T_{24} = -Y_4T_{14}g_{11} - T_{12}g_{17} - T_{24}g_{19} - T_{12}g_{20}$
$S_{8,27} = Y_3T_3T_{12}T_{24} - Y_4T_2T_{14}T_{34} + Y_4T_3T_{12}T_{34} - Y_4T_3T_{14}T_{24} + T_{14}T_{24}T_{34} = -Y_4T_{34}g_{11} + T_{24}g_{17}$

$S_{9,11} = Y_2T_3^2T_{12} - Y_4T_2^2T_{14} + T_2T_{14}T_{24} = T_3T_{12}g_8 + (-Y_4T_2 + T_{24})g_{11}$
$S_{9,12} = Y_2T_3^2T_{14} - Y_4T_2^2T_{13} + T_2T_{13}T_{24} = T_3T_{14}g_8 + (-Y_4T_2 + T_{24})g_{12}$
$S_{9,14} = -Y_1Y_4T_2^2T_{24} + Y_1T_2T_{24}^2 + Y_2^2T_3T_{14} - Y_2Y_4T_3T_{12} = -Y_4T_{24}g_4 + Y_2T_{14}g_8 + Y_2Y_4g_{11} + T_{24}g_{14}$
$S_{9,15} = -Y_1Y_4T_2^2T_{34} + Y_1T_2T_{24}T_{34} + Y_2Y_3T_3T_{12} - Y_2Y_4T_3T_{14} = -Y_4T_{34}g_4 + (Y_3T_{12} - Y_4T_{14})g_8 + Y_4T_{12}g_9 - Y_4^2g_{11} + T_{34}g_{14} + g_{27}$
$S_{9,17} = Y_2Y_4T_3T_{14} - Y_2T_{14}T_{34} - Y_3Y_4T_2T_{12} + Y_3T_{12}T_{24} = Y_4T_{14}g_8 - Y_4T_{12}g_9 + Y_4^2g_{11} - g_{27}$
$S_{9,18} = Y_2Y_3T_3T_{14}T_{24} + Y_2Y_4T_3T_{13}T_{24} + Y_2Y_4T_3T_{14}T_{34} - Y_2T_3T_4T_{34} - Y_3Y_4T_2T_{12}T_{34} + Y_3T_{12}T_{24}T_{34} = (-Y_3T_{14}T_{24} + Y_4T_{13}T_{24} + Y_4T_{14}T_{34} - T_4T_{34})g_8 + (-Y_4T_{12}T_{34} - Y_4T_{14}T_{24})g_9 + Y_4^2T_{34}g_{11} + Y_4^2T_{24}g_{12} + T_{24}g_{18} - Y_4T_{34}g_{21}$
$S_{9,21} = Y_2T_3T_{12}T_{34} + Y_2T_3T_{14}T_{24} - Y_4T_2^2T_4 + T_2T_4T_{24} = (T_{12}T_{34} + T_{14}T_{24})g_8 + (-Y_4T_2 + T_{24})g_{21}$
$S_{9,22} = Y_2Y_4T_3T_{13} - Y_2T_{13}T_{34} - Y_3Y_4T_2T_{14} + Y_3T_{14}T_{24} = Y_4T_{13}g_8 - Y_4T_{14}g_9 - T_{34}g_{10} + Y_4^2g_{12} + g_{18}$
$S_{9,28} = -Y_2Y_3T_3T_{14}^2T_{34} + Y_2Y_4T_3T_{13}^2T_{24} + Y_2Y_4T_3T_{13}T_{14}T_{34} - Y_2T_3T_4T_{13}T_{34} - Y_3Y_4T_2T_{13}T_{14}T_{24} + Y_3T_{13}T_{14}T_{24}^2 = (-Y_3T_{14}^2T_{34} + Y_4T_{13}^2T_{24} + Y_4T_{13}T_{14}T_{34})g_8 - (Y_4T_{13}T_{14}T_{24} + Y_4T_{14}^2T_{34})g_9 + Y_4^2T_{13}T_{34}g_{11} + (Y_4^2T_{13}T_{24} - Y_4T_4T_{34})g_{12} + Y_4^2T_3T_{34}g_{19} - Y_4T_{14}T_{34}g_{20} + T_{24}g_{28}$

$S_{10,12} = T_2T_3T_{14} + Y_3T_2T_{12} - 2Y_4T_2T_{14} + T_2T_4 = T_{14}g_8 + T_{12}g_9 - Y_4g_{11} + g_{21}$
$S_{10,19} = Y_2T_{14}^2 + Y_3T_{12}^2 - 2Y_4T_{12}T_{14} + T_4T_{12} = g_{25}$
$S_{10,25} = -Y_3T_{12}^2T_{13} + Y_3T_{12}T_{14}^2 + 2Y_4T_{12}T_{13}T_{14} - 2Y_4T_{14}^3 - T_4T_{12}T_{13} + T_4T_{14}^2 = (-Y_3T_{12} + 2Y_4T_{14} - T_4)g_{19}$
$S_{10,27} = Y_3T_{12}T_{13}T_{24} + Y_3T_{12}T_{14}T_{34} + Y_4T_{12}T_{13}T_{34} - Y_4T_{13}T_{14}T_{24} - 2Y_4T_{14}^2T_{34} + T_4T_{14}T_{34} = T_{14}g_{18} + (Y_3T_{24} + Y_4T_{34})g_{19}$
$S_{10,28} = -Y_2Y_3T_{14}^2T_{34} + Y_2Y_4T_{13}^2T_{24} + Y_2Y_4T_{13}T_{14}T_{34} - Y_2T_4T_{13}T_{34} + Y_3^2T_{12}T_{14}T_{24} - 2Y_3Y_4T_{14}^2T_{24} + Y_3T_4T_{14}T_{24} = (Y_4T_{13}T_{24} + Y_4T_{14}T_{34} - T_4T_{34})g_{10} + (Y_3T_{12} - 2Y_4T_{14} + T_4)g_{18} - Y_3T_{34}g_{25}$

$S_{11,12} = -T_3T_{12}T_{13} + T_3T_{14}^2 = -T_3g_{19}$
$S_{11,14} = -Y_1T_3T_{12}T_{24} + Y_2T_{14}^2 - Y_4T_{12}T_{14} = -T_{12}g_{13} + g_{25}$
$S_{11,15} = -Y_1T_3T_{12}T_{34} + Y_3T_{12}T_{14} - Y_4T_{14}^2 = -T_{12}g_{16} + Y_4g_{19}$
$S_{11,21} = -T_3T_4T_{12} + T_{12}T_{14}T_{34} + T_{14}^2T_{24} = -T_{24}g_{19} - T_{12}g_{20}$
$S_{11,22} = -Y_3T_{12}T_3^2 + Y_4T_2T_3T_{13} - T_2T_{13}T_{34} = (Y_4T_3 - T_{34})g_{12} - T_3g_{17}$
$S_{11,27} = -Y_2T_3T_{12}T_{34} + Y_3T_2T_{12}T_{24} + Y_4T_2T_{12}T_{34} - Y_4T_2T_{14}T_{24} = -T_{12}T_{34}g_8 + T_{12}T_{24}g_9 - Y_4T_{24}g_{11}$
$S_{11,28} = -Y_3T_2T_{14}^2T_{34} - Y_3T_3T_{12}T_{13}T_{24} + Y_4T_2T_{13}^2T_{24} + Y_4T_2T_{13}T_{14}T_{34} - T_2T_4T_{13}T_{34} = -T_{14}^2T_{34}g_9 + Y_4T_{13}T_{34}g_{11} + (Y_4T_{13}T_{24} - T_4T_{34})g_{12} - T_{13}T_{24}g_{17} + Y_4T_3T_{34}g_{19} - T_{14}T_{34}g_{20}$

$S_{12,14} = -Y_1T_3T_{14}T_{24} + Y_2T_{13}T_{14} - Y_4T_{12}T_{13} = T_{14}g_{10} - T_{14}g_{13} - Y_4g_{19}$
$S_{12,15} = -Y_1T_3T_{14}T_{34} + Y_3T_{12}T_{13} - Y_4T_{13}T_{14} = -T_{14}g_{16} + Y_3g_{19}$
$S_{12,19} = T_2T_{14}^2 - T_3T_{12}T_{14} = T_{14}g_{11}$
$S_{12,21} = -T_3T_4T_{14} + T_{12}T_{13}T_{34} + T_{13}T_{14}T_{24}$
$S_{12,28} = -Y_3T_2T_{14}^2T_{34} - Y_3T_3T_{14}^2T_{24} + Y_4T_2T_{13}^2T_{24} + Y_4T_2T_{13}T_{14}T_{34} - T_2T_4T_{13}T_{34} = -T_{14}^2T_{34}g_9 + Y_4T_{13}T_{34}g_{11} + (Y_4T_{13}T_{34} - T_4T_{34})g_{12} + Y_4T_3T_{34}g_{19} - T_{14}T_{34}g_{20} - T_{14}T_{24}g_{22}$

$S_{13,14} = Y_2T_3T_{14} + Y_3T_2T_{13} - Y_4T_2T_{14} - Y_4T_3T_{12} + T_2T_4 = T_{14}g_8 + T_{12}g_9 + g_{21}$

$S_{13,15} = Y_3T_2T_{12}T_{34} + Y_3T_3T_{12}T_{24} - Y_4T_2T_{14}T_{34} - Y_4T_3T_{14}T_{24} + T_2T_4T_{34} = T_{12}T_{34}g_9 - Y_4T_{34}g_{11} + T_{24}g_{17} + T_{34}g_{21}$

$S_{13,16} = Y_3T_{12}T_{34} + Y_3T_{14}T_{24} - Y_4T_{13}T_{24} - Y_4T_{14}T_{34} + T_4T_{34} = g_{18}$

$S_{13,17} = Y_1Y_4T_3T_{14}T_{24} - Y_1T_{14}T_{24}T_{34} + Y_3^2T_{12}^2 - Y_3Y_4T_{12}T_{14} + Y_3T_4T_{12} = Y_4T_{14}g_{13} - T_{14}g_{24} + Y_3g_{25}$

$S_{13,20} = Y_1T_{13}T_{24}^2 + Y_1T_{14}T_{24}T_{34} + Y_3T_4T_{12} - Y_4T_4T_{14} + T_4^2 = (Y_3T_{12} + T_4)g_{10} - Y_4^2g_{19} + T_{13}g_{23} + T_{14}g_{24} - Y_3g_{25}$

$S_{13,22} = Y_1Y_4T_3T_{13}T_{24} - Y_1T_{13}T_{24}T_{34} + Y_3^2T_{12}T_{14} - Y_3Y_4T_{14}^2 + Y_3T_4T_{14} = Y_3T_{14}g_{10} + Y_4T_{13}g_{13} - Y_3Y_4g_{19} - T_{13}g_{24}$

$S_{13,23} = Y_2Y_3T_3T_{12} + Y_2T_3T_4 + Y_3T_{12}T_{24} - Y_4^2T_3T_{12} - Y_4T_{14}T_{24} + T_4T_{24} = (Y_3T_{12} + T_4)g_8 + Y_4T_{12}g_9 + Y_4g_{21}$

$S_{13,24} = -Y_2Y_3T_3T_{14} + Y_3T_{12}T_{34} + Y_4^2T_3T_{14} - Y_4T_3T_4 - Y_4T_{14}T_{34} + T_4T_{34} = -Y_3T_{14}g_8 - Y_4T_{14}g_9 + g_{18} - Y_4g_{20}$

$S_{13,26} = -Y_3^2T_3T_{12}T_{24} + 2Y_3Y_4T_3T_{14}T_{24} + Y_3T_{12}T_{34}^2 - Y_4^2T_3T_{13}T_{24} - Y_4T_{14}T_{34}^2 + T_4T_{34}^2 = -Y_3T_{24}g_{17} + T_{34}g_{18} + Y_4T_{24}g_{22}$

$S_{13,28} = -Y_1Y_3T_3T_{14}^2T_{34} + Y_1Y_4T_3T_{13}^2T_{24} + Y_1Y_4T_3T_{13}T_{14}T_{34} - Y_1T_3T_4T_{13}T_{34} + Y_3^2T_{12}T_{13}T_{14} - Y_3Y_4T_{13}T_{14}^2 + Y_3T_4T_{13}T_{14} = Y_4T_{13}^2g_{13} + (-Y_3T_{14}^2 + Y_4T_{13}T_{14} - T_4T_{13})g_{16} + (Y_3^2T_{14} - Y_3Y_4T_{13})g_{19}$

$S_{14,15} = -Y_2T_{14}T_{34} + Y_3T_{12}T_{24} + Y_4T_{12}T_{34} - Y_4T_{14}T_{24} = -g_{27}$

$S_{14,16} = -Y_2T_3T_{14}T_{34} + Y_3T_2T_{14}T_{24} - Y_4T_2T_{13}T_{24} + Y_4T_3T_{12}T_{34} = -T_{14}T_{34}g_8 + T_{14}T_{24}g_9 - Y_4T_{24}g_{11} - Y_4T_{24}g_{12}$

$S_{14,21} = Y_1T_{12}T_{24}T_{34} + Y_1T_{14}T_{24}^2 - Y_2T_4T_{14} + Y_4T_4T_{12} = T_{14}g_{23} + T_{12}g_{24}$

$S_{14,23} = Y_2Y_3T_2T_{12} + Y_2T_2T_4 - Y_2T_{14}T_{24} - Y_4^2T_2T_{12} + Y_4T_{12}T_{24} = Y_4T_2g_8 + Y_2T_{12}g_9 + Y_2g_{21}$

$S_{14,24} = -Y_2Y_3T_2T_{14} - Y_2T_{14}T_{34} + Y_4^2T_2T_{14} - Y_4T_2T_4 + Y_4T_{12}T_{34} = -Y_4T_{14}g_8 - Y_2T_{14}g_9 - Y_4g_{21}$

$S_{14,26} = -Y_2T_{14}T_{34}^2 - Y_3^2T_2T_{12}T_{24} + 2Y_3Y_4T_2T_{14}T_{24} - Y_4^2T_2T_{13}T_{24} + Y_4T_{12}T_{34}^2 = (-Y_3T_{12}T_{24} + 2Y_4T_{14}T_{24})g_9 - Y_4^2T_{24}g_{12} - Y_4T_{24}g_{17} - T_{34}g_{27}$

$S_{14,28} = -Y_1Y_3T_2T_{14}^2T_{34} + Y_1Y_4T_2T_{13}^2T_{24} + Y_1Y_4T_2T_{13}T_{14}T_{34} - Y_1T_2T_4T_{13}T_{34} - Y_2Y_3T_{13}T_{14}^2 + Y_3Y_4T_{12}T_{13}T_{14} = -Y_1T_{14}^2T_{34}g_9 - Y_3T_{14}^2g_{10} + Y_1Y_4T_{13}T_{34}g_{11} + ((Y_1Y_4T_{13}T_{24} - Y_1T_4T_{34})g_{12} +$

$Y_4T_{13}T_{14}g_{13} + (Y_4T_{12}T_{13} - Y_4T_{14}^2 - T_4T_{14})g_{16} + (Y_3Y_4T_{14} - Y_4^2T_{13})g_{19} + T_{14}^2g_{26}$

$S_{15,21} = Y_1T_{12}T_{34}^2 + Y_1T_{14}T_{24}T_{34} - Y_3T_4T_{12} + Y_4T_4T_{14} = -Y_4^2g_{19} + T_{14}g_{24} - Y_3g_{25} + T_{12}g_{26}$
$S_{15,26} = -Y_3^2T_2T_{12} + 2Y_3Y_4T_2T_{14} - Y_3T_{12}T_{34} - Y_4^2T_2T_{13} + Y_4T_{14}T_{34} = (-Y_3T_{12} + 2Y_4T_{14})g_9 - Y_4^2g_{12} - Y_4g_{17}$

$S_{16,17} = Y_1Y_4T_3T_{14}T_{34} - Y_1T_{14}T_{34}^2 - Y_3^2T_{12}T_{14} + Y_3Y_4T_{12}T_{13} = Y_4T_{14}g_{16} + Y_3Y_4g_{19} - T_{14}g_{26}$
$S_{16,18} = -Y_1Y_3T_3T_{14}T_{24} + Y_1Y_4T_3T_{13}T_{24} + Y_1Y_4T_3T_{14}T_{34} - Y_1T_3T_4T_{34} - Y_3^2T_{12}T_{14} + Y_3Y_4T_{12}T_{13} = (-Y_3T_{14} + Y_4T_{13})g_{13} + (Y_4T_{14} - T_4)g_{16}$
$S_{16,20} = Y_1T_{13}T_{24}T_{34} + Y_1T_{14}T_{34}^2 - Y_3T_4T_{14} + Y_4T_4T_{13} = -Y_3T_{14}g_{10} + T_{13}g_{24} + T_{14}g_{26}$
$S_{16,22} = Y_1Y_4T_3T_{13}T_{34} - Y_1T_{13}T_{34}^2 - Y_3^2T_{14}^2 + Y_3Y_4T_{13}T_{14} = Y_4T_{13}g_{16} + Y_3^2g_{19} - T_{13}g_{26}$
$S_{16,26} = -Y_3^2T_3T_{12} + 2Y_3Y_4T_3T_{14} - Y_3T_{14}T_{34} - Y_4^2T_3T_{13} + Y_4T_{13}T_{34} = -Y_3g_{17} + Y_4g_{22}$

$S_{17,18} = -Y_3T_3T_{14}T_{24} + Y_4T_3T_{13}T_{24} - T_3T_4T_{34} + T_{14}T_{34}^2 = -T_{34}g_{20} - T_{24}g_{22}$
$S_{17,19} = Y_3T_3T_{14}^2 - Y_4T_3T_{13}T_{14} + T_{13}T_{14}T_{34} = T_{14}g_{22}$
$S_{17,20} = Y_3T_{12}T_{13}T_{24} + Y_3T_{12}T_{14}T_{34} - Y_4T_3T_4T_{14} + T_4T_{14}T_{34} = T_{14}g_{18} + Y_3T_{24}g_{19} - Y_4T_{14}g_{20}$
$S_{17,22} = Y_4T_3T_{12}T_{13} - Y_4T_3T_{14}^2 - T_{12}T_{13}T_{34} + T_{14}^2T_{34} = (Y_4T_3 - T_{34})g_{19}$
$S_{17,28} = -Y_3T_3T_{12}T_{14}^2T_{34} + Y_4T_3T_{12}T_{13}^2T_{24} + Y_4T_3T_{12}T_{13}T_{14}T_{34} - Y_4T_3T_{13}T_{14}^2T_{24} - T_4T_3T_{12}T_{13}T_{34} + T_{13}T_{14}^2T_{24}T_{34} = -T_{14}^2T_{34}g_{17} + (Y_4T_3T_{13}T_{24} + Y_4T_3T_{14}T_{34} - T_3T_4T_{34})g_{19} - T_{14}^2T_{34}g_{20}$

$S_{18,19} = Y_3T_{13}T_{14}T_{24} + Y_3T_{14}^2T_{34} - Y_4T_{13}^2T_{24} - Y_4T_{13}T_{14}T_{34} + T_4T_{13}T_{34} = g_{28}$
$S_{18,24} = Y_1Y_3T_{14}T_{24}^2 - Y_1Y_4T_{13}T_{24}^2 - Y_1Y_4T_{14}T_{24}T_{34} + Y_1T_4T_{24}T_{34} - Y_2Y_3^2T_{12}T_{14} - Y_3Y_4^2T_{12}T_{14} - Y_3Y_4T_4T_{12} = -(Y_3Y_4T_{12} + Y_4T_4)g_{10} + Y_4^3g_{19} + (Y_3T_{14} - Y_4T_{13})g_{23} + (-Y_4T_{14} + T_4)g_{24} + Y_3Y_4g_{25}$
$S_{18,26} = Y_1Y_3T_{14}T_{24}T_{34} - Y_1Y_4T_{13}T_{24}T_{34} - Y_1Y_4T_{14}T_{34}^2 + Y_1T_4T_{34}^2 - Y_3^3T_{12}^2 + 2Y_3^2Y_4T_{12}T_{14} - Y_3Y_4^2T_{12}T_{13} = Y_3Y_4T_{14}g_{10} - Y_3Y_4^2g_{19} + (Y_3T_{14} - Y_4T_{13})g_{24} - Y_3^2g_{25} + (-Y_4T_{14} + T_4)g_{26}$

$S_{18,27} = Y_2Y_3T_{14}^2T_{24} - Y_2Y_4T_{13}T_{14}T_{24} - Y_2Y_4T_{14}^2T_{34} + Y_2T_4T_{14}T_{34} + Y_3Y_4T_{12}^2T_{34} + Y_3^2T_{12}^2T_{24} - Y_3Y_4T_{12}T_{14}T_{24} = -Y_4T_{14}T_{24}g_{10} + 2Y_4T_{12}g_{18} + 2Y_4^2T_{24}g_{19} + (Y_3T_{24} - Y_4T_{34})g_{25} + T_4g_{27}$

$S_{18,28} = -Y_3T_{12}T_{14}^2T_{34}^2 + Y_3T_{13}T_{14}^2T_{24}^2 + Y_4T_{12}T_{13}^2T_{24}T_{34} + Y_4T_{12}T_{13}T_{14}T_{34}^2 - Y_4T_{13}^2T_{14}T_{24}^2 - Y_4T_{13}T_{14}^2T_{24}T_{34} - T_4T_{12}T_{13}T_{34}^2 + T_4T_{13}T_{14}T_{24}T_{34} = -T_{14}^2T_{34}g_{18} + (Y_4T_{13}T_{24}T_{34} + Y_4T_{14}T_{34}^2 - T_4T_{34}^2)g_{19} + T_{14}T_{24}g_{28}$

$S_{19,28} = -Y_3T_{12}T_{14}^2T_{34} - Y_3T_{14}^3T_{24} + Y_4T_{12}T_{13}^2T_{24} + Y_4T_{12}T_{13}T_{14}T_{34} - T_4T_{12}T_{13}T_{34} = -T_{14}^2g_{18} + (Y_4T_{13}T_{24} + Y_4T_{14}T_{34} - T_4T_{34})g_{19}$

$S_{20,22} = -Y_3T_{13}T_{14}T_{24} - Y_3T_{14}^2T_{34} + Y_4T_3T_4T_{13} - T_4T_{13}T_{34} = Y_4T_{13}g_{20} - g_{28}$

$S_{22,28} = -Y_3T_3T_{14}^2T_{34} + Y_4T_3T_{13}T_{14}T_{34} - T_3T_4T_{13}T_{34} + T_{13}^2T_{24}T_{34} = -T_{13}T_{34}g_{20} - T_{14}T_{34}g_{22}$

$S_{23,24} = -Y_2Y_3T_{12}T_{34} - Y_2Y_3T_{14}T_{24} - Y_2T_4T_{34} + Y_4^2T_{12}T_{34} + Y_4^2T_{14}T_{24} - Y_4T_4T_{24} = -Y_4T_{24}g_{10} - Y_2g_{18} - Y_4g_{27}$

$S_{23,26} = -Y_2Y_3T_{12}T_{34}^2 - Y_2T_4T_{34}^2 - Y_3^2T_{12}T_{24}^2 + 2Y_3Y_4T_{14}T_{24}^2 + Y_4^2T_{12}T_{34}^2 - Y_4^2T_{13}T_{24}^2 = -Y_4T_{24}T_{34}g_{10} + (-Y_2T_{34} + Y_4T_{24})g_{18} + (Y_3T_{24} - Y_4T_{34})g_{27}$

$S_{23,28} = -Y_1Y_3T_{14}^2T_{24}T_{34} + Y_1Y_4T_{13}^2T_{24}^2 + Y_1Y_4T_{13}T_{14}T_{24}T_{34} - Y_1T_4T_{13}T_{24}T_{34} - Y_2Y_3^2T_{12}T_{13}T_{14} - Y_2Y_3T_4T_{13}T_{14} + Y_3Y_4^2T_{12}T_{13}T_{14} = (-Y_3^2T_{12}T_{14} + Y_3Y_4T_{12}T_{13} - Y_3Y_4T_{14}^2 + Y_4T_4T_{13})g_{10} + (-Y_3^2Y_4T_{12} + 3Y_3Y_4^2T_{14} - 2Y_3Y_4T_4 - Y_4^3T_{13})g_{19} + Y_4T_{13}^2g_{23} + (-Y_3T_{14}^2 + Y_4T_{13}T_{14} - T_4T_{13})g_{24} + Y_3^2T_{14}g_{25}$

$S_{24,26} = Y_2Y_3T_{14}T_{34} - Y_3^2T_{12}T_{24} + 2Y_3Y_4T_{14}T_{24} - Y_4^2T_{13}T_{24} + Y_4T_4T_{34} = Y_4g_{18} + Y_3g_{27}$

$S_{24,27} = Y_1Y_3T_{12}T_{24}^2 + Y_1Y_4T_{12}T_{24}T_{34} - Y_1Y_4T_{14}T_{24}^2 + Y_2^2Y_3T_{14}^2 - Y_2Y_4^2T_{14}^2 + Y_2Y_4T_4T_{14} = (Y_3T_{12} - Y_4T_{14})g_{23} + Y_4T_{12}g_{24} + (Y_2Y_3 - Y_4^2)g_{25}$

$S_{24,28} = -Y_1Y_3T_{14}^2T_{34}^2 + Y_1Y_4T_{13}^2T_{24}T_{34} + Y_1Y_4T_{13}T_{14}T_{34}^2 - Y_1T_4T_{13}T_{34}^2 + Y_2Y_3^2T_{13}T_{14}^2 - Y_3Y_4^2T_{13}T_{14}^2 + Y_3Y_4T_4T_{13}T_{14} = (Y_3^2T_{14}^2 - Y_3Y_4T_{13}T_{14})g_{10} + Y_3^2T_4g_{19} + Y_4T_{13}^2g_{24} + (-Y_3T_{14}^2 + Y_4T_{13}T_{14} - T_4T_{13})g_{26}$

$S_{25,27} = Y_3T_{12}^2T_{34} + Y_3T_{12}T_{14}T_{24} - Y_4T_{12}T_{14}T_{34} - Y_4T_{14}^2T_{24} + T_4T_{12}T_{34} = T_{12}g_{18} + Y_4T_{24}g_{19}$
$S_{25,28} = -Y_2Y_3T_{14}^3T_{34} + Y_2Y_4T_{13}^2T_{24} + Y_2Y_4T_{13}T_{14}^2T_{34} - Y_2T_4T_{13}T_{14}T_{34} + Y_3^2T_{12}^2T_{13}T_{24} - 2Y_3Y_4T_{12}T_{13}T_{14}T_{24} + Y_3T_4T_{12}T_{13}T_{24} = (Y_4T_{13}T_{14}T_{24} + Y_4T_{14}^2T_{34} - T_4T_{14}T_{34})g_{10} + (Y_3T_{12}T_{14} - 2Y_4T_{14}^2 + T_4T_{14})g_{18} + (Y_3^2T_{12}T_{24} - 2Y_3Y_4T_{14}T_{24} + Y_3T_4T_{34})g_{19} - Y_3T_{14}T_{34}g_{25}$

$S_{26,27} = Y_1Y_3T_{12}T_{24}T_{34} + Y_1Y_4T_{12}T_{34}^2 - Y_1Y_4T_{14}T_{24}T_{34} + Y_2Y_3^2T_{12}T_{14} - 2Y_2Y_3Y_4T_{14}^2 + Y_2Y_4^2T_{13}T_{14} = (Y_3Y_4T_{12} + Y_4^2T_{14} + Y_4T_4)g_{10} + Y_1T_{24}g_{18} - 2Y_4^3g_{19} + (-Y_3T_{14} + Y_4T_{13})g_{23} - T_4g_{24} - 2Y_3Y_4g_{25} + Y_4T_{12}g_{26}$

$S_{27,28} = -Y_2Y_3T_{14}^2T_{34}^2 + Y_2Y_4T_{13}^2T_{24}T_{34} + Y_2Y_4T_{13}T_{14}T_{34}^2 - Y_2T_4T_{13}T_{34}^2 - Y_3^2T_{12}T_{13}T_{24}^2 - Y_3Y_4T_{12}T_{13}T_{24}T_{34} + Y_3Y_4T_{13}T_{14}T_{24}^2 = (Y_4T_{13}T_{24}T_{34} + Y_4T_{14}T_{34}^2 - T_4T_{34}^2)g_{10} + (Y_3T_{12}T_{34} - Y_3T_{14}T_{24} - Y_4T_{13}T_{24} - 2Y_4T_{14}T_{34} + T_4T_{34})g_{18} - Y_3^2T_{24}^2g_{19} - Y_3T_{34}^2g_{25} + Y_4T_{24}g_{28}$.

This finishes the proof of the lemma. ■

Let $k[T]$ denote the polynomial ring $k[T_1, T_2, T_3, T_4, T_{12}, T_{13}, T_{14}T_{24}, T_{34}]$. Using the elimination theory, we know that the set $\Sigma = \{g_7, g_{11}, g_{12}, g_{19}, g_{20}, g_{21}\}$ generates the ideal $I \cap k[T]$ of $k[T]$. Let $\psi : k[X, Y, T] \to k[X, Y, T]$ be the k-endomorphism sending X_i to X_i, Y_i to Y_i $(i = 1, 2, 3, 4)$, T_i to X_i $(i = 1, 2, 3)$, T_4 to f, and T_{ij} to L_{ij}, then we have

$\psi(g_7) = X_1$
$\psi(g_{11}) = -X_3L_{12} + X_2L_{14} = X_1^2L_{24}$
$\psi(g_{12}) = -X_3L_{14} + X_2L_{13} = -X_1^2L_{34}$
$\psi(g_{19}) = -L_{14}^2 + L_{12}L_{13} = X_1^2f$
$\psi(g_{20}) = -L_{14}L_{34} + X_3f - L_{13}L_{24} = 0$

$\psi(g_{21}) = X_2 f - L_{14}L_{24} - L_{12}L_{34} = 0.$

This shows that there are no new generators and so

$$\ker D = k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}, f].$$

The next two lemmas show that $\ker D$ cannot be generated over $k[X_1, X_2, X_3]$ by linear forms in the Y_i's.

Lemma 3.4.3. *With the above notation, if L is an element of $\ker D$ of the form*

$$L = \alpha_1 Y_1 + \cdots + \alpha_4 Y_4$$

for some $\alpha_1, \ldots \alpha_4 \in k[X_1, X_2, X_3]$, then

$$L \in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$$

.

Proof

If L is a linear generator for $\ker D$ over $k[X_1, X_2, X_3]$, then L has the form

$$L = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4$$

where $\alpha_i \in k[X_1, X_2, X_3]$ $i \in \{1, 2, 3, 4\}$. Since $L \in \ker D$, we have

$$\alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 + \alpha_4 X_2 X_3 = 0. \tag{112}$$

Let $\phi = \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2$, then equation (112) shows that both X_2 and X_3 are divisors of ϕ. Taking equation (112) modulo X_2 gives that

$$X_1^2 \alpha_{12} + X_3^2 \alpha_{32} = 0 \tag{113}$$

where $\alpha_{12} = \alpha_1 \mid_{X_2=0}$ and $\alpha_{32} = \alpha_3 \mid_{X_2=0}$. Since X_1 and X_3 are relatively prime, equation (113) implies that $\alpha_1 = -X_3^2\beta_{32} + X_2\beta_1$ and $\alpha_3 = X_1^2\beta_{32} + X_2\beta_3$ for some $\beta_1, \beta_3 \in k[X_1, X_2, X_3]$ and β_{32} in $k[X_1, X_3]$. After simplification we find

$$\phi = X_1^2 X_2 \beta_1 + X_2 X_3^2 \beta_3 + \alpha_2 X_2^2. \tag{114}$$

Since X_3 is a divisor of ϕ, equation (114) implies that

$$X_1^2X_2\beta_1\mid_{X_3=0}+X_2^2\alpha_2\mid_{X_3=0}=0.$$

Consequently, $\alpha_2 = X_1^2u + X_3v$ and $\beta_1 = -X_2u + X_3w$ for some $u \in k[X_1, X_2]$ and $v, w \in k[X_1, X_2, X_3]$. Replacing these values of α_2 and β_1 in the expression (114) of ϕ, we get

$$\phi = X_2X_3(X_1^2w + X_3\beta_3 + X_2v)$$

and consequently $\alpha_4 = -\phi/(X_2X_3) = -(X_1^2w + X_3\beta_3 + X_2v)$. Hence,

$$\begin{aligned}
\alpha_1 &= -X_2^2u - X_3^2\beta_{32} + X_2X_3w\\
\alpha_2 &= X_1^2u + X_3v\\
\alpha_3 &= X_1^2\beta_{32} + X_2\beta_3\\
\alpha_4 &= -(X_1^2w + X_3\beta_3 + X_2v)
\end{aligned}$$

and so

$$\begin{aligned}
L &= \alpha_1Y_1 + \alpha_2Y_2 + \alpha_3Y_3 + \alpha_4Y_4\\
&= u(X_1^2Y_2 - X_2^2Y_1) + \beta_{32}(X_1^2Y_3 - X_3^2Y_1)\\
&+ v(X_3Y_4 - X_2Y_3) - w(X_1^2Y_4 - X_2X_3Y_1)\\
&+ \beta_3(X_2Y_4 - X_3Y_2)\\
&\in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}].
\end{aligned}$$

■

Lemma 3.4.4. *With the above notation,* $f \notin k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$.

Proof

If $f \in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$, we can choose a polynomial $\Phi \in E := k[X_1, X_2, X_3, U_1, U_2, U_3, U_4, U_5]$ such that

$$f = \Phi(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}). \tag{115}$$

Consider the $\mathbb{N}^2$-grading on $k[X,Y]$ defined by declaring $k \subseteq k[X,Y]_{(0,0)}$ and $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$ for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$. Also define a similar $\mathbb{N}^2$-grading on E by $k \subseteq E_{(0,0)}$ and $\deg(X_i) = (1,0)$, $\deg(U_j) = (2,1)$ for $j \in \{1,2,3\}$, and $\deg(U_4) = \deg(U_5) = (1,1)$. Write

$$\Phi = \Phi_{d_1} + \Phi_{d_2} + \cdots \Phi_{d_r}$$

where Φ_{d_i} is the homogeneous component of Φ of degree $d_i \in \mathbb{N}^2$. Since the elements $L_{12}, L_{13}, L_{14}, L_{24}, L_{34}$ are all homogeneous with respect to the $\mathbb{N}^2$-grading on $k[X,Y]$ defined above, it is easy to check that

$$\deg\left(\Phi_{d_i}(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34})\right)$$

is either zero or homogeneous of degree d_i, for all $i \in \{1, \ldots, r\}$. Also, since f is a homogeneous element of degree $(2,2)$ of $k[X,Y]$, equation (115) implies that

$$f = \Phi_{(2,2)}(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34})$$

and this can only happen if

$$f = aL_{24}^2 + bL_{34}^2 + cL_{24}L_{34} \tag{116}$$

for some $a, b, c \in k$. Indeed, a homogeneous element of degree $(2,2)$ of E can only be a linear combination of U_4^2, U_5^2 and U_4U_5 because of the degrees of the X_i's and the U_i's defined above.

Now equation (116) implies that $f \in k[X_2, X_3, Y_2, Y_3, Y_4]$, which is absurd. ∎

Theorem 3.4.1 is now a direct consequence of the above two lemmas.

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