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BRAIDED FROBENIUS ALGEBRAS

By
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October 2006

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Abstract

Monoidal categories have proven to be especially useful in the analysis of both algebraic structures such as associative algebras and geometric structures such as knots and braids. In this paper, we consider Frobenius algebras. These are algebraic structures consisting of an associative algebra and a coassociative coalgebra, satisfying a compatibility relation. Frobenius algebras have many applications in algebra and computer science. They have also been shown to characterize low-dimensional topological quantum field theories.

They are traditionally considered in symmetric monoidal categories. But we generalize the theory to braided monoidal categories. In the process, we obtain a number of new examples of this algebraic notion. Our examples arise in categories of crossed G -sets, categories of representations of quasitriangular Hopf algebras and several geometric categories.

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Dedication

I dedicate this work to my mother and father.

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Chapter 1

Introduction

1.1 History

Monoidal categories have become a very active area of research ever since they were introduced in 1963 by Mac Lane in [14] and by Bénabou in [5]. Roughly, a monoidal category is a category equipped with a bifunctor called tensor \otimes , a distinguished object I called the tensor unit, and natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $I \otimes A \cong A$ and $A \otimes I \cong A$ which satisfy some coherence conditions. The idea is that the tensor of two objects is multiplication, which is associative up to natural isomorphism, and I is a unit for the multiplication up to natural isomorphism. Some important examples include the category of sets and functions **Set** where \otimes is the cartesian product \times and $I = \{*\}$, $\mathbf{Vect}_{\mathbb{k}}$ where $\otimes = \otimes_{\mathbb{k}}$ and $I = \mathbb{k}$, $G\text{-Mod}_{\mathbb{k}}$ the category of linear representations of a fixed group G where $\otimes = \otimes_{\mathbb{k}}$ and $I = \mathbb{k}$, and the category of sets and relations **Rel**, where $\otimes =$ cartesian product and $I = \{*\}$.

Monoidal categories also play a prominent role in topological quantum field theory (TQFT). Their axiomatization which is due to Atiyah [2] translates nicely into the language of monoidal categories. From a categorical point of view the formulation is very neat and tidy. TQFTs were first considered by Witten [18], where he showed how the Jones polynomial and knot invariants are related to 3-dimensional TQFTs [3]. They have since been extensively studied by many others including Atiyah, Baez, Lawrence, etc. Briefly, an n -dimensional TQFT is a strict symmetric monoidal functor

from the category of n -dimensional cobordisms to the category of vector spaces. The 2-dimensional case has been completely determined. 2-dimensional TQFTs are in one-to-one correspondence with commutative Frobenius algebras. In higher dimensions no such characterization is known.

Another important concept that arises in algebra and physics is that of a braided monoidal category. These were invented by Joyal and Street in 1985/1986 and first appeared in their paper entitled *Braided Monoidal Categories* [8]. Since then there has been much research on braided monoidal categories. For instance, Freyd and Yetter showed that their category of framed tangles is the free braided compact closed category on one object [7]. Braided monoidal categories also have a close connection with quantum groups, the Yang-Baxter equation, and quasitriangular Hopf algebras. The category of H -modules of a quasitriangular Hopf algebra H is braided. Also, for a finite-dimensional Hopf algebra, Drinfeld's quantum double has a simple categorical interpretation [10].

The primary goal and contribution of this thesis is to extend the notion of Frobenius algebras to the braided setting as suggested by Kock in [12]. In doing so we obtain the notion of a universal braided Frobenius object which leads us to new geometric and algebraic structures.

1.2 Thesis Outline

In chapter 2 we start by reviewing some important concepts from monoidal category theory. Namely we give the definition of a monoidal category and provide some examples that will be important to us later on. We then state the coherence theorem for monoidal categories [15], which establishes the commutativity of a large class of diagrams. We define the notions of a monoid object and a comonoid object and present examples. These two concepts play an integral part in generalizing the definition of Frobenius algebras to abstract monoidal categories. Next we introduce another important concept related to Frobenius algebras, namely duality in monoidal categories. Frobenius algebras have the property that they are self-dual in the above sense. After this, we review the definition of a braided monoidal category and examine in depth

two significant examples: the category of crossed G -sets, with G a group, and the category $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ whose arrows are matrices with rows and columns indexed by crossed G -sets. These categories are both due to Freyd and Yetter [7]. In this paper they also consider a certain generalization of compact closed categories to braided monoidal categories, which they call pivotal categories. We explain what it means for a category to be pivotal and state Freyd and Yetter's theorem which asserts that $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is pivotal.

Chapter 3 begins with a brief overview of the traditional theory of Frobenius algebras. A Frobenius algebra is an associative \mathbb{k} -algebra A of finite dimension, where \mathbb{k} is a field, equipped with a linear functional $\varepsilon : A \rightarrow \mathbb{k}$ such that

$$\varepsilon(ab) = 0 \text{ for all } a \in A \text{ implies } b = 0.$$

We provide numerous examples showing that Frobenius algebras pervade mathematics. Next we state a result which Kock attributes to Lawvere [13], that provides a characterization of any given Frobenius algebra in terms of a coalgebra structure that is compatible with the algebra structure. This compatibility condition is known as the Frobenius relation. Following Kock we use this result to define the notion of a Frobenius object in an abstract monoidal category. Subsequently we state and prove a theorem which establishes five equivalent conditions for a monoid to be a Frobenius object. This theorem is an adaptation of one found in [17]. We conclude the chapter by showing that the category of Frobenius objects is a groupoid.

We discuss oriented cobordisms and 2-dimensional TQFTs in Chapter 4. We first introduce the concept of oriented cobordisms and develop the strict symmetric monoidal category $\mathbf{2Cob}$ of 2-dimensional cobordisms. We then state a proposition which gives a set of generators for a skeletal version of $\mathbf{2Cob}$. As well, we give a set of relations and we briefly indicate which of them are sufficient. Next we define what an n -dimensional TQFT is and state the main theorem which classifies them in the $n = 2$ case. The theorem says that the category of TQFTs and symmetric monoidal natural transformations, $\mathbf{SymMonCat}(\mathbf{2Cob}, \mathbf{Vect}_{\mathbb{k}})$, is equivalent to the category of commutative Frobenius algebras, $\mathbf{cFrob}(\mathbf{Vect}_{\mathbb{k}})$. More generally if \mathcal{V} is any symmetric monoidal category then $\mathbf{SymMonCat}(\mathbf{2Cob}, \mathcal{V})$ is equivalent to the

category of commutative Frobenius objects in \mathcal{V} , $\mathbf{cFrob}(\mathcal{V})$.

In chapters 3 and 4 we give complete details of several results which are only sketched in other sources.

In the final chapter we define the notion of a braided Frobenius object; it is simply a Frobenius object that lives in a braided monoidal category. We then prove some facts about the categories $\mathbb{X}\mathbf{G}\text{-Set}$ and $\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R$. Using these results we are able to obtain a large class of braided Frobenius objects in $\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R$. We also show that in any pivotal category, $A \otimes A^*$ can be given a Frobenius structure for any object A . We then consider the centre construction which, given a monoidal category \mathcal{C} , produces a braided monoidal category $\mathcal{Z}(\mathcal{C})$. We show how braided Frobenius objects in $\mathcal{Z}(\mathcal{C})$ relate to Frobenius objects in \mathcal{C} . The next place that we look for examples is in the category of H -modules, where H is a quasitriangular Hopf algebra. A braided Frobenius object in this instance is an H -module that is equipped with the structure of a Frobenius algebra in the usual sense with the additional requirement that the structure maps be H -linear. We give an explicit description of the free braided monoidal category generated by a braided Frobenius object. We then describe a category which we conjecture to be equivalent in which morphisms have a purely geometric description. We then discuss the difficulties in establishing this conjecture.

Chapter 2

Monoidal Category Theory

2.1 Monoidal Categories

We start by collecting some important notions from category theory.

Definition 2.1.1. A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category \mathcal{C} , a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $I \in |\mathcal{C}|$, and natural isomorphisms $\forall A, B, C \in |\mathcal{C}|$:

$$\lambda_A : I \otimes A \rightarrow A$$

$$\rho_A : A \otimes I \rightarrow A$$

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

such that the following diagrams commute

$$\begin{array}{ccc}
 & (A \otimes B) \otimes (C \otimes D) & \\
 \alpha \nearrow & & \searrow \alpha \\
 ((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\
 \alpha \otimes 1_D \searrow & & \nearrow 1_A \otimes \alpha \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha} & A \otimes ((B \otimes C) \otimes D)
 \end{array} \tag{M1}$$

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha} & A \otimes (I \otimes B) \\
 \rho \otimes 1_B \searrow & & \nearrow 1_A \otimes \lambda \\
 & A \otimes B &
 \end{array} \tag{M2}$$

$$\lambda = \rho : I \otimes I \longrightarrow I \tag{M3}$$

We say that a monoidal category is **strict** if all components of α , λ , and ρ are identity maps.

Example 2.1.2. Let M be a monoid in the usual sense of the word. Then we can view M as a category whose objects are elements of M and the only arrows are identity arrows. Let \otimes denote the multiplication in M and let $I \in M$ be the unit for this multiplication. Then (M, \otimes, I) is a strict monoidal category. This follows from the monoid axioms.

Example 2.1.3. Any category with all finite products (resp. coproducts) is a monoidal category where $\otimes = \times$ (resp. $+$), $I = 1$ (resp. 0). The isomorphisms α , λ , and ρ are determined by the universal property of product (resp. coproduct). In particular the category **Set** is a monoidal category. Unless stated otherwise we will take the monoidal structure on **Set** to be the one given by cartesian product.

Example 2.1.4. Let \mathbb{k} be a fixed field and consider the category $\mathbf{Vect}_{\mathbb{k}}$ whose objects are vector spaces over \mathbb{k} and arrows are \mathbb{k} -linear maps. Then $\mathbf{Vect}_{\mathbb{k}}$ is a monoidal category where:

$$\begin{aligned} V \otimes W &= V \otimes_{\mathbb{k}} W \\ I &= \mathbb{k} \end{aligned}$$

and $\alpha : (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, $\lambda : \mathbb{k} \otimes U \cong U$, and $\rho : U \otimes \mathbb{k} \cong U$ are the usual vector space isomorphisms.

Example 2.1.5. Let \mathcal{C} be a category and consider the category $\mathbf{Func}(\mathcal{C})$ whose objects are endofunctors on \mathcal{C} and arrows are natural transformations between such functors. Then $\mathbf{Func}(\mathcal{C})$ is a strict monoidal category where $F \otimes G = F \circ G$ for functors F and G and $I = id_{\mathcal{C}}$.

2.2 Coherence for Monoidal Categories

We now state a coherence theorem for monoidal categories which can be found in [15]. We give the version from Kock's book.

Theorem 2.2.1 (Mac Lane's Coherence Theorem). *Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. Every diagram that can be built out of the components of α , λ , and ρ , and identity maps, using composition and monoidal operations, commutes.*

I will use this theorem extensively to establish the commutativity of diagrams.

2.3 Monoid Objects

Definition 2.3.1. *Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A **monoid** (M, μ, η) in \mathcal{C} is an object $M \in |\mathcal{C}|$ together with two arrows $\mu : M \otimes M \longrightarrow M$, $\eta : I \longrightarrow M$*

such that

$$\begin{array}{ccc}
 (M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{1_M \otimes \mu} & M \otimes M \\
 \mu \otimes 1_M \downarrow & & & & \downarrow \mu \\
 M \otimes M & \xrightarrow{\mu} & & & M
 \end{array} \tag{Mon1}$$

and

$$\begin{array}{ccccc}
 I \otimes M & \xrightarrow{\eta \otimes 1_M} & M \otimes M & \xleftarrow{1_M \otimes \eta} & M \otimes I \\
 & \searrow \lambda & \downarrow \mu & \swarrow \rho & \\
 & & M & &
 \end{array} \tag{Mon2}$$

commute. The map μ is called multiplication and η is called the unit. Equation (Mon1) is called the associativity axiom and (Mon2) is called the unit axiom [15].

Example 2.3.2. As one might expect monoids in the monoidal category **Set** are monoids in the usual sense. That is, a set equipped with an associative binary operation called multiplication, and a unit element for that operation.

Example 2.3.3. Monoids in the monoidal category \mathbf{Vect}_k are associative k -algebras.

Example 2.3.4. Monoids in the monoidal category $\mathbf{Func}(\mathcal{C})$ are monads [15].

Definition 2.3.5. A *morphism of monoids* $f : (M, \mu, \eta) \longrightarrow (M', \mu', \eta')$ is an arrow $f : M \longrightarrow M'$ such that following diagrams commute:

$$\begin{array}{ccc}
 M \otimes M & \xrightarrow{f \otimes f} & M' \otimes M' \\
 \mu \downarrow & & \downarrow \mu' \\
 M & \xrightarrow{f} & M'
 \end{array}
 \qquad
 \begin{array}{ccc}
 M & \xrightarrow{f} & M' \\
 \eta \swarrow & & \swarrow \eta' \\
 & I &
 \end{array}$$

The following lemma is immediate and can be found in [15].

Lemma 2.3.6. Given a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ the monoids in \mathcal{C} form a category $\mathbf{Mon}_{\mathcal{C}}$.

2.4 Comonoid Objects

We now consider the dual notion to that of a monoid in a monoidal category.

Definition 2.4.1. Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A **comonoid** (C, Δ, ϵ) in \mathcal{C} is an object $C \in |\mathcal{C}|$ together with two arrows $\Delta : C \rightarrow C \otimes C$, $\epsilon : C \rightarrow I$ such that

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \xrightarrow{1_C \otimes \Delta} C \otimes (C \otimes C) \\
 \Delta \downarrow & & \downarrow \alpha^{-1} \\
 C \otimes C & \xrightarrow{\Delta \otimes 1_C} & (C \otimes C) \otimes C
 \end{array} \tag{CoM1}$$

and

$$\begin{array}{ccccc}
 I \otimes C & \xleftarrow{\epsilon \otimes 1_M} & C \otimes C & \xrightarrow{1_M \otimes \epsilon} & C \otimes I \\
 & \searrow \lambda & \uparrow \Delta & \swarrow \rho & \\
 & & C & &
 \end{array} \tag{CoM2}$$

commute. The map Δ is called *comultiplication* and ϵ is called the *counit*. Equation (CoM1) is called the *coassociativity axiom* and (CoM2) is called the *counit axiom* [15].

Example 2.4.2. Let $X \neq \emptyset$ be a set and let $I = \{*\}$ be the one point set. Define $\Delta : X \rightarrow X \times X$ by $x \mapsto (x, x)$ and $\epsilon : X \rightarrow I$ by $x \mapsto *$. Then (X, Δ, ϵ) is a comonoid in **Set**. In fact, one can show that this is the only possible comonoid structure on X .

Example 2.4.3. Comonoids in \mathbf{Vect}_k are coassociative k -coalgebras.

Example 2.4.4. Comonoids in the monoidal category $\mathbf{Func}(\mathcal{C})$ are comonads [15].

A morphism of comonoids is defined in a similar fashion to Definition 2.3.5, and we have a result similar to Lemma 2.3.6. We use $CoMon_{\mathcal{C}}$ to denote the category of comonoids in a monoidal category \mathcal{C} .

Remark 2.4.5. By duality we have that monoids (resp. comonoids) in \mathcal{C} are comonoids (resp. monoids) in \mathcal{C}^{op} .

2.5 Hopf Algebras

For this section we follow Kassel [10], and we fix a base field \mathbb{k} , H a vector space over \mathbb{k} equipped with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε) . We give the vector space $H \otimes H$ the algebra structure $(H \otimes H, \mu', \eta')$ where

$$\mu' = (\mu \otimes \mu) \circ (1 \otimes \tau_{H,H} \otimes 1)$$

where $\tau_{V,W}$ is the unique linear map such that $\tau_{V,W}(v \otimes w) = w \otimes v \forall v \in V, w \in W$ for vector spaces V and W . The unit for $H \otimes H$ is

$$\eta' = \eta \otimes \eta.$$

We also give $H \otimes H$ its canonical coalgebra structure $(H \otimes H, \Delta', \varepsilon')$. Namely

$$\Delta' = \Delta \otimes \Delta$$

$$\varepsilon' = \varepsilon \otimes \varepsilon.$$

We then have the following theorem which can be found in [10].

Theorem 2.5.1. *The following statements are equivalent.*

- (i) *The maps μ and η are morphisms of coalgebras.*
- (ii) *The maps Δ and ε are morphisms of algebras.*

Whenever H has this kind of compatibility between its algebra coalgebra structure then we give it a name.

Definition 2.5.2. A **bialgebra** is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra satisfying the equivalent conditions of Theorem 2.5.1. A **morphism of bialgebras** is a morphism for the underlying algebra and coalgebra structures.

Before we define what a Hopf algebra is we need the following:

Definition 2.5.3. If (A, μ, η) is an algebra and (C, Δ, ε) is a coalgebra then $\text{Hom}(C, A)$, the set of linear maps, is a vector space. For $f, g \in \text{Hom}(C, A)$ we define their **convolution** $f \star g$ to be the following composite:

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

\star is in fact bilinear.

We then have the following nice result.

Proposition 2.5.4. (i) *The triple $(\text{Hom}(C, A), \star, \eta \circ \varepsilon)$ is an algebra.*

(ii) *The evident map $\lambda_{C,A} : A \otimes C^* \longrightarrow \text{Hom}(C, A)$ is morphism of algebras where $A \otimes C^*$ is the tensor product of the algebra A with the algebra C^* dual to the coalgebra C .*

The proof of this result can be found in [10]. We are now ready for:

Definition 2.5.5. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then an element $S \in \text{Hom}(H, H)$ is called an **antipode** for the bialgebra H if*

$$S \star id_H = id_H \star S = \eta \circ \varepsilon$$

*A **Hopf algebra** is a bialgebra with an antipode. Then a morphism of Hopf algebras is morphism of the underlying bialgebras which commutes with the antipode.*

If a bialgebra has an antipode then it is unique. For the next example we follow Majid [16].

Example 2.5.6. Let G be a finite group, \mathbb{k} a field, and let $\mathbb{k}G$ be the group algebra defined by $\mathbb{k}G = \{x = \sum_{g \in G} x(g)g\}$ where $x(g) \in \mathbb{k}$ for all $g \in G$. We make $\mathbb{k}G$ into a \mathbb{k} -algebra as follows: $\mu : \mathbb{k}G \otimes \mathbb{k}G \longrightarrow \mathbb{k}G$ is given by $g \otimes h \mapsto gh$ for all $g, h \in G$. The unit $\eta : \mathbb{k} \longrightarrow \mathbb{k}G$ is defined by $\eta(1) = e$, where $e \in G$ is the unit for multiplication in G . We extend μ and η by linearity. To make $\mathbb{k}G$ a Hopf algebra we define $\Delta : \mathbb{k}G \longrightarrow \mathbb{k}G \otimes \mathbb{k}G$ by $\Delta(g) = g \otimes g$ for all $g \in G$. We define $\varepsilon : \mathbb{k}G \longrightarrow \mathbb{k}$ by $\varepsilon(g) = 1$ and $S : \mathbb{k}G \longrightarrow \mathbb{k}G$ by $S(g) = g^{-1}$ for all $g \in G$. We extend these maps by linearity. When equipped with these five maps $\mathbb{k}G$ becomes a (cocommutative) Hopf algebra.

Remark 2.5.7. We notice that in this instance $S^2 = id$, this need not be the case in general.

Let A be an associative algebra over \mathbb{k} .

Definition 2.5.8. A **right A -module** is a vector space M with a \mathbb{k} -linear map $\chi_M : M \otimes A \rightarrow M$, called the **right action of A on M** , such that

$$\begin{array}{ccc}
 & (M \otimes A) \otimes A & \\
 \alpha \swarrow & & \searrow \chi_M \otimes 1_A \\
 M \otimes (A \otimes A) & & M \otimes A \\
 \downarrow 1_M \otimes \mu & & \downarrow \chi_M \\
 M \otimes A & \xrightarrow{\chi_M} & M
 \end{array}
 \qquad
 \begin{array}{ccc}
 M \otimes A & \xleftarrow{1_M \otimes \eta} & M \otimes \mathbb{k} \\
 \downarrow \chi_M & & \swarrow \lambda \\
 M & &
 \end{array}$$

both commute. If M and N are right A -modules, a \mathbb{k} -linear map $\phi : M \rightarrow N$ is called a **right A -homomorphism** or **right A -linear** if

$$\begin{array}{ccc}
 M \otimes A & \xrightarrow{\phi \otimes 1_A} & N \otimes A \\
 \downarrow \chi_M & & \downarrow \chi_N \\
 M & \xrightarrow{\phi} & N
 \end{array}$$

commutes.

If we write $\chi_M(m \otimes a) = m \cdot a$ and $\mu(a \otimes b) = ab$ and $\eta(1) = 1$ then we can interpret the diagrams of the above definition in terms of elements as follows:

$$\begin{aligned}
 (m \cdot a) \cdot b &= m \cdot (ab) & \forall a, b \in A, \forall m \in M \\
 m \cdot 1 &= m & \forall m \in M \\
 \phi(m \cdot a) &= (\phi(m)) \cdot a & \forall a \in A, \forall m \in M
 \end{aligned}$$

We can also define left A -modules and left A -homomorphisms in a similar manner.

Now for a given bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ the category of left H -modules is monoidal. Given two H -modules U and V then we can give $U \otimes V$ a left H -module structure as follows

$$a \cdot (u \otimes v) = \Delta(a) \cdot (u \otimes v)$$

where a given $\sum_{i=1}^n x_i \otimes y_i \in H \otimes H$ acts on $U \otimes V$ as follows $\sum_{i=1}^n x_i \otimes y_i(u \otimes v) = \sum_{i=1}^n x_i u \otimes y_i v$. When H is the group algebra of some finite group G the H -module structure on $U \otimes V$ is compatible with the one used in group representation theory. We give \mathbb{k} the following H -module structure

$$x \cdot a = \varepsilon(x)a \quad \forall x \in H, a \in \mathbb{k}.$$

Then it can be shown that the canonical isomorphisms from $(V \otimes U) \otimes W \cong V \otimes (U \otimes W)$, $\mathbb{k} \otimes V \cong V$, and $V \otimes \mathbb{k} \cong V$ are maps of H -modules.

2.6 Monoidal Functors and Natural Transformations

Definition 2.6.1. A *monoidal functor* $(F, F_2, F_0) : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories \mathcal{C} and \mathcal{D} consists of the following three items:

- a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between categories;
- $\forall A, B \in |\mathcal{C}|$, morphisms $F_2(A, B) : F(A) \otimes F(B) \rightarrow F(A \otimes B)$ in \mathcal{D} which are natural in A and B ;
- for units I and I' , a morphism $F_0 : I' \rightarrow F(I)$ in \mathcal{D} .

These must make the following three diagrams, involving the structural maps α , λ , and ρ , commute in \mathcal{D} :

$$\begin{array}{ccc}
 (F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\alpha'} & F(A) \otimes (F(B) \otimes F(C)) \\
 \downarrow F_2 \otimes 1 & & \downarrow 1 \otimes F_2 \\
 F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\
 \downarrow F_2 & & \downarrow F_2 \\
 F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha)} & F(A \otimes (B \otimes C))
 \end{array} \tag{MF1}$$

$$\begin{array}{ccc}
 F(B) \otimes I' & \xrightarrow{\rho'} & F(B) \\
 1 \otimes F_0 \downarrow & & \uparrow F(\rho) \\
 F(B) \otimes F(I) & \xrightarrow{F_2} & F(B \otimes I)
 \end{array} \quad (\text{MF2})$$

$$\begin{array}{ccc}
 I' \otimes F(B) & \xrightarrow{\lambda'} & F(B) \\
 F_0 \otimes 1 \downarrow & & \uparrow F(\lambda) \\
 F(I) \otimes F(B) & \xrightarrow{F_2} & F(I \otimes B)
 \end{array} \quad (\text{MF3})$$

A monoidal functor is said to be **strong** when F_0 and all the $F_2(A, B)$ are isomorphisms, and **strict** when F_0 and all the $F_2(A, B)$ are identities [15].

Remark 2.6.2. If (F, F_2, F_0) is a strict monoidal functor then Definition 2.6.1 reduces to the following:

- a functor $F : M \rightarrow M'$;
- $F(\alpha) = \alpha'$, $F(\lambda) = \lambda'$, $F(\rho) = \rho'$;
- $F(f \otimes g) = F(f) \otimes F(g)$ for all arrows f and g in M .

Remark 2.6.3. Let $F : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \rightarrow (\mathcal{D}, \otimes', I', \alpha', \lambda', \rho')$ be a strict monoidal functor. If a diagram built from arrows and objects in \mathcal{C} using composition and tensor commutes, then the diagram built by replacing each object and each arrow by its image under F will also commute in \mathcal{D} . This follows from the previous remark. In particular if (M, μ, η) is a monoid in \mathcal{C} then $(F(M), F(\mu), F(\eta))$ is automatically a monoid in \mathcal{D} , and similarly for comonoids.

Next we consider what morphisms of monoidal functors should be.

Definition 2.6.4. A **monoidal natural transformation** $\theta : (F, F_2, F_0) \rightarrow (G, G_2, G_0)$ between monoidal functors $(F, F_2, F_0), (G, G_2, G_0) : M \rightarrow M'$ is a natural transformation $\theta : F \rightarrow G$ between the functors F and G such that the following

two diagrams

$$\begin{array}{ccc}
 F(A) \otimes F(B) & \xrightarrow{F_2} & F(A \otimes B) \\
 \theta_A \otimes \theta_B \downarrow & & \downarrow \theta_{A \otimes B} \\
 G(A) \otimes G(B) & \xrightarrow{G_2} & G(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(I) & \xrightarrow{\theta_I} & G(I) \\
 F_0 \swarrow & & \nearrow G_0 \\
 & I' &
 \end{array}$$

commute [15].

This definition allows us to consider categories whose objects are monoidal functors and arrows are monoidal NTs. These concepts will be used in sections 4.3 and 5.3.

2.7 Duality in Monoidal Categories

We now turn our attention to a key notion that arises in the theory of Frobenius algebras.

Definition 2.7.1. A **duality** $A \dashv B$ between two objects A and B in a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a pair of morphisms

$$\psi : A \otimes B \longrightarrow I \qquad \text{and} \qquad \varphi : I \longrightarrow B \otimes A$$

called the **counit** and **unit** respectively, such that

$$\begin{array}{ccc}
 A & \xrightarrow{\rho_A^{-1}} & A \otimes I \xrightarrow{1 \otimes \varphi} & A \otimes (B \otimes A) \xrightarrow{\alpha^{-1}} & (A \otimes B) \otimes A \\
 & & & & \downarrow \psi \otimes 1 \\
 & & & & I \otimes A \\
 & & & & \downarrow \lambda_A \\
 & & & & A
 \end{array}
 \qquad (D1)$$

1_A

$$\begin{array}{ccccc}
 B & \xrightarrow{\lambda_B^{-1}} & I \otimes B & \xrightarrow{\varphi \otimes 1} & (B \otimes A) \otimes B & \xrightarrow{\alpha} & B \otimes (A \otimes B) \\
 & & & & & & \downarrow 1 \otimes \psi \\
 & & & & & & B \otimes I \\
 & & & & & & \downarrow \rho_B \\
 & & & & & & B \\
 & \searrow 1_B & & & & & \\
 & & & & & &
 \end{array} \tag{D2}$$

both commute. In this case we say that A is **left dual** (or **left adjoint**) to B and B is **right dual** (or **right adjoint**) to A [9].

Before giving some examples we prove a useful proposition.

Proposition 2.7.2. *If A and B are objects in a monoidal category \mathcal{C} such that $B \dashv A$ then $A \otimes B$ can be made into a monoid in \mathcal{C} .*

Proof. $B \dashv A$ means that \exists morphisms $a' : B \otimes A \longrightarrow I$ and $b' : I \longrightarrow A \otimes B$ making diagrams analogous to (D1) and (D2) commute. We refer to these diagrams as $\overline{(D1)}$ and $\overline{(D2)}$ respectively. Now let's make $A \otimes B$ into a monoid:

We need a unit $\eta : I \longrightarrow A \otimes B$, so we take $\eta = b'$. We also need a multiplication $\mu : (A \otimes B) \otimes (A \otimes B) \longrightarrow A \otimes B$. So we define μ as follows

$$\begin{array}{ccccc}
 (A \otimes B) \otimes (A \otimes B) & \xrightarrow{\alpha^{-1}} & ((A \otimes B) \otimes A) \otimes B & \xrightarrow{\alpha \otimes 1} & (A \otimes (B \otimes A)) \otimes B \\
 & & & & \downarrow (1 \otimes a') \otimes 1 \\
 & & & & (A \otimes I) \otimes B \\
 & & & & \downarrow \rho_A \otimes 1 \\
 & & & & A \otimes B \\
 & \searrow \mu & & & \\
 & & & &
 \end{array}$$

We now check that both unit triangles in Definition 2.3.1 commute. We start with

the left one.

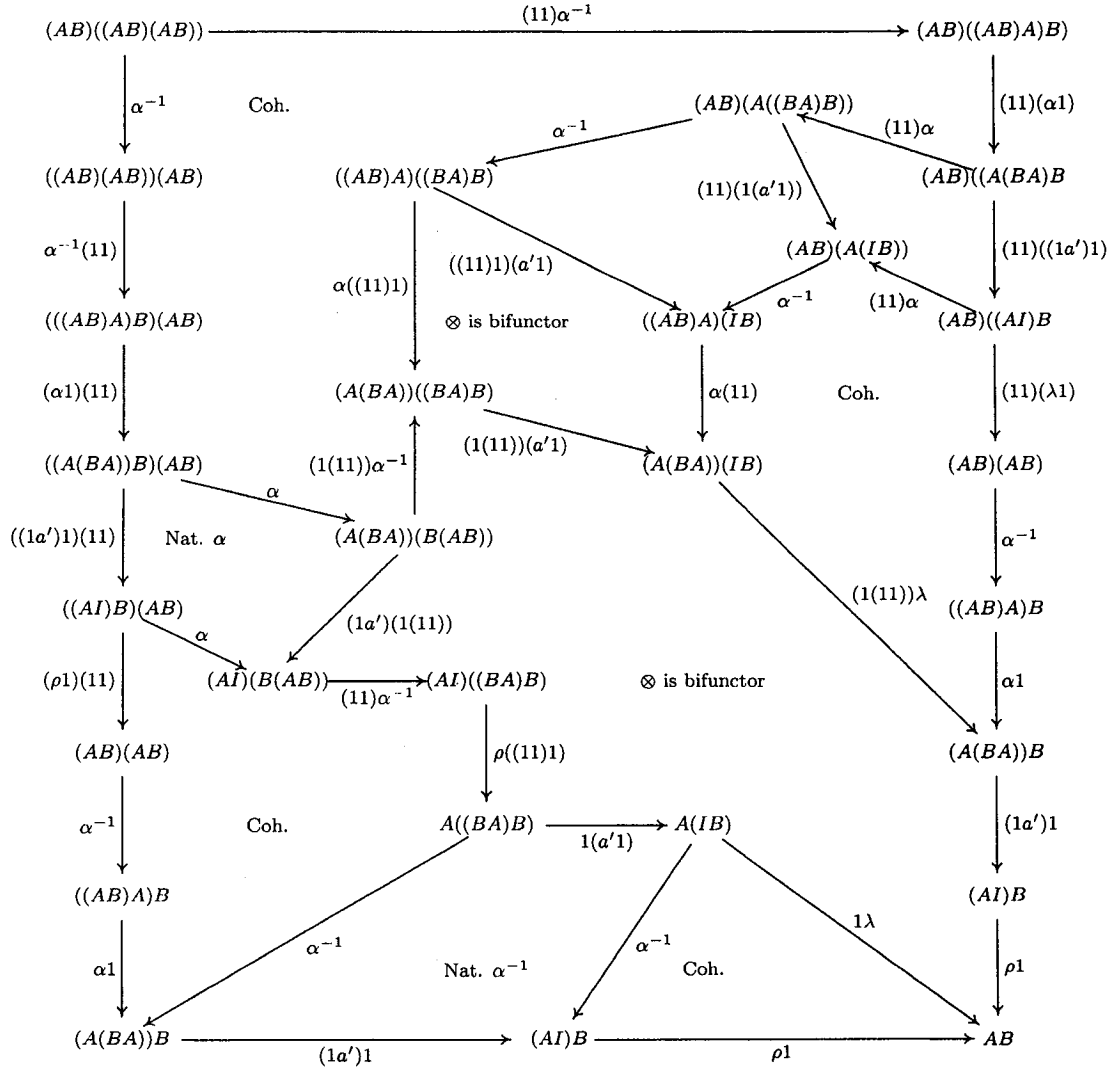
$$\begin{array}{ccccc}
 I \otimes (A \otimes B) & \xrightarrow{b' \otimes (1 \otimes 1)} & (A \otimes B) \otimes (A \otimes B) & \xrightarrow{\alpha^{-1}} & ((A \otimes B) \otimes A) \otimes B \\
 \searrow \alpha^{-1} & & \searrow \alpha^{-1} & & \downarrow \alpha \otimes 1 \\
 & \text{Nat. } \alpha^{-1} & & & \\
 (I \otimes A) \otimes B & \xrightarrow{(b' \otimes 1) \otimes 1} & ((A \otimes B) \otimes A) \otimes B & \xrightarrow{\alpha \otimes 1} & (A \otimes (B \otimes A)) \otimes B \\
 \uparrow \lambda_{A \otimes B}^{-1} & \text{Coh.} & & & \downarrow (1 \otimes a') \otimes 1 \\
 & \lambda_A^{-1} \otimes 1 & & & (A \otimes I) \otimes B \\
 & & & & \downarrow \rho_A \otimes 1 \\
 A \otimes B & \xrightarrow{1} & & & A \otimes B
 \end{array}
 \tag{i}$$

Equation (i) holds since this is just $\overline{(D2)}$ tensored with $1_B : B \rightarrow B$. We now check that the right unit triangle commutes.

$$\begin{array}{ccccc}
 (A \otimes B) \otimes I & \xrightarrow{(1 \otimes 1) \otimes b'} & (A \otimes B) \otimes (A \otimes B) & \xrightarrow{\alpha^{-1}} & ((A \otimes B) \otimes A) \otimes B \\
 \searrow \alpha & & \searrow \alpha & & \downarrow \alpha \otimes 1 \\
 & \text{Nat. } \alpha & & & \\
 A \otimes (B \otimes I) & \xrightarrow{1 \otimes (1 \otimes b')} & A \otimes (B \otimes (A \otimes B)) & & (A \otimes (B \otimes A)) \otimes B \\
 \uparrow \rho_{A \otimes B}^{-1} & \text{Coh.} & \downarrow 1 \otimes \alpha^{-1} & \nearrow \alpha^{-1} & \downarrow (1 \otimes a') \otimes 1 \\
 & & A \otimes ((B \otimes A) \otimes B) & \text{Nat. } \alpha^{-1} & (A \otimes I) \otimes B \\
 & & \downarrow 1 \otimes (a' \otimes 1) & \nearrow \alpha^{-1} & \downarrow \rho_A \otimes 1 \\
 & & A \otimes (I \otimes B) & \text{Coh.} & \\
 & & \downarrow 1 \otimes \lambda_B & & \\
 A \otimes B & \xrightarrow{1} & & & A \otimes B
 \end{array}
 \tag{ii}$$

Equation (ii) holds since this is just $\overline{(D1)}$ tensored with $1_A : A \rightarrow A$.

Finally we show that (Mon1) in Definition 2.3.1 commutes. In this diagram we save space by omitting the \otimes symbols.



The two squares in the upper right hand corner of the diagram commute by naturality. Hence $A \otimes B$ is a monoid in \mathcal{C} . \square

Example 2.7.3. In \mathbf{Set} $X \dashv Y$ if and only if X and Y are singletons. More generally the tensor unit in any monoidal category is always self dual.

Example 2.7.4. In \mathbf{Vect}_k $V \dashv W$ if and only if V and W are finite-dimensional and $W \cong V^*$. Let $\{e_i\}_{i=1}^n$ be a basis of V and let $f : W \rightarrow V^*$ be an isomorphism. Define $g : V \otimes V^* \rightarrow k$ by $g(v \otimes \Lambda) = \Lambda(v)$ for all $v \in V$ and $\Lambda \in V^*$. Similarly define $h : k \rightarrow V^* \otimes V$ by $h(1) = \sum_{i=1}^n e_i^* \otimes e_i$ where $\{e_i^*\}_{i=1}^n$ is the dual basis. Then the maps ψ and φ of Definition 2.7.1 are given by $\psi = g \circ 1_V \otimes f$ and $\varphi = f^{-1} \otimes 1_V \circ h$.

Example 2.7.5. In $\mathbf{Func}(\mathcal{C})$ $F \dashv U$ if and only if F is left adjoint to U . The unit and counit for the duality are the unit and counit of the adjunction respectively and (D1) and (D2) express that these functors form an adjoint pair.

2.8 Braided Monoidal Categories

Braided monoidal categories form an interesting class of monoidal categories.

Definition 2.8.1. A natural isomorphism $\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$ in a monoidal category, $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, is a **braiding** if the following diagrams commute:

$$\begin{array}{ccc}
 A \otimes (B \otimes C) & \xrightarrow{\sigma} & (B \otimes C) \otimes A \\
 \alpha \nearrow & & \searrow \alpha \\
 (A \otimes B) \otimes C & & B \otimes (C \otimes A) \\
 \sigma \otimes 1_C \searrow & & \nearrow 1_B \otimes \sigma \\
 (B \otimes A) \otimes C & \xrightarrow{\alpha} & B \otimes (A \otimes C)
 \end{array} \tag{B1}$$

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{\sigma} & C \otimes (A \otimes B) \\
 \alpha^{-1} \nearrow & & \searrow \alpha^{-1} \\
 A \otimes (B \otimes C) & & (C \otimes A) \otimes B \\
 1_A \otimes \sigma \searrow & & \nearrow \sigma \otimes 1_B \\
 A \otimes (C \otimes B) & \xrightarrow{\alpha^{-1}} & (A \otimes C) \otimes B
 \end{array} \tag{B2}$$

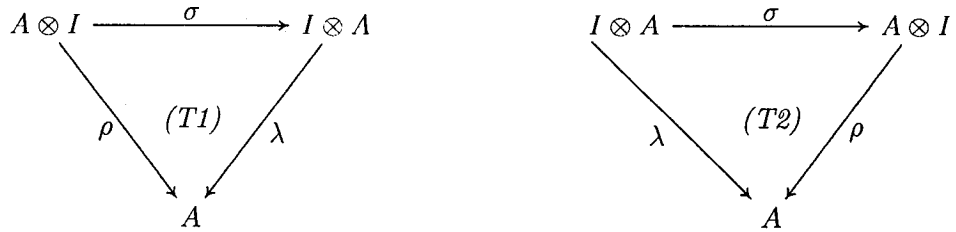
A monoidal category equipped with a braiding is called a **braided monoidal category** [9].

Remark 2.8.2. If $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a braided monoidal category then it may happen that $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B}$ for all objects A and B in \mathcal{C} . In this case we say that $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a **symmetric monoidal category**.

Lemma 2.8.3. *If \mathcal{C} is a symmetric monoidal category then $\text{Mon}_{\mathcal{C}}$ is monoidal.*

The following is due to Joyal and Street [9].

Proposition 2.8.4. *The following diagrams are commutative in any braided monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$:*



We now state a result of Kelly and Laplaza found in [11].

Proposition 2.8.5. *If \mathcal{C} is any monoidal category with tensor unit I , then the monoid $\text{Hom}(I, I)$ is commutative. Furthermore the value of the composite $I \cong I \otimes I \xrightarrow{f \otimes g} I \otimes I \cong I$ is $f \circ g = g \circ f$.*

Remark 2.8.6. We note that $\text{Hom}(I, I)$ is sometimes called **the set of scalars**.

Let's consider a few examples from some familiar monoidal categories.

Example 2.8.7. For the category $\text{Vect}_{\mathbb{k}}$ we have that $\text{Hom}(\mathbb{k}, \mathbb{k}) \cong \mathbb{k}$ since linear maps from \mathbb{k} to \mathbb{k} are the same thing as scalars. In Rel , the category of sets and relations, $\otimes =$ cartesian product and the tensor unit is $I = \{*\}$. Here we have that $\text{Hom}(I, I) = \{\emptyset, \{(*, *)\}\}$. If H is a Hopf algebra then its category of left H -modules is monoidal with $I = \mathbb{k}$ the base field. The action on \mathbb{k} is given by the counit ε for H . Namely $a \cdot x = \varepsilon(a)x$ for all $a \in H$ and $x \in \mathbb{k}$. Just as we had for $\text{Vect}_{\mathbb{k}}$, we have $\text{Hom}(\mathbb{k}, \mathbb{k}) \cong \mathbb{k}$.

We should also mention the appropriate notion for a functor between braided monoidal categories.

Definition 2.8.8. A **braided monoidal functor** from $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ to $(\mathcal{C}', \otimes', I', \alpha', \lambda', \rho', \sigma')$ is a monoidal functor (F, F_2, F_0) between the underlying monoidal categories with the additional requirement that

$$\begin{array}{ccc}
 F(A) \otimes' F(B) & \xrightarrow{\sigma'_{F(A), F(B)}} & F(B) \otimes' F(A) \\
 F_2(A, B) \downarrow & & \downarrow F_2(B, A) \\
 F(A \otimes B) & \xrightarrow{F(\sigma_{A, B})} & F(B \otimes A)
 \end{array} \quad (\text{BMF})$$

commutes. In the case where the category is symmetric, the functor is called a **symmetric monoidal functor** [15].

Before we consider some examples of braided monoidal categories we make a few definitions. In what follows G is a fixed group.

Definition 2.8.9. A (finite) **crossed G -set** is a (finite) G -set X and a function $|\cdot| : X \rightarrow G$ such that $|g \cdot x| = g|x|g^{-1} \forall x \in X, g \in G$ [7].

Definition 2.8.10. A **map of crossed G -sets**, $f : X \rightarrow Y$, is an equivariant map of G -sets such that $|f(x)| = |x| \forall x \in X$ [7].

Notation 2.8.11. Given crossed G -sets X and Y we will write $|\cdot|_X$ and $|\cdot|_Y$ for $|\cdot| : X \rightarrow G$ and $|\cdot| : Y \rightarrow G$ respectively when we wish to distinguish between these two functions.

We will provide examples of crossed G -sets later when we define the related notion of crossed G -monoids in section 5.1. Let's now record an important fact:

Lemma 2.8.12. $\mathbb{X}G\text{-Set}$ is a category. The objects of $\mathbb{X}G\text{-Set}$ are all finite crossed G -sets and the arrows are all crossed G -set maps between them.

We now prove an important theorem which is due to Freyd and Yetter.

Theorem 2.8.13 (Freyd and Yetter [7]). For any group, G , $\mathbb{X}G\text{-Set}$ is a braided monoidal category, where $X \otimes Y$ is the cartesian product $X \times Y$ of the underlying sets with the component-wise G -action, and $|(x, y)| = |x|_X |y|_Y$. The braiding $\sigma_{X, Y} : X \otimes Y \rightarrow Y \otimes X$ is defined by $\sigma_{X, Y}(x, y) = (|x| \cdot y, x)$.

Proof. We first show that $\mathbb{XG}\text{-Set}$ is a monoidal category. Let $f \in \text{Hom}_{\mathbb{XG}\text{-Set}}(X, Y)$ and $g \in \text{Hom}_{\mathbb{XG}\text{-Set}}(X', Y')$. Define $f \otimes g \in \text{Hom}_{\mathbb{XG}\text{-Set}}(X \otimes X', Y \otimes Y')$ as follows

$$(f \otimes g)(x, x') = (f(x), g(x')) \quad \forall x \in X, x' \in X'.$$

With this definition of $f \otimes g$ it is easy to see that $\otimes : \mathbb{XG}\text{-Set} \times \mathbb{XG}\text{-Set} \rightarrow \mathbb{XG}\text{-Set}$ is a functor. Now consider $I = \{*\}$ with $g \cdot * = * \quad \forall g \in G$ and define $| | : I \rightarrow G$ by $* \mapsto 1_G$. Then this makes I into a crossed G -set. For each $A, B, C \in |\mathbb{XG}\text{-Set}|$ define

$$\begin{aligned} \alpha_{A,B,C} : (A \otimes B) \otimes C &\longrightarrow A \otimes (B \otimes C) && \text{by } ((a, b), c) \longmapsto (a, (b, c)) \\ \lambda_A : I \otimes A &\longrightarrow A && \text{by } (*, a) \longmapsto a \\ \rho_A : A \otimes I &\longrightarrow A && \text{by } (a, *) \longmapsto a \end{aligned}$$

Then it follows that $\mathbb{XG}\text{-Set}$ is a monoidal category. We now show that $\sigma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ as defined above is a braiding in $\mathbb{XG}\text{-Set}$. We first check that $\sigma_{X,Y}$ is an arrow in $\mathbb{XG}\text{-Set}$. Let $g \in G$ be arbitrary and $(x, y) \in X \otimes Y$ arbitrary. Then

$$\begin{aligned} \sigma_{X,Y}(g \cdot (x, y)) &= \sigma_{X,Y}(g \cdot x, g \cdot y) \\ &= (|g \cdot x| \cdot (g \cdot y), g \cdot x) \\ &= (g|x|g^{-1} \cdot (g \cdot y), g \cdot x) \\ &= ((g|x|gg^{-1}) \cdot y, g \cdot x) \\ &= ((g|x|) \cdot y, g \cdot x) \\ &= (g \cdot (|x| \cdot y), g \cdot x) \\ &= g \cdot (|x| \cdot y, x) \\ &= g \cdot \sigma_{X,Y}(x, y). \end{aligned}$$

Hence $\sigma_{X,Y}$ is equivariant. Moreover

$$\begin{aligned} |\sigma_{X,Y}(x, y)| &= (|x| \cdot y, x) \\ &= |x| \cdot y \cdot |x| \\ &= |x| |y| |x|^{-1} |x| \\ &= |x| |y| \\ &= |(x, y)|. \end{aligned}$$

Thus $\sigma_{X,Y}$ is a map of crossed G -sets. Now we show that $\sigma_{X,Y}$ is a natural isomorphism. To see that it is an isomorphism we will exhibit its inverse. Define

$$\sigma_{X,Y}^{-1} : Y \otimes X \longrightarrow X \otimes Y \quad \text{by } (y, x) \longmapsto (x, |x|^{-1} \cdot y).$$

Then one easily checks that this defines a map of crossed G -sets and that $\sigma_{X,Y} \sigma_{X,Y}^{-1} = id_{Y \otimes X}$ and $\sigma_{X,Y}^{-1} \sigma_{X,Y} = id_{X \otimes Y}$. Next we check that σ is natural. Let $f \in Hom_{\mathbf{xG}\text{-Set}}(X, X')$ and $g \in Hom_{\mathbf{xG}\text{-Set}}(Y, Y')$, then we must show that the following diagram commutes

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\sigma_{X,Y}} & Y \otimes X \\ f \otimes g \downarrow & & \downarrow g \otimes f \\ X' \otimes Y' & \xrightarrow{\sigma_{X',Y'}} & Y' \otimes X' \end{array}$$

Indeed

$$\begin{array}{ccccc} (x, y) & \xrightarrow{\sigma_{X,Y}} & (|x| \cdot y, x) & \xrightarrow{g \otimes f} & (g(|x| \cdot y), f(x)) \\ f \otimes g \downarrow & & & & \parallel \text{ } g \text{ is equivariant} \\ (f(x), g(y)) & \xrightarrow{\sigma_{X',Y'}} & (|f(x)| \cdot g(y), f(x)) & \xrightarrow{\text{by Def.2.8.10}} & (|x| \cdot g(y), f(x)) \end{array}$$

Finally we must check that diagrams (B1) and (B2) from Definition 2.8.1 commute. We have

$$\begin{array}{ccccc}
 & & (a, (b, c)) & \xrightarrow{\sigma} & (|a| \cdot b, |a| \cdot c, a) \\
 & \nearrow \alpha & & & \searrow \alpha \\
 ((a, b), c) & & & & (|a| \cdot b, (|a| \cdot c, a)) \\
 & \searrow \sigma \otimes 1_C & & & \nearrow 1_B \otimes \sigma \\
 & & (|a| \cdot b, a, c) & \xrightarrow{\alpha} & (|a| \cdot b, (a, c))
 \end{array}$$

$$\begin{array}{ccccc}
 & & ((a, b), c) & \xrightarrow{\sigma} & (|a||b| \cdot c, (a, b)) \\
 & \nearrow \alpha^{-1} & & & \searrow \alpha^{-1} \\
 (a, (b, c)) & & & & (|a||b| \cdot c, a, b) \\
 & \searrow 1_A \otimes \sigma & & & \nearrow \sigma \otimes 1_B \\
 & & (a, (|b| \cdot c, b)) & \xrightarrow{\alpha^{-1}} & ((a, |b| \cdot c), b)
 \end{array}$$

Hence σ is a braiding in $\mathbb{X}\mathbf{G}\text{-Set}$. \square

Definition 2.8.14 (Freyd and Yetter). *Let R be a fixed commutative ring with 1. The category $\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R$ consists of the following data. The objects are finite crossed G -sets, and $\text{Hom}_{\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R}(X, Y)$ is the set of all matrices M over R indexed by pairs $(x, y) \in X \times Y$, and satisfying*

$$M_{x,y} = M_{g \cdot x, g \cdot y} \quad \forall g \in G \quad \forall (x, y) \in X \times Y \quad (1)$$

$$M_{x,y} = \delta_{|x|, |y|} M_{x,y} \quad \forall (x, y) \in X \times Y \quad (2)$$

Lemma 2.8.15. $\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R$ is a category.

Proof. We first need to define composition of arrows in $\mathbf{G}\text{-}\mathbb{X}\mathbf{Mat}/R$. Given $M : X \rightarrow Y$ and $N : Y \rightarrow Z$ define $P := N \circ M : X \rightarrow Z$ as follows:

$$P_{x,z} = \sum_{y \in Y} M_{x,y} N_{y,z} \quad (\text{matrix multiplication}).$$

Then it is straightforward to check that the entries of P satisfy conditions (1) and (2). Next define $1_X : X \rightarrow X$ by

$$(1_X)_{x,x'} := \delta_{x,x'} \quad (\text{identity matrix}).$$

The associativity and identity axioms for a category follow from the properties of matrix multiplication. \square

We will now spend some time proving some important facts about $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. These proofs will involve numerous calculations so we introduce a new notation to handle these.

Notation 2.8.16 (A Tensor Calculus For $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$). *Given $f : X \longrightarrow Y$ in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$, where $f = [F_{x,y}]$ we write the indices coming from $X = \text{dom}f$ as subscripts and indices coming from $Y = \text{codom}f$ as superscripts. In the new notation $f = [F_x^y]$.*

The second convention that we will adopt is that whenever we have an up index and a down index repeated, we will assume that we are summing over the repeated index. For example in the old notation we had $f = [F_{x,y}] : X \longrightarrow Y$ and $g = [G_{y,z}] : Y \longrightarrow Z$ then $(g \circ f)_{x,z} = \sum_{y \in Y} F_{x,y} G_{y,z}$. In the new notation $(g \circ f)_{x,z} = F_x^y G_y^z$, i.e. $\sum_{y \in Y} F_{x,y} G_{y,z} = F_x^y G_y^z$. We will switch back and forth between notations without comment, using the one that best suits our needs at the time.

This next result is due to Freyd and Yetter however we fill in the details which are left to the reader in [7].

Theorem 2.8.17. *There exists a functor $\otimes : (\mathbf{G} - \mathbb{X}\mathbf{Mat}/R) \times (\mathbf{G} - \mathbb{X}\mathbf{Mat}/R) \longrightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$, an object $I \in |\mathbf{G} - \mathbb{X}\mathbf{Mat}/R|$, and natural isomorphisms α , λ , and ρ such that $(\mathbf{G} - \mathbb{X}\mathbf{Mat}/R, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category.*

Proof. We define $\otimes : (\mathbf{G} - \mathbb{X}\mathbf{Mat}/R) \times (\mathbf{G} - \mathbb{X}\mathbf{Mat}/R) \longrightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ as follows: Given X and Y , crossed G -sets, define $X \otimes Y$ as in Theorem 2.8.13. For arrows $M \in \text{Hom}_{\mathbf{G} - \mathbb{X}\mathbf{Mat}/R}(X, Y)$ and $M' \in \text{Hom}_{\mathbf{G} - \mathbb{X}\mathbf{Mat}/R}(X', Y')$ where $M = [M_x^y]$ and $M' = [M_{x'}^{y'}]$ define

$$(M \otimes M')_{(x,x')}^{(y,y')} = M_x^y M_{x'}^{y'} \quad \forall (x, x') \in X \otimes X', (y, y') \in Y \otimes Y'.$$

We check that $M \otimes M' : X \otimes X' \longrightarrow Y \otimes Y'$ is in fact an arrow in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$.

Let $g \in G$ then

$$\begin{aligned}
 (M \otimes M')_{g \cdot (x, x')}^{g \cdot (y, y')} &= (M \otimes M')_{(g \cdot x, g \cdot x')}^{(g \cdot y, g \cdot y')} \\
 &= M_{g \cdot x}^{g \cdot y} M'_{g \cdot x'}^{g \cdot y'} \\
 &= M_x^y M'_{x'}^{y'} \\
 &= (M \otimes M')_{(x, x')}^{(y, y')}.
 \end{aligned}$$

Hence $(M \otimes M')_{g \cdot (x, x')}^{g \cdot (y, y')} = (M \otimes M')_{(x, x')}^{(y, y')} \quad \forall (x, x') \in X \otimes X', (y, y') \in Y \otimes Y', g \in G.$

Lastly we must show that

$$\begin{aligned}
 (M \otimes M')_{(x, x')}^{(y, y')} &= \delta_{|(x, x')|, |(y, y')|} (M \otimes M')_{(x, x')}^{(y, y')} \\
 &= \delta_{|x||x'|, |y||y'|} (M \otimes M')_{(x, x')}^{(y, y')}.
 \end{aligned}$$

So we must show that if $|x||x'| \neq |y||y'|$ then $(M \otimes M')_{(x, x')}^{(y, y')} = 0.$

Case 1: $|x| \neq |y|$

Then

$$\begin{aligned}
 (M \otimes M')_{(x, x')}^{(y, y')} &= M_x^y M'_{x'}^{y'} \\
 &= \delta_{|x|, |y|} M_x^y M'_{x'}^{y'} \\
 &= 0
 \end{aligned}$$

as $|x| \neq |y|.$

Case 2: $|x| = |y|$

Then $|x||x'| = |y||x'|.$ Since we are assuming that $|x||x'| \neq |y||y'|$ it follows that $|x'| \neq |y'|.$ Hence

$$\begin{aligned}
 (M \otimes M')_{(x, x')}^{(y, y')} &= M_x^y M'_{x'}^{y'} \\
 &= M_x^y \delta_{|x'|, |y'|} M'_{x'}^{y'} \\
 &= 0
 \end{aligned}$$

as $|x'| \neq |y'|.$ Hence $M \otimes M'$ is an arrow in $\mathbf{G} - \mathbf{XMat}/R.$ To show \otimes is a functor it remains to show that it preserves identities and composition.

For $1_X : X \rightarrow X$ and $1_Y : Y \rightarrow Y$ we check that $1_X \otimes 1_Y = 1_{X \otimes Y}$. Indeed

$$\begin{aligned} (1_X \otimes 1_Y)_{(x,y)}^{(x',y')} &= (1_X)_x^y (1_Y)_{x'}^{y'} \\ &= \delta_{x,y} \delta_{x',y'} \\ &= \delta_{(x,x'),(y,y')} \\ &= (1_{X \otimes Y})_{(x,y)}^{(x',y')}. \end{aligned}$$

Let $F = [F_x^y] : X \rightarrow Y$, $G = [G_y^z] : Y \rightarrow Z$, $H = [H_u^v] : U \rightarrow V$, and $K = [K_v^w] : V \rightarrow W$ be arrows in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. Then we must check that $G \circ F \otimes K \circ H = G \otimes K \circ F \otimes H$. We have

$$\begin{aligned} (G \circ F \otimes K \circ H)_{(x,v)}^{(z,w)} &= (G \circ F)_x^z (K \circ H)_v^w \\ &= F_x^y G_y^z H_v^u K_u^w \\ &= F_x^y H_v^u G_y^z K_u^w && (R \text{ commutative}) \\ &= (F \otimes H)_{(x,v)}^{(y,u)} (G \otimes K)_{(y,u)}^{(z,w)} \\ &= (G \otimes K \circ F \otimes H)_{(x,v)}^{(z,w)}. \end{aligned}$$

Therefore \otimes is a functor. Next we define the tensor unit $I = \{1\}$ where $g \cdot 1 = 1 \forall g \in G$ and $|| : I \rightarrow G$ is given by $|1| = 1_G$. This makes I into a crossed G -set, and hence an object in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. Next, we define the isomorphisms α , λ , and ρ , which will of course be different from those in Theorem 2.8.13. For $A, B, C \in |\mathbf{G} - \mathbb{X}\mathbf{Mat}/R|$ define $\alpha = \alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$ by

$$\alpha_{((a,b),c)}^{(a',(b',c'))} = \delta_{a,a'} \delta_{b,b'} \delta_{c,c'}.$$

We define $\lambda = \lambda_A : I \otimes A \rightarrow A$ by

$$\lambda_{(1,a)}^b = \delta_{a,b}$$

and $\rho = \rho_A : A \otimes I \rightarrow A$ by

$$\rho_{(a,1)}^b = \delta_{a,b}.$$

The fact that α , λ , and ρ are arrows in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ follows easily from properties of the Kronecker delta. To see that α , λ , and ρ are isomorphisms we will exhibit their

inverses.

$\alpha^{-1} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$ is given by

$$(\alpha^{-1})_{(a', (b', c'))}^{((a, b), c)} = \delta_{a', a} \delta_{b', b} \delta_{c', c}.$$

$\lambda^{-1} : A \longrightarrow I \otimes A$ is given by

$$(\lambda^{-1})_b^{(1, a)} = \delta_{b, a}.$$

$\rho^{-1} : A \longrightarrow A \otimes I$ is given by

$$(\rho^{-1})_b^{(a, 1)} = \delta_{b, a}.$$

Next we check that these maps are natural.

naturality of α :

Given $P = [P_a^{a'}] : A \longrightarrow A'$, $S = [S_b^{b'}] : B \longrightarrow B'$, $T = [T_c^{c'}] : C \longrightarrow C'$ we must show

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{(P \otimes S) \otimes T} & (A' \otimes B') \otimes C' \\ \downarrow \alpha & & \downarrow \alpha \\ A \otimes (B \otimes C) & \xrightarrow{P \otimes (S \otimes T)} & A' \otimes (B' \otimes C') \end{array}$$

commutes. We have

$$\begin{aligned} (\alpha \circ (P \otimes S) \otimes T)_{(a, b), c}^{(a', (b', c'))} &= ((P \otimes S) \otimes T)_{(a, b), c}^{((\bar{a}, \bar{b}), \bar{c})} \alpha_{(\bar{a}, \bar{b}), \bar{c}}^{(a', (b', c'))} \\ &= ((P \otimes S) \otimes T)_{(a, b), c}^{(a', (b', c'))} \\ &= (P_a^{a'} S_b^{b'}) T_c^{c'}. \end{aligned}$$

On the other hand going around the bottom leg we have

$$\begin{aligned} (P \otimes (S \otimes T) \circ \alpha)_{(a, b), c}^{(a', (b', c'))} &= \alpha_{(\bar{a}, (\bar{b}, \bar{c}))}^{(a', (b', c'))} (P \otimes (S \otimes T))_{(\bar{a}, (\bar{b}, \bar{c}))}^{(a', (b', c'))} \\ &= (P \otimes (S \otimes T))_{(a, (b, c))}^{(a', (b', c'))} \\ &= P_a^{a'} (S_b^{b'} T_c^{c'}) \\ &= (P_a^{a'} S_b^{b'}) T_c^{c'}. \end{aligned}$$

Hence the required diagram commutes, and α is natural.

naturality of ρ :

Given $T = [T_a^b] : A \longrightarrow B$ we must show

$$\begin{array}{ccc} A \otimes I & \xrightarrow{\rho} & A \\ T \otimes 1_I \downarrow & & \downarrow T \\ B \otimes I & \xrightarrow{\rho} & B \end{array}$$

commutes. The top leg of the diagram gives

$$\begin{aligned} (T \circ \rho)_{(a,1)}^b &= \rho_{(a,1)}^{a'} T_{a'}^b \\ &= T_a^b \end{aligned}$$

while the bottom leg gives

$$\begin{aligned} (\rho \circ T \otimes 1_I)_{(a,1)}^b &= (T \otimes 1_I)_{(a,1)}^{(b',1)} \rho_{(b',1)}^b \\ &= (T \otimes 1_I)_{(a,1)}^{(b,1)} \\ &= T_a^b \delta_{1,1} \\ &= T_a^b. \end{aligned}$$

Hence ρ is natural. A similar calculation shows that λ is also natural. To show that $(\mathbf{G} - \mathbb{X}\mathbf{Mat}/R, \otimes, I, \alpha, \lambda, \rho)$ is a monoidal category it remains to show that (M1), (M2), and (M3) in Definition 2.1.1 commute. We notice that, as matrices, all the arrows involved in the diagrams (M1) and (M2) are identities, and so it is immediate that these diagrams commute. Equation (M3) obviously holds by definition of λ and ρ . \square

We now go on to prove the theorem, due to Freyd and Yetter, that $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is braided. Our method of proof is distinct from theirs, using a construction we stumbled upon in working with this category. Our first step is thus the construction of a strict monoidal functor $Mat : \mathbb{X}\mathbf{G} - \mathbf{Set} \longrightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$.

Theorem 2.8.18. *The map $Mat : \mathbb{X}\mathbf{G} - \mathbf{Set} \longrightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ defined by $Mat(X) = X$ and $(Mat(f))_x^y = \delta_{f(x),y}$ for $f : X \longrightarrow Y$ is a strict monoidal functor.*

Proof. First we show that $Mat : \mathbb{X}\mathbf{G}\text{-Set} \rightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is a functor. Let $f \in Hom_{\mathbb{X}\mathbf{G}\text{-Set}}(X, Y)$, then we must show that $Mat(f) \in Hom_{\mathbf{G} - \mathbb{X}\mathbf{Mat}/R}(X, Y)$. We must show that $Mat(f)$ satisfies the two conditions of Definition 2.8.14. Let $g \in G$ arbitrary then

$$\begin{aligned} (Mat(f))_{g \cdot x}^{g \cdot y} &= \delta_{f(g \cdot x), g \cdot y} \\ &= \delta_{g \cdot f(x), g \cdot y} \\ &= \delta_{f(x), y} \\ &= (Mat(f))_x^y \quad \forall x \in X, y \in Y. \end{aligned}$$

For the second condition we must show that if $|x| \neq |y|$ then $(Mat(f))_x^y = 0$. Assume $|x| \neq |y|$ then as f is a map of crossed G -sets we have that $|x| = |f(x)|$. Hence $|f(x)| \neq |y|$ as $|x| \neq |y|$ and so $f(x) \neq y$. So $(Mat(f))_x^y = 0$. Thus $Mat(f)$ is an arrow in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$.

Now let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be arrows in $\mathbb{X}\mathbf{G}\text{-Set}$ then we check that $Mat(g \circ f) = Mat(g) \circ Mat(f)$. Indeed

$$\begin{aligned} (Mat(g) \circ Mat(f))_x^z &= (Mat(f))_x^y (Mat(g))_y^z \\ &= \sum_{y \in Y} \delta_{f(x), y} \delta_{g(y), z} \\ &= \delta_{g(f(x)), z} \\ &= (Mat(g \circ f))_x^z \quad \forall x \in X, y \in Y. \end{aligned}$$

Finally, it's clear that $Mat(1_X) = 1_X$, and thus Mat is a functor. To avoid any confusion let us label the monoidal categories involved as follows: $(\mathbb{X}\mathbf{G}\text{-Set}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathbf{G} - \mathbb{X}\mathbf{Mat}/R, \otimes, I, \alpha', \lambda', \rho')$. Then it is clear from the definition of the functor Mat that $Mat(\alpha) = \alpha'$, $Mat(\lambda) = \lambda'$, and $Mat(\rho) = \rho'$. Finally we check that $Mat(f \otimes g) = Mat(f) \otimes Mat(g)$ for arrows $f : X \rightarrow Y$ and $g : U \rightarrow V$.

$$\begin{aligned} (Mat(f \otimes g))_{(x, u)}^{(y, v)} &= \delta_{(f(x), g(u)), (y, v)} \\ &= \delta_{f(x), y} \delta_{g(u), v} \\ &= (Mat(f))_x^y (Mat(g))_u^v \\ &= (Mat(f) \otimes Mat(g))_{(x, u)}^{(y, v)}. \end{aligned}$$

□

We can now prove the main theorem for this section.

Theorem 2.8.19. *For any group G , the monoidal category $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is braided. The braiding $\sigma'_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ is given by $\sigma'_{X,Y} := \text{Mat}(\sigma_{X,Y})$, where $\sigma_{X,Y}$ is the braiding in $\mathbb{X}\mathbf{G} - \mathbf{Set}$.*

Proof. Since Mat is a strict monoidal functor and $\sigma_{X,Y}$ is a braiding for $\mathbb{X}\mathbf{G} - \mathbf{Set}$, Remark 2.6.2 guarantees that diagrams (B1) and (B2) of Definition 2.8.1 commute in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. Since functors preserve isomorphisms we also have that $\sigma'_{X,Y} = \text{Mat}(\sigma_{X,Y})$ is an isomorphism. So it remains to show that $\sigma'_{X,Y}$ is natural.

naturality of $\sigma'_{X,Y}$:

Given $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ check that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\sigma'_{X,Y}} & Y \otimes X \\ f \otimes g \downarrow & & \downarrow g \otimes f \\ X' \otimes Y' & \xrightarrow{\sigma'_{X',Y'}} & Y' \otimes X' \end{array}$$

commutes.

Let $f = [F_x^{x'}]$ and $g = [G_y^{y'}]$ then the top leg of the diagram is:

$$\begin{aligned} ((g \otimes f) \circ \sigma')_{(x,y)}^{(y',x')} &= (\sigma')_{(x,y)}^{(y,\bar{x})} (g \otimes f)_{(y,\bar{x})}^{(y',x')} \\ &= \sum_{\bar{x} \in X, \bar{y} \in Y} \delta_{(|x| \cdot y, x), (y, \bar{x})} G_{\bar{y}}^{y'} F_{\bar{x}}^{x'} \\ &= G_{|x| \cdot y}^{y'} F_x^{x'}. \end{aligned} \tag{*}$$

The bottom leg of the diagram is:

$$\begin{aligned}
 (\sigma' \circ (f \otimes g))_{(x,y)}^{(y',x')} &= (f \otimes g)_{(x,y)}^{(\bar{x},\bar{y})} (\sigma')_{(x,y)}^{(y',x')} \\
 &= \sum_{\bar{x} \in X, \bar{y} \in Y} F_x^{\bar{x}} G_y^{\bar{y}} \delta_{(|\bar{x}|, \bar{y}, \bar{x}), (y', x')} \\
 &= F_x^{x'} G_y^{|x'|^{-1} \cdot y'} \\
 &= F_x^{x'} G_{|x'| \cdot y}^{|x'| \cdot (|x'|^{-1} \cdot y')} \\
 &= F_x^{x'} G_{|x'| \cdot y}^{y'}. \tag{**}
 \end{aligned}$$

Then if $|x| = |x'|$, (*) and (**) are equal. If $|x| \neq |x'|$ then $F_x^{x'} = 0$ and so (*) is again equal to (**). Thus σ' is natural. \square

2.9 Pivotal Categories

Definition 2.9.1. A (strict) *pivotal category* is a monoidal category, \mathcal{C} , equipped with a contravariant functor $(-)^*$ satisfying $(f \otimes g)^* = g^* \otimes f^*$, $I = I^*$, and $(-)^{**} = id_{\mathcal{C}}$, and a family of maps $\epsilon_A : A \otimes A^* \rightarrow I$ satisfying the following three diagrams

$$\begin{array}{ccc}
 (A \otimes B) \otimes (B^* \otimes A^*) & \xrightarrow{\alpha} & A \otimes (B \otimes (B^* \otimes A^*)) \\
 & & \downarrow 1_A \otimes \alpha^{-1} \\
 & & A \otimes ((B \otimes B^*) \otimes A^*) \\
 & & \downarrow 1_A \otimes (\epsilon_B \otimes 1_{A^*}) \\
 & & A \otimes (I \otimes A^*) \\
 & & \downarrow 1_A \otimes \lambda \\
 & & A \otimes A^* \\
 & & \downarrow \epsilon_A \\
 & & I
 \end{array}
 \tag{P1}$$

$\epsilon_{A \otimes B}$

$$\begin{array}{ccc}
 I \otimes I^* & \xlongequal{\quad} & I \otimes I \\
 \epsilon_I \searrow & & \swarrow \lambda \\
 & I &
 \end{array}$$

(P2)

and

$$\begin{array}{ccc}
 (A^* \otimes A) \otimes B^* & \xrightarrow{(1_{A^*} \otimes f) \otimes 1_{B^*}} & (A^* \otimes B) \otimes B^* \xrightarrow{\alpha} A^* \otimes (B \otimes B^*) \\
 \eta_A \otimes 1_{B^*} \uparrow & & \downarrow 1_{A^*} \otimes \epsilon_B \\
 I \otimes B^* & & A^* \otimes I \\
 \lambda^{-1} \uparrow & & \downarrow \rho \\
 B^* & \xrightarrow{f^*} & A^* \\
 \rho^{-1} \downarrow & & \uparrow \lambda \\
 B^* \otimes I & & I \otimes A^* \\
 1_{B^*} \otimes \eta_{A^*} \downarrow & & \uparrow \epsilon_{B^*} \otimes 1_{A^*} \\
 B^* \otimes (A \otimes A^*) & \xrightarrow[1_{B^*} \otimes (f \otimes 1_{A^*})]{} & B^* \otimes (B \otimes A^*) \xrightarrow{\alpha^{-1}} (B^* \otimes B) \otimes A^*
 \end{array}$$

(P3)

where $\eta_A = (\epsilon_{A^*})^*$ [7].

We state this next result without proof. The details can be found in [7].

Theorem 2.9.2. $\mathbf{G} - \mathbf{XMat}/R$ is pivotal with X^* having the same underlying G -set as X , but $| \cdot |$ replaced by $| \cdot |^{-1}$, and with η and ϵ determined by

$$(\eta_X)_1^{(x,y)} = \delta_{x,y}$$

and

$$(\epsilon_X)_{(x,y)}^1 = \delta_{x,y}.$$

Similarly β is **nondegenerate in the variable W** if there exists a copairing $\gamma : \mathbb{k} \rightarrow W \otimes V$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 W & \xrightarrow{\lambda_W^{-1}} & \mathbb{k} \otimes W & \xrightarrow{\gamma \otimes 1} & (W \otimes V) \otimes W & \xrightarrow{\alpha} & W \otimes (V \otimes W) \\
 & & & & & & \downarrow 1 \otimes \beta \\
 & & & & & & W \otimes \mathbb{k} \\
 & & & & & & \downarrow \rho_W \\
 & & & & & & W \\
 & \searrow 1_W & & & & & \\
 & & & & & &
 \end{array}$$

Finally we say that the pairing β is **nondegenerate** if it is nondegenerate in both variables [12].

It is not immediately obvious however we do have the following result.

Lemma 3.1.2. *If the pairing $\beta : V \otimes W \rightarrow \mathbb{k}$ is nondegenerate then the copairings agree [12].*

Remark 3.1.3. Saying that the pairing $\beta : V \otimes W \rightarrow \mathbb{k}$ is nondegenerate is just another way of saying that $V \dashv W$ in the monoidal category $\mathbf{Vect}_{\mathbb{k}}$.

Remark 3.1.4. Given a pairing $\beta : V \otimes W \rightarrow \mathbb{k}$, for each fixed $v \in V$ we have a linear map ${}_v\beta$ induced by β which is given by

$$\begin{aligned}
 {}_v\beta : W &\longrightarrow \mathbb{k} \\
 w &\longmapsto \langle v|w \rangle.
 \end{aligned}$$

So ${}_v\beta \in W^* \forall v \in V$. Now as β is linear we obtain a linear map $\beta_{right} : V \rightarrow W^*$ given by $v \mapsto {}_v\beta = \langle v|-\rangle$. Similarly β induces a linear map $\beta_{left} : W \rightarrow V^*$ given by $w \mapsto \beta_w = \langle -|w \rangle$. Probably a more familiar notion of nondegeneracy is the requirement that β_{right} and β_{left} be injective, which is a priori weaker than our notion.

We now record some results which help us characterize when these two notions coincide.

Lemma 3.1.5. *The pairing $\beta : V \otimes W \longrightarrow \mathbb{k}$ is nondegenerate in W if and only if W is finite-dimensional and the induced map $\beta_{\text{left}} : W \longrightarrow V^*$ is injective. Similarly, nondegeneracy in V is equivalent to finite-dimensionality of V plus injectivity of $\beta_{\text{right}} : V \longrightarrow W^*$.*

Lemma 3.1.6. *Let V and W be finite-dimensional vector spaces, and $\beta : V \otimes W \longrightarrow \mathbb{k}$ a pairing. Then β_{right} is the dual map of β_{left} modulo the identification $V \cong V^{**}$. Also β_{left} is the dual of the map of β_{right} modulo the identification $W \cong W^{**}$.*

Lemma 3.1.7. *Given a pairing $\beta : V \otimes W \longrightarrow \mathbb{k}$ between finite-dimensional vector spaces, the following are equivalent*

- (i) β is nondegenerate.
- (ii) The induced map $\beta_{\text{left}} : W \rightarrow V^*$ is an isomorphism.
- (iii) The induced map $\beta_{\text{right}} : V \rightarrow W^*$ is an isomorphism.

If we know in advance that V and W have the same dimension then being an isomorphism in the lemma is equivalent to being injective, so in that case nondegeneracy can be characterized by each of the following conditions

- (i)' $\langle v|w \rangle = 0 \forall v \in V \implies w = 0$;
- (ii)' $\langle v|w \rangle = 0 \forall w \in W \implies v = 0$.

For the remainder of this section we let (A, μ, η) be associative \mathbb{k} -algebra.

Now suppose that M is a right A -module. Then $M^* = \text{Hom}(M, \mathbb{k})$ has a natural left A -module structure given by:

$$A \otimes M^* \longrightarrow M^* \quad a \otimes f \mapsto a \cdot f := [m \mapsto f(m \cdot a)].$$

Similarly, if M is a left A -module, then M^* has a right A -module structure given by:

$$M^* \otimes A \longrightarrow M^* \quad f \otimes a \mapsto f \cdot a := [m \mapsto f(a \cdot m)].$$

If M and N are right (resp. left) A -modules and $\psi : M \rightarrow N$ is a right (resp. left) A -homomorphism the dual map:

$$\begin{aligned} N^* &\xrightarrow{\psi^*} M^* \\ \phi &\longmapsto \phi \circ \psi \end{aligned}$$

is a left (resp. right) A -homomorphism. It is then easy to show that $(\)^*$ is a contravariant functor from the category of right A -modules to left A -modules and the same holds with left and right switched.

Definition 3.1.8. Given a pairing $\beta : M \otimes N \rightarrow \mathbb{k}$ where M is a right A -module and N is a left A -module we say that β is **associative** if:

$$\begin{array}{ccc} (M \otimes A) \otimes N & \xrightarrow{\alpha} & M \otimes (A \otimes N) \\ \chi_M \otimes 1_N \downarrow & & \downarrow 1_M \otimes \chi_N \\ M \otimes N & & M \otimes N \\ & \searrow \beta & \swarrow \beta \\ & \mathbb{k} & \end{array}$$

commutes. i.e. $\langle x \cdot a | y \rangle = \langle x | a \cdot y \rangle \quad \forall x \in M, \forall y \in N, \forall a \in A.$

Lemma 3.1.9. For a pairing $\beta : M \otimes N \rightarrow \mathbb{k}$ with M a right A -module and N a left A -module, the following are equivalent:

- (i) $\beta : M \otimes N \rightarrow \mathbb{k}$ is associative.
- (ii) $\beta_{\text{left}} : N \rightarrow M^*$ is left A -linear.
- (iii) $\beta_{\text{right}} : M \rightarrow N^*$ is right A -linear.

We now focus on the setting where the A -modules in consideration are the \mathbb{k} -algebra itself. For the remainder of this section we assume A to be finite-dimensional. Note that A has both a left and a right A -module structure. We are now ready to define the key concept of this section.

Definition 3.1.10. A **Frobenius algebra** is a \mathbb{k} -algebra A of finite dimension, equipped with a linear functional $\varepsilon : A \rightarrow \mathbb{k}$ whose nullspace $\text{Null}(\varepsilon) = \{x \in A \mid \varepsilon(x) = 0\}$ contains no nontrivial left ideals. The map $\varepsilon \in A^*$ is called a **Frobenius form**.

Remark 3.1.11. Having no nontrivial left ideals in $\text{Null}(\varepsilon)$ is equivalent to having no nontrivial principal left ideals in $\text{Null}(\varepsilon)$. So we restate this condition like this:

$$\varepsilon(Ay) = 0 \Rightarrow y = 0.$$

Remark 3.1.12. There is a one-to-one correspondence between linear functionals on A and associative pairings. Given $\varepsilon : A \rightarrow \mathbb{k}$ define $\beta_\varepsilon : A \otimes A \rightarrow \mathbb{k}$ by the following composite

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\varepsilon} \mathbb{k}.$$

Given an associative pairing $\beta : A \otimes A \rightarrow \mathbb{k}$ define $\varepsilon_\beta : A \rightarrow \mathbb{k}$ by $a \mapsto \langle a|1_A \rangle = \langle 1_A|a \rangle$. One easily checks that these processes are inverse.

Lemma 3.1.13. Let $\varepsilon : A \rightarrow \mathbb{k}$ be a linear functional and let $\langle \mid \rangle$ denote the corresponding associative pairing $A \otimes A \rightarrow \mathbb{k}$. The following are equivalent:

- (i) The pairing is nondegenerate.
- (ii) $\text{Null}(\varepsilon)$ contains no nontrivial left ideals.
- (iii) $\text{Null}(\varepsilon)$ contains no nontrivial right ideals.

This lemma allows us to give an equivalent characterization of a Frobenius algebra.

Corollary 3.1.14. If A is a \mathbb{k} -algebra of finite dimension equipped with an associative nondegenerate pairing $\beta : A \otimes A \rightarrow \mathbb{k}$, then (A, ε_β) is a Frobenius algebra. We call the pairing β the **Frobenius pairing**.

This corollary then quickly leads to:

Corollary 3.1.15. If A is a finite-dimensional \mathbb{k} -algebra equipped with a left A -isomorphism ϕ to its dual then $(A, \phi(1_A))$ is a Frobenius algebra. Equivalently if A is equipped with a right A -isomorphism ψ to its dual then $(A, \psi(1_A))$ is a Frobenius algebra.

Proposition 3.1.16. *If (A, ε) is a Frobenius algebra then the following conditions are equivalent:*

- (i) *The Frobenius form $\varepsilon : A \rightarrow \mathbb{k}$ is central. i.e. $\varepsilon(ab) = \varepsilon(ba) \forall a, b \in A$.*
- (ii) *The pairing $\langle | \rangle$ is symmetric. i.e. $\langle a|b \rangle = \langle b|a \rangle \forall a, b \in A$.*
- (iii) *The left A -isomorphism $A \xrightarrow{\cong} A^*$ is also right A -linear.*
- (iv) *The right A -isomorphism $A \xrightarrow{\cong} A^*$ is also left A -linear.*

Definition 3.1.17. *If (A, ε) is a Frobenius algebra we say that it is **symmetric** if one of the equivalent conditions of Proposition 3.1.16 holds.*

Example 3.1.18. If $A = \mathbb{k}$, then taking $\varepsilon : A \rightarrow \mathbb{k}$ to be the identity makes (A, ε) a Frobenius algebra since $\ker(\varepsilon) = 0$ is trivial. Hence \mathbb{k} is a symmetric Frobenius algebra.

Example 3.1.19. Let A be a finite field extension of \mathbb{k} . Since fields have no nontrivial ideals, any nonzero \mathbb{k} -linear map $A \rightarrow \mathbb{k}$ will be a Frobenius form. Hence A is a Frobenius algebra; it is symmetric since A is commutative.

Example 3.1.20. The ring of $n \times n$ matrices over \mathbb{k} , $Mat_n(\mathbb{k})$, is a Frobenius algebra with Frobenius form given by the trace map:

$$\varepsilon = Tr : Mat_n(\mathbb{k}) \rightarrow \mathbb{k} \quad (a_{i,j}) \mapsto \sum_{i=1}^n a_{i,i}.$$

To show that $(Mat_n(\mathbb{k}), \varepsilon)$ is a Frobenius algebra we show that the pairing β_ε induced by ε is nondegenerate. We will show that the contrapositive of (i)' in Lemma 3.1.7 holds. It is enough to show this using a (linear) basis of $Mat_n(\mathbb{k})$. Let $E_{i,j}$ be the matrix with only one nonzero entry $e_{i,j} = 1$. Then these matrices $E_{i,j}$ where $i, j \in (1, \dots, n)$ form a basis of $Mat_n(\mathbb{k})$. Note that the (k,l) entry of $E_{i,j}$ is $(E_{i,j})_{k,l} = \delta_{k,i} \delta_{l,j}$.

We claim that $\beta_\varepsilon(E_{i,j} \otimes E_{j,i}) \neq 0$. Indeed

$$\begin{aligned}
\beta_\varepsilon(E_{i,j} \otimes E_{j,i}) &= \text{Tr}(E_{i,j}E_{j,i}) \\
&= \sum_{p=1}^n (E_{i,j}E_{j,i})_{p,p} \\
&= \sum_{p=1}^n \sum_{l=1}^n (E_{i,j})_{p,l} (E_{j,i})_{l,p} \\
&= \sum_{p=1}^n \sum_{l=1}^n \delta_{p,i} \delta_{l,j} \delta_{l,j} \delta_{p,i} \\
&= \sum_{p=1}^n \delta_{p,i} \\
&= 1 \neq 0.
\end{aligned}$$

Hence $(\text{Mat}_n(\mathbb{k}), \text{Tr})$ is a Frobenius algebra. In fact it is symmetric since $\text{Tr}(AB) = \text{Tr}(BA)$.

Example 3.1.21. Let $G = \{t_0, \dots, t_n\}$ be a finite group with unit element t_0 . The group algebra $\mathbb{k}G = \{\sum_{i=0}^n a_i t_i \mid a_i \in \mathbb{k}\}$ with multiplication given by multiplication in G is an associative \mathbb{k} -algebra. We define a Frobenius form as follows:

$$\varepsilon : \mathbb{k}G \longrightarrow \mathbb{k} \quad t_i \mapsto \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

and extend by linearity. So the induced pairing sends $g \otimes h$ to $\varepsilon(gh)$. This pairing is nondegenerate as $\varepsilon(t_i t_i^{-1}) = \varepsilon(t_0) = 1 \neq 0 \forall t_i \in G$.

3.2 Frobenius Algebras and Coalgebras

We now state two propositions which can be found in Kock's book. Together these results establish the connection between Frobenius algebras and associative \mathbb{k} -algebras of finite dimension equipped with a coalgebra structure and satisfying the so called **Frobenius relation**.

Proposition 3.2.1. *Given a Frobenius algebra (A, ε) there exists a unique comultiplication δ whose counit is ε and which satisfies the Frobenius relation:*

$$\begin{array}{ccccc}
 A \otimes A & \xrightarrow{1_A \otimes \delta} & A \otimes (A \otimes A) & \xrightarrow{\alpha^{-1}} & (A \otimes A) \otimes A \\
 \downarrow \delta \otimes 1_A & \searrow \mu & & & \downarrow \mu \otimes 1_A \\
 & & A & & A \\
 & & \searrow \delta & & \\
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{1_A \otimes \mu} & A \otimes A
 \end{array} \quad (\text{Frob. rel.})$$

and this comultiplication is coassociative.

Example 3.2.2. If G is a finite group and \mathbb{k} is a field then the group algebra $A = \mathbb{k}G$ can be made into a Frobenius algebra as seen in Example 3.1.21. The comultiplication given by the previous proposition is

$$\delta(g) = \sum_{hk=g} h \otimes k$$

for all $g \in G$ which is then extended by linearity. It is then a routine exercise to show that δ satisfies the required equations in the above proposition.

Proposition 3.2.3. *Let A denote a vector space equipped with a multiplication map $\mu : A \otimes A \rightarrow A$, with unit map $\eta : \mathbb{k} \rightarrow A$, a comultiplication $\delta : A \rightarrow A \otimes A$ with counit $\varepsilon : A \rightarrow \mathbb{k}$, and suppose that the Frobenius relation holds. Then*

- (i) *the vector space A is of finite dimension,*
- (ii) *the multiplication μ is associative, and thus A is an associative \mathbb{k} -algebra of finite dimension (also, the comultiplication is coassociative),*
- (iii) *the counit ε is a Frobenius form, and thus (A, ε) is a Frobenius algebra.*

With these two propositions we can easily see how to generalize the definition of Frobenius algebra to appropriately endowed monoidal categories.

3.3 Abstract Theory of Frobenius Algebras: Frobenius Objects

Almost everything that we have done so far can be generalized to the setting of an abstract monoidal category with many of the same results remaining true.

Definition 3.3.1. Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A **Frobenius object** in \mathcal{C} is an object A equipped with four maps:

$$\eta : I \longrightarrow A, \quad \mu : A \otimes A \longrightarrow A, \quad \delta : A \longrightarrow A \otimes A, \quad \text{and} \quad \varepsilon : A \longrightarrow I$$

such that the diagrams below, all commute:

$$\begin{array}{ccccc}
 I \otimes A & \xrightarrow{\eta \otimes 1_A} & A \otimes A & \xleftarrow{1_A \otimes \eta} & A \otimes I \\
 & \searrow \lambda & \downarrow \mu & & \swarrow \rho \\
 & & A & &
 \end{array} \tag{FA1}$$

$$\begin{array}{ccccc}
 I \otimes A & \xleftarrow{\varepsilon \otimes 1_A} & A \otimes A & \xrightarrow{1_A \otimes \varepsilon} & A \otimes I \\
 & \searrow \lambda & \uparrow \delta & & \swarrow \rho \\
 & & A & &
 \end{array} \tag{FA2}$$

$$\begin{array}{ccccccc}
 A \otimes A & \xrightarrow{1_A \otimes \delta} & A \otimes (A \otimes A) & \xrightarrow{\alpha^{-1}} & (A \otimes A) \otimes A & & \\
 \downarrow \delta \otimes 1_A & & \searrow \mu & & \downarrow \mu \otimes 1_A & & \\
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{1_A \otimes \mu} & A \otimes A & & \\
 & & \swarrow \delta & & & &
 \end{array} \tag{FA3}$$

This definition is taken from [12].

Remark 3.3.2. Frobenius objects in the monoidal category \mathbf{Vect}_k are precisely Frobenius algebras in the usual sense.

Lemma 3.3.3. *If $(A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object then (A, μ, η) is a monoid and (A, δ, ε) is a comonoid [12].*

We now state and prove a theorem, a variation of which can be found in a paper by Street [17]. This establishes some equivalent conditions characterizing Frobenius objects.

Theorem 3.3.4. *Suppose that (A, μ, η) is a monoid in a monoidal category \mathcal{C} . Then the following are equivalent:*

(a) $\exists \varepsilon : A \rightarrow I$ and $\varphi : I \rightarrow A \otimes A$ such that

$$\begin{array}{ccccccc}
 A & \xrightarrow{\lambda^{-1}} & I \otimes A & \xrightarrow{\varphi \otimes 1_A} & (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) \\
 \rho^{-1} \downarrow & & & & & & \downarrow 1_A \otimes \mu \\
 A \otimes I & \xrightarrow{1_A \otimes \varphi} & A \otimes (A \otimes A) & \xrightarrow{\alpha^{-1}} & (A \otimes A) \otimes A & \xrightarrow{\mu \otimes 1_A} & A \otimes A
 \end{array} \quad (\text{a.1})$$

and

$$\begin{array}{ccccc}
 I \otimes A & \xleftarrow{\varepsilon \otimes 1_A} & A \otimes A & \xrightarrow{1_A \otimes \varepsilon} & A \otimes I \\
 \lambda \downarrow & & \uparrow \varphi & & \downarrow \rho \\
 A & \xleftarrow{\eta} & I & \xrightarrow{\eta} & A
 \end{array} \quad (\text{a.2})$$

commute.

(b) $\exists \varepsilon : A \rightarrow I$ and $\delta : A \rightarrow A \otimes A$ such that

$$\begin{array}{ccccc}
 A \otimes A & \xrightarrow{1_A \otimes \delta} & A \otimes (A \otimes A) & \xrightarrow{\alpha^{-1}} & (A \otimes A) \otimes A \\
 \downarrow \delta \otimes 1_A & \searrow \mu & & & \downarrow \mu \otimes 1_A \\
 & & A & & \\
 & & \searrow \delta & & \\
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{1_A \otimes \mu} & A \otimes A
 \end{array} \tag{b.1}$$

and

$$\begin{array}{ccccc}
 I \otimes A & \xleftarrow{\varepsilon \otimes 1_A} & A \otimes A & \xrightarrow{1_A \otimes \varepsilon} & A \otimes I \\
 & \searrow \lambda & \uparrow \delta & \swarrow \rho & \\
 & & A & &
 \end{array} \tag{b.2}$$

commute, i.e. A is a Frobenius object.

(c) $\exists \varepsilon : A \rightarrow I$ and $\delta : A \rightarrow A \otimes A$ such that (A, δ, ε) is comonoid and

$$\begin{array}{ccccc}
 A \otimes A & \xrightarrow{1_A \otimes \delta} & A \otimes (A \otimes A) & \xrightarrow{\alpha^{-1}} & (A \otimes A) \otimes A \\
 \downarrow \delta \otimes 1_A & \searrow \mu & & & \downarrow \mu \otimes 1_A \\
 & & A & & \\
 & & \searrow \delta & & \\
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{1_A \otimes \mu} & A \otimes A
 \end{array} \tag{c.1}$$

commutes.

(d) $\exists \varepsilon : A \rightarrow I$ such that $\sigma := \varepsilon \circ \mu$ is a counit for $A \dashv A$.

(e) \exists a counit $\sigma : A \otimes A \longrightarrow I$ for a duality $A \dashv A$ such that

$$\begin{array}{ccc}
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) \xrightarrow{1_A \otimes \mu} A \otimes A \\
 \mu \otimes 1_A \downarrow & & \downarrow \sigma \\
 A \otimes A & \xrightarrow{\sigma} & I
 \end{array} \tag{e.1}$$

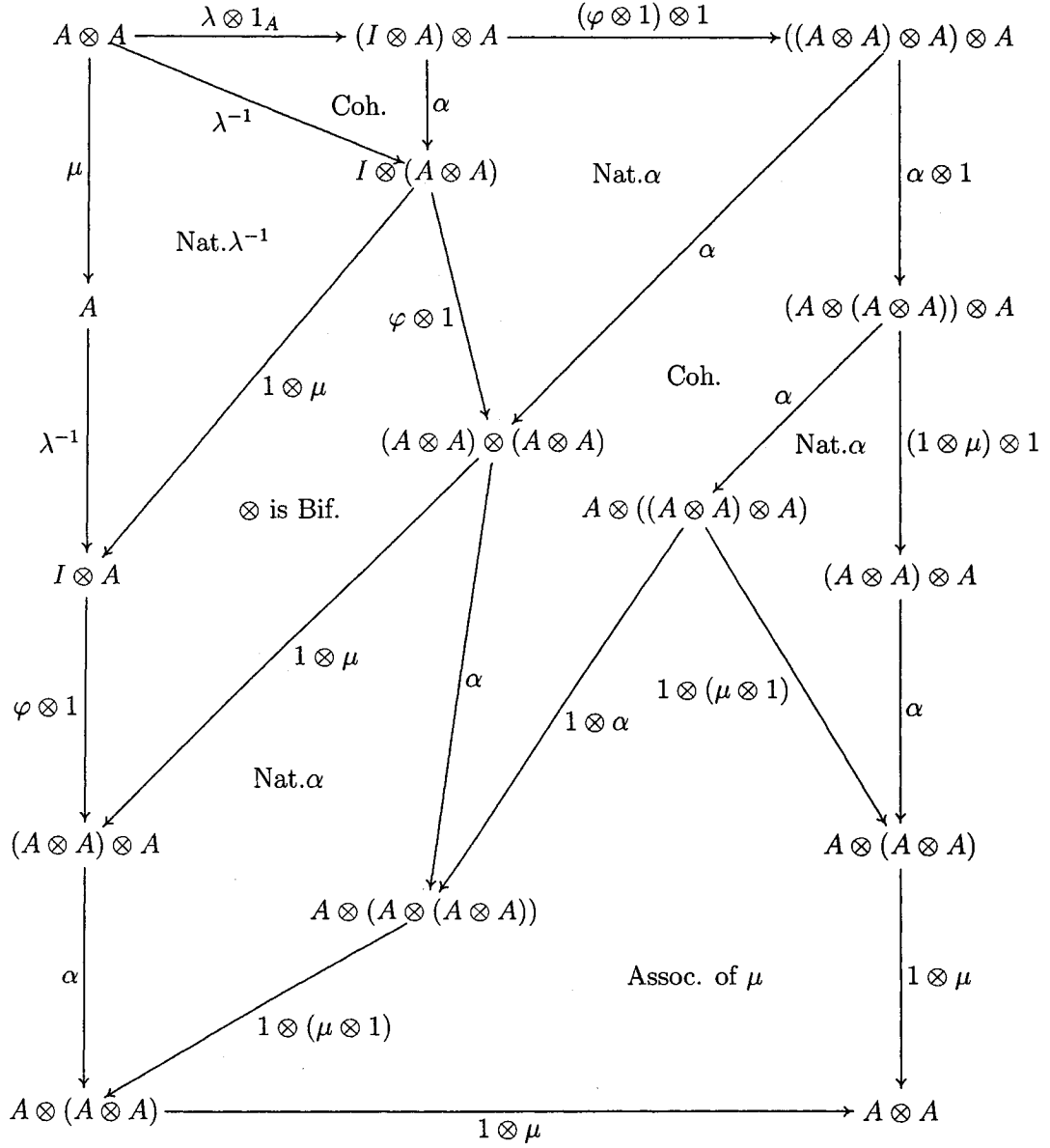
commutes.

Proof. [(a) \implies (b)]: assume $\exists \varepsilon : A \longrightarrow I$ and $\varphi : I \longrightarrow A \otimes A$ such that (a.1) and (a.2) commute. Define $\delta : A \longrightarrow A \otimes A$ to be

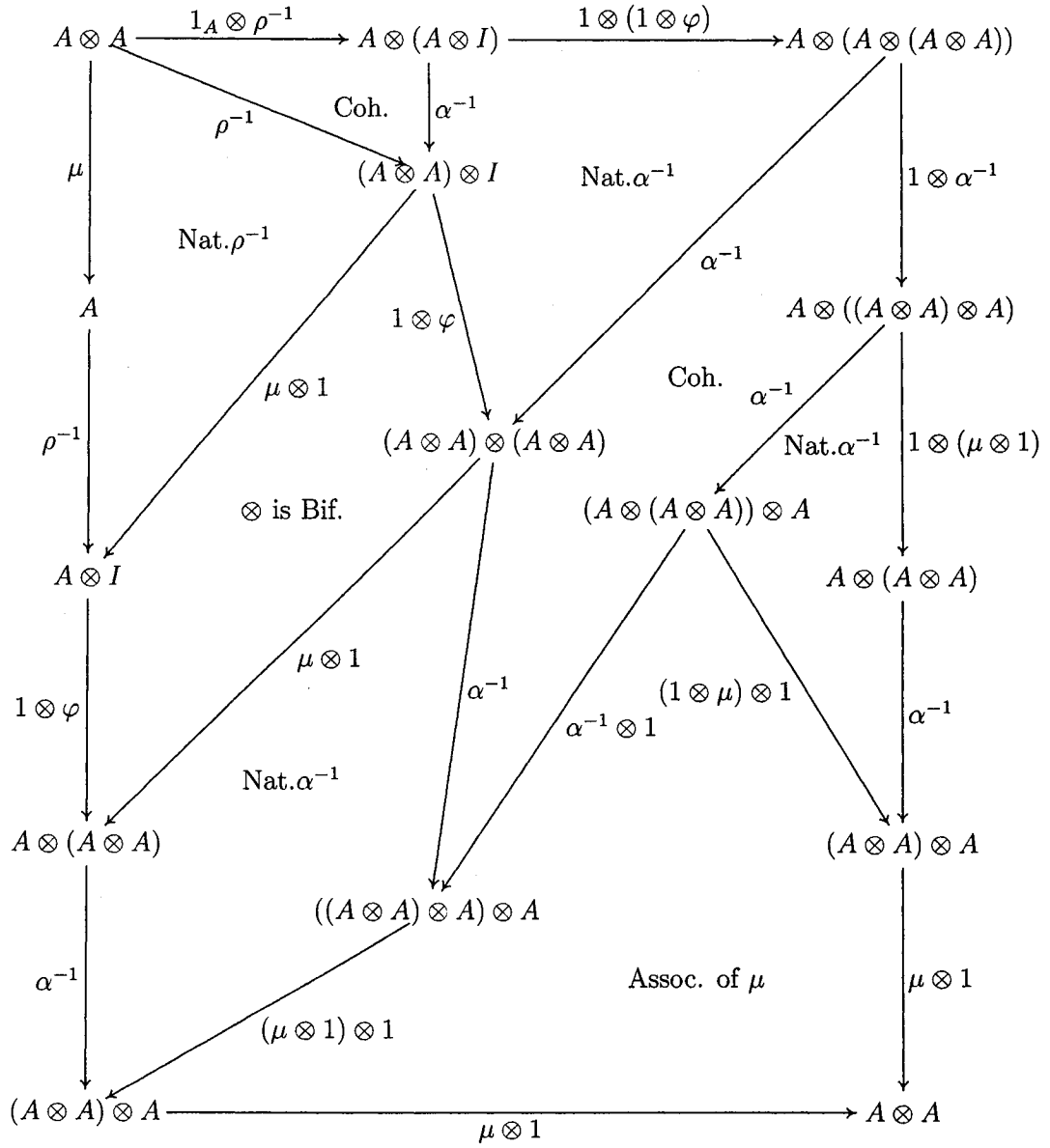
$$\delta := (1_A \otimes \mu) \circ \alpha \circ (\varphi \otimes 1_A) \circ \lambda^{-1} = (\mu \otimes 1_A) \circ \alpha^{-1} \circ (1_A \otimes \varphi) \circ \rho^{-1}$$

i.e. δ is the map $A \longrightarrow A \otimes A$ in diagram (a.1). Now let's show that both diagrams

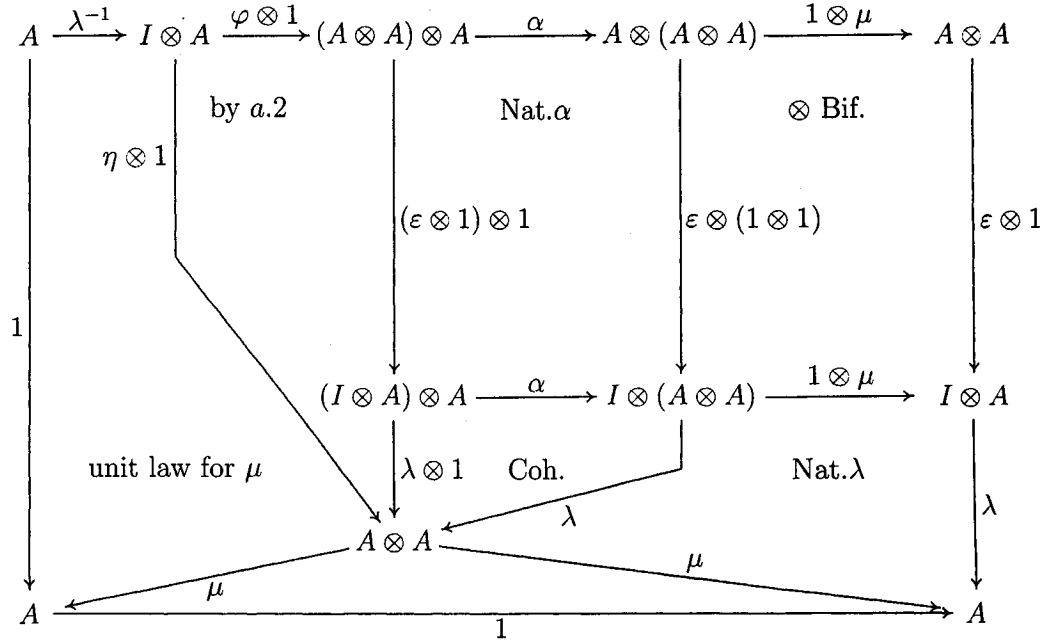
in (b.1) commute. We start with the upper triangle first.



Now for the lower triangle we have



Next we show that the left triangle in (b.2) commutes.

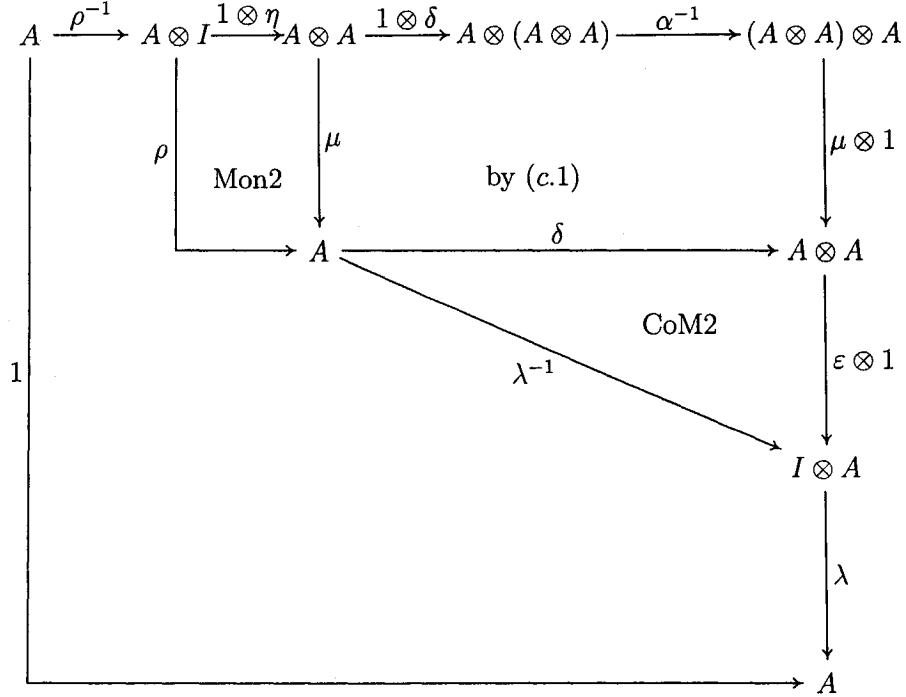


The right triangle's commutativity is shown in the same way. Hence (a) \implies (b).

[(b) \implies (c)]: Assume $\exists \varepsilon : A \rightarrow I$ and $\delta : A \rightarrow A \otimes A$ such that (b.1) and (b.2) commute. Then let's show that (A, δ, ε) is a comonoid. So we have to check (CoM1) and (CoM2) of Definition 2.4.1. (CoM1) follows from Lemma 3.3.3 and (b.2) is the same equation as (CoM2). Hence (b) \implies (c).

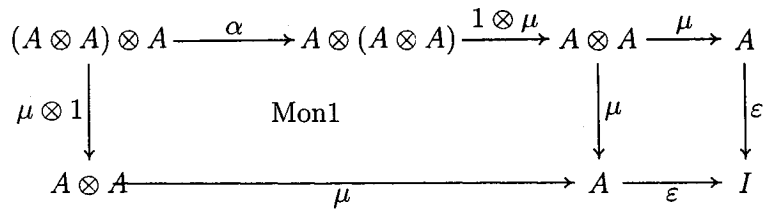
[(c) \implies (d)]: Assume $\exists \varepsilon : A \rightarrow I$ and $\delta : A \rightarrow A \otimes A$ such that (A, δ, ε) is a comonoid and (c.1) commutes. Define $\sigma : A \otimes A \rightarrow I$ by $\sigma = \varepsilon \circ \mu$ and define $\varphi : I \rightarrow A \otimes A$ by $\varphi = \delta \circ \eta$. We show that $A \dashv A$ with unit φ and counit σ . We

check diagram (D1) of Definition 2.7.1 first.



The diagram (D2) in Definition 2.7.1 commutes by a similar argument, and thus $A \dashv A$ with counit $\sigma = \varepsilon \circ \mu$ and unit $\varphi = \delta \circ \eta$.

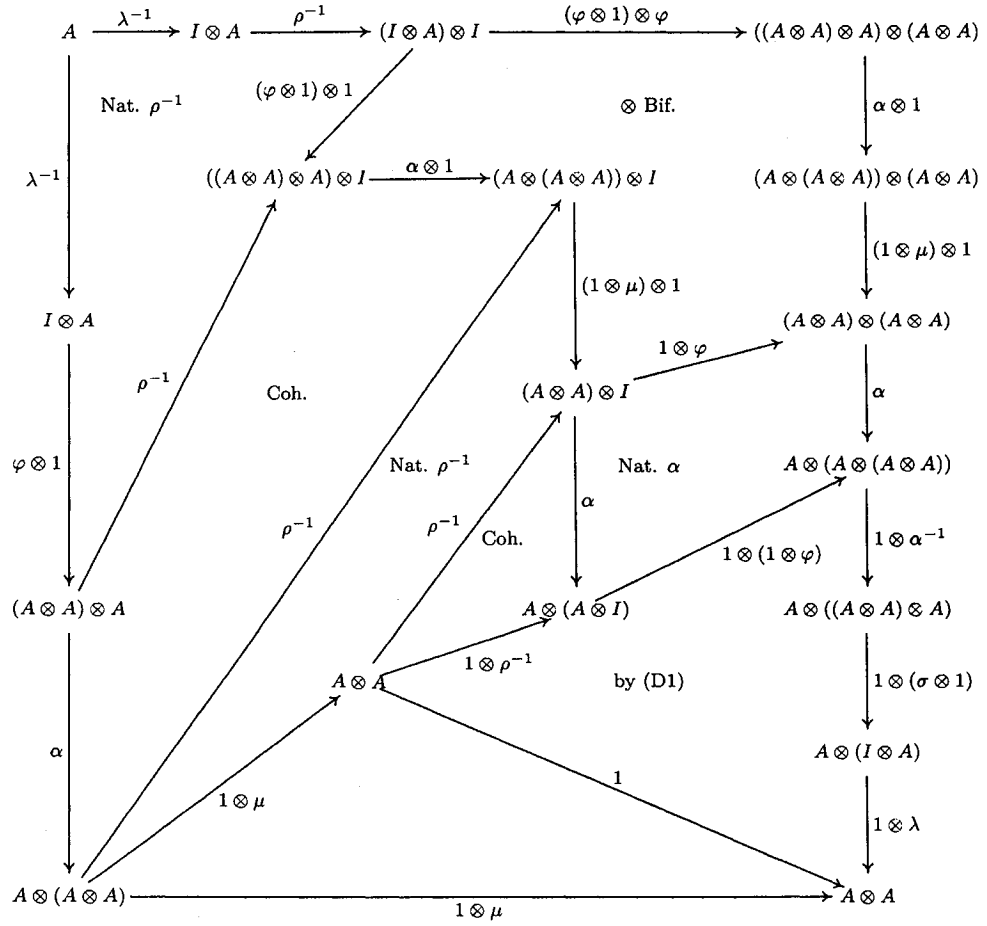
[(d) \implies (e)]: Assume $\exists \varepsilon : A \rightarrow I$ such that $\sigma = \varepsilon \circ \mu$ is a counit for $A \dashv A$. Then all we need show is that (e.1) commutes. Indeed



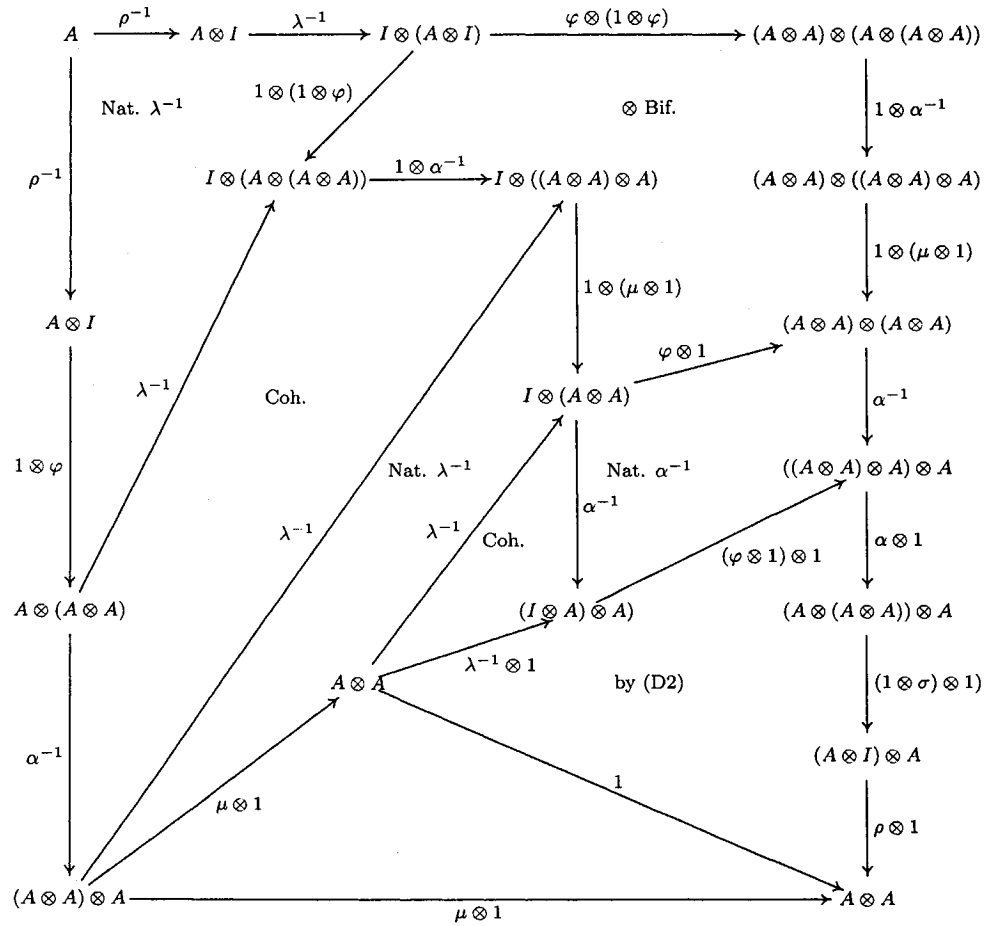
[(e) \implies (a)]: Assume \exists a counit $\sigma : A \otimes A \rightarrow I$ for a duality $A \dashv A$ such that (e.1) commutes. Let $\varphi : I \rightarrow A \otimes A$ be a unit for the duality. So σ and φ make diagrams (D1) and (D2) from Definition 2.7.1 commute.

We now establish a series of commutative diagrams in order to show that (a.1) is

commutative.



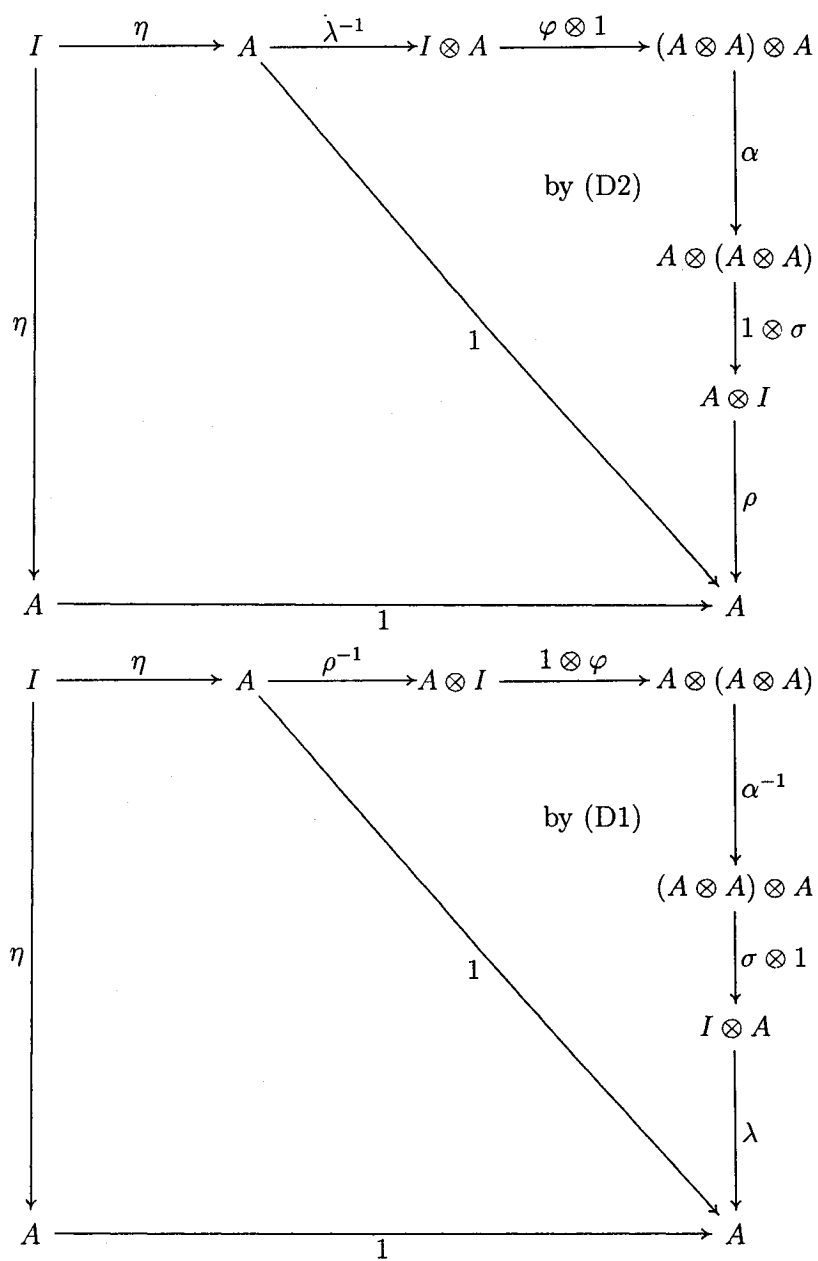
Next we require the following:

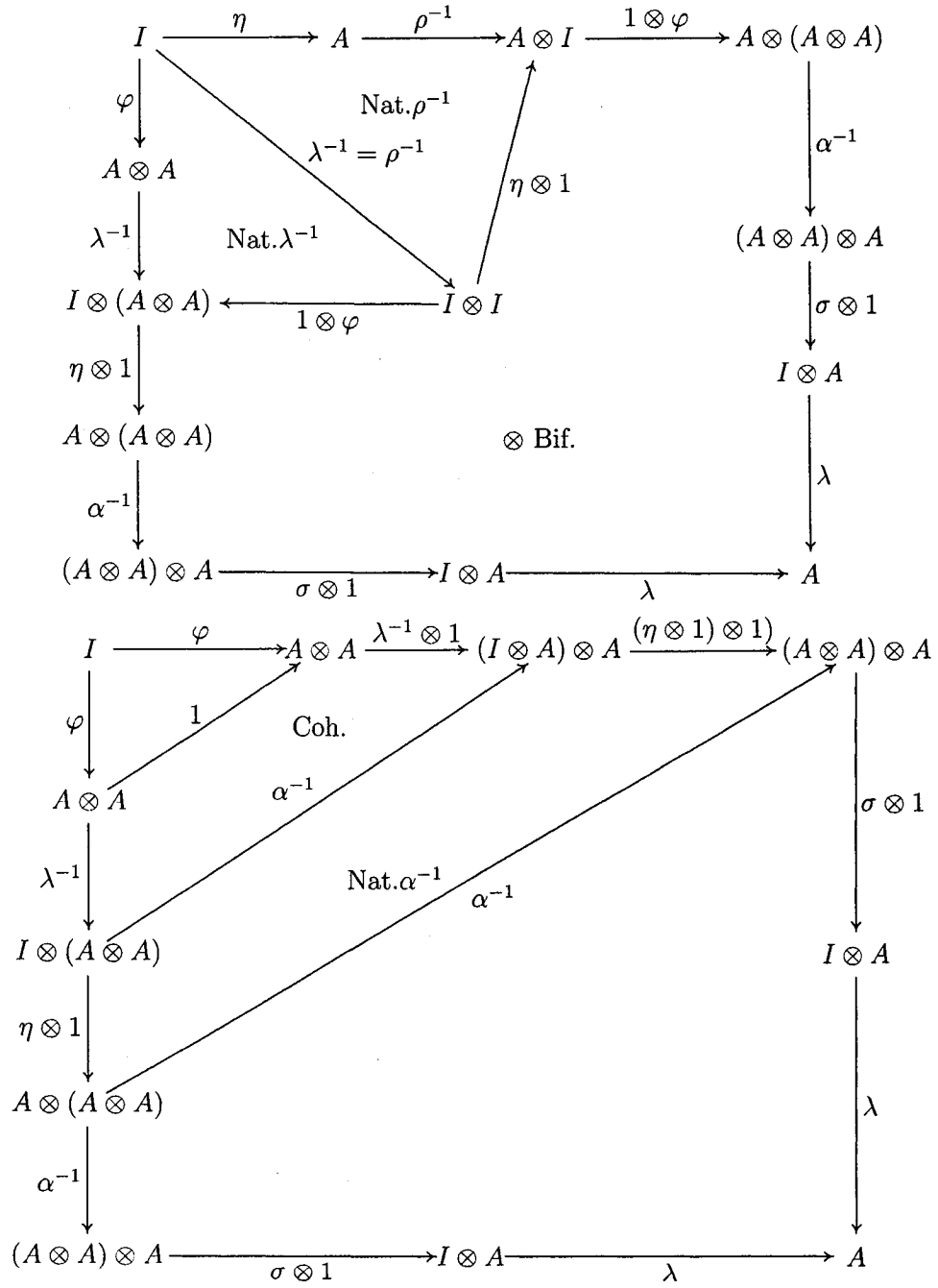


Now we establish the existence of a morphism $\varepsilon : A \rightarrow I$

$$\begin{array}{ccccc}
 A & \xrightarrow{\lambda^{-1}} & I \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A \\
 \rho^{-1} \downarrow & \searrow 1 & & & \downarrow \sigma \\
 A \otimes I & & & & \\
 1 \otimes \eta \downarrow & \text{Mon2} & & & \\
 A \otimes A & \xrightarrow{\mu} & A & \xrightarrow{\lambda^{-1}} & I \otimes A \xrightarrow{\eta \otimes 1} A \otimes A \xrightarrow{\sigma} I
 \end{array}$$

$$\begin{array}{ccccc}
 A & \xrightarrow{\rho^{-1}} & A \otimes I & \xrightarrow{1 \otimes \eta} & A \otimes A \\
 \lambda^{-1} \downarrow & \searrow 1 & & & \downarrow \sigma \\
 I \otimes A & & & & \\
 \eta \otimes 1 \downarrow & \text{Mon2} & & & \\
 A \otimes A & \xrightarrow{\mu} & A & \xrightarrow{\rho^{-1}} & A \otimes I \xrightarrow{1 \otimes \eta} A \otimes A \xrightarrow{\sigma} I
 \end{array}$$





Hence (a.2) commutes and this completes the proof. □

This theorem gives us the following result.

Corollary 3.3.5. *If A and B are both objects in a monoidal category \mathcal{C} such that $B \dashv A$ and $A \dashv B$ then $A \otimes B$ can be made into a Frobenius object in \mathcal{C} .*

Proof. By the previous theorem it is enough to show that $A \otimes B \dashv A \otimes B$. This diagram chase is similar and is omitted. \square

Now suppose that we have two Frobenius objects. Then one question we could ask is what is an appropriate notion of a morphism from one to the other.

Definition 3.3.6. *A morphism of Frobenius objects from $(A, \mu, \eta, \delta, \varepsilon)$ to $(A', \mu', \eta', \delta', \varepsilon')$ in a monoidal category \mathcal{C} , is morphism in \mathcal{C} $\varphi : A \rightarrow A'$ which is a morphism of monoids from (A, μ, η) to (A', μ', η') and comonoids from (A, δ, ε) to $(A', \delta', \varepsilon')$.*

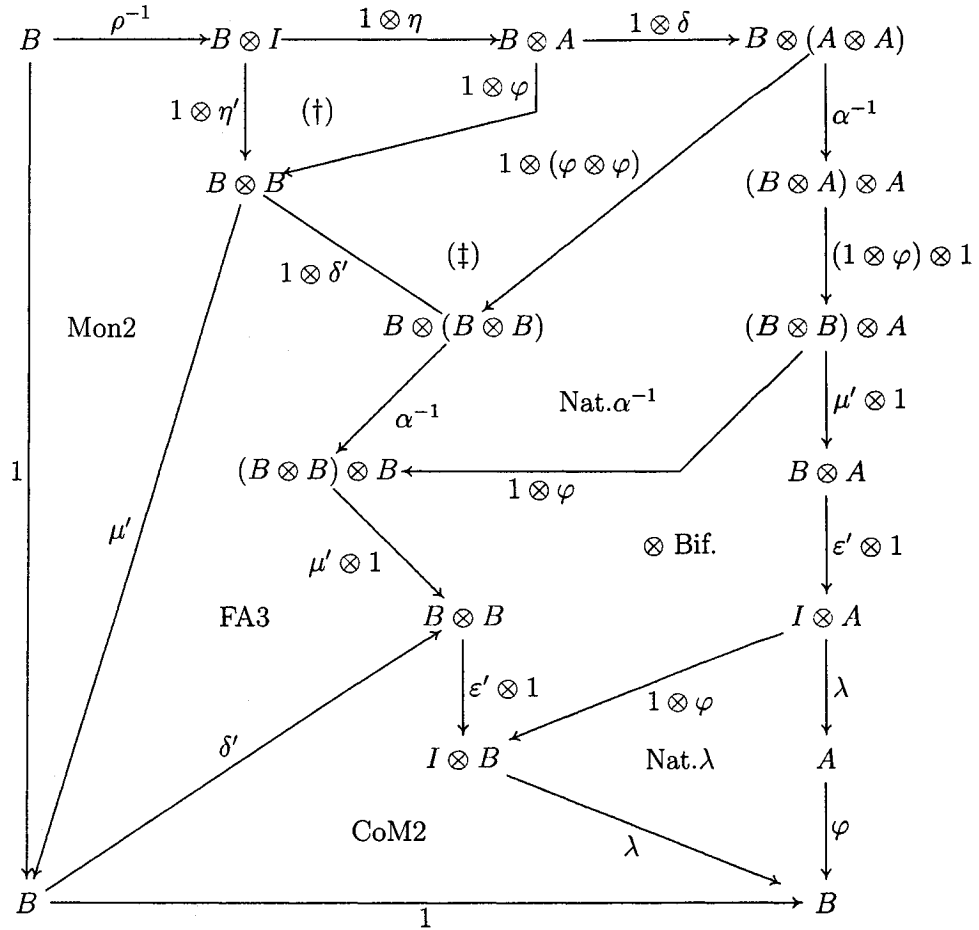
With this definition it is easy to see that given a monoidal category \mathcal{C} there is a new category, $\mathbf{Frob}(\mathcal{C})$ whose objects are Frobenius objects and morphisms are as defined above. $\mathbf{Frob}(\mathcal{C})$ has the following amazing property.

Lemma 3.3.7. *$\mathbf{Frob}(\mathcal{C})$ is a groupoid, i.e. all morphisms are isomorphisms.*

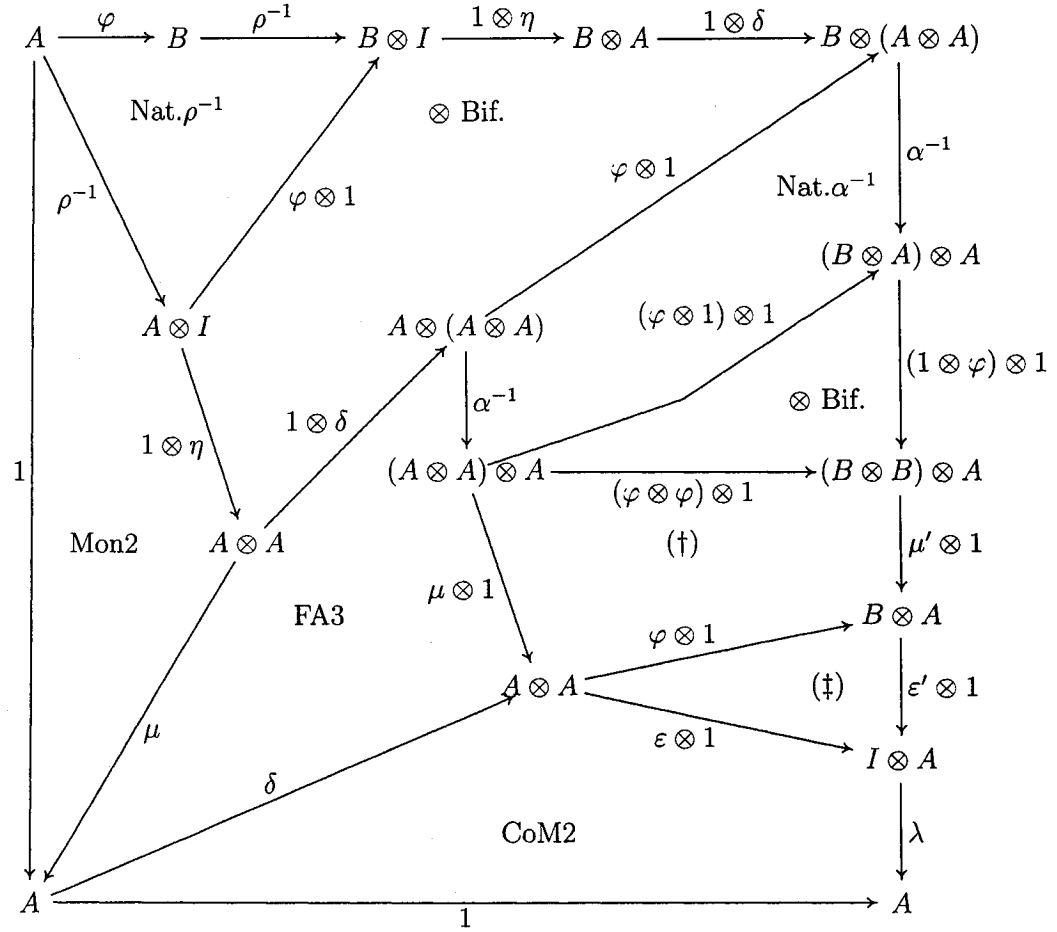
Proof. Let φ be a morphism from $(A, \mu, \eta, \delta, \varepsilon)$ to $(B, \mu', \eta', \delta', \varepsilon')$ in $\mathbf{Frob}(\mathcal{C})$, then we define its inverse as follows

$$\begin{array}{ccccccc}
 B & \xrightarrow{\rho^{-1}} & B \otimes I & \xrightarrow{1 \otimes \eta} & B \otimes A & \xrightarrow{1 \otimes \delta} & B \otimes (A \otimes A) \\
 & & & & & & \downarrow \alpha^{-1} \\
 & & & & & & (B \otimes A) \otimes A \\
 & & & & & & \downarrow (1 \otimes \varphi) \otimes 1 \\
 & & & & & & (B \otimes B) \otimes A \\
 & & & & & & \downarrow \mu' \otimes 1 \\
 & & & & & & B \otimes A \\
 & & & & & & \downarrow \varepsilon' \otimes 1 \\
 & & & & & & I \otimes A \\
 & & & & & & \downarrow \lambda \\
 & & & & & & A \\
 & \searrow \varphi^{-1} & & & & & \\
 & & & & & &
 \end{array}$$

Now we must show that $\varphi^{-1} \circ \varphi = 1_A$ and $\varphi \circ \varphi^{-1} = 1_B$.



The triangle \dagger commutes since φ is a morphism of monoids and, the quadrilateral \ddagger commutes as φ is a morphism of comonoids. Hence $\varphi \circ \varphi^{-1} = 1_B$. For the other composite we have



Again \dagger commutes as φ is a morphism of monoids and \ddagger commutes as φ is a morphism of comonoids. Hence $\varphi^{-1} \circ \varphi = 1_A$. It remains to show that φ^{-1} is a morphism of monoids and comonoids. First we show that it is a monoid morphism.

To show it preserves multiplication, we note that:

$$\begin{aligned} \varphi \circ \mu &= \mu' \circ \varphi \otimes \varphi && \text{as } \varphi \text{ is monoid morphism} \\ \varphi^{-1} \circ \varphi \circ \mu &= \varphi^{-1} \circ \mu' \circ \varphi \otimes \varphi && \text{compose both sides with } \varphi^{-1} \\ \mu &= \varphi^{-1} \circ \mu' \circ \varphi \otimes \varphi \\ \mu \circ \varphi^{-1} \otimes \varphi^{-1} &= \varphi^{-1} \circ \mu' \circ \varphi \otimes \varphi \circ \varphi^{-1} \otimes \varphi^{-1} && \text{compose both sides with } \varphi^{-1} \otimes \varphi^{-1} \\ \mu \circ \varphi^{-1} \otimes \varphi^{-1} &= \varphi^{-1} \circ \mu'. \end{aligned}$$

So φ^{-1} preserves multiplication, next we show that it preserves the unit.

$$\begin{aligned} \varphi \circ \eta &= \eta' && \text{as } \varphi \text{ is monoid morphism} \\ \varphi^{-1} \circ \varphi \circ \eta &= \varphi^{-1} \circ \eta' && \text{compose both sides with } \varphi^{-1} \\ \eta &= \varphi^{-1} \circ \eta'. \end{aligned}$$

So φ^{-1} preserves the unit and thus it is a monoid morphism. Similarly one shows that φ^{-1} is a comonoid morphism. \square

Chapter 4

2Cob: The Category of 2D Cobordisms

All manifolds that we will consider in this chapter will be assumed to be smooth and compact, but not necessarily connected. We also do not assume them to be embedded in any Euclidean space. By **closed manifold** we will mean a compact manifold without boundary. All of what follows in this chapter can be found in [12].

4.1 Oriented Cobordisms in 2D

Before defining the notion of oriented cobordism we first need some kind of gadget that will tell us where a cobordism starts and ends.

Definition 4.1.1. *Suppose M is a manifold of dimension n and Σ is a closed submanifold of dimension $n-1$. In addition assume that both are oriented. Now let $[v_1, \dots, v_{n-1}]$ be a positive basis for $T_x\Sigma$, where x is a point in Σ and $T_x\Sigma$ denotes the tangent space of Σ at x . If $w \in T_xM$ is such that $[v_1, \dots, v_{n-1}, w]$ is a positive basis for T_xM we call w a **positive normal**.*

Let M , Σ , and w be as above with the additional assumption that Σ is a connected component of the boundary of M .

Definition 4.1.2. *If w is inward pointing with respect to M then we call Σ an **in-boundary**. Similarly if the positive normal w is outward pointing with respect to M then we call Σ an **out-boundary**.*

Now at first glance it is not clear that the notions of in-boundary and out-boundary are well-defined as we have chosen a specific positive normal w and point $x \in \Sigma$ to define them. However Kock proves the following lemma.

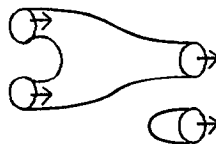
Lemma 4.1.3. *If some positive normal points inwards at some $x \in \Sigma$ then any other positive normal at any other point $y \in \Sigma$ also points inwards.*

We have the same result for outward pointing positive normals as well and hence in-boundary and out-boundary are well-defined concepts. Then the boundary of a manifold M will be a union of its in-boundaries and out-boundaries. The in-boundary of M may be empty as could its out-boundary. We are now ready to define the main concept of this section.

Definition 4.1.4. *Let Σ_0 and Σ_1 be closed oriented $(n-1)$ -manifolds. An **oriented cobordism** from Σ_0 to Σ_1 is a compact oriented manifold M together with smooth maps $\Sigma_0 \rightarrow M \leftarrow \Sigma_1$ such that Σ_0 maps diffeomorphically preserving orientation onto the in-boundary of M , and Σ_1 maps diffeomorphically preserving orientation onto the out-boundary of M . We write $M : \Sigma_0 \rightarrow \Sigma_1$ to denote that M is an oriented cobordism from Σ_0 to Σ_1 .*

When depicting oriented cobordisms for graphical convenience we will, whenever possible, put all in-boundaries on the left and all out-boundaries on the right.

Example 4.1.5. Consider the case when $n = 2$. Then the following is a cobordism from a disjoint union of two circles to two circles. The arrows depict the positive normal and hence show us the in-boundaries and out-boundaries.



We will typically omit the positive normal when depicting cobordisms as it is understood in-boundaries appear on the left and out-boundaries on the right. This next example we consider will be important when we want to define cobordism categories, it will play a key role in defining the identity morphisms.

Example 4.1.6. Suppose Σ is an oriented, closed, $(n-1)$ -manifold, let $I = [0, 1]$ be given its standard orientation. Then consider the product manifold $\Sigma \times I$ with its standard orientation and with standard orientation on the boundary points. Then the in-boundary is $\Sigma \times \{0\}$ and the out-boundary is $\Sigma \times \{1\}$. Moreover the following maps are orientation preserving diffeomorphisms (OPDs)

$$\Sigma \longrightarrow \Sigma \times \{0\} \subset \Sigma \times I \quad \text{and} \quad \Sigma \longrightarrow \Sigma \times \{1\} \subset \Sigma \times I.$$

In this way $\Sigma \times I$ is an oriented cobordism from Σ to itself, this construction is called the cylinder construction. Now suppose we are given an oriented n -manifold M with boundary such that there is an orientation preserving diffeomorphism, say ψ , from $\Sigma \times I$ to M with $\psi(\partial(\Sigma \times I)) = \partial M$. Then ψ takes in-boundaries to in-boundaries and similarly for out-boundaries. So we have the following maps

$$\Sigma \longrightarrow \Sigma \times \{0\} \subset \Sigma \times I \xrightarrow{\psi} M \quad \text{and} \quad \Sigma \longrightarrow \Sigma \times \{1\} \subset \Sigma \times I \xrightarrow{\psi} M$$

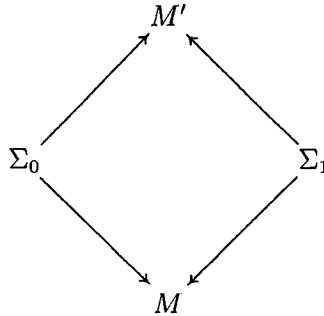
which are OPDs onto the in-boundary and out-boundary of M respectively. Hence M is also an oriented cobordism from Σ to itself.

Example 4.1.7. Suppose that we are given three closed oriented $(n-1)$ -manifolds Σ_0 and Σ_1 and Σ such that Σ_0 and Σ_1 are both diffeomorphic to Σ . Then the cylinder construction in Example 4.1.6 provides us with an oriented cobordism from Σ_0 to Σ_1 as follows

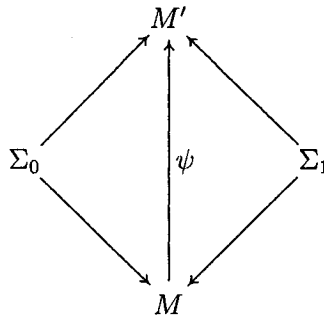
$$\begin{aligned} \Sigma_0 &\longrightarrow \Sigma \longrightarrow \Sigma \times \{0\} \subset \Sigma \times I; \\ \Sigma_1 &\longrightarrow \Sigma \longrightarrow \Sigma \times \{1\} \subset \Sigma \times I. \end{aligned}$$

We see that in Example 4.1.6 the cylinder cobordism and M are related by a diffeomorphism and so define essentially the same cobordism. We will have need of this special relationship later on and so we make the following definition which captures what is going on here.

Definition 4.1.8. Let Σ_0 and Σ_1 be two oriented, closed, $(n-1)$ -manifolds and suppose we are given two oriented cobordisms from Σ_0 to Σ_1 ,



We say that they are equivalent if there is an orientation preserving diffeomorphism $\psi : M \xrightarrow{\sim} M'$ making the following diagram commute:



So the cylinder cobordism in Example 4.1.6 is equivalent to M in the sense just defined. The last thing we would like to mention before closing this section is that given two oriented cobordisms say $M_0 : \Sigma_0 \rightarrow \Sigma$ and $M_1 : \Sigma \rightarrow \Sigma_1$ we can produce a new oriented cobordism from Σ_0 to Σ_1 . We do this by gluing the manifolds M_0 and M_1 along the common boundary Σ , this manifold is denoted M_0M_1 . The details of this construction can be found in [12]. We should note that there is no canonical way to glue two manifolds with boundary together, however gluing does exist and we have the following result which can also be found in [12].

Theorem 4.1.9. Let Σ be an out-boundary of M_0 and an in-boundary of M_1 , and consider the topological manifold M_0M_1 . Let α and β be two smooth structures on

M_0M_1 which both induce the original structure on M_0 and M_1 . Then there is a diffeomorphism $\phi : (M_0M_1, \alpha) \longrightarrow (M_0M_1, \beta)$ such that $\phi|_{\Sigma} = id_{\Sigma}$.

So the smooth structure is unique up to diffeomorphism.

4.2 The Category 2Cob

We are now ready to define the categories \mathbf{nCob} , for $n \geq 1$, of which 2Cob will be of particular interest.

Definition 4.2.1. *There is category, denoted \mathbf{nCob} , whose objects consist of closed oriented $(n-1)$ -manifolds, and given two closed oriented $(n-1)$ -manifolds say Σ and Σ' then $\text{Hom}_{\mathbf{nCob}}(\Sigma, \Sigma') = \{ \text{equivalence classes of oriented cobordisms from } \Sigma \text{ to } \Sigma' \}$. The identity arrows are the cobordism classes of the cylinder and composition is given by gluing two representatives together and then taking the equivalence class of the resulting oriented cobordism.*

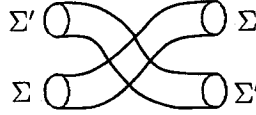
The proof that \mathbf{nCob} forms a category can be found in [12]. For our purposes we will only be concerned with the category $\mathbf{2Cob}$, that is when $n = 2$. However before setting $n = 2$ we make a couple more observations about \mathbf{nCob} .

Lemma 4.2.2. *Disjoint union in the category of (oriented) n -manifolds is a coproduct and the initial object is the empty n -manifold. Moreover $\tau_{\Sigma, \Sigma'} : \Sigma \amalg \Sigma' \longrightarrow \Sigma' \amalg \Sigma$, which is given by the universal property of coproduct, is an orientation preserving diffeomorphism. So the category of (oriented) n -manifolds is a (strict) symmetric monoidal category.*

We note that given two oriented n -manifolds Σ and Σ' then their disjoint union, $\Sigma \amalg \Sigma'$ is given the unique orientation such that the inclusions of Σ and Σ' are orientation preserving. So now given two oriented cobordisms $M_0 : \Sigma_0 \longrightarrow \Sigma_1$ and $M'_0 : \Sigma'_0 \longrightarrow \Sigma'_1$ then their disjoint union $M_0 \amalg M'_0 : \Sigma_0 \amalg \Sigma'_0 \longrightarrow \Sigma_1 \amalg \Sigma'_1$ is an oriented cobordism from $\Sigma_0 \amalg \Sigma'_0$ to $\Sigma_1 \amalg \Sigma'_1$. This follows from the orientation given on the disjoint union of oriented manifolds. So we have the following lemma whose proof may be found in [12].

Lemma 4.2.3. *Disjoint union is well-defined on cobordism classes, and makes $(\mathbf{nCob}, \amalg, \emptyset)$ a strict monoidal category.*

Moreover by Example 4.1.7 we see that the orientation preserving diffeomorphism $\tau_{\Sigma, \Sigma'} : \Sigma \amalg \Sigma' \rightarrow \Sigma' \amalg \Sigma$ in Lemma 4.2.2 induces a cobordism which we denote as $T_{\Sigma, \Sigma'} : \Sigma \amalg \Sigma' \rightarrow \Sigma' \amalg \Sigma$. We will draw this cobordism as follows



Notice the order the boundaries appear in the diagram, that is left to right = bottom to top. We will always use this convention when drawing cobordisms. Now as our manifolds are abstract, it makes no sense to say that one crosses over or under the other. That is why in the diagram we make it ambiguous. This next lemma along with its proof can also be found in [12].

Lemma 4.2.4. *For any two cobordisms $M : \Sigma_0 \rightarrow \Sigma_1$ and $M' : \Sigma'_0 \rightarrow \Sigma'_1$ the following square commutes*

$$\begin{array}{ccc}
 \Sigma_0 \amalg \Sigma'_0 & \xrightarrow{M \amalg M'} & \Sigma_1 \amalg \Sigma'_1 \\
 T_{\Sigma_0, \Sigma'_0} \downarrow & & \downarrow T_{\Sigma_1, \Sigma'_1} \\
 \Sigma'_0 \amalg \Sigma_0 & \xrightarrow{M' \amalg M} & \Sigma'_1 \amalg \Sigma_1
 \end{array}$$

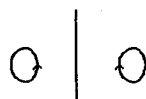
i.e. T_{Σ_1, Σ'_1} is a natural in Σ_1 and Σ'_1 .

Hence $(\mathbf{nCob}, \amalg, T, \emptyset)$ is a strict symmetric monoidal category.

We now focus on the case where $n = 2$. What we want to do is find a description of a **skeletal** version of $\mathbf{2Cob}$. A skeleton of a category is a full subcategory which has exactly one object for each isomorphism class. A classic example is the category of finite-dimensional (real) vector spaces. A skeleton of this category would consist of objects \mathbb{R}^n ($n \in \mathbb{N}$) and linear maps as arrows. To describe the skeleton of $\mathbf{2Cob}$ we must classify the invertible cobordisms. The first result we need is the following lemma which is found in [12].

Lemma 4.2.5. *If $M : \Sigma_0 \longrightarrow \Sigma_1$ is invertible then Σ_0 and Σ_1 have the same number of connected components.*

The next observation we make is that every closed oriented 1-manifold is diffeomorphic to a finite disjoint union of circles, one for each connected component. Moreover suppose we are given a circle with a fixed orientation and a copy of it with reverse orientation, then there is an orientation preserving diffeomorphism between them. Reflection in the line below is an orientation-preserving diffeomorphism.



The next result we state gives a necessary and sufficient condition for the existence of invertible cobordisms and is found in [12].

Proposition 4.2.6. *Two closed oriented 1-manifolds Σ_0 and Σ_1 are diffeomorphic if and only if there is an invertible cobordism between them.*

For the proof we proceed as in [12].

Proof. [\implies] Suppose Σ_0 and Σ_1 are diffeomorphic then we can use the cylinder construction in Example 4.1.7 to produce an invertible cobordism.

[\impliedby] Conversely suppose that there is an invertible cobordism between Σ_0 and Σ_1 then to show that these two closed oriented 1-manifolds are diffeomorphic it is enough to show that they have the same number of connected components since by a previous remark any closed oriented 1-manifold is diffeomorphic to a disjoint union of circles, one circle for each connected component. By Lemma 4.2.5 they have the same number of connected components. \square

So we have established that an equivalent condition to the existence of invertible cobordisms is that the in-boundary and the out-boundary have the same number of connected components. In fact every invertible cobordism in $\mathbf{2Cob}$ is induced by a diffeomorphism. We will now classify the diffeomorphisms between closed oriented

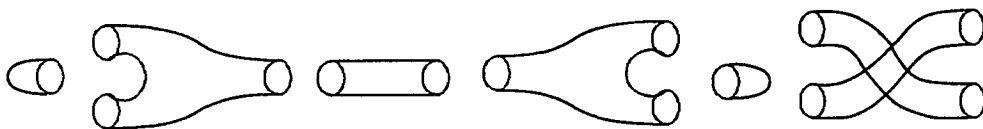
1-manifolds and hence all the invertible cobordisms in $\mathbf{2Cob}$. For this we need the following result, from [12].

Proposition 4.2.7. *Two diffeomorphisms $\Sigma_0 \rightrightarrows \Sigma_1$ induce the same cobordism class $\Sigma_0 \longrightarrow \Sigma_1$ if and only if they are smoothly homotopic.*

Now in dimension 1 every orientation-preserving diffeomorphism from the circle to itself is homotopic to the identity. Thus for any closed oriented 1-manifold the only orientation-preserving diffeomorphisms, up to homotopy, are the permutations of its connected components. So the only invertible cobordisms in $\mathbf{2Cob}$ are the permutation cobordisms including the identity cobordism which corresponds to the identity permutation diffeomorphism.

We are now able to describe the skeleton of $\mathbf{2Cob}$. Let $\mathbf{0}$ denote the empty 1-manifold, let $\mathbf{1}$ denote a given circle Σ , and let \mathbf{n} denote the disjoint union of n copies of Σ . Then the full subcategory $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$ is a skeleton of $\mathbf{2Cob}$ which we will also call $\mathbf{2Cob}$. From here on it will be understood that when we refer to $\mathbf{2Cob}$ we mean the skeleton just described. We can now state the main theorem of this section, found in [12], it provides us with a very precise description of $\mathbf{2Cob}$.

Proposition 4.2.8. *The symmetric monoidal category $\mathbf{2Cob}$ is generated under composition and disjoint union by the following six cobordisms*

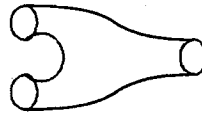


In [12] two different proofs are given. One uses the classification of surfaces and the other relies on some results from Morse theory. We will present the former. We first describe Kock's classification of surfaces with in and out-boundaries.

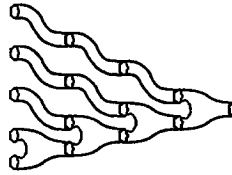
Proposition 4.2.9. *Two compact, connected oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus and the same number of in-boundaries and the same number of out-boundaries.*

We now proceed with the proof of Proposition 4.2.8.

Proof. The first ingredient is to define a normal form for connected cobordisms. The idea is that a given connected cobordism $\mathbf{m} \rightarrow \mathbf{n}$ of genus g will be equivalent to any other connected cobordism $\mathbf{m} \rightarrow \mathbf{n}$ with the same genus. So we will build a cobordism with these required properties we do this by constructing three cobordisms whose composition is a cobordism from \mathbf{m} to \mathbf{n} with these properties; the in-part $\mathbf{m} \rightarrow \mathbf{1}$, the topological part $\mathbf{1} \rightarrow \mathbf{1}$, and the out-part $\mathbf{1} \rightarrow \mathbf{n}$. If $m > 0$ then take $m - 1$ copies of



and an appropriate number of cylinders and compose them so that the output hole of the pants lines up with the bottom input hole of the next pair of pants then connect the cylinders to the remaining input holes of the pants. As an example suppose $m = 5$ then this procedure produces the following cobordism $\mathbf{m} \rightarrow \mathbf{1}$:



If $m = 0$ then the in-part consists of a single copy of

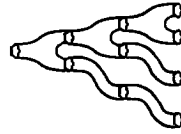


The out-part is described similarly where if $n > 0$ then we take $n - 1$ copies of



and an appropriate number of cylinders and compose them so that the top output hole of the pants joins up with the input hole of the following pair of pants. We

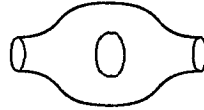
then connect the cylinders to the remaining output holes of the pants. For example if $n = 4$ then the out-part is the following cobordism:



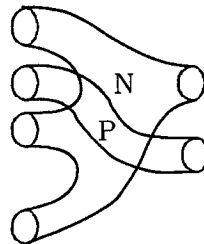
If $n = 0$ then the out-part is given by taking one copy of



The topological part is given by taking g copies of the handle cobordism



We must now deal with the disconnected cobordisms. Given a cobordism say $M : \mathbf{m} \rightarrow \mathbf{n}$ then as a manifold it is a disjoint union of its connected components however in general it will not be a disjoint union of connected cobordisms. The reason being that the in-boundaries/out-boundaries of the connected components may not appear in an order in M which is attainable by taking a disjoint union of the connected components. For example consider the following:

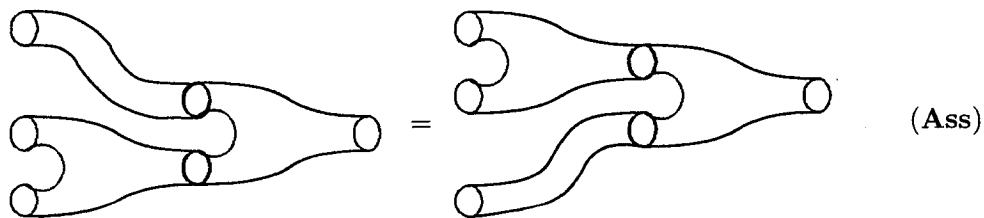


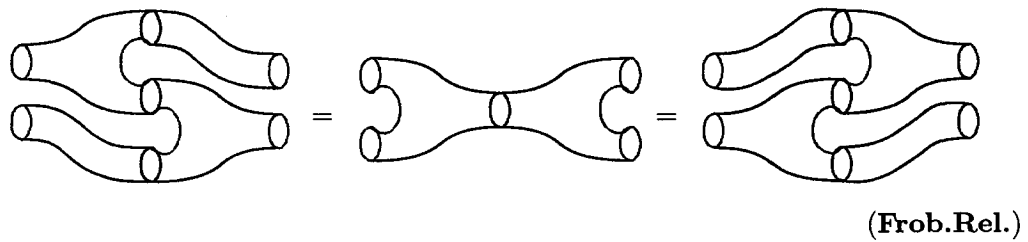
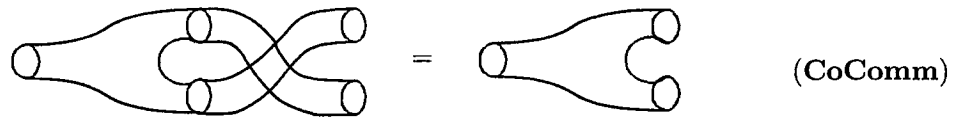
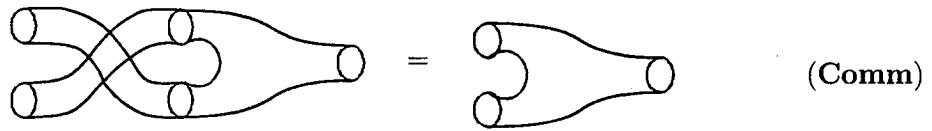
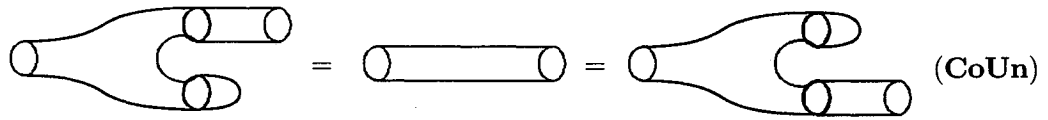
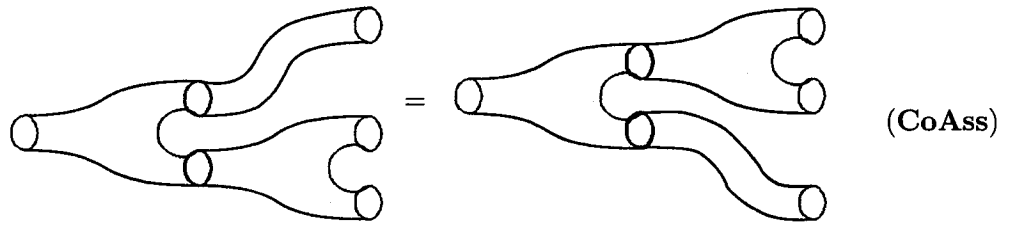
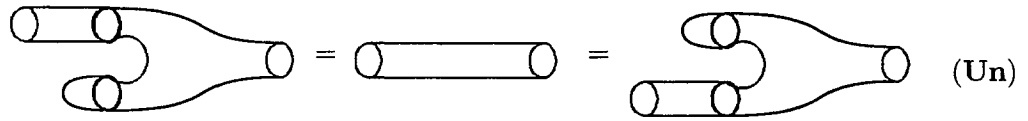
in which we see that this cobordism, say M , is a disjoint union of the manifolds N and P but when viewing N and P as cobordisms that neither $N \amalg P$ nor $P \amalg N$ is the above cobordism since their in/out-boundaries do not appear in the same order as in M . It will however be possible to decompose every cobordism in $\mathbf{2Cob}$

into a permutation cobordism followed by disjoint union of connected cobordisms followed by another permutation cobordism. Indeed suppose that $M : \mathbf{m} \rightarrow \mathbf{n}$ has q connected components M_1, \dots, M_q . This means that the in-boundary of M is $\Sigma_1 \amalg \dots \amalg \Sigma_m$ and the out-boundary is $\Sigma'_1 \amalg \dots \amalg \Sigma'_n$ where the sigmas are all copies of the same circle, we have labelled them to keep track of their positions within the disjoint union. Now let p_1, \dots, p_q denote the in-boundaries of M_1, \dots, M_q respectively, then p_1, \dots, p_q are pairwise disjoint and $\cup_{i=1}^q p_i = \mathbf{m}$. Then there is a permutation diffeomorphism of \mathbf{m} which places the subsets: p_1 before p_2 , p_2 before p_3, \dots, p_{q-1} before p_q . This diffeomorphism induces an invertible cobordism $S : \mathbf{m} \rightarrow \mathbf{m}$. We now consider the new cobordism $M \circ S := SM$ which has q connected components $(SM)_1, \dots, (SM)_q$ and with the property that the in-boundary of SM is a disjoint union of the in-boundaries of $(SM)_1, \dots, (SM)_q$. Similarly we can apply the same procedure to the out-boundary of SM to obtain a permutation cobordism $T : \mathbf{n} \rightarrow \mathbf{n}$ such that the out-boundary of SMT is a disjoint union of the out-boundaries of the connected cobordisms $(SMT)_1, \dots, (SMT)_q$. So the cobordism $SMT : \mathbf{m} \rightarrow \mathbf{n}$ is a disjoint union of the connected cobordisms $(SMT)_1, \dots, (SMT)_q$. Thus $M = S^{-1}(SMT)T^{-1}$ where S^{-1} and T^{-1} are permutation cobordisms and SMT is a disjoint union of connected cobordisms.

The last part of this proof is to simply note that any permutation cobordism is generated by cylinders and the twist cobordisms. □

Now that we have established a generating set for $2\mathbf{Cob}$, it is natural to see what relations hold between them. The following relations hold as a direct consequence of Proposition 4.2.9.





In addition to these relations are the relations expressing the naturality of the twist cobordism and the fact that it satisfies the axioms required of a symmetry. It

is shown in [12] that the relations **Un**, **CoUn**, **Comm**, and **Frob.Rel.** along with the relations expressing that the twist cobordism makes **2Cob** a symmetric monoidal category are sufficient. This means that all relations in **2Cob** are consequences of these ones.

Remark 4.2.10. The listed relations for **2Cob** imply **1** is a commutative Frobenius object in this category.

4.3 2D TQFTs and Frobenius Algebras

Topological quantum field theories (TQFTs) were first considered by Witten [18] in 1988 and shortly after they were given a mathematical axiomatization [2] by Atiyah. In the article [2] Atiyah's axioms do not mention cobordisms however it can be shown that his definition is equivalent to the one given in terms of the symmetric monoidal category **2Cob**. We first give a definition of TQFTs in the style of Atiyah and then we provide an alternate definition which is more categorical. We should first give some kind of indication of the why this is related to physics. Following Kock [12] the general idea is that space is represented by $(n-1)$ closed oriented manifolds and space-time by oriented cobordisms whose in-boundary represents space at time zero and the out-boundary represents space at some later time. Then a TQFT assigns to a $(n-1)$ manifold a state space, that is vector space over some fixed field, and to an n -cobordism an evolution operator, that is a linear map from the initial state space to the final state space. In addition the cylinder is sent to the identity linear map, which means that the state of the system does not change if the topology of the space-time does not change. This is what it means for the theory to be topological. The following definition can be found in [12].

An n -dimensional TQFT is a rule \mathcal{T} that assigns to any closed oriented $(n-1)$ manifold Σ a vector space $\mathcal{T}\Sigma$ and to any n -cobordism $M : \Sigma \longrightarrow \Sigma'$ a linear map $\mathcal{T}M : \mathcal{T}\Sigma \longrightarrow \mathcal{T}\Sigma'$, and satisfies the following five axioms:

- If M and M' are equivalent cobordisms then $\mathcal{T}M = \mathcal{T}M'$
- The cylinder cobordism $\Sigma \times I : \Sigma \longrightarrow \Sigma$ is sent to the identity. i.e. $\mathcal{T}(\Sigma \times I) = 1_{\mathcal{T}\Sigma}$
- Given a decomposition of a cobordism M into a composition of two cobordisms

M' and M'' , i.e. $M = M'M''$, then $\mathcal{T}M = \mathcal{T}M'\mathcal{T}M''$ where the right hand side is the composition of linear maps written in diagrammatic order.

- If $\Sigma = \Sigma' \amalg \Sigma''$ then $\mathcal{T}\Sigma = \mathcal{T}\Sigma' \otimes \mathcal{T}\Sigma''$. Similarly for cobordisms M and N we require that $\mathcal{T}(M \amalg N) = \mathcal{T}M \otimes \mathcal{T}N$. So disjoint union is sent to tensor product.
- If $\Sigma = \emptyset$ viewed as an $(n-1)$ closed oriented manifold then $\mathcal{T}\Sigma = \mathbb{k}$ is the ground field.

As we mentioned previously the first two axioms say that the theory is topological, while the fourth axiom says that the state space of independent systems is given by the tensor product of the state spaces which agrees with the formalism of quantum mechanics.

From a mathematical point of view we see that the first three axioms amount to saying that \mathcal{T} is a functor from \mathbf{nCob} to $\mathbf{Vect}_{\mathbb{k}}$. The last two axioms express that it is in fact a strict monoidal functor. Now in the above discussion we see that Atiyah does not require the functor \mathcal{T} preserve the symmetry in \mathbf{nCob} and $\mathbf{Vect}_{\mathbb{k}}$ however from a categorical point of view it is desirable that \mathcal{T} preserve the symmetry. For a discussion on this issue please see [12] page 174. In [12] Kock also alludes to the fact that the issue of symmetry is one that has been swept under the rug by many. In any case for our purposes we will define an n -dimensional TQFT as in [12].

Definition 4.3.1. *An n -dimensional TQFT is a strict symmetric monoidal functor from $(\mathbf{nCob}, \amalg, T, \emptyset)$ to $(\mathbf{Vect}_{\mathbb{k}}, \otimes, \tau, \mathbb{k})$*

Before stating the main theorem which classifies the 2-dimensional TQFTs we need some notation. Suppose that $(\mathcal{C}, \otimes, I, \tau)$ and $(\mathcal{K}, \odot, I', \sigma)$ are symmetric monoidal categories. Then we can define a new category whose objects are symmetric monoidal functors from \mathcal{C} to \mathcal{K} and arrows are monoidal natural transformations. This category will be denoted $\mathbf{SymmMonCat}(\mathcal{C}, \mathcal{K})$, and we see in this notation that $\mathbf{SymmMonCat}(\mathbf{nCob}, \mathbf{Vect}_{\mathbb{k}})$ has as objects n -dimensional TQFTs. We denote this category by $\mathbf{nTQFT}_{\mathbb{k}}$. With this new notation we can now state the main theorem of this section.

Theorem 4.3.2. *There is a canonical equivalence of categories*

$$\mathbf{2TQFT}_{\mathbb{k}} \simeq \mathbf{cFA}_{\mathbb{k}}$$

where \mathbf{cFA}_k is the category of commutative Frobenius algebras.

This means that 2-dimensional TQFTs are essentially the same thing as commutative Frobenius algebras, an observation first made by Dijkgraaf in his Ph.D. thesis [6]. Others have since given more precise proofs with Abrams [1] having given the most detailed proof to date. The proof found in Kock uses a graphical language for Frobenius algebras which is identical to the graphical language of $\mathbf{2Cob}$. The key here is that the algebraic equations for a commutative Frobenius algebra when expressed graphically actually hold as equations in $\mathbf{2Cob}$. So the axioms for a commutative Frobenius algebra are topological. One can easily adapt this proof to obtain the more general theorem which is explicitly stated by Kock but is implicit in the work of Abrams [1] and others.

Theorem 4.3.3. *If $(\mathcal{C}, \otimes, I, \tau)$ is a symmetric monoidal category, then there is a canonical equivalence of categories*

$$\mathbf{SymmMonCat}(\mathbf{2Cob}, \mathcal{C}) \simeq \mathbf{cFrob}(\mathcal{C})$$

where $\mathbf{cFrob}(\mathcal{C})$ is the category of commutative Frobenius objects in \mathcal{C} .

Chapter 5

Braided Frobenius Algebras

5.1 Braided Frobenius Objects

Up until now all of the Frobenius objects we have considered were from symmetric monoidal categories. We would like to consider the more general case where the category happens to be a braided monoidal category.

Definition 5.1.1. *Suppose $A = (A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object in a braided monoidal category, then A is called a **braided Frobenius object**. We say that A is **braided commutative** if it is braided and $\mu \circ \sigma_{A,A} = \mu$ where σ is the braiding.*

The first place that we will look for braided Frobenius objects is in the category $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. We don't waste our time looking in $\mathbb{X}\mathbf{G} - \mathbf{Set}$, since by Example 2.4.2 we know each crossed G -set X has only one comonoid structure. Moreover in order for the comultiplication map to be a map of crossed G -sets we are forced to define $|x| = 1_G$ for all $x \in X$. To find examples of braided Frobenius objects in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ we need several intermediate results. The first result we need is a classification of monoid objects in $\mathbb{X}\mathbf{G} - \mathbf{Set}$.

Proposition 5.1.2. *Let C be a crossed G -set, then $(C, | \cdot |, \mu, \eta)$ is a monoid in $\mathbb{X}\mathbf{G} - \mathbf{Set}$ if and only if*

(i) (C, μ, η) is a monoid in \mathbf{Set} .

(ii) $| \cdot | : C \longrightarrow G$ is a monoid homomorphism.

(iii) $g \cdot () : C \longrightarrow C$ is a monoid homomorphism for all $g \in G$.

Proof. $[\implies]$ Assume that $(C, | \cdot |, \mu, \eta)$ is a monoid in $\mathbb{X}G\text{-Set}$. Now as the forgetful functor from $\mathbb{X}G\text{-Set}$ to \mathbf{Set} is a strict monoidal functor we have that (i) holds. Now as μ and η are arrows in $\mathbb{X}G\text{-Set}$ we also have:

$$|\eta(1)| = |1| = 1_G \quad (*)$$

$$g \cdot \eta(1) = \eta(g \cdot 1) = \eta(1) \quad \forall g \in G \quad (**)$$

$$|\mu(c, d)| = |c||d| \quad \forall (c, d) \in C \otimes C \quad (***)$$

$$g \cdot \mu(c, d) = \mu(g \cdot c, g \cdot d) \quad (****)$$

Then $(*)$ and $(***)$ imply (ii) and $(**)$ and $(****)$ imply (iii).

$[\impliedby]$ (ii) and (iii) imply that μ and η are arrows in $\mathbb{X}G\text{-Set}$. (i) implies that (Mon1) and (Mon2) of Definition 2.3.1 commute in \mathbf{Set} and hence as α, λ, ρ and η are arrows in $\mathbb{X}G\text{-Set}$ then these diagrams also commute in $\mathbb{X}G\text{-Set}$. Thus $(C, | \cdot |, \mu, \eta)$ is a monoid in $\mathbb{X}G\text{-Set}$. \square

We shall give these objects a name.

Definition 5.1.3. Any crossed G -set C equipped with maps $\mu : C \otimes C \longrightarrow C$, $\eta : I \longrightarrow C$ satisfying (i), (ii), and (iii) of the previous proposition will be called a *crossed G -monoid*.

Example 5.1.4. Let G be any abelian group and M a monoid equipped with a monoid homomorphism $f : M \longrightarrow G$. We make M into a crossed G -monoid as follows.

$$\begin{aligned} g \cdot m &= m & \forall m \in M, g \in G \\ |m| &= f(m) & \forall m \in M. \end{aligned}$$

Example 5.1.5. Let G an arbitrary group and M any monoid. Define $g \cdot m = m$ and $|m| = 1_G$ for all $m \in M$ and $g \in G$. Then $(M, | \cdot |)$ is a crossed G -monoid.

Example 5.1.6. Consider $G = \mathbb{Z}/2 = \mathbb{Z}/2\mathbb{Z}$ the additive group of integers mod 2. Let $M = (\mathbb{N}, +)$ be the commutative monoid of natural numbers. Define $\bar{a} \cdot n = n$ and $|n| = \bar{n}$ for all $n \in \mathbb{N}$ and $\bar{a} \in \mathbb{Z}/2$. Then clearly the G action is a monoid homomorphism and that $| |$ is one as well follows from the equation $\overline{ab} = \overline{a}\overline{b}$. This example is a special case of Example 5.1.4.

Example 5.1.7. Let p be prime and let $G = GL_2(\mathbb{Z}/p)$ be the set of 2×2 invertible matrices over \mathbb{Z}/p . Let $M = \mathbb{Z}/p \setminus \{\overline{0}\}$ be the multiplicative group of units of the field \mathbb{Z}/p . Then define $A \cdot \overline{m} = \overline{m}$ and $|\overline{m}| = \overline{m}I_2$ for all $\overline{m} \in \mathbb{Z}/p$ and $A \in GL_2(\mathbb{Z}/p)$. Then $| |$ is clearly group homomorphism and $|A \cdot \overline{m}| = A|\overline{m}|A^{-1}$ as $|\overline{m}|$ is central. Thus $(M, | |)$ is a crossed G -monoid.

Example 5.1.8. Let $G = \mathbb{Z}/8^\times$ be the multiplicative group of units of $\mathbb{Z}/8$. So $G = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ and $g^2 = \overline{1}$ for all $g \in G$. Let $M = \mathbb{Z}/8$ and we define $\overline{n} \cdot \overline{m} = \overline{nm}$ and $|\overline{1}| = \overline{3}$ for all $\overline{n} \in G$ and $\overline{m} \in M$. It is then a routine exercise to show $| |$ is a well-defined group homomorphism and that $g \cdot ()$ is also a group homomorphism. Thus $(M, | |)$ is a crossed G -monoid where neither the G action nor the map $| |$ are trivial.

More generally we suspect the theory of crossed modules may yield further examples.

Proposition 5.1.9. Let $(C, | |, \mu, \eta)$ be a crossed G -monoid. Define $U : I \longrightarrow C$, $M : C \otimes C \longrightarrow C$ in $\mathbf{G} - \mathbf{XMat}/R$ as follows:

$$U_1^c = \delta_{\eta(1),c} \quad M_{(a,b)}^c = \delta_{\mu(a,b),c} \quad \forall a, b, c \in C.$$

Then (C, M, U) is a monoid in $\mathbf{G} - \mathbf{XMat}/R$.

Proof. Since $(C, | |, \mu, \eta)$ is monoid in $\mathbf{XG} - \mathbf{Set}$, and as $M = \text{Mat}(\mu)$ and $U = \text{Mat}(\eta)$ it follows that (C, M, U) is the image of the monoid $(C, | |, \mu, \eta)$ under the strict monoidal functor Mat hence (C, M, U) is also a monoid. \square

We will now develop a “process” which can take a monoid (resp. comonoid) in $\mathbf{G} - \mathbf{XMat}/R$ and produce a comonoid (resp. monoid) in $\mathbf{G} - \mathbf{XMat}/R$. To do this we will need the following results.

Lemma 5.1.10. *There is a contravariant functor $(-)^t : \mathbf{G} - \mathbb{X}\text{Mat}/R \longrightarrow \mathbf{G} - \mathbb{X}\text{Mat}/R$ which satisfies the following equations:*

$$(i) (\alpha)^t = \alpha^{-1}.$$

$$(ii) (\lambda)^t = \lambda^{-1}.$$

$$(iii) (\rho)^t = \rho^{-1}.$$

$$(iv) (f \otimes g)^t = f^t \otimes g^t \quad \forall \text{ arrows } f, g \text{ in } \mathbf{G} - \mathbb{X}\text{Mat}/R.$$

(i.e. $(-)^t : (\mathbf{G} - \mathbb{X}\text{Mat}/R)^{op} \longrightarrow \mathbf{G} - \mathbb{X}\text{Mat}/R$ is a strict monoidal functor.)

Proof. First recall from the proof of Theorem 2.8.17 that for each $A, B, C \in |\mathbf{G} - \mathbb{X}\text{Mat}/R|$ $\alpha = \alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ is defined by

$$\alpha_{((a,b),c)}^{(a',(b',c'))} = \delta_{a,a'} \delta_{b,b'} \delta_{c,c'}$$

and $\lambda = \lambda_A : I \otimes A \longrightarrow A$ is defined by

$$\lambda_{(1,a)}^b = \delta_{a,b}$$

and $\rho = \rho_A : A \otimes I \longrightarrow A$ by

$$\rho_{(a,1)}^b = \delta_{a,b}.$$

Now given an $A \in |\mathbf{G} - \mathbb{X}\text{Mat}/R|$ define $(A)^t = A$ and for each arrow $F : X \longrightarrow Y$ in $\mathbf{G} - \mathbb{X}\text{Mat}/R$ define $F^t : Y \longrightarrow X$ as follows $(F^t)_y^x = F_x^y \quad \forall x \in X, y \in Y$ i.e. the transpose of the matrix. We must check that equations (1) and (2) from Definition 2.8.14 hold.

(1): Let $g \in G$ then

$$\begin{aligned} (F^t)_{g,y}^{g,x} &= F_{g,x}^{g,y} \\ &= F_x^y \quad \text{since } F \text{ is an arrow in } \mathbf{G} - \mathbb{X}\text{Mat}/R \\ &= (F^t)_y^x \end{aligned}$$

as required.

(2):

$$\begin{aligned} (F^t)_y^x &= F_x^y \\ &= \delta_{|x|,|y|} F_x^y \quad \text{since } F \text{ is an arrow in } \mathbf{G} - \mathbb{X}\text{Mat}/R \\ &= \delta_{|y|,|x|} F_x^y \\ &= \delta_{|y|,|x|} (F^t)_y^x \end{aligned}$$

as required. Thus F^t is an arrow in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. It is clear that $(1_X)^t = 1_X$, and that given $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ then $(G \circ F)^t = F^t \circ G^t$. These equations follow from properties of matrix multiplication and transposes of matrices. Therefore $(-)^t$ is a contravariant functor.

Now let's check that $(\alpha)^t = \alpha^{-1}$:

We know that

$$(\alpha^{-1})_{(a', (b', c'))}^{((a, b), c)} = \delta_{a', a} \delta_{b', b} \delta_{c', c}.$$

On the other hand

$$\begin{aligned} (\alpha^t)_{(a', (b', c'))}^{((a, b), c)} &= \alpha_{(a, b), c}^{(a', (b', c'))} \\ &= \delta_{a, a'} \delta_{b, b'} \delta_{c, c'} \\ &= \delta_{a', a} \delta_{b', b} \delta_{c', c} \\ &= (\alpha^{-1})_{(a', (b', c'))}^{((a, b), c)} \end{aligned}$$

hence $(\alpha)^t = \alpha^{-1}$. (ii) and (iii) are similar. Now let's show that (iv) $(f \otimes g)^t = f^t \otimes g^t$ for arrows $f : A \longrightarrow B$ and $g : C \longrightarrow D$ in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$.

$$\begin{aligned} ((f \otimes g)^t)_{(b, d)}^{(a, c)} &= (f \otimes g)_{(a, c)}^{(b, d)} \\ &= f_a^b g_c^d. \end{aligned}$$

On the other hand

$$\begin{aligned} (f^t \otimes g^t)_{(b, d)}^{(a, c)} &= (f^t)_b^a (g^t)_d^c \\ &= f_a^b g_c^d \\ &= ((f \otimes g)^t)_{(b, d)}^{(a, c)} \end{aligned}$$

as required. □

Remark 5.1.11. By Remark 2.4.5 we know that any monoid say (M, μ, η) in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is a comonoid in $(\mathbf{G} - \mathbb{X}\mathbf{Mat}/R)^{op}$ the opposite category. Now as $(-)^t : (\mathbf{G} - \mathbb{X}\mathbf{Mat}/R)^{op} \longrightarrow \mathbf{G} - \mathbb{X}\mathbf{Mat}/R$ is a strict monoidal functor, we then have by Remark 2.6.3 that (M, μ^t, η^t) is a comonoid in $\mathbf{G} - \mathbb{X}\mathbf{Mat}/R$. So this transpose functor takes monoids and turns them into comonoids. Thus Proposition 5.1.9 gives

us a class of monoids in $\mathbf{G} - \mathbb{X}\text{Mat}/R$ which by the previous remark can also be given a comonoid structure by taking transposes.

We would now like to characterize which of these objects also satisfy the Frobenius relation and hence telling us which of these are braided Frobenius objects.

This leads to our first main result, giving a large class of braided Frobenius objects in the category $\mathbf{G} - \mathbb{X}\text{Mat}/R$.

Theorem 5.1.12. *Let (A, μ, η) be a crossed G -monoid. Let (A, M, U) be the monoid in $\mathbf{G} - \mathbb{X}\text{Mat}/R$ given by $M = \text{Mat}(\mu)$, and $U = \text{Mat}(\eta)$. Let (A, Δ, ε) be the comonoid in $\mathbf{G} - \mathbb{X}\text{Mat}/R$ given by $\Delta = M^t$ and $\varepsilon = U^t$. If A is a group then $(A, M, U, \Delta, \varepsilon)$ is a braided Frobenius object in $\mathbf{G} - \mathbb{X}\text{Mat}/R$.*

Proof. Let (A, μ, η) , and $(A, M, U, \Delta, \varepsilon)$ be as above. We will write $\mu(a, b) = ab$ and $\eta(1) = e_A \quad \forall a, b \in A$. As (A, M, U) is a monoid and (A, Δ, ε) is a comonoid we have that (FA1) and (FA2) commute. So it remains to show that the Frobenius relation, (FA3), holds. First we check that $(1_A \otimes M) \circ \alpha \circ (\Delta \otimes 1_A) = \Delta \circ M$

$$\begin{aligned}
(1_A \otimes M \circ \alpha \circ \Delta \otimes 1_A)_{(a,b)}^{(e',f')} &= (\Delta \otimes 1_A)_{(a,b)}^{((a',b'),c')} \alpha_{((a',b'),c')}^{(e,(f,g))} (1_A \otimes M)_{(e,(f,g))}^{(e',f')} \\
&= (\Delta \otimes 1_A)_{(a,b)}^{((e,f),g)} (1_A \otimes M)_{(e,(f,g))}^{(e',f')} \\
&= \Delta_a^{(e,f)} (1_A)_b^g (1_A)_e^{e'} M_{(f,g)}^{f'} \\
&= \sum_{e,f,g} \delta_{ef,a} \delta_{b,g} \delta_{e,e'} \delta_{fg,f'} \\
&= \sum_f \delta_{e'f,a} \delta_{fb,f'} \\
&= \sum_f \delta_{f,e'^{-1}a} \delta_{fb,f'} \\
&= \delta_{e'^{-1}ab,f'} \\
&= \delta_{ab,e'f'}.
\end{aligned}$$

Moreover

$$\begin{aligned}
 (\Delta \circ M)_{(a,b)}^{(e',f')} &= M_{(a,b)}^f \Delta_f^{(e',f')} \\
 &= \sum_f \delta_{ab,f} \delta_{e'f',f} \\
 &= \delta_{ab,e'f'}
 \end{aligned}$$

as required. The proof that $(M \otimes 1_A) \circ \alpha^{-1} \circ (1_A \otimes \Delta) = \Delta \circ M$ is similar. \square

Another good place to look for braided Frobenius objects is in a pivotal category.

Lemma 5.1.13. *If \mathcal{C} is a pivotal category then for every object $A \in |\mathcal{C}|$ we have that $A \otimes A^*$ is a braided Frobenius object.*

Proof. In diagram (P3) of Definition 2.9.1 first taking $f = 1_A$ and then taking $f = 1_{A^*}$ yields four commutative diagrams which imply that $A \dashv A^*$ and $A^* \dashv A$ and so by Corollary 3.3.5 $A \otimes A^*$ is a braided Frobenius object. \square

Of course $A^* \otimes A$ will also be a braided Frobenius object.

Example 5.1.14. $\mathbf{G}\text{-XMat}/R$ is a pivotal category hence we can apply the previous lemma to obtain another class of braided Frobenius objects in $\mathbf{G} - \text{XMat}/R$.

Following [10] we now consider the category $\mathbf{Vect}_{gr}(\mathbb{k})$ of non-negatively graded vector spaces and \mathbb{k} -linear maps which respect the grading. More precisely an object V in $\mathbf{Vect}_{gr}(\mathbb{k})$ is a vector space over \mathbb{k} along with subspaces $(V_i)_{i \in \mathbb{N}}$ such that $V = \bigoplus_{i \in \mathbb{N}} V_i$. Elements of V_i are called homogeneous of degree i . We give \mathbb{k} the trivial grading $\mathbb{k} = \mathbb{k}_0$. Given a pair of graded vector spaces V and W then a morphism from V to W is a linear map $f : V \rightarrow W$ such that $f(V_i) \subseteq W_i$. Next given graded vector spaces V and W then their tensor product can be given a grading as follows $V \otimes W = \bigoplus_{n \in \mathbb{N}} (V \otimes W)_n$ where $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. It is then easy to show that the natural isomorphisms α , λ , and ρ from $\mathbf{Vect}_{\mathbb{k}}$ which make it monoidal are also maps in $\mathbf{Vect}_{gr}(\mathbb{k})$. Thus we have

Lemma 5.1.15. *The category $(\mathbf{Vect}_{gr}(\mathbb{k}), \otimes, \mathbb{k}, \alpha, \lambda, \rho)$ is monoidal.*

The following result can be found in [10] as an exercise. We give a slightly less general version of the result.

Lemma 5.1.16. *Let $\xi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{k} \setminus \{0\}$ be a function such that*

$$\begin{aligned}\xi(m, n+p) &= \xi(m, p)\xi(m, n) \\ \xi(m+n, p) &= \xi(m, p)\xi(n, p) \quad \forall m, n, p \in \mathbb{N}\end{aligned}$$

Then the morphism $\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V$ defined by

$$\sigma_{V,W}(v \otimes w) = \xi(n, p)(w \otimes v),$$

where $v \in V$ and $w \in W$ are homogeneous of degrees n and p respectively, is a braiding for $(\mathbf{Vect}_{gr}(\mathbb{k}), \otimes, \mathbb{k}, \alpha, \lambda, \rho)$.

Now as the forgetful functor $U : (\mathbf{Vect}_{gr}(\mathbb{k}), \otimes, \mathbb{k}, \alpha, \lambda, \rho) \longrightarrow (\mathbf{Vect}_{\mathbb{k}}, \otimes, \mathbb{k}, \alpha, \lambda, \rho)$ is a strict monoidal functor it follows by Remark 2.6.3 that if $(A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object in $\mathbf{Vect}_{gr}(\mathbb{k})$ then $(UA, U\mu, U\eta, U\delta, U\varepsilon) = (A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object in $\mathbf{Vect}_{\mathbb{k}}$. So in particular A must be a Frobenius algebra in the usual sense. Now let $\xi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{k} \setminus \{0\}$ and $\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V$ be as in Lemma 5.1.16. Thus we have just shown:

Proposition 5.1.17. *The quintuple $(A, \mu, \eta, \delta, \varepsilon)$ is a braided Frobenius object in $(\mathbf{Vect}_{gr}(\mathbb{k}), \otimes, \mathbb{k}, \alpha, \lambda, \rho, \sigma)$ if and only if $(A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius algebra in the usual sense, (A, μ, η) is a graded \mathbb{k} -algebra, and (A, δ, ε) is a graded \mathbb{k} -coalgebra.*

Example 5.1.18. We fix $x \in \mathbb{R} \setminus \{0\}$ and define $\xi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R} \setminus \{0\}$ by $\xi(n, p) = x^{np}$ for all $n, p \in \mathbb{N}$. Then ξ satisfies the equations of Lemma 5.1.16 and so $\mathbf{Vect}_{gr}(\mathbb{R})$ is braided. The remainder of this example is taken from [12]. Let X be a compact oriented n -manifold, and let $H^*(X) = \bigoplus_{i=0}^n H^i(X)$ be the de Rham cohomology. Here $H^i(X)$ is the set of closed differentiable i -forms modulo the exact ones. Then $H^*(X)$ is a ring under wedge product and integration over X with respect to a chosen volume form provides a linear map $H^*(X) \longrightarrow \mathbb{R}$. The Poincaré duality states the corresponding bilinear pairing $H^*(X) \otimes H^*(X) \longrightarrow \mathbb{R}$ is nondegenerate. So it is a Frobenius algebra over \mathbb{R} . Moreover $H^*(X)$ is in fact graded i.e. for $\alpha \in H^p(X)$ and

$\beta \in H^q(X)$ then their wedge product satisfies $\alpha \wedge \beta \in H^{p+q}(X)$. Thus $H^*(X)$ is a braided Frobenius object in $\mathbf{Vect}_{gr}(\mathbb{R})$. In addition we also have that, for α and β as before, $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$. Thus if we let $x = -1$ then

$$\begin{aligned} \mu(\sigma_{H^*(X), H^*(X)}(\alpha \otimes \beta)) &= \mu(\xi(p, q)(\beta \otimes \alpha)) \\ &= (-1)^{pq} \beta \wedge \alpha \\ &= \alpha \wedge \beta \\ &= \mu(\alpha \otimes \beta) \end{aligned}$$

where μ is the ring multiplication. So we see that $H^*(X)$ is braided commutative.

Next we consider a construction which to any (strict) monoidal category $(\mathcal{C}, \otimes, I)$ assigns a braided monoidal category $\mathcal{Z}(\mathcal{C})$. This is called the centre construction. We follow Kassel for this definition [10].

Definition 5.1.19. *An object of $\mathcal{Z}(\mathcal{C})$ is a pair $(V, c_{-,V})$ where V is an object of \mathcal{C} and $c_{-,V}$ is a family of natural isomorphisms*

$$c_{X,V} : X \otimes V \longrightarrow V \otimes X$$

defined for all objects X in \mathcal{C} such that for all objects X, Y in \mathcal{C} the diagram

$$\begin{array}{ccc} X \otimes Y \otimes V & \xrightarrow{1_X \otimes c_{Y,V}} & X \otimes V \otimes Y \\ & \searrow c_{X \otimes Y, V} & \downarrow c_{X,V} \otimes 1_Y \\ & & V \otimes X \otimes Y \end{array}$$

commutes. A morphism from $(V, c_{-,V})$ to $(W, c_{-,W})$ is a morphism $f : V \longrightarrow W$ in \mathcal{C} such that for each object X in \mathcal{C} we have the commutative diagram

$$\begin{array}{ccc} X \otimes V & \xrightarrow{1_X \otimes f} & X \otimes W \\ c_{X,V} \downarrow & & \downarrow c_{X,W} \\ V \otimes X & \xrightarrow{f \otimes 1_X} & W \otimes X \end{array}$$

Remark 5.1.20. Naturality of $c_{-,V}$ in the previous definition means that given any arrow $g : X \longrightarrow Y$ in \mathcal{C} we have that $c_{Y,V} \circ g \otimes 1_V = 1_V \otimes g \circ c_{X,V}$.

Remark 5.1.21. We can generalize the centre the construction to include monoidal categories that aren't necessarily strict.

This next theorem can also be found in [10] along with its proof.

Theorem 5.1.22. *Let $(\mathcal{C}, \otimes, I)$ be a strict monoidal category. Then $\mathcal{Z}(\mathcal{C})$ is a strict braided monoidal category where*

(i) *the tensor unit is $(I, 1_I)$,*

(ii) *the tensor product of $(V, c_{-,V})$ and $(W, c_{-,W})$ is given by*

$$(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V \otimes W})$$

where $c_{X,V \otimes W} : X \otimes V \otimes W \longrightarrow W \otimes V \otimes X$ is given by

$$c_{X,V \otimes W} = (1_V \otimes c_{X,W}) \circ (c_{X,V} \otimes 1_W),$$

(iii) *and the braiding is given by*

$$c_{V,W} : (V, c_{-,V}) \otimes (W, c_{-,W}) \longrightarrow (W, c_{-,W}) \otimes (V, c_{-,V}).$$

Lemma 5.1.23. *For any strict monoidal category $(\mathcal{C}, \otimes, I)$ the functor $\Pi : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$ given by $\Pi(V, c_{-,V}) = V$ and is the identity on arrows is a strict monoidal functor [10].*

Using this lemma we obtain our next new result.

Theorem 5.1.24. *Let $(\mathcal{C}, \otimes, I)$ be a strict monoidal category. Then $((A, c_{-,A}), \mu, \eta, \delta, \varepsilon)$ is a braided Frobenius object in $\mathcal{Z}(\mathcal{C})$ if and only if*

(i) *$(A, c_{-,A})$ is an object in $\mathcal{Z}(\mathcal{C})$,*

(ii) *$(A, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object in \mathcal{C} , and*

(iii) *$\mu : A \otimes A \longrightarrow A$, $\eta : I \longrightarrow A$, $\delta : A \longrightarrow A \otimes A$, and $\varepsilon : A \longrightarrow I$ are arrows in $\mathcal{Z}(\mathcal{C})$.*

Proof. $[\implies]$ Since $\Pi : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$ is a strict monoidal functor we have that

$$(\Pi(A, c_{-,A}), \Pi\mu, \Pi\eta, \Pi\delta, \Pi\varepsilon) = (A, \mu, \eta, \delta, \varepsilon)$$

is Frobenius object in \mathcal{C} , so (ii) holds. (i) and (iii) obviously hold.

$[\impliedby]$ By (i) and (iii) $(A, c_{-,A})$ is an object in $\mathcal{Z}(\mathcal{C})$ and $\mu, \eta, \delta, \varepsilon$ are arrows in $\mathcal{Z}(\mathcal{C})$ hence all diagrams involving these arrows which commute in \mathcal{C} also commute in $\mathcal{Z}(\mathcal{C})$ with A replaced by $(A, c_{-,A})$. \square

5.2 Quasitriangular Hopf Algebras

We shall now give an outline of when the category of H -modules, for a bialgebra or Hopf algebra, is braided monoidal. We follow the treatment presented in [10].

Definition 5.2.1. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We call it **quasi-cocommutative** if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$ we have

$$\tau_{H,H} \circ \Delta = R\Delta(x)R^{-1}.$$

The element R is called a **universal R -matrix**. A quasi-cocommutative Hopf algebra is Hopf algebra whose underlying bialgebra is quasi-cocommutative.

Definition 5.2.2. A quasi-cocommutative bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$ or a quasi-cocommutative Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ is called **quasitriangular** if the universal R -matrix $R = \sum_{i=1}^n s_i \otimes t_i$ satisfies the two relations

$$(\Delta \otimes id_H)(R) = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j \tag{3}$$

$$(id_H \otimes \Delta)(R) = \sum_{i,j} s_i s_j \otimes t_j \otimes t_i. \tag{4}$$

We should mention that in [10] Kassel calls quasitriangular bialgebras braided bialgebras, and similarly for Hopf algebras.

Example 5.2.3. We note that every cocommutative bialgebra is quasitriangular with universal R -matrix $R = 1 \otimes 1$.

We now present an example as found in [16] of a quasitriangular Hopf algebra.

Example 5.2.4. Let $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n written multiplicatively. Let g denote the generator of this group, then $g^n = 1$. By Example 2.5.6 the group algebra $\mathbb{C}\mathbb{Z}/n$ is a Hopf algebra, moreover the element $\mathcal{R} \in \mathbb{C}\mathbb{Z}/n \otimes \mathbb{C}\mathbb{Z}/n$ defined by

$$\mathcal{R} = \frac{1}{n} \sum_{a,b=0}^{n-1} e^{\frac{-2\pi i ab}{n}} g^a \otimes g^b$$

is a universal \mathcal{R} -matrix which satisfies the equations of Definition 5.2.2. Hence $\mathbb{C}\mathbb{Z}/n$ is a quasitriangular Hopf algebra. In [16] Majid calls the Hopf algebra $\mathbb{C}\mathbb{Z}/n$ with this quasitriangular structure the anyon-generating quantum group and is denoted \mathbb{Z}'/n .

Now suppose that $(H, \mu, \eta, \Delta, \varepsilon, R)$ is a quasitriangular bialgebra and consider the monoidal category $H\text{-Mod}$ of left H -modules. Then using the universal R -matrix we can build a map of H -modules $c_{V,W}^R : V \otimes W \longrightarrow W \otimes V$ which is a natural isomorphism. Namely

$$c_{V,W}^R(v \otimes w) = \tau_{V,W}(R(v \otimes w)) = \sum_i t_i w \otimes s_i v$$

where $R = \sum_i s_i \otimes t_i$ and $(c_{V,W}^R)^{-1}(w \otimes v) = R^{-1}(v \otimes w)$. We then have the following proposition which is found in [10].

Proposition 5.2.5. (a) *The map $c_{V,W}^R$ is an isomorphism of H -modules, and*
 (b) *for any triple (U, V, W) of H -modules we have*

$$c_{U \otimes V, W}^R = (c_{U,W}^R \otimes id_V) \circ (id_U \otimes c_{V,W}^R) \tag{5}$$

$$c_{U, V \otimes W}^R = (id_V \otimes c_{U,W}^R) \circ (c_{U,V}^R \otimes id_W). \tag{6}$$

The equations in Proposition 5.2.5 are exactly the equations (B1) and (B2) from Definition 2.8.1 where α is the identity. Hence the proposition shows that $H\text{-Mod}$ is a (strict) braided monoidal category. We can in fact say even more.

Proposition 5.2.6. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then the monoidal category $H\text{-Mod}$ is braided if and only if the bialgebra H is quasitriangular [10].*

We now give a description of the braided monoidal category of H -modules for the the anyon-generating quantum group \mathbb{Z}'_n . This result is found in [16].

Example 5.2.7. Let \mathbb{Z}'_n be the anyon-generating quantum group. Its category \mathcal{C}_n of modules has as objects vector spaces which are $\mathbb{Z}/n\mathbb{Z}$ -graded and the morphisms are linear maps that preserve the grading. The category is a braided monoidal one with the tensor product defined by adding the grading modulo n and with braiding

$$\sigma(v \otimes w) = e^{\frac{2\pi i |v||w|}{n}} w \otimes v$$

for homogeneous elements v and w of degree $|v|$ and $|w|$ respectively. For $n > 2$ $\sigma^2 \neq id$.

Now suppose that $(H, \mu, \eta, \Delta, \varepsilon, R)$ is a quasitriangular bialgebra. Then we know that $H\text{-Mod}$ is braided monoidal, and so we can ask what are the braided Frobenius objects in this category. If $(A, \mu', \eta', \delta', \varepsilon')$ is a braided Frobenius object in $H\text{-Mod}$ then in particular it is a Frobenius algebra in the classical sense. This follows from the fact that the forgetful functor from $H\text{-Mod}$ to $\mathbf{Vect}_{\mathbb{k}}$ is a strict monoidal functor. This brings us to our next new result.

Theorem 5.2.8. *$(A, \mu', \eta', \delta', \varepsilon')$ is a braided Frobenius object in $H\text{-Mod}$ if and only if $(A, \mu', \eta', \delta', \varepsilon')$ is a Frobenius algebra in $\mathbf{Vect}_{\mathbb{k}}$, A is an H -module and $\mu', \eta', \delta', \varepsilon'$ are maps of H -modules.*

Following [10] suppose that $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ is a quasitriangular Hopf algebra and let V be a left H -module. Then the dual space $V^* = \text{Hom}(V, \mathbb{k})$ can be given a left H -module structure as follows. Let $f \in V^*$, $v \in V$ and $a \in H$ then $a \cdot f \in V^*$ is the linear functional $(a \cdot f)(v) = f(S(a) \cdot v)$. Moreover if V is finite-dimensional then the evaluation map $d_V : V^* \otimes V \rightarrow \mathbb{k}$ and the coevaluation map $b_V : \mathbb{k} \rightarrow V \otimes V^*$ are maps of left H -modules and are the counit and unit for the duality $V^* \dashv V$. There is another left H -module structure that we can put on the dual space of V . Namely let *V be the same vector space as $\text{Hom}(V, \mathbb{k})$ but with the following H -module structure. Let $f \in {}^*V$, $v \in V$, and $a \in H$ then $a \cdot f$ is the linear functional $(a \cdot f)(v) = f(S^{-1}(a) \cdot v)$. Then the obvious maps $b'_V : \mathbb{k} \rightarrow {}^*V \otimes V$ and

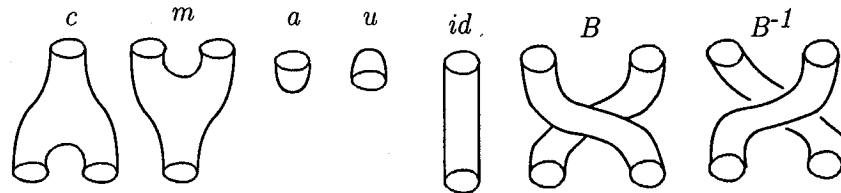
$d'_V : V \otimes^* V \longrightarrow \mathbb{k}$ are maps of left H -modules and are the unit and counit for the duality $V \dashv^* V$, when V is finite-dimensional. When $S = S^{-1}$, i.e. $S^2 = id_H$, we have that $V^* = {}^*V$ as left H -modules.

Example 5.2.9. Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ be a quasitriangular Hopf algebra with $S^2 = id_H$ and let V be a finite-dimensional left H -module. Then by the above discussion $V \dashv V^*$ and $V^* \dashv V$. Hence by Corollary 3.3.5 $V \otimes V^*$ is a braided Frobenius object in $H\text{-Mod}$.

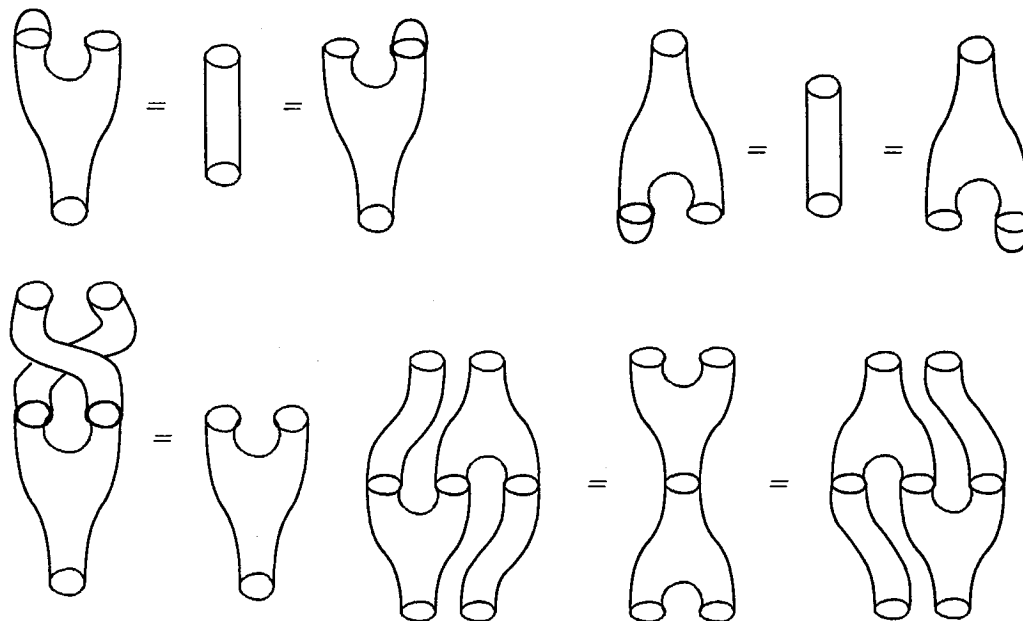
5.3 Free Braided Monoidal Category on a Braided Frobenius Object

We begin by describing the free braided monoidal category generated by a braided Frobenius object. We give a formal definition in terms of generators and relations. We use geometric notation even though this is a formal construction.

Definition 5.3.1. *The braided monoidal category \mathbf{BrFrob} is generated monoidally by the following seven morphisms.*



These morphisms are subject to the following equations



We also require that B induce a braiding, say β .

Theorem 5.3.2. *Let $(\mathcal{V}, \square, I, \sigma)$ be a braided monoidal category and we denote by $\mathbf{BrMonCat}(\mathbf{BrFrob}, \mathcal{V})$ the category whose objects are braided strict monoidal functors from \mathbf{BrFrob} to \mathcal{V} and arrows are monoidal natural transformations. Then $(\mathbf{BrFrob}, \otimes, 0, \beta)$ is the free braided monoidal category on a braided (co)commutative Frobenius object. Thus we have a canonical equivalence of categories*

$$\mathbf{BrMonCat}(\mathbf{BrFrob}, \mathcal{V}) \simeq \mathbf{BrCFrob}(\mathcal{V})$$

where $\mathbf{BrCFrob}(\mathcal{V})$ is the category of braided commutative Frobenius objects in \mathcal{V} .

Proof. The proof is basically trivial since this category is defined formally in terms of its generators and relations. \square

This theorem is not particularly useful since the relation on morphisms is difficult to work with. We would like a more geometric description of the free category. We can conjecture that the following structure is correct.

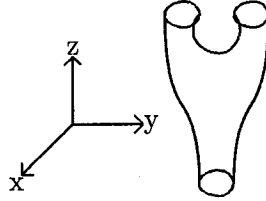
We introduce a category which is an embedded version of **2Cob**. A similar structure is suggested in the work of Bartlett [4]. From now until further notice we set X to be the space $X = \mathbb{R}^2 \times I$ where $I = [0, 1]$ and we identify X with the subspace $\{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq 1\}$. We give X the smooth structure inherited from \mathbb{R}^3 .

Definition 5.3.3. An *X-cobordism* M is a compact orientable smooth 2-submanifold of X with boundary ∂M , possibly $\partial M = \emptyset$. Moreover $\partial M = \partial M_1 \amalg \partial M_0$ where $\partial M_k \subset \mathbb{R}^2 \times \{k\}$ for $k = 0, 1$ and

$$\partial M_k = \bigcup_{i_k=1}^{n_k} C_k(i_k, 1/3)$$

where $C_k(i_k, 1/3) =$ circle of radius $1/3$ centered at $(0, i_k, k)$. We orient M and ∂M so that ∂M_1 is an in-boundary and ∂M_0 is an out-boundary. In this case we say that M is an *X-cobordism* from n_1 to n_0 . If $\partial M_k = \emptyset$ then $n_k = 0$.

The following is an example of X-cobordism from 2 to 1:

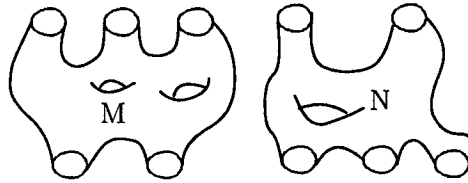


When depicting X-cobordisms we will usually omit the coordinate axes from our drawing. Next given two X-cobordisms M and N from n to p we say that they are equivalent, $M \approx N$, if there exists a smooth orientation-preserving map $H : X \times I \rightarrow X$ such that

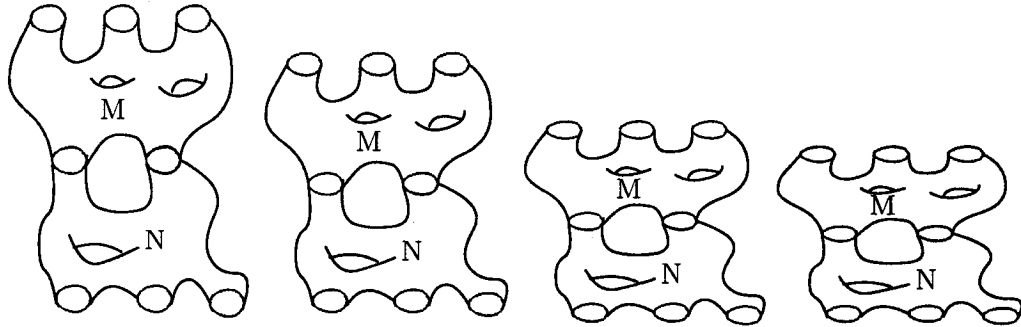
- $H(-, t)$ is a diffeomorphism for all $t \in I$,
- $H(-, 0) = id_X$,
- $H(M, 1) = N$,
- $H|_{(\partial X) \times I} = \pi_{\partial X}$ where $\pi_{\partial X}$ is projection onto ∂X .

Given two X-cobordisms M from n to p and N from m to q we can form a new X-cobordism $M \otimes N$ from $n + m$ to $p + q$ by juxtaposing N alongside M . This may

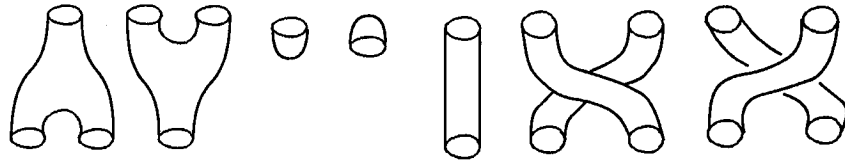
require deforming M and N so that they don't overlap. Pictorially $M \otimes N$ looks like this



If we are given X-cobordisms M from n to m and N from m to q then we can form a new X-cobordism MN which is obtained by gluing M and N together along ∂M_0 and ∂N_1 and then rescaling it. Schematically this looks like

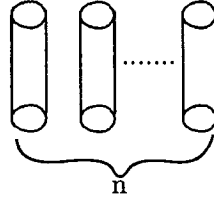


So with these tools we can now build a category which we call **TangleCob**. It has as objects non-negative integers, and an arrow from n to m is an equivalence class $[M]$ of X-cobordisms from n to m generated by gluing and tensoring of the following seven X-cobordisms:

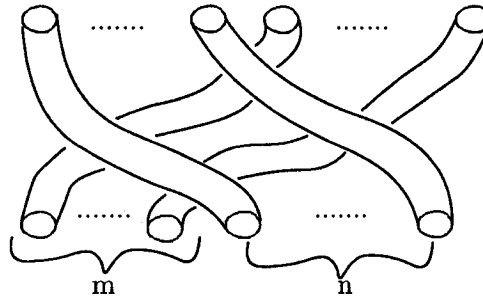


Given $[M] : n \rightarrow m$ and $[N] : m \rightarrow p$ then $[N] \circ [M] := [MN]$ where $[MN]$ is the equivalence class of M glued to N as described above. Given $n, m \geq 0$ we define $n \otimes m := n + m$ and given $[M] : n \rightarrow p$ $[M'] : m \rightarrow q$ we define $[M] \otimes [M']$ to be

$[M \otimes M'] : n + m \longrightarrow p + q$. The identity on n is the equivalence class of the following X-cobordism



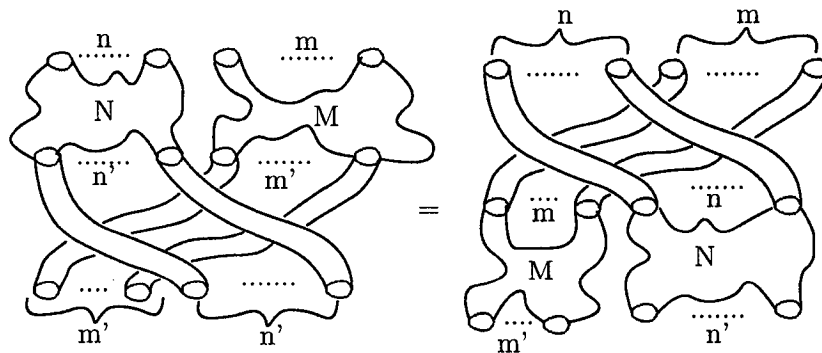
It is then easy to see that **TangleCob** is a category and that \otimes is a bifunctor. Moreover the object 0 satisfies $n \otimes 0 = n = 0 \otimes n$ for all objects n , and $(n \otimes m) \otimes p = n \otimes (m \otimes p)$ for all objects n, m , and p . Hence **TangleCob** is a strict monoidal category. In addition, for all pairs of objects n and m we have a morphism $[T_{n,m}] : n + m \longrightarrow m + n$, where $T_{n,m}$ is the following X-cobordism:



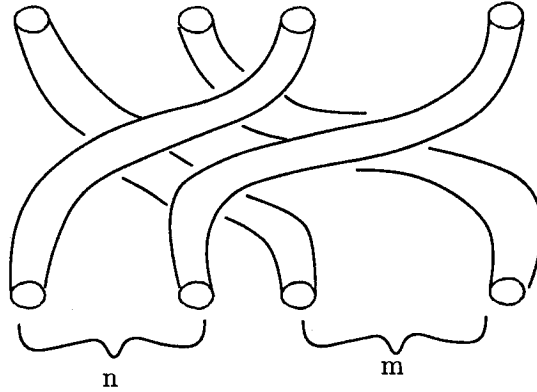
Moreover given morphisms $[N] : n \longrightarrow n'$ and $[M] : m \longrightarrow m'$, we have

$$[T_{n',m'}] \circ [N \otimes M] = [M \otimes N] \circ [T_{n,m}]$$

This equation can be seen to hold by the following diagram.



The equations which express that $[T]$ is braiding are easy to verify. The inverse of $[T_{n,m}]$ is depicted by the following cobordism



Thus $(\mathbf{TangleCob}, 0, \otimes, [T])$ is a strict braided monoidal category.

Given any morphism $[M] : n \rightarrow m$ in $\mathbf{TangleCob}$ we can view M as a morphism in $\mathbf{2Cob}$ where its in-boundary is ∂M_1 and its out-boundary is ∂M_0 . Now given another $M' \in [M]$ it is clear that M' , viewed as a morphism in $\mathbf{2Cob}$, will be equivalent to M . Thus we can define a functor \mathcal{COR} from $\mathbf{TangleCob}$ to $\mathbf{2Cob}$ by sending n to \mathbf{n} and for morphisms forgetting the embedding into X . This functor is in fact a braided strict monoidal functor which is surjective on objects and arrows.

We believe that this correctly captures the free structure as stated in the following conjecture.

Conjecture 5.3.4. *We have the following equivalence of categories*

$$\mathbf{TangleCob} \simeq \mathbf{BrFrob}.$$

Since \mathbf{BrFrob} is free, it is clear that every equation that holds in \mathbf{BrFrob} holds in $\mathbf{TangleCob}$. To establish the converse suppose that $[M]$ and $[N]$ are morphisms in $\mathbf{TangleCob}$ with $[M] = [N]$. By applying the \mathcal{COR} functor, we obtain two equal $\mathbf{2Cob}$ morphisms. Given the structure of $\mathbf{2Cob}$, the images of $[M]$ and $[N]$ must be related by a series of basic equations. If this series did not use the symmetry equation, we would be finished. However we do not see how to establish that the symmetry equation never arises.

Chapter 6

Future Work

Obviously the first order of business is to establish a proof of the conjecture. This may yield interesting geometric insights into the theory of braided categories.

Another interesting open question is to find a geometric description of the free braided or symmetric monoidal category generated by a noncommutative Frobenius object.

Every commutative Frobenius \mathbb{k} -algebra A can be viewed as a Frobenius object in the braided category of graded vector spaces where we give A the trivial grading. Hence our theory can be viewed as a natural extension of the more well-known theory. Therefore it would be interesting to view the applications of Frobenius algebras in representation theory, physics, and computer science in this more general theory.

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