

## INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

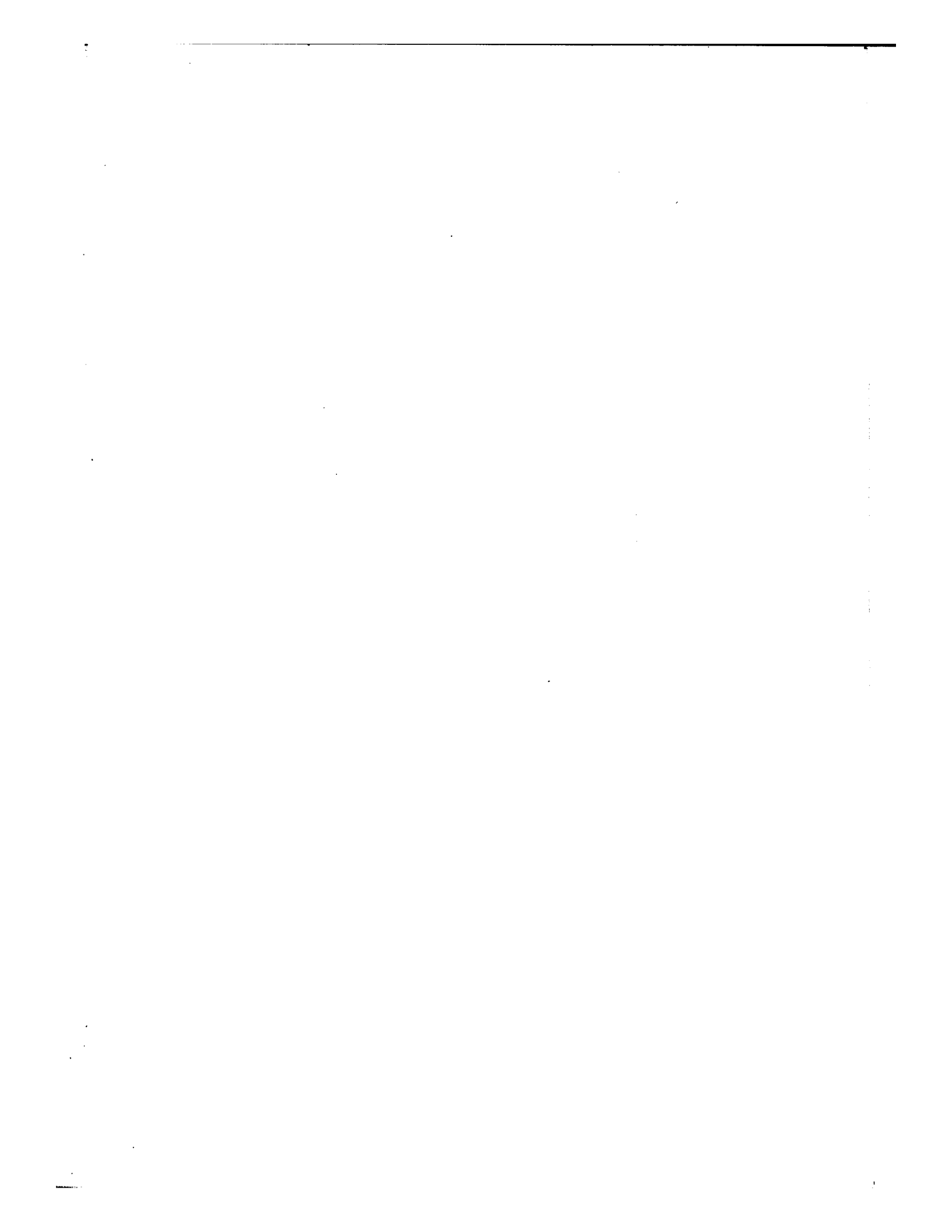
**The quality of this reproduction is dependent upon the quality of the copy submitted.** Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

ProQuest Information and Learning  
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA  
800-521-0600

UMI<sup>®</sup>



5c

ANALYTIC IMMERSIONS OF PARABOLIC  
RIEMANN SURFACES

A thesis submitted

by

Paul Andre Vincent

to

the Faculty of Pure and Applied Science  
of the University of Ottawa

in partial fulfillment of the requirements  
for the degree of  
Master of Science  
in the subject of  
Mathematics

May, 1967.



UMI Number: EC52195

### INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

**UMI<sup>®</sup>**

---

UMI Microform EC52195  
Copyright 2007 by ProQuest LLC  
All rights reserved. This microform edition is protected against  
unauthorized copying under Title 17, United States Code.

---

ProQuest LLC  
789 East Eisenhower Parkway  
P.O. Box 1346  
Ann Arbor, MI 48106-1346

CONTENTS

	Contents	i
	Acknowledgments	ii
	Abstract	iii
SECTION 1	Parallel Concepts for Manifolds and Riemann Surfaces	1
SECTION 2	Parallel Concepts of Covering Manifolds	5
SECTION 3	Analytic Immersions of Simply Connected Riemann Surfaces	14
SECTION 4	Analytic Immersions of Parabolic Riemann Surfaces	20
SECTION 5	APPENDIX : Further Results on Immersion of Torii into Torii	44
	BIBLIOGRAPHY	51

### ACKNOWLEDGMENTS

I am in debt to and sincerely wish to thank Dr. H.G. Helfenstein, who, as thesis advisor, proposed this work and with continuous patience and insight was the determining factor in its completion.

It is also a privilege to acknowledge the assistance of an Ontario Graduate Fellowship granted by the Ontario Government.

ABSTRACT

---

Under conformal equivalence the set of all parabolic Riemann surfaces is divided into equivalence classes. Since a conformal equivalence is an analytic immersion, then the study of analytic immersions between all pairs of parabolic Riemann surfaces reduces to the study of immersions between all equivalence classes by representatives. The complex plane, the "cylinder", and the set of all torii modulo conformal equivalence form a useful set of representatives which enables us to determine easily whether analytic immersions exist or not. Where they do exist the use of the fiber map theorem permits us to give these analytic immersions as analytic immersions between complex planes.

In particular, in the case of a pair of torii, we have been able to find a necessary and sufficient criterion to determine the existence of analytic immersions. Furthermore if there exist any then we can determine all immersions.

# ANALYTIC IMMERSIONS OF PARABOLIC RIEMANN SURFACES

## SECTION 1

### PARALLEL CONCEPTS FOR MANIFOLDS AND RIEMANN SURFACES

Definition 1.1 - An  $m$  - dimensional differentiable manifold  $M$  of class  $C^r$ ,  $r > 0$  is a connected Hausdorff space together with a fixed complete atlas with respect to  $E^m$  compatible with  $\Gamma^r(E^m)$ , the pseudo group of all local transformations  $f$  of any open subset of  $E^m$  onto another open subset of  $E^m$  such that  $f$  and  $f^{-1}$  are of class  $C^r$ .

Definition 1.2 - A Riemann surface is a connected Hausdorff space together with a fixed complete atlas with respect to  $\mathbb{C}$ , the complex numbers, compatible with  $\Gamma(\mathbb{C})$ , the set of all local homeomorphisms  $\psi$  of  $\mathbb{C}$  such that  $\psi$  is holomorphic.

Definition 1.3 - Let  $M$  be a differentiable manifold.  $\mathcal{F}$  is defined to be the collection of all functions  $f$  such that :

(1)  $f : M \rightarrow \mathbb{R}$ ,

(2) for all  $p \in M$  and chart  $(U, \psi)$  such that  $p \in U$  then

$$f \circ \psi^{-1} : \psi(U) \rightarrow \mathbb{R} \text{ is differentiable}$$

And  $\mathcal{F}(p)$  is defined similarly except that  $f \circ \psi^{-1}$  is taken to be differentiable in a neighbourhood of  $\psi(p)$ .

Definition 1.4 - A map  $\Phi$  of the differentiable manifold  $M$  of dimension  $m$  into the differentiable manifold  $N$  of dimension  $n$  is differentiable at  $p \in M$  if  $f \circ \Phi \in \mathcal{F}(p)$  for all  $f \in \mathcal{F}[\Phi(p)]$  and is differentiable on  $M$  if it is differentiable for all  $p \in M$ .

An analytic map between Riemann surfaces is defined similarly with  $\mathbb{R}$  in the definition 1.3 being replaced by  $\mathbb{C}$  and holomorphy replacing differentiability throughout.

Definition 1.5 - Let  $M$  and  $N$  be two manifolds of the same dimension. A map  $\Phi$  of  $M$  into  $N$  is said to be a local homeomorphism if for all  $p \in M$  there exist two neighbourhoods  $N(p)$  and  $N(\Phi(p))$  such that  $\Phi|_{N(p)}$  is a homeomorphism of  $N(p)$  onto  $N(\Phi(p))$ .

Definition 1.6 - Let  $M$  and  $N$  be two differentiable manifolds of the same dimension and  $\Phi$  a map such that  $\Phi: M \rightarrow N$ . If  $\Phi$  is 1:1 and if both  $\Phi$  and  $\Phi^{-1}$  are differentiable then  $\Phi$  is called a diffeomorphism.

Definition 1.7 - Let  $R_1$  and  $R_2$  be two Riemann surfaces and  $\Phi$  a map such that  $\Phi: R_1 \rightarrow R_2$ . If  $\Phi$  is 1:1 and if  $\Phi$  is analytic (and consequently so is  $\Phi^{-1}$ ) then  $\Phi$  is called a conformal equivalence.

Definition 1.8 - An analytic map  $f: R \rightarrow R_1$ ,  $R, R_1$  two Riemann surfaces, is an analytic immersion if its local representation  $w = g(z)$  for any two corresponding local charts satisfies  $g'(z) \neq 0$ , i.e.  $g(z)$  is locally 1:1.

THEOREM 1.1 - Let  $R_1, R_2, R'_1$  and  $R'_2$  be four Riemann surfaces such that  $R_1$  is conformally equivalent to  $R'_1$  and  $R_2$  is conformally equivalent to  $R'_2$ . Then there is a 1:1 correspondence between the analytic immersions of  $R_1 \rightarrow R_2$  and those of  $R'_1 \rightarrow R'_2$ .

PROOF : Let  $f: R_1 \rightarrow R'_1$  and  $g: R_2 \rightarrow R'_2$  be the respective conformal equivalences. Let  $\Phi$  be an immersion of  $R_1 \rightarrow R_2$ .

It is clear that since  $g, \Phi$  and  $f$  are analytic and locally 1:1 we have the composition  $g \circ \Phi \circ f^{-1}$  is also

$$\begin{array}{ccc} R_1 & \xrightarrow{\Phi} & R_2 \\ f^{-1} \uparrow & & \downarrow g \\ R'_1 & & R'_2 \end{array}$$

analytic and locally 1:1 and hence an analytic

immersion of  $R'_1$  into  $R'_2$ ; therefore to every distinct analytic immersion

$\Phi$  of  $R_1 \longrightarrow R_2$  there corresponds a distinct analytic immersion  $g \circ \Phi \circ f^{-1}$

of  $R'_1 \longrightarrow R'_2$  and vice versa to every distinct analytic immersion

$\Psi$  of  $R'_1 \longrightarrow R'_2$  there corresponds a distinct analytic immersion  $f \circ \Psi \circ g^{-1}$

of  $R_1 \longrightarrow R_2$ . This shows that there is a 1:1 correspondence between the

analytic immersion of  $R_1 \longrightarrow R_2$  and of  $R'_1 \longrightarrow R'_2$ .

SECTION 2PARALLEL CONCEPTS OF COVERING MANIFOLDS

Definition 2.1 - Let  $B$  be a path connected space. A

connected topological space  $E$  is said to be a topological covering space over  $B$  relative to a map  $p:E \longrightarrow B$  if the following conditions are satisfied :

- (1)  $p$  maps  $E$  onto  $B$
- (2) For all  $b \in B$ , there exists a connected open neighbourhood  $V$  of  $b$  in  $B$  such that each component of  $p^{-1}(V)$  is open in  $E$  and is mapped homeomorphically onto  $V$  by  $p$ .

Proposition 2.1 - (i)  $p$  is an open map

(ii)  $E$  is a topological manifold if  $B$  is.

PROOF : (i)  $p:E \longrightarrow B$  is an open map since, if  $U$  is an open set of  $E$ , then for all  $x \in U$ ,  $p(x)$  has an open connected neighbourhood  $V$  in  $B$  and hence there exists an open component  $W_x$  of  $p^{-1}(V)$  containing  $x$ , which is mapped homeomorphically onto  $V$ . Hence we have that  $U \cap W_x$  is open and since

$p|_{W_x}$  is a homeomorphism  $p(U \cap W_x)$  is open in  $B$ .

But  $U = \bigcup_x (U \cap W_x)$  so that  $p(U)$  is open in  $B$ .

(ii)  $E$  also is a topological manifold.  $E$  was given to be connected so that we show that  $E$  is Hausdorff and locally Euclidean.

Let  $x_1, x_2 \in E$  be two distinct points. If  $p(x_1) = p(x_2) =$

$b$  then  $x_1$  and  $x_2$  must come from two different open

components of  $p^{-1}(V)$  where  $V$  is a suitable open connected neighbourhood of  $b$  since  $p$  is a homeomorphism on any one

component. Hence  $x_1$  and  $x_2$  belongs to two disjoint open subsets of  $E$ .

If  $p(x_1) \neq p(x_2)$  then since  $B$  is Hausdorff then there are

two disjoint open neighbourhoods  $N_1$  and  $N_2$  of  $p(x_1)$  and

$p(x_2)$  respectively. If  $V_1$  and  $V_2$  are suitable open

connected neighbourhoods of  $p(x_1)$  and  $p(x_2)$  respectively

then  $V_1 \cap N_1$  and  $V_2 \cap N_2$  are disjoint and so  $p^{-1}(V_1 \cap N_1)$

and  $p^{-1}(V_2 \cap N_2)$  are disjoint open sets containing  $x_1$  and

$x_2$  respectively.

$E$  is clearly locally Euclidean since each  $x \in E$  has a neighbourhood  $N$  which is mapped homeomorphically by  $p$  onto a neighbourhood  $V$  of  $p(x)$  and  $p(x)$  has a neighbourhood  $W$  which is homeomorphic to an open subset of a Euclidean space. Hence  $(p|N)^{-1}(V \cap W)$  is a neighbourhood of  $x$  homeomorphic to an open subset of a Euclidean space.

Example 2.1 - Let  $B = S^1$  the unit circle and  $E = \mathbb{R}$  the real line and let  $p$  be the exponential map. Then  $p : \mathbb{R} \rightarrow S^1$  such that  $p(x) = e^{2\pi ix}$  with  $x \in \mathbb{R}$  and it is clear that for every proper connected subspace  $U$  of  $S^1$ ,  $p$  carries every component of  $p^{-1}(U)$  homeomorphically onto  $U$ .

Definition 2.2 - Let  $M$  be a differentiable manifold.  $M^*$  is a differentiable covering manifold of  $M$  relative to the differentiable map  $f: M^* \rightarrow M$  if :

- (1)  $f$  maps  $M^*$  onto  $M$
- (2) For all  $p \in M$  there exist an open connected neighbourhood  $V$  of  $p$  in  $M$  such that each component of  $f^{-1}(V)$  is open in  $M^*$  and is mapped diffeomorphically onto  $V$  by  $f$ .

The example 2.1 above provides also an example of a differentiable covering manifold since  $S^1$  and  $R$  are differentiable manifolds and  $p$  is a differentiable map.

Definition 2.3 - Let  $R$  be a Riemann surface. The Riemann surface  $R^*$  is an analytic covering Riemann surface relative to the analytic map  $f: R^* \rightarrow R$  if :

- (1)  $f$  maps  $R^*$  onto  $R$
- (2) For all points  $p \in R$  there exists an open connected neighbourhood  $V$  of  $p$  in  $R$  such that each component of  $f^{-1}(V)$  is open in  $R^*$  and is conformally equivalent to  $V$  under  $f$ .

Example 2.2 - Let  $R^* = \{ z : 0 < \Re(z) < 1, -\infty < \Im(z) < \infty \}$

and  $R = \{ w : 1 < |w| < e \}$

Let  $f(z) = w$  be defined as  $f(z) = e^z$

Then (1)  $f: R^* \xrightarrow{\text{onto}} R$

- (2) for each point  $p \in R$  it is clear that we can

find an open connected neighbourhood  $V$  of  $p$  in  $R$  such that each component of  $f^{-1}(V)$  is open and conformally equivalent to  $V$  since  $e^z$  is locally a conformal equivalence.

THEOREM 2.1 - Every analytic covering map is an analytic immersion.

PROOF : Let  $R^*$  be the analytic covering Riemann surface of the Riemann surface  $R$ , relative to  $f:R^* \rightarrow R$ . Let  $p^* \in R^*$  be a point which lies over  $p \in R$ . Then by definition  $p$  has an open connected neighbourhood  $N(p)$  and  $p^*$  has an open neighbourhood  $N(p^*)$  in which  $f$  is a conformal equivalence, i.e.  $f|_{N(p^*)}$  in local representation is 1 : 1 and analytic and so by definition  $f$  is an analytic immersion Q.E.D.

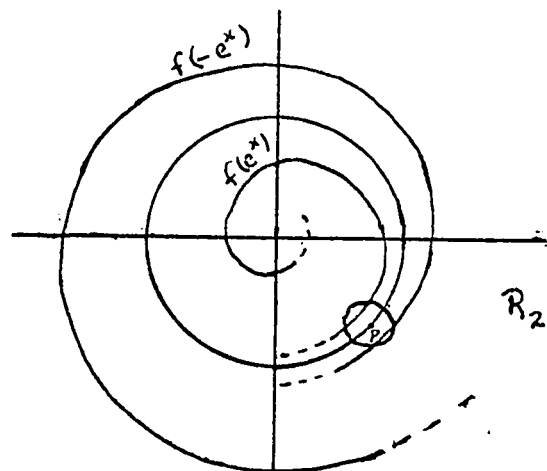
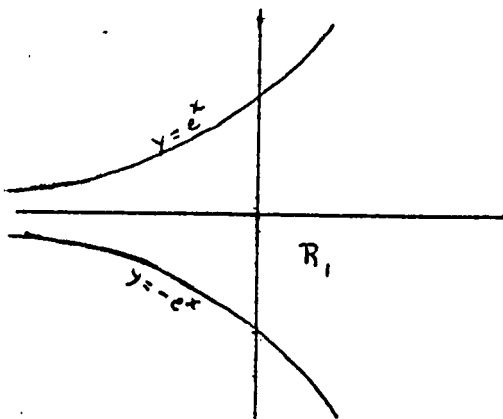
However not every analytic immersion is a covering map.

Consider the following counter example.

Example 2.3 - Let  $R_1 = \{ z = x + iy : |y| < e^x \}$

$$R_2 = \{ w : 0 < |w| < \infty \}$$

Consider the map  $f(z) = e^{2\pi iz}$  which takes  $R_1$  onto  $R_2$



i.e. Let  $z = x + iy$

For  $y = 0$ ,  $z = x$  is the real axis and  $f(z) = e^{2\pi i x}$

the unit circle.

$$\begin{aligned} \text{For } y = e^x \quad f(z) &= e^{2\pi i(x + ie^x)} = e^{2\pi(ix - e^x)} \\ &= e^{-2\pi e^x} \cdot e^{2\pi ix} \end{aligned}$$

which is a spiral emanating from the unit circle and spiralling in about the origin.

For  $y = -e^x$

$$f(z) = e^{2\pi e^x} \cdot e^{2\pi ix}$$

which is a spiral emanating from the unit circle and tending to infinity.

Hence  $f(z) = e^{2\pi iz}$  maps  $R_1$  onto  $R_2$  and is analytic

and locally 1 : 1 and thus  $f$  is an immersion.

However if we consider a point  $p \in R_2$  which lies on the unit circle then there exists no open connected neighbourhood  $V$  such that each component of  $f^{-1}(V)$  is conformally equivalent to  $V$ . So that  $f$  is not an analytic covering map.

Definition 2.4 - A topological space is simply connected if it is arcwise connected and if its fundamental group reduces to the identity.

We want to consider now the notion of a universal covering surface. For this we adopt the results that Hu uses ([3], p. 90).

Let  $A$  be a covering space of  $B$  relative to the projection  $p : A \rightarrow B$  and  $A'$  a covering space of  $B'$  relative to  $p' : A' \rightarrow B'$  and let  $f : B \rightarrow B'$  be a continuous map.

Let  $a_0 \in A$ ,  $b_0 \in B$ ,  $a'_0 \in A'$ ,  $b'_0 \in B'$  be given such that  $p(a_0) = b_0$ ,  $p'(a'_0) = b'_0$  and  $f(b_0) = b'_0$ , then  $p$ ,  $p'$  and  $f$  induce homomorphisms  $p_*$ ,  $p'_*$ ,  $f_*$  on the fundamental groups,

$$\text{i.e.} \quad \begin{array}{ccc} \pi(A, a_0) & & \pi(A', a'_0) \\ p_* \downarrow & & \downarrow p'_* \\ \pi(B, b_0) & \xrightarrow{f_*} & \pi(B', b'_0) \end{array}$$

THEOREM 2.2 - (Fiber map theorem). There exist a unique continuous map  $g : A \rightarrow A'$  such that  $g(a_0) = a'_0$  and  $p'_*g = f_*p_*$  iff  $f_*$  carries the image of  $p_*$  into that of  $p'_*$ .

Let  $A$  and  $A'$  be two covering spaces over the same base space  $B$  with respect to  $p$  and  $p'$ . If  $a_o \in A$  and  $a'_o \in A'$  such that

$$p(a_o) = b_o = p'(a'_o)$$

$$\text{and } p(\pi(A, a_o)) \subset p'(\pi(A', a'_o))$$

then the fiber map theorem gives a unique map  $g : A \rightarrow A'$  such that

$$g(a_o) = a'_o \text{ and } p' \circ g = p.$$

Hu shows furthermore that  $g$  is a covering map of  $A$  over  $A'$ .

For example if  $A$  is simply connected then  $\pi(A) = 1$  and so for any covering space  $A'$  of  $B$  we have a unique covering map  $g$  of  $A$  over  $A'$  such that  $p' \circ g = p$ ; a covering space with this property is called universal.

In particular for a topological manifold  $M$  one shows ([3], p. 93-95) that  $M$  has always a universal covering space (and up to homeomorphism only one) which is simply connected.

We may then adopt the following definition.

Definition 2.5 - The universal covering space  $U$  of the topological manifold  $M$  is the unique (up to homeomorphism) simply connected covering space of  $M$ .

It is clear that analogous results to the fiber map theorem, and definition 2.5 hold in the case of differentiable covering manifolds or analytic covering manifolds; so that reference will be made to these results accordingly.

Finally the next result will prove to be useful ( [4] , p.178).

THEOREM 2.3 - Let  $M$  and  $M^*$  be differentiable manifolds of the same dimension and let  $p : M^* \rightarrow M$  be an immersion. If  $M^*$  is compact so is  $M$  and  $p$  is a covering map.

SECTION 3ANALYTIC IMMERSIONS OF SIMPLY CONNECTED RIEMANN SURFACES

By theorem 1.1 it is clear that in our study of analytic immersions of Riemann surfaces we can replace each surface by the most convenient conformally equivalent surface. Hence we use :

RIEMANN MAPPING THEOREM - Every simply connected Riemann surface is conformally equivalent to exactly one of the following three surfaces :

- (1) the unit sphere  $S$
- (2) the Euclidean plane  $E$
- (3) the unit disk  $H = \{ z : |z| < 1 \}$

(for a proof see [6] p.225)

To study analytic immersions of simply connected Riemann surfaces, by the above theorem, it suffices to investigate all the combinations of immersions between  $S$ ,  $E$  and  $H$  into each other.

First of all let us consider the immersions that are possible of  $S$ ,  $E$  and  $H$  into  $S$ . To this end we use the following result.

THEOREM 3.1 - Let  $R$  be an arbitrary Riemann surface and  $f : R \rightarrow S$  an analytic immersion. Then at every point  $p$  in  $R$  such that  $f(p) = \infty$  in a local representation on  $S$ ,  $f$  is represented by an analytic function with a pole of first order.

PROOF:  $S$  may be represented by the extended  $W$  - plane; with the finite part of the  $W$  - plane and the part  $|W| > 0$  described by the complex variable  $w = 1/W$ ,  $S$  is covered by two charts with the neighbourhood relation  $w = 1/W$ , so that  $w = 0$  corresponds to the point at infinity of the closed  $W$  - plane.

Now let  $f : R \rightarrow S$  be an analytic immersion and  $p \in R$  such that  $f(p) = \infty$ . By definition  $f$  is locally  $1 : 1$  at  $p$ , making  $p$  an isolated singularity; that is there is a neighbourhood  $N(p)$  of  $p$  in  $R$  such that at all  $q \in N(p)$ ,  $q \neq p$ ,  $f(q)$  is regular.

Choosing a chart  $z$  in  $N(p)$  we may represent  $f$  by an analytic function  $W = F(z)$  such that  $F(0) = \infty$ . Hence  $z = 0$  is a pole of  $F(z)$ . Let  $k$  be its multiplicity so that the Laurent Expansion of  $F(z)$  gives us a function  $G(z)$  holomorphic in some neighbourhood of  $z = 0$  and

$$W = F(z) = \frac{G(z)}{z^k}, \quad G(0) \neq 0$$

Hence  $w = 1/W = \frac{z^k}{G(z)}$  becomes the local representation of  $f$  in the charts

$z, w$  and as above  $f$  must be locally 1 : 1 in this representation.

Therefore

$$\left. \frac{dw}{dz} \right|_{z=0} \neq 0$$

But this implies that  $k = 1$

Q.E.D.

Now we apply this theorem by letting  $R = S, E, H$  in three separate cases:

Case (a)  $f : R = S \longrightarrow S$  ;  $R$  being equal to  $S$  may be represented

by a closed  $z$  - plane and  $f$  represented in local coordinates,

$W = F(z)$ , will be analytic except for poles in the closed

$z$  - plane. Hence  $F(z)$  is meromorphic in the extended plane

and consequently is a rational function, i.e. a quotient of

two polynomials ( [1] p. 217).

Since  $S$  is compact and  $f$  is an immersion then  $f$  is a covering

map by theorem 2.3. But  $S$  is also simply connected. Hence  $S$

becomes the universal covering surface of  $S$ . And since the

universal covering is unique up to homeomorphism  $f$  must be a

homeomorphism and hence globally 1 : 1.

But the only transformations which are 1 : 1 and rational are the linear fractionals,

$$F(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

Case (b)  $f : R = E \longrightarrow S$  : Here  $f$  is represented by a function  $F(z)$  where  $E$  is taken to be the finite  $z$  - plane. Then  $F(z)$  having at most poles of order 1 is a meromorphic function with poles at most of first order :

$$\text{e.g. } F(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad e^z, \quad \Gamma(z).$$

Case (c)  $f : R = H \longrightarrow S$  : Then  $f$  has the same properties as in case (b)

$$\text{e.g. } F(z) = e^{1/(z-1)}, \quad \log \frac{1}{1+z}, \quad \text{with a}$$

suitable determination of  $\log$ .

Secondly, let us consider the analytic immersions of  $S$ ,  $E$  and  $H$  into  $E$

Case (a)  $f : S \longrightarrow E$  : Then the local representation  $F(z)$  of  $f$  must be analytic in the extended  $z$  - plane which by

Liouville's theorem is then a constant. But this is

impossible for then  $F'(z) = 0$  for all  $z$ . Hence there are

no immersions of  $S$  into  $E$ .

Case (b)  $f : E_1 \longrightarrow E_2$  : Here any function  $f$  which is analytic and locally 1 : 1 in the finite plane will do.

$$\text{e.g. } f(z) = e^z ,$$

Case (c)  $f : H \longrightarrow E$  : Any function which is analytic and locally 1 : 1 in  $H$  will do.

$$\text{e.g. } f(z) = e^z , e^{1/(z-1)} , \log(z-2)$$

with a suitable determination for  $\log, \frac{az+b}{cz+d}$

where  $ad - cb \neq 0$  and if  $c \neq 0$ ,  $|\frac{d}{c}| > 1$

Finally consider all possible analytic immersions of  $S$ ,  $E$  and  $H$  into  $H$

Case (a)  $f : S \longrightarrow H$  : Again we may represent  $S$  by the extended plane;  $f$  then must be analytic in the extended plane. Thus by Liouville's theorem  $f$  must be a constant. Therefore there are no analytic immersions.

Case (b)  $f : E \longrightarrow H$  :  $f$  clearly must be entire and bounded since  $f(E) \subset H$  and hence reduces to a constant by Liouville's theorem. Then there are no analytic immersions.

Case (c)  $f : H \rightarrow H$  : Any function  $f$  which is analytic in  $H$  and

$f'(z) \neq 0$  as well as  $|f(z)| < 1$  for all  $z \in H$ .

e.g.  $e^{(iz - 1)}$ ,  $1/(2-z)$ .

SECTION 4ANALYTIC IMMERSIONS OF PARABOLIC RIEMANN SURFACES

Since the universal covering surface  $U$  of a Riemann surface  $R$  is simply connected, then by the Riemann mapping theorem  $U$  is conformally equivalent to either the sphere  $S$ , the plane  $E$  or the unit disk  $H$ . We now proceed to study the case where  $U$  is conformally equivalent to  $E$ .

Definition 4.1 - A Riemann surface is called parabolic if its universal covering surface  $U$  is conformally equivalent to the Euclidean plane  $E$ .

Since  $U$  is conformally equivalent to  $E$ , for parabolic Riemann surfaces we may even take  $E$  as the universal covering surface. To simplify the study program we use the result ([6] P. 229-230) that every parabolic Riemann surface is conformally equivalent to exactly one of the following types of surfaces :

- 1 -  $E$  itself
- 2 - All cylinders  $C$ , where the "Cylinder" is defined to be any homeomorphic image of a special right

circular cylinder. In fact, it is a theorem that up to conformal equivalence there is only one type of cylinder of parabolic type; two representations are the following :

- (a) The cylinder is homeomorphic to the finite punctured

$$\text{plane } \mathring{E} = \{z : 0 < |z| < \infty\}$$

- (b) Topologically the cylinder may be regarded as an

infinite strip in the plane with the two sides

identified. This gives rise to the following

construction : Let  $\Omega$  be a complex number. Let  $G$

be the group of all maps of the form  $T_n(z) = z + n\Omega$

of the  $z$  - plane  $E$  where  $n$  is an integer. Now take

the orbit space  $R$  defined by  $G$ ; that is  $R = E/G$  where

two points  $z_1$  and  $z_2$  are equivalent if there is an

integer  $n$  such that  $T_n(z_1) = z_2$ .  $E$  is thus divided

up into equivalence classes consisting of strips of

width  $|\Omega|$  having a boundary line on one side only.

$R$  is obtained by identifying equivalent points of each

strip and hence  $R$  is homeomorphic to such a strip and

hence to the cylinder.

3 - All torii  $T$ , i.e. compact surfaces of genus one. The torus being topologically a parallelogram with opposite sides identified can then be characterized similarly to the cylinder as follows :

Let  $\Omega$  and  $\Omega'$  be 2 complex numbers such that  $\Im m\left(\frac{\Omega'}{\Omega}\right) \neq 0$  and define  $G$  as the group of all maps of the form

$$T_{m,m'}(z) = z + m\Omega + m'\Omega'$$

of the  $z$  - plane  $E$  where  $m$  and  $m'$  are integers. Then

take the orbit space  $R$  defined by  $G$ , i.e.  $T = E/G$

where two points  $z_1$  and  $z_2$  are equivalent if there

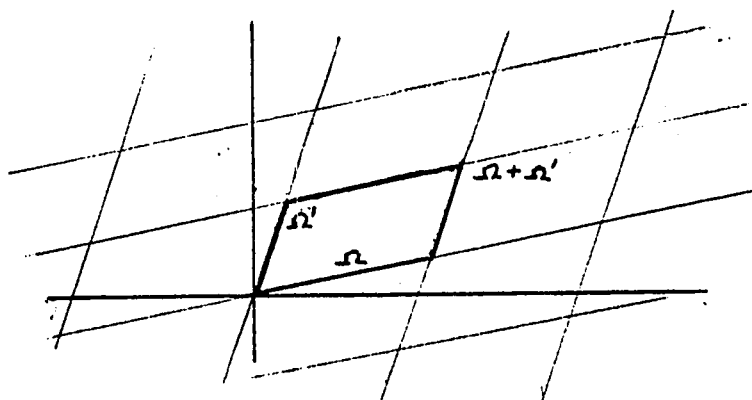
exist two integers  $m$  and  $m'$  such that  $T_{m,m'}(z_1) = z_2$ .

$E$  is thus divided into equivalence classes of paral-

lelograms of which the principal one would be the

interior of the parallelogram with vertices  $0, \Omega, \Omega'$  and  $\Omega + \Omega'$ .

as well as the two edges containing the origin  $0$ .



R is obtained by identifying equivalent points of each parallelogram. Thus R is homeomorphic to such a parallelogram and thus to a torus.

This also shows that for any other pair of complex numbers  $\Omega_1, \Omega_1'$ ,  $R_1$  obtained similarly is homeomorphic to R.

However it can be shown that R and  $R_1$  are conformally equivalent iff there exist four integers a, b, c, d, such that

$$\frac{\Omega}{\Omega'} = \frac{a \frac{\Omega}{\Omega_1} + b}{c \frac{\Omega}{\Omega_1} + d}, \quad ad - bc = 1$$

([5] p. 251). Hence there exist infinitely many different torii from the conformal point of view.

We can now reduce the study of all possible analytic immersions of parabolic Riemann surfaces, by theorem 1.1, to those of T, C and E. Since E is the universal covering surface for T, C and E it would then be convenient if we could lift the immersions f to be studied to immersions of E into E by the fiber map theorem.

Two questions arise; is the fiber map applicable, and if so are the "lifts" also immersions ?

First of all, it can be seen that the fiber map theorem is in fact applicable since the universal covering surface  $E$  is simply connected and so  $\pi_1(E) = 1$ . So that  $p_*$ ,  $p'_*$  and  $f_*$  being homomorphisms we have  $f_* (p_* (\pi_1(E))) = 1 = p'_* (\pi_1(E))$ , i.e.  $f_* (\text{Im } p_*) = \text{Im } p'_*$ .

Secondly, since, by theorem 2.1, every covering map is an immersion, then  $p$  and  $p'$  are immersions. So that  $f$  is an analytic immersion iff the lift  $g$  is an analytic immersion.

First we consider the three cases of analytic immersions of  $E$ ,  $C$ ,  $T$  into  $E$  :

Case (a)  $f : E_1 \rightarrow E_2$  has already been discussed.

Case (b)  $f : C \rightarrow E$ ; we may take  $\mathring{E}$  for  $C$ . Then we require all functions  $f$  such that (1)  $f$  is analytic for all  $z \neq 0$

(2)  $f'(z) \neq 0$  for all  $z \neq 0$

e.g.  $f(z) = 1/z, z^2, e^{1/z^3}$ .

Case (c)  $f : T \rightarrow E$ . Since  $T$  is compact and  $E$  is not compact then by theorem 2.3 analytic immersions are impossible.

Next we consider the three cases of possible analytic immersions of  $E$ ,  $C$ ,  $T$  into  $C$  :

Case (a)  $f : E \longrightarrow C$  ; take  $\mathring{E}$  for  $C$ . Hence we are looking for

functions  $f$  such that (1)  $f$  is entire

(2)  $f(z) \neq 0$  for all  $z \in E$

(3)  $f'(z) \neq 0$  for all  $z \in E$

e.g.  $f(z) = e^z$

Case (b)  $f : C_1 \longrightarrow C_2$  ; we represent  $C_i$  by  $\mathring{E}_i$ ,  $i = 1, 2$ , and thus

$f : \mathring{E}_1 \longrightarrow \mathring{E}_2$  : Hence we are looking for functions  $f$  such

that (1)  $f$  is analytic for all  $z \neq 0$

(2)  $f(z) \neq 0$  for all  $z \neq 0$

(3)  $f'(z) \neq 0$  for all  $z \neq 0$

e.g.  $f(z) = z^n$ ,  $n$  an integer  $\neq 0$ ,  $e^z$ ,  $1/e^{z^3}$ .

Case (c)  $f : T \longrightarrow C$  ; again since  $T$  is compact and  $C$  is not

compact by theorem 2.3 no analytic immersions are possible.

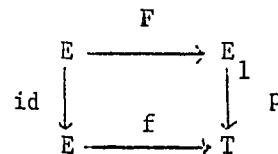
Finally we consider all possible analytic immersions of

$E$ ,  $C$ ,  $T$  into  $T$  :

Case (a)  $f : E \longrightarrow T$  ; let  $T = E_1/G$  where  $G$  is generated by two

complex numbers  $\alpha, \alpha'$ . Let  $p$  be

the projection  $p : E_1 \longrightarrow E_1/G$ .



Then by the fiber map theorem there

is a unique map  $F : E \longrightarrow E_1$  such that

$f = p \circ F$ . Hence we are looking for entire

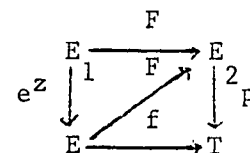
functions  $F$  such that  $F'(z) \neq 0$  for all  $z \in E$ .

e.g.  $F(z) = e^z, Cz + D$  ,  $C, D$  complex numbers  $C \neq 0$

Case (b)  $f : C \longrightarrow T$ ; let  $C$  be represented by  $\check{E}$  and  $T$  by  $E_2/G$ .

Let  $e^z$  project  $E$ , onto  $\check{E}$  and  $p$  project  $E_2$  onto  $T$ .

Hence we want all immersions  $F$  of  $E_1$  into  $E_2$



which is the lift of an immersion  $f$  of  $\check{E}$  into  $T$

such that  $f(e^z) = p(F(z))$ , Immersions of  $\check{E}$  into  $T$

do exist, for let  $F_1$  be an immersion of  $\check{E}$  into  $E_2$

then  $p \circ F_1$  is an immersion of  $\check{E}$  into  $T$ . e.g.  $F(z) = e^z,$

$z^n$ ,  $n$  a non zero integer.

Case (c)  $f : T_1 \longrightarrow T_2$ ; the results of this are a little more

involved and can be formalized in the following theorems :

THEOREM 4.1 - Let  $T_1$  and  $T_2$  be two torii such that  $T_1 = E_1/G_1$ ,

$T_2 = E_2/G_2$ ,  $G_1$  generated by  $(\Omega, \Omega')$  and  $G_2$  generated by  $(\Omega_2, \Omega_2')$ .

Then there exists an analytic immersion of  $T_1$  into  $T_2$  iff there exist

four integers  $a, b, c, d$  such that

$$\frac{\Omega_2}{\Omega_2'} = \frac{a \frac{\Omega}{\Omega'} + b}{c \frac{\Omega}{\Omega'} + d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

and  $\text{g.c.d.}(a, b, c, d) = 1$ .

PROOF : (a) Necessity:

Assume there exists an analytic immersion  $f : T_1 \rightarrow T_2$ .

Hence by the fiber map theorem there is

a unique immersion  $F : E_1 \rightarrow E_2$  such that

$$f \circ p_1 = p_2 \circ F$$

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ T_1 & \xrightarrow{f} & T_2 \end{array}$$

$p_1$  and  $p_2$  being the respective projections

of  $E_1$  and  $E_2$  onto  $T_1$  and  $T_2$ .

Now  $z \in E_1$  and  $z + m\Omega + m'\Omega' \in E_1$ , for some integers  $m$

and  $m'$ , are taken onto the same point of  $T_1$  by  $f \circ p_1$ .

Hence  $p_2(F(z + m\Omega + m'\Omega')) = p_2(F(z))$

and so there are two integers  $n$  and  $n'$  such that

$$F(z + m\Omega + m'\Omega') = F(z) + n\Omega + n'\Omega'$$

and  $n$  and  $n'$  depend on  $m$  and  $m'$ .

Therefore

$$(*) \quad F(z + m\Omega + m'\Omega') = F(z) + n(m, m')\Omega + n'(m, m')\Omega'$$

for all  $z \in E_1$ . Differentiating both sides we have

$$F'(z + m\Omega + m'\Omega') = F'(z).$$

$F'$  is then elliptic and since it is entire it reduces to a

non zero constant, ([2] p. 125), i.e.  $F'(z) = \beta \neq 0$ ,

$\beta$  complex no.

Then  $F(z) = \beta z + \gamma$ ,  $\gamma$  complex no.,

and (\*) becomes

$$\beta z + \beta(m\Omega + m'\Omega') + \gamma = \beta z + \gamma + n(m, m')\Omega + n'(m, m')\Omega'$$

$$\beta(m\Omega + m'\Omega') = n(m, m')\Omega + n'(m, m')\Omega'$$

Let  $m = 1$  and  $m' = 0$ ; then

$$\beta\Omega = n(1, 0)\Omega + n'(1, 0)\Omega'$$

Let  $m = 0$  and  $m' = 1$ ; then

$$\beta\Omega' = n(0, 1)\Omega + n'(0, 1)\Omega'$$

And now letting  $n(1,0) = a$ ,  $n'(1,0) = b$ ,  $n(0,1) = c$  and

$n'(0,1) = d$  we have

$$\frac{\Omega}{\Omega'} = \frac{\beta\Omega}{\beta\Omega'} = \frac{a\Omega_1 + b\Omega'_1}{c\Omega_1 + d\Omega'_1} = \frac{\frac{a}{\beta}\frac{\Omega_1}{\beta} + \frac{b}{\beta}}{\frac{c}{\beta}\frac{\Omega_1}{\beta} + \frac{d}{\beta}}$$

and  $ad - bc \neq 0$  since  $\det\left(\frac{\Omega}{\Omega'}\right) \neq 0$ . If  $\text{g.c.d.}(a,b,c,d) \neq 1$ ,

divide top and bottom of R.H.S. of the above equation by the

$\text{g.c.d.}(a,b,c,d)$ .

(b) Sufficiency :

Now assume there are four integers  $a, b, c, d$  such that

$$\frac{\Omega}{\Omega'} = \frac{\frac{a}{\beta}\frac{\Omega_1}{\beta} + \frac{b}{\beta}}{\frac{c}{\beta}\frac{\Omega_1}{\beta} + \frac{d}{\beta}} = \frac{a\Omega_1 + b\Omega'_1}{c\Omega_1 + d\Omega'_1}$$

with  $ad - bc \neq 0$  and  $\text{g.c.d.}(a,b,c,d) = 1$ . Then there

exists a unique complex number  $C$  such that

$$C\Omega = a\Omega_1 + b\Omega'_1$$

$$C\Omega' = c\Omega_1 + d\Omega'_1$$

Define  $F : E_1 \rightarrow E_2$  by  $F(z) = Cz$ . Let  $m$  and  $m'$  be

any integers; then

$$\begin{aligned}
 F(z + m\Omega + m'\Omega') &= Cz + mC\Omega + m'C\Omega' \\
 &= Cz + m(a\Omega + b\Omega') + m'(c\Omega + d\Omega') \\
 &= Cz + (ma + m'c)\Omega + (mb + m'd)\Omega'
 \end{aligned}$$

Now  $n(m, m') = ma + m'c$  and  $n'(m, m') = mb + m'd$  are

integral functions of  $m$  and  $m'$  and we have

$$F(z + m\Omega + m'\Omega') = F(z) + n(m, m')\Omega + n'(m, m')\Omega'$$

But this means that there is an  $f$  such that  $f : T_1 \rightarrow T_2$  and

$$f \circ p_1 = p_2 \circ F$$

Now  $F$ ,  $p_1$  and  $p_2$  are locally 1-1 and analytic and hence

so is  $f$ . Thus  $f$  is an analytic immersion of  $T_1$  into  $T_2$

Q.E.D.

Notice that there exist immersions between two torii without the torii being conformally equivalent. If  $a, b, c, d$  are four integers such that  $ad - bc \neq 1, 0$  and  $\Omega$  and  $\Omega'$  are complex numbers generating a torus  $T_2$  then

$$q = \frac{a \frac{\Omega}{\Omega'} + b}{c \frac{\Omega}{\Omega'} + d}$$

is a complex number such that  $\text{Im}(q) \neq 0$ . The complex numbers  $\Omega'$

and  $\Omega = \Omega' q$  generate a torus  $T_1$  since  $\Im m\left(\frac{\Omega}{\Omega'}\right) = \Im m(q) \neq 0$ . Thus

$T_1$  and  $T_2$  are not conformally equivalent but by the above theorem there

exist immersions  $T_1 \longrightarrow T_2$ .

Let  $T_1$  and  $T_2$  be two tori given by  $(\Omega, \Omega')$  and  $(\Omega_1, \Omega'_1)$

respectively and let there be four integers  $a, b, c, d$  such that

$$\frac{\Omega}{\Omega'} = \frac{a \frac{\Omega_1}{\Omega'_1} + b}{c \frac{\Omega_1}{\Omega'_1} + d}, \quad ad - bc \neq 0$$

and  $\gcd(a, b, c, d) = 1$

We want now to find all analytic immersions  $f : T_1 \longrightarrow T_2$ .

In part (a) of the preceding proof we saw that the lift  $F : E_1 \longrightarrow E_2$

of an immersion  $f$  necessarily had the form  $F(z) = Cz + D, C \neq 0$ .

Hence we are looking for all complex numbers  $C$  and  $D$  which makes  $F(z)$

the lift of an immersion.

We found also that to each pair  $(m, m')$  of integers there exists another pair  $(n(m, m'), n'(m, m'))$  such that

$$F(z + m\Omega + m'\Omega') = F(z) + n(m, m')\Omega + n'(m, m')\Omega'$$

$$(1) \quad \text{and that } \begin{cases} C\Omega = n(1, 0)\Omega + n'(1, 0)\Omega' \\ C\Omega' = n(0, 1)\Omega + n'(0, 1)\Omega' \end{cases}$$

$$(2) \text{ then } \frac{\Omega}{\Omega'} = \frac{n(1,0)\frac{\Omega}{\Omega'} + n'(1,0)}{n(0,1)\frac{\Omega}{\Omega'} + n'(0,1)} = \frac{a\frac{\Omega}{\Omega'} + b}{c\frac{\Omega}{\Omega'} + d}$$

By (1) then to find C reduces to finding  $n(1,0)$ ,  $n'(1,0)$ ,  $n(0,1)$ ,  $n'(0,1)$ .

D of course is arbitrary since translations are analytic immersions.

$$\text{Let } \frac{\Omega}{\Omega'} = q_1 \text{ and}$$

$$L_1 = \begin{pmatrix} n(1,0) & n'(1,0) \\ n(0,1) & n'(0,1) \end{pmatrix}, \quad L_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write (2) in terms of  $L_1$  and  $L_2$ , i.e.

$$L_1(q_1) = L_2(q_1)$$

and  $T(q_1) = L_2^{-1} \circ L_1(q_1) = q_1$  so that  $q_1$  is a fixed point  $T = L_2^{-1} \circ L_1$

We are then looking for  $L$  having been given  $L_2$  and  $q_1$ . Let

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{i.e. } T(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

with  $\alpha, \beta, \gamma, \delta$  integers such that  $\alpha\delta - \beta\gamma \neq 0$  and g.c.d.

$(\alpha, \beta, \gamma, \delta) = 1$ . If  $q_1$  is a fixed point then

$$q_1 = \frac{\alpha q_1 + \beta}{\gamma q_1 + \delta}$$

$$(3) \quad \text{or} \quad \gamma q_1^2 + (\delta - \alpha) q_1 - \beta = 0$$

We distinguish now two cases

Case (1) - If  $\gamma = 0$  then  $\delta = \alpha \neq 0$  and  $\beta = 0$  for otherwise  $q_1 = \frac{\beta}{\delta - \alpha}$

which contradicts the fact that  $\Im m\left(\frac{\Omega_1}{\Omega'_1}\right) \neq 0$ . Hence

$$L_1 = L_2 \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} a\alpha & b\alpha \\ c\alpha & d\alpha \end{pmatrix}$$

$$\text{and} \quad C(\alpha) = \frac{n(1,0)\Omega_1 + n'(1,0)\Omega'_1}{\Omega} = \frac{\alpha(a\Omega_1 + b\Omega'_1)}{\Omega}$$

No further restrictions on  $\alpha$  are necessary as shown by a

similar argument as that in the sufficiency part of theorem 4.1

Hence letting  $A = \frac{a\Omega_1 + b\Omega'_1}{\Omega}$ , then we have a two

parameter family of immersions

$$F(z) = (\alpha A) z + D$$

Case (2) - If  $\gamma \neq 0$  then  $\bar{q}_1$  and  $q_1$  are two roots of (3). Hence

$$q_1 + \bar{q}_1 = 2 \Re(q_1) = \frac{(\alpha - \delta)}{\gamma}$$

$$q_1 \cdot \bar{q}_1 = q_1^2 = -\frac{\beta}{\gamma}$$

Hence  $\gamma \neq 0 \Rightarrow \Re(q_1)$  and  $|q_1|^2$  are rational numbers. Note that

$\alpha\delta - \beta\gamma \neq 0$  is a necessary condition for the roots of

$$(4) \quad \gamma x^2 + (\delta - \alpha)x - \beta = 0$$

to be not real, for if  $\alpha\delta - \beta\gamma = 0$  then the discriminant

$$\begin{aligned} (\delta - \alpha)^2 - 4\gamma(-\beta) &= \delta^2 - 2\delta\alpha + \alpha^2 + 4\gamma\beta \\ &= \delta^2 + 2\delta\alpha + \alpha^2 - 4\delta\alpha + 4\gamma\beta \\ &= (\delta + \alpha)^2 - 4(\alpha\delta - \beta\gamma) \\ &= (\delta + \alpha)^2 > 0 \end{aligned}$$

which is impossible if  $q_1$  is to be not real.

Our task once again is to find the set of all  $\alpha, \beta, \delta, \gamma$ 's

which makes  $q_1$  and  $\bar{q}_1$  the two roots of equation (4).

Since this case of  $\gamma \neq 0$  implies  $\Re(q_1)$  and  $|q_1|^2$

being rational then let

$$2\Re(q_1) = p/q, \quad q \text{ and } p \text{ integers such that } q > 0$$

$$\text{and g.c.d. } (p, q) = 1$$

$$|q_1|^2 = r/s, \quad r \text{ and } s \text{ integers such that } r > 0$$

$$s > 0 \text{ and g.c.d. } (r, s) = 1$$

But  $(x - q_1)(x - \bar{q}_1) = 0$

$$x^2 - 2 \Re(q_1) x + |q_1|^2 = 0$$

ans so  $x^2 - \frac{p}{q} x + \frac{r}{s} = 0$

or  $qs x^2 - ps x + rq = 0$

Now to ensure that the coefficients are in the lowest terms possible divide through by  $g = \text{g.c.d.}(q, s)$ . Hence

$$(5) \quad q's x^2 - ps' x + rq' = 0$$

where  $q' = q/g$  and  $s' = s/g$ .

Therefore (4) must be a non zero integral multiple  $\nu$  of (5), i.e.

$$(6) \quad \nu q' s x^2 - \nu ps' x + \nu rq' = 0$$

Comparing (4) and (6) then we obtain

$$\gamma = \nu q's$$

$$\delta - \alpha = -\nu ps'$$

$$\beta = -\nu rq'$$

or equivalently

$$\alpha = \delta + \nu s'p$$

$$\beta = -\nu q'r$$

$$\gamma = \nu q's$$

$$\delta = \delta$$

so that besides  $\nu \neq 0$  the integers  $(\delta, \nu)$  are arbitrary. It can be checked independently that for  $\nu = 0$  we also get immersions. This gives us the set of all  $\alpha, \beta, \gamma, \delta$ . We have then  $L_1$ , i.e.

$$L_1 = L_2 \circ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta + \nu s'p & -\nu q'r \\ \nu q's & \delta \end{pmatrix}$$

$$\text{and } n(1,0) = a(\delta + \nu s'p) + b\nu q's$$

$$n'(1,0) = -a\nu q'r + b\delta$$

$$n(0,1) = c(\delta + \nu s'p) + d\nu q's$$

$$n'(0,1) = -c\nu q'r + d\delta$$

Using the first relation of (1) we can find C,

$$\begin{aligned} C &= \frac{n(1,0)\Omega_1 + n'(1,0)\Omega'_1}{\Omega} \\ &= \frac{1}{\Omega} \left\{ a(\delta + \nu s'p)\Omega_1 + b\nu q's\Omega_1 - (a\nu q'r)\Omega'_1 + b\delta\Omega'_1 \right\} \end{aligned}$$

$$\begin{aligned}
&= \delta \frac{(a \Omega_1 + b \Omega'_1)}{\Omega} + \nu \frac{(as'p \Omega_1 + bq's \Omega_1 - aq'r \Omega'_1)}{\Omega} \\
&= \delta A + \nu B
\end{aligned}$$

where A is as in case (1) and

$$B = \frac{as'p \Omega_1 + bq's \Omega_1 - aq'r \Omega'_1}{\Omega}$$

No further restrictions on  $\delta$  and  $\nu$  are necessary as shown by a similar argument to the one in the sufficiency part of theorem 4.1. To show that C can't be reduced to one parameter we must show that B is not a real multiple of A. Consider

$$\frac{B}{A} = \frac{(as'p + bq's)\Omega_1 - aq'r \Omega'_1}{a \Omega_1 + b \Omega'_1}$$

Hence we must check that the determinant

$$\begin{vmatrix} as'p + bq's & -aq'r \\ a & b \end{vmatrix} \neq 0$$

$$\begin{aligned}
\text{i e. } (as'p + bq's) b + a^2 q'r &= \frac{qs}{g} (abp + b^2 + a^2 \frac{r}{s}) \\
&= \frac{qs}{g} (a^2 |q_1|^2 + 2ab \Re(q_1) + b^2) \\
&= \frac{qs}{g} |a q_1 + b|^2 \neq 0
\end{aligned}$$

Hence the values  $C(\delta, \nu) = \delta A + \nu B$  form a parallelogram

lattice spanned by A and B and so

$$(7) \quad F(z) = (\delta A + \nu B) z + D$$

is a three parameter family of immersions. Notice that should we choose  $\nu = 0$  then  $\gamma = 0$ , i.e. the first case and (7) reduces to the solution of case 1. Hence both cases may be represented by (7).

Before formalizing these results as a theorem we make the following definition :

Let P be a parallelogram lattice generated by the complex numbers  $(\Omega, \Omega')$ . Then

$$\Omega_1 = \xi \Omega + \eta \Omega'$$

$$\Omega'_1 = \lambda \Omega + \zeta \Omega'$$

with  $\xi\zeta - \eta\lambda = \pm 1$ , where  $\xi, \eta, \lambda, \zeta$  are integers, are also generators of P, since with  $\xi\zeta - \eta\lambda = \pm 1$  we can find four integers a, b, c, d such that

$$(8) \quad \begin{cases} \Omega = a \Omega_1 + b \Omega'_1 \\ \Omega' = c \Omega_1 + d \Omega'_1 \end{cases}$$

so that any point of P described by  $(\Omega, \Omega')$  can also be described by  $(\Omega_1, \Omega'_1)$ .

Furthermore if  $\Re\left(\frac{\Omega}{\Omega'}\right)$  and  $\left|\frac{\Omega}{\Omega'}\right|^2$  are rational then  $\Re\left(\frac{\Omega'}{\Omega}\right)$  and  $\left|\frac{\Omega'}{\Omega}\right|^2$  are also rational as can easily be shown by elementary manipulations of (8).

Definition 4.2 - A parallelogram lattice is said to be ample if any two generators  $(\Omega, \Omega')$  satisfy the condition :

$$\Re\left(\frac{\Omega}{\Omega'}\right) \quad \text{and} \quad \left|\frac{\Omega}{\Omega'}\right|^2 \quad \text{are both rational}$$

The remark preceding the definition shows independence of choice of generators here.

Another characterization of an ample lattice is described in the following theorem.

THEOREM 4.2 - A parallelogram lattice is ample iff there exists a positive real number  $\phi$  such that the squares of the distances of all lattice points from a fixed lattice point are rational multiples of  $\phi$ .

PROOF: (a) Let  $(\Omega, \Omega')$  be generators of an ample lattice.

Then

$$\begin{aligned} |m\Omega + m'\Omega'|^2 &= (m\Omega + m'\Omega') (m\bar{\Omega} + m'\bar{\Omega}') \\ &= |\Omega|^2 \left\{ m^2 \frac{r}{s} + mm' \frac{p}{q} + m'^2 \right\} \end{aligned}$$

Where  $p, q, r, s$  are defined as before. Hence take the first factor as  $\phi$  and the second factor is the rational number required.

(b) Conversely if  $|m\Omega + m'\Omega'|^2 = \phi R(m, m')$  where  $\phi$  is a positive real number independent of  $m, m'$ , and  $R(m, m')$  a rational number depending on  $m, m'$ , then  $|\Omega|^2 = \phi R(1, 0)$  and  $|\Omega'|^2 = \phi R(0, 1)$  and so

$$\left| \frac{\Omega}{\Omega'} \right|^2 = \frac{R(1, 0)}{R(0, 1)}$$

which is rational. Also

$$\left| \frac{\Omega}{\Omega'} \right|^2 + 2\Re\left(\frac{\Omega}{\Omega'}\right) + 1 = \left| \frac{\Omega}{\Omega'} + 1 \right|^2 = \frac{|\Omega + \Omega'|^2}{|\Omega'|^2} = \frac{R(1, 1)}{R(0, 1)}$$

Hence  $\Re\left(\frac{\Omega}{\Omega'}\right)$  must be rational and consequently  $(\Omega, \Omega')$  forms an ample lattice.

Q.E.D.

Definition 4.3 - A torus  $T$  is said to be ample if any two of its generators  $(\Omega, \Omega')$  form an ample lattice.

Using these definitions and the preceding results and notation we have

THEOREM 4.3 - Let there exist an analytic immersion  $T \longrightarrow T_1$ . If  $T_1$  is not ample then there exists a two parameter family  $F(z) = (\alpha A) z + D$  of immersions. If  $T_1$  is ample then there is a three parameter family  $F(z) = (\delta A + \nu B) z + D$  of immersions.

We state now a couple of immediate results in the following theorem.

THEOREM 4.4 - If there exists an immersion of  $T \longrightarrow T'$  then there is an immersion of  $T' \longrightarrow T$ . Furthermore either both  $T$  and  $T'$  are ample or else neither are.

PROOF: We found that a necessary and sufficient condition for the existence of an immersion of  $T \longrightarrow T'$  was that there exist four integers  $a, b, c, d$  such that

$$(1) \quad q = \frac{a q_1 + b}{c q_1 + d}, \quad ad - bc \neq 0$$

where  $q = \frac{\Omega}{\Omega'}$ ,  $q_1 = \frac{\Omega_1}{\Omega'_1}$  and  $(\Omega, \Omega')$  generates  $T$

and  $(\Omega_1, \Omega'_1)$  generates  $T'$

then

$$\begin{aligned} & d \left( \frac{aq_1 + b}{cq_1 + d} \right) - b \\ \frac{dq - b}{-cq + a} &= \frac{-c \left( \frac{aq_1 + b}{cq_1 + d} \right) + a}{daq_1 + bd - (cq_1 + d)} \\ &= \frac{-caq_1 - cb + a(cq_1 + b)}{(da - bc)q_1} \\ &= \frac{(da - bc)q_1}{(ad - bc)} \\ &= q_1 \end{aligned}$$

(2)

which is a necessary and sufficient condition for the

existence of an immersion of  $T' \rightarrow T$

It is easy to verify from (1) that if  $q_1$  is such that

$\Re(q_1)$  and  $|q_1|^2$  are rational then it is so for  $q$ ; and

from (2) that if  $q$  satisfies this then so does  $q_1$ .

Q.E.D.

Conversely however, if we are given two ample torii  $T$  and  $T'$  then there do not necessarily exist immersions between  $T$  and  $T'$ : namely there need not exist an  $L_2$  such that

$$q = L_2(q_1)$$

For example let  $q = 1/2 + i\sqrt{3}/2$ ,  $q_1 = 1 + i$  which satisfy

all requirements, then for any integers  $a, b, c, d$  we have that

$\operatorname{Im}\left(\frac{aq_1 + b}{cq_1 + d}\right)$  is rational and  $\operatorname{Im}(q)$  is irrational so that

$$q = \frac{aq_1 + b}{cq_1 + d}$$

is never possible.

SECTION 5APPENDIXFURTHER RESULTS ON IMMERSIONS OF TORII INTO TORII

Let  $T$  and  $T_1$  be two torii defined by  $(\Omega, \Omega')$  and  $(\Omega_1, \Omega'_1)$  of complex numbers respectively. Let there exist four integers  $a, b, c$  and  $d$  such that

$$\frac{\Omega}{\Omega'} = \frac{a \frac{\Omega_1}{\Omega'_1} + b}{c \frac{\Omega_1}{\Omega'_1} + d}, \quad ad - bc \neq 0 \text{ and } (a, b, c, d) = 1$$

Let  $f$  be an immersion of  $T$  into  $T_1$ . Since  $T$  is compact then  $f$  is a covering map by theorem 2.3. Using the same notation as section four's study of immersions of torii into torii we show :

THEOREM 5.1 - If  $T_1$  is not ample then the number of sheets of  $f$  is  $|ad - bc| \alpha^2$ . If  $T_1$  is ample then the number of sheets of  $f$  is  $|ad - bc| \cdot |(\delta^2 + (ps') \delta \nu + (rsq'^2) \nu^2)|$

PROOF: We make a few reductions without loss of generality.

We use the same representation as before i.e.  $T = E/G$ ,

$T_1 = E_1/G_1$  where  $G$  is generated by  $(\Omega, \Omega')$  and  $G_1$  by  $(\Omega_1, \Omega'_1)$ .

Since equivalent points of each parallelogram of the net  $(\Omega, \Omega')$

are identified it suffices then to consider the lift

$$F(z) = Cz + D$$

on the parallelogram  $0, \Omega, \Omega + \Omega', \Omega'$  with sides  $\overline{0\Omega}$  and  $\overline{0\Omega'}$  only. Here  $C = \delta A + \nu B$  or  $\alpha A$  depending on whether or not  $T_1$  is ample and  $D$  is an arbitrary complex number.

The image of this parallelogram will lie in  $E_1$  touching certain parallelograms in the net generated by  $(\Omega, \Omega')$ .

Since equivalent points in all parallelograms, which are covered, are identified and since  $F(z) = Cz + D$  is 1 - 1 then these points identified as one in  $T_1$  will have in its inverse image as many distinct points as parallelograms in the net of  $(\Omega, \Omega')$  in which they were lying. Hence it is clear that since we are dealing with a covering map each set of identified points in the image must have the same number of points. Consequently the area of this image in  $E$  must be some positive integer  $N$  times the area of the parallelogram  $0, \Omega, \Omega + \Omega', \Omega'$ ;  $N$  is thus the number of sheets required.

To get at the precise image and final result we may simplify

$F$  by taking  $D = 0$  since it plays no role in the above mentioned

determination; for translations won't affect the area of the

image of  $F$ .

Describe the parallelogram  $0, \Omega, \Omega + \Omega', \Omega'$ , by

$$\{x\Omega + y\Omega' : 0 \leq x, y < 1\}$$

and observe its image under  $F(z) = Cz$

$$F(x\Omega + y\Omega') = C(x\Omega + y\Omega') = xC\Omega + yC\Omega'$$

But we have  $C\Omega = n(1,0)\Omega_1 + n'(1,0)\Omega'_1$

$$C\Omega' = n(0,1)\Omega_1 + n'(0,1)\Omega'_1$$

so that  $F(x\Omega + y\Omega') = x(n(1,0)\Omega_1 + n'(1,0)\Omega'_1)$

$$+ y(n(0,1)\Omega_1 + n'(0,1)\Omega'_1)$$

Hence the image is a parallelogram  $0, n(1,0)\Omega_1 + n'(1,0)\Omega'_1,$

$(n(1,0) + n(0,1))\Omega_1 + (n'(1,0) + n'(0,1))\Omega'_1, n(0,1)\Omega_1 +$

$n'(0,1)\Omega'_1$

If we consider now  $\Omega$  and  $\Omega'$  as vectors in the plane say  $\vec{\Omega}_1$

and  $\vec{\Omega}'_1$  then the area of the parallelogram  $0, \Omega, \Omega + \Omega',$

$\Omega'$  is  $|\vec{\Omega}_1 \times \vec{\Omega}'_1|$  and the area of the image parallelogram

is  $\left| [n(1,0)\vec{\Omega}_1 + n'(1,0)\vec{\Omega}'_1] \times [n(0,1)\vec{\Omega}_1 + n'(0,1)\vec{\Omega}'_1] \right|$

$$\begin{aligned}
&= |n(1,0) \quad n'(0,1) \quad (\vec{n}_1 \times \vec{n}'_1) + n(0,1) \quad n'(1,0) \quad (\vec{n}_1 \times \vec{n}'_1)| \\
&= |n(1,0) \quad n'(0,1) - n(0,1) \quad n'(1,0)| |\vec{n}_1 \times \vec{n}'_1| \\
&= |\det(L_1)| |\vec{n}_1 \times \vec{n}'_1|
\end{aligned}$$

Hence the number of sheets is then  $|\det(L_1)|$ . If  $T_1$  is not ample then

$$\begin{aligned}
|\det(L_1)| &= |\det(L_2 \cdot T)| = |\det L_2| \cdot |\det T| \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} \alpha & 0 \\ 0 & \alpha \end{vmatrix} \\
&= |ad - bc| \alpha^2
\end{aligned}$$

If  $T_1$  is ample then

$$\begin{aligned}
|\det L_1| &= |\det L_2| \cdot |\det T| \\
&= |ad - bc| \begin{vmatrix} \delta + \nu s'p & -\nu q'r \\ \nu q's & \delta \end{vmatrix} \\
&= |ad - bc| |\delta^2 + (ps')\delta\nu + (rsq'^2)\nu^2|
\end{aligned}$$

Q.E.D.

It is clear that in the case where  $T_1$  is not ample the set of all possible number of sheets depends only on  $|ad - bc|$ .

In the case where  $T_1$  is ample let  $N$  be the number of sheets

$$\begin{aligned} \text{i.e. } N &= |ad - bc| \left| \delta^2 + (ps') \delta \nu + (rsq'^2) \nu^2 \right| \\ &= |ad - bc| \left( \delta + sq' \nu q_1 \right) \left( \delta + sq' \nu \bar{q}_1 \right). \end{aligned}$$

Consider the negative of the discriminant of the quadratic form in the formula for  $N$

$$\begin{aligned} \Delta &= -p^2 s'^2 + 4rsq'^2 \\ &= -p^2 s'^2 + 4 \frac{r}{s} (s'g) q'^2 \\ &= 4 s'^2 \left( -\frac{p^2}{4} + \frac{r}{s} q'^2 \right) \\ &= 4 s'^2 q'^2 \left\{ \frac{r}{s} - \left( \frac{p}{2q} \right)^2 \right\} \\ &= 4 s'^2 q'^2 \left( |q_1|^2 - (\mathcal{R}(q_1))^2 \right) \\ &= 4 s'^2 q'^2 (\Im(q_1))^2 = (2 s' q' g \Im(q_1))^2 > 0 \end{aligned}$$

If  $ps'$  is even then we can write

$$\begin{aligned} N &= |ad - bc| \left| \delta^2 + (ps') \delta \nu + \left( \frac{1}{4} p^2 s'^2 \nu^2 - \frac{1}{4} p^2 s'^2 \nu^2 \right) + rsq'^2 \nu^2 \right| \\ &= |ad - bc| \left\{ \left( \delta + \frac{1}{2} ps' \nu \right)^2 + \frac{1}{4} \Delta \nu^2 \right\} \\ &= |ad - bc| \left\{ \delta_1^2 + \frac{1}{4} \Delta \nu^2 \right\} \end{aligned}$$

where  $\delta_1 = \delta + \frac{1}{2} ps' \nu$  and  $\frac{\Delta}{4}$  is a positive integer. If  $(\delta, \nu)$

assume all integral values then so do  $(\delta_1, \nu)$ . Hence the set of all

possible numbers of sheets depends only on  $\Delta$  and  $|ad - bc|$ .

If on the other hand  $ps'$  is odd we have

$$\begin{aligned}
 N &= |ad - bc| \left| \delta^2 + (ps' - 1)\delta\nu + \frac{(ps' - 1)^2}{4}\nu^2 + \delta\nu \right. \\
 &\quad \left. - \frac{(ps' - 1)^2}{4}\nu^2 + rsq'^2\nu^2 \right| \\
 &= |ad - bc| \left| \left( \delta + \frac{(ps' - 1)\nu}{2} \right)^2 + \delta\nu - \left( ps'^2 - \frac{ps' + 1}{2} \right)\nu^2 + rsq'^2\nu^2 \right| \\
 &= |ad - bc| \left| \left( \delta + \frac{(ps' - 1)\nu}{2} \right)^2 + \delta\nu - \frac{1}{4}(\Delta\nu^2 + 2ps'\nu^2 \right. \\
 &\quad \left. - 2\nu^2 + \nu^2) \right| \\
 &= |ad - bc| \left| \left( \delta + \frac{(ps' - 1)\nu}{2} \right)^2 + \delta\nu + \frac{(ps' - 1)\nu^2}{2} + \frac{1}{4}(\Delta + 1)\nu^2 \right| \\
 &= |ad - bc| \left| (\delta_2)^2 + \delta_2\nu + \frac{1}{4}(\Delta + 1)\nu^2 \right|
 \end{aligned}$$

where  $\delta_2 = \delta + \frac{(ps' - 1)\nu}{2}$  and  $\frac{1}{4}(\Delta + 1)$  are positive integers

since

$$\begin{aligned}
 \Delta + 1 &= 1 - (ps')^2 + 4rsq'^2 \\
 &= (1 + ps')(1 - ps') + 4rsq'^2,
 \end{aligned}$$

then  $ps'$  being odd the first summand is divisible by 4.

Again if  $(\delta, \nu)$  run through all integers then so do  $(\delta_2, \nu)$ .

We have then established the following theorem :

THEOREM 5.2 - The set of all possible number of sheets for all immersions between two given torii  $T \rightarrow T_1$  depends only on  $|ad - bc|$  if  $T_1$  is not ample and only on  $|ad - bc|$  and  $\Delta$  when  $T_1$  is ample.

Note that although the lifts  $F(z) = \alpha Az + D$  and  $F(z) = (\delta + \nu B)z + D$  are one-to-one maps between  $E$  and  $E_1$  the immersions themselves are not 1 - 1 since this would imply that  $T$  and  $T_1$  are conformally equivalent which is not always the case. Nor is the inverse  $F^{-1}$  in general the lift of an immersion  $f : T' \rightarrow T$  since if  $w = F(z) = Cz + D$  then  $z = F^{-1}(w) = \frac{w - D}{C} = \frac{1}{C}w - \frac{D}{C}$ . But the set of all  $\frac{1}{C(\alpha)}$  or all  $\frac{1}{C(\delta, \nu)}$  does not form a lattice.

Consider now the case  $T = T_1$ , that is self-immersions of a torus. In this case we can choose  $\Omega = \Omega_1, \Omega' = \Omega'_1$  i.e.  $a = d = 1, b = c = 0$ .

In theorem 4.2 we have  $A = \frac{\Omega_1}{\Omega} = 1$  and  $B = \frac{s'p\Omega_1 - q'r\Omega'_1}{\Omega} = \frac{\Omega'_1 qs}{g\Omega} q_1^2$   
 $= \frac{\Omega_1 qs}{g\Omega} q_1 = \frac{qs}{g} q_1$ .

So if  $T$  is ample then the family of self immersion is given by

$$F(z) = \left( \delta + \nu \frac{qs}{g} q_1 \right) z + D,$$

otherwise they are of the form  $F(z) = \delta z + D$

BIBLIOGRAPHY

- [1] Hille, Einar, "Analytic Function Theory", Vol. I,  
Ginn and Company (1959).
  
- [2] Hille, Einar, "Analytic Function Theory", Vol. II,  
Ginn and Company (1962).
  
- [3] Hu, Sze-Tsen, "Homotopy Theory", Academic Press (1959).
  
- [4] Kobayashi and Nomizu, "Foundation of Differential  
Geometry", Interscience Publishers (1963).
  
- [5] Nevanlinna, R., "Uniformisierung", Berlin (1953).
  
- [6] Springer, George, "Introduction to Riemann Surfaces"  
Addison - Wesley Publishing Company (1957)