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State of a Network when one Node Overloads

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STATE OF A NETWORK WHEN ONE NODE OVERLOADS

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Abstract

We delve into a couple of topics in the theory of Markov chains and stochastic networks. The properties of a stable Markov chain $X = (X_1, \hat{X})$ will be investigated when X_1 tends to infinity. We derive the distribution of $\hat{X}$ when X_1 passes a threshold for the first time as the threshold tends to infinity. Moreover, the exact asymptotics of the mean time until X_1 reaches the threshold is given.

In addition, we present a new approach to determine the exact asymptotics of the X's steady state. The results are applied to an open modified Jackson network with two partially coupled processors.

Finally, a ratio limit property is established for a Markovian kernel which has unbounded jumps.

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Next, my thanks and appreciation to my family. My wife whose support and assistance was essential during my studies. My father's never ending encouragement and belief in life kept me working. I am grateful to all members of my extended family for their care.

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Dedication

To my wife, *Elham*.

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Introduction

What is the thesis about?

Queueing theory and the theory of queueing networks are mathematically interesting branches of probability theory. However, the outstanding characteristic of this theory is its interaction with real world models. It is one of the mathematical theories where most new results are generated by applications. It is worthwhile to mention that the whole study of queueing theory started with the works of Erlang (1909) [Erl09, Erl17] and Engset (1918) [Eng18] in telephony. Erlang was a Danish engineer who worked for the Copenhagen Telephone Exchange and Engset was a manager of the Norwegian Telecommunication Authority.

Analyzing congestion is one of the main goals of queueing theory. Congestion happens when the number of customers requesting a specific service exceeds the service provider's ability to process demands. As a result, customers join a queue and wait in a waiting room until their service starts. Obviously, in the real world the waiting room capacity is finite. The waiting room can be the buffer of a processor in a laptop or simply the physical capacity of a bank branch. As the queue size exceeds the maximum capacity threshold, the customers will be ignored by the system or they voluntarily leave the network. As a result, data will be lost in a network of processors, patients will have to wait for an unreasonable period of time or customers of a cable company will be disappointed, [Res03].

The main objective of this monograph is to shed light on some aspects of congestion in a network. Specifically, we will investigate the behavior of the network when one specific processor overloads, i.e. its queue size tends to infinity. The approach used in this sequel is an extension of the approach first published in [MCD99]. The aforementioned reference establishes a new approach in studying rare events for Markov chains. The method was later standardized and its relations with large deviation techniques were looked into, [Fol05a, Fol05b].

As an application, we will mainly focus on a modified Jackson network with two partially coupled processors.

Outline of the monograph and contribution review

The discussions are mostly divided into developing a general theory for Markov chains and subsequently applying the results to the modified Jackson network with partially coupled processors.

Chapter 1 gives the general framework of the Markov chains which will be dealt with in the sequel. In addition, it introduces the modified Jackson network (MJN).

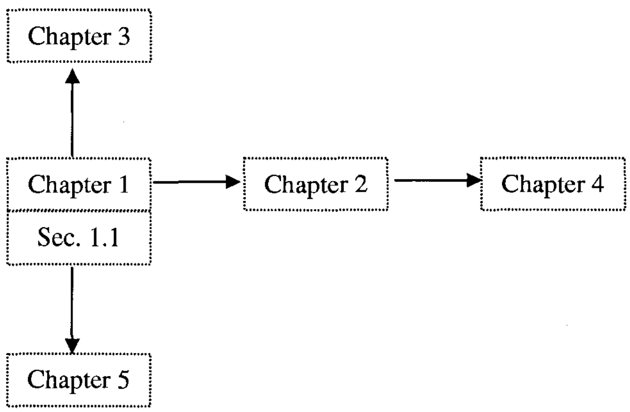
The method of Foley & McDonald to find the exact asymptotics of the steady state is reviewed in Chapter 2. In Proposition 2.1, Condition 5 of [MCD99] is relaxed for this approach.

The third chapter introduces a new approach to determine the asymptotics of the steady state. By results in [Fol05b], the modified Jackson network has three large deviation paths to overload the level classified as *bridge*, *jitter* and *cascade*. The asymptotics of the stationary distribution, π, is given in [Fol05b] for the jitter and the bridge cases. Chapter 3 provides the asymptotics of π in the cascade cases.

Chapter 4 investigates the distribution of a network of queues when the size of one queue tends to infinity. We will improve the results in [MCD99, Fol01] and provide the limiting hitting distribution of the remaining queues (phase) when the first queue (level) gets large for a Markov chain. The asymptotics of the corresponding mean hitting time is also given. Surprisingly, our results imply the existence of the distribution when the cheapest large deviations path of MJN is a bridge; the bridge cases happen when large deviations path tries to avoid the horizontal axis.

The last chapter is an independent exposition from Chapters 2, 3 and to some extent from Chapter 4. We study a Markov chain on $\mathbb{Z} \times \mathbb{Z}_+$ which is shift invariant with respect to the first coordinate and deliver the hitting distribution of the second coordinate when the first tends to infinity. In addition, ratio limit property is discussed for a class of Markovian kernels with unbounded jumps. Chapter 5 can be studied right after Section 1.1. We succeed in partially generalizing the results of [Kes74].

Dependence among chapters is illustrated below.



Our approach versus others

In order to study the asymptotic behavior of a stochastic network, the other major approaches are Complex analysis approach and Matrix analytic approach. The approach used in this sequel stems from the works of Feller, Doob, Kesten and McDonald. As it is illustrated in the thesis, our method is capable of addressing a wide range of problems regarding rare events of Markov chains and overloading networks. This includes finding exact asymptotics of the steady state and determining the hitting distribution of some queues while other queues are overloaded.

We provide a short explanation of other approaches, and describe how they find the exact asymptotics of the steady state, π. Other rare event problems are addressed similarly.

$\diamondsuit$ *Complex analysis approach* uses Kolmogrov equations of $\pi(\ell,\hat{x})$, to derive a recursion formula for the generating function $F(s,t)=\sum_\ell\sum_{\hat{x}}\pi(\ell,\hat{x})s^\ell t^{\hat{x}}$. By virtue of the complex analysis techniques, it is possible to study the properties of F on the complex plane. As a result, the coefficients of F, which are π's, could be targeted. In this method, the computations are straightforward but tedious and obviously there is limitation regarding the complexity of the system. [Fay79] uses this method to derive the decay rate of the stationary distribution for a network with two parallel coupled processors.

$\Diamond$ *Matrix analytic approach* is stemmed from the celebrated work of Neuts [Neu81]. Focusing on networks where level is nearest neighbor, with a suitable choice of states, the transition matrix of the joint queue length is block partitioned. The new block partitioned matrix usually looks like the kernel matrix of an $M/M/1$ queue. By using the properties of the block matrixes, stability conditions and the asymptotics of the steady state are studied. For some recent developments, refer to [Sak06, Li07, Miy04, He08].

The disadvantage of the Markov additive method, which is employed here, with respect to the other approaches is lack of power to determine the exact value of all involving parameters, like f in Theorem 2.3. Though, for most networks the parameters could be found by fast simulation. This being said, the method has at least the following benefits:

1. Heuristically, the approach is more practical since we will be working with three associated chains and while pursuing an argument the chain that fits better is implemented.

2. The calculations are simple.

3. Higher dimensional processes (networks with more than two servers) can be dealt with as long as we have a clear idea about the Green function. On the other hand, notice that the complex method can not go past two dimensional chains.

Chapter 1

PRELIMINARIES

1.1 Markov Chains

Consider a stable Markov chain $X = (X_1, \hat{X})$ in $S = \mathbb{Z}_+ \times \hat{S}$ with kernel K, where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\hat{S}$ is a countable set. The first and second coordinate will be referred to as level and phase of the chain, respectively. We may think of X as the joint queue length of a network with n processors, where X_1 represents the number of customers waiting or in service by the first processor and $\hat{X}$ gives the joint queue length of the remaining processors. Another example of X is a random walk in the first closed orthant where the jumps out of the closed region are suppressed. Since X is stable, there exists an invariant probability measure or steady state π, i.e. for any fixed $x = (x_1, \hat{x}) \in S$, we have

$$\sum_{y \in S} \pi(y) K(y; x) = \pi(x).$$

If there is no ambiguity, we will use ℓ as the first coordinate of any $x \in S$. Moreover, use the convention that $\pi(\ell, \hat{x})$ has exact asymptotics $f(\ell)$ if and only if $\lim_{\ell \to \infty} \pi(\ell, \hat{x})/f(\ell) = 1$. Notation $f(\ell) \sim g(\ell)$ will be used for the functions f and g whenever $\lim_{\ell \to \infty} f(\ell)/g(\ell) = 1$.

Remark 1.1. The properties of a Markov jump process can be studied through a discrete time Markov chain using minimal construction method and by ignoring the intensities of the states, [Asm87]. Although, the models that will be discussed later deal with continuous time but we will only focus on the embedded chain with discrete time. Therefore, define the kernel, K, to be the matrix of the jump probabilities, i.e. $K(x;y)$ is the probability that the chain after leaving x jumps to y.

For most models in stochastic networks, the state space S can be partitioned into two regions, Δ, the boundary, and its compliment, the interior, such that X is shift invariant on $\Delta^c \equiv S - \Delta$ with respect to its first coordinate. Usually, we may think of Δ as

$$\Delta \equiv \{(x_1, \hat{x}) |\ 0 \leq x_1 \leq M_\Delta\}, \tag{1.1}$$

for a fixed integer $M_\Delta \geq 0$, see Figure 1.1. However, based on the model other forms of Δ can be considered, see [Fol01] where for analyzing a network with "join the shortest queue" policy different boundaries are used.

Therefore, X is limited to the cases where an $M_\Delta \geq 0$ and the set of functions $\{p_\ell(\cdot\ ;\ \cdot); \ell \in \mathbb{Z}\}$ exist such that for any $x_1, y_1 > M_\Delta$, the transition possibility from $(x_1, \hat{x})$ to $(y_1, \hat{y})$ in one step is given by

$$K((x_1, \hat{x}); (y_1, \hat{y})) = p_{y_1 - x_1}(\hat{x}; \hat{y}). \tag{1.2}$$

The existence of the functions $p_\ell(\cdot\ ;\ \cdot)$ guaranties the level homogeneity of the chain after a fixed level. Obviously, this Markov chain can be embedded in a free Markov additive chain $X^\infty = (X_1^\infty, \hat{X}^\infty)$ with kernel K^∞ and state space $S^\infty \equiv \mathbb{Z} \times \hat{S}$. X^∞ is Markov additive in the sense that for any levels x_1 and $y_1 \in \mathbb{Z}$,

$$K^\infty((x_1, \hat{x}); (y_1, \hat{y})) = p_{y_1 - x_1}(\hat{x}, \hat{y}), \tag{1.3}$$

hence,

$$K^\infty((x_1, \hat{x}); (y_1, \hat{y})) = K^\infty((0, \hat{x}); (y_1 - x_1, \hat{y})).$$

We further impose the condition that one step transition probabilities of X and X^∞ agree from states on the boundary to the states in the interior, i.e.

$$K^\infty(x;y) = K(x;y) \quad \text{for any } x \in \Delta \text{ and } y \in S - \Delta.$$

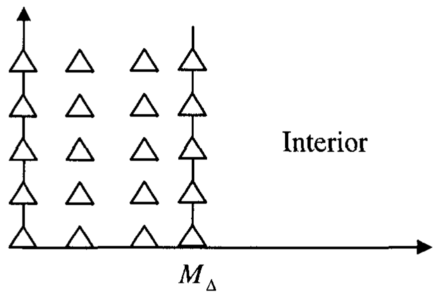


Figure 1.1: Boundary versus the interior

Remark 1.2. The above assumption about jumps from the boundary to the interior could be relaxed by imposing assumptions on the tail of the jumps that start from the boundary and end up in the interior, [MCD99].

Remark 1.3. Notice that any Markov chain can be considered as a space-time Markov additive chain. Let $\hat{X}$ be a Markov Chain on $\hat{S}$ with transition kernel $Q(.\,;\,.)$. Define $X^\infty = (X_1, \hat{X})$ on $\mathbb{Z} \times \hat{S}$ with kernel q as

$$q((x_1, \hat{x}); (y_1, \hat{y})) = \begin{cases} Q(\hat{x}; \hat{y}) & \text{if } y_1 - x_1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $q((x_1, \hat{x}); (y_1, \hat{y})) = q((0, \hat{x}); (y_1 - x_1, \hat{y}))$.

The notation $\blacktriangle$ will be used for the points in S^∞ which are out of the interior. Therefore, S^∞ is partitioned to the sets $\blacktriangle$ and $S - \Delta$.

Next, suppose that K^∞ has a harmonic function $h(x_1, \hat{x})$. h can be interpreted as a right eigenvector for the matrix $[K^\infty(x; y)]_{x,y \in S^\infty}$. For any fixed $x \in S^\infty$,

$$\sum_{y \in S^\infty} K^\infty(x; y) h(y) = h(x).$$

Further, impose the condition that the harmonic function is of the form

$$h(x_1, \hat{x}) \equiv \exp(\alpha x_1) \hat{h}(\hat{x}), \tag{1.4}$$

for some strictly positive α and in addition, $\hat{h} > 0$ is not a function of the level.

Remark 1.4. The existence of h in the above form merits further investigation. For the networks and chains considered in the sequel it is straightforward to determine α and $\hat{h}$. However, generally finding the above representation could be complicated. By the discussion in [Woe00, Section 25], if a random walk on a graph is uniformly irreducible, then the minimal harmonic functions in the cone of the positive harmonic functions have the exponential form same as h in (1.4).

The next step would be to amend the transition probabilities of the Markov additive chain, X^∞ by the harmonic function h. To do so, we perform the Doob's h-transform (h-twist) on the kernel, K^∞ to obtain the new kernel, $\mathcal{K}$, i.e. for any $x, y \in S^\infty$

$$\mathcal{K}(x; y) = K^\infty(x; y)h(y)/h(x). \tag{1.5}$$

It is evident that $\mathcal{K}$ is a kernel on S^∞. Let $\mathcal{X}$ be the associated twisted Markov chain with kernel $\mathcal{K}$.

Proposition 1.5. *The above defined $\mathcal{X}$ is Markov additive with respect to the first coordinate.*

Proof. Directly check the additive property of the kernel $\mathcal{K}$,

$$\begin{aligned}
&\mathcal{K}((x_1, \hat{x}); (y_1, \hat{y})) \\
&= K^\infty((x_1, \hat{x}); (y_1, \hat{y}))e^{\alpha y_1}\hat{h}(\hat{y})/e^{\alpha x_1}\hat{h}(\hat{x}) && \text{by definition of } \mathcal{K} \\
&= K^\infty((0, \hat{x}); (y_1 - x_1, \hat{y}))e^{\alpha(y_1 - x_1)}\hat{h}(\hat{y})/\hat{h}(\hat{x}) && \text{by additive property of } K^\infty \\
&= \mathcal{K}((0, \hat{x}); (y_1 - x_1, \hat{y})).
\end{aligned}$$

□

So far we have been dealing with three Markov chains X, X^∞ and $\mathcal{X}$ which will be called *original*, *free* and *twisted* Markov chains, respectively.

The next step is to shed some light on the Markovian part of the twisted chain. Define the marginal kernel of the second coordinate as

$$\hat{\mathcal{K}}(\hat{x};\hat{y}) = \sum_{\ell\in\mathbb{Z}} \mathcal{K}((0,\hat{x});(\ell,\hat{y})).$$

$\hat{\mathcal{K}}$ is well-defined by Proposition 1.5 and the Markov chain with the kernel $\hat{\mathcal{K}}$ is a chain on $\hat{S}$. This chain also can be classified as positive, null recurrent or transient.

Remark 1.6. Throughout this sequel, we will investigate the properties of the original Markov chain, X, in different cases where the Markovian part of the twisted chain is positive, null-recurrent or transient.

From now on, we assume that $\hat{\mathcal{K}}$ has an invariant measure φ which is defined on $\hat{S}$ and is unique up to constant multiples, i.e. for any fixed $\hat{y}$

$$\sum_{\hat{x}\in\hat{S}} \varphi(\hat{x})\hat{\mathcal{K}}(\hat{x};\hat{y}) = \varphi(\hat{y}).$$

Notice that φ is not necessarily a probability distribution.

Remark 1.7. The existence of φ is not immediate in general. In positive recurrent cases the invariant measure exists and it can be considered as a probability measure after some manipulation, i.e. $\sum_{\hat{x}\in\hat{S}} \varphi(\hat{x}) = 1$. On the other hand, for null recurrent case, the measure always exists but it is not a probability measure, [Der53]. Most of the complications arise in the transient case. Harris [Har57] gave a sufficient condition for the existence of the measure and six years later Veech [Vee63] proved the necessity of Harris' condition. However, in many cases checking the condition is highly demanding or impossible. On the other hand, for application problems where only Markov chains with bounded jumps from each state are considered the Harris condition applies easily and the invariant measure exists even in the transient cases. Another remarkable article on this topic is [Kes95] by Kesten.

The next step is to define the time reversals of the original and the twisted chains. For the original chain, the time reversal is given by,

$$K^*((x_1,\hat{x});(y_1,\hat{y})) = \frac{\pi(y_1,\hat{y})}{\pi(x_1,\hat{x})} K((y_1,\hat{y});(x_1,\hat{x})).$$

In most cases π is unknown, so we prefer to work with K^*'s approximation, $\overleftarrow{\mathcal{K}}$, defined as the time reversal of $\mathcal{K}$ in the following way,

$$\overleftarrow{\mathcal{K}}((x_1,\hat{x});(y_1,\hat{y})) = \varphi(\hat{y})\mathcal{K}((y_1,\hat{y});(x_1,\hat{x}))/\varphi(\hat{x}).$$

The rational behind the approximation is discussed in Proposition 2.9. The next proposition ensures that $\overleftarrow{\mathcal{K}}$ is well-defined.

Proposition 1.8. *φ is an invariant measure for $\mathcal{K}$.*

Proof. We must check that $\sum_{(y_1,\hat{y})} \varphi(\hat{y})\mathcal{K}((y_1,\hat{y});(x_1,\hat{x})) = \varphi(\hat{x})$.

$$\begin{aligned}
&\sum_{(y_1,\hat{y})} \varphi(\hat{y})\mathcal{K}((y_1,\hat{y});(x_1,\hat{x})) \\
&= \sum_{y_1,\hat{y}} \varphi(\hat{y})\mathcal{K}((0,\hat{y});(x_1-y_1,\hat{x})) && \text{by being additive} \\
&= \sum_{\hat{y}}\sum_{y_1} \varphi(\hat{y})\mathcal{K}((0,\hat{y});(x_1-y_1,\hat{x})) \\
&= \sum_{\hat{y}} \varphi(\hat{y})\hat{\mathcal{K}}(\hat{y};\hat{x}) && x_1-y_1 \text{ ranges over all } \mathbb{Z} \\
&= \varphi(\hat{x}) && \text{by definition of invariant measure.}
\end{aligned}$$

□

Proposition 1.9. *$\overleftarrow{\mathcal{K}}$ is additive with respect to the first coordinate.*

Proof. Keep in mind that, by Proposition 1.5, $\mathcal{K}$ is additive. For any fixed $(\ell_1,\hat{x})$ and $(\ell_2,\hat{y})$,

$$\begin{aligned}
\overleftarrow{\mathcal{K}}((\ell_1,\hat{x});(\ell_2,\hat{y})) &= \frac{\varphi(\hat{y})}{\varphi(\hat{x})}\mathcal{K}((\ell_2,\hat{y});(\ell_1,\hat{x})) \\
&= \frac{\varphi(\hat{y})}{\varphi(\hat{x})}\mathcal{K}((0,\hat{y});(\ell_1-\ell_2,\hat{x})) \\
&= \frac{\varphi(\hat{y})}{\varphi(\hat{x})}\mathcal{K}((\ell_2-\ell_1,\hat{y});(0,\hat{x}))
\end{aligned}$$

$$= \overleftarrow{\mathcal{K}}((0,\hat{x});(\ell_2-\ell_1,\hat{y})).$$

□

We will use the Green function on several occasions, defined as

$$G(x;y) = \sum_{n\in\mathbb{Z}_+} K^{(n)}(x;y). \tag{1.6}$$

$G(x;y)$ is the expected number of visits to y starting from x by X. Analogously, $G^\infty, \mathcal{G}$ and $\overleftarrow{\mathcal{G}}$ are defined using $K^\infty, \mathcal{K}$ and $\overleftarrow{\mathcal{K}}$, respectively.

Taboo kernels are noted by ${}_AK^{(n)}(x;y)$, which gives the probability that starting from the state x, X reaches y in n steps without hitting the set A at any time k that $0 < k < n$. The corresponding taboo Green function is defined as

$${}_AG(x;y) = \sum_{n\in\mathbb{Z}_+} {}_AK^{(n)}(x;y). \tag{1.7}$$

Likewise ${}_AG^\infty, {}_A\mathcal{G}$ and ${}_A\overleftarrow{\mathcal{G}}$ are defined using corresponding taboo kernels.

Example 1.10. Consider an $M/M/1$ queue with first come first serve policy. Suppose that the arrival rate of the customers is λ and the service rate is μ. For simplicity, assume that $\lambda+\mu=1$. Let $X[n]$ be the embedded discrete time queue size consisting of customers waiting for or in service at time n. Since the second coordinate is a single state, we are dealing with the positive recurrent case and X can be considered one dimensional. X is a Markov chain on $S=\mathbb{Z}_+$. Starting from any $k>0$, X jumps to $k+1$ and $k-1$ with transition probabilities λ and μ, respectively. Away from 0, X has an additive behavior. Let $\Delta=\{0\}$ and $\blacktriangle$ to be $\{0,-1,-2,\cdots\}$, so X can be embedded in the free chain X^∞ on $S^\infty=\mathbb{Z}$ with kernel K^∞. For any $k,\ell\in\mathbb{Z}$,

$$K^\infty(k;\ell) = \begin{cases} \mu & \text{if } \ell = k-1, \\ \lambda & \text{if } \ell = k+1. \end{cases}$$

X^∞ is a transient chain since X is stable and therefore $\mu > \lambda$.

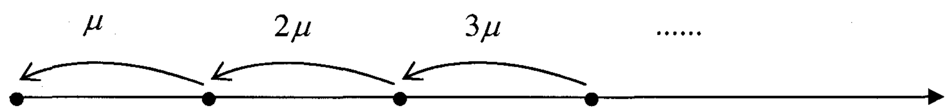


Figure 1.2: $M/M/\infty$ queue can not be embedded in an additive chain.

Define

$$h(\ell) = (\frac{\mu}{\lambda})^\ell = e^{\ell \ln(\mu/\lambda)}.$$

h is harmonic for K^∞ and $\alpha = \ln(\mu/\lambda) > 0$, as required. Notice that $\hat{h} \equiv 1$. Moreover, the twisted chain has the following transitions,

$$\mathcal{K}(k;\ell) = \begin{cases} \lambda & \text{if } \ell = k-1, \\ \mu & \text{if } \ell = k+1. \end{cases}$$

Clearly, $\varphi \equiv 1$.

Example 1.11. Not every network can be embedded in a free Markov additive chain. Consider an $M/M/\infty$ system of infinite independent processors. X, the queue size, is defined on $\mathbb{Z}_+$. Here the arrival rate is λ but the rate of jumping from n to $n-1$ is $n\mu$, where μ is the service rate of each processor. Since the rates are level dependent for any state n, X can not be analyzed through an additive chain. See Figure 1.2. However, it is possible to implement the h-transform method to some extent. The kernel of X can be twisted by the function

$$h(\ell) = \ell!(\frac{\mu}{\lambda})^\ell.$$

Notice that $\ell!$ and $(\frac{\mu}{\lambda})^\ell$ are elements of the steady state distribution which is Poisson given by

$$\pi(\ell) = e^{-\lambda/\mu}\frac{(\lambda/\mu)^\ell}{\ell!}.$$

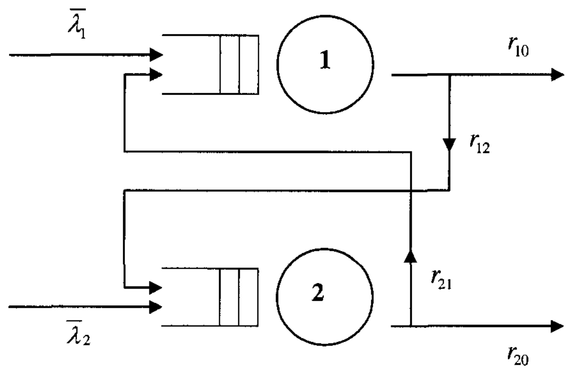


Figure 1.3: Open Jackson network

1.2 The Model: Modified Jackson Network

Jackson network.

Consider a network of two service providers, called processors. The first processor has its own dedicated stream of customers that arrive according to a Poisson distribution with rate $\bar{\lambda}_1$. Customers arriving at this processor will be served on a first come first serve policy (FCFS). The first processor will treat each customer's demand according to an exponential distribution with mean $1/\mu_1$. Upon service completion, the customer leaves the system with probability r_{10} or joins the queue of the second processor with probability r_{12}. The routings happen with no delay. The same happens for the second processor which has its own dedicated stream of customers arriving according to a Poisson process with rate $\bar{\lambda}_2$. The second processor serves customers in an exponentially distributed time with mean $1/\mu_2$. After service completion customers leave the system or join the queue of the first processor with probabilities r_{20} and r_{21}, respectively. The routings and rates are shown in Figure 1.3.

We assume that all arrivals, services and routing processes are independent. So far this is exactly the structure of the famous *Open Jackson Network*. It is referred to as *open*, because of the arrivals and departures to and from the system. On the contrary, the system is *closed* if the number of customers in the system is fixed, say L; henceforth arrivals in and departures out of the system are banned, [Asm87, page 120]. In addition, *semi-open* is also an interesting case where the system is open but at each moment only L customers are present in the system, where L is fixed, [Che01, page 21]. In this monograph, we focus only on an open network.

Notice that $r_{ii} > 0$ conveys that the processing time at processor i is a geometrically sum of exponential random variables, which is again exponentially distributed; henceforth without loss of generality, we assume $r_{11} = r_{22} = 0$.

Let λ_i be the total stream rate of customers at processor i. Considering the customers routed to processor $i = 1, 2$ from processor $j = 2, 1$, respectively, λ_i's would satisfy

$$\begin{cases} \lambda_1 & = \bar{\lambda}_1 + \lambda_2 r_{21} \\ \lambda_2 & = \bar{\lambda}_2 + \lambda_1 r_{12}. \end{cases}$$

The only solution to this system of equations is

$$(\lambda_1, \lambda_2) = (\frac{\bar{\lambda}_1 + \bar{\lambda}_2 r_{21}}{1 - r_{12} r_{21}}, \frac{\bar{\lambda}_2 + \bar{\lambda}_1 r_{12}}{1 - r_{12} r_{21}}).$$

Let $X_1(t)$ and $\hat{X}(t)$ be the number of customers at processor one and two, respectively, at time t. $X_1(t)$ and $\hat{X}(t)$ are non-negative for all $t \geq 0$. $X(t) = (X_1(t), \hat{X}(t))$ is a Markov process on $\mathbb{Z}_+ \times \mathbb{Z}_+$. It can be shown that X is positive recurrent if $\rho_i < 1$ for both $i = 1, 2$, where $\rho_i = \lambda_i / \mu_i$. If X is positive recurrent then it has a stationary distribution, say π such that

$$P\{X[t] = x\} = \pi(x),$$
$$\sum_x \pi(x) = 1,$$
$$\lim_{t \to \infty} P\{X[t] = x |\ X[0] = y\} = \pi(x), \tag{1.8}$$

where y and $x = (x_1, \hat{x})$ are in $\mathbb{Z}_+ \times \mathbb{Z}_+$. [Jac57] proves that the stationary distribution of the joint queue length surprisingly has the following product form:

$$\pi(x_1, \hat{x}) = (1 - \rho_1) \rho_1^{x_1} \times (1 - \rho_2) \rho_2^{\hat{x}}, \tag{1.9}$$

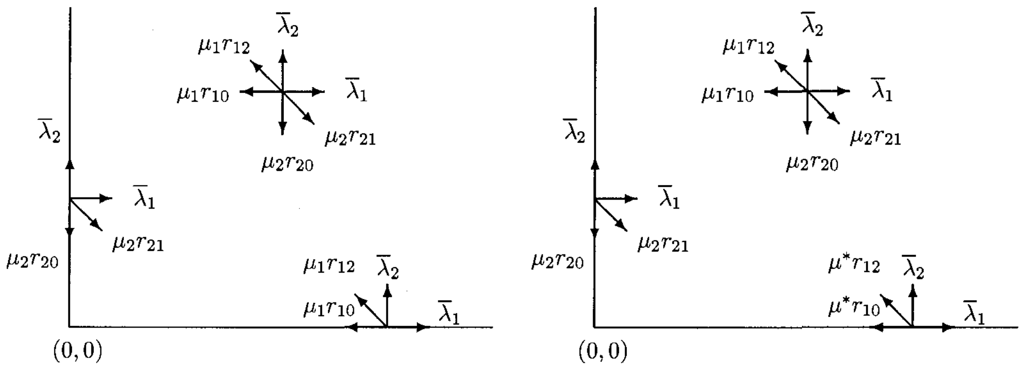


Figure 1.4: Open Jackson network rates versus modified Jackson network's.

which in turn by (1.8) means that after a long period of time ($t \longrightarrow \infty$), the steady state of this system at time t behaves as if the two processors operate like two independent $M/M/1$ queues with constant arrival (birth) rates λ_1, λ_2 and service (death) rates μ_1 and μ_2, respectively. We highlighted the word "as if" above, since there is a common misconception about Jackson's result. It can be shown in sharp contrast to the $M/M/1$ queue that the stream of *all* customers at each processor does not follow a Poisson distribution in general, [Dis81].

Modified Jackson network (MJN).

The network considered in this monograph is a modification of the above well-known Jackson network. Let's generalize the above network in the sense that when processor two is idle and has no customers it helps processor one. Obviously, this will result in a better performance and shorter joint queue length. This is what is known as the Jackson network with *partially coupled processors*. The analysis of this network happens to be more complicated than the pure Jackson network and the stationary distribution is not of the product form (1.9). Intuitively, the two processors are not independent since there is a joint effort to process the customer demands at the first queue when queue two is empty. Take μ^* to be the joint effort of the two processors.

Without loss of generality assume that

$$\mu^* + \mu_2 + \sum_i \lambda_i = 1.$$

Remark 1.12. As it was mentioned in Remark 1.1, we choose to work with the discrete chain embedded in X. Therefore, we regard the transition rates as jump probabilities.

Let K be the kernel of the embedded Markov chain for the partially coupled system. Using $e_1 = (1,0)$ and $e_2 = (0,1)$, starting from the point $(x_1, \hat{x})$ where $x_1 > 0$, the transition probabilities for K are

Jump direction	Probability	
e_1	$\bar{\lambda}_1$	
e_2	$\bar{\lambda}_2$	
$-e_1 + e_2$	$\begin{cases} \mu_1 r_{12} & \hat{x} > 0 \\ \mu^* r_{12} & \hat{x} = 0 \end{cases}$	
$-e_1$	$\begin{cases} \mu_1 r_{10} & \hat{x} > 0 \\ \mu^* r_{10} & \hat{x} = 0 \end{cases}$	
$-e_2$	$\mu_2 r_{20}$	$\hat{x} > 0$
$e_1 - e_2$	$\mu_2 r_{21}$	$\hat{x} > 0,$

where for example jumps toward e_1 and $-e_1 + e_2$ are $K((x_1, \hat{x}); (x_1 + 1, \hat{x}))$ and $K((x_1, \hat{x}); (x_1 - 1, \hat{x} + 1))$, respectively; the rest of jumps have analogous interpretation. The transition rates of our modified Jackson network (MJN) versus the original Jackson network are given in Figure 1.4.

A generalization of the partially coupled network, which will not be discussed in this thesis, is the *coupled processors* case where any idle processor helps the other one, for analysis of a special case refer to [Res03].

In order to embed X into a Markov additive chain, simply take $M_\Delta = 0$. The interior is the set of all states where the level is strictly positive and $\blacktriangle$ is the set of

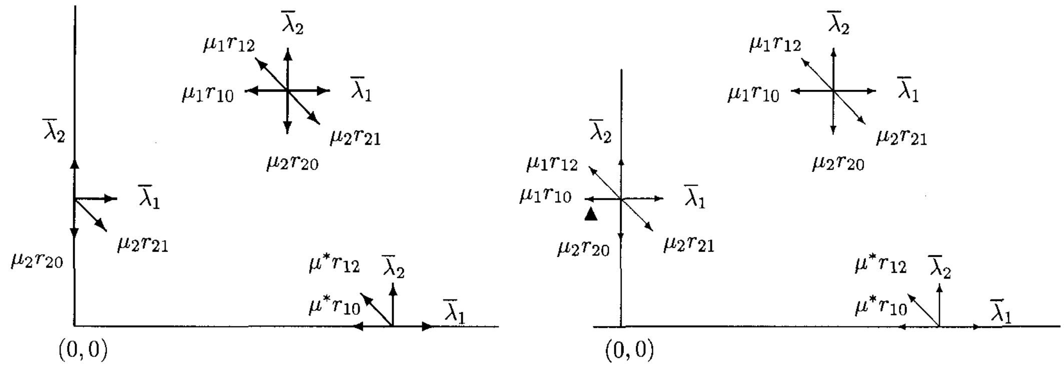


Figure 1.5: Modified Jackson network versus associated free Markov additive chain

all states where the level is non-positive, Figure 1.5. The rest of the properties are immediate, we will determine the harmonic function h and the invariant measure φ later. Notice that φ exists and is unique up to multiple constants since there are only finite number of states to be reached from each state by one jump.

Some known results.

The necessary and sufficient conditions for the stability of MJN can be proved using a coupling argument, [Fol05b].

Proposition 1.13. *The joint queue length process of the MJN is positive recurrent if and only if*

$$\lambda_2 < \mu_2 \tag{1.10}$$

and

$$\lambda_1 < \rho_2\mu_1 + (1-\rho_2)\mu^*, \tag{1.11}$$

where $\rho_i = \lambda_i/\mu_i$.

It is well known that the Jackson network is stable and positive recurrent if and only if for $i = 1, 2$, $\rho_i < 1$. On the contrary, for the MJN ρ_1 can be larger than 1 but ρ_2 is still less than one. To see this, divide (1.11) by μ_1 to get $\rho_1 < \rho_2 + (1-\rho_2)\mu^*/\mu_1$ and the result is immediate since $\mu^*/\mu_1 > 1$.

Lemma 1 on Page 524 of [Fol05b] is also crucial for our analysis.

Proposition 1.14. *For the stable MJN,*

$$\sum_{\hat{y}\geq\hat{y}_0} \pi(0,\hat{y}) \leq c\rho_2^{\hat{y}_0},$$

for some constant c.

[Fol05b] uses the flat boundary theory of [Shw95] to show that there are three large deviations paths to overload the first processor.

The large deviations theory deserves an explanation. A good reference is [Shw95], which will be used here. Another recent book with applications in queueing theory is [Man07]. [Dem98] is also a standard book for the general large deviations theory.

Define $F_\ell = \{(x_1,\hat{x}) \in S|\ x_1 \geq \ell\}$. It is said that the rare event of overloading happens with *large deviations rate* α if and only if

$$\lim_{\ell\to\infty} \frac{1}{\ell} \ln P\{X \in F_\ell|\ X(0) = (0,0)\} = -\alpha.$$

For MJN, ignoring what happens on the axes, X is a Markovian chain where all of its possible jump directions are $\{d_1 = e_1, d_2 = e_2, d_3 = -e_1 + e_2, d_4 = -e_1, d_5 = -e_2, d_6 = e_1 - e_2\}$. Suppose that the rate of a jump in d_i direction is J_i; this notation will be used only in the current explanation. The probabilities of our desired rare event are governed by the function

$$\Lambda(y) \equiv \sup_{\theta=(\theta_1,\theta_2)} [\langle\theta, y\rangle - M^+(\theta)],$$

where $\langle .,. \rangle$ stands for dot product of two vectors and

$$M^+(\theta) = \sum_{i=1}^{6} J_i \left(e^{\langle\theta,d_i\rangle} - 1\right).$$

Let $F(\ell)$ be the set of paths in S that start at the origin, $(0,0)$, and end at F_ℓ. If we speed up jump rates by a factor ℓ, but reduce the jumps by a factor $1/\ell$, then we get the scaled process X_ℓ. From the theory in [Shw95, Chapter 5],

$$\lim_{\ell\to\infty} \frac{1}{\ell} \ln P\{X \in F_\ell|\ X(0) = (0,0)\ \}$$

$$= \lim_{\ell\to\infty} \frac{1}{\ell} \ln P\{X \in F(\ell) |\ X(0) = (0,0)\ \}$$
$$= \lim_{\ell\to\infty} \frac{1}{\ell} \ln P\{X_\ell \in F |\ X_\ell(0) = (0,0)\ \}$$
$$= - \inf_{p\in F}[I(p)], \tag{1.12}$$

where $I(p) = \int_0^T \Lambda(p(s), p'(s))ds$ and F is the set of absolutely continuous paths starting from the origin and hitting $\{(1, \hat{y}) \mid \hat{y} \geq 0\}$ at some time T before returning to the origin. $\inf_{p\in F}[I(p)]$ is called the *minimum action.* Moreover, it is known that the inf in (1.12) is attained by a path which consists of line segments with constant speed and changes direction only on the y-axis. This is called the *cheapest path.*

For the MJN, equality (1.12) can not handle all the possible cases. Therefore, an extension of the above argument is required, which is the subject of [Fol05b]. It is worthwhile to mention that [Ign01b] gives another characterization of the large deviations rate.

The cheapest path for the MJN can be any of the following:

- Jitter: This is the path that spends a positive proportion of time on the level axis while overloading it. It is a constant speed path parallel to the first axis until hitting $(1, 0)$.

- Bridge: This case happens when the cheapest path moves parallel to the first axis but the expected time spent on the x-axis is zero until hitting $(1, 0)$, i.e. the trajectory mostly skims above the x-axis.

- Cascade: This type of trajectory first jitters along the y-axis and then leaves the vertical axis toward the horizontal axis to overload it. After leaving the y-axis, it heads for the point $(1, 0)$ at constant velocity.

To derive the above results, [Fol05b] reduces the problem to a differentiable, constrained nonlinear optimization problem and uses Karush-Kuhn-Tucker conditions to calculate the minimal action in each of the three cases. In addition, [Fol05a] provides the exact asymptotics of the stationary distribution in the jitter and bridge cases but not for the cascade case.

Definition 1.15. Let $M^+(\theta_1, \theta_2)$ be the log moment generating function of the compound Poisson process associated with the jumps in the interior,

$$M^+(\theta_1,\theta_2) = \bar{\lambda}_1(e^{\theta_1}-1) + \bar{\lambda}_2(e^{\theta_2}-1) + \mu_1 r_{12}(e^{\theta_2-\theta_1}-1) + \mu_1 r_{10}(e^{-\theta_1}-1) + \mu_2 r_{20}(e^{-\theta_2}-1) + \mu_2 r_{21}(e^{\theta_1-\theta_2}-1).$$

Definition 1.16. Let $M^-(\theta_1, \theta_2)$ be the log moment generating function of the compound Poisson process associated with the jumps starting from a point on the x-axis,

$$M^-(\theta_1,\theta_2) = \bar{\lambda}_1(e^{\theta_1}-1) + \bar{\lambda}_2(e^{\theta_2}-1) + \mu^* r_{12}(e^{\theta_2-\theta_1}-1) + \mu^* r_{10}(e^{-\theta_1}-1).$$

Staying put rates are relaxed in M^+ and M^- since their coefficient is zero. MJN and a pure Jackson network (where the processors are not coupled) have the same M^+.

Let θ^b be the easternmost point on the graph of $M^+(\theta_1,\theta_2)=0$. Moreover, let θ^j be the non-zero intersection of $M^+(\theta_1,\theta_2)=0$ and $M^-(\theta_1,\theta_2)=0$ and finally define $\theta^c = (\ln(\rho_1^{-1}), \ln(\rho_2^{-1}))$. See Figure 1.6.

The following characterizations are essential for our analysis:

Theorem 1.17. *For the MJN:*

1. If $\theta_2^j < \min\{\ln(\rho_2^{-1}), \theta_2^b\}$, then the minimum action is θ_1^j and the minimal action path is a jitter.

2. If $\ln(\rho_2^{-1}) < \min\{\theta_2^j, \theta_2^b\}$, the minimum action is $\theta_1^c = \ln(\rho_1^{-1})$ and the minimal action path is a cascade.

3. If $\theta_2^b \le \theta_2^j$ and $\theta_2^b < \ln(\rho_2^{-1})$, then the minimum action is θ_1^b and the minimal action path is a bridge.

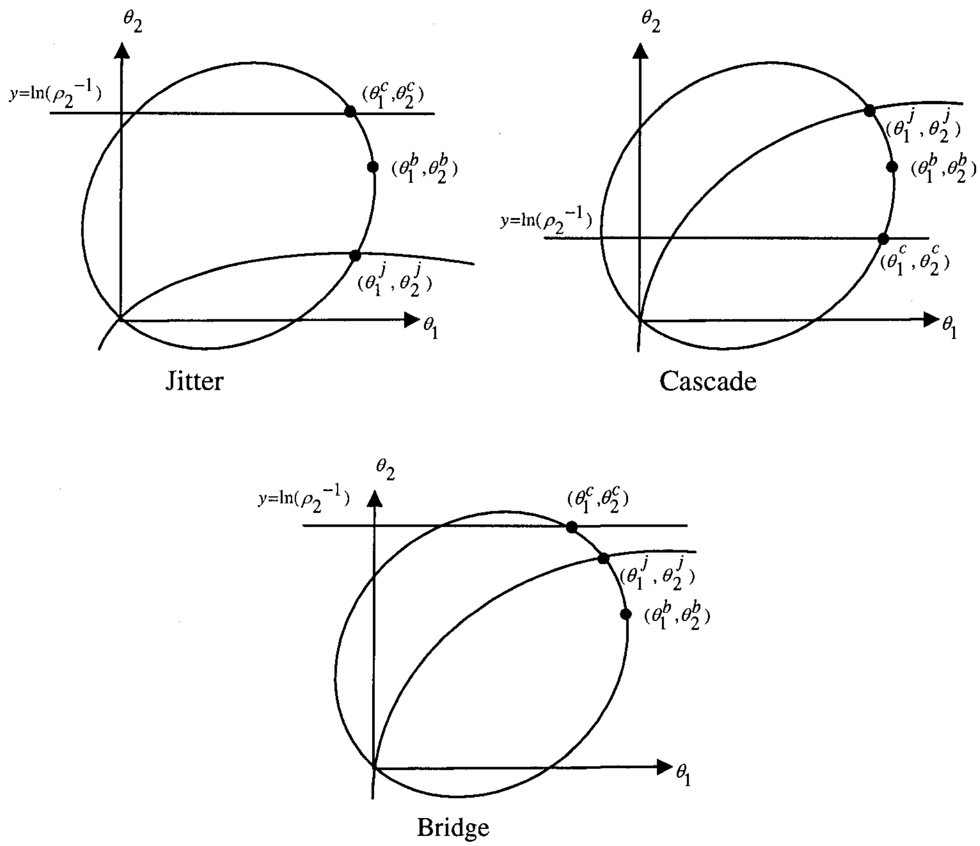


Figure 1.6: A rough sketch of $M^+ = 0$ (the egg shape) and $M^- = 0$ (the curve passing through the origin); top left to the right are possible graphs for jitter, cascade and bridge cases, respectively.

Chapter 2

Exact Asymptotics of the Steady State: Markov Additive Approach

In order to derive the exact asymptotics of the steady state as the level tends to infinity, we will introduce the Markov additive approach stemmed from the work of McDonald in [MCD99]. We will apply the model to the modified Jackson network (MJN) and quote some of the existing results.

Consider a stable Markov chain $X = (X_1, \hat{X})$ with kernel K in the state space $S \equiv \mathbb{Z}_+ \times \hat{S}$ where $\hat{S}$ is a countable space. For example, $\hat{S}$ can be $\mathbb{Z}_+^n$; so there is no restriction on the dimension of X. We assume that X satisfies the conditions described in the previous chapter, i.e. X can be embedded in a free Markov additive chain. π, K^*, S^∞, $X^\infty = (X_1^\infty, \hat{X}^\infty)$, K^∞, $h(x_1, \hat{x}) = \exp(\alpha x_1)\hat{h}(\hat{x})$, $\mathcal{X} = (\mathcal{X}_1, \hat{\mathcal{X}})$, $\mathcal{K}$, φ and $\overleftarrow{\mathcal{K}}$ have the same definitions as before.

There is one general approach to study all cases, as in [Fol05a], however many of the assumptions are unnecessary for the positive recurrent case. Therefore, we will go through the positive recurrent case in the next section and then discuss the approach in its general form.

2.1 Positive Recurrent Case

Suppose that $\hat{\mathcal{K}}$ is the kernel of a positive recurrent Markov chain. Hence, φ can be considered as a probability measure. We will go through the method to derive the asymptotics of the stationary distribution and relax one of the existing assumptions.

To obtain the exact asymptotics via Markov additive approach, we need the following result about φ and $\hat{h}$,

Proposition 2.1. *If the marginal Markov chain with kernel $\hat{\mathcal{K}}$ is positive recurrent then*

$$\sum_{\hat{x}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty.$$

Proof. As in [Ney87], define

$$K_{\alpha}^{\infty}((x_1,\hat{x});(y_1,\hat{y})) = K^{\infty}((x_1,\hat{x});(y_1,\hat{y}))\ e^{\alpha(y_1-x_1)}$$

for the fixed $\alpha > 0$, from the harmonic function h. Notice that

$$\begin{aligned}
&\sum_{(y_1,\hat{y})\in S^{\infty}} K_{\alpha}^{\infty}((x_1,\hat{x});(y_1,\hat{y}))\hat{h}(\hat{y}) \\
&= \sum_{(y_1,\hat{y})\in S^{\infty}} K^{\infty}((x_1,\hat{x});(y_1,\hat{y}))e^{\alpha(y_1-x_1)}\hat{h}(\hat{y}) \\
&= \hat{h}(\hat{x}) \sum_{(y_1,\hat{y})\in S^{\infty}} K^{\infty}((x_1,\hat{x});(y_1,\hat{y}))\frac{e^{\alpha y_1}\hat{h}(\hat{y})}{e^{\alpha x_1}\hat{h}(\hat{x})} \\
&= \hat{h}(\hat{x}) \sum_{(y_1,\hat{y})\in S^{\infty}} \mathcal{K}((x_1,\hat{x});(y_1,\hat{y})) = \hat{h}(\hat{x}),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{(x_1,\hat{x})\in S^{\infty}} (\varphi\hat{h}^{-1})(\hat{x})K_{\alpha}^{\infty}((x_1,\hat{x});(y_1,\hat{y})) \\
&= \hat{h}^{-1}(\hat{y}) \sum_{(x_1,\hat{x})\in S^{\infty}} \varphi(\hat{x})K^{\infty}((x_1,\hat{x});(y_1,\hat{y}))\frac{e^{\alpha y_1}\hat{h}(\hat{y})}{e^{\alpha x_1}\hat{h}(\hat{x})} \\
&= \hat{h}^{-1}(\hat{y}) \sum_{(x_1,\hat{x})\in S^{\infty}} \varphi(\hat{x})\mathcal{K}((x_1,\hat{x});(y_1,\hat{y})) = \hat{h}^{-1}(\hat{y})\varphi(\hat{y}).
\end{aligned}$$

Therefore, $\hat{h}$ and $\varphi\hat{h}^{-1}$ are respectively the right invariant vector and left invariant measure for K_α^∞. Since $\hat{\mathcal{K}}$ is 1-positive recurrent, we have $|\varphi| = \sum_{\hat{x}\in\hat{S}} \varphi(\hat{x}) < \infty$. But since the inner product of the left and right invariant vectors of K_α^∞ is finite, i.e.

$$|(\varphi\hat{h}^{-1})\cdot\hat{h}| = \sum_{\hat{x}}(\varphi\hat{h}^{-1})(\hat{x})\hat{h}(\hat{x}) = |\varphi| < \infty,$$

K_α^∞ is 1-positive recurrent by *Criterion III* in [Ver67]. Being 1-positive recurrent implies that the invariant measure is finite, therefore $\sum_{\hat{x}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty$. □

To push forward with the approach, the following assumptions are required:

A1. $0 < d < \infty$ where

$$d = \sum_{\hat{x}\in\hat{S}} \varphi(\hat{x}) E\{\mathcal{X}_1[1]\ |\mathcal{X}[0] = (0,\hat{x})\},$$

i.e. the first coordinate of the stationary version of the twisted chain has a finite, strictly positive drift.

We point out the fact that the mean d is well defined since φ is a distribution.

A2. $\sum_{z\in\Delta} \pi(z) P_z\{\mathcal{T}_\blacktriangle = \infty\} > 0$, where

$$\mathcal{T}_\blacktriangle = \min\{n > 0|\ \mathcal{X}[n] \in \blacktriangle\}.$$

This implies that there is a positive probability that the twisted chain starting from Δ never again hits $\blacktriangle$.

Notice that for $z \in \Delta$, $P_z(\mathcal{T}_\blacktriangle = \infty) > 0$ can happen if at the first step $\mathcal{X}$ jumps into the interior. Therefore, the sum in A2 only needs to be considered over those z's on the boundary that $\mathcal{K}(z; S - \Delta) > 0$.

A3. $\sum_{z\in\Delta} \pi(z) h(z) I\{K(z; S - \Delta) > 0\} < \infty$.

A3 restricts the trajectories that overload the level. Refer to page 51 for more detail.

Example 2.2 (Example 1.10 cont'd). Consider the $M/M/1$ queue described in Example 1.10 and let's check the above assumptions. A1 holds since the drift of $\mathcal{X}$ is $d = \mu - \lambda$ and this is strictly positive. $P_0(\mathcal{T}_\blacktriangle = \infty) > 0$, by the fact that $\mathcal{X}$ is transient and using A1. Therefore, A2 holds. $A3$ is also immediate since boundary consists of only one point.

Theorem 2.3 ([Fol01] page 582, Theorem 5). *Let $\hat{\mathcal{K}}$ be positive recurrent. Suppose that $\mathcal{X}_1$ has period $p = 1$. If X satisfies A1-A3 then as $\ell \to \infty$,*

$$\pi(\ell, \hat{y}) \sim f \frac{1}{d} e^{(-\alpha\ell)} \hat{h}^{-1}(\hat{y}) \varphi(\hat{y}),$$

where $f = \sum_{z \in \Delta} \pi(z) h(z) P_z\{\mathcal{T}_\blacktriangle = \infty\}$.

Sketch of the proof. First, notice that $f > 0$ by A2. The invariant probability measure at any point $(\ell, \hat{y})$ can be expressed by employing the number of visits to that state before returning to a fixed set. Let the fixed set be the boundary.

$$\begin{aligned} \pi(\ell, \hat{y}) &= \sum_{z \in \Delta} \pi(z)_\Delta G(z; (\ell, \hat{y})) \\ &= \sum_{z \in \Delta} \pi(z)_\blacktriangle G^\infty(z; (\ell, \hat{y})), \end{aligned} \tag{2.1}$$

definition of $_\Delta G$ is given in (1.7) and the last equality is valid since X and X^∞ have the same transitions if the destination state is in the interior. (2.1) only involves the free chain, for that reason we may twist the free chain,

$$e^{-\alpha\ell} \hat{h}^{-1}(\hat{y}) \sum_{z \in \Delta} \pi(z) h(z)_\blacktriangle \mathcal{G}(z; (\ell, \hat{y})). \tag{2.2}$$

(2.2) will serve an essential role on several occasions. $_\blacktriangle\mathcal{G}(z; (\ell, \hat{y}))$ is the expected number of visits by $\mathcal{X}$ to $(\ell, \hat{y})$ prior to the first return to $\blacktriangle$ starting from z. Condition this expected value on the first overshoot of level ℓ occurring at some state $(\ell + s, \hat{w})$,

$$_\blacktriangle\mathcal{G}(z; (\ell, \hat{y})) = \sum_{s \geq 0,\ \hat{w} \in \hat{S}} P_z\{\mathcal{X}[\mathcal{T}_\ell] = (\ell + s, \hat{w}), \mathcal{T}_\ell < \mathcal{T}_\blacktriangle\}_\blacktriangle \mathcal{G}((\ell + s, \hat{w}); (\ell, \hat{y})), \tag{2.3}$$

where

$$\mathcal{T}_\ell = \min\{n > 0 | \ \mathcal{X}[n] \in F_\ell\},$$

and $\mathcal{T}_\blacktriangle$ is the first return time to $\blacktriangle$ by $\mathcal{X}$.

In (2.3) as ℓ tends to infinity $\{\mathcal{T}_\ell < \mathcal{T}_\blacktriangle\}$ is independent from $\{\mathcal{X}[\mathcal{T}_\ell] = (\ell+s, \hat{w})\}$. By the argument in [Kes74], the latter has an invariant measure $\mu(s, \hat{w})$ (note that this is true only under the positive recurrent property). It follows that,

$$P_z\{\mathcal{X}[\mathcal{T}_\ell] = (\ell + s, \hat{w}), \mathcal{T}_\ell < \mathcal{T}_\blacktriangle\} \to P_z\{\mathcal{T}_\blacktriangle = \infty\}\mu(s, \hat{w}). \tag{2.4}$$

Also, ${}_\blacktriangle\mathcal{G}((\ell+s, \hat{w}); (\ell, \hat{y})) \to \mathcal{G}((\ell+s, \hat{w}); (\ell, \hat{y})) \leq \mathcal{G}((0, \hat{y}); (0, \hat{y}))$, applying dominated convergence theorem, A2 and A3, we have

$$e^{\alpha\ell}\pi(\ell, \hat{y}) \sim f\hat{h}^{-1}(\hat{y}) \sum_{s\geq 0,\ \hat{w}\in\hat{S}} \mu(s, \hat{w})\mathcal{G}((s, \hat{w}); (0, \hat{y})).$$

Bringing into play Lemma 5 in [Fol01], the above sum is equal to $\varphi(\hat{y})/d$. Finally, we get

$$e^{\alpha\ell}\pi(\ell, \hat{y}) \sim f\frac{1}{d}\hat{h}^{-1}(\hat{y})\varphi(\hat{y}).$$

□

Remark 2.4. As in [Fol01], the condition $p = 1$ can be relaxed, but then more technicalities are required. Throughout, we assume $p = 1$ for simplicity.

Example 2.5 (Example 2.2 cont'd). In the $M/M/1$ queue assume that $\mu + \lambda = 1$. So the transition rates can be looked as probabilities. $\hat{h}$ and φ are both constants, say equal to 1. We recognize the elements in the asymptotics of $\pi(\ell)$ as $f = \pi(0)P_0\{\mathcal{T}_\blacktriangle = \infty\} = \mu - \lambda$, $d = \mu - \lambda$ and $e^\alpha = \mu/\lambda$, therefore

$$\pi(\ell) \sim \pi(0)(\frac{\lambda}{\mu})^\ell,$$

which is the well-known result for birth and death chains.

Application to MJN.

We pursue the model introduced in Section 1.2. In order to find a function that is harmonic for the free chain both on the x-axis and above it, choose $h(\ell, \hat{x})$ to be $\exp(\theta_1^j \ell)\exp(\theta_2^j \hat{x})$, where (θ_1^j, θ_2^j) gives the coordinates of the non-zero intersection of $M^+ = 0$ and $M^- = 0$, see Figure 1.6. h is harmonic by the definition of $M^+ = 0$ and $M^- = 0$. Recognize α as θ_1^j and $\hat{h}(\hat{x})$ as $\exp(\theta_2^j \hat{x})$.

Following the twist of the kernel, the transitions for $\mathcal{K}$ are

Jump direction	Transition	
e_1	$\bar{\lambda}_1 e^{\theta_1^j}$	
e_2	$\bar{\lambda}_2 e^{\theta_2^j}$	
$-e_1 + e_2$	$\begin{cases} \mu_1 r_{12} e^{-\theta_1^j + \theta_2^j} & \hat{x} > 0 \\ \mu^* r_{12} e^{-\theta_1^j + \theta_2^j} & \hat{x} = 0 \end{cases}$	
$-e_1$	$\begin{cases} \mu_1 r_{10} e^{-\theta_1^j} & \hat{x} > 0 \\ \mu^* r_{10} e^{-\theta_1^j} & \hat{x} = 0 \end{cases}$	
$-e_2$	$\mu_2 r_{20} e^{-\theta_2^j}$	$\hat{x} > 0$
$e_1 - e_2$	$\mu_2 r_{21} e^{\theta_1^j - \theta_2^j}$	$\hat{x} > 0.$

Consequently, the Markovian part of the twisted chain has the following transitions if $\hat{x} > 0$,

$$\begin{aligned}
\hat{\mathcal{K}}(\hat{x}; \hat{x}+1) =& \bar{\lambda}_2 e^{\theta_2^j} + \mu_1 r_{12} e^{-\theta_1^j + \theta_2^j} \\
\hat{\mathcal{K}}(\hat{x}; \hat{x}-1) =& \mu_2 r_{20} e^{-\theta_2^j} + \mu_2 r_{21} e^{\theta_1^j - \theta_2^j} \\
\hat{\mathcal{K}}(\hat{x}; \hat{x}) =& \bar{\lambda}_1 e^{\theta_1^j} + \mu_1 r_{10} e^{-\theta_1^j} + (\mu^* - \mu_1),
\end{aligned}$$

on the other hand, if $\hat{x} = 0$,

$$\begin{aligned}
\hat{\mathcal{K}}(\hat{x}; \hat{x}+1) =& \bar{\lambda}_2 e^{\theta_2^j} + \mu^* r_{12} e^{-\theta_1^j + \theta_2^j} \\
\hat{\mathcal{K}}(\hat{x}; \hat{x}) =& \bar{\lambda}_1 e^{\theta_1^j} + \mu^* r_{10} e^{-\theta_1^j}.
\end{aligned}$$

Visibly, $\hat{\mathcal{K}}$ is positive recurrent if and only if

$$\rho \equiv \frac{\bar{\lambda}_2 e^{\theta_2^j} + \mu_1 r_{12} e^{-\theta_1^j + \theta_2^j}}{\mu_2 r_{20} e^{-\theta_2^j} + \mu_2 r_{21} e^{\theta_1^j - \theta_2^j}} < 1. \tag{2.5}$$

In this case, φ becomes

$$\varphi(\hat{x}) = \begin{cases} \left(1 + \frac{r}{1-\rho}\right)^{-1} r\rho^{\hat{x}-1} & \text{if } \hat{x} > 0 \\ \left(1 + \frac{r}{1-\rho}\right)^{-1} & \text{if } \hat{x} = 0, \end{cases}$$

where $r = \left(\bar{\lambda}_2 e^{\theta_2^j} + \mu^* r_{12} e^{-\theta_1^j + \theta_2^j}\right) / \left(\mu_2 r_{20} e^{\theta_2^j} + \mu_2 r_{21} e^{\theta_1^j - \theta_2^j}\right)$.

We now check assumptions A1-A3. Pick the parameters of the network such that A1 is satisfied. A2 holds automatically. In order to show A3, the network parameter ρ_2 needs to satisfy

$$\rho_2 < \exp(-\theta_2^j), \tag{2.6}$$

which is true due to part 1 of Theorem 1.17.

Using Proposition 1.14,

$$\begin{aligned} \sum_{z \in \Delta} \pi(z) h(z) I\{K(z; S - \Delta) > 0\} &= \sum_{\hat{z}} \pi(0, \hat{z}) e^{\theta_2^j \hat{z}} \\ &\le c \sum_{\hat{z}} \rho_2^{\hat{z}} e^{\theta_2^j \hat{z}} < \infty, \end{aligned} \tag{2.7}$$

for some constant c. Notice that the last series is convergent by (2.6).

Therefore, Theorem 2.3 gives the asymptotics of the steady state for the MJN in the jitter positive recurrent cases,

$$\pi(\ell, \hat{y}) \sim f \frac{1}{d} e^{(-\theta_1^j \ell)} e^{(-\theta_2^j \hat{y})} \varphi(\hat{y}). \tag{2.8}$$

2.2 A General Theory

Throughout this section, we ease the condition that φ is a distribution. As a result, we may have $\sum_{\hat{x} \in \hat{S}} \varphi(\hat{x}) = \infty$ and $\hat{\mathcal{X}}$ can possibly be null recurrent or transient. This relaxation would end the existence of the invariant distribution μ in the proof of

Theorem 2.3, henceforth we have to pursue a new methodology. To do so, the results in [Fol05a] are reviewed briefly.

The subsequent assumptions are required.

D1. h is chosen such that $P\{\lim_{n\to\infty} \mathcal{X}_1[n] = \infty \mid \mathcal{X}[0] = z\} = 1$ for all $z \in S^\infty$.

This assumption substitutes A1 in the previous section. Notice that A1 can not be used at this point since the stationary version of the twisted chain does not exist in general and the mean drift may have no meaning.

D2. There exists an $\alpha_0 > \alpha$ and M such that

$$E\{e^{\alpha_0(X_1^\infty[1]-X_1^\infty[0])} | \, X^\infty[0], \hat{X}^\infty[1]\} < M,$$

for any $X^\infty[0]$ and $\hat{X}^\infty[1]$.

This assumption is necessary to make sure that the level jumps have exponentially light tail.

D3. $\mathcal{K}$ is irreducible in the sense that the probability $p((0,\hat{x});(0,\hat{y}))$ that $\mathcal{X}$ goes from $(0,\hat{x})$ to $(0,\hat{y})$ is positive for any $\hat{x},\hat{y} \in \hat{S}$. Moreover, there exists an integer N and $\delta > 0$ fixed such that for any $\hat{y}$ there exists an integer $m = m(\hat{y})$ such that $1 \le m \le N$ and $\mathcal{K}^{(m)}((0,\hat{y});(1,\hat{y})) \ge \delta$.

D4. Define $T_{\hat{\sigma}}^\infty$ to be the number of steps for $\hat{X}^\infty$ to return to some state $\hat{\sigma} \in \hat{S}$. We assume that the distribution of $P_{(0,\hat{\sigma})}(X_1^\infty[T_{\hat{\sigma}}^\infty] = \cdot)$ is not concentrated on a proper subgroup of the integers.

This assumption implies that $\mathcal{X}_1$ is aperiodic which turns out to be the hypothesis in Theorem 2.3.

D5. $\hat{\mathcal{K}}$ has spectral radius 1, i.e. for any $\hat{x}$ and $\hat{y}$,

$$\lim_{n\to\infty}[\hat{\mathcal{K}}^{(n)}(\hat{x};\hat{y})]^{1/n} = 1$$

or

the radius of convergence of the series $\sum_n \hat{\mathcal{K}}^{(n)}(\hat{x};\hat{y})z^n$ is 1.

D6. We assume that constant functions are the unique non-negative harmonic functions for $\hat{\mathcal{K}}$.

This is the strong Liouville property; see [Woe00, page 265] . Note that this condition automatically holds when $\hat{\mathcal{K}}$ is recurrent.

D7a. There exists a state $\hat{a}$ and a function $\hat{f}$ such that uniformly in ℓ,

$$\mathcal{G}\left((0,\hat{x});(\ell,\hat{a})\right)/\mathcal{G}\left((0,\hat{a});(\ell,\hat{a})\right) \le \hat{f}(\hat{x})$$

where

$$\sum_{\hat{x}\in\hat{S}} \hat{\mathcal{K}}(\hat{z};\hat{x})\hat{f}(\hat{x}) < \infty$$

for all states $\hat{z}$.

D7b. Part (a) holds for the Green function of $(-\overleftarrow{\mathcal{X}}_1, \overleftarrow{\hat{\mathcal{X}}})$ where $(\overleftarrow{\mathcal{X}}_1, \overleftarrow{\hat{\mathcal{X}}})$ is the time reversal of $(\mathcal{X}_1, \hat{\mathcal{X}})$ with respect to φ (see Proposition 1.8).

A sufficient condition for the two preceding assumptions is that $(X_1^\infty, \hat{X}^\infty)$ has bounded jumps. To see this, let $p(x;a)$ be the probability of ever going from x to a by $\mathcal{X}$. Clearly,

$$p((0,\hat{a});(\ell,\hat{a})) \ge p((0,\hat{a});(0,\hat{x}))p((0,\hat{x});(\ell,\hat{a})).$$

Hence

$$\frac{\mathcal{G}\left((0,\hat{x});(\ell,\hat{a})\right)}{\mathcal{G}\left((0,\hat{a});(\ell,\hat{a})\right)} = \frac{p\left((0,\hat{x});(\ell,\hat{a})\right)}{p\left((0,\hat{a});(\ell,\hat{a})\right)} \le \frac{1}{p((0,\hat{a});(0,\hat{x}))}.$$

But

$$\sum_{\hat{x}\in\hat{S}} \hat{\mathcal{K}}(\hat{z};\hat{x})\frac{1}{p((0,\hat{a});(0,\hat{x}))} < \infty$$

because starting from $\hat{z}$, $\hat{\mathcal{K}}$ has finite range.

For the previous condition to hold, it therefore suffices that the range of $\mathcal{K}((0,\hat{z});\ (\cdot,\cdot))$ is finite for all $\hat{z}$.

The above assumptions make it feasible to prove the following theorem:

Theorem 2.6 ([Fol05a] page 559, Theorem 1). *Under assumptions D1-D7 with fixed $z, w, \hat{y}$ and $\hat{x}$,*

$$\frac{\mathcal{G}(z;(\ell,\hat{x}))}{\mathcal{G}(w;(\ell,\hat{y}))} \sim \frac{\varphi(\hat{x})}{\varphi(\hat{y})}.$$

The above result will play a crucial role in obtaining the asymptotics of the steady state. Note that in the non-positive recurrent cases as $\ell \to \infty$, $\mathcal{G}(z;(\ell,\hat{x}))$ tends to zero for any fixed z and $\hat{x}$. Therefore, the key step (2.4) in the proof of Theorem 2.3 is not applicable in general. So we use the ratio of the Green functions in this case.

In order to obtain the asymptotics, the following assumptions are required as well.

D8. We assume that for some fixed state $\hat{a}$,

$$\frac{\mathcal{G}(w;(\ell,\hat{a}))}{\mathcal{G}((0,\hat{a});(\ell,\hat{a}))} \text{ is bounded uniformly in } w \in \blacktriangle \text{ for } \ell \text{ sufficiently large.}$$

D9. There exists a subset $C \subseteq \Delta$ such that $\pi(C) > 0$ and such that, for $z \in C$, $P_z(\mathcal{T}_\blacktriangle = \infty) > 0$ where $\mathcal{T}_\blacktriangle$ is the first return time to $\blacktriangle$ by $\mathcal{X}$.

This is assumption A2 in the positive recurrent case.

D10. $\lambda(x) \equiv \pi(x)h(x)\chi\{x \in \Delta\}$ is a finite measure.

We recognize D10 as assumption A3 in the previous section.

D10 has more implications regarding overloading trajectories of a network that will be discussed after the Theorem. Notice that some of the assumptions are used to cope with the possibility of unbounded jumps for the level and the phase.

Theorem 2.7 ([Fol05a] page 573, Theorem 4). *Assuming D1-D10, for any phases $\hat{y}$ and $\hat{b}$ such that $\varphi(\hat{b}) > 0$,*

$$e^{\alpha\ell}\pi(\ell,\hat{y}) \sim f\hat{h}^{-1}(\hat{y})\varphi(\hat{y})\frac{\mathcal{G}((0,\hat{b});(\ell,\hat{b}))}{\varphi(\hat{b})} \tag{2.9}$$

where $f = \sum_{z\in\Delta} \pi(z)h(z)P_z(\mathcal{T}_\blacktriangle = \infty)$.

Sketch of the proof. Notice that f is finite because of D10. The proof starts the same as proof of Theorem 2.3 up to the (2.2), which is equal to

$$\begin{aligned} & e^{-\alpha\ell}\hat{h}^{-1}(\hat{y})\sum_{z\in\Delta}\pi(z)h(z)_{\blacktriangle}\mathcal{G}(z;(\ell,\hat{y})) \\ =& e^{-\alpha\ell}\hat{h}^{-1}(\hat{y})\mathcal{G}((0,\hat{y});(\ell,\hat{y}))\sum_{z\in\Delta}\pi(z)h(z)\frac{{}_{\blacktriangle}\mathcal{G}(z;(\ell,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell,\hat{y}))}. \end{aligned}$$

As a result of D8, $\frac{{}_{\blacktriangle}\mathcal{G}(z;(\ell,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell,\hat{y}))}$ is bounded uniformly in ℓ and $z \in \Delta$. The dominated convergence theorem can be used for the series to show that

$$\sum_{z\in\Delta}\pi(z)h(z)\frac{{}_{\blacktriangle}\mathcal{G}(z;(\ell,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell,\hat{y}))} \to f.$$

On the other hand, using Theorem 2.6, if $\varphi(\hat{b}) > 0$ then

$$\mathcal{G}((0,\hat{y});(\ell,\hat{y})) \sim \varphi(\hat{y})\frac{\mathcal{G}((0,\hat{b});(\ell,\hat{b}))}{\varphi(\hat{b})},$$

and this finishes the proof. □

Remark 2.8. If $\hat{\mathcal{K}}$ is positive recurrent, Theorem 2.3 and the previous theorem together imply that

$$\frac{\varphi(\hat{b})}{\mathcal{G}((0,\hat{b});(\ell,\hat{b}))} \to d,$$

where d is the mean drift of $\mathcal{X}_1$ introduced in A1.

Theorem 2.7 brings us one step closer to determining the asymptotics of the steady state. The only unknown decaying term in (2.9) is the Green function. On the other hand, assumption D10 is restrictive. Considering a two dimensional network, this assumption fails for the cases where the overloading trajectories first climb the y-axis and then overload the x-axis, as in Theorem 1.17 part 2. See page 51 for details.

Although the next proposition will not be used later, it shows the potential of the current approach through the fact that $\overleftarrow{\mathcal{K}}$ approximates K^*.

Proposition 2.9. *Under assumptions of Theorem 2.7, for fixed phases $\hat{x}$ and $\hat{y}$ and fixed displacement $m \geq 0$:*

$$\lim_{\substack{x_1, y_1 \to \infty \\ |x_1 - y_1| = m}} \frac{\overleftarrow{\mathcal{K}}((y_1,\hat{y});(x_1,\hat{x}))}{K^*((y_1,\hat{y});(x_1,\hat{x}))} = 1$$

Proof. Begin with reversing $\overleftarrow{\mathcal{K}}$ using φ,

$$\begin{aligned}
\overleftarrow{\mathcal{K}}((y_1,\hat{y});(x_1,\hat{x})) &= \frac{\varphi(\hat{x})\mathcal{K}((x_1,\hat{x});(y_1,\hat{y}))}{\varphi(\hat{y})} \\
&= \frac{\varphi(\hat{x})K^\infty((x_1,\hat{x});(y_1,\hat{y}))h(y_1,\hat{y})}{\varphi(\hat{y})h(x_1,\hat{x})} \\
&= \frac{\varphi(\hat{x})K((x_1,\hat{x});(y_1,\hat{y}))h(y_1,\hat{y})}{\varphi(\hat{y})h(x_1,\hat{x})} \qquad K = K^\infty \text{ on } S - \Delta \\
&= \frac{\varphi(\hat{x})K((x_1,\hat{x});(y_1,\hat{y}))e^{-\alpha x_1}\hat{h}^{-1}(\hat{x})}{\varphi(\hat{y})e^{-\alpha_1 y_1}\hat{h}^{-1}(\hat{y})}.
\end{aligned}$$

For the K^* we have:

$$\begin{aligned}
K^*((y_1,\hat{y});(x_1,\hat{x})) &= \frac{\pi(x_1,\hat{x})K((x_1,\hat{x});(y_1,\hat{y}))}{\pi(y_1,\hat{y})} \\
&\sim \frac{\mathcal{G}((0,\hat{b});(x_1,\hat{b}))}{\mathcal{G}((0,\hat{b});(y_1,\hat{b}))} \times \frac{\varphi(\hat{x})K((x_1,\hat{x});(y_1,\hat{y}))e^{-\alpha x_1}\hat{h}^{-1}(\hat{x})}{\varphi(\hat{y})e^{-\alpha y_1}\hat{h}^{-1}(\hat{y})},
\end{aligned}$$

for sufficiently large x_1 and y_1 and using representation (2.9). This leaves us with

$$K^*((y_1,\hat{y});(x_1,\hat{x})) \sim \frac{\mathcal{G}((0,\hat{b});(x_1,\hat{b}))}{\mathcal{G}((0,\hat{b});(y_1,\hat{b}))} \times \overleftarrow{\mathcal{K}}((y_1,\hat{y});(x_1,\hat{x})),$$

but $\mathcal{G}((0,\hat{b});(x_1,\hat{b}))/\mathcal{G}((0,\hat{b});(y_1,\hat{b}))$ converges to 1 using Theorem 2.6. □

Application to MJN.

We will only address the non-positive recurrent cases. The positive recurrent case can be done analogously and it will crudely match the results in Section 2.1.

Referring to the last part of Theorem 1.17, consider the case where the least action path is a bridge and the minimal action is θ_1^b. Observe Figure 1.6 for the position of $\theta^b = (\theta_1^b, \theta_2^b)$ on the graph of $M^+ = 0$. Define

$$h_0(\ell, \hat{x}) = e^{\theta_1^b \ell} e^{\theta_2^b \hat{x}}.$$

By the definition of M^+ and since θ^b satisfies $M^+ = 0$, h_0 is harmonic for K^∞ above the level axis, i.e. for any $x = (x_1, \hat{x})$ that $\hat{x} > 0$,

$$\sum_{y=(y_1,\hat{y})} K^\infty(x; y) h_0(y) = h_0(x).$$

However, this may not be the case on the first axis as there is no guaranty that $M^- = 0$ passes through θ^b. Henceforth, the following two cases should be dealt with:

Case I: On the x-axis, the h_0-twist results in a killing probability, κ, which is strictly positive. To be precise, define

$$\mathcal{K}_0(x; y) = K^\infty(x; y) \times \frac{h_0(y)}{h_0(x)},$$

so the current case happens if $\kappa = 1 - \sum_y \mathcal{K}_0((\ell, 0); y) > 0$, for any $\ell \in \mathbb{Z}$.

Case II: h_0 is also harmonic on the level axis, i.e. the above defined κ is equal to zero.

The next step would be to introduce the harmonic function, h, and the invariant measure, φ, for the above cases. Define

$$\hat{\mathcal{K}}_0(\hat{x}; \hat{y}) = \sum_\ell \mathcal{K}_0((0, \hat{x}); (\ell, \hat{y})).$$

Let $p_0 = \hat{\mathcal{K}}_0(0; 1)$ and $u = \hat{\mathcal{K}}_0(\hat{x}; \hat{x} + 1)$ for $\hat{x} > 0$. Note that u is also equal to $d = \hat{\mathcal{K}}_0(\hat{x}; \hat{x} - 1)$ for any $\hat{x} > 0$. This is true since θ^b is exactly the easternmost point of $M^+ = 0$ and furthermore,

$$\frac{\partial M^+(\theta^b)}{\partial \theta_2} = 0.$$

For the first case where $\kappa > 0$, direct calculation implies that h is harmonic for K^∞ where,

$$h(\ell, \hat{x}) = h_0(\ell, \hat{x}) \times (1 + \kappa \hat{x} / p_0)$$

$$= \exp(\theta_1^b \ell) \exp(\theta_2^b \hat{x})(1 + \kappa \hat{x}/p_0). \tag{2.10}$$

Similar to the aforementioned notations

$$\mathcal{K}(x; y) = K^\infty(x; y) \times \frac{h(y)}{h(x)},$$

and

$$\hat{\mathcal{K}}(\hat{x}; \hat{y}) = \sum_\ell \mathcal{K}((0, \hat{x}); (\ell, \hat{y})),$$

hence,

$$\varphi(\hat{x}) = \begin{cases} 1 & \text{if } \hat{x} = 0, \\ \frac{p_0}{u}(1 + \kappa \hat{x}/p_0)^2 & \text{otherwise.} \end{cases} \tag{2.11}$$

Notice that $\sum_{\hat{x}} \varphi(\hat{x}) = \infty$.

For the second case, obviously h can be taken equal to h_0 and $\varphi \equiv 1$. Similar to the first case, φ is not summable. Moreover, $\hat{\mathcal{K}}(\hat{x}; \hat{x}+1) = \hat{\mathcal{K}}(\hat{x}; \hat{x}-1)$ for any $\hat{x} > 0$ and therefore in the second scenario $\hat{\mathcal{K}}$ is null recurrent.

At this instance, we check assumptions D1 to D10. Assumption D1 is satisfied by a choice of parameters.

D2 is immediate since X^∞ has bounded jumps both in level and phase direction.

For D3, note that $p((0, \hat{x}); (0, \hat{y}))$ is larger than the probability of going from $(0, \hat{x})$ to $(0, \hat{y})$ along the y-axis which is strictly positive. For the second property in the assumption, observe that $\mathcal{K}((0, \hat{y}); (1, \hat{y})) \geq \bar{\lambda}_1 e^{\theta_1^b} > 0$, for any $\hat{y}$. This also proves D4 since $\hat{\sigma}$ can be reached in one step.

As for D5, observe that if we ignore staying put then $\hat{\mathcal{K}}_0$ is a symmetric nearest neighbor walk and therefore has spectral radius 1, [Woe00, Lemma 1.26]. Now, applying the last step of the twist, we get the same spectral radius for $\hat{\mathcal{K}}$. Refer to [Ign01a] for a detailed discussion on how our choice of θ^b secures D5.

Similar to D5, the same property implies D6. In detail, $\hat{\mathcal{K}}_0$ is a symmetric nearest neighbor walk and therefore its only harmonic functions are constants. Adding the last linear term $(1+\kappa\hat{\cdot}/p_0)$ to the twist does not alter the restriction on harmonic functions.

Since the chain can jump only to finite number of states starting from any point, D7 is immediate by the argument after D7b.

[Fol05a, page 582] gives the detailed reason for D8 using a coupling argument. We will not reproduce the argument here, but remark that intuitively $\mathcal{G}(w;(\ell,\hat{a}))$ is less than $\mathcal{G}((0,\hat{a});(\ell,\hat{a}))$ for large ℓ's, since starting from w the Markovian part of the twisted chain has an upward tendency.

D9 is immediate from the fact that the level has strictly positive drift starting from any state.

For D10 to hold,

$$\sum_{z\in\Delta}\pi(z)h(z)I\{K(z;S-\Delta)\}=\sum_{\hat{z}}\pi(0,\hat{z})e^{\theta_2^b\hat{z}}(1+\kappa\hat{z}/p_0),$$

remembering the inequality in Proposition 1.14, this is finite if and only if

$$\sum_{\hat{z}}\rho_2^{\hat{z}}e^{\theta_2^b\hat{z}}<\infty,$$

this is equivalent to having $e^{\theta_2^b}<\rho_2^{-1}$ which is true by part 3 of Theorem 1.17.

Previous to applying Theorem 2.7, we provide the asymptotics of the Green function for the MJN quoting [Fol05a, Proposition 1],

$$\mathcal{G}((0,0);(\ell,0))\sim\begin{cases}C_+\ell^{-3/2} & \text{for case I in which } \kappa>0,\\ C_0\ell^{-1/2} & \text{for case II in which } \kappa=0,\end{cases}\tag{2.12}$$

where C_+ and C_0 are explicitly given.

The asymptotics of the steady state for the non-positive recurrent cases, when cascade trajectories are forbidden, is

$$\pi(\ell,\hat{y})\sim\begin{cases}fC_+e^{-\theta_1^b\ell}\ell^{-3/2}e^{-\theta_2^b\hat{y}}\frac{p_0}{u}(1+\kappa\hat{y}/p_0) & \text{if } \kappa>0,\\ fC_0e^{-\theta_1^b\ell}\ell^{-1/2}e^{-\theta_2^b\hat{y}} & \text{if } \kappa=0.\end{cases}\tag{2.13}$$

2.3 Restriction

A restriction of the Markov additive approach is assumption D10 (or A3 equivalently). This assumptions happens to fail for the cascade cases in a two dimensional network, where the overloading trajectories first climb the y-axis and then cascade onto the x-axis to overload the level. Refer to page 51 for details. We will introduce a new approach in the next chapter to resolve this issue and give the asymptotics of the steady state when D10 does not hold.

Chapter 3

Exact Asymptotics: a New Approach

We will introduce a new approach to investigate the asymptotics of the steady state when one coordinate gets large, i.e. determine $\lim_{\ell\to\infty}\pi(\ell,\hat{y})$ for any fixed $\hat{y}$. The methodology is derived from the technique we use in the next chapter to determine the asymptotic distribution of the phase and happens to be very effective. This new method can be applied to a wide range of networks and in addition does not require condition D10 of the preceding chapter. In spite of this, we will only illustrate how it can be applied to the cascade case of the modified Jackson network which was described in Section 1.2. We will also show that the network under analysis does not satisfy D10, and hence the Markov additive approach does not apply. For another recent exposition of the approach refer to [Ada05], where the technique is applied to a Markov chain arising in a production context.

3.1 The Cascade Case

Consider the MJN. Let $X = (X_1, \hat{X})$ in $\mathbb{Z}_+ \times \mathbb{Z}_+$ be the joint queue length of customers waiting or being served with invariant probability measure (steady state) $\pi(\ell, \hat{x})$. Suppose the minimum action path is a cascade. Shortly, it will be shown that the minimum action path first jitters along the y-axis and then overloads the level. This circumstance happens if $\ln(\rho_2^{-1}) < \min\{\theta_2^j, \theta_2^b\}$ and henceforth the minimum action shall be $\theta_1^c = \ln(\rho_1^{-1})$, refer to Theorem 1.17 and Figure 1.6.

Analysis of the parameters.

Theorem 3.1 ([Fol05b] page 538, Theorem 3). *θ_1^c is the minimum action if and only if*

$$\rho_1^{-1} > r_{10} + r_{12}\rho_2^{-1} \tag{3.1}$$

$$\ln(\rho_2^{-1}) < \theta_2^b. \tag{3.2}$$

In addition to the above theorem, the following result will be required. It presents the necessary and sufficient conditions for the cascade case expressed only in terms of the system's traffic parameters:

Theorem 3.2. *The minimum action path is a cascade if and only if*

$$\rho_1^{-1} > r_{10} + r_{12}\rho_2^{-1}$$

and

$$\rho_2^{-1} < r_{20} + r_{21}\rho_1^{-1}. \tag{3.3}$$

Notice that the first inequality is (3.1).

Proof. Suppose the inequalities in the Theorem hold, we show that the minimum action is a cascade. By Theorem 3.1, the minimum action is a cascade if $\rho_2^{-1} < \exp(\theta_2^b)$. Assume the contrary,

$$\exp(\theta_2^b) \leq \rho_2^{-1} < r_{20} + r_{21}\rho_1^{-1}. \tag{3.4}$$

Consider θ_1 as a function of θ_2, $\theta_1 = f(\theta_2)$, defined by $M^+(f(\theta_2), \theta_2) = 0$. Take Θ to be the supremum of the values such that f is a function on $[0, \Theta]$. By [Fol05b, Theorem 1] and as it can be seen in Figure 1.6, θ^b is the unique point that $\theta_1 > 0$ and

$$\frac{df}{d\theta_2}(\theta_2^b) = 0.$$

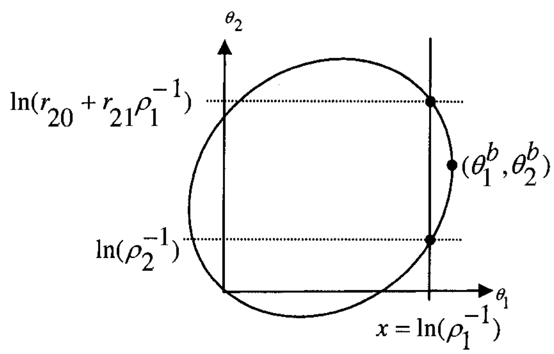
A direct calculation shows that the equation $M^+(\ln(\rho_1^{-1}), x) = 0$ has two solutions $x = \ln(\rho_2^{-1})$ and $x = \ln(r_{20} + r_{21}\rho_1^{-1})$. But since

$$\ln(\rho_1^{-1}) = f(\ln(\rho_2^{-1})) = f(\ln(r_{20} + r_{21}\rho_1^{-1})),$$

there exists another point $\theta^* = (\theta_1^*, \theta_2^*)$ on $M^+ = 0$, such that $\frac{df}{d\theta_2}(\theta_2^*) = 0$ and

$$\rho_2^{-1} < \exp(\theta_2^*) < r_{20} + r_{21}\rho_1^{-1}.$$

By (3.4), $\theta_2^b < \theta_2^*$ and moreover, $\frac{df}{d\theta_2}(\theta_2^b) = \frac{df}{d\theta_2}(\theta_2^*) = 0$ and this contradicts the uniqueness of θ^b. The correct position of the parameters is shown below.



Hence, $\exp(\theta_2^b) \leq \rho_2^{-1}$ is impossible. $\exp(\theta_2^b) > \rho_2^{-1}$ together with the first inequality of the Theorem imply that the minimum action path is a cascade.

Now, suppose the minimum action is a cascade. The first inequality is (3.1). To prove (3.3), repeat the above reasoning for the function f, the only possible outcomes are,

$$\rho_2^{-1} < e^{\theta_2^b} < r_{20} + r_{21}\rho_1^{-1},$$

$$\rho_2^{-1} = r_{20} + r_{21}\rho_1^{-1}.$$

To rule out the equality, notice that $\ln(\rho_1^{-1}) \le \theta_1^b$ where θ^b is the eastern most point on $M^+ = 0$. Therefore, the line $x = \ln(\rho_1^{-1})$ naturally crosses $M^+ = 0$ at two points. Solving $M^+(\ln(\rho_1^{-1}), \theta_2) = 0$, θ_2 can be either $r_{20} + r_{21}\rho_1^{-1}$ or ρ_2^{-1}. Hence the equality happens if $\ln(\rho_1^{-1}) = \theta_1^b$ and $\rho_2^{-1} = r_{20} + r_{21}\rho_1^{-1} = \exp(\theta_2^b)$, which again contradicts the fact that the minimum path is a cascade. □

Lemma 3.3. *If the minimum action path is a cascade then* $\rho_1 < 1$.

Proof. Stability Proposition 1.13 implies that $\rho_2 < 1$. Therefore, using (3.1),

$$\begin{aligned}\rho_1^{-1} &> r_{1,0} + r_{1,2}\rho_2^{-1} \\ &> r_{1,0} + r_{1,2} = 1.\end{aligned}$$

□

Asymptotics of $\pi(0, \hat{y})$ as $\hat{y} \to \infty$.

Our objective at this point is to find the asymptotics of $\pi(0, \hat{y})$ when horizontally the minimum action path is a cascade or equivalently derive a function $A(\hat{y})$ such that

$$\lim_{\hat{y}\to\infty} \frac{\pi(0, \hat{y})}{A(\hat{y})} = 1.$$

To do so, we will use the Markov additive approach developed in the previous chapter. The following result from [Fol01] is essential.

Theorem 3.4. $\theta_2^j < \theta_2^b$ *if and only if* $\rho < 1$ *where*

$$\rho = \frac{\bar{\lambda}_2 e^{\theta_2^j} + \mu_1 r_{12} e^{-\theta_1^j + \theta_2^j}}{\mu_2 r_{20} e^{-\theta_2^j} + \mu_2 r_{21} e^{\theta_1^j - \theta_2^j}}. \tag{3.5}$$

We recognize ρ as the same parameter we encountered in (2.5), where the positive recurrent case was under investigation.

Corollary 3.5. *For the pure Jackson network (where* $\mu^* = \mu_1$*), a jitter path is better than a bridge (the cheapest path is a jitter rather than a bridge) if*

$$\rho_2^{-1} > r_{20} + r_{21}\rho_1^{-1}. \tag{3.6}$$

Our immediate task is to show that a minimal cascade path can not consist of a bridge up the y-axis followed by a cascade. The rational is that such a path is not optimal since this would give a nonlinear large deviations path in a domain with constant jump rates.

Corollary 3.6. *If the minimal action path is a cascade then the first part of the path jitters along the y-axis and does not form a bridge.*

Proof. Look at the system as if the second queue is overloading. Using index v for the vertical viewpoint, $M_v^+ = 0$ and $M_v^- = 0$ equations are exactly the same as those of a pure Jackson network. Therefore, Corollary 3.5 can be used directly, the only alternation is to switch the parameters since we are looking at the network vertically. Hence, vertically system jitters along the y-axis and does not form a bridge if $\rho_1^{-1} > r_{10}+r_{12}\rho_2^{-1}$, which is inequality (3.1) and this holds since the minimum action path is a cascade. $\square$

In view of the fact that we are interested in $\pi(0,\hat{y})$ as $\hat{y}$ tends to infinity, we apply the Markov additive approach vertically to the network. Define the boundary to be

$$\Delta_v = \{(\ell,0)|\ \ell \in \mathbb{Z}_+\} \cup \{(\ell,1)|\ \ell \in \mathbb{Z}_+\}.$$

Removing this boundary the chain can be imbedded in a Markov additive chain, X_v^{∞}, on $\mathbb{Z}_+ \times \mathbb{Z}$. The free chain is precisely that of a pure Jackson network. The harmonic function for this chain can be found by looking at the non-zero intersection of $M_v^+ = 0$ and $M_v^- = 0$, which is $(\ln(\rho_2^{-1}), \ln(r_{10}+r_{12}\rho_2^{-1}))$. Therefore, the harmonic function is

$$h_v(x_1,\hat{x}) = \rho_2^{-\hat{x}}(r_{10}+r_{12}\rho_2^{-1})^{x_1}.$$

We recognize $\ln(\rho_2^{-1})$ as α of the preceding chapters and notice that as required $\alpha_v = \ln(\rho_2^{-1}) > 0$ since by Theorem 1.13, $\rho_2 < 1$. Let $\mathcal{X}_v$ be the twist of X_v^{∞} with respect to $h_v(x_1,\hat{x})$.

Corollary 3.6 ensures that positive recurrent results discussed in Section 2.1 can be used. However, we need to check the conditions A1-A3. Having in mind the notations used on page 29, define

$$r_v = \frac{\bar{\lambda}_1(r_{10} + r_{12}\rho_2^{-1}) + \mu_2 r_{21}\frac{r_{10}+r_{12}\rho_2^{-1}}{\rho_2^{-1}}}{\mu_1 r_{10}(r_{10} + r_{12}\rho_2^{-1}) + \mu_1 r_{12}\frac{\rho_2^{-1}}{r_{10}+r_{12}\rho_2^{-1}}}.$$

and

$$\varphi_v(0) = \left(1 + \frac{r}{1-r}\right)^{-1}.$$

In addition, let d_v be the following mean drift:

$$d_v = \varphi_v(0)(\bar{\lambda}_2\rho_2^{-1} - \mu_2 r_{21}\frac{r_{10} + r_{12}\rho_2^{-1}}{\rho_2^{-1}} - \mu_2 r_{20}\frac{1}{\rho_2^{-1}})$$
$$+ (1 - \varphi_v(0))(\bar{\lambda}_2\rho_2^{-1} + \mu_1 r_{12}\frac{\rho_2^{-1}}{r_{10} + r_{12}\rho_2^{-1}} - \mu_2 r_{21}\frac{r_{10} + r_{12}\rho_2^{-1}}{\rho_2^{-1}} - \mu_2 r_{20}\frac{1}{\rho_2^{-1}}).$$

Notice that $d_v > 0$ by the choice of $\alpha = \ln(\rho_2^{-1}) > 0$ and therefore A1 holds. To see A2, having in mind the discussion after the assumption and the proof of Theorem 2.3, we do not use the whole boundary Δ_v in the definition of f_v, rather we take into account only those states that can reach the interior in one step, hence

$$f_v = \sum_{\hat{z}} \pi(1, \hat{z}) P_{(1,\hat{z})}\{\mathcal{T}_{\Delta_v} = \infty\},$$

where $\mathcal{T}_{\Delta_v}$ is the first return time of $\mathcal{X}_v$ to Δ_v. Automatically, A2 holds. For A3, a vertical reproduction of Proposition 1.14 implies that $\pi(\ell, 1) \leq c\rho_1^\ell$ for some constant c, consequently

$$\begin{aligned}\sum_{z\in\Delta_v} \pi(z)h_v(z)I\{K(z; S - \Delta_v)\} &= \sum_{\ell\in\mathbb{Z}_+} \pi(\ell, 1)h_v(\ell, 1)\\ &= \sum_{\ell\in\mathbb{Z}_+} \pi(\ell, 1)\rho_2^{-1}(r_{10} + r_{12}\rho_2^{-1})^\ell\\ &\leq \sum_{\ell\in\mathbb{Z}_+} c\rho_1^\ell(r_{10} + r_{12}\rho_2^{-1})^\ell\end{aligned}$$

and this is finite since $\rho_1(r_{10} + r_{12}\rho_2^{-1}) < 1$ by (3.1). Thus all the assumptions hold.

Finally, the asymptotics of $\pi(0, \hat{y})$ is given by Theorem 2.3 as

$$\pi(0, \hat{y}) \sim f_v \frac{1}{d_v}\varphi_v(0)\rho_2^{\hat{y}} \quad \text{as } \hat{y} \text{ tends to } \infty. \tag{3.7}$$

Finally the asymptotics of $\pi(\ell, \hat{y})$ as $\ell \to \infty$.

We are back to our horizontal analysis. Let Δ be the y-axis. Embed K into K^{∞} which is defined on $\mathbb{Z} \times \mathbb{Z}_{+}$. Take $\mathcal{K}_0$ to be the twisted free kernel using

$$h_0(x_1, \hat{x}) = \rho_1^{-x_1} \rho_2^{-\hat{x}}.$$

Note that $(\rho_1^{-1}, \rho_2^{-1})$ is a point on $M^{+} = 0$ and therefore it shall be straightforward to show that h_0 is harmonic above the x-axis. Let β, δ and σ be the rates of birth, death and staying put for $\hat{\mathcal{K}}_0$ away from zero (off the x-axis), respectively.

$$\begin{aligned}
\beta &= (\bar{\lambda}_2 + \mu_1 r_{12} \rho_1) \rho_2^{-1} \\
\delta &= (\mu_2 r_{20} + \mu_2 r_{21} \rho_1^{-1}) \rho_2 \\
\sigma &= \bar{\lambda}_1 \rho_1^{-1} + \mu_1 r_{10} \rho_1 + (\mu^* - \mu_1).
\end{aligned}$$

The same notations with index 0 will be used starting from zero. Clearly, $\delta_0 = 0$ and the parameters can be chosen such that $\beta + \delta + \sigma = 1$. Let κ_0 be the rate of killing at 0 for $\hat{\mathcal{K}}_0$, i.e. $\kappa_0 = 1 - (\beta_0 + \sigma_0)$ where,

$$\begin{aligned}
\beta_0 &= (\bar{\lambda}_2 + \mu^* r_{12} \rho_1) \rho_2^{-1} \\
\sigma_0 &= \bar{\lambda}_1 \rho_1^{-1} + \mu^* r_{10} \rho_1 + \mu_2.
\end{aligned}$$

Although, h_0 is harmonic above the x-axis, but still we need to analyze its behavior on the level axis.

Proposition 3.7. *$\hat{\mathcal{K}}_0$ has a killing at zero, i.e. $\kappa_0 > 0$.*

Proof. We simply show that $\beta_0 + \sigma_0 < 1$. Knowing that $\beta + \sigma + \delta = 1$,

$$\begin{aligned}
&1 - \beta_0 - \sigma_0 \\
&= (\beta - \beta_0) + (\sigma - \sigma_0) + \delta \\
&= [(\mu_1 - \mu^*) r_{12} \rho_1 \rho_2^{-1}] + [(\mu^* - \mu_1) + (\mu_1 - \mu^*) r_{10} \rho_1 - \mu_2] + (\mu_2 r_{20} + \mu_2 r_{21} \rho_1^{-1}) \rho_2 \\
&= [(\mu_1 - \mu^*) r_{12} \rho_1 \rho_2^{-1}] + [(\mu^* - \mu_1) + (\mu_1 - \mu^*) r_{10} \rho_1] + [(\mu_2 r_{20} + \mu_2 r_{21} \rho_1^{-1}) \rho_2 - \mu_2],
\end{aligned}$$

harking back to the cascade condition (3.3) on page 40 that $\rho_2^{-1} < r_{20} + r_{21}\rho_1^{-1}$, the last bracket is

$$\begin{aligned}&(\mu_2 r_{20} + \mu_2 r_{21}\rho_1^{-1})\rho_2 - \mu_2\\ &= \mu_2(r_{20} + r_{21}\rho_1^{-1})\rho_2 - \mu_2\\ &> \mu_2\rho_2^{-1}\rho_2 - \mu - 2 > 0.\end{aligned}$$

As a result κ_0 is strictly larger than the sum of the first two brackets

$$[(\mu_1 - \mu^*)r_{12}\rho_1\rho_2^{-1}] + [(\mu^* - \mu_1) + (\mu_1 - \mu^*)r_{10}\rho_1],$$

factorizing the positive term $(\mu^* - \mu_1)$, this is equal to

$$(\mu^* - \mu_1)[1 - \rho_1(\rho_2^{-1}r_{12} + r_{10})],$$

and this is strictly positive by the other cascade inequality (3.1). □

Proposition 3.8. *Away from zero, $\hat{\mathcal{K}}_0$ has a negative drift, i.e. $\beta < \delta$.*

Proof. Use the key inequality (3.3) in the denominator at the first line,

$$\begin{aligned}\frac{\beta}{\delta} &= \frac{\bar{\lambda}_2\rho_2^{-1} + \mu_1 r_{12}\rho_1\rho_2^{-1}}{\mu_2\rho_2(r_{20} + r_{21}\rho_1^{-1})}\\ &< \frac{\bar{\lambda}_2\rho_2^{-1} + \mu_1 r_{12}\rho_1\rho_2^{-1}}{\mu_2}\\ &= \frac{\bar{\lambda}_2\frac{\mu_2}{\lambda_2} + \mu_1 r_{12}\frac{\lambda_1}{\mu_1}\frac{\mu_2}{\lambda_2}}{\mu_2}\\ &= \frac{\bar{\lambda}_2 + \lambda_1 r_{12}}{\lambda_2}\\ &= \frac{\bar{\lambda}_2 + (\frac{\bar{\lambda}_1 + \bar{\lambda}_2 r_{21}}{1 - r_{12}r_{21}})r_{12}}{\lambda_2}\\ &= \frac{\bar{\lambda}_2 + \bar{\lambda}_1 r_{12}}{\lambda_2} = 1.\end{aligned}$$

□

Theorem 3.9. *If the minimal action path is a cascade that initially climbs the y-axis, i.e.* $\ln(\rho_2^{-1}) < \min\{\theta_2^j, \theta_2^b\}$, *then*

$$\pi(\ell,\hat{y}) \sim \quad C[\kappa_0\frac{\delta}{\beta}]^{-1}\ \rho_1^{\ell}\ \rho_2^{\hat{y}}\ [A(\frac{\beta}{\delta})^{\hat{y}-1} + \kappa_0\frac{\delta}{\beta}],$$

where $C \equiv f_v\frac{1}{d_v}\varphi_v(0)$ *is the constant in* (3.7) *and* $A = (\delta/\beta - 1)\beta_0 - \kappa_0$.

Proof. Define $h_1(\hat{y}) = A + \kappa_0(\frac{\delta}{\beta})^{\hat{y}}$ for any $\hat{y} \geq 0$, where $A = (\frac{\delta}{\beta} - 1)\beta_0 - \kappa_0$. Notice that $h_1(\hat{y}) > 0$ for any $\hat{y}$. The function

$$h(\ell,\hat{y}) = h_0(\ell,\hat{y})h_1(\hat{y}) = \rho_1^{-\ell}\rho_2^{-\hat{y}}(A + \kappa_0(\frac{\delta}{\beta})^{\hat{y}}), \tag{3.8}$$

is harmonic for the free kernel. We now h-twist K^∞ to get kernel $\mathcal{K}$. We simultaneously prove that h is harmonic for K^∞ and therefore $\sum_{(\ell',\hat{x})\in\mathbb{Z}\times\mathbb{Z}_+}\mathcal{K}((\ell,\hat{y});(\ell',\hat{x})) = 1$. Notice that

$$\mathcal{K}((\ell,\hat{y});(\ell',\hat{x})) = \mathcal{K}_0((\ell,\hat{y});(\ell',\hat{x}))h_1(\hat{x})/h_1(\hat{y}).$$

If $\hat{y} > 0$ starting from $(\ell,\hat{y})$,

$$\begin{aligned}
&\sum_{(\ell',\hat{x})}\mathcal{K}((\ell,\hat{y});(\ell',\hat{x}))\\
&= \sum_{\ell'}\mathcal{K}((\ell,\hat{y});(\ell',\hat{y}+1)) + \sum_{\ell'}\mathcal{K}((\ell,\hat{y});(\ell',\hat{y}-1)) + \sum_{\ell'}\mathcal{K}((\ell,\hat{y});(\ell',\hat{y}))\\
&= \beta\frac{h_1(\hat{y}+1)}{h_1(\hat{y})} + \delta\frac{h_1(\hat{y}-1)}{h_1(\hat{y})} + \sigma\\
&= \beta\frac{(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}+1})}{(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})} + \delta\frac{(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}+1})}{(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})} + \sigma\\
&= \frac{(\beta+\delta)A + (\delta+\beta)\kappa_0(\frac{\delta}{\beta})^{\hat{y}}}{(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})} + \sigma = 1.
\end{aligned}$$

Moreover, initiating from $(\ell, 0)$,

$$\begin{aligned}
\sum_{(\ell',\hat{x})}\mathcal{K}((\ell,0);(\ell',\hat{x})) &= \beta_0\frac{h_1(1)}{h_1(0)} + \sigma_0\\
&= \beta_0\frac{A+\kappa_0(\frac{\delta}{\beta})}{A+\kappa_0} + \sigma_0
\end{aligned}$$

$$= \frac{(\frac{\delta}{\beta}-1)\beta_0 - \kappa_0 + \kappa_0(\frac{\delta}{\beta})}{(\frac{\delta}{\beta}-1)} + \sigma_0$$
$$= \frac{(\frac{\delta}{\beta}-1)\beta_0 + \kappa_0(\frac{\delta}{\beta}-1)}{(\frac{\delta}{\beta}-1)} + \sigma_0$$
$$= \beta_0 + \delta_0 + \sigma_0 = 1.$$

From the standard results for the birth-death processes the invariant measure for $\hat{\mathcal{K}}$ is

$$\varphi(0) = 1 \qquad \text{if } \hat{y} > 0, \quad \varphi(\hat{y}) = \frac{\beta_0 \frac{h_1(1)}{h_1(0)} \times \beta \frac{h_1(2)}{h_1(1)} \times \cdots \times \beta \frac{h_1(\hat{y})}{h_1(\hat{y}-1)}}{\delta \frac{h_1(0)}{h_1(1)} \times \cdots \times \delta \frac{h_1(\hat{y}-1)}{h_1(\hat{y})}} \tag{3.9}$$
$$= \frac{\beta_0 \beta^{\hat{y}-1} \times h_1^2(\hat{y})}{\delta^{\hat{y}} \times h_1^2(0)}$$
$$= \frac{1}{(\frac{\delta}{\beta}-1)^2 \beta_0 \delta} (\frac{\beta}{\delta})^{\hat{y}-1} \times (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}}) h_1(\hat{y}). \tag{3.10}$$

Take the time reversal of the twisted chain, $\mathcal{X}$, with respect to φ to be $\overleftarrow{\mathcal{X}}$. $\overleftarrow{\mathcal{X}}$ is well-defined by Proposition 1.8 and moreover $\overleftarrow{\mathcal{X}}$ has eventually a positive vertical drift away from the x-axis. This is true, since the transition of an upward move starting from $\hat{y} > 0$ is

$$u = \delta \frac{h_1(\hat{y})}{h_1(\hat{y}+1)} \frac{\varphi(\hat{y}+1)}{\varphi(\hat{y})} = \delta \frac{(\frac{\beta}{\delta})^{\hat{y}} (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}+1})}{(\frac{\beta}{\delta})^{\hat{y}-1} (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}})},$$

while for a downward jump,

$$d = \beta \frac{h_1(\hat{y})}{h_1(\hat{y}-1)} \frac{\varphi(\hat{y}-1)}{\varphi(\hat{y})} = \beta \frac{(\frac{\beta}{\delta})^{\hat{y}-2} (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}-1})}{(\frac{\beta}{\delta})^{\hat{y}-1} (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}})}.$$

Consequently, for a fixed y_0,

$$\frac{u}{d} = \frac{\frac{\beta}{\delta} (A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}+1})}{(A + \kappa_0 (\frac{\delta}{\beta})^{\hat{y}-1})} > 1 \quad \text{for all } \hat{y} > y_0, \tag{3.11}$$

where the last inequality holds since by Proposition 3.8, $\frac{\delta}{\beta} > 1$.

Therefore, conditioning on starting in $(\ell, \hat{y})$, $\overleftarrow{\mathcal{X}}(\overleftarrow{\mathcal{T}}_{\blacktriangle})$ tends to ∞ as $\ell \to \infty$. This brings us to the situation where we are able to calculate $\pi(\ell, \hat{y})$:

$$\begin{aligned}
\pi(\ell, \hat{y}) &= \sum_{v \in \Delta} \pi(v)_{\Delta}G(v; (\ell, \hat{y})) \\
&= \sum_{v \in \Delta} \pi(v)_{\blacktriangle}G^{\infty}(v; (\ell, \hat{y})) \\
&= \frac{1}{h(\ell, \hat{y})} \sum_{v \in \Delta} \pi(v) h(v)_{\blacktriangle}\mathcal{G}(v; (\ell, \hat{y})) \\
&= \frac{1}{h(\ell, \hat{y})} \sum_{v \in \Delta} \pi(v) h(v) (\sum_{n} {}_{\blacktriangle}\mathcal{K}^n(v; (\ell, \hat{y}))) \\
&= \frac{\varphi(\hat{y})}{h(\ell, \hat{y})} \sum_{v \in \Delta} \frac{\pi(v) h(v)}{\varphi(v)} (\sum_{n} P_{(\ell, \hat{y})}\{\overleftarrow{\mathcal{X}}[n] = v, \overleftarrow{\mathcal{T}}_{\blacktriangle} = n\}) \\
&= \frac{\varphi(\hat{y})}{h(\ell, \hat{y})} \sum_{v \in \Delta} \sum_{n} \frac{\pi(v) h(v)}{\varphi(v)} P_{(\ell, \hat{y})}\{\overleftarrow{\mathcal{X}}[n] = v, \overleftarrow{\mathcal{T}}_{\blacktriangle} = n\} \\
&= \frac{\varphi(\hat{y})}{h(\ell, \hat{y})} E_{(\ell, \hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_{\blacktriangle}])\},
\end{aligned} \tag{3.12}$$

where $\zeta(\cdot) =: \pi(\cdot) h(\cdot)/\varphi(\cdot)$ and it is zero on $\blacktriangle - \Delta$.

Therefore,

$$\pi(\ell, \hat{y}) \frac{h(\ell, \hat{y})}{\varphi(\hat{y})} = E_{(\ell, \hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_{\blacktriangle}])\} = E_{(\ell, \hat{y})} \left\{ \pi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_{\Delta}]) \frac{h(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_{\Delta}])}{\varphi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_{\Delta}])} \right\}. \tag{3.13}$$

In the last term, $\blacktriangle$ is replaced with Δ since for the network under investigation starting from any state in the interior it is not possible that $\overleftarrow{\mathcal{X}}$ hits $\blacktriangle$ for the first time at $\blacktriangle - \Delta$.

Now, let's show that the integrand in (3.13) is bounded. For a typical hitting point $(0, \hat{z})$,

$$\begin{aligned}
\pi(0, \hat{z}) \frac{h(0, \hat{z})}{\varphi(\hat{z})} &= \pi(0, \hat{z}) \frac{\rho_1^0 \rho_2^{-\hat{z}} h_1(\hat{z})}{\varphi(\hat{z})} \\
&\leq (\pi(0, \hat{z}) \rho_2^{-\hat{z}}) \times \frac{1}{\frac{1}{(\frac{\delta}{\beta}-1)^2 \beta_0 \delta} \times (A(\frac{\beta}{\delta})^{\hat{z}-1} + \kappa_0(\frac{\delta}{\beta}))},
\end{aligned}$$

the first parenthesis is bounded by the same argument as in (2.7) and additionally as $\hat{z} \to \infty$ the denominator of the fraction tends to a constant since by Proposition 3.8,

$\frac{\delta}{\beta} > 1$.

On the other hand, as $\ell \to \infty$, $\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta]$ hits the boundary at a typical point $\pi(0, \hat{z})$ where $\hat{z}$ is a large phase. Therefore, given that we start at $(\ell, \hat{y})$, as $\ell \to \infty$, we can use (3.7) and the values of h and φ to prove the following limit

$$\pi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])\frac{h(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}{\varphi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])} \sim f_v\frac{1}{d_v}\varphi_v(0)\ \rho_2^z\ \rho_2^{-z}(\frac{\delta}{\beta} - 1)^2\beta_0\delta[A(\frac{\beta}{\delta})^{z-1} + \kappa_0\frac{\delta}{\beta}]^{-1}$$
$$\sim f_v\frac{1}{d_v}\varphi_v(0)\ (\frac{\delta}{\beta} - 1)^2\beta_0\delta[\kappa_0\frac{\delta}{\beta}]^{-1}$$

Since the integrand is bounded and convergent, the dominated convergence theorem can be used to obtain the following limit. As ℓ tends to infinity and taking $C = f_v\frac{1}{d_v}\varphi_v(0)$,

$$E_{(\ell,\hat{y})}\left\{\pi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])\frac{h(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}{\varphi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}\right\} \sim C(\frac{\delta}{\beta} - 1)^2\beta_0\delta[\kappa_0\frac{\delta}{\beta}]^{-1},$$

which is free of $\hat{y}$.

Finally, employing the vital equality (3.13),

$$\pi(\ell, \hat{y}) = \frac{\varphi(\hat{y})}{h(\ell, \hat{y})}E_{(\ell,\hat{y})}\left\{\pi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])\frac{h(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}{\varphi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}\right\}$$
$$\sim C[\kappa_0\frac{\delta}{\beta}]^{-1}\ \rho_1^\ell\ \rho_2^{\hat{y}}\ [A(\frac{\beta}{\delta})^{\hat{y}-1} + \kappa_0\frac{\delta}{\beta}], \tag{3.14}$$

and this completes the proof. □

Specific parameters.

It is note worthy to give at least one modified Jackson network where the large deviations path overloading the first node is a cascade. The following transition rates provide the case in which the overloading path is a cascade:

$$\mu_1 = 0.4, \quad \mu_2 = 0.3, \quad \mu^* = 0.5$$
$$\bar{\lambda}_1 = 0.1, \quad \bar{\lambda}_2 = 0.1, \quad r_{12} = 0.75$$
$$r_{10} = 0.25, \quad r_{21} = 0.25, \quad r_{20} = 0.75.$$

D10 in the cascade case.

As was mentioned before the Markov additive approach developed in the previous chapter does not apply to the cascade case of the modified Jackson network. We will show that this case does not comply with D10.

D10 stated that $\sum_{\hat{y}\in\hat{S}} \pi(0,\hat{y})\hat{h}^{-1}(\hat{y}) < \infty$. To prove that the series diverges, it is sufficient to show that

$$\lim_{\hat{y}\to\infty} \pi(0,\hat{y})\hat{h}^{-1}(\hat{y}) \neq 0.$$

Using (3.7) and (3.8),

$$\begin{aligned}\pi(0,\hat{y})\hat{h}^{-1}(\hat{y}) &\sim f_v\frac{1}{d_v}\varphi_v(0)\rho_2^{\hat{y}} \times \rho_2^{-\hat{y}}(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})\\ &= f_v\frac{1}{d_v}\varphi_v(0)(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}}),\end{aligned}$$

as $\hat{y}\to\infty$ this does not converge to zero since by Proposition 3.8, $\delta/\beta > 1$.

3.2 Revisiting the Jitter and Bridge Cases

The above method can be applied to the jitter and bridge cases. We will not generate all the details, however illustrate how the new approach would work in these situations.

The jitter (positive recurrent) case.

If the Markovian part of the twisted chain is positive recurrent with the stationary distribution φ and cascade case is impossible by D10 then starting from any point $(\ell,\hat{y})$, $\overleftarrow{\mathcal{X}}$ drifts to the left and hits $\blacktriangle$ for the first time at some state in Δ. Employing [Kes74, Theorem 1], as ℓ tends to infinity

$$P_{(\ell,\hat{y})}\{\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta]=z\} \to \frac{1}{d^*}\varphi(z)\mu^*(z), \tag{3.15}$$

for some finite measure $\mu^*(z)$ on $\mathbb{Z}_+$ and where d^* is the mean horizontal drift of $\overleftarrow{\mathcal{X}}$. Consequently,

$$\begin{aligned}
&\lim_{\ell\to\infty} E_{(\ell,\hat{y})}\left\{\pi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])\frac{h(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}{\varphi(\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta])}\right\} \\
&= \lim_{\ell\to\infty}\sum_{z\in\Delta} P_{(\ell,\hat{y})}\{\overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\Delta]=z\}\pi(z)\frac{h(z)}{\varphi(z)} \\
&= \sum_{z\in\Delta}\frac{1}{d^*}\varphi(z)\mu^*(z)\pi(z)\frac{h(z)}{\varphi(z)} \\
&= \frac{1}{d^*}\sum_{z\in\Delta}\mu^*(z)\pi(z)h(z)
\end{aligned}$$

Use representation (3.13) to obtain

$$\begin{aligned}
\pi(\ell,\hat{y}) \sim& \frac{1}{d^*}\sum_{z\in\Delta}\mu^*(z)\pi(z)h(z)\times h^{-1}(\ell,\hat{y})\varphi(\hat{y}) \\
=& \frac{1}{d^*}\sum_{z\in\Delta}\mu^*(z)\pi(z)h(z)\times e^{-\theta_1^j\ell}e^{-\theta_2^j\hat{y}}\varphi(\hat{y}),
\end{aligned}$$

which gives the same decay rate as our previous finding (2.8) in Section 2.1. Moreover, $\frac{1}{d^*}\sum_{z\in\Delta}\mu^*(z)\pi(z)h(z)$ can be interpreted as $\frac{1}{d}f$.

The bridge case.

In this case φ is not a distribution and equivalently the Markovian part of the twisted chain is not positive recurrent, the limiting result (3.15) does not hold and hence we are back at the same situation as the cascade case. Unfortunately, the problem turns out to be tricky and our new approach is not better than the Markov additive method discussed previously. The reason is that as $\hat{z}$ in $(0,\hat{z})$ tends to infinity the function $\zeta(0,\hat{z})$ tends to zero. Therefore to obtain an asymptotic result, the distribution of the hitting point at Δ will be required and this takes us to the similar difficulties as in the Markov additive method.

3.3 Future Work

The new method seems to be applicable to higher dimensional networks. In addition, it is capable to generate a unified theory addressing asymptotic problems of networks.

A further application would be deriving the asymptotics of $\pi(\ell, \hat{y})$ by simulation. The current approaches are based on tracking trajectories that eventually hit $(\ell, \hat{y})$. However, simulation by means of the technique introduced in the current chapter has the benefit of following trajectories starting at $\pi(\ell, \hat{y})$ and hitting the boundary. Intuitively, the new simulation seems to be faster and more efficient for large ℓ's.

Chapter 4

Limiting Hitting Distribution of the Phase and Mean Hitting Time

In this chapter, the limiting distribution of the phase when the level for the first time passes a large threshold is derived. The results are established for a Markov chain $X = (X_1, \hat{X})$ in a closed orthant that satisfies assumptions D1-D10 from Chapter 2. X is typically defined on $S \equiv \mathbb{Z}_+ \times \hat{S}$; we may think of $\hat{S}$ as $\mathbb{Z}_+^n$. Our findings extend the results in [MCD99] and [Fol01]. The restriction that the Markovian part of the twisted chain is positive recurrent is relaxed and the theory will be able to analyze the non-positive recurrent cases. Subsequently, the theory is applied to the modified Jackson network with partially coupled servers introduced in Section 1.2.

More precisely, we will investigate the exact asymptotics of the following hitting distribution of the set $F_\ell = \{(x_1, \hat{x})|\ x_1 \geq \ell\}$, Figure 4.1. For an initial state $\sigma \in S$, define T_σ to be the first return time to σ by X and T_ℓ the first time X_1 reaches a level $\geq \ell$, i.e.

$$T_\ell = \inf\{n > 0|\ X_1[n] \geq \ell\},$$
$$T_\sigma = \inf\{n > 0|\ X[n] = \sigma\}.$$

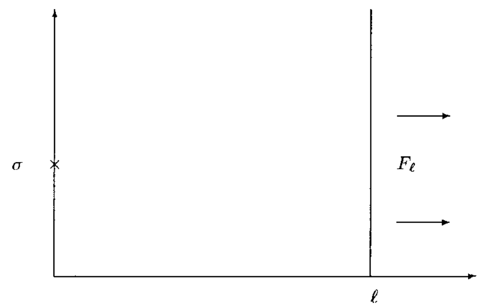


Figure 4.1: View of F_ℓ and the level tending to infinity.

As ℓ tends to infinity, the limit of

$$P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x} |\ T_\ell < T_\sigma\} \tag{4.1}$$

for fixed $s \geq 0$ and fixed phase $\hat{x} \in \hat{S}$ is established. Notice that in (4.1), X reaches F_ℓ for the first time at $(\ell + s, \hat{x})$. Moreover, the exact asymptotics of the mean time to reach F_ℓ, $E_\sigma(T_\ell)$ is given.

4.1 Literature Review

Matrix analytic results.

[Kro04] addresses a similar problem in the context of a general QBD with possibly infinitely many phases. Let kernel $Q(\hat{x}, \hat{y})$ to be probability that X starting at $(1, \hat{x})$ hits level 2 at $\hat{y}$ before exiting the system from level 0. It can be shown that Q satisfies the usual recursion equalities in matrix geometric approach and hence the usual tools can be used. They also state their results in the context of two $M/M/1$ queues in tandem. They manage to prove that in some cases, $Q^{(n)}(\hat{x}, \hat{y})$ decays at a rate which is not necessarily the same as the decay rate for the stationary distribution.

Their technique compared to the method that will be developed here is very complicated and obviously can not address the situations where the chain has unbounded overshoots over level ℓ to hit F_ℓ for the first time. Moreover, it is not clear to us if it can handle networks with coupled processors. For further references consult [Kro04].

Foley - McDonald results: Positive recurrent case.

[MCD99] and subsequently [Fol01] consider the same problem and obtain the limit of the hitting distribution (4.1) and mean time asymptotics for the cases where the Markovian part of the twisted chain is positive recurrent, though their results include the cases in which unbounded jumps are possible for the level and the phase. These results were achieved through the following technical lemma.

Lemma 4.1 ([MCD99] page 120, Lemma 1.8 (Comparison Lemma)). *If the Markovian part of the twisted chain is positive recurrent and moreover, suppose assumptions A1-A3 in Section 2.1 hold, then for any fixed state* $\sigma \in \Delta$,

$$\pi(\sigma)P_\sigma\{T_\ell < T_\sigma\} \sim \sum_{z\in\Delta} \pi(z)P_z\{T_\ell < T_\Delta\}.$$

Sketch of the proof. The above asymptotic result is equivalent to

$$\lim_{\ell\to\infty} \frac{\sum_{z\in\Delta} \pi(z)P_z\{T_\ell < T_\Delta\} - \pi(\sigma)P_\sigma\{T_\ell < T_\sigma\}}{\sum_{z\in\Delta} \pi(z)P_z\{T_\ell < T_\Delta\}} = 0. \tag{4.2}$$

Using the positive recurrent criteria, the decay rate of the denominator is canceled with that of the numerator. Hence by the existence of a strictly positive limit for the remaining terms in the denominator, it only remains to show that the numerator approaches zero as ℓ tends to infinity. To do so, we use the following realities which are guaranteed by [Bac00],

$$\sum_{z\in\Delta} \pi(z)P_z\{T_\ell < T_\Delta\} = \sum_{y\in F_\ell} \pi(y)P_y\{T_\Delta < T_\ell\},$$
$$\pi(\sigma)P_\sigma\{T_\ell < T_\sigma\} = \sum_{y\in F_\ell} \pi(y)P_y\{T_\sigma < T_\ell\}.$$

Therefore, the numerator in (4.2) shall become

$$\sum_{y\in F_\ell} \pi(y)P_y\{T_\Delta < T_\ell < T_\sigma\},$$

conditioning on where X hits Δ for the first time and using the existence of the overshoot distribution discussed on page 27 the sum tends to zero and the lemma is established. □

Remark 4.2. Notice that this proof is not valid when the Markovian part of the twisted chain is non-positive recurrent since the decay factors in the denominator of (4.2) do not cancel with those of the numerator unless we have some restricting assumptions. In addition, the overshoot distribution does not exist. However, it can be proved that in general the numerator tends to zero.

Observe that starting from any state $\sigma \notin F_\ell$, in order to hit F_ℓ for the first time, X can overshoot level ℓ and reach F_ℓ in $(\ell + s, \hat{y})$ for any $s \geq 0$ and $\hat{y} \in \hat{S}$.

It is now time to quote the Hitting distribution theorem from [MCD99] and [Fol01].

Theorem 4.3 ([MCD99] page 121, Lemma 1.11). *Let $\hat{\mathcal{K}}$ be positive recurrent. Suppose that $\mathcal{X}_1$ has period 1. If X satisfies A1-A3, of Section 2.1, then for any fixed state $\sigma \in \Delta$ with $P_\sigma\{T_\blacktriangle = \infty\} > 0$, $s \geq 0$ and as $\ell \to \infty$,*

$$P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x} | \; T_\ell < T_\sigma\} \sim Ce^{-\alpha s}\hat{h}^{-1}(\hat{x})\mu(s, \hat{x}),$$

where C is the normalization constant

$$\sum_{\hat{y}} \sum_{t \geq 0} e^{-\alpha t}\hat{h}^{-1}(\hat{y})\mu(t, \hat{y})$$

and μ is the overshoot distribution of the twisted chain (see page 27).

Clearly, the theorem and its proof do not apply to the cases where the Markovian part of the twisted chain is not positive recurrent and hence the overshoot distribution μ does not exist.

Having Theorem 4.3, it is possible to obtain the asymptotics of the mean hitting time.

4.2 An Extension

In this section, we relax the restriction that $\hat{\mathcal{K}}$ is positive recurrent and through the theory of Markov chains, derive the limit of (4.1) as ℓ tends to infinity.

Throughout, we assume that we are in the same situation as in Section 2.2. Therefore, X, X^∞, $\mathcal{X}$ have the same meanings as before and in addition, they satisfy D1-D10.

Specifically, h is chosen such that

$$P\{\lim_{n\to\infty} \mathcal{X}_1[n] = \infty \mid \mathcal{X}[0] = z\} = 1, \tag{4.3}$$

for all $z \in S^\infty$. Also, there is no restriction that φ is summable, i.e. it is possible to have $\sum_{\hat{x}} \varphi(\hat{x}) = \infty$.

To handle all the possible cases, we introduce a function of the level called a sub-rate function. Suppose that a sub-rate function $r(\ell)$ exists such that the asymptotics of the steady state for any fixed $\hat{y}$ is given by

$$\pi(\ell, \hat{y}) \sim Ce^{-\alpha\ell} r(\ell)(\varphi\hat{h}^{-1})(\hat{y}), \tag{4.4}$$

for some constant C (independent from $\hat{y}$) and in a way that for any fixed non-negative s and t,

$$\lim_{\ell\to\infty} \frac{r(\ell+t)}{r(\ell+s)} = 1, \tag{4.5}$$

and moreover, for any fixed non-negative s, and all except finite number of ℓ's,

$$\sum_{t\geq 0} e^{-\alpha t}\frac{r(\ell+t)}{r(\ell+s)} < \infty. \tag{4.6}$$

The sub-rate function as viewed in Theorem 2.3 is 1 when the Markovian part of the twisted chain is positive recurrent. However, for the modified Jackson network, it can also be the inverse of a polynomial as in (2.13) where $r(\ell)$ is either $\ell^{-1/2}$ or $\ell^{-3/2}$. In general non-cascade cases, by (2.9), $r(\ell) = \mathcal{G}((0,\hat{b});(\ell,\hat{b}))/\varphi(\hat{b})$ for any $\hat{b}$ with $\varphi(\hat{b}) > 0$. In either case, r satisfies (4.5), see Theorem 2.6. On the other hand, (4.6) is necessary to tackle Markov chains with possible unbounded jumps. If $r \equiv 1$, this property is obvious. In the cases where $r(\ell) = \mathcal{G}((0,\hat{b});(\ell,\hat{b}))/\varphi(\hat{b})$, without loss of generality take $\hat{b}$ to be the fixed phase, $\hat{a}$, introduced in assumption D8. Remembering that $\mathcal{G}$ is shift invariant with respect to the first coordinate,

$$\frac{\mathcal{G}((0,\hat{a});(\ell+t,\hat{a}))}{\mathcal{G}((0,\hat{a});(\ell+s,\hat{a}))}=\frac{\mathcal{G}((s-t,\hat{a});(\ell+s,\hat{a}))}{\mathcal{G}((0,\hat{a});(\ell+s,\hat{a}))}.$$

For all, except finitely many, non-negative t's, $w=(s-t,\hat{a})$ is a point in $\blacktriangle$. Hence, by D8 the fraction on the right is bounded for large ℓ's, and the series in (4.6) is finite for large choices of ℓ, i.e.

$$\sum_{t\geq 0} e^{-\alpha t}\frac{\mathcal{G}((0,\hat{a});(\ell+t,\hat{a}))}{\mathcal{G}((0,\hat{a});(\ell+s,\hat{a}))}<\infty,\quad \text{for } \ell \text{ sufficiently large.}$$

Notice that (4.4) provides the asymptotics of $\pi(\ell+t,\hat{y})/\pi(\ell+s,\hat{x})$ for the non-negative s and t and any fixed phases $\hat{x}$ and $\hat{y}$. Precisely, as $\ell\to\infty$,

$$\frac{\pi(\ell+s,\hat{y})}{\pi(\ell+t,\hat{x})}\sim\frac{e^{-\alpha s}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})}{e^{-\alpha t}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})}. \tag{4.7}$$

Unfortunately, to obtain the limiting hitting distribution of the phase the aforementioned asymptotics is not sufficient and further assumptions are required.

The following notion is one of the parameters used to establish the limiting hitting distribution. The definition stems from the work of H. Kesten in [Kes74].

Definition 4.4. For any fixed $(y_1,\hat{y})\in S$ (where $y_1\geq 0$), define $k^*(y_1,\hat{y})$ to be the probability that the time reversed free chain (with respect to φ), $\overleftarrow{\mathcal{X}}$, starting from $(y_1,\hat{y})$ leaves F_0 forever at the first step and its level drifts toward minus infinity.

In view of (4.3), k^* is not zero everywhere on its domain. In case $k^*(y_1,\hat{y})=0$ for any $y_1>0$ and should there not be any ambiguity, we will drop the first variable in the notation and use $k^*(\hat{y})$ rather than $k^*(0,\hat{y})$.

The main theorem of this section is

Theorem 4.5 (Hitting distribution). *Let X be a Markov chain in S satisfying assumptions D1-D10. Furthermore, suppose the following assumptions hold,*

H1. $\sum_{\hat{x}\in\hat{S}}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})<\infty.$

H2. For some fixed $\hat{x}_0$, the sequence of measures

$$\nu_\ell(s, \hat{y}) := \frac{\pi(\ell + s, \hat{y})}{\pi(\ell, \hat{x}_0)}$$

on S is a tight sequence.

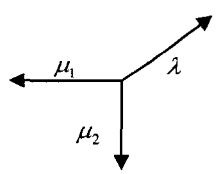
Then for any fixed $\sigma \in S$,

$$\lim_{\ell \to \infty} P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x} | \; T_\ell < T_\sigma\} = \frac{e^{-\alpha s}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})k^*(s, \hat{x})}{\sum_{t \geq 0}\sum_{\hat{y}}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})e^{-\alpha t}k^*(t, \hat{y})}. \tag{4.8}$$

Notice that k^* can be obtained by fast simulation.

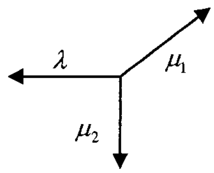
H1 is immediate in the positive recurrent cases as was proved in Proposition 2.1 and it appears to be a necessary condition. The next example describes a situation where H1 does not hold and the limit in (4.8) does not exist.

Example 4.6. Consider the Flatto-Hahn model as in [Fla84] where two processors with service rates μ_1 at processor one and μ_2 at the second processor, serve stream of Poisson arrivals with rate λ. This model exhibits the case where jobs arrive according to λ and immediately require two different classes of services each provided by one of the processors. This is similar to couples' arrival at a shopping center and their immediate visit to the ladies' and men's room, and in addition, the stream of singles is suppressed. For the sake of regularity, take $\mu_1 + \mu_2 + \lambda = 1$. X, the joint queue size of the customers either in line or in service is a Markov chain in $\mathbb{Z}_+ \times \mathbb{Z}_+$. Starting from any $(x_1, \hat{x})$ where $x_1 > 0$ and $\hat{x} > 0$ the transitions are,



Taking Δ to be the y-axis, and considering the homogenous behavior of the system for positive levels, X can be embedded in X^∞. X is stable if and only if

$\lambda < \min(\mu_1, \mu_2)$. A harmonic function for X^∞ is $h(\ell, \hat{x}) = (\mu_1/\lambda)^\ell$. Obviously, $\alpha = \ln(\mu_1/\lambda) > 0$ and $\hat{h} \equiv 1$. Applying the twist to X^∞ will lead us to $\mathcal{X}$, with the following transitions. Starting from any $(x_1, \hat{x})$ where $x_1 > 0$ and $\hat{x} > 0$,



Ignoring the transitions that stay put, the Markovian part of the twisted chain behaves like an $M/M/1$ queue and therefore its invariant measure $\varphi(\hat{x})$ is a constant multiple of $(\mu_1/\mu_2)^{\hat{x}}$. Clearly, if

$$\mu_1 > \mu_2, \tag{4.9}$$

then

$$\sum_{\hat{x} \in \hat{S}} \varphi(\hat{x}) \hat{h}^{-1}(\hat{x}) = \infty,$$

and H1 does not hold.

We show that assuming (4.9) the limiting hitting distribution of the phase does not exist as level overloads. Intuitively, under (4.9), it is more probable that the trajectories of $\mathcal{X}$ follow a north eastern route and hence as ℓ becomes large $\hat{X}[T_\ell]$ tends to infinity. Interestingly, this matches with the results in [Fla84], under (4.9) and as $\ell \to \infty$,

$$\hat{X}[T_\ell] \sim \left(\frac{\mu_1 - \mu_2}{\mu_1 - \lambda} \right) \ell,$$

implying that the limit in (4.8) is out of question.

Back to the Theorem's assumptions; H2 is hard to check and we had no luck checking it for the Markov chains of dimension three or more. This being said, we were able to use H2 for 2-dimensional cases and specifically verify it for the modified Jackson network.

Notice that we used fixed state $(\ell, \hat{x}_0)$ in the denominator of the measure ν_ℓ. However, by Lemma 4.7, H2 can be stated using any other fixed state $(\ell + s_0, \hat{x}_0)$.

In order to prove the Hitting Distribution Theorem, the natural steps are

$$
\begin{aligned}
&P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x} | \; T_\ell < T_\sigma\} \\
&= \frac{P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x}, T_\ell < T_\sigma\}}{P_\sigma\{T_\ell < T_\sigma\}} \\
&= \frac{\pi(\sigma) P_\sigma\{X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x}, T_\ell < T_\sigma\}}{\sum_{y=(y_1,\hat{y}) \in F_\ell} \pi(\sigma) P_\sigma\{X_1[T_\ell] = y_1, \hat{X}[T_\ell] = \hat{y}, T_\ell < T_\sigma\}} \\
&= \frac{\pi(\ell + s, \hat{x}) P_{(\ell+s,\hat{x})}\{T_\sigma^* < T_\ell^*\}}{\sum_{y \in F_\ell} \pi(y_1, \hat{y}) P_{(y_1,\hat{y})}\{T_\sigma^* < T_\ell^*\}} \quad \text{time reversal with respect to } \pi,
\end{aligned}
$$

where T_A^* is the first time that X^* returns to the set A. Dividing the numerator and denominator by $\pi(\ell + s, \hat{x})$,

$$
\frac{P_{(\ell+s,\hat{x})}\{T_\sigma^* < T_\ell^*\}}{\sum_{y \in F_\ell} \frac{\pi(y_1,\hat{y})}{\pi(\ell+s,\hat{x})} P_{(y_1,\hat{y})}\{T_\sigma^* < T_\ell^*\}} \tag{4.10}
$$

H2 is needed to ensure that $\sum \pi(y_1, \hat{y})/\pi(\ell + s, \hat{x})$ can be summed to the limit, moreover, we have to prove that $P_{(\ell+s,\hat{x})}\{T_\sigma^* < T_\ell^*\}$ has a positive limit. We shall address these issues in the coming subsection entitled Technicalities.

Technicalities.

Part I.

Notice that by H2, the following notion is a well-defined and non-empty set. For the fixed $\hat{x}_0$, let $\beta_\epsilon := \beta_\epsilon(0, \hat{x}_0) \subseteq S$ be the set such that

1. $\beta_\epsilon^c = S - \beta_\epsilon$ is a finite set, and
2. for all $\ell \geq 0$, $\sum_{(s,\hat{y}) \in \beta_\epsilon} \nu_\ell(s, \hat{y}) < \epsilon$.

In H2, we assumed that $\frac{\pi(\ell+\cdot,\cdot)}{\pi(\ell,\hat{x}_0)}$ is tight for fixed $\hat{x}_0$, the next lemma shows that this choice can be replaced by any other fixed state.

Lemma 4.7. *If H2 holds, then for any $s \geq 0$ and $\hat{x}$, the sequence of measures*

$$\left\{ \frac{\pi(\ell + \cdot, \cdot)}{\pi(\ell + s, \hat{x})} \right\}_{\ell \geq 0}$$

is a tight sequence on S.

Proof. Fix s and $\hat{x}$ and let $\hat{x}_0$ be as in H2. By (4.7), the sequence $\frac{\pi(\ell+s,\hat{x})}{\pi(\ell,\hat{x}_0)}$ is convergent and therefore it has an upper bound, say M. For $\epsilon > 0$, choose the set $\beta_\epsilon \subseteq S$, which is independent of ℓ, such that β_ϵ^c is finite and

$$\sum_{y \in \beta_\epsilon} \frac{\pi(y)}{\pi(\ell, \hat{x}_0)} < \epsilon / M.$$

Hence,

$$\begin{aligned} \sum_{y \in \beta_\epsilon} \frac{\pi(y)}{\pi(\ell + s, \hat{x})} &= \sum_{y \in \beta_\epsilon} \frac{\pi(y)}{\pi(\ell, \hat{x}_0)} \times \frac{\pi(\ell, \hat{x}_0)}{\pi(\ell + s, \hat{x})} \\ &< M \sum_{y \in \beta_\epsilon} \frac{\pi(y)}{\pi(\ell, \hat{x}_0)} < M(\epsilon/M) = \epsilon. \end{aligned}$$

Therefore, the sequence $\left\{ \frac{\pi(\ell+\cdot,\cdot)}{\pi(\ell+s,\hat{x})} \right\}_{\ell \geq 0}$ is tight. □

Corollary 4.8. *Fix $\hat{x}$ and $s \geq 0$. Assuming H1-H2, we have*

$$\lim_{\ell \to \infty} \sum_{t \geq 0} \sum_{\hat{y}} \frac{\pi(\ell + t, \hat{y})}{\pi(\ell + s, \hat{x})} = \sum_{t \geq 0} \sum_{\hat{y}} \frac{e^{-\alpha t} \varphi(\hat{y}) \hat{h}^{-1}(\hat{y})}{e^{-\alpha s} \varphi(\hat{x}) \hat{h}^{-1}(\hat{x})}.$$

Proof. Under assumption H2 which implies the previous Lemma, and using (4.7) the series on the left can be summed to the limit. In addition, the double series on the right hand side is finite by H1. □

Part II.

In this part we deal with the term $P_{(\ell+s,\hat{x})}\{T_\sigma^* < T_\ell^*\}$ in (4.10).

Lemma 4.9. *Assuming D1-D10, for any phase $\hat{y}$, $s \geq 0$ and $\sigma \in \Delta$,*

$$|P_{(\ell+s,\hat{y})}\{T_\Delta^* < T_\ell^*\} - P_{(\ell+s,\hat{y})}\{T_\sigma^* < T_\ell^*\}| \longrightarrow 0 \quad \textit{as } \ell \textit{ tends to infinity.}$$

The proof does not work for the cascade cases of the MJN, where $\sum_{z\in\Delta}\pi(z)h(z)=\infty$ and therefore D10 does not hold. Notice that $P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\}$ involves trajectories that leave F_ℓ on the first step and do not return before hitting Δ. By Lemma 4.10, for some $\hat{y}$'s as ℓ increases this probability does not tend to zero.

Proof. Notice that asymptotically π can be represented as in (2.9).

By the results of [Fol05a, Lemma 6], if D8 holds, for any $\hat{y}$ and for ℓ sufficiently large, then

$$\frac{\mathcal{G}\left(z;(\ell,\hat{y})\right)}{\mathcal{G}\left((0,\hat{y});(\ell,\hat{y})\right)} \text{ is bounded uniformly in } z\in\Delta. \tag{4.11}$$

Notice that $P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\} \geq P_{(\ell+s,\hat{y})}\{T^*_\sigma < T^*_\ell\}$, hence

$$\begin{aligned}
&|P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\} - P_{(\ell+s,\hat{y})}\{T^*_\sigma < T^*_\ell\}|\\
&= P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\} - P_{(\ell+s,\hat{y})}\{T^*_\sigma < T^*_\ell\}\\
&= \sum_{v\in\Delta} P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell, X^*[T^*_\Delta]=v\}P_v\{T^*_\ell < T^*_\sigma\} &(4.12)\\
&= \frac{1}{\pi(\ell+s,\hat{y})}\sum_{v\in\Delta}\pi(v)P_v\{T_\ell < T_\Delta, X[T_\ell]=(\ell+s,\hat{y})\}P_v\{T^*_\ell < T^*_\sigma\} &(4.13)\\
&= \frac{1}{\pi(\ell+s,\hat{y})}\sum_{v\in\Delta}\pi(v)P_v\{T^\infty_\ell < T^\infty_\blacktriangle, X^\infty[T^\infty_\ell]=(\ell+s,\hat{y})\}P_v\{T^*_\ell < T^*_\sigma\}\\
&= \frac{1}{\pi(\ell+s,\hat{y})\hat{h}(\hat{y})e^{\alpha(\ell+s)}}\times\\
&\qquad \sum_{v\in\Delta}\pi(v)h(v)P_v\{\mathcal{T}_\ell < \mathcal{T}_\blacktriangle, \mathcal{X}(\mathcal{T}_\ell)=(\ell+s,\hat{y})\}P_v\{T^*_\ell < T^*_\sigma\}\\
&\leq \frac{1}{\pi(\ell+s,\hat{y})\hat{h}(\hat{y})e^{\alpha(\ell+s)}}\sum_{v\in\Delta}\pi(v)h(v)_\blacktriangle\mathcal{G}(v;(\ell+s,\hat{y}))P_v\{T^*_\ell < T^*_\sigma\}\\
&= \frac{1}{\pi(\ell+s,\hat{y})\hat{h}(\hat{y})e^{\alpha(\ell+s)}\{\mathcal{G}((0,\hat{y});(\ell+s,\hat{y}))\}^{-1}}\times\\
&\qquad \sum_{v\in\Delta}\pi(v)h(v)\frac{{}_\blacktriangle\mathcal{G}(v;(\ell+s,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell+s,\hat{y}))}P_v\{T^*_\ell < T^*_\sigma\}. &(4.14)
\end{aligned}$$

(4.12) is obtained by conditioning on the first state that X^* hits Δ. Time reversal of the chain in the first probability gives (4.13). For the next equalities, notice that the trajectories in the first probability avoid the boundary so they are the same as the trajectories of the free chain; once we have the free chain, we can twist the kernel.

In (4.14)

$$\frac{{}_{\blacktriangle}\mathcal{G}(v;(\ell+s,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell+s,\hat{y}))} \leq \frac{\mathcal{G}(v;(\ell+s,\hat{y}))}{\mathcal{G}((0,\hat{y});(\ell+s,\hat{y}))}$$

is bounded uniformly in $v \in \Delta$ for ℓ sufficiently large by (4.11), $P_v\{T^*_\ell < T^*_\sigma\} \leq 1$ and $\sum_{v\in\Delta} \pi(v)h(v) < \infty$ by (D10). By (2.9) the fraction before the series in (4.14) converges to $1/f$ which is positive by (D9). Moreover, using the argument in the proof of Theorem 4 in [Fol05a], $\sum_{v\in\Delta} \pi(v)h(v)_{\blacktriangle}\mathcal{G}(v;(\ell,\hat{y}))/\mathcal{G}((0,\hat{y});(\ell,\hat{y}))$ tends to f. Finally,

$$\lim_{\ell\to\infty} P_v\{T^*_\ell < T^*_\sigma\} = 0,$$

due to the fact that X^* is positive recurrent and therefore intuitively starting from $v \in \Delta$ the chain first hits the fixed state σ and then drives away to level $\ell = \infty$.

The result of the lemma follows by applying the dominated convergence theorem to the series in (4.14). □

Lemma 4.10. *Assuming D1-D10 and H1-H2, for a fixed phase $\hat{y}$ and $s \geq 0$,*

$$|P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\} - k^*(s,\hat{y})| \longrightarrow 0 \quad \textit{as } \ell \textit{ tends to infinity.} \tag{4.15}$$

Proof. First,

$$|P_{(\ell+s,\hat{y})}\{T^*_\Delta < T^*_\ell\} - k^*(s,\hat{y})| = |P_{(\ell+s,\hat{y})}\{T^*_\ell \leq T^*_\Delta\} - (1-k^*(s,\hat{y}))|. \tag{4.16}$$

Notice that $F_\ell \cap \Delta = \varnothing$ and there is no chance for T^*_ℓ to be equal to T^*_Δ. Therefore, we may replace $P_{(\ell+s,\hat{y})}\{T^*_\ell \leq T^*_\Delta\}$ with $P_{(\ell+s,\hat{y})}\{T^*_\ell < T^*_\Delta\}$

Investigating the terms on the right hand side of (4.16),

$$\begin{aligned}
&P_{(\ell+s,\hat{y})}\{T^*_\ell < T^*_\Delta\} \\
&= \sum_{(x_1,\hat{x})\in S} P_{(\ell+s,\hat{y})}\{T^*_\ell < T^*_\Delta, X_1[T^*_\ell] - \ell = x_1, \hat{X}^*[T^*_\ell] = \hat{x}\} \\
&= \sum_{(x_1,\hat{x})\in S} \frac{\pi(\ell+x_1,\hat{x})}{\pi(\ell+s,\hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell < T_\Delta, X[T_\ell] = (\ell+s,\hat{y})\ \}
\end{aligned}$$

$$= \sum_{(x_1,\hat{x})\in S} \frac{\pi(\ell + x_1, \hat{x})}{\pi(\ell + s, \hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < T_\blacktriangle^\infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\ \}.$$

By definition $k^*(s, \hat{y}) \equiv P_{(s,\hat{y})}\{\overleftarrow{\mathcal{T}}_0 = \infty\} = P_{(\ell+s,\hat{y})}\{\overleftarrow{\mathcal{T}}_\ell = \infty\}$. Hence,

$$\begin{aligned} 1 - k^*(s, \hat{y}) &= P_{(\ell+s,\hat{y})}\{\overleftarrow{\mathcal{T}}_\ell < \infty\} \\ &= \sum_{(x_1,\hat{x})\in S} P_{(\ell+s,\hat{y})}\{\overleftarrow{\mathcal{T}}_\ell < \infty, \overleftarrow{\mathcal{X}}[\overleftarrow{\mathcal{T}}_\ell] = (\ell + x_1, \hat{x})\} \\ &= \sum_{(x_1,\hat{x})\in S} \frac{\varphi(\hat{x})}{\varphi(\hat{y})} P_{(\ell+x_1,\hat{x})}\{\mathcal{T}_\ell < \infty, \mathcal{X}[\mathcal{T}_\ell] = (\ell + s, \hat{y})\} \\ &= \sum_{(x_1,\hat{x})\in S} \frac{e^{-\alpha x_1}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})}{e^{-\alpha s}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < \infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\}. \end{aligned}$$

Pick β_ϵ to be the infinite set associated with s and $\hat{y}$ such that

$$\sum_{(x_1,\hat{x})\in\beta_\epsilon} \frac{\pi(\ell + x_1, \hat{x})}{\pi(\ell + s, \hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < T_\blacktriangle^\infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\} < \epsilon/2$$

and since $\sum_{\hat{x}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty$,

$$\sum_{(x_1,\hat{x})\in\beta_\epsilon} \frac{e^{-\alpha x_1}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})}{e^{-\alpha s}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < T_\blacktriangle^\infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\} < \epsilon/2$$

As a result,

$$\begin{aligned} &|P_{(\ell+s,\hat{y})}\{T_\ell^* < T_\Delta^*\} - (1 - k^*(s, \hat{y}))| \\ &= |\sum_{(x_1,\hat{x})\in\beta_\epsilon^c} \frac{\pi(\ell + x_1, \hat{x})}{\pi(\ell + s, \hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < T_\blacktriangle^\infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\} - \\ &\sum_{(x_1,\hat{x})\in\beta_\epsilon^c} \frac{e^{-\alpha x_1}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})}{e^{-\alpha s}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} P_{(\ell+x_1,\hat{x})}\{T_\ell^\infty < \infty, X^\infty[T_\ell^\infty] = (\ell + s, \hat{y})\}| + \epsilon \end{aligned} \quad (4.17)$$

Remember M_Δ as the level specifying the boundary on page 6 and take $\blacktriangle_{-\ell} := \blacktriangle - M_\Delta - \ell$ to be the horizontal translation of $\blacktriangle$ to the left by $M_\Delta + \ell$, i.e. $\blacktriangle_{-\ell} = \{(z_1 - M_\Delta - \ell, \hat{z})|\ (z_1, \hat{z}) \in \blacktriangle\}$. Starting from any point $(x_1, \hat{x}) \in F_0$, $T_{\blacktriangle_{-\ell}}^\infty < \infty$. This is true since the free chain, starting from level ≥ 0, behaves the same as the original chain before hitting $\blacktriangle_{-\ell}$ and the original chain is positive recurrent.

However, starting from level ≥ 0 as ℓ tends to infinity, $T^{\infty}_{\blacktriangle_{-\ell}} \uparrow \infty$. Hence

$$\{T_0^{\infty} < T^{\infty}_{\blacktriangle_{-\ell}}\} \uparrow \{T_0^{\infty} < \infty\},$$

so

$$\{T_0^{\infty} < T^{\infty}_{\blacktriangle_{-\ell}}, X^{\infty}[T_0^{\infty}] = (s, \hat{y})\} \uparrow \{T_0^{\infty} < \infty, X^{\infty}[T_0^{\infty}] = (s, \hat{y})\}.$$

Translating back the expressions to the right again, we get

$$P_{(\ell+x_1,\hat{x})}\{T_\ell^{\infty} < T_{\blacktriangle}^{\infty}, X^{\infty}[T_\ell^{\infty}] = (\ell+s, \hat{y})\} \rightarrow$$
$$P_{(\ell+x_1,\hat{x})}\{T_\ell^{\infty} < \infty, X^{\infty}[T_\ell^{\infty}] = (\ell+s, \hat{y})\}$$

as $\ell \rightarrow \infty$. By (4.7), (4.17) implies that

$$\lim_{\ell\rightarrow\infty} |P_{(\ell+s,\hat{y})}\{T_\Delta^* < T_\ell^*\} - k^*(s,\hat{y})| \leq \epsilon$$

where ϵ is arbitrary and the result of the lemma follows. □

Proof of the Hitting Distribution Theorem.

Proof of Theorem 4.5. Without loss of generality, let σ to be a state in Δ. Pursue the representation (4.10). We deal with the rest of the proof in two steps.

Firstly, in (4.10), for any fixed t and $\hat{y}$,

$$|P_{(\ell+t,\hat{y})}(T_\sigma^* < T_\ell^*) - k^*(t,\hat{y})|$$
$$\leq |P_{(\ell+t,\hat{y})}(T_\sigma^* < T_\ell^*) - P_{(\ell+t,\hat{y})}(T_\Delta^* < T_\ell^*)| + |P_{(\ell+t,\hat{y})}(T_\Delta^* < T_\ell^*) - k^*(t,\hat{y})|,$$

therefore, by Lemma 4.9 and Lemma 4.10,

$$\lim_{\ell\rightarrow\infty} P_{(\ell+t,\hat{y})}(T_\sigma^* < T_\ell^*) = k^*(t,\hat{y}). \tag{4.18}$$

Secondly, using Lemma 4.8 and the fact that $P_{(\ell+t,\hat{y})}(T_\sigma^* < T_\ell^*) \leq 1$, the denominator can be summed to the limit. Hence,

$$P_\sigma(X_1[T_\ell] - \ell = s, \hat{X}[T_\ell] = \hat{x} | \ T_\ell < T_\sigma)$$

$$\sim \frac{e^{-\alpha s}\varphi(\hat{x})\hat{h}^{-1}(\hat{x})k^*(s,\hat{x})}{\sum_{t\geq 0}\sum_{\hat{y}}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})e^{-\alpha t}k^*(t,\hat{y})\frac{r(\ell+t)}{r(\ell+s)}}.$$

However, by (4.5),

$$\frac{r(\ell+t)}{r(\ell+s)} \to 1,$$

Having (4.6) and knowing that $e^{-\alpha t}k^*(t,\hat{y}) < 1$, dominated convergence theorem can be used to establish the result of the theorem. □

Asymptotics of the Mean Hitting time.

Theorem 4.11. *Under assumptions D1-D10 and H1-H2, the asymptotics of the mean hitting time for any fixed σ is given by:*

$$\{E_\sigma(T_\ell)\}^{-1} \sim Ce^{-\alpha\ell}r(\ell)\sum_{(y_1,\hat{y})\in S} e^{-\alpha y_1}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})k^*(y_1,\hat{y}),$$

where r is the sub-rate function defined on page 59 and C is the constant in (4.4).

Proof. By Theorem 1 in [Bac00],

$$\{E_\sigma(T_\ell)\}^{-1} \sim \pi(\sigma)P_\sigma\{T_\ell < T_\sigma\}.$$

We have,

$$\begin{aligned}
&\pi(\sigma)P_\sigma\{T_\ell < T_\sigma\} \\
&= \pi(\sigma)\sum_{(y_1,\hat{y})\in S} P_\sigma\{T_\ell < T_\sigma, X[T_\ell] = (\ell+y_1,\hat{y})\} \\
&= \sum_{(y_1,\hat{y})\in S} \pi(\ell+y_1,\hat{y})P_{(\ell+y_1,\hat{y})}\{T^*_\sigma < T^*_\ell\}, \qquad (4.19)
\end{aligned}$$

the first equality is obtained by conditioning on the state that X hits F_ℓ for the first time and the second equality is attained by time reversing X with respect to the steady state π. (4.19) can be written as

$$\pi(\ell,\hat{b})\sum_{(y_1,\hat{y})\in S}\frac{\pi(\ell+y_1,\hat{y})}{\pi(\ell,\hat{b})}P_{(\ell+y_1,\hat{y})}\{T^*_\sigma < T^*_\ell\},$$

for some $\hat{b}$ that $\varphi(\hat{b}) > 0$.

Having H1 and H2, the results in the Technicalities apply and therefore (4.19) is asymptotic to

$$Ce^{-\alpha\ell}r(\ell)\varphi(\hat{b})\hat{h}^{-1}(\hat{b})\sum_{(y_1,\hat{y})\in S}\frac{e^{-\alpha y_1}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})}{\varphi(\hat{b})\hat{h}^{-1}(\hat{b})}k^*(y_1,\hat{y}), \tag{4.20}$$

simplifying the similar terms provides the result of the theorem. □

One more interesting issue is the decay rate of $P_\sigma\{T_\ell < T_\sigma\}$, which can be addressed similarly to the previous proof.

Corollary 4.12. *Under assumptions D1-D10 and H1-H2, $P_\sigma\{T_\ell < T_\sigma\}$ has the same decay rate as $\pi(\ell,\cdot)$ when $\ell \to \infty$.*

Proof. Suppose that the decay rate of $\pi(\ell,\cdot)$ is $e^{-\alpha\ell}r(\ell)$. Partition $P_\sigma\{T_\ell < T_\sigma\}$ on the point X hits F_ℓ for the first time to obtain an expression similar to (4.19). Following the previous proof will lead us to (4.20) with decay rate $e^{-\alpha\ell}r(\ell)$. □

How to verify H2?

A vital ingredient in the Hitting Distribution proof is the fact that the series

$$\sum_{y\in F_\ell}\frac{\pi(y_1,\hat{y})}{\pi(\ell+s,\hat{x})}$$

in the denominator of (4.10) can be summed to the limit. This stage in the proof was achieved by assumption H2. Nonetheless, the question remains on how to check and verify H2 for a specific Markov chain or a stochastic network. We develop a coupling argument which works for the MJN and similar 2-dimensional networks. The argument also leads us to a new technique to find the asymptotics of the steady state, discussed in Chapter 3.

Bearing in mind the goal of applying the theory to 2-dimensional networks like the MJN, hereafter, we limit our results to the cases where the level chain is nearest neighbor. Hence in H2 the excess beyond level ℓ is not necessary and there is no chance to hit F_ℓ for the first time at levels higher than ℓ.

We have found H2$'$ to be checkable for 2-dimensional chains.

H2$'$. Define $\zeta(z) := \frac{\pi h}{\phi}(z)$. Assume that for a fixed $\hat{x}_0$, there exists a function g and a set $\beta \subset \hat{S}$ such that, β^c is finite and

$$\frac{E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{T}_\Delta))\}}{E_{(\ell,\hat{x}_0)}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{T}_\Delta))\}} \leq g(\hat{y}) \quad \forall\, \hat{y} \in \beta \text{ and for } \ell \text{ sufficiently large,}$$

where $\sum_{\hat{y}} g(\hat{y})\phi(\hat{y})\hat{h}^{-1}(\hat{y}) < \infty$ and $\overleftarrow{\mathcal{X}}$ is the time reversal of $\mathcal{X}$ with respect to ϕ.

Notice that in view of representation (3.13),

$$\frac{\varphi(\hat{x}_0)\hat{h}^{-1}(\hat{x}_0)}{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} \times \frac{\pi(\ell,\hat{y})}{\pi(\ell,\hat{x}_0)} = \frac{E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{T}_\Delta))\}}{E_{(\ell,\hat{x}_0)}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{T}_\Delta))\}} \leq g(\hat{y}) \tag{4.21}$$

where by H2$'$ the last inequality holds for $\hat{y} \in \beta$ and $\sum_{\hat{y}} g(\hat{y})\phi(\hat{y})\hat{h}^{-1}(\hat{y}) < \infty$.

The existence of g and the fact that β^c is finite imply that the sequence of functions

$$q_\ell(\hat{y}) =: \frac{\varphi(\hat{x}_0)\hat{h}^{-1}(\hat{x}_0)}{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} \times \frac{\pi(\ell,\hat{y})}{\pi(\ell,\hat{x}_0)}$$

is uniformly integrable with respect to the finite measure $\frac{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})}{\varphi(\hat{x}_0)\hat{h}^{-1}(\hat{x}_0)}$. For $\epsilon > 0$, $\beta_\epsilon \subset \hat{S}$ can be chosen such that β_ϵ^c is finite and for all ℓ,

$$\sum_{\hat{y}\in\beta} \frac{\pi(\ell,\hat{y})}{\pi(\ell,\hat{x}_0)} = \sum_{\hat{y}\in\beta} \frac{\varphi(\hat{x}_0)\hat{h}^{-1}(\hat{x}_0)}{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})} \frac{\pi(\ell,\hat{y})}{\pi(\ell,\hat{x}_0)} \times \frac{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})}{\varphi(\hat{x}_0)\hat{h}^{-1}(\hat{x}_0)} < \epsilon.$$

The above argument shows that H2$'$ implies H2. The appearance of H2$'$ seems to be strange, but it is more applicable since its terms can be analyzed through trajectories starting at level ℓ and eventually hitting the boundary.

4.3 Application

We now apply the results in the previous Section to the modified Jackson network with partially coupled processors introduced in Section 1.2.

Jitter case.

Consider the MJN as in Section 2.1. By Proposition 2.1, assumption H1 automatically holds. On the other hand, assumption H2 holds since the Foley-McDonald results discussed in Section 4.1 apply. Therefore, the following result is automatic by comparing Theorems 4.3 and 4.5.

Corollary 4.13. *Consider the modified Jackson network. If the least action path to overload the level is a jitter path, i.e.* $\theta_2^j < \min\{\ln(\rho_2^{-1}), \theta_2^b\}$, *then for any fixed* σ,

$$\lim_{\ell\to\infty} P_\sigma\{\hat{X}[T_\ell] = \hat{x}|\ T_\ell < T_\sigma\} = \frac{\varphi(\hat{x})e^{-\theta_2^j\hat{x}}k^*(\hat{x})}{\sum_{\hat{y}\geq 0}\varphi(\hat{y})e^{-\theta_2^j\hat{y}}k^*(\hat{y})},$$

where φ *is given on page 29.*

In addition for any fixed σ, *the decay rate of* $\{E_\sigma(T_\ell)\}^{-1}$ *and* $P_\sigma\{T_\ell < T_\sigma\}$ *is* $e^{-\theta_1^j\ell}$.

Notice that for the MJN, X hits F_ℓ for the first time only at level ℓ. Consequently, the first coordinate of k^* is always zero and moreover the term $e^{-\alpha}$ of (4.8) is not present.

Bridge case.

Suppose that the least action path to overload the level is a bridge. In this case by Theorem 1.17, $\theta_2^b \leq \theta_2^j$ and $\theta_2^b < \ln(\rho_2^{-1})$. Moreover, the asymptotics of the steady state was given in (2.13). To show the existence of the limiting hitting distribution of the phase, we check the assumptions.

For H1 to hold, from (2.11) and (2.10), $\varphi(\hat{x}) = \frac{p_0}{u}(1+\kappa\hat{x}/p_0)^2$ and $\hat{h}(\hat{x}) = \exp(\theta_2^b\hat{x})(1+\kappa\hat{x}/p_0)$. Hence, H1 holds if and only if

$$\sum_{\hat{x}\in\hat{S}}\varphi(\hat{x})\hat{h}^{-1}(\hat{x}) = \frac{p_0}{u}\sum_{\hat{x}}\exp(-\theta_2^b\hat{x})(1+\kappa\hat{x}/p_0) < \infty,$$

and this happens if θ_2^b is strictly positive.

To verify H2, we use a coupling argument to prove H2′. For this reason, some information is required on how $\overleftarrow{\mathcal{X}}$ hits the boundary for the first time, given that it starts in F_ℓ. We are again in the situation of Chapter 3 dealing with ζ. Using (3.7), we proved that if

$$\rho_1^{-1} > r_{10} + r_{12}\rho_2^{-1}, \tag{4.22}$$

then strictly positive D exists such that

$$\lim_{\hat{y}\longrightarrow\infty} \rho_2^{-\hat{y}}\pi(0,\hat{y}) = D. \tag{4.23}$$

But remarkably, (4.22) resembles the properties of the bridge. [Fol05b, Theorem 3] states that if either one of the following holds,

$$\rho_1^{-1} < r_{10} + r_{12}\rho_2^{-1} \tag{4.24}$$

$$\rho_2^{-1} > e^{\theta_2^b}, \tag{4.25}$$

then the minimum action is a bridge. Gathering all this information, in order to have (4.23) and to ensure the bridge phenomena, we may keep (4.25) but switch the inequality in (4.24).

Proposition 4.14. *For the modified Jackson network, assume that H1 holds. If* (4.22) *holds then*

$$\frac{E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{\mathcal{T}}_\Delta))\}}{E_{(\ell,0)}\{\zeta(\overleftarrow{\mathcal{X}}(\overleftarrow{\mathcal{T}}_\Delta))\}}$$

is bounded by a constant uniformly in ℓ and $\hat{y}$, i.e. $g(\cdot)$ from H2′ is a constant function. Hence the network satisfies H2′ and equivalently H2.

Proof. From (4.25), $\rho_2 e^{\theta_2^b} < 1$. Pick $\epsilon > 0$ such that $r := \rho_2 e^{\theta_2^b}\frac{D+\epsilon}{D-\epsilon} < 1$. Using (4.23), choose $N > 0$ such that

$$(D-\epsilon) \leq \rho_2^{-\hat{y}}\pi(0,\hat{y}) \leq (D+\epsilon)$$

for $\hat{y} \geq N$. Hence,

$$\frac{\pi(0,\hat{y}+t)}{\pi(0,\hat{y})} \leq \rho_2^t\frac{D+\epsilon}{D-\epsilon} \quad \text{for } t \geq 0 \text{ and } \hat{y} \geq N.$$

Also notice that $\phi(\hat{y})\hat{h}^{-1}(\hat{y}) = \frac{p_0}{u}\exp(-\theta_2^b\hat{y})(1+\kappa\hat{y}/p_0)$. As a result, for any $t > 0$ and $\hat{y} \geq N$,

$$\begin{aligned}\frac{\zeta(0,\hat{y}+t)}{\zeta(0,\hat{y})} =& \frac{\pi(0,\hat{y}+t)}{\pi(0,\hat{y})} \times \frac{\phi(\hat{y})\hat{h}^{-1}(\hat{y})}{\phi(\hat{y}+t)\hat{h}^{-1}(\hat{y}+t)} \\ \leq& \rho_2^t \frac{D+\epsilon}{D-\epsilon} \times e^{\theta_2^b t} \frac{1+\kappa\hat{y}/p_0}{1+\kappa(\hat{y}+t)/p_0} \\ \leq& (\rho_2 e^{\theta_2^b})^t \frac{D+\epsilon}{D-\epsilon} \leq r < 1, \end{aligned} \tag{4.26}$$

For $\hat{y} < N$ and any non-negative t,

$$\frac{\zeta(0,\hat{y}+t)}{\zeta(0,\hat{y})} = \frac{\zeta(0,N)}{\zeta(0,\hat{y})} \times \frac{\zeta(0,\hat{y}+t)}{\zeta(0,N)}.$$

However, for all but a finite number of $\hat{y}+t$'s the second fraction satisfies (4.26). Moreover, there are only finite number of possible terms in the first fraction. Therefore, for some constant D_1,

$$\frac{\zeta(\hat{y}+t)}{\zeta(\hat{y})} \leq D_1 \quad \text{for all } t \geq 0, \tag{4.27}$$

where D_1 is uniform in $\hat{y}$.

Now consider the product space of all paths w of $\overleftarrow{\mathcal{X}}$, which start from $(\ell,\hat{y})$ times paths w' which start from $(\ell,0)$. On this product space we can define a coupled path starting from $(\ell,0)$, which follows w' until w' hits the path w and then follows w. Define H_ℓ to be $\{w' : \overleftarrow{\mathcal{X}}_1[n] < \ell, \text{ for all } n > 0\}$. Let $\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w))$ be the value of the function ζ at the first state that $\overleftarrow{\mathcal{X}}$ hits the boundary following the trajectory w. Hence,

$$\begin{aligned} &E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w))\} \times E_{(\ell,0)}\{\chi_{H_\ell}(w')\} \\ &= E_{(\ell,\hat{y})} \otimes E_{(\ell,0)}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w)).\chi_{H_\ell}(w')\} \\ &\leq D_1\ E_{(\ell,\hat{y})} \otimes E_{(\ell,0)}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w')).\chi_{H_\ell}(w')\} \end{aligned} \tag{4.28}$$

where (4.28) is obtained using (4.27) and the fact that if the path w starting from $(\ell,\hat{y})$ does not couple, it will hit the boundary at a higher phase. Also, notice that any member of H_ℓ starting from $(\ell,0)$ is trapped between the level axis and a path starting from $(\ell,\hat{y})$, since there are no northeast and southwest jumps for the MJN.

This implies that

$$\frac{E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w))\}}{E_{(\ell,0)}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w'))\}} \leq \frac{E_{(\ell,\hat{y})}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w))\}}{E_{(\ell,0)}\{\zeta(\overleftarrow{\mathcal{X}}_{\overleftarrow{T}_\Delta}(w'))\chi_{H_\ell}(w')\}}$$
$$\leq \frac{D_1}{P_{(\ell,0)}(w' \in H_\ell)}$$
$$= \frac{D_1}{k^*(0)}.$$

It is obvious that $k^*(0)$ is non-zero for the modified Jackson network and this finishes the proof. □

Theorem 4.15. *Suppose the least action path for the modified Jackson network is a bridge. As the number of customers in the first queue tends to infinity, the limiting hitting distribution of the phase exists if the following inequalities hold:*

$$(1) \quad \theta_2^b > 0,$$
$$(2) \quad \rho_1^{-1} > r_{1,0} + r_{1,2}\, \rho_2^{-1}.$$

Moreover,

$$\lim_{\ell\to\infty} P_\sigma\{\hat{X}[T_\ell] = \hat{x}|\ T_\ell < T_\sigma\} = \frac{(1+\kappa\hat{x}/p_0)e^{-\theta_2^b\hat{x}}k^*(\hat{x})}{\sum_{\hat{y}\geq 0}(1+\kappa\hat{y}/p_0)e^{-\theta_2^b\hat{y}}k^*(\hat{y})}.$$

In addition for any fixed σ, the decay rate of $\{E_\sigma(T_\ell)\}^{-1}$ and $P_\sigma\{T_\ell < T_\sigma\}$ is given by

$$\begin{cases} e^{-\theta_1^b\ell}\ell^{-3/2} & \text{if } \kappa > 0, \\ e^{-\theta_1^b\ell}\ell^{-1/2} & \text{if } \kappa = 0, \end{cases}$$

where κ is the same parameter as in (2.13).

Proof. (1) is the property that ensured H1 and (2) is the inequality that led us to (4.23), which in turn was used in the proof of Proposition 4.14. Due to the fact that the network satisfies H1 and H2, the limiting hitting distribution of the phase exists and it can be explicitly given by the values of φ and $\hat{h}$ in (2.11) and (2.10). □

Cascades.

Suppose the cheapest path to overload the level is a cascade for the modified Jackson network. It was shown on page 51, that D10 does not hold in this case. Therefore, the proof of Lemma 4.9 is not valid and we can not apply Theorems 4.5 and 4.11 to the cascade cases. However, H1 and H2$'$ hold. To verify H1, remember the tools developed in Theorem 3.9,

$$h(\ell,\hat{y}) = \rho_1^{-\ell} \times \underbrace{\rho_2^{-\hat{y}}(A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})}_{\hat{h}(\hat{y})}$$

$$\varphi(\hat{y}) = \frac{1}{(\frac{\delta}{\beta}-1)^2\beta_0\delta}(\frac{\beta}{\delta})^{\hat{y}-1} \times (A+\kappa_0(\frac{\delta}{\beta})^{\hat{y}})^2 \text{ if } \hat{y} > 0.$$

For H1 to hold, $\sum \varphi \hat{h}^{-1}(\hat{y}) < \infty$ if and only if

$$\sum \rho_2^{\hat{y}}[A(\frac{\beta}{\delta})^{\hat{y}-1} + \kappa_0(\frac{\delta}{\beta})] < \infty,$$

but this is true since by Theorem 1.13, $\rho_2 < 1$. Moreover, by Proposition 3.8, $\beta/\delta < 1$.

To verify H2$'$, notice that by the argument in the proof of Corollary 3.6, (4.22) holds. Therefore, if we replace $e^{\theta_2^b}$ with ρ_2^{-1}, which is the analogous parameter in the cascade case, Proposition 4.14 is in effect .

4.4 Future Work

The nature of assumption H2, which ensured our limiting results, is still unclear and a more general criterion for checking the assumption is needed. This is particularly necessary for high dimensional networks.

We conjecture that Lemma 4.9 is valid even if D10 fails. Finding a proof of the Lemma while D10 is relaxed will enable us to apply the asymptotic results of this chapter to cascade cases of the MJN.

As aforementioned, the Matrix Analytic Method can solve problems close to the ones in this chapter, [Kro04]. However, the method can not match our Hitting

Distribution and Mean Hitting time results. Addressing these issues remains an interesting unanswered topic.

Chapter 5

Ratio Limit Property For a Markovian Kernel with Unbounded Jumps

Consider an irreducible additive Markov chain $W = (V, Z)$ on $S^\infty \equiv \mathbb{Z} \times \hat{S}$ with kernel J. As was observed in Section 1.1, being additive means that for any $(x_1, \hat{x})$ and $(y_1, \hat{y})$ in S^∞,

$$J((x_1, \hat{x}); (y_1, \hat{y})) = J((0, \hat{x}); (y_1 - x_1, , \hat{y})).$$

Our approach is capable of addressing any countable $\hat{S}$, however we will limit $\hat{S}$ to be equal to $\mathbb{Z}_+$ to be able to compare our findings with those in the literature. We shall call the elements of $\hat{S}$ the phase. We will call the first coordinate the level, assume that V is a nearest neighbor chain and furthermore presuppose that starting from any point $(x_1, \hat{x}) \in S^\infty$, the level drifts to $-\infty$, i.e.

$$P\left\{\lim_{n\to\infty} V[n] = -\infty | \; W[0] = (x_1, \hat{x})\right\} = 1. \tag{5.1}$$

Obviously, W is a transient chain.

We shall reserve the hatted letters for the phase and as usual use $\ell, m, ...$ for the level when there is no ambiguity.

Our primary goal is to provide the limiting hitting distribution of Z when V for the first time passes a threshold which is tending to infinity. In particular, we will establish the following limit:

$$\lim_{m\to\infty} P_\sigma\{Z[R_m] = \hat{x} \mid R_m < \infty\}, \tag{5.2}$$

where $R_m = \inf\{n > 0 \mid V[n] = m\}$, σ is a fixed state and $\hat{x}$ is a fixed phase.

Comparing with the material discussed in preceding chapters, W and J are the same as the free chain X^∞ and its kernel K^∞, respectively, on page 6. Moreover, (5.2) is closely related to (4.1). Let R_σ be the first return time to σ. Clearly, $R_m < R_\sigma$ is replaced with $R_m < \infty$, since in the context of our Markov additive chain W may leave σ, never reach level m and drift to $-\infty$.

Taking into account that W starting from any phase on level m may never reach level $m+1$, illustrates that the problem under investigation is connected with the existence of a quasi-stationary measure. To make the problem tangible, define the Markovian kernel D on $\hat{S}$ as,

$$D(\hat{x};\hat{y}) = P_{(0,\hat{x})}\{Z[R_1] = \hat{y}\ , R_1 < \infty\}.$$

D has possible unbounded jumps since it is possible to hit the next level at any phase. In addition, it is a defective kernel, i.e. for any fixed $\hat{x}$,

$$\sum_{\hat{y}} D(\hat{x};\hat{y}) < 1.$$

Due to unbounded jumps, D does not satisfy condition (1.4) in [Kes95], therefore there is no guarantee that the ratio limit property holds or the Yaglom limit exists. However, we will prove that under some assumptions the ratio limit property holds. Explicitly, Corollary 5.7 shows that as n tends to infinity the limit of

$$\frac{D^{(m)}(\hat{x};\hat{y})}{D^{(m)}(\hat{u};\hat{z})},$$

exists for any fixed phases $\hat{x}, \hat{y}, \hat{u}$ and $\hat{z}$.

5.1 Some Background

[Kes74] considers the same problem and establishes (5.2). In the context of that reference, let $Z_0, Z_1, Z_2, \cdots$ be a Markov chain with separable metric state space, and V at time n to be the partial sum of identically distributed random variables $\{u_i\}_{i\geq 0}$, in the sense that,

$$V[n] = \sum_{0\leq i\leq n-1} u_i. \tag{5.3}$$

Further, the distribution of u_n only depends on the state of Z at times n and $n+1$.

Assume that Z is a positive recurrent Markov chain, so there exists a finite invariant distribution φ for Z. Moreover, the following condition is in effect, $\lim_{n\to\infty} V[n]/n$ exists a.e. with respect to P_x for any x in the state space and the limit is strictly positive and finite. Theorem 1 on page 359 of [Kes74] shows the existence of (5.2) and considering some regularities provides the limit. Afterwards in Section 4, the case of negative drift is considered when $n^{-1}V[n]$ has a limit a.s. but the limit is strictly negative. This case is converted to the previous situation with positive drift using the method of Chapter XI.6 in [Fel71]. The method basically uses a harmonic function and the Doob's transform to change the kernel of the chain in the desired way. We previously employed this approach in (1.5). We will extend the results of [Kes74] to cover the cases where unique φ exists but it is not necessarily a finite measure. Thus, Z could also be null recurrent or transient.

From another viewpoint, our problem is close to the material in [Kes95]. There is always a chance that W starting from any level drifts to $-\infty$ and never hits an upper level. To situate the discussion in the framework of the reference, consider drifting to $-\infty$ as a new state, δ. Clearly, the kernel D can be extended to the state space $\hat{S}_\delta = \{\delta\} \cup \hat{S}$ to be a stochastic kernel, i.e. for any phase $\hat{x}$,

$$\sum_{\hat{y}\in\hat{S}_\delta} D(\hat{x};\hat{y}) = D(\hat{x};\delta) + \sum_{\hat{y}\in\hat{S}} D(\hat{x};\hat{y}) = 1.$$

We may think of δ as the absorbing state.

We will state some of the results in [Kes95]. However, to avoid the intricacies only a simplified version of the assumptions are presented.

Theorem 5.1. *Assume that*

1. *for any $\hat{x} \in \hat{S}$, $D(\hat{x};\hat{y})$ is non-zero only for a finite number of $\hat{y}$'s.*

2. *there exists some strictly positive constant c such that $D(\hat{x};\hat{x}) > c$ for all $\hat{x} \in \hat{S}$.*

Let the spectral radius of D to be r, i.e.

$$\lim_{n\to\infty} (D^{(n)}(\hat{x};\hat{y}))^{1/n} = r^{-1},$$

then D has r-invariant measure μ and r-harmonic function f, which are unique up to a constant multiple and D satisfies the ratio limit property, that is to say

$$\lim_{m\to\infty} \frac{D^{(m)}(\hat{x};\hat{y})}{D^{(m)}(\hat{u};\hat{z})} = \frac{f(\hat{x})\mu(\hat{y})}{f(\hat{u})\mu(\hat{z})}.$$

In addition, if we assume that $L < \infty$ exists such that

$$\sum_{\hat{y}\in\hat{S}} D(\hat{x};\hat{y}) = 1 \ \text{ for } \hat{x} \geq L, \tag{5.4}$$

and starting from some fixed phase the chain underlined by D eventually hits δ and $r > 1$, then μ can be normalized as a probability measure and

$$\lim_{m\to\infty} \frac{D^{(m)}(\hat{x};\hat{y})}{\sum_{\hat{z}\in\hat{S}} D^{(m)}(\hat{u};\hat{z})} = \frac{f(\hat{x})}{f(\hat{u})}\mu(\hat{y}). \tag{5.5}$$

Notice that our level crossing kernel D does not satisfy assumption 1 of the theorem. This is true since W starting from state $(0,\hat{x})$ can hit level 1 for the first time at any phase $\hat{y}$, and therefore $D(\hat{x};\hat{y})$ is strictly positive for infinite number of $\hat{y}$'s.

Remark 5.2. Despite the fact that D does not comply with assumption 1, there is still some connection. Only for this remark, let Z to be a nearest neighbor chain (we already assumed this property for the level). Suppose that the chain starts from $(0,\hat{x})$, the conditional probability that W hits level 1 given that after n steps it has not hit level 1 decreases as n becomes large. This is true since W has a negative drift in the first coordinate. Therefore, starting from $(0,\hat{x})$ if $R_1 < \infty$, then W should hit level 1 at a phase not far from $\hat{x}$.

D does not satisfy (5.4) either. Therefore, it is not clear if the Yaglom limit, (5.5), exists or not. The assumption states that only finitely many states can reach the absorbing point in one step, however in our situation the chain can leave level zero from any phase and never again hit level 1. Hence, the absorbing state can be reached in one step from infinitely many states. Observe that (5.2) can be written as

$$\lim_{m\to\infty} \frac{P_\sigma\{Z[R_m] = \hat{x}\ ,\ R_m < \infty\}}{P_\sigma\{R_m < \infty\}} = \lim_{m\to\infty} \frac{D^{(m)}(\hat{\sigma};\hat{x})}{\sum_{\hat{z}\in\hat{S}} D^{(m)}(\hat{\sigma};\hat{z})},$$

therefore, our results are closely related to the Yaglom limit and the existence of quasi-stationary measures. For further references consult [Fer95].

One more interesting issue is the resemblance between (5.2) and the hitting distribution of the phase discussed in Chapter 4. To be precise, compare (4.1) with the questions currently under investigation. One way to explore the hitting distribution problems for X is to remove the boundary Δ and work with X^∞ and then prove that the two chains give the same results. Although we did some research in this direction but it appeared to be a difficult task.

5.2 Theory and Results

Our approach stems from the techniques used in the previous chapters.

Suppose $\alpha > 0$ exists such that $h(m,\hat{x}) = e^{\alpha m}\hat{h}(\hat{x})$ and take $\mathcal{W} = (\mathcal{V},\mathcal{Z})$ to be the twisted chain with kernel $\mathcal{J}$ as in (1.5), that is

$$\mathcal{J}(x;y) = J(x;y)h(y)/h(x), \tag{5.6}$$

for any $x,y \in S^\infty$.

As in (1.6), for any $x, y \in S^\infty$ define the Green function

$$\mathcal{G}(z;x) = \sum_{n=0}^{\infty} \mathcal{J}^{(n)}(z;x)$$

of $\mathcal{J}$. Moreover, throughout

$$\hat{\mathcal{J}}(\hat{z};\hat{x}) = \sum_{m\in\mathbb{Z}} \mathcal{J}((0,\hat{z});(m,\hat{x})),$$

the kernel of the Markovian part of the twisted chain will be used. To avoid complications consider $\hat{\mathcal{J}}$ to be irreducible.

The existence of an invariant measure for $\hat{\mathcal{J}}$ is in question. We can not be sure it exists since we are not out ruling unbounded jumps for the phase. Therefore, to push through the approach we limit W to the cases where the Markovian part of the twisted chain has an invariant measure φ,

$$\sum_{\hat{x}\in\hat{S}} \varphi(\hat{x})\hat{\mathcal{J}}(\hat{x};\hat{y}) = \varphi(\hat{y}).$$

This brings us to the same setting as in Proposition 1.8. The time reversal of $\mathcal{J}$ can be obtained with respect to φ, i.e. for any $(s,\hat{x})$ and $(m,\hat{y})$ in S^∞, we can define

$$\overleftarrow{\mathcal{J}}((s,\hat{x});(m,\hat{y})) = \mathcal{J}((m,\hat{y});(s,\hat{x})) \times \frac{\varphi(\hat{y})}{\varphi(\hat{x})}.$$

Take $\overleftarrow{\mathcal{G}}$ to be the Green function associated with this reversed kernel,

$$\overleftarrow{\mathcal{G}}(z;x) = \sum_{n=0}^{\infty} \overleftarrow{\mathcal{J}}^{(n)}(z;x), \text{ for any } x, z \in S^\infty.$$

The assumptions we require to establish the main results are stated below.

F1. h is chosen such that $P\{\lim_{n\to\infty} \mathcal{V}[n] = \infty \mid \mathcal{W}[0] = z\} = 1$ for all $z \in S^\infty$.

F2. The probability that W reaches $(0,\hat{y})$ starting from $(0,\hat{x})$ is positive for any $\hat{x}$ and $\hat{y}$ in $\hat{S}$. Moreover, there exists an integer N and $\omega > 0$ such that for any $\hat{x} \in \hat{S}$, there exists $m = m(\hat{x}) \leq N$ such that $\mathcal{J}^{(m)}((0,\hat{x});(1,\hat{x})) \geq \omega$.

F3. Let $R_{\hat{x}}$ be the first return time of Z to $\hat{x}$. Assume that $P_{(0,\hat{x})}\{V[R_{\hat{x}}] = \cdot\}$ is not concentrated on a subgroup of the integers.

F4. $\hat{\mathcal{J}}$ has spectral radius one.

F5. The only harmonic functions for $\hat{\mathcal{J}}$ are the constant functions.

F6a. For a fixed $\hat{x}_0$, consider the sequence of functions g_s as

$$g_s(\hat{x}) = \mathcal{G}((0,\hat{x});(s,\hat{x}_0))/\mathcal{G}((0,\hat{x}_0);(s,\hat{x}_0))$$

We assume that dominated convergence theorem can be applied to the following series:

$$\sum_{\hat{x}\in\hat{S}} \hat{\mathcal{J}}(\hat{y};\hat{x})g_s(\hat{x})$$

for any $\hat{y}$. That is to say,

$$\lim_{s\to\infty} \sum_{\hat{x}\in\hat{S}} \hat{\mathcal{J}}(\hat{y};\hat{x})g_s(\hat{x}) = \sum_{\hat{x}\in\hat{S}} \hat{\mathcal{J}}(\hat{y};\hat{x}) \lim_{s\to\infty} g_s(\hat{x}).$$

F6b. $(-\overleftarrow{\mathcal{V}}, \overleftarrow{\mathcal{Z}})$ satisfies the condition in part (a).

F7. There exists a phase $\hat{\sigma}$ such that,

$$\frac{\overleftarrow{\mathcal{G}}((0,\hat{z});(-m,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))}$$

is bounded uniformly in $\hat{z}$ for any positive m sufficiently large.

It is easy to show that $(-\overleftarrow{\mathcal{V}}, \overleftarrow{\mathcal{Z}})$ satisfies F1-F4. Let's compare these assumptions with the assumptions in Section 2.2. Note that D2 is not required since V is considered to be a nearest neighbor chain and therefore its one step jumps are bounded. F1, F2, F3, F4 , F5 and F6 are the same as D1, D3, D4, D5, D6 and D7, respectively. We only restate the fact that if the chain has bounded jumps in the phase direction then it would satisfy F6. F7 looks the same as D8, however it is stated for the time reversal.

Example 5.3. A Markov additive chain satisfying the above assumptions is the free chain associated with the modified Jackson network. All the assumptions were verified in Section 2.2 except F7.

We pursue a coupling argument to show F7. Take $\hat{\sigma} = 0$. We will show that $\overleftarrow{\mathcal{G}}((0,\hat{z});(-m,0))/\overleftarrow{\mathcal{G}}((0,0);(-m,0))$ is bounded uniformly in $\hat{z}$ as m tends to ∞. Let w and w' show trajectories starting from $(0,\hat{z})$ and $(0,0)$, respectively. Define $N(w) \equiv N_{(-m,0)}(w)$ to be the number of visits to $(-m,0)$ by $\overleftarrow{\mathcal{X}}$ following the trajectory w. A similar notion can be defined for w'. Let $H = \{w' \mid \overleftarrow{\mathcal{X}}_1[n] \leq 0, \text{ for all } n\}$.

Form the product space of all paths w and w'. On the product space, define the following coupling: stay on w' until it hits the path w and then follow w. Notice that if w hits $(-m,0)$ and $w' \in H$, then w' must hit w since w' is fenced in by w and the level axis. In fact, there is no way for w' to jump over w since there are no northeast or southwest jump directions for the reversed MJN.

Let $N(w,w')$ to be the number of visits to $(-m,0)$ by the coupled path. For any $w' \in H$, w can miss $(-m,0)$ while w' may hit the target, therefore,

$$N(w)\chi_H(w') \leq N(w,w')\chi_H(w'),$$

where χ is the characteristic function.

$$\begin{aligned}
\overleftarrow{\mathcal{G}}((0,\hat{z});(-m,0)) \cdot P_{(0,0)}\{w' \in H\} &= E_{(0,\hat{z})}\{N(w)\} \cdot P_{(0,0)}\{w' \in H\} \\
&= E_{(0,\hat{z})}\{N(w)\} \cdot E_{(0,0)}\{\chi_H(w')\} \\
&= E_{(0,\hat{z})} \otimes E_{(0,0)}\{N(w)\chi_H(w')\} \\
&\leq E_{(0,\hat{z})} \otimes E_{(0,0)}\{N(w,w')\chi_H(w')\} \\
&\leq E_{(0,\hat{z})} \otimes E_{(0,0)}\{N(w,w')\} \\
&= E_{(0,0)}\{N(w')\} \\
&= \overleftarrow{\mathcal{G}}((0,0);(-m,0)).
\end{aligned}$$

This implies that $\overleftarrow{\mathcal{G}}((0,\hat{z});(-m,0))/\overleftarrow{\mathcal{G}}((0,0);(-m,0))$ is less than or equal to the inverse of $P_{(0,0)}\{w' \in H\}$, which is uniform in terms of the phase $\hat{z}$ and the level m. Therefore, F7 holds for this model.

Similar to Definition 4.4, we need the following notion. Notice that the first variable of k^* is omitted since the level chain is limited to be nearest neighbor.

Definition 5.4. Let $k^*(\hat{x})$ be the probability that $\overleftarrow{\mathcal{W}}$ starting from $(0, \hat{x})$, leaves level zero and $\overleftarrow{\mathcal{V}}$ stays negative forever.

Clearly, the definition would have the same implication if defined as starting from $(\ell, \hat{x})$ and tracking the trajectories that never return to level ℓ afterwards.

Our main result is the following theorem.

Theorem 5.5. *Suppose the chain $W = (V, Z)$ satisfies F1-F7. If*

$$\sum_{\hat{x} \in \hat{S}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty, \tag{5.7}$$

then for any fixed state $\sigma \in S^\infty$ and fixed phase $\hat{x} \in \hat{S}$,

$$\lim_{m \to \infty} P_\sigma\{Z[R_m] = \hat{x} | R_m < \infty\} = \frac{\varphi(\hat{x})\hat{h}^{-1}(\hat{x})k^*(\hat{x})}{\sum_{\hat{y} \in \hat{S}} \varphi(\hat{y})\hat{h}^{-1}(\hat{y})k^*(\hat{y})}. \tag{5.8}$$

We defer the proof of the theorem until later.

By Proposition 2.1, (5.7) is fulfilled whenever the Markovian part of the twisted chain is positive recurrent, i.e. if $\sum_{\hat{x} \in \hat{S}} \varphi(\hat{x}) < \infty$, then the condition holds.

Comparing (5.8) with (4.8) delivers an interesting result for the Markov chains in the first orthant. Suppose the positive recurrent, irreducible Markov chain $X = (X_1, \hat{X})$ is defined on the state space $S = \mathbb{Z}_+ \times \hat{S}$. Using our usual embedding techniques X can be embedded in $S^\infty = \mathbb{Z} \times \hat{S}$. Call the new free chain $X^\infty = (X_1^\infty, \hat{X}^\infty)$. X_1^∞ and $\hat{X}^\infty$ play the role of V and Z, respectively. We have,

Corollary 5.6. *If the Markov chain X satisfies conditions of Theorems 4.5 and 5.5, then using the notation of the preceding chapters, for any fixed state $\sigma \in S^\infty$ and fixed phase $\hat{x} \in \hat{S}$,*

$$\lim_{m \to \infty} P_\sigma\{\hat{X}[T_m] = \hat{x} | T_m < T_\sigma\} = \lim_{m \to \infty} P_\sigma\{\hat{X}^\infty[T_m^\infty] = \hat{x} | T_m^\infty < \infty\}.$$

This is not a clear-cut conclusion since starting from state σ the trajectories of X and X^∞ could be very different. While X_1^∞ is allowed to travel to any negative level before reaching level m, X_1 is sandwiched between levels 0 and m and has no access to negative levels.

In addition, it can be shown that the ratio limit property holds for the level crossing kernel and the Yaglom limit exists.

Corollary 5.7. *Under the assumptions of Theorem 5.5,*

$$\lim_{m\to\infty} \frac{D^{(m)}(\hat{x};\hat{y})}{D^{(m)}(\hat{u};\hat{z})} = \frac{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})k^*(\hat{y})}{\varphi(\hat{z})\hat{h}^{-1}(\hat{z})k^*(\hat{z})}, \tag{5.9}$$

and

$$\lim_{m\to\infty} \frac{D^{(m)}(\hat{x};\hat{y})}{\sum_{\hat{z}\in\hat{S}} D^{(m)}(\hat{u};\hat{z})} = \frac{\varphi(\hat{y})\hat{h}^{-1}(\hat{y})k^*(\hat{y})}{\sum_{\hat{z}\in\hat{S}} \varphi(\hat{z})\hat{h}^{-1}(\hat{z})k^*(\hat{z})}. \tag{5.10}$$

In order to prove the results, the following lemma is required.

Lemma 5.8. *Under assumptions F1-F6, for fixed states $(n,\hat{x}),(s,\hat{u})\in S^\infty$ and fixed phases $\hat{y},\hat{z}\in\hat{S}$,*

$$\lim_{m\to\infty} \frac{\overleftarrow{\mathcal{G}}((n,\hat{x});(m,\hat{y}))}{\overleftarrow{\mathcal{G}}((s,\hat{u});(m,\hat{z}))} = \frac{\varphi(\hat{y})}{\varphi(\hat{z})}.$$

Proof.

$$\begin{aligned}&\frac{\overleftarrow{\mathcal{G}}((n,\hat{x});(m,\hat{y}))}{\overleftarrow{\mathcal{G}}((s,\hat{u});(m,\hat{z}))}\\ &= \frac{\varphi(\hat{y})}{\varphi(\hat{z})} \times \left\{\frac{\varphi(\hat{u})}{\varphi(\hat{x})} \times \frac{\mathcal{G}((m,\hat{y});(n,\hat{x}))}{\mathcal{G}((m,\hat{z});(s,\hat{u}))}\right\} \to \frac{\varphi(\hat{y})}{\varphi(\hat{z})}\times 1,\end{aligned}$$

as m tends to infinity. The limit is obtained by Theorem 2.6. □

In particular, we shall require

$$\lim_{m\to\infty} \frac{\overleftarrow{\mathcal{G}}((n,\hat{x});(m,\hat{y}))}{\overleftarrow{\mathcal{G}}((s,\hat{u});(m,\hat{y}))} = 1. \tag{5.11}$$

Proof of Theorem 5.5. For simplicity let σ to be $(0, \hat{\sigma})$.

$$\begin{aligned}
&P_\sigma\{Z[R_m] = \hat{x} \mid R_m < \infty\} \\
&= \frac{P_\sigma\{Z[R_m] = \hat{x},\ R_m < \infty\}}{P_\sigma\{R_m < \infty\}} \\
&= \frac{P_\sigma\{Z[R_m] = \hat{x},\ R_m < \infty\}}{\sum_{\hat{y}\in\hat{S}} P_\sigma\{Z[R_m] = \hat{y},\ R_m < \infty\}} && W \text{ first reaches } \ell \text{ in } \hat{y} \\
&= \frac{\hat{h}^{-1}(\hat{x}) P_\sigma\{\mathcal{Z}[\mathcal{R}_m] = \hat{x},\ \mathcal{R}_m < \infty\}}{\sum_{\hat{y}\in\hat{S}} \hat{h}^{-1}(\hat{y}) P_\sigma\{\mathcal{Z}[\mathcal{R}_m] = \hat{y},\ \mathcal{R}_m < \infty\}} && h\text{-transform.} \qquad (5.12)
\end{aligned}$$

For the probability in the last term,

$$\begin{aligned}
&P_\sigma\{\mathcal{Z}[\mathcal{R}_m] = \hat{x},\ \mathcal{R}_m < \infty\} \\
&= \sum_{p\geq 1} P_\sigma\{\mathcal{W}[p] = (m, \hat{x}), \mathcal{R}_m = p\} && \text{condition on the values of } R_m \\
&= \sum_{p} \frac{\varphi(\hat{x})}{\varphi(\hat{\sigma})} P_{(m,\hat{x})}\{\overleftarrow{\mathcal{W}}[p] = \sigma, \overleftarrow{\mathcal{R}}_m > p\} && \text{reverse time w.r.t. } \varphi \\
&= \sum_{p} \frac{\varphi(\hat{x})}{\varphi(\hat{\sigma})}\ {}_m\overleftarrow{\mathcal{J}}^{(p)}((m, \hat{x}); \sigma) \\
&= \frac{\varphi(\hat{x})}{\varphi(\hat{\sigma})}\ {}_m\overleftarrow{\mathcal{G}}((m, \hat{x}); \sigma),
\end{aligned}$$

where $\hat{\sigma}$ is σ's coordinate in $\hat{S}$ and ${}_m\overleftarrow{\mathcal{J}}$ is the taboo kernel avoiding level m. In addition, similar to (1.7), define ${}_m\overleftarrow{\mathcal{G}}$ as the Green function which stays away from level m.

Plug in the above representation into (5.12) to get

$$\frac{(\hat{h}^{-1}\varphi)(\hat{x})\ {}_m\overleftarrow{\mathcal{G}}((m, \hat{x}); \sigma)}{\sum_{\hat{y}} (\hat{h}^{-1}\varphi)(\hat{y})\ {}_m\overleftarrow{\mathcal{G}}((m, \hat{y}); \sigma)}. \tag{5.13}$$

Now, we investigate ${}_m\overleftarrow{\mathcal{G}}$. First, observe that by Proposition 1.9, ${}_m\overleftarrow{\mathcal{G}}$ is shift invariant with respect to the first coordinate. Hence,

$${}_m\overleftarrow{\mathcal{G}}((m, \hat{x}); (0, \hat{\sigma})) = {}_0\overleftarrow{\mathcal{G}}((0, \hat{x}); (-m, \hat{\sigma})),$$

so without loss of generality translate the level by $-m$ in the arguments, i.e. we start at level 0 and end up at the point $(-m, \hat{\sigma})$ while avoiding level 0.

Fix level n between levels $-m$ and 0, i.e. $-m < n < 0$. In order to reach level $-m$ starting from level 0, the chain should cross level n at some time. Let χ to be the characteristic function and $\overleftarrow{\mathcal{R}}_m$ the first return time to level m by $\overleftarrow{\mathcal{W}}$. Compare

$$\begin{aligned}
&\frac{{}_0\overleftarrow{\mathcal{G}}((0,\hat{x});(-m,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} \\
&= E_{(0,\hat{x})}\left\{\sum_{p=0}^{\overleftarrow{\mathcal{R}}_0-1}\chi(\overleftarrow{\mathcal{W}}[p]=(-m,\hat{\sigma})\ )\right\}/\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma})) \\
&= E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n<\overleftarrow{\mathcal{R}}_0)\sum_{p=0}^{\overleftarrow{\mathcal{R}}_0-1}\chi(\overleftarrow{\mathcal{W}}[p]=(-m,\hat{\sigma}))\right\}/\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))
\end{aligned} \tag{5.14}$$

and

$$E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n<\overleftarrow{\mathcal{R}}_0)\sum_{p=0}^{\infty}\chi(\overleftarrow{\mathcal{W}}[p]=(-m,\hat{\sigma}))\right\}/\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma})), \tag{5.15}$$

where in the latter term, the trajectories are allowed to hit level 0 prior to hitting level $-m$. The difference of the two terms does not exceed

$$\begin{aligned}
&E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n<\overleftarrow{\mathcal{R}}_0<\infty)\sum_{p\geq\overleftarrow{\mathcal{R}}_0}\chi(\overleftarrow{\mathcal{W}}[p]=(-m,\hat{\sigma}))\right\}/\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma})) \\
&\leq \sum_{\hat{k}} E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n<\overleftarrow{\mathcal{R}}_0<\infty)\overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_0]=\hat{k}\right\}\frac{\overleftarrow{\mathcal{G}}((0,\hat{k});(-m,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} \\
&\leq C\sum_{\hat{k}} E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n<\overleftarrow{\mathcal{R}}_0<\infty)\overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_0]=\hat{k}\right\} \qquad \text{by F7} \\
&= \frac{C}{(\hat{h}^{-1}\varphi)(\hat{x})}\sum_{\hat{k}}(\hat{h}^{-1}\varphi)(\hat{k})E_{(0,\hat{k})}\left\{\chi(R_n<R_0<\infty)Z[R_0]=\hat{x}\right\} \\
&\leq \frac{C}{(\hat{h}^{-1}\varphi)(\hat{x})}\sum_{\hat{k}}(\hat{h}^{-1}\varphi)(\hat{k})E_{(0,\hat{k})}\left\{\chi(R_n<R_0<\infty)\right\} \\
&= \frac{C}{(\hat{h}^{-1}\varphi)(\hat{x})}\sum_{\hat{k}}(\hat{h}^{-1}\varphi)(\hat{k})P_{(0,\hat{k})}\left\{R_n<R_0<\infty\right\}.
\end{aligned} \tag{5.16}$$

In (5.16), the series is finite by (5.7). Moreover, by (5.1) when W leaves level 0 and hits level n, as n becomes smaller it gets harder for the chain to get back to level 0.

In detail, $\{R_n < R_0 < \infty\}$ is a decreasing sequence of events converging to the null set, therefore for any $(0,\hat{k})$ and $n \to -\infty$,

$$P_{(0,\hat{k})}\{R_n < R_0 < \infty\} \to 0.$$

Hence, by the dominated convergence theorem, (5.16) approaches zero as n tends to $-\infty$. Therefore, for any choice of $\epsilon > 0$, n can be found such that

$$\frac{C}{(\hat{h}^{-1}\varphi)(\hat{x})}\sum_{\hat{k}}(\hat{h}^{-1}\varphi)(\hat{k})P_{(0,\hat{k})}\{R_n < R_0 < \infty\} < \epsilon.$$

As a consequence, we shall replace (5.14) with (5.15) in the rest of the proof. In (5.15) condition on the phase that we reach level n for the first time

$$\sum_{\hat{k}} E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0), \overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_n] = \hat{k}, \sum_{p=0}^{\infty}\chi(\overleftarrow{\mathcal{W}}[p] = (-m,\hat{\sigma}))\right\} / \overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma})).$$

Using strong Markov property, this is equal to

$$\begin{aligned}
&\sum_{\hat{k}} E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0), \overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_n] = \hat{k}\right\}\frac{\overleftarrow{\mathcal{G}}((n,\hat{k});(-m,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} \\
&= \sum_{\hat{k}} E_{(0,\hat{x})}\left\{\chi(\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0), \overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_n] = \hat{k}\right\}\frac{\overleftarrow{\mathcal{G}}((0,\hat{k});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))}. \qquad (5.17)
\end{aligned}$$

By (5.11)

$$\frac{\overleftarrow{\mathcal{G}}((0,\hat{k});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} \to 1,$$

as $m \to \infty$. In addition,

$$\frac{\overleftarrow{\mathcal{G}}((0,\hat{k});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} = \frac{\overleftarrow{\mathcal{G}}((0,\hat{k});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))} \times \frac{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))}.$$

The first fraction is bounded uniformly in $\hat{k}$ for large m's by F7 and for the second fraction we have

$$\frac{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))}$$

$$
\begin{aligned}
&\leq \frac{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))}{\{\text{probability of reaching } (n,\hat{\sigma}) \text{ starting from } (0,\hat{\sigma})\} \times \overleftarrow{\mathcal{G}}((n,\hat{\sigma});(-m,\hat{\sigma}))} \\
&= \frac{1}{\text{probability of reaching } (n,\hat{\sigma}) \text{ starting from } (0,\hat{\sigma})} \times \frac{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m-n,\hat{\sigma}))} \\
&= \frac{1}{\text{probability of reaching } (n,\hat{\sigma}) \text{ starting from } (0,\hat{\sigma})},
\end{aligned}
$$

and this does not involve $\hat{k}$ or m. Using dominated convergence theorem, as $m \to \infty$, (5.17) tends to

$$\sum_{\hat{k}} E_{(0,\hat{x})} \left\{ \chi(\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0), \overleftarrow{\mathcal{Z}}[\overleftarrow{\mathcal{R}}_n] = \hat{k} \right\},$$

which in turn is identical to

$$P_{(0,\hat{x})}\{\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0\}. \tag{5.18}$$

Notice that $\{\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0\}$ for $n = -1, -2, -3, \cdots$ is a decreasing sequence of events. Partition the sample space to $\{\overleftarrow{\mathcal{R}}_0 < \infty\}$ and $\{\overleftarrow{\mathcal{R}}_0 = \infty\}$. We have

$$\bigcap_{n \leq -1} \{\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0\} = \bigcap_{n \leq -1} \{\overleftarrow{\mathcal{R}}_n < \overleftarrow{\mathcal{R}}_0 = \infty\},$$

bearing in mind that $\overleftarrow{\mathcal{W}}$ has a negative drift for the level, the above intersection is equal to $\{\overleftarrow{\mathcal{R}}_0 = \infty\}$. Therefore, as $n \to -\infty$, (5.18) converges to $k^*(\hat{x})$.

So it is proved that as $m \to \infty$

$$\frac{{}_m\overleftarrow{\mathcal{G}}((m,\hat{x});(0,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((m,\hat{\sigma});(0,\hat{\sigma}))} = \frac{{}_0\overleftarrow{\mathcal{G}}((0,\hat{x});(-m,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((0,\hat{\sigma});(-m,\hat{\sigma}))} \to k^*(\hat{x}) \tag{5.19}$$

We are currently in a position to evaluate (5.13) as $m \to \infty$,

$$
\begin{aligned}
&\frac{(\hat{h}^{-1}\varphi)(\hat{x})\ {}_m\overleftarrow{\mathcal{G}}((m,\hat{x});\sigma)}{\sum_{\hat{y}}(\hat{h}^{-1}\varphi)(\hat{y})\ {}_m\overleftarrow{\mathcal{G}}((m,\hat{y});\sigma)} \\
&\quad = \frac{(\hat{h}^{-1}\varphi)(\hat{x})\ {}_m\overleftarrow{\mathcal{G}}((m,\hat{x});(0,\hat{\sigma}))/\overleftarrow{\mathcal{G}}((m,\hat{\sigma});(0,\hat{\sigma}))}{\sum_{\hat{y}}(\hat{h}^{-1}\varphi)(\hat{y})\ {}_m\overleftarrow{\mathcal{G}}((m,\hat{y});(0,\hat{\sigma}))/\overleftarrow{\mathcal{G}}((m,\hat{\sigma});(0,\hat{\sigma}))},
\end{aligned}
\tag{5.20}
$$

By (5.7), $\sum_{\hat{y}}(\hat{h}^{-1}\varphi)(\hat{y}) < \infty$ and moreover,

$$\frac{{}_m\overleftarrow{\mathcal{G}}((m,\hat{y});(0,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((m,\hat{\sigma});(0,\hat{\sigma}))} \leq \frac{\overleftarrow{\mathcal{G}}((m,\hat{y});(0,\hat{\sigma}))}{\overleftarrow{\mathcal{G}}((m,\hat{\sigma});(0,\hat{\sigma}))},$$

which by F7 is uniformly bounded in $\hat{y}$ for large m. Using the dominated convergence theorem one more time as $m \to \infty$, (5.20) converges to

$$\frac{\varphi(\hat{x})\hat{h}^{-1}(\hat{x})k^*(\hat{x})}{\sum_{\hat{y}\in\hat{S}}\varphi(\hat{y})\hat{h}^{-1}(\hat{y})k^*(\hat{y})},$$

and this ends the proof. □

Proof of Corollary 5.7. Observe that

$$D^{(n)}(\hat{x};\hat{y}) = P_{(0,\hat{x})}\{Z[R_n] = \hat{y}, R_n < \infty\},$$

and

$$\frac{D^{(n)}(\hat{x};\hat{y})}{\sum_{\hat{z}\in\hat{S}} D^{(n)}(\hat{u};\hat{z})} = P_{(0,\hat{x})}\{Z[R_n] = \hat{y} \mid R_n < \infty\},$$

therefore the results are immediate by Theorem 5.5. □

Invariant vectors of D.

Define the twisted kernel associated with D as

$$\mathcal{D}(\hat{x};\hat{y}) = D(\hat{x};\hat{y}) \times \frac{e^{\alpha}\hat{h}(\hat{y})}{\hat{h}(\hat{x})}. \tag{5.21}$$

$\mathcal{D}$ is well-defined since,

$$\begin{aligned} D(\hat{x};\hat{y}) \times \frac{e^{\alpha}\hat{h}(\hat{y})}{\hat{h}(\hat{x})} &= P_{(0,\hat{x})}\{Z[R_1] = \hat{y}, R_1 < \infty\} \times \frac{e^{\alpha}\hat{h}(\hat{y})}{\hat{h}(\hat{x})} \\ &= P_{(0,\hat{x})}\{\mathcal{Z}[\mathcal{R}_1] = \hat{y}, \mathcal{R}_1 < \infty\}. \end{aligned}$$

So it is clear that $\mathcal{D}(\hat{x};\hat{y})$ is the probability that the twisted chain $\mathcal{W}$ with kernel $\mathcal{J}$ leaves $(\ell,\hat{x})$ and hits level $\ell+1$ for the first time at $\hat{y}$.

Theorem 5.9. *$\iota = \varphi k^*$ on $\hat{S}$ is an invariant measure for $\mathcal{D}$.*

Proof. For $\hat{y}$ fixed,

$$\sum_{\hat{x}} \varphi(\hat{x})k^*(\hat{x})\mathcal{D}(\hat{x};\hat{y})$$

$$= \sum_{\hat{x}} k^*(\hat{x})[\sum_{n\geq 1} \sum_{\substack{y_0=(m,\hat{x}),y_1,\cdots,y_{n-1},y_n=(m+1,\hat{y}) \\ \forall p\leq n-1\ ;\ level(y_p)\leq m}} \varphi(\hat{x})\mathcal{J}(\hat{y}_0;\hat{y}_1)\cdots\mathcal{J}(\hat{y}_{n-1};\hat{y}_n)\ ]$$

$$= \sum_{\hat{x}} k^*(\hat{x})[\sum_{n\geq 1} \sum_{\substack{y_n=(m+1,\hat{y}),y_{n-1},\cdots,y_1,y_0=(m,\hat{x}) \\ \forall p\leq n-1\ ;\ level(y_p)\leq m}} \varphi(\hat{y})\overleftarrow{\mathcal{J}}(\hat{y}_n;\hat{y}_{n-1})\cdots\overleftarrow{\mathcal{J}}(\hat{y}_1;\hat{y}_0)\ ]$$

$$= \varphi(\hat{y})\sum_{\hat{x}}[\sum_{n\geq 1} \sum_{\substack{y_n=(m+1,\hat{y}),y_{n-1},\cdots,y_1,y_0=(m,\hat{x}) \\ \forall p\leq n-1\ ;\ level(y_p)\leq m}} \overleftarrow{\mathcal{J}}((m+1,\hat{y});\hat{y}_{n-1})\cdots\overleftarrow{\mathcal{J}}(\hat{y}_1;(m,\hat{x}))k^*(\hat{x})\ ]$$

$$= \varphi(\hat{y})k^*(\hat{y}).$$

To justify the last equality, notice that the expression in the bracket represents the probability that the time reversed Markov chain leaves level $m+1$ from $\hat{y}$ forever, visits level m for the last time at $\hat{x}$ and leaves it from this point. Therefore the summation of the expression in the bracket over $\hat{x}$ gives $k^*(\hat{y})$. □

Corollary 5.10. *If $\sum_{\hat{x}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty$, then $\mu = a^{-1}\varphi k^*\hat{h}^{-1}$ is an $e^{-\alpha}$-invariant distribution for D, where $a = \sum_{\hat{x}\in\hat{S}}(\varphi k^*\hat{h}^{-1})(\hat{x})$. To be precise, for any fixed $\hat{y}$,*

$$\sum_{\hat{x}\in\hat{S}} \mu(\hat{x})D(\hat{x};\hat{y}) = e^{-\alpha}\mu(\hat{y}).$$

Proof. The normalization factor, a, is finite since using the assumption,

$$a \leq \sum_{\hat{x}} \varphi(\hat{x})\hat{h}^{-1}(\hat{x}) < \infty.$$

Therefore, μ is a probability distribution.

For any $\hat{y}\in\hat{S}$,

$$\sum_{\hat{x}\in\hat{S}}(\varphi k^*\hat{h}^{-1})(\hat{x})D(\hat{x};\hat{y}) = \sum_{\hat{x}\in\hat{S}}(\varphi k^*)(\hat{x})\left(\hat{h}^{-1}(\hat{x})D(\hat{x};\hat{y})\right)$$

$$= e^{-\alpha}\hat{h}^{-1}(\hat{y})\sum_{\hat{x}\in\hat{S}}(\varphi k^*)(\hat{x})\mathcal{D}(\hat{x};\hat{y}) \qquad \text{by (5.21)}$$

$$= e^{-\alpha}\hat{h}^{-1}(\hat{y})(\varphi k^*)(\hat{y}) \qquad \text{using Theorem 5.9.}$$

□

In accordance with Corollary 5.10, $\hat{h}$ is an $e^{-\alpha}$-invariant vector for D. For any fixed $\hat{x}$,

$$\sum_{\hat{y}\in\hat{S}} D(\hat{x};\hat{y})\hat{h}(\hat{y}) = e^{-\alpha}\hat{h}^{-1}(\hat{x})\sum_{\hat{y}\in\hat{S}}\mathcal{D}(\hat{x};\hat{y}) \qquad \text{by (5.21)}$$

$$= e^{-\alpha}\hat{h}^{-1}(\hat{x}) \times 1, \qquad \text{using F1.}$$

We prompt the following definitions,

(i) D is $e^{-\alpha}$-recurrent or transient if $\sum_{n\geq 0} e^{-\alpha n} D^{(n)}(\hat{x};\hat{y})$ is divergent or convergent, respectively for any $\hat{x}$ and $\hat{y}$.

(ii) The $e^{-\alpha}$-recurrent D is null or positive if $e^{-\alpha n} D^{(n)}(\hat{x};\hat{y})$ converges or does not converge to zero, respectively for any choice of $\hat{x}$ and $\hat{y}$.

Proposition 5.11. *Assume that*

1. the probability k^* can be bounded away from zero, i.e. $\varepsilon > 0$ and a phase $\hat{x}_0$ exist such that for any $\hat{x} \geq \hat{x}_0$,

$$k^*(\hat{x}) \geq \varepsilon > 0. \tag{5.22}$$

2. $\sum_{\hat{x}}(\varphi\hat{h}^{-1})(\hat{x}) < \infty$,

3. $\sum_{\hat{x}}\varphi(\hat{x}) = \infty$,

then the matrix D is not $e^{-\alpha}$-positive recurrent, though it has an $e^{-\alpha}$-invariant finite measure.

An example where condition (5.22) holds is the Markov additive chain X^∞ associated with X which was the joint queue length of the modified Jackson network; refer to Section 1.2.

Proof of Proposition 5.11. Corollary 5.10 shows that under the second assumption of the proposition $\mu = a^{-1}\varphi k^* \hat{h}^{-1}$ is a finite $e^{-\alpha}$-invariant measure. Also, we proved that $\hat{h}$ is a right $e^{-\alpha}$-invariant vector.

By the well-known Theorem D in [Ver67], D is not $e^{-\alpha}$-positive recurrent if and only if $|\mu\hat{h}| = \sum_{\hat{x}}(\mu\hat{h})(\hat{x}) = \infty$. But

$$\begin{aligned}\sum_{\hat{x}}(\mu\hat{h})(\hat{x}) &= \sum_{\hat{x}}(a^{-1}\varphi k^*\hat{h}^{-1})(\hat{x})\hat{h}(\hat{x}) \\ &= a^{-1}\sum_{\hat{x}}(\varphi k^*)(\hat{x}) \\ &\geq a^{-1}\varepsilon\sum_{\hat{x}}\varphi(\hat{x}) = \infty, \qquad \text{by (5.22)}\end{aligned}$$

where the last equality is obtained by the third assumption and is valid for all but possibly a finite number of $\hat{x}$'s. □

This is a fascinating result illustrating the complexity of the topic under investigation. Being 1-positive recurrent is equivalent to having a finite invariant measure which is unique up to constant multiples. However, the above Proposition shows that this may not be true for $e^{-\alpha}$-positive recurrent chains where $e^{-\alpha} < 1$.

5.3 Final Remarks

The results of this chapter prove that there is room left to improve H. Kesten's findings in [Kes95] more than a decade ago. However, this happens to be a knotty problem and even some published results, like [Kij92], have failed to tackle the obstacles; see also [Kij93].

From another scene, the true nature of conditions under which the Markovian part of $\mathcal{W}$ has a unique invariant measure, φ, remain unexplained. The necessary and sufficient conditions for such an existence can be found in [Har57, Vee63, Yan71] and [Pru64].

One more issue to talk about is the existence of a quasi-stationary measure for D. The current conditions to ensure the existence of the measure do not apply to D. For example D does not satisfy condition (1.4) in Theorem 1.1 of [Fer95]. In detail and in accordance with the notation used in this section, take U_m to be $Z[R_m]$ and let ξ be the hitting time that U hits the absorbing state, δ, then

$$\lim_{\hat{x}\to\infty} P_{\hat{x}}(\xi < t) \neq 0 \quad \text{for any } t \geq 0.$$

However, we know that there is a chance that the quasi-stationary measure exists even though the known sufficient conditions do not hold, [Hog97, Pak95].

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Notations.