Option Pricing with Long Memory Stochastic Volatility Models

Zhigang Tong

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

In this thesis, we propose two continuous time stochastic volatility models with long memory that generalize two existing models. More importantly, we provide analytical formulae that allow us to study option prices numerically, rather than by means of simulation. We are not aware about analytical results in continuous time long memory case. In both models, we allow for the non-zero correlation between the stochastic volatility and stock price processes. We numerically study the effects of long memory on the option prices. We show that the fractional integration parameter has the opposite effect to that of volatility of volatility parameter in short memory models. We also find that long memory models have the potential to accommodate the short term options and the decay of volatility skew better than the corresponding short memory stochastic volatility models.

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Chapter 1

Introduction

The story of modeling financial markets with stochastic processes dates back as far as 1900 with studies of Bachelier. He modeled the stock prices as a Brownian motion with drift. A more appropriate model is based on geometric Brownian motion. Black and Scholes (1973) demonstrate how to price options under this assumption. Today this model is known as the Black-Scholes model and remains one of the most successful and widely used derivative pricing models available.

The main drawback of the Black-Scholes model is the rather strong assumption that the volatility of stock returns is constant. Under the assumption, when the implied volatility calculated from the empirical option data is plotted against option's strike price and time to maturity, the resulting graph should be a flat surface. However, in practice, the implied volatility surface is not flat and the implied volatility tends to vary with the strike price and time to maturity. This disparity is known as the volatility skew.

This consequently leads to development of dynamic volatility modeling. A natural extension is so-called stochastic volatility model in which the volatility is a function of some stochastic process. We have a variety of stochastic volatility models. The representative models are Hull and White model (1987), Heston model (1993) and

Schöbel-Zhu model (1999). The analytical formulae are known for the latter two models. It is now well known that these models are able to reproduce some empirical stylized facts regarding derivative securities and implied volatilities.

The main problem with these standard stochastic volatility models is that they cannot capture the well-documented evidence of volatility persistence and particularly occurrence of fairly pronounced implied volatility skew effects even for rather long maturity options. In practice, a decrease of the skew amplitude when time to maturity increases turns out to be much slower than it goes according to the standard stochastic volatility model. One way to solve this problem is to model volatility as a long memory stochastic process. The idea of long memory stochastic volatility is not new in the literature. It has been empirically observed that the autocorrelation function of the squared returns is usually characterized by its slow decay towards zero. This decay is neither exponential, as in short memory processes, nor implies a unit root, as in integrated processes. Consequently, it has been suggested that the squared returns may be modeled as a long memory process, whose autocorrelations decay at a hyperbolic rate. In this direction, Comte and Renault (1998) propose a continuous time fractional stochastic volatility model. They assume that the stochastic volatility is driven by fractional Ornstein-Uhlenbeck process; that is the standard Ornstein-Uhlenbeck process where the Brownian motion is replaced by a fractional Brownian Motion. Comte, Coutin and Renault (2003) consider a fractional affine stochastic volatility model, where the volatility process is driven by a fractional square root process. In both models they assume that return process is independent of the volatility process. Due to the complex structures of the long memory stochastic processes, they cannot derive the analytical formulae for option pricing. Instead, they introduce some discretization schemes and price options using Monte-Carlo simulations. Chronopoulou and Viens (2012a) study the stochastic volatility model of Comte and Renault (1998). Chronopoulou and Viens (2012b) also study two discrete time models: a discretization of the continuous model of Comte and Renault (1998) via an

Euler scheme and a discrete time model in which the returns are a zero mean i.i.d. sequence where the volatility is exponential of a fractional ARIMA process. In order to deal with the pricing problem, Chronopoulou and Viens (2012a, 2012b) construct a multinomial recombining tree using sampled values of the volatility.

In this thesis, we extend the works of Comte and Renault (1998) and Comte, Coutin and Renault (2003). We propose two continuous time long memory stochastic volatility models. The first model is the fractional Heston model where we model the volatility as a fractional square root process, as in Comte, Coutin and Renault (2003). However, we allow the return process to be correlated with the volatility process. We use Fourier inversion techniques to obtain the closed-form solutions for option prices. The second model is the fractional Schöbel-Zhu model, where we model the volatility as a fractional Ornstein-Uhlenbeck process, as in Comte and Renault (1998). Unfortunately, we cannot find the closed-form solution for this continuous time model. Instead, we discretize the original model and then derive the analytical formula for option pricing based on the resulting discrete time model.

We numerically study the effects of long memory on the option prices. Without the closed-form solutions for option prices, this would be a time-consuming task. We show that the fractional integration parameter has the opposite effect to that of volatility of volatility parameter. In the fractional Heston model, the lower integration parameter will increase the kurtosis of returns and this has the effect of raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices. In the fractional Schöbel-Zhu model, the lower integration parameter will increase the option prices. We also find the long memory stochastic volatility models can capture the well-documented evidence of volatility persistence. Long memory models have the potential to accommodate the short term options and the decay of volatility skew better than the corresponding short memory stochastic volatility models.

The structure of this thesis is as follows. In Chapter 2, we provide a brief introduction to stochastic processes and some mathematical tools. We introduce the concepts of Brownian motion and stochastic integrals. We also include important lemmas and theorems such as Itô's Lemma, Feynman-Kac theorem and Girsanov's theorem. We apply these theorems to some important stochastic process, including geometric Brownian motion, Ornstein-Uhlenbeck process and square root process. In Chapter 3, we explain the concepts of self-financing, no arbitrage and equivalent martingale measure. We furthermore show under which conditions an economy is free of arbitrage opportunities and how prices of derivatives can be calculated. As an example, we analyze the Black-Scholes option pricing model. In Chapter 4, we review three representative stochastic volatility models, namely Hull and White model (1987), Heston model (1993) and Schöbel-Zhu model (1999). We show how to compute the option prices under these models. In Chapter 5, we discuss long memory processes and show several aspects of their behavior. We introduce the definitions of long memory process, self-similar processes and fractional Brownian motion. We briefly discuss the concept of fractional integration and fractional calculus. We also mention how to generalize fractional Brownian motion to fractionally integrated processes. We give two examples of fractionally integrated processes: fractional Ornstein-Uhlenbeck process and fractional square root process. In chapter 6, we introduce two fractional stochastic volatility models: fractional Heston model and fractional Schöbel-Zhu model. We show how to obtain the analytical solution for option prices under these models. We also numerically investigate the effects of long memory on the option pricing. We summarize the thesis and discuss possible future extensions in Chapter 7.

In summary, in this thesis we propose two stochastic volatility models with long memory that generalize two existing models. More importantly, we provide analytical formulae that allow us to study option prices numerically, rather than by means of simulation. We are not aware about analytical results in continuous time long memory case.

Chapter 2

Stochastic Processes and Stochastic Calculus for Option Pricing

This chapter provides a brief introduction to stochastic processes and the so-called stochastic calculus. We will omit some technical details that are not crucial for a reasonable level of understanding and focus on processes and results that will become important in later chapters. The recommended references in this area are Björk (2009), Karatzas and Shreve (1991), Mikosch (1999), Øksendal (2010), Shreve (2004) and Zhu (2009).

2.1 Brownian Motion

Brownian motion plays a central role in probability theory, theory of stochastic processes, and also in finance. We start with a definition of this important process. Then we will list some of its elementary properties.

Definition 2.1.1 *A stochastic process $B(t)$ is called a Brownian motion or a Wiener* *process if it satisfies the following conditions:*

- $B(0) = 0$;
- $B(t)$ *has independent increments. In other words,* $B(u) - B(t)$ *and* $B(s) - B(r)$ *are independent for* $r < s \leq t < u$;
- $B(t)$ *has continuous trajectories;*
- $B(t) - B(s) \sim N(0, t - s)$ *for* $s < t$.

The finite dimensional distributions of Brownian motion are multivariate Gaussian, hence $B(t)$ is a Gaussian process. From the definition, we know that $B(t) - B(s)$ has the same distribution as $B(t - s) - B(0) = B(t - s)$, which is normal with mean zero and variance $t - s$.

It is immediate from the definition that Brownian motion has expectation function

$$\mathrm{E(B(t)) = 0.}$$

It has covariance function

$$\begin{aligned}\mathrm{Cov(B(t), B(s))} &= \mathrm{E\left[(B(t) - B(s) + B(s))B(s)\right] = E\left[(B(t) - B(s))B(s)\right] + E(B^2(s))} \\ &= \mathrm{E(B(t) - B(s))E(B(s)) + s = 0 + s = s, \qquad s < t.}\end{aligned}$$

Hence,

$$\mathrm{Cov(B(t), B(s)) = min(s, t).}$$

The defining characteristics of a standard Brownian motion look very nice, but they have some drastic consequences. It can be shown that the paths of a standard Brownian motion are nowhere differentiable, which roughly means that the paths change a shape in a neighborhood of any point in a completely non-predictable way.

2.2 Stochastic Integrals

We now turn to the construction of the stochastic integral. For that purpose, we consider as given a Brownian motion $B(t)$ and another stochastic process $X(t)$. We assume that both processes live in a probability space Ω. In order to guarantee the existence of the stochastic integral we have to introduce the idea of filtration and the class $\mathcal{L}^2$.

Assume that $\{\mathcal{F}_t\}$ is a collection of σ-fields on the same probability space Ω and that all $\{\mathcal{F}_t\}$s are subsets of a larger σ-field $\mathcal{F}$ on Ω.

Definition 2.2.1 *The collection $\{\mathcal{F}_t\}$ of σ-fields on Ω is called a filtration if*

$$\mathcal{F}_s \subset \mathcal{F}_t, \quad \textit{for all} \quad s < t.$$

Thus, informally speaking, a filtration is an increasing stream of information. For applications, a filtration is usually linked to a stochastic process.

Definition 2.2.2 *The stochastic $X(t)$ is said to be adapted to the filtration $\{\mathcal{F}_t\}$ if*

$$\sigma(X(t)) \subset \mathcal{F}_t.$$

In particular, $\sigma(X(s),\ s \leq t) \subset \mathcal{F}_t$.

We now define the class $\mathcal{L}^2$.

Definition 2.2.3 *A stochastic process $X(t)$ belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied:*

- *$X(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$;*
- $\int_a^b \mathrm{E}[\mathrm{X}^2(\mathrm{s})]\mathrm{ds} < \infty$.

Suppose $X(t)$ is a stochastic process that belongs to the class $\mathcal{L}^2[0,T]$. For a given partition

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$$

and $t \in [t_{k-1}, t_k]$, let $s_n(t)$ be a Riemann-Stieltjes sum defined by

$$s_n(t) = \sum_{i=1}^{k-1} X(t_{i-1})(B(t_i) - B(t_{i-1})) + X(t_{k-1})(B(t) - B(t_{k-1})).$$

Let $I_t(X)$ denote the mean square limit of $s_n(t)$ (if exists):

$$\lim_{n\to\infty} \mathrm{E}[(\mathrm{s_n(t)} - \mathrm{I_t(X)})^2] = 0.$$

Definition 2.2.4 *The mean square limit $I_t(X)$ is called the Itô stochastic integral of $X(t)$. It is denoted by*

$$I_t(X) = \int_0^t X(s)dB(s), \quad t \in [0,T].$$

The Itô stochastic integral $I_t(X) = \int_0^t X(s)dB(s),\ t \in [0,T]$, constitutes a stochastic process.

Next, we introduce the concept of a martingale.

Definition 2.2.5 *A stochastic process $(X(t),\ t \geq 0)$ is called a martingale with respect to a filtration $\{\mathcal{F}_t,\ t \geq 0\}$ if*

- $\mathrm{E}[|\mathrm{X(t)}|] < \infty$ *for each t;*
- *$X(t)$ is adapted to $\{\mathcal{F}_t\}$;*
- $\mathrm{E}[\mathrm{X(t)}|\mathcal{F}_\mathrm{s}] = \mathrm{X(s)}$ *for all s and t with $s \leq t$.*

With the concepts of stochastic integrals and martingales, we can see that the Itô stochastic integral $I_t(X) = \int_0^t X(s)dB(s)$ has the following properties. To state them, assume that $X(t) \in \mathcal{L}^2[0,T]$.

- $I_t(X)$ for $t \in [0,T]$ is a martingale with respect to the natural Brownian filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$, that is

$$\mathrm{E}\left[\int_0^t \mathrm{X(s)dB(s)}|\mathcal{F}_\mathrm{s}\right] = \int_0^\mathrm{s} \mathrm{X(s)dB(s)},\ \text{for s} \leq \mathrm{t};$$

- $I_t(X)$ has expectation zero;

- $\mathrm{E}\left[\int_0^t \mathrm{X(s)dB(s)}\right]^2 = \int_0^t \mathrm{E[X^2(s)]ds},\ t \in [0,T]$;

- For $X(t)$ and $Y(t)$ in $\mathcal{L}^2[0,T]$, we have

$$\mathrm{E}\left[\int_0^t \mathrm{X(s)dB(s)} \int_0^t \mathrm{Y(s)dB(s)}\right] = \int_0^t \mathrm{E[X(s)Y(s)]ds},\ \mathrm{t} \in [0,\mathrm{T}].$$

2.3 Itô's Lemma

Let $X(t)$ be a stochastic process and suppose that there exists a real number $x(0)$ and two adapted processes $\mu(t)$ and $\sigma(t)$ such that the following relation holds for all $t \geq 0$.

$$X(t) = x(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s). \tag{2.3.1}$$

We will often write equation (2.3.1) in the following form

$$dX(t) = \mu(t)dt + \sigma(t)dB(t), \tag{2.3.2}$$

$$X(0) = x(0). \tag{2.3.3}$$

In this case we say that $X(t)$ has a stochastic differential given by (2.3.2) with the initial condition given by (2.3.3). Note that the formal notation $dX(t) = \mu(t)dt +$

$\sigma(t)dB(t)$ has no particular meaning. It is simply a shorthand version of the expression (2.3.1) above.

In pricing options, we often take as given a stochastic differential equation (SDE) for some basic quantity such as stock price. Many other quantities of interest will be functions of that basic process. To determine the dynamics of these other processes, we shall apply Itô's Lemma, which is basically the chain rule for stochastic processes.

Theorem 2.3.1 *(Itô's Lemma) Assume that $X(t)$ is a stochastic process with the stochastic differential given by*

$$dX(t) = \mu(t)dt + \sigma(t)dB(t),$$

where $\mu(t)$ and $\sigma(t)$ are adapted processes to a filtration $\{\mathcal{F}_t\}$. Let $Y(t)$ be a new process defined by $Y(t) = f(X(t), t)$ where $f(x, t)$ is a function twice differentiable in its first argument and once in its second. Then $Y(t)$ has the stochastic differential:

$$dY(t) = \left(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma(t)\frac{\partial f}{\partial X}dB(t),$$

where $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x}(x,t)|\{x = X(t)\}$ and $\frac{\partial^2 f}{\partial X^2} = \frac{\partial^2 f}{\partial x^2}(x,t)|\{x = X(t)\}$.

The proof is based on a Taylor expansion of $f(X(t), t)$ combined with appropriate limits. The formal proof can be found in Øksendal (2010) and similar textbooks. In the following section, we will give examples of applications of Itô's Lemma.

2.4 Important Stochastic Processes in Finance

In this section we will discuss particular examples of stochastic processes that are frequently applied in financial models. Most of these processes are built using a Brownian motion introduced in section 2.1.

2.4.1 Geometric Brownian Motion

A stochastic process $X(t)$ is said to be a geometric Brownian motion if it is a solution to a stochastic differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t), \tag{2.4.1}$$

for given constants $\mu \in \mathbb{R}$ and $\sigma > 0$. The initial value for the process is assumed to be positive, $x(0) > 0$.

To find a solution to the stochastic differential equation (2.4.1), we apply Itô's Lemma with a function $f(x,t) = \ln(x)$ and define the process $Y(t) = f(X(t),t) = \ln(X(t))$. Since

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2},$$

we get from Itô's Lemma that by setting $\mu(t) = \mu X(t)$ and $\sigma(t) = \sigma X(t)$,

$$\begin{aligned} dY(t) &= \left(0 + \frac{1}{X(t)}\mu X(t) - \frac{1}{2}\frac{1}{X^2(t)}\sigma^2 X^2(t)\right) dt + \frac{1}{X(t)}\sigma X(t)dB(t) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB(t). \end{aligned} \tag{2.4.2}$$

Hence, we have

$$Y(t) = y(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t), \tag{2.4.3}$$

which implies that

$$\ln(X(t)) = \ln(x(0)) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t).$$

Taking exponentials on both sides, we get

$$X(t) = x(0) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t)\right]. \tag{2.4.4}$$

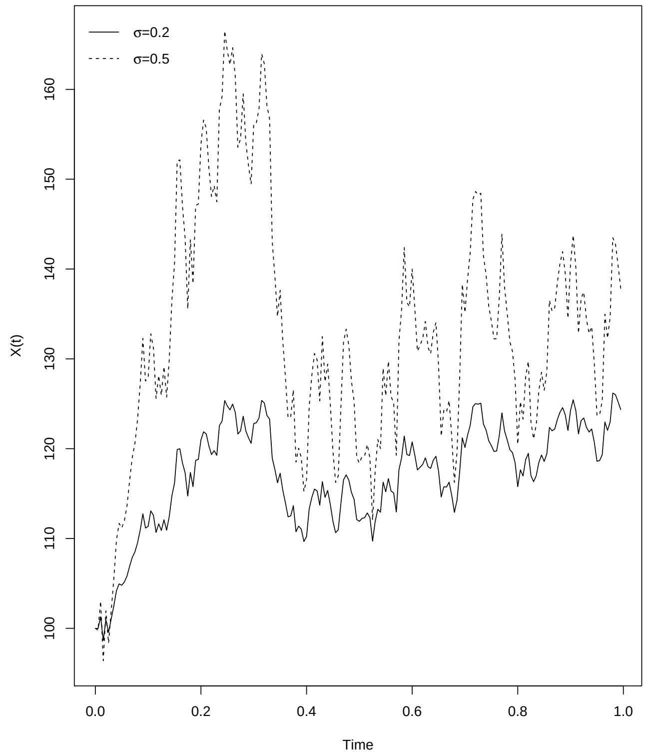

Figure 2.1: Simulation of geometric Brown motion for different volatility of volatility parameters. $dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$. $\mu = 0.1$.

Since $B(t) \sim N(0, t)$, we see from (2.4.4) that $X(t)$ given $X(0) = x(0)$ will be log-normally distributed. The paths of $X(t)$ can be simulated based on (2.4.2) by computing

$$Y(t_i) = Y(t_{i-1}) + \left(\mu - \frac{1}{2}\sigma^2\right)(t_i - t_{i-1}) + \sigma Z(t_i)\sqrt{t_i - t_{i-1}},$$

and

$$X(t_i) = \exp(Y(t_i)),$$

where $Z(t_i)$'s are i.i.d. and $Z(t_i) \sim N(0, 1)$.

Figure 2.1 shows a single simulated path for $\sigma = 0.2$ and a path for $\sigma = 0.5$. For both paths we have used $\mu = 0.1$ and $x(0) = 100$, and the same sequence of random numbers.

2.4.2 Ornstein-Uhlenbeck Process

A stochastic process $X(t)$ is said to be an Ornstein-Uhlenbeck process if its dynamics is of the form

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dB(t), \tag{2.4.5}$$

where κ, θ and σ are constants with $\kappa > 0$ and $\sigma > 0$. An Ornstein-Uhlenbeck process exhibits mean reversion in the sense that the drift is positive when $X(t) < \theta$ and negative when $X(t) > \theta$. The process is therefore always pulled towards a long-term level of θ. However, the random shock to the process through the term $\sigma dB(t)$ may cause the process to move further away from θ. The parameter κ controls the size of the expected adjustment towards the long-term level and is often referred to as the mean reversion parameter or the speed of adjustment.

To find a solution to the stochastic equation (2.4.5), we apply Itô's Lemma with the function $f(x,t) = \exp(\kappa t)x$ and define the process $Y(t) = f(X(t),t) = \exp(\kappa t)X(t)$. Since

$$\frac{\partial f}{\partial t} = \kappa \exp(\kappa t)x, \quad \frac{\partial f}{\partial x} = \exp(\kappa t), \quad \frac{\partial^2 f}{\partial x^2} = 0,$$

we get from Itô's Lemma that by setting $\mu(t) = \kappa(\theta - X(t))$ and $\sigma(t) = \sigma$,

$$\begin{aligned} dY(t) &= (\kappa \exp(\kappa t)X(t) + \exp(\kappa t)\kappa(\theta - X(t)) + 0)\, dt + \sigma \exp(\kappa t)dB(t) \\ &= \kappa\theta \exp(\kappa t)dt + \sigma \exp(\kappa t)dB(t). \end{aligned} \tag{2.4.6}$$

Hence, we have

$$Y(t) = y(0) + \int_0^t \kappa\theta \exp(\kappa u)du + \int_0^t \sigma \exp(\kappa u)dB(u).$$

After substitution of the definition of $Y(t)$ and a multiplication by $\exp(-\kappa t)$, we arrive at the expression

$$X(t) = \exp(-\kappa t)x(0) + \theta(1 - \exp(-\kappa t)) + \int_0^t \sigma \exp(-\kappa(t-u))dB(u). \qquad (2.4.7)$$

From the properties of the stochastic integral, we know that the integral $\int_0^t \sigma \exp(-\kappa(t-u))dB(u)$ is normally distributed with mean zero and variance

$$\mathrm{Var}\left[\int_0^{\mathrm{t}} \sigma \exp(-\kappa(\mathrm{t}-\mathrm{u}))\mathrm{dB(u)}\right] = \int_0^{\mathrm{t}} \sigma^2 \exp(-2\kappa(\mathrm{t}-\mathrm{u}))\mathrm{du} = \frac{\sigma^2}{2\kappa}(1-\exp(-2\kappa\mathrm{t})).$$

We can thus conclude that $X(t)$ given $X(0) = x(0)$ is normally distributed, with mean and variance given by

$$\mathrm{E}[\mathrm{X(t)}|\mathrm{X(0)} = \mathrm{x(0)}] = \exp(-\kappa\mathrm{t})\mathrm{x(0)} + \theta(1-\exp(-\kappa\mathrm{t})),$$

$$\mathrm{Var}[\mathrm{X(t)}|\mathrm{X(0)} = \mathrm{x(0)}] = \frac{\sigma^2}{2\kappa}\left(1-\exp(-2\kappa\mathrm{t})\right).$$

Ornstein-Uhlenbeck Process takes its values in $\mathbb{R}$. For $t \to \infty$, we get the unconditional mean and variance

$$\mathrm{E}[\mathrm{X(t)}] = \theta, \qquad (2.4.8)$$

$$\mathrm{Var}[\mathrm{X(t)}] = \frac{\sigma^2}{2\kappa}. \qquad (2.4.9)$$

The paths of $X(t)$ can be simulated by informally discretizing the Ornstein-Uhlenbeck process

$$X(t_i) = X(t_{i-1}) + \kappa(\theta - X(t_{i-1}))(t_i - t_{i-1}) + \sigma Z(t_i)\sqrt{t_i - t_{i-1}},$$

where $Z(t_i)$'s are i.i.d. and $Z(t_i) \sim N(0,1)$.

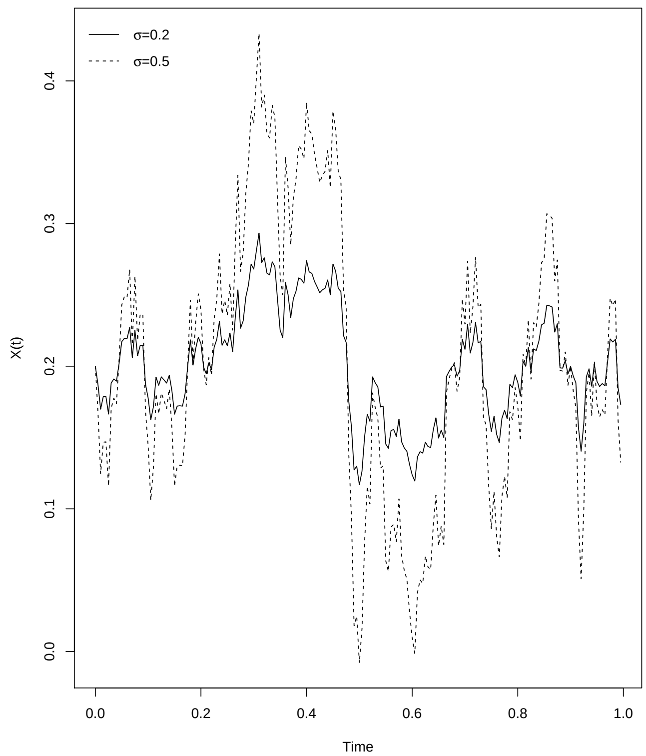

Figure 2.2: Simulation of Ornstein-Uhlenbeck process for different volatility of volatility parameters. $dX(t) = \kappa(\theta - X(t))dt + \sigma dB(t)$. $\theta = 0.2$ and $\kappa = 4$.

Another way of simulation is from the solution of the Ornstein-Uhlenbeck process (see (2.4.7)). We get

$$X(t_i) = \exp(-\kappa(t_i - t_{i-1}))X(t_{i-1}) + \theta(1 - \exp(-\kappa(t_i - t_{i-1}))) + \int_{t_{i-1}}^{t_i} \sigma \exp(-\kappa(t_i - u))dB(u),$$

or

$$X(t_i) = \exp(-\kappa(t_i - t_{i-1}))X(t_{i-1}) + \theta(1 - \exp(-\kappa(t_i - t_{i-1}))) + \sigma\sqrt{\frac{1 - \exp(-2\kappa(t_i - t_{i-1}))}{2\kappa}}Z(t_i),$$

where $Z(t_i)$'s are i.i.d. and $Z(t_i) \sim N(0, 1)$.

In our simulation studies, we find these two methods produce very similar results.

Figure 2.2 shows a single simulated path for $\sigma = 0.2$ and a path for $\sigma = 0.5$. For both paths we have used $\kappa = 4$, $\theta = 0.2$ and $x(0) = 0.2$, and the same sequence of random numbers.

2.4.3 Square Root Process

A one-dimensional stochastic process $X(t)$ is said to be a square root process if its dynamics is of the form

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dB(t), \tag{2.4.10}$$

where κ, θ and σ are constants with $\kappa > 0$ and $\sigma > 0$. Like an Ornstein-Uhlenbeck process, square process also exhibits mean reversion. The only difference to the dynamics of an Ornstein-Uhlenbeck process is the term $\sqrt{X(t)}$ in the volatility. The conditional variance rate is now $\sigma^2 X(t)$ which is proportional to the level of the process.

A square root can only take on non-negative values. To see this, note that if the value should become zero, then the drift is positive and the volatility zero, and therefore the value of the process will become positive immediately after. It can be shown if $2\kappa\theta > \sigma^2$ and $x(0) > 0$, $X(t)$ will be always positive and the process given in (2.4.10) is then well-defined.[1]

To find a solution to the stochastic equation (2.4.10), we try the same trick as for the Ornstein-Uhlenbeck process, that is we look at $Y(t) = f(X(t), t) = \exp(\kappa t)X(t)$. Since

$$\frac{\partial f}{\partial t} = \kappa \exp(\kappa t)x, \quad \frac{\partial f}{\partial x} = \exp(\kappa t), \quad \frac{\partial^2 f}{\partial x^2} = 0,$$

[1] see e.g. Lamberton and Lapeyre (1996) for a proof.

by Itô's Lemma and setting $\mu(t) = \kappa(\theta - X(t))$ and $\sigma(t) = \sigma\sqrt{X(t)}$,

$$\begin{aligned} dY(t) &= (\kappa \exp(\kappa t) X(t) + \exp(\kappa t)\kappa(\theta - X(t)) + 0)\, dt + \sigma \exp(\kappa t)\sqrt{X(t)} dB(t) \\ &= \kappa\theta \exp(\kappa t) dt + \sigma \exp(\kappa t)\sqrt{X(t)} dB(t). \end{aligned} \tag{2.4.11}$$

Hence, we have

$$Y(t) = y(0) + \int_0^t \kappa\theta \exp(\kappa u) du + \int_0^t \sigma \exp(\kappa u)\sqrt{X(u)} dB(u).$$

Computing the ordinary integral and substituting the definition of $Y(t)$, we get

$$X(t) = \exp(-\kappa t)x(0) + \theta(1 - \exp(-\kappa t)) + \int_0^t \sigma \exp(-\kappa(t-u))\sqrt{X(u)} dB(u). \tag{2.4.12}$$

It can be shown that $X(t)$ given $X(0) = x(0)$ is non-centrally χ^2 distributed. From the properties of stochastic integral, we can compute the conditional mean and variance of $X(t)$ as

$$\mathrm{E[X(t)|X(0) = x(0)] = \exp(-\kappa t)x(0) + \theta(1 - \exp(-\kappa t))},$$

$$\begin{aligned} \mathrm{Var[X(t)|X(0) = x(0)]} &= \frac{\sigma^2}{\kappa}\left(\exp(-\kappa t) - \exp(-2\kappa t)\right) x(0) \\ &\quad + \frac{\sigma^2}{2\kappa}\left(1 - \exp(-2\kappa t)\right)\theta. \end{aligned}$$

For $t \to \infty$, we get the unconditional mean and variance

$$\mathrm{E[X(t)]} = \theta, \tag{2.4.13}$$

$$\mathrm{Var[X(t)]} = \frac{\sigma^2\theta}{2\kappa}. \tag{2.4.14}$$

The paths of $X(t)$ can be simulated by

$$X(t_i) = X(t_{i-1}) + \kappa(\theta - X(t_{i-1}))(t_i - t_{i-1}) + \sigma\sqrt{X(t_{i-1})}Z(t_i)\sqrt{t_i - t_{i-1}},$$

where $Z(t_i)$'s are i.i.d. and $Z(t_i) \sim N(0, 1)$.

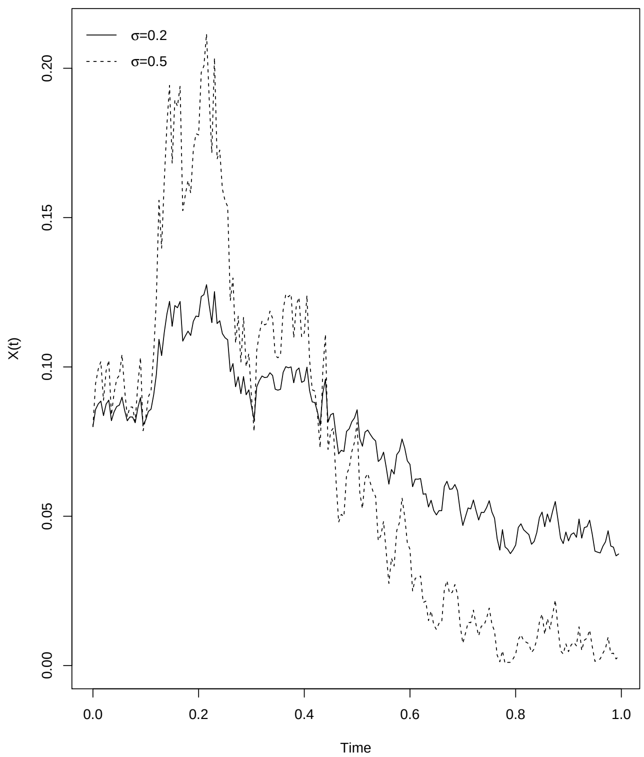

Figure 2.3: Simulation of square root process for different volatility of volatility parameters. $dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dB(t)$. $\theta = 0.08$ and $\kappa = 2$.

Figure 2.3 shows a single simulated path for $\sigma = 0.2$ and a path for $\sigma = 0.5$. For both paths we have used $\kappa = 2$, $\theta = 0.08$ and $x(0) = 0.08$, and the same sequence of random numbers.

2.5 Feynman-Kac Theorem

In pricing options, we often need to calculate an expected value. Feynman-Kac theorem provides a link between a partial differential equation (PDE) and a conditional expectation of a diffusion. This is useful if we have difficulty in calculating the expected value, we can at least obtain it by numerically solving the PDE, as the Feynman-Kac theorem states.

Theorem 2.5.1 *(Feynman-Kac Theorem) Let $X(t)$ be a stochastic process driven by a stochastic differential equation*

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dB(t),$$

with an initial value at initial time t,

$$X(t) = x,$$

and let $Y(t,x) \in \mathcal{L}^2$ be a deterministic function which satisfies

$$\int_t^T \mathrm{E}\left[\sigma(\mathrm{s}, \mathrm{X}(\mathrm{s}))\frac{\partial \mathrm{Y}}{\partial \mathrm{x}}(\mathrm{s}, \mathrm{X}(\mathrm{s}))\right]^2 \mathrm{ds} < \infty$$

with boundary condition

$$Y(T, X(T)) = f(X(T)).$$

If the function $Y(t,x)$ is a solution to the boundary value problem

$$\frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 Y}{\partial x^2} + \mu(t,x)\frac{\partial Y}{\partial x} - g(t,x)Y(t,x) = 0, \tag{2.5.1}$$

then Y has the representation:

$$Y(t,x) = \mathrm{E}\left[\exp\left(-\int_{t}^{T} g(s, X(s))ds\right) f(X(T))|X(t) = x\right]. \qquad (2.5.2)$$

Vice versa, if the expected value of (2.5.2) exists, then the PDE (2.5.1) holds.

From Feynman-Kac theorem, we know that computing the expected value is equivalent to solving a corresponding PDE. We will provide the examples of the application of Feynman-Kac theorem in the following chapters.

2.6 Girsanov's Theorem

Assume we have the probability space $\{\Omega, \mathcal{F}, P\}$. Then a change of measure from P to Q means we have probability space $\{\Omega, \mathcal{F}, Q\}$.

Definition 2.6.1 *Two measures P and Q are equivalent if*

$$P(A) > 0 \Rightarrow Q(A) > 0, \ \ for\ all\ A \in \Omega,$$

and

$$P(A) = 0 \Rightarrow Q(A) = 0, \ \ for\ all\ A \in \Omega.$$

Using two equivalent measures, we can define a Radon-Nikodym derivative,

$$M(t) = \frac{dQ}{dP}(t),$$

which enables us to change a measure to another. It follows that for any random element X

$$\mathrm{E}^{P}[XM] = \int_{\Omega} X(\omega)M(t,\omega)dP(\omega) = \int_{\Omega} X(\omega)dQ(\omega) = \mathrm{E}^{Q}[X].$$

This interchangeability of the expected values under two different measures confirms the important role of a Radon-Nikodym derivative as intermediate link between two measures.

The Girsanov's theorem gives us some concrete instructions to change the measures for stochastic processes.

Theorem 2.6.2 *(Girsanov's Theorem) Suppose we have a filtration $\mathcal{F}_t$ over a period $[0,T]$ where $T<\infty$. Define a random process $M(t)$:*

$$M(t)=\exp\left[-\int_0^t \lambda(u)dB^P(u)-\frac{1}{2}\int_0^t \lambda^2(u)du\right],\ t\in[0,T],$$

where $B^P(t)$ is a Brownian motion under probability measure P and $\lambda(t)$ is an $\mathcal{F}_t$-measurable process that satisfies a condition

$$\mathrm{E}\left\{\exp\left[\frac{1}{2}\int_0^{\mathrm{t}} \lambda^2(\mathrm{u})\mathrm{du}\right]\right\}<\infty,\ \mathrm{t}\in[0,\mathrm{T}].$$

If we define B^Q by

$$B^Q(t)=B^P(t)+\int_0^t \lambda(u)du,\ t\in[0,T],$$

then we have the following results:

- *$M(t)$ defines a Radon-Nikodym derivative*

$$M(t)=\frac{dQ}{dP}(t);$$

- *B^Q is a Brownian motion with respect to $\mathcal{F}_t$ under probability measure Q.*

To change the measures for multidimensional stochastic differential equations, we require a multidimensional Girsanov's theorem, which is very similar to the one-dimension version.

Theorem 2.6.3 *(Multidimensional Girsanov's Theorem) Suppose we have a filtration $\mathcal{F}_t$ over a period $[0,T]$ where $T < \infty$. Let $\Lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$ be an n-dimensional process that is $\mathcal{F}_t$-measurable and satisfies a condition*

$$\mathrm{E}\left\{\exp\left[\frac{1}{2}\int_0^{\mathrm{t}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}^2(\mathrm{u})\mathrm{du}\right]\right\} < \infty,\ \mathrm{t} \in [0,\mathrm{T}].$$

We define a random process $M(t)$:

$$M(t) = \exp\left[\sum_{i=1}^{n}\left(-\int_0^t \lambda_i(u)dB_i^P(u) - \frac{1}{2}\int_0^t \lambda_i^2(u)du\right)\right],\ t \in [0,T],$$

where $B_i^P(t)$ for $i = 1, \ldots, n$ is an n-dimensional Brownian motion under probability measure P. If we define B_i^Q by

$$B_i^Q(t) = B_i^P(t) + \int_0^t \lambda_i(u)du,\ for\ i = 1, 2, \ldots, n,$$

then we have the following results:

- *$M(t)$ defines a Radon-Nikodym derivative*

$$M(t) = \frac{dQ}{dP}(t);$$

- *B_i^Q for $i = 1, \ldots, n$ is a multidimensional Brownian motion with respect to $\mathcal{F}_t$ under probability measure Q.*

The Girsanov's theorem is of fundamental importance in pricing options. We will illustrate its importance in the following chapters.

Chapter 3

Option Pricing with Black-Scholes Model

The cornerstone of option pricing theory is the assumption that economy is free of arbitrage opportunities and there exists an equivalent martingale measure such that under this measure, the discounted prices of financial securities should follow a martingale. To understand this important result, we explain the concepts of self-financing, no arbitrage and equivalent martingale measure.[1] We furthermore show under which conditions an economy is free of arbitrage opportunities and how prices of derivatives can be calculated. As an example, we analyze the Black-Scholes model. We also point out the limitation of Black-Scholes model.

3.1 Self-financing and No Arbitrage

Let $\{\Omega, \mathcal{F}, P\}$ denote a probability space. Let us consider a financial market consisting of n assets with prices $Z_1(t), \ldots, Z_n(t)$, which under probability measure P are

[1] The more detailed explanations can be found in Björk (2009), Mikosch (1999) and Shreve (2004).

governed by the following stochastic differential equations:

$$dZ_i(t) = \mu_i(t)dt + \sigma_i(t)dB_i(t), \; i = 1, 2, \ldots, n,$$

where $B_i(t)$ for $i = 1, 2, \ldots, n$ is a Brownian motion.

Next, we denote an n-dimensional stochastic process $\delta(t) = (\delta_1(t), \ldots, \delta_n(t))$ as a trading strategy, where $\delta_i(t)$ denotes the holdings in asset i at time t. The value $V(\delta, t)$ at time t of a trading strategy δ is given by

$$V(\delta, t) = \sum_{i=1}^{n} \delta_i(t) Z_i(t).$$

Definition 3.1.1 *A self-financing trading strategy is a strategy δ with the property:*

$$V(\delta, t) = V(\delta, 0) + \sum_{i=1}^{n} \int_0^t \delta_i(s) dZ_i(s), \; t \in [0, T].$$

Hence, a self-financing trading strategy is a trading strategy that requires nor generates funds between time 0 and time T. In other words, any profit/loss is generated by buying or selling one of the assets Z_i.

Definition 3.1.2 *An arbitrage opportunity is a self-financing trading strategy δ, with*

- $V(\delta, 0) \leq 0$*;*
- $V(\delta, T) \geq 0$ *almost surely;*
- $\mathrm{E}[\mathrm{V}(\delta, \mathrm{T})] \geq 0.$

In words, arbitrage is a situation where it is possible to make a profit without the possibility of incurring a loss.

Definition 3.1.3 *A derivative security (also known as a contingent claim) is a financial contract whose value at expiration time (maturity time) T is precisely determined* *by the prices of the underlying assets at time T.*

The most important derivative is the European call option.

Definition 3.1.4 *A European call with exercise price (or strike price) K and time of maturity T on the underlying asset S is a contract defined by the following clauses:*

- *The holder of the option has, at time T, the right to buy one share of the underlying stock at the price K from the underwriter of the option;*
- *The holder of the option is in no way obliged to buy the underlying stock;*
- *The right to buy the underlying stock at the price K can only be exercised at the precise time T.*

Definition 3.1.5 *A derivative security with pay-off $H(T)$ at time T is said to be attainable if there is a self-financing strategy δ such that $V(\delta, T) = H(T)$.*

Definition 3.1.6 *An economy is called complete if all the derivative securities are attainable.*

If no arbitrage opportunities exist in an economy, we should have a unique price for the attainable derivative $H(T)$. This is a fair price because it is free from arbitrage. However, this raises two questions. First, under which conditions is a continuous trading economy free of arbitrage opportunities? Second, under which conditions is the economy complete?

3.2 Equivalent Martingale Measure

The questions of no-arbitrage and completeness were first addressed mathematically in the papers of Harrison and Kreps (1979) and Harrison and Pliska (1981). They showed that both questions can be solved at once using the notion of a martingale measure.

Definition 3.2.1 *An asset is called a numeraire if it has strictly positive prices for all* $t \in [0, T]$*.*

We can use numeraire to denominate all prices in an economy.

Let $\{\Omega, \mathcal{F}, P\}$ denote the probability space from the previous section. Consider now a numeraire $N(t)$ and a probability measure P_N that is associated with $N(t)$.

Definition 3.2.2 *The measure* P_N *is called equivalent martingale measure if*

- P_N *is equivalent to* P*;*
- *For any self-financing portfolio* $V(\delta, t)$*,* $V(\delta, t)/N(t)$ *is a martingale under* P_N*, i.e.*

$$\mathrm{E}^{\mathrm{P_N}}\left[\frac{\mathrm{V}(\delta,\mathrm{t})}{\mathrm{N}(\mathrm{t})}|\mathcal{F}_{\mathrm{s}}\right] = \frac{\mathrm{V}(\delta,\mathrm{s})}{\mathrm{N}(\mathrm{s})},\ \mathrm{s} \leq \mathrm{t}.$$

Subject to the definitions given above, we are now in a position to state two key theorems of financial mathematics.

Theorem 3.2.3 *(First Fundamental Theorem of Finance) The market is arbitrage free if and only if there exists an equivalent martingale measure.*

Theorem 3.2.4 *(Second Fundamental Theorem of Finance) Assume that the market is arbitrage free. The market is then complete if and only if for every choice of numeraire there exists a unique equivalent martingale measure.*

3.3 Black-Scholes Model

Let us now consider the Black and Scholes (1973) option pricing model. In the Black-Scholes economy, there are two assets: a riskless money-market account H, and a stock with price process S.

The dynamics of H is

$$dH(t) = rH(t)dt, \tag{3.3.1}$$

with $H(0) = 1$. r is a constant with $r > 0$, denoting the riskless interest rate.

Hence, $H(t)$ is value of one dollar compounded at a fixed (risk-free) rate r. From (3.3.1), we can see

$$H(t) = \exp(rt).$$

We assume that under the physical probability measure P, the stock price S is given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{3.3.2}$$

where $B(t)$ is a Brownian motion and μ, σ are constants with $\sigma > 0$. Hence, $S(t)$ is a geometric Brownian motion process.

The value of the money-market account $H(t)$ is strictly positive and can serve as a numeraire. Hence, we obtain the relative (discounted) price $S'(t) = S(t)/H(t)$. From Itô's Lemma we know that the relative price process follows

$$dS'(t) = (\mu - r)S'(t)dt + \sigma S'(t)dB(t). \tag{3.3.3}$$

To identify equivalent martingale measure corresponding to the numeraire H, we can apply Girsanov's theorem. For $\lambda(t) \equiv -(\mu - r)/\sigma$ we obtain the new measure Q where the process S' follows

$$\begin{aligned} dS'(t) &= (\mu - r)S'(t)dt + \sigma S'(t)\left(dB^Q(t) - \frac{\mu - r}{\sigma}\right) \\ &= \sigma S'(t)dB^Q(t), \end{aligned} \tag{3.3.4}$$

which is a martingale. For $\sigma \neq 0$, this is the only measure which turns the relative prices $S(t)/H(t)$ into martingale, and the measure Q is unique. Therefore, from the second fundamental theorem of finance, the Black-Scholes economy is arbitrage-free and complete for $\sigma \neq 0$.

Under the measure Q, the original price process S follows the process

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)\left(dB^Q(t) - \frac{\mu - r}{\sigma}\right) \\ &= rS(t)dt + \sigma S(t)dB^Q(t). \end{aligned} \tag{3.3.5}$$

We see that under the equivalent martingale measure the drift μ of the process S is replaced by the interest rate r. For this reason, Q is also known as risk neutral measure and pricing under this measure is known as risk neutral valuation.

The solution to the stochastic differential equation (3.3.5) can be expressed as

$$S(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B^Q(t)\right], \tag{3.3.6}$$

where $B^Q(t)$ is the value of the Brownian motion at time t under the risk neutral measure. The random variable $B^Q(t)$ has a normal distribution with mean 0 and variance t.

In summary, we start with process $S(t)$ under measure P (see (3.3.2)). The discounted process $S'(t)$ follows the dynamics in (3.3.3) under measure P. Girsanov's theorem leads to measure Q so that $S'(t)$ follows the dynamics in (3.3.4) under Q. Finally, we can go back to the original process $S(t)$ under measure Q in (3.3.5). The solution is given by (3.3.6) under Q.

We take as given the Black-Scholes model and now we approach the main problem to be studied in this thesis, namely the pricing of options. The price of a European call option in the Black-Scholes model can be calculated from the Black-Scholes formula.

Theorem 3.3.1 *(Black-Scholes Formula) Assume under the measure Q the stock price S follows the dynamics*

$$dS(t) = rS(t)dt + \sigma S(t)dB^Q(t),$$

where r *and* σ *are non-negative constants.* $B^Q(t)$ *is a Brownian motion under measure* Q.

Denote $C^{BS}(t; r, K, T, \sigma, S(t))$ *the time* t *price of a European call with exercise price* K *and time of maturity* T *on the underlying asset* $S(t)$ *calculated based on Black-Scholes model. We have*

$$C^{BS}(0; r, K, T, \sigma, S(0)) = S(0)\Phi(d_1) - \exp(-rT)K\Phi(d_2), \tag{3.3.7}$$

where $\Phi(x)$ *is the cumulated normal distribution function,*

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \tag{3.3.8}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Proof: From the definition, we can see

$$C^{BS}(T; r, K, T, \sigma, S(T)) = \max[S(T) - K, 0].$$

For time $t = 0$, from the second fundamental theorem of finance, we know

$$\begin{aligned} &C^{BS}(0; r, K, T, \sigma, S(0)) \\ &= \mathrm{E^Q[exp(-rT)C^{BS}(T; r, K, T, \sigma, S(T))|\mathcal{F}_0]} \\ &= \int_{-\infty}^{\infty} \exp(-rT) \max\left\{S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma y\right] - K, 0\right\} \frac{\exp(-\frac{1}{2}\frac{y^2}{T})}{\sqrt{2\pi T}} dy \\ &= \int_{-d_2}^{\infty} \left\{S(0)\exp\left[-\frac{1}{2}\sigma^2 T + \sigma y\sqrt{T}\right] - K\exp(-rT)\right\} \frac{\exp(-\frac{1}{2}\frac{y^2}{T})}{\sqrt{2\pi T}} dy \\ &= S(0)\Phi(d_1) - \exp(-rT)K\Phi(d_2), \end{aligned} \tag{3.3.9}$$

where $\Phi(x)$ is the cumulated normal distribution function,

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \tag{3.3.10}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

∎

3.4 Implied Volatility

Using Black-Scholes option pricing model, the price of a call option is the function of the spot (current) price $S(0)$, interest rate r, the strike K, the constant volatility σ and the maturity T. Except for the volatility σ, all the other variables are observable. Since the quoted option price C^{obs} is observable, using the Black-Scholes formula we can therefore calculate or imply the volatility that is consistent with the quoted historical option prices and observed variables. We can therefore define implied volatility σ_{impl} by

$$C^{BS}(0; r, K, T, \sigma_{impl}, S(0)) = C^{obs}$$

where C^{BS} is the option price calculated by the Black-Scholes formula (equation (3.3.9)). Implied volatility surfaces are graphs plotting σ_{impl} for each call option's strike K and expiration T. Theoretically, options whose underlying is governed by the geometric Brownian motion should have a flat implied volatility surface, since volatility is a constant. However, in practice, the implied volatility surface is not flat and σ_{impl} varies with K and T. This disparity is known as the volatility skew. There are several patterns for the volatility skew:

- Volatility Smile. Implied volatilities plotted against strike prices tend to vary in a U-shape relationship resembling a smile. This pattern is commonly seen in near-term equity options[2] and options in the forex (foreign exchange) market;
- Reverse Skew (Volatility Smirk). The implied volatilities for options at the lower strikes are higher than those at higher strikes. The reverse skew pattern typically appears for longer term equity options and index options.
- Forward Skew. The implied volatilities for options at the lower strikes are lower than those at higher strikes. The forward skew pattern is common for options in the commodities market.

Figure 3.1 gives a general picture of three patterns observed in the market. The volatility skew may produce various biases in option pricing or hedging. This consequently led to a development of dynamic volatility modeling which we will turn to in the next chapter.

[2] options that expire very soon, usually within next few weeks or months.

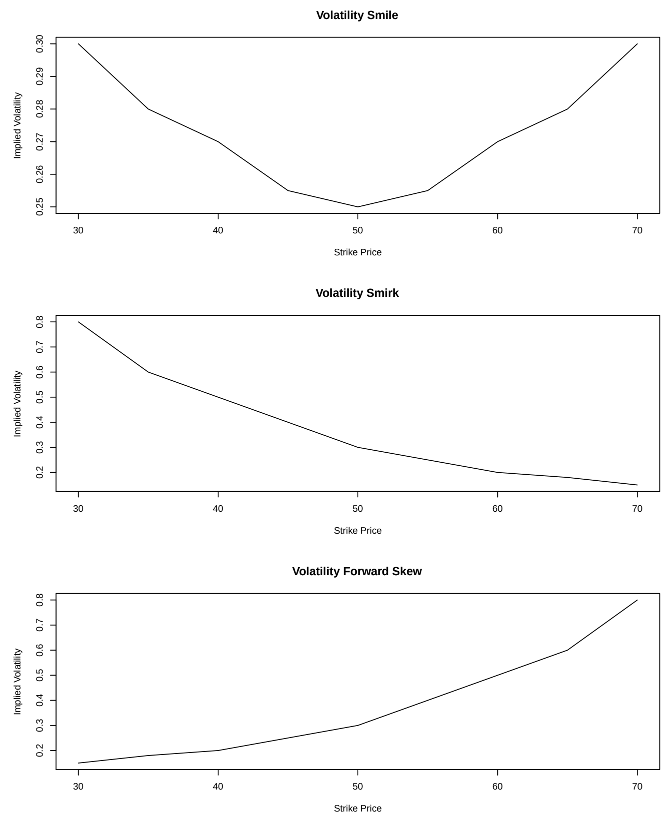

Figure 3.1: Patterns of volatility skew.

Chapter 4

Option Pricing with Stochastic Volatility Models

4.1 Introduction

Since the Black-Scholes formula was derived, a number of empirical studies have concluded that the assumption of constant volatility is inadequate to describe the stock returns. The volatility has been observed to exhibit consistently some empirical characteristics:

- Volatility tends to revert around some long term value;
- Volatility clusters with time: large (small) price changes tend to follow large (small) price changes;
- Volatility is correlated with stock returns.

The stochastic volatility models have been put forward to model the variability of volatility and to capture the volatility skew. A general stochastic volatility model under physical probability measure P is defined as

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dB_1(t),$$

$$\sigma^2(t) = f(Y(t)),$$

$$dY(t) = \mu_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dB_2(t),$$

$$dB_1(t)dB_2(t) = \rho dt,$$

where $S(t)$ is the asset price, $f(\cdot)$ is some deterministic function and $B_1(t)$ and $B_2(t)$ are two Brownian motions with correlation ρ.

We note that while $S(t)$ is observable, this is not the case for $Y(t)$. Because of the extra source of randomness - the second Brownian motion in the volatility process, option pricing with stochastic volatility models is more difficult. It is now a multi-dimensional problem to construct a risk neutral measure and use the risk neutral pricing principles with a stochastic volatility model.

Assume that r is a risk-free interest rate. We define a random process $M(t)$:

$$M(t) = \exp\left[\sum_{i=1}^{2}\left(-\int_0^t \lambda_i(u)dB_i(u) - \frac{1}{2}\int_0^t \lambda_i^2(u)du\right)\right],\ t \in [0, T],$$

where $\lambda_1(t) = (\mu - r)/\sigma(t)$ and $\lambda_2(t)$ is a process associated with the volatility process.

We define B_i^Q for $i = 1, 2$ by

$$B_i^Q(t) = B_i(t) + \int_0^t \lambda_i(u)du,\ for\ i = 1, 2,$$

Let $\mathcal{F}_t$ be the filtration generated by $B_1^Q(t)$ and $B_2^Q(t)$. Then from multidimensional Girsanov's theorem, we know that $(B_1^Q,\ B_2^Q)$ is a bivariate Brownian motion with respect to $\mathcal{F}_t$ under probability measure Q. Under this new probability measure Q,

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t),$$

$$\sigma^2(t) = f(Y(t)),$$

$$dY(t) = (\mu_Y(t, Y(t)) - \sigma_Y(t, Y(t))\lambda_2(t))dt + \sigma_Y(t, Y(t))dB_2^Q(t),$$

$$dB_1^Q(t)dB_2^Q(t) = \rho dt.$$

As in the one-dimensional Black-Scholes model, the market price of risk, $\lambda_1(t)$, is chosen to make the rate of return of the stock under the new measure equal to the riskless interest rate (see equation (3.3.5)). With a role similar to $\lambda_1(t)$, $\lambda_2(t)$ is called the market price of volatility risk. $\lambda_2(t)$, however, can not be determined as easily as $\lambda_1(t)$, because of the fact that volatility is neither directly observable nor traded so that we do not know immediately the risk neutral rate of return that is appropriate for volatility. Therefore, in the stochastic volatility models, the market is incomplete and we have a variety of no arbitrage option prices since different market price of volatility risk will produce a different martingale measures and each measure will produce a different price, in general. The market price of volatility risk is determined on the market, by the agents in the market, and this means that if we assume a particular structure of the market price of risk, then we have implicitly made an assumption about the preferences on the market (see Björk (2009) and Shreve (2004) for more discussions on the incompleteness of stochastic volatility models).

Different forms of the market price of volatility risk have been explored in research. A common and simple assumption is $\lambda_2(t) = 0$. This simplified assumption is used often when the volatility process is complicated and other convenient forms of $\lambda_2(t)$ are unavailable. This assumption indicates that the volatility process is the same after the change of measure. In this thesis, we will maintain this assumption throughout.

There is no generally accepted canonical stochastic volatility model. In this chapter, we first introduce two general approaches to pricing options under the stochastic volatility models: characteristic function approach and Hull-White formula. Then we review three most significant models: Heston Model, Schöbel-Zhu Model and Hull-White Model.

4.2 Pricing European Options: Characteristic Function Approach

Denote the time t price of a European call with exercise price K and time of maturity T on the underlying asset $S(t)$ by $C(t; K, T, S(t))$. From the first fundamental theorem of finance, we know that

$$\begin{aligned} & C(t; K, T, S(t)) \\ & = \mathrm{E^Q}\left[\exp\left(-r(T-t)\right)\left(S(T)-K\right) 1_{(S(T)>K)} | \mathcal{F}_t\right] \\ & = \mathrm{E^Q}\left[\exp\left(-r(T-t)\right) S(T)\, 1_{(S(T)>K)} | \mathcal{F}_t\right] - \mathrm{E^Q}\left[\exp\left(-r(T-t)dt\right) K\, 1_{(S(T)>K)} | \mathcal{F}_t\right]. \end{aligned} \tag{4.2.1}$$

For the first term in the second equality, we can choose the stock price $S(t)$ as numeraire and switch the measure Q to a measure Q_1, and for the second term we use the zero-coupon bond[1] to switch Q to the so-called $T-$forward measure Q_2.

We first consider the change of the risk-neutral measure Q to the new measure Q_1. According to the Girsanov theorem, we construct a Radon-Nikodym derivative using the corresponding numeraire,

$$\frac{dQ_1}{dQ} = \frac{S(T)H(t)}{H(T)S(t)} = \exp(-r(T-t))\frac{S(T)}{S(t)},$$

[1]Zero-coupon bond is the bond that does not pay coupons or interest payments to the bondholder. The bondholder only receives the face value of the bond at maturity.

where $H(t) = \exp(rt)$ is the money-market account at time t.

The second change of measure takes place between the money-market account $H(t)$ and the zero-coupon bond $B(t,T)$. Given the riskless interest rate r,

$$B(t,T) = \exp(-r(T-t)).$$

The Radon-Nikodym derivative for the change of the risk-neutral measure Q to a new measure Q_2 is given by

$$\frac{dQ_2}{dQ} = \frac{B(T,T)H(t)}{B(t,T)H(T)} = 1.$$

Under these new measures, the option pricing representation (4.2.1) can be restated as

$$\begin{aligned} &C(t; K, T, S(t)) \\ &= S_t \mathrm{E}^{\mathrm{Q_1}}\left[1_{(\mathrm{X(T)>\ln K})}|\mathcal{F}_{\mathrm{t}}\right] - \exp(-\mathrm{r(T-t)})\mathrm{K}\mathrm{E}^{\mathrm{Q_2}}\left[1_{(\mathrm{X(T)>\ln K})}|\mathcal{F}_{\mathrm{t}}\right] \\ &= S_t Q_1(X(T) > \ln K|\mathcal{F}_t) - \exp(-r(T-t))KQ_2(X(T) > \ln K|\mathcal{F}_t), \end{aligned} \tag{4.2.2}$$

where $X(t) = \ln(S(t))$.

We can express the probabilities in the last line by the Fourier transform. The characteristic functions of $X(T)$ under Q_j are defined by

$$f_j(\phi) = \mathrm{E}^{\mathrm{Q_j}}[\exp(\mathrm{i}\phi \mathrm{X(T)})|\mathcal{F}_{\mathrm{t}}] \text{ for j} = 1, 2,$$

so that using definition of $\frac{dQ_1}{dQ}$,

$$\begin{aligned} f_1(\phi) &= \mathrm{E}^{\mathrm{Q}_1}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}] \\ &= \mathrm{E}^{\mathrm{Q}}\left[\frac{\mathrm{dQ}_1}{\mathrm{dQ}}\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}\right] \\ &= \mathrm{E}^{\mathrm{Q}}\left[\exp(-\mathrm{r}(\mathrm{T}-\mathrm{t}))\frac{\mathrm{S}(\mathrm{T})}{\mathrm{S}(\mathrm{t})}\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}\right] \\ &= \mathrm{E}^{\mathrm{Q}}[\exp((1+\mathrm{i}\phi)\mathrm{X}(\mathrm{T})-\mathrm{r}(\mathrm{T}-\mathrm{t})-\mathrm{X}(\mathrm{t}))|\mathcal{F}_\mathrm{t}] \\ &= \exp(-r(T-t)-X(t))\mathrm{E}^{\mathrm{Q}}[\exp((1+\mathrm{i}\phi)\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}]. \end{aligned}$$

Using the definition of $\frac{dQ_2}{dQ}$,

$$\begin{aligned} f_2(\phi) &= \mathrm{E}^{\mathrm{Q}_2}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}] \\ &= \mathrm{E}^{\mathrm{Q}}\left[\frac{\mathrm{dQ}_2}{\mathrm{dQ}}\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}\right] \\ &= \mathrm{E}^{\mathrm{Q}}\left[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}\right]. \end{aligned}$$

If we define $f(\phi)$ as the characteristic function of $X(T)$ under Q

$$f(\phi) = \mathrm{E}^{\mathrm{Q}}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{T}))|\mathcal{F}_\mathrm{t}],$$

then

$$f_1(\phi) = \exp(-r(T-t)-X(t))f(-i+\phi), \tag{4.2.3}$$

$$f_2(\phi) = f(\phi). \tag{4.2.4}$$

In other words, we have simple formulae that link characteristic functions under Q_j, $j = 1, 2$ and under Q. If we can compute $f(\phi)$ in closed-form, we can also calculate the characteristic functions $f_j,\ j = 1, 2$ from (4.2.3) and (4.2.4). Then we can obtain the probabilities

$$Q_j(X(T) > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\mathrm{f_j}(\phi)\frac{\exp(-\mathrm{i}\phi\ln\mathrm{K})}{\mathrm{i}\phi}\right)\mathrm{d}\phi, \quad \mathrm{j} = 1, 2. \tag{4.2.5}$$

Plugging $Q_j,\ j = 1, 2$ into (4.2.2), we obtain the option pricing formula expressed in terms of the characteristic functions.

4.3 Pricing European Options: Hull-White Formula

Hull-White formula is applicable to the general stochastic volatility models characterized by

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t),$$

$$\sigma^2(t) = f(Y(t)),$$

$$dY(t) = (\mu_Y(t, Y(t)) - \sigma_Y(t, Y(t))\lambda_2(t))dt + \sigma_Y(t, Y(t))dB_2^Q(t),$$

$$dB_1^Q(t)dB_2^Q(t) = 0,$$

where $S(t)$ is the asset price, $f(\cdot)$ is some deterministic function and $B_1^Q(t)$ and $B_2^Q(t)$ are two Brownian motions under risk neutral probability measure Q.

From the first fundamental theorem of finance, we know that the European call option of the underlying stock S with the strike price K and time of maturity T can be evaluated as

$$C(t; K, T, S(t)) = \mathrm{E^Q[\exp(-r(T-t))C(T; K, T, S(T))|\mathcal{F}_t]},$$

where the filtration $\mathcal{F}_t$ is generated by two Brownian motions $B_1^Q(t)$ and $B_2^Q(t)$.

Conditioning on the volatility path and using iterated expectations, we have

$$C(t;K,T,S(t)) = \mathrm{E^Q}\left\{\mathrm{E^Q}\left[\exp(-r(T-t))C(T;K,T,S(T))|\mathcal{F}_t,\sigma_s,s\in[t,T]\right]|\mathcal{F}_t\right\}.$$

The inner expectation is the price of the call option when volatility is time-dependent but deterministic. In this situation, the solution to $S(T)$ becomes

$$S(T) = S(t)\exp\left[r(T-t) - \frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dB_1^Q(s)\right],$$

where $\sigma(t)$ is a deterministic function of time. From the properties of the stochastic integrals, we know that $\ln(S(T)/S(t))$ has a normal distribution with mean $(r - \frac{1}{2}\overline{\sigma^2})(T-t)$ and variance $\overline{\sigma^2}(T-t)$, where $\overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$ is the mean squared volatility. This distribution is the same as the risk neutral price distribution in a Black-Scholes model if volatility equals $\sqrt{\overline{\sigma^2}}$. Hence, the option price in the Hull-White model is given by

$$C(t;K,T,S(t)) = \mathrm{E^Q}\left[C^{BS}\left(t;r,K,\sqrt{\overline{\sigma^2}},T,S(t)\right)|\mathcal{F}_t\right],$$

where $C^{BS}(t;r,K,T,\sqrt{\overline{\sigma^2}},S(t))$ can be computed from the Black-Scholes formula (see (3.3.9)).

The Hull-White pricing formula is valid for any stochastic volatility process provided the correlation between $B_1^Q(t)$ and $B_2^Q(t)$ is zero. For correlated volatility, option prices have to be obtained using Monte Carlo simulation.

4.4 Heston Model

Heston's Model (1993) stands out from other stochastic volatility models because there exists an analytical solution for European options that takes into account correlation between stock price process and volatility process. This model is characterized by

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t), \tag{4.4.1}$$

or

$$dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t), \tag{4.4.2}$$

$$d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t), \tag{4.4.3}$$

with

$$dB_1^Q(t)dB_2^Q(t) = \rho dt.$$

where $X(t) = \ln(S(t))$. $B_1^Q(t)$ and $B_2^Q(t)$ are two Brownian motions under risk neutral probability measure Q. It is clear that $\sigma^2(t)$ is a square root process (see (2.4.10)) and $S(t)$ is a geometric Brownian motion given $\sigma^2(t)$.

We can compute the characteristic function $f(\phi)$ of $X(T)$ under Q

$$\begin{aligned} f(\phi) &= \mathrm{E^Q[exp(i\phi X(T))|\mathcal{F}_t]} \\ &= \mathrm{E^Q}\left[\mathrm{exp}\left(\mathrm{i}\phi\left(\mathrm{X(t)} + \int_\mathrm{t}^\mathrm{T}\left(\mathrm{r} - \frac{1}{2}\sigma^2(\mathrm{s})\right)\mathrm{ds} + \int_\mathrm{t}^\mathrm{T}\sigma(\mathrm{s})\mathrm{dB_1^Q(s)}\right)\right)|\mathcal{F}_\mathrm{t}\right] \\ &= \exp(i\phi(X(t) + r(T-t))) \\ &\quad \times \mathrm{E^Q}\left[\mathrm{exp}\left(\mathrm{i}\phi\left(-\frac{1}{2}\int_\mathrm{t}^\mathrm{T}\sigma^2(\mathrm{s})\mathrm{ds} + \int_\mathrm{t}^\mathrm{T}\sigma(\mathrm{s})\mathrm{dB_1^Q(s)}\right)\right)|\mathcal{F}_\mathrm{t}\right]. \end{aligned}$$

We can decompose B_1^Q into two parts

$$B_1^Q = \rho B_2^Q + \sqrt{1-\rho^2}B^Q,$$

where B^Q is a Brownian motion that is not correlated with B_2^Q. Now we can rewrite the above equation as

$$\begin{aligned} f(\phi) &= \exp(i\phi(X(t)+r(T-t)))\mathrm{E}^{\mathrm{Q}}\left[\exp\left(\mathrm{i}\phi\left(-\frac{1}{2}\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds}\right.\right.\right. \\ &\quad \left.\left.\left. +\rho\int_t^T\sigma(s)dB_2^Q(s)+\sqrt{1-\rho^2}\int_t^T\sigma(s)dB^Q(s)\right)\right)|\mathcal{F}_t\right] \\ &= \exp(i\phi(X(t)+r(T-t)))\mathrm{E}^{\mathrm{Q}}\left\{\mathrm{E}^{\mathrm{Q}}\left[\exp\left(\mathrm{i}\phi\left(-\frac{1}{2}\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds}\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. +\rho\int_t^T\sigma(s)dB_2^Q(s)+\sqrt{1-\rho^2}\int_t^T\sigma(s)dB^Q(s)\right)\right)|\mathcal{F}_t,\sigma(s),s\in[t,T]\right]|\mathcal{F}_t\right\} \\ &= \exp(i\phi(X(t)+r(T-t)))\mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds}\right.\right. \\ &\quad \left.\left. +i\phi\rho\int_t^T\sigma(s)dB_2^Q(s)+\frac{1}{2}(1-\rho^2)(i\phi)^2\int_t^T\sigma^2(s)ds\right)|\mathcal{F}_t\right]. \end{aligned} \tag{4.4.4}$$

We note that $\int_t^T \sigma(s)dB_2^Q(s)$ remains a function of B_2^Q after conditioning since $\sigma(\cdot)$ is defined in terms of $B_2^Q(\cdot)$. Now, we use the structure of volatility process $\sigma^2(t)$ (see (4.4.3)). We can integrate $\sigma^2(t)$ and obtain

$$\sigma^2(T)-\sigma^2(t)=\kappa\theta(T-t)-\kappa\int_t^T\sigma^2(s)ds+\gamma\int_t^T\sigma(s)dB_2^Q(s).$$

Rearrange this equation to obtain

$$\int_t^T\sigma(s)dB_2^Q(s)=\frac{1}{\gamma}\left[\sigma^2(T)-\sigma^2(t)-\kappa\theta(T-t)-\kappa\int_t^T\sigma^2(s)ds\right].$$

By inserting $\int_t^T \sigma(s)dB_2^Q(s)$ into (4.4.4) we have

$$\begin{aligned} f(\phi) = \exp(i\phi(X(t)+r(T-t)))\mathrm{E}^{\mathrm{Q}}\Bigg[\exp\Bigg(-\frac{\mathrm{i}\phi}{2}\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds} \\ + i\phi\frac{\rho}{\gamma}\left(\sigma^2(T)-\sigma^2(t)-\kappa\theta(T-t)+\kappa\int_t^T\sigma^2(s)ds\right) \\ +\frac{1-\rho^2}{2}(i\phi)^2\int_t^T\sigma^2(s)ds\Bigg)|\mathcal{F}_t\Bigg] \\ = \exp[i\phi(X(t)+r(T-t))-s_2(\sigma^2(t)+\kappa\theta(T-t))] \\ \times\,\mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\mathrm{s}_1\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds}+\mathrm{s}_2\sigma^2(\mathrm{T})\right)|\mathcal{F}_{\mathrm{t}}\right], \end{aligned} \tag{4.4.5}$$

where

$$\begin{aligned} s_1 &= -i\phi\left[\frac{\rho\kappa}{\gamma}-\frac{1}{2}+\frac{1}{2}i\phi(1-\rho^2)\right], \\ s_2 &= i\phi\frac{\rho}{\gamma}. \end{aligned}$$

To obtain the final form of $f(\phi)$, we need to calculate the following expression

$$Y(t,\sigma^2(t)) = \mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\mathrm{s}_1\int_{\mathrm{t}}^{\mathrm{T}}\sigma^2(\mathrm{s})\mathrm{ds}+\mathrm{s}_2\sigma^2(\mathrm{T})\right)|\mathcal{F}_{\mathrm{t}}\right].$$

Let $V(t)=\sigma^2(t)$. According to the Feynman-Kac theorem, $Y(t,V(t))$ should fulfill the following one-dimensional PDE,

$$-\frac{\partial Y}{\partial t}+\frac{1}{2}\gamma^2V(t)\frac{\partial^2Y}{\partial V^2}+\kappa(\theta-V)\frac{\partial Y}{\partial V}-s_1VY=0,$$

with boundary condition

$$Y(T,V(T))=\exp(s_2V(T)).$$

The solution to this PDE (see Zhu (2009)) is

$$Y(t, V(t)) = \exp[A(\tau) + B(\tau)V(t)], \tag{4.4.6}$$

where $\tau = T - t$, and

$$A(\tau) = \frac{2\kappa\theta}{\gamma^2} \ln\left[\frac{2\beta_1}{\beta_2} \exp\left(\frac{1}{2}(\kappa - \beta_1)\tau\right)\right],$$

$$B(\tau) = \frac{1}{\beta_2}\left[\beta_1 s_2(1 + \exp(-\beta_1\tau)) - (1 - \exp(-\beta_1\tau))(2s_1 + \kappa s_2)\right],$$

where

$$\beta_1 = \sqrt{\kappa^2 + 2\gamma^2 s_1},$$

$$\beta_2 = 2\beta_1 \exp(-\beta_1\tau) + (\kappa + \beta_1 - \gamma^2 s_2)(1 - \exp(-\beta_1\tau)).$$

Finally, using (4.4.5) and (4.4.6), we can compute the characteristic function $f(\phi)$ as

$$f(\phi) = \exp[i\phi(X(t) + r\tau) - s_2(\sigma^2(t) + \kappa\theta\tau) + A(\tau) + B(\tau)\sigma^2(t)].$$

Given the characteristic function $f(\phi)$, we obtain Heston option pricing formula expressed in terms of the characteristic functions (see section 4.2).

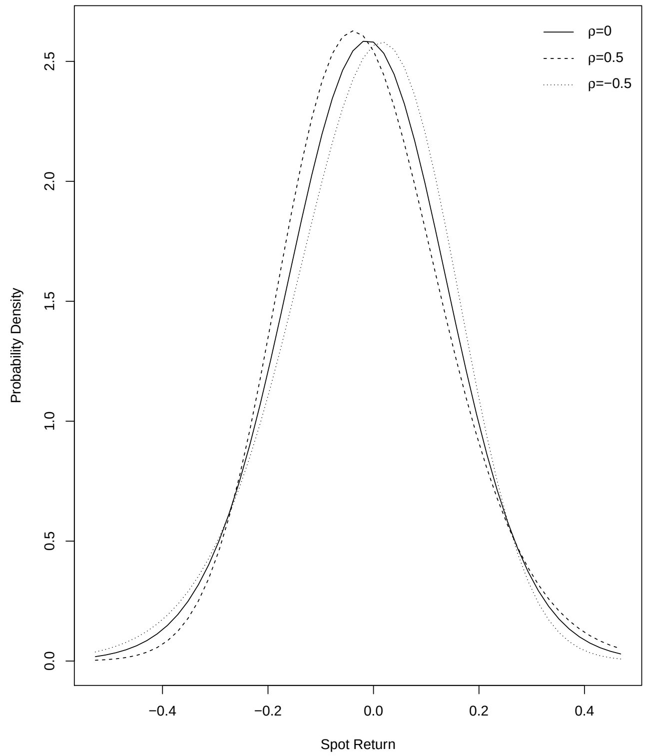

Figure 4.1: Conditional probability density of the spot returns over a six-month horizon for different correlation parameters for the Heston model. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.05$, $\kappa = 2$, $\gamma = 0.2$, $x(0) = \log(1000)$ and $\sigma^2(0) = 0.05$.

With the analytical form of characteristic function $f(\phi)$ under the probability measure Q, we can compute the density function $p(x(t))$ of $x(t)$ simply from the inverse Fourier transform

$$p(x(t)) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\phi)\exp(-i\phi x(t))d\phi.$$

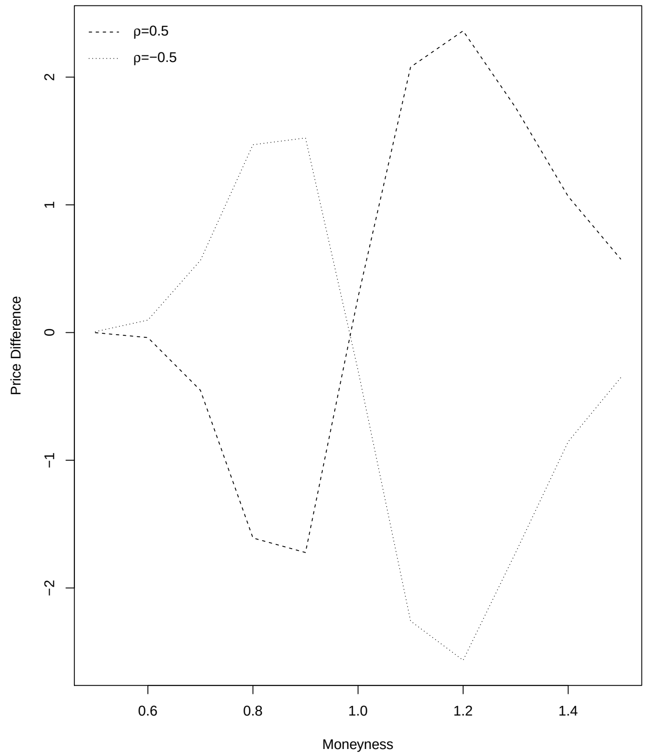

Figure 4.2: Option prices from the Heston model with non-zero correlation minus that with zero correlation. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.05$, $\kappa = 2$, $\gamma = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma^2(0) = 0.05$.

The Heston stochastic volatility model can conveniently explain properties of option prices in terms of the underlying distribution of spot returns. The correlation parameter ρ positively affects the skewness of spot returns. A positive correlation results in high variance when the spot asset rises, and this spreads the right tail of the probability density. Conversely, the left tail is associated with low variance and is not spread out. Figure 4.1 shows how a positive correlation creates a fat right tail and a thin left tail in the distribution of spot returns. Figure 4.2 plots the option prices with non-zero correlation minus that with zero correlation against the moneyness.[2] It shows that a positive correlation increases the prices of out-of-money options and decreases the prices of in-the-money option relative to the zero-correlation model.[3] A negative correlation has completely opposite effects. It decreases the prices of out-of-the-money options relative to in-the-money options.

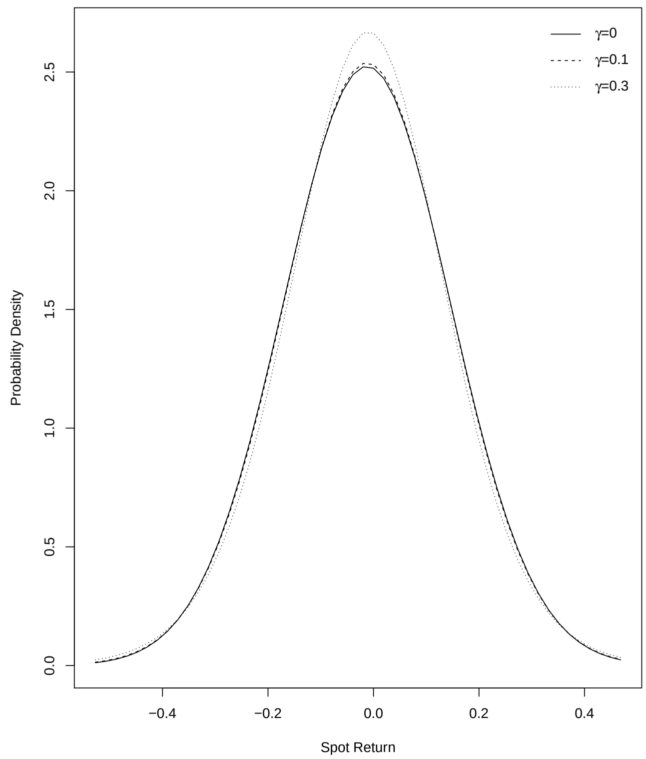

Figure 4.3: Conditional probability density of the spot returns over a six-month horizon for different volatility of volatility parameters for the Heston model. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.05$, $\kappa = 2$, $\rho = 0$, $x(0) = \log(1000)$ and $\sigma^2(0) = 0.05$.

[2] Moneyness is defined as the ratio of exercise price to spot price.

[3] An option is at the money if the strike price is the same as the current spot price of the underlying security. A call option is in the money when the strike price is below the spot price. A call option is out of the money when the strike price is above the spot price.

The parameter γ controls the volatility of volatility. When γ is zero, the volatility is deterministic, and spot returns have a normal distribution. Otherwise, γ increases the kurtosis of returns. Figure 4.3 shows how this creates the fat tails in the distribution of returns. As Figure 4.4 shows, this has the effect of raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices.[4]

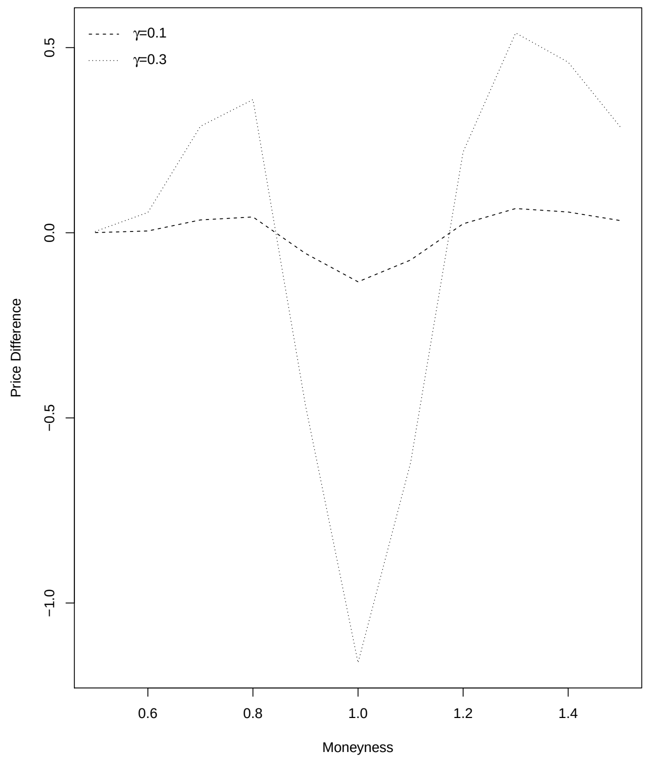

Figure 4.4: Option prices from the Heston model with different volatility of volatility parameters minus that from Black-Scholes model. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.05$, $\kappa = 2$, $\rho = 0$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma^2(0) = 0.05$.

[4]A call option with an exercise price significantly below the market price of the underlying security is called far-in-the-money. A call option with an exercise price significantly above the market price of the underlying security is called far-out-of-the-money. A call option with an exercise price close to the market price of the underlying security is called near-the-money.

Figure 4.5 shows that Heston model can produce various patterns of volatility skew. We will get volatility smile for zero correlation, volatility smirk for positive correlation and forward skew for negative correlation.

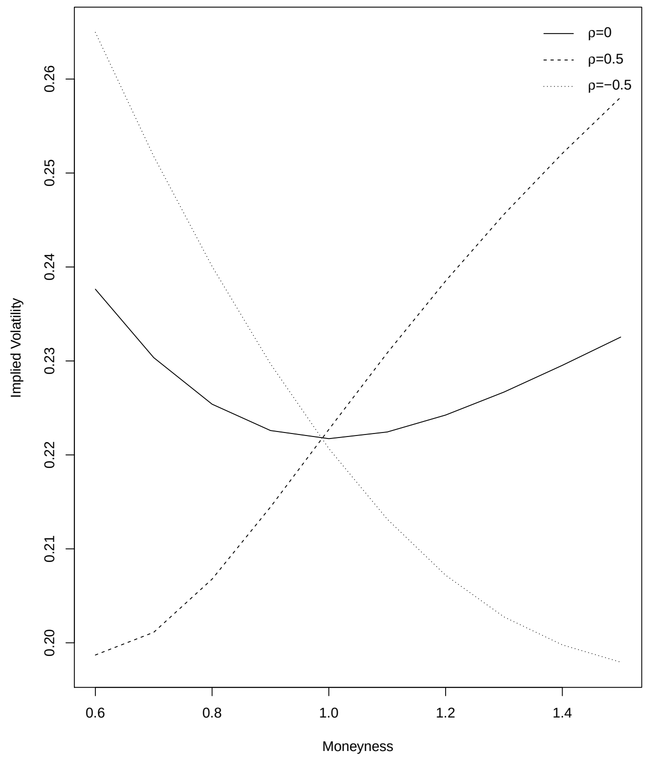

Figure 4.5: Implied volatility plot from the Heston model with different correlation parameters. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.05$, $\kappa = 2$, $\gamma = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma^2(0) = 0.05$.

4.5 Schöbel-Zhu Model

Schöbel and Zhu (1999) extended the Stein and Stein's (1991) formulation to a general case and derived an analytic solution for option prices. The Schöbel-Zhu model consists of the following two correlated processes,

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t), \tag{4.5.1}$$

or

$$dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t), \tag{4.5.2}$$

$$d\sigma(t) = \kappa(\theta - \sigma(t))dt + \gamma dB_2^Q(t), \tag{4.5.3}$$

with

$$dB_1^Q(t)dB_2^Q(t) = \rho dt,$$

where $X(t) = \ln(S(t))$. $B_1^Q(t)$ and $B_2^Q(t)$ are two Brownian motions under the risk neutral probability measure Q. It is clear that $\sigma(t)$ is an Ornstein-Uhlenbeck process and $S(t)$ is a geometric Brownian motion given $\sigma(t)$.

As shown in Schöbel and Zhu (1999) (computations are similar to those of Heston model), the characteristic function $f(\phi)$ of $X(T)$ under Q is given by

$$\begin{aligned} f(\phi) &= \exp\left[i\phi(X(t) + r(T-t)) - s_2\sigma^2(t) - \frac{1}{2}i\phi\rho\gamma(T-t)\right] \\ &\quad \times \mathrm{E^Q}\left[\exp\left(-\mathrm{s_1}\int_\mathrm{t}^\mathrm{T}\sigma^2(\mathrm{s})\mathrm{ds} - \mathrm{s_2}\int_\mathrm{t}^\mathrm{T}\sigma(\mathrm{s})\mathrm{ds} + \mathrm{s_3}\sigma^2(\mathrm{T}) + \mathrm{s_4}\sigma(\mathrm{T})\right)|\mathcal{F}_\mathrm{t}\right] \\ &= \exp\left[i\phi(X(t) + r\tau) + A(\tau) + B(\tau)\sigma(t) + \frac{1}{2}C(\tau)\sigma^2(t)\right], \end{aligned}$$

where $\tau = T - t$, and

$$s_1 = -\frac{1}{2}i\phi\left[i\phi(1-\rho^2) - 1 + \frac{2\rho\kappa}{\gamma}\right],$$

$$s_2 = \frac{\rho\kappa\theta}{\gamma}i\phi,$$

$$s_3 = \frac{\rho}{2\gamma}i\phi, \quad s_4 = 0.$$

The functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ are dependent on s_1, s_2, s_3 and s_4 and are given by

$$\begin{aligned}A(\tau) = &-\frac{1}{2}\ln(\beta_4) + \frac{[(\kappa\theta\beta_1 - \beta_2\beta_3)^2 - \beta_3^2(1-\beta_2^2)]\sinh(\beta_1\tau)}{2\beta_1^3\beta_4\gamma^2} \\ &+ \frac{(\kappa\theta\beta_1 - \beta_2\beta_3)\beta_3(\beta_4 - 1)}{\beta_1^3\gamma^2\beta_4} + \frac{\tau}{2\beta_1^2\gamma^2}[\kappa\beta_1^2(\gamma^2 - \kappa\theta^2) + \beta_3^2] \\ &+ \frac{s_4}{\beta_1^2\beta_4}\left[\beta_3(\beta_4 - 1) + \left(\kappa\theta\beta_1 + \frac{1}{2}\gamma^2\beta_1 s_4 + \beta_2\beta_3\right)\sinh(\beta_1\tau)\right] - s_3\gamma^2\tau,\end{aligned}$$

$$B(\tau) = \frac{(\kappa\theta\beta_1 - \beta_2\beta_3)(1 - \cosh(\beta_1\tau)) - (\kappa\theta\beta_1\beta_2 - \beta_3)\sinh(\beta_1\tau)}{\beta_1\beta_4\gamma^2} + \frac{s_4}{\beta_4},$$

$$C(\tau) = \frac{\kappa}{\gamma^2} - \frac{\beta_1}{\gamma^2}\frac{\sinh(\beta_1\tau) + \beta_2\cosh(\beta_1\tau)}{\beta_4} - 2s_3,$$

where

$$\beta_1 = \sqrt{2\gamma^2 s_1 + \kappa^2}, \quad \beta_2 = \frac{\kappa - 2\gamma^2 s_3}{\beta_1},$$

$$\beta_3 = \kappa^2\theta - s_2\gamma^2, \quad \beta_4 = \cosh(\beta_1\tau) + \beta_2\sinh(\beta_1\tau).$$

Given the characteristic function $f(\phi)$, we can compute the option prices as in Heston's model.

To find the stochastic process followed by $\sigma^2(t)$ under the Schöbel-Zhu model, we apply Itô's Lemma with the function $f(u,t) = u^2$ and define $U(t) = \sigma(t)$ and $V(t) = f(U(t),t) = U^2(t) = \sigma^2(t)$. Since

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial u} = 2u, \quad \frac{\partial^2 f}{\partial u^2} = 2,$$

we get from Itô's Lemma that

$$\begin{aligned}dV(t) &= \left(0 + 2U(t)\kappa(\theta - U(t)) + \frac{1}{2}2\gamma^2\right)dt + 2\gamma U(t)dB_2^Q(t) \\ &= (\gamma^2 + 2\kappa(\theta - U(t))U(t))dt + 2\gamma U(t)dB_2^Q.\end{aligned}$$

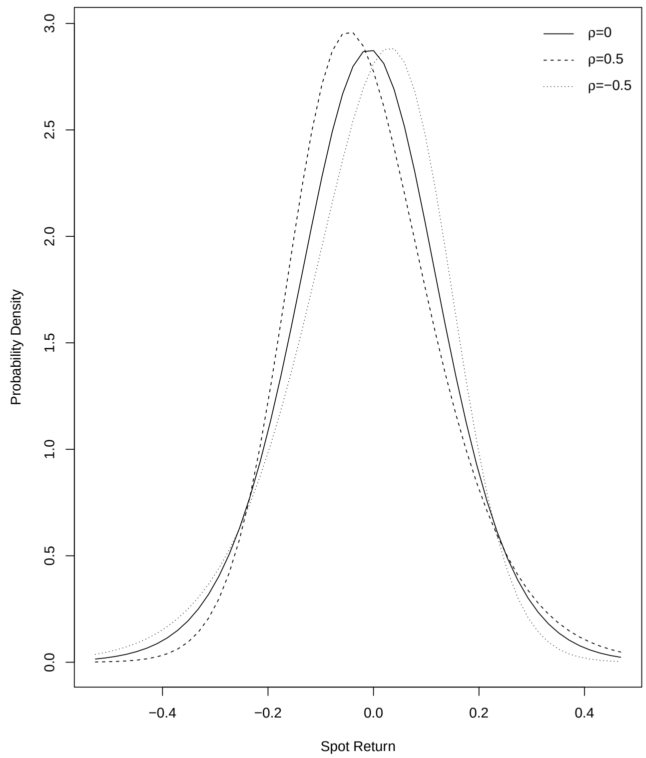

Figure 4.6: Conditional probability density of the spot returns over a six-month horizon for different correlation parameters for the Schöbel-Zhu model. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma(t) = \kappa(\theta - \sigma(t))dt + \gamma dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $x(0) = \log(1000)$ and $\sigma(0) = 0.2$.

Hence,

$$d\sigma^2(t) = (\gamma^2 + 2\kappa(\theta - \sigma(t))\sigma(t))dt + 2\gamma\sigma(t)dB_2^Q.$$

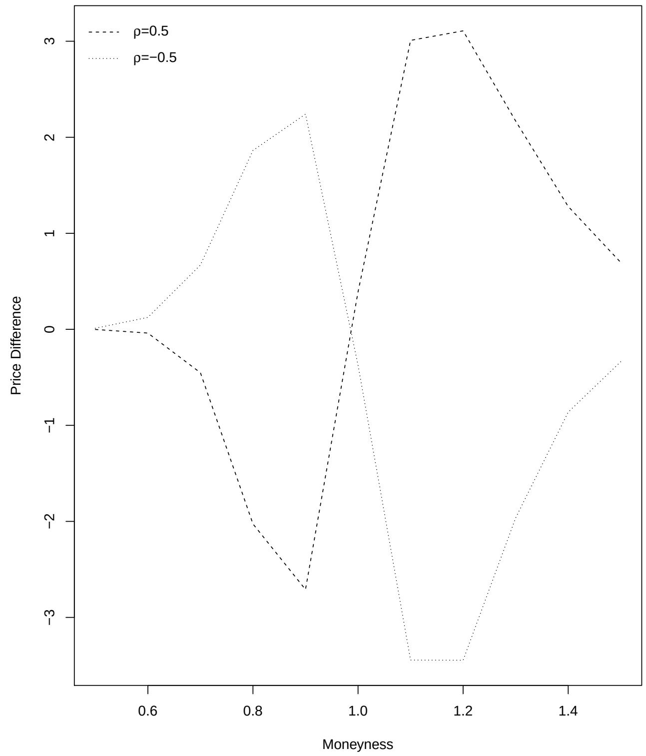

Figure 4.7: Option prices from the Schöbel-Zhu model with non-zero correlation minus that with zero correlation. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma(t) = \kappa(\theta - \sigma(t))dt + \gamma dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma(0) = 0.2$.

The unconditional mean and variance of $\sigma^2(t)$ are

$$\mathrm{E}(\sigma^2(\mathrm{t})) = \theta^2 + \frac{\gamma^2}{2\kappa},$$

$$\mathrm{Var}(\sigma^2(\mathrm{t})) = \frac{2\theta^2\gamma^2}{\kappa} + \frac{\gamma^4}{2\kappa^2}.$$

We can see that the conditional variance of $d\sigma^2(t)$ is a linear function of $\sigma(t)$, as in the Heston model. However, the conditional mean is a quadratic function of $\sigma(t)$, different from the Heston model.

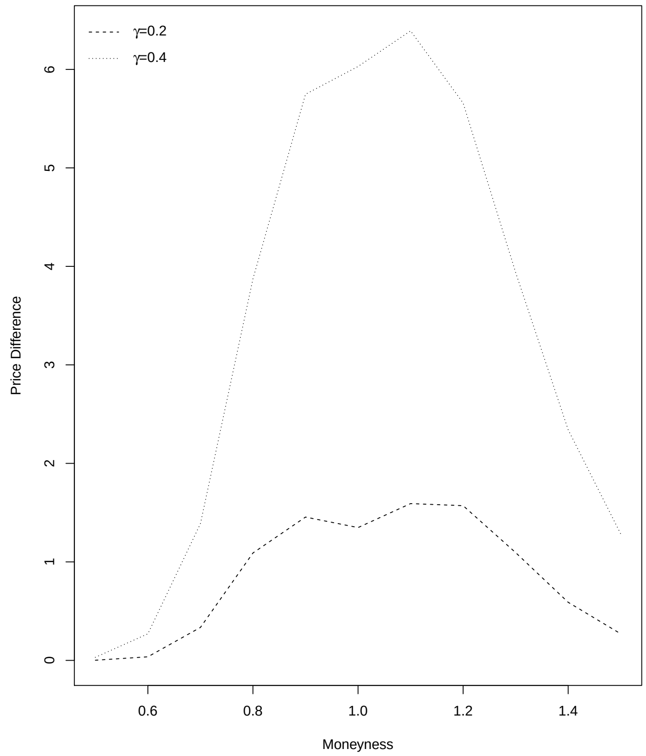

Figure 4.8: Option prices from the Schöbel-Zhu model with different volatility of volatility parameters minus that from Black-Scholes model. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma(t) = \kappa(\theta - \sigma(t))dt + \gamma dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\rho = 0$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma(0) = 0.2$.

We can examine the effects of stochastic volatility on option prices. Figure 4.6 and Figure 4.7 show that, like the Heston model, the correlation parameter ρ positively affects the skewness of spot returns. Figure 4.8 shows that unlike the Heston model, an increase in volatility of volatility parameter γ always leads to a higher option prices, which is expected because an increase in γ increases the long-run mean of $\sigma^2(t)$, hence also increases the option prices. Non-zero correlation can produce volatility smirk or forward skew, as shown in Figure 4.9.

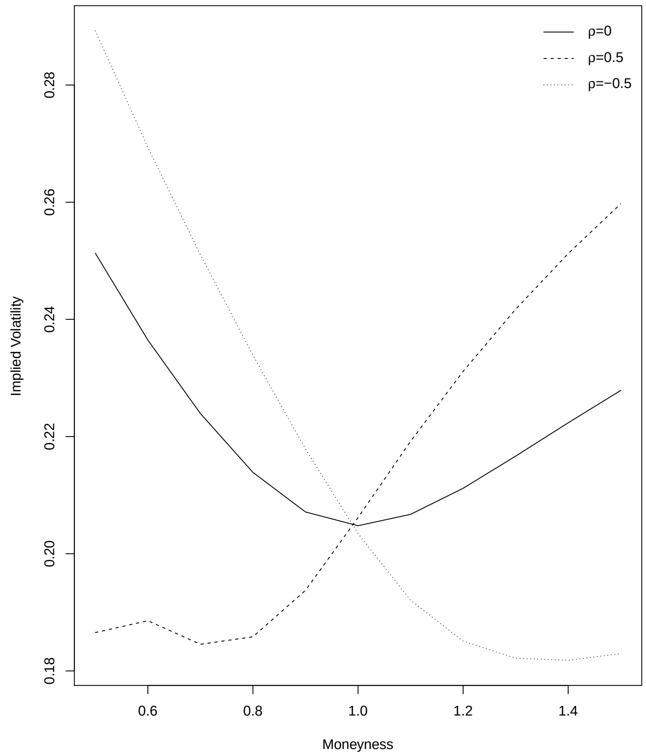

Figure 4.9: Implied volatility plot from the Schöbel-Zhu model with different correlation parameters. $dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t)$, $d\sigma(t) = \kappa(\theta - \sigma(t))dt + \gamma dB_2^Q(t)$ and $dB_1^2(t)dB_2^2(t) = \rho dt$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma(0) = 0.2$.

4.6 Hull-White Model

The Hull-White model (1987) is characterized by the following two processes:

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t), \tag{4.6.1}$$

$$d\sigma^2(t) = \mu\sigma^2(t)dt + \gamma\sigma^2(t)dB_2^Q(t), \tag{4.6.2}$$

where $B_1^Q(t)$ and $B_2^Q(t)$ are two Brownian motions under risk neutral probability measure Q. $dB_1^Q(t)dB_2^Q(t) = 0$. It is clear that $\sigma^2(t)$ is a geometric Brownian motion and $S(t)$ is also a geometric Brownian motion given $\sigma^2(t)$.

Unlike Heston model and Schöbel and Zhu (1999) model, we cannot compute the characteristic function $f(\phi)$ of $X(T) = \ln(S(T)$ under Q in closed-form for Hull-White model. The option prices can be computed using the Hull-White formula (see section 4.3).

Chapter 5

Long Memory Processes

In this chapter, we introduce long memory processes and discuss several aspects of their behaviors. First, we define a long memory process. Then we introduce self-similar processes and define fractional Brownian motion as a special example of a self-similar process. We also briefly discuss a concept of fractional integration and fractional calculus. Finally, we generalize fractional Brownian motion to fractionally integrated processes.

5.1 Definition

Let $X(t),\ t \in \mathbb{Z}$, be a real-valued time series. The autocovariance function $\gamma(h,t)$ is defined by

$$\gamma(h,t) = \mathrm{Cov(X(t), X(t+h))},\ \mathrm{t, h} \in \mathbb{Z}.$$

If $\gamma(h,t)$ is independent of t and $\mathrm{Var(X(t))} < \infty$, we say $X(t)$ is second-order stationary and we can define $\gamma(h) = \gamma(h,t)$. For second-order stationary processes, we also define the autocorrelation function (ACF) $\rho(h)$ of $X(t)$ by

$$\rho(h) = \mathrm{Corr(X(t), X(t+h))} = \frac{\gamma(\mathrm{h})}{\gamma(0)},\ \mathrm{t, h} \in \mathbb{Z}.$$

The autocorrelation of most time series observed in reality will decay to zero as the lag increases. However, the speed of decay can be very different. For a large family of processes including ARMA process, the autocorrelation decays exponentially

$$|\rho(h)| \leq Cr^h, \ C > 0, \ 0 < r < 1. \tag{5.1.1}$$

These processes are said to have short memory.

The key defining characteristic of a long memory process is that its autocorrelation decays slower than that specified in (5.1.1). Beran (1994) gives the following definition of the long memory process:

Definition 5.1.1 *A long memory process is a stationary process with a hyperbolically decaying autocorrelation function,*

$$\rho(h) \sim C_\rho h^{2d-1}, \ C_\rho \neq 0, \ 0 < d < \frac{1}{2}, \ \textit{as } h \to \infty. \tag{5.1.2}$$

d is the so-called (long) memory parameter which controls the speed of decay of the autocorrelation.

A time series is more persistent when d is closer to $\frac{1}{2}$. Sometimes, the Hurst parameter $H = d + \frac{1}{2}$ is used in place of d. Thus, for long memory processes, $\frac{1}{2} < H < 1$.

The above definitions of long memory process are equivalent. There also exist alternative definitions of long memory process. These definitions along with the ones we introduced earlier are similar but not exactly equivalent. The following definitions are also used in the literature (see, e.g., Palma (2007)):

- $\sum_{h=-\infty}^{\infty} |\gamma(h)| = \infty;$
- $\sum_{h=-n}^{n} |\gamma(h)| \sim n^{2d} L_1(n), \ \textit{as } n \to \infty, \ 0 < d < \frac{1}{2};$
- $\gamma(h) \sim h^{2d-1} L_2(h), \ \textit{as } h \to \infty, \ 0 < d < \frac{1}{2}.$

Here, $L_i(x),\ i = 1, 2$ are slowly varying functions in the sense that $L_i(cx)/L_i(x) \to 1$ as $x \to \infty$ for any $c > 0$.

5.2 Self-Similar Processes

Definition 5.2.1 *A real-valued stochastic process $X(t),\ t \in R$ is self-similar with index $H > 0$ or H-ss, if, for any $a > 0$,*

$$X(at) \stackrel{d}{=} a^H X(t),$$

where $\stackrel{d}{=}$ denotes the equality of the finite-dimensional distributions.

Definition 5.2.2 *A real-valued stochastic process $X(t),\ t \in R$ is H-sssi process (H-ss process with stationary increments) if it is H-ss and $X(t+h) - X(t) \stackrel{d}{=} X(t) - X(0)$ holds for any $h \in \mathbb{R}$.*

The H-sssi process has the following basic properties:

- $X(0) = 0$. This follows from $X(0) = X(a \cdot 0) \stackrel{d}{=} a^H X(0)$, for any $a > 0$.
- If $H \neq 1$, then $\mathrm{E(X(t)) = 0}$. This follows from

  $$\mathrm{E(X(2t)) = E(X(2t) - X(t)) + E(X(t)) = 2E(X(t))},$$

  while, on the other hand, $\mathrm{E(X(2t)) = 2^H E(X(t))}$. We then have $\mathrm{2E(X(t)) = 2^H E(X(t))}$.
- $X(t) \stackrel{d}{=} -X(-t)$. This follows from $X(t) - X(0) \stackrel{d}{=} X(0) - X(-t)$ and $X(0) = 0$.
- Let $\sigma^2 = E(X(1)^2) < \infty$, then $\mathrm{E(X^2(t)) = |t|^{2H}}\sigma^2$. We have

  $$\mathrm{E(X^2(t)) = E(X^2(|t| sign(t))) = |t|^{2H} E(X^2(sign(t))) = |t|^{2H} E(X^2(1))}.$$

- $\mathrm{Cov(X(s),X(t))} = \frac{\sigma^2}{2}\left[|\mathrm{t}|^{2\mathrm{H}} + |\mathrm{s}|^{2\mathrm{H}} - |\mathrm{t}-\mathrm{s}|^{2\mathrm{H}}\right]$. This follows from

$$\begin{aligned}
2\mathrm{Cov(X(s),X(t))} &= \mathrm{Var(X(t))} + \mathrm{Var(X(s))} - \mathrm{Var(X(t)-X(s))}\\
&= \mathrm{Var(X(t))} + \mathrm{Var(X(s))} - \mathrm{Var(X(t-s))}.
\end{aligned}$$

Implicit in the definition of H-sssi process is the restriction $H \leq 1$. This follows from

$$\mathrm{E(|X(2t)|) = E(|X(2t) - X(t) + X(t)|) \leq E(|X(2t)-X(t)|) + E(|X(t|) = 2E(|X(t|)},$$

while, on the other hand, $\mathrm{E(|X(2t)|) = 2^H E(|X(t)|)}$. We then have $\mathrm{2E(X(t)) \leq 2^H E(|X(t)|)}$, so $2^H \leq 2$.

Now define

$$Y(k) = X(k+1) - X(k),\ k \in \mathbb{Z}$$

We can show that $Y(k)$ has the following properties:

- $\mathrm{E(Y(k))} = 0$.
- $\gamma(k) = \mathrm{Cov(Y(i),Y(i+k))} = \frac{\sigma^2}{2}(|\mathrm{k}+1|^{2\mathrm{H}} - 2|\mathrm{k}|^{2\mathrm{H}} + |\mathrm{k}-1|^{2\mathrm{H}})$. This follows from

$$\begin{aligned}
\mathrm{Cov(Y(i),Y(i+k))} &= \mathrm{Cov(Y(i+1)-Y(i), Y(i+k+1)-Y(i+k))}\\
&= \mathrm{Cov(Y(i+1),Y(i+k+1)) + Cov(Y(i),Y(i+k))}\\
&\quad - \mathrm{Cov(Y(i+1),Y(i+k)) - Cov(Y(i),Y(i+k+1))}\\
&= \frac{\sigma^2}{2}\left[|i+1|^{2H} + |i+k+1|^{2H} - |k|^{2H} + |i|^{2H} + |i+k|^{2H}\right.\\
&\quad - |k|^{2H} - |i+1|^{2H} - |i+k|^{2H} + |k-1|^{2H} - |i|^{2H}\\
&\quad \left. -|i+k+1|^{2H} + |k+1|^{2H}\right]\\
&= \frac{\sigma^2}{2}\left[|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right].
\end{aligned}$$

Clearly, if $H = \frac{1}{2}$, then $\gamma(k) = 0$. Otherwise, $\gamma(k) \neq 0$.

- If $H \neq \frac{1}{2}$,

$$\gamma(k) \sim \sigma^2 H(2H-1)|k|^{2H-2} \quad as \quad k \to \infty.$$

We can show this by considering positive k only since $\gamma(k) = \gamma(-k)$. For $k \geq 1$,

$$\begin{aligned}
\gamma(k) &= \frac{\sigma^2}{2}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H})\right] \\
&= \frac{\sigma^2}{2}k^{2H}\left[\left(1+\frac{1}{k}\right)^{2H} - 2 + \left(1-\frac{1}{k}\right)^{2H}\right] \\
&= \frac{\sigma^2}{2}k^{2H}g(1/k),
\end{aligned}$$

where $g(x) =: (1+x)^{2H} - 2 + (1-x)^{2H}$.

Using Taylor expansion we can derive the asymptotic behavior of $\gamma(k)$. First notice that if $H \neq \frac{1}{2}$

$$g'(x) = 2H(1+x)^{2H-1} - 2H(1-x)^{2H-1},$$

and

$$g''(x) = 2H(2H-1)(1+x)^{2H-2} + 2H(2H-1)(1-x)^{2H-2}.$$

We have

$$g(x) \approx g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 = 2H(2H-1)x^2, \quad as\ x \to 0.$$

Therefore,

$$\gamma(k) \sim \sigma^2 H(2H-1)|k|^{2H-2} \quad as \quad k \to \infty.$$

For the autocorrelation function $\rho(k)$ of $Y(k)$, we have

- If $H = \frac{1}{2}$, then $\rho(k) = 0,\ k \in \mathbb{Z}$. Hence, the observation $Y(k)$ are uncorrelated.

- If $0 < H < \frac{1}{2}$, then the correlations are summable. In fact we have

$$\sum_{k=-\infty}^{\infty} \rho(k) = 0.$$

- If $\frac{1}{2} < H < 1$, then the correlations decay at such a slow rate that

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty.$$

  In this case the process $Y(k),\ k \in \mathbb{Z}$ is a long memory process.

- If $H = 1$, then $\rho(k) = 1,\ k \in \mathbb{Z}$. This case is not relevant in practice.

5.3 Fractional Brownian Motion

Definition 5.3.1 *A fractional Brownian motion $B_H(t)$ (fBm) is a Gaussian self-similar process with index $H \in (0,1)$ and stationary increments. It is called standard if $\sigma^2 = \mathrm{Var}(\mathrm{B_H(1)}) = 1$.*

Since $B_H(t)$ is an H-sssi process, it has the following basic properties:

- $B_H(0) = 0$;
- $\mathrm{E(B_H(t))} = 0$;
- $B_H(t) \stackrel{d}{=} -B_H(-t)$;
- $\mathrm{Cov(B_H(s), B_H(t))} = \frac{\sigma^2}{2}\left[|\mathrm{t}|^{2\mathrm{H}} + |\mathrm{s}|^{2\mathrm{H}} - |\mathrm{t}-\mathrm{s}|^{2\mathrm{H}}\right]$.

When $H = \frac{1}{2}$, $\mathrm{Cov(B_H(s), B_H(t))} = \sigma^2 \min(|\mathrm{s}|, |\mathrm{t}|)$, therefore, $B_{\frac{1}{2}}(t)$ is actually a Brownian motion.

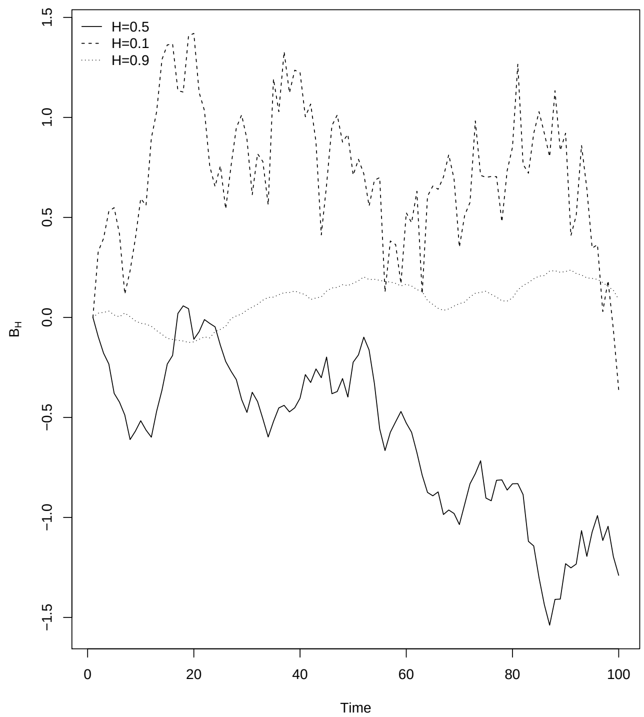

Figure 5.1: Simulation of fractional Brownian motion for different Hurst parameters.

Figure 5.1 depicts the realizations of fractional Brownian motion for $H = 0.1$, $H = 0.5$ and $H = 0.9$. It is clear that the smaller the Hurst parameter, the rougher the corresponding path.

A fractional Brownian motion always can be represented as a convolution of some deterministic function w.r.t. Brownian motion. This representation is also often taken directly as a definition of fractional Brownian motion.

Define

$$s_+ = \begin{cases} s & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}, \quad s_- = \begin{cases} -s & \text{if } s < 0 \\ 0 & \text{if } s \geq 0 \end{cases}.$$

Consider the kernel

$$Q_{u,1}(x;H) = c_1\left[(u-x)_+^{H-1/2} - (-x)_+^{H-1/2}\right] + c_2\left[(u-x)_-^{H-1/2} - (-x)_-^{H-1/2}\right]$$
$$=: c_1 Q_{u,1}^+(x;H) + c_2 Q_{u,1}^-(x;H),$$

where c_1, c_2 are real constants.

We note that the kernel $Q_{u,1}(x;H)$ is square integrable. Indeed, the first integrand $(u-x)_+^{H-1/2} - (-x)_+^{H-1/2}$ behaves like $(H-1/2)(-x)^{H-3/2}$ as $x \to \infty$ and $(u-x)_+^{H-1/2}$ as $x \to u$. Therefore, as long as $0 < H < 1$, $Q_{u,1}(x;H)$ is square integrable.

Theorem 5.3.2 *Let $B(u)$, $u \in \mathbb{R}$ be a standard Brownian motion on $\mathbb{R}$. Define*

$$B_H(u) = \int_{-\infty}^{\infty} Q_{u,1}(x;H)dB(x). \tag{5.3.1}$$

Then $B_H(u)$, $u \in \mathbb{R}$, is a fractional Brownian motion.

Proof: We want to show that $B_H(u)$, $u \in \mathbb{R}$, defined in (5.3.1) is a fractional Brownian motion in the sense of Definition 5.3.1. First, we know that the stochastic integral $\int_{-\infty}^{\infty} Q_{u,1}(x;H)dB(x)$ is normal with mean zero. Also, one can argue that the multivariate distributions are normal. Hence, the process defined in (5.3.1) is Gaussian. Moreover, one can argue that the process $B_H(u)$ is H-sssi. Therefore, $B_H(0) = 0$ almost surely.

Hence, it is sufficient to check that the covariance function of the process defined in (5.3.1) agrees with the covariance function of fBm.

In order to do this, note that due to stationarity of increments we have for $u < v$,

$$\mathrm{E[B_H(u)B_H(v)] = \frac{1}{2}(EB_H^2(u) + EB_H^2(v) - E(B_H(v-u) - B_H(0))^2).}$$

Thus, in order to evaluate covariance, we will evaluate the variance of $B_H(u)$ first.

For $u > 0$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (Q_{u,1}^{+}(x;H))^2 dx \\
&= \int_{-\infty}^{\infty} \left[(u-x)_{+}^{H-1/2} - (-x)_{+}^{H-1/2}\right]^2 dx \\
&= \int_{0}^{u} (u-x)^{2H-1} dx + \int_{-\infty}^{0} \left[(u-x)^{H-1/2} - (-x)^{H-1/2}\right]^2 dx \\
&= \frac{1}{2H} u^{2H} + u^{2H-1} \int_{-\infty}^{0} \left[(1-x/u)^{H-1/2} - (-x/u)^{H-1/2}\right]^2 dx.
\end{aligned}$$

Substitution $v = x/u$ yields

$$\begin{aligned}
&\int_{-\infty}^{\infty} (Q_{u,1}^{+}(x;H))^2 dx \\
&= \frac{1}{2H} u^{2H} + u^{2H} \int_{-\infty}^{0} \left[(1-v)^{H-1/2} - (-v)^{H-1/2}\right]^2 dv \\
&= u^{2H} \left\{ \frac{1}{2H} + \int_{0}^{\infty} \left[(1+v)^{H-1/2} - v^{H-1/2}\right]^2 dv \right\} =: u^{2H} C_1^2(H).
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\int_{-\infty}^{\infty} (Q_{u,1}^{-}(x;H))^2 dx \\
&= \int_{-\infty}^{\infty} \left[(u-x)_{-}^{H-1/2} - (-x)_{-}^{H-1/2}\right]^2 dx \\
&= \int_{u}^{\infty} \left[(x-u)^{H-1/2} - x^{H-1/2}\right]^2 dx - \int_{0}^{u} x^{2H-1} dx \\
&= u^{2H} \left\{ \int_{1}^{\infty} \left[(v-1)^{H-1/2} - v^{H-1/2}\right]^2 dv - \frac{1}{2H} \right\} =: u^{2H} C_2^2(H).
\end{aligned}$$

Furthermore,

$$\int_{-\infty}^{\infty}(Q^{+}_{u,1}(x;H))(Q^{-}_{u,1}(x;H))dx$$
$$= -\int_0^u (u-x)^{H-\frac{1}{2}}x^{H-\frac{1}{2}}dx$$
$$= -u^{2H}\int_0^1 (1-v)^{H-\frac{1}{2}}v^{H-\frac{1}{2}}dv =: -u^{2H}C_3(H).$$

Similar computation holds for $u < 0$. Therefore,

$$\mathrm{Var(B_H(u)) = u^{2H}\left(c_1^2C_1^2(H) + c_2^2C_2^2(H) - 2c_1c_2C_3(H)\right) =: u^{2H}C_4(H),}$$

and

$$\mathrm{E[B_H(u)B_H(v)] = \frac{1}{2}\left(|u|^{2H} + |v|^{2H} - |u-v|^{2H}\right)C_4(H),}$$

where the constant $C_4(H)$ is equal to $\mathrm{Var(B_H(1))}$.

Therefore, we have proved that $B_H(u),\ u \in \mathbb{R}$ is indeed a fractional Brownian motion. ∎

Depending on the values of c_1 and c_2, we obtain different representation of fractional Brownian motion. Let us consider two examples.

- If $c_1 = c_2 = 1$, then $B_H(u) = \int_{-\infty}^{\infty}(|u-x|^{H-1/2} - |x|^{H-1/2})dB(x)$ is the so-called well-balanced representation of fBm.

- Let
$$c_1 = \frac{1}{C_1(H)} = \left\{\frac{1}{2H} + \int_0^{\infty}\left[(1+v)^{H-1/2} - v^{H-1/2}\right]^2 dv\right\}^{-1/2},$$
and $c_2 = 0$. Then the integral
$$B_H(u) = \frac{1}{C_1(H)}\int_{-\infty}^{\infty}\left[(u-x)_+^{H-1/2} - (-x)_+^{H-1/2}\right]dB(x)$$

defines a standard fractional Brownian motion.

Another representation of fBm is given in Lévy (1953):

$$B_H(u) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^u (u - x)^{H-1/2} dB(x). \tag{5.3.2}$$

This is not a standard Brownian motion because $\mathrm{E}(\mathrm{B}_\mathrm{H}^2(1)) = \frac{1}{2\mathrm{H}\Gamma^2(\mathrm{H}+\frac{1}{2})}$. Note that the representation does not follow from (5.3.1), however, one can argue that the above formula defines a fractional Brownian motion. This representation is a tool for easy $\mathcal{L}^2$-definition of integrals with respect to fractional Brownian motion. In this thesis, we are going to focus on this representation since option pricing theory in continuous-time long memory stochastic volatility models can be based upon it.

Since $B_H(t)$ is a H-sssi process, its increments are stationary. We can define fractional Gaussian noise as follows.

Definition 5.3.3 *A process $Z(k)$ is called fractional Gaussian noise if*

$$Z(k) = B_H(k + 1) - B_H(k), \; k \in \mathbb{Z},$$

where $B_H(t)$ is a fractional Brownian motion.

Since $Z(k)$ is an increment of $B_H(t)$, it has the following properties:

- $Z(k) \sim N(0, \sigma^2)$;
- $\gamma(k) = \mathrm{Cov}(\mathrm{Z(i)}, \mathrm{Z(i+k)}) = \frac{\sigma^2}{2}(|\mathrm{k}+1|^{2\mathrm{H}} - 2|\mathrm{k}|^{2\mathrm{H}} + |\mathrm{k}-1|^{2\mathrm{H}})$;
- If $H = \frac{1}{2}$, then $\gamma(k) = 0, \; k \in \mathbb{Z}\backslash\{0\}$.
- If $0 < H < \frac{1}{2}$, then the covariances are summable.
- If $\frac{1}{2} < H < 1$, then

$$\sum_{k=-\infty}^{\infty} \gamma(k) = \infty.$$

Thus, when $\frac{1}{2} < H < 1$, $Z(k),\ k \in \mathbb{Z}$, is a long memory process.

5.4 Fractional Calculus and Fractional Integration

In this section, we provide the definitions of fractional calculus and integration, which we will require in the following discussions.

Definition 5.4.1 *Let $\phi \in \mathcal{L}^1[a,b]$ and $d > 0$, the Riemann-Liouville left- and right-sided fractional integrals on (a,b) of order d are defined by*

$$(I^d_{a+}\phi)(s) = \frac{1}{\Gamma(d)} \int_a^s \phi(u)(s-u)^{d-1} du,\ a < s < b,$$

and

$$(I^d_{b-}\phi)(s) = \frac{1}{\Gamma(d)} \int_s^b \phi(u)(u-s)^{d-1} du,\ a < s < b.$$

Definition 5.4.2 *Let $\phi \in \mathcal{L}^1[a,b]$ and $d > 0$, the Riemann-Liouville fractional integrals on $\mathbb{R}$ are defined as*

$$(I^d_{+}\phi)(s) = \frac{1}{\Gamma(d)} \int_{-\infty}^s \phi(u)(s-u)^{d-1} du,$$

and

$$(I^d_{-}\phi)(s) = \frac{1}{\Gamma(d)} \int_s^\infty \phi(u)(u-s)^{d-1} du.$$

If we denote $f(s)$ by

$$f(s) = (I^d_{a+}\phi)(s),$$

and solve it for ϕ, we have

$$\int_a^u \phi(z)dz = \frac{1}{\Gamma(1-d)} \int_a^u f(s)(u-s)^{-d} ds.$$

Finally by differentiating both sides with respect to u, we get

$$\phi(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_a^u f(s)(u-s)^{-d} ds.$$

Definition 5.4.3 *Let $0 < d < 1$, the Riemann-Liouville fractional derivatives on interval (a, b) can be defined as*

$$(D_{a+}^d f)(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_a^u f(s)(u-s)^{-d} ds, \ a < s < b,$$

and

$$(D_{b-}^d f)(u) = -\frac{1}{\Gamma(1-d)} \frac{d}{du} \int_u^b f(s)(s-u)^{-d} ds, \ a < s < b.$$

Definition 5.4.4 *Let $0 < d < 1$, the Riemann-Liouville fractional derivatives on $\mathbb{R}$ can be defined as*

$$(D_{+}^d f)(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_{-\infty}^u f(s)(u-s)^{-d} ds,$$

and

$$(D_{b-}^d f)(u) = -\frac{1}{\Gamma(1-d)} \frac{d}{du} \int_u^{\infty} f(s)(s-u)^{-d} ds.$$

Let $0 < d < 1$. Following Samko et al. (1993), we know that the fractional derivative D_{a+}^d and fractional integration I_{a+}^d have the following properties:

- For any $\phi \in \mathcal{L}^1[a, b]$, we have

$$D_{a+}^d I_{a+}^d \phi = \phi;$$

- For any f such that $f = I_{a+}^d \phi$, we have

$$I_{a+}^d D_{a+}^d f = f;$$

- If the function $I_{a+}^{1-d} f$ is absolutely continuous, then

$$(I_{a+}^d D_{a+}^d) f(s) = f(s) - \frac{(I_{a+}^{1-d} f)(a)}{\Gamma(d)} (s-d)^{d-1}, \ s \in (a,b),$$

  where $(I_{a+}^{1-d} f)(a) = \lim_{s \downarrow a} (I_{a+}^{1-d} f)(s)$.

The derivative operator D_{b-}^d has properties corresponding to those of D_{a+}^d.

5.5 Fractionally Integrated Processes

To be able to model long-memory processes in continuous time, we must generalize fractional Brownian motion. In this section we will study a class of linear continuous-time processes that exhibit long memory which was first developed by Comte and Renault (1996).

Comte and Renault (1996) start with the representation of fBm given in Lévy (1953) (see (5.3.2)):

$$B_H(u) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^u (u-x)^{H-1/2} dB(x),$$

where $B(\cdot)$ is a Brownian motion.

According to the notation used by Comte and Renault (1996), we rewrite the above representation by

$$W_d(t) = \frac{1}{\Gamma(d+1)} \int_0^t (t-s)^d dB(s), \tag{5.5.1}$$

where $-\frac{1}{2} < d < \frac{1}{2}$ and $d = H - \frac{1}{2}$. $B(t)$ is a Brownian motion.

Comte and Renault (1996) extend definition (5.5.1) to a class of fractionally integrated process.

Definition 5.5.1 *A fractionally integrated process of order d, $-\frac{1}{2} < d < \frac{1}{2}$ is defined* *as*

$$X(t) = \int_0^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s), \tag{5.5.2}$$

where $\tilde{A}(t)$ is a deterministic function of class $\mathcal{C}^1$ on $[0,\infty)$ and $B(t)$ is a Brownian motion.

Such process can be shown to be asymptotically equivalent to the stationary process $Y(t)$ in the sense that $\lim_{t\to\infty} \mathrm{E[X(t) - Y(t)]^2} = 0$, where

$$Y(t) = \int_{-\infty}^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s), \tag{5.5.3}$$

with

$$\mathrm{Var(Y)} = \int_0^\infty \mathrm{A^2(x)dx} < \infty, \tag{5.5.4}$$

where $A(x) =: \frac{x^d}{\Gamma(d+1)} \tilde{A}(x)$.

Comte and Renault (1996) then prove the following theorem:

Theorem 5.5.2 *Let*

$$X(t) = \int_0^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s)$$

be a fractionally integrated process of order d with $0 < d < \frac{1}{2}$ and

$$\lim_{x\to\infty} x\tilde{A}(x) = A_\infty \neq 0,$$

then

$$Y(t) = \int_{-\infty}^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s)$$

is weakly stationary process, asymptotically equivalent to $X(t)$, which verifies

$$\lim_{h\to\infty} h^{1-2d} \gamma_Y(h) = \frac{\Gamma(1-2d)\Gamma(d)}{\Gamma(1-d)\Gamma(1+d)^2} A_\infty^2,$$

where $\gamma_Y(h) = \mathrm{Cov(Y(t), Y(t+h))}$ *is the autocovariance function of* $Y(t)$*.*

As $Y(t)$ is asymptotically equivalent to $X(t)$, the covariance between $X(t)$ and $X(t+h)$ decreases towards zero as $h \to \infty$ at the same rate as h^{2d-1}. Comte and Renault (1996) therefore refer to all processes $X(t)$ verifying (5.5.4) with $0 < d < \frac{1}{2}$ as continuous-time long memory processes of order d.

Continuous-time fractionally integrated processes as defined in (5.5.2) admit several representations. In particular, Comte and Renault (1996) prove the following representation.

Theorem 5.5.3 *If* $X(t)$ *is a fractionally integrated process of order* d*,* $-\frac{1}{2} < d < \frac{1}{2}$*, defined by*

$$X(t) = \int_0^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s), \ t \in [0, T], \tag{5.5.5}$$

with $\tilde{A}(\cdot)$ *being* $\mathcal{C}^1$ *on* $[0, T]$*, then* $X(t)$ *can be written as*

$$X(t) = \int_0^t C(t-s) dW_d(s), \ t \in [0, T], \tag{5.5.6}$$

with C *continuous on* $[0, T]$*, where*

$$C(x) = \frac{1}{\Gamma(1-d)\Gamma(1+d)} \frac{d}{dx} \left(\int_0^x (x-s)^{-d} s^d \tilde{A}(s) ds \right),$$

and $W_d(\cdot)$ *is defined in (5.5.1).*

The reciprocal is true if C *is supposed* $\mathcal{C}^1$*, and then the resulting* $\tilde{A}$ *function is continuous and*

$$\tilde{A}(x) = C(0) + \int_0^x C'(u) \left(1 - \frac{u}{x}\right)^d du.$$

Thus, there is a one-to-one correspondence between $C(\cdot)$ and $\tilde{A}(\cdot)$. In other words, integration w.r.t. Brownian motion (5.5.5) can be replaced with integration w.r.t. fBm (5.5.6). Conversely, (5.5.5) can be treated as the definition of stochastic integration w.r.t. fBm.

Comte and Renault (1996) also define a fractional derivation/integration of order d that yields a usual asymptotically stationary short memory process.

Theorem 5.5.4 *If $X(t)$ is a fractionally integrated process of order d, $-\frac{1}{2} < d < \frac{1}{2}$,*

$$X(t) = \int_0^t \frac{(t-s)^d}{\Gamma(d+1)} \tilde{A}(t-s) dB(s),\ t \in [0,T],$$

then

$$X^{(-d)}(t) = \frac{d}{dt}\left[\int_0^t \frac{(t-s)^{-d}}{\Gamma(1-d)} X(s) ds\right] = \int_0^t \frac{(t-s)^{-d}}{\Gamma(1-d)} dX(s)$$

is well-defined and mean square continuous. If, moreover, $\tilde{A}(0)$ is invertible, and $\tilde{A}$ $\mathcal{C}^2$ on $[0,T]$, then $X^{(-d)}$ admits the $MA(\infty)$ representation:

$$X^{(-d)}(t) = \int_0^t C(t-s) dB(s),$$

where $\tilde{A}$ and C are one-to-one related as in theorem 5.5.3. That is, $X^{(-d)}(t)$ doesn't depend on d.

We can see that $X^{(-d)}(t)$ is in fact the Riemann-Liouville fractional derivatives of $X(t)$:

$$X^{(-d)}(t) = (D^d_{0+}X)(t) = \frac{1}{\Gamma(1-d)} \frac{d}{dt} \int_0^t X(s)(t-s)^{-d} ds.$$

Finally, Comte and Renault (1996) prove the invariance property of the fractional derivative/integration.

Theorem 5.5.5 *If a process $X(t)$ satisfies*

$$dX(t) = -\kappa X(t) dt + \sigma dW_d(t),\ X(0) = 0,$$

where $-\frac{1}{2} < d < \frac{1}{2}$, κ and σ are constants and $W_d(t)$ is an fBm, then its fractional *derivative $X^{(-d)}(t)$ of order d satisfies the Ornstein-Uhlenbeck equation*

$$dX^{(-d)}(t) = -\kappa X^{(-d)}(t)dt + \sigma dB(t),\ X^{(-d)}(0) = 0,$$

where $B(t)$ is a Brownian motion and

$$X^{(-d)}(t) = \frac{d}{dt}\left[\int_0^t \frac{(t-s)^{-d}}{\Gamma(1-d)}X(s)ds\right] = \int_0^t \frac{(t-s)^{-d}}{\Gamma(1-d)}dX(s).$$

Conversely, if $Y(t)$ satisfies

$$dY(t) = -\kappa Y(t)dt + \sigma dB(t),\ Y(0) = 0,$$

then its fractional integral $Y^{(d)}(t)$ of order d satisfies

$$dY^{(d)}(t) = -\kappa Y^{(d)}(t)dt + \sigma dW_d(t),\ Y^{(d)}(0) = 0,$$

where

$$Y^{(d)}(t) = \frac{1}{\Gamma(d)}\int_0^t (t-s)^{d-1}Y(s)ds.$$

Now we give two examples that apply the results in this section.

5.6 Fractional Ornstein-Uhlenbeck Process

A stochastic process $X(t)$ is said to be a fractional Ornstein-Uhlenbeck process if its dynamics is of the form

$$X(t) = Y^{(d)}(t) + \theta, \tag{5.6.1}$$

where

$$Y^{(d)}(t) = \frac{1}{\Gamma(d)}\int_0^t (t-s)^{d-1}Y(s)ds = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)}dY(s), \tag{5.6.2}$$

where $-\frac{1}{2} < d < \frac{1}{2}$ and

$$dY(t) = -\kappa Y(t) + \sigma dB(t),\ \kappa > 0,\ Y(0) = 0, \tag{5.6.3}$$

where $B(t)$ is a standard Brownian motion.

We can see that $Y^{(d)}(t)$ is in fact the Riemann-Liouville order-d fractional integration of $Y(t)$. From theorem 5.5.5, we know $Y^{(d)}(t)$ can be alternatively represented as

$$dY^{(d)}(t) = -\kappa Y^{(d)}(t)dt + \sigma dW_d(t),\ \kappa > 0,\ Y^{(d)}(0) = 0,$$

where $W_d(t)$ is a fractional Brownian motion with Hurst parameter $H = d + \frac{1}{2}$.

The solution to the last equation is

$$Y^{(d)}(t) = \int_0^t \exp(-\kappa(t-s))\sigma dW_d(s).$$

Thus, from theorem 5.5.3, the fractional Ornstein-Uhlenbeck process $X(t)$ can be expressed in two ways:

$$X(t) = \theta + \int_0^t A(t-s)dB(s) = \theta + \int_0^t \frac{(t-s)^d}{\Gamma(d+1)}\tilde{A}(t-s)dB(s),$$

and

$$X(t) = \theta + \int_0^t C(t-s)dW_d(s),$$

where obviously

$$C(x) = \exp(-\kappa x)\sigma,$$

and

$$\begin{aligned} A(x) &= \frac{\sigma}{\Gamma(1+d)}\frac{d}{dx}\left[\int_0^x \exp(-\kappa s)(x-s)ds\right] \\ &= \frac{\sigma}{\Gamma(1+d)}\left(x^d - \kappa\exp(-\kappa x)\int_0^x \exp(\kappa s)s^d ds\right). \end{aligned}$$

Moreover, it can be checked that for $0 < d < \frac{1}{2}$, the process $X(t)$ satisfies the long memory condition in theorem 5.5.2[1]

$$\lim_{x \to \infty} x\tilde{A}(x) = A_\infty,$$

with

$$A_\infty = \frac{\sigma}{\kappa} d.$$

We can simulate the process $Y^{(d)}(t)$, $X(t)$ and numerically evaluate the integral in equation (5.6.2) using only the involved processes $Y(s)$ and $B(s)$ on a discrete partition of $[0, t]$: $j/n, j = 0, 1, \ldots, [nt]$.[2] A natural way to obtain such approximations is to approximate the integrands by step functions:

$$\begin{aligned} Y^{(d)}(t_i) &= \int_0^{t_i} \frac{(t_i - [ns]/n)^d}{\Gamma(1+d)} dY(s) \\ &\approx \sum_{j=1}^{i} \frac{(t_i - (j-1)/n)^d}{\Gamma(1+d)} (Y(t_j) - Y(t_{j-1})) \\ &= \left[\sum_{j=0}^{i-1} \frac{(j+1)^d - j^d}{n^d \Gamma(1+d)} L^j \right] Y(t_i) \end{aligned} \tag{5.6.4}$$

where $t_i = i/n$ for $0 \le i \le [nt]$ and $L^j Y(t_i) = Y(t_{i-j})$.

$Y(t_i)$ is an AR(1) process[3]

$$Y(t_i) = \rho_n Y(t_{i-1}) + \gamma_n \varepsilon_2(t_i), \tag{5.6.5}$$

where $\rho_n = 1 - \frac{\kappa}{n}$, $\gamma_n = \frac{\gamma}{\sqrt{n}}$ and $\varepsilon_2(t_i) \sim N(0, 1)$, $i \ge 0$ are i.i.d..

[1]See Comte and Renault (1996) for the proof.

[2]$[z]$ is the integer k such that $k \le z < k + 1$.

[3]Recall from section 2.4.2, $Y(t_i)$ is a simple discretization of continuous time Ornstein-Uhlenbeck process $Y(t)$.

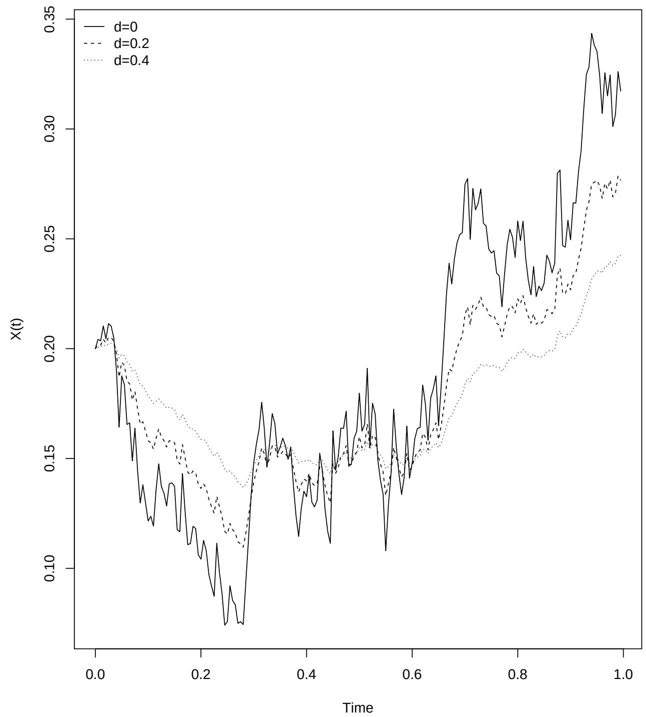

Figure 5.2: Simulation of fractional Ornstein-Uhlenbeck process for different integration parameters. $X(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \sigma dB(t)$. $\theta = 0.2$, $\kappa = 4$, $\sigma = 0.2$ and $y(0) = 0$.

Figure 5.2 shows a single simulated path of X for $d = 0$, $d = 0.2$ and $d = 0.4$ respectively. For three paths we have used $\theta = 0.2$, $\kappa = 4$, $\sigma = 0.2$ and the same sequence of random numbers. We find that the integration parameter d influences the smoothness of the volatility process. The greater d is, the smoother the path of X is.

5.7 Fractional Square Root Process

Let $\hat{Y}(t)$ be a square root stochastic process (see section 2.4.3):

$$d\hat{Y}(t) = \kappa(\theta - \hat{Y}(t))dt + \sigma\sqrt{\hat{Y}(t)}dB(t),\ \hat{Y}(0) = \theta, \tag{5.7.1}$$

where $\kappa > 0$, $\sigma > 0$ and $B(t)$ is the standard Brownian motion. We also impose the restriction $\kappa\theta \geq \frac{\sigma^2}{2}$. Under this restriction, Lamberton and Lapeyre (1996) show that $\hat{Y}(t)$ starting from a positive value has a zero-probability to hit the barrier zero within a finite time. In other words, $\hat{Y}(t)$ is positive almost surely over any finite time horizon.

Denote by $Y(t)$ the centered version of $\hat{Y}(t)$:

$$Y(t) = \hat{Y}(t) - \theta.$$

The process of $Y(t)$ then becomes

$$dY(t) = -\kappa Y(t)dt + \sigma\sqrt{\theta + Y(t)}dB(t). \tag{5.7.2}$$

A stochastic process $X(t)$ is said to be a fractional square root process if its dynamics is of the form

$$X(t) = \theta + Y^{(d)}(t), \tag{5.7.3}$$

where $-\frac{1}{2} < d < \frac{1}{2}$ and $Y^{(d)}(t)$ is the Riemann-Liouville order-d fractional integration of $Y(t)$:

$$Y^{(d)}(t) = (I^d_{0+}Y)(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} Y(s)ds. \tag{5.7.4}$$

Note that if $d = 0$, then formally, $X(t) = \hat{Y}(t)$ and is a square root process.[4]

[4] Since $Y^{(d)}(t)$ is not bounded from below, the positivity of $X(t)$ will never be guaranteed.

Let

$$\widetilde{X}(t) = \theta + \widetilde{Y}^{(d)}(t),$$

where $-\frac{1}{2} < d < \frac{1}{2}$ and $\widetilde{Y}^{(d)}(t)$ is the Riemann-Liouville order-d fractional integration of $Y(t)$:

$$\widetilde{Y}^{(d)}(t) = (I_+^d Y)(t) = \int_{-\infty}^{t} \frac{(t-s)^{d-1}}{\Gamma(d)} Y(s) ds.$$

Comte, Coutin and Renault (2003) prove the following theorem:

Theorem 5.7.1 *For $0 \le d < \frac{1}{2}$, and if Y is mean-square stationary and with an exponentially decaying autocovariance function ($|\gamma_Y(u)| \le \gamma_Y(0)\exp(-\kappa|u|)$), then $\widetilde{Y}^{(d)} = (I_+^d Y)(t)$ is mean square stationary and for $t \to \infty$: $||\widetilde{Y}^{(d)} - \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} Y(s) ds||_2 = O(t^{d-1/2})$.*

From this theorem, we know that $X(t)$ is asymptotically equivalent (in quadratic mean) to the stationary process $\widetilde{X}(t)$.

Comte, Coutin and Renault (2003) also prove the following theorems:

Theorem 5.7.2 *For $0 < d < \frac{1}{2}$, $\mathrm{Var}(\widetilde{\mathrm{X}}(\mathrm{t})) = \frac{\theta\sigma^2}{\kappa^{2\mathrm{d}+1}} \frac{\Gamma(1-2\mathrm{d})\Gamma(2\mathrm{d})}{\Gamma(1-\mathrm{d})\Gamma(\mathrm{d})}$.*

Theorem 5.7.3 *For $0 < d < \frac{1}{2}$,*

$$\frac{\gamma_{\widetilde{X}}(h)}{\gamma_{\widetilde{X}}(0)} = 1 - \frac{(\kappa h)^{2d+1}}{2d(2d+1)\Gamma(2d)} + O(h^2) \text{ when } h \to 0,$$

$$\frac{\gamma_{\widetilde{X}}(h)}{\gamma_{\widetilde{X}}(0)} \sim \frac{(\kappa h)^{2d-1}}{\Gamma(2d)} \text{ when } h \to \infty.$$

From the above theorems, we can see that $\widetilde{X}(t)$ is the long memory process. In the very short term, the autocorrelation function of $\widetilde{X}(t)$ reaches 1 with the speed of $(\kappa h)^{2d+1}$ instead of κh.

Since the covariance function of $X(t)$ is asymptotically approaching $\widetilde{X}(t)$, we can see that $X(t)$ is also a long memory process. In the very short term, the autocorrelation function of $X(t)$ reaches 1 with the speed of $(\kappa h)^{2d+1}$ instead of κh.

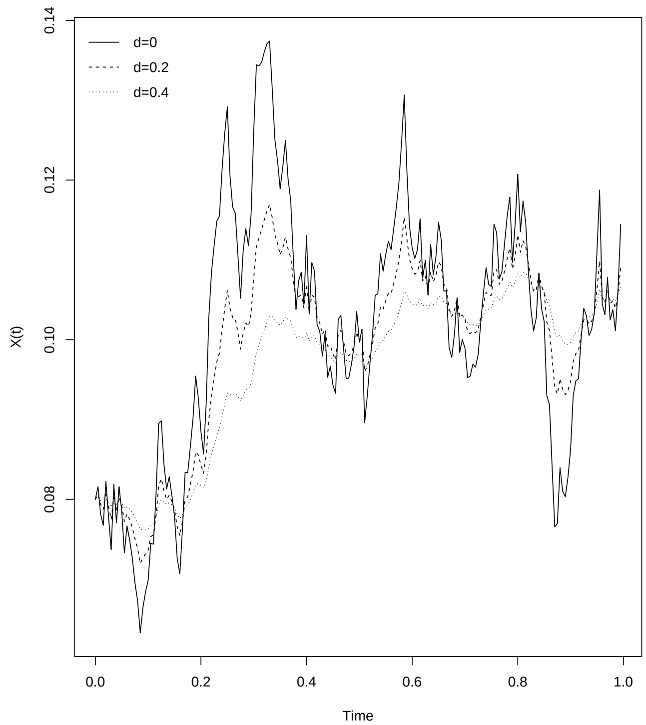

Figure 5.3: Simulation of fractional square root process for different integration parameters. $X(t) = \theta + Y^{(d)}(t)$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} Y(s)ds$ and $dY(t) = -\kappa Y(t)dt + \sigma\sqrt{\theta + Y(t)}dB(t)$. $\theta = 0.08$, $\kappa = 0.2$, $\sigma = 0.2$ and $y(0) = 0$.

We can simulate the fractional square root process in the similar way to the fractional Ornstein-Uhlenbeck process. Figure 5.3 shows a single simulated path of X for $d = 0$, $d = 0.2$ and $d = 0.4$ respectively. For three paths we have used $\theta = 0.08$, $\kappa = 0.2$, $\sigma = 0.2$ and the same sequence of random numbers. We find that the integration parameter d influences the smoothness of the volatility process. The greater d is, the smoother the path of X is.

Chapter 6

Option Pricing with Long Memory Stochastic Volatility Models

6.1 Introduction

In chapter 4, we introduced stochastic volatility models. It is widely believed that volatility skew can be explained to a great extent by such models. However, recent evidence documents the long memory property in various volatility measures. For example, Ding et al. (1993), De Lima and Crato (1994), and Breidt et al. (1998), among others, observe that the squared returns of market indexes have the long memory property. Furthermore, Bollerslev and Mikkelsen (1999) document the fact that volatility skew effects are significant for very long term options. However, intuitively, under the assumption of short memory for the volatility processes, by a simple application of the law of large numbers to volatility process, the effects of the randomness of the volatility should vanish when the time to maturity of the option increases and therefore the volatility skew should be erased. Sundaresan (2000) points out this as the so-called term structure of volatility smiles puzzle.

To better reconcile the short term and long term observed patterns of the term structure of implied volatilities, Comte and Renault (1998) introduce a stochastic volatility model with long memory. Their model can be defined as

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB^Q(t),$$

or

$$dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dB^Q(t),$$

$$\sigma^2(t) = \exp(Y(t)),$$

$$dY(t) = \kappa(\theta - Y(t))dt + \gamma dB_H^Q(t),$$

where $X(t) = \ln(S(t))$, $B^Q(t)$ is Brownian motion under the risk neutral probability measure Q and $B_H^Q(t)$ is fractional Brownian motion under the risk neutral probability measure Q. It is clear that $Y(t)$ is a fractional Ornstein-Uhlenbeck process (see section 5.6) and $S(t)$ is a geometric Brownian motion given $Y(\cdot)$.

Since the closed-form solution for option pricing does not exist, Comte and Renault (1998) provide discrete approximation to this fractional stochastic volatility model and compute option prices based on Monte-Carlo simulation. Chronopoulou and Viens (2012a) also study this stochastic volatility model. In order to deal with the pricing problem, they construct a multinomial recombining tree using sampled values of the volatility. Besides this continuous-time model, Chronopoulou and Viens (2012b) also study two discrete time models: a discretization of the continuous model via an Euler scheme and a discrete time model in which the returns are a zero mean i.i.d. sequence where the volatility is exponential of a fractional ARIMA process.

Comte, Coutin and Renault (2003) propose an affine fractional stochastic volatility model where they specify the volatility process as a square root process and then perform a fractional integration of it. Their model can be described as

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB_1^Q(t),$$

or

$$dX(t) = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_1^Q(t),$$

$$\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t),$$

$$(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds.$$

$$d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t).$$

where $X(t) = \ln(S(t))$. $B_1^Q(t)$ and $B_2^Q(t)$ are two independent Brownian motions under the risk neutral probability measure Q. It is clear that $\sigma^2(t)$ is a fractional square root process (see section 5.7) and $S(t)$ is a generalized geometric Brownian motion given $\sigma^2(\cdot)$.

Comte, Coutin and Renault (2003) provide a recursive algorithm of discretization of fractional integrals in order to compute the option prices through simulations. They show that the volatility process in the affine fractional stochastic volatility model appears to be not much more persistent in the very short run than any standard diffusion volatility process while it is infinitely more persistent in the long run: the autocovariance function of the volatility process decreases at a hyperbolic rate for infinitely large lags instead of the standard exponential rate.

In this chapter, we extend the fractional stochastic volatility models developed by Comte and Renault (1998) and Comte, Coutin and Renault (2003). First, we introduce both fractional Heston model and fractional Schöbel-Zhu model where we also allow the non-zero correlation between volatility and stock price processes. Second, we derive the closed-form solution for the price of options under fractional Heston model. We also provide an approximate formula for fractional Schöbel-Zhu model. To date, no closed-form solutions for option prices under continuous-time long memory volatility models exist. Accordingly, our result is the first attempt at this. Third, we numerically explore the effects of long memory on option prices. Without the closed-form solutions, this will be a computationally intensive task.

6.2 Fractional Heston Model

6.2.1 Analytical Formula for Characteristic Function

We define the fractional Heston model by

$$dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t) dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)} dB_2^Q(t), \tag{6.2.1}$$

$$\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t), \tag{6.2.2}$$

$$(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} \sigma_c^2(s) ds, \tag{6.2.3}$$

$$d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)} dB_2^Q(t), \tag{6.2.4}$$

where $X(t) = \ln(S(t))$. $0 < d < \frac{1}{2}$, $B_1^Q(t)$ and $B_2^Q(t)$ are two independent Brownian motions under the risk neutral probability measure Q. ρ is a constant to induce the correlation between the volatility and stock price processes. It is clear that $\sigma^2(t)$ is a fractional square root process.

This model as formulated above has never been considered in literature. By imposing the constraint that $\rho = 0$, we obtain the affine fractional stochastic volatility model of Comte, Coutin and Renault (2003). [1] By imposing the constraint that $d = 0$, we will have the Heston stochastic volatility model.

Let $\mathcal{F}_t$ be the filtration generated by $B_1^Q(\cdot)$ and $B_2^Q(\cdot)$. To obtain the analytical formula for option pricing, we need to find the conditional characteristic function $f(\phi)$ of $X(t+\tau)$ ($f(\phi) = \mathrm{E^Q[exp(i\phi X(t+\tau))|\mathcal{F}_t]}$) under the probability measure Q, as in section 4.4. This will be achieved by representing the fractional square root process in terms of square root process via fractional integration. We can compute $f(\phi)$ from

[1] Note that, since $(\sigma_c^2)^{(d)}(t)$ is not lower bounded, the positivity of $\sigma^2(t)$ will never be guaranteed, irrespective of the value of θ. However, Comte, Coutin and Renault (2003) find that positivity is indeed preserved in simulations for relevant parameter values.

the following theorem:

Theorem 6.2.1 *The conditional characteristic function $f(\phi)$ of $X(t+\tau)$ under the probability measure Q for the fractional Heston model is given by*

$$f(\phi) = \exp\left[i\phi\tau r - \frac{1}{2}i\phi\rho^2\gamma^2\theta\tau - \frac{1}{2}\phi(i+\phi)\theta\tau - \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\int_0^t \left((t+\tau-s)^d \right.\right.$$
$$\left.\left. -(t-s)^d\right)\sigma_c^2(s)ds - A(\tau) - (i\phi\rho + B(\tau))\sigma_c^2(t) + i\phi X(t)\right], \tag{6.2.5}$$

where the functions $A(\tau)$ and $B(\tau)$ can be computed by numerically solving the following system of ODEs:

$$\dot{A}(\tau) = -\frac{1}{2}\gamma^2\theta B^2(\tau),$$
$$\dot{B}(\tau) = -\kappa B(\tau) - \frac{1}{2}\gamma^2 B^2(\tau) + \frac{1}{2}i\phi\rho^2\gamma^2 + \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\tau^d - i\phi\rho\kappa,$$

with boundary conditions $A(0) = 0$ and $B(0) = -i\phi\rho$.

Proof: From (6.2.2) and (6.2.3) we can decompose the integrated volatility as

$$\begin{aligned}
\int_t^{t+\tau}\sigma^2(s)ds &= \theta\tau + \int_0^{t+\tau}(\sigma_c^2)^{(d)}(s)ds - \int_0^t(\sigma_c^2)^{(d)}(s)ds \\
&= \theta\tau + (\sigma_c^2)^{(d+1)}(t+\tau) - (\sigma_c^2)^{(d+1)}(t) \\
&= \theta\tau + \int_0^{t+\tau}\frac{(t+\tau-s)^d}{\Gamma(d+1)}\sigma_c^2(s)ds - \int_0^t\frac{(t-s)^d}{\Gamma(d+1)}\sigma_c^2(s)ds \\
&= \theta\tau + \frac{1}{\Gamma(d+1)}\int_0^t[(t+\tau-s)^d - (t-s)^d]\sigma_c^2(s)ds \\
&\quad + \frac{1}{\Gamma(d+1)}\int_t^{t+\tau}(t+\tau-s)^d\sigma_c^2(s)ds,
\end{aligned} \tag{6.2.6}$$

where we have used the fact $(\sigma_c^2)^{(d+1)}(t) = \int_0^t(\sigma_c^2)^{(d)}(s)ds$ and (6.2.3) in (6.2.6).

The equation (6.2.4) can be written as

$$\sigma_c^2(t+\tau) = \sigma_c^2(t) - \kappa \int_t^{t+\tau} \sigma_c^2(s)ds + \int_t^{t+\tau} \gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s). \tag{6.2.7}$$

Equation (6.2.7) can be equivalently written as

$$\int_t^{t+\tau} \gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s) = \sigma_c^2(t+\tau) - \sigma_c^2(t) + k\int_t^{t+\tau} \sigma_c^2(s)ds. \tag{6.2.8}$$

The equation (6.2.1) can be written as

$$\begin{aligned} X(t+\tau) = X(t) &+ \int_t^{t+\tau} \left(r - \frac{1}{2}\sigma^2(s) - \frac{1}{2}\rho^2\gamma^2(\theta+\sigma_c^2(s))\right) ds \\ &+ \int_t^{t+\tau} \sigma(s)dB_1^Q(s) + \int_t^{t+\tau} \rho\gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s). \end{aligned} \tag{6.2.9}$$

We can compute $f(\phi)$ by

$$\begin{aligned} f(\phi) &= \mathrm{E^Q}[\exp(\mathrm{i}\phi \mathrm{X(t+\tau)})|\mathcal{F}_\mathrm{t}] \\ &= \mathrm{E^Q}\left[\exp\left(\mathrm{i}\phi\mathrm{X(t)} + \mathrm{i}\phi\int_\mathrm{t}^{\mathrm{t}+\tau}\left(\mathrm{r} - \frac{1}{2}\sigma^2(\mathrm{s}) - \frac{1}{2}\rho^2\gamma^2(\theta+\sigma_\mathrm{c}^2(\mathrm{s}))\right)\mathrm{ds}\right.\right. \\ &\quad \left.\left. + i\phi\int_t^{t+\tau}\sigma(s)dB_1^Q(s) + i\phi\int_t^{t+\tau}\rho\gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s)\right)|\mathcal{F}_t\right] \\ &= \exp(i\phi X(t) + i\phi\tau r)\mathrm{E^Q}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\int_\mathrm{t}^{\mathrm{t}+\tau}\rho^2\gamma^2(\theta+\sigma_\mathrm{c}^2(\mathrm{s}))\mathrm{ds}\right.\right. \\ &\quad \left.\left. -\frac{1}{2}i\phi\int_t^{t+\tau}\sigma^2(s)ds + i\phi\int_t^{t+\tau}\sigma(s)dB_1^Q(s) + i\phi\int_t^{t+\tau}\rho\gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s)\right)|\mathcal{F}_t\right] \\ &= \exp(i\phi X(t) + i\phi\tau r)\mathrm{E^Q}\left\{\mathrm{E^Q}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\int_\mathrm{t}^{\mathrm{t}+\tau}\rho^2\gamma^2(\theta+\sigma_\mathrm{c}^2(\mathrm{s}))\mathrm{ds}\right.\right.\right. \\ &\quad -\frac{1}{2}i\phi\int_t^{t+\tau}\sigma^2(s)ds + i\phi\int_t^{t+\tau}\sigma(s)dB_1^Q(s) \\ &\quad \left.\left.\left. + i\phi\int_t^{t+\tau}\rho\gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s)\right)|\mathcal{F}_t, B_2^Q(s) : s\in[t,t+\tau]\right]|\mathcal{F}_t\right\}. \end{aligned} \tag{6.2.10}$$

Noting that

$$\begin{aligned}
&\mathrm{E}^{\mathrm{Q}}\left[\exp\left(\mathrm{i}\phi\int_{\mathrm{t}}^{\mathrm{t}+\tau}\sigma(\mathrm{s})\mathrm{dB}_1^{\mathrm{Q}}(\mathrm{s})\right)|\mathcal{F}_{\mathrm{t}},\mathrm{B}_2^{\mathrm{Q}}(\mathrm{s}):\mathrm{s}\in[\mathrm{t},\mathrm{t}+\tau]\right]\\
&=\mathrm{E}^{\mathrm{Q}}\left[-\frac{1}{2}\phi^2\int_{\mathrm{t}}^{\mathrm{t}+\tau}\sigma^2(\mathrm{s})\mathrm{ds}|\mathcal{F}_{\mathrm{t}}\right],
\end{aligned}$$

we have

$$\begin{aligned}
f(\phi)&=\mathrm{E}^{\mathrm{Q}}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{t}+\tau))|\mathcal{F}_{\mathrm{t}}]\\
&=\exp(i\phi X(t)+i\phi\tau r)\mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\int_{\mathrm{t}}^{\mathrm{t}+\tau}\rho^2\gamma^2(\theta+\sigma_{\mathrm{c}}^2(\mathrm{s}))\mathrm{ds}\right.\right.\\
&\quad\left.\left.-\frac{1}{2}i\phi\int_t^{t+\tau}\sigma^2(s)ds-\frac{1}{2}\phi^2\int_t^{t+\tau}\sigma^2(s)ds+i\phi\int_t^{t+\tau}\rho\gamma\sqrt{\theta+\sigma_c^2(s)}dB_2^Q(s)\right)|\mathcal{F}_t\right].
\end{aligned}$$

Using (6.2.8) and (6.2.6), we have

$$
\begin{aligned}
f(\phi) &= \mathrm{E}^{\mathrm{Q}}[\exp(\mathrm{i}\phi \mathrm{X}(\mathrm{t}+\tau))|\mathcal{F}_{\mathrm{t}}] \\
&= \exp(i\phi X(t) + i\phi\tau r)\mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\rho^2\gamma^2\theta\tau - \frac{1}{2}\mathrm{i}\phi\int_{\mathrm{t}}^{\mathrm{t}+\tau}\rho^2\gamma^2\sigma_{\mathrm{c}}^2(\mathrm{s})\mathrm{ds}\right.\right. \\
&\quad - \frac{1}{2}i\phi\int_t^{t+\tau}\sigma^2(s)ds - \frac{1}{2}\phi^2\int_t^{t+\tau}\sigma^2(s)ds \\
&\quad \left.\left. + i\phi\rho\left(\sigma_c^2(t+\tau) - \sigma_c^2(t) + \kappa\int_t^{t+\tau}\sigma_c^2(s)ds\right)\right)|\mathcal{F}_t\right] \\
&= \exp\left(i\phi X(t) + i\phi\tau r - \frac{1}{2}i\phi\rho^2\gamma^2\theta\tau - i\phi\rho\sigma_c^2(t)\right)\mathrm{E}^{\mathrm{Q}}\left[\exp\left(-\frac{1}{2}\mathrm{i}\phi\int_{\mathrm{t}}^{\mathrm{t}+\tau}\rho^2\gamma^2\sigma_{\mathrm{c}}^2(\mathrm{s})\mathrm{ds}\right.\right. \\
&\quad - \frac{1}{2}\phi(i+\phi)\left(\theta\tau + \frac{1}{\Gamma(d+1)}\int_0^t[(t+\tau-s)^d - (t-s)^d]\sigma_c^2(s)ds\right. \\
&\quad \left.\left.\left. + \frac{1}{\Gamma(d+1)}\int_t^{t+\tau}(t+\tau-s)^d\sigma_c^2(s)ds\right) + i\phi\rho\kappa\int_t^{t+\tau}\sigma^2(s)ds + i\phi\rho\sigma_c^2(t+\tau)\right)\right] \\
&= \exp\left(i\phi X(t) + i\phi\tau r - \frac{1}{2}i\phi\rho^2\gamma^2\theta\tau - i\phi\rho\sigma_c^2(t) - \frac{1}{2}\phi(i+\phi)\theta\tau\right. \\
&\quad \left. - \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\int_0^t[(t+\tau-s)^d - (t-s)^d]\sigma_c^2(s)ds\right) \\
&\quad \times \mathrm{E}^{\mathrm{Q}}\left[\exp\left(\int_{\mathrm{t}}^{\mathrm{t}+\tau}\left(-\frac{1}{2}\mathrm{i}\phi\rho^2\gamma^2 - \frac{1}{2}\phi(\mathrm{i}+\phi)\frac{1}{\Gamma(\mathrm{d}+1)}(\mathrm{t}+\tau-\mathrm{s})^{\mathrm{d}}\right.\right.\right. \\
&\quad \left.\left. + i\phi\rho\kappa\right)\sigma_c^2(s)ds\right)\exp(i\phi\rho\sigma_c^2(t+\tau))|\mathcal{F}_t\Big].
\end{aligned}
$$

In other words, the original formula (6.2.10) for $f(\phi)$ involves long memory process $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, whereas the formula above involves a standard square root process $\sigma_c^2(t)$ only (no long memory process anymore).

Let $V(t) = \sigma_c^2(t)$. To obtain the final form of $f(\phi)$, we need to calculate the following expression

$$
\begin{aligned}
Y(t, V(t)) = \mathrm{E}^{\mathrm{Q}}\Bigg[\exp\Bigg(\int_{\mathrm{t}}^{\mathrm{t}+\tau}\Bigg(&-\frac{1}{2}\mathrm{i}\phi\rho^2\gamma^2 - \frac{1}{2}\phi(\mathrm{i}+\phi)\frac{1}{\Gamma(\mathrm{d}+1)}(\mathrm{t}+\tau-\mathrm{s})^{\mathrm{d}} \\
&+ i\phi\rho\kappa\Bigg)V(s)ds\Bigg)\exp(i\phi\rho V(t+\tau))|\mathcal{F}_t\Bigg].
\end{aligned}
$$

According to the Feynman-Kac theorem, $Y(t, V(t))$ should fulfill the following one-dimensional PDE,

$$\frac{\partial Y}{\partial t} + \frac{1}{2}\nu^2(t,V)\frac{\partial^2 Y}{\partial V^2} + \mu(t,V)\frac{\partial Y}{\partial V} - g(t,V)Y(t,V) = 0, \tag{6.2.11}$$

where

$$\mu(t, V(t)) = -\kappa V(t),$$
$$\nu(t, V(t)) = \gamma\sqrt{\theta + V(t)},$$
$$g(t, V(t)) = \left(\frac{1}{2}i\phi\rho^2\gamma^2 + \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\tau^d - i\phi\rho\kappa\right)V(t),$$

with boundary condition

$$Y(T, V(T)) = \exp(i\phi\rho V(T)).$$

The solution to this PDE is

$$Y(t, V(t)) = \exp(-A(\tau) - B(\tau)V(t)),$$

where $\tau = T - t$, and

$$\dot{A}(\tau) = -\frac{1}{2}\gamma^2\theta B^2(\tau),$$
$$\dot{B}(\tau) = -\kappa B(\tau) - \frac{1}{2}\gamma^2 B^2(\tau) + \frac{1}{2}i\phi\rho^2\gamma^2 + \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\tau^d - i\phi\rho\kappa,$$

with boundary conditions $A(0) = 0$ and $B(0) = -i\phi\rho$. The functions $A(\tau)$ and $B(\tau)$ can be computed by numerically solving the system of ODEs.

Finally, the characteristic function $f(\phi)$ is

$$f(\phi) = \exp\left[i\phi\tau r - \frac{1}{2}i\phi\rho^2\gamma^2\theta\tau - \frac{1}{2}\phi(i+\phi)\theta\tau - \frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\int_0^t \left((t+\tau-s)^d \right.\right.$$
$$\left.\left. -(t-s)^d\right)\sigma_c^2(s)ds - A(\tau) - (i\phi\rho + B(\tau))\sigma_c^2(t) + i\phi X(t)\right]. \tag{6.2.12}$$

∎

Note that the characteristic function of the fractional Heston model is inherently different from that of the Heston model in that it is non-Markovian. As a result, the characteristic function will be dependent on the history of the volatility through the term $-\frac{1}{2}\phi(i+\phi)\frac{1}{\Gamma(d+1)}\int_0^t \left((t+\tau-s)^d - (t-s)^d\right)\sigma_c^2(s)ds$. When $d = 0$, this term will disappear, then the characteristic function will be only dependent on the current volatility.

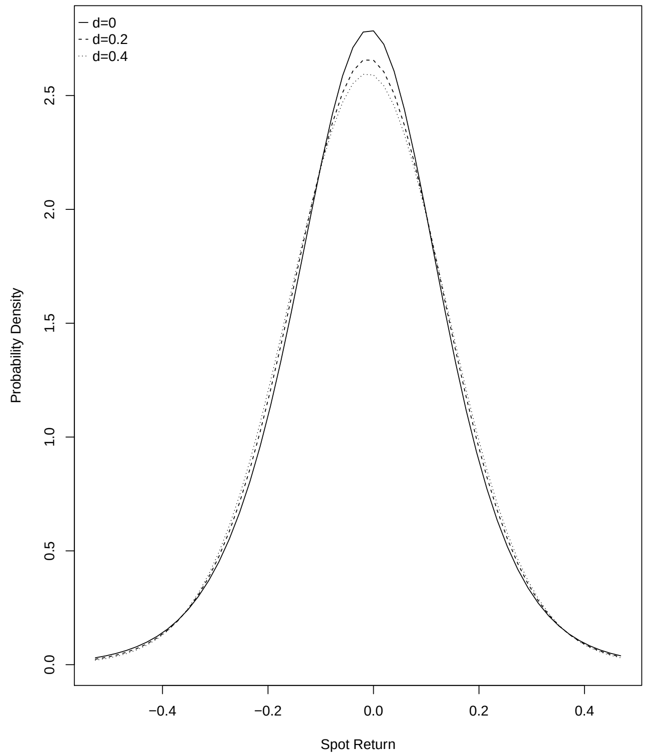

Figure 6.1: Conditional probability density of the spot returns over a six-month horizon for different long memory parameters for the fractional Heston model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t) dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)} dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} \sigma_c^2(s) ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)} dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\gamma = 0.2$, $\rho = 0$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

6.2.2 Numerical Results

With the analytical form of the characteristic function $f(\phi)$ under the probability measure Q given in theorem 6.2.1, we can compute the density function $p(x(t))$ of $x(t)$ simply from the inverse Fourier transform

$$p(x(t)) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\phi) \exp(-i\phi x(t)) d\phi.$$

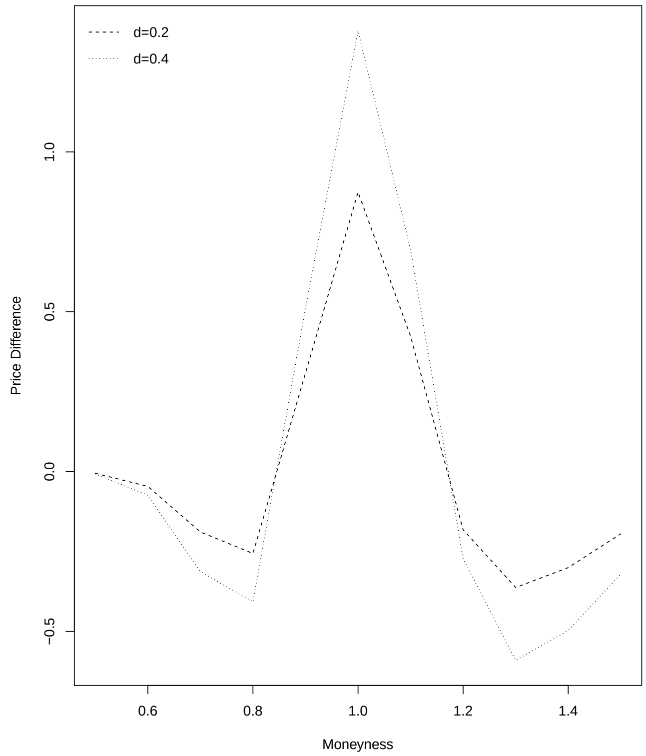

Figure 6.2: Option prices from the fractional Heston model with different long memory parameters minus that from Heston model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t)dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\gamma = 0.2$, $\rho = 0$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

The fractional Heston stochastic volatility model can conveniently explain properties of option prices in terms of the underlying distribution of spot returns. The parameter d influences the smoothness of the volatility process. The greater d is, the smoother the path of the volatility process is. Therefore, d has the opposite effect to the volatility of volatility parameter γ in Heston model. As we know, γ in Heston model increases the kurtosis of returns. This has the effect of raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices. Since d

has the opposite effect to γ, we speculate that lower d will increase the kurtosis and this has the effect of raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices.[2] Figure 6.1 and Figure 6.2 conform this speculation.

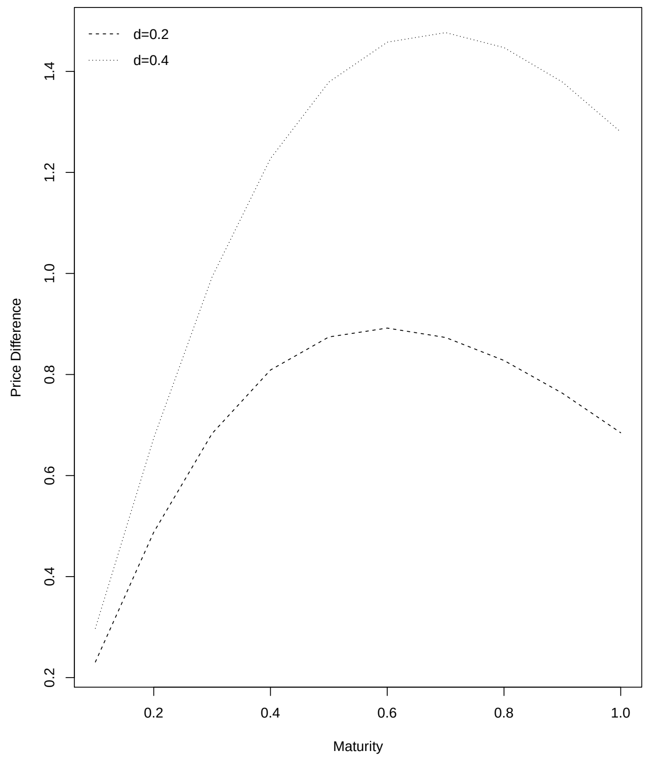

Figure 6.3: Effects of time to maturity on the option price differences between the fractional Heston model and Heston model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t)dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\gamma = 0.2$, $\rho = 0$, $K = 1000$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

[2] A call option with an exercise price significantly below the market price of the underlying security is called far-in-the-money. A call option with an exercise price significantly above the market price of the underlying security is called far-out-of-the-money. A call option with an exercise price close to the market price of the underlying security is called near-the-money.

Figure 6.3 shows that for the at-the-money options, the effect of d on the option prices is not a linear function of maturity. Instead, the term structure of the price effect seems to have a concave shape. Figure 6.4 shows that for the at-the-money options, the larger the volatility of volatility parameter γ is, the larger effect d has on the price of options.

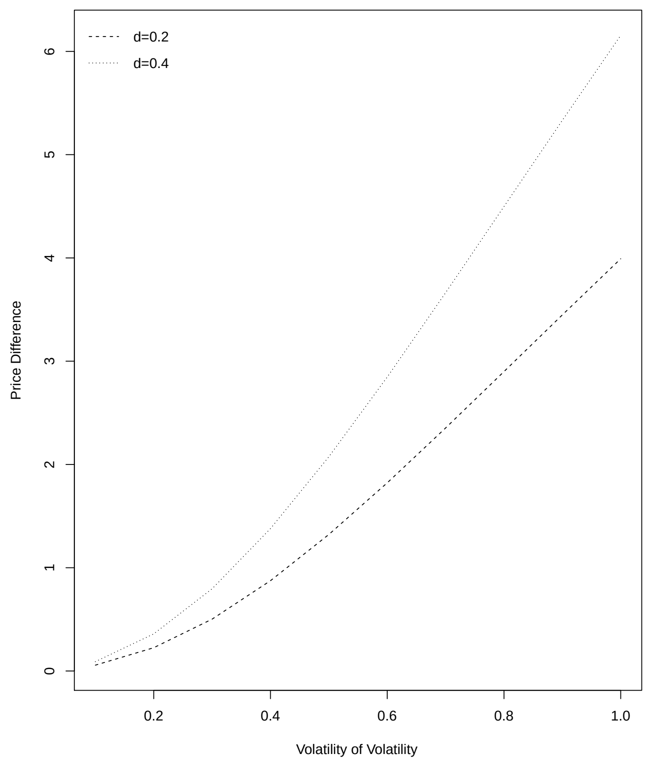

Figure 6.4: Effects of volatility of volatility on the option price differences between the fractional Heston model and Heston model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t)dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\rho = 0$, $\tau = 0.5$, $K = 1000$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

Figure 6.5 plots the implied volatility as a function of maturity and d. It seems the higher d results in a much less steep volatility skew at the short maturities. For each d, the volatility skew flattens out as the maturity is increased and the final steepness is roughly the same. It is clear that the flattening out is slower for the model with higher d than the model with lower d. This is consistent with the findings of Comte, Coutin and Renault (2003). This also shows that the short memory models which are able to reproduce the slow decay of the volatility skew must have a very large level of persistence which in turn results in short term options with skews which are too pronounced relative to the reality. The long memory models, on the other hand, can accommodate both the short term options and the decay at the same time. Figure 6.5 clearly shows the difference between short memory and long memory models.

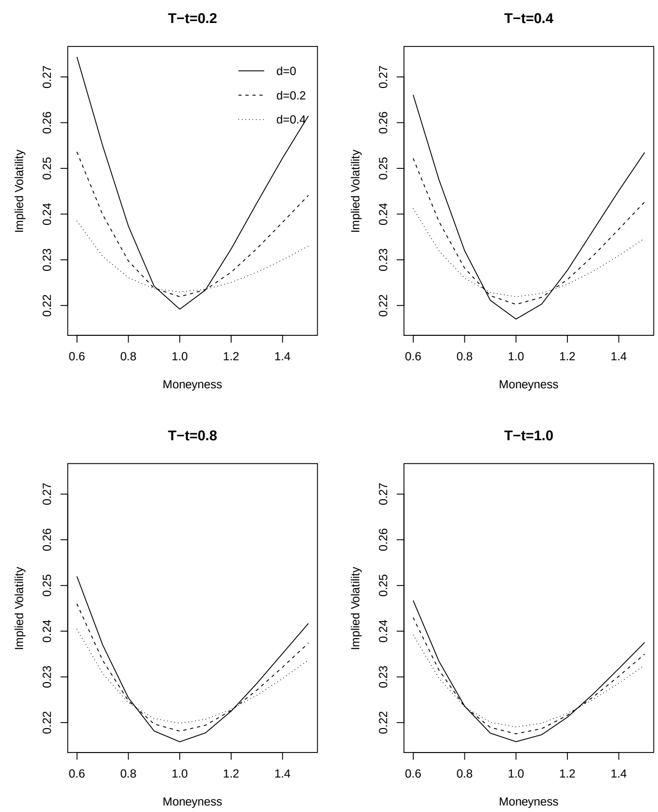

Figure 6.5: Implied volatility plots from the fractional Heston model with different long memory parameters. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t)dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\gamma = 0.2$, $\rho = 0$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

Figure 6.6 shows that in fractional Heston model, zero-correlation between volatility and stock price processes only can produce a volatility smile. For more general volatility skew, we need to extend the model of Comte, Coutin and Renault (2003) to introduce non-zero correlation.

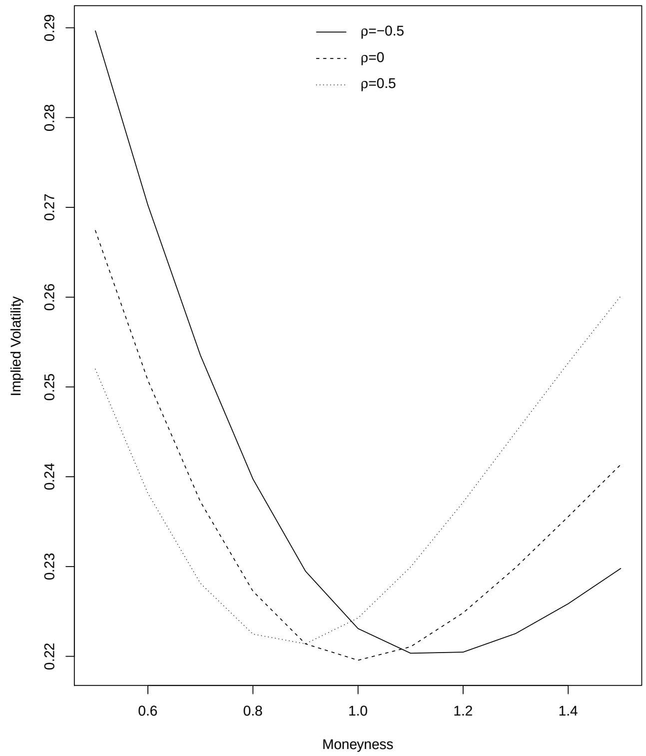

Figure 6.6: Implied volatility plots from the fractional Heston model with different correlation parameters. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t) - \frac{1}{2}\rho^2\gamma^2(\theta + \sigma_c^2(t))\right) dt + \sigma(t)dB_1^Q(t) + \rho\gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$, $\sigma^2(t) = \theta + (\sigma_c^2)^{(d)}(t)$, $(\sigma_c^2)^{(d)}(t) = \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}\sigma_c^2(s)ds$ and $d\sigma_c^2(t) = -\kappa\sigma_c^2(t)dt + \gamma\sqrt{\theta + \sigma_c^2(t)}dB_2^Q(t)$. $r = 0$, $\theta = 0.08$, $\kappa = 0.2$, $\gamma = 0.2$, $d = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $\sigma_c^2(0) = 0$.

6.3 Fractional Schöbel-Zhu Model

6.3.1 Approximate Analytical Formula for Characteristic Function

The fractional Schöbel-Zhu model is characterized by

$$dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dB_1^Q(t), \tag{6.3.1}$$

$$\sigma(t) = Y^{(d)}(t) + \theta, \tag{6.3.2}$$

$$Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s), \tag{6.3.3}$$

$$dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t),\ \kappa > 0,\ Y(0) = 0, \tag{6.3.4}$$

where $X(t) = \ln(S(t))$. $0 < d < \frac{1}{2}$. $B_1^Q(t)$ and $B_2^Q(t)$ are two standard Brownian motions under the risk neutral probability measure Q and $dB_1^Q(t)dB_2^Q(t) = \rho dt$.

We see that the volatility process $\sigma(t)$ is driven by a fractional Ornstein-Uhlenbeck process.[3] This model, as formulated above, has never been considered in the literature. If $d = 0$, $Y^{(d)}(t)$ will become an Ornstein-Uhlenbeck process and we will get back the Schöbel-Zhu stochastic volatility model.

Unfortunately, unlike the fractional Heston model, we can not obtain the closed-form solution for option prices under the fractional Schöbel-Zhu Model. We propose to approximate the continuous-time model by a discretized model and then derive the analytical formula based on the discrete-time model.

Suppose we want to price an option which expires at time T. We can approximate

[3] In Comte and Renault (1998), volatility process $\sigma(t)$ is driven by an exponential of fractional Ornstein-Uhlenbeck process. This specification ensures the positivity of $\sigma(t)$, however, the attractiveness of obtaining an analytical solution to option prices is also lost. When $\sigma(t)$ is negative, it should not be interpreted as the volatility of the underlying asset. Instead, it is merely a latent variable which drives the true volatility of the asset, the true volatility being defined as the square root of the variance. See Haastrecht, Lord and Pelsser (2009) for a complete discussion.

the volatility process and numerically evaluate the integral in equation (6.3.3) using only the involved process $Y(s)$ and $B_2^Q(s)$ on a discrete partition of $[0,T]$: $j/n, j = 0, 1, \ldots, [nT]$. A natural way to obtain such approximations is to approximate the integrands by step functions (see section 5.6):

$$\begin{aligned} Y^{(d)}(t_i) &= \int_0^{t_i} \frac{(t_i - [ns]/n)^d}{\Gamma(1+d)} dY(s) \\ &\approx \left[\sum_{j=0}^{i-1} \frac{(j+1)^d - j^d}{n^d \Gamma(1+d)} L^j \right] Y(t_i) \\ &= \left[\sum_{j=0}^{[nT]} \frac{(j+1)^d - j^d}{n^d \Gamma(1+d)} L^j \right] Y(t_i) \end{aligned} \tag{6.3.5}$$

where $t_i = i/n$ for $0 \leq i \leq [nT]$ and $L^j Y(t_i) = Y(t_{i-j})$. $Y(t_i) = 0$ for $i < 0$.

Note that $Y(t_i)$ is an AR(1) process

$$Y(t_i) = \rho_n Y(t_{i-1}) + \gamma_n \varepsilon_2(t_i), \tag{6.3.6}$$

where $\rho_n = 1 - \frac{\kappa}{n}$, $\gamma_n = \frac{\gamma}{\sqrt{n}}$ and $\varepsilon_2(t_i) \sim N(0,1)$, $i \geq 0$ are i.i.d..

A discretized approximation $X(t_i)$ can be obtained by

$$X(t_i) = X(t_{i-1}) + \left(r - \frac{1}{2}\sigma^2(t_{i-1})\right)\frac{1}{n} + \sigma(t_{i-1})\left(\rho\varepsilon_2(t_i) + \sqrt{1-\rho^2}\varepsilon_1(t_i)\right)\sqrt{\frac{1}{n}}, \tag{6.3.7}$$

where $\sigma(t_i) = Y^{(d)}(t_i) + \theta$, $\varepsilon_1(t_i)$, $i \geq 0$ is a sequence of i.i.d. $N(0,1)$ random variables and is independent of $\varepsilon_2(t_i)$.

Let $\widetilde{\sigma}(t_i) = \sqrt{\frac{1}{n}}\sigma(t_i)$, we can rewrite equation (6.3.7) as

$$X(t_i) = X(t_{i-1}) + r_n - \frac{1}{2}\widetilde{\sigma}^2(t_{i-1}) + \widetilde{\sigma}(t_{i-1})\left(\rho\varepsilon_2(t_i) + \sqrt{1-\rho^2}\varepsilon_1(t_i)\right), \tag{6.3.8}$$

where $r_n = \frac{r}{n}$.

Let

$$\widetilde{Y}(t_i) = \begin{pmatrix} \sqrt{\frac{1}{n}}Y(t_i) \\ \sqrt{\frac{1}{n}}Y(t_{i-1}) \\ \vdots \\ \sqrt{\frac{1}{n}}Y(t_{i-l+1}) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \rho_n & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \Sigma(t_i) = \begin{pmatrix} \varepsilon_2(t_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\beta = (\beta_0, \beta_1, \beta_2, \ldots, \beta_{l-1})', \quad \iota = (1, 0, \ldots, 0)',$$

where $l = [nT] + 1$. $\beta_j = \frac{(j+1)^d - j^d}{n^d \Gamma(1+d)}$ for $0 \leq j \leq [nT]$.

We can then rewrite $\widetilde{\sigma}(t_i)$ in matrix form

$$\widetilde{Y}(t_i) = \Psi \widetilde{Y}(t_{i-1}) + \widetilde{\gamma}_n \Sigma(t_i), \tag{6.3.9}$$

$$\widetilde{\sigma}(t_i) = \beta' \widetilde{Y}(t_i) + \widetilde{\theta}_n, \tag{6.3.10}$$

where $\widetilde{\gamma}_n = \gamma_n \sqrt{\frac{1}{n}}$ and $\widetilde{\theta}_n = \theta \sqrt{\frac{1}{n}}$.

Let $\mathcal{F}_{t_i}$ be the filtration generated by $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$. The characteristic function of $X(t_{[nT]})$ under risk-neutral probability measure Q can be computed from the following theorem:

Theorem 6.3.1 *The conditional characteristic function $f(\phi)$ of $X(t_{[nT]})$ under the risk-neutral probability measure Q for the discretized fractional Schöbel-Zhu model is given by*

$$\begin{aligned} f(\phi) &= \mathrm{E^Q[exp(i\phi X(t_{[nT]}))|\mathcal{F}_{t_i}]} \\ &= \exp[-A(\tau) + i\phi X(t_i) - B(\tau)'\widetilde{Y}(t_i) - \widetilde{Y}(t_i)'C(\tau)\widetilde{Y}(t_i)]. \end{aligned} \tag{6.3.11}$$

where $\tau = [nT] - t_i$ and the functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ can be computed by *recursively solving the following system of difference equations:*

$$\begin{aligned} A(\tau) &= A(\tau-1) - i\phi r_n + \frac{1}{2} i\phi\widetilde{\theta}_n^2 + \frac{1}{2}\phi^2\widetilde{\theta}_n^2(1-\rho^2) + \frac{1}{2}\ln(1+2\widetilde{\gamma}_n^2\iota' C(\tau-1)\iota) \\ &- \frac{1}{2+4\widetilde{\gamma}_n^2\iota' C(\tau-1)\iota}(\iota' B(\tau-1)\widetilde{\gamma}_n - i\phi\widetilde{\theta}_n\rho)^2, \end{aligned} \tag{6.3.12}$$

$$\begin{aligned} B(\tau) &= i\phi\widetilde{\theta}_n\beta + \Psi' B(\tau-1) + (1-\rho^2)\phi^2\beta\widetilde{\theta}_n - \frac{1}{1+2\widetilde{\gamma}_n^2\iota' C(\tau-1)\iota} \\ &(\iota' B(\tau-1)\widetilde{\gamma}_n - i\phi\widetilde{\theta}_n\rho)(-i\phi\rho\beta + 2\widetilde{\gamma}_n\Psi' C(\tau-1)\iota), \end{aligned} \tag{6.3.13}$$

$$\begin{aligned} C(\tau) &= \frac{i\phi}{2}\beta\beta' + \Psi' C(\tau-1)\Psi + \frac{1}{2}\phi^2(1-\rho^2)\beta\beta' - \frac{1}{2+4\gamma_n^2\iota' C(\tau-1)\iota} \\ &(-i\phi\rho\beta + 2\gamma_n\Psi' C(\tau-1)\iota)(-i\phi\rho\beta + 2\gamma_n\Psi' C(\tau-1)\iota)', \end{aligned} \tag{6.3.14}$$

with the initial conditions $A(0) = 0$*,* $B(0) = 0$ *and* $C(0) = 0$*.*

Proof: We guess the characteristic function of $X(t_{[nT]})$ under risk-neutral probability measure Q takes the following form

$$\begin{aligned} f(\phi) &= \mathrm{E^Q}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{t_{[nT]}}))|\mathcal{F}_{\mathrm{t_i}}] \\ &= \exp[-A(\tau) + i\phi X(t_i) - B(\tau)'\widetilde{Y}(t_i) - \widetilde{Y}(t_i)' C(\tau)\widetilde{Y}(t_i)]. \end{aligned} \tag{6.3.15}$$

We can solve $A(\tau)$, $B(\tau)$ and $C(\tau)$ by taking the conditional expectation.

$$\begin{aligned} f(\phi) &= \mathrm{E^Q}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{t_i}+\tau))|\mathcal{F}_{\mathrm{t_i}}] \\ &= \mathrm{E^Q}\{\mathrm{E^Q}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{t_i}+\tau))|\mathcal{F}_{\mathrm{t_{i+1}}}]|\mathcal{F}_{\mathrm{t_i}}\}. \end{aligned}$$

Using (6.3.15), we have

$$\begin{aligned} f(\phi) &= \mathrm{E^Q}\{\exp[-\mathrm{A}(\tau-1) + \mathrm{i}\phi\mathrm{X}(\mathrm{t_{i+1}}) - \mathrm{B}(\tau-1)'\widetilde{\mathrm{Y}}(\mathrm{t_{i+1}}) \\ &- \widetilde{Y}(t_{i+1})' C(\tau-1)\widetilde{Y}(t_{i+1})]|\mathcal{F}_{t_i}\}. \end{aligned}$$

Making use of (6.3.8), (6.3.9) and (6.3.10), we have

$$\begin{aligned}
f(\phi) &= \mathrm{E}^{\mathrm{Q}}\Big\{\exp\Big[-\mathrm{A}(\tau-1)+\mathrm{i}\phi\mathrm{X}(\mathrm{t_i})+\mathrm{i}\phi(\mathrm{r_n}-\frac{1}{2}(\beta'\widetilde{\mathrm{Y}}(\mathrm{t_i})+\widetilde{\theta}_{\mathrm{n}})^2) \\
&\quad + i\phi(\beta'\widetilde{Y}(t_i)+\widetilde{\theta}_n)(\rho\varepsilon_2(t_{i+1})+\sqrt{1-\rho^2}\varepsilon_1(t_{i+1})) - B(\tau-1)'\Psi\widetilde{Y}(t_i) \\
&\quad - B(\tau-1)'\widetilde{\gamma}_n\Sigma(t_{i+1}) - (\Psi\widetilde{Y}(t_i)+\widetilde{\gamma}_n\Sigma(t_{i+1}))'C(\tau-1) \\
&\quad (\Psi\widetilde{Y}(t_i)+\widetilde{\gamma}_n\Sigma(t_{i+1}))\Big]\,|\mathcal{F}_{t_i}\Big\} \\
&= \exp\Big[-A(\tau-1)+i\phi X(t_i)+i\phi(r_n-\frac{1}{2}(\beta'\widetilde{Y}(t_i)+\widetilde{\theta}_n)^2) \\
&\quad -B(\tau-1)'\Psi\widetilde{Y}(t_i)-\widetilde{Y}(t_i)'\Psi'C(\tau-1)\Psi\widetilde{Y}(t_i)\Big] \\
&\quad \mathrm{E}^{\mathrm{Q}}\{\exp[\mathrm{i}\phi(\beta'\widetilde{\mathrm{Y}}(\mathrm{t_i})+\widetilde{\theta}_{\mathrm{n}})\sqrt{1-\rho^2}\varepsilon_1(\mathrm{t_{i+1}})+\mathrm{i}\phi(\beta'\widetilde{\mathrm{Y}}(\mathrm{t_i})+\widetilde{\theta}_{\mathrm{n}})\rho\varepsilon_2(\mathrm{t_{i+1}}) \\
&\quad - \widetilde{Y}(t_i)'\Psi'C(\tau-1)\widetilde{\gamma}_n\Sigma(t_{i+1}) - \widetilde{\gamma}_n\Sigma(t_{i+1})'C(\tau-1)\Psi\widetilde{Y}(t_i) \\
&\quad - \widetilde{\gamma}_n^2\Sigma(t_{i+1})'C(\tau-1)\Sigma(t_{i+1})]|\mathcal{F}_{t_i}\} \\
&= \exp\Big[-A(\tau-1)+i\phi X(t_i)+i\phi r_n-\frac{1}{2}i\phi\widetilde{\theta}_n^2-i\phi\widetilde{\theta}_n\beta'\widetilde{Y}(t_i)-\frac{1}{2}i\phi\widetilde{Y}(t_i)'\beta\beta'\widetilde{Y}(t_i) \\
&\quad -B(\tau-1)'\Psi\widetilde{Y}(t_i)-\widetilde{Y}(t_i)'\Psi'C(\tau-1)\Psi\widetilde{Y}(t_i)-\frac{1}{2}\phi^2(\beta'\widetilde{Y}(t_i)+\widetilde{\theta}_n)^2(1-\rho^2)\Big] \\
&\quad \mathrm{E}^{\mathrm{Q}}\{\exp[\mathrm{i}\phi((\beta'\widetilde{\mathrm{Y}}(\mathrm{t_i})+\widetilde{\theta}_{\mathrm{n}})\rho\varepsilon_2(\mathrm{t_{i+1}})-\iota'\mathrm{B}(\tau-1)\widetilde{\gamma}_{\mathrm{n}}\varepsilon_2(\mathrm{t_{i+1}}) \\
&\quad - \widetilde{Y}(t_i)'\Psi'C(\tau-1)\iota\widetilde{\gamma}_n\varepsilon_2(t_{i+1}) - \iota'C(\tau-1)\Psi\widetilde{Y}(t_i)\widetilde{\gamma}_n\varepsilon_2(t_{i+1}) \\
&\quad - \widetilde{\gamma}_n^2\iota'C(\tau-1)\iota\varepsilon_2(t_{i+1})^2]|\mathcal{F}_{t_i}\}.
\end{aligned}$$

Using the fact that for a standard normal variable U (Heston and Nandi (2000)),

$$\mathrm{E}[\exp(\mathrm{aU}+\mathrm{bU}^2)] = \exp\left[-\frac{1}{2}\ln(1-2\mathrm{b})+\frac{\mathrm{a}^2}{2-4\mathrm{b}}\right],$$

we get

$$
\begin{aligned}
f(\phi) = \exp\Bigg[&-A(\tau-1) + i\phi X(t_i) + i\phi r_n - \frac{1}{2} i\phi\widetilde{\theta}_n^2 - i\phi\widetilde{\theta}_n\beta'\widetilde{Y}(t_i) \\
&- \frac{1}{2} i\phi\widetilde{Y}(t_i)'\beta\beta'\widetilde{Y}(t_i) - B(\tau-1)'\Psi\widetilde{Y}(t_i) - \widetilde{Y}(t_i)'\Psi' C(\tau-1)\Psi\widetilde{Y}(t_i) \\
&- \frac{1}{2}\phi^2(\beta'\widetilde{Y}(t_i) + \widetilde{\theta}_n)^2(1-\rho^2) - \frac{1}{2}\ln(1 + 2\widetilde{\gamma}_n^2\iota' C(\tau-1)\iota) \\
&+ \frac{1}{2 + 4\widetilde{\gamma}_n^2\iota' C(\tau-1)\iota}(-\iota' B(\tau-1)\widetilde{\gamma}_n + i\phi\widetilde{\theta}_n\rho + (i\phi\rho\beta' - 2\widetilde{\gamma}_n\iota' C(\tau-1)'\Psi)\widetilde{Y}(t_i))^2\Bigg].
\end{aligned}
\tag{6.3.16}
$$

By equalling the coefficients of equations (6.3.15) and (6.3.16), we get the system of difference equations for $A(\tau)$, $B(\tau)$ and $C(\tau)$ as in (6.3.12), (6.3.13) and (6.3.17) respectively. ∎

Let $C(t; K, T, S(t))$ be the price of an European call option of the underlying stock with the price $S(t)$, the strike price K and time of maturity T. We can approximate $C(t; K, T, S(t))$ by $C(t_i; K, T, S(t_i))$ with $t_i = [nt]$ and

$$
\begin{aligned}
C(t_i; K, T, S(t_i)) &= \mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})(\exp(\mathrm{X(t_{[nT]})}) - \mathrm{K})1_{(\exp(\mathrm{X(t_{[nT]})})>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&= \mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})\exp(\mathrm{X(t_i+\tau)})1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&\quad - \mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})\mathrm{K}1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&= \mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})\exp(\mathrm{X(t_i+\tau)})|\mathcal{F}_{\mathrm{t_i}}\right] \\
&\quad \times \mathrm{E^Q}\left[\frac{\exp(-\tau \mathrm{r_n})\exp(\mathrm{X(t_i+\tau)})1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}}{\mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})\exp(\mathrm{X(t_i+\tau)})|\mathcal{F}_{\mathrm{t_i}}\right]}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&\quad - \mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})|\mathcal{F}_{\mathrm{t_i}}\right]\mathrm{E^Q}\left[\frac{\exp(-\tau \mathrm{r_n})\mathrm{K}1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}}{\mathrm{E^Q}\left[\exp(-\tau \mathrm{r_n})|\mathcal{F}_{\mathrm{t_i}}\right]}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&= S(t_i)\mathrm{E^Q}\left[\frac{\exp(-\tau \mathrm{r_n})\exp(\mathrm{X(t_i+\tau)})1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}}{\mathrm{S(t_i)}}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&\quad - \exp(-\tau r_n)K\mathrm{E^Q}\left[1_{(\exp(\mathrm{X(t_i+\tau)})>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right].
\end{aligned}
\tag{6.3.17}
$$

Let $p(x|t_i)$ be the conditional probability density of $X(t_{[nT]})$ given filtration $\mathcal{F}_{t_i}$ under risk neutral probability measure Q. We know

$$\begin{aligned}
&\mathrm{E}^{\mathrm{Q}}\left[\frac{\exp(-\tau \mathrm{r_n})\exp(\mathrm{X}(\mathrm{t_i}+\tau)).1_{(\exp(\mathrm{X}(\mathrm{t_i}+\tau))>\mathrm{K})}}{\mathrm{S}(\mathrm{t_i})}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&= \int_{\ln(K)}^{\infty}\frac{\exp(-\tau r_n)\exp(x)}{S(t_i)}p(x|t_i)dx \\
&= \int_{\ln(K)}^{\infty} p^*(x|ti)dx,
\end{aligned}$$

where $p^*(x|t_i) =: \frac{\exp(-\tau r_n)\exp(x)}{S(t_i)}p(x|t_i)$.

It is easy to see that $p^*(x|t_i)$ is a valid probability density because it is non-negative and $\int_{-\infty}^{\infty}\frac{\exp(-\tau r_n)\exp(x)}{S(t_i)}p(x|t_i)dx = 1$. Hence, we can rewrite $C(t_i; K, T, S(t_i))$ as

$$\begin{aligned}
C(t_i; K, T, S(t_i)) &= S(t_i)\mathrm{E}^{\mathrm{Q_1}}\left[1_{(\exp(\mathrm{X}(\mathrm{t_i}+\tau))>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right] - \exp(-\tau\mathrm{r_n})\mathrm{K}\mathrm{E}^{\mathrm{Q}}\left[1_{(\exp(\mathrm{X}(\mathrm{t_i}+\tau))>\mathrm{K})}|\mathcal{F}_{\mathrm{t_i}}\right] \\
&= S(t_i)Q_1(X(t_i+\tau) > \ln K) - \exp(-\tau r_n)KQ(X(t_i+\tau) > \ln K),
\end{aligned} \tag{6.3.18}$$

where Q and Q_1 are two probability measures.

We can express the probabilities in the last line by the Fourier transform. The characteristic function of $X(t_i+\tau)$ under Q_1 is defined by

$$\begin{aligned}
f_1(\phi) &= \mathrm{E}^{\mathrm{Q_1}}[\exp(\mathrm{i}\phi\mathrm{X}(\mathrm{t_i}+\tau))|\mathcal{F}_{\mathrm{t_i}}] \\
&= \int_{-\infty}^{\infty}\exp(i\phi x)p^*(x|t_i)dx \\
&= \int_{-\infty}^{\infty}\exp(i\phi x)\frac{\exp(-\tau r_n)\exp(x)}{S(t_i)}p(x|t_i)dx \\
&= \exp(-\tau r_n - X(t_i))\int_{-\infty}^{\infty}\exp(i\phi x)\exp(x)p(x|t_i)dx \\
&= \exp(-\tau r_n - X(t_i))\mathrm{E}^{\mathrm{Q}}[\exp((1+\mathrm{i}\phi)\mathrm{X}(\mathrm{t_i}+\tau))|\mathcal{F}_{\mathrm{t_i}}] \\
&= \exp(-\tau r_n - X(t_i))f(-i+\phi),
\end{aligned} \tag{6.3.19}$$

where $f(\phi)$ is given in theorem 6.3.1.

Given the characteristic functions f_1 and f, we can obtain the probabilities

$$Q_1(X(t_i+\tau) > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\mathrm{f}_1(\phi)\frac{\exp(-\mathrm{i}\phi\ln\mathrm{K})}{\mathrm{i}\phi}\right)\mathrm{d}\phi, \tag{6.3.20}$$

and

$$Q(X(t_i+\tau) > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\mathrm{f}(\phi)\frac{\exp(-\mathrm{i}\phi\ln\mathrm{K})}{\mathrm{i}\phi}\right)\mathrm{d}\phi. \tag{6.3.21}$$

Plugging $Q_1(\cdot)$ and $Q(\cdot)$ into (6.3.18), we get the approximate closed-form formula for option prices expressed in terms of the characteristic functions.

6.3.2 Numerical Results

We investigate the influence of parameter d on the option prices. We know that the parameter d influences the smoothness of the volatility process. The greater d is, the smoother the path of the volatility process is. Therefore, d has the opposite effect to the volatility of volatility parameter γ in Schöbel-Zhu model.

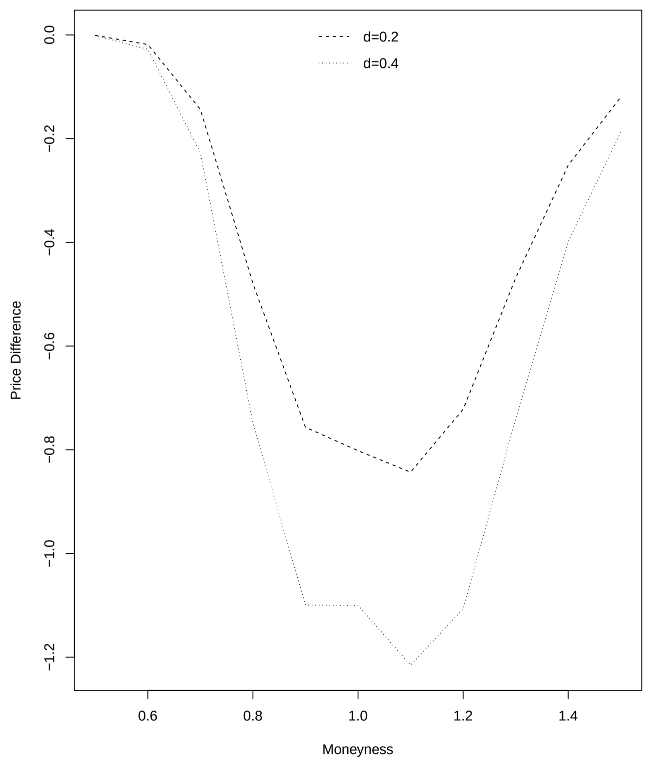

Figure 6.7: Option prices from the fractional Schöbel-Zhu model with different long memory parameters minus that from Schöbel-Zhu model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dB_1^Q(t)$, $\sigma(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t)$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $\rho = 0$, $\tau = 0.5$, $x(0) = \log(1000)$ and $y(0) = 0$.

As we know, an increase in volatility of volatility parameter γ in Schöbel-Zhu model always leads to a higher option prices, which is a consequence of the fact that an increase in γ increases the long-run mean of the volatility, hence also increases the option prices. Since d has the opposite effect to γ, we speculate that higher d will decrease the option prices.[4] Figure 6.7 conforms this speculation.

[4]This is different from fractional Heston model where lower d will increase the kurtosis and this has the effect of raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices.

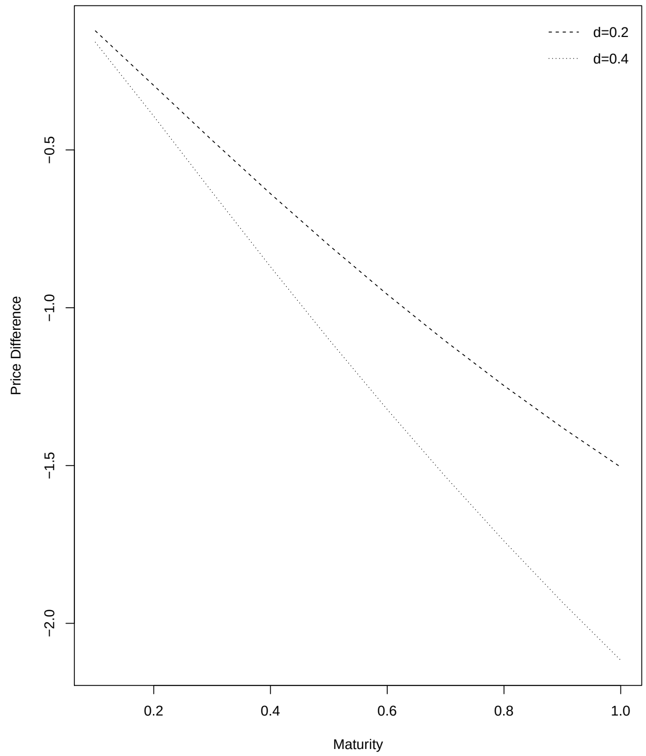

Figure 6.8: Effects of time to maturity on the option price differences between the fractional Schöbel-Zhu model and Schöbel-Zhu model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dB_1^Q(t)$, $\sigma(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t)$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $\rho = 0$, $K = 1000$, $x(0) = \log(1000)$ and $y(0) = 0$.

Figure 6.8 shows that for the at-the-money options, the effect of d on the option prices is a linear function of maturity. Figure 6.9 shows that for the at-the-money options, the larger the volatility of volatility parameter γ is, the larger effect d has on the price of options.

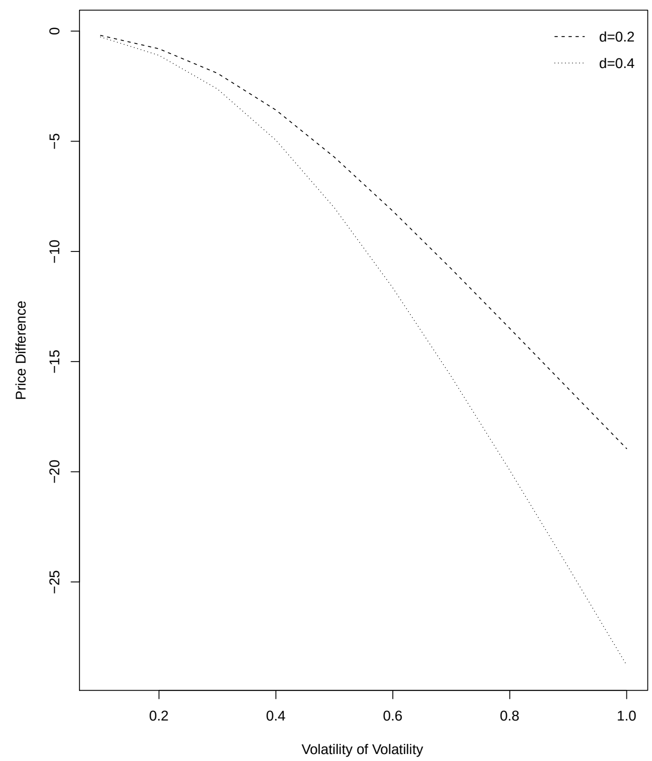

Figure 6.9: Effects of volatility of volatility on the option price differences between the fractional Schöbel-Zhu model and Schöbel-Zhu model. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dB_1^Q(t)$, $\sigma(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t)$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\rho = 0$, $\tau = 0.5$, $K = 1000$, $x(0) = \log(1000)$ and $y(0) = 0$.

Figure 6.10 plots the implied volatility as a function of maturity and d. It seems the higher d results in a much less steep volatility skew at the short maturities, although the final steepness is almost the same. This indicates the fractional Schöbel-Zhu model, like the fractional Heston model, has the potential to accommodate both the short term options and the decay at the same time better than the corresponding short memory stochastic volatility models.

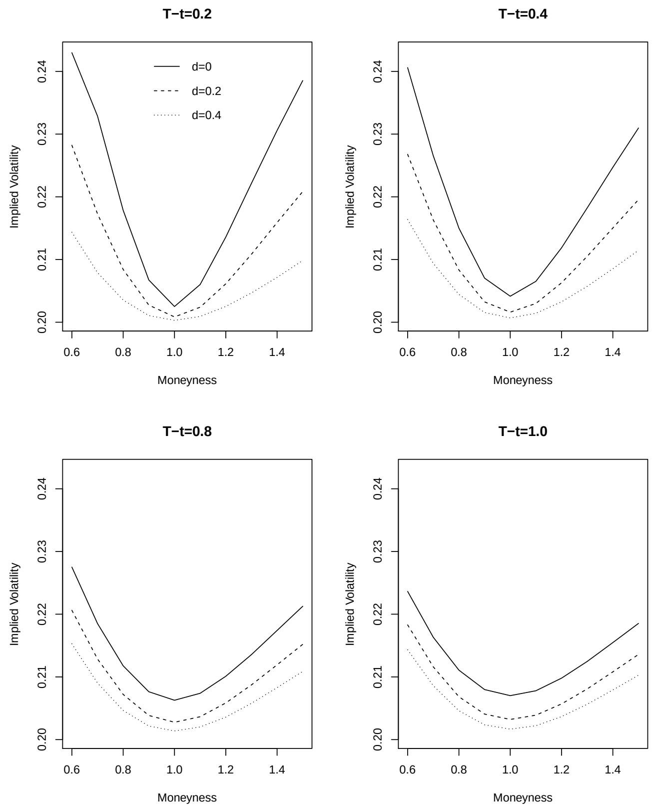

Figure 6.10: Implied volatility plots from the fractional Schöbel-Zhu model with different long memory parameters. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dB_1^Q(t)$, $\sigma(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t)$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $\rho = 0$, $x(0) = \log(1000)$ and $y(0) = 0$.

Figure 6.11 shows that in the fractional Schöbel-Zhu model, a non-zero correlation between the volatility and stock price processes is necessary in order to produce the more general shapes of volatility skew.

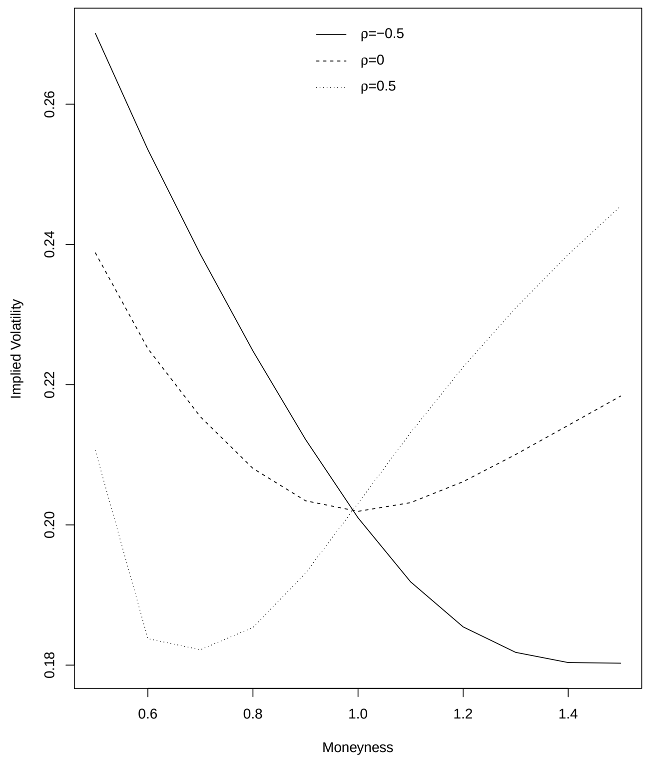

Figure 6.11: Implied volatility plots from the fractional Schöbel-Zhu model with different correlation parameters. $dX(t) = \left(r - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t) dB_1^Q(t)$, $\sigma(t) = Y^{(d)}(t) + \theta$, $Y^{(d)}(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s)$ and $dY(t) = -\kappa Y(t) + \gamma dB_2^Q(t)$. $r = 0$, $\theta = 0.2$, $\kappa = 4$, $\gamma = 0.2$, $d = 0.2$, $\tau = 0.5$, $x(0) = \log(1000)$ and $y(0) = 0$.

Chapter 7

Conclusion and Future Extensions

7.1 Conclusion

In this thesis we have developed the methods for pricing options under the long memory stochastic volatility models. Although there has been ample research on option pricing with short memory stochastic volatility, there is much less research on option pricing with a long memory stochastic volatility framework.

We have proposed two continuous-time long memory stochastic volatility models. The first model is the fractional Heston model which is an extension of the popular Heston (1993) model and has been studied by Comte, Coutin and Renault (2003). The second model is the fractional Schöbel-Zhu model which is built on the models developed by Schöbel-Zhu (1999) and Comte and Renault (1998). In both models, we allow for the non-zero correlation between the stochastic volatility and stock price processes. Due to the complicated structure of the long memory stochastic volatility process, pricing options is challenging. Indeed, the previous studies on option prices with long memory stochastic volatility models are based on the time-consuming Monte Carlo simulations. To overcome this problem, we propose to use Fourier inversion techniques to obtain the closed-form solutions for option pricing.

We derive the analytical solution to the option pricing for the fractional Heston model. Since it is not feasible to directly obtain the closed-form solution to the continuous time fractional Schöbel-Zhu model, we discretize the original model and then derive the analytical solution to the resulting discrete time option pricing model. We numerically study the effects of long memory on the option prices. We show that the higher integration parameter has the similar effect as the lower volatility of volatility parameter. We also find that the long memory models have the potential to accommodate the short term options and the decay of volatility skew better than the corresponding short memory stochastic volatility models.

Our goal in this thesis is to study option pricing with long memory stochastic volatility models. We haven't touched on the topic of the parameter estimation under the long memory stochastic volatility models. As we know, there are several existing techniques for estimating the parameters, especially the long memory parameter. Robinson (1995) initially introduce the log-periodogram regression and Geweke and Porter-Hudak (1983) use the GPH estimator for estimating the long-memory parameter. Moreover, Fox and Taqqu (1986) introduce the Whittle-based approach to estimate the long memory parameter, together with the remaining parameters of the model. This approach has been adapted to the long memory stochastic volatility models by Gao et al. (20001) and Casas and Gao (2008). Another approach to obtain the long memory parameter is introduced by Chronopoulou and Viens (2012a), who obtain the parameter by calibrating the model with the realized option prices. In the future, we would like to apply these techniques to estimate the parameters in our newly developed models. We also want to compare the performance of each model based on the real-life option data.

7.2 Future Extensions

There are a number of directions in which this thesis work could be extended. We only discuss two possible extensions here.

7.2.1 Fractionally Integrated CARMA Stochastic Volatility Models

So far, in all of our fractional stochastic volatility models, the process of the volatility is governed by 1-st order stochastic differential equation. We can extend it to p-th order stochastic differential equation.

For example, we can model the stochastic process for volatility $\sigma(t)$ as

$$\sigma(t) = Y(t) + \theta,$$

where

$$Y(t)^{(p)} + a_1 Y(t)^{(p-1)} + \ldots + a_p Y(t) = \gamma \left[b_0 W_d^{(1)}(t) + b_1 W_d^{(2)}(t) + \ldots + b_q W_d^{(q+1)}(t) \right],\ t \geq 0,$$

where the superscript $^{(j)}$ denotes the j-fold differentiation with respect to t. $W_d(t)$ is fractional Brownian motion with $0 < d < \frac{1}{2}$, $\gamma > 0$, $b_q \neq 0$ and $0 \leq q < p$. As the fractional Brownian motion is nowhere differentiable, the above differential equation is interpreted as being equivalent to the observation and state equations

$$Y(t) = \gamma b' Z(t),$$

and

$$dZ(t) = AZ(t)dt + e dW_d(t),\ t \geq 0,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} b_0 \\ b_1 \\ . \\ . \\ . \\ b_{p-2} \\ b_{p-1} \end{pmatrix}, \quad Z(t) = \begin{pmatrix} Z(t) \\ Z^{(1)}(t) \\ . \\ . \\ . \\ Z^{(p-1)}(t) \end{pmatrix},$$

where $b_j = 0$ for $q < j \leq p$.

Given this representation, we can use similar technique as in the fractional Schöbel-Zhu model to derive the characteristic function and hence the closed-form formula for option prices.

7.2.2 Lévy-driven Fractionally Integrated CARMA Stochastic Volatility Models

The works of this thesis are built on the somewhat restrictive setting of stochastic integration of deterministic integrands with fractional Brownian motion integrators. We can extend the works of Barndorff-Nielsen and Shephard (2001) and consider the fractionally integrated CARMA stochastic volatility driven by fractional Lévy process. The Gaussian fractionally CARMA stochastic volatility models then become special case of the Lévy-driven fractionally integrated CARMA stochastic volatility models. Following Brockwell and Marquardt (2005), we define the fractional Lévy process as

Definition 7.2.1 *Let $L(t)$ be a two-sided Lévy process on $\mathbb{R}$ without Brownian motion component and satisfying*

$$\mathrm{E}[\mathrm{L}(1)^{\alpha}] < \infty \text{ for some } 1 < \alpha < 2.$$

For fractional integration parameter $0 < d < 1 - \frac{1}{\alpha}$, *a stochastic process*

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] dL(ds), \ t \in \mathbb{R},$$

is called a fractional Lévy process.

We can then build a Lévy-driven fractional CARMA stochastic volatility model. For example, we can model the stochastic process for volatility $\sigma^2(t)$ as

$$\sigma^2(t) = Y(t) + \theta,$$

where

$$Y(t)^{(p)} + a_1 Y(t)^{(p-1)} + \ldots + a_p Y(t) = \gamma \left[b_0 M_d^{(1)}(t) + b_1 M_d^{(2)}(t) + \ldots + b_q M_d^{(q+1)}(t) \right],$$

where the superscript $^{(j)}$ denotes the j-fold differentiation with respect to t. $M_d(t)$ is fractional Lévy process with $0 < d < \frac{1}{2}$, $\gamma > 0$ and $b_q \neq 0$.

We can use similar technique as in the fractional Heston model to derive the characteristic function and hence the closed-form formula for option prices.

Appendix A

R Codes for Simulations

A.1 Simulation of Geometric Brown Motion

This code was used to create Figure 2.1 for $\sigma = 0.2$ and $\sigma = 0.5$.

```
mu=0.1;
sigma=0.2;
x=rep(log(100),200);
err=rnorm(200);
for (i in 2:200){
   x[i]=x[i-1]+(mu-1/2*sigma^2)/200+sigma*sqrt(1/200)*err[i];
}
y=exp(x);
```

A.2 Simulation of Ornstein-Uhlenbeck Process

This code was used to create Figure 2.2 for $\sigma = 0.2$ and $\sigma = 0.5$.

```
theta=0.2;
sigma=0.2;
k=4;
x=rep(0.2,200);
err=rnorm(200);
for (i in 2:200){
   x[i]=x[i-1]+k*(theta-x[i-1])/200+sigma*sqrt(1/200)*err[i];
}
y=x;
```

A.3 Simulation of Square Root Process

This code was used to create Figure 2.3 for $\sigma = 0.2$ and $\sigma = 0.5$.

```
theta=0.08;
sigma=0.2;
k=2;
x=rep(0.08,200);
err=rnorm(200);
for (i in 2:200){
   x[i]=x[i-1]+k*(theta-x[i-1])/200+sigma*sqrt(x[i-1])*sqrt(1/200)*err[i];
}
y=x;
```

A.4 Simulation of Fractional Brownian Motion

This code was used to create Figure 5.1 for $H = 0.1$, $H = 0.5$ and $H = 0.9$.

```
library(fArma);

x=fbmSim(n=100,H=0.1,method=c("mvn","chol","lev","circ","wave"),
waveJ=7,doplot=TRUE,fgn=FALSE);
```

A.5 Simulation of Fractional Ornstein-Uhlenbeck Process

This code was used to create Figure 5.2 for $d = 0$, $d = 0.2$ and $d = 0.4$.

```
theta=0.2;
sigma=0.2;
k=4;
alpha=0.2;
err=rnorm(200);
x=rep(0.2,200);
x_alpha=rep(0,200);

for (i in 2:200){
    x_alpha[i]=x_alpha[i-1]-k*x_alpha[i-1]/200+sigma*sqrt(1/200)*err[i];
}

for (i in 2:200){
```

```
   x[i]=0;
   for (j in 0:(i-2)){
      x[i]=((j+1)^alpha-j^alpha)/(200^alpha*gamma(1+alpha))*x_alpha[i-j]+x[i];
   }
   x[i]=x[i]+theta
}
y=x;
```

A.6 Simulation of Fractional Square Root Process

This code was used to create Figure 5.3 for $d = 0$, $d = 0.2$ and $d = 0.4$.

```
theta=0.08;
sigma=0.2;
k=2;
alpha=0.2;
err=rnorm(200);
x=rep(0.08,200);
x_alpha=rep(0,200);

for (i in 2:200){
    x_alpha[i]=x_alpha[i-1]-k*x_alpha[i-1]/200+sigma*sqrt(x_alpha[i-1]
    +theta)*sqrt(1/200)*err[i];
}

for (i in 2:200){
   x[i]=0;
   for (j in 0:(i-2)){
      x[i]=((j+1)^alpha-j^alpha)/(200^alpha*gamma(1+alpha))*x_alpha[i-j]+x[i];
   }
   x[i]=x[i]+theta
}
y=x;
```

Appendix B

R Codes for Option Pricing

B.1 Helper Functions

This function is used to compute the implied volatility from Black-Scholes model.

```
imp_BS = function(x, a, x0, T, K, r){

   d2=1/(x*sqrt(T))*(log(exp(x0)/K)+(r-1/2*x^2)*T);
   d1=d2+x*sqrt(T);
   exp(x0)*pnorm(d1)-exp(-r*T)*K*pnorm(d2)-a;
}
```

This function is used to calculate the density from a characteristic function.

```
characteristic_function_to_density <- function(
  phi, # characteristic function; should be vectorized
  n,   # Number of points, ideally a power of 2
  a, b # Evaluate the density on [a,b[
  ) {
  i <- 0:(n-1)            # Indices
  dx <- (b-a)/n           # Step size, for the density
  x <- a + i * dx         # Grid, for the density
  dt <- 2*pi / ( n * dx ) # Step size, frequency space
  c <- -n/2 * dt          # Evaluate the characteristic function on [c,d]
  d <-  n/2 * dt          # (center the interval on zero)
  t <- c + i * dt         # Grid, frequency space
  phi_t <- phi(t)
  X <- exp( -(0+1i) * i * dt * a ) * phi_t
  Y <- fft(X)
  density <- dt / (2*pi) * exp( - (0+1i) * c * x ) * Y
```

```
  data.frame(
    i = i,
    t = t,
    characteristic_function = phi_t,
    x = x,
    density = Re(density)
  )
}
```

This function is to compute gamma function.

```
gamma2 = function(x){

  if (x > 0) return(gamma(x))
  else if (x ==0) return(1e10)
  else return(1/x*gamma2(1+x))
}
```

B.2 Option Pricing with the Heston Model

This function was used to compute option prices using the characteristic function of Heston model.

```
opt_cir=function(para){

T=para$T;
r=para$r;
x0=para$x0;
y0=para$y0;
K=para$K;
k=para$k;
theta=para$theta;
sigma=para$sigma;
rho=para$rho;

#characteristic function

fff  = function(pphi){

  s1=-pphi*1i*(rho*k/sigma-1/2+1/2*1i*pphi*(1-rho^2));
  s2=1i*pphi*rho/sigma;
```

```
  gamma1=sqrt(k^2+2*sigma^2*s1);
  gamma2=2*gamma1*exp(-gamma1*T)+(k+gamma1-sigma^2*s2)*(1-exp(-gamma1*T));

  H1=1/gamma2*(gamma1*s2*(1+exp(-gamma1*T))-(1-exp(-gamma1*T))*(2*s1+k*s2))-s2;
  H2=2*k*theta/sigma^2*log(2*gamma1/gamma2*exp(1/2*(k-gamma1)*T))-s2*k*theta*T;

  return(exp(1i*pphi*(x0+r*T)+H1*y0+H2));
}

#Integral I

ff1 = function(pphi){

       Re(exp(complex(real=0, imaginary=-pphi*log(K)))*exp(-T*r)/exp(x0)*
       mapply(fff,complex(r=pphi,i=-1))/complex(real=0, imaginary=pphi));
}

#Integral II

ff2 = function(pphi){

       Re(exp(complex(real=0, imaginary=-pphi*log(K)))*mapply(fff,pphi)/
       complex(real=0, imaginary=pphi));
}

#Option Pricing Formula

arma_sv = function () {

    exp(x0)*(1/2+1/pi*integrate(ff1, lower=0, upper=Inf)$value)-
    K*exp(-r*T)*(1/2+1/pi*integrate(ff2, lower=0, upper=Inf)$value);

}
sv=arma_sv();
imp_vol=uniroot(imp_BS,c(-10,10),a=sv,x0=x0,T=para$T,K=para$K,r=para$r);

#Calculate the conditional density

lower=x0-10;
upper=x0+10;
nn=2^10;
```

```
p_lower=trunc(nn*9.5/20);
p_upper=trunc(nn*10.5/20);

den_sv <- characteristic_function_to_density(
  function(t) mapply(fff,t),nn,lower, upper);

list(opt=sv,imp_vol=imp_vol$root,den=list(x=den_sv$x[p_lower:p_upper]-x0,
     density=den_sv$density[p_lower:p_upper]));

}
para_list=list(T=0.5,r=0.0,y0=0.05,x0=log(1000),K=1000,k=2,theta=0.05,
               sigma=0.2,rho=0);
temp1=opt_cir(para_list)
temp1$opt;
temp1$imp_vol;
```

B.3 Option Pricing with Schöbel and Zhu Model

This function was used to compute option prices using the characteristic function of Schöbel and Zhu model.

```
opt_zhu=function(para){

T=para$T;
r=para$r;
y0=para$y0;
x0=para$x0;
K=para$K;
k=para$k;
theta=para$theta;
sigma=para$sigma;
rho=para$rho;

#characteristic function

fff  = function(pphi){

     s1=-1/2*pphi*1i*(pphi*1i*(1-rho^2)-1+2*rho*k/sigma);
     s2=rho*k*theta/sigma*pphi*1i;
     s3=rho/(2*sigma)*pphi*1i;
```

```
     s4=0;

     gamma1=sqrt(2*sigma^2*s1+k^2);
     gamma2=(k-2*sigma^2*s3)/gamma1;
     gamma3=k^2*theta-s2*sigma^2;
     gamma4=cosh(gamma1*T)+gamma2*sinh(gamma1*T);

     H3=k/sigma^2-gamma1/sigma^2*(sinh(gamma1*T)+
        gamma2*cosh(gamma1*T))/gamma4;
     H4=((k*theta*gamma1-gamma2*gamma3)*(1-cosh(gamma1*T))-
        (k*theta*gamma1*gamma2-gamma3)*sinh(gamma1*T))/
        (gamma1*gamma4*sigma^2);
     H5=-1/2*log(gamma4)+((k*theta*gamma1-gamma2*gamma3)^2-
        gamma3^2*(1-gamma2^2))*
        sinh(gamma1*T)/(2*gamma1^3*gamma4*sigma^2)+
        (k*theta*gamma1-gamma2*gamma3)*gamma3*(gamma4-1)/
        (gamma1^3*sigma^2*gamma4)+T/(2*gamma1^2*sigma^2)*
         (k*gamma1^2*(sigma^2-k*theta^2)+gamma3^2);

     return(exp(1i*pphi*(x0+r*T)+1/2*H3*y0^2+H4*y0+H5));
}

#Integral I

ff1 = function(pphi){

Re(exp(complex(real=0, imaginary=-pphi*log(K)))*exp(-T*r)/exp(x0)*
mapply(fff,complex(r=pphi,i=-1))/complex(real=0, imaginary=pphi));
}

#Integral II

ff2 = function(pphi){

Re(exp(complex(real=0, imaginary=-pphi*log(K)))*mapply(fff,pphi)/
complex(real=0, imaginary=pphi));
}

#Option Pricing Formula

arma_sv = function () {
```

```
exp(x0)*(1/2+1/pi*integrate(ff1, lower=0, upper=Inf)$value)-
K*exp(-r*T)*(1/2+1/pi*integrate(ff2, lower=0, upper=Inf)$value);

}
sv=arma_sv();
imp_vol=uniroot(imp_BS,c(-10,10),a=sv,x0=x0,T=para$T,K=para$K,r=para$r);

#Calculate the conditional density

lower=x0-10;
upper=x0+10;
nn=2^10;

p_lower=trunc(nn*9.5/20);
p_upper=trunc(nn*10.5/20);

den_sv <- characteristic_function_to_density(
  function(t) mapply(fff,t),nn,lower, upper);

list(opt=sv,imp_vol=imp_vol$root,den=list(x=den_sv$x[p_lower:p_upper]-x0,
density=den_sv$density[p_lower:p_upper]));

}
para_list=list(r = 0.0,T = 0.5,rho = 0.5,k = 4,theta = 0.2,sigma = 0.1,
               x0 = log(1000),y0=0.2,K = 1000);
temp=opt_zhu(para_list)
temp$opt;
temp$imp_vol;
```

B.4 Option Pricing with the Fractional Heston Model

This function was used to compute option prices using the characteristic function of the fractional Heston model.

```
library(deSolve)

opt_lmcir = function(para){

T=para$T;
r=para$r;
```

```
y0=para$y0;
x0=para$x0;
K=para$K;
alpha=para$alpha;
k=para$k;
theta=para$theta;
sigma=para$sigma;
rho=para$rho;

#characteristic function

fff  = function(pphi){

     #solve the ODE numerically

     ZODE2 = function(Time, State, Pars) {
             with(as.list(State), {

               df = -1/2*sigma^2*theta*g*g
               dg = -k*g-1/2*sigma^2*g*g+1/2*pphi*rho^2*sigma^2*1i+
                    1/2*(pphi*1i+pphi^2)/gamma(1+alpha)*Time^alpha-
                    pphi*rho*k*1i;

               return(list(c(df, dg)));
               })
       }

       yini = c(f = 0+0i, g =-pphi*rho*1i);
       times = seq(0, T, length = 2);
       out = zvode(func = ZODE2, y = yini, parms = NULL,
       times = times,atol = 1e-10, rtol = 1e-10);

       return(exp(pphi*T*r*1i-1/2*pphi*rho^2*sigma^2*theta*T*1i-
       1/2*(pphi*1i+pphi^2)*theta*T-
       out[2,2]-(pphi*rho*1i+out[2,3])*y0+pphi*x0*1i));
}

#Integral I

ff1 = function(pphi){
```

```
        Re(exp(complex(real=0, imaginary=-pphi*log(K)))*exp(-T*r)/exp(x0)*
        mapply(fff,complex(r=pphi,i=-1))/complex(real=0, imaginary=pphi));
}

#Integral II

ff2 = function(pphi){

        Re(exp(complex(real=0, imaginary=-pphi*log(K)))*mapply(fff,pphi)/
        complex(real=0, imaginary=pphi));
}

#Option Pricing Formula

arma_sv = function () {

    exp(x0)*(1/2+1/pi*integrate(ff1, lower=0, upper=Inf)$value)-
    K*exp(-r*T)*(1/2+1/pi*integrate(ff2, lower=0, upper=Inf)$value);

}
sv=arma_sv();
imp_vol=uniroot(imp_BS,c(-10,10),a=sv,x0=x0,T=T,K=K,r=r);

#Calculate the conditional density

lower=x0-10;
upper=x0+10;
nn=2^10;

p_lower=trunc(nn*9.5/20);
p_upper=trunc(nn*10.5/20);

den_sv <- characteristic_function_to_density(
  function(t) mapply(fff,t),nn,lower, upper);

list(opt=sv,imp_vol=imp_vol$root,den=list(x=den_sv$x[p_lower:p_upper]-x0,
density=den_sv$density[p_lower:p_upper]));

}
para_list=list(T=0.5,r=0.0,y0=0,x0=log(1000),K=1000,alpha=0.2,
               k=2,theta=0.05,sigma=0.4,rho=0);
```

```
temp=opt_lmcir(para_list)
temp$opt;
temp$imp_vol;
```

B.5 Option Pricing with the Fractional Schöbel-Zhu Model

This function was used to compute option prices using the approximate characteristic function of the fractional Schöbel-Zhu model.

```
opt_lmzhu=function(para){

n = para$n;
m = para$m;
r = para$r/n;
T = trunc(para$T*n);
rho = para$rho;
dd = para$dd;
q = m*para$T;
k = para$k;
ma_q = rep(0,q);

for (i in 1:q){
   ma_q[i] = (i^dd-(i-1)^dd)/(n^dd*gamma(1+dd));
}

theta = para$theta*sqrt(1/n);
sigma = para$sigma*sqrt((1-exp(-2*k/n))/(2*k))*sqrt(1/n);
tao = T+1;
ll = array(c(1, rep(0, q-1)), c(q,1));
Beta = array(ma_q, c(q,1));
Phi=rbind(cbind(exp(-k/n),array(0,c(1,q-1))),cbind(diag(rep(1,q-1)),
rep(0,q-1)));
x0 = para$x0;
y0 = array(0, c(q,1));
K = para$K;

#characteristic function

fff  = function(pphi){
```

```
     A1 = array(0, c(1, 2));
     B1 = array(-pphi, c(1, 2));
     B2 = array(0, c(q, 1, 2));
     B3= array(0, c(q, q, 2));

     for (jj in 2:tao){

          i = 2-jj %% 2;
          j = 2-(jj-1) %% 2;
          temp1 = 1+2*sigma^2*t(ll)%*%B3[,,j]%*%ll;
          temp2 = B1[,j]*rho*Beta+sigma*t(Phi)%*%(B3[,,j]+t(B3[,,j]))%*%ll;
          temp3 = sigma*t(ll)%*%B2[,,j]+B1[,j]*theta*rho;

          A1[,i] = A1[,j]+B1[,j]*r-1/2*theta^2*B1[,j]-1/2*(1-rho^2)*
          theta^2*B1[,j]^2+1/2*log(temp1)-1/(2*temp1)*temp3^2;

          B2[,,i] = -(B1[,j]+(1-rho^2)*B1[,j]^2)*theta*Beta+t(Phi)%*%B2[,,j]-
          as.vector(1/temp1*temp3)*temp2;

          B3[,,i] = -1/2*(B1[,j]+B1[,j]^2*(1-rho^2))*Beta%*%t(Beta)+
          t(Phi)%*%B3[,,j]%*%Phi-as.vector(1/(2*temp1))*(temp2%*%t(temp2));
    }
    exp(-A1[,i]-B1[,i]*x0-t(B2[,,i])%*%y0-t(y0)%*%B3[,,i]%*%y0);
}

#Integral I

ff1 = function(pphi){

Re(exp(complex(real=0, imaginary=-pphi*log(K)))*exp(-T*r)/
exp(x0)*mapply(fff,complex(real=1, imaginary=pphi))/
complex(real=0, imaginary=pphi));
}

#Integral II

ff2 = function(pphi){

Re(exp(complex(real=0, imaginary=-pphi*log(K)))*
mapply(fff,complex(real=0, imaginary=pphi))/
complex(real=0, imaginary=pphi));
```

```
}

#Option Pricing Formula

arma_sv = function () {

exp(x0)*(1/2+1/pi*integrate(ff1, lower=0, upper=Inf)$value)-
K*exp(-r*T)*(1/2+1/pi*integrate(ff2, lower=0, upper=Inf)$value);

}
sv=arma_sv();
imp_vol=uniroot(imp_BS,c(-10,10),a=sv,x0=x0,T=para$T,K=para$K,r=para$r);

#Calculate the conditional density

lower=x0-10;
upper=x0+10;
nn=2^10;

p_lower=trunc(nn*9.5/20);
p_upper=trunc(nn*10.5/20);

den_sv <- characteristic_function_to_density(
  function(t) mapply(fff,complex(r=0,i=t)),nn,lower, upper);

list(opt=sv,imp_vol=imp_vol$root,den=list(x=den_sv$x[p_lower:p_upper]-x0,
density=den_sv$density[p_lower:p_upper]));

}
para_list=list(n=50,m=50,r = 0.0,T = 0.5,rho = 0,dd = 0.2,k = 4,
              theta = 0.2,sigma = 0.2,x0 = log(1000),K = 1000);
temp=opt_lmzhu(para_list)
temp$opt;
temp$imp_vol;
```

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