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SET-MARKOV PROCESSES

By

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A Thesis

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Abstract

This thesis introduces a type of Markov property, called the ‘set-Markov’ property, that can be defined for set-indexed processes, and in particular for multiparameter processes. This property is stronger than the ‘sharp Markov’ property that has been introduced earlier in the literature. For processes indexed by the real line, the set-Markov property coincides with the classical Markov property.

An important class of set-Markov processes are ‘ $\mathcal{Q}$ -Markov’ processes, where $\mathcal{Q}$ is a family of transition probabilities satisfying a Chapman-Kolmogorov type relationship. Two constructions are indicated for a $\mathcal{Q}$ -Markov process, as applications of Kolmogorov’s extension theorem; the first construction is valid only under a certain geometrical condition.

A $\mathcal{Q}$ -Markov process can be associated to a ‘suitably indexed’ collection of classical Markov processes; therefore, ‘the generator’ of the process is defined as the collection of all the generators of these classical Markov processes. It is proved that under certain conditions, one can construct a $\mathcal{Q}$ -Markov process knowing its generator.

Processes with independent increments are $\mathcal{Q}$ -Markov; they are characterized by ‘convolution systems’ of distributions. In particular, Lévy processes have a double characterization in terms of their characteristic functions and in terms of their generators. Some other examples of $\mathcal{Q}$ -Markov processes are considered, including empirical processes. For each of these examples, the generator provides us with a means of constructing the process.

‘Adapted sets’ and ‘optional sets’ generalize the classical notions of stopping time and optional time. Using these sets, the strong Markov properties associated to a $\mathcal{Q}$ -Markov process, respectively a sharp Markov process, are considered.

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Dedication

I dedicate this work to my parents for their love and support.

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Chapter 1

Introduction

The Markov property is without doubt one of the most appealing notions that exists in the theory of stochastic processes and many processes modelling physical phenomena enjoy it, simply because it is the natural expression of the law under which various systems evolve. To explain it in a few words, this law requires that the future behaviour of the system be completely independent of its past, given its present status, or equivalently, in order to predict the future development of certain events, one does not need to know anything about the past, the present being all that is needed.

The first attempt in trying to generalize the notions of ‘time’ and ‘Markov property’ (which are intrinsically related) was made in [51] (as early as 1948): a process $(X_z)_{z \in \mathbf{R}_+^2}$ indexed by the positive quadrant of the plane, is said to be ‘Markov with respect to a set $A \subseteq \mathbf{R}_+^2$ ’ if $\mathcal{F}_A$ and $\mathcal{F}_{A^c}$ are conditionally independent given $\mathcal{F}_{\partial A}$, where $\mathcal{F}_D = \sigma(X_z; z \in D)$ for any $D \subseteq \mathbf{R}_+^2$. This definition (which is now known as **sharp Markov property**) began to become popular as further steps were taken in the study of two-parameter processes and processes such as the Brownian sheet (apparently, first introduced by a statistician, in [45]) and the Poisson sheet were found to be natural generalizations of the classical Brownian motion and Poisson process. The big question which stayed open for quite a few decades was: what is the largest class of sets $\mathcal{A}$ for which the Brownian sheet, the Poisson sheet, or some of their associated processes are sharp Markov?

But let us forget for the moment about the Markov property and ask ourselves a

natural question: leaving aside the pure theoretical aspect, why would anyone go into the study of processes indexed by the plane, by $\mathbf{R}^d$, or even by more general objects? The answer to this question comes in different forms. A possible application to statistics of the Brownian sheet was suggested in [44]. In [4] it is claimed that some two-parameter jump processes are known to have application in reliability theory. Cabaña's famous problem of the vibrating string is the most cited instance where one can actually identify a concrete object (i.e. the displacement of the point x of the string at moment t) with a two-parameter process. More recently, in [19] it is pointed out that the stochastic wave equation in two spatial dimensions driven by a special type of noise is known in the literature as having some physical [11] and geophysical [58] applications. This is a particular case of the following general problem: suppose one is given a physical system governed by a partial differential equation; suppose then that the system is perturbed randomly, perhaps by some sort of white noise and we want to find out how it evolves in time. The solution will be a multiparameter stochastic process.

Returning to our discussion about the Markov property, a quick review of the literature to date on this subject reveals the following.

At least four possible definitions were proposed for a Markov process in the continuous, multiparameter case.

The first one, which is the oldest one, the sharp Markov property, has produced the most research activity and due to the constant efforts made in this area, there aren't too many questions waiting for an answer here. But let us proceed chronologically.

In 1976, it was proved that: 1. the Brownian sheet fails to have the sharp Markov property with respect to a very simple region, the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ (see [69]); and 2. a process indexed by the plane which is Euclidean invariant and ergodic cannot satisfy the sharp Markov property for all open sets (see [18]).

This led to the conclusion that, instead of the sharp σ -field $\mathcal{F}_{\partial A}$ one has to consider a larger one, called the germ σ -field, which is defined as $\mathcal{G}_{\partial A} = \cap_G \mathcal{F}_G$ where the intersection is taken over all open sets G containing ∂A . The new Markov property, for which $\mathcal{F}_A$ and $\mathcal{F}_{A^c}$ are conditionally independent given $\mathcal{G}_{\partial A}$ was called **the Lévy Markov property** or **germ Markov property** (this definition was first introduced

in [55] and it was extensively examined in the Gaussian case, for instance in [62], [42]; some of its properties are discussed in [52]).

In 1984, it was proved that: 1. the Brownian sheet satisfies the germ Markov property with respect to all open sets (see [59]), which, according to [53], is equivalent to saying that the germ Markov property holds with respect to all sets; and 2. all processes with independent increments are sharp Markov with respect to finite unions of rectangles (see [64]).

For more than a decade it was thought that, in fact, for the Brownian sheet, the finite unions of rectangles are the only sets for which the sharp Markov property holds. An intuitive reason for this is given in [70], using the propagation of ‘singularities’ (points where the law of the iterated logarithm breaks down) in the sheet. The first instance where the sharp Markov property was shown to hold for a domain whose boundary contains no vertical or horizontal segment was the result of [21] for domains bounded by ‘separation lines’ (continuous non-increasing curves that meet both coordinate axes). For these sets it is actually showed that the sharp σ -field coincides with the germ field. With this, the subject was revived and pursued with even more fervour. By now, one thing was clear: from the point of view of the sharp Markov property, Gaussian processes behave completely differently from point processes. Intuitively, one can compare the Poisson sheet whose discontinuities propagate along horizontal and vertical lines and for which one can get a lot of information just looking at the boundary of a set, with the Brownian sheet, whose singularities criss-cross the plane horizontally and vertically in a symmetric way so that it is impossible to say which way they are travelling looking only at the boundary of the set.

For point processes a partial answer was given in [56]: under certain assumptions a point process with independent increments satisfies the sharp Markov property with respect to a certain class of bounded open sets, a class which contains the triangles and the lower layers (a set $A \subseteq \mathbf{R}_+^2$ is a lower layer if $z \in A \Rightarrow [0, z] \subseteq A$). This result was an extension of a result of [16] (which appears also in [17]) in which it is proved that the Poisson sheet satisfies the sharp Markov property with respect to the class of bounded, relatively convex open sets.

The complete answer was given in 1992 in the form of two thorough papers written by Dalang and Walsh.

In [23], the result of [21] is extended to domains whose boundaries are singular curves of bounded variation; in this paper it is also shown that the Brownian sheet satisfies the sharp Markov property with respect to ‘almost all’ Jordan curves, where ‘almost all’ has to be understood in the sense of Baire category.

Multiparameter processes with independent increments have been in the literature for many years, their definition being a straightforward generalization of the classical one: they have to assume independent values over disjoint hyper-parallelepipeds. It was proved in [1] that most of the classical properties of processes with independent increments on the real line can be transferred to multiparameter processes, including the Lévy-Itô path decomposition of the stochastically continuous part. However, it was not so straightforward to identify the appropriate type of Markov property satisfied by these processes. The main result of [22] states that for jump processes with independent increments which have *only* positive jumps, the sharp Markov property holds for all bounded open sets; in the case of a process which has negative jumps as well, the minimal splitting field turns out to be larger than the sharp field.

The third definition for a Markov process indexed by the plane was introduced in [61] and it was first studied in the Gaussian case. It was proved there that a Gaussian process is Markov (in the specified sense) if and only if it is an integral process with respect to the Brownian sheet (the stochastic integral in the plane had been introduced and studied in [15]). The authors of [34] used the same definition of the Markov property, which was generalized and studied thoroughly in [46]. The basic idea in this third Markov property, which for simplicity will be called the **KLM-Markov property** (‘KLM’ is an abbreviation for the names of the three authors of [46]), is the separation of parameters, i.e. the simultaneous definition of a horizontal Markov property and a vertical Markov property. (Clearly, the partial order $z = (s, t) \leq z' = (s', t')$ defined by $s \leq s'$ and $t \leq t'$, induced by the cartesian coordinates plays a crucial role.) What is interesting about this definition is the fact that it has two equivalent formulations, in terms of the strict-sense history $\mathcal{F}_z = \sigma(X_{z'}; z' \in R_z)$ and the wide-sense history $\mathcal{F}_z^* = \sigma(X_{z'}; z' \in R_z^*)$ associated to the point z in the plane

(here $R_z = \{z'; z' \leq z\}$ and $R_z^* = \{z'; z \not\leq z'\}$). Note that these are the two histories with respect to which the weak martingales and respectively strong martingales are defined in the plane.

To be precise, we say that a two-parameter process $(X_z)_{z \in \mathbb{R}_+^2}$ is **KLM-*Markov**, if for any z the σ -fields $\mathcal{F}_z^*$ and $\sigma(X_{z'}; z' \in R_z^{*c})$ are conditionally independent given $\sigma(X_{z'}; z' \in \partial R_z^*)$, and moreover the predictor $E[h(X_{z_1}, \dots, X_{z_n}) | \mathcal{F}_z^*]$ for $z_1, \dots, z_n \in R_z^{*c}$ and h measurable and bounded, depends only on the values $X_{s_i, t_i}, X_{s_i, t}, X_z$ of the process at the points obtained by projecting $z_i, i = 1, \dots, n$ on the boundary of R_z^* . The definition is analogous in the strict-sense formulation: we say that a two-parameter process $(X_z)_{z \in \mathbb{R}_+^2}$ is **KLM-Markov**, if for any z the σ -fields $\mathcal{F}_z$ and $\sigma(X_{z'}; z' \in R_z^c)$ are conditionally independent given $\sigma(X_{z'}; z' \in \partial R_z)$, and moreover the predictor $E[h(X_{z_1}, \dots, X_{z_n}) | \mathcal{F}_z]$ for $z_1, \dots, z_n \in R_z^c$ and h measurable and bounded, depends only on the values $X_{z \wedge z_i}$ of the process at the points obtained by projecting $z_i, i = 1, \dots, n$ on the boundary of R_z . It is proved in [46] that in the definition of the KLM-*Markov property we can consider $n = 1$ and that the KLM-*Markov property is equivalent to the KLM-Markov property.

An easy but important exercise, not mentioned by the authors of the paper, is to show that all processes with independent increments are KLM-Markov. (By an ‘increment’ of a two-parameter process X we understand the value $\Delta_{[z, z']}X$ of the process over the rectangle $[z, z']$, defined by $\Delta_{[z, z']}X = X_z - X_{s't'} - X_{s't} + X_{z'}$.) To prove this, let $z' \in R_z^{*c}$ and write $X_{z'} = \Delta_{[z, z']}X + X_{s't'} + X_{s't} - X_z$. The result follows since $\Delta_{[z, z']}X$ is independent of $\mathcal{F}_z^*$.

On the other hand the authors proved that any KLM-Markov process is sharp Markov with respect to the finite unions of rectangles and germ Markov with respect to the relatively convex bounded open sets.

Given a two-parameter homogeneous semigroup $P = (P_z)_{z \in \mathbb{R}_+^2}$ we say that the KLM-Markov process X defined on the probability space $(\Omega, \mathcal{F}, P_x)$ is a **realization** of the semigroup P if $X_0 = x$ a.s. and $E_x[h(X_z)] = P_z h(x)$ for any z and for any measurable and bounded function h . In [54] it is proved that such a realization exists, and moreover, it can be chosen such that its trajectories are cadlag: i.e. right-continuous and with quadrantale limits.

The fourth definition of a Markov property in the plane was introduced in [72]. However, in this paper the problem is treated from a completely different point of view since the authors consider processes parametrized by smooth curves in $\mathbf{R}_+^2$. Such a path-parametrized process $Y = (Y_\gamma)_\gamma$ indexed over all continuous curves in $\mathbf{R}_+^2$ that are either increasing or decreasing is said to be γ **Markov** if for every connected open set D with a piecewise monotone boundary the σ -fields $\sigma(Y_\gamma; \gamma \subseteq D)$ and $\sigma(Y_\gamma; \gamma \subseteq D^c)$ are conditionally independent given $\sigma(Y_\gamma; \gamma \subseteq \partial D)$. On the other hand, if $X = (X_z)_{z \in \mathbf{R}_+^2}$ is an arbitrary process indexed by points in $\mathbf{R}_+^2$ and Y is a path-parametrized process then the process $\tilde{Y} = (\tilde{Y}_\gamma)_\gamma$ defined as $\tilde{Y}_\gamma = (Y_\gamma, X_{\gamma_0}, X_{\gamma_1})$ (where γ_0, γ_1 are the end points of γ) is said to be γ **+Markov** if for every connected open set D with a piecewise monotone boundary the σ -fields $\sigma(\tilde{Y}_\gamma; \gamma \subseteq D)$ and $\sigma(\tilde{Y}_\gamma; \gamma \subseteq D^c)$ are conditionally independent given $\sigma(\tilde{Y}_\gamma; \gamma \subseteq \partial D)$. Examples of such path-parametrized processes are the following: let W be the planar white noise with $W([0, z]) = W_z$ and for each smooth curve γ , let $A(\gamma), B(\gamma)$ be its vertical and horizontal shadows; set $Y_\gamma := (W(A(\gamma)), W(B(\gamma)))$ and $\tilde{Y}_\gamma := (Y_\gamma, W_{\gamma_0}, W_{\gamma_1})$; then Y is γ Markov and $\tilde{Y}$ is γ +Markov.

The case of Markov processes indexed by a discrete partially ordered set was considered by various authors ([14], [48]) for solving stochastic optimal control problems, the Markovian nature of the model allowing us to discard any unnecessary information when formulating such a problem. More precisely, if $(S, \leq)$ is a countable partially ordered set (with a minimal element) such that each $s \in S$ has a finite set $U(s) = \{\delta_1(s), \dots, \delta_d(s)\}$ of direct successors and finitely many predecessors (think of it as being $\mathbf{N}^d$) and $(\mathcal{F}_s)_{s \in S}$ is an S -indexed filtration, then a **stopping point** is a random point ν in S satisfying $\{\nu = s\} \in \mathcal{F}_s \forall s$ and a **strategy** is an increasing sequence $(\sigma_n)_n$ of random points with $\tau = \inf\{n; \sigma_n = \sigma_{n+1}\}$, $\sigma_{n+1} \in U(\sigma_n)$ if $n < \tau$, $\sigma_n = \sigma_\tau$ if $n \geq \tau$ and σ_{n+1} is $\mathcal{F}_{\sigma_n}$ -measurable. We will consider only stopping points ν which are accessible by means of a strategy i.e. $\nu = \sigma_\tau$.

If for each $s \in S$, the reward obtained if we stop at s is given by the random variable Z_s and there is no advantage in taking either one of the paths that lead to s , to solve the optimal stopping problem associated to $\{Z_s; s \in S\}$ means to determine the accessible stopping point ν^* which maximizes the expected reward $E[Z_s]$. If in

addition, there is a running reward which is obtained any time we take the decision of going from s to its direct successor u , reward given by the random variable $Z_{s,u}$, then we are facing an optimal control problem for which we will have to find the best strategy σ^* which maximizes $E[\sum_{n=0}^{\tau-1} Z_{\sigma_n, \sigma_{n+1}} + Z_{\sigma_\tau}]$. (Note that this also gives us the best stopping point $\nu^* = \sigma_\tau^*$.) The usefulness of the Markov property in this set-up is the following: Let $Y = (Y_s)_{s \in S}$ be a fixed S -indexed process. If Y is a Markov chain (in the sense that the σ -fields $\sigma(Y_u; u \leq s)$ and $\sigma(Y_u; u \geq s)$ are conditionally independent given $\sigma(Y_s)$ for any $s \in S$) then for any choice of the rewards $Z_s, Z_{s,u}, s \in S, u \in U(s)$ such that Z_s is $\sigma(Y_s)$ -measurable and $Z_{s,u}$ is $\sigma(Y_s, Y_u)$ -measurable, the optimal control problem associated to them is solved by a strategy σ^* satisfying the condition: σ_{n+1}^* is $\sigma(Y_{\sigma_n^*}, \sigma_n^*)$ -measurable $\forall n$ (any strategy satisfying this condition is called a **Markov strategy**). In other words, in the case when the underlying probabilistic model is a Markov chain, it is enough to solve the optimal control problem over the restricted class of Markov strategies. This result (whose converse is also true) is proved in [48]. In the same context, the authors of [14] considered Markov chains $(Y_s)_{s \in S}$ for which the mechanism of transition is known: $P(Y_{\delta_i(s)} = y | Y_u, u \leq s) = P_i(Y_s, y), i = 1, \dots, d$ where $P_1, \dots, P_d$ are transition functions.

The literature is rather scarce when it comes to the strong Markov property for processes indexed by partially ordered sets. In the case of the processes indexed by the plane, the obvious notion of **stopping point** (introduced for the first time in [57], in complete analogy with the classical case) does not prove to be fruitful for the Markov property. Instead, various authors were considering stopping lines. A random decreasing line L is called a **stopping line** with respect to the filtration $(\mathcal{F}_z)_z$, if for any $z \in \mathbf{R}_+^2$ we have $\{z \leq L\} \in \mathcal{F}_z$. (A ‘decreasing’ line is the image of continuous curve γ on $\mathbf{R}_+^2$ such that $\gamma(0)$ belongs to the y -axis, $\gamma(1)$ belongs to the x -axis and $\theta \leq \theta'$ implies $\gamma_1(\theta) \leq \gamma_1(\theta')$ and $\gamma_2(\theta) \geq \gamma_2(\theta')$; to each decreasing line l one can associate the set $D(l) := \{z; \exists z' \in l, z \leq z'\}$.)

In [56] it is proved that every strictly simple point process N which is sharp Markov with respect to the sets $D(l)$ for every decreasing line l , and whose intensity is absolutely continuous with respect to the Lebesgue measure, is strong Markov in the sense

that the σ -fields $\mathcal{F}_{D(L)}$ and $\mathcal{F}_{D(L)^c}$ are conditionally independent given $\mathcal{F}_L$ for any stopping line L , where $\mathcal{F}_A := \sigma(\{N_z 1_A(z), 1_A(z); z \in \mathbf{R}_+^2\})$ for an arbitrary random set A . They also proved that $\mathcal{F}_{D(L)} = \{F \in \mathcal{F}; F \cap \{L \leq l\} \in \mathcal{F}_{D(l)} \text{ for any decreasing line } l\}$, which is similar to the definition of the stopped σ -field in the classical case.

The same problem was studied in the last section of [72], but for path-parametrized processes. The path-parametrized process $\tilde{Y}$ is said to be strong Markov if for any stopping line L , the σ -fields $\mathcal{F}_{D(L)}^*$ and $\mathcal{F}_{D(L)^c}^*$ are conditionally independent given $\mathcal{F}_L^*$, where $\mathcal{F}_{D(L)}^* = \sigma(\{\tilde{Y}_{\gamma \cap D(L)}; \gamma \subseteq \mathbf{R}_+^2\})$, $\mathcal{F}_{D(L)^c}^* = \sigma(\{\tilde{Y}_{\gamma \cap D(L)^c}; \gamma \subseteq \mathbf{R}_+^2\})$ and $\mathcal{F}_L = \sigma(\{\tilde{Y}_\gamma; \gamma \subseteq L\})$.

A different type of strong Markov property was introduced in [31] (as early as 1977), in the general context of ‘random fields’, which in this paper are increasing collections $(\mathcal{F}_V)_V$ of σ -fields, indexed over the closed subsets V of an n -dimensional space. Such a random field is said to be **locally Markov** if for any closed sets V and W with $V \cap W = \emptyset$, $\mathcal{F}_V$ and $\mathcal{F}_W$ are conditionally independent given $\mathcal{F}_{\partial V}$. A random set T whose values are closed sets is said to be a **Markov random set** if $\{T \subseteq V\} \in \mathcal{F}_V$ for any closed set V . Two σ -fields are associated to each Markov random set T : $\mathcal{A}_T = \bigcap_{\epsilon > 0} \mathcal{A}_T^\epsilon$ and $\mathcal{B}_T = \bigcap_{\epsilon > 0} \mathcal{B}_T^\epsilon$, where $\mathcal{A}_T^\epsilon = \sigma(\{F \cap \{T_\epsilon \supseteq V\}; F \in \mathcal{F}_V, V \text{ closed}\}) \vee \sigma(T)$ and $\mathcal{B}_T^\epsilon = \sigma(\{F \cap \{(\partial T)_\epsilon \supseteq V\}; F \in \mathcal{F}_V, V \text{ closed}\}) \vee \sigma(T)$. The main result is that for a locally Markov random field $(\mathcal{F}_V)_V$, for any Markov random set T whose values are compact sets, if V is an arbitrary closed set and $F \in \mathcal{F}_V$, we have $P[F|\mathcal{A}_T] = P[F|\mathcal{B}_T]$ a.s. ($T \cap V = \emptyset$).

In a much more recent paper [33], another type of strong Markov property is introduced for processes indexed by a partially ordered discrete set T . Namely, a T -indexed process X is said to be **Markov** if for any $t \in T$, the σ -fields $\sigma(X_{t'}; t' \leq t)$ and $\sigma(X_{t'}; t' \geq t)$ are conditionally independent given X_t . When T is a finite or countable set, a random element τ in T is called a **splitting element** if for any $t \in T$ we can write $\{\tau = t\} = F_1 \cap F_2$ with $F_1 \in \sigma(X_{t'}; t' \leq t)$ and $F_2 \in \sigma(X_{t'}; t' \geq t)$. To each splitting element τ one can associate the σ -fields $\mathcal{A}_1(\tau) = \sigma(\{F \cap \{\tau = t\}; F \in \sigma(X_{t'}; t' \leq t), t \in T\})$ and $\mathcal{A}_2(\tau) = \sigma(\{F \cap \{\tau = t\}; F \in \sigma(X_{t'}; t' \geq t), t \in T\})$. It is stipulated there that for a Markov process X , the σ -fields $\mathcal{A}_1(\tau)$ and $\mathcal{A}_2(\tau)$ are conditionally independent given (τ, X_τ) if and only if τ is a splitting element.

In the 1990's several authors have begun again to be interested in the sharp or germ Markov property, this time for processes which are solutions of stochastic partial differential equations (PDE). In [60] it is proved that the solution of the Cauchy problem for a quasi-linear parabolic stochastic PDE driven by a space-time white noise is germ Markov and, if the Cauchy problem is replaced by a problem with some periodic boundary conditions then the germ Markov property holds only in the linear case. A similar result was proved in [24] for the elliptical equation.

Because stochastic PDE's in more than one spatial dimension driven by white noise do not have real-valued process solutions (see [71]), there has been little study of Markov properties of solutions to these equations and, in particular, no study of the sharp Markov property. However, the Markov property indicates how information flows through space-time, so it would be interesting to study this property for equations which do have real-valued solutions. Therefore, equations driven by some other types of noise were taken into consideration. Partially motivated by some physical applications, the authors of [19] considered the wave equation in two spatial dimensions driven by a noise which is white in time and has smooth spatial covariance. The same wave equation in two spatial dimensions, but this time driven by a Lévy point process, was studied in [20]. They showed that the solution is sharp Markov for domains bounded by the plane if and only if the angle between the plane and the time axis is at least $\pi/4$; the sharp Markov property also holds for bounded polyhedra.

The final remark which will conclude our review of the literature on this subject is the following.

The next level of generality in the theory of stochastic processes indexed by partially ordered sets was to consider the class of processes with independent increments, indexed by a collection $\mathcal{A}$ of closed subsets of the d -dimensional unit cube $[0, 1]^d$. These processes, called 'set-indexed processes with independent increments', have to be finitely additive, and to assume independent values over disjoint sets. They were also called 'random measures' in the terminology of [2], or 'ID processes' in the terminology of [8] (which assumed in addition the stationarity of the increments). In [2] the (set-indexed) *Lévy processes* are introduced as 'stochastically continuous' processes with independent increments, and the set convergence is defined using the

metric $d_\lambda(A, B) := \lambda(A \Delta B)$, where λ is the Lebesgue measure; it is proved that the distribution of such a process is completely characterized by a system of infinitely divisible distributions on the real line. Concentrating only on the jump part of a (stationary) Lévy process, the authors of [8] and [2] obtained in 1984 (independently of each other), some very similar conditions under which a ‘cadlag’ version exists; these conditions relate the underlying Lévy measure of the process, with the metric entropy of the class $\mathcal{A}$.

A completely new approach in the modern theory of stochastic processes indexed by partially ordered sets was initiated and developed by Ivanoff and Merzbach for the martingale case (see for example [36], [25], [37], [38], [41]), powerful results like Doob-Meyer decomposition and martingale representations finding their analogues in this context. In this framework, the processes that are taken into consideration are indexed by a collection $\mathcal{A}$ of closed subsets of a topological space T . If we identify this class with the class of all rectangles $[0, z]$, $z \in \mathbf{R}^d$ and we denote with X_z the value of the set-indexed process at the rectangle $[0, z]$, then $(X_z)_z$ will be a d -dimensional process; hence, we can view this theory as a generalization of the theory of multiparameter processes.

In 1998 a new paper [39] was written by Ivanoff and Merzbach, this time dealing with a certain Markov property for processes indexed by the collection $\mathcal{A}$. This paper motivated the beginning of the present research in the area of certain Markov properties for set-indexed processes and led to the writing of this work.

When trying to define a Markov property for processes indexed by the collection $\mathcal{A}$ of sets (which will be introduced formally in Chapter 2), there is a natural question that comes to mind: is it possible to find an analogous sharp Markov property in this general context? The answer to this question is positive and the analogy to the multiparameter theory is complete in the sense that, if we identify our set-indexed process with a multiparameter process, then the sharp Markov property defined in the general framework becomes exactly the sharp Markov property for finite unions of rectangles.

To make things more precise, it is said in [39] that an $\mathcal{A}$ -indexed process X is **sharp Markov** if $\mathcal{F}_B \perp \mathcal{F}_{B^c} \mid \mathcal{F}_{\partial B}$ for any $B \in \mathcal{A}(u)$, where $\perp$ denotes conditional

independence (see Appendix A.1) and $\mathcal{F}_B = \sigma(\{X_A; A \in \mathcal{A}\})$, $\mathcal{F}_{B^c} = \sigma(\{X_A; A \in \mathcal{A}, A \not\subseteq B\})$ and $\mathcal{F}_{\partial B} = \sigma(\{X_A; A \in \mathcal{A}, A \subseteq B, A \not\subseteq B^0\})$. (Throughout this work we will denote with $\mathcal{A}(u)$ the class of all finite unions of sets in $\mathcal{A}$; this collection is a natural object to work with if we think that in the theory of 2-parameter processes the increments are usually defined over the rectangles with the sides parallel to the coordinate axes but which do not necessarily have the origin as a vertex and any such rectangle C can be written as the difference $C = A \setminus B$ where A is a rectangle in $\mathcal{A}$ and B is the union of two rectangles from $\mathcal{A}$ which are contained in A ; in other words, for an additive set-indexed process X , the value X_C can be identified exactly with the increment $X_A - X_B$.)

A set-indexed process $(X_A)_{A \in \mathcal{A}}$ is said to have **independent increments** if $X_{C_1}, \dots, X_{C_n}$ are independent whenever $C_1 := A_1 \setminus B_1, \dots, C_n := A_n \setminus B_n$ are pairwise disjoint increments with $A_i \in \mathcal{A}, B_i \in \mathcal{A}(u)$. The authors of [39] considered also another type of Markov property (the set-indexed analogue of the KLM-Markov property), stronger than the sharp Markov property, and which is satisfied by all processes with independent increments: a process $X = (X_A)_{A \in \mathcal{A}}$ is called **Markov** if $\forall B \in \mathcal{A}(u), \forall A_1, \dots, A_k \in \mathcal{A}, A_i \not\subseteq B$ and for any measurable bounded function $h : \mathbf{R}^k \rightarrow \mathbf{R}$ we have

$$E[h(X_{A_1}, \dots, X_{A_k}) | \mathcal{F}_B] = E[h(X_{A_1}, \dots, X_{A_k}) | \mathcal{F}_{\partial B} \cap \mathcal{F}_{\cup_{i=1}^k A_i}].$$

It is proved there that under certain circumstances the sharp Markov property and the Markov property are equivalent. Moreover, a sharp Markov process is also strong Markov with respect to any ‘stopping set’ in a sense that will be specified later on, and consequently, any sharp Markov point process is strong Markov (roughly speaking, a stopping set is any random set ξ which satisfies $\{A \subseteq \xi\} \in \mathcal{F}_A$ for all $A \in \mathcal{A}$).

As in the case of two-parameter processes, constructing a general process which is sharp Markov (even with respect to such a small class of sets as the class of finite unions of rectangles) is not an easy task and the question of existence of such a process is not a trivial matter. For planar processes the problem was solved by the author of [54] who constructed a KLM-Markov process which, as we know (Theorem 3.7 [46]) is sharp Markov with respect to the finite unions of rectangles. The trajectories of

the process constructed in [54] also have nice regularity properties.

The main type of Markov property that we will consider throughout this work will be different from the ones introduced in [39] and will be called ‘set-Markov’, to emphasize the fact that it can be defined in the general context of set-indexed processes (in fact, the set-Markov property implies the sharp Markov property, but not the Markov property). This definition requires that the value of the process $X_{A \setminus B}$ over the increment $A \setminus B$ be conditionally independent of the history $\mathcal{F}_B$ given the present status X_B . We believe that this is a natural definition because it captures the essence of the Markov property in terms of the increments, allowing us to have a perfect analogy with the classical case (for which we know that for a Markov process X , the increment $X_t - X_s$, $s < t$ is conditionally independent of the past history $\mathcal{F}_s$ given X_s). Moreover, from the technical point of view, since the collection $\mathcal{C}$ of all increments is a semialgebra (and hence $\mathcal{C}(u)$, the collection of all finite unions of sets in $\mathcal{C}$, is an algebra of sets, in other words it is a collection for which the usual operations with sets are permitted) this definition will enable us to make all the computations with objects indexed by the same family.

This work is organized as follows:

In Chapter 2 we will present the general definitions and properties that are specific to the set-indexed framework; in particular, we will examine the additivity property of a set-indexed process and we will formally introduce the notion of ‘flow’ as an increasing map from an interval $[0, a]$ of the real line to the class $\mathcal{A}(u)$ of sets.

In Chapter 3 we will first define the set-Markov property and we will examine some of its basic properties: all processes with independent increments are set-Markov; a set-Markov process becomes Markov in the classical sense, when it is transported by a flow (it is clear that by using different flows, time can get very easily compressed or decompressed, without affecting in any way the ‘distance’ between the arrival sets; so, the notion of homogeneity is meaningless for set-Markov processes). Next we will introduce a sub-class of set-Markov processes, called $\mathcal{Q}$ -Markov, for which there exists a ‘transition system’ $\mathcal{Q}$ characterizing the transitions from a state $B \in \mathcal{A}(u)$ to a another state of the form $B' \in \mathcal{A}(u)$ with $B \subseteq B'$. For $\mathcal{Q}$ -Markov processes we will indicate two types of constructions as applications of Kolmogorov’s extension theorem.

The first construction is valid only if the indexing collection $\mathcal{A}$ satisfies a certain geometric property called SHAPE (in particular, this construction is perfectly suited for multiparameter processes), while the second construction is valid in the general case. For both constructions one has to impose a natural ‘consistency’ assumption on the transition system $\mathcal{Q}$, which basically requires that the finite-dimensional distributions of the process be uniquely defined, up to a permutation.

In Chapter 4 we will define ‘the generator’ of a $\mathcal{Q}$ -Markov process as the collection of all the generators of classical Markov processes obtained as ‘traces’ of the original set-indexed process over some (suitably chosen) flows. This definition will permit us to translate the previous ‘consistency’ assumption in terms of the generators, and therefore to identify the necessary and sufficient conditions that have to be satisfied by a collection of one-dimensional generators so that they become the generator of a set-indexed $\mathcal{Q}$ -Markov process.

In Chapter 5 we will examine the class of set-indexed processes with independent increments, in the light of their set-Markov property. The object that will allow us to reconstruct such a process will no longer be the transition system $\mathcal{Q}$ but a ‘convolution’ family $(F_C)_{C \in \mathcal{C}}$ of probability measures on $\mathbf{R}$ which will be identified with the distributions of the process over the sets in $\mathcal{C}$. The sub-class of Lévy processes will be introduced very much in the spirit of the classical theory, and the starting point will be the existing theory of multidimensional Lévy processes. In particular, for these processes the consistency condition satisfied by the generators has a very simple form.

In Chapter 6 we will examine three examples of $\mathcal{Q}$ -Markov processes which do not have independent increments: the empirical process, a bivariate process and a simple example of a ‘jump’ process. Again, for these processes, the consistency condition satisfied by the generator is much simpler than the one we derived in the general case.

In Chapter 7 we introduce the notions of ‘adapted set’ and ‘optional set’ as natural generalizations of the notions of ‘stopping time’ and ‘optional time’. Using these objects we will examine the strong Markov properties that can be associated to the sharp Markov property and $\mathcal{Q}$ -Markov property.

Finally, at the end of this work we have included two appendix chapters which

contain generally known facts that have been used occasionally for deriving certain results: while Chapter A contains some classical results about conditioning, in Chapter B we have gathered some results about stochastic processes indexed by the real line.

Chapter 2

The Set-Indexed Framework

This chapter introduces the general definitions, properties and assumptions that are used in the theory of set-indexed processes.

2.1 The Indexing Collection

Let T be a compact Hausdorff topological space.

Definition 2.1.1 *A collection $\mathcal{A}$ of compact subsets of T is called an indexing collection if it satisfies the following properties:*

1. *it contains $\emptyset$ and T ;*
2. *$\forall A, B \in \mathcal{A}; A, B \neq \emptyset \Rightarrow A \cap B \neq \emptyset$;*
3. *it is closed under arbitrary intersections; and*
4. *(Separability from above) there exist an increasing sequence of finite sub-semilattices $(\mathcal{A}_n)_n$ of $\mathcal{A}$ which are closed under intersections (and contain $\emptyset$ and T) and a sequence $(g_n)_n$ of functions $g_n : \mathcal{A} \rightarrow \mathcal{A}_n(u)$ (where $\mathcal{A}_n(u)$ is the class of unions of sets in $\mathcal{A}_n$) such that:*
 - (a) *g_n preserves arbitrary intersections and finite unions i.e. if $(A_\alpha)_{\alpha \in \Lambda}$ is an arbitrary collection of sets in $\mathcal{A}$, then $g_n(\cap_{\alpha \in \Lambda} A_\alpha) = \cap_{\alpha \in \Lambda} g_n(A_\alpha)$ and if*

$A_1, \dots, A_k, A'_1, \dots, A'_m \in \mathcal{A}$ are such that $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j$, then
 $\cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j)$;

- (b) $A \subseteq g_n(A)^0$ for any $A \in \mathcal{A}$;
- (c) $g_{n+1}(A) \subseteq g_n(A)$ for any $A \in \mathcal{A}$;
- (d) $A = \cap_n g_n(A)$ for any $A \in \mathcal{A}$;
- (e) $g_n(\emptyset) = \emptyset$.

Comments 2.1.2 1. The sequence $(g_n)_n$ will play the role of the ‘dyadic’ approximations for the indexing sets. They will permit us to approximate each set in $\mathcal{A}$ ‘strictly from above’.

2. If $A \in \mathcal{A}$ and $A \neq T$, then there exists a rank N such that $g_n(A) \neq T$ for all $n \geq N$. Also, if $A \in \mathcal{A}$, $A \neq \emptyset$ then $g_n(A) \neq \emptyset$ for all n . This is true because $A = \cap_n g_n(A)$.

3. Let $\emptyset' := \cap_{A \in \mathcal{A} \setminus \{\emptyset\}} A$ be the minimal set in $\mathcal{A}$. The role played by the set $\emptyset'$ will be similar to that played by 0 in the classical case of processes indexed by $\mathbf{R}_+$. Note that we can also write $\emptyset' = \cap_n \cap_{A \in \mathcal{A}_n \setminus \{\emptyset\}} A$: let x be such that x lies in any set $A \in \mathcal{A}_n \setminus \{\emptyset\}$ for any n ; then x lies in any set $B \in \mathcal{A}_n(u) \setminus \{\emptyset\}$ for any n , in particular, $x \in g_n(A) \forall n, \forall A \in \mathcal{A}$ and hence $x \in \cap_n g_n(A) = A \forall A \in \mathcal{A}$. Denoting $A_n = \cap_{A \in \mathcal{A}_n \setminus \{\emptyset\}} A$, $(A_n)_n$ will be a decreasing sequence of compact nonempty sets with intersection $\emptyset'$. By the finite intersection property, $\emptyset' \neq \emptyset$.

We will always assume that $\emptyset' \in \mathcal{A}'$, if $\mathcal{A}'$ is a finite sub-semilattice of $\mathcal{A}$. (In particular $\emptyset' \in \mathcal{A}_n$ for any n .)

4. Usually the finite sub-semilattices $\mathcal{A}_n$ and the functions g_n are chosen such that

$$g_n(A) = \cap_{D \in \mathcal{A}_n(u), A \subseteq D^0} D.$$

5. The function g_n has the monotone property i.e. if $A, A' \in \mathcal{A}$ are such that $A \subseteq A'$ then $g_n(A) \subseteq g_n(A')$.

(To see this write $A' = A \cup A'$ and then, because g_n preserves finite unions, $g_n(A') = g_n(A) \cup g_n(A') \supseteq g_n(A)$.)

Let $\mathcal{A}(u)$ be the class of all finite unions of sets in $\mathcal{A}$. Note that $\mathcal{A}(u)$ is closed under finite intersections and finite unions. Each function g_n has a unique extension to $\mathcal{A}(u)$ defined by:

$$g_n(B) := \bigcup_{A \in \mathcal{A}, A \subseteq B} g_n(A).$$

Note that if $B = \bigcup_{i=1}^k A_i$ is an arbitrary set in $\mathcal{A}(u)$ with $A_i \in \mathcal{A}$ then $g_n(B) := \bigcup_{i=1}^k g_n(A_i)$.

Lemma 2.1.3 (a) *The functions g_n preserve finite intersections and finite unions on $\mathcal{A}(u)$.*

(b) (Lemma 2.1.2, [41]) *For any $B \in \mathcal{A}(u)$ we have $B = \bigcap_n g_n(B)$, with $g_{n+1}(B) \subseteq g_n(B)$.*

(c) *The functions g_n have the monotone property on $\mathcal{A}(u)$.*

Proof: (a) It is clear that g_n preserves finite unions of sets in $\mathcal{A}(u)$. To show that g_n also preserves finite intersections, let $B_i = \bigcup_{j=1}^{m_i} A_{ij}, i = 1, \dots, k$ be sets in $\mathcal{A}(u)$ with $A_{ij} \in \mathcal{A}$. Then $\bigcap_{i=1}^k B_i = \bigcap_{i=1}^k \bigcup_{j=1}^{m_i} A_{ij} = \bigcup_{j_1=1}^{m_1} \dots \bigcup_{j_k=1}^{m_k} \bigcap_{i=1}^k A_{ij_i}$ and hence $g_n(\bigcap_{i=1}^k B_i) = \bigcup_{j_1=1}^{m_1} \dots \bigcup_{j_k=1}^{m_k} \bigcap_{i=1}^k g_n(A_{ij_i}) = \bigcap_{i=1}^k \bigcup_{j=1}^{m_i} g_n(A_{ij}) = \bigcap_{i=1}^k g_n(B_i)$.

(c) Let $B, B' \in \mathcal{A}(u)$ be such that $B \subseteq B'$. Each set $A \in \mathcal{A}$ which is contained in B will also be contained in B' and by the definition of g_n it follows that $g_n(B) \subseteq g_n(B')$.

□

A very important role in the theory of set-indexed processes will be played by the following collection of sets:

$$\mathcal{C} := \{C = A \setminus B; A \in \mathcal{A}, B \in \mathcal{A}(u)\}.$$

Writing an arbitrary set C in $\mathcal{C}$ as $C = A \setminus (A \cap B)$ we can always assume that a set in $\mathcal{C}$ is of the form $C = A \setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u), B \subseteq A$. The value $X_C = X_A - X_B$ of an (additive !) set-indexed process X over such a set C will play the role of the increment $X_t - X_s, s < t$ from the classical theory of processes indexed by $\mathbf{R}_+$.

Lemma 2.1.4 *Every set $C \in \mathcal{C}$ admits a representation $C = A_C \setminus B_C$ with $A_C \in \mathcal{A}$, $A_C \cap B_C \in \mathcal{A}(u)$ (called ‘the maximal representation’) such that for any other representation $C = A \setminus B$, $A \in \mathcal{A}$, $B \in \mathcal{A}(u)$ we have $A_C \subseteq A$ and $B_C \supseteq B$.*

Proof: Let us consider all the pairs (A_i, B_i) , $i \in I$ with the property that $A_i \in \mathcal{A}$, $B_i \in \mathcal{A}(u)$ and $C = A_i \setminus B_i$. Then $C = A_C \setminus B_C$ where $A_C := \bigcap_{i \in I} A_i$ and $B_C := \bigcup_{i \in I} B_i$. Since $\mathcal{A}$ is closed under arbitrary intersections, $A_C \in \mathcal{A}$. We will prove next that $A_C \cap B_C \in \mathcal{A}(u)$. Note that for every $i \in I$ we have $C = A_i \setminus B_i \supseteq A_C \setminus B_i \supseteq A_C \setminus B_C = C$. Hence $C = A_C \setminus B_i = A_C \setminus (A_C \cap B_i)$ for every $i \in I$ i.e. $A_C \cap B_i$ is the complement of C in A_C for every $i \in I$. Denote $B := A_C \cap B_i$ (regardless of which $i \in I$ we are considering). Clearly $B \in \mathcal{A}(u)$. Since $B = A_C \cap B_i$ for every $i \in I$ we also have $B = A_C \cap \bigcup_{i \in I} B_i = A_C \cap B_C$.

□

In complete analogy with the class $\mathcal{C}$ we introduce its sub-classes

$$\mathcal{C}_n := \{C = A \setminus B; A \in \mathcal{A}_n, B \in \mathcal{A}_n(u)\}.$$

Note that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A}_n \subseteq \mathcal{C}_n$.

Lemma 2.1.5 (a) *Each set in $\mathcal{A}(u)$ (respectively $\mathcal{A}_n(u)$) can be expressed as a union of pairwise disjoint sets in $\mathcal{C}$ (respectively $\mathcal{C}_n$).*

(b) *The collections $\mathcal{C}$ and $\mathcal{C}_n$ are semi-algebras i.e. they are closed under finite intersections and the complement of every set in $\mathcal{C}$ (respectively $\mathcal{C}_n$) can be expressed as the finite union of some pairwise disjoint sets in $\mathcal{C}$ (respectively $\mathcal{C}_n$).*

(c) *The algebra generated by $\mathcal{C}$ (i.e. the minimal algebra containing $\mathcal{C}$) is exactly $\mathcal{C}(u)$, the class of finite unions of sets in $\mathcal{C}$.*

(d) *Each set in $\mathcal{C}(u)$ can be expressed as a finite union of pairwise disjoint sets in $\mathcal{C}$.*

Proof: (a) Let $B = \bigcup_{i=1}^k A_i \in \mathcal{A}(u)$ with $A_i \in \mathcal{A}$. If we denote $C_i := A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for $i = 1, \dots, k$ then $C_1, \dots, C_k$ are pairwise disjoint sets in $\mathcal{C}$ whose union is B .

(b) We will prove the statement only for $\mathcal{C}$, the same arguments working for $\mathcal{C}_n$ as well. If $C = A \setminus B$, $C' = A' \setminus B'$ are two sets in $\mathcal{C}$, with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{A}(u)$

then $C \cap C' = (A \cap A') \setminus (B \cup B')$ is again a set in $\mathcal{C}$. Let $C = A \setminus B$ be an arbitrary set in $\mathcal{C}$ with $A \in \mathcal{A}, B \in \mathcal{A}(u), B \subseteq A$. Then $C^c = A^c \cup B$ and $A^c \cap B = \emptyset$. Note that $A^c = T \setminus A \in \mathcal{C}$ and B can be expressed as a finite union of disjoint sets in $\mathcal{C}$.

(c) It is clear that $\mathcal{C}(u)$ is contained in any algebra which contains $\mathcal{C}$, hence $\mathcal{C}(u)$ is contained in the algebra generated by $\mathcal{C}$. On the other hand, $\mathcal{C}(u)$ itself is an algebra: clearly $\emptyset \in \mathcal{C}(u)$ and $\mathcal{C}(u)$ is closed under finite unions; to see that $\mathcal{C}(u)$ is also closed under complements, let $C = \bigcup_{i=1}^k C_i$ be an arbitrary set in $\mathcal{C}(u)$ with $C_i \in \mathcal{C}$; then, since $\mathcal{C}$ is a semialgebra, $C^c = \bigcap_{i=1}^k C_i^c = \bigcap_{i=1}^k \bigcup_{j=1}^{m_i} C_{ij} = \bigcup_{j_1=1}^{m_1} \cdots \bigcup_{j_k=1}^{m_k} \bigcap_{i=1}^k C_{ij_i} \in \mathcal{C}(u)$.

(d) This is a general property of any algebra of sets.

□

Definition 2.1.6 (a) A representation $B = \bigcup_{i=1}^k A_i$ with $A_i \in \mathcal{A}$ is called **extremal** if $A_i \not\subseteq \bigcup_{j \neq i} A_j$ for each $i = 1, \dots, k$.

(b) A representation $C = A \setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u)$ is called **extremal** if the representation of B is extremal.

Note that extremal representations always exist for sets in $\mathcal{A}(u)$ and $\mathcal{C}$ although they might not be unique. In order to examine the uniqueness of the extremal representations, one has to consider the following geometric assumption.

Assumption 2.1.7 (SHAPE) If $A, A_1, \dots, A_k \in \mathcal{A}$ and $A \subseteq \bigcup_{i=1}^k A_i$ then there exists an index $i = 1, \dots, k$ such that $A \subseteq A_i$.

Comments 2.1.8 1. Note that if SHAPE holds and $A \in \mathcal{A}$ is such that $A = \bigcup_{i=1}^k A_i$ for some $A_1, \dots, A_k \in \mathcal{A}$, then there exists an index $i = 1, \dots, k$ such that $A = A_i$.

2. Without loss of generality we can assume that any finite sub-semilattice of $\mathcal{A}$ (in particular $\mathcal{A}_n$) satisfies SHAPE: if $A_0 \subseteq \bigcup_{i=1}^k A_i$, then $A_0 = \bigcup_{i=1}^k (A_0 \cap A_i)$; replace A_0 in the finite sub-semilattice by $A_0 \cap A_i; i = 1, \dots, k$.

Lemma 2.1.9 (a) If $\mathcal{A}$ satisfies SHAPE then any set in $\mathcal{A}(u)$ has a unique extremal representation.

(b) If $\mathcal{A}$ satisfies SHAPE then any set in $\mathcal{C}$ has a unique extremal representation of the form $C = A \setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u)$ and $B \subseteq A$; moreover, this representation ‘almost’ coincides with the maximal representation in the sense that $A = A_C$ and $B = A_C \cap B_C$.

Proof: (a) Say $B = \cup_{i=1}^k A_i = \cup_{j=1}^m A'_j$, $A_i, A'_j \in \mathcal{A}$ are two extremal representations of the set $B \in \mathcal{A}(u)$. By SHAPE each A_i has to be contained in some A'_{j_i} with $j_i = 1, \dots, m$ and each A'_{j_i} in turn, has to be contained into some A_{l_i} . Hence $A_i \subseteq A_{l_i}$ and because the representation $B = \cup_{i=1}^k A_i$ is extremal, we must have $i = l_i$. This proves that $A_i = A'_{j_i}$ for each $i = 1, \dots, k$. We get $\cup_{i=1}^k A'_{j_i} = \cup_{i=1}^k A_i = \cup_{j=1}^m A'_j$ which forces $\{j_1, \dots, j_k\} = \{1, \dots, m\}$ since the representation $B = \cup_{j=1}^m A'_j$ is extremal. Hence the two representations are the same.

(b) To avoid trivialities assume that $C \neq \emptyset$. Let $C = A \setminus B$ be an arbitrary representation of C with $A \in \mathcal{A}, B \in \mathcal{A}(u), B \subseteq A$. By maximality $B \subseteq B_C$ and hence $A \setminus B_C \subseteq A \setminus B = C = A_C \setminus B_C \subseteq A \setminus B_C$. It follows that $C = A \setminus B_C$. Therefore $A = (A \setminus B_C) \cup (A \cap B_C) = C \cup (A \cap B_C) \subseteq A_C \cup B_C$. Because SHAPE holds and A cannot be contained in B_C (this would imply that $C = A \setminus B_C = \emptyset$) we get $A \subseteq A_C$ and hence $A = A_C$ by the definition of A_C . Finally $B = A_C \cap B_C$ as these sets are both the complement of C in $A = A_C$.

□

Definition 2.1.10 Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ and $A \in \mathcal{A}'$. The set

$$C := A \setminus \cup_{A' \in \mathcal{A}', A \not\subseteq A'} A'$$

is called the **left neighbourhood** of A in $\mathcal{A}'$.

Lemma 2.1.11 Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ and $A \in \mathcal{A}'$. If C is the left neighbourhood of A in $\mathcal{A}'$ then

$$C = A \setminus \cup_{A'' \in \mathcal{A}', A'' \subset A} A''$$

where $\subset$ denotes strict inclusion.

Proof: It is clear that if $A'' \subset A$ then $A \not\subseteq A''$. On the other hand, taking an arbitrary set $A' \in \mathcal{A}'$ such that $A \not\subseteq A'$ then $A'' := A \cap A' \in \mathcal{A}'$ and $A'' \subset A$.

□

Since the finite sub-semilattices of the indexing collection $\mathcal{A}$ will play a very important role in the theory of set-indexed Markov processes, having an ordering on the sets of a finite sub-semilattice proves to be a very useful tool in handling these objects. The ordering that we have in mind will be defined in such a manner that a set is never numbered before any of its subsets.

Definition 2.1.12 *Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ which has n distinct sets. The ordering $\{A_1, A_2, \dots, A_n\}$ of the sets in $\mathcal{A}'$ is called **consistent** if $A_j \subset A_i \Rightarrow j < i$. By convention, we will always assume that $\emptyset' \in \mathcal{A}'$ and we set $A_0 := \emptyset'$.*

It is clear that a consistent ordering of a finite semilattice $\mathcal{A}'$ always exists, but in general it may not be unique. (To prove the existence we proceed inductively: assume that the sets $A_1, \dots, A_i \in \mathcal{A}'$ have already been counted and choose $A_{i+1} \in \mathcal{A}' \setminus \{A_1, \dots, A_i\}$ such that all of its subsets that are in $\mathcal{A}'$ have already been counted i.e. if there exists a set $A \in \mathcal{A}'$ with $A \subset A_{i+1}$ then $A = A_j$ for some $j \leq i$).

Lemma 2.1.13 (a) *Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ and $\{A_0 = \emptyset', A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$. If C_i is the left neighbourhood of A_i , then $C_0 = A_0$ and for each $i = 1, \dots, n$*

$$C_i = A_i \setminus (\cup_{j=1}^{i-1} A_j) \quad \text{and} \quad \cup_{j=1}^i C_j = \cup_{j=1}^i A_j.$$

(b) *If $\{A_0 = \emptyset', A_1, \dots, A_n\}$, $\{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$, C_i, C'_i are the left neighbourhoods of A_i, A'_i and $A_i = A'_{\pi(i)}$ for a permutation π of $\{1, \dots, n\}$ with $\pi(1) = 1$, then*

$$C_i = C'_{\pi(i)} \quad \forall i = 1, \dots, n.$$

Proof: (a) By the definition of the consistent ordering, $\cup_{j=0}^{i-1} A_j \subseteq \cup_{A \in \mathcal{A}', A_i \not\subseteq A} A$ and $\cup_{A \in \mathcal{A}', A \subset A_i} A \subseteq \cup_{j=0}^{i-1} A_j$; the result follows using the two equivalent formulas for the left neighbourhood of the set A_i .

(b) This is clear because both C_i and $C'_{\pi(i)}$ are the left-neighbourhoods of the same set $A_i = A'_{\pi(i)}$.

□

The following result will prove to be an extremely powerful tool when working with finite semilattices for which we have to choose the most ‘appropriate’ consistent ordering.

Proposition 2.1.14 (a) *Let $B_1 \subseteq \dots \subseteq B_m$ be sets in $\mathcal{A}(u)$. There exists a finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and a consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$ such that $B_l = \cup_{j=0}^{i_l} A_j; l = 1, \dots, m$, for some indices $0 < i_1 \leq \dots \leq i_m = n$.*

(b) *Let $\mathcal{A}' \subseteq \mathcal{A}''$ be finite sub-semilattices of $\mathcal{A}$. For any consistent ordering $\{A'_0 = \emptyset', A'_1, \dots, A'_m\}$ of $\mathcal{A}'$ there exists a consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}''$ such that, if $A'_l = A_{i_l}; l = 1, \dots, m$ for some indices $i_1 \leq \dots \leq i_m$, then $\cup_{s=0}^l A'_s = \cup_{j=0}^{i_l} A_j; l = 1, \dots, m$.*

Proof: (a) Let $B_l = \cup_{k=1}^{n_l} A_{kl}, A_{kl} \in \mathcal{A}$ be an extremal representation and $\mathcal{A}'$ be the minimal finite sub-semilattice which contains all the sets $A_{kl}, l = 1, \dots, m; k = 1, \dots, n_l$. Consider first all the sets in $\mathcal{A}'$ that are contained in B_1 and order them consistently. Say these sets are $A_0 = \emptyset', A_1, \dots, A_{i_1}$; then $B_1 = \cup_{j=0}^{i_1} A_j$. Next order consistently all the sets in $\mathcal{A}'$ that are contained in B_2 and have not been numbered yet. If we denote these sets with $A_{i_1+1}, \dots, A_{i_2}$, then the ordering $\{A_0 = \emptyset', A_1, \dots, A_{i_2}\}$ will be consistent: we cannot find a set A_i with $i_1 < i \leq i_2$ which is contained in another set A_j with $j \leq i_1$ because A_j is contained in B_1 and A_i is not. We have $B_2 = \cup_{j=0}^{i_2} A_j$. Continue this procedure until all the sets in $\mathcal{A}'$ have been ordered.

(b) In the first step, order consistently all the sets in $\mathcal{A}''$ which are contained in A'_1 . Say these sets are $A_0 = \emptyset', A_1, \dots, A_{i_1} = A'_1$; then $A_{i_1} = \cup_{j=0}^{i_1} A_j$. Next order consistently all the sets in $\mathcal{A}''$ which are contained in A'_2 and have not been numbered yet. If we denote these sets with $A_{i_1+1}, \dots, A_{i_2} = A'_2$ then the ordering $\{A_0 = \emptyset', A_1, A_2, \dots, A_{i_2}\}$ will be consistent and $A_{i_1} \cup A_{i_2} = \cup_{j=0}^{i_2} A_j$. Continue in the same manner until at the last step all the sets in $\mathcal{A}''$ have been ordered. Finally order consistently the remaining sets in $\mathcal{A}''$.

□

The next result will be used in the proof of Theorem 2.3.3, which states that if the indexing collection $\mathcal{A}$ satisfies SHAPE, then any set-indexed process can be extended in a unique additive manner to the class $\mathcal{C}(u)$.

Proposition 2.1.15 *Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$, $\{A_0 = \emptyset', A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$ and C_i the left neighbourhood of A_i ; $i = 0, \dots, n$. Assume that there exist some indices $j_1 < \dots < j_k$ such that*

$$C := C_{j_1} \cup \dots \cup C_{j_k} \in \mathcal{C}.$$

Then we can find a consistent ordering $\{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ of $\mathcal{A}'$ such that, if C'_i is the left neighbourhood of A'_i , then $C_{j_1} = C'_{m+1}, C_{j_2} = C'_{m+2}, \dots, C_{j_k} = C'_{m+k}$ for some index m (i.e. $C_{j_1}, \dots, C_{j_k}$ become consecutive left neighbourhoods).

Proof: Let $C = A \setminus B, A \in \mathcal{A}, B \in \mathcal{A}(u), B \subseteq A$ be a representation of C such that $A_{j_i} \subseteq A$ for all $i = 1, \dots, k$. Since $A \setminus B = C = C_{j_1} \cup \dots \cup C_{j_k}$ it follows that $B \cap C_{j_i} = \emptyset$ for all $i = 1, \dots, k$. But $C_{j_i} \subseteq A_{j_i}$; hence $A_{j_i} \not\subseteq B$ for any i . This means that we can order the sets of the semilattice $\mathcal{A}'$ that are in B before ordering the sets $A_{j_1}, \dots, A_{j_k}$.

We claim that for each $i = 1, \dots, k$ it is impossible to find a set A_l of the semilattice $\mathcal{A}'$ such that $A_{j_i} \subseteq A_l \subseteq A_{j_{i+1}}$.

Proof of the claim: Assume that this is the case. Note that

$$C_l \subseteq A_l \setminus B$$

(To see this remember that $C_l = A_l \setminus \cup_{A' \in \mathcal{A}', A_l \not\subseteq A'} A'$; let $B = \cup_{i=1}^m A'_i, A'_i \in \mathcal{A}'$ be a representation of B . If $A_l \subseteq A'_i$ for some $i = 1, \dots, m$ then $A_{j_i} \subset A_l \subseteq A'_i \subseteq B$, which is a contradiction. Hence $A_l \not\subseteq A'_i$ for any $i = 1, \dots, m$ and therefore $B = \cup_{i=1}^m A'_i \subseteq \cup_{A' \in \mathcal{A}', A_l \not\subseteq A'} A'$.)

On the other hand $A_l \setminus B \subseteq A \setminus B = C$ because $A_l \subseteq A_{j_{i+1}} \subseteq A$. It follows that $C_l \subseteq C$ i.e. $l \in \{j_1, \dots, j_k\}$. But because the ordering is consistent and $A_{j_i} \subset A_l \subset A_{j_{i+1}}$ we must have $j_i < l < j_{i+1}$ and this is a contradiction. The proof of the claim is complete.

Taking in account these facts we can consider the following consistent ordering of $\mathcal{A}'$: first order consistently the sets of $\mathcal{A}'$ that are not contained in B ; next order *consecutively* the sets $A_{j_1}, \dots, A_{j_k}$; finally order consistently the remaining sets of $\mathcal{A}'$.

□

2.2 Examples of Indexing Collections

In this section we will discuss several examples of topological spaces T endowed with an indexing collection $\mathcal{A}$.

Example 2.2.1 Let $T = [0, 1]^d$ and $\mathcal{A} = \{[0, x]; x \in T\} \cup \{\emptyset\}$ be the collection of the hyper-parallelipeds in T with sides parallel to the axis and having 0 as a vertex (these are called ‘rectangles’). Clearly the intersection of any two ‘rectangles’ is non-empty and $\mathcal{A}$ is closed under arbitrary intersections: $\bigcap_{\alpha \in \Lambda} [0, x_\alpha] = [0, \inf_{\alpha \in \Lambda} x_\alpha]$. (Here $\inf_{\alpha \in \Lambda} x_\alpha = (\inf_{\alpha \in \Lambda} x_{\alpha,i})_{i=1, \dots, d}$ is defined with respect to the partial order induced by the coordinate axis: $x \leq y$ means $x_i \leq y_i, i = 1, \dots, d$.)

To show that $\mathcal{A}$ is an indexing collection it remains to prove that $\mathcal{A}$ satisfies the ‘separability from above’ assumption. Take $\mathcal{A}_n = \{[0, x]; x_i = \frac{k_i}{2^n}, 0 \leq k_i \leq 2^n, i = 1, \dots, d\} \cup \{\emptyset\}$ be the ‘dyadic’ approximation semilattice. Then $(\mathcal{A}_n)_n$ is an increasing sequence of finite sub-semilattices which are closed under arbitrary intersections. Define $g_n : \mathcal{A} \rightarrow \mathcal{A}_n$ by taking $g_n([0, x])$ to be the smallest ‘rectangle’ with dyadic vertices which contains $[0, x]$ in its interior (i.e. $g_n([0, x]) := [0, (\frac{k_1}{2^n}, \dots, \frac{k_d}{2^n})]$ where k_i is chosen such that $\frac{k_i-1}{2^n} \leq x_i < \frac{k_i}{2^n}$ if $x_i < 1$, or $k_i = 2^n$ if $x_i = 1$). Clearly $g_n([0, x]) = T$. By definition set $g_n(\emptyset) := \emptyset$. It can be easily checked that the functions g_n have the desired properties.

Note that $\emptyset' = \{0\}$ and the left neighbourhoods associated to the finite sub-semilattice $\mathcal{A}_n$ are the $\frac{1}{2^n}$ -sided squares in $[0, 1]^d$. In the case of this example the geometric property SHAPE holds: assume that $[0, x] \subseteq \bigcup_{j=1}^m [0, x_j]$; then there must be some index $j = 1, \dots, m$ such that $x \in [0, x_j]$. But this implies $x \leq x_j$ and hence $[0, x] \subseteq [0, x_j]$.

Example 2.2.2 Let $T = [-1, 1]^d$ and $\mathcal{A} = \{[0, x]; x \in T\} \cup \{\emptyset, T\}$ be the collection of all 2^d -dimensional ‘rectangles’ with sides parallel to the axis and having 0 as a vertex (here $[0, x] := \prod_{i=1}^d [0, x_i]$ and we make the convention that if $a, b \in \mathbf{R}$ then $[a, b] = [b, a] = [a \wedge b, a \vee b]$). Clearly $\mathcal{A}$ is closed under arbitrary intersections.

Consider $\mathcal{A}_n = \{[0, x]; x_i = \frac{k_i}{2^n}, -2^n \leq k_i \leq 2^n, i = 1, \dots, d\} \cup \{\emptyset, T\}$. In order to have $[0, x] \subseteq (g_n([0, x]))^0$ we must define the function g_n in such a way that its values will be sets in $\mathcal{A}_n(u)$ and not in $\mathcal{A}_n$ as in the case of the previous example. More precisely, if $|x_i| < 1$ for each $i = 1, \dots, d$ define $g_n([0, x])$ to be the smallest 2^d -dimensional ‘rectangle’ with dyadic vertices of order n which contains $[0, x]$ in its interior (for example, if $d = 2$ and $0 < x_i < 1, i = 1, 2$ with $\frac{k_i-1}{2^n} \leq x_i < \frac{k_i}{2^n}$ we define $g_n([0, x]) := [0, (\frac{k_1}{2^n}, \frac{k_2}{2^n})] \cup [0, (-\frac{1}{2^n}, \frac{k_2}{2^n})] \cup [0, (\frac{k_1}{2^n}, -\frac{1}{2^n})] \cup [0, (-\frac{1}{2^n}, -\frac{1}{2^n})]$). Clearly $g_n(T) = T$. Set by definition $g_n(\emptyset) := \emptyset$.

Note that in this case we have again $\emptyset' = \{0\}$, the left neighbourhoods associated to the finite sub-semilattice $\mathcal{A}_n$ are the $\frac{1}{2^n}$ -sided squares in $[-1, 1]^d$ and assumption SHAPE holds too.

Example 2.2.3 Let $T = [0, 1]^d$ and $\mathcal{A}$ be the collection of all compact lower layers of T together with $\{\emptyset\}$. (Recall that a set $A \subseteq \mathbf{R}^d$ is called a **lower layer** if $[0, z] \subseteq A$ whenever $z \in A$.) It is not difficult to observe that $\mathcal{A}$ is closed under arbitrary intersections and that in this case $\mathcal{A}(u) = \mathcal{A}$. In order to be able to approximate from above a lower layer we will have to consider $\mathcal{A}_n$ as the collection of finite unions of ‘rectangles’ having all the vertices with dyadic coordinates of order n . We have $\mathcal{A}_n(u) = \mathcal{A}_n$. Define $g_n(A) := \cap_{B \in \mathcal{A}_n, A \subseteq B} B$.

Again $\emptyset' = \{0\}$ but in this case the assumption SHAPE does not hold: we could very easily think of three lower layers A_1, A_2, A_3 such that $A_1 \subseteq A_2 \cup A_3$ but $A_1 \not\subseteq A_2, A_1 \not\subseteq A_3$. Note that in this case $\mathcal{A}$ is a lattice with $\vee = \cup$ and $\wedge = \cap$.

Example 2.2.4 Let $T = [-1, 1]^d$ and $\mathcal{A}$ be the collection of all compact lower layers contained in T , together with $\{\emptyset\}$. This example is completely similar to the Example 2.2.3.

Examples 2.2.5 Let $T = \overline{B(0, t_0)}$ (compact ball in $\mathbf{R}^3$) and $\mathcal{A} = \{A_{R,t}; R = [a, b] \times$

$[c, d]', 0 \leq a < b < 2\pi, -\pi \leq c < d \leq \pi, t \in [0, t_0]\}$, where the set

$$A_{R,t} := \{(\tau \cos \theta \cos \tau, \tau \sin \theta \cos \tau, \tau \sin \tau); \theta \in [a, b], \tau \in [c, d], \tau \in [0, t]\}$$

can be interpreted as the history of the region

$$R := '[a, b] \times [c, d]' = \{(\cos \theta \cos \tau, \sin \theta \cos \tau, \sin \tau); \theta \in [a, b], \tau \in [c, d]\}$$

of the Earth from the beginning until time t (here θ represents the longitude of a generic point in the region R , while τ is the latitude). Hence, the collection $\mathcal{A}$ can be identified with the history of the world until time t_0 .

This example is similar to Example 2.2.1.

2.3 Set-Indexed Processes

The basic definitions and properties associated with set-indexed processes are discussed in this section, with special emphasis on the additivity property.

Definition 2.3.1 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ has a unique additive extension to:*

(a) **the class $\mathcal{A}(u)$** if $\forall A_1, \dots, A_n, A'_1, \dots, A'_m \in \mathcal{A}$ with $\cup_{i=1}^n A_i = \cup_{j=1}^m A'_j$

$$\begin{aligned} \sum_{i=1}^n X_{A_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{A_{i_1} \cap A_{i_2}} + \dots + (-1)^{n+1} X_{A_1 \cap \dots \cap A_n} = \\ \sum_{j=1}^m X_{A'_j} - \sum_{1 \leq j_1 < j_2 \leq m} X_{A'_{j_1} \cap A'_{j_2}} + \dots + (-1)^{m+1} X_{A'_1 \cap \dots \cap A'_m} \quad a.s. \end{aligned}$$

(by the inclusion-exclusion principle);

(b) **the class $\mathcal{C}$** if $\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{A}(u)$ with $A \setminus B = A' \setminus B'$

$$X_A - X_{A \cap B} = X_{A'} - X_{A' \cap B'} \quad a.s.$$

(c) **the class $\mathcal{C}(u)$** if $\forall C_1, \dots, C_n, C'_1, \dots, C'_m \in \mathcal{C}$ with $\cup_{i=1}^n C_i = \cup_{j=1}^m C'_j$

$$\begin{aligned} \sum_{i=1}^n X_{C_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{C_{i_1} \cap C_{i_2}} + \dots + (-1)^{n+1} X_{C_1 \cap \dots \cap C_n} = \\ \sum_{j=1}^m X_{C'_j} - \sum_{1 \leq j_1 < j_2 \leq m} X_{C'_{j_1} \cap C'_{j_2}} + \dots + (-1)^{m+1} X_{C'_1 \cap \dots \cap C'_m} \quad a.s. \end{aligned}$$

A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ is said to have a unique additive extension to $\mathcal{A}(u), \mathcal{C}, \mathcal{C}(u)$ if the previous respective relationships hold (everywhere this time).

Lemma 2.3.2 *Let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process.*

(a) *If for every $A, A_1, \dots, A_n \in \mathcal{A}$ with $A = \cup_{i=1}^n A_i$ we have*

$$X_A = \sum_{i=1}^n X_{A_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{A_{i_1} \cap A_{i_2}} + \dots + (-1)^{n+1} X_{A_1 \cap \dots \cap A_n} \quad \text{a.s.} \quad (1)$$

then X has a unique additive extension to $\mathcal{A}(u)$.

(b) *If X has a unique additive extension to $\mathcal{C}$ and for every $C, C_1, \dots, C_n \in \mathcal{C}$ with $C = \cup_{i=1}^n C_i$ we have*

$$X_C = \sum_{i=1}^n X_{C_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{C_{i_1} \cap C_{i_2}} + \dots + (-1)^{n+1} X_{C_1 \cap \dots \cap C_n} \quad \text{a.s.} \quad (2)$$

then X has a unique additive extension to $\mathcal{C}(u)$.

Proof: (a) Let $A_1, \dots, A_k, A'_1, \dots, A'_m \in \mathcal{A}$ be such that $\cup_{i=1}^n A_i = \cup_{j=1}^m A'_j$. We want to prove that

$$\sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} X_{A_{i_1} \cap \dots \cap A_{i_k}} = \sum_{l=1}^m (-1)^{l+1} \sum_{1 \leq j_1 < \dots < j_l \leq m} X_{A'_{j_1} \cap \dots \cap A'_{j_l}} \quad \text{a.s.}$$

Because $A_{i_1} \cap \dots \cap A_{i_k} = \cup_{j=1}^m (A_{i_1} \cap \dots \cap A_{i_k} \cap A'_j)$ (where each of the sets $A_{i_1} \cap \dots \cap A_{i_k} \cap A'_j$ is in $\mathcal{A}$ because $\mathcal{A}$ is closed under intersections) and X is additive on $\mathcal{A}$ we have

$$X_{A_{i_1} \cap \dots \cap A_{i_k}} = \sum_{l=1}^m (-1)^{l+1} \sum_{1 \leq j_1 < \dots < j_l \leq m} X_{A_{i_1} \cap \dots \cap A_{i_k} \cap A'_{j_1} \cap \dots \cap A'_{j_l}} \quad \text{a.s.}$$

Hence

$$\begin{aligned} & \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} X_{A_{i_1} \cap \dots \cap A_{i_k}} = \\ & \sum_{k=1}^n (-1)^{k+1} \sum_{l=1}^m (-1)^{l+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{1 \leq j_1 < \dots < j_l \leq m} X_{A_{i_1} \cap \dots \cap A_{i_k} \cap A'_{j_1} \cap \dots \cap A'_{j_l}} \quad \text{a.s.} \end{aligned}$$

and the result follows since the expression on the right hand side is symmetric in A_i 's and A'_j 's.

(b) The argument is identical to the one used for proving (a), by replacing the sets in $\mathcal{A}$ with sets in $\mathcal{C}$.

□

Theorem 2.3.3 *If the indexing collection $\mathcal{A}$ satisfies SHAPE, then any set-indexed process has a unique additive extension to $\mathcal{C}(u)$. Moreover, in this case, the additivity relation holds everywhere.*

Proof: Let $X := (X_A)_{A \in \mathcal{A}}$ be an arbitrary set-indexed process. We will proceed in three steps.

(a) We will prove first that relation (1) holds. By Lemma 2.3.2.(a) it will follow that the process X has a unique additive extension to $\mathcal{A}(u)$.

Let $A = \cup_{i=1}^n A_i$ with $A, A_1, \dots, A_n \in \mathcal{A}$. Since SHAPE holds, there exists an index j such that $A = A_j$. Reindexing the sets $A_1, \dots, A_n$ we can assume that $A = A_n$. Then each A_i with $i < n$ will be contained in A_n and hence the intersection $A_i \cap A_n$ will be simply A_i . In other words the sets $A_{i_1} \cap \dots \cap A_{i_k}$ with $i_j < n$ remain unchanged when intersected with A_n . Hence

$$\begin{aligned}
 & \sum_{i=1}^n X_{A_i} - \sum_{1 \leq i_1 < i_2 \leq n} X_{A_{i_1} \cap A_{i_2}} + \dots + (-1)^{n+1} X_{\cap_{i=1}^n A_i} = \\
 & \left(\sum_{i=1}^{n-1} X_{A_i} + X_{A_n} \right) - \left(\sum_{1 \leq i_1 < i_2 \leq n-1} X_{A_{i_1} \cap A_{i_2}} + \sum_{i=1}^{n-1} X_{A_i \cap A_n} \right) + \\
 & \left(\sum_{1 \leq i_1 < i_2 < i_3 \leq n-1} X_{A_{i_1} \cap A_{i_2} \cap A_{i_3}} + \sum_{1 \leq i_1 < i_2 \leq n-1} X_{A_{i_1} \cap A_{i_2} \cap A_n} \right) + \dots + \\
 & (-1)^{n+1} X_{\cap_{i=1}^{n-1} A_i \cap A_n} = \\
 & \left(\sum_{i=1}^{n-1} X_{A_i} + X_{A_n} \right) - \left(\sum_{1 \leq i_1 < i_2 \leq n-1} X_{A_{i_1} \cap A_{i_2}} + \sum_{i=1}^{n-1} X_{A_i} \right) + \\
 & \left(\sum_{1 \leq i_1 < i_2 < i_3 \leq n-1} X_{A_{i_1} \cap A_{i_2} \cap A_{i_3}} + \sum_{1 \leq i_1 < i_2 \leq n-1} X_{A_{i_1} \cap A_{i_2}} \right) + \dots + \\
 & (-1)^{n+1} X_{\cap_{i=1}^{n-1} A_i} = X_{A_n}.
 \end{aligned}$$

(b) Since SHAPE holds, any set in $\mathcal{C}$ has a unique extremal representation of the form $C = A \setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u), B \subseteq A$ (according to Lemma 2.1.9, (b)). Hence we can extend the process X to $\mathcal{C}$ in a unique way, namely by defining $X_C := X_A - X_B$.

(c) We will prove that relation (2) holds. By Lemma 2.3.2.(b) it will follow that the process X has a unique additive extension to $\mathcal{C}(u)$.

Let $C = \cup_{i=1}^n C_i$ with $C, C_1, \dots, C_n \in \mathcal{C}$. Without loss of generality we can assume that $C \neq \emptyset$. We will proceed in two sub-steps.

(c1) Assume that $C_1, \dots, C_n$ are consecutive left neighbourhoods in a finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$. Let $C = A_C \setminus B_C$ be the maximal representation of C and $B'_C = A_C \cap B_C$. Say $\{A_0 = \emptyset', A'_1, \dots, A'_m\}$ is a consistent ordering of $\mathcal{A}'$ such that if C'_i is the left neighbourhood of the set A'_i , then $B'_C = \cup_{i=1}^p C'_i$ and $C_1 = C'_{p+1}, C_2 = C'_{p+2}, \dots, C_n = C'_{p+n}$.

Then $A_C = B'_C \cup C = B'_C \cup \cup_{i=p+1}^{p+n} C'_i = \cup_{i=1}^{p+n} C'_i = \cup_{i=1}^{p+n} A'_i = B'_C \cup \cup_{i=p+1}^{p+n} A'_i$. Clearly $A_C \not\subseteq B'_C$. Also $A_C \not\subseteq A'_i$ for any $p+1 \leq i < p+n$ (if this would be the case, then $\cup_{i=p+1}^{p+n} C'_i = C = A_C \setminus B'_C \subseteq A'_i \setminus B'_C = (A'_i \cup B'_C) \setminus B'_C \subseteq (B'_C \cup \cup_{j=p+1}^i A'_j) \setminus B'_C = (\cup_{j=1}^i C'_j) \setminus (\cup_{j=1}^p C'_j) = \cup_{j=p+1}^i C'_j$, which is a contradiction). Using SHAPE and Comment 2.1.8.1 we can conclude that $A_C = A'_{p+n}$. It follows that $A'_i \subseteq A'_{p+n}$ for every $1 \leq i < p+n$. Using the fact that each $C'_i = A'_i \setminus \cup_{j=1}^{i-1} (A'_i \cap A'_j)$ and knowing that X has a unique additive extension to $\mathcal{A}(u)$ and $\mathcal{C}$ we get

$$\begin{aligned} \sum_{i=1}^n X_{C_i} &= \sum_{i=p+1}^{p+n} X_{C'_i} = \sum_{i=p+1}^{p+n} X_{A'_i} - \sum_{i=p+1}^{p+n} \sum_{j=1}^{i-1} X_{A'_i \cap A'_j} + \\ &+ \sum_{i=p+1}^{p+n} \sum_{1 \leq j_1 < j_2 < i} X_{A'_i \cap A'_{j_1} \cap A'_{j_2}} - \dots + \sum_{i=p+1}^{p+n} (-1)^{i+1} X_{A'_i \cap \dots \cap A'_1} \end{aligned}$$

We start processing the terms which compose the last member of the previous equality. Write the first sum as $\sum_{i=p+1}^{p+n-1} X_{A'_i} + X_{A'_{p+n}}$. Because $A'_j \subseteq A'_{p+n}$ for $1 \leq j < p+n$ we can write the second sum as

$$\begin{aligned} \sum_{i=p+1}^{p+n-1} \sum_{j=1}^{i-1} X_{A'_i \cap A'_j} &+ \sum_{j=1}^{p+n-1} X_{A'_{p+n} \cap A'_j} = \\ \sum_{i=p+1}^{p+n-1} \sum_{j=1}^{i-1} X_{A'_i \cap A'_j} &+ \sum_{j=1}^p X_{A'_j} + \sum_{j=p+1}^{p+n-1} X_{A'_j} \end{aligned}$$

and the first term of the first sum gets cancelled by the last term of the second sum. From the first sum we are left with $X_{A'_{p+n}}$. Next, because $A'_{j_1} \cap A'_{j_2} \subseteq A'_{p+n}$ for any

$1 \leq j_1 < j_2 < p+n$ we can write the third sum as

$$\begin{aligned}
 & \sum_{i=p+1}^{p+n-1} \sum_{1 \leq j_1 < j_2 \leq i-1} X_{A'_i \cap A'_{j_1} \cap A'_{j_2}} + \sum_{1 \leq j_1 < j_2 \leq p+n-1} X_{A'_{p+n} \cap A'_{j_1} \cap A'_{j_2}} = \\
 & \sum_{i=p+1}^{p+n-1} \sum_{1 \leq j_1 < j_2 \leq i-1} X_{A'_i \cap A'_{j_1} \cap A'_{j_2}} + \sum_{j_2=2}^{p+n-1} \sum_{j_1=1}^{j_2-1} X_{A'_{j_1} \cap A'_{j_2}} = \\
 & \sum_{i=p+1}^{p+n-1} \sum_{1 \leq j_1 < j_2 \leq i-1} X_{A'_i \cap A'_{j_1} \cap A'_{j_2}} + \sum_{j_2=2}^p \sum_{j_1=1}^{j_2-1} X_{A'_{j_1} \cap A'_{j_2}} + \\
 & \sum_{j_2=p+1}^{p+n-1} \sum_{j_1=1}^{j_2-1} X_{A'_{j_1} \cap A'_{j_2}}.
 \end{aligned}$$

The first term of the second sum gets cancelled by the last term of the third sum. From the second sum we are left with $\sum_{j=1}^p X_{A'_j}$. Continuing inductively we can conclude that $\sum_{i=1}^n X_{C_i}$ is equal to

$$\begin{aligned}
 X_{A'_{p+n}} & - \sum_{j=1}^p X_{A'_j} + \sum_{1 \leq j_1 < j_2 \leq p} X_{A'_{j_1} \cap A'_{j_2}} - \dots + (-1)^p X_{A'_1 \cap \dots \cap A'_p} = \\
 X_{A'_{p+n}} & - X_{\cup_{j=1}^p A'_j} = X_{A_C} - X_{B'_C} = X_C
 \end{aligned}$$

using the fact that X has a unique additive extension to $\mathcal{A}(u)$ and $\mathcal{C}$.

(c2) Assume that $C_1, \dots, C_n$ are arbitrary, not necessarily left neighbourhoods. Let $C_i = A_i \setminus B_i$, $B_i = \cup_{j=1}^{m_i} A_{ij}$; $i = 1, \dots, n$ be the extremal representations of C_i with $A_i, A_{ij} \in \mathcal{A}$, $B_i \subseteq A_i$, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij} and $\{A'_0 = \emptyset', A'_1, \dots, A'_m\}$ a consistent ordering of $\mathcal{A}'$. Let C'_i be the left neighbourhood of the set A'_i .

Then each $C_i = \cup_{j \in J_i} C'_j$ for some $J_i \subseteq \{1, \dots, m\}$ and hence each $C_{i_1} \cap \dots \cap C_{i_k} = \cup_{j \in J_{i_1, \dots, i_k}} C'_j$ with $J_{i_1, \dots, i_k} = J_{i_1} \cap \dots \cap J_{i_k}$.

Fix a k -tuple $1 \leq i_1 < \dots < i_k \leq n$. Using Proposition 2.1.15, we can find a consistent ordering of $\mathcal{A}'$ such that the set $C_{i_1} \cap \dots \cap C_{i_k} \in \mathcal{C}$ becomes the union of some consecutive left neighbourhoods of $\mathcal{A}'$.

Hence $X_{C_{i_1} \cap \dots \cap C_{i_k}} = \sum_{j \in J_{i_1, \dots, i_k}} X_{C'_j}$ by the previous step (c1). We have

$$\sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} X_{C_{i_1} \cap \dots \cap C_{i_k}} = \sum_{k=1}^n (-1)^{k+1} \sum_{j \in J_{i_1, \dots, i_k}} X_{C'_j}$$

which is the same thing as $\sum_{j \in J} X_{C'_j}$ with $J = \cup_{i=1}^m J_i$.

On the other hand, $C = \cup_{i=1}^n C_i = \cup_{i=1}^n \cup_{j \in J_i} C'_j = \cup_{j \in J} C'_j$ and using again Proposition 2.1.15 we can find a consistent ordering of $\mathcal{A}'$ such that the sets $C'_j; j \in J$ become consecutive left neighbourhoods ; hence $X_C = \sum_{j \in J} X_{C'_j}$ by step (c1), and the conclusion follows.

□

Important Note: Throughout this work, all the equalities between the values of a process which has an additive extension to $\mathcal{C}(u)$ should be understood as almost sure equalities, even if this will not be usually mentioned. The only instance when these equalities actually hold everywhere is when the indexing collection $\mathcal{A}$ satisfies assumption SHAPE.

We will introduce now three different types of equivalence relations between set-indexed processes. The definitions are not specific to the set-indexed case, they are encountered in the context of processes with arbitrary index set; we include them only for the sake of completeness.

Definition 2.3.4 *Let $X := (X_A)_{A \in \mathcal{A}}$ and $Y := (Y_A)_{A \in \mathcal{A}}$ be two set-indexed processes. We say that*

- (a) *X and Y are **indistinguishable** if $P(X_A = Y_A \ \forall A \in \mathcal{A}) = 1$.*
- (b) *X and Y are **modifications** of each other if $P(X_A = Y_A) = 1 \ \forall A \in \mathcal{A}$.*
- (c) *X and Y are **versions** of each other if they have the same finite dimensional distributions.*

Indistinguishability is the strongest form of equivalence among processes, while the fact that they are versions of each others is the weakest; the fact that they are modifications of each others lies between the two forms of equivalence.

A sufficient condition for two processes X and Y to be versions of each other is that they have the same finite dimensional distributions over all finite sub-semilattices of $\mathcal{A}$. To see this, note that given any finite sub-collection of $\mathcal{A}$ there exists a (unique) minimal finite sub-semilattice containing it.

In what follows we will examine the information structure that can be associated to a set-indexed process.

Definition 2.3.5 Let $(\Omega, \mathcal{F})$ be a measurable space. A collection $(\mathcal{F}_A)_{A \in \mathcal{A}}$ of sub- σ -fields of $\mathcal{F}$ is called a **filtration** if $\mathcal{F}_A \subseteq \mathcal{F}_{A'}$ whenever $A, A' \in \mathcal{A}, A \subseteq A'$.

We will always consider filtrations which are *complete* i.e. $\{A \in \mathcal{F}; P(A) = 0\} \subseteq \mathcal{F}_{\emptyset'}$.

Given a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ we can extend it to a filtration indexed by $\mathcal{A}(u)$ by setting

$$\mathcal{F}_B := \bigvee_{A \in \mathcal{A}(B)} \mathcal{F}_A, B \in \mathcal{A}(u) \quad (3)$$

where $\mathcal{A}(B) := \{A \in \mathcal{A}; A \subseteq B\}$. Then $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ is also a *filtration* i.e. if $B_1, B_2 \in \mathcal{A}(u), B_1 \subseteq B_2$ then $\mathcal{F}_{B_1} \subseteq \mathcal{F}_{B_2}$. (This is true because $\mathcal{A}(B_1) \subseteq \mathcal{A}(B_2)$.)

Definition 2.3.6 A filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ is **outer-continuous** if for every $B \in \mathcal{A}(u)$ we have $\mathcal{F}_B = \cap_n \mathcal{F}_{B_n}$ for any decreasing sequence $(B_n)_{n \geq 1}$ in $\mathcal{A}(u)$ such that $B = \cap_n B_n$.

Note that even if we assume that the original $\mathcal{A}$ -indexed filtration is outer-continuous (i.e. for every $A \in \mathcal{A}$ we have $\mathcal{F}_A = \cap_n \mathcal{F}_{A_n}$ for any decreasing sequence $(A_n)_{n \geq 1}$ in $\mathcal{A}$ such that $A = \cap_n A_n$), the extended filtration to $\mathcal{A}(u)$ may not be outer-continuous. However, we do have the following result.

Lemma 2.3.7 (Lemma 1.4.1, [41]) Let $(\mathcal{F}_A)_{A \in \mathcal{A}}$ be an outer-continuous filtration, $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ the extended filtration to $\mathcal{A}(u)$ and define

$$\mathcal{F}_B^r := \cap_n \mathcal{F}_{g_n(B)}, B \in \mathcal{A}(u).$$

The family $(\mathcal{F}_B^r)_{B \in \mathcal{A}(u)}$ is an outer-continuous filtration.

Definition 2.3.8 A set-indexed process X is called **adapted** with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if X_A is $\mathcal{F}_A$ -measurable for every $A \in \mathcal{A}$.

Lemma 2.3.9 If $X := (X_A)_{A \in \mathcal{A}}$ is a set-indexed process which is adapted with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$, then X_B is $\mathcal{F}_B$ -measurable for any $B \in \mathcal{A}(u)$.

Proof: Let $B = \cup_{i=1}^n A_i$ with $A_i \in \mathcal{A}$. Since X has a unique additive extension to $\mathcal{A}(u)$, we have $X_B = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} X_{A_{i_1} \cap \dots \cap A_{i_k}}$. The result follows since all the variables $X_{A_{i_1} \cap \dots \cap A_{i_k}}$ are $\mathcal{F}_B$ -measurable.

□

At times it is useful to consider filtrations satisfying certain orthogonality properties.

Definition 2.3.10 A filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$

(a) satisfies condition (CI) if $\forall B_1, B_2 \in \mathcal{A}(u)$ $\mathcal{F}_{B_1} \perp \mathcal{F}_{B_2} | \mathcal{F}_{B_1 \cap B_2}$

(b) satisfies condition (CO) if $\forall B_1, B_2 \in \mathcal{A}(u)$, $\mathcal{F}_{B_1} \perp \mathcal{F}_{B_2} | \mathcal{F}_{B_1 \cap B_2}$

((CI) and (CO) stand for ‘conditional independence’, respectively ‘conditional orthogonality’.)

Lemma 2.3.11 If a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ satisfies condition (CI), then

$$\mathcal{F}_{B_1} \cap \mathcal{F}_{B_2} = \mathcal{F}_{B_1 \cap B_2} \quad \forall B_1, B_2 \in \mathcal{A}(u).$$

In particular condition (CI) implies condition (CO).

Proof: Clearly $\mathcal{F}_{B_1 \cap B_2} \subseteq \mathcal{F}_{B_i}$, $i = 1, 2$. Conversely, take $F \in \mathcal{F}_{B_1} \cap \mathcal{F}_{B_2}$; then on one hand $\mathbf{1}_F = E[\mathbf{1}_F | \mathcal{F}_{B_1}]$ since $\mathbf{1}_F$ is $\mathcal{F}_{B_1}$ -measurable and on the other hand, by the (CI) property, since $\mathbf{1}_F$ is also $\mathcal{F}_{B_2}$ -measurable we have $\mathbf{1}_F = E[\mathbf{1}_F | \mathcal{F}_{B_1}] = E[\mathbf{1}_F | \mathcal{F}_{B_1 \cap B_2}]$; hence $\mathbf{1}_F$ is $\mathcal{F}_{B_1 \cap B_2}$ -measurable i.e. $F \in \mathcal{F}_{B_1 \cap B_2}$.

□

For a fixed set-indexed process $X := (X_A)_{A \in \mathcal{A}}$, the minimal filtration with respect to which X is adapted is given by $\mathcal{F}_A := \sigma(\{X_{A'}; A' \in \mathcal{A}(A)\})$. Note that if we extend this filtration to $\mathcal{A}(u)$ by the formula (3), then we also have $\mathcal{F}_B := \sigma(\{X_A; A \in \mathcal{A}(B)\})$, $\forall B \in \mathcal{A}(u)$. Moreover,

$$\mathcal{F}_{B_1 \cup B_2} = \mathcal{F}_{B_1} \vee \mathcal{F}_{B_2} \quad \forall B_1, B_2 \in \mathcal{A}(u).$$

(Clearly $\mathcal{F}_{B_i} \subseteq \mathcal{F}_{B_1 \cup B_2}$, $i = 1, 2$; conversely, if $A \in \mathcal{A}(B_1 \cup B_2)$ then $A = (A \cap B_1) \cup (A \cap B_2)$ and $X_A = X_{A \cap B_1} + X_{A \cap B_2} - X_{A \cap B_1 \cap B_2}$ (a.s.) is $\mathcal{F}_{B_1} \vee \mathcal{F}_{B_2}$ -measurable.)

In analogy with the minimal filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ we define the following σ -fields, for an arbitrary set $B \in \mathcal{A}(u)$:

$$\begin{aligned}\mathcal{F}_{\partial B} &:= \sigma(\{X_A; A \in \mathcal{A}(\partial B)\}) \\ \mathcal{F}_{B^c} &:= \sigma(\{X_A; A \in \mathcal{A}(B^c)\})\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}(\partial B) &:= \{A \in \mathcal{A}; A \subseteq B, A \not\subseteq B^0\} \\ \mathcal{A}(B^c) &:= \{A \in \mathcal{A}; A \not\subseteq B\}.\end{aligned}$$

In what follows we will consider two types of Markov properties for set-indexed processes. These definitions have been first introduced in [39]; we just extend them for arbitrary filtrations.

Definition 2.3.12 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called*

(a) **sharp Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$** *if it is adapted and*
 $\forall B \in \mathcal{A}(u)$

$$\mathcal{F}_B \perp \mathcal{F}_{B^c} | \mathcal{F}_{\partial B};$$

(b) **Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$** *if it is adapted and $\forall B \in \mathcal{A}(u), \forall A_i \in \mathcal{A}, A_i \not\subseteq B, i = 1, \dots, k$ and for every bounded measurable function $h : \mathbb{R}^k \rightarrow \mathbb{R}$*

$$E[h(X_{A_1}, \dots, X_{A_k}) | \mathcal{F}_B] = E[h(X_{A_1}, \dots, X_{A_k}) | \mathcal{F}_{\partial B} \cap \mathcal{F}_{\cup_{i=1}^k A_i}].$$

*The process X is called simply **sharp Markov** (respectively **Markov**) if it is sharp Markov (respectively Markov) with respect to its minimal filtration.*

Note that on the real line, the sharp Markov property is equivalent to the usual Markov property.

From the definition it is clear that any Markov process with respect to a certain filtration is sharp Markov with respect to the same filtration. Conversely, we have the following result.

Theorem 2.3.13 (Theorem 3.1, [39]) *If the filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ satisfies condition (CO), then any sharp Markov process with respect to $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ is also a Markov process with respect to $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$.*

The definition of a set-indexed process with independent increments is similar to the definition of such a process on the plane; the role of the rectangles on the plane is taken by the sets in $\mathcal{C}$.

Definition 2.3.14 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is said to have independent increments if $X_\emptyset = X_{\emptyset'} = 0$ a.s. and $X_{C_1}, \dots, X_{C_k}$ are independent whenever $C_1, \dots, C_k \in \mathcal{C}$ are pairwise disjoint.*

Note that if X is a process with independent increments then $X_{C_1}, \dots, X_{C_k}$ are independent whenever $C_1, \dots, C_k \in \mathcal{C}(u)$ are pairwise disjoint. (Write each C_i as the disjoint union of some sets $C_{ij}, j = 1, \dots, m_i$ in $\mathcal{C}$; hence X_{C_i} is equal to $\sum_{j=1}^{m_i} X_{C_{ij}}$ and we use the associativity of independence.)

We conclude this section with a two general results from the theory of set-indexed processes.

Lemma 2.3.15 (Lemma 2.4, [39]) *For any $A \in \mathcal{A}, B \in \mathcal{A}(u)$ with $A \not\subseteq B$, $X_{A \cap B}$ can be expressed as a linear combination of terms of the form X_D with $D \in \mathcal{A}(\partial B) \cap \mathcal{A}(A)$; hence $X_{A \cap B}$ is $\mathcal{F}_{\partial B} \cap \mathcal{F}_A$ -measurable.*

Theorem 2.3.16 (Theorem 2.3, [39]) *Any process with independent increments is Markov.*

2.4 Sample Path Properties. Flows

In this section we will introduce various notions of regularity for the sample paths of a set-indexed process.

Definition 2.4.1 *A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ is called*

- (a) **monotone outer-continuous** if for any decreasing sequence $(A_n)_n \subseteq \mathcal{A}$ with $A := \bigcap_n A_n$, we have $\lim_{n \rightarrow \infty} x(A_n) = x(A)$;
- (b) **monotone inner-continuous** if for any increasing sequence $(A_n)_n \subseteq \mathcal{A}$ with $A := \overline{\bigcup_n A_n} \in \mathcal{A}$, we have $\lim_{n \rightarrow \infty} x(A_n) = x(A)$;

A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called **monotone outer-continuous** (respectively **monotone inner-continuous**) if all its sample paths are monotone outer-continuous (respectively monotone inner-continuous).

A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ with a unique additive extension to $\mathcal{C}(u)$ is said to be monotone outer-continuous (respectively monotone inner-continuous) ‘on $\mathcal{C}(u)$ ’ if for any decreasing (respectively increasing) sequence $(C_n)_n \subseteq \mathcal{C}(u)$ such that $C := \bigcap_n C_n \in \mathcal{C}(u)$ (respectively $C := \overline{\bigcup_n C_n} \in \mathcal{C}(u)$), we have $\lim_n x(C_n) = x(C)$.

Definition 2.4.2 A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ with a unique additive extension to $\mathcal{C}(u)$ is called **increasing** if $x(\emptyset) = 0$ and $x(C) \geq 0$ for every $C \in \mathcal{C}$.

Note that a monotone outer-continuous function which has a unique additive extension to $\mathcal{C}(u)$ may not be monotone outer-continuous on $\mathcal{C}(u)$. However, we do have the following result.

Proposition 2.4.3 (Proposition 1.4.8, [41]) If $x : \mathcal{A} \rightarrow \mathbf{R}$ is a monotone outer-continuous increasing set-function, then x is monotone outer-continuous on $\mathcal{C}(u)$ at $\emptyset$, i.e. for every decreasing sequence $(C_n)_n \subseteq \mathcal{C}(u)$ with $\bigcap_n C_n = \emptyset$, we have $\lim_{n \rightarrow \infty} x(C_n) = 0$. Hence, any increasing monotone outer-continuous set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ is countably additive on $\mathcal{C}(u)$ and therefore, it has a unique additive extension to a measure on $\sigma(\mathcal{A})$.

Let us now formally introduce an object named ‘flow’, which will prove to be extremely useful in exploiting the Markov property considered throughout this work.

Definition 2.4.4 Any map $f : [0, a] \rightarrow \mathcal{A}(u)$ which is increasing with respect to the partial order induced by the set-inclusion is called a **flow**. A flow f is called

- (a) **right-continuous** (respectively **left-continuous**) if for any $t \in [0, a]$ and for any decreasing (respectively increasing) sequence $(t_n)_n$ with $\lim_{n \rightarrow \infty} t_n = t$ we have $f(t) = \cap_n f(t_n)$ (respectively $f(t) = \overline{\cup_n f(t_n)}$);
- (b) **continuous** if it is right-continuous and left-continuous;
- (c) **simple** if it is continuous and there exists a partition $0 = t_0 < t_1 < \dots < t_n = a$ and some $\mathcal{A}$ -valued (continuous) flows $f_i : [t_i, t_{i+1}] \rightarrow \mathcal{A}$, $i = 0, \dots, n-1$ such that $f(0) = \emptyset'$ and

$$f(t) = \cup_{j=0}^i f_j(t_j) \cup f_i(t), t \in [t_i, t_{i+1}], i = 0, \dots, n.$$

Comment 2.4.5 If $f : [0, a] \rightarrow \mathcal{A}(u)$ is a right-continuous (respectively left-continuous) flow and $x : \mathcal{A} \rightarrow \mathbf{R}$ is a set-function with a unique additive extension to $\mathcal{C}(u)$ which is monotone outer-continuous (respectively monotone inner-continuous) on $\mathcal{C}(u)$, then $x \circ f$ is right-continuous (respectively left-continuous) on $[0, a]$. (In particular, if f is a right-continuous flow and Λ is a finite positive measure on $\sigma(\mathcal{A})$, then $\Lambda \circ f$ becomes a distribution function on $[0, a]$.)

The preceding comment may not be true if the set-function x is monotone outer-continuous (respectively monotone inner-continuous) on $\mathcal{A}$ but not on $\mathcal{C}(u)$. However, the comment is true if the flow f is simple. In order to prove this result, we need the following lemma, whose proof relies on the separability from above of the indexing collection $\mathcal{A}$.

Lemma 2.4.6 (Lemma 5.1.6, [41]) *Let $A, B \in \mathcal{A}_1$, $B \subset A$ such that B is minimal in A (in $\mathcal{A}_1$). Then, there exists a continuous flow $f : [0, 1] \rightarrow \mathcal{A}$ such that $f(0) = A$, $f(1) = B$.*

More precisely, we have the following result.

Proposition 2.4.7 *Let $x : \mathcal{A} \rightarrow \mathbf{R}$ be a set-function with a unique additive extension to $\mathcal{C}(u)$. Then x is monotone outer-continuous (respectively monotone inner-continuous) if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the function $x^f := x \circ f$ is right-continuous (respectively left-continuous). (For the sufficiency part it is enough to consider only the $\mathcal{A}$ -valued continuous flows.)*

Proof: For necessity, let $f : [0, a] \rightarrow \mathcal{A}(u)$ be a simple flow of the form $f(t) = \bigcup_{j=0}^i f_j(a_j) \cup f_i(t), t \in [a_i, a_{i+1}]$ where $0 = a_0 < a_1 < \dots < a_m = a$ is a partition of $[0, a]$ and $f_i : [a_i, a_{i+1}] \rightarrow \mathcal{A}$ are some $\mathcal{A}$ -valued (continuous) flows. Let $t, t_n \in [0, a]$ be such that $t_n \rightarrow t$. Suppose that $t \in [a_i, a_{i+1}]$ for some i ; then $f(t) = B \cup A$ for $A := f_i(t) \in \mathcal{A}$ and some $B \in \mathcal{A}(u)$. For n large enough, $t_n \in [a_i, a_{i+1}]$ and $f(t_n) = B \cup A_n$ for $A_n := f_i(t_n) \in \mathcal{A}$.

If the sequence $(t_n)_n$ is decreasing, then $(A_n)_n$ is a decreasing sequence of sets in $\mathcal{A}$ with intersection A , using the right-continuity of the flow f_i . Hence $x(A_n) \rightarrow x(A)$. Using the additivity of the set-function x , we can conclude that $x_{t_n}^f = x(B \cup A_n) = x(B) + x(A_n) - x(B \cap A_n) \rightarrow x(B) + x(A) - x(B \cap A) = x(B \cup A) = x_t^f$. (Say $B = \bigcup_{j=1}^m A'_j, A'_j \in \mathcal{A}$. Then $x(B \cap A_n) = \sum_{k=1}^m (-1)^{k+1} \sum_{j_1 < \dots < j_k} x(A'_{j_1} \cap \dots \cap A'_{j_k} \cap A_n) \rightarrow \sum_{k=1}^m (-1)^{k+1} \sum_{j_1 < \dots < j_k} x(A'_{j_1} \cap \dots \cap A'_{j_k} \cap A) = x(B \cap A)$ because each $(A'_{j_1} \cap \dots \cap A'_{j_k} \cap A_n)_n$ is a decreasing sequence of sets in $\mathcal{A}$ with intersection $A'_{j_1} \cap \dots \cap A'_{j_k} \cap A$.)

If the sequence $(t_n)_n$ is increasing we get the same conclusion using the left-continuity of the flow f_i .

For sufficiency, let $(A_n)_n$ be a decreasing sequence in $\mathcal{A}$ with intersection A . Let $A_0 := T$. For each $A_{n+1} \subseteq A_n \subseteq A_{n-1}$ there exists two continuous $\mathcal{A}$ -valued flows f_{n-1}, f'_n such that $f_{n-1}(t_{n-1}) = A_{n-1}, f_{n-1}(t_n) = A_n$ and $f'_n(t'_n) = A_n, f'_n(t'_{n+1}) = A_{n+1}$ for some $t_{n-1} \leq t_n$ and $t'_n \leq t'_{n+1}$ (using Lemma 2.4.6). We can certainly find a linear rescaling f_n of the flow f'_n such that $f_n(t_n) = A_n$, that is the set A_n is the image of the same point t_n under the two different flows. The sequence $(t_n)_n$ is decreasing and let $t_* := \lim_n t_n$. Let $f : [0, t_0] \rightarrow \mathcal{A}$ be defined by

$$f(t) := \begin{cases} f_n(t) & \text{if } t \in [t_n, t_{n+1}] \\ g(t) & \text{if } t \in [0, t_*] \end{cases}$$

where $g : [0, t_*] \rightarrow \mathcal{A}$ is a continuous flow with $g(0) = \emptyset', g(t_*) = A$. Clearly f is a continuous $\mathcal{A}$ -valued flow. Since x^f is right-continuous it follows that $x(A_n) = x_{t_n}^f \rightarrow x_{t_*}^f = x(A)$.

If the sequence $(A_n)_n$ is increasing and $\overline{\bigcup_n A_n} := A \in \mathcal{A}$ then we start with $A_0 := \emptyset'$ and construct a continuous $\mathcal{A}$ -valued flow f such that $f(t_n) = A_n \forall n$ for an increasing sequence $(t_n)_n$ with $t^* = \lim_n t_n$. We can set $f(t^*) := A$. The left-continuity of x^f will give us again the desired conclusion $x(A_n) \rightarrow x(A)$.

□

The next result will be occasionally used in this work.

Lemma 2.4.8 (Lemma 5.1.7, [41]) *For any finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and for any consistent ordering $\{\emptyset' = A_0, A_1, \dots, A_n\}$ of $\mathcal{A}'$, there exists a simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ which ‘connects the sets of $\mathcal{A}'$ in the sense of the ordering ord’, i.e. $f(t_i) = \cup_{j=0}^i A_j \forall i = 0, \dots, n$, where $t_0 = 0 < t_1 < \dots < t_n = a$ is the partition corresponding to f .*

The previous result has the following consequence.

Lemma 2.4.9 *Given $B_1, \dots, B_m \in \mathcal{A}(u)$ such that $B_1 \subseteq \dots \subseteq B_m$, there exists a simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ and $t_1, \dots, t_m \in [0, a], t_1 \leq \dots \leq t_m$ such that $B_i = f(t_i), i = 1, \dots, m$.*

Proof: Using Proposition 2.1.14.(a), there exists a finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and a consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$ such that $B_l = \cup_{j=0}^{i_l} A_j; l = 1, \dots, m$ for some $1 \leq i_1 \leq \dots \leq i_m = n$. By Lemma 2.4.8, there exists a simple flow f corresponding to a partition $0 = t_0 \leq t_1 \leq \dots \leq t_n = a$ such that $f(t_i) = \cup_{j=0}^i A_j$ for all $i = 0, \dots, n$. Hence $f(t_{i_l}) = B_l$ for $l = 1, \dots, m$.

□

The second type of regularity property for the sample paths of a set-indexed process is introduced using the Hausdorff distance d_H ; for this we have to assume that T is *metrizable*.

Assume that (T, d) is a compact complete separable metric space and let $\mathcal{K}$ be the collection of all compact subsets of T . For each $A \in \mathcal{K} \setminus \{\emptyset\}, \epsilon > 0$ let

$$A^\epsilon := \{t \in T; d(A, t) \leq \epsilon\}$$

where $d(A, t) = \inf_{s \in A} d(s, t)$ is the distance from A to t . The Hausdorff metric d_H is defined on $\mathcal{K} \setminus \{\emptyset\}$ by the formula:

$$d_H(A, B) = \inf\{\epsilon > 0; A \subseteq B^\epsilon, B \subseteq A^\epsilon\}.$$

The following result says that monotone convergence is stronger than d_H -convergence (this result is valid only for compact spaces T).

Lemma 2.4.10 (Lemma 1.3.1, [41]) *Let $(A_n)_n$ be a monotone sequence of closed subsets of T .*

(a) *If $(A_n)_n$ is decreasing with $A = \bigcap_n A_n$, then $\lim_{n \rightarrow \infty} d_H(A_n, A) = 0$.*

(b) *If $(A_n)_n$ is increasing with $A = \overline{\bigcup_n A_n}$, then $\lim_{n \rightarrow \infty} d_H(A_n, A) = 0$.*

A classical result says that $\mathcal{K} \setminus \{\emptyset\}$ is d_H -compact (because T is compact). The next result says that under certain (quite general) conditions the collection $\mathcal{A} \setminus \{\emptyset\}$ will be d_H -compact.

Lemma 2.4.11 (Lemma 1.3.3, [41]) *Assume that the functions g_n are $\mathcal{A}$ -valued and that they satisfy the following condition:*

$$d_H(A, g_n(A)) \leq \epsilon_n \quad \forall A \in \mathcal{A}, \forall n \geq 1$$

for a certain sequence $(\epsilon_n)_n$ converging monotonically to 0. Then $\mathcal{A} \setminus \{\emptyset\}$ is d_H -closed (and hence d_H -compact) in $\mathcal{K} \setminus \{\emptyset\}$.

Here are the regularity properties that we have mentioned.

Definition 2.4.12 *A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ is called*

- (a) **outer-continuous** *if for every sequence $(A_n)_n \subseteq \mathcal{A}$ with $\lim_n d_H(A_n, A) = 0$, $A \in \mathcal{A}$ and $A_n \supseteq A, \forall n$, we have $\lim_n x(A_n) = x(A)$;*
- (b) **inner-continuous** *if for every sequence $(A_n)_n \subseteq \mathcal{A}$ with $\lim_n d_H(A_n, A) = 0$, $A \in \mathcal{A}$ and $A_n \subseteq A^0, \forall n$, we have $\lim_n x(A_n) = x(A)$;*
- (c) **d_H -continuous** *if for every sequence $(A_n)_n \subseteq \mathcal{A}$ with $\lim_n d_H(A_n, A) = 0$, $A \in \mathcal{A}$, we have $\lim_n x(A_n) = x(A)$;*
- (d) **cadlag** *if it is outer-continuous and it 'has inner limits' i.e. for every sequence $(A_n)_n \subseteq \mathcal{A}$ with $\lim_n d_H(A_n, A) = 0$, $A \in \mathcal{A}$ and $A_n \subseteq A^0 \forall n$, $(x(A_n))_n$ converges.*

A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called **outer-continuous** (respectively **inner-continuous**, or **d_H -continuous**, or **cadlag**) if all its sample paths are outer-continuous (respectively inner-continuous, or d_H -continuous, or cadlag).

Notation: The space of all cadlag set-functions $x : \mathcal{A} \rightarrow \mathbf{R}$ is denoted with $\mathcal{D}(\mathcal{A})$.

Remark: Various authors considered similar notions for cadlaguity in the d -dimensional euclidean cube I_d replacing the convergence in the Hausdorff distance with other types of convergences. The authors of [2] considered the convergence in the metric d_λ defined by $d_\lambda(A, B) := \lambda(A \Delta B)$, where λ denotes the Lebesgue measure on I_d and $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the usual symmetric difference of two sets; the authors of [8] considered the convergence $A = \liminf A_n = \limsup A_n$ instead of $d_H(A_n, A) \rightarrow 0$ where $\liminf A_n := \bigcup_m \bigcap_{n \geq m} A_n$, $\limsup A_n := \bigcap_m \bigcup_{n \geq m} A_n$.

Clearly any d_H -continuous function is both outer-continuous and inner-continuous. On the other hand, by Lemma 2.4.10 it follows immediately that any outer-continuous function is monotone outer-continuous. An inner-continuous function may not be monotone-inner continuous: if $(A_n)_n$ is an increasing sequence of sets in $\mathcal{A}$ and $A := \overline{\bigcup_n A_n} \in \mathcal{A}$ we cannot conclude that $A_n \subseteq A^0$. However, if $\mathcal{A}$ satisfies the following assumption

Assumption 2.4.13 (Separability from below) For any $A \in \mathcal{A}$ with a non-empty interior, for any $\epsilon > 0$ there exists a set $A_\epsilon \in \mathcal{A}$ such that $A_\epsilon \subseteq A^0$ and $d_H(A_\epsilon, A) < \epsilon$.

then any inner-continuous set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ which vanishes over all sets A with an empty interior, is monotone inner-continuous (Section 7.1, [41]).

An intuitive candidate for a cadlag function would be a ‘purely atomic’ set-function, that is a linear combination of purely atomic measures,

$$x(A) = \sum_{j=1}^n a_j I_{\{z_j \in A\}}$$

for some $a_j \in \mathbf{R}$ and for some distinct points $z_j \in T, j = 1, \dots, k$. The z_j ’s are called the ‘locations of the atoms’ and the a_j ’s are the ‘masses of the atoms’. Any purely atomic function can be extended to a unique finite signed measure on $\sigma(\mathcal{A})$; if the

masses of all its atoms are positive then this extension is a finite positive measure (hence it is monotone outer-continuous on $\sigma(\mathcal{A})$).

As observed by the author of [67], our intuition breaks down if one of the atoms fall exactly on the boundary of our compact space T . As a counterexample consider the case when T is the unit square, $\mathcal{A} = \{[0, z]; z \in T\}$ and x has a single atom of mass 1 located on the boundary of T , say $(1, 1/2)$ is this atom. Then x does not have an inner limit at any point $z^* = (1, v)$ with $v \in (1/2, 1)$: if $A_n = [0, z_n = (s_n, t_n)]$ and $z_n \rightarrow z^*$ then $x(A_n) = 0$ if $s_n < 1$ and $x(A_n) = 1$ if $s_n = 1$. It was not noticed in [2] or [8] that a general purely atomic function may not be cadlag.

To circumvent this difficulty we will adopt the technique introduced for the first time by Ivanoff and Merzbach in [40].

Given $D \subseteq T$ and $\epsilon > 0$, define $D^{-\epsilon} := \cap_{A \in \mathcal{A}, D \subseteq A^\epsilon} A$.

Definition 2.4.14 A set $A \in \mathcal{A}$ is **proper** if there exists an $\epsilon_0 > 0$ such that $(A^\epsilon)^{-\epsilon} = A \forall 0 < \epsilon < \epsilon_0$.

We note that by definition, $(A^\epsilon)^{-\epsilon} \subseteq A$ for any $A \in \mathcal{A}$. It is not true that every set in $\mathcal{A}$ is proper. For example, if $T = [0, 1]^2$ and $\mathcal{A} = \{[0, z]; z \in T\}$ then $[0, z]$ is proper if and only if $z = (s, t)$ where $s \neq 1$ and $t \neq 1$.

Now, for $z \in T$, define

$$A_z = \cap_{A \in \mathcal{A}, z \in A} A$$

and let $T^* := \{z \in T; A_z \text{ is proper}\}$. Let $\mathcal{P}(\mathcal{A})$ be the space of all purely atomic functions with atoms located in T^* . The next result appears in [67].

Theorem 2.4.15 Any purely atomic function with atoms located in T^* is cadlag i.e. $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$.

In what follows we will introduce a sub-class of purely-atomic functions with positive masses of atoms and the corresponding class of set-indexed processes.

Definition 2.4.16 A set-function $x : \mathcal{A} \rightarrow \mathbf{R}$ is called a **point measure** if there exists a finite subset $\{z_1, \dots, z_n\} \subseteq T$ such that $x(A) = \sum_{j=1}^n I_{\{z_j \in A\}}$ for every $A \in \mathcal{A}$. A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called a **point process** if all its sample paths are point measures.

In other words, a point measure is a purely atomic set-function whose atoms have masses equal to 1. We will denote with $\mathcal{D}^1(\mathcal{A})$ the space of all point measures with atoms located in T^* . Clearly $\mathcal{D}^1(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$.

Proposition 2.4.17 *If $x : \mathcal{A} \rightarrow \mathbf{R}$ is a purely atomic set-function, then for every flow $f : [0, a] \rightarrow \mathcal{A}(u)$ with $f(a) = T$, the function $x^f := x \circ f$ is also purely atomic.*

Proof: Assume $x(A) = \sum_{j=1}^n a_j I_{\{z_j \in A\}}$, $A \in \mathcal{A}$ and let f be an arbitrary flow which reaches T . Set $z_j^f := \min\{t \in [0, a]; z_j \in f(t)\}$. Then $z_j \in f(t)$ if and only if $z_j^f \in [0, t]$ and hence $x_t^f = \sum_{j=1}^n a_j I_{\{z_j^f \in [0, t]\}}$.
□

Note: If x is a point measure, it might be possible that $x^f = x \circ f$ is not a point measure, for a certain flow f . To see this, consider the following example: let $x(A) := \sum_{j=1}^2 I_{\{z_j \in A\}}$, $A \in \mathcal{A}$ be a point measure such that $z_1, z_2 \in \partial A$ for some set $A \in \mathcal{A}$; let f be a flow such that $f(t) = A$ and $f(s) \subseteq A^0, \forall s < t$; then $\Delta x^f(t) = 2$ and x^f is not a point measure.

The third type of regularity property for the sample paths of a set-indexed process uses the notion of ‘flow’.

Notation: The space of all set-indexed processes $X := (X_A)_{A \in \mathcal{A}}$ with the property that for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$, the process $X^f := (X_{f(t)})_{t \in [0, a]}$ has a cadlag modification, is denoted with $\mathcal{D}[S(\mathcal{A})]$.

The space $\mathcal{D}[S(\mathcal{A})]$ contains many important classes of set-indexed processes. Processes with purely atomic sample paths (in particular point processes) and processes which are monotone outer-continuous and monotone inner-continuous are cadlag on every simple flow. On the other hand, the ‘strong martingales’ (defined in [41]) or the ‘Levy processes’ (which will be introduced in Chapter 5 of this work) possess cadlag modifications on every simple flow.

We will conclude this section with a note on ‘separability’.

By the definition of the indexing collection (property ‘Separability from above’) and Lemma 2.4.10, it follows that the indexing collection $\mathcal{A} \setminus \{\emptyset\}$ is separable with

respect to the metric d_H : its countable dense subset is the collection $\{g_n(A); A \in \mathcal{A}, A \neq \emptyset, n \geq 1\}$.

We have the following general definition.

Definition 2.4.18 *Let T be a separable metric space. An $\overline{\mathbf{R}}$ -valued process $(X_t)_{t \in T}$ (defined on a probability space $(\Omega, \mathcal{F}, P)$) is **separable** if there exists a countable dense subset $D \subseteq T$ and a set $N \in \mathcal{F}, P(N) = 0$ such that $\forall \omega \notin N, \forall t \in T, \exists (t_n)_n \subseteq D, \lim_n t_n = t$ such that $\lim_n X_{t_n}(\omega) = X_t(\omega)$.*

Following carefully the results given in Section 38, [10] for the case $T = [0, \infty)$, we realize that the same arguments will work for no matter what separable metric space T , by replacing the intervals with the balls. Therefore, we have the following analogue of Theorem 38.1, [10] which will be used in Section 5.5 of this work.

Theorem 2.4.19 *Let T be a separable metric space. Any $\mathbf{R}$ -valued process $(X_t)_{t \in T}$ has an $\overline{\mathbf{R}}$ -valued separable modification and a $\mathbf{R}$ -valued separable version.*

2.5 The Collection of Projected Flows

Let $f_0 : [0, 1] \rightarrow \mathcal{A}$ be a fixed continuous flow such that $f_0(0) = \emptyset'$ and $f_0(1) = T$. For each finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and for each consistent ordering ord of $\mathcal{A}'$ we will define a simple flow $f_{\mathcal{A}', \text{ord}}$ which will be regarded as the ‘projection’ of the original flow f_0 on the semilattice, in the sense of the prescribed ordering.

The collection of these simple flows will prove to be an useful tool in Section 4.3 where we derive an alternative version for the necessary and sufficient conditions that are imposed on the ‘generator’ of a $\mathcal{Q}$ -Markov process. The results of this section are not used anywhere else in this work.

Let $\mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ and $\text{ord} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$. For each $i = 1, \dots, n - 1$ let

$$\tau_i := \inf\{t \in [0, 1]; f_0(t) \cap A_{i+1} \not\subseteq \cup_{j=1}^i A_j\}.$$

Define the following continuous $\mathcal{A}$ -valued flow

$$f_i(t) := f_0(t) \cap A_i, \quad t \in [0, 1]$$

and note that $f_{i+1}(t) \subseteq \cup_{j=1}^i A_j$ for $t < \tau_i$. Therefore, we will consider the flow f_{i+1} only on the region $[\tau_i, 1]$.

The simple flow $f := f_{\mathcal{A}', \text{ord}}$ that we will construct will ‘connect’ the sets of the semilattice $\mathcal{A}'$ in the sense of the ordering ord and will have the same behaviour between the sets $\cup_{j=1}^i A_j$ and $\cup_{j=1}^{i+1} A_j$ as the flow f_{i+1} .

For $t \in [0, 1]$ let $f(t) := f_1(t)$. We have $f(0) = \emptyset, f(1) = A_1$ and the flow f is continuous and $\mathcal{A}$ -valued between these values. We would like next to go from A_1 to $A_1 \cup A_2$ by means of a continuous flow which adds only sets in $\mathcal{A}$ to the initial set A_1 . The best candidate is the flow $f'_2(s) := A_1 \cup f_2(s), s \in [\tau_1, 1]$, the only problem being that we have to be able to paste this flow immediately after the flow f_1 . The solution will be a change of scale: we shift the interval $[\tau_1, 1]$ by $1 - \tau_1$ (in other words, instead of $s \in [\tau_1, 1]$ we will consider $s + (1 - \tau_1)$) so that τ_1 becomes 1 and 1 becomes $1 + (1 - \tau_1)$. For $t \in [1, 1 + (1 - \tau_1)]$ let $f(t) := A_1 \cup f_2(t - (1 - \tau_1)) = A_1 \cup (f_0(t - (1 - \tau_1)) \cap A_2)$.

We can continue in the same manner until we reach the set $\cup_{j=1}^n A_j$. In general we have for each $i = 0, \dots, n - 1$

$$f(t) := (\cup_{j=1}^i A_j) \cup [f_0(t - \sum_{j=1}^i (1 - \tau_j)) \cap A_{i+1}], \quad t \in [t_i, t_{i+1}]$$

where $t_0 = 0, t_1 = 1$ and $t_i = 1 + \sum_{j=1}^{i-1} (1 - \tau_j), i = 2, \dots, n$.

The flow f will be denoted $f_{\mathcal{A}', \text{ord}}$ (since it depends on both the semilattice $\mathcal{A}'$ and the ordering ord) and will be called **the projection** of the flow f_0 on the semilattice $\mathcal{A}'$, with respect to the ordering ord .

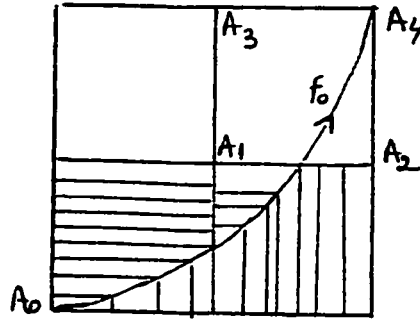
Example 2.5.1 Let $T = [0, 1]^2, \mathcal{A} = \{[0, z]; z \in T\}, \mathcal{A}'$ be a finite sub-semilattice of $\mathcal{A}$ and $\text{ord} = \{A_0 = \emptyset, A_1, A_2, A_3, A_4 = T\}$ a consistent ordering of $\mathcal{A}'$, where $A_1 = [0, (1/2, 1/2)], A_2 = [0, (1, 1/2)], A_3 = [0, (1/2, 1)]$. Suppose that $f_0(t) = [0, (t, t^2)], t \in [0, 1]$.

Then the projected flow $f_{\mathcal{A}', \text{ord}}$ has the following definition:

$$f_{\mathcal{A}', \text{ord}}(t) := \begin{cases} f_0(t) \cap A_1 & \text{if } t \in [0, 1] \\ A_1 \cup [f_0(s) \cap A_2] & \text{if } s := t - (1 - \tau_1) \in [\tau_1, 1] \\ (A_1 \cup A_2) \cup [f_0(s) \cap A_3] & \text{if } s := t - (1 - \tau_1) - (1 - \tau_2) \in [\tau_2, 1] \\ (A_1 \cup A_2 \cup A_3) \cup f_0(s) & \text{if } s := t - (1 - \tau_1) - (1 - \tau_2) - (1 - \tau_3) \in [\tau_3, 1] \end{cases}$$

where $\tau_i := \inf\{t; f_0(t) \cap A_{i+1} \subseteq \bigcup_{j=1}^i A_j\}; i = 1, 2, 3$.

The following picture shows the path of the flow $f_{\mathcal{A}', \text{ord}}$ in this case:



The following result says that until a certain moment, the two flows f_0 and f have exactly the same path and after that moment, any increment on the path of the flow f_0 (which is contained in $\bigcup_{j=1}^n A_j$) can be recovered as the disjoint union of a finite number of increments on the path of the projected flow f .

Proposition 2.5.2 *Let $s_i := \sup\{t \in [0, 1]; f_0(t) \subseteq \bigcup_{j=1}^i A_j\}, i = 1, \dots, n$. Define $u_1 := s_1$ and $u_i := s_1 + \sum_{j=1}^{i-1} (1 - \tau_j), i = 2, \dots, n$. Then $f_0(t) = f(t)$ for any $t \leq s_1$ and*

$$f_0(s_1 + \epsilon) \setminus f_0(s_1) = \dot{\bigcup}_{j=1}^i [f(u_j + \epsilon) \setminus f(u_j)]$$

for any $\epsilon \in (0, s_i - s_1), i = 2, \dots, n$.

Proof: The first assertion is clear. Let C_i be the left neighbourhood of A_i in $\mathcal{A}'$. Let $i = 2, \dots, n$ be fixed. For any $\epsilon < s_i - s_1, f_0(s_1 + \epsilon) \subseteq \bigcup_{j=1}^i A_j = \bigcup_{j=1}^i C_j$ and hence we have

$$f_0(s_1 + \epsilon) \setminus f_0(s_1) = \dot{\bigcup}_{j=1}^i [(f_0(s_1 + \epsilon) \setminus f_0(s_1)) \cap C_j]$$

On the other hand

$$(f_0(s_1 + \epsilon) \setminus f_0(s_1)) \cap C_j = (f_0(s_1 + \epsilon) \cap A_j) \setminus [(\bigcup_{k=1}^{j-1} A_k) \cup (f_0(s_1) \cap A_j)]$$

$$\begin{aligned}
 &= [(\cup_{k=1}^{j-1} A_k) \cup (f_0(s_1 + \epsilon) \cap A_j)] \setminus [(\cup_{k=1}^{j-1} A_k) \cup (f_0(s_1) \cap A_j)] \\
 &= f(s_1 + \sum_{k=1}^{j-1} (1 - \tau_k) + \epsilon) \setminus f(s_1 + \sum_{k=1}^{j-1} (1 - \tau_k))
 \end{aligned}$$

by the definition of the projected flow f between the sets $\cup_{k=1}^{j-1} A_k$ and $\cup_{k=1}^j A_k$.

□

A consequence of the above proposition is the following: let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process which has a unique additive extension to $\mathcal{C}(u)$. Denote with $X_t^0 := X_{f_0(t)}$, $t \in [0, 1]$ and $X_t := X_{f(t)}$, $t \in [0, t_n]$ the projections of the process X over the flows f_0, f . Then $X_t^0 = X_t$ for any $t \leq s_1$ and

$$X_{s_1+\epsilon}^0 - X_{s_1}^0 = \sum_{j=1}^i (X_{u_j+\epsilon} - X_{u_j}) \text{ for any } \epsilon \in (0, s_i - s_1), i = 2, \dots, n.$$

The next proposition describes the relationship between the projections f and g of the flow f_0 on the same finite semilattice $\mathcal{A}'$, but with respect to two different orderings.

Proposition 2.5.3 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$, with $A_i = A'_{\pi(i)}$, $\forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$ and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then*

- (a) $t_1 = u_1 = 1$ and $t_i - t_{i-1} = u_{\pi(i)} - u_{\pi(i)-1}$ for every $i = 2, \dots, n$;
- (b) $f(t) = g(t)$ for any $t \in [0, t_1]$ and

$$f(t_{i-1} + \epsilon) \setminus f(t_{i-1}) = g(u_{\pi(i)-1} + \epsilon) \setminus g(u_{\pi(i)-1})$$

for every $\epsilon \in (0, t_i - t_{i-1})$ and for every $i = 2, \dots, n$.

Proof: (a) From the definition of the projected flow we have $t_1 = u_1 = 1$ and $t_i = 1 + \sum_{j=1}^{i-1} (1 - \tau_j)$, $u_i = 1 + \sum_{j=1}^{i-1} (1 - \tau'_j)$ for $i = 2, \dots, n$, where

$$\tau_i := \inf\{t \in [0, 1]; A_{i+1} \cap f_0(t) \not\subseteq \cup_{j=1}^i A_j\}$$

$$\tau'_i := \inf\{t \in [0, 1]; A'_{i+1} \cap f_0(t) \not\subseteq \cup_{j=1}^i A'_j\}.$$

Hence $t_i - t_{i-1} = 1 - \tau_{i-1}$ and $u_{\pi(i)} - u_{\pi(i)-1} = 1 - \tau'_{\pi(i)-1}$. We will prove that

$$\tau_{i-1} = \tau'_{\pi(i)-1} \text{ for any } i = 2, \dots, n.$$

Note that τ_{i-1} is the first instance when $A_i \cap f_0(t) \not\subseteq \cup_{j=1}^{i-1} A_j$. But $A_i \cap f_0(t) \subseteq \cup_{j=1}^{i-1} A_j$ if and only if $A_i \cap f_0(t) \subseteq \cup_{j=1}^{i-1} (A_j \cap A_i) = \cup_{A \in \mathcal{A}', A \subseteq A_i} A$. Similarly $\tau'_{\pi(i)-1}$ is the first instance when $A'_{\pi(i)} \cap f_0(t) \not\subseteq \cup_{j=1}^{\pi(i)-1} A'_j$; but $A'_{\pi(i)} \cap f_0(t) \subseteq \cup_{j=1}^{\pi(i)-1} A'_j$ if and only if $A'_{\pi(i)} \cap f_0(t) \subseteq \cup_{A \in \mathcal{A}', A \subseteq A'_{\pi(i)}} A$. The equality follows since $A_i = A'_{\pi(i)}$.

(b) Clearly $f(t) = g(t) = f_0(t) \cap A_1$ for any $t \in [0, t_1]$. By the definition of f we have that for each $i = 2, \dots, n$

$$\begin{aligned} f(t_{i-1} + \epsilon) \setminus f(t_{i-1}) &= [(\cup_{j=1}^{i-1} A_j) \cup (f_0(t_{i-1} + \epsilon - \sum_{j=1}^{i-1} (1 - \tau_j)) \cap A_i)] \setminus (\cup_{j=1}^{i-1} A_j) \\ &= (f_0(\tau_{i-1} + \epsilon) \cap A_i) \setminus (\cup_{j=1}^{i-1} A_j) = f_0(\tau_{i-1} + \epsilon) \cap C_i. \end{aligned}$$

Similarly

$$g(u_{\pi(i)-1} + \epsilon) \setminus g(u_{\pi(i)-1}) = f_0(\tau'_{\pi(i)-1} + \epsilon) \cap C'_{\pi(i)}.$$

The result follows since $\tau_{i-1} = \tau'_{\pi(i)-1}$ and $C_i = C'_{\pi(i)}$.

□

A consequence of the above proposition is the following: let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process which has a unique additive extension to $\mathcal{C}(u)$. Denote with $X'_t := X_{f(t)}$, $t \in [0, t_n]$ and $X''_t := X_{g(t)}$, $t \in [0, u_n]$ the projections of the process X over the flows f, g . Then $X'_t = X''_t$ for any $t \leq t_1$ and

$$X'_{t_{i-1} + \epsilon} - X'_{t_{i-1}} = X''_{u_{\pi(i)-1} + \epsilon} - X''_{u_{\pi(i)-1}}$$

for any $\epsilon \in (0, t_i - t_{i-1})$, $i = 2, \dots, n$. In other words, the trajectory of the process X' can be constructed from the trajectory of the process X'' by permuting (according to the permutation π) the $n - 1$ pieces determined by the partition $u_1 < u_2 < \dots < u_n$.

Chapter 3

Set-Markov Processes

Defining a Markov property for processes indexed by a partially ordered set is not a problem that can be tackled very easily when someone has the declared goal of proving that processes with such a property actually exist.

For processes indexed by the plane, different types of definitions were proposed, the only one for which a constructive result exists being Lévy's sharp Markov property (see the introductory Chapter 1). For $\mathcal{A}$ -indexed processes, where $\mathcal{A}$ is an indexing collection, there are already in the literature (see [41]) two possible definitions for the Markov property called 'sharp Markov property' (analogous to the sharp Markov property on the plane) and 'Markov property' (analogous to the Markov property introduced by the authors of [46] on the plane). Having the merit of being as general as possible, these definitions have also overcome the difficulty of working with processes indexed by one class of objects (the points) and with a Markov property considered with respect to another class of objects (the sets), because in their case everything is defined in terms of the sets of the indexing collection. However, an existence result is not yet available for these types of Markov properties in their full generality, although we know that any process with independent increments is Markov (in the specified sense).

Here we will consider a different type of definition for a set-indexed Markov process, such a process being called 'set-Markov'. The biggest advantage of this definition is, in our opinion, the fact that, not only it is completely analogous to the Markov

property on the line, but we can also very easily make use of the rich theory of Markov processes on the line. In particular, exploiting this definition we will develop without any difficulty a transition system theory paralleling the classical one and we will be able to prove the desired existence result. The key ingredient to this promising theory is the object named ‘flow’, introduced in Section 2.4, which will transport a closed interval of the real line into our indexing class $\mathcal{A}(u)$.

In this chapter we will introduce the formal definition of set-Markov processes and we will prove some of their properties. In particular we will prove that all processes with independent increments are set-Markov and that the class of set-Markov processes is a sub-class of the class of sharp Markov processes. Given a transition system $\mathcal{Q}$ we will restrict our attention to those set-Markov processes for which the transition from one state to another is dictated by $\mathcal{Q}$. We will call these processes $\mathcal{Q}$ -Markov and we will construct them in two cases: when the geometrical property SHAPE holds and when it does not. Finally, we will see that to each transition system $\mathcal{Q}$ one can associate a transition semigroup of bounded linear operators, very much in the spirit of the classical theory.

Most of the results of Section 3.1 and Section 3.3 appear in [7].

3.1 Set-Markov Processes. General Properties

In this section we will give the definition of a set-Markov process and we will discuss its properties.

If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are three sub- σ -fields of the same probability space, we will use the notation $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ if $\mathcal{F}$ and $\mathcal{H}$ are *conditionally independent* given $\mathcal{G}$ i.e, for every bounded $\mathcal{H}$ -measurable random variable Y , we have $E[Y \mid \mathcal{F} \vee \mathcal{G}] = E[Y \mid \mathcal{G}]$. (Basic properties of conditional independence are discussed in Appendix A.1)

Definition 3.1.1 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called **set-Markov with respect to a filtration** $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if it is adapted and $\forall A \in \mathcal{A}, \forall B \in \mathcal{A}(u)$*

$$\mathcal{F}_B \perp \sigma(X_{A \setminus B}) \mid \sigma(X_B) \quad (4)$$

The process X is called simply **set-Markov** if it is set-Markov with respect to its minimal filtration.

Comments 3.1.2 1. In the above definition we can assume without loss of generality that $A \not\subseteq B$: if $A \subseteq B$, then $X_{A \setminus B} = 0$ a.s.

2. It is easy to see that in the classical case (i.e. for the processes indexed by the real line) the set-Markov property is equivalent to the usual Markov property: a process $X := (X_t)_{t \in [0, a]}$ is Markov if and only if $\forall s, t \in [0, 1], s < t$ we have $\mathcal{F}_s \perp \sigma(X_t - X_s) \mid \sigma(X_s)$.

We have the following immediate consequence of the definition.

Proposition 3.1.3 *Any process with independent increments is set-Markov.*

Proof: Let $X := (X_A)_{A \in \mathcal{A}}$ be a process with independent increments and $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ its minimal filtration. Using a monotone class argument, it is not difficult to see that for every $A \in \mathcal{A}, B \in \mathcal{A}(u)$, $X_{A \setminus B}$ is independent of $\mathcal{F}_B$ (and hence $X_{A \setminus B}$ is also independent of X_B).

□

The next result gives some equivalent definitions for the set-Markov property.

Proposition 3.1.4 (Characterization Properties) *Let $(\mathcal{F}_A)_{A \in \mathcal{A}}$ be a set-indexed filtration and $X := (X_A)_{A \in \mathcal{A}}$ an adapted set-indexed process. The following statements are equivalent:*

1. *the process X is set-Markov with respect to $(\mathcal{F}_A)_{A \in \mathcal{A}}$;*
2. *$\forall A \in \mathcal{A}, B \in \mathcal{A}(u), \mathcal{F}_B \perp \sigma(X_{A \cup B}) \mid \sigma(X_B)$;*
3. *$\forall B, B' \in \mathcal{A}(u), B \subseteq B', \mathcal{F}_B \perp \sigma(X_{B'}) \mid \sigma(X_B)$.*

Proof: The fact that 1 is equivalent to 2 follows from Lemma A.1.3, Appendix A.1 since $X_{A \cup B} = X_{A \setminus B} + X_B$ a.s. and X_B is $\mathcal{F}_B$ -measurable. Clearly 3 implies 2; it remains to check that 2 implies 3. Let $B, B' \in \mathcal{A}(u)$ be such that $B \subseteq B'$. Say

$B' = \cup_{i=1}^k A_i$, $A_i \in \mathcal{A}$ is an arbitrary representation. Then $B' = B \cup B' = B \cup \cup_{i=1}^k A_i$ and the result follows by induction on k : assume that the statement is true for k . Then

$$\begin{aligned} E[h(X_{B \cup \cup_{i=1}^{k+1} A_i}) | \mathcal{F}_B] &= E[E[h(X_{B \cup \cup_{i=1}^{k+1} A_i}) | \mathcal{F}_{B \cup \cup_{i=1}^k A_i}] | \mathcal{F}_B] = \\ &= E[E[h(X_{B \cup \cup_{i=1}^{k+1} A_i}) | X_{B \cup \cup_{i=1}^k A_i}] | \mathcal{F}_B] = E[h'(X_{B \cup \cup_{i=1}^k A_i}) | \mathcal{F}_B] \end{aligned}$$

which is $\sigma(X_B)$ -measurable, by the induction hypothesis.

□

Comment 3.1.5 Using a monotone class argument it is not difficult to see that a set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is set-Markov if and only if $\forall A \in \mathcal{A}, \forall B \in \mathcal{A}(u), \forall A_i \in \mathcal{A}, A_i \subseteq B; i = 1, \dots, m$

$$\sigma(X_{A_1}, \dots, X_{A_m}) \perp \sigma(X_{A \setminus B}) \mid \sigma(X_B).$$

(According to the above characterization, we can replace $A \setminus B$ with $A \cup B$, or with $B' \in \mathcal{A}(u), B \subseteq B'$.) This property virtually says that we can ‘discretize’ the past history of a set-Markov process, exactly as in the classical case.

The following result says that a set-Markov process becomes Markov in the usual sense along any flow (i.e. along any increasing collection of sets in $\mathcal{A}(u)$, indexed by a closed interval of the real line). Moreover when we restrict our attention only to the ‘simple’ flows, i.e. to those flows which are piecewise $\mathcal{A}$ -valued, this becomes in fact a sufficient condition for the set-Markov nature of the process. This result, while simple, is crucial since it provides us with the means to define the generator of a set-Markov process; this will be done in Chapter 4.

Proposition 3.1.6 *Let $(\mathcal{F}_A)_{A \in \mathcal{A}}$ be a set-indexed filtration and $X := (X_A)_{A \in \mathcal{A}}$ a set-indexed process. Then X is set-Markov with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is Markov with respect to the filtration $(\mathcal{F}_{f(t)})_{t \in [0, a]}$. (For the necessity part we can consider any flow, not only the simple ones.)*

Comment 3.1.7 The filtration $(\mathcal{F}_{f(t)})_{t \in [0, a]}$ is not the minimal filtration associated to the process X^f even if the set-indexed filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ is the minimal filtration associated to the process X .

Proof: From the classical theory of Markov processes, the process X^f is Markov with respect to the filtration $(\mathcal{F}_{f(t)})_t$ if and only if it is adapted and for every $s < t$ we have

$$\mathcal{F}_{f(s)} \perp \sigma(X_{f(t)}) \mid \sigma(X_{f(s)}).$$

This is equivalent to the characterization property 3, Proposition 3.1.4, since we know that whenever the sets $B, B' \in \mathcal{A}(u)$ are such that $B \subseteq B'$ there exists a simple flow f and some $s < t$ such that $f(s) = B$ and $f(t) = B'$ (Lemma 2.4.9).

□

Comment 3.1.8 To emphasize the importance of the above result, let us record that a similar phenomenon happens in the martingale case. More precisely, a set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is a ‘strong martingale’ with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is a martingale with respect to the filtration $(\mathcal{F}_{f(t)})_{t \in [0, a]}$ (see Lemma 5.1.2, [41]). A ‘strong martingale’ is an adapted set-indexed process X with a unique additive extension to $\mathcal{C}(u)$ which satisfies the condition $E[X_C | \mathcal{G}_C^*] = 0$ for all $C \in \mathcal{C}$, where $\mathcal{G}_C^* := \bigvee_{B \in \mathcal{A}(u), B \cap C = \emptyset} \mathcal{F}_B$.

The next lemma shows that we can consider more than one set in the definition of the set-Markov property. In particular, this will imply that the set-Markov property is stronger than the sharp Markov property.

Lemma 3.1.9 *Let X be a set-Markov process with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$. Then $\forall B \in \mathcal{A}(u), \forall A_1, \dots, A_k \in \mathcal{A}$*

$$\mathcal{F}_B \perp \sigma(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) \mid \sigma(X_B).$$

Proof: Without loss of generality we can assume that $A_i \not\subseteq B \ \forall i$. Let $h : \mathbb{R}^k \rightarrow \mathbb{R}$ be an arbitrary bounded measurable function.

Say $B = \cup_{j=k+1}^m A_j$, $A_j \in \mathcal{A}$. Let $\mathcal{A}'$ be the minimal finite semilattice which contains the sets $A_1, \dots, A_m$, $\{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ a consistent ordering of $\mathcal{A}'$ and C'_i the left neighborhood of A'_i in $\mathcal{A}'$ for $i = 1, \dots, n$.

Say $A_j = A'_{i_j}$ for $j = 1 \dots k$; then $A_j \setminus B = A'_{i_j} \setminus B = \cup_{i \in I_j} C'_i$ with $I_j \subseteq \{1, \dots, i_j\}$ and $X_{A_j \setminus B} = \sum_{i \in I_j} X_{C'_i}$.

Therefore we can say that $h(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) = h_1(X_{C'_{l_1}}, \dots, X_{C'_{l_s}})$, for a certain bounded measurable function $h_1 : \mathbf{R}^s \rightarrow \mathbf{R}$ and some $l_1 \leq \dots \leq l_s$, $C'_{l_i} \not\subseteq B$. In order to simplify the notation, let us denote $D_i := C'_{l_i}$, $i = 1, \dots, s$. Let $B_i = B \cup \cup_{k=1}^i D_k$ for $i = 1, \dots, s$. Then $D_i = B_i \setminus B_{i-1}$ and $X_{D_i} = X_{B_i} - X_{B_{i-1}}$; hence $X_{D_1}, \dots, X_{D_{s-1}}$ are $\mathcal{F}_{B_{s-1}}$ -measurable. Because each D_i is the left neighbourhood of A'_{l_i} , we also have $B_i = B \cup \cup_{k=1}^i A'_{l_k}$ and therefore $D_i = (A'_{l_i} \cup B_{i-1}) \setminus B_{i-1} = A'_{l_i} \setminus B_{i-1}$. Using Proposition 3.1.4 (property 3), we have $E[f(X_{D_s}) | \mathcal{F}_{B_{s-1}}] = E[f(X_{D_s}) | X_{B_{s-1}}]$ for any bounded measurable function f . At this point we may apply Lemma A.1.3, Appendix A.1 to get

$$E[h_1(X_{D_1}, \dots, X_{D_s}) | \mathcal{F}_{B_{s-1}}] = E[h_1(X_{D_1}, \dots, X_{D_s}) | X_{B_{s-1}}, X_{D_1}, \dots, X_{D_{s-1}}]$$

So

$$\begin{aligned} E[h(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) | \mathcal{F}_B] &= E[E[h_1(X_{D_1}, \dots, X_{D_s}) | \mathcal{F}_{B_{s-1}}] | \mathcal{F}_B] = \\ &= E[E[h_1(X_{D_1}, \dots, X_{D_s}) | X_{B_{s-1}}, X_{D_1}, \dots, X_{D_{s-1}}] | \mathcal{F}_B]. \end{aligned}$$

Writing $X_{B_{s-1}} = X_B + \sum_{k=1}^{s-1} X_{D_k}$ we get

$$E[h(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) | \mathcal{F}_B] = E[h_2(X_{D_1}, \dots, X_{D_{s-1}}, X_B) | \mathcal{F}_B].$$

Continuing in the same manner, reducing at each step another set D_i , we finally get

$$E[h(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) | \mathcal{F}_B] = E[h(X_{A_1 \setminus B}, \dots, X_{A_k \setminus B}) | X_B].$$

□

Proposition 3.1.10 *If $X := (X_A)_{A \in \mathcal{A}}$ is a set-Markov process with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$, then it also sharp Markov with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$.*

Proof: Let $B \in \mathcal{A}(u)$, $A_i \in \mathcal{A}(B^c)$; $i = 1, \dots, k$ and $h : \mathbf{R}^k \rightarrow \mathbf{R}$ a bounded and measurable function. Then, writing $X_{A_i} = X_{A_i \cap B} + X_{A_i \setminus B}$ and using Lemma A.1.3, Appendix A.1, we get

$$E[h(X_{A_1}, \dots, X_{A_n}) | \mathcal{F}_B] = E[h(X_{A_1}, \dots, X_{A_n}) | X_B, X_{A_i \cap B}, i = 1, \dots, n]$$

which is $\mathcal{F}_{\partial B}$ -measurable, by Lemma 2.3.15.

□

In what follows we will give an important characterization of the set-Markov processes that will be instrumental for the construction of these processes.

Proposition 3.1.11 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is set-Markov if and only if for every finite semilattice $\mathcal{A}'$, for every consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$ and for every $i = 1, \dots, n - 1$*

$$\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=0}^{i+1} A_j}) \mid \sigma(X_{\cup_{j=0}^i A_j}). \quad (5)$$

Proof: We will use the characterization of the set-Markov property given by Proposition 3.1.4 (property 3). Necessity follows immediately.

For sufficiency note first that equation (5) implies a similar equation where the union of the first $i + 1$ sets is replaced by the union of the first $i + p$ sets. In fact it is easily shown by induction on p that for every $i \leq n - p$

$$\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=0}^{i+k} A_j}, k = 1, \dots, p) \mid \sigma(X_{\cup_{j=0}^i A_j}). \quad (6)$$

(The statement for $p = 1$ is true by hypothesis. Assume next that the statement is true for p . The result for $p + 1$ follows by Lemma A.1.4, Appendix A.1 since for each fixed $i \leq n - p - 1$, $\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=0}^{i+1} A_j}) \mid \sigma(X_{\cup_{j=0}^i A_j})$ and $\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^i A_j}) \perp \sigma(X_{\cup_{j=0}^{i+k} A_j}, k = 2, \dots, p + 1) \mid \sigma(X_{\cup_{j=0}^{i+1} A_j}).$)

Consider now arbitrary sets $B, B' \in \mathcal{A}(u)$ with $B \subseteq B'$. Using Comment 3.1.5, it is enough to show that $\forall A'_l \in \mathcal{A}, A'_l \subseteq B; l = 1, \dots, m$, $\sigma(X_{A'_1}, \dots, X_{A'_m}) \perp \sigma(X_{B'}) \mid \sigma(X_B)$. Without loss of generality we can assume that the sets $A'_1, \dots, A'_m$ form a finite semilattice (if not, there exists a finite semilattice $\mathcal{A}'$ which contains them) and the ordering $\{A'_0 = \emptyset', A'_1, \dots, A'_m\}$ is consistent.

There exists a bijective map ψ such that $(X_{A'_1}, X_{A'_2}, \dots, X_{A'_m}) = \psi(X_{A'_1}, X_{A'_1 \cup A'_2}, \dots, X_{\cup_{l=1}^m A'_l})$ a.s. Consequently, we have

$$\sigma(X_{A'_1}, X_{A'_2}, \dots, X_{A'_m}) = \sigma(X_{A'_1}, X_{A'_1 \cup A'_2}, \dots, X_{\cup_{l=1}^m A'_l}).$$

By Proposition 2.1.14.(a), there exists there exists a finite semilattice $\mathcal{A}'$ and a consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$ such that $A'_1 = \cup_{j=0}^{i_1} A_j$, $A'_1 \cup A'_2 = \cup_{j=0}^{i_2} A_j$, $\dots$, $\cup_{l=1}^m A'_l = \cup_{j=0}^{i_m} A_j$, $B = \cup_{j=0}^{i_{m+1}} A_j$ and $B' = \cup_{j=0}^{i_{m+2}} A_j$ for some $i_1 \leq i_2 \leq \dots \leq i_{m+2}$. Using (6), it follows that $\sigma(X_{A'_1}, X_{A'_1 \cup A'_2}, \dots, X_{\cup_{l=1}^m A'_l}) \perp \sigma(X_{B'}) \mid \sigma(X_B)$: take $i := i_{m+1}$ and $p := i_{m+2} - i_{m+1}$.

□

3.2 Q-Markov Processes. The Construction under SHAPE

In this section we will introduce a special class of set-Markov processes, called $\mathcal{Q}$ -Markov processes, for which the mechanism of transition from one state to another is completely known. This theory should be viewed as a generalization of the classical theory of Markov processes corresponding to a certain transition system.

We will assume that *the indexing collection $\mathcal{A}$ satisfies SHAPE* and we will prove that, under a certain consistency assumption, there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{A}}, \mathcal{B}(\mathbf{R})^{\mathcal{A}})$ under which the coordinate-variable process $X := (X_A)_{A \in \mathcal{A}}$ defined by $X_A(x) := x_A$ is $\mathcal{Q}$ -Markov. In particular, this proves the existence of a multiparameter process, which is sharp Markov with respect to the class of all finite unions of rectangles of type $[0, z]$.

We recall that by a *transition probability* on $\mathbf{R}$ we mean a function $Q(x; \Gamma)$, $x \in \mathbf{R}$, $\Gamma \in \mathcal{B}(\mathbf{R})$ which is measurable in x and a probability measure in Γ . Given two random variables X and Y , a transition probability Q is said to be a version of the *conditional distribution* of Y given X the if $P[Y \in \Gamma \mid X = x] = Q(x; \Gamma)$ for almost all x (with respect to the distribution of X). Such a version always exists, but it is

clearly not unique. (Properties of conditional distributions are discussed in Appendix A.2.)

Note that if $X := (X_t)_{t \in [0, a]}$ is a classical Markov process with initial distribution μ and for each $s \leq t$, Q_{st} is a version of the conditional distribution of X_t given X_s , then the finite dimensional distribution of X (over an ordered $n + 1$ -tuple of the form $0 < t_1 < t_2 < \dots < t_n$) is given by

$$P(X_0 \in \Gamma_0, X_{t_1} \in \Gamma_1, X_{t_2} \in \Gamma_2, \dots, X_{t_n} \in \Gamma_n) = \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) \prod_{i=1}^n I_{\Gamma_i}(x_i) Q_{t_{n-1}t_n}(x_{n-1}; dx_n) \dots Q_{t_1t_2}(x_1; dx_2) Q_{0t_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Using this observation, it would seem that all we need to construct a Markov process is the initial distribution μ and the conditional distributions Q_{st} . However, for any $s \leq t \leq u$ the conditional distribution of X_u given X_s has two versions: $Q_{su}(x; \Gamma)$ and $Q_{st}Q_{tu}(x; \Gamma) := \int_{\mathbf{R}} Q_{tu}(y; \Gamma) Q_{st}(x; dy)$ (by Proposition A.2.3, Appendix A.2); hence, for each $\Gamma \in \mathcal{B}(\mathbf{R})$ fixed, these versions will coincide for almost all x (with respect to the distribution of X_s), the negligible set depending on the set Γ .

In theory, one is confronted with the problem of constructing a Markov process $X := (X_t)_{t \in [0, a]}$ (and implicitly the probability space $(\Omega, \mathcal{F}, P)$ on which it is defined) knowing the initial distribution μ and a system $\mathcal{Q}$ of transition probabilities Q_{st} (each Q_{st} will be later identified as a version of the conditional distribution of X_t given X_s). However, because the probability measure P is not defined yet, we have to require that for $s \leq t \leq u$, the versions Q_{su} and $Q_{st}Q_{tu}$ coincide everywhere, i.e.

$$Q_{su}(x; \Gamma) = \int_{\mathbf{R}} Q_{tu}(y; \Gamma) Q_{st}(x; dy) \quad \forall x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R}).$$

In this case, the family $\mathcal{Q}$ is called a (one-parameter) *transition system*. A Markov process $X := (X_t)_{t \in [0, a]}$ for which Q_{st} is a version of the conditional distribution of X_t given X_s for every $s \leq t$, is called *$\mathcal{Q}$ -Markov*.

In what follows we will indicate how we can transfer these ideas to the set-indexed case.

Definition 3.2.1 (a) For each $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$ let $Q_{BB'}$ be a transition probability on $\mathbf{R}$. The family $\mathcal{Q} := (Q_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ is called a **transition system** if $\forall B \in \mathcal{A}(u)$, $Q_{BB}(x; \cdot) = \delta_x$ and $\forall B, B', B'' \in \mathcal{A}(u)$, $B \subseteq B' \subseteq B''$

$$Q_{BB''}(x; \Gamma) = \int_{\mathbf{R}} Q_{B'B''}(y; \Gamma) Q_{BB'}(x; dy) \quad \forall x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$$

(b) Let $\mathcal{Q} := (Q_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ be a transition system. A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called **$\mathcal{Q}$ -Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$** if it is adapted and $\forall B, B' \in \mathcal{A}(u)$, $B \subseteq B'$

$$P[X_{B'} \in \Gamma | \mathcal{F}_B] = Q_{BB'}(X_B; \Gamma) \text{ a.s. } \forall \Gamma \in \mathcal{B}(\mathbf{R}).$$

The process X is called **simply $\mathcal{Q}$ -Markov** if it is $\mathcal{Q}$ -Markov with respect to its minimal filtration.

Note that a $\mathcal{Q}$ -Markov process is a set-Markov process for which $Q_{BB'}$ is a version of the conditional distribution of $X_{B'}$ given X_B , for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$: from the definition of the $\mathcal{Q}$ -Markov property, $\forall B, B' \in \mathcal{A}(u)$, $B \subseteq B'$, $P[X_{B'} \in \Gamma | \mathcal{F}_B] = P[X_{B'} \in \Gamma | X_B]$, i.e. $\mathcal{F}_B \perp \sigma(X_{B'}) \mid \sigma(X_B)$.

According to the preceding discussion, if X is set-Markov and $Q_{BB'}$ is an arbitrary version of the conditional distribution of $X_{B'}$ given X_B for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$, then X is not necessarily $\mathcal{Q}$ -Markov, simply because the family $\mathcal{Q}$ of all $Q_{BB'}$'s may not be a transition system.

Let $\mathcal{Q} := (Q_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ be a fixed transition system.

Comment 3.2.2 Using a monotone class argument, it is not difficult to see that a set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov if and only if $\forall A \in \mathcal{A}$, $\forall B \in \mathcal{A}(u)$ and for every partition $B = \cup_{i=1}^p C_i$, $C_i \in \mathcal{C}$

$$P[X_{A \cup B} \in \Gamma | X_{C_1}, \dots, X_{C_p}] = Q_{B, A \cup B}(X_B; \Gamma) \text{ a.s. } \forall \Gamma \in \mathcal{B}(\mathbf{R}).$$

This property will be used in Chapter 6 in order to show that certain processes are $\mathcal{Q}$ -Markov.

The next result follows exactly as Proposition 3.1.6 and gives the expected correspondence via flows.

Proposition 3.2.3 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is $\mathcal{Q}^f$ -Markov with respect to the filtration $(\mathcal{F}_{f(t)})_{t \in [0, a]}$, where $\mathcal{Q}_{st}^f := \mathcal{Q}_{f(s), f(t)}$. (For the necessity part we can consider any flow, not only the simple ones.)*

The following result is the counterpart of Proposition 3.1.11; in particular, this result will allow us to write down the finite dimensional distributions of a set-indexed $\mathcal{Q}$ -Markov process in a close form that is very similar to the classical case.

Proposition 3.2.4 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov if and only if for every finite semilattice $\mathcal{A}'$, for every consistent ordering $\{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$ and for every $i = 1, \dots, n - 1$*

$$\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=0}^{i+1} A_j}) \mid \sigma(X_{\cup_{j=0}^i A_j}) \text{ and}$$

$$Q_{\cup_{j=1}^i A_j, \cup_{j=1}^{i+1} A_j} \text{ is a version of the c.d. of } X_{\cup_{j=1}^{i+1} A_j} \text{ given } X_{\cup_{j=1}^i A_j}$$

(c.d. stands for ‘conditional distribution’).

Proof: Necessity follows immediately by the definition of a $\mathcal{Q}$ -Markov process. For sufficiency, note first that the process X is set-Markov by Proposition 3.1.11. Consider now arbitrary sets $B, B' \in \mathcal{A}(u)$ with $B \subseteq B'$. We want to prove that $Q_{BB'}$ is a version of the c.d. of $X_{B'}$ given X_B .

By Proposition 2.1.14.(a), there exists a finite semilattice $\mathcal{A}'$ and a consistent ordering $\{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$ such that $B = \cup_{j=0}^{i_1} A_j$ and $B' = \cup_{j=0}^{i_2} A_j$ for some $i_1 \leq i_2$.

We can easily prove by induction on p that for every $i \leq n - p$

$$Q_{\cup_{j=0}^i A_j, \cup_{j=0}^{i+p} A_j} \text{ is a version of the c.d. of } X_{\cup_{j=0}^{i+p} A_j} \text{ given } X_{\cup_{j=0}^i A_j}.$$

(The statement for $p = 1$ is true by hypothesis. Assume next that the statement is true for p . The result for $p + 1$ follows by Proposition A.2.3, Appendix A.2, since

for each fixed $i \leq n - p - 1$, $Q_{\cup_{j=0}^i A_j, \cup_{j=0}^{i+p} A_j}$ is a version of the c.d. of $X_{\cup_{j=0}^{i+p} A_j}$ given $X_{\cup_{j=0}^i A_j}$, $Q_{\cup_{j=0}^{i+p} A_j, \cup_{j=1}^{i+p+1} A_j}$ is a version of the c.d. of $X_{\cup_{j=0}^{i+p+1} A_j}$ given $X_{\cup_{j=0}^{i+p} A_j}$, and $\sigma(X_{\cup_{j=0}^i A_j}) \perp \sigma(X_{\cup_{j=0}^{i+p+1} A_j}) \mid \sigma(X_{\cup_{j=0}^{i+p} A_j})$.

In particular, this implies that $Q_{BB'}$ is a version of the conditional distribution of $X_{B'}$ given X_B : take $i := i_1$ and $p := i_2 - i_1$.

□

The consequence of the previous proposition is that we can characterize a set-indexed $\mathcal{Q}$ -Markov process in terms of its finite dimensional distributions. More precisely, we have the following result.

Proposition 3.2.5 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with initial distribution μ if and only for every finite semilattice $\mathcal{A}'$ and for every consistent ordering $\{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$*

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n) = \quad (7)$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) \prod_{i=1}^n I_{\Gamma_i}(x_i) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^n A_j}(x_{n-1}; dx_n) \dots \\ Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Proof: We will use the characterization of the $\mathcal{Q}$ -Markov property given by Proposition 3.2.4.

For necessity, we have

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n) = \\ \int_{\Gamma_0} P[X_{A_1} \in \Gamma_1, X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n \mid X_{A_0} = x_0] \mu(dx_0) = \\ \int_{\Gamma_0} \int_{\Gamma_1} P[X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n \mid X_{A_1} = x_1] Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0)$$

because $Q_{\emptyset, A_1}$ is a version of the c.d. of X_{A_1} given $X_{\emptyset}$ and

$$P[X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n \mid X_{A_1} = x_1, X_{A_0} = x_0] =$$

$$P[X_{A_1 \cup A_2} \in \Gamma_2, \dots, X_{\cup_{j=1}^n A_j} \in \Gamma_n | X_{A_1} = x_1]$$

(since $\sigma(X_{A_0}) \perp \sigma(X_{A_1 \cup A_2}, \dots, X_{\cup_{j=1}^n A_j}) \mid \sigma(X_{A_1})$).

Continuing inductively in the same manner, at the last step we get the desired relationship.

For sufficiency, we note first that μ is the distribution of $X_{\emptyset}$ (take $\Gamma_i = \mathbf{R}; i = 1, \dots, n$ in (7)). Taking $\Gamma_2 = \dots = \Gamma_n = \mathbf{R}$ in (7) we obtain

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1) = \int_{\Gamma_0} \int_{\Gamma_1} Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0) \quad \forall \Gamma_0, \Gamma_1 \in \mathcal{B}(\mathbf{R})$$

which proves that $Q_{\emptyset, A_1}$ is a version of the c.d. of X_{A_1} given $X_{\emptyset}$.

For every $\Gamma_0, \Gamma_1, \Gamma_2 \in \mathcal{B}(\mathbf{R})$

$$\int_{\Gamma_0} \int_{\Gamma_1} P[X_{A_1 \cup A_2} \in \Gamma_2 | X_{A_1} = x_1, X_{A_0} = x_0] Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0) =$$

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_1 \cup A_2} \in \Gamma_2) =$$

$$\int_{\Gamma_0} \int_{\Gamma_1} Q_{A_1, A_1 \cup A_2}(x_1; \Gamma_2) Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0)$$

using equation (7) with $\Gamma_3 = \dots = \Gamma_n = \mathbf{R}$

This implies that

$$P[X_{A_1 \cup A_2} \in \Gamma_2 | X_{A_1} = x_1, X_{A_0} = x_0] = Q_{A_1, A_1 \cup A_2}(x_1; \Gamma_2)$$

$$(\text{depends only on } x_1) = P[X_{A_1 \cup A_2} \in \Gamma_2 | X_{A_1} = x_1]$$

for $P \circ (X_{A_0}, X_{A_1})^{-1}$ -almost all (x_0, x_1) and for every $\Gamma_2 \in \mathcal{B}(\mathbf{R})$. Consequently, $\sigma(X_{A_0}) \perp \sigma(X_{A_1 \cup A_2}) \mid \sigma(X_{A_1})$ and $Q_{A_1, A_1 \cup A_2}$ is a version of the c.d. of $X_{A_1 \cup A_2}$ given X_{A_1} .

Finally, using an inductive argument, we can conclude that for every $i \leq n-1$, $\sigma(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^{i-1} A_j}) \perp \sigma(X_{\cup_{j=0}^{i+1} A_j}) \mid \sigma(X_{\cup_{j=0}^i A_j})$ and $Q_{\cup_{j=1}^i A_j, \cup_{j=1}^{i+1} A_j}$ is a version of the c.d. of $X_{\cup_{j=1}^{i+1} A_j}$ given $X_{\cup_{j=1}^i A_j}$, i.e. X is a $\mathcal{Q}$ -Markov process, according to Proposition 3.2.4.

□

The next result says that the $\mathcal{Q}$ -Markov property of a set-indexed process depends only on the finite-dimensional distributions of the process. Consequently, any version of a $\mathcal{Q}$ -Markov process will automatically be $\mathcal{Q}$ -Markov.

Proposition 3.2.6 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with initial distribution μ if and only for every finite semilattice $\mathcal{A}'$ and for every consistent ordering $\{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$, if we suppose that $A_i = \cup_{j \in I_i} C_j$; $i = 2, \dots, n$ for some subset $I_i \subseteq \{1, \dots, i\}$, where C_i the left neighbourhood of A_i for each $i = 1, \dots, n$, then*

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_2} \in \Gamma_2, \dots, X_{A_n} \in \Gamma_n) = \\ \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(\sum_{j \in I_i} (x_j - x_{j-1})) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^n A_j}(x_{n-1}; dx_n) \dots \\ Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Proof: For each $i = 2, \dots, n$, $A_i = \cup_{j \in I_i} C_j = \cup_{j \in I_i} [(\cup_{l=1}^j A_l) \setminus (\cup_{l=1}^{j-1} A_l)]$ and

$$X_{A_i} = \sum_{j \in I_i} X_{C_j} = \sum_{j \in I_i} (X_{\cup_{l=1}^j A_l} - X_{\cup_{l=1}^{j-1} A_l}) \text{ a.s.}$$

□

If the indexing collection $\mathcal{A}$ satisfies SHAPE, the construction of a set-indexed $\mathcal{Q}$ -Markov process will be made using only the sets in $\mathcal{A}$.

The following assumption requires that the distribution of the process over the sets $A_0, A_1, \dots, A_n$ of a finite semilattice, does not depend on the consistent ordering of the semilattice.

Assumption 3.2.7 *If $\text{ord}1 = \{A_0 = \emptyset, A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset, A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$, with $A_i = A'_{\pi(i)}$, $\forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and we suppose that $A_i = \cup_{j \in I_i} C_j$ for some subset $I_i \subseteq \{1, \dots, i\}$, where C_i is the left neighbourhood of A_i for each $i = 1, \dots, n$, then*

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(\sum_{j \in I_i} (x_j - x_{j-1})) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^n A_j}(x_{n-1}; dx_n) \dots \quad (8)$$

$$Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0) =$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i} \left(\sum_{j \in I_i} (y_{\pi(j)} - y_{\pi(j)-1}) \right) Q_{\cup_{j=1}^{n-1} A'_j, \cup_{j=1}^n A'_j} (y_{n-1}; dy_n) \dots$$

$$Q_{A'_1, A'_1 \cup A'_2} (y_1; dy_2) Q_{\emptyset, A'_1} (y_0; dy_1) \mu(dy_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Proposition 3.2.8 *Assumption 3.2.7 is a necessary condition for the existence of a $\mathcal{Q}$ -Markov process (with initial distribution μ).*

Proof: Suppose that there exists a set-indexed $\mathcal{Q}$ -Markov process $X := (X_A)_{A \in \mathcal{A}}$ (with initial distribution μ). Let $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$ be arbitrary. Then

$$P(X_{A_0} \in \Gamma_0, X_{A_1} \in \Gamma_1, X_{A_2} \in \Gamma_2, \dots, X_{A_n} \in \Gamma_n) =$$

$$P(X_{A'_0} \in \Gamma_0, X_{A'_1} \in \Gamma_1, X_{A'_2} \in \Gamma_{\pi^{-1}(2)}, \dots, X_{A'_n} \in \Gamma_{\pi^{-1}(n)})$$

By Proposition 3.2.6, the left-hand side of the previous equation is equal to the left-hand side of equation (8); its right-hand side is equal to

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_{\pi^{-1}(i)}} \left(\sum_{j \in I'_i} (y_j - y_{j-1}) \right) Q_{\cup_{j=1}^{n-1} A'_j, \cup_{j=1}^n A'_j} (y_{n-1}; dy_n) \dots$$

$$Q_{A'_1, A'_1 \cup A'_2} (y_1; dy_2) Q_{\emptyset, A'_1} (y_0; dy_1) \mu(dy_0)$$

where $A'_i = \cup_{j \in I'_i} C'_j$ for some subset $I'_i \subseteq \{1, \dots, i\}$, and C'_i is the left neighbourhood of A'_i for each $i = 1, \dots, n$.

Finally $\cup_{j \in I'_{\pi(i)}} C'_j = A'_{\pi(i)} = A_i = \cup_{j \in I_i} C_j = \cup_{j \in I_i} C'_{\pi(j)}$, using Lemma 2.1.13.(b). Consequently, we have $I'_{\pi(i)} = \{\pi(j); j \in I_i\}$; hence

$$\prod_{i=2}^n I_{\Gamma_{\pi^{-1}(i)}} \left(\sum_{j \in I'_i} (y_j - y_{j-1}) \right) = \prod_{i=2}^n I_{\Gamma_i} \left(\sum_{j \in I'_{\pi(i)}} (y_j - y_{j-1}) \right) = \prod_{i=2}^n I_{\Gamma_i} \left(\sum_{j \in I_i} (y_{\pi(j)} - y_{\pi(j)-1}) \right)$$

and the result follows.

□

We are now ready to prove the main result of this section which says that, if the indexing collection satisfies *SHAPE*, then Assumption 3.2.7 is also sufficient to construct a $\mathcal{Q}$ -Markov process.

Theorem 3.2.9 *Suppose that the indexing collection $\mathcal{A}$ satisfies SHAPE.*

If μ is a probability measure on $\mathbf{R}$ and $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system which satisfies Assumption 3.2.7, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{A}}, \mathcal{B}(\mathbf{R})^{\mathcal{A}})$ under which the coordinate-variable process $X := (X_A)_{A \in \mathcal{A}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ .

Proof: For each k -tuple $(A_1, \dots, A_k)$ of distinct sets in $\mathcal{A}$ we will define a probability measure $\mu_{A_1 \dots A_k}$ on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R})^k)$ such that the system of all these probability measures satisfy the following two consistency conditions:

(C1) If $(A'_1 \dots A'_k)$ is another ordering of the k -tuple $(A_1 \dots A_k)$, say $A_i = A'_{\pi(i)}$, where π is a permutation of $\{1, \dots, k\}$, then

$$\mu_{A_1 \dots A_k}(\Gamma_1 \times \dots \times \Gamma_k) = \mu_{A'_1 \dots A'_k}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)})$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

(C2) If $A_1, \dots, A_k, A_{k+1}$ are $k+1$ distinct sets in $\mathcal{A}$, then

$$\mu_{A_1 \dots A_k}(\Gamma_1 \times \dots \times \Gamma_k) = \mu_{A_1 \dots A_k A_{k+1}}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R})$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

By Kolmogorov's extension theorem (Theorem 36.1, [10]) there exists a probability measure P on $(\mathbf{R}^{\mathcal{A}}, \mathcal{R}^{\mathcal{A}})$ under which the coordinate-variable process $X := (X_A)_{A \in \mathcal{A}}$ defined by $X_A(x) := x_A$ has the measures $\mu_{A_1 \dots A_k}$ as its finite dimensional distributions. The process X will have a unique additive extension to $\mathcal{C}(u)$ by Theorem 2.3.3 (and the additivity relationships will hold everywhere). Finally, the $\mathcal{Q}$ -Markov property of this process will follow from the way we choose its finite dimensional distributions $\mu_{A_0 A_1 \dots A_n}$ over the consistently ordered sets of a finite semilattice, according to Proposition 3.2.6.

Step 1 (Construction of the finite dimensional distributions)

Let $(A_1, \dots, A_k)$ be a k -tuple of distinct sets in $\mathcal{A}$. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A_1, \dots, A_k$, $\{B_0 = \emptyset, B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and suppose that $B_j = \cup_{v \in J_j} D_v$; $j = 2, \dots, n$ for some

subset $J_j \subseteq \{1, \dots, j\}$, where D_j is the left neighbourhood of the set B_j for each $j = 1, \dots, n$. Define

$$\mu_{B_0 B_1 \dots B_n}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_n) := \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{j=2}^n I_{\Gamma_j} \left(\sum_{v \in J_j} (x_v - x_{v-1}) \right)$$

$$Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^n B_j}(x_{n-1}; dx_n) \dots Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset', B_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Say $A_i = B_{j_i}$ for some $j_i \in \{1, \dots, n\}; i = 1, \dots, k$, let $\alpha : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k$, $\alpha(x_0, x_1, \dots, x_n) := (x_{j_1}, \dots, x_{j_k})$ and define

$$\mu_{A_1 \dots A_k} := \mu_{B_0 B_1 \dots B_n} \circ \alpha^{-1}.$$

The fact that $\mu_{A_1 \dots A_k}$ does not depend on the ordering of the semilattice $\mathcal{A}'$ is a consequence of Assumption 3.2.7.

Step 2 (Consistency condition (C1))

Let $(A'_1 \dots A'_k)$ be another ordering of the k -tuple $(A_1 \dots A_k)$, with $A_i = A'_{\pi(i)}$, π being a permutation of $\{1, \dots, k\}$. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A_1, \dots, A_k$, $\{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and suppose that $B_j = \cup_{v \in J_j} D_v$ for some subset $J_j \subseteq \{1, \dots, j\}$, where D_j is the left neighbourhood of the set B_j for each $j = 1, \dots, n$. Say $A_i = B_{j_i}, A'_i = B_{j'_i}$ and let $\alpha(x_0, x_1, \dots, x_n) := (x_{j_1}, \dots, x_{j_k}), \beta(x_0, x_1, \dots, x_n) := (x_{j'_1}, \dots, x_{j'_k})$. We have $j_i = j'_{\pi(i)}$. Hence

$$\begin{aligned} \alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k) &= \{(x_0, x_1, \dots, x_n); x_{j_1} \in \Gamma_1, \dots, x_{j_k} \in \Gamma_k\} \\ &= \{(x_0, x_1, \dots, x_n); x_{j'_1} \in \Gamma_{\pi^{-1}(1)}, \dots, x_{j'_k} \in \Gamma_{\pi^{-1}(k)}\} \\ &= \beta^{-1}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)}) \end{aligned}$$

and

$$\begin{aligned} \mu_{A_1 \dots A_k}(\Gamma_1 \times \dots \times \Gamma_k) &= \mu_{B_0 B_1 \dots B_n}(\alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) \\ &= \mu_{B_0 B_1 \dots B_n}(\beta^{-1}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)})) \\ &= \mu_{A'_1 \dots A'_k}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)}) \end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

Step 3 (Consistency Condition (C2))

Let $A_1, \dots, A_k, A_{k+1}$ be $k+1$ distinct sets in $\mathcal{A}$. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contain the sets $A_1, \dots, A_k$, respectively $A_1, \dots, A_k, A_{k+1}$. Clearly $\mathcal{A}' \subseteq \mathcal{A}''$. Using Proposition 2.1.14.(b), there exist a consistent ordering $\{B_0 = \emptyset, B_1, \dots, B_n\}$ of $\mathcal{A}''$ and a consistent ordering $\{B'_0 = \emptyset, B'_1, \dots, B'_m\}$ of $\mathcal{A}'$ such that, if $B'_l = B_{i_l}; l = 1, \dots, m$ for some indices $i_1 < i_2 < \dots < i_m$, then $\cup_{s=1}^l B'_s = \cup_{j=1}^{i_l} B_j$ for all $l = 1, \dots, m$. Let $\gamma(x_0, x_1, \dots, x_n) = (x_0, x_{i_1}, \dots, x_{i_m})$ and suppose for the moment that

$$\mu_{B'_0 B'_1 \dots B'_m} = \mu_{B_0 B_1 \dots B_n} \circ \gamma^{-1}. \quad (9)$$

Say $A_i = B_{j_i}; i = 1, \dots, k+1$ for some $j_i \in \{1, \dots, n\}$ and define $\alpha(x_0, x_1, \dots, x_n) := (x_{j_1}, \dots, x_{j_{k+1}})$. On the other hand, if we say that for each $i = 1, \dots, k$ we have $A_i = B'_{l_i}$ for some $l_i \in \{1, \dots, m\}$, then $j_1 = i_{l_1}, \dots, j_k = i_{l_k}$; define $\beta(y_0, y_1, \dots, y_m) = (y_{l_1}, \dots, y_{l_k})$. Then

$$\begin{aligned} \beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k) &= \{(y_0, y_1, \dots, y_m); y_{l_1} \in \Gamma_1, \dots, y_{l_k} \in \Gamma_k\} \\ \gamma^{-1}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) &= \{(x_0, x_1, \dots, x_n); x_{i_{l_1}} \in \Gamma_1, \dots, x_{i_{l_k}} \in \Gamma_k\} \\ &= \{(x_0, x_1, \dots, x_n); x_{j_1} \in \Gamma_1, \dots, x_{j_k} \in \Gamma_k\} \\ &= \alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R}) \end{aligned}$$

and

$$\begin{aligned} \mu_{A_1 \dots A_k}(\Gamma_1 \times \dots \times \Gamma_k) &\stackrel{\text{def}}{=} \mu_{B'_0 B'_1 \dots B'_m}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) \\ &= \mu_{B_0 B_1 \dots B_n}(\gamma^{-1}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k))) \\ &= \mu_{B_0 B_1 \dots B_n}(\alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R})) \\ &\stackrel{\text{def}}{=} \mu_{A_1 \dots A_k A_{k+1}}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R}) \end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

In order to prove (9) let $\Gamma_0, \Gamma_1, \dots, \Gamma_m \in \mathcal{B}(\mathbf{R})$ be arbitrary. Suppose that $B_j = \cup_{v \in J_j} D_v; j = 2, \dots, n$ for some subset $J_j \subseteq \{1, \dots, j\}$, where D_j is the left neighbourhood of the set B_j for each $j = 1, \dots, n$. Then

$$\mu_{B_0 B_1 \dots B_n}(\gamma^{-1}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m)) = \quad (10)$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m}(\gamma(x_0, x_1, \sum_{j \in J_2} (x_j - x_{j-1}), \dots, \sum_{j \in J_n} (x_j - x_{j-1}))) \\ Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^n B_j}(x_{n-1}; dx_n) \dots Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset, B_1}(x_0; dx_1) \mu(dx_0).$$

Note that $\gamma(x_0, x_1, \sum_{j \in J_2} (x_j - x_{j-1}), \dots, \sum_{j \in J_n} (x_j - x_{j-1})) = (x_0, \sum_{j \in J_{i_1}} (x_j - x_{j-1}), \dots, \sum_{j \in J_{i_m}} (x_j - x_{j-1}))$. On the other hand, suppose that $B'_l = \cup_{w \in L_l} D'_w$; $l = 1, \dots, m$ for some subset $L_l \subseteq \{1, \dots, l\}$, where D'_l is the left neighbourhood of the set B'_l for each $l = 1, \dots, m$. Note that $D'_l = (\cup_{w=1}^l B'_w) \setminus (\cup_{w=1}^{l-1} B'_w) = (\cup_{j=1}^{i_l} B_j) \setminus (\cup_{j=1}^{i_l-1} B_j) = \cup_{j=i_{l-1}+1}^{i_l} B_j$ and hence we have

$$\cup_{v \in J_{i_l}} D_v = B_{i_l} = B'_l = \cup_{w \in L_l} D'_w = \cup_{w \in L_l} \cup_{v=i_{w-1}+1}^{i_w} D_v$$

This implies that

$$J_{i_l} = \cup_{w \in L_l} \{i_{w-1} + 1, i_{w-1} + 2, \dots, i_w\} \quad \forall l = 1, \dots, m$$

and $\sum_{j \in J_{i_l}} (x_j - x_{j-1}) = \sum_{w \in L_l} \sum_{j=i_{w-1}+1}^{i_w} (x_j - x_{j-1}) = \sum_{w \in L_l} (x_{i_w} - x_{i_{w-1}})$. Therefore the integrand of (10) does not depend on the variables $x_j, j \notin \{i_0, i_1, \dots, i_m\}$. By the definition of the transition system, the integral of (10) collapses to

$$\int_{\mathbf{R}^{m+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_{i_1}) \prod_{l=2}^m I_{\Gamma_l}(\sum_{w \in L_l} (x_{i_w} - x_{i_{w-1}})) Q_{\cup_{j=1}^{i_{m-1}-1} B_j, \cup_{j=1}^{i_m} B_j}(x_{i_{m-1}}; dx_{i_m}) \dots \\ Q_{\cup_{j=1}^{i_1-1} B_j, \cup_{j=1}^{i_2} B_j}(x_{i_1}; dx_{i_2}) Q_{\emptyset, \cup_{j=1}^{i_1} B_j}(x_0; dx_{i_1}) \mu(dx_0)$$

which is exactly the definition of $\mu_{B'_0 B'_1 \dots B'_m}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m)$ because $\cup_{j=1}^{i_l} B_j = \cup_{s=1}^l B'_s$ for every $l = 1, \dots, m$. This concludes the proof of (9).

□

Finally we will translate the preceding result in terms of flows. Let μ be an arbitrary probability measure on $\mathbf{R}$.

Definition 3.2.10 For each finite sub-semilattice $\mathcal{A}'$ and for each consistent ordering $\text{ord} = \{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$ pick one simple flow $f := f_{\mathcal{A}', \text{ord}}$ which ‘connects the sets of $\mathcal{A}'$ in the sense of the ordering ord ’ (see Lemma 2.4.8). Let $\mathcal{S}$ be the collection of all flows $f_{\mathcal{A}', \text{ord}}$.

The next assumption will provide a necessary and sufficient condition which will allow us to reconstruct a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ from a class of transition systems $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$, if the indexing collection $\mathcal{A}$ satisfies SHAPE. It basically says that whenever we have two simple flows $f, g \in \mathcal{S}$ such that $f(s) = g(u)$, $f(t) = g(v)$ for some $s < t, u < v$ we must have $Q_{st}^f = Q_{uv}^g$.

Assumption 3.2.11 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$, such that $A_1, \dots, A_k$ is a permutation of $A'_1, \dots, A'_k$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then*

$$Q_{0t_1}^f = Q_{0u_1}^g \quad \text{and} \quad Q_{t_k t_n}^f = Q_{u_k u_n}^g.$$

The following assumption is easily seen to be equivalent to Assumption 3.2.7.

Assumption 3.2.12 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, we suppose that $A_i = \cup_{j \in I_i} C_j$ for some subset $I_i \subseteq \{1, \dots, i\}$, where C_i is the left neighbourhood of A_i for each $i = 1, \dots, n$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then*

$$\begin{aligned} & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(\sum_{j \in I_i} (x_j - x_{j-1})) Q_{t_{n-1} t_n}^f(x_{n-1}; dx_n) \dots \quad (11) \\ & Q_{t_1 t_2}^f(x_1; dx_2) Q_{t_0 t_1}^f(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i}(\sum_{j \in I_i} (y_{\pi(j)} - y_{\pi(j)-1})) Q_{u_{n-1} u_n}^g(y_{n-1}; dy_n) \dots \\ & Q_{u_1 u_2}^g(y_1; dy_2) Q_{u_0 u_1}^g(y_0; dy_1) \mu(dy_0) \end{aligned}$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

The following result is immediate.

Corollary 3.2.13 *Suppose that the indexing collection $\mathcal{A}$ satisfies SHAPE.*

If μ is a probability measure on $\mathbf{R}$ and $\{Q^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ is a collection of transition systems which satisfies Assumption 3.2.11 and Assumption 3.2.12, then there exist a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^{\mathcal{A}}, \mathcal{B}(\mathbf{R})^{\mathcal{A}})$ under which:

1. the coordinate-variable process $X := (X_A)_{A \in \mathcal{A}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ ; and
2. $\forall f \in \mathcal{S}$, $X^f := (X_{f(t)})_t$ is $\mathcal{Q}^f$ -Markov.

Proof: Let $B, B' \in \mathcal{A}(u)$ be such that $B \subseteq B'$. Let $B = \cup_{j=1}^m A'_j$, $B' = \cup_{l=1}^p A''_l$ be their *unique* extremal representations, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A'_j, A''_l and $\text{ord} = \{A_0 = \emptyset, A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$ such that $B = \cup_{j=1}^k A_j$ and $B' = \cup_{j=1}^n A_j$. Denote $f := f_{\mathcal{A}', \text{ord}}$ with $f(t_i) = \cup_{j=1}^i A_j$. We define $Q_{BB'} := Q_{t_k t_n}^f$.

The definition of $Q_{BB'}$ does not depend on the ordering of the semilattice $\mathcal{A}'$, because of Assumption 3.2.11.

The family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ of all these transition probabilities is a transition system: if $B \subseteq B' \subseteq B'' \in \mathcal{A}(u)$ are such that $B \subseteq B' \subseteq B''$, then there exists a finite semilattice $\mathcal{A}'$ and a consistent ordering $\text{ord} = \{A_0 = \emptyset, A_1, \dots, A_n\}$ of $\mathcal{A}'$, such that $B = \cup_{j=1}^k A_j$, $B' = \cup_{j=1}^l A_j$, $B'' = \cup_{j=1}^m A_j$ for some $k \leq l \leq m$; if we denote $f := f_{\mathcal{A}', \text{ord}}$ with $f(t_i) = \cup_{j=1}^i A_j$, then

$$Q_{BB'} Q_{B'B''} = Q_{st}^f Q_{tu}^f = Q_{su}^f = Q_{BB''}$$

because $\mathcal{Q}^f$ is a one-dimensional transition system.

Finally, it is not hard to see that the transition system $\mathcal{Q}$ satisfies Assumption 3.2.7 and the conclusion follows by Theorem 3.2.9.

□

3.3 Q-Markov Processes. The General Construction

If the indexing collection $\mathcal{A}$ does not satisfy SHAPE, then the process that we constructed in the previous section may not have a unique additive extension to $\mathcal{C}(u)$; therefore this construction is not valid in general. In this section we will give the construction of a $\mathcal{Q}$ -Markov process in the general case, when assumption SHAPE does not necessarily hold. The construction will be made using increments (i.e. sets in $\mathcal{C}$) instead of sets in $\mathcal{A}$; more precisely, we will prove that there exists a probability measure P on the space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined by $X_C(x) := x_C, C \in \mathcal{C}$ is $\mathcal{Q}$ -Markov. The (almost sure) additivity of this process will follow from the way we choose its finite dimensional distributions.

Let μ be a probability measure on $\mathbf{R}$ and $\mathcal{Q} := (Q_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ a fixed transition system.

We begin by giving another characterization of a $\mathcal{Q}$ -Markov process in terms of its finite dimensional distributions over the sets in $\mathcal{C}$.

Proposition 3.3.1 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with initial distribution μ if and only for every finite semilattice $\mathcal{A}'$ and for every consistent ordering $\{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$, if we denote with C_i the left neighbourhood of A_i for each $i = 0, \dots, n$, then*

$$P(X_{C_0} \in \Gamma_0, X_{C_1} \in \Gamma_1, X_{C_2} \in \Gamma_2, \dots, X_{C_n} \in \Gamma_n) = \quad (12)$$

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(x_i - x_{i-1}) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^n A_j}(x_{n-1}; dx_n) \dots \\ Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset, A_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Proof: For each $i = 2, \dots, n$, $C_i = (\cup_{j=1}^i A_j) \setminus (\cup_{j=1}^{i-1} A_j)$ and

$$X_{C_i} = X_{\cup_{j=1}^i A_j} - X_{\cup_{j=1}^{i-1} A_j} \quad \text{a.s.}$$

The result follows by Proposition 3.2.5.

□

In order to be able to construct a $\mathcal{Q}$ -Markov process in the general case we must impose another consistency assumption on the transition system $\mathcal{Q}$, which requires that the distribution of the process over the left neighbourhoods $C_0, C_1, \dots, C_n$ of a finite sub-semilattice does not depend on the consistent ordering of the semilattice.

Assumption 3.3.2 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}, \forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, then*

$$\begin{aligned} & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(x_i - x_{i-1}) Q_{\cup_{j=1}^{n-1} A_j, \cup_{j=1}^n A_j}(x_{n-1}; dx_n) \dots \\ & Q_{A_1, A_1 \cup A_2}(x_1; dx_2) Q_{\emptyset' A_1}(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{\cup_{j=1}^{n-1} A'_j, \cup_{j=1}^n A'_j}(y_{n-1}; dy_n) \dots \\ & Q_{A'_1, A'_1 \cup A'_2}(y_1; dy_2) Q_{\emptyset' A'_1}(y_0; dy_1) \mu(dy_0) \end{aligned} \quad (13)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Note: In Section 3.2, we have seen that Assumption 3.2.7 (which involves only the finite dimensional distributions over the sets in $\mathcal{A}$) is a necessary condition for the existence of a $\mathcal{Q}$ -Markov process, and we have proved that, if SHAPE holds, then this assumption is also sufficient. In this section, we will prove that Assumption 3.3.2 is also a necessary and sufficient condition for the existence of a $\mathcal{Q}$ -Markov process. Therefore, in general Assumption 3.3.2 is stronger than Assumption 3.2.7, but if SHAPE holds, then the two assumptions are equivalent.

The technical difference between these two assumptions is that they rely on the two different bijections which relate the vector $(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^i A_j})$ with the vector $(X_{A_0}, X_{A_1}, \dots, X_{A_n})$ on one hand, for Assumption 3.2.7, respectively with the vector $(X_{C_0}, X_{C_1}, \dots, X_{C_n})$ on the other hand, for Assumption 3.3.2.

Proposition 3.3.3 *Assumption 3.3.2 is a necessary condition for the existence of a $\mathcal{Q}$ -Markov process (with initial distribution μ).*

Proof: Suppose that there exists a set-indexed $\mathcal{Q}$ -Markov process $X := (X_A)_{A \in \mathcal{A}}$ (with initial distribution μ). Let $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$ be arbitrary. By Lemma 2.1.13 we have $C_i = C'_{\pi(i)}$. Then

$$P(X_{C_0} \in \Gamma_0, X_{C_1} \in \Gamma_1, X_{C_2} \in \Gamma_2, \dots, X_{C_n} \in \Gamma_n) =$$

$$P(X_{C'_0} \in \Gamma_0, X_{C'_1} \in \Gamma_1, X_{C'_2} \in \Gamma_{\pi^{-1}(2)}, \dots, X_{C'_n} \in \Gamma_{\pi^{-1}(n)}).$$

By Proposition 3.3.1, the left-hand side of the previous equation is equal to the left-hand side of equation (13); its right-hand side is equal to

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_{\pi^{-1}(i)}}(y_i - y_{i-1}) Q_{\cup_{j=1}^{n-1} A'_j, \cup_{j=1}^n A'_j}(y_{n-1}; dy_n) \dots \\ Q_{A'_1, A'_1 \cup A'_2}(y_1; dy_2) Q_{\emptyset, A'_1}(y_0; dy_1) \mu(dy_0).$$

Finally $\prod_{j=2}^n I_{\Gamma_{\pi^{-1}(i)}}(y_i - y_{i-1}) = \prod_{i=2}^n I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1})$ and the result follows.

□

The finite dimensional distributions of a $\mathcal{Q}$ -Markov process over the sets in $\mathcal{C}$ have to be defined so that they ensure the (almost sure) additivity of the process. The next result gives the definition of the finite dimensional distribution (of a $\mathcal{Q}$ -Markov process) over an arbitrary k -tuple of sets in $\mathcal{C}$ and shows that, if the transition system satisfies Assumption 3.3.2, then the definition will not depend on the extremal representations of these sets.

Lemma 3.3.4 *Let $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ be a transition system satisfying Assumption 3.3.2. Let $(C_1, \dots, C_k)$ be a k -tuple of distinct sets in $\mathcal{C}$ and suppose that each set C_i admits two extremal representations $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij} = A'_i \setminus \cup_{j=1}^{m_i} A'_{ij}$. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattices of $\mathcal{A}$ which contain the sets A_i, A_{ij} , respectively A'_i, A'_{ij} , $\{B_0 = \emptyset, B_1, \dots, B_n\}, \{B'_0 = \emptyset, B'_1, \dots, B'_m\}$ two consistent orderings of $\mathcal{A}', \mathcal{A}''$ and D_j, D'_l the left neighbourhoods of the sets B_j, B'_l for $j = 1, \dots, n; l = 1, \dots, m$. If each set $C_i; i = 1, \dots, k$ can be written as $C_i = \cup_{j \in J_i} D_j = \cup_{l \in L_i} D'_l$ for some $J_i \subseteq \{1, \dots, n\}, L_i \subseteq \{1, \dots, m\}$, then*

$$\int_{\mathbf{R}^{n+1}} \prod_{i=1}^k I_{\Gamma_i}(\sum_{j \in J_i} (x_j - x_{j-1})) Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^n B_j}(x_{n-1}; dx_n) \dots \quad (14)$$

$$\begin{aligned}
& Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset' B_1}(x; dx_1) \mu(dx) = \\
& \int_{\mathbf{R}^{m+1}} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{l \in L_i} (y_l - y_{l-1}) \right) Q_{\cup_{l=1}^{m-1} B'_l, \cup_{l=1}^m B'_l}(y_{m-1}; dy_m) \dots \\
& Q_{B'_1, B'_1 \cup B'_2}(y_1; dy_2) Q_{\emptyset' B'_1}(y; dy_1) \mu(dy)
\end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$, with the convention $x_0 = y_0 = 0$.

Proof: Let $\tilde{\mathcal{A}}$ be the minimal finite sub-semilattice of $\mathcal{A}$ determined by the sets in $\mathcal{A}'$ and $\mathcal{A}''$; clearly $\mathcal{A}' \subseteq \tilde{\mathcal{A}}, \mathcal{A}'' \subseteq \tilde{\mathcal{A}}$. Let $\text{ord}^1 = \{E_0 = \emptyset', E_1, \dots, E_N\}$ and $\text{ord}^2 = \{E'_0 = \emptyset', E'_1, \dots, E'_N\}$ be two consistent orderings of $\tilde{\mathcal{A}}$ such that, if $B_j = E_{i_j}; j = 1, \dots, n$ and $B'_l = E'_{k_l}; l = 1, \dots, m$ for some indices $i_1 < i_2 < \dots < i_n$, respectively $k_1 < k_2 < \dots < k_m$, then

$$\begin{aligned}
B_1 &= \cup_{p=1}^{i_1} E_p, \quad B_1 \cup B_2 = \cup_{p=1}^{i_2} E_p, \dots, \cup_{j=1}^n B_j = \cup_{p=1}^{i_n} E_p \\
B'_1 &= \cup_{q=1}^{k_1} E'_q, \quad B'_1 \cup B'_2 = \cup_{q=1}^{k_2} E'_q, \dots, \cup_{l=1}^m B'_l = \cup_{q=1}^{k_m} E'_q.
\end{aligned}$$

Let π be the permutation of $\{1, \dots, N\}$ such that $E_p = E'_{\pi(p)}; p = 1, \dots, N$. Denote by H_p, H'_q the left neighbourhoods of E_p, E'_q with respect to the orderings $\text{ord}^1, \text{ord}^2$; clearly $H_p = H'_{\pi(p)}, \forall p = 1, \dots, N$. Note that $D_j = (\cup_{v=1}^j B_v) \setminus (\cup_{v=1}^{j-1} B_v) = (\cup_{p=1}^{i_j} E_p) \setminus (\cup_{p=1}^{i_{j-1}} E_p) = \cup_{p=i_{j-1}+1}^{i_j} H_p; j = 1, \dots, n$ and similarly $D'_l = \cup_{q=k_{l-1}+1}^{k_l} H'_q; l = 1, \dots, m$. Hence

$$\begin{aligned}
C_i &= \cup_{j \in J_i} D_j = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H_p = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H'_{\pi(p)} \\
&= \cup_{l \in L_i} D'_l = \cup_{l \in L_i} \cup_{q=k_{l-1}+1}^{k_l} H'_q
\end{aligned}$$

and we can conclude that

$$\{\pi(p); p \in \cup_{j \in J_i} \{i_{j-1} + 1, i_{j-1} + 2, \dots, i_j\}\} = \cup_{l \in L_i} \{k_{l-1} + 1, k_{l-1} + 2, \dots, k_l\}.$$

Assumption 3.3.2 combined with the previous relationship implies that

$$\begin{aligned}
& \int_{\mathbf{R}^{N+1}} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{j \in J_i} \sum_{p=i_{j-1}+1}^{i_j} (x_p - x_{p-1}) \right) Q_{\cup_{p=1}^{N-1} E_p, \cup_{p=1}^N E_p}(x_{N-1}; dx_N) \dots \\
& Q_{E_1, E_1 \cup E_2}(x_1; dx_2) Q_{\emptyset' E_1}(x; dx_1) \mu(dx) =
\end{aligned}$$

$$\begin{aligned}
& \int_{\mathbf{R}^{N+1}} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{j \in J_i} \sum_{p=i_{j-1}+1}^{i_j} (y_{\pi(p)} - y_{\pi(p)-1}) \right) Q_{\cup_{q=1}^{N-1} E'_q, \cup_{q=1}^N E'_q} (y_{N-1}; dy_N) \dots \\
& \quad Q_{E'_1, E'_1 \cup E'_2} (y_1; dy_2) Q_{\emptyset' E'_1} (y; dy_1) \mu(dy) = \\
& \int_{\mathbf{R}^{N+1}} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{l \in L_i} \sum_{q=k_{l-1}+1}^{k_l} (y_q - y_{q-1}) \right) Q_{\cup_{q=1}^{N-1} E'_q, \cup_{q=1}^N E'_q} (y_{N-1}; dy_N) \dots \\
& \quad Q_{E'_1, E'_1 \cup E'_2} (y_1; dy_2) Q_{\emptyset' E'_1} (y; dy_1) \mu(dy)
\end{aligned}$$

with the convention $x_0 = y_0 = 0$. This gives us the desired relationship, because $\sum_{p=i_{j-1}+1}^{i_j} (x_p - x_{p-1}) = x_{i_j} - x_{i_{j-1}}$, the left-hand side integral collapses to an integral with respect to $Q_{\cup_{j=1}^n B_j, \cup_{j=1}^{n-1} B_j} (x_{i_n}; dx_{i_n}) \dots Q_{B_1, B_1 \cup B_2} (x_{i_1}; dx_{i_2}) Q_{\emptyset' B_1} (x; dx_{i_1}) \mu(dx)$, and a similar phenomenon happens in the right-hand side.

□

We are now ready to prove the main result of this section which says that Assumption 3.3.2 is also sufficient to construct a $\mathcal{Q}$ -Markov process.

Theorem 3.3.5 *If μ is a probability measure on $\mathbf{R}$ and $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system which satisfies Assumption 3.3.2, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ .*

Proof: For each k -tuple $(C_1, \dots, C_k)$ of distinct sets in $\mathcal{C}$ we will define a probability measure $\mu_{C_1 \dots C_k}$ on $(\mathbf{R}^k, \mathcal{B}(\mathbf{R})^k)$ such that the system of all these probability measures satisfy the following two consistency conditions:

(C1) If $(C'_1 \dots C'_k)$ is another ordering of the k -tuple $(C_1 \dots C_k)$, say $C_i = C'_{\pi(i)}$, where π is a permutation of $\{1, \dots, k\}$, then

$$\mu_{C_1 \dots C_k} (\Gamma_1 \times \dots \times \Gamma_k) = \mu_{C'_1 \dots C'_k} (\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)})$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

(C2) If $C_1, \dots, C_k, C_{k+1}$ are $k+1$ distinct sets in $\mathcal{C}$, then

$$\mu_{C_1 \dots C_k} (\Gamma_1 \times \dots \times \Gamma_k) = \mu_{C_1 \dots C_k C_{k+1}} (\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R})$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

By Kolmogorov's extension theorem (Theorem 36.1, [10]), there exists a probability measure P on $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined by $X_C(x) := x_C$ has the measures $\mu_{C_1 \dots C_k}$ as its finite dimensional distributions. We will prove that the process X has an (almost surely) unique additive extension to $\mathcal{C}(u)$. The $\mathcal{Q}$ -Markov property of this process will follow from the way we choose its finite dimensional distributions $\mu_{C_0 C_1 \dots C_n}$ over the left neighbourhoods of a finite sub-semilattice, according to Proposition 3.3.1.

Step 1 (Construction of the finite dimensional distributions)

Let $(C_1, \dots, C_k)$ be a k -tuple of distinct sets in $\mathcal{C}$ and $C_i = A_i \setminus \bigcup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k$ some extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A_i, A_{ij}, \{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of the set B_j for $j = 1, \dots, n$. Define

$$\mu_{D_0 D_1 \dots D_n}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_n) := \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{j=2}^n I_{\Gamma_j}(x_j - x_{j-1})$$

$$Q_{\bigcup_{j=1}^{n-1} B_j, \bigcup_{j=1}^n B_j}(x_{n-1}; dx_n) \dots Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset', B_1}(x_0; dx_1) \mu(dx_0)$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

Say $C_i = \bigcup_{j \in J_i} D_j$ for some $J_i \subseteq \{1, \dots, n\}; i = 1, \dots, k$, let $\alpha : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^k, \alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_k} x_j)$ and define

$$\mu_{C_1 \dots C_k} := \mu_{D_0 D_1 \dots D_n} \circ \alpha^{-1}.$$

The fact that $\mu_{C_1 \dots C_k}$ does not depend on the ordering of the semilattice $\mathcal{A}'$ is a consequence of Assumption 3.3.2. The fact that $\mu_{C_1 \dots C_k}$ does not depend on the extremal representations of the sets C_i is also a consequence of Assumption 3.3.2, using Lemma 3.3.4. Finally, we observe that the finite dimensional distributions are additive.

Step 2 (Consistency condition (C1))

Let $(C'_1 \dots C'_k)$ be another ordering of the k -tuple $(C_1 \dots C_k)$, with $C_i = C'_{\pi(i)}$, π being a permutation of $\{1, \dots, k\}$. Let $C_i = A_i \setminus \bigcup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k$ be extremal representations, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A_i, A_{ij}, \{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of the set B_j for each $j = 1, \dots, n$. Say $C_i = \bigcup_{j \in J_i} D_j, C'_i = \bigcup_{j \in J'_i} D_j$ and

let $\alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_k} x_j)$, $\beta(x_0, x_1, \dots, x_n) := (\sum_{j \in J'_1} x_j, \dots, \sum_{j \in J'_k} x_j)$. We have $J_i = J'_{\pi(i)}$. Hence

$$\begin{aligned} \alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k) &= \{(x_0, x_1, \dots, x_n); \sum_{j \in J_1} x_j \in \Gamma_1, \dots, \sum_{j \in J_k} x_j \in \Gamma_k\} \\ &= \{(x_0, x_1, \dots, x_n); \sum_{j \in J'_1} x_j \in \Gamma_{\pi^{-1}(1)}, \dots, \sum_{j \in J'_k} x_j \in \Gamma_{\pi^{-1}(k)}\} \\ &= \beta^{-1}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)}) \end{aligned}$$

and

$$\begin{aligned} \mu_{C_1 \dots C_k}(\Gamma_1 \times \dots \times \Gamma_k) &= \mu_{D_0 D_1 \dots D_n}(\alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) \\ &= \mu_{D_0 D_1 \dots D_n}(\beta^{-1}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)})) \\ &= \mu_{C'_1 \dots C'_k}(\Gamma_{\pi^{-1}(1)} \times \dots \times \Gamma_{\pi^{-1}(k)}) \end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

Step 3 (Consistency Condition (C2))

Let $C_1, \dots, C_k, C_{k+1}$ be $k+1$ distinct sets in $\mathcal{C}$ and $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k+1$ some extremal representations. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattices of $\mathcal{A}$ which contain the sets $A_i, A_{ij}; i = 1, \dots, k; j = 1, \dots, n_i$, respectively $A_i, A_{ij}; i = 1, \dots, k+1; j = 1, \dots, n_i$. Clearly $\mathcal{A}' \subseteq \mathcal{A}''$. Using Proposition 2.1.14.(b), there exist a consistent ordering $\{B_0 = \emptyset, B_1, \dots, B_n\}$ of $\mathcal{A}''$ and a consistent ordering $\{B'_0 = \emptyset, B'_1, \dots, B'_m\}$ of $\mathcal{A}'$ such that, if $B'_l = B_{i_l}; l = 1, \dots, m$ for some indices $i_1 < i_2 < \dots < i_m$, then $\cup_{s=1}^l B'_s = \cup_{j=1}^{i_l} B_j$ for all $l = 1, \dots, m$. For each $j = 1, \dots, n; l = 1, \dots, m$, let D_j, D'_l be the left neighbourhoods of B_j in $\mathcal{A}''$, respectively of B'_l in $\mathcal{A}'$ and recall that $D'_l = \cup_{j=i_{l-1}+1}^{i_l} D_j$ for each $l = 1, \dots, m$. Let $\gamma(x_0, x_1, \dots, x_n) := (x_0, \sum_{j=1}^{i_1} x_j, \sum_{j=i_1+1}^{i_2} x_j, \dots, \sum_{j=i_{m-1}+1}^{i_m} x_j)$ and for the moment suppose that

$$\mu_{D'_0 D'_1 \dots D'_m} = \mu_{D_0 D_1 \dots D_n} \circ \gamma^{-1}. \quad (15)$$

Say $C_i = \cup_{j \in J_i} D_j; i = 1, \dots, k+1$ for some $J_i \subseteq \{1, \dots, n\}$ and define $\alpha(x_0, x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_{k+1}} x_j)$. On the other hand, if we say that for each $i = 1, \dots, k$ we have $C_i = \cup_{l \in L_i} D'_l$ for some $L_i \subseteq \{1, \dots, m\}$, then $J_i = \cup_{l \in L_i} \{i_{l-1} + 1, i_{l-1} + 2, \dots, i_l\};$

define $\beta(y_0, y_1, \dots, y_m) := (\sum_{l \in L_1} y_l, \dots, \sum_{l \in L_k} y_l)$. Then

$$\begin{aligned} \beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k) &= \{(y_0, y_1, \dots, y_m); \sum_{l \in L_1} y_l \in \Gamma_1, \dots, \sum_{l \in L_k} y_l \in \Gamma_k\} \\ \gamma^{-1}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) &= \{(x_0, x_1, \dots, x_n); \sum_{l \in L_i} \sum_{j=i_{l-1}+1}^{i_l} x_j \in \Gamma_i \quad \forall i \leq k\} \\ &= \{(x_0, x_1, \dots, x_n); \sum_{j \in J_1} x_j \in \Gamma_1, \dots, \sum_{j \in J_k} x_j \in \Gamma_k\} \\ &= \alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R}) \end{aligned}$$

and

$$\begin{aligned} \mu_{C_1 \dots C_k}(\Gamma_1 \times \dots \times \Gamma_k) &\stackrel{\text{def}}{=} \mu_{D'_0 D'_1 \dots D'_m}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k)) \\ &= \mu_{D_0 D_1 \dots D_n}(\gamma^{-1}(\beta^{-1}(\Gamma_1 \times \dots \times \Gamma_k))) \\ &= \mu_{D_0 D_1 \dots D_n}(\alpha^{-1}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R})) \\ &\stackrel{\text{def}}{=} \mu_{C_1 \dots C_k C_{k+1}}(\Gamma_1 \times \dots \times \Gamma_k \times \mathbf{R}) \end{aligned}$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

In order to prove (15) let $\Gamma_0, \Gamma_1, \dots, \Gamma_m \in \mathcal{B}(\mathbf{R})$ be arbitrary. Then

$$\begin{aligned} \mu_{D_0 D_1 \dots D_n}(\gamma^{-1}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m)) &= \\ \int_{\mathbf{R}^{n+1}} I_{\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m}(\gamma(x_0, x_1, x_2 - x_1, \dots, x_n - x_{n-1})) \\ Q_{\cup_{j=1}^{n-1} B_j, \cup_{j=1}^n B_j}(x_{n-1}; dx_n) \dots Q_{B_1, B_1 \cup B_2}(x_1; dx_2) Q_{\emptyset, B_1}(x_0; dx_1) \mu(dx_0). \end{aligned}$$

Note that $\gamma(x_0, x_1, x_2 - x_1, \dots, x_n - x_{n-1}) = (x_0, x_{i_1}, x_{i_2} - x_{i_1}, \dots, x_{i_m} - x_{i_{m-1}})$ and hence the integrand does not depend on the variables $x_j, j \notin \{i_0, i_1, \dots, i_m\}$. By the definition of the transition system, the above integral collapses to

$$\begin{aligned} \int_{\mathbf{R}^{m+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_{i_1}) \prod_{l=2}^m I_{\Gamma_l}(x_{i_l} - x_{i_{l-1}}) Q_{\cup_{j=1}^{i_m-1} B_j, \cup_{j=1}^{i_m} B_j}(x_{i_{m-1}}; dx_{i_m}) \dots \\ Q_{\cup_{j=1}^{i_1} B_j, \cup_{j=1}^{i_2} B_j}(x_{i_1}; dx_{i_2}) Q_{\emptyset, \cup_{j=1}^{i_1} B_j}(x_0; dx_{i_1}) \mu(dx_0) \end{aligned}$$

which is exactly the definition of $\mu_{E_0 E_1 \dots E_m}(\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_m)$ because $\cup_{j=1}^{i_l} B_j = \cup_{s=1}^l B_{i_s}$ for every $s = 1, \dots, m$. This concludes the proof of (15).

Step 4 (Almost Sure Additivity of the Canonical Process)

We will show that the canonical process X on the space $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}})$ has an (almost surely) unique additive extension to $\mathcal{C}(u)$ (with respect to the probability measure P given by Kolmogorov's extension theorem). Let $C, C_1, \dots, C_k \in \mathcal{C}$ be such that $C = \bigcup_{i=1}^k C_i$, and suppose that $C_i = A_i \setminus \bigcup_{j=1}^{n_i} A_{ij}; i = 1, \dots, k$ are extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A_i, A_{ij}, \{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of B_j . Assume that each $C_i = \bigcup_{j \in J_i} D_j$ for some $J_i \subseteq \{1, \dots, n\}$. Because the finite dimensional distributions of X were chosen in an additive way, we have $X_{C_i} = \sum_{j \in J_i} X_{D_j}$ a.s., $X_{C_{i_1} \cap C_{i_2}} = \sum_{j \in J_{i_1} \cap J_{i_2}} X_{D_j}$ a.s., $\dots$, $X_{\bigcap_{i=1}^k C_i} = \sum_{j \in \bigcap_{i=1}^k J_i} X_{D_j}$ a.s., $X_C = \sum_{j \in \bigcup_{i=1}^k J_i} X_{D_j}$ a.s. Hence $\sum_{i=1}^k X_{C_i} - \sum_{i_1 < i_2} X_{C_{i_1} \cap C_{i_2}} + \dots + (-1)^k X_{\bigcap_{i=1}^k C_i} = \sum_{j \in \bigcup_{i=1}^k J_i} X_{D_j} = X_C$ a.s.

□

Finally we will translate the preceding result in terms of flows. Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

The next assumption will provide a necessary and sufficient condition which will allow us to reconstruct a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ from a class of transition systems $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$. It requires that whenever we have two simple flows $f, g \in \mathcal{S}$ such that $f(s) = g(u), f(t) = g(v)$ for some $s < t, u < v$, we must have $Q_{st}^f = Q_{uv}^g$.

Assumption 3.3.6 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_m\}$ are two consistent orderings of some finite semilattices $\mathcal{A}', \mathcal{A}''$ such that $\bigcup_{j=1}^n A_j = \bigcup_{j=1}^m A'_j$, $\bigcup_{j=1}^k A_j = \bigcup_{j=1}^l A'_j$ for some $k < n, l < m$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}'', \text{ord}2}$ with $f(t_i) = \bigcup_{j=1}^i A_j$, $g(u_i) = \bigcup_{j=1}^i A'_j$, then*

$$Q_{0t_1}^f = Q_{0u_1}^g \quad \text{and} \quad Q_{t_k t_n}^f = Q_{u_l u_m}^g.$$

The following assumption is easily seen to be equivalent to Assumption 3.3.2. In general, this assumption is stronger than the former Assumption 3.2.12 (which involves only the finite dimensional distributions over the sets in $\mathcal{A}$), but if SHAPE holds, then the two assumptions are equivalent.

Assumption 3.3.7 If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, $\forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then

$$\int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(x_i - x_{i-1}) Q_{t_{n-1}t_n}^f(x_{n-1}; dx_n) \dots \quad (16)$$

$$\begin{aligned} & Q_{t_1t_2}^f(x_1; dx_2) Q_{0t_1}^f(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{u_{n-1}u_n}^g(y_{n-1}; dy_n) \dots \\ & Q_{u_1u_2}^g(y_1; dy_2) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0) \end{aligned}$$

for every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

The following result is immediate.

Corollary 3.3.8 If μ is a probability measure on $\mathbf{R}$ and $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ is a collection of transition systems which satisfies Assumption 3.3.6 and Assumption 3.3.7, then there exist a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:

1. the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ ; and
2. $\forall f \in \mathcal{S}$, $X^f := (X_{f(t)})_t$ is $\mathcal{Q}^f$ -Markov.

Proof: Let $B, B' \in \mathcal{A}(u)$ be such that $B \subseteq B'$. Let $B = \cup_{j=1}^m A'_j$, $B' = \cup_{l=1}^p A''_l$ be some extremal representations, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A'_j, A''_l and $\text{ord} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$ such that $B = \cup_{j=1}^k A_j$ and $B' = \cup_{j=1}^n A_j$. Denote $f := f_{\mathcal{A}', \text{ord}}$ with $f(t_i) = \cup_{j=1}^i A_j$. We define $Q_{BB'} := Q_{t_k t_n}^f$.

The definition of $Q_{BB'}$ does not depend on the extremal representations of B, B' , because of Assumption 3.3.6. Using the same argument as in the proof of Corollary 3.2.13, it follows that the family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system.

Finally, it is not hard to see that the transition system $\mathcal{Q}$ satisfies Assumption 3.3.2 and the conclusion follows by Theorem 3.3.5.

□

3.4 Transition Semigroups

Let $B(\mathbf{R})$ be the Banach space of all bounded measurable functions $h : \mathbf{R} \rightarrow \mathbf{R}$, with the supremum norm $\|h\| := \sup_{x \in \mathbf{R}} h(x)$. Let L be a fixed closed subspace of $B(\mathbf{R})$ and I the identity operator on L .

We recall that a *bounded linear operator* on L is a linear function $T : L \rightarrow L$ which has the property that $\|T\| := \sup_{h \in L, h \neq 0} \frac{\|Th\|}{\|h\|} < \infty$. One example of a bounded linear operator on the whole space $B(\mathbf{R})$ is the operator associated to a transition probability $Q(x; dy)$ on $\mathbf{R}$, defined by

$$(Th)(x) := \int_{\mathbf{R}} h(y)Q(x; dy), \quad x \in \mathbf{R}.$$

A family $\mathcal{T} := (T_{st})_{s, t \in [0, a]; s < t}$ of bounded linear operators on L is called a (one-parameter) *semigroup on L* if $T_{ss} = I \forall s$ and $T_{st}T_{tu} = T_{su} \forall s < t < u$. Clearly, if $\mathcal{Q} := (Q_{st})_{s < t}$ is a transition system on $\mathbf{R}$ and we denote with T_{st} the operator associated to Q_{st} , then the family $(T_{st})_{s < t}$ will be an example of a semigroup on the whole space $B(\mathbf{R})$.

Given a semigroup $\mathcal{T}$ on L , a classical Markov process $(X_t)_{t \in [0, a]}$ with respect to a filtration $(\mathcal{F}_s)_s$ is called *$\mathcal{T}$ -Markov with respect to the filtration $(\mathcal{F}_s)_s$* if $\forall s < t$

$$E[h(X_t)|\mathcal{F}_s] = (T_{st}h)(X_s) \quad \forall h \in L.$$

This notion clearly generalizes the notion of $\mathcal{Q}$ -Markov process on $\mathbf{R}_+$.

In this section we will take the same approach as above but for set-indexed processes. More precisely, we will introduce a more general class of set-Markov processes, called $\mathcal{T}$ -Markov and we will do some investigation to see if this class is indeed non-empty. In other words, we will give an answer to the following question: what are the conditions that have to be imposed on the semigroup $\mathcal{T}$ which will ensure the

existence of a set-indexed $\mathcal{T}$ -Markov process? We will discover that the existence of a $\mathcal{T}$ -Markov process forces the semigroup $\mathcal{T}$ to be *almost surely* the one associated to a transition system, and therefore we will be in the position to invoke the results of Sections 3.2 and 3.3.

We begin with a definition.

Definition 3.4.1 (a) For each $B, B' \in \mathcal{A}(u)$ with $B \subseteq B'$ let $T_{BB'}$ be a bounded linear operator on L . The family $\mathcal{T} := (T_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ is called a **semigroup on L** if $\forall B \in \mathcal{A}(u)$, $T_{BB} = I$ and $\forall B, B', B'' \in \mathcal{A}(u)$, $B \subseteq B' \subseteq B''$

$$T_{BB'}T_{B'B''} = T_{BB''}.$$

(b) Let $\mathcal{T} := (T_{BB'})_{B, B' \in \mathcal{A}(u); B \subseteq B'}$ be a semigroup on L . A set-Markov process $X := (X_A)_{A \in \mathcal{A}}$ with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ is called **$\mathcal{T}$ -Markov with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$** if $\forall B, B' \in \mathcal{A}(u)$, $B \subseteq B'$

$$E[h(X_{B'})|\mathcal{F}_B] = (T_{BB'}h)(X_B) \quad \forall h \in L.$$

The process X is called **simply $\mathcal{T}$ -Markov** if it is $\mathcal{T}$ -Markov with respect to its minimal filtration.

In what follows we will obtain the finite-dimensional distribution of a $\mathcal{T}$ -Markov process X , over an arbitrary finite semilattice $\mathcal{A}' = \{A_0 = \emptyset, A_1, \dots, A_n\}$ ordered consistently. Because of the additivity property of the process, it is enough to specify its finite-dimensional distribution over the collection of first i -th unions of sets in $\mathcal{A}'$, $i = 1, \dots, n$. This distribution will be obtained inductively. Assume that the initial distribution of the process is X . At the first step, we get: $\forall h_0 \in B(\mathbf{R})$

$$E[h_0(X_{A_0})] = \int_{\mathbf{R}} h_0 d\mu.$$

At the second step, we get: $\forall h_0 \in B(\mathbf{R}), \forall h_1 \in L$

$$\begin{aligned} E[h_0(X_{A_0})h_1(X_{A_0 \cup A_1})] &= E[h_0(X_{\emptyset})E[h_1(X_{A_1})|\mathcal{F}_{\emptyset}]] = \\ &= E[h_0(X_{\emptyset})(T_{\emptyset A_1}h_1)(X_{\emptyset})] = \int_{\mathbf{R}} h_0 \cdot T_{\emptyset A_1}h_1 d\mu. \end{aligned}$$

In general, we get: $\forall h_0 \in B(\mathbf{R}), \forall h_1, \dots, h_n \in L$

$$E[h_0(X_{A_0})h_1(X_{A_0 \cup A_1}) \dots h_n(X_{\cup_{j=0}^n A_j})] = \int_{\mathbf{R}} h_0 \cdot T_{\emptyset, A_1}(h_1 \cdot T_{A_1, A_1 \cup A_2}(\dots (h_{n-1} \cdot T_{\cup_{j=0}^{n-1} A_j, \cup_{j=0}^n A_j} h_n) \dots)) d\mu.$$

The next result says that, if the subspace L is large enough, then the transition semigroup $\mathcal{T}$ and the initial distribution μ determine uniquely a $\mathcal{T}$ -Markov process. We say that a closed subspace L of $B(\mathbf{R})$ is **separating** if for any probability measures P, Q on $\mathbf{R}$

$$\int_{\mathbf{R}} h dP = \int_{\mathbf{R}} h dQ \quad \forall h \in L \Rightarrow P = Q.$$

An example of a separating subspace of the space $B(\mathbf{R})$ is the space $C_0(\mathbf{R})$ of all continuous functions $h : \mathbf{R} \rightarrow \mathbf{R}$ which vanish at infinity.

Proposition 3.4.2 *Assume that L is a separating subspace of $B(\mathbf{R})$. Let $\mathcal{T} := (T_{BB'})_{B \subseteq B'}$ be a transition semigroup on L and μ a probability measure on $\mathbf{R}$. Any $\mathcal{T}$ -Markov processes with initial distribution μ have the same finite dimensional distributions.*

Proof: Let $X := (X_A)_{A \in \mathcal{A}}$ and $Y := (Y_A)_{A \in \mathcal{A}}$ be two $\mathcal{T}$ -Markov processes with initial distribution μ . It is enough to prove that they have the same finite-dimensional distributions over any finite semilattice, or even over the collection of the first i -th unions of the sets of any finite semilattice (by the additivity property of the two processes). Let $\mathcal{A}' = \{A_0 = \emptyset, A_1, \dots, A_n\}$ be a finite semilattice ordered consistently. According to the previous formula we have

$$E[h_0(X_{A_0})h_1(X_{A_0 \cup A_1}) \dots h_n(X_{\cup_{j=0}^n A_j})] = E[h_0(Y_{A_0})h_1(Y_{A_0 \cup A_1}) \dots h_n(Y_{\cup_{j=0}^n A_j})]$$

for any $h_0 \in B(\mathbf{R}), h_1, \dots, h_n \in L$. Since L is separating, this implies that the distribution of $(X_{A_0}, X_{A_0 \cup A_1}, \dots, X_{\cup_{j=0}^n A_j})$ coincides with the distribution of $(Y_{A_0}, Y_{A_0 \cup A_1}, \dots, Y_{\cup_{j=0}^n A_j})$, by Proposition III.4.6, [30].

□

The following result gives us the desired correspondence in terms of flows.

Proposition 3.4.3 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{T}$ -Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is $\mathcal{T}^f$ -Markov with respect to the filtration $(\mathcal{F}_{f(t)})_{t \in [0, a]}$, where $T_{st}^f := T_{f(s), f(t)}$. (For the necessity part we can consider any flow, not only the simple ones.)*

Note that if $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system on $\mathbf{R}$ and $T_{BB'}$ is the operator associated to $Q_{BB'}$, then the family $\mathcal{T} := (T_{BB'})_{B \subseteq B'}$ of these operators is a semigroup on the whole space $B(\mathbf{R})$. It is clear that any $\mathcal{Q}$ -Markov process will be $\mathcal{T}$ -Markov and vice-versa.

The next result says that in fact any transition semigroup has to be *almost surely* of this form, if there exists a $\mathcal{T}$ -Markov process associated to it.

Proposition 3.4.4 *Let $\mathcal{T} := (T_{BB'})_{B \subseteq B'}$ be a semigroup on a closed subspace L and assume that there exists a $\mathcal{T}$ -Markov process $X := (X_A)_{A \in \mathcal{A}}$ associated to it. Then each operator $T_{BB'}$ is ‘almost surely’ the restriction to L of the operator associated to a transition probability $Q_{BB'}$ on $B(\mathbf{R})$, in the sense that $\forall h \in L$*

$$(T_{BB'}h)(x) = \int_{\mathbf{R}} h(y) Q_{BB'}(x; dy) \quad P \circ X_B^{-1} \text{ a.s.}(x).$$

Proof: For each $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$ let $Q_{BB'}(x; dy)$ be a version of the conditional distribution of $X_{B'}$ given X_B . By the definition of the $\mathcal{T}$ -Markov property, for each $h \in L$ we have

$$(T_{BB'}h)(x) = E[h(X_{B'}) | X_B = x] = \int_{\mathbf{R}} h(y) Q_{BB'}(x; dy) \quad P \circ X_B^{-1} \text{ a.s.}(x).$$

□

The preceding proposition gives us a necessary condition that has to be satisfied by a transition semigroup $\mathcal{T}$ in order to ensure the existence of a $\mathcal{T}$ -Markov process. This condition clearly cannot be sufficient since it involves an ‘almost sure’ statement, which refers in fact to the probability distribution of X_B , which is not constructed yet. To be able to construct a $\mathcal{T}$ -Markov process we have in fact to assume that each $T_{BB'}$ is the operator associated to a transition probability on $\mathbf{R}$. The next result (which is an immediate consequence of Theorem 3.3.5) gives sufficient conditions for the existence of a $\mathcal{T}$ -Markov process.

Theorem 3.4.5 *Let μ be a probability measure on $\mathbf{R}$ and $\mathcal{T} := (T_{BB'})_{B \subseteq B'}$ a semigroup on $B(\mathbf{R})$ such that each $T_{BB'}$ is the operator associated to a transition probability $Q_{BB'}$ on $\mathbf{R}$.*

If the transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ satisfies Assumption 3.3.2, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{T}$ -Markov with initial distribution μ .

For the sake of completeness we will also include the corresponding result in terms of flows. Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

Corollary 3.4.6 *Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{T}^f := (T_{st}^f)_{s,t}; f \in \mathcal{S}\}$ a collection of semigroups on $B(\mathbf{R})$ such that each T_{st}^f is the operator associated to a transition probability Q_{st}^f on $\mathbf{R}$.*

If the collection $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of transition systems satisfies Assumption 3.3.6 and Assumption 3.3.7, then there exist a semigroup $\mathcal{T} := (T_{BB'})_{B \subseteq B'}$ on $B(\mathbf{R})$ and a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{T}$ -Markov with initial distribution μ ; and*
2. *$\forall f \in \mathcal{S}$, $X^f := (X_{f(t)})_t$ is $\mathcal{T}^f$ -Markov.*

Chapter 4

The Generator

In this chapter we will make use of flows to introduce the generator of a set-indexed $\mathcal{Q}$ -Markov process. Corollary 3.3.8 allows us to characterize a set-indexed $\mathcal{Q}$ -Markov process by a class $\{\mathcal{Q}^f := (Q_{st}^f; f \in \mathcal{S})$ of transition systems, where $\mathcal{S}$ is a collection of simple flows as in Definition 3.2.10. These transition systems in turn are characterized by their corresponding generators. This observation permits us to define a generator for a set-indexed $\mathcal{Q}$ -Markov process as a class of generators indexed by the suitable class $\mathcal{S}$ of flows. There are two main results in this chapter. The first is that the generator defined in this way allows us to reconstruct the finite dimensional distributions of the process. The second gives a sufficient condition for an arbitrary class of generators be the generator of a set-indexed $\mathcal{Q}$ -Markov process.

4.1 The Definition of the Generator

In this section we will introduce the formal definition of ‘the generator’ of a set-indexed $\mathcal{Q}$ -Markov process. Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

Let $B(\mathbf{R})$ be the Banach space of all bounded measurable functions $h : \mathbf{R} \rightarrow \mathbf{R}$, with the supremum norm $\|h\| := \sup_{x \in \mathbf{R}} h(x)$.

Let $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ be a transition system and $X := (X_A)_{A \in \mathcal{A}}$ a $\mathcal{Q}$ -Markov

process (with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$) with initial distribution μ .

For each flow $f \in \mathcal{S}$, the process X^f is $\mathcal{Q}^f$ -Markov (with respect to the filtration $(\mathcal{F}_{f(t)})_t$); let $\mathcal{T}^f := (T_{st}^f)_{s < t}$ be its corresponding semigroup.

The backward generator of the process X^f at time s is a linear operator $\mathcal{G}_s^f$ on the Banach space $B(\mathbf{R})$, defined by

$$\mathcal{G}_s^f h := \lim_{\epsilon \searrow 0} \frac{T_{s-\epsilon, s}^f h - h}{\epsilon} = -\frac{\partial^-}{\partial r} T_{rs}^f h|_{r=s}$$

whose domain is the subspace $\mathcal{D}(\mathcal{G}_s^f)$ of those functions h for which the limit exists in the supremum norm topology. It is not difficult to verify that *Kolmogorov's backward equation* holds:

$$-\frac{\partial^-}{\partial s} T_{st}^f h = \mathcal{G}_s^f T_{st}^f h, \quad \text{if } T_{st}^f h \in \mathcal{D}(\mathcal{G}_s^f).$$

If the function $T_{st}^f h$ is strongly continuously differentiable in s , then Kolmogorov's backward equation has the following integral form:

$$T_{st}^f h - h = \int_s^t \mathcal{G}_u^f T_{ut}^f h du, \quad \text{if } T_{ut}^f h \in \mathcal{D}(\mathcal{G}_u^f) \quad \forall u \in [s, t].$$

The forward generator of the process X^f at time s is defined by

$$\mathcal{G}_s^{*f} h := \lim_{\epsilon \searrow 0} \frac{T_{s, s+\epsilon}^f h - h}{\epsilon} = \frac{\partial^+}{\partial t} T_{st}^f h|_{t=s}$$

whose domain is the subspace $\mathcal{D}(\mathcal{G}_s^{*f})$ of those functions h for which the limit exists in the supremum norm topology. *Kolmogorov's forward equation* is:

$$\frac{\partial^+}{\partial t} T_{st}^f h = T_{st}^f \mathcal{G}_s^{*f} h, \quad \text{if } h \in \mathcal{D}(\mathcal{G}_s^{*f}).$$

If the function $T_{st}^f h$ is strongly continuously differentiable in t , then Kolmogorov's forward equation has the following integral form:

$$T_{st}^f h - h = \int_s^t T_{su}^f \mathcal{G}_u^{*f} h du, \quad \text{if } h \in \mathcal{D}(\mathcal{G}_u^{*f}) \quad \forall u \in [s, t].$$

We will make the usual assumption that all the domains $\mathcal{D}(\mathcal{G}_s^f)$ and $\mathcal{D}(\mathcal{G}_s^{*f})$ have a common subspace $\mathcal{D}$, which is dense in $B(\mathbf{R})$, such that for every $s < t$, $T_{st}^f(\mathcal{D}) \subseteq \mathcal{D}$

and for every $h \in \mathcal{D}$, the function $T_{st}^f h$ is strongly continuously differentiable with respect to s and t with

$$-(\frac{\partial^-}{\partial r} T_{rs}^f h)|_{r=s} = (\frac{\partial^+}{\partial t} T_{st}^f h)|_{t=s} \quad \forall s.$$

The operator $\mathcal{G}_s^f = \mathcal{G}_s^{*f}$ defined on $\mathcal{D}$, is called *the generator* of the process X^f at time s . A consequence of Kolmogorov's equations is that

$$-\frac{\partial}{\partial s} T_{st}^f h = \frac{\partial}{\partial t} T_{st}^f h = T_{st}^f \mathcal{G}_t^f h = \mathcal{G}_s^f T_{st}^f h \quad \forall h \in \mathcal{D}.$$

Hence

$$T_{st}^f h - h = \int_s^t (\frac{\partial}{\partial v} T_{sv}^f h) dv = \int_s^t \mathcal{G}_s^f T_{sv}^f h dv = \int_s^t T_{sv}^f \mathcal{G}_v^f h dv \quad \forall h \in \mathcal{D}$$

and

$$T_{st}^f h - h = - \int_s^t (\frac{\partial}{\partial v} T_{vt}^f h) dv = \int_s^t T_{vt}^f \mathcal{G}_t^f h dv = \int_s^t \mathcal{G}_v^f T_{vt}^f h dv \quad \forall h \in \mathcal{D}.$$

For each $h \in \mathcal{D}$ we have

$$\begin{aligned} E[h(X_{f(s+\epsilon)}) - h(X_{f(s)}) | \mathcal{F}_{f(s)}] &= (T_{f(s), f(s+\epsilon)}^f h)(X_{f(s)}) - h(X_{f(s)}) = \\ &= \epsilon(\mathcal{G}_{f(s)}^f h)(X_{f(s)}) + o(\epsilon). \end{aligned}$$

The generator on the simple flow f appears as a means of describing how the process moves along the flow from one set to another set, which is ‘infinitesimally close’.

The fact that each set in $\mathcal{A}(u)$ can be approached from different directions, according to the flow that we are looking at, which passes through that set, makes us think about the possibility defining “the” generator of a set-indexed $\mathcal{Q}$ -Markov process as the collection of all generators of the process along a large enough collection of simple flows. (Intuitively, we can identify the set-indexed generator at the set $B \in \mathcal{A}(u)$ with the collection of the ‘directional’ derivatives of the Banach-valued map $\mathcal{A}(u) \ni B' \mapsto T_{BB'} h$.) More precisely, we have the following definition.

Definition 4.1.1 *Let $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ be a transition system. The generator of a $\mathcal{Q}$ -Markov process $X := (X_A)_{A \in \mathcal{A}}$ is the collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$, where $\mathcal{G}_s^f$ is the generator of the process $X^f := (X_{f(t)})_t$ at time s .*

Note that if we know the generator $\mathcal{G}^f := (\mathcal{G}_s^f)_s$ of a set-indexed $\mathcal{Q}$ -Markov process on a certain flow f , we can in fact write down the generator of the process over any flow g which has the same path as f . More precisely, if two flows $f : [0, a] \rightarrow \mathcal{A}(u)$ and $g : [0, b] \rightarrow \mathcal{A}(u)$ are such that $f(t) = g(\alpha(t)) \forall t \in [0, a]$ for a certain continuous one-to-one map $\alpha : [0, a] \rightarrow [0, b]$ which has a non-vanishing derivative denoted α' , then

$$\mathcal{G}_s^f = \alpha'(s) \mathcal{G}_{\alpha(s)}^g \quad \text{and} \quad \mathcal{D}(\mathcal{G}_s^f) = \mathcal{D}(\mathcal{G}_{\alpha(s)}^g).$$

This is a consequence of the following lemma.

Lemma 4.1.2 *Let $X^1 := (X_t^1)_{t \in [0, a]}$ and $X^2 := (X_t^2)_{t \in [0, b]}$ be two real-valued processes such that $X_t^2 := X_{\alpha(t)}^1$ for all $t \in [0, a]$, where $\alpha : [0, a] \rightarrow [0, b]$ is a continuous one-to-one map. Let $\mathcal{Q}^1 := (\mathcal{Q}_{st}^1)_{s, t \in [0, a]; s < t}$ be a transition system and define $\mathcal{Q}_{st}^2 := \mathcal{Q}_{\alpha(s), \alpha(t)}^1$. Then $\mathcal{Q}^2 := (\mathcal{Q}_{st}^2)_{s < t}$ is also a transition system and X^1 is $\mathcal{Q}^1$ -Markov if and only if X^2 is $\mathcal{Q}^2$ -Markov. In this case, if in addition the map α has a non-vanishing derivative α' , and we denote with $\mathcal{G}_s^2, \mathcal{G}_{\alpha(s)}^1$ the generators of X^2, X^1 at times s , respectively $\alpha(s)$, and with $\mathcal{D}(\mathcal{G}_s^2), \mathcal{D}(\mathcal{G}_{\alpha(s)}^1)$ their respective domains, then*

$$\mathcal{G}_s^2 = \alpha'(s) \mathcal{G}_{\alpha(s)}^1 \quad \text{and} \quad \mathcal{D}(\mathcal{G}_s^2) = \mathcal{D}(\mathcal{G}_{\alpha(s)}^1).$$

Proof: Clearly $\mathcal{Q}^2$ is a transition system: for any $s < t < u$ we have

$$\mathcal{Q}_{st}^2 \mathcal{Q}_{tu}^2 = \mathcal{Q}_{\alpha(s), \alpha(t)}^1 \mathcal{Q}_{\alpha(t), \alpha(u)}^1 = \mathcal{Q}_{\alpha(s), \alpha(u)}^1 = \mathcal{Q}_{su}^2.$$

Assume that X^1 is $\mathcal{Q}^1$ -Markov and let $s < t, x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$ be arbitrary. Then

$$P[X_t^2 \in \Gamma | X_s^2 = x] = P[X_{\alpha(t)}^1 \in \Gamma | X_{\alpha(s)}^1 = x] = \mathcal{Q}_{\alpha(s), \alpha(t)}^1(x; \Gamma) = \mathcal{Q}_{st}^2(x; \Gamma)$$

i.e. X^2 is $\mathcal{Q}^2$ -Markov.

Suppose next that α has a non-vanishing derivative α' . We will prove that $\mathcal{D}(\mathcal{G}_s^2) \subseteq \mathcal{D}(\mathcal{G}_{\alpha(s)}^1)$: if $h \in \mathcal{D}(\mathcal{G}_s^2)$, then $h \in \mathcal{D}(\mathcal{G}_{\alpha(s)}^1)$ since the limit

$$\begin{aligned} (\mathcal{G}_{\alpha(s)}^1 h)(x) &= \lim_{u \searrow \alpha(s)} \frac{\int_{\mathbf{R}} (h(y) - h(x)) \mathcal{Q}_{\alpha(s), u}^1(x; dy)}{u - \alpha(s)} = \lim_{t \searrow s} \frac{\int_{\mathbf{R}} (h(y) - h(x)) \mathcal{Q}_{\alpha(s), \alpha(t)}^1(x; dy)}{\alpha(t) - \alpha(s)} = \\ &= \lim_{t \searrow s} \frac{t - s}{\alpha(t) - \alpha(s)} \cdot \lim_{t \searrow s} \frac{1}{t - s} \int_{\mathbf{R}} (h(y) - h(x)) \mathcal{Q}_{st}^2(x; dy) = \frac{1}{\alpha'(s)} \cdot (\mathcal{G}_s^2 h)(x) \end{aligned}$$

exists, uniformly in $x \in \mathbf{R}$. Similarly $\mathcal{D}(\mathcal{G}_{\alpha(s)}^1) \subseteq \mathcal{D}(\mathcal{G}_s^2)$.

□

The general problem of finding the conditions that have to be imposed on a family $(\mathcal{G}_s)_s$ of linear operators on $B(\mathbf{R})$ so that each $\mathcal{G}_s$ becomes the generator at time s of a classical $\mathcal{Q}$ -Markov process is not straightforward. In the homogeneous case, the Hille-Yosida theorem (Theorem I.2.6. [30]) gives the necessary and sufficient conditions for a linear operator $\mathcal{G}$ on $B(\mathbf{R})$ to be the generator of a (strongly continuous positive contraction) semigroup $\mathcal{T} := (T_t)_t$; however, there is no guarantee that each T_t is the operator associated to some transition probability Q_t . In the inhomogeneous case, things become even more complicated.

Therefore, in trying to determine when a given collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ of families of linear operators on $B(\mathbf{R})$ actually form the generator of a set-indexed $\mathcal{Q}$ -Markov process, for ease of exposition we will assume that each operator $\mathcal{G}_s^f$ is the generator at time s of a semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .

The main result of this section is an immediate consequence of Corollary 3.3.8. It basically says that the generator allows us to reconstruct the distribution of a set-indexed $\mathcal{Q}$ -Markov process (or, equivalently, its finite dimensional distributions) if certain consistency assumptions hold.

Theorem 4.1.3 *Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$ such that each operator $\mathcal{G}_s^f$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .*

If the collection $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of transition systems satisfies Assumption 3.3.6 and Assumption 3.3.7, then there exist a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ ; and*

2. $\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is an extension of the operator $\mathcal{G}_s^f$.

In Chapter 5 and Chapter 6 we will focus on some particular examples of set-indexed $\mathcal{Q}$ -Markov processes (processes with independent increments, respectively empirical processes) and we will see that in their case, the consistency assumptions in terms of generators become much more simple.

4.2 The Construction using the Generator

In this section, we will find equivalent formulations in terms of the generators $\mathcal{G}_s^f$ for Assumption 3.3.6 and Assumption 3.3.7, in the case of a general set-indexed $\mathcal{Q}$ -Markov process. These new assumptions, even if notationally complex, will provide us with a set of necessary and sufficient conditions that have to be imposed on a collection of generators so that they become the generator of a set-indexed $\mathcal{Q}$ -Markov process. Most of the results of this section appear as the last section of [7].

Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$ such that each operator $\mathcal{G}_s^f$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .

Recall from the previous section that, in this case, the functions $T_{st}^f h$ with $h \in \mathcal{D}$ are assumed to be strongly continuously differentiable with respect to s and t ; consequently, we have a set of four integral equations from which we retain the following:

$$T_{st}^f h - h = \int_s^t \mathcal{G}_v^f T_{vt}^f h dv \quad \forall h \in \mathcal{D}. \quad (17)$$

The goal is to find the conditions that have to be satisfied by the collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ such that the collection $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of transition systems satisfies Assumption 3.3.6 and Assumption 3.3.7. By invoking Theorem 4.1.3 we will be able to conclude next that there exist a set-indexed transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^c, \mathcal{B}(\mathbf{R})^c)$ under

which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ (defined on this space) is $\mathcal{Q}$ -Markov with generator $\{\mathcal{G}^f; f \in \mathcal{S}\}$ and initial distribution μ .

Using the integral equation (17), we will first give the equivalent form in terms of generators of Assumption 3.3.6.

Assumption 4.2.1 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_m\}$ are two consistent orderings of some finite semilattices $\mathcal{A}', \mathcal{A}''$, such that $\cup_{j=1}^n A_j = \cup_{j=1}^m A'_j$, $\cup_{j=1}^k A_j = \cup_{j=1}^l A'_j$ for some $k < n, l < m$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then for every $h \in \mathcal{D}$*

$$\int_0^{t_1} \mathcal{G}_v^f T_{vt_1}^f h dv = \int_0^{u_1} \mathcal{G}_v^g T_{vu_1}^g h dv \quad \text{and} \quad \int_{t_k}^{t_n} \mathcal{G}_v^f T_{vt_n}^f h dv = \int_{u_l}^{u_m} \mathcal{G}_v^g T_{vu_m}^g h dv.$$

We will prove next that the condition of Assumption 3.3.7 can also be written in a certain form which admits an expression in terms of generators.

Let us introduce the following notational convention.

If $Q_1(x_1; dx_2), Q_2(x_2; dx_3), \dots, Q_{n-1}(x_{n-1}; dx_n)$ are transition probabilities on $\mathbf{R}$, $T_1, T_2, \dots, T_{n-1}$ are their associated bounded linear operators, and $h : \mathbf{R}^n \rightarrow \mathbf{R}$ is a bounded measurable function, then

$$T_1 T_2 \dots T_{n-1} [h(x_1, x_2, \dots, x_n)](x_1) \stackrel{\text{def}}{=} \int_{\mathbf{R}^{n-1}} h(x_1, x_2, \dots, x_n) Q_{n-1}(x_{n-1}; dx_n) \dots Q_2(x_2; dx_3) Q_1(x_1; dx_2).$$

If in addition, $\mathcal{G}$ is a linear operator on $B(\mathbf{R})$ with domain $\mathcal{D}(\mathcal{G})$, and the function h is chosen such that for every $x_1, \dots, x_k \in \mathbf{R}$, the function

$$\int_{\mathbf{R}^{n-k}} h(x_1, x_2, \dots, x_n) Q_{n-1}(x_{n-1}; dx_n) \dots Q_{k+1}(x_{k+1}; dx_{k+2}) Q_k(\cdot; dx_{k+1})$$

is in $\mathcal{D}(\mathcal{G})$, then

$$\begin{aligned} & T_1 T_2 \dots T_{k-1} \mathcal{G} T_k \dots T_{n-1} [h(x_1, x_2, \dots, x_n)](x_1) \stackrel{\text{def}}{=} \\ & \int_{\mathbf{R}^{k-1}} (\mathcal{G} \int_{\mathbf{R}^{n-k}} h(x_1, x_2, \dots, x_n) Q_{n-1}(x_{n-1}; dx_n) \dots Q_{k+1}(x_{k+1}; dx_{k+2}) \\ & \quad Q_k(\cdot; dx_{k+1}))(x_k) Q_{k-1}(x_{k-1}; dx_k) \dots Q_2(x_2; dx_3) Q_1(x_1; dx_2). \end{aligned}$$

Let $\mu_{t_1}^f$ be the probability measure defined by $\mu_{t_1}^f(\Gamma) := \int_{\mathbf{R}} Q_{0t_1}^f(x; \Gamma) \mu(dx)$. The following result is crucial. It shows that condition (16) of the consistency Assumption 3.3.7 has an equivalent form in which both the left-hand side and the right-hand side are integrated with respect to the same transition probabilities $Q_{u_{\pi(j)-1}, u_{\pi(j)}}^g$; and moreover, this equivalent form can be translated in terms of the generators $(\mathcal{G}_w^f)_{w \in [t_{i-1}, t_i]}$ and $(\mathcal{G}_v^g)_{v \in [u_{\pi(i)-1}, u_{\pi(i)}]}$.

Lemma 4.2.2 *Let $\text{ord}1 = \{A_0 = \emptyset, A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset, A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}, \forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$.*

Suppose that $Q_{0t_1}^f = Q_{0u_1}^g$. The following statements are equivalent:

(a) *For every $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$*

$$\begin{aligned} & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{i=2}^n I_{\Gamma_i}(x_i - x_{i-1}) Q_{t_{i-1}t_i}^f(x_{i-1}; dx_i) \dots \quad (18) \\ & Q_{t_1t_2}^f(x_1; dx_2) Q_{0t_1}^f(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{n+1}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{i=2}^n I_{\Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{u_{n-1}u_n}^g(y_{n-1}; dy_n) \dots \\ & Q_{u_1u_2}^g(y_1; dy_2) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0); \end{aligned}$$

(b) *for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $h_2, \dots, h_i \in \mathcal{B}(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1*

$$\begin{aligned} & T_{u_1u_{l_1}}^g T_{u_{l_1}u_{l_2}}^g \dots T_{u_{l_{p-3}}u_{l_{p-2}}}^g T_{u_{l_{p-2}}u_{l_{p+1}}}^g T_{u_{l_{p+1}}u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}^g \quad (19) \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (T_{t_{i-1}t_i}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right](x_1) = \\ & T_{u_1u_{l_1}}^g T_{u_{l_1}u_{l_2}}^g \dots T_{u_{l_{2(i-1)-1}}u_{l_{2(i-1)}}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1})) \right](x_1); \end{aligned}$$

(c) for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $h_2, \dots, h_i \in \mathcal{D}$ and for $\mu_{l_1}^f$ -almost all x_1

(c1) if $p = 2(i - 1)$ we have

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (\mathcal{G}_w^f T_{w t_i}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right] (x_1) dw = \\ & \int_{u_{l_{p-1}}}^{u_{l_p}} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \mathcal{G}_v^g T_{v u_{l_p}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1})) \right] (x_1) dv; \end{aligned}$$

(c2) if $p < 2(i - 1)$ we have

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} T_{u_1 u_{l_1}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (\mathcal{G}_w^f T_{w t_i}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right] (x_1) dw = \\ & \int_{u_{l_{p-1}}}^{u_{l_p}} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \mathcal{G}_v^g T_{v u_{l_p}}^g \\ & T_{u_{l_p} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \left\{ \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) \cdot \right. \\ & \left. [h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1})) - h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))] \right\} (x_1) dv. \end{aligned}$$

Proof: (a) $\Rightarrow$ (b): Let X be a $\mathcal{Q}^f$ -Markov process and Y a $\mathcal{Q}^g$ -Markov process with the same initial distribution μ . Then (a) is equivalent to saying that the distribution of $(X_0, X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ coincide with the distribution of $(Y_0, Y_{u_1}, Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}}, \dots, Y_{u_{\pi(n)}} - Y_{u_{\pi(n)-1}})$.

Since $Q_{0t_1}^f = Q_{0u_1}^g$, both X_{t_1} and Y_{u_1} have the same distribution $\mu_{t_1}^f$. Hence (a) is equivalent to saying that for $\mu_{t_1}^f$ -almost all x_1 , the conditional distribution of $(X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ given $X_{t_1} = x_1$ coincide with the conditional distribution of $(Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}}, \dots, Y_{u_{\pi(n)}} - Y_{u_{\pi(n)-1}})$ given $Y_{u_1} = x_1$.

For $i = 2$ we will use the following relationship: for every $\Gamma_2 \in \mathcal{B}(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1

$$P[X_{t_2} - X_{t_1} \in \Gamma_2 | X_{t_1} = x_1] = P[Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}} \in \Gamma_2 | Y_{u_1} = x_1].$$

Using Lemma A.2.4, Appendix A.2, the left-hand side is $\int_{\mathbf{R}} I_{\Gamma_2+x_1}(x_2) Q_{t_1 t_2}^f(x_1; dx_2)$, whereas on the right-hand side we have

$$\int_{\mathbf{R}^2} I_{\Gamma_2}(y_{u_{\pi(2)}} - y_{u_{\pi(2)-1}}) Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}).$$

By a monotone class argument we can conclude that for every $h_2 \in B(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1

$$\begin{aligned} \int_{\mathbf{R}} h_2(x_2) Q_{t_1 t_2}^f(x_1; dx_2) &= \int_{\mathbf{R}^2} h_2(x_1 + y_{u_{\pi(2)}} - y_{u_{\pi(2)-1}}) \\ &\quad Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) \end{aligned} \quad (20)$$

which is the desired relationship for $i = 2$.

For $i = 3$ we will use the following relationship: for every $\Gamma_2, \Gamma_3 \in \mathcal{B}(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1

$$\begin{aligned} P[X_{t_2} - X_{t_1} \in \Gamma_2, X_{t_3} - X_{t_2} \in \Gamma_3 | X_{t_1} = x_1] &= \\ P[Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}} \in \Gamma_2, Y_{u_{\pi(3)}} - Y_{u_{\pi(3)-1}} \in \Gamma_3 | Y_{u_1} = x_1]. \end{aligned}$$

Using Lemma A.2.4, Appendix A.2, the left-hand side can be written as

$$\int_{\mathbf{R}^2} I_{\Gamma_2+x_1}(x_2) I_{\Gamma_3+x_2}(x_3) Q_{t_2 t_3}^f(x_2; dx_3) Q_{t_1 t_2}^f(x_1; dx_2)$$

which becomes

$$\begin{aligned} \int_{\mathbf{R}^3} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3+x_1+y_{\pi(2)}-y_{\pi(2)-1}}(x_3) Q_{t_2 t_3}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3) \\ Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)})^g Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) \end{aligned}$$

using equation (20).

Using also Lemma A.2.4, Appendix A.2, the right-hand side can be written as

$$\int_{\mathbf{R}^4} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y_{\pi(3)} - y_{\pi(3)-1}) Q_{u_{l_3}, u_{l_4}}^g(y_{l_3}; dy_{l_4}) Q_{u_{l_2}, u_{l_3}}^g(y_{l_2}; dy_{l_3}) \\ Q_{u_{l_1}, u_{l_2}}^g(y_{l_1}; dy_{l_2}) Q_{u_1, u_{l_1}}^g(x_1; dy_{l_1})$$

where $l_1 \leq l_2 \leq l_3 \leq l_4$ is the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3)$.

By a monotone class argument we can conclude that for every $h_2, h_3 \in \mathcal{B}(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1

$$\int_{\mathbf{R}^3} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_3) Q_{t_2 t_3}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3) \quad (21) \\ Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) = \\ \int_{\mathbf{R}^4} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y_{\pi(3)} - y_{\pi(3)-1}) Q_{u_{l_3}, u_{l_4}}^g(y_{l_3}; dy_{l_4}) \\ Q_{u_{l_2}, u_{l_3}}^g(y_{l_2}; dy_{l_3}) Q_{u_{l_1}, u_{l_2}}^g(y_{l_1}; dy_{l_2}) Q_{u_1, u_{l_1}}^g(x_1; dy_{l_1})$$

which is the desired relationship for $i = 3$.

The inductive argument will be omitted since it is identical, but notationally complex.

(b) $\Rightarrow$ (a): Let $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$ be arbitrary. Using the fact that $Q_{0t_1}^f = Q_{0u_1}^g$ and equation (20), the left-hand side of (18) becomes

$$\int_{\mathbf{R}^{n+2}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3+y_1+y_{\pi(2)}-y_{\pi(2)-1}}(x_3) I_{\Gamma_4+x_3}(x_4) \dots \\ I_{\Gamma_n+x_{n-1}}(x_n) Q_{t_{n-1} t_n}^f(x_{n-1}; dx_n) \dots Q_{t_3 t_4}^f(x_3; dx_4) Q_{t_2 t_3}^f(y_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3) \\ Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1}) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0)$$

which in turn can be written as

$$\int_{\mathbf{R}^{n+3}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y_{\pi(3)} - y_{\pi(3)-1}) \\ I_{\Gamma_4+y_1+\sum_{j=2}^3(y_{\pi(j)}-y_{\pi(j)-1})}(x_4) \dots I_{\Gamma_n+x_{n-1}}(x_n) Q_{t_{n-1} t_n}^f(x_{n-1}; dx_n) \dots$$

$$Q_{t_3 t_4}^f(y_1 + \sum_{j=2}^3 (y_{\pi(j)} - y_{\pi(j)-1}); dx_4) Q_{u_{l_3}, u_{l_4}}^g(y_{l_3}; dy_{l_4}) Q_{u_{l_2}, u_{l_3}}^g(y_{l_2}; dy_{l_3}) \\ Q_{u_{l_1}, u_{l_2}}^g(y_{l_1}; dy_{l_2}) Q_{u_1, u_{l_1}}^g(y_1; dy_{l_1}) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0)$$

using equation (21), where $l_1 \leq l_2 \leq l_3 \leq l_4$ is the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3)$.

Continuing in the same manner at the last step we will get exactly the desired right-hand side of (18), since the increasing ordering of the values $\pi(2) - 1, \pi(2), \dots, \pi(n) - 1, \pi(n)$ is exactly $1 \leq 2 \leq \dots \leq n$.

(b) $\Leftrightarrow$ (c): The basic ingredient will be equation (17), which gives the integral expression of a semigroup in terms of its generator.

Since $\mathcal{D}$ is dense we can assume that the functions $h_2, \dots, h_i$ are in $\mathcal{D}$ in the expression given by **(b)**. Subtract

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\ [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))](x_1)$$

from both sides of this expression.

On the left-hand side we will have

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\ [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (T_{t_{i-1} t_i}^f h_i - h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))](x_1)$$

which can be written as

$$\int_{t_{i-1}}^{t_i} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\ [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (\mathcal{G}_w^f T_{wt_i}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))](x_1) dw$$

using the integral expression (17) and Fubini's theorem.

On the right-hand side we have

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g$$

$$\begin{aligned}
& \{T_{u_{l_{p-1}} u_{l_p}}^g T_{u_{l_p} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\
& [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1}))](y_{l_{p-1}}) - \\
& - T_{u_{l_{p-1}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\
& [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))](y_{l_{p-1}})\}(x_1).
\end{aligned}$$

If we denote

$$\begin{aligned}
h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_p}) &:= \\
& \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1})) \\
& Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g(y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \cdots Q_{u_{l_{p+1}} u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) \\
& Q_{u_{l_p} u_{l_{p+1}}}^g(y_{l_p}; dy_{l_{p+1}})
\end{aligned}$$

and

$$\begin{aligned}
h(x_1, y_{l_1}, \dots, y_{l_{p-1}}) &:= \\
& \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \\
& Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g(y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \cdots Q_{u_{l_{p+1}} u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) \\
& Q_{u_{l_{p-1}} u_{l_{p+1}}}^g(y_{l_{p-1}}; dy_{l_{p+1}})
\end{aligned}$$

then the right-hand side becomes

$$\begin{aligned}
& T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \\
& [(T_{u_{l_{p-1}} u_{l_p}}^g h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}}) - h(x_1, y_{l_1}, \dots, y_{l_{p-1}})](x_1) = \\
& T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \\
& [h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_{p-1}}) - h(x_1, y_{l_1}, \dots, y_{l_{p-1}})] + \\
& + \int_{u_{l_{p-1}}}^{u_{l_p}} (\mathcal{G}_v^g T_{v u_{l_p}}^g h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}}) dv](x_1)
\end{aligned}$$

since $h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot) \in \mathcal{D}$ for every $x_1, y_{l_1}, \dots, y_{l_{p-1}}$. We have two cases:

Case 1) If $p = 2(i - 1)$, then all the integrals with respect to $Q_{u_{l_p} u_{l_{p+1}}}^g$, $Q_{u_{l_{p+1}} u_{l_{p+2}}}^g, \dots, Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g$ disappear in the preceding expressions. Hence

$$h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_{p-1}}) = h(x_1, y_{l_1}, \dots, y_{l_{p-1}})$$

and the result follows.

Case 2) If $p < 2(i - 1)$, then

$$h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_{p-1}}) = (T_{u_{l_p} u_{l_{p+1}}}^g H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}})$$

$$h(x_1, y_{l_1}, \dots, y_{l_{p-1}}) = (T_{u_{l_{p-1}} u_{l_{p+1}}}^g H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}})$$

where

$$\begin{aligned} H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_{p+1}}) := \\ \int_{\mathbf{R}^{2(i-1)-p-1}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \\ Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g (y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \dots Q_{u_{l_{p+1}} u_{l_{p+2}}}^g (y_{l_{p+1}}; dy_{l_{p+2}}). \end{aligned}$$

To simplify the notation we will omit the arguments of the function H . Hence

$$\begin{aligned} h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_{p-1}}) - h(x_1, y_{l_1}, \dots, y_{l_{p-1}}) = \\ (T_{u_{l_p} u_{l_{p+1}}}^g H)(y_{l_{p-1}}) - [T_{u_{l_{p-1}} u_{l_p}}^g (T_{u_{l_p} u_{l_{p+1}}}^g H)](y_{l_{p-1}}) = \\ - \int_{u_{l_{p-1}}}^{u_{l_p}} (\mathcal{G}_v^g T_{vu_{l_p}}^g (T_{u_{l_p} u_{l_{p+1}}}^g H))(y_{l_{p-1}}) dv \end{aligned}$$

since $H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot) \in \mathcal{D}$, for every $x_1, y_{l_1}, \dots, y_{l_{p-1}}$.

Using Fubini's theorem, the right-hand side becomes

$$\begin{aligned} \int_{u_{l_{p-1}}}^{u_{l_p}} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \\ [(\mathcal{G}_v^g T_{vu_{l_p}}^g U(x_1, y_{l_1}, \dots, y_{l_{p-1}}, \cdot))(y_{l_{p-1}})](x_1) dv \end{aligned}$$

where

$$\begin{aligned} U(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_p}) := \\ h'(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y_{l_p}) - (T_{u_{l_p} u_{l_{p+1}}}^g H(x_1, y_{l_1}, y_{l_2}, \dots, y_{l_{p-1}}, \cdot))(y_{l_p}) = \end{aligned}$$

$$\int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1})$$

$$[h_i(x_1 + \sum_{j=2}^i (y_{\pi(j)} - y_{\pi(j)-1})) - h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))]$$

$$Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g(y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \cdots Q_{u_{l_p} u_{l_{p+1}}}^g(y_{l_p}; dy_{l_{p+1}}).$$

This concludes the proof.

□

The following assumption is the equivalent form in terms of generators of Assumption 3.3.7.

Assumption 4.2.3 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then the generators $\mathcal{G}^f$ and $\mathcal{G}^g$ satisfy condition (c) stated in Lemma 4.2.2.*

Clearly Assumptions 4.2.1 and 4.2.3 are necessary conditions satisfied by the generator of any set-indexed $\mathcal{Q}$ -Markov process. The following result says that in fact, these two assumptions are also sufficient for the construction of the process.

Theorem 4.2.4 *Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$ such that each operator $\mathcal{G}_s^f$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .*

If the collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ satisfies Assumption 4.2.1 and Assumption 4.2.3, then there exist a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ ; and*
2. *$\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is an extension of the operator $\mathcal{G}_s^f$.*

Proof: The collection $\{Q^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of transition systems satisfies Assumption 3.3.6 and Assumption 3.3.7 (using Lemma 4.2.2). The proof is immediate using Theorem 4.1.3.

□

4.3 The Construction using the Generator along the Projected Flows

In this section we will develop a new set of consistency conditions for the generator of a set-indexed $\mathcal{Q}$ -Markov process (which are stronger than those of the previous section) which will allow us to reconstruct the process, if $\mathcal{S}$ is the collection of projected flows introduced in Section 2.5. The new consistency conditions become even more complex than in the previous section. However, we are able to prove that in this case, (the distributions of) the projections of the constructed process over a semilattice ordered consistently in two different ways can be obtained from each other, via a permutation. The results of this section are not required anywhere else in this work.

Let $f_0 : [0, 1] \rightarrow \mathcal{A}$ be a fixed continuous flow with $f_0(0) = \emptyset'$ and $f_0(1) = T$. For each finite sub-semilattice $\mathcal{A}'$ of $\mathcal{A}$ and for each consistent ordering $\text{ord} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ of $\mathcal{A}'$, let $f_{\mathcal{A}', \text{ord}}$ be the projection of the flow f_0 on the semilattice $\mathcal{A}'$ with respect to the ordering ord . Let $\mathcal{S}$ be the collection of all simple flows $f_{\mathcal{A}', \text{ord}}$.

Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$, such that each operator $\mathcal{G}_s^f$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .

The first step is to find the relationship between the transition probabilities $Q_{t_{i-1}, t_{i-1}+\epsilon}^f$ and $Q_{u_{\pi(i)-1}, u_{\pi(i)-1}+\epsilon}^g$, and then to write down the new relationship between the generators $(\mathcal{G}_w^f)_{w \in [t_{i-1}, t_{i-1}+\epsilon]}$ and $(\mathcal{G}_v^g)_{v \in [u_{\pi(i)-1}, u_{\pi(i)-1}+\epsilon]}$, for every $0 < \epsilon \leq t_i - t_{i-1}$.

Let $\mu_{t_1}^f$ be the probability measure defined by $\mu_{t_1}^f(\Gamma) := \int_{\mathbf{R}} Q_{0t_1}^f(x; \Gamma) \mu(dx)$.

Lemma 4.3.1 *Let $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$, with $A_i = A'_{\pi(i)}, \forall i$, where*

π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{A', \text{ord}1}$, $g := f_{A', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$.

Suppose that $Q_{0t_1}^f = Q_{0u_1}^g$. The following statements are equivalent:

- (a) for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $0 < \epsilon \leq t_i - t_{i-1}$, and for every $\Gamma_0, \Gamma_1, \dots, \Gamma_i \in \mathcal{B}(\mathbf{R})$

$$\begin{aligned} & \int_{\mathbf{R}^{i+1}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{j=2}^{i-1} I_{\Gamma_j}(x_j - x_{j-1}) I_{\Gamma_i}(x'_{i-1} - x_{i-1}) \quad (22) \\ & Q_{t_{i-1}, t_{i-1}+\epsilon}^f(x_{i-1}; dx'_{i-1}) Q_{t_{i-2}, t_{i-1}}^f(x_{i-2}; dx_{i-1}) \dots Q_{t_1, t_2}^f(x_1; dx_2) \\ & Q_{0t_1}^f(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{2(i-1)+2}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{j=2}^{i-1} I_{\Gamma_j}(y_{\pi(j)} - y_{\pi(j)-1}) I_{\Gamma_i}(y'_{\pi(i)-1} - y_{\pi(i)-1}) \\ & Q_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g(y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \dots Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) \\ & Q_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g(y'_{l_{p-1}}; dy_{l_{p+1}}) Q_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g(y_{l_{p-1}}; dy'_{l_{p-1}}) \\ & Q_{u_{l_{p-2}}, u_{l_{p-1}}}^g(y_{l_{p-2}}; dy_{l_{p-1}}) \dots Q_{u_{l_1}, u_{l_2}}^g(y_{l_1}; dy_{l_2}) Q_{u_1, u_{l_1}}^g(y_1; dy_{l_1}) \\ & Q_{0u_1}^g(y_0; dy_1) \mu(dy_0); \end{aligned}$$

- (b) for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $0 < \epsilon \leq t_i - t_{i-1}$, for every $h_2, \dots, h_i \in B(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1

$$\begin{aligned} & T_{u_1, u_{l_1}}^g T_{u_{l_1}, u_{l_2}}^g \dots T_{u_{l_{p-3}}, u_{l_{p-2}}}^g T_{u_{l_{p-2}}, u_{l_{p+1}}}^g T_{u_{l_{p+1}}, u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \quad (23) \\ & [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (T_{t_{i-1}, t_{i-1}+\epsilon}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}))](x_1) = \\ & T_{u_1, u_{l_1}}^g T_{u_{l_1}, u_{l_2}}^g \dots T_{u_{l_{p-2}}, u_{l_{p-1}}}^g \\ & T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g T_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g T_{u_{l_{p+1}}, u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \\ & [\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + (y'_{\pi(i)-1} - y_{\pi(i)-1}))](x_1); \end{aligned}$$

(c) for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}$, $\pi(i) = l_p$, then for every $0 < \epsilon \leq t_i - t_{i-1}$, for every $h_2, \dots, h_i \in \mathcal{D}$ and for $\mu_{t_1}^f$ -almost all x_1

(c1) if $p = 2(i - 1)$ we have

$$\begin{aligned} & \int_{t_{i-1}}^{t_{i-1}+\epsilon} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (\mathcal{G}_w^f T_{w, t_{i-1}+\epsilon}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right] (x_1) dw = \\ & \int_{u_{l_{p-1}}}^{u_{l_{p-1}}+\epsilon} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \mathcal{G}_v^g T_{v, u_{l_{p-1}}+\epsilon}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + (y'_{\pi(i)-1} - y_{\pi(i)-1})) \right] (x_1) dv; \end{aligned}$$

(c2) if $p < 2(i - 1)$ we have

$$\begin{aligned} & \int_{t_{i-1}}^{t_{i-1}+\epsilon} T_{u_1 u_{l_1}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \\ & \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) (\mathcal{G}_w^f T_{w, t_{i-1}+\epsilon}^f h_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right] (x_1) dw = \\ & \int_{u_{l_{p-1}}}^{u_{l_{p-1}}+\epsilon} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-3}} u_{l_{p-2}}}^g T_{u_{l_{p-2}} u_{l_{p-1}}}^g \mathcal{G}_v^g T_{v, u_{l_{p-1}}+\epsilon}^g \\ & T_{u_{l_{p-1}+\epsilon} u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots T_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g \left\{ \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) \cdot \right. \\ & \left. [h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + (y'_{\pi(i)-1} - y_{\pi(i)-1})) - \right. \\ & \left. \left. h_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1})) \right] \right\} (x_1) dv. \end{aligned}$$

Proof: The proof is identical to the proof of Lemma 4.2.2; the only difference is that we are replacing t_i by $t_{i-1} + \epsilon$.

□

Note: We will give below the explicit form of the relationship given at (b) for $i = 2, 3$. For $i = 2$ we have

$$\begin{aligned} \int_{\mathbf{R}} h_2(x'_1) Q_{t_1, t_1 + \epsilon}^f(x_1; dx'_1) &= \\ \int_{\mathbf{R}^2} h_2(x_1 + y'_{u_{\pi(2)-1}} - y_{u_{\pi(2)-1}}) Q_{u_{\pi(2)-1} u_{\pi(2)-1} + \epsilon}^g(y_{\pi(2)-1}; dy'_{\pi(2)-1}) & \\ Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}). & \end{aligned} \quad (24)$$

For $i = 3$ we have two cases: if $\pi(2) < \pi(3)$, then

$$\begin{aligned} \int_{\mathbf{R}^3} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x'_2) Q_{t_2, t_2 + \epsilon}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx'_2) &= \\ Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) &= \\ \int_{\mathbf{R}^4} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}) & \\ Q_{u_{\pi(3)-1}, u_{\pi(3)-1} + \epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) Q_{u_{\pi(2)}, u_{\pi(3)-1}}^g(y_{\pi(2)}; dy_{\pi(3)-1}) & \\ Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}); & \end{aligned} \quad (25)$$

if $\pi(3) < \pi(2)$, then

$$\begin{aligned} \int_{\mathbf{R}^3} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x'_2) Q_{t_2, t_2 + \epsilon}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx'_2) &= \\ Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) &= \\ \int_{\mathbf{R}^4} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}) & \\ Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_{\pi(3)-1} + \epsilon, u_{\pi(2)-1}}^g(dy'_{\pi(3)-1}; y_{\pi(2)-1}) & \\ Q_{u_{\pi(3)-1}, u_{\pi(3)-1} + \epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) Q_{u_1, u_{\pi(3)-1}}^g(x_1; dy_{\pi(3)-1}). & \end{aligned} \quad (26)$$

The second step will be to find the relationship between the transition probabilities $Q_{t_{i-1} + \epsilon, t_{i-1} + \epsilon'}^f$ and $Q_{u_{\pi(i)-1} + \epsilon, u_{\pi(i)-1} + \epsilon'}^g$ and then to write down the relationship between the generators $(\mathcal{G}_w^f)_{w \in [t_{i-1} + \epsilon, t_{i-1} + \epsilon']}$ and $(\mathcal{G}_v^g)_{v \in [u_{\pi(i)-1} + \epsilon, u_{\pi(i)-1} + \epsilon']}$ for arbitrary $0 \leq \epsilon < \epsilon' \leq t_i - t_{i-1}$.

Lemma 4.3.2 *Let $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$, with $A_i = A'_{\pi(i)}$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$.*

Suppose that $Q_{0t_1}^f = Q_{0u_1}^g$. The following statements are equivalent:

- (a) *for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $0 \leq \epsilon < \epsilon' \leq t_i - t_{i-1}$, and for every $\Gamma_0, \Gamma_1, \dots, \Gamma_i, \Gamma'_i \in \mathcal{B}(\mathbf{R})$*

$$\int_{\mathbf{R}^{i+2}} I_{\Gamma_0}(x_0) I_{\Gamma_1}(x_1) \prod_{j=2}^{i-1} I_{\Gamma_j}(x_j - x_{j-1}) \quad (27)$$

$$\begin{aligned} & I_{\Gamma_i}(x'_{i-1} - x_{i-1}) I_{\Gamma'_i}(x''_{i-1} - x'_{i-1}) Q_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f(x'_{i-1}; dx''_{i-1}) Q_{t_{i-1}, t_{i-1}+\epsilon}^f(x_{i-1}; dx'_{i-1}) \\ & Q_{t_{i-2}t_{i-1}}^f(x_{i-2}; dx_{i-1}) \dots Q_{t_1t_2}^f(x_1; dx_2) Q_{0t_1}^f(x_0; dx_1) \mu(dx_0) = \\ & \int_{\mathbf{R}^{2(i-1)+3}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{j=2}^{i-1} I_{\Gamma_j}(y_{\pi(j)} - y_{\pi(j)-1}) I_{\Gamma_i}(y'_{\pi(i)-1} - y_{\pi(i)-1}) \\ & I_{\Gamma'_i}(y''_{\pi(i)-1} - y'_{\pi(i)-1}) Q_{u_{l_2(i-1)-1} u_{l_2(i-1)}}^g(y_{l_2(i-1)-1}; dy_{l_2(i-1)}) \dots Q_{u_{l_{p+1}} u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) \\ & Q_{u_{l_{p-1}+\epsilon'}, u_{l_{p+1}}}^g(y''_{l_{p-1}}; dy_{l_{p+1}}) Q_{u_{l_{p-1}+\epsilon}, u_{l_{p-1}+\epsilon'}}^g(y'_{l_{p-1}}; dy''_{l_{p-1}}) \\ & Q_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g(y_{l_{p-1}}; dy'_{l_{p-1}}) Q_{u_{l_{p-2}} u_{l_{p-1}}}^g(y_{l_{p-2}}; dy_{l_{p-1}}) \dots Q_{u_{l_1} u_{l_2}}^g(y_{l_1}; dy_{l_2}) \\ & Q_{u_1 u_{l_1}}^g(y_1; dy_{l_1}) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0); \end{aligned}$$

- (b) *for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$, and with p the index for which $\pi(i) - 1 = l_{p-1}, \pi(i) = l_p$, then for every $0 \leq \epsilon < \epsilon' \leq t_i - t_{i-1}$, for every $h_2, \dots, h_i, h'_i \in \mathcal{B}(\mathbf{R})$ and for $\mu_{t_1}^f$ -almost all x_1*

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g T_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \dots \quad (28)$$

$$T_{u_{l_2(i-1)-1}, u_{l_2(i-1)}}^g \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right]$$

$$\begin{aligned}
& (T_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f h'_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}))(x_1) = \\
& \quad T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \\
& \quad T_{u_{l_{p-1}+\epsilon}, u_{l_{p-1}+\epsilon'}}^g T_{u_{l_{p-1}+\epsilon'}, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \\
& \quad \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
& \quad \left. h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1}) \right](x_1);
\end{aligned}$$

(c) for each $i = 2, \dots, n$, if we denote with $l_1 \leq l_2 \leq \dots \leq l_{2(i-1)}$ the increasing ordering of the values $\pi(2) - 1, \pi(2), \pi(3) - 1, \pi(3), \dots, \pi(i) - 1, \pi(i)$ and with p the index for which $\pi(i) - 1 = l_{p-1}$, $\pi(i) = l_p$, then for every $0 \leq \epsilon < \epsilon' \leq t_i - t_{i-1}$, for every $h_2, \dots, h_i, h'_i \in \mathcal{D}$ and for $\mu_{t_1}^f$ -almost all x_1

(c1) if $p = 2(i - 1)$ we have

$$\begin{aligned}
& \int_{t_{i-1}+\epsilon}^{t_{i-1}+\epsilon'} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \\
& \quad \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
& \quad \left. (\mathcal{G}_w^f T_{w, t_{i-1}+\epsilon'}^f h'_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}))(x_1) dw = \right. \\
& \quad \int_{u_{l_{p-1}+\epsilon}}^{u_{l_{p-1}+\epsilon'}} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \mathcal{G}_v^g T_{v, u_{l_{p-1}+\epsilon'}}^g \\
& \quad \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
& \quad \left. h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1}) \right](x_1) dv;
\end{aligned}$$

(c2) if $p < 2(i - 1)$ we have

$$\int_{t_{i-1}+\epsilon}^{t_{i-1}+\epsilon'} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g T_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots$$

$$\begin{aligned}
& T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
& \left. (\mathcal{G}_w^f T_{w, t_{i-1}+\epsilon'}^f h_i')(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}))(x_1) dw = \right. \\
& \int_{u_{l_{p-1}+\epsilon}}^{u_{l_{p-1}+\epsilon'}} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \mathcal{G}_v^g T_{v, u_{l_{p-1}+\epsilon'}}^g \\
& T_{u_{l_{p-1}+\epsilon'}, u_{l_{p+1}}}^g T_{u_{l_{p+1}}, u_{l_{p+2}}}^g \cdots T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \\
& \left. \left\{ \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right\} \right. \\
& \left. [h_i'(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1}) - \right. \\
& \left. - h_i'(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}))](x_1) dv. \right.
\end{aligned}$$

Proof: (a) $\Rightarrow$ (b): Let X be a $\mathcal{Q}^f$ -Markov process and Y a $\mathcal{Q}^g$ -Markov process with the same initial distribution μ . Then (a) is equivalent to saying that the distribution of $(X_0, X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_{i-1}} - X_{t_{i-2}}, X_{t_{i-1}+\epsilon} - X_{t_{i-1}}, X_{t_{i-1}+\epsilon'} - X_{t_{i-1}+\epsilon})$ coincides with the distribution of $(Y_0, Y_{u_1}, Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}}, \dots, Y_{u_{\pi(i-1)}} - Y_{u_{\pi(i-1)-1}}, Y_{u_{\pi(i)-1}+\epsilon} - Y_{u_{\pi(i)-1}}, Y_{u_{\pi(i)-1}+\epsilon'} - Y_{u_{\pi(i)-1}+\epsilon})$.

Since $\mathcal{Q}_{0t_1}^f = \mathcal{Q}_{0u_1}^g$, both X_{t_1} and Y_{u_1} have the same distribution $\mu_{t_1}^f$. Hence (a) is equivalent to saying that for $\mu_{t_1}^f$ -almost all x_1 , the conditional distribution of $(X_{t_2} - X_{t_1}, \dots, X_{t_{i-1}} - X_{t_{i-2}}, X_{t_{i-1}+\epsilon} - X_{t_{i-1}}, X_{t_{i-1}+\epsilon'} - X_{t_{i-1}+\epsilon})$ given $X_{t_1} = x_1$ coincide with the conditional distribution of $(Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}}, \dots, Y_{u_{\pi(i-1)}} - Y_{u_{\pi(i-1)-1}}, Y_{u_{\pi(i)-1}+\epsilon} - Y_{u_{\pi(i)-1}}, Y_{u_{\pi(i)-1}+\epsilon'} - Y_{u_{\pi(i)-1}+\epsilon})$ given $Y_{u_1} = x_1$.

For $i = 2$ we will use the following relationship: for every $0 \leq \epsilon < \epsilon' \leq t_2 - t_1$ and for every Borel sets Γ_2, Γ'_2

$$\begin{aligned}
& P[X_{t_1+\epsilon} - X_{t_1} \in \Gamma_2, X_{t_1+\epsilon'} - X_{t_1+\epsilon} \in \Gamma'_2 | X_{t_1} = x_1] = \\
& P[Y_{u_{\pi(2)-1}+\epsilon} - Y_{u_{\pi(2)-1}} \in \Gamma_2, Y_{u_{\pi(2)-1}+\epsilon'} - Y_{u_{\pi(2)-1}+\epsilon} \in \Gamma'_2 | Y_{u_1} = x_1].
\end{aligned}$$

Using relationship (24) (which can be derived from equation (27) for $i = 2, \epsilon = \epsilon'$), the left-hand side can be written as

$$\begin{aligned} \int_{\mathbf{R}^2} I_{\Gamma_2+x_1}(x'_1) I_{\Gamma'_2+x'_1}(x''_1) Q_{t_1+\epsilon, t_1+\epsilon'}^f(x'_1; dx''_1) Q_{t_1, t_1+\epsilon}^f(x_1; dx'_1) = \\ \int_{\mathbf{R}^3} I_{\Gamma_2}(y'_{\pi(2)-1} - y_{\pi(2)-1}) I_{\Gamma'_2+x_1+y'_{\pi(2)-1}-y_{\pi(2)-1}}(x''_1) \\ Q_{t_1+\epsilon, t_1+\epsilon'}^f(x_1 + y'_{\pi(2)-1} - y_{\pi(2)-1}; dx''_1) Q_{u_{\pi(2)-1}, u_{\pi(2)-1}+\epsilon}^g(y_{\pi(2)-1}; dy'_{\pi(2)-1}) \\ Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}). \end{aligned}$$

The right-hand side is

$$\begin{aligned} \int_{\mathbf{R}^3} I_{\Gamma_2}(y'_{\pi(2)-1} - y_{\pi(2)-1}) I_{\Gamma'_2}(y''_{\pi(2)-1} - y'_{\pi(2)-1}) \\ Q_{u_{\pi(2)-1}+\epsilon, u_{\pi(2)-1}+\epsilon'}^g(y'_{\pi(2)-1}; dy''_{\pi(2)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)-1}+\epsilon}^g(y_{\pi(2)-1}; dy'_{\pi(2)-1}) \\ Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}). \end{aligned}$$

By a monotone class argument we can conclude that for every $0 \leq \epsilon < \epsilon' \leq t_2 - t_1$ and for every $h_2, h'_2 \in B(\mathbf{R})$

$$\begin{aligned} \int_{\mathbf{R}^3} h_2(y'_{\pi(2)-1} - y_{\pi(2)-1}) h'_2(x''_1) Q_{t_1+\epsilon, t_1+\epsilon'}^f(x_1 + y'_{\pi(2)-1} - y_{\pi(2)-1}; dx''_1) \\ Q_{u_{\pi(2)-1}, u_{\pi(2)-1}+\epsilon}^g(y_{\pi(2)-1}; dy'_{\pi(2)-1}) Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) = \\ \int_{\mathbf{R}^3} h_2(y'_{\pi(2)-1} - y_{\pi(2)-1}) h'_2(x_1 + y''_{\pi(2)-1} - y'_{\pi(2)-1}) \\ Q_{u_{\pi(2)-1}+\epsilon, u_{\pi(2)-1}+\epsilon'}^g(y'_{\pi(2)-1}; dy''_{\pi(2)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)-1}+\epsilon}^g(y_{\pi(2)-1}; dy'_{\pi(2)-1}) \\ Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}) \end{aligned}$$

which is the desired relationship for $i = 2$.

To prove the relationship for $i = 3$ we will use the following relationship: for every $0 \leq \epsilon < \epsilon' \leq t_3 - t_2$ and for every Borel sets $\Gamma_2, \Gamma_3, \Gamma'_3$

$$P[X_{t_2} - X_{t_1} \in \Gamma_2, X_{t_2+\epsilon} - X_{t_2} \in \Gamma_3, X_{t_2+\epsilon'} - X_{t_2+\epsilon} \in \Gamma'_3 | X_{t_1} = x_1] = \quad (29)$$

$$\begin{aligned} P[Y_{u_{\pi(2)}} - Y_{u_{\pi(2)-1}} \in \Gamma_2, Y_{u_{\pi(3)-1}+\epsilon} - Y_{u_{\pi(3)-1}} \in \Gamma_3, \\ Y_{u_{\pi(3)-1}+\epsilon'} - Y_{u_{\pi(3)-1}+\epsilon} \in \Gamma'_3 | Y_{u_1} = x_1]. \end{aligned}$$

The left-hand side is

$$\int_{\mathbf{R}^3} I_{\Gamma_2+x_1}(x_2) I_{\Gamma_3+x_2}(x'_2) I_{\Gamma'_3+x'_2}(x''_2) Q_{t_2+\epsilon, t_2+\epsilon'}^f(x'_2; dx''_2) \\ Q_{t_2, t_2+\epsilon}^f(x_2; dx'_2) Q_{t_1, t_2}^f(x_1; dx_2).$$

Using equation (20) (which can be derived from equation (27) for $i = 2, \epsilon = 0, \epsilon' = t_2 - t_1$), this can be written as

$$\int_{\mathbf{R}^2} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3+x_1+y_{\pi(2)}-y_{\pi(2)-1}}(x'_2) I_{\Gamma'_3+x'_2}(x''_2) \quad (30)$$

$$Q_{t_2+\epsilon, t_2+\epsilon'}^f(x'_2; dx''_2) Q_{t_2, t_2+\epsilon}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx'_2) Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) \\ Q_{u_1, u_{\pi(2)-1}}^g(x_1; dy_{\pi(2)-1}).$$

At this point we may apply equation (27) for $i = 3$.

Case 1 Suppose that $\pi(2) < \pi(3)$. Using equation (25) (which can be derived from equation (27) for $i = 3, \epsilon = \epsilon'$, if $\pi(2) < \pi(3)$), integral (30) can be written as

$$\int_{\mathbf{R}^5} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y'_{\pi(3)-1} - y_{\pi(3)-1}) \\ I_{\Gamma'_3+x_1+y_{\pi(2)}-y_{\pi(2)-1}+y'_{\pi(3)-1}-y_{\pi(3)-1}}(x''_2) \\ Q_{t_2+\epsilon, t_2+\epsilon'}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}; dx''_2) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\ Q_{u_{\pi(2)}, u_{\pi(3)-1}}^g(y_{\pi(2)}; dy_{\pi(3)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1, u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1}).$$

The right-hand side of (29) can be written as

$$\int_{\mathbf{R}^5} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y'_{\pi(3)-1} - y_{\pi(3)-1}) I_{\Gamma'_3}(y''_{\pi(3)-1} - y'_{\pi(3)-1}) \\ Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(3)-1}+\epsilon'}^g(y'_{\pi(3)-1}; dy''_{\pi(3)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\ Q_{u_{\pi(2)}, u_{\pi(3)-1}}^g(y_{\pi(2)}; dy_{\pi(3)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1, u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1}).$$

By a monotone class argument we can conclude that for every $0 \leq \epsilon < \epsilon' \leq t_3 - t_2$ and for every $h_2, h_3, h'_3 \in B(\mathbf{R})$ we have

$$\int_{\mathbf{R}^5} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(y'_{\pi(3)-1} - y_{\pi(3)-1}) h'_3(x''_2)$$

$$\begin{aligned}
& Q_{t_2+\epsilon, t_2+\epsilon'}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}; dx_2'') Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
& Q_{u_{\pi(2)}, u_{\pi(3)-1}}^g(y_{\pi(2)}; dy_{\pi(3)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1, u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1}) = \\
& \int_{\mathbf{R}^5} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(y'_{\pi(3)-1} - y_{\pi(3)-1}) \\
& h'_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y''_{\pi(3)-1} - y_{\pi(3)-1}) \\
& Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(3)-1}+\epsilon'}^g(y'_{\pi(3)-1}; dy''_{\pi(3)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
& Q_{u_{\pi(2)}, u_{\pi(3)-1}}^g(y_{\pi(2)}; dy_{\pi(3)-1}) Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1, u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1})
\end{aligned}$$

which is the desired relationship for $i = 3$, if $\pi(2) < \pi(3)$.

Case 2 Suppose that $\pi(3) < \pi(2)$. Using equation (26) (which can be derived from equation (27) for $i = 3, \epsilon = \epsilon'$, if $\pi(3) < \pi(2)$), integral (30) can be written as

$$\begin{aligned}
& \int_{\mathbf{R}^5} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y'_{\pi(3)-1} - y_{\pi(3)-1}) \\
& I_{\Gamma'_3+x_1+y_{\pi(2)}-y_{\pi(2)-1}+y'_{\pi(3)-1}-y_{\pi(3)-1}}(x_2'') \\
& Q_{t_2+\epsilon, t_2+\epsilon'}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}; dx_2'') Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) \\
& Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(2)-1}}^g(dy'_{\pi(3)-1}; y_{\pi(2)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
& Q_{u_1, u_{\pi(3)-1}}^g(y_1; dy_{\pi(3)-1}).
\end{aligned}$$

The right-hand side of (29) can be written as

$$\begin{aligned}
& \int_{\mathbf{R}^5} I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3}(y'_{\pi(3)-1} - y_{\pi(3)-1}) I_{\Gamma'_3}(y''_{\pi(3)-1} - y'_{\pi(3)-1}) \\
& Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_{\pi(3)-1}+\epsilon', u_{\pi(2)-1}}^g(y''_{\pi(3)-1}; dy_{\pi(2)-1}) \\
& Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(3)-1}+\epsilon'}^g(y'_{\pi(3)-1}; dy''_{\pi(3)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
& Q_{u_1, u_{\pi(3)-1}}^g(x_1; dy_{\pi(3)-1}).
\end{aligned}$$

By a monotone class argument we can conclude that for every $0 \leq \epsilon < \epsilon' \leq t_3 - t_2$ and for every $h_2, h_3, h'_3 \in B(\mathbf{R})$ we have

$$\begin{aligned}
& \int_{\mathbf{R}^5} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(y'_{\pi(3)-1} - y_{\pi(3)-1}) h'_3(x_2'') \\
& Q_{t_2+\epsilon, t_2+\epsilon'}^f(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y'_{\pi(3)-1} - y_{\pi(3)-1}; dx_2'') Q_{u_{\pi(2)-1}, u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)})
\end{aligned}$$

$$\begin{aligned}
 & Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(2)-1}}^g(dy'_{\pi(3)-1}; y_{\pi(2)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
 & Q_{u_1, u_{\pi(3)-1}}^g(y_1; dy_{\pi(3)-1}) = \\
 & \int_{\mathbf{R}^5} h_2(y_{\pi(2)} - y_{\pi(2)-1}) h_3(y'_{\pi(3)-1} - y_{\pi(3)-1}) \\
 & h'_3(x_1 + y_{\pi(2)} - y_{\pi(2)-1} + y''_{\pi(3)-1} - y_{\pi(3)-1}) \\
 & Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_{\pi(3)-1}+\epsilon', u_{\pi(2)-1}}^g(y''_{\pi(3)-1}; dy_{\pi(2)-1}) \\
 & Q_{u_{\pi(3)-1}+\epsilon, u_{\pi(3)-1}+\epsilon'}^g(y'_{\pi(3)-1}; dy''_{\pi(3)-1}) Q_{u_{\pi(3)-1}, u_{\pi(3)-1}+\epsilon}^g(y_{\pi(3)-1}; dy'_{\pi(3)-1}) \\
 & Q_{u_1 u_{\pi(3)-1}}^g(x_1; dy_{\pi(3)-1})
 \end{aligned}$$

which is the desired relationship for $i = 3$, if $\pi(3) < \pi(2)$.

The general argument will be omitted since it is identical to the argument for the case $i = 3$, but notationally more complex.

(b) $\Rightarrow$ (a): Let $\Gamma_0, \Gamma_1, \dots, \Gamma_i, \Gamma'_i \in \mathcal{B}(\mathbf{R})$ be arbitrary. Using the fact that $Q_{0t_1}^f = Q_{0u_1}^g$ and equation (20) (which can be derived from equation (28) for $i = 2, \epsilon = 0, \epsilon' = t_2 - t_1$), the left-hand side of (27) becomes

$$\begin{aligned}
 & \int_{\mathbf{R}^{i+3}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) I_{\Gamma_2}(y_{\pi(2)} - y_{\pi(2)-1}) I_{\Gamma_3+y_1+y_{\pi(2)}-y_{\pi(2)-1}}(x_3) I_{\Gamma_4+x_3}(x_4) \dots \\
 & I_{\Gamma_{i-1}+x_{i-2}}(x_{i-1}) I_{\Gamma_i+x_{i-1}}(x'_{i-1}) I_{\Gamma'_i+x'_{i-1}}(x''_{i-1}) Q_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f(x'_{i-1}; dx''_{i-1}) \\
 & Q_{t_{i-1}, t_{i-1}+\epsilon}(x_{i-1}; dx'_{i-1}) Q_{t_{i-2}t_{i-1}}^f(x_{i-2}; dx_{i-1}) \dots Q_{t_3t_4}^f(x_3; dx_4) \\
 & Q_{t_2t_3}^f(y_1 + y_{\pi(2)} - y_{\pi(2)-1}; dx_3) Q_{u_{\pi(2)-1} u_{\pi(2)}}^g(y_{\pi(2)-1}; dy_{\pi(2)}) Q_{u_1 u_{\pi(2)-1}}^g(y_1; dy_{\pi(2)-1}) \\
 & Q_{0u_1}^g(y_0; dy_1) \mu(dy_0)
 \end{aligned}$$

(we have formally replaced x_2 by $y_1 + y_{\pi(2)} - y_{\pi(2)-1}$).

We continue in the same manner and we use successively equation (28) first for $i = 3, \epsilon = 0, \epsilon' = t_3 - t_2$, then for $i = 4, \epsilon = 0, \epsilon' = t_4 - t_3$, and so on until we reach $i - 1$, taking $\epsilon = 0, \epsilon' = t_{i-1} - t_{i-2}$. The left-hand side of (27) becomes

$$\begin{aligned}
 & \int_{\mathbf{R}^{2(i-1)+3}} I_{\Gamma_0}(y_0) I_{\Gamma_1}(y_1) \prod_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) I_{\Gamma_i+y_1+\sum_{j=2}^{i-1}(y_{\pi(j)}-y_{\pi(j)-1})}(x'_{i-1}) \\
 & I_{\Gamma'_i+x'_{i-1}}(x''_{i-1}) Q_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f(x'_{i-1}; dx''_{i-1}) Q_{t_{i-1}, t_{i-1}+\epsilon}^f(y_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}); dx'_{i-1})
 \end{aligned}$$

$$\begin{aligned}
 & Q_{u_{l_{2(i-1)-1}} u_{l_{2(i-1)}}}^g(y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \cdots Q_{u_{l_{p+1}} u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) \\
 & Q_{u_{l_{p-1}} u_{l_{p+1}}}^g(y_{l_{p-1}}; dy_{l_{p+1}}) Q_{u_{l_{p-2}} u_{l_{p-1}}}^g(y_{l_{p-2}}; dy_{l_{p-1}}) \cdots Q_{u_{l_1} u_{l_2}}^g(y_{l_1}; dy_{l_2}) \\
 & Q_{u_1 u_{l_1}}^g(y_1; dy_1) Q_{0u_1}^g(y_0; dy_1) \mu(dy_0).
 \end{aligned}$$

We use next equation (28) for i taking $\epsilon = \epsilon'$ and we 'eliminate' the operator $Q_{t_{i-1}, t_{i-1}+\epsilon}^f$ by inserting the operator $Q_{u_{l_{p-1}}, u_{l_{p-1}}+\epsilon}^g$ at the appropriate place (formally we replace x'_{i-1} by $y_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + (y'_{\pi(i)-1} - y_{\pi(i)-1}))$). Finally, by using equation (28) for i, ϵ, ϵ' , we 'eliminate' the operator $Q_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f$ and we get exactly the desired right-hand side of (27) (formally we replace x''_{i-1} by $y_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + (y''_{\pi(i)-1} - y_{\pi(i)-1}))$).

(b) $\Leftrightarrow$ (c): The basic ingredient will be equation (17), which gives the integral expression of a semigroup in terms of its generator.

Since $\mathcal{D}$ is dense we can assume that the functions $h_2, \dots, h_i$ are in $\mathcal{D}$ in the expression given by (b). Subtract

$$\begin{aligned}
 & T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}}+\epsilon}^g T_{u_{l_{p-1}}+\epsilon, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots \\
 & T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
 & \left. h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}) \right]
 \end{aligned}$$

in both members of this expression.

On the left hand-side we will have

$$\begin{aligned}
 & T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}}+\epsilon}^g T_{u_{l_{p-1}}+\epsilon, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots \\
 & T_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\
 & \left. (T_{t_{i-1}+\epsilon, t_{i-1}+\epsilon'}^f h'_i - h'_i)(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}) \right]
 \end{aligned}$$

which can be written as

$$\int_{t_{i-1}+\epsilon}^{t_{i-1}+\epsilon'} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}}+\epsilon}^g T_{u_{l_{p-1}}+\epsilon, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots$$

$$T_{u_{l_2(i-1)-1}, u_{l_2(i-1)}}^g \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\ \left. (\mathcal{G}_w^f T_{w, l_{i-1}+\epsilon'}^f h_i')(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1})) dv \right]$$

using the integral expression (17) and Fubini's theorem.

On the right-hand side we have

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \cdots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}}+\epsilon}^g \\ \{ T_{u_{l_{p-1}}+\epsilon, u_{l_{p-1}}+\epsilon'}^g T_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots T_{u_{l_2(i-1)-1}, u_{l_2(i-1)}}^g \\ \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right. \\ \left. h_i'(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1})) (y'_{l_{p-1}}) - \right. \\ \left. - T_{u_{l_{p-1}}+\epsilon, u_{l_{p+1}}}^g T_{u_{l_{p+1}} u_{l_{p+2}}}^g \cdots T_{u_{l_2(i-1)-1}, u_{l_2(i-1)}}^g \right. \\ \left. \left[\prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \right] \right. \\ \left. h_i'(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1})) (y'_{l_{p-1}}) \}.$$

If we denote

$$h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y''_{l_{p-1}}) := \\ \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \\ h_i'(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1}) Q_{u_{l_2(i-1)-1}, u_{l_2(i-1)}}^g(y_{l_2(i-1)-1}; dy_{l_2(i-1)}) \\ \cdots Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g(y_{l_{p+1}}; dy_{l_{p+2}}) Q_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g(y'_{l_{p-1}}; dy_{l_{p+1}})$$

and

$$h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}) := \\ \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1})$$

$$h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}) Q_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g (y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \\ \dots Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g (y_{l_{p+1}}; dy_{l_{p+2}}) Q_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g (y'_{l_{p-1}}; dy_{l_{p+1}})$$

then the right-hand side becomes

$$T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \\ [(T_{u_{l_{p-1}+\epsilon}, u_{l_{p-1}+\epsilon'}}^g h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y'_{l_{p-1}}) - h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}})] = \\ T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}}, u_{l_{p-1}+\epsilon}}^g \\ [h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y'_{l_{p-1}}) - h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}) + \\ + \int_{u_{l_{p-1}+\epsilon}}^{u_{l_{p-1}+\epsilon'}} (G_v^g T_{v, u_{l_{p-1}+\epsilon'}}^g h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y'_{l_{p-1}}) dv](x_1)$$

since $h_{\epsilon'}(x_1, y_{l_1}, y_{l_2}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot) \in \mathcal{D}$ for every $x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}$. We have two cases:

Case 1) If $p = 2(i-1)$, then all the integrals with respect to $Q_{u_{l_{p-1}+\epsilon}, u_{l_{p-1}+\epsilon'}}^g$, $Q_{u_{l_{p-1}+\epsilon'}, u_{l_{p+1}}}^g$, $Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g, \dots, Q_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g$ disappear in the preceding expressions. Hence $h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y'_{l_{p-1}}) = h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}})$ and the result follows.

Case 2) If $p < 2(i-1)$, then

$$h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y'_{l_{p-1}}) = (T_{u_{l_{p-1}+\epsilon'}, u_{l_{p+1}}}^g H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y'_{l_{p-1}}) \\ h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}) = (T_{u_{l_{p-1}+\epsilon}, u_{l_{p+1}}}^g H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y'_{l_{p-1}})$$

where

$$H(x_1, y_{l_1}, y_{l_2}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y_{l_{p+1}}) := \\ \int_{\mathbf{R}^{2(i-1)-p-1}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \\ h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1}) Q_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g (y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \\ \dots Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g (y_{l_{p+1}}; dy_{l_{p+2}}).$$

To simplify the notation we will omit the arguments of the function H . Hence

$$\begin{aligned} & h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y'_{l_{p-1}}) - h_{\epsilon}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}) = \\ & (T_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g H)(y'_{l_{p-1}}) - [T_{u_{l_{p-1}}+\epsilon, u_{l_{p-1}}+\epsilon'}^g (T_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g H)](y'_{l_{p-1}}) = \\ & - \int_{u_{l_{p-1}}+\epsilon}^{u_{l_{p-1}}+\epsilon'} (\mathcal{G}_v^g T_{v, u_{l_{p-1}}+\epsilon'}^g (T_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g H))(y'_{l_{p-1}}) dv \end{aligned}$$

since $H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot) \in \mathcal{D}$ for every $x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}$.

Using Fubini theorem's, the right-hand side becomes

$$\begin{aligned} & \int_{u_{l_{p-1}}+\epsilon}^{u_{l_{p-1}}+\epsilon'} T_{u_1 u_{l_1}}^g T_{u_{l_1} u_{l_2}}^g \dots T_{u_{l_{p-2}} u_{l_{p-1}}}^g T_{u_{l_{p-1}} u_{l_{p-1}}+\epsilon}^g \\ & [(\mathcal{G}_v^g T_{v, u_{l_{p-1}}+\epsilon'}^g U_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y'_{l_{p-1}})](x_1) dv \end{aligned}$$

where

$$\begin{aligned} & U_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y''_{l_{p-1}}) := \\ & h_{\epsilon'}(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, y''_{l_{p-1}}) - (T_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g H(x_1, y_{l_1}, \dots, y_{l_{p-1}}, y'_{l_{p-1}}, \cdot))(y''_{l_{p-1}}) = \\ & \int_{\mathbf{R}^{2(i-1)-p}} \prod_{j=2}^{i-1} h_j(y_{\pi(j)} - y_{\pi(j)-1}) h_i(y'_{\pi(i)-1} - y_{\pi(i)-1}) \\ & [h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y''_{\pi(i)-1} - y_{\pi(i)-1}) - \\ & - h'_i(x_1 + \sum_{j=2}^{i-1} (y_{\pi(j)} - y_{\pi(j)-1}) + y'_{\pi(i)-1} - y_{\pi(i)-1})] \\ & Q_{u_{l_{2(i-1)-1}}, u_{l_{2(i-1)}}}^g (y_{l_{2(i-1)-1}}; dy_{l_{2(i-1)}}) \dots Q_{u_{l_{p+1}}, u_{l_{p+2}}}^g (y_{l_{p+1}}; dy_{l_{p+2}}) \\ & Q_{u_{l_{p-1}}+\epsilon', u_{l_{p+1}}}^g (y''_{l_{p-1}}; dy_{l_{p+1}}). \end{aligned}$$

The concludes the proof.

□

We have the following assumption.

Assumption 4.3.3 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}, \forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then the generators $\mathcal{G}^f$ and $\mathcal{G}^g$ satisfy condition (c) stated in Lemma 4.3.2.*

Here is the main result of this section.

Theorem 4.3.4 *Let μ be a probability measure on $\mathbf{R}$ and $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ a collection of families of linear operators on $B(\mathbf{R})$ such that each operator $\mathcal{G}_s^f$ is defined on a dense subspace $\mathcal{D}$ of $B(\mathbf{R})$ and is the generator at time s of a semigroup $T^f := (T_{st}^f)_{s < t}$, each T_{st}^f being the operator associated to a transition probability Q_{st}^f .*

If the collection $\{\mathcal{G}^f; f \in \mathcal{S}\}$ satisfies Assumption 4.2.1 and Assumption 4.3.3, then there exist a transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ and a probability measure P on the product space $(\mathbf{R}^c, \mathcal{B}(\mathbf{R})^c)$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is $\mathcal{Q}$ -Markov with initial distribution μ ;*
2. *$\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is an extension of the operator $\mathcal{G}_s^f$;*
3. *if $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_n\}$ are two consistent orderings of the same finite semilattice $\mathcal{A}'$, with $A_i = A'_{\pi(i)}, \forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$, then*

$$P \circ (X_{t_{i-1}+\epsilon'}^f - X_{t_{i-1}+\epsilon}^f)^{-1} = P \circ (X_{u_{\pi(i)-1}+\epsilon'}^g - X_{u_{\pi(i)-1}+\epsilon}^g)^{-1}$$

for every $i = 2, \dots, n$ and for every $0 \leq \epsilon < \epsilon' \leq t_i - t_{i-1}$.

Proof: The collection $\{\mathcal{Q}^f := (Q_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of transition systems satisfies Assumption 3.3.6 and Assumption 3.3.7 (using Lemma 4.3.2). The proof is immediate using Theorem 4.1.3.

□

Chapter 5

Processes with Independent Increments

On the real line, processes with independent increments constitute a fundamental class of stochastic processes; typical examples include Brownian motion, compound Poisson processes, and stable processes. Historically, their systematic study began in 1934, with the characterization of the infinitely divisible distributions given by Lévy's impressive paper [49]. It turned out that any such process can be written as the sum of a non-random function, a discrete process, and a stochastically continuous process. Moreover, any *Lévy process* (i.e. a stochastically continuous process with independent increments) has a cadlag version, which can be constructed as the sum of a continuous Gaussian process and an independent jump process, obtained as a limit of a sequence of independent compound Poisson processes.

On the other hand, processes with independent increments are prototypes of important classes of processes, like Markov processes and semimartingales. An immense amount of literature has been dedicated to them, including two excellent (and quite recent) monographs [9], [65]; a good all-time reference remains Chapter IV, [32].

In the present chapter, we will take the first step in the study of set-indexed processes with independent increments and we will show that these processes can also be described by means of convolution systems of distributions on the real line. We want to emphasize that for most of the chapter, the underlying space T need not

be metrizable.

Most of the results of this chapter appear in [6].

5.1 General Properties

In this section we will give some general properties of processes with independent increments, with special emphasis on the set-Markov property. We will also introduce the two fundamental examples of such processes: the Brownian motion and the Poisson process.

We begin by recalling that a set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is said to have *independent increments* if $X_\emptyset = X_{\emptyset'} = 0$ a.s., and $X_{C_1}, \dots, X_{C_k}$ are independent whenever $C_1, \dots, C_k \in \mathcal{C}$ are pairwise disjoint (Definition 2.3.14).

Comment 5.1.1 Since each set in $\mathcal{C}(u)$ can be written as the finite union of some pairwise disjoint sets in $\mathcal{C}$, by the associativity of independence, it follows that the above property can be extended to the sets in $\mathcal{C}(u)$; namely, if X is a process with independent increments, then $X_{C_1}, \dots, X_{C_k}$ are independent whenever $C_1, \dots, C_k \in \mathcal{C}(u)$ are pairwise disjoint.

Let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process with independent increments. If we denote with F_C the distribution of X_C for each $C \in \mathcal{C}(u)$, then the family $(F_C)_{C \in \mathcal{C}(u)}$ of these probability measures satisfies the following conditions:

1. $F_\emptyset = F_{\emptyset'} = \delta_0$; and
2. $F_{C_1} * \dots * F_{C_n} = F_{C'_1} * \dots * F_{C'_m}$ whenever $C_1, \dots, C_n, C'_1, \dots, C'_m \in \mathcal{C}$ are such that $\dot{\cup}_{i=1}^n C_i = \dot{\cup}_{j=1}^m C'_j$. (Here $*$ denotes the convolution operator and $\dot{\cup}$ denotes a pairwise disjoint union.)

Any such family of distributions will be called a *convolution system*.

Proposition 5.1.2 *Any process with independent increments is $\mathcal{Q}$ -Markov, with the transition system $\mathcal{Q}$ given by*

$$Q_{BB'}(x; \Gamma) := F_{B' \setminus B}(\Gamma - x) \quad \forall x \in \mathbf{R}, \forall \Gamma \in \mathcal{B}(\mathbf{R})$$

for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$, where $(F_C)_{C \in \mathcal{C}(u)}$ is the convolution system of the process.

Proof: We will prove first that the family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system: let $B, B', B'' \in \mathcal{A}(u)$ be such that $B \subseteq B' \subseteq B''$ and $x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$ arbitrary; then

$$\begin{aligned} \int_{\mathbf{R}} Q_{B'B''}(y; \Gamma) Q_{BB'}(x; dy) &= \int_{\mathbf{R}} F_{B'' \setminus B'}(\Gamma - y) F_{B' \setminus B}(dy - x) \\ &= \int_{\mathbf{R}} F_{B'' \setminus B'}(\Gamma - x - z) F_{B' \setminus B}(dz) \\ &= (F_{B' \setminus B} * F_{B'' \setminus B})(\Gamma - x) \\ &= F_{B'' \setminus B}(\Gamma - x) \\ &= Q_{BB''}(x; \Gamma) \end{aligned}$$

because $(B' \setminus B) \cup (B'' \setminus B') = B'' \setminus B$.

Let $X := (X_A)_{A \in \mathcal{A}}$ be a process with independent increments. We recall that the process is automatically set-Markov (Proposition 3.1.3). We will prove that $Q_{BB'}$ is a version of the conditional distribution of $X_{B'}$ given X_B , for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$ (see Definition A.2.1, Appendix A.2). Let $x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$ be arbitrary; then

$$\begin{aligned} P[X_{B'} \in \Gamma | X_B = x] &= P[X_{B' \setminus B} \in \Gamma - x | X_B = x] \\ &= P(X_{B' \setminus B} \in \Gamma - x) \\ &= F_{B' \setminus B}(\Gamma - x) \\ &= Q_{BB'}(x; \Gamma) \end{aligned}$$

by the independence of $X_{B' \setminus B}$ and X_B .

□

We have the following conjecture.

Conjecture 5.1.3 *If $X := (X_A)_{A \in \mathcal{A}}$ is a $\mathcal{Q}$ -Markov process with the transition system $\mathcal{Q}$ given by the relationship $Q_{BB'}(x; \Gamma) := F_{B' \setminus B}(\Gamma - x)$; $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$ where $(F_C)_{C \in \mathcal{C}(u)}$ is a convolution system, then X has independent increments.*

Attempt of proof: First note that the distribution of $X_{B' \setminus B}$ is $F_{B' \setminus B}$, $\forall B, B' \in \mathcal{A}(u)$, $B \subseteq B'$:

$$P[X_{B' \setminus B} \in \Gamma | X_B = x] = Q_{BB'}(x; \Gamma + x)$$

$$\begin{aligned}
 &= F_{B' \setminus B}(\Gamma) \text{ (does not depend on } x) \\
 &= P(X_{B' \setminus B} \in \Gamma).
 \end{aligned}$$

We suppose that $X_{C_1}, \dots, X_{C_n}$ are independent for any pairwise disjoint sets $C_1, \dots, C_n \in \mathcal{C}$ such that $\cup_{i=1}^n C_i = B' \setminus B$ for some $B \subseteq B'$ in $\mathcal{A}(u)$. Note that *this is not clear* even if we know that $P \circ X_{\cup_{i=1}^n C_i}^{-1} = F_{\cup_{i=1}^n C_i} = *_{i=1}^n F_{C_i} = *_{i=1}^n (P \circ X_{C_i}^{-1})$; this is where the conjecture lies.

Finally, if $C_1, \dots, C_n$ are arbitrary pairwise disjoint sets in $\mathcal{C}$, $C_i = A_i \setminus \cup_{j=1}^{n_i} A_{ij}$ are some extremal representations, $\mathcal{A}'$ is the minimal finite sub-semilattice of $\mathcal{A}$ determined by the sets A_i, A_{ij} , $\{B_0 = \emptyset', B_1, \dots, B_m\}$ is a consistent ordering of $\mathcal{A}'$ and we denote with D_j the left neighbourhood of B_j in $\mathcal{A}'$, then $C_i = \cup_{j \in J_i} D_j$ and $X_{C_i} = \sum_{j \in J_i} X_{D_j}$ a.s. If the previous conjectured fact is true then the variables $X_{D_1}, \dots, X_{D_m}$ will be independent since we can write $\cup_{j=1}^m D_j = \cup_{j=1}^m B_j = B' \setminus B$ with $B = \emptyset'$ and $B' = \cup_{j=1}^m B_j$. By the associativity of the independence, the variables $X_{C_1}, \dots, X_{C_n}$ will be independent.

□

Comment 5.1.4 The semigroup associated to a process with independent increments with convolution system $(F_C)_{C \in \mathcal{C}(u)}$ is given by

$$(T_{BB'}h)(x) = \int_{\mathbf{R}} h(x+y) F_{B' \setminus B}(dy), \quad h \in B(\mathbf{R}), x \in \mathbf{R}$$

for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$.

Let Λ be a finite positive measure on $\sigma(\mathcal{A})$. Here are the two basic examples of set-indexed processes with independent increments.

Examples 5.1.5 1. A process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments for which the distribution F_C of each increment $X_C, C \in \mathcal{C}$ is normal with mean 0 and variance Λ_C , is called a *Brownian motion* (with variance measure Λ).

Each probability measure $Q_{BB'}(x; \cdot)$ is also normal with mean x and variance $\Lambda_{B' \setminus B}$. The associated semigroup $\mathcal{T}$ is given by:

$$(T_{BB'}h)(x) = \frac{1}{\sqrt{2\pi\Lambda_{B' \setminus B}}} \int_{\mathbf{R}} h(y) \exp \left\{ -\frac{(y-x)^2}{2\Lambda_{B' \setminus B}} \right\} dy, \quad x \in \mathbf{R}, h \in B(\mathbf{R}).$$

2. A process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments for which the distribution F_C of each increment $X_C, C \in \mathcal{C}$ is Poisson with mean Λ_C is called a *Poisson process* with variance measure Λ .

Each probability measure $Q_{BB'}(k; \cdot)$ is Poisson:

$$Q_{BB'}(k; \cdot) = e^{-\Lambda_{B' \setminus B}} \sum_{n \geq 0} \frac{\Lambda_{B' \setminus B}^n}{n!} \delta_{k+n}, \quad k = 0, 1, 2, \dots$$

The semigroup $\mathcal{T}$ is given by

$$(T_{BB'}h)(k) = e^{-\Lambda_{B' \setminus B}} \sum_{n \geq 0} h(k+n) \frac{\Lambda_{B' \setminus B}^n}{n!}, \quad k = 0, 1, 2, \dots$$

for every bounded function $h : \{0, 1, 2, \dots\} \rightarrow \mathbf{R}$.

Note: Alternatively, a Brownian motion with variance measure Λ is a centered Gaussian process $X := (X_A)_{A \in \mathcal{A}}$ (with an additive extension to $\mathcal{C}(u)$), whose covariance function is given by

$$\text{cov}(X_{A_1}, X_{A_2}) = \Lambda_{A_1 \cap A_2}, \quad \forall A_1, A_2 \in \mathcal{A}.$$

The covariance function can be extended to $\mathcal{C}(u)$ by a similar relation.

In what follows we will show that our main tool from the previous chapters, the flow, becomes also extremely useful in this set-up.

Proposition 5.1.6 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}(u)}$ has independent increments if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ has independent increments. (For the necessity part we can consider any flow, not only the simple ones.)*

Proof: For necessity, let $f : [0, a] \rightarrow \mathcal{A}(u)$ be an arbitrary flow (not necessarily simple) and $s, t \in [0, a], s < t$. Then $X_t^f - X_s^f = X_{f(t) \setminus f(s)}$ is independent of $\mathcal{F}_{f(s)} := \sigma(\{X_A; A \in \mathcal{A}, A \subseteq f(s)\})$ and therefore it is independent of $\sigma(\{X_{f(u)}; u \leq s\})$ (which is contained in $\mathcal{F}_{f(s)}$). This is enough to conclude that the process X^f has independent increments.

For sufficiency, let $C_1, \dots, C_k \in \mathcal{C}$ be pairwise disjoint and $C_i = A_i \setminus \bigcup_{j=1}^{n_i} A_{ij}$; $i = 1, \dots, k$ some extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij} , $\{B_0 = \emptyset, B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$ and D_j the left neighbourhood of B_j for each $j = 1, \dots, n$.

Say $C_i = \bigcup_{j \in J_i} D_j$ for a subset J_i of $\{1, \dots, n\}$. By Lemma 2.4.8, there exists a simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ such that $f(t_i) = \bigcup_{j=0}^i B_j$, where $t_0 = 0 < t_1 < \dots < t_n \leq a$ is the partition corresponding to f . Hence, for every $i = 1, \dots, k$

$$X_{C_i} = \sum_{j \in J_i} X_{D_j} = \sum_{j \in J_i} (X_{t_j}^f - X_{t_{j-1}}^f) \text{ a.s.}$$

These variables are independent by the associativity of independence.

□

Let us consider our two basic examples, the Brownian motion and the Poisson process in the light of the previous proposition. Let Λ be a finite positive measure on $\sigma(\mathcal{A})$.

1. A set-indexed process $X := (X_A)_{A \in \mathcal{A}(u)}$ is a Brownian motion with variance measure Λ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is a Brownian motion with variance function $\Lambda^f := \Lambda \circ f$. If the function Λ^f is differentiable, then the generator of the process X^f at time s is given by

$$(\mathcal{G}_s h)(x) = \frac{1}{2} (\Lambda_s^f)' h''(x), \quad x \in \mathbf{R}$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ contains the space $C_b^2(\mathbf{R})$ of all twice continuously differentiable functions $h : \mathbf{R} \rightarrow \mathbf{R}$ which have the property that h, h', h'' are bounded (see Example B.1.4.1, Appendix B.1).

2. A set-indexed process $X := (X_A)_{A \in \mathcal{A}(u)}$ is Poisson with variance measure Λ if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is Poisson with variance function $\Lambda^f := \Lambda \circ f$. If the function Λ^f is differentiable, then the generator of the process X^f at time s is given by

$$(\mathcal{G}_s h)(k) = (\Lambda_s^f)' \cdot (h(k+1) - h(k)), \quad k = 0, 1, 2, \dots$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ is the space $B(\{0, 1, 2, \dots\})$ of all bounded functions $h : \{0, 1, 2, \dots\} \rightarrow \mathbf{R}$ (see Example B.1.4.2, Appendix B.1).

For the remainder of this section we will concentrate on the variance function Λ of an arbitrary set-indexed process with independent increments. We need the following definition.

Definition 5.1.7 *A process $X := (X_A)_{A \in \mathcal{A}}$ is square-integrable if*

$$E(X_A^2) < \infty \quad \forall A \in \mathcal{A}.$$

We prove that the variance function Λ has a unique finite additive extension to $\mathcal{C}(u)$ (although, in general, we may not be able to extend it to a measure on $\sigma(\mathcal{A})$).

Proposition 5.1.8 *Let $X := (X_A)_{A \in \mathcal{A}}$ be a square-integrable process with independent increments and let $\Lambda_A := \text{Var}(X_A)$, $A \in \mathcal{A}$. Then Λ has a unique additive extension to $\mathcal{C}(u)$ and $\Lambda_C = \text{Var}(X_C)$, $\forall C \in \mathcal{C}(u)$.*

Proof: We will show first that Λ has a unique additive extension to $\mathcal{A}(u)$ by proving inductively that for every $A_1, \dots, A_n \in \mathcal{A}$ we have

$$\text{Var}(X_{\cup_{i=1}^n A_i}) = \sum_{i=1}^n \text{Var}(X_{A_i}) - \sum_{i < j} \text{Var}(X_{A_i \cap A_j}) + \dots + (-1)^{n+1} \text{Var}(X_{\cap_{i=1}^n A_i})$$

The case $n = 1$ is trivial. Assume that the statement is true for $n - 1$. Let $A_1, \dots, A_n$ be some arbitrary sets in $\mathcal{A}$. We have $\text{Var}(X_{\cup_{i=1}^n A_i}) = \text{Var}(X_{A_1 \setminus \cup_{i=2}^n A_i}) + \text{Var}(X_{\cup_{i=2}^n A_i}) = \text{Var}(X_{A_1}) - \text{Var}(X_{\cup_{i=2}^n (A_1 \cap A_i)}) + \text{Var}(X_{\cup_{i=2}^n A_i})$, by independence. Now we use the induction hypothesis for $\text{Var}(X_{\cup_{i=2}^n (A_1 \cap A_i)})$ and $\text{Var}(X_{\cup_{i=2}^n A_i})$. We set

$$\Lambda_{\cup_{i=1}^n A_i} \stackrel{\text{def}}{=} \sum_{i=1}^n \Lambda_{A_i} - \sum_{i < j} \Lambda_{A_i \cap A_j} + \dots + (-1)^{n+1} \Lambda_{\cap_{i=1}^n A_i}.$$

Trivially, this definition does not depend on the representation of $B = \cup_{i=1}^n A_i$.

To show that Λ has a unique additive extension to $\mathcal{C}$ we will use again the independence: let $A, B \in \mathcal{A}(u)$; then $\text{Var}(X_{A \setminus B}) = \text{Var}(X_A) - \text{Var}(X_{A \cap B}) \stackrel{\text{def}}{=} \Lambda_{A \setminus B}$. The existence of the unique additive extension of Λ to $\mathcal{C}(u)$ is proved exactly as for $\mathcal{A}(u)$.

□

Comment 5.1.9 Note that if the variance function Λ is monotone outer-continuous, then it can be extended to a measure on $\sigma(\mathcal{A})$ by Proposition 2.4.3.

We will conclude this section with a few remarks about the ‘homogeneity’ of the projections of a process with independent increments over a suitable class of flows.

If $X := (X_A)_{A \in \mathcal{A}}$ is a square integrable process with independent increments whose variance function Λ is monotone outer-continuous and monotone inner-continuous and $f : [0, a] \rightarrow \mathcal{A}(u)$ is a simple flow, then by Proposition 2.4.7, the function $\Lambda \circ f : [0, a] \rightarrow [0, b]$ is continuous (where $b = \Lambda(f(a))$). Therefore, for each $s \in [0, b]$, there exists some $t_s \in [0, a]$ such that $(\Lambda \circ f)(t_s) = s$. The function

$$g(s) := f(t_s), \quad s \in [0, b]$$

is clearly a flow. What is important is that the variance function $\Lambda \circ g$ of the projected process X^g is equal to the identity map, i.e. $\text{Var}(X_s^g) = s, \forall s \in [0, b]$. If the distribution of each increment $X_C, C \in \mathcal{C}$ is completely determined by its variance Λ_C (as it is the case of the Brownian motion and Poisson process), then we can conclude that the projected process X^g is ‘homogenous’, i.e. the distribution of each increment $X_t^g - X_s^g, s < t$ depends on s, t only through $t - s$.

If in addition, the function $\Lambda \circ f$ is strictly increasing, then it is one-to-one and

$$g = f \circ (\Lambda \circ f)^{-1}.$$

In this case, the flow g is simple too: say $f(t) = \cup_{j=0}^i A_j \cup f_i(t), t \in [t_i, t_{i+1}], i = 0, \dots, n$ for some $t_0 = 0 < t_1 < \dots < t_n = a$ and some $\mathcal{A}$ -valued flows $f_i : [t_i, t_{i+1}] \rightarrow \mathcal{A}$; denote $\Lambda(\cup_{j=0}^i A_j) := s_i$ and $g_i := f_i \circ (\Lambda \circ f)^{-1} : [s_i, s_{i+1}] \rightarrow \mathcal{A}$; for each $s \in [s_i, s_{i+1}]$ we have $g(s) = f((\Lambda \circ f)^{-1}(s)) = \cup_{j=0}^i A_j \cup f_i((\Lambda \circ f)^{-1}(s)) = \cup_{j=0}^i A_j \cup g_i(s)$.

Note also that the flow g follows exactly the same path as f .

5.2 The Compound Poisson Process

In this section we will introduce an important class of set-indexed processes with independent increments, the compound Poisson processes (see Appendix B.1 for the definition of the classical compound Poisson process).

We begin with the definition of a set-indexed compound Poisson process.

Definition 5.2.1 Let $\Pi(dz; dx)$ be a finite positive measure on $(T \times \mathbf{R}, \sigma(\mathcal{A}) \times \mathcal{B}(\mathbf{R}))$ and let $\Pi_A(\Gamma) := \Pi(A \times \Gamma)$, $A \in \sigma(\mathcal{A})$, $\Gamma \in \mathcal{B}(\mathbf{R})$. A process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments, for which the distribution F_C of each increment X_C , $C \in \mathcal{C}$ has the log-characteristic function

$$\psi_C(u) = \int_{\mathbf{R}} (e^{iuy} - 1) \Pi_C(dy), \quad u \in \mathbf{R}$$

is called a **compound Poisson process** (corresponding to the measure Π).

Note: If the measure Π can be written as a product of measures $\Pi(dz; dx) = \Lambda(dz) \times G(dx)$ for some probability measure G on $\mathbf{R}$, then the process X is said to have *stationary increments* (with respect to Λ), i.e. the distribution of each increment X_C , $C \in \mathcal{C}$ depends on C only through Λ_C . In particular, if $G = \delta_1$, then X is a Poisson process with variance measure Λ .

Comments 5.2.2 1. If $X := (X_A)_{A \in \mathcal{A}}$ is a compound Poisson process (corresponding to a measure Π), then each increment X_C , $C \in \mathcal{C}$ has a compound Poisson distribution (corresponding to the measure Π_C).

2. A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is compound Poisson (corresponding to a measure Π) if and only if for every simple flow $f : [0, a] \rightarrow \mathcal{A}(u)$ the process $X^f := (X_{f(t)})_{t \in [0, a]}$ is compound Poisson (corresponding to the measure Π^f defined by $\Pi^f([0, t]; \Gamma) := \Pi(f(t) \times \Gamma)$).

In this case, if the collection $\mathcal{S}$ can be chosen such that for every flow $f \in \mathcal{S}$ the function $\Pi_t^f(\Gamma)$ is differentiable with respect to t (for every fixed Borel set $\Gamma \subseteq \mathbf{R}$), then the generator of the process X^f at time s is given by

$$(\mathcal{G}_s^f h)(x) = \int_{\mathbf{R}} (h(x+y) - h(x)) (\Pi_s^f)'(dy), \quad x \in \mathbf{R}$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ is equal to the space $B(\mathbf{R})$ (see Example B.1.4.2, Appendix B.1).

We will conclude this section with the simultaneous construction of a compound Poisson process, corresponding to a measure $\Pi(dz; dx)$, and a Poisson process, whose variance measure Λ is the first marginal of Π .

Theorem 5.2.3 *Let $\Pi(dz; dx)$ be a finite positive measure on $(T \times \mathbf{R}, \sigma(\mathcal{A}) \times \mathcal{B}(\mathbf{R}))$ and let $\Lambda_A := \Pi(A \times \mathbf{R})$, $A \in \sigma(\mathcal{A})$. If N is a Poisson random variable with mean Λ_T and $(Z_j, X_j)_{j \geq 1}$ is an independent sequence of independent identically distributed random variables on $T \times \mathbf{R}$ with distribution $\frac{1}{\Pi(T \times \mathbf{R})} \cdot \Pi(dz; dx)$, then the processes*

$$\begin{aligned} X_A &:= \sum_{j=1}^N X_j I_{\{Z_j \in A\}}, \quad A \in \mathcal{A} \\ W_A &:= \sum_{j=1}^N I_{\{Z_j \in A\}}, \quad A \in \mathcal{A} \end{aligned}$$

are a compound Poisson process (corresponding to the measure Π), respectively a Poisson process with variance measure Λ .

Proof: Clearly both X and W have unique countably additive extensions to $\sigma(\mathcal{A})$. In other words, for each fixed ω the set-functions $A \mapsto X_A(\omega)$ and $A \mapsto W_A(\omega)$ are a finite signed measure, respectively a finite positive measure on $\sigma(\mathcal{A})$.

The simplest way to show that X is a compound Poisson process is by proving that the joint characteristic function corresponding to a finite number of increments $X_{C_1}, \dots, X_{C_m}$ over disjoint sets $C_1, \dots, C_m \in \mathcal{C}$ is the product of the m characteristic functions corresponding to $C_1, \dots, C_m$, i.e.

$$E[\exp \{i \sum_{k=1}^m u_k X_{C_k}\}] = \prod_{k=1}^m \exp \left\{ \int_{\mathbf{R}} (e^{iu_k x} - 1) \Pi_{C_k}(dx) \right\}.$$

The argument is classical and the extension to the set-indexed case poses no particular problems (see for instance the argument on page 14-15 of [1] for the planar case). The argument for the Poisson process is identical.

□

Comment 5.2.4 In the construction described above, both processes X and W have purely atomic sample paths, if $\Lambda(\{z\}) = 0, \forall z \in T$. As noted in Section 2.4 not all

purely atomic set-functions are cadlag, and a counterexample can be given even in the case of processes indexed by the unit square. Fortunately, we know that if the atoms of a purely atomic function fall in the ‘well-behaved’ part

$$T^* := \{z \in T; A_z \text{ is proper}\}$$

of the space T (introduced in Section 2.4), then the function will be cadlag. In the case of our previous construction it suffices to require that the measure $\Pi(dz; dx)$ is concentrated on $T^* \times \mathbf{R}$ in order to obtain the cadlaguity of the processes X and W .

5.3 Construction of a Process with Independent Increments

In this section we will give two ways of constructing a process with independent increments: 1. using Kolomogorov’s extension theorem; 2. using the property that any set-indexed process with independent increments can be associated with a ‘large enough’ and ‘suitably indexed’ collection of classical processes with independent increments (i.e processes indexed by the real line).

The finite dimensional distributions of a process with independent increments over the sets in $\mathcal{C}$ have to be defined so that they ensure the (almost sure) additivity of the process. Our first result gives the definition of the finite dimensional distribution (of a process with independent increments) over an arbitrary k -tuple of sets in $\mathcal{C}$ and shows that, if the family $(F_C)_{C \in \mathcal{C}}$ is a convolution system, then the definition will not depend on the extremal representations of these sets. This result is a version of Lemma 3.3.4, in the case of processes with independent increments.

Lemma 5.3.1 *Let $(F_C)_{C \in \mathcal{C}}$ be a convolution system. Let $(C_1, \dots, C_k)$ be a k -tuple of distinct sets in $\mathcal{C}$ and suppose that each set C_i admits two extremal representations $C_i = A_i \setminus \bigcup_{j=1}^{n_i} A_{ij} = A'_i \setminus \bigcup_{j=1}^{m_i} A'_{ij}$. Let $\mathcal{A}', \mathcal{A}''$ be the minimal finite sub-semilattices of $\mathcal{A}$ which contain the sets A_i, A_{ij} , respectively A'_i, A'_{ij} , $\{B_0 = \emptyset', B_1, \dots, B_n\}$, $\{B'_0 = \emptyset', B'_1, \dots, B'_m\}$ two consistent orderings of $\mathcal{A}', \mathcal{A}''$ and D_j, D'_l the left neighbourhoods*

of the sets B_j, B'_l for $j = 1, \dots, n; l = 1, \dots, m$. If each set $C_i; i = 1, \dots, k$ can be written as $C_i = \cup_{j \in J_i} D_j = \cup_{l \in L_i} D'_l$ for some $J_i \subseteq \{1, \dots, n\}, L_i \subseteq \{1, \dots, m\}$, then

$$\int_{\mathbf{R}^n} \prod_{i=1}^k I_{\Gamma_i}(\sum_{j \in J_i} x_j) F_{D_n}(x_n) \dots F_{D_1}(x_1) = \int_{\mathbf{R}^m} \prod_{i=1}^k I_{\Gamma_i}(\sum_{l \in L_i} y_l) F_{D'_m}(y_m) \dots F_{D'_1}(y_1) \quad (31)$$

for every $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbf{R})$.

Proof: Let $\tilde{\mathcal{A}}$ be the minimal finite sub-semilattice of $\mathcal{A}$ determined by the sets in $\mathcal{A}'$ and $\mathcal{A}''$; clearly $\mathcal{A}' \subseteq \tilde{\mathcal{A}}, \mathcal{A}'' \subseteq \tilde{\mathcal{A}}$. Let $\text{ord}^1 = \{E_0 = \emptyset', E_1, \dots, E_N\}$ and $\text{ord}^2 = \{E'_0 = \emptyset', E'_1, \dots, E'_N\}$ be two consistent orderings of $\tilde{\mathcal{A}}$ such that, if $B_j = E_{i_j}; j = 1, \dots, n$ and $B'_l = E'_{k_l}; l = 1, \dots, m$ for some indices $i_1 < i_2 < \dots < i_n$, respectively $k_1 < k_2 < \dots < k_m$, then

$$B_1 = \cup_{p=1}^{i_1} E_p, \quad B_1 \cup B_2 = \cup_{p=1}^{i_2} E_p, \dots, \cup_{j=1}^n B_j = \cup_{p=1}^{i_n} E_p$$

$$B'_1 = \cup_{q=1}^{k_1} E'_q, \quad B'_1 \cup B'_2 = \cup_{q=1}^{k_2} E'_q, \dots, \cup_{l=1}^m B'_l = \cup_{q=1}^{k_m} E'_q.$$

Let π be the permutation of $\{1, \dots, N\}$ such that $E_p = E'_{\pi(p)}; p = 1, \dots, N$. Denote by H_p, H'_q the left neighbourhoods of E_p, E'_q with respect to the orderings $\text{ord}^1, \text{ord}^2$; clearly $H_p = H'_{\pi(p)}, \forall p = 1, \dots, N$. Note that $D_j = (\cup_{v=1}^j B_v) \setminus (\cup_{v=1}^{j-1} B_v) = (\cup_{p=1}^{i_j} E_p) \setminus (\cup_{p=1}^{i_{j-1}} E_p) = \cup_{p=i_{j-1}+1}^{i_j} H_p; j = 1, \dots, n$ and similarly $D'_l = \cup_{q=k_{l-1}+1}^{k_l} H'_q; l = 1, \dots, m$. Hence

$$\begin{aligned} C_i &= \cup_{j \in J_i} D_j = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H_p = \cup_{j \in J_i} \cup_{p=i_{j-1}+1}^{i_j} H'_{\pi(p)} \\ &= \cup_{l \in L_i} D'_l = \cup_{l \in L_i} \cup_{q=k_{l-1}+1}^{k_l} H'_q \end{aligned}$$

and we can conclude that

$$\{\pi(p); p \in \cup_{j \in J_i} \{i_{j-1} + 1, i_{j-1} + 2, \dots, i_j\}\} = \cup_{l \in L_i} \{k_{l-1} + 1, k_{l-1} + 2, \dots, k_l\}.$$

This relationship, combined with the fact that

$$\begin{aligned} \int_{\mathbf{R}^N} h(x_1, x_2, \dots, x_n) F_{H_N}(dx_N) \dots F_{H_2}(dx_2) F_{H_1}(dx_1) = \\ \int_{\mathbf{R}^N} h(y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(N)}) F_{H'_N}(dy_N) \dots F_{H'_2}(dy_2) F_{H'_1}(dy_1) \end{aligned}$$

for any bounded measurable function h , implies that

$$\begin{aligned} \int_{\mathbf{R}^N} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{j \in J_i} \sum_{p=i_{j-1}+1}^{i_j} x_p \right) F_{H_N}(dx_N) \dots F_{H_2}(dx_2) F_{H_1}(dx_1) = \\ \int_{\mathbf{R}^N} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{j \in J_i} \sum_{p=i_{j-1}+1}^{i_j} y_{\pi(p)} \right) F_{H'_N}(dy_N) \dots F_{H'_2}(dy_2) F_{H'_1}(dy_1) = \\ \int_{\mathbf{R}^N} \prod_{i=1}^k I_{\Gamma_i} \left(\sum_{l \in L_i} \sum_{q=k_{l-1}+1}^{k_l} y_q \right) F_{H'_N}(dy_N) \dots F_{H'_2}(dy_2) F_{H'_1}(dy_1). \end{aligned}$$

This gives us the desired relationship, because

$$F_{D_j} = *_{p=i_{j-1}+1}^{i_j} F_{H_p}; j = 1, \dots, n$$

$$F_{D'_k} = *_{q=k_{l-1}+1}^{k_l} F_{H'_q}; k = 1, \dots, m$$

and for every probability measures F, G on $\mathbf{R}$ and for every $h \in B(\mathbf{R})$

$$\int_{\mathbf{R}^2} h(x+y) F(dx) G(dy) = \int_{\mathbf{R}} h(z) (F * G)(dz).$$

□

Here is the main result of this section, which says that the only ingredient that is needed for the construction of a set-indexed process with independent increments is a convolution system. This result is a simplified version of Theorem 3.3.5, in the case of processes with independent increments.

Theorem 5.3.2 *If $(F_C)_{C \in \mathcal{C}}$ is a convolution system, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a process with independent increments and the distribution of each increment $X_C, C \in \mathcal{C}$ is F_C .*

Proof: As expected, the proof will be an application of Kolmogorov's extension theorem. The proof in the classical case can be found on pages 35-36 of [65]; in the d -dimensional case, a straightforward proof is mentioned on page 12 of [1].

For each k -tuple $(C_1, \dots, C_k)$ of distinct sets in $\mathcal{C}$ we will define a probability measure $\mu_{C_1 \dots C_k}$ on $(\mathbf{R}^k, \mathcal{R}^k)$ in the following manner: let $C_i = A_i \setminus \bigcup_{j=1}^{i-1} A_{ij}; i =$

$1, \dots, k$ be extremal representations of these sets, $\mathcal{A}'$ the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets A_i, A_{ij} , $\{B_0 = \emptyset', B_1, \dots, B_n\}$ a consistent ordering of $\mathcal{A}'$, and D_j the left neighbourhood of B_j for $j = 1, \dots, n$; if $C_i = \dot{\cup}_{j \in J_i} D_j$ for some $J_i \subseteq \{1, \dots, n\}; i = 1, \dots, k$, then define

$$\mu_{C_1 \dots C_k} := (F_{D_1} \times \dots \times F_{D_n}) \circ \alpha^{-1}$$

where $\alpha(x_1, \dots, x_n) := (\sum_{j \in J_1} x_j, \dots, \sum_{j \in J_k} x_j)$.

It is clear that $\mu_{C_1 \dots C_k}$ does not depend on the ordering of the semilattice $\mathcal{A}'$; on the other hand, Lemma 5.3.1 shows that $\mu_{C_1 \dots C_k}$ does not depend on the extremal representations of the sets C_i .

The fact that the family of all probability measures $\mu_{C_1 \dots C_k}$ satisfies the two consistency conditions of Kolmogorov's extension theorem, follows exactly as in the proof of Theorem 3.3.5. Therefore, there exists a probability measure P on $(\mathbf{R}^{\mathcal{C}}, \mathcal{R}^{\mathcal{C}})$ such that the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined by $X_C(x) := x_C$ has the measures $\mu_{C_1 \dots C_k}$ as its finite dimensional distributions. Moreover, this process has an (almost surely) unique additive extension to $\mathcal{C}(u)$, since its finite dimensional distributions were chosen in an additive way.

Finally, it is not difficult to see that the process X has independent increments: if $C_1, \dots, C_k$ are arbitrary pairwise disjoint sets in $\mathcal{C}$, then each C_i can be written as $C_i = \dot{\cup}_{j \in J_i} D_j$, where $\{D_0 = \emptyset', D_1, \dots, D_n\}$ is the collection of all left neighbourhoods associated to a certain finite sub-semilattice $\mathcal{A}'$; hence $X_{C_i} = \sum_{j \in J_i} X_{D_j}, i = 1, \dots, k$ are independent, by the associativity of independence (the variables $X_{D_j}; j = 1, \dots, n$ are independent since they are defined as projections under the product measure $F_{D_1} \times \dots \times F_{D_n}$).

□

Note: *The requirement that the family $(F_C)_{C \in \mathcal{C}}$ be a convolution system is more than sufficient for the proof of the previous theorem!* All we need for the construction of a process with independent increments is a family $(F_C)_{C \in \mathcal{C}}$ which satisfies property 2 (from the definition of a convolution system) in a very particular instance only. More precisely, it is enough to assume that for any finite sub-semilattices $\mathcal{A}' \subseteq \tilde{\mathcal{A}}$, and for any consistent orderings $\text{ord} = \{B_0 = \emptyset', B_1, \dots, B_n\}$, $\text{ord}^1 = \{E_0 = \emptyset', E_1, \dots, E_N\}$

of $\mathcal{A}'$, respectively $\tilde{\mathcal{A}}$ which are chosen such that, if $B_j = E_{i_j}; j = 1, \dots, n$ then $\cup_{v=1}^j B_v = \cup_{p=1}^{i_j} E_p; j = 1, \dots, n$, we have

$$F_{D_j} = *_{p=i_{j-1}+1}^{i_j} F_{H_p}, \quad \forall j = 1, \dots, n$$

where D_j, H_p are the left neighbourhoods of B_j, E_p in $\mathcal{A}'$, respectively $\tilde{\mathcal{A}}$.

We will conclude this section by giving the second construction of a process with independent increments which is suggested by the fact that the projection of such process on any simple flow is also a process with independent increments (indexed by the real line).

Before giving this second construction, we should mention that a family $(F_{st})_{s < t}$ of probability measures on $\mathbf{R}$ is called a *convolution system* if $F_{tt} = \delta_0 \forall t$ and $F_{st} * F_{tu} = F_{su}$ for every $s < t < u$.

Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

The next assumption will provide a necessary and sufficient condition which will allow us to reconstruct a convolution system $(F_C)_{C \in \mathcal{C}}$ from a collection $\{(F_{st}^f)_{s < t}; f \in \mathcal{S}\}$ of convolution systems. It requires that whenever we have two flows $f, g \in \mathcal{S}$ such that $f(t) \setminus f(s) = g(v) \setminus g(u)$ for some $s < t, u < v$, we must have $F_{st}^f = F_{uv}^g$. This assumption is a version of Assumption 3.3.6 in the case of processes with independent increments.

Assumption 5.3.3 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_m\}$ are two consistent orderings of some finite semilattices $\mathcal{A}', \mathcal{A}''$ such that $(\cup_{j=1}^n A_j) \setminus (\cup_{j=1}^k A_j) = (\cup_{j=1}^m A'_j) \setminus (\cup_{j=1}^l A'_j)$ for some $k < n, l < m$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}'', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j, g(u_i) = \cup_{j=1}^i A'_j$, then*

$$F_{t_k t_n}^f = F_{u_l u_m}^g$$

The following result is immediate. This result is a simplified version of Corollary 3.3.8 in the case of processes with independent increments.

Corollary 5.3.4 *If $\{(F_{st}^f)_{s < t}; f \in \mathcal{S}\}$ is a collection of convolution systems which satisfies Assumption 5.3.3, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:*

1. the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a process with independent increments; and
2. $\forall f \in \mathcal{S}$, the distribution of each increment $X_t^f - X_s^f, s < t$ is F_{st}^f .

Proof: Let C be an arbitrary set in $\mathcal{C}$ and $C = A \setminus B, A \in \mathcal{A}, B = \cup_{j=1}^m A'_j \in \mathcal{A}(u)$ one of its extremal representations. Let $\mathcal{A}'$ be the minimal finite sub-semilattice of $\mathcal{A}$ which contains the sets $A, A'_1, \dots, A'_m$, and $\text{ord} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ a consistent ordering of $\mathcal{A}'$ such that $B = \cup_{j=0}^k A_j, A \cup B = \cup_{j=0}^n A_j$ for some $k < n$; let $f := f_{\mathcal{A}', \text{ord}}$ with $f(t_i) = \cup_{j=1}^n A_j$. Define

$$F_C := F_{t_k t_n}^f$$

Note that this definition does not depend on the extremal representation of the set C , because of Assumption 5.3.3.

Clearly $F_\emptyset = F_{\emptyset'} = \delta_0$. Finally, it is not difficult to see that the family $(F_C)_{C \in \mathcal{C}}$ satisfies property 2 (from the definition of a convolution system) in the particular instance mentioned in the note following Theorem 5.3.2: denote $f := f_{\tilde{\mathcal{A}}, \text{ord}^1}, g := f_{\mathcal{A}', \text{ord}}$ with $f(t_i) = \cup_{p=1}^i E_p; i = 1, \dots, N$ and $g(u_j) = \cup_{v=1}^j B_v; j = 1, \dots, n$; then

$$F_{D_j} = F_{u_{j-1} u_j}^g = F_{t_{i_{j-1}} t_{i_j}}^f = *_{p=i_{j-1}+1}^{i_j} F_{t_{p-1} t_p}^f = *_{p=i_{j-1}+1}^{i_j} F_{H_p}$$

(using Assumption 5.3.3).

□

5.4 Lévy Processes

In this section, we will define a set-indexed Lévy process as a process with independent increments, which is monotone outer-continuous in probability and monotone inner continuous in probability (see Appendix B.1 for the definition and some properties of the classical Lévy process).

We will first show that the distribution of such a process is completely determined by a system of infinitely divisible distributions. Then, using the construction given by Corollary 5.3.4 for a process with independent increments, we will be able to

show that a Lévy process is also characterized by a ‘consistent’ collection of Lévy generators.

We begin with the definition of a set-indexed Lévy process. We want to emphasize that this definition does not require the existence of a metric on the underlying space T .

Definition 5.4.1 *A process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments is called a Lévy process if $X_{A_n} \xrightarrow{P} X_A$ whenever the sequence $(A_n)_n \subseteq \mathcal{A}$ is chosen such that either $A_n \supseteq A_{n+1} \forall n$ and $A = \bigcap_n A_n$, or $A_n \subseteq A_{n+1} \forall n$ and $A = \overline{\bigcup_n A_n} \in \mathcal{A}$.*

Note that this definition coincides with the definition of a Lévy process on the real line, but as soon as we get to higher dimensions, we obtain a more general concept than the one that exists in the literature. For instance, in [1] a Lévy (multiparameter) process is defined as a process with independent increments which is continuous in probability when a point z is approached from any direction, not only from the ‘upper-right’ or ‘lower-left’ quadrants. Similarly, the set-indexed Lévy process considered in [2] is continuous in probability when a Borel set A is approached by an arbitrary sequence $(A_n)_n$, using the distance $d_\lambda(A, B) := \lambda(A \Delta B)$ (λ is the Lebesgue measure).

In our general case, by considering the monotone convergence of sets, we avoid having to introduce either a metric or a measure on the space T (that would play the role of the Lebesgue measure on the Euclidean space $[0, 1]^d$). The good news is that we are still able to obtain infinitely divisible distributions for the increments of our Lévy process. (In [8], the stochastic continuity aspect is simply ignored and a set-indexed Lévy process is defined as a process with independent and infinitely divisible increments.) The difference between our approach and that of other authors is that we will obtain *monotone* continuity properties of the generating triplets $(\gamma_A, \Lambda_A, \Pi_A)$; in [1], the usual continuity properties (i.e. a point is approached from *all* the quadrants) have been obtained; in [2], the continuity properties are obtained in terms of the metric d_λ .

The following result gives the expected correspondence via flows.

Proposition 5.4.2 *A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is a Lévy process if and*

only if for every simple flow f the process $X^f := (X_{f(t)})_t$ is Lévy. (For the sufficiency part it is enough to consider only $\mathcal{A}$ -valued continuous flows.)

Proof: By Proposition 5.1.6, a necessary and sufficient condition for a set-indexed process to have independent increments is that it also has independent increments on every simple flow.

The same argument that we used for proving Proposition 2.4.7 can be used here for showing that the set-indexed process X is monotone outer-continuous in probability and monotone inner-continuous in probability if and only if for every simple flow f , the process X^f is continuous in probability. The only difference is that now we are dealing with the convergence in probability, instead of pointwise convergence. We do not repeat this argument here.

□

Remark: Although it might not have been clear until now why we need our indexing collection $\mathcal{A}$ to be *separable from above*, it is necessary to point out that it is exactly on this property that the proof of Lemma 2.4.6 (that is used in the argument of the previous proof) relies.

Here is the main result of this section.

Theorem 5.4.3 (Characterization) (i) If $X := (X_A)_{A \in \mathcal{A}}$ is a Lévy process, then the distribution of each $X_C, C \in \mathcal{C}(u)$ is infinitely divisible.

The generating triplet $(\gamma_C, \Lambda_C, \Pi_C)$ has the following properties:

- (a) $\gamma_\emptyset = 0$; the function $C \mapsto \gamma_C$ is additive on $\mathcal{C}(u)$, monotone outer-continuous and monotone inner-continuous;
- (b) $\Lambda_\emptyset = 0$; the function $C \mapsto \Lambda_C$ has a unique additive extension to a measure on $\sigma(\mathcal{A})$;
- (c) $\Pi_\emptyset = 0$; for any fixed Borel set Γ contained in some set $\{y; |y| > \epsilon\}, \epsilon > 0$, the function $C \mapsto \Pi_C(\Gamma)$ has a unique extension to a measure on $\sigma(\mathcal{A})$.

(ii) Conversely, given any system $(F_C)_{C \in \mathcal{C}}$ of infinitely divisible distributions on $\mathbf{R}$ with generating triplets $(\gamma_C, \Lambda_C, \Pi_C)$ satisfying the conditions (a), (b), (c), there

exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a Lévy process and the distribution of each increment $X_C, C \in \mathcal{C}$ is F_C .

Proof: (i) Instead of trying to prove directly that the distribution of each increment $X_C, C \in \mathcal{C}$ is infinitely divisible, we will cheat a little bit and we will bring in the heavy machinery of *simple* flows (which makes use of the additional structure that we have for our indexing collection $\mathcal{A}$).

Let $C \in \mathcal{C}$ be arbitrary, say $C = A \setminus B$ with $A \in \mathcal{A}, B \in \mathcal{A}(u)$. By Lemma 2.4.9, there exists a *simple* flow f such that $f(s) = B, f(t) = A \cup B$ for some $s < t$; the process $X^f := (X_{f(t)})_t$ is Lévy, and hence the distribution of $X_C = X_t^f - X_s^f$ is infinitely divisible (see Appendix B.1). If $C \in \mathcal{C}(u)$, the distribution of X_C is also infinitely divisible, using the fact that the process X has independent increments.

Clearly, the functions $A \mapsto \gamma_A, A \mapsto \Lambda_A, A \mapsto \Pi_A(\Gamma)$ (for any Borel set $\Gamma \subseteq \mathbf{R}$) are additive on $\mathcal{C}(u)$. Moreover, these functions are continuous over any simple flow f , i.e. the functions $t \mapsto \gamma_t^f := \gamma_{f(t)}, t \mapsto \Lambda_t^f := \Lambda_{f(t)}, t \mapsto \Pi_t^f(\Gamma) := \Pi_{f(t)}(\Gamma)$ are continuous, where the set Γ is contained in some set $\{y; |y| > \epsilon\}, \epsilon > 0$ (see Appendix B.1). Therefore, by Proposition 2.4.7, these functions are monotone outer-continuous and monotone inner-continuous.

The functions Λ, Π are also increasing, i.e. $\Lambda_C \geq 0, \Pi_C(\Gamma) \geq 0, \forall C \in \mathcal{C}$; using Proposition 2.4.3, it follows that they have unique additive extensions to some measures on $\sigma(\mathcal{A})$.

(ii) Conversely, let $(F_C)_{C \in \mathcal{C}}$ be system of infinitely divisible distributions on $\mathbf{R}$ with generating triplets $(\gamma_C, \Lambda_C, \Pi_C)$ satisfying the conditions (a), (b), (c). By the additivity of the functions γ, Λ, Π , it follows that the family $(F_C)_{C \in \mathcal{C}}$ is a convolution system. Using Theorem 5.3.2, there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a process with independent increments and the distribution of each increment $X_C, C \in \mathcal{C}$ is F_C .

This process will automatically be a Lévy process: let $(A_n)_n$ be a decreasing sequence of sets in $\mathcal{A}$ with intersection A ; then $(A_n \setminus A)_n$ is a decreasing sequence of sets in $\mathcal{C}$ with empty intersection and by the monotone outer-continuity of the

functions γ, Λ, Π it follows that $X_{A_n \setminus A} \xrightarrow{D} 0$ i.e. $X_{A_n} \xrightarrow{P} X_A$. (A similar argument may be used for an increasing sequence $(A_n)_n$ using the monotone inner-continuity of the functions γ, Λ, Π .)

□

Corollary 5.4.4 *A process $X := (X_A)_{A \in \mathcal{A}}$ with independent increments is Lévy if and only if the distribution F_C of each increment $X_C, C \in \mathcal{C}$ is infinitely divisible and its characterizing triplet $(\gamma_C, \Lambda_C, \Pi_C)$ satisfies properties (a), (b) and (c) stated in Theorem 5.4.3.*

Fundamental examples of Lévy processes are the Brownian motion and the compound Poisson process.

Note: If the probability measure F defined by $F(A) := \frac{1}{\Lambda_T} \cdot \Lambda_A, A \in \sigma(\mathcal{A})$ satisfies $\gamma_A = F(A) \cdot \gamma_T, \Pi_A = F(A) \cdot \Pi_T, \forall A \in \mathcal{A}$ then the process X is said to have *stationary increments* (with respect to Λ), i.e. the distribution of each increment $X_C, C \in \mathcal{C}$ depends on C only through Λ_C .

The construction of a general $\mathcal{Q}$ -Markov process, knowing the generators of the process along a suitable class of simple flows, is given by Theorem 4.2.4. In what follows, we will give the simplified version of this result, in the case of processes with independent increments.

Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10, and denote by f_A the simple flow in the class $\mathcal{S}$ which connects the sets of the semilattice $\mathcal{A}' := \{A_0 = \emptyset', A_1 = A\}$ with $f(t_A) = A$.

The generator of a set-indexed Lévy process, which is weakly differentiable on any simple flow $f \in \mathcal{S}$, has a known form (see Appendix B.1). The next result says that conversely, given a collection $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ of families of Lévy generators, it is possible to reconstruct a set-indexed Lévy process, providing certain consistency conditions hold.

Let $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ be a collection of families of Lévy generators with domains $\mathcal{D}(\mathcal{G}_s^f) = C_b^2(\mathbf{R})$ and generating triplets $(\gamma_s^f, \Lambda_s^f, \Pi_s^f)$. The following assumption is an equivalent form in terms of generators of Assumption 5.3.3.

Assumption 5.4.5 *If $\text{ord}1 = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord}2 = \{A'_0 = \emptyset', A'_1, \dots, A'_m\}$ are two consistent orderings of some finite semilattices $\mathcal{A}', \mathcal{A}''$ such that $(\cup_{j=1}^n A_j) \setminus (\cup_{j=1}^k A_j) = (\cup_{j=1}^m A'_j) \setminus (\cup_{j=1}^l A'_j)$ for some $k < n, l < m$, and we denote $f := f_{\mathcal{A}', \text{ord}1}$, $g := f_{\mathcal{A}'', \text{ord}2}$ with $f(t_i) = \cup_{j=1}^i A_j, g(u_i) = \cup_{j=1}^i A'_j$, then*

$$\gamma_{t_k t_n}^f = \gamma_{u_l u_m}^g, \quad \Lambda_{t_k t_n}^f = \Lambda_{u_l u_m}^g, \quad \Pi_{t_k t_n}^f = \Pi_{u_l u_m}^g$$

where $\gamma_{st}^f := \gamma_t^f - \gamma_s^f, \Lambda_{st}^f := \Lambda_t^f - \Lambda_s^f, \Pi_{st}^f := \Pi_t^f - \Pi_s^f$.

Clearly Assumption 5.4.5 is a necessary condition satisfied by any process with independent increments which has Lévy generators over the class $\mathcal{S}$ of simple flows. The following result is an immediate consequence of Corrolary 5.3.4 which says that in fact, this assumption is also sufficient for the construction of such a process.

Theorem 5.4.6 *If $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s; f \in \mathcal{S}\}$ is a collection of families of Lévy generators with domains $\mathcal{D}(\mathcal{G}_s^f) = C_b^2(\mathbf{R})$ and generating triplets $(\gamma_s^f, \Lambda_s^f, \Pi_s^f)$, which satisfies Assumption 5.4.5, then there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which:*

1. *the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a process with independent increments; and*
2. *$\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is an extension of the operator $\mathcal{G}_s^f$.*

The process X is Lévy if the following additional condition holds: for any decreasing (respectively increasing) sequence $(A_n)_n \subseteq \mathcal{A}$ with $A := \cap_n A_n$ (respectively $A := \overline{\cup_n A_n} \in \mathcal{A}$) we have

$$\gamma_n^{f_n} \rightarrow \gamma_t^f, \quad \Lambda_n^{f_n} \rightarrow \Lambda_t^f, \quad \Pi_n^{f_n}(\Gamma) \rightarrow \Pi_t^f(\Gamma) \quad (32)$$

where $t_n = t_{A_n}, f_n = f_{A_n}, \forall n$ and $t = t_A, f = f_A$.

Proof: For each flow $f \in \mathcal{S}$, $f : [0, a] \rightarrow \mathcal{A}(u)$ and for each $s, t \in [0, a], s < t$, let F_{st}^f be the infinitely divisible distribution with generating triplet $(\gamma_{st}^f, \Lambda_{st}^f, \Pi_{st}^f)$. Note that

each family $(F_{st}^f)_{s<t}$ is a convolution system. Let $\mathcal{Q}^f := (Q_{st}^f)_{s<t}$ be the associated transition system, defined by $Q_{st}^f(x; \Gamma) := F_{st}^f(\Gamma - x)$, $x \in \mathbf{R}$, $\Gamma \in \mathcal{B}(\mathbf{R})$, and denote with T_{st}^f the operator associated to Q_{st}^f . Then the generator at time s of the semigroup $\mathcal{T} := (T_{st}^f)_{s<t}$ is an extension of the operator $\mathcal{G}_s^f$.

Moreover, by Assumption 5.4.5, the collection $\{(F_{st}^f)_{s<t}; f \in \mathcal{S}\}$ of convolution systems satisfies Assumption 5.3.3; hence, by Corollary 5.3.4, there exists a probability measure P on the product space $(\mathbf{R}^{\mathcal{C}}, \mathcal{B}(\mathbf{R})^{\mathcal{C}})$ under which: 1. the coordinate-variable process $X := (X_C)_{C \in \mathcal{C}}$ defined on this space is a process with independent increments; and 2. for every flow $f \in \mathcal{S}$ the distribution of each increment $X_{f(t)} - X_{f(s)}$, $s < t$ is exactly F_{st}^f .

Hence we can define some set-indexed functions $(\gamma_A)_{A \in \mathcal{A}}$, $(\Lambda_A)_{A \in \mathcal{A}}$, $(\Pi_A)_{A \in \mathcal{A}}$ (with unique additive extensions to $\mathcal{C}(u)$) such that the distribution of each increment X_C , $C \in \mathcal{C}$ is infinitely divisible with generating triplet $(\gamma_C, \Lambda_C, \Pi_C)$. These functions will be monotone outer-continuous and monotone inner-continuous on $\mathcal{A}$ (which is equivalent to saying that the process X is Lévy) if (32) holds.

□

Our final remark is that set-indexed Lévy processes belong to the class $\mathcal{D}[\mathcal{S}(\mathcal{A})]$ (introduced in Section 2.4) since for every simple flow f , the (Lévy) process $X^f := (X_{f(t)})_t$ has a cadlag modification (see Appendix B.1).

5.5 Existence of a Cadlag Version of a Lévy process

In this section we will include several remarks about the existence of a cadlag version of a set-indexed Lévy process. These results are not new; rather it is an attempt to incorporate all the existing results from the literature into our context. We also include a slight extension of Theorem 3.1, [8] about the existence of a cadlag version of a jump Lévy process, with two minor corrections (one to the statement, the other to the proof) of this result. We will make use of the notions of ‘metric entropy’ and ‘Vapnik-Cervonenkis class’, which are treated in Appendix B.3.

We recall that our notion of ‘cadlag’ is defined using the Hausdorff metric. Therefore, in this section we assume that T is a metric space.

Given a system $(F_C)_{C \in \mathcal{C}}$ of infinitely divisible distributions on $\mathbf{R}$ with generating triplets $(\gamma_C, \Lambda_C, \Pi_C)$ satisfying the conditions (a), (b), (c) stated in Theorem 5.4.3, we want to construct an (almost surely) cadlag Lévy process $X := (X_A)_{A \in \mathcal{A}}$ such the distribution of each increment $X_C, C \in \mathcal{C}$ is F_C .

We will consider separately the Gaussian and the jump part. The building blocks for the two constructions will be the Brownian motion and the compound Poisson process.

1. The Gaussian Case

Theorem 5.5.1 *Let $X := (X_A)_{A \in \mathcal{A}}$ be a Lévy process with Gaussian increments (i.e. $\Pi_A = 0, \forall A \in \mathcal{A}$, where $(\gamma_A, \Lambda_A, \Pi_A)$ denotes its generating triplet). If*

1. *all the approximation functions g_n are $\mathcal{A}$ -valued and $d_H(A, g_n(A)) \leq \epsilon_n, \forall n$ for some $\epsilon_n \searrow 0$;*
2. *the collection $\mathcal{A}$ is totally bounded with respect to the pseudo-metric d_Λ defined by $d_\Lambda(A, B) := \Lambda_{A \Delta B}$ and the metric entropy H_{d_Λ} of $\mathcal{A}$ with respect to d_Λ satisfies*

$$\int_0^1 (H_{d_\Lambda}(\epsilon))^{1/2} d\epsilon < \infty; \text{ and} \quad (33)$$

3. *the functions γ and Λ are d_H -continuous on $\mathcal{A} \setminus \{\emptyset\}$*

then the process X has a d_H -continuous (and hence cadlag) version.

Proof: Since the function γ is d_H -continuous, we can assume without loss of generality that $\gamma_C = 0, \forall C \in \mathcal{C}$, i.e. X is a Brownian motion with variance measure Λ . Moreover, we know that any set-indexed process has a separable version (Theorem 2.4.19); hence, without loss of generality, we will suppose that X is separable.

On one hand, because condition (33) holds, the process X is d_Λ -continuous, using Theorem 1.1, [3]. On the other hand, using the argument on page 3 of [3], this is

equivalent to saying that the process X is also d_H -continuous, because its variance measure Λ is d_H -continuous (this argument requires that the indexing space $\mathcal{A} \setminus \{\emptyset\}$ be d_H -compact; this condition is satisfied by Lemma 2.4.11).

□

Note: The fact that $\mathcal{A}$ is a Vapnik-Cervonenkis class in $\sigma(\mathcal{A})$ is:

- a sufficient condition for the existence of a d_H -continuous version of a Brownian motion indexed by $\mathcal{A}$, with d_H -continuous variance measure Λ (condition (33) is satisfied if $\mathcal{A}$ is a Vapnik-Cervonenkis class in $\sigma(\mathcal{A})$, according to Theorem 1.10, [3]); and
- a necessary condition for the existence of a d_Λ -bounded and uniformly continuous version of a Brownian motion indexed by $\mathcal{A}$, with arbitrary variance measure Λ (according to the argument on page 125 of [29]).

2. The Non-Gaussian Case

The following result extends and gives a minor correction to Theorem 3.1, [8]. Our generalization has to be understood in two directions: in the first place, our indexing sets lie in an arbitrary metric space T ; in the second place the stationarity condition is replaced by a more general condition which requires that the Lévy measure of the process be bounded by a stationary measure (so that it does not need to be stationary itself). One correction is that we require that the measure Π be concentrated on the well-behaved part T^* of the space T (introduced in Section 2.4); without this requirement, the constructed process may fail to be cadlag. The second correction is a minor technical point encountered in the proof.

Theorem 5.5.2 *Let $X := (X_A)_{A \in \mathcal{A}}$ be a Lévy process with no Gaussian part (i.e. $\gamma_C = 0, \Lambda_C = 0, \forall C \in \mathcal{C}$), whose Lévy measure $\Pi(dz; dx)$ is concentrated on $T^* \times \mathbf{R}$, its first marginal $\Lambda(A) := \Pi(A \times \mathbf{R})$ has no atoms (i.e. $\Lambda(\{z\}) = 0, \forall z \in T$) and suppose that Π is bounded from above by a stationary Lévy measure $\Pi^1(dz; dx) = F^1(dz) \times \Pi_T^1(dx)$ i.e.*

$$\Pi(dz; dx) \leq \Pi^1(dz; dx) = F^1(dz) \times \Pi_T^1(dx)$$

where F^1 is a probability measure on $\sigma(\mathcal{A})$. Denote $M^1(x) := \Pi_T^1((-\infty, -x) \cup (x, \infty))$ and $G^1(y) := y \cdot \inf\{x > 0; M^1(y) < x\}$.

Suppose that the Lévy measure Π_T^1 satisfies the following conditions:

$$(B1) \quad \limsup_{x \rightarrow 0} x^2 |\ln x|^9 M^1(x) < \infty$$

$$(B2) \quad \text{for some } \tau > 0, x^\tau M^1(x) \text{ increases as } x \searrow 0$$

and

$$\int_{\{|x| \leq 1\}} |x| \Pi_T^1(dx) = \infty.$$

Assume that the indexing collection $\mathcal{A}$ is totally bounded with inclusion (in $\sigma(\mathcal{A})$) with respect to the probability measure F^1 , and that its metric entropy with inclusion $H_I(\epsilon)$ satisfies the following condition:

(A1) $H_I(\epsilon) = \epsilon^{-c_0} L(\epsilon)$ for some constant $c_0 > 1$, where L is a slowly varying function near 0 such that $\epsilon^{(c_0+1)/2} H_I(\epsilon)$ increases as $\epsilon \searrow 0$.

If

$$\int_0^1 G^1\left(\frac{H_I(\epsilon)}{\epsilon}\right) d\epsilon < \infty \quad (34)$$

then the process X possesses a cadlag version on $\mathcal{A}$.

Proof: For the sake of completeness we include the proof, even if the arguments are essentially the same as those used to prove Theorem 3.1, [8].

For each $n \geq 0$, let $X^n := (X_A^n)_{A \in \mathcal{A}}$ be a cadlag compound Poisson process with Lévy measure $\Pi^n(A \times \Gamma) := \Pi(A \times (\Gamma \cap \{y; a_n < |y| \leq a_{n-1}\}))$, $n \geq 1$; $\Pi^0(A \times \Gamma) := \Pi(A \times (\Gamma \cap \{y; |y| > 1\}))$, using Comment 5.2.4. (Here $(a_n)_{n \geq 1}$ is a decreasing sequence of positive constants converging to 0, $a_0 = 1$.) We assume that the processes $(X^n)_{n \geq 0}$ are independent (by redefining them on a product space, if necessary).

Note that each $\Pi^n(dz; dx)$ is a finite Lévy measure concentrated on $T^* \times \mathbf{R}$:

$$\mu_n := \Pi^n(T \times \mathbf{R}) = \int_{\{y; a_n < |y| \leq a_{n-1}\}} \Pi_T(dy) \leq \frac{1}{a_n^2} \int_{|y| \leq 1} y^2 \Pi_T(dy) < \infty; \quad n \geq 1$$

$$\mu_0 := \Pi^0(T \times \mathbf{R}) = \int_{\{|y| > 1\}} \Pi_T(dy) < \infty.$$

Let $Z_A^n := X_A^n - E[X_A^n]$, $n \geq 1$ and $\|Z^n\|_{\mathcal{A}} := \sup_{A \in \mathcal{A}} |Z_A^n|$. Note that the log-characteristic function of X_A^0 is

$$\psi_A^0(u) = \int_{\{|y|>1\}} (e^{iuy} - 1) \Pi_A(dy)$$

whereas the log-characteristic function of Z_A^n is

$$\psi_A^n(u) = \int_{\{y; a_n < |y| \leq a_{n-1}\}} (e^{iuy} - 1 - iuy) \Pi_A(dy).$$

Therefore, if we can define

$$X_A := X_A^0 + \sum_{n \geq 1} Z_A^n \quad \text{uniformly in } A \in \mathcal{A}$$

then the process $X := (X_A)_{A \in \mathcal{A}}$ will be a cadlag Lévy process with no Gaussian part and Lévy measure Π .

In order to show that $\sum_{n \geq 1} \|Z^n\|_{\mathcal{A}}$ converges a.s. it is enough to prove that there exists a summable sequence $(\alpha_n)_n$ of positive real numbers such that

$$\sum_{n \geq 1} P^*(\|Z^n\|_{\mathcal{A}} \geq \alpha_n) < \infty$$

using an extension of Borel-Cantelli Lemma (P^* denotes the outer measure induced by the underlying probability measure P ; we use this measure because $\|Z^n\|_{\mathcal{A}}$ may not be measurable). Our constants α_n will be of the form $\alpha_n := 4\eta_n$, where η_n will be chosen later such that

$$\sum_{n \geq 1} \eta_n < \infty. \tag{35}$$

Let $\beta \in (0, 1)$ (its value will be chosen later) and set $\delta_n := \beta^n$, $n \geq 1$. For each $n \geq 1$ let $\mathcal{A}_n := \mathcal{A}_{\delta_n}$ denote a collection of measurable sets from $\sigma(\mathcal{A})$ such that for any $A \in \mathcal{A}$ there exist two sets $A_n^-, A_n^+ \in \mathcal{A}_n$ with the following properties: $A_n^- \subseteq A \subseteq A_n^+$ and $F(A_n^+ \setminus A_n^-) \leq \delta_n$. Without loss of generality we can assume that $F(A_n^- \Delta A_{n-1}^-) \leq \delta_{n-1}$ for every n . Recall that the collection $\mathcal{A}_n$ can be chosen such that it has at most $\exp(2H_I(\delta_n))$ elements.

Let $(\gamma_j)_{j \geq 1}$ be a sequence (to be chosen later) such that

$$\sum_{j \geq 1} \gamma_j = 1. \tag{36}$$

Let $\eta_{nj} = \eta_n \gamma_j$. Let $M_n^1 := M^1(a_n)$ and $Q_{n-1}^1 := \int_{a_n < |x| \leq a_{n-1}} x^2 \Pi_T^1(dx)$. Clearly $Q_{n-1} \leq a_{n-1}^2 M_n$.

Let $(k_n)_{n \geq 1}$ be an increasing sequence of positive integers (to be chosen later) such that $\lim_{n \rightarrow \infty} k_n = \infty$. For each $A \in \mathcal{A}$ we may write

$$Z_A^n = \sum_{j=1}^{k_n} (Z_{A_j^-}^n - Z_{A_{j-1}^-}^n) + (Z_A^n - Z_{A_{k_n}^-}^n) \quad (37)$$

where we considered $A_0^- := \emptyset$.

Hence $P^*(\|Z^n\|_{\mathcal{A}} > 4\eta_n) = P^*(\exists A \in \mathcal{A} : |Z_A^n| > 4\eta_n) \leq P_{(I)} + P_{(II)}$ where

$$P_{(I)} := \sum_{j=1}^{k_n} P^*(\exists A_j^- \in \mathcal{A}_j, A_{j-1}^- \in \mathcal{A}_{j-1}, F(A_j^- \Delta A_{j-1}^-) \leq \delta_{j-1} : |Z_{A_j^-}^n - Z_{A_{j-1}^-}^n| > 2\eta_{nj})$$

and

$$P_{(II)} := \sum_{A_{k_n}^-, A_{k_n}^+ \in \mathcal{A}_{k_n}, F(A_{k_n}^+ \setminus A_{k_n}^-) \leq \delta_{k_n}} P^*(\exists A \in \mathcal{A}, A_{k_n}^- \subseteq A \subseteq A_{k_n}^+ : |Z_A^n \setminus A_{k_n}^-| > 2\eta_n).$$

Note that

$$P_{(I)} \leq \sum_{j=1}^{k_n} \exp(2H_I(\delta_j)) r_{nj} \text{ and } P_{(II)} \leq \exp(2H_I(\delta_{k_n})) r_n$$

where

$$r_{nj} := \max_{A_j^- \in \mathcal{A}_j, A_{j-1}^- \in \mathcal{A}_{j-1}, F(A_j^- \Delta A_{j-1}^-) \leq \delta_{j-1}} \{P(|Z_{A_j^-}^n \setminus A_{j-1}^-| > \eta_{nj}) + P(|Z_{A_{j-1}^-}^n \setminus A_j^-| > \eta_{nj})\}$$

and

$$r_n := \max_{A_{k_n}^-, A_{k_n}^+ \in \mathcal{A}_{k_n}, F(A_{k_n}^+ \setminus A_{k_n}^-) \leq \delta_{k_n}} P^*(\sup_{A \in \mathcal{A} : A_{k_n}^- \subseteq A \subseteq A_{k_n}^+} |Z_A^n \setminus A_{k_n}^-| > 2\eta_n).$$

In order to find an upper bound for r_{nj} we will use an inequality specific to infinitely divisible distributions. This inequality is given by Lemma 2.3, [8], except that we will use it for Lévy measures whose domain is $\{x; a' < |x| \leq a\}$. (This is in fact the inequality that the authors of [8] are using too, even if they preferred to prove it only in the case when $a' = 0$; the same proof works for arbitrary $a' > 0$.)

In our case each Z_A^n is a random variable with an infinitely divisible distribution whose log-characteristic function is

$$\psi_A^n(u) = \int_{a_n < |x| \leq a_{n-1}} (e^{iux} - 1 - iux) \Pi_A(dx)$$

Then $\theta_n := \int_{a_n < |x| \leq a_{n-1}} x^2 \Pi_A(dx) \leq F(A) Q_{n-1}^1$ and, according to the previous lemma,

$$P(|Z_A^n| > x) \leq 2 \exp\left\{-\frac{x^2}{2(F(A) Q_{n-1}^1 + \frac{a_{n-1}x}{3})}\right\}$$

for every $x > 0$. Using this inequality for both $A_j^- \setminus A_{j-1}^-$ and $A_{j-1}^- \setminus A_j^-$ we get $r_{nj} \leq 2p_{nj}$ where

$$p_{nj} := 2 \exp\left\{-\frac{\eta_{nj}^2}{2(\delta_{j-1} Q_{n-1}^1 + \frac{a_{n-1}\eta_{nj}}{3})}\right\}.$$

Suppose that for $j = 1, \dots, k_n$,

$$2H_I(\delta_j) \leq \frac{\eta_{nj}^2}{8 \max(\delta_{j-1} Q_{n-1}^1, \frac{a_{n-1}\eta_{nj}}{3})} \quad (38)$$

and

$$\frac{\eta_{nj}^2}{8 \max(\delta_{j-1} Q_{n-1}^1, \frac{a_{n-1}\eta_{nj}}{3})} \geq 2 \log n + \log k_n. \quad (39)$$

Then

$$\exp(2H_I(\delta_j)) p_{nj} \leq 2 \exp(-2 \log n - \log k_n)$$

and

$$\sum_{j=1}^{k_n} \exp(2H_I(\delta_j)) p_{nj} \leq 2k_n \exp(-2 \log n - \log k_n) = 2n^{-2}$$

which is summable.

We will find now a summable upper bound for the second term $P_{(II)}$.

Note first that for any set $A \in \sigma(\mathcal{A})$ we have

$$|Z_A^n| \leq |X_A^n| + \int_{a_n < |x| \leq a_{n-1}} |x| \Pi_A(dx) \leq a_{n-1} W_A^n + a_{n-1} F(A) M_n^1$$

where $(W_A^n)_{A \in \mathcal{A}}$ is the set-indexed Poisson process with mean $E(W_A^n) = \Pi_A(\Gamma_n)$.

We begin by taking a closer look at r_n . Fix $A_{k_n}^-, A_{k_n}^+ \in \mathcal{A}_{k_n}$ such that $A_{k_n}^- \subseteq A_{k_n}^+$ and $F(A_{k_n}^+ \setminus A_{k_n}^-) \leq \delta_{k_n}$. For each $A \in \mathcal{A}$, $A_{k_n}^- \subseteq A \subseteq A_{k_n}^+$ we have

$$|Z_{A \setminus A_{k_n}^-}^n| \leq a_{n-1} W_{A_{k_n}^+ \setminus A_{k_n}^-}^n + a_{n-1} \delta_{k_n} M_n^1$$

Suppose that

$$\eta_n > e^2 a_{n-1} \delta_{k_n} M_n^1 \quad (40)$$

Then $\sup_{A \in \mathcal{A}, A_{k_n}^- \subseteq A \subseteq A_{k_n}^+} |Z_{A \setminus A_{k_n}^-}^n| \leq a_{n-1} W_{A_{k_n}^+ \setminus A_{k_n}^-}^n + \eta_n$ and hence

$$P^* \left(\sup_{A \in \mathcal{A}, A_{k_n}^- \subseteq A \subseteq A_{k_n}^+} |Z_{A \setminus A_{k_n}^-}^n| > 2\eta_n \right) \leq P(W_{A_{k_n}^+ \setminus A_{k_n}^-}^n > \eta_n / a_{n-1})$$

At this point we will use an inequality which gives a bound on the tail of a Poisson distribution. This inequality is given by Lemma 2.2, [8].

In our case $W_{A_{k_n}^+ \setminus A_{k_n}^-}^n$ is a Poisson random variable with mean $\Pi_{A_{k_n}^+ \setminus A_{k_n}^-}(\Gamma_n) \leq F(A_{k_n}^+ \setminus A_{k_n}^-) \Pi_T^1(\Gamma_n) \leq \delta_{k_n} M_n^1$. Using the previous lemma with $s = \eta_n / a_{n-1} > e^2 \delta_{k_n} M_n^1$ we get

$$r_n \leq \exp(-\eta_n / a_{n-1}).$$

Suppose that

$$H_I(\delta_{k_n}) \leq \frac{\eta_n}{4a_{n-1}} \quad (41)$$

and

$$\frac{\eta_n}{4a_{n-1}} \geq \log n. \quad (42)$$

Then $\exp(2H_I(\delta_{k_n}))r_n \leq \exp(-\eta_n / 2a_{n-1}) \leq \exp(-2 \log n) = n^{-2}$ which is summable.

It remains to choose the constants $a_n, \eta_n, \beta, \gamma_j, k_n$ such that conditions (35), (36), (38), (39), (40), (41), (42) hold. But this follows exactly as in the proof of Theorem 3.1, [8].

□

Comment 5.5.3 The preceding proof contains also a correction to the original proof of Theorem 3.1, [8]. In this paper it is proved that $P^*(\|Z^n\|_{\mathcal{A}} > 6\eta_n) \leq P_{(I)} + P'_{(II)}$ where $P'_{(II)}$ has exactly the same definition as our $P_{(II)}$ with the constant $2\eta_n$ replaced by $4\eta_n$. An unnecessarily large upper bound for $P'_{(II)}$ is used, namely

$$P'_{(II)} \leq \exp(2H_I(\delta_{k_n})) \cdot (2q_n)$$

where

$$q_n := \max_{A_{k_n}^-, A_{k_n}^+ \in \mathcal{A}_{k_n}, F(A_{k_n}^+ \setminus A_{k_n}^-) \leq \delta_{k_n}} P^* \left(\sup_{A_{k_n}^- \subseteq B \subseteq A_{k_n}^+} |Z_B^n| > 2\eta_n \right).$$

On page 168 of [8] it is claimed that

$$\sup_{A_{k_n}^- \subseteq B \subseteq A_{k_n}^+} |Z_B^n| \leq a_{n-1} W_{A_{k_n}^+ \setminus A_{k_n}^-}^n + a_{n-1} M_n |A_{k_n}^+ \setminus A_{k_n}^-|$$

which may not be true. Our approach makes use of the upper bound τ_n (instead of q_n) and circumvents this difficulty.

Chapter 6

Examples

In this chapter we will see some examples of $\mathcal{Q}$ -Markov processes which do not have independent increments.

6.1 The Empirical Process

In this section we will examine the set-indexed empirical process as a $\mathcal{Q}$ -Markov process. In particular, we will prove that in this case the consistency condition imposed on the generator has a very simple form.

Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

Proposition 6.1.1 *Let $(Z_j)_{j=1,\dots,n}$ be a sequence of independent identically distributed random variables on T with distribution F . Let*

$$X_A := \sum_{j=1}^n I_{\{Z_j \in A\}}, \quad A \in \mathcal{A}.$$

Then:

(i) *the process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with the transition system $\mathcal{Q}$ given by*

$$Q_{BB'}(k; \{m\}) := \binom{n-k}{m-k} \left(\frac{F(B' \setminus B)}{1 - F(B)} \right)^{m-k} \left(1 - \frac{F(B' \setminus B)}{1 - F(B)} \right)^{n-m}$$

for every $k, m \in \{0, 1, \dots, n\}, k \leq m$ and for every $B, B' \in \mathcal{A}(u), B \subseteq B'$;

- (ii) for every right-continuous flow $f : [0, a] \rightarrow \mathcal{A}(u)$ with $f(a) = T$, the process $X^f := (X_{f(t)})_{t \in [0, a]}$ can be written as

$$X_t^f = \sum_{j=1}^n I_{\{Z_j^f \leq t\}}, \quad t \in [0, a]$$

where $(Z_j^f)_{j=1, \dots, n}$ is a sequence of independent identically distributed random variables on $[0, a]$ with distribution $F^f := F \circ f$; and

- (iii) if the collection $\mathcal{S}$ can be chosen such that for every flow $f \in \mathcal{S}$ the function F^f is differentiable, then the generator of the process X^f at time s is

$$(\mathcal{G}_s^f h)(k) = \begin{cases} \lambda_s(k) \cdot (h(k+1) - h(k)) & \text{if } k = 0, 1, \dots, n-1 \\ 0 & \text{if } k = n \end{cases}$$

with

$$\lambda_s(k) := (n-k) \frac{(F^f)'(s)}{1 - F^f(s)}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to $B(\{0, 1, \dots, n\})$, the space of all functions $h : \{0, 1, \dots, n\} \rightarrow \mathbb{R}$.

Comments 6.1.2 1. The process X is called *the empirical process* (of size n) associated to the measure F ; this process can be extended uniquely to a finite positive measure on $\sigma(\mathcal{A})$, called *the empirical measure*.

2. According to (ii), each process $X^f, f \in \mathcal{S}$ is a one-parameter empirical process (of size n) associated to the measure F^f .

3. If $F(\{z\}) = 0, \forall z \in T$ then X is a point process since $P(Z_i = Z_j) = 0, \forall i \neq j$.

Proof: (i) It is not difficult to see that the family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system. According to Comment 3.2.2, it is enough to prove that for every $A \in \mathcal{A}, B \in \mathcal{A}(u)$, for every partition $B = \cup_{i=1}^p C_i, C_i \in \mathcal{C}$ and for every $k_i, m \in \{0, 1, \dots, n\}$ with $k := \sum_{i=1}^p k_i \leq m$

$$P[X_{A \cup B} = m | X_{C_1} = k_1, \dots, X_{C_p} = k_p] = Q_{B, A \cup B}(k; \{m\}).$$

We have

$$P(X_{C_1} = k_1, \dots, X_{C_p} = k_p, X_{A \cup B} = m) = \frac{n!}{\prod_{i=1}^p k_i! (m-k)! (n-m)!} \prod_{i=1}^p (F(C_i))^{k_i} (F(A \setminus B))^{m-k} (1 - F(A \cup B))^{n-m}$$

and

$$P(X_{C_1} = k_1, \dots, X_{C_p} = k_p) = \frac{n!}{\prod_{i=1}^p k_i! (n-k)!} \prod_{i=1}^p (F(C_i))^{k_i} (1 - F(B))^{n-k}.$$

By dividing these two relationships we get

$$P[X_{A \cup B} = m | X_{C_1} = k_1, \dots, X_{C_p} = k_p] = \frac{(n-k)!}{(m-k)! (n-m)!} \cdot \frac{(F(A \setminus B))^{m-k} (1 - F(A \cup B))^{n-m}}{(1 - F(B))^{n-k}}$$

which concludes the proof of (i).

(ii) For each right-continuous flow $f : [0, a] \rightarrow \mathcal{A}(u)$ with $f(a) = T$, the function F^f is right-continuous and increasing with $F^f(a) = F(T) = 1$; let

$$Z_j^f := \min\{t \in [0, a]; Z_j \in f(t)\}.$$

Note that $Z_j^f \leq t$ if and only if $Z_j \in f(t)$. Hence

$$X_t^f = X_{f(t)} = \sum_{j=1}^n I_{\{Z_j \in f(t)\}} = \sum_{j=1}^n I_{\{Z_j^f \leq t\}}$$

where the random variables $(Z_j^f)_{j \geq 1}$ are independent identically distributed with distribution F^f .

(iii) For each $f \in \mathcal{S}$, let $Q_{st}^f := Q_{f(s)f(t)}$ and T_{st}^f the bounded linear operator associated to Q_{st}^f . We have

$$(T_{st}^f h)(k) - h(k) = \sum_{m=k+1}^n (h(m) - h(k)) Q_{st}^f(k; \{m\}) = \sum_{m=k+1}^n (h(m) - h(k)) \cdot \binom{n-k}{m-k} (F^f(t) - F^f(s))^{m-k} \cdot \frac{(1 - F^f(t))^{n-m}}{(1 - F^f(s))^{n-k}}$$

for every function $h \in B(\{0, 1, \dots, n\})$. By definition, the generator of the process X^f at time s is

$$(\mathcal{G}_s^f h)(k) := \lim_{t \rightarrow s} \frac{(T_{st}^f h)(k) - h(k)}{t - s} \text{ uniformly in } k.$$

The result follows since

$$\lim_{t \rightarrow s} \frac{(F^f(t) - F^f(s))^{m-k}}{t - s} = \begin{cases} (F^f)'(s) & \text{if } m = k + 1 \\ 0 & \text{if } m > k + 1 \end{cases}.$$

□

The next theorem shows that the finite dimensional distributions of a set-indexed empirical process are completely characterized by its generator, provided that a certain consistency assumption holds.

Theorem 6.1.3 *For each $f \in \mathcal{S}$, $f : [0, a] \rightarrow \mathcal{A}(u)$, let F^f be a differentiable probability distribution function on $[0, a]$.*

If $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s ; f \in \mathcal{S}\}$ is a collection of families of linear operators on $B(\{0, 1, \dots, n\})$ given by

$$(\mathcal{G}_s^f h)(k) = \begin{cases} \lambda_s(k) \cdot (h(k+1) - h(k)) & \text{if } k = 0, 1, \dots, n-1 \\ 0 & \text{if } k = n \end{cases}$$

with

$$\lambda_s(k) := (n - k) \frac{(F^f)'(s)}{1 - F^f(s)}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to the space $B(\{0, 1, \dots, n\})$, and we assume that there exists a probability measure F on $(T, \sigma(\mathcal{A}))$ such that

$$F \circ f = F^f, \quad \forall f \in \mathcal{S}$$

then there exists a probability measure P on the product space $\{0, 1, \dots, n\}^c$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in c}$ defined on this space is a version of the empirical process (of size n) associated to the measure F ; and*

2. $\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is $\mathcal{G}_s^f$.

Proof: For each $f \in \mathcal{S}$, the operator $\mathcal{G}_s^f$ is the generator at time s of the semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$ associated to the transition system $\mathcal{Q}^f := (Q_{st}^f)_{s < t}$ defined by

$$Q_{st}^f(k; \{m\}) := \binom{n-k}{m-k} \left(\frac{F^f(t) - F^f(s)}{1 - F^f(s)} \right)^{m-k} \left(1 - \frac{F^f(t) - F^f(s)}{1 - F^f(s)} \right)^{n-m}$$

for every $k, m \in \{0, 1, \dots, n\}$, $k \leq m$. The result will follow by Theorem 4.1.3, provided that the conditions of this theorem are satisfied.

We will prove first that the collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.6: let $f, g \in \mathcal{S}$ be such that $f(s) = g(u)$ and $f(t) = g(v)$; we have

$$F^f(s) = F(f(s)) = F(g(u)) = F^g(u)$$

$$F^f(t) - F^f(s) = F(f(t)) - F(f(s)) = F(g(v)) - F(g(u)) = F^g(v) - F^g(u)$$

and hence $Q_{st}^f = Q_{uv}^g$.

We will prove next that the collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.7 with $\mu := \delta_0$. Let $\text{ord1} = \{A_0 = \emptyset, A_1, \dots, A_n\}$ and $\text{ord2} = \{A_0 = \emptyset, A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, $\forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{\mathcal{A}', \text{ord1}}$, $g := f_{\mathcal{A}', \text{ord2}}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$ and $t_0 = u_0 = 0$. We want to prove that

$$\int_{\mathbf{R}^n} I_{\{k_1\}}(x_1) \prod_{i=2}^n I_{\{k_i\}}(x_i - x_{i-1}) Q_{t_{n-1}t_n}^f(x_{n-1}; dx_n) \dots Q_{0t_1}^f(0; dx_1) = \quad (43)$$

$$\int_{\mathbf{R}^n} I_{\{k_1\}}(y_1) \prod_{i=2}^n I_{\{k_i\}}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{u_{n-1}u_n}^g(y_{n-1}; dy_n) \dots Q_{0u_1}^g(0; dy_1)$$

for every $k_i \in \{0, 1, \dots, n\}$ with $\sum_{i=1}^n k_i \leq n$.

The left-hand side of (43) is equal to

$$\frac{n!}{\prod_{i=1}^n k_i! (n - \sum_{i=1}^n k_i)!} \prod_{i=1}^n (F^f(t_i) - F^f(t_{i-1}))^{k_i} (1 - F^f(t_n))^{n - \sum_{i=1}^n k_i} \quad (44)$$

since

$$Q_{0t_1}^f(0; \{k_1\}) = \frac{n!}{k_1! (n - k_1)!} (F^f(t_1))^{k_1} (1 - F^f(t_1))^{n - k_1};$$

$$Q_{t_1 t_2}^f(k_1; \{k_1 + k_2\}) = \frac{(n - k_1)!}{k_2!(n - k_1 - k_2)!} \cdot \frac{(F^f(t_2) - F^f(t_1))^{k_2} (1 - F^f(t_2))^{n - k_1 - k_2}}{(1 - F^f(t_1))^{n - k_1}};$$

etc. On the right-hand side of (43) we write

$$\prod_{i=2}^n I_{\{k_i\}}(y_{\pi(i)} - y_{\pi(i)-1}) = \prod_{i=2}^n I_{\{l_i\}}(y_i - y_{i-1})$$

with $l_i := k_{\pi^{-1}(i)}$, and therefore this side becomes

$$\frac{n!}{\prod_{i=1}^n l_i! (n - \sum_{i=1}^n l_i)!} \prod_{i=1}^n (F^g(u_i) - F^g(u_{i-1}))^{l_i} (1 - F^g(u_n))^{n - \sum_{i=1}^n l_i}. \quad (45)$$

Finally, we observe that (44) is equal to (45) since

$$F^f(t_i) - F^f(t_{i-1}) = F(C_i) = F(C'_{\pi(i)}) = F^g(u_{\pi(i)}) - F^g(u_{\pi(i)-1})$$

(where C_i, C'_i are the left neighbourhoods of A_i, A'_i) and hence

$$(F^f(t_i) - F^f(t_{i-1}))^{k_i} = (F^g(u_{\pi(i)}) - F^g(u_{\pi(i)-1}))^{l_{\pi(i)}}$$

for every $i = 1, \dots, n$.

□

6.2 A Bivariate Process

In this section we will examine a bivariate $\mathcal{Q}$ -Markov process, whose construction is based on the empirical process.

Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

Proposition 6.2.1 *Let $(Z_j, X_j)_{j=1, \dots, n}$ be a sequence of independent identically distributed random variables on $T \times \mathbf{R}$ with distribution $F \times G$. Let*

$$\begin{aligned} X_A &:= \sum_{j=1}^n I_{\{Z_j \in A\}}, \quad A \in \mathcal{A} \\ Y_A &:= \sum_{j=1}^n X_j I_{\{Z_j \in A\}}, \quad A \in \mathcal{A}. \end{aligned}$$

Then:

- (i) the process $(X, Y) := (X_A, Y_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with the transition system $\mathcal{Q}$ given by

$$Q_{BB'}((k, y); \{m\} \times \Gamma) = Q_{BB'}(k; \{m\} \times \Gamma) := \binom{n-k}{m-k} \left(\frac{F(B' \setminus B)}{1 - F(B)} \right)^{m-k} \left(1 - \frac{F(B' \setminus B)}{1 - F(B)} \right)^{n-m} G^{*m}(\Gamma)$$

for every $k, m \in \{0, 1, \dots, n\}$, $k \leq m$, for every $y \in \mathbf{R}$, $\Gamma \in \mathcal{B}(\mathbf{R})$ and for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$ (here G^{*m} denotes $\underbrace{G * \dots * G}_{m \text{ times}}$):

- (ii) for every right-continuous flow $f : [0, a] \rightarrow \mathcal{A}(u)$ with $f(a) = T$, the processes $X^f := (X_{f(t)})_{t \in [0, a]}$, $Y^f := (Y_{f(t)})_{t \in [0, a]}$ can be written as

$$\begin{aligned} X_t^f &= \sum_{j=1}^n I_{\{Z_j^f \leq t\}}, \quad t \in [0, a] \\ Y_t^f &= \sum_{j=1}^n X_j I_{\{Z_j^f \leq t\}}, \quad t \in [0, a] \end{aligned}$$

where $(Z_j^f)_{j=1, \dots, n}$ is a sequence of independent identically distributed random variables on $[0, a]$ with distribution $F^f := F \circ f$; and

- (iii) if the collection $\mathcal{S}$ can be chosen such that for every flow $f \in \mathcal{S}$ the function F^f is differentiable, then the generator of the process (X^f, Y^f) at time s is

$$(\mathcal{G}_s^f h)(k, y) = \lambda_s(k) \cdot \int_{\mathbf{R}} (h(k+1, z) - h(k, y)) G^{*(k+1)}(dz)$$

for $k = 0, 1, \dots, n-1$; $y \in \mathbf{R}$, with

$$\lambda_s(k) := (n-k) \frac{(F^f)'(s)}{1 - F^f(s)}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to $B(\{0, 1, \dots, n\} \times \mathbf{R})$, the space of all bounded functions $h : \{0, 1, \dots, n\} \times \mathbf{R} \rightarrow \mathbf{R}$.

Comment 6.2.2 The process Y may not be set-Markov, as it can be seen from the following example: let $T = [0, 1]$, $\mathcal{A} = \{[0, t]; t \in T\}$, X_1, X_2 some independent

random variables with common distribution $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ and Z_1, Z_2 some i.i.d. random variables on T , independent of X_1, X_2 ; the process $Y := (Y_t)_{t \in T}$ defined by

$$Y_t := \sum_{j=1}^2 X_j I_{\{Z_j \in [0, t]\}}, \quad t \in [0, 1]$$

is not Markov: $P[Y_1 > 2 | Y_{1/2} = 2] > 0$ but $P[Y_1 > 2 | Y_{1/2} = 2, Y_{1/4} = 1] = 0$.

Proof: (i) It is not difficult to see that the family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system. According to Comment 3.2.2, it is enough to prove that for every $A \in \mathcal{A}$, $B \in \mathcal{A}(u)$, for every partition $B = \cup_{i=1}^p C_i$, $C_i \in \mathcal{C}$, for every $k_i, m \in \{0, 1, \dots, n\}$ with $k := \sum_{i=1}^p k_i \leq m$, for every $y_i \in \mathbf{R}$ and for every $\Gamma_i, \Gamma \in \mathcal{B}(\mathbf{R})$

$$P[X_{A \cup B} = m, Y_{A \cup B} \in \Gamma | X_{C_1} = k_1, Y_{C_1} = y_1, \dots, X_{C_p} = k_p, Y_{C_p} = y_p] =$$

$$Q_{B, A \cup B}((k, \sum_{i=1}^p y_i); \{m\} \times \Gamma)$$

or, equivalently

$$P(X_{A \cup B} = m, Y_{A \cup B} \in \Gamma, X_{C_1} = k_1, Y_{C_1} \in \Gamma_1, \dots, X_{C_p} = k_p, Y_{C_p} \in \Gamma_p) = \quad (46)$$

$$\int_{\prod_{i=1}^p \Gamma_i} Q_{B, A \cup B}((k, \sum_{i=1}^p y_i); \{m\} \times \Gamma) P_{(X_{C_1}, Y_{C_1}, \dots, X_{C_p}, Y_{C_p})}(k_1, dy_1, \dots, k_p, dy_p)$$

for every $\Gamma_1, \dots, \Gamma_p \in \mathcal{B}(\mathbf{R})$ (here $P_{(X_{C_1}, Y_{C_1}, \dots, X_{C_p}, Y_{C_p})}$ denotes the distribution of the random vector $(X_{C_1}, Y_{C_1}, \dots, X_{C_p}, Y_{C_p})$).

The result follows since the left-hand side of (46) is equal to

$$\frac{n!}{\prod_{i=1}^p k_i! (m - k)! (n - m)!} \prod_{i=1}^p (F(C_i))^{k_i} (F(A \setminus B))^{m-k} (1 - F(A \cup B))^{n-m} \quad (47)$$

$$G^{*k_1}(\Gamma_1) \dots G^{*k_p}(\Gamma_p) \cdot G^{*m}(\Gamma)$$

the right-hand side is equal to

$$\frac{n!}{\prod_{i=1}^p k_i! (n - k)!} \prod_{i=1}^p (F(C_i))^{k_i} (1 - F(B))^{n-k} \quad (48)$$

$$\int_{\prod_{i=1}^p \Gamma_i} Q_{B, A \cup B}((k, \sum_{i=1}^p y_i); \{m\} \times \Gamma) G^{*k_1}(dy_1) \dots G^{*k_p}(dy_p)$$

and (47) and (48) are easily seen to be equal, using the definition of $Q_{B, A \cup B}$.

(ii) The variables Z_j^f are the same as in the proof of Proposition 6.1.1.

(iii) For each $f \in \mathcal{S}$, let $Q_{st}^f := Q_{f(s)f(t)}$ and T_{st}^f the bounded linear operator associated to Q_{st}^f . We have

$$(T_{st}^f h)(k, y) - h(k, y) = \sum_{m=k}^n \int_{\mathbf{R}} (h(m, z) - h(k, y)) Q_{st}^f((k, y); \{m\} \times dz) =$$

$$\sum_{m=k}^n \binom{n-k}{m-k} (F^f(t) - F^f(s))^{m-k} \cdot \frac{(1 - F^f(t))^{n-m}}{(1 - F^f(s))^{n-k}} \int_{\mathbf{R}} (h(m, z) - h(k, y)) G^{*m}(dz)$$

for every function $h \in B(\{0, 1, \dots, n\} \times \mathbf{R})$. By definition, the generator of the process X^f at time s is

$$(\mathcal{G}_s^f h)(k, y) := \lim_{t \rightarrow s} \frac{(T_{st}^f h)(k, y) - h(k, y)}{t - s} \text{ uniformly in } k, y.$$

The result follows since

$$\lim_{t \rightarrow s} \frac{(F^f(t) - F^f(s))^{m-k}}{t - s} = \begin{cases} (F^f)'(s) & \text{if } m = k + 1 \\ 0 & \text{if } m > k + 1 \end{cases}.$$

□

The next theorem shows that the finite dimensional distributions of the set-indexed process constructed in the previous proposition are completely characterized by its generator, provided that a certain consistency assumption holds.

Theorem 6.2.3 *For each $f \in \mathcal{S}$, $f : [0, a] \rightarrow \mathcal{A}(u)$, let F^f be a differentiable probability distribution function on $[0, a]$.*

If $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s ; f \in \mathcal{S}\}$ is a collection of families of linear operators on $B(\{0, 1, \dots, n\} \times \mathbf{R})$ given by

$$(\mathcal{G}_s^f h)(k, y) = \lambda_s(k) \cdot \int_{\mathbf{R}} (h(k+1, z) - h(k, y)) G^{*(k+1)}(dz)$$

for $k = 0, 1, \dots, n-1; y \in \mathbf{R}$, with

$$\lambda_s(k) := (n-k) \frac{(F^f)'(s)}{1 - F^f(s)}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to the space $B(\{0, 1, \dots, n\} \times \mathbf{R})$, and we assume that there exists a probability measure F on $(T, \sigma(\mathcal{A}))$ such that

$$F \circ f = F^f, \quad \forall f \in \mathcal{S}$$

then there exists a probability measure P on the product space $(\{0, 1, \dots, n\} \times \mathbf{R})^c$ under which:

1. the coordinate-variable process $(X, Y) := (X_C, Y_C)_{C \in c}$ defined on this space is a version of the process constructed in Proposition 6.2.1; and
2. $\forall f \in \mathcal{S}$, the generator of the process $(X^f, Y^f) := (X_{f(t)}, Y_{f(t)})_t$ at time s is $\mathcal{G}_s^f$.

Proof: For each $f \in \mathcal{S}$, the operator $\mathcal{G}_s^f$ is the generator at time s of the semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$ associated to the transition system $\mathcal{Q}^f := (Q_{st}^f)_{s < t}$ defined by

$$Q_{st}^f((k, y); \{m\} \times \Gamma) = Q_{st}^f(k; \{m\} \times \Gamma) := \binom{n-k}{m-k} \left(\frac{F^f(t) - F^f(s)}{1 - F^f(s)} \right)^{m-k} \left(1 - \frac{F^f(t) - F^f(s)}{1 - F^f(s)} \right)^{n-m} G^{*m}(\Gamma)$$

for every $k, m \in \{0, 1, \dots, n\}, k \leq m$ and for every $y \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$. The result will follow by Theorem 4.1.3, provided that the conditions of this theorem are satisfied.

The fact that the collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.6 follows as for the empirical process (see the proof of Theorem 6.1.3) since Q_{st}^f depends only on f, s, t only through $F^f(s)$ and $F^f(t)$.

It remains to prove that the collection $\{\mathcal{Q}^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.7 with $\mu := \delta_{(0,0)}$. Let $\text{ord1} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord2} = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}, \forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{\mathcal{A}', \text{ord1}}, g := f_{\mathcal{A}', \text{ord2}}$ with $f(t_i) = \cup_{j=1}^i A_j, g(u_i) = \cup_{j=1}^i A'_j$ and $t_0 = u_0 = 0$. We want to prove that

$$\int_{\mathbf{R}^{2n}} I_{\{k_1\} \times \Gamma_1}(x_1, x'_1) \prod_{i=2}^n I_{\{k_i\} \times \Gamma_i}(x_i - x_{i-1}, x'_i - x'_{i-1}) \quad (49)$$

$$Q_{t_{n-1}t_n}^f((x_{n-1}, x'_{n-1}); dx_n \times dx'_n) \dots Q_{0t_1}^f((0, 0); dx_1 \times dx'_1) =$$

$$\int_{\mathbf{R}^{2n}} I_{\{k_1\} \times \Gamma_1}(y_1, y'_1) \prod_{i=2}^n I_{\{k_i\} \times \Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}, y'_{\pi(i)} - y'_{\pi(i)-1}) \\ Q_{u_{n-1}u_n}^g((y_{n-1}, y'_{n-1}); dy_n \times dy'_n) \dots Q_{0u_1}^g((0, 0); dy_1 \times dy'_1)$$

for every $k_i \in \{0, 1, \dots, n\}$ with $\sum_{i=1}^n k_i \leq n$ and for every $\Gamma_1, \dots, \Gamma_n \in \mathcal{B}(\mathbf{R})$.

The left-hand side of (49) is equal to

$$\frac{n!}{\prod_{i=1}^n k_i!(n - \sum_{i=1}^n k_i)!} \prod_{i=1}^n (F^f(t_i) - F^f(t_{i-1}))^{k_i} (1 - F^f(t_n))^{n - \sum_{i=1}^n k_i} \quad (50) \\ \cdot \int_{\mathbf{R}^n} I_{\Gamma_1}(x'_1) \prod_{i=2}^n I_{\Gamma_i}(x'_i - x'_{i-1}) G^{*k_n}(dx'_n) \dots G^{*k_1}(dx'_1)$$

since

$$Q_{0t_1}^f(0; \{k_1\} \times dx'_1) = \frac{n!}{k_1!(n - k_1)!} (F^f(t_1))^{k_1} (1 - F^f(t_1))^{n - k_1} \cdot G^{*k_1}(dx'_1); \\ Q_{t_1t_2}^f(k_1; \{k_1 + k_2\} \times dx'_2) = \frac{(n - k_1)!}{k_2!(n - k_1 - k_2)!} \cdot \frac{(F^f(t_2) - F^f(t_1))^{k_2} (1 - F^f(t_2))^{n - k_1 - k_2}}{(1 - F^f(t_1))^{n - k_1}} \\ \cdot G^{*k_2}(dx'_2);$$

etc. On the right-hand side of (49) we write

$$\prod_{i=2}^n I_{\{k_i\} \times \Gamma_i}(y_{\pi(i)} - y_{\pi(i)-1}, y'_{\pi(i)} - y'_{\pi(i)-1}) = \prod_{i=2}^n I_{\{l_i\} \times \Sigma_i}(y_i - y_{i-1}, y'_i - y'_{i-1})$$

with $l_i := k_{\pi^{-1}(i)}$, $\Sigma_i := \Gamma_{\pi^{-1}(i)}$, and therefore this side becomes

$$\frac{n!}{\prod_{i=1}^n l_i!(n - \sum_{i=1}^n l_i)!} \prod_{i=1}^n (F^g(u_i) - F^g(u_{i-1}))^{l_i} (1 - F^g(u_n))^{n - \sum_{i=1}^n l_i} \quad (51) \\ \cdot \int_{\mathbf{R}^n} I_{\Sigma_1}(y'_1) \prod_{i=2}^n I_{\Sigma_i}(y'_i - y'_{i-1}) G^{*l_n}(dy'_n) \dots G^{*l_1}(dy'_1).$$

In order to prove that (50) is equal to (51) we recall that

$$(F^f(t_i) - F^f(t_{i-1}))^{k_i} = (F^g(u_{\pi(i)}) - F^g(u_{\pi(i)-1}))^{l_{\pi(i)}}$$

for every $i = 1, \dots, n$ (see the proof of Theorem 6.1.3).

Finally, let $s_i := \sum_{j=1}^i k_j$, $r_i := \sum_{j=1}^i l_j$; $i = 1, \dots, n$ and $s_0 = r_0 := 0$. We have $s_1 = r_1 = k_1$ and $s_n = r_n = k := \sum_{i=1}^n k_i$.

We consider the permutation ρ of $\{1, 2, \dots, k\}$ defined by

$$\rho(s_{\pi^{-1}(i)-1} + a) = r_{i-1} + a$$

for every $a = 1, 2, \dots, k_{\pi^{-1}(i)}$; $i = 1, \dots, n$ (ρ is indeed a bijective map since $k_{\pi^{-1}(i)} = l_i = s_{\pi^{-1}(i)} - s_{\pi^{-1}(i)-1} = r_i - r_{i-1}$).

Knowing that in general $\int_{\mathbf{R}} h(z)(F * G)(dz) = \int_{\mathbf{R}^2} h(x+y)F(dx)G(dy)$ and using the change of variable $u_j := v_{\rho(j)}$, we have

$$\begin{aligned} & \int_{\mathbf{R}^n} I_{\Gamma_1}(x'_1) \prod_{i=2}^n I_{\Gamma_i}(x'_i - x'_{i-1}) G^{*k_n}(dx'_n) \dots G^{*k_1}(dx'_1) = \\ & \int_{\mathbf{R}^k} I_{\Gamma_1}\left(\sum_{j=1}^{s_1} u_j\right) \prod_{i=2}^n I_{\Gamma_i}\left(\sum_{j=s_{i-1}+1}^{s_i} u_j - \sum_{j=s_{i-2}+1}^{s_{i-1}} u_j\right) G(du_k) \dots G(du_1) = \\ & \int_{\mathbf{R}^k} I_{\Gamma_1}\left(\sum_{j=1}^{s_1} v_{\rho(j)}\right) \prod_{i=2}^n I_{\Gamma_i}\left(\sum_{j=s_{i-1}+1}^{s_i} v_{\rho(j)} - \sum_{j=s_{i-2}+1}^{s_{i-1}} v_{\rho(j)}\right) G(dv_k) \dots G(dv_1) = \\ & \int_{\mathbf{R}^k} I_{\Gamma_1}\left(\sum_{j=1}^{s_1} v_{\rho(j)}\right) \prod_{i=2}^n I_{\Gamma_{\pi^{-1}(i)}}\left(\sum_{j=s_{\pi^{-1}(i)-1}+1}^{s_{\pi^{-1}(i)}} v_{\rho(j)} - \sum_{j=s_{\pi^{-1}(i)-2}+1}^{s_{\pi^{-1}(i)-1}} v_{\rho(j)}\right) G(dv_k) \dots G(dv_1) = \\ & \int_{\mathbf{R}^k} I_{\Sigma_1}\left(\sum_{l=1}^{r_1} v_l\right) \prod_{i=2}^n I_{\Sigma_i}\left(\sum_{l=r_{i-1}+1}^{r_i} v_l - \sum_{l=r_{i-2}+1}^{r_{i-1}} v_l\right) G(dv_k) \dots G(dv_1) = \\ & \int_{\mathbf{R}^n} I_{\Sigma_1}(y'_1) \prod_{i=2}^n I_{\Sigma_i}(y'_i - y'_{i-1}) G^{*l_n}(dy'_n) \dots G^{*l_1}(dy'_1) \end{aligned}$$

This concludes the proof that (50) is equal to (51).

□

6.3 A Jump Process

In this section we will construct a simple $\mathcal{Q}$ -Markov process with ‘jumps’ and we will see what consistency condition has to be imposed on the generator of this process.

Let $\mathcal{S}$ be a collection of simple flows as in Definition 3.2.10.

Proposition 6.3.1 *Let $(Z_j)_{j \geq 1}$ be a sequence of independent identically distributed random variables on T with distribution F , Λ an independent random variable with density $f_\Lambda : (0, \infty) \rightarrow \mathbf{R}$, and N a random variable such that:*

- N is conditionally independent of $(Z_j)_{j \geq 1}$ given Λ ; and
- the conditional distribution of N given Λ is Poisson with mean Λ .

Let

$$X_A := \sum_{j=1}^N I_{\{Z_j \in A\}}, \quad A \in \mathcal{A}.$$

Then:

- (i) the process $X := (X_A)_{A \in \mathcal{A}}$ is $\mathcal{Q}$ -Markov with the transition system $\mathcal{Q}$ given by

$$Q_{BB'}(k; \{m\}) := \frac{(F(B' \setminus B))^{m-k}}{(m-k)!} \cdot \frac{\int_0^\infty e^{-\lambda F(B')} \lambda^m f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F(B)} \lambda^k f_\Lambda(\lambda) d\lambda}$$

for every $k, m \in \{0, 1, 2, \dots\}$, $k \leq m$ and for every $B, B' \in \mathcal{A}(u)$, $B \subseteq B'$;

- (ii) for every right-continuous flow $f : [0, a] \rightarrow \mathcal{A}(u)$ with $f(a) = T$, the process $X^f := (X_{f(t)})_{t \in [0, a]}$ can be written as

$$X_t^f = \sum_{j=1}^N I_{\{Z_j^f \leq t\}}, \quad t \in [0, a]$$

where $(Z_j^f)_{j \geq 1}$ is a sequence of independent identically distributed random variables on $[0, a]$ with distribution $F^f := F \circ f$; and

- (iii) if the collection $\mathcal{S}$ can be chosen such that for every flow $f \in \mathcal{S}$ the function F^f is differentiable, then the generator of the process X^f at time s is

$$(\mathcal{G}_s^f h)(k) = \lambda_s(k) \cdot (h(k+1) - h(k)), \quad k = 0, 1, 2, \dots$$

with

$$\lambda_s(k) := (F^f)'(s) \cdot \frac{\int_0^\infty e^{-\lambda F^f(s)} \lambda^{k+1} f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F^f(s)} \lambda^k f_\Lambda(\lambda) d\lambda}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to the space $B(\{0, 1, 2, \dots\})$ of all bounded functions $h : \{0, 1, 2, \dots\} \rightarrow \mathbf{R}$.

Proof: (i) We know that for any disjoint sets $C_1, \dots, C_p \in \mathcal{C}$, the variables $X_{C_1}, \dots, X_{C_p}$ are conditionally independent given Λ , and the distribution of each $X_C, C \in \mathcal{C}$ given $\Lambda = \lambda$ is Poisson with mean $\lambda F(C)$.

From here we can conclude that $\forall A \in \mathcal{A}, \forall B \in \mathcal{A}(u)$

$$\mathcal{F}_B \perp \sigma(X_{A \setminus B}) \mid \sigma(\Lambda)$$

where $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ is the minimal filtration of the process X .

(In order to show that $E[h(X_{A \setminus B}) \cdot Y \mid \Lambda] = E[h(X_{A \setminus B}) \mid \Lambda] E[Y \mid \Lambda]$ for any $\mathcal{F}_B$ -measurable function Y , it is enough to assume that $Y = \prod_{i=1}^p h_i(X_{C_i})$ for some disjoint $C_i \in \mathcal{C}, C_i \subseteq B$ and some $h_i \in B(\mathbf{R})$.)

Let $h \in B(\mathbf{R})$ be arbitrary. Then $\forall A \in \mathcal{A}, \forall B \in \mathcal{A}(u)$

$$\begin{aligned} E[h(X_{A \setminus B}) \mid \mathcal{F}_B] &= E[E[h(X_{A \setminus B}) \mid \mathcal{F}_B \vee \sigma(\Lambda)] \mid \mathcal{F}_B] \\ &= E[E[h(X_{A \setminus B}) \mid \sigma(\Lambda)] \mid \mathcal{F}_B] \\ &= E[h'(\Lambda) \mid \mathcal{F}_B]. \end{aligned}$$

In order to prove the process X is set-Markov, it is enough to show that $\mathcal{F}_B \perp \sigma(\Lambda) \mid \sigma(X_B)$ or, equivalently, for any partition $B = \cup_{i=1}^p C_i, C_i \in \mathcal{C}$

$$E[h'(\Lambda) \mid X_{C_1}, \dots, X_{C_p}] = E[h'(\Lambda) \mid X_B].$$

Since Λ has an absolutely continuous distribution (with density $f_\Lambda(\lambda)$), the distribution of Λ given X_C , for $C \in \mathcal{C}(u)$ is also absolutely continuous; we denote with $f_{\Lambda \mid X_C}(\lambda \mid X_C)$ its density function.

Using Bayes theorem, if $k = \sum_{i=1}^p k_i$ we have

$$\begin{aligned} f_{\Lambda \mid X_{C_1}, \dots, X_{C_p}}(\lambda \mid k_1, \dots, k_p) &= \frac{f_{X_{C_1}, \dots, X_{C_p} \mid \Lambda}(k_1, \dots, k_p \mid \lambda) f_\Lambda(\lambda)}{\int_0^\infty f_{X_{C_1}, \dots, X_{C_p} \mid \Lambda}(k_1, \dots, k_p \mid \lambda) f_\Lambda(\lambda) d\lambda} \\ &= \frac{\prod_{i=1}^p [e^{-\lambda F(C_i)} \cdot \frac{1}{k_i!} \cdot (\lambda F(C_i))^{k_i}] \cdot f_\Lambda(\lambda)}{\int_0^\infty \prod_{i=1}^p [e^{-\lambda F(C_i)} \cdot \frac{1}{k_i!} \cdot (\lambda F(C_i))^{k_i}] \cdot f_\Lambda(\lambda) d\lambda} \\ &= \frac{e^{-\lambda F(B)} \cdot \frac{(\lambda F(B))^k}{k!} \cdot f_\Lambda(\lambda)}{\int_0^\infty e^{-\lambda F(B)} \cdot \frac{(\lambda F(B))^k}{k!} \cdot f_\Lambda(\lambda) d\lambda} \\ &= \frac{f_{X_B \mid \Lambda}(k \mid \lambda) f_\Lambda(\lambda)}{\int_0^\infty f_{X_B \mid \Lambda}(k \mid \lambda) f_\Lambda(\lambda) d\lambda} \\ &= f_{\Lambda \mid X_B}(\lambda \mid k). \end{aligned}$$

Hence

$$\begin{aligned} E[h'(\Lambda)|X_{C_1}, \dots, X_{C_p}] &= \int_{\mathbf{R}} h'(\lambda) f_{\Lambda|X_{C_1}, \dots, X_{C_p}}(\lambda|X_{C_1}, \dots, X_{C_p}) d\lambda \\ &= \int_{\mathbf{R}} h'(\lambda) f_{\Lambda|X_B}(\lambda|X_B) d\lambda = E[h'(\Lambda)|X_B]. \end{aligned}$$

It is not difficult to see that the family $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is a transition system. We prove now that $\forall A \in \mathcal{A}, B \in \mathcal{A}(u)$, $Q_{B, A \cup B}$ is a version of the conditional distribution of $X_{A \cup B}$ given X_B : for $0 \leq k \leq m$

$$\begin{aligned} P[X_{A \cup B} = m | X_B = k] &= \frac{P(X_B = k, X_{A \setminus B} = m - k)}{P(X_B = k)} \\ &= \frac{\int_0^\infty P[X_B = k | \Lambda = \lambda] \cdot P[X_{A \setminus B} = m - k | \Lambda = \lambda] \cdot f_\Lambda(\lambda) d\lambda}{\int_0^\infty P[X_B = k | \Lambda = \lambda] \cdot f_\Lambda(\lambda) d\lambda} \\ &= \frac{\int_0^\infty [e^{-\lambda F(B)} \cdot \frac{1}{k!} \cdot (\lambda F(B))^k] \cdot [e^{-\lambda F(A \setminus B)} \cdot \frac{1}{(m-k)!} \cdot (\lambda F(A \setminus B))^{m-k}] \cdot f_\Lambda(\lambda) d\lambda}{\int_0^\infty [e^{-\lambda F(B)} \cdot \frac{1}{k!} \cdot (\lambda F(B))^k] \cdot f_\Lambda(\lambda) d\lambda} \\ &= \frac{(F(A \setminus B))^{m-k}}{(m-k)!} \cdot \frac{\int_0^\infty e^{-\lambda F(A \cup B)} \lambda^m f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F(B)} \lambda^k f_\Lambda(\lambda) d\lambda}. \end{aligned}$$

(ii) The argument is the same as for the empirical process (see the proof of Proposition 6.1.1).

(iii) For each $f \in \mathcal{S}$, let $Q_{st}^f := Q_{f(s)f(t)}$ and T_{st}^f the bounded linear operator associated to Q_{st}^f . We have

$$\begin{aligned} (T_{st}^f h)(k) - h(k) &= \sum_{m \geq k+1} (h(m) - h(k)) Q_{st}^f(k; \{m\}) = \\ &= \sum_{m \geq k+1} (h(m) - h(k)) \cdot \frac{(F^f(t) - F^f(s))^{m-k}}{(m-k)!} \cdot \frac{\int_0^\infty e^{-\lambda F^f(t)} \lambda^m f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F^f(s)} \lambda^k f_\Lambda(\lambda) d\lambda} \end{aligned}$$

for every function $h \in B(\{0, 1, 2, \dots\})$. By definition, the generator of the process X^f at time s is

$$(\mathcal{G}_s^f h)(k) := \lim_{t \rightarrow s} \frac{(T_{st}^f h)(k) - h(k)}{t - s} \text{ uniformly in } k.$$

The result follows since

$$\lim_{t \rightarrow s} \frac{(F^f(t) - F^f(s))^{m-k}}{t - s} = \begin{cases} (F^f)'(s) & \text{if } m = k + 1 \\ 0 & \text{if } m > k + 1 \end{cases}.$$

□

The next theorem shows that the finite dimensional distributions of the process constructed in the previous proposition are completely characterized by its generator, provided that a certain consistency assumption holds. This assumption is the same as for the empirical process.

Theorem 6.3.2 *For each $f \in \mathcal{S}$, $f : [0, a] \rightarrow \mathcal{A}(u)$, let F^f be a differentiable probability distribution function on $[0, a]$.*

If $\{\mathcal{G}^f := (\mathcal{G}_s^f)_s ; f \in \mathcal{S}\}$ is a collection of families of linear operators on $B(\{0, 1, 2, \dots\})$ given by

$$(\mathcal{G}_s^f h)(k) = \lambda_s(k) \cdot (h(k+1) - h(k)), \quad k = 0, 1, 2, \dots$$

with

$$\lambda_s(k) = (F^f)'(s) \cdot \frac{\int_0^\infty e^{-\lambda F^f(s)} \lambda^{k+1} f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F^f(s)} \lambda^k f_\Lambda(\lambda) d\lambda}$$

and domain $\mathcal{D}(\mathcal{G}_s^f)$ equal to the space $B(\{0, 1, 2, \dots\})$, and we assume that there exists a probability measure F on $(T, \sigma(\mathcal{A}))$ such that

$$F \circ f = F^f, \quad \forall f \in \mathcal{S}$$

then there exists a probability measure P on the product space $(\{0, 1, 2, \dots\})^c$ under which:

1. *the coordinate-variable process $X := (X_C)_{C \in c}$ defined on this space is a version of the process constructed in Proposition 6.3.1; and*
2. *$\forall f \in \mathcal{S}$, the generator of the process $X^f := (X_{f(t)})_t$ at time s is $\mathcal{G}_s^f$.*

Proof: For each $f \in \mathcal{S}$, the operator $\mathcal{G}_s^f$ is the generator at time s of the semigroup $\mathcal{T}^f := (T_{st}^f)_{s < t}$ associated to the transition system $\mathcal{Q}^f := (Q_{st}^f)_{s < t}$ defined by

$$Q_{st}^f(k; \{m\}) := \frac{(F^f(t) - F^f(s))^{m-k}}{(m-k)!} \cdot \frac{\int_0^\infty e^{-\lambda F^f(t)} \lambda^m f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F^f(s)} \lambda^k f_\Lambda(\lambda) d\lambda}$$

for every $k, m \in \{0, 1, 2, \dots\}, k \leq m$. The result will follow by Theorem 4.1.3, provided that the conditions of this theorem are satisfied.

The fact that the collection $\{Q^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.6 follows as for the empirical process (see the proof of Theorem 6.1.3) since Q_{st}^f depends only on f, s, t only through $F^f(s)$ and $F^f(t)$.

It remains to prove that the collection $\{Q^f; f \in \mathcal{S}\}$ satisfies Assumption 3.3.7 with $\mu := \delta_0$. Let $\text{ord1} = \{A_0 = \emptyset', A_1, \dots, A_n\}$ and $\text{ord2} = \{A_0 = \emptyset', A'_1, \dots, A'_n\}$ be two consistent orderings of the same finite semilattice $\mathcal{A}'$ with $A_i = A'_{\pi(i)}$, $\forall i$, where π is a permutation of $\{1, \dots, n\}$ with $\pi(1) = 1$, and denote $f := f_{\mathcal{A}', \text{ord1}}$, $g := f_{\mathcal{A}', \text{ord2}}$ with $f(t_i) = \cup_{j=1}^i A_j$, $g(u_i) = \cup_{j=1}^i A'_j$ and $t_0 = u_0 = 0$. We want to prove that

$$\int_{\mathbf{R}^n} I_{\{k_1\}}(x_1) \prod_{i=2}^n I_{\{k_i\}}(x_i - x_{i-1}) Q_{t_{n-1}t_n}^f(x_{n-1}; dx_n) \dots Q_{0t_1}^f(0; dx_1) = \quad (52)$$

$$\int_{\mathbf{R}^n} I_{\{k_1\}}(y_1) \prod_{i=2}^n I_{\{k_i\}}(y_{\pi(i)} - y_{\pi(i)-1}) Q_{u_{n-1}u_n}^g(y_{n-1}; dy_n) \dots Q_{0u_1}^g(0; dy_1)$$

for every $k_i \in \{0, 1, 2, \dots\}$.

The left-hand side of (52) is equal to

$$\frac{1}{\prod_{i=1}^n k_i!} \prod_{i=1}^n (F^f(t_i) - F^f(t_{i-1}))^{k_i} \cdot \int_0^\infty e^{-\lambda F^f(t_n)} \lambda^{\sum_{i=1}^n k_i} \cdot f_\Lambda(\lambda) d\lambda \quad (53)$$

since

$$Q_{0t_1}^f(0; \{k_1\}) = \frac{(F^f(t_1))^{k_1}}{k_1!} \cdot \int_0^\infty e^{-\lambda F^f(t_1)} \lambda^{k_1} f_\Lambda(\lambda) d\lambda;$$

$$Q_{t_1t_2}^f(k_1; \{k_1 + k_2\}) = \frac{(F^f(t_2) - F^f(t_1))^{k_2}}{k_2!} \cdot \frac{\int_0^\infty e^{-\lambda F^f(t_2)} \lambda^{k_1+k_2} f_\Lambda(\lambda) d\lambda}{\int_0^\infty e^{-\lambda F^f(t_1)} \lambda^{k_1} f_\Lambda(\lambda) d\lambda};$$

etc. On the right-hand side of (52) we write

$$\prod_{i=2}^n I_{\{k_i\}}(y_{\pi(i)} - y_{\pi(i)-1}) = \prod_{i=2}^n I_{\{l_i\}}(y_i - y_{i-1})$$

with $l_i := k_{\pi^{-1}(i)}$, and therefore this side becomes

$$\frac{1}{\prod_{i=1}^n l_i!} \prod_{i=1}^n (F^g(u_i) - F^g(u_{i-1}))^{l_i} \cdot \int_0^\infty e^{-\lambda F^g(u_n)} \lambda^{\sum_{i=1}^n l_i} \cdot f_\Lambda(\lambda) d\lambda. \quad (54)$$

Finally, we observe that (53) is equal to (54) by the same argument used in the proof of Theorem 6.1.3 for the empirical process.

□

Chapter 7

Strong Markov Properties

In this chapter we will introduce two different types of strong Markov properties for set-indexed processes, which can be associated to the sharp Markov property, respectively the $\mathcal{Q}$ -Markov property. In the first section, we will introduce the notions of ‘adapted set’ and ‘optional set’ and we will define their associated σ -fields.

Most of the results of this chapter (except Section 7.3) appear in [5].

7.1 Adapted Sets and Optional Sets

In this section we will introduce adapted sets and optional sets. The ‘adapted set’ is a generalization of the classical notion of ‘stopping time’, the total ordering of the real line being replaced by set-inclusion; we are not using the term ‘stopping set’ for this notion, since this term has already been introduced in the literature ([38], [39]) for a different object. The ‘optional set’ is a generalization of the classical notion of ‘optional time’. (See Appendix B.2 for the definition and some properties of the stopping times and optional times.)

We will assume that the approximating functions g_n have the following definition:

$$g_n(B) := \cap_{D \in \mathcal{A}_n(u); B \subseteq D^0} D, \quad \forall B \in \mathcal{A}(u).$$

This is the case of many examples of indexing collections, including the lower layers of $[0, 1]^d$ (or $[-1, 1]^d$) and the rectangles $[0, z]$, $z \in [0, 1]^d$ (or $z \in [-1, 1]^d$).

From the separability from above of the indexing collection $\mathcal{A}$ we have

$$\forall D, B \in \mathcal{A}(u), D \subseteq B^0 \Rightarrow \exists n \text{ such that } g_n(D) \subseteq B^0 \quad (55)$$

since $D \subseteq B^0$ if and only if $(B^0)^c \subseteq D^c = \cup_n (g_n(D))^c$ and $(B^0)^c$ is a compact set.

Let $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ be the extension to $\mathcal{A}(u)$ of a set-indexed filtration and $(\mathcal{F}_B^r)_{B \in \mathcal{A}(u)}$ its minimal outer-continuous filtration, defined by $\mathcal{F}_D^r := \cap_n \mathcal{F}_{g_n(D)}$, $D \in \mathcal{A}(u)$. Then

$$\mathcal{F}_D^r = \cap_{B \in \mathcal{A}(u), D \subseteq B^0} \mathcal{F}_B, \quad D \in \mathcal{A}(u). \quad (56)$$

The following assumption gives the approximation from below for a set in $\mathcal{A}(u)$.

Assumption 7.1.1 *For any $B \in \mathcal{A}(u)$ there exists a monotone increasing sequence $(d_n(B))_{n \geq 1} \subseteq \mathcal{A}(u)$ such that $B^0 = \cup_n d_n(B)$, $d_n(B) \subseteq d_{n+1}(B)^0 \forall n$.*

Consequently,

$$\forall D, B \in \mathcal{A}(u), D \subseteq B^0 \Rightarrow \exists n \text{ such that } D \subseteq d_n(B)^0. \quad (57)$$

A *random set* α is a function with values in $\mathcal{A}(u)$, defined on a measurable space $(\Omega, \mathcal{F})$.

Definition 7.1.2 *A random set α is called an **adapted set** of a filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ if $\{\alpha \subseteq B\} \in \mathcal{F}_B$ for every $B \in \mathcal{A}(u)$; it is called an **optional set** of a filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ if $\{\alpha \subseteq B^0\} \in \mathcal{F}_B$ for every $B \in \mathcal{A}(u)$.*

To any adapted set α we can associate the σ -field:

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)\}.$$

To any optional set α we can associate the σ -field:

$$\mathcal{F}_\alpha^r \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B^0\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)\}.$$

The following facts are completely analogous to the classical case.

Lemma 7.1.3 (a) *If $\alpha \equiv B \in \mathcal{A}(u)$, then α is an adapted set and $\mathcal{F}_\alpha = \mathcal{F}_B$, $\mathcal{F}_\alpha^r = \mathcal{F}_B^r$.*

- (b) A random set is an optional set of the filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ if and only if it is an adapted set of the filtration $(\mathcal{F}_B^r)_{B \in \mathcal{A}(u)}$. In particular, any adapted set is an optional set.
- (c) If α is an optional set, then $\mathcal{F}_\alpha^r = \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B\} \in \mathcal{F}_B^r \forall B \in \mathcal{A}(u)\}$. In particular, if α is an adapted set then $\mathcal{F}_\alpha \subseteq \mathcal{F}_\alpha^r$.
- (d) If α is an optional set and β is an adapted set such that $\alpha \subseteq \beta^0$, then $\mathcal{F}_\alpha^r \subseteq \mathcal{F}_\beta$.
- (e) If α and β are adapted sets (respectively optional sets) such that $\alpha \subseteq \beta$, then $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ (respectively $\mathcal{F}_\alpha^r \subseteq \mathcal{F}_\beta^r$).

Proof: (a) Clear.

(b) If α is an optional set of the filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$, then $\{\alpha \subseteq B\} = \bigcap_{n \geq m} \{\alpha \subseteq g_n(B)^0\} \in \mathcal{F}_{g_m(B)} \forall m \geq 1$ and hence $\{\alpha \subseteq B\} \in \bigcap_m \mathcal{F}_{g_m(B)} = \mathcal{F}_B^r$. Conversely, if α is an adapted set of the filtration $(\mathcal{F}_B^r)_{B \in \mathcal{A}(u)}$, then $\{\alpha \subseteq B^0\} = \bigcup_n \{\alpha \subseteq d_n(B)\} \in \mathcal{F}_B$, because $\{\alpha \subseteq d_n(B)\} \in \mathcal{F}_{d_n(B)}^r \subseteq \mathcal{F}_B$ (using property (56) and the fact that $d_n(B) \subseteq B^0$).

(c) Same type as argument as (b).

(d) For each $F \in \mathcal{F}_\alpha^r$ we have $F \cap \{\beta \subseteq B\} = (F \cap \{\alpha \subseteq B^0\}) \cap \{\beta \subseteq B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)$ i.e. $F \in \mathcal{F}_\beta$.

(e) Suppose that both α and β are adapted sets. For each $F \in \mathcal{F}_\alpha$ we have $F \cap \{\beta \subseteq B\} = (F \cap \{\alpha \subseteq B\}) \cap \{\beta \subseteq B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)$ i.e. $F \in \mathcal{F}_\beta$. The case when both α and β are optional sets is similar.

□

Comment 7.1.4 If α is a discrete random set i.e., it takes on only countably many configurations, then $\{\alpha = B\} = \{\alpha \subseteq B\} \setminus \bigcup_{D \in \text{range}(\alpha), D \subset B} \{\alpha \subseteq D\}$ (where $\subset$ denotes strict inclusion) and $\{\alpha \subseteq B\} = \bigcup_{D \in \text{range}(\alpha), D \subseteq B} \{\alpha = D\}$.

Hence a discrete random set α is an adapted set (respectively an optional set) if and only if $\{\alpha = B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)$ (respectively $\{\alpha = B\} \in \mathcal{F}_B^r \forall B \in \mathcal{A}(u)$). In this case

$$\mathcal{F}_\alpha = \{F \in \mathcal{F} : F \cap \{\alpha = B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)\}$$

(respectively $\mathcal{F}_\alpha^r = \{F \in \mathcal{F} : F \cap \{\alpha = B\} \in \mathcal{F}_B^r \forall B \in \mathcal{A}(u)\}$).

The next result allows us to get around the difficulty of defining ‘progressive measurability’ in the set-indexed case.

Proposition 7.1.5 *If $X := (X_A)_{A \in \mathcal{A}}$ is an adapted monotone outer-continuous process, then*

- (a) X_α is $\mathcal{F}_\alpha$ -measurable for any discrete adapted set α ; and
- (b) X_α is $\mathcal{F}_\alpha^+$ -measurable for any optional set α .

Proof: (a) For any arbitrary $a \in \mathbf{R}$, $\{X_\alpha < a\} \cap \{\alpha = B\} = \{X_B < a\} \cap \{\alpha = B\} \in \mathcal{F}_B, \forall B \in \mathcal{A}(u)$ i.e., $\{X_\alpha < a\} \in \mathcal{F}_\alpha$.

(b) By the monotone outer-continuity of the process, $X_\alpha = \lim_n X_{g_n(\alpha)}$ and

$$\{X_\alpha < a\} = \bigcup_{N \geq m} \bigcap_{n \geq N} \{X_{g_n(\alpha)} < a\} \in \mathcal{F}_{g_m(\alpha)}, \quad \forall m \geq 1$$

using (a), since $g_n(\alpha)$ is a discrete adapted set. Hence $\{X_\alpha < a\} \in \bigcap_m \mathcal{F}_{g_m(\alpha)} = \mathcal{F}_\alpha^+$.

□

Proposition 7.1.6 *If α is an optional set then $g_n(\alpha)$ is a (discrete) adapted set and $\mathcal{F}_\alpha^+ = \bigcap_n \mathcal{F}_{g_n(\alpha)}$.*

Proof: We claim that the following relation holds:

$$\{g_n(\alpha) = B\} = \{\alpha \subseteq B^0\} \setminus \bigcup_{D \in \mathcal{A}_n(u), D \subset B} \{\alpha \subseteq D^0\} \quad \forall B \in \mathcal{A}(u) \quad (58)$$

where $\subset$ denotes strict inclusion.

To see this, suppose that $g_n(\alpha) = B$. Clearly $\alpha \subseteq g_n(\alpha)^0 = B^0$. If there were some $D \in \mathcal{A}_n(u), D \subset B$ such that $\alpha \subseteq D^0$, then by the definition of g_n we would have $g_n(\alpha) \subseteq D$, which is impossible. Conversely, suppose that $\alpha \subseteq B^0$ and $\alpha \not\subseteq D^0 \forall D \in \mathcal{A}_n(u), D \subset B$. Since $\alpha \subseteq B^0$, there exists n such that $g_n(\alpha) \subseteq B$. Since $g_n(\alpha) \in \mathcal{A}(u)$ and $\alpha \subseteq g_n(\alpha)^0$, we cannot have $g_n(\alpha) \subset B$; hence $g_n(\alpha) = B$.

From (58) it follows that $\{g_n(\alpha) = B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)$ i.e., $g_n(\alpha)$ is an adapted set.

Let us prove now that $\mathcal{F}_\alpha^\tau = \cap_n \mathcal{F}_{g_n(\alpha)}$. Since $\alpha \subseteq g_n(\alpha)^0$, we have $\mathcal{F}_\alpha^\tau \subseteq \mathcal{F}_{g_n(\alpha)}$ by Lemma 7.1.3, (d). Conversely, let $F \in \cap_n \mathcal{F}_{g_n(\alpha)}$. For each $B \in \mathcal{A}(u)$

$$F \cap \{\alpha \subseteq B\} = \cap_{n \geq m} (F \cap \{g_n(\alpha) \subseteq g_n(B)\}) \in \mathcal{F}_{g_m(B)}, \quad \forall m \geq 1.$$

Hence $F \cap \{\alpha \subseteq B\} \in \cap_m \mathcal{F}_{g_m(B)} = \mathcal{F}_B^\tau$ i.e. $F \in \mathcal{F}_\alpha^\tau$.

□

We conclude this section with some additional properties of adapted sets and optional sets.

Proposition 7.1.7 (a) *If α is an adapted set (respectively an optional set), then for every $B \in \mathcal{A}(u)$, $\{\alpha \subseteq B\} \in \mathcal{F}_\alpha$ (respectively $\{\alpha \subseteq B\} \in \mathcal{F}_\alpha^\tau$).*

(b) *If α is a discrete adapted set, then for every $B \in \mathcal{A}(u)$, $\{\alpha = B\} \in \mathcal{F}_\alpha$ and*

$$\mathcal{F}_\alpha|_{\{\alpha=B\}} = \mathcal{F}_B|_{\{\alpha=B\}}.$$

(c) *If α is an optional set, then for every $B \in \mathcal{A}(u)$, $\{\alpha = B\} \in \mathcal{F}_B^\tau$, $F \cap \{\alpha = B\} \in \mathcal{F}_B^\tau, \forall F \in \mathcal{F}_\alpha^\tau$ and*

$$\mathcal{F}_\alpha^\tau|_{\{\alpha=B\}} = \mathcal{F}_B^\tau|_{\{\alpha=B\}}.$$

Proof: (a) Let α be an adapted set and let $B \in \mathcal{A}(u)$ be arbitrary. Then $\{\alpha \subseteq B\} \cap \{\alpha \subseteq B'\} = \{\alpha \subseteq B \cap B'\} \in \mathcal{F}_{B \cap B'} \subseteq \mathcal{F}_{B'}$ for every $B' \in \mathcal{A}(u)$ i.e. $\{\alpha \subseteq B\} \in \mathcal{F}_\alpha$. The case when α is an optional set is similar.

(b) Recall the definition of a conditional σ -field $\mathcal{F}|_S := \{F \cap S; F \in \mathcal{F}\}$; note that $\mathcal{F}|_S = \{F \in \mathcal{F}; F \subseteq S\}$ if $S \in \mathcal{F}$.

Let $B, B' \in \mathcal{A}(u)$ be arbitrary. The set $\{\alpha = B\} \cap \{\alpha \subseteq B'\}$ coincides with the set $\{\alpha = B\}$ if $B \subseteq B'$ and it is $\emptyset$ otherwise; in both cases it belongs to $\mathcal{F}_{B'}$. Since this happens for any $B' \in \mathcal{A}(u)$, by the definition of $\mathcal{F}_\alpha$ it follows that $\{\alpha = B\} \in \mathcal{F}_\alpha$.

As for the second equality, consider first $F \in \mathcal{F}_\alpha$; then $F \cap \{\alpha = B\} \in \mathcal{F}_B$. Conversely, take $F \in \mathcal{F}_B, F \subseteq \{\alpha = B\}$ (this is the general form of an element of $\mathcal{F}_B|_{\{\alpha=B\}}$ since $\{\alpha = B\} \in \mathcal{F}_B$). Because $\{\alpha = B\} \in \mathcal{F}_\alpha$ too, it is enough to show that $F \in \mathcal{F}_\alpha$: let $B' \in \mathcal{A}(u)$ be arbitrary and note that the set $F \cap \{\alpha \subseteq B'\}$ coincides with F if $B \subseteq B'$ and it is $\emptyset$ otherwise; in both cases it belongs to $\mathcal{F}_{B'}$.

(c) For arbitrary $m \geq 1$ we have $\{\alpha = B\} = \bigcap_{n \geq m} \{g_n(\alpha) = g_n(B)\} \in \mathcal{F}_{g_m(B)}$. Hence $\{\alpha = B\} \in \bigcap_m \mathcal{F}_{g_m(B)} = \mathcal{F}_B$.

If $F \in \mathcal{F}_\alpha$, then $F \cap \{\alpha = B\} = \bigcap_{n \geq m} (F \cap \{g_n(\alpha) = g_n(B)\}) \in \mathcal{F}_{g_m(B)} \forall m \geq 1$. Hence $F \cap \{\alpha = B\} \in \bigcap_m \mathcal{F}_{g_m(B)} = \mathcal{F}_B$.

Finally, to prove the last equality, consider first $F \in \mathcal{F}_\alpha$. Clearly $F \cap \{\alpha = B\} \in \mathcal{F}_B$. Since $\{\alpha = B\} \in \mathcal{F}_B$ an arbitrary element of $\mathcal{F}_B|_{\{\alpha=B\}}$ is of the form $F \in \mathcal{F}_B$ with $F \subseteq \{\alpha = B\}$. Let us consider such an F . Note that $F \in \mathcal{F}_\alpha$ since for arbitrary $B' \in \mathcal{A}(u)$ we have $F \cap \{\alpha \subseteq B'\} = F \cap \{\alpha = B\} \cap \{\alpha \subseteq B'\}$ which is $\emptyset$ if $B \not\subseteq B'$ and F if $B \subseteq B'$ and in either case it is in $\mathcal{F}_{B'}$. Finally $F = F \cap \{\alpha = B\} \in \mathcal{F}_\alpha|_{\{\alpha=B\}}$.

□

7.2 The Strong Sharp Markov Property

In this section we will consider a strong Markov property for set-indexed processes which can be associated to the sharp Markov property, using the optional sets. Our approach is very similar to the one considered in [39] for stopping sets, the advantage of our method being that we do not need to assume outer-continuity of either the filtration or the process.

Let $X := (X_A)_{A \in \mathcal{A}}$ be a set-indexed process and $(\mathcal{F}_A)_{A \in \mathcal{A}}$ its minimal filtration. The following σ -fields have been introduced in [39] for any random set α :

$$\begin{aligned} \mathcal{F}_\alpha^X & \stackrel{\text{def}}{=} \sigma(\{X_A I_{\{A \subseteq \alpha\}}, I_{\{A \subseteq \alpha\}}; A \in \mathcal{A}\}) \\ \mathcal{F}_{\partial\alpha}^X & \stackrel{\text{def}}{=} \sigma(\{X_A I_{\{A \subseteq \alpha, A \not\subseteq \alpha^0\}}, I_{\{A \subseteq \alpha, A \not\subseteq \alpha^0\}}; A \in \mathcal{A}\}) \\ \mathcal{F}_{\alpha^c}^X & \stackrel{\text{def}}{=} \sigma(\{X_A I_{\{A \not\subseteq \alpha\}}, I_{\{A \not\subseteq \alpha\}}; A \in \mathcal{A}\}). \end{aligned}$$

Note: We have $\{B \subseteq \alpha\} \in \mathcal{F}_\alpha^X \cap \mathcal{F}_{\alpha^c}^X, \forall B \in \mathcal{A}(u)$: if $B = \bigcup_{i=1}^k A_i, A_i \in \mathcal{A}$, then $\{B \subseteq \alpha\} = \bigcap_{i=1}^k \{A_i \subseteq \alpha\}$.

Lemma 7.2.1 (Lemma 4.1, [39]) (a) If $\alpha \equiv B \in \mathcal{A}(u)$ then $\mathcal{F}_\alpha^X = \mathcal{F}_B, \mathcal{F}_{\partial\alpha}^X = \mathcal{F}_{\partial B}, \mathcal{F}_{\alpha^c}^X = \mathcal{F}_{B^c}$.

(b) For any random set α we have $\mathcal{F}_{\partial\alpha}^X \subseteq \mathcal{F}_\alpha^X$.

Lemma 7.2.2 (Lemma 4.6, [39]) *If α is a discrete random set, then $\{\alpha = B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)$ and $\mathcal{F}_\alpha^X|_{\{\alpha=B\}} \subseteq \mathcal{F}_\alpha^X$.*

Lemma 7.2.3 (a) *For any discrete adapted set α we have $\mathcal{F}_\alpha^X = \mathcal{F}_\alpha$.*

(b) *For any optional set α we have $\mathcal{F}_\alpha^r = \cap_n \mathcal{F}_{g_n(\alpha)}^X, \mathcal{F}_{g_{n+1}(\alpha)}^X \subseteq \mathcal{F}_{g_n(\alpha)}^X \forall n$.*

(c) *For any optional set α we have $\mathcal{F}_\alpha^X \subseteq \mathcal{F}_\alpha^r$.*

Proof: (a) To prove that $\{A \subseteq \alpha\} \in \mathcal{F}_\alpha$, note that $\{A \subseteq \alpha\} \cap \{\alpha = B\}$ is $\emptyset$ if $A \not\subseteq B$ and it is exactly $\{\alpha = B\}$ otherwise; in either case this intersection lies in $\mathcal{F}_B$. Similarly $\{X_A \in \Gamma\} \cap \{A \subseteq \alpha\} \in \mathcal{F}_\alpha$ for every $\Gamma \in \mathcal{B}(\mathbf{R})$. Conversely, let $F \in \mathcal{F}_\alpha$. Using Lemma 7.2.2, $F \cap \{\alpha = B\} \in \mathcal{F}_B|_{\{\alpha=B\}} = \mathcal{F}_\alpha^X|_{\{\alpha=B\}} \subseteq \mathcal{F}_\alpha^X$ since $\{\alpha = B\} \in \mathcal{F}_{\partial\alpha}^X \subseteq \mathcal{F}_\alpha^X$. Finally $F = \cup_{B \in \text{range}(\alpha)} (F \cap \{\alpha = B\}) \in \mathcal{F}_\alpha^X$.

(b) Using (a), $\mathcal{F}_{g_n(\alpha)}^X = \mathcal{F}_{g_n(\alpha)}$ and hence $\mathcal{F}_\alpha^r = \cap_n \mathcal{F}_{g_n(\alpha)} = \cap_n \mathcal{F}_{g_n(\alpha)}^X$.

(c) For each $A \in \mathcal{A}$ we have $I_{\{A \subseteq \alpha\}} = \lim_n I_{\{A \subseteq g_n(\alpha)\}}$ and $X_A I_{\{A \subseteq \alpha\}} = \lim_n X_A I_{\{A \subseteq g_n(\alpha)\}}$. Since $I_{\{A \subseteq g_n(\alpha)\}}$ and $X_A I_{\{A \subseteq g_n(\alpha)\}}$ are $\mathcal{F}_{g_n(\alpha)}^X$ -measurable and the σ -fields $(\mathcal{F}_{g_n(\alpha)}^X)_n$ are decreasing, it follows that the limits $I_{\{A \subseteq \alpha\}}$ and $X_A I_{\{A \subseteq \alpha\}}$ are measurable with respect to the intersection $\cap_n \mathcal{F}_{g_n(\alpha)}^X = \mathcal{F}_\alpha^r$.

□

The following theorem generalizes a classical result (see Proposition 4.1.3, [30]).

Theorem 7.2.4 (Theorem 4.7, [39]) *If $X := (X_A)_{A \in \mathcal{A}}$ is a sharp Markov process, then*

$$\mathcal{F}_\alpha^X \perp \mathcal{F}_{\alpha^c}^X \mid \mathcal{F}_{\partial\alpha}^X$$

for any discrete adapted set α .

Ideally, we would like to say that a sharp Markov process $X := (X_A)_{A \in \mathcal{A}}$ is strong Markov if for any optional set α , we have

$$\mathcal{F}_\alpha^r \perp \mathcal{F}_{\alpha^c}^X \mid \mathcal{F}_{\partial\alpha}^X \tag{59}$$

which would imply that $\mathcal{F}_\alpha^X \perp \mathcal{F}_{\alpha^c}^X \mid \mathcal{F}_{\partial\alpha}^X$, using Lemma 7.2.3.(c). However, we would rather not introduce a new terminology, since we could not find an example of a process satisfying this property.

In what follows we will see how close we can get of the desired conditional independence (59) for an arbitrary sharp Markov process. The following σ -field has been introduced in [39] for any random set α :

$$\mathcal{F}_{[\alpha, g_n(\alpha)]}^X \stackrel{\text{def}}{=} \sigma(\{X_A 1_{\{A \subseteq g_n(\alpha), A \not\subseteq \alpha^0\}}, 1_{\{A \subseteq g_n(\alpha), A \not\subseteq \alpha^0\}}; A \in \mathcal{A}\}).$$

Lemma 7.2.5 *Suppose that $g_n(A) \in \mathcal{A}_n$, $\forall A \in \mathcal{A}$, $\forall n \geq 1$. Then for every optional set α , $\mathcal{F}_{\partial g_n(\alpha)}^X \subseteq \mathcal{F}_{[\alpha, g_n(\alpha)]}^X \subseteq \mathcal{F}_{g_n(\alpha)}^X$.*

Proof: The first inclusion is valid for any random set α . To prove it, it will be enough to show that the generators of $\mathcal{F}_{\partial g_n(\alpha)}^X$ are in $\mathcal{F}_{[\alpha, g_n(\alpha)]}^X$. Note that the set $\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq g_n(\alpha)^0\}$ can be written as the difference

$$(\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}) \setminus (\{A \subseteq g_n(\alpha)^0\} \cap \{A \not\subseteq \alpha^0\})$$

and the first set $\{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}$ lies in $\mathcal{F}_{[\alpha, g_n(\alpha)]}$. As for the second set $\{A \subseteq g_n(\alpha)^0\} \cap \{A \not\subseteq \alpha^0\}$, (using (55) and the fact that $A \subseteq g_k(A)^0$) it can be written as

$$\cup_k (\{g_k(A) \subseteq g_n(\alpha)\} \cap \{A \not\subseteq \alpha^0\}) = \cup_k \cap_{m \geq k} (\{g_m(A) \subseteq g_n(\alpha)\} \cap \{g_m(A) \not\subseteq \alpha^0\})$$

which belongs to $\mathcal{F}_{[\alpha, g_n(\alpha)]}^X$ since $g_m(A) \in \mathcal{A}$. A similar argument shows that $\{X_A \in \Gamma\} \cap \{A \subseteq g_n(\alpha)\} \cap \{A \not\subseteq g_n(\alpha)^0\} \in \mathcal{F}_{[\alpha, g_n(\alpha)]}$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$.

To prove the second inclusion we will assume that α is an optional set and we will show that the generators of $\mathcal{F}_{[\alpha, g_n(\alpha)]}^X$ are in $\mathcal{F}_{g_n(\alpha)}^X$. Note that $\{A \not\subseteq \alpha^0\} = \cap_k \{g_k(A) \not\subseteq \alpha\} \in \mathcal{F}_\alpha^X \subseteq \mathcal{F}_\alpha^* \subseteq \mathcal{F}_{g_n(\alpha)}^X$ by Lemma 7.2.3. Hence $\{A \not\subseteq \alpha^0\} \cap \{A \subseteq g_n(\alpha)\} \in \mathcal{F}_{g_n(\alpha)}^X$. Finally, a similar argument shows that $\{X_A \in \Gamma\} \cap \{A \not\subseteq \alpha^0\} \cap \{A \subseteq g_n(\alpha)\} \in \mathcal{F}_{g_n(\alpha)}^X$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$.

□

Note that the proof of the first inclusion in the above lemma works only for those of our examples for which we have $g_n : \mathcal{A} \rightarrow \mathcal{A}_n$ for every $n \geq 1$. For one of our 2^d -dimensional examples, this inclusion is still valid. More precisely, we have the following result.

Lemma 7.2.6 *If $T = [-1, 1]^d$ and $\mathcal{A} = \{[0, x]; x \in T\}$, then for every optional set α , $\mathcal{F}_{\partial g_n(\alpha)}^X \subseteq \mathcal{F}_{[\alpha, g_n(\alpha)]}^X \subseteq \mathcal{F}_{g_n(\alpha)}^X$.*

Proof: We will prove only the first inclusion. Using the same argument as in the proof of the previous lemma, it is enough to show that

$$\cap_{m \geq n} \{g_m(A) \in [\alpha, g_n(\alpha)]\} \in \mathcal{F}_{[\alpha, g_n(\alpha)]}^X$$

for every $A \in \mathcal{A}$, $n \geq 1$.

Let $A \in \mathcal{A}$, $n \geq 1$ be fixed. Assume that A is in the first quadrant. Then, if we write $g_m(A) = \cup_1^{2^d} A_i^m$ where $A_i^m \in \mathcal{A}$ is the component of $g_m(A)$ that is in the i -th quadrant, we have $A \subseteq A_1^m$.

We claim that the following relation holds:

$$\cap_{m \geq n} \{g_m(A) \in [\alpha, g_n(\alpha)]\} = \{A \in [\alpha, g_n(\alpha)]\} \cap \cap_{m \geq n} \{A_1^m \in [\alpha, g_n(\alpha)]\}.$$

Assume that $g_m(A) \in [\alpha, g_n(\alpha)]$ for every $m \geq n$. Then $A \not\subseteq \alpha^0$ and $A \subseteq g_m(A) \subseteq g_n(\alpha)$. For each $m \geq n$ we have $A_1^m \not\subseteq \alpha^0$ since $A \subseteq A_1^m$ and $A \not\subseteq \alpha^0$. Also $A_1^m \subseteq g_m(A) \subseteq g_n(\alpha)$. Hence $A_1^m \in [\alpha, g_n(\alpha)]$.

Conversely, let $m \geq n$ be fixed. Since $A \not\subseteq \alpha^0$ and $A \subseteq g_m(A)$ we have $g_m(A) \not\subseteq \alpha^0$. Because $A_1^m \subseteq g_n(\alpha)$ we must have $g_m(A) \subseteq g_n(\alpha)$. Hence $g_m(A) \in [\alpha, g_n(\alpha)]$ and the proof is complete.

□

Since the σ -fields $(\mathcal{F}_{[\alpha, g_n(\alpha)]}^X)_n$ may not necessarily be monotonically decreasing, we let

$$\mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m} := \bigvee_{N \geq n} \mathcal{F}_{[\alpha, g_N(\alpha)]}^X \text{ and } \overline{\mathcal{F}}_{\partial \alpha}^X := \cap_n \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}.$$

Note that $\mathcal{F}_{\partial \alpha}^X \subseteq \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}$ since $\{A \subseteq \alpha, A \not\subseteq \alpha^0\} = \cap_{k \geq n} \{A \subseteq g_k(\alpha), A \not\subseteq \alpha^0\} \in \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}$, and similarly $\{X_A \in \Gamma, A \subseteq \alpha, A \not\subseteq \alpha^0\} \in \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}$ for any $\Gamma \in \mathcal{B}(\mathbf{R})$; hence $\mathcal{F}_{\partial \alpha}^X \subseteq \overline{\mathcal{F}}_{\partial \alpha}^X$. Then, for every optional set α

$$\mathcal{F}_{\partial g_n(\alpha)}^X \subseteq \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m} \subseteq \mathcal{F}_{g_n(\alpha)}^X. \quad (60)$$

Theorem 7.2.7 *Suppose that either $g_n(A) \in \mathcal{A}_n$, $\forall A \in \mathcal{A}$, $\forall n \geq 1$ or $T = [-1, 1]^d$ and $\mathcal{A} = \{[0, x]; x \in T\}$. If $X := (X_A)_{A \in \mathcal{A}}$ is a sharp Markov process, then for every optional set α , $\mathcal{F}_\alpha^X \perp \mathcal{F}_{\alpha^c}^X \mid \overline{\mathcal{F}}_{\partial \alpha}^X$.*

Proof: The argument is similar to the one used in the proof of Theorem 4.19, [39]. Let Y be a bounded, $\mathcal{F}_{\alpha^c}^X$ -measurable random variable. By a monotone class argument it is enough to assume that

$$Y = \prod_{i=1}^k h_i(X_{A_i} I_{\{A_i \subseteq \alpha\}}, I_{\{A_i \subseteq \alpha\}})$$

where $A_i \in \mathcal{A}$, $h_i : \mathbf{R}^2 \rightarrow \mathbf{R}$ continuous and bounded.

Let $Y_n = \prod_{i=1}^k h_i(X_{A_i} I_{\{A_i \subseteq g_n(\alpha)\}}, I_{\{A_i \subseteq g_n(\alpha)\}})$. Note that Y_n is $\mathcal{F}_{g_n(\alpha)^c}^X$ -measurable and bounded. Moreover, since $\{A_i \subseteq g_n(\alpha)\}_{n \geq 1}$ is a decreasing sequence with $\cap_n \{A_i \subseteq g_n(\alpha)\} = \{A_i \subseteq \alpha\}$ we have $Y = \lim_n Y_n$.

Using Theorem 7.2.4 and (60), $E[Y_n | \mathcal{F}_{g_n(\alpha)}^X] = E[Y_n | \mathcal{F}_{\partial g_n(\alpha)}^X] = E[Y_n | \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}]$. Since $\mathcal{F}_{\alpha}^r = \cap_n \mathcal{F}_{g_n(\alpha)}^X$ and $\overline{\mathcal{F}_{\partial \alpha}^X} = \cap_n \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}$ we get

$$E[Y | \mathcal{F}_{\alpha}^r] = \lim_n E[Y_n | \mathcal{F}_{g_n(\alpha)}^X] = \lim_n E[Y_n | \mathcal{F}_{[\alpha, g_n(\alpha)]}^{X,m}] = E[Y | \overline{\mathcal{F}_{\partial \alpha}^X}]$$

using a generalized form of the Martingale Convergence Theorem.

□

Corollary 7.2.8 *Suppose that either $g_n(A) \in \mathcal{A}_n$, $\forall A \in \mathcal{A}$, $\forall n \geq 1$ or $T = [-1, 1]^d$ and $\mathcal{A} = \{[0, x]; x \in T\}$. If $X := (X_A)_{A \in \mathcal{A}}$ is a sharp Markov process, then for every optional set α , $\mathcal{F}_{\alpha}^X \perp \mathcal{F}_{\alpha^c}^X \mid \overline{\mathcal{F}_{\partial \alpha}^X}$.*

Comment 7.2.9 The applicability of the preceding results extends to all sharp Markov processes, without any requirement for regularity of sample paths or filtrations. Therefore, this holds for all processes with independent increments, including the Brownian motion on the lower layers, which is known to be a.s. unbounded, and therefore a.s. discontinuous.

7.3 The Strong $\mathcal{Q}$ -Markov Property

In this section we will introduce the strong $\mathcal{Q}$ -Markov property in the set-indexed case. See Appendix B.2 for the strong $\mathcal{Q}$ -Markov property for processes indexed by the real line.

We begin with the definition of a set-indexed strong $\mathcal{Q}$ -Markov process.

Definition 7.3.1 (a) A transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is called **measurable** if for every flow f , the transition system $\mathcal{Q}^f := (Q_{st}^f)_{s < t}$ is measurable, where $Q_{st}^f := Q_{f(s), f(t)}$.

(b) A transition system $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ is called **weakly monotone outer-continuous** in B' if $\forall B, B' \in \mathcal{A}(u), B \subseteq B', \forall (B'_n)_n \subseteq \mathcal{A}(u)$ such that $B'_{n+1} \subseteq B'_n, \forall n$ and $\cap_n B'_n = B'$

$$Q_{BB'_n}(x; \cdot) \xrightarrow{w} Q_{BB'}(x; \cdot).$$

(c) Let $\mathcal{Q} := (Q_{BB'})_{B \subseteq B'}$ be a measurable transition system, which is weakly monotone outer-continuous in B' . A set-indexed process $X := (X_A)_{A \in \mathcal{A}}$ is called **strong $\mathcal{Q}$ -Markov with respect to a filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$** if it is adapted and monotone outer-continuous and for every adapted set $\alpha = f(\tau)$ with values on the path of a strictly increasing flow f , for every random set $\alpha' = f(\tau')$ with values on the path of the same flow f , such that $\alpha \subseteq \alpha'$ and τ' is $\mathcal{F}_\alpha$ -measurable, and for every bounded continuous function $h : \mathbf{R} \rightarrow \mathbf{R}$

$$E[h(X_{\alpha'}) | \mathcal{F}_\alpha] = \int_{\mathbf{R}} h(y) Q_{\alpha\alpha'}(X_\alpha; dy).$$

The process X is called **simply strong $\mathcal{Q}$ -Markov** if it is strong $\mathcal{Q}$ -Markov with respect to its minimal filtration.

The next result gives us the form of an adapted set which takes values on the path of a strictly increasing flow f .

Lemma 7.3.2 A random set $\alpha := f(\tau)$ with values on the path of a strictly increasing flow f , is an adapted set with respect to a filtration $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ if and only if τ is a stopping time with respect to the filtration $(\mathcal{F}_{f(t)})_t$. In this case, $\mathcal{F}_\alpha = \mathcal{F}_\tau^f$.

Proof: For necessity, we have $\{\tau \leq t\} = \{f(\tau) \subseteq f(t)\} \in \mathcal{F}_{f(t)}$ for every $t \in [0, a]$ (using the fact that f is strictly increasing). For sufficiency, let $B \in \mathcal{A}(u)$ be arbitrary and $t_B := \inf\{t \in [0, a]; f(t) \not\subseteq B\}$; note that $t \leq t_B$ if and only if $f(t) \subseteq B$. Hence $\{\alpha \subseteq B\} = \{f(\tau) \subseteq B\} = \{\tau \leq t_B\} \in \mathcal{F}_{f(t_B)} \subseteq \mathcal{F}_B$.

The equality of the two stopped σ -fields is proved similarly.

□

The next result gives the expected correspondence via flows.

Proposition 7.3.3 *Let $X := (X_A)_{A \in \mathcal{A}}$ be a monotone outer-continuous process, adapted with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$.*

The process X is strong $\mathcal{Q}$ -Markov with respect to the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if and only if for every strictly increasing right-continuous flow f , the process $X^f := (X_{f(t)})_t$ is strong $\mathcal{Q}^f$ -Markov with respect to the filtration $(\mathcal{F}_{f(t)})_t$, where $\mathcal{Q}_{st}^f := \mathcal{Q}_{f(s), f(t)}$.

Proof: For every flow f , the transition system $\mathcal{Q}^f$ is measurable and weakly right-continuous in t and the process X^f is right-continuous and adapted with respect to the filtration $(\mathcal{F}_{f(t)})_t$. The result follows using Lemma 7.3.2 and Proposition B.2.5, Appendix B.2.

□

Here are some consequences of the previous proposition.

- Examples 7.3.4**
1. A monotone outer-continuous Brownian motion with variance measure Λ is strong $\mathcal{Q}$ -Markov, where $Q_{BB'}(x; \cdot)$ is the normal distribution with mean 0 and variance $\Lambda_{B' \setminus B}$ (see Proposition B.2.3.(a), Appendix B.2).
 2. A monotone outer-continuous stationary compound Poisson process corresponding to the product measure $\Pi(dz; dx) = \Lambda(dz) \times G(dx)$ (where Λ is a finite positive measure on T and G is a probability measure on $\mathbf{R}$) is strong $\mathcal{Q}$ -Markov, where

$$Q_{BB'}(x; \Gamma) := \sum_{n \geq 0} \frac{\Lambda_{B' \setminus B}^n}{n!} G^{*n}(\Gamma - x)$$

where G^{*n} denotes $\underbrace{G * \dots * G}_{n \text{ times}}$ (see Proposition B.2.3.(b), Appendix B.2).

We will conclude this section with another type of strong $\mathcal{Q}$ -Markov property which is satisfied by the Brownian motion and the stationary compound Poisson process.

A random set α is called an $\mathcal{A}$ -valued stopping set if $\alpha(\omega) \in \mathcal{A}, \forall \omega$ and $\{\alpha \supseteq A\} \in \mathcal{F}_A, \forall A \in \mathcal{A}$; the associated σ -field is defined by

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap \{\alpha \subseteq B\} \in \mathcal{F}_B \forall B \in \mathcal{A}(u)\}.$$

Proposition 7.3.5 *If $X := (X_A)_{A \in \mathcal{A}}$ is either a monotone outer-continuous Brownian motion with variance measure Λ or a monotone outer-continuous stationary compound Poisson process corresponding to a measure $\Pi(dz; dx) := \Lambda(dz) \times G(dx)$, then for all $\mathcal{A}$ -valued stopping sets α and α' such that $\alpha \subseteq \alpha'$ and $\Lambda_{\alpha'}$ is $\mathcal{F}_\alpha$ -measurable, and for every $h \in B(\mathbf{R})$*

$$E[h(X_{\alpha'}) | \mathcal{F}_\alpha] = \int_{\mathbf{R}} h(y) Q_{\alpha\alpha'}(X_\alpha; dy)$$

Proof: The idea is essentially the same as in the classical case (see Proposition B.2.3, Appendix B.2).

(a) Let $X := (X_A)_{A \in \mathcal{A}}$ be a monotone outer-continuous Brownian motion with variance measure Λ .

For each $u \in \mathbf{R}$ fixed, the process

$$M_A := e^{iuX_A + \frac{1}{2}u^2\Lambda_A}, \quad A \in \mathcal{A}$$

is a martingale on $\mathcal{A}$: let $A, A' \in \mathcal{A}$ be such that $A \subseteq A'$; then

$$\begin{aligned} E[M_{A'} | \mathcal{F}_A] &= M_A \cdot E[e^{iuX_{A' \setminus A} + \frac{1}{2}u^2\Lambda_{A' \setminus A}} | \mathcal{F}_A] \\ &= M_A \cdot E[e^{iuX_{A' \setminus A}}] \cdot e^{\frac{1}{2}u^2\Lambda_{A' \setminus A}} \\ &= M_A \end{aligned}$$

(Note that the extension of the process M to $\mathcal{A}(u)$ is not of exponential type.)

Using the Optional Sampling Theorem for martingales indexed by directed sets (Theorem 2.15, [47]) we can conclude that for any $\mathcal{A}$ -valued stopping sets α and α' with $\alpha \subseteq \alpha'$ we have $E[M_{\alpha'} | \mathcal{F}_\alpha] = M_\alpha$; because $\Lambda_{\alpha'}$ is $\mathcal{F}_\alpha$ -measurable

$$E[e^{iuX_{\alpha'}} | \mathcal{F}_\alpha] = e^{iuX_\alpha - \frac{1}{2}u^2\Lambda_{\alpha'} \setminus \alpha} = \int_{\mathbf{R}} e^{iuy} Q_{\alpha\alpha'}(X_\alpha; dy).$$

and the result follows using Lemma 2.6.13, [43].

(b) Let $X := (X_A)_{A \in \mathcal{A}}$ be a monotone outer-continuous stationary compound Poisson process corresponding to a measure $\Pi(dz; dx) := \Lambda(dz) \times G(dx)$, where Λ is a positive measure on T and G is a probability measure on $\mathbf{R}$. Denote with φ_G the characteristic function of G .

For each $u \in \mathbf{R}$ fixed, the process

$$M_A := e^{iuX_A - \Lambda_A(\varphi_G(u)-1)}, \quad A \in \mathcal{A}$$

is a martingale on $\mathcal{A}$: if $A, A' \in \mathcal{A}$ are such that $A \subseteq A'$, then

$$\begin{aligned} E[M_{A'} | \mathcal{F}_A] &= M_A \cdot E[e^{iuX_{A' \setminus A} - \Lambda_{A' \setminus A}(\varphi_G(u)-1)} | \mathcal{F}_A] \\ &= M_A \cdot E[e^{iuX_{A' \setminus A}}] \cdot e^{-\Lambda_{A' \setminus A}(\varphi_G(u)-1)} \\ &= M_A. \end{aligned}$$

For any $\mathcal{A}$ -valued stopping sets α and α' with $\alpha \subseteq \alpha'$, we have $E[M_{\alpha'} | \mathcal{F}_\alpha] = M_\alpha$; because $\Lambda_{\alpha'}$ is $\mathcal{F}_\alpha$ -measurable

$$E[e^{iuX_{\alpha'}} | \mathcal{F}_\alpha] = e^{iuX_\alpha + \Lambda_{\alpha' \setminus \alpha}(\varphi_G(u)-1)} = \int_{\mathbf{R}} e^{iuy} Q_{\alpha\alpha'}(X_\alpha; dy)$$

and the result follows.

□

Appendix A

Conditioning

In this appendix chapter we will discuss the notions of conditional independence and conditional distribution, which are essential in the study of any Markov property. These will rely on the notion of conditional expectation, as it is defined in the classical literature (Section 34, [10]; Section 15-1, [12], etc.).

A.1 Conditional Independence

In this section we will discuss the notion of ‘conditional independence’ which is essential in the study of any type of Markov property.

We begin with a definition.

Definition A.1.1 *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sub- σ -fields of a probability space $(\Omega, \mathcal{K}, P)$. We say that $\mathcal{F}$ and $\mathcal{H}$ are **conditionally independent given $\mathcal{G}$** , and we write $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ if*

$$E[Y|\mathcal{F} \vee \mathcal{G}] = E[Y|\mathcal{G}]$$

for every bounded $\mathcal{H}$ -measurable random variable Y .

The next result gives an equivalent definition of the conditional independence; it says that the σ -fields $\mathcal{F}$ and $\mathcal{H}$ are symmetric.

Lemma A.1.2 (Lemma 1.1, [53]) *$\mathcal{F}$ and $\mathcal{H}$ are conditionally independent given $\mathcal{G}$ if and only if*

$$E[XY|\mathcal{G}] = E[X|\mathcal{G}]E[Y|\mathcal{G}]$$

for every bounded $\mathcal{F}$ -measurable random variable X and for every bounded $\mathcal{H}$ -measurable random variable Y .

The next result proved to be extremely useful in exploiting the properties of the set-Markov processes.

Lemma A.1.3 *Let $\mathcal{G}' \subseteq \mathcal{G}$ be two sub- σ -fields in a certain probability space $(\Omega, \mathcal{F}, P)$ and X, Y two random variables on this space such that Y is $\mathcal{G}$ -measurable. If $\mathcal{G} \perp \sigma(X) \mid \mathcal{G}'$, then $\mathcal{G} \perp \sigma(X, Y) \mid \mathcal{G}' \vee \sigma(Y)$. (Here X, Y are allowed to be multidimensional, that is $X : \Omega \rightarrow \mathbf{R}^d, Y : \Omega \rightarrow \mathbf{R}^k$.)*

Proof: By a monotone class argument it is enough to assume that $h(x, y) = f(x)g(y)$ where f and g are bounded and measurable. Then

$$E[f(X)g(Y)|\mathcal{G}] = g(Y)E[f(X)|\mathcal{G}] = g(Y)E[f(X)|\mathcal{G}']$$

which is $\mathcal{G}' \vee \sigma(Y)$ -measurable.

□

The following result has been occasionally used in this work.

Lemma A.1.4 *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}$ be four σ -fields in the same probability space. If $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ and $(\mathcal{F} \vee \mathcal{G}) \perp \mathcal{K} \mid \mathcal{H}$, then $\mathcal{F} \perp (\mathcal{H} \vee \mathcal{K}) \mid \mathcal{G}$.*

Proof: Let Z be a bounded $\mathcal{H} \vee \mathcal{K}$ -measurable random variable. By a monotone class argument we can suppose that $Z = XY$ where X is a bounded $\mathcal{H}$ -measurable random variable and Y is a bounded $\mathcal{K}$ -measurable random variable. We have

$$\begin{aligned} E[XY|\mathcal{F} \vee \mathcal{G}] &= E[X \cdot E[Y|\mathcal{F} \vee \mathcal{G} \vee \mathcal{H}]|\mathcal{F} \vee \mathcal{G}] \\ &= E[X \cdot E[Y|\mathcal{H}]|\mathcal{F} \vee \mathcal{G}] \\ &= E[X \cdot E[Y|\mathcal{H}]|\mathcal{G}] \end{aligned}$$

which is $\mathcal{G}$ -measurable.

□

Here is a list of other properties of the conditional independence.

- Comments A.1.5**
1. If $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ and $\mathcal{F}' \subseteq \mathcal{F}, \mathcal{H}' \subseteq \mathcal{H}$, then $\mathcal{F}' \perp \mathcal{H}' \mid \mathcal{G}$.
 2. If $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$, then $(\mathcal{F} \vee \mathcal{G}) \perp (\mathcal{H} \vee \mathcal{G}) \mid \mathcal{G}$.
 3. If $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}$ and $\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{F} \vee \mathcal{G}$, then $\mathcal{F} \perp \mathcal{H} \mid \mathcal{G}'$.

A.2 Conditional Distribution

In this section we will introduce the notion of conditional distribution and we will discuss some of its properties.

Definition A.2.1 (a) *A function $Q(x; \Gamma), x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$ is called a **transition probability** if $Q(\cdot; \Gamma)$ is a measurable function for every $\Gamma \in \mathcal{B}(\mathbf{R})$ and $Q(x; \cdot)$ is a probability measure for every $x \in \mathbf{R}$.*

(b) *Let X, Y be random variables. A transition probability Q is called a **version of the conditional distribution of Y given X** , if for every $\Gamma \in \mathcal{B}(\mathbf{R})$*

$$P[Y \in \Gamma | X = x] = Q(x; \Gamma) \quad P_X \text{ a.s.}(x)$$

where P_X is the distribution of X .

Note: By a monotone class argument, if Q is a version of the conditional distribution of Y given X , then for every $h \in B(\mathbf{R})$

$$E[h(Y) | X = x] = \int_{\mathbf{R}} h(y) Q(x; dy) \quad P_X \text{ a.s.}(x). \quad (61)$$

Remark: A version of the conditional distribution always exists, for any random variables X and Y (page 396, [12]).

The following property allows us to calculate the joint distribution of (X, Y) knowing a version of a conditional distribution.

Theorem A.2.2 (Theorem 15-3C, [12]) *If P_X is the distribution of X and Q is a version of the conditional distribution of Y given X , then $\forall \Gamma_1, \Gamma_2 \in \mathcal{B}(\mathbf{R})$*

$$P(X \in \Gamma_1, Y \in \Gamma_2) = \int_{\Gamma_1} Q(x; \Gamma_2) P_X(dx).$$

The following result is related to the Markov property and has been used several times in this work.

Proposition A.2.3 *Let X, Y, Z be some random variables such that*

$$\sigma(X) \perp \sigma(Z) \mid \sigma(Y). \quad (62)$$

If Q_1 is a version of the conditional distribution of Y given X and Q_2 is a version of the conditional distribution of Z given Y , then the transition probability Q defined by

$$Q(x; \Gamma) := \int_{\mathbf{R}} Q_2(y; \Gamma) Q_1(x; dy), \quad x \in \mathbf{R}, \Gamma \in \mathcal{B}(\mathbf{R})$$

is a version of the conditional distribution of Z given X .

Proof: Using equations (61) and (62) we have

$$\begin{aligned} P[Z \in \Gamma | X] &= P[P[Z \in \Gamma | X, Y] | X] = P[P[Z \in \Gamma | Y] | X] \\ &= P[Q_2(Y; \Gamma) | X] = \int_{\mathbf{R}} Q_2(Y; \Gamma) Q_1(X; dy). \end{aligned}$$

□

The following result was also used in this work.

Lemma A.2.4 *Let X, Y, Z be three random variables such that $\sigma(X) \perp \sigma(Z) \mid \sigma(Y)$. If we denote by $Q_{Y|X}$ a version of the conditional distribution of Y given X , then for every bounded measurable function $h : \mathbf{R}^2 \rightarrow \mathbf{R}$*

$$E[h(Y, Z) | X = x] = \int_{\mathbf{R}} E[h(y, Z) | Y = y] Q_{Y|X}(x; dy) \quad P \circ X^{-1} \text{ a.s.}(x). \quad (63)$$

Proof: Recall that by definition (see page 390 of [12]), $E[h(Y, Z) | X = x] := \alpha(x)$ is the unique $(P \circ X^{-1})$ almost surely measurable function $\alpha : \mathbf{R} \rightarrow \mathbf{R}$ which satisfies

$$\int_{\Gamma_1} \alpha(x) (P \circ X^{-1})(dx) = \int_{X^{-1}(\Gamma_1)} h(Y, Z) dP \quad \forall \Gamma_1 \in \mathcal{B}(\mathbf{R}).$$

By a monotone class argument, it is enough to consider the case when $h(y, z) = I_{\Gamma_2}(y)I_{\Gamma_3}(z)$ for some Borel sets Γ_2, Γ_3 . We will prove that the right-hand side of equation (63) verifies the definition of $E[h(Y, Z)|X = x]$. Let $\Gamma_1 \in \mathcal{B}(\mathbf{R})$ be arbitrary. Denote $P_X := P \circ X^{-1}$ and let $Q_{Z|Y}$ be a version of the conditional distribution of Z given Y . We have

$$\begin{aligned} \int_{\Gamma_1} \int_{\mathbf{R}} E[I_{\Gamma_2}(y)I_{\Gamma_3}(Z)|Y = y]Q_{Y|X}(x; dy)P_X(dx) &= \\ \int_{\Gamma_1} \int_{\Gamma_2} E[I_{\Gamma_3}(Z)|Y = y]Q_{Y|X}(x; dy)P_X(dx) &= \\ \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} Q_{Z|Y}(y; dz)Q_{Y|X}(x; dy)P_X(dx). \end{aligned}$$

This in turn is equal to

$$P(X \in \Gamma_1, Y \in \Gamma_2, Z \in \Gamma_3) = \int_{X^{-1}(\Gamma_1)} I_{\Gamma_2}(Y)I_{\Gamma_3}(Z)dP$$

using the fact that $\sigma(X) \perp \sigma(Z) \mid \sigma(Y)$.

□

Appendix B

Stochastic Processes on the Real Line

In this appendix chapter we have gathered some classical results related to the theory of stochastic processes indexed by the real line.

B.1 Lévy Processes

In this section we will introduce the Lévy processes indexed by the real line and we will discuss some of their properties.

We begin by recalling the classical *Lévy-Khintchine formula* (Theorem 2.8.1, [65]): if F is an infinitely divisible distribution on $\mathbf{R}$, then its log-characteristic function $\psi(u) := \log \int_{\mathbf{R}} e^{iuy} F(dy)$, $u \in \mathbf{R}$ can be written uniquely as

$$\psi(u) = i\gamma u - \frac{1}{2}\Lambda u^2 + \int_{\{|y|>1\}} (e^{iuy} - 1)\Pi(dy) + \int_{\{0<|y|\leq 1\}} (e^{iuy} - 1 - iuy)\Pi(dy)$$

where $\gamma \in \mathbf{R}$, $\Lambda \geq 0$, and Π is a Lévy measure on $\mathbf{R}$, i.e. $\Pi(\{0\}) = 0$ and $\int_{\mathbf{R}} (y^2 \wedge 1)\Pi(dy) < \infty$. (γ, Λ, Π) is called *the generating triplet* of F .

In particular, an infinitely divisible distribution on $\mathbf{R}$, whose log-characteristic function is given by

$$\psi(u) = \int_{\mathbf{R}} (e^{iuy} - 1)\Pi(dy)$$

where Π is a finite positive measure on $\mathbf{R}$, is called a *compound-Poisson distribution* (corresponding to the measure Π).

Definition B.1.1 A process $(X_t)_{t \in [0, a]}$ with independent increments, which is continuous in probability (i.e. $X_{t_n} \xrightarrow{P} X_t$ whenever $\lim_n t_n = t$) is called a **Lévy process**.

If $(X_t)_{t \in [0, a]}$ is a Lévy process, then:

1. the distribution F_{st} of each increment $X_t - X_s$, $s < t$ is infinitely divisible (Theorem 2.9.1, [65]); in particular, if each F_{st} is a compound Poisson distribution, then X is called a *compound Poisson process*;
2. if we denote with $(\gamma_t, \Lambda_t, \Pi_t)$ the generating triplet of the distribution F_{0t} , then the functions $\gamma_t, \Lambda_t, \Pi_t(\Gamma)$ (for every fixed Borel set Γ contained in some set $\{y; |y| > \epsilon\}, \epsilon > 0$) are continuous with respect to t (Theorem 2.9.8, [65]);
3. the process X has a cadlag modification (Theorem 19.2, [65]).

Note: If the probability distribution function F defined by $F(t) := \frac{1}{\Lambda_a} \cdot \Lambda_t, t \in [0, a]$ satisfies $\gamma_t = F(t) \cdot \gamma_a, \Pi_t = F(t) \cdot \Pi_a, \forall t \in [0, a]$, then the Lévy process $(X_t)_{t \in [0, a]}$ is said to have *stationary increments* (with respect to F), i.e. the distribution of each increment $X_t - X_s, s < t$ depends on s, t only through the value $F(t) - F(s)$.

Definition B.1.2 A Lévy process $(X_t)_{t \in [0, a]}$ for which the log-characteristic function $\psi_t(u) := \log E[e^{iuX_t}], u \in \mathbf{R}$ is differentiable in t , uniformly in u on compact intervals, is called **weakly differentiable**.

Proposition B.1.3 The generator of a weakly differentiable Lévy process with characterizing triplet $(\gamma_t, \Lambda_t, \Pi_t)$ is

$$(\mathcal{G}_s h)(x) = \gamma'_s h'(x) + \frac{1}{2} \Lambda'_s h''(x) + \int_{\{|y| > 1\}} (h(x+y) - h(x)) \Pi'_s(dy) + \int_{\{|y| \leq 1\}} (h(x+y) - h(x) - yh'(x)) \Pi'_s(dy)$$

where $\gamma'_s, \Lambda'_s, \Pi'_s(\Gamma)$ denote the derivatives at t of the functions $\gamma_t, \Lambda_t, \Pi_t(\Gamma)$ (for any fixed Borel set Γ contained in some set $\{y; |y| > \epsilon\}, \epsilon > 0$). The domain $\mathcal{D}(\mathcal{G}_s)$ contains the space $C_b^2(\mathbf{R})$ of all twice continuously differentiable functions $h : \mathbf{R} \rightarrow \mathbf{R}$ which have the property that h, h', h'' are bounded.

Proof: The homogeneous version of this result is treated in many standard textbooks (e.g on pages 291-292 of [32], or on pages 285-286 of [63]). For the inhomogeneous version we could not find a proof anywhere in the literature and therefore, for completeness, we include this proof here.

Let F_{st} be the distribution of $X_t - X_s$ and $(T_{st}h)(x) := \int_{\mathbf{R}} h(x+y)F_{st}(dy)$ the associated operator.

Let $h \in C_b^2(\mathbf{R})$ be arbitrary. Without loss of generality we can assume that h can be represented as $h(x) = \int_{\mathbf{R}} e^{iux} \tilde{h}(u) du$ for a certain integrable function $\tilde{h}$ with $\int_{\mathbf{R}} u^2 |\tilde{h}(u)| du < \infty$. (The space of functions which admit this kind of representation is dense in $C_b^2(\mathbf{R})$; the operator $\mathcal{G}_s$ is closed.) If we denote with $\psi_{st}(u)$ the log-characteristic function of $X_t - X_s$, then

$$(T_{st}h)(x) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{iu(x+y)} \tilde{h}(u) du F_{st}(dy) = \int_{\mathbf{R}} e^{iux} \tilde{h}(u) e^{\psi_{st}(u)} du$$

$$(T_{st}h)(x) - h(x) = \int_{\mathbf{R}} e^{iux} \tilde{h}(u) (e^{\psi_{st}(u)} - 1) du.$$

By definition $(\mathcal{G}_s h)(x) = \lim_{t \searrow s} \frac{(T_{st}h)(x) - h(x)}{t-s}$ uniformly in x ; hence

$$(\mathcal{G}_s h)(x) = \lim_{t \searrow s} \int_{\mathbf{R}} e^{iux} \tilde{h}(u) \cdot \frac{e^{\psi_{st}(u)} - 1}{t-s} du = \int_{\mathbf{R}} e^{iux} \tilde{h}(u) \cdot \left(\frac{\partial^+}{\partial t} e^{\psi_{st}(u)} \right) \Big|_{t=s} du =$$

$$\int_{\mathbf{R}} e^{iux} \tilde{h}(u) \left(\frac{\partial^+}{\partial t} \psi_{st}(u) \right) \Big|_{t=s} du$$

since $e^{\psi_{st}(u)}|_{t=s} = 1$. The result follows since

$$\frac{\partial^+}{\partial t} \psi_{st}(u) \Big|_{t=s} = \psi'_s(u) = i\gamma'_s u - \frac{1}{2} \Lambda'_s u^2 + \int_{\{|y|>1\}} (e^{iuy} - 1) \Pi'_s(dy) +$$

$$\int_{\{|y|\leq 1\}} (e^{iuy} - 1 - iuy) \Pi'_s(dy)$$

and $h'(x) = \int_{\mathbf{R}} iue^{iux} \tilde{h}(u) du$, $h''(x) = -\int_{\mathbf{R}} u^2 e^{iux} \tilde{h}(u) du$.

□

Here are some consequences of the previous result.

Examples B.1.4 1. If X is a Brownian motion whose variance function Λ is differentiable, then the generator of the process X at time s is

$$(\mathcal{G}_s h)(x) = \frac{1}{2} \Lambda'_s h''(x), \quad x \in \mathbf{R}.$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ contains the space $C_b^2(\mathbf{R})$.

2. If X is a compound Poisson process corresponding to the measure Π such that the function $\Pi_t(\Gamma)$ is differentiable with respect to t (for any fixed Borel set $\Gamma \subseteq \mathbf{R}$), then the generator of the process X at time s is

$$(\mathcal{G}_s h)(x) = \int_{\mathbf{R}} (h(x+y) - h(x)) \Pi'_s(dy), \quad x \in \mathbf{R}.$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ is equal to the space $B(\mathbf{R})$.

In particular, if X is a Poisson process whose variance measure Λ is differentiable, then the generator of the process X at time s is

$$(\mathcal{G}_s h)(k) = \Lambda'_s \cdot (h(k+1) - h(k)), \quad k = 0, 1, 2, \dots$$

whose domain $\mathcal{D}(\mathcal{G}_s^f)$ is the space $B(\{0, 1, 2, \dots\})$ of all bounded functions $h : \{0, 1, 2, \dots\} \rightarrow \mathbf{R}$.

B.2 The Strong Q-Markov Property

In this section we will introduce the notions of stopping time and optional time and the strong $\mathcal{Q}$ -Markov property for processes indexed by the real line.

Definition B.2.1 A random time $\tau : \Omega \rightarrow [0, a]$ is called a **stopping time** of a filtration $(\mathcal{F}_t)_{t \in [0, a]}$ if $\{\tau \leq t\} \in \mathcal{F}_t, \forall t \in [0, a]$; it is called an **optional time** of a filtration $(\mathcal{F}_t)_{t \in [0, a]}$ if $\{\tau < t\} \in \mathcal{F}_t, \forall t \in [0, a]$.

To any stopping time τ one can associate the σ -field

$$\mathcal{F}_\tau := \{F : F \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \in [0, a]\}.$$

To any optional time τ one can associate the σ -field

$$\mathcal{F}_{\tau+} := \{F : F \cap \{\tau < t\} \in \mathcal{F}_t \ \forall t \in [0, a]\}.$$

Here are some basic properties of stopping times and optional times.

- τ is an optional time of the filtration $(\mathcal{F}_t)_{t \in [0, a]}$ if and only if it is a stopping time of the filtration $(\mathcal{F}_{t+})_{t \in [0, a]}$, defined by $\mathcal{F}_{t+} := \cap_{s > t} \mathcal{F}_s$ (Corollary 1.2.4, [43]);
- if τ is a stopping time of the filtration $(\mathcal{F}_t)_{t \in [0, a]}$, then τ is $\mathcal{F}_\tau$ -measurable (Problem 1.2.13, [43]);
- if τ is an optional time, then $\mathcal{F}_{\tau+} = \{F : F \cap \{\tau \leq t\} \in \mathcal{F}_{t+} \ \forall t \in [0, a]\}$ (Problem 1.2.21, [43]);
- if $(X_t)_{t \in [0, a]}$ is a progressively measurable process, then X_τ is $\mathcal{F}_\tau$ -measurable for any stopping time τ (Proposition 1.2.18, [43]); consequently X_τ is $\mathcal{F}_{\tau+}$ -measurable for any optional time τ ;
- for any optional time τ , there exists a decreasing sequence $(\tau_n)_n$ of discrete stopping times with $\lim_n \tau_n = \tau$, such that $\mathcal{F}_{\tau+} = \cap_n \mathcal{F}_{\tau_n}$ (Problems 1.2.23, 1.2.24, [43]).

Definition B.2.2 (a) *A transition system $\mathcal{Q} := (Q_{st})_{s < t}$ is called **measurable** if for every Borel set Γ , the function $(s, t, x) \mapsto Q_{st}(x; \Gamma)$ is measurable.*

(b) *Let $\mathcal{Q} := (Q_{st})_{s, t \in [0, a]; s < t}$ be a measurable transition system. A process $(X_t)_{t \in [0, a]}$ is called **strong $\mathcal{Q}$ -Markov** with respect to a filtration $(\mathcal{F}_t)_{t \in [0, a]}$ if it is progressively measurable and $\forall t \in [0, a]$, for every stopping time $\tau \leq a - t$ and $\forall h \in B(\mathbf{R})$*

$$E[h(X_{\tau+t}) | \mathcal{F}_\tau] = \int_{\mathbf{R}} h(y) Q_{\tau, \tau+t}(X_\tau; dy).$$

We recall that an adapted right-continuous process is progressively measurable (Proposition 1.1.13, [43]). Here are some examples of strong $\mathcal{Q}$ -Markov processes.

Proposition B.2.3 (a) *A right-continuous Brownian motion with variance measure Λ is strong $\mathcal{Q}$ -Markov, where $Q_{st}(x; \cdot)$ is the normal distribution with mean 0 and variance $\Lambda((s, t])$.*

(b) *A right-continuous stationary compound Poisson process corresponding to the product measure $\Pi(dz; dx) = \Lambda(dz) \times G(dx)$ (where Λ is a finite positive measure on $[0, a]$ and G is a probability measure on $\mathbf{R}$) is strong $\mathcal{Q}$ -Markov, where*

$$Q_{st}(x; \Gamma) := \sum_{n \geq 0} \frac{\Lambda((s, t])^n}{n!} G^{*n}(\Gamma - x)$$

and G^{*n} denotes $\underbrace{G * \dots * G}_{n \text{ times}}$.

Proof: **(a)** The homogeneous version of this result is Proposition 2.6.15, [43]. For the inhomogeneous version we could not find a proof anywhere in the literature and therefore, for completeness, we include this proof here.

Let $(X_t)_{t \in [0, a]}$ be a right-continuous Brownian motion with variance measure Λ . Denote $\Lambda(t) := \Lambda((0, t])$. For each $u \in \mathbf{R}$ fixed, the process

$$M_t := e^{iuX_t + \frac{1}{2}u^2\Lambda(t)}, \quad t \in [0, a]$$

is a martingale: for each $s < t$

$$\begin{aligned} E[M_t | \mathcal{F}_s] &= M_s \cdot E[e^{iu(X_t - X_s) + \frac{1}{2}u^2\Lambda((s, t])} | \mathcal{F}_s] \\ &= M_s \cdot E[e^{iu(X_t - X_s)}] \cdot e^{\frac{1}{2}u^2\Lambda((s, t])} \\ &= M_s. \end{aligned}$$

Using the Optional Sampling Theorem (Theorem 1.3.22, [43]) we can conclude that for any $t \in [0, a]$ and for any stopping time $\tau \leq a - t$, we have $E[M_{\tau+t} | \mathcal{F}_\tau] = M_\tau$; because $\Lambda(\tau + t)$ is $\mathcal{F}_\tau$ -measurable

$$E[e^{iuX_{\tau+t}} | \mathcal{F}_\tau] = e^{iuX_\tau - \frac{1}{2}u^2\Lambda((\tau, \tau+t])} = \int_{\mathbf{R}} e^{iuy} Q_{\tau, \tau+t}(X_\tau; dy).$$

and the result follows using Lemma 2.6.13, [43].

(b) Let $(X_t)_{t \in [0, a]}$ be a stationary compound Poisson process corresponding to the product measure $\Pi(dz; dx) := \Lambda(dz) \times G(dx)$, where Λ is a positive measure on T and G is a probability measure on $\mathbf{R}$. Denote by φ_G the characteristic function of G . Denote $\Lambda(t) := \Lambda((0, t])$. For each $u \in \mathbf{R}$ fixed, the process

$$M_t := e^{iuX_t - \Lambda(t) \cdot (\varphi_G(u) - 1)}, \quad t \in [0, a]$$

is a martingale: for each $s < t$

$$\begin{aligned} E[M_t | \mathcal{F}_s] &= M_s \cdot E[e^{iu(X_t - X_s) - \Lambda((s, t]) \cdot (\varphi_G(u) - 1)} | \mathcal{F}_s] \\ &= M_s \cdot E[e^{iu(X_t - X_s)}] \cdot e^{-\Lambda((s, t]) \cdot (\varphi_G(u) - 1)} \\ &= M_s. \end{aligned}$$

For any $t \in [0, a]$ and for any stopping time $\tau \leq a - t$, we have $E[M_{\tau+t} | \mathcal{F}_\tau] = M_\tau$; because $\Lambda(\tau + t)$ is $\mathcal{F}_\tau$ -measurable

$$E[e^{iuX_{\tau+t}} | \mathcal{F}_\tau] = e^{iuX_\tau + \Lambda((\tau, \tau+t]) \cdot (\varphi_G(u) - 1)} = \int_{\mathbf{R}} e^{iuy} Q_{\tau, \tau+t}(X_\tau; dy)$$

and the result follows.

□

Definition B.2.4 A transition system $\mathcal{Q} := (Q_{st})_{s < t}$ is called **weakly right-continuous in t** if $\forall x \in \mathbf{R}, \forall s < t, \forall t_n \searrow t$, we have $Q_{st_n}(x; \cdot) \xrightarrow{w} Q_{st}(x; \cdot)$.

Proposition B.2.5 Let $\mathcal{Q} := (Q_{st})_{s < t}$ a measurable transition system, which is weakly right-continuous in t and $(X_t)_{t \in [0, a]}$ a right-continuous process, adapted with respect to a filtration $(\mathcal{F}_t)_{t \in [0, a]}$.

The process $(X_t)_{t \in [0, a]}$ is strong $\mathcal{Q}$ -Markov with respect to the filtration $(\mathcal{F}_t)_{t \in [0, a]}$ if and only if for every stopping time τ , for every random time $\tau' \leq a$ such that $\tau \leq \tau'$ and τ' is $\mathcal{F}_\tau$ -measurable, and for every bounded continuous function $h : \mathbf{R} \rightarrow \mathbf{R}$

$$E[h(X_{\tau'}) | \mathcal{F}_\tau] = \int_{\mathbf{R}} h(y) Q_{\tau, \tau'}(X_\tau; dy).$$

Proof: The homogeneous version of this result is Proposition 2.6.17, [43].

Define $\tau'_n := \tau + \frac{k}{2^n}$ on the set $A_k := \{\tau + \frac{k-1}{2^n} \leq \tau' < \tau + \frac{k}{2^n}\}$; note that $\tau'_n \searrow \tau'$. By the strong $\mathcal{Q}$ -Markov property, for every bounded continuous function $h : \mathbf{R} \rightarrow \mathbf{R}$

$$E[h(X_{\tau+\frac{k}{2^n}})|\mathcal{F}_\tau] = \int_{\mathbf{R}} h(y)Q_{\tau, \tau+\frac{k}{2^n}}(X_\tau; dy).$$

Since $X_{\tau'_n} = X_{\tau+\frac{k}{2^n}}$ on the set $A_k \in \mathcal{F}_\tau$, we have (using Problem 2.6.9, [43])

$$E[h(X_{\tau'_n})|\mathcal{F}_\tau] = E[h(X_{\tau+\frac{k}{2^n}})|\mathcal{F}_\tau] \text{ on } A_k.$$

Also $\int_{\mathbf{R}} h(y)Q_{\tau\tau'_n}(X_\tau; dy) = \int_{\mathbf{R}} h(y)Q_{\tau, \tau+\frac{k}{2^n}}(X_\tau; dy)$ on A_k and therefore

$$E[h(X_{\tau'_n})|\mathcal{F}_\tau] = \int_{\mathbf{R}} h(y)Q_{\tau\tau'_n}(X_\tau; dy) \text{ on } A_k.$$

Since $(A_k)_k$ forms a partition of the whole space Ω , we have

$$E[h(X_{\tau'_n})|\mathcal{F}_\tau] = \int_{\mathbf{R}} h(y)Q_{\tau\tau'_n}(X_\tau; dy).$$

Take the limit as $n \rightarrow \infty$ in this equality. The result follows using the Bounded Convergence Theorem for Conditional Expectations (Theorem 15-1D, [12]), the right-continuity of X and the weak right-continuity in t of $\mathcal{Q}$.

□

B.3 Metric Entropy of Classes of Sets

In this appendix section we will introduce the notions of metric entropy and metric entropy with inclusion.

Definition B.3.1 Let (E, τ) be a pseudo-metric space.

For any $\epsilon > 0$ we say that $E_\epsilon \subseteq E$ is an ϵ -net if any point in E is within ϵ -distance of a point in E_ϵ (i.e. $E \subseteq \cup_{e \in E_\epsilon} \overline{B_\epsilon(e)}$, where $\overline{B_\epsilon(e)}$ denotes the closed ball of radius ϵ centered at e .) The space E is **totally bounded** with respect to τ if for every $\epsilon > 0$ there exists a finite ϵ -net.

Denote by $N_\tau(\epsilon)$ the cardinality of the smallest ϵ -net i.e. $N_\tau(\epsilon)$ is the smallest number of closed balls of radius ϵ that cover E . The function

$$H_\tau(\epsilon) := \log N_\tau(\epsilon)$$

is called the *metric entropy* of E with respect to the metric τ .

Note that $H_\tau(\epsilon)$ takes on only positive values and it increases as ϵ gets small; also, if $\epsilon > \text{diam}(T)$ then $N_\tau(\epsilon) = 1$ and $H_\tau(\epsilon) = 0$. It is therefore only the behaviour of $H_\tau(\epsilon)$ for small ϵ that will be of interest.

The second notion of metric entropy is defined in the following context:

Definition B.3.2 Let $(T, \mathcal{B}, \mu)$ be a finite measure space and $\mathcal{A} \subseteq \mathcal{B}$ be an arbitrary class of measurable sets. The class $\mathcal{A}$ is **totally bounded with inclusion** (in $\mathcal{B}$) with respect to the measure μ if for every $\epsilon > 0$ there exists a finite collection $\mathcal{A}_\epsilon \subseteq \mathcal{B}$ such that for any $A \in \mathcal{A}$ there exists $A_\epsilon^-, A_\epsilon^+ \in \mathcal{A}_\epsilon$ with $A_\epsilon^- \subseteq A \subseteq A_\epsilon^+$ and $\mu(A_\epsilon^+ \setminus A_\epsilon^-) \leq \epsilon$.

Denote by $N_I(\epsilon)$ the cardinality of the smallest collection $\mathcal{A}_\epsilon$. The function

$$H_I(\epsilon) := \log N_I(\epsilon)$$

is called the *metric entropy with inclusion* of $\mathcal{A}$ (in $\mathcal{B}$) with respect to μ .

If $\mathcal{A}$ is totally bounded with inclusion with respect to the measure μ , then it is also totally bounded with respect to the pseudo-metric d_μ defined by

$$d_\mu(A, B) := (\mu(A \Delta B))^{1/2}, \quad A, B \in \mathcal{A}$$

(where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the two sets). The collection $\mathcal{A}_\epsilon$ becomes an ϵ -net in the pseudo-metric space $(\mathcal{A}, d_\mu)$. Therefore, the cardinality $N_{d_\mu}(\epsilon)$ of the smallest ϵ -net (with respect to d_μ) cannot exceed the cardinality of the smallest $\mathcal{A}_\epsilon$. We have

$$N_{d_\mu}(\epsilon) \leq N_I(\epsilon) \text{ and } H_{d_\mu}(\epsilon) \leq H_I(\epsilon).$$

We conclude this section with two examples of classes of sets for which we can place some bounds on the metric entropy with inclusion.

The first example is *Dudley's class* $I(d, \alpha, M)$ of sets in I_d (the d -dimensional unit cube) whose boundaries are given by functions from the sphere S^{d-1} into R^d with

derivatives of order $\leq \alpha$. (For a precise definition of $I(d, \alpha, M)$ see [27].) For this class of sets, the metric entropy with inclusion (in the class of all Borel sets of I_d) with respect to the Lebesgue measure is

$$H_I(\epsilon) = O(1/\epsilon)^{(d-1)/\alpha}.$$

(Note that $I(d, 2, M)$ give us the class of all closed convex subsets of I_d .)

The second example is introduced using the following notation. Let T be an arbitrary set and $\mathcal{A}$ a collection of subsets of T . For any finite subset $F \subseteq T$ let

$$\Delta^{\mathcal{A}}(F) := \text{card}\{A \cap F; A \in \mathcal{A}\}$$

$$m^{\mathcal{A}}(n) := \max\{\Delta^{\mathcal{A}}(F); \text{card}(F) = n\}.$$

Note that if F has n elements then $\Delta^{\mathcal{A}}(F) \leq 2^n$; hence $m^{\mathcal{A}}(n) \leq 2^n$. Set

$$V(\mathcal{A}) = \inf\{n; m^{\mathcal{A}}(n) < 2^n\} \quad (\inf \emptyset = \infty).$$

The number $V(\mathcal{A})$ is called the Vapnik-Cervonenkis (VC) index of the class $\mathcal{A}$. The class $\mathcal{A}$ is called a *Vapnik-Cervonenkis class* if its VC index is finite.

An example of such a class is the class $\mathcal{A} = \{[a, b]; a < b\}$ of rectangles in $\mathbf{R}^d$ for which $V(\mathcal{A}) = 2^d + 1$.

Lemma B.3.3 (Lemma 7.13, [28]) *Let $(T, \mathcal{B}, F)$ be a probability space and $\mathcal{A} \subseteq \mathcal{B}$ a class of sets which is totally bounded with inclusion (in $\mathcal{B}$) with respect to the probability measure F ; let $H_I(\epsilon)$ be its metric entropy with inclusion. If $\mathcal{A}$ is a Vapnik-Cervonenkis class in $\mathcal{B}$ with VC index v , then there exists a constant $K = K(v)$ (not depending on F) such that for every $0 < \epsilon \leq \frac{1}{2}$,*

$$H_I(\epsilon) \leq K - v \log \epsilon - v \log |\log \epsilon|.$$

Hence for a Vapnik-Cervonenkis class we have

$$H_I(\epsilon) = O(\log 1/\epsilon).$$

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