The Relative Nuclear Dimension of C^*-algebras, and the Nuclear Dimension of Generalised Toeplitz Algebras

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Abstract

We consider the class of *generalised Toeplitz algebras*; those C^*-algebras that can be expressed as an extension of $C(X)$ by the compact operators $\mathcal{K}$, for some compact metrizable space X. We show that one can generalise the result of Brake and Winter, that the nuclear dimension of the Toeplitz algebra is 1, to show that for any generalised Toeplitz algebra its nuclear dimension must be equal to $\dim_{\mathrm{nuc}} C(X)$. This shows that Brake and Winter's *dimension reduction phenomenon* is applicable to a much wider class of algebras. We also introduce our definition for the relative nuclear dimension of a C^*-algebra. This is a modification to the definition of nuclear dimension that requires us to factor through algebras of the form $F \otimes B$ for F finite dimensional and B some fixed algebra we are working relative to. We explore various properties satisfied by the relative nuclear dimension with a particular eye to its being a modification of nuclear dimension.

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Dedications

This thesis is dedicated to my father, Brian Gardner, who sadly could not be here to see the finished product. To my mother, Elizabeth, and to my all my friends and family who have been so supporting over the years. I could not have done this without you, thank you.

Contents

Chapter 1

Introduction

The nuclear dimension of a C^*-algebra, as defined by Winter and Zacharias [50], is a non-commutative generalisation of topological covering dimension that had been introduced to improve upon an earlier non-commutative generalisation that is the decomposition rank [30]. There have been many different non-commutative generalisations besides these, such as the stable rank [40], and real rank [14]; individually these prior notions have been highly useful though their applications are restricted somewhat. In particular decomposition rank has had great applications to the classification of stably finite, separable, simple, nuclear C^*-algebras, but it limited by the fact that an algebra has finite decomposition rank only if it is quasidiagonal [30, Thm. 4.4]. Thus the nuclear dimension was introduced as a particular notion that would be applicable to a larger class of nuclear C^*-algebras, while having strong relations to many other important regularity properties of non-commutative spaces.

For the following definition we recall that a completely positive, or cp, map $\phi : A \to B$ is one that sends positive elements of A to positive elements of B and such that each amplification map $\phi_n : M_n(A) \to M_n(B)$ is positive in the same sense for all $n \in \mathbb{N}$. If a cp map is, in addition, contractive we refer to it as a cpc map. A cpc map is order 0 if it sends orthogonal elements to orthogonal elements.

Definition 1.0.1. [50, Def. 2.1] A C^*-algebra A has nuclear dimension at most n, written $\dim_{\text{nuc}} A \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$, with F_λ finite dimensional, $\psi_\lambda : A \to F_\lambda$ cpc, and $\varphi_\lambda : F_\lambda \to A$ cp such that

i) $\varphi_\lambda \psi_\lambda(a) \to a$ uniformly on finite subsets of A,

ii) F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that for all $i = 0, \ldots, n$ the restriction $\varphi_\lambda^{(i)} := \varphi_\lambda \restriction_{F_\lambda^{(i)}}$ is cpc order 0.

Then $\dim_{\text{nuc}} A = n$ if n is the least such number for which this is true.

The intuitive picture we consider with this definition is to fit the system of approximations into diagrams of the form

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi_\lambda"'] & & A \\ & F_\lambda \arrow[ur, "\varphi_\lambda"'] & \end{tikzcd}$$

This definition differs from that of the decomposition rank, [30, Def. 3.1], in quite a mild fashion. Decomposition rank is defined the same as above except for the fact the maps up, the φ_λ, need to be contractive. A small change but one which opens up the class of nuclear algebras for which finite dimension makes sense; for example the Toeplitz algebra $\mathcal{T}$ is known to have infinite decomposition rank, since it is not quasidiagonal [30, Ex. 6.1 iv)], but its nuclear dimension was originally known to be 1 or 2 [50, Ex. 6.3].

By now the most well known relation for nuclear dimension is between finite nuclear dimension and $\mathcal{Z}$-stability, that is $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra [26]. This is represented by the Toms-Winter conjecture, [50, Conj. 9.3], that states that $\mathcal{Z}$-stability, finite nuclear dimension, and strict comparison of positive elements are equivalent for simple, separable, unital, nuclear C^*-algebras. While this is not yet known to be true in complete generality, particularly it is not known that strict comparison should imply $\mathcal{Z}$-stability, we do at least now know that for simple, separable, nuclear algebras finite nuclear dimension is equivalent to $\mathcal{Z}$-stability, [16, Thm. A]. Moreover this comes with the consequence that for simple, unital C^*-algebras there are only three possible values for the nuclear dimension, $0, 1$, or ∞, [16, Cor. C]. Then for such $\mathcal{Z}$-stable algebras the distinction between nuclear dimension 0 and 1 depends on whether the algebra is AF, since $\dim_{\mathrm{nuc}} A = 0$ if and only if A is an AF algebra [50, Rem. 2.2 iii)]. When we are dealing with non-simple algebras then it is possible that the nuclear dimension stores more information about an algebra.

One particular example for us to consider is the Toeplitz algebra, whose precise value for nuclear dimension has only recently been determined to be 1 by Brake and Winter [10]. This is a well understood algebra, to the point that determining its nuclear dimension was less about gleaning any insight into the algebra and more about presenting a "new instance of dimension reduction phenomenon" in their own words; but the proof they present, their dimension reduction, ends up being applicable to a larger class of algebras that the Toeplitz algebra sits inside. The Toeplitz algebra $\mathcal{T}$ can be realised as the extension of $C(S^1)$ by the compact operators $\mathcal{K}(\mathcal{H})$ over a separable infinite dimensional Hilbert space, that is we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0.$$

As such it is quite immediately possible to obtain a fairly narrow estimate for the nuclear dimension of $\mathcal{T}$ by [50, Prop 2.9] which states that for an extension

$$0 \to J \to E \to A \to 0$$

we obtain the following estimate for the nuclear dimensions of the algebras involved

$$\max \{\dim_{\text{nuc}} J, \dim_{\text{nuc}} A\} \leq \dim_{\text{nuc}} E \leq \dim_{\text{nuc}} J + \dim_{\text{nuc}} A + 1.$$

Since $\mathcal{K}$ is AF, thus $\dim_{\text{nuc}}\mathcal{K} = 0$ it is quite immediate to see that $1 \leq \dim_{\text{nuc}}\mathcal{T} \leq 2$ but Winter and Zacharias were unable to determine which specific value the nuclear dimension must satisfy, indeed this was left as one of their problems [50, Prob 9.2]. If we are given a compact metrizable space X then we can certainly consider extensions of $C(X)$ by $\mathcal{K}(\mathcal{H})$ which describes a class of extension algebras first discussed by Brown, Douglas, and Fillmore [12] and which, for our purposes, we have elected to describe as *generalised Toeplitz algebras*. Then for any algebra $\mathcal{T}_X$ that exists as an extension of $C(X)$ by the compact operators $\mathcal{K}(\mathcal{H})$, see definition 3.1.1, we will have a similar estimate to that of the Toeplitz algebra by [50, Prop 2.9]; specifically we will have

$$\dim_{\text{nuc}} C(X) \leq \dim_{\text{nuc}} \mathcal{T}_X \leq \dim_{\text{nuc}} C(X) + 1.$$

A natural question to consider with this class of algebras is if we are able to employ a similar argument to Brake and Winter's to be able to find the exact value of the nuclear dimension. This is the content of our chapter 3 on these generalised Toeplitz algebras, to show that we can in fact generalise, with some work, the proof in [10] to be applicable to every single algebra in this class.

If $\dim_{\text{nuc}} A = n < \infty$ then the maps up, the φ_λ, are not necessarily contractive but are n-decomposable, thus each φ_λ exists as the sum of $n+1$ cpc maps $\varphi_\lambda^{(i)}$. We can thus see that we have a bound $\|\varphi_\lambda\| \leq n+1$, by which it is still possible to show that A must satisfy the completely positive approximation property, and so be nuclear [50, Rem. 2.2 i)]. Conversely though, nuclearity of A is not enough to show $\dim_{\text{nuc}} A < \infty$. This can be seen to be an issue with the nuclear dimension in the sense that we cannot use it to glean any information about certain algebras that otherwise might be particularly nice; algebras that are simple, separable, and nuclear might not have finite nuclear dimension. An example of nuclear algebras that have infinite nuclear dimension is a particular class of Villadsen algebras, defined first by Villadsen [46] and expanded upon by Toms and Winter [45], where it is the latter paper that our own understanding comes from. Given a compact Hausdorff space X, and sequences $(n_i)_{\mathbb{N}}$, $(m_i)_{\mathbb{N}}$ of natural numbers with $n_1 = 1$ then a unital algebra A is a Villadsen algebra of the first type (a VI algebra), [45, Def 3.2], if it is an inductive limit of the form

$$A \cong \lim_{i \to \infty} (M_{m_i} \otimes C(X^{\times n_i}), \phi_i),$$

where the ϕ_i are VI maps. We refer to our section 2.3 for a more thorough understanding of these algebras. It is possible however, by choosing X to be a finite dimensional

CW-complex and by making certain requirements of the connecting maps, to construct a class of these VI algebras that are simple, unital, nuclear, but have infinite nuclear dimension. Inspired by the fact nuclear dimension is a modification to decomposition rank that is applicable to a wider range of algebras we seek to define a notion of relative nuclear dimension that is a modification to the nuclear dimension that will hopefully deal with algebras such as these Villadsen algebras that are nuclear with finite nuclear dimension. We consider nuclear dimension of A relative to some C^*-algebra B by factoring instead through algebras of the form $F_\lambda \otimes B$ for F_λ finite dimensional. We also make a slight modification to the notion of decomposability in the definition of nuclear dimension, where instead of relying on the restrictions $\varphi_\lambda^{(i)}$ being order 0 we relax this so that they need to only be approximately order 0. Our chapter 4 is devoted to providing a rigorous definition to our notion of relative nuclear dimension, and to exploring the various properties satisfied by it.

Chapter 2

Nuclear Dimension

In this section we present more explicitly some of the results for nuclear dimension and order 0 maps that we make use of or generalise in some fashion. We also look at the class of Villadsen algebras that are unital, simple, nuclear, but which do not have finite nuclear dimension.

2.1 Order 0 Maps

First recall that a map $\phi : A \to B$ is completely positive, or cp, if for all $n \in \mathbb{N}$ the map $\phi_n : M_n(A) \to M_n(B)$ defined by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ is positive in the sense that positive matrices are mapped to positive matrices. If a cp map is in addition contractive we refer to it as a cpc map.

Two elements a, b of a C^*-algebra A are said to be orthogonal, $a \perp b$ if $ab = ba = a^*b = ab^* = 0$. Note that if a and b are self-adjoint then $a \perp b$ if and only if $ab = 0$. We now present some relevant facts about order 0 maps, which are those maps that preserve orthogonality. Such maps are very much the building blocks of nuclear dimension, and indeed decomposition rank before it and exist as a special, much more useful case, of the notion of the strict order of a cpc map, see [47, Def 3.1 b)]. We refer to [49] as giving the most important structural theorems of order 0 maps.

Definition 2.1.1. [49, Def. 1.3] Let A, B be C^*-algebras and let $\varphi : A \to B$ be a cpc map. We say φ has order 0 if, for $a, b \in A$

$$a \perp b \implies \varphi(a) \perp \varphi(b).$$

Trivially any $*$-homomorphism $\varphi : A \to B$ must be order 0 since for $a \perp b$ we have $\varphi(a)\varphi(b) = \varphi(ab) = 0$. For an example of a cpc order 0 map that is not itself a $*$-homomorphism consider a compact space X with a finite open cover $(U_j)_{j=1}^n$. Let $(h_j)_{j=1}^n$ be a partition of unity subordinate to this cover. Choose a collection $(Y_k)_{k=1}^s$

of sets from this open cover that are pairwise disjoint and let h_k^Y be the corresponding functions from the partition. Then one may define a cpc order 0 map $\varphi : \mathbb{C}^s \to C(X)$ by setting $\varphi((a_k)_{k=1}^s) = \sum_{k=1}^s a_k h_k^Y$.

We then come to the main structure theorem of [49]. Recall that $\mathcal{M}(C)$ is the multiplier algebra of C and C' the commutant.

Theorem 2.1.2. [49, Thm. 2.3] Let A and B be C^*-algebras and $\varphi : A \to B$ a cpc order 0 map. Set $C := C^*(\varphi(A)) \subset B$. There are a positive element $h \in \mathcal{M}(C) \cap C'$ with $\|h\| = \|\varphi\|$ and a $*$-homomorphism

$$\pi_\varphi : A \to \mathcal{M}(C) \cap \{h'\}$$

such that

$$\pi_\varphi(a)h = \varphi(a)$$

for all $a \in A$. If A is unital then $h = \varphi(1_A)$.

For a unital C^*-algebra A one can then easily check if an order 0 map $\varphi : A \to B$ is in fact a $*$-homomorphism, since if $\varphi(1_A)$ is a projection in B then φ must be a $*$-homomorphism. To see this assume $\varphi(1_A)$ is a projection, then we have

$$\varphi(ab) = \pi_\varphi(ab)\varphi(1_A) = \pi_\varphi(a)\pi_\varphi(b)\varphi(1_A) = \pi_\varphi(a)\varphi(1_A)\pi_\varphi(b)\varphi(1_A) = \varphi(a)\varphi(b)$$

for all $a, b \in A$. In particular any ucp, that is completely positive and unital, order 0 map $A \to B$ is automatically a $*$-homomorphism.

While there are many important corollaries to this theorem the two in particular that are the most applicable to us in our context are the following.

Corollary 2.1.3. [49, Cor. 3.1] Let A, B be C^*-algebras and $\varphi : A \to B$ a cpc order 0 map. Then the map given by $\pi(\mathrm{id}_{(0,1]} \otimes a) := \varphi(a)$ for $a \in A$ induces a $*$-homomorphism $\pi : C((0,1]) \otimes A \to B$.

Conversely a $*$-homomorphism $\pi : C((0,1]) \otimes A \to B$ induces a cpc order 0 map $\varphi : A \to B$ via $\varphi(a) := \pi(\mathrm{id}_{(0,1]} \otimes a)$.

Recall that an algebra A is said to be projective if for every $*$-homomorphism $A \to \frac{B}{J}$, for $J \triangleleft B$ an ideal, there exists a lift $A \to B$ that is a $*$-homomorphism. By [34, Thm 10.1.11] and [34, Thm. 10.2.1] we have that $C((0,1]) \otimes F$ is projective for all finite dimensional F. By combining this with the above corollary we get that any cpc order 0 map $\varphi : F \to \frac{A}{J}$, for $J \triangleleft A$ an ideal, is able to be lifted to a cpc order 0 map $\phi : F \to A$.

We shall also be making use of the following result regarding the tensors or cpc order 0 maps.

Corollary 2.1.4. [49, Cor. 3.3] Let A, B, C, D be C^*-algebras and $\varphi : A \to B$, $\psi : C \to D$ be cpc order 0 maps. Then the induced cpc map

$$\varphi \otimes_\mu \psi : A \otimes_\mu C \to B \otimes_\mu D$$

has order 0, where μ denotes the minimal or maximal tensor product. In particular for any $k \in \mathbb{N}$ the amplification

$$\varphi^{(k)} : M_k(A) \to M_k(B)$$

has order 0.

2.2 Relevant Results for Nuclear Dimension

First let us restate the definition for the nuclear dimension.

Definition 2.2.1. [50, Def. 2.1] A C^*-algebra A has nuclear dimension at most n, written $\dim_{\mathrm{nuc}} A \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$, with F_λ a finite dimensional C^*-algebra, $\psi_\lambda : A \to F_\lambda$ cpc, and $\varphi_\lambda : F_\lambda \to A$ cp such that

i) $\varphi_\lambda \psi_\lambda(a) \to a$ uniformly on finite subsets of A,

ii) F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that for all $i = 0, \ldots, n$ the restriction $\varphi_\lambda^{(i)} := \varphi_\lambda \upharpoonright_{F_\lambda^{(i)}}$ is cpc order 0.

Then $\dim_{\mathrm{nuc}} A = n$ if n is the least such number for which this is true.

In [50] the triples $(F_\lambda, \psi_\lambda, \varphi_\lambda)$ are referred to as piecewise contractive n-decomposable cp approximations and the net $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ a system of piecewise contractive n-decomposable cp approximations for A. For brevity we choose not to use this full name instead referring to such objects as n-decomposable cp approximations and systems of n-decomposable cp approximations with the understanding that when we say a map $\varphi : F \to A$ is n-decomposable we specifically mean that it satisfies condition $ii)$ of the above definition.

It will often be useful for us to refer to the decomposition rank, so we present its definition.

Definition 2.2.2. [30, Def 3.1] A C^*-algebra has decomposition rank at most n, written $\mathrm{dr} A \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ with F_λ a finite dimensional C^*-algebra, $\psi_\lambda : A \to F_\lambda$ cpc and $\varphi_\lambda : F_\lambda \to A$ cpc such that

i) $\varphi_\lambda \psi_\lambda(a) \to a$ uniformly on finite subsets of A,

ii) F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that for all $i = 0, \ldots, n$ the restriction $\varphi_\lambda^{(i)} := \varphi_\lambda \restriction_{F_\lambda^{(i)}}$ is cpc order 0.

Then $\mathrm{dr} A = n$ if n is the least such number for which this is true.

Note that the only difference between decomposition rank and nuclear dimension is that decomposition rank requires that the φ_λ must be contractive itself, whereas they only need to be completely positive for nuclear dimension. This immediately shows that $\dim_{\mathrm{nuc}} A \leq \mathrm{dr} A$ for all C^*-algebras A since any system of approximations for the decomposition rank of A necessarily satisfy the conditions to be a system of approximations for the nuclear dimension of A. In general this is not an equality; for example the Toeplitz algebra has nuclear dimension 1 [10] but its decomposition rank is infinite [30, Ex. 6.1 iv)].

In practice in order to make use of the nuclear dimension rather than considering a general net of such approximations we rather focus on the equivalent definition using an $(\mathscr{F}, \varepsilon)$ net.

Proposition 2.2.3. Let A be a C^*-algebra. Then $\dim_{\mathrm{nuc}} A \leq n$ if for all $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ there exists an n-decomposable cp approximation (F, ψ, φ) for $\mathscr{F}$ within ε. That is a triple with F finite dimensional, $\psi : A \to F$ cpc, and $\varphi : F \to A$ is completely positive and n-decomposable in the sense of condition *ii)* of 2.2.1, such that $\|\varphi\psi(a) - a\| < \varepsilon$ for all $a \in \mathscr{F}$.

Proof: Assume $\dim_{\mathrm{nuc}} A \leq n$ and $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ is a system of n-decomposable cp approximations for A from definition 2.2.1. Let $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ be given. Since $\varphi_\lambda\psi_\lambda(a) \to a$ uniformly on finite sets of A we must be able to find some $\lambda \in \Lambda$ such that $\|\varphi_\lambda\psi_\lambda(a) - a\| < \varepsilon$ for all $a \in \mathscr{F}$. Then $(F_\lambda, \psi_\lambda, \varphi)$ is an n-decomposable cp approximation that works for $(\mathscr{F}, \varepsilon)$. Conversely we may create a net Λ indexed by finite sets $\mathscr{F} \subset A$ and $\varepsilon > 0$ with an order defined by $(\mathscr{F}, \varepsilon) \leq (\mathscr{G}, \eta)$ if $\mathscr{F} \subset \mathscr{G}$ and $\varepsilon > \eta$. Then if $(F_{(\mathscr{F},\varepsilon)}, \psi_{(\mathscr{F},\varepsilon)}, \varphi_{(\mathscr{F},\varepsilon)})$ is the n-decomposable cp approximation for $\mathscr{F} \subset A$ within $\varepsilon > 0$ we can see that $(F_{(\mathscr{F},\varepsilon)}, \psi_{(\mathscr{F},\varepsilon)}, \varphi_{(\mathscr{F},\varepsilon)})_\Lambda$ is a system of n-decomposable cp approximations for A. Thus $\dim_{\mathrm{nuc}} A \leq n$. ∎

Remark 2.2.4. While each φ_λ in a system of approximations for the nuclear dimension of A are not contractive we are still able to get a uniform bound for each of them. If $\dim_{\mathrm{nuc}} A = n < \infty$ and $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ is a system of n-decomposable cp approximations, then since each φ_λ is n-decomposable it can be written as a sum of $n+1$ cpc maps $\varphi_\lambda^{(i)}$, so we can see that for each φ_λ we have $\|\varphi_\lambda\| \leq n+1$. With this in mind it can still be shown that A satisfies the completely positive approximation property so that A is nuclear. Thus finite nuclear dimension implies nuclearity, see [50, Rem. 2.2 i)].

Remark 2.2.5. It is the case that we have $\dim_{\text{nuc}} A = 0$ if and only if A is AF, [50, Rem. 2.2 iii)]. If A is AF then it exists as an inductive limit $A \cong \lim_{n\to\infty} F_n$ for finite dimensional C^*-algebras F_n. Then for any finite set $\mathscr{F} \subset A$ one can find an approximation $\mathscr{F}' \subset F_n$ within a given $\varepsilon > 0$ for some n. Set $\psi : A \to F_n$ to be the extension of $\mathrm{id} : F_n \to F_n$ by Arveson's extension theorem and let $\varphi : F_n \to A$ be the induced $*$-homomorphism from the inductive system. Then (F_n, ψ, φ) would be a 0-decomposable cp approximation for $\mathscr{F}$ within ε. Conversely if $\dim_{\text{nuc}} A = 0$ there exists a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ of 0-decomposable cp approximations for A. In particular this means the φ_λ are necessarily cpc order 0. If we know each of the φ_λ are unital then by 2.1.2 they are in fact $*$-homomorphisms. If they are not unital then it is harder to do, but still possible to show that one may find a system of 0-decomposable approximations where the φ_λ are $*$-homomorphisms. Since they are part of a system of approximations it can be seen that for all finite sets $\mathscr{F} \subset A$ and $\varepsilon > 0$ there exists some λ such that $\mathrm{dist}(a, \varphi_\lambda(F_\lambda)) < \varepsilon$ for all $a \in \mathscr{F}$; thus A is AF through Bratteli's characterisation of AF algebras [11].

A particular result of note is the relation between the nuclear dimension of $C_0(X)$ and the topological covering dimension of a locally compact Hausdorff space.

Proposition 2.2.6. [50, Prop. 2.4] Let X be a locally compact Hausdorff space. Then

$$\dim_{\text{nuc}} C_0(X) = \mathrm{dr} C_0(X).$$

In particular if X is second countable we have

$$\dim_{\text{nuc}} C_0(X) = \dim X.$$

The definition for topological covering dimension of X is given by [38, Def. 3.1.1]. We won't be using this specific definition though, opting instead to use an equivalent condition of n-decomposability given by [30, Prop 1.6].

Remark 2.2.7. This result is useful to consider in the context of generalised Toeplitz algebras since the goal is to show that $\dim_{\text{nuc}} \mathcal{T}_X = \dim_{\text{nuc}} C(X)$. Particularly we are concerned with the direction $\dim_{\text{nuc}} C_0(X) \le \dim X$ which can be shown by making use of partitions of unity. We know by [30, Prop. 1.6] that $\dim X \le n$ if and only if every finite open cover of X has an open refinement that is n-decomposable. A family of subsets $(U_\lambda)_\Lambda$ of X is n-decomposable, [30, Def 1.4], if there is a decomposition $\Lambda = \bigsqcup_{i=o}^n \Lambda_j$ such that for all $i = 0, \ldots, n$ and $\lambda, \lambda' \in \Lambda_i$ we have $U_\lambda \cap U_{\lambda'} = \varnothing$. Assume $\dim X = n$ and let $\mathscr{F} \subset C(X)$ finite and $\varepsilon > 0$ be given. We may find a finite cover $(U_j)_{j=1}^r$ of X for which $|f(x) - f(y)| < \frac{2\varepsilon}{3}$ for all $x, y \in U_j$ for any $f \in \mathscr{F}$ and any j. Let $(U_{k,j})_{0\le k\le n, j\in j_k}$ be an n-decomposable refinement of this cover, thus $U_{k,j} \cap U_{k,j'} = \varnothing$ for all k and any $j, j' \in J_k$, and choose a collection of points $\{x_{k,j}\}$ such that $x_{k,j} \in U_{k,j}$. Let $(h_{k,j})_{0\le k\le n, j\in J_k}$ be a partition of unity subordinate to this refinement, and set

$x(k) = |J_k|$; we then define, for each k, a cpc order 0 map $\varphi^{(k)} : \mathbb{C}^{x(k)} \to C(X)$ by setting $\varphi^{(k)}((a_j)_{j\in J_k}) = \sum_{j\in J_k} a_j h_{k,j}$. Taking the sum of these maps $\varphi = \sum_{k=0}^{n}$ we obtain an n-decomposable cp map. Set $F = \bigoplus_{k=0}^{n} \mathbb{C}^{x(k)}$ and define $\psi : C(X) \to F$ by $\psi(f) = (f(x_{k,j}))_{0\leq k\leq n, j\in J_k}$; by construction one can see $\|\varphi\psi(f) - f\| < \varepsilon$ for all $f \in \mathscr{F}$. Hence (F, ψ, φ) is an n-decomposable cp approximation for $\mathscr{F}$ within ε and so it can be seen that $\dim_{\text{nuc}} C(X) \leq \dim X$. Our goal in using this viewpoint with generalised Toeplitz algebras will be to find a particularly good n-decomposable cover of X that allows us to define a useful system of n-decomposable cp approximations for $C(X)$.

The following permanence properties have some importance to us for the purpose of defining relative nuclear dimension.

Proposition 2.2.8. [50, Prop 2.3] Let A, B, C, D, E be C^*-algebras. Suppose $C = \lim_{i\to\infty} C_i$ is an inductive limit of C^*-algebras and D a quotient of E. Then

(i) $\dim_{\text{nuc}}(A \oplus B) = \max(\dim_{\text{nuc}} A, \dim_{\text{nuc}} B)$,

(ii) $\dim_{\text{nuc}}(A \otimes B) \leq (\dim_{\text{nuc}} A + 1)(\dim_{\text{nuc}} B + 1) - 1$; if B is AF then $\dim_{\text{nuc}}(A \otimes B) \leq \dim_{\text{nuc}} A$,

(iii) $\dim_{\text{nuc}} C \leq \liminf_{i\to\infty}(\dim_{\text{nuc}} C_i)$,

(iv) $\dim_{\text{nuc}} D \leq \dim_{\text{nuc}} E$.

The importance of these properties for us comes from the fact that they can all be derived in the same way one can derive them for the decomposition rank [30, Sec. 3], and indeed for the completely positive rank before it [47, Sec. 3]. Then as nuclear dimension is itself a mild alteration to the decomposition rank these properties provide for us some level of measuring stick by which we can consider our proposed notion of relative nuclear dimension. Since we present our relative nuclear dimension as itself a *mild* and reasonable alteration to the nuclear dimension we seek to ensure that these properties are satisfied to at least some degree.

In a similar vein we consider the following proposition, that is not one we use directly but which we take as reference in order to make a similar statement true for the relative nuclear dimension. Note that $\prod_\Lambda F_\lambda$ is the algebra of bounded functions $(f_\lambda)_\Lambda$ such that $f_\lambda \in F_\lambda$, and $\bigoplus_\Lambda F_\lambda$ those bounded functions that go to 0 at infinity.

Proposition 2.2.9. [50, Prop. 3.2] Let A be a C^*-algebra with $\dim_{\text{nuc}} A = n < \infty$. Then there is a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ of n-decomposable cp approximations for A such that the map

$$\tilde{\psi} : A \to \frac{\prod_\Lambda F_\lambda}{\bigoplus_\Lambda F_\lambda}$$

induced by the ψ_λ has order 0.

Moreover we wish to be able to show a relative nuclear dimension version of the following.

Proposition 2.2.10. [30, Prop 4.2] Let A be a separable C^*-algebra, then $\mathrm{dr}A = n < \infty$ if and only if there exists a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ of n-decomposable cpc approximations $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ for A in terms of the decomposition rank such that the map

$$\tilde{\psi} : A \to \frac{\prod_\Lambda F_\lambda}{\bigoplus_\Lambda F_\lambda}$$

induced by the ψ_λ is multiplicative.

While this is in terms of the decomposition rank this is still applicable to the nuclear dimension case, at least when $\dim_{\mathrm{nuc}} A = 0$ since then one gets a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ of 0-decomposable cp approximations for A where in particular the φ_λ are all necessarily cpc order 0. Thus we seek to obtain a result not dissimilar to this for the relative nuclear dimension when it is 0.

As we have already noted extension algebras have particularly nice estimates in relation to their short exact sequence.

Proposition 2.2.11. [50, Prop 2.9] Let $0 \to J \to E \to A \to 0$ be an exact sequence of C^*-algebras. Then,

$$\max\{\dim_{\mathrm{nuc}} A, \dim_{\mathrm{nuc}} J\} \le \dim_{\mathrm{nuc}} E \le \dim_{\mathrm{nuc}} A + \dim_{\mathrm{nuc}} J + 1.$$

Thus for all generalised Toeplitz algebras $\mathcal{T}_X$, that is the class of algebras for which we have a short exact sequence $0 \to \mathcal{K} \to \mathcal{T}_X \to C(X) \to 0$ for compact metrizable X, we immediately know that $\dim_{\mathrm{nuc}} C(X) \le \dim_{\mathrm{nuc}} \mathcal{T}_X \le \dim_{\mathrm{nuc}} C(X) + 1$. Thus our goal with these generalised Toeplitz algebras is to show that $\dim_{\mathrm{nuc}} \mathcal{T}_X \le \dim_{\mathrm{nuc}} C(X)$ so that this above proposition immediately tells us that $\dim_{\mathrm{nuc}} \mathcal{T}_X = \dim_{\mathrm{nuc}} C(X)$ for all such algebras.

We also wish to bring to attention Robert's result of the relation between the nuclear dimension of A and comparison of the Cuntz semigroup of A. The stabilised Cuntz semigroup of A, $Cu(A)$ originally introduced by Coward, Elliott, and Ivanescu, is a positively ordered abelian monoid defined as a quotient of $M_\infty(A \otimes \mathcal{K})_+$ by the Cuntz equivalence. For $a, b \in M_\infty(A \otimes \mathcal{K})_+$ we say a is Cuntz subequivalent to b, $a \precsim b$ if there exists a sequence $(v_n)_{\mathbb{N}}$ in $M_\infty(A \otimes \mathcal{K})$ such that $\|v_n b v_n^* - a\| \to 0$. If $a \precsim b$ and $b \precsim a$ then a and b are said to be Cuntz equivalent, $a \sim b$. The partial order on $Cu(A)$ is the Cuntz subequivalence, written $[a] \le [b] \iff a \precsim b$. The n-comparison property is able to be defined for any ordered abelian semigroup W and requires a further relation $\le_s$ defined on W by $x \le_s y$ if there exists $k \in \mathbb{N}$ such that $(k+1)x \le ky$. Then W has the n-comparison property if for $x, y_0, \dots, y_n \in W$ with $x \le_s y_i$ for all i then $x \le \sum_{i=0}^n y_i$. We refer to section 4.4 for a more complete treatment.

Theorem 2.2.12. [41, Thm. 1] If A has nuclear dimension n then $Cu(A)$ has the n-comparison property.

This is an important instance, for the realms of classification, of nuclear dimension having nice relations to other regularity properties of non-commutative spaces. We would expect a similar property to this for an appropriately defined notion of relative nuclear dimension.

In terms of the relation between the nuclear dimension and other nice regularity properties of non-commutative spaces the most well known, and important in terms of classification, is the Toms-Winter conjecture.

Conjecture 2.2.13. [50, Conj. 9.3] For a separable, simple, unital, infinite, and nuclear C^*-algebra A, the following are equivalent:

(i) A has finite nuclear dimension.

(ii) A is $\mathcal{Z}$-stable.

(iii) A has strict comparison of positive elements.

(iv) A has almost unperforated Cuntz semigroup.

Here $\mathcal{Z}$ is the Jiang-Su algebra [26], which is unital, simple, separable, nuclear C^*-algebra that is infinite dimensional with unique tracial state, and whose K-theory coincides with the K-theory of $\mathbb{C}$. Moreover $\mathcal{Z}$ is the unique algebra among all inductive limits of dimension drop algebras that satisfies all of these conditions. The notion of strict comparison is equivalent to that of 0-comparison in the Cuntz semigroup, and it is known that strict comparison of A is equivalent to the almost unperforation of the Cuntz semigroup $Cu(A)$, see [1, Prop. 5.2.20]. It has been shown by Winter that $(i) \implies (ii)$, [48], and more recently it has been shown by Castillejos, Evington, Tikuisis, Winter, and White that $(ii) \implies (i)$ so that we know $\mathcal{Z}$-stability and finite nuclear dimension are equivalent in this setting, [16, Thm. A]. That $(ii) \implies (iii)$ has been shown by Rørdam in the finite case [42]. The main thing missing for this to be a theorem however is a general result that $(iii) \implies (ii)$. It has been shown though in the case where A has only one tracial state [35], and further has been generalised where the tracial state space of A has compact and finite dimensional extreme boundary [29], [44]. Notably this equivalence between finite nuclear dimension and $\mathcal{Z}$-stability show that for all simple unital C^*-algebras the possible values for nuclear dimension are $0, 1$, or ∞, [16, Cor. C]. Thus it is the non-simple case that we are concerned with finding precise values of the nuclear dimension since in that setting it may well hold more information, as is the case for the class of generalised Toeplitz algebras.

2.3 Villadsen Algebras

While we know by [16, Cor. C] that all simple unital algebras must have nuclear dimension $0, 1$ or ∞ it does still lead to the possibility of unital, simple, nuclear algebras that have infinite nuclear dimension. Indeed one may see there is an entire class of Villadsen algebras that are unital, simple, and even nuclear, but which do not have finite nuclear dimension. Originally this class was shown to not be $\mathcal{Z}$-stable through use of an early version of the Toms-Winter conjecture [45, Thm. 3.4]. To be able to say much on this we should first fully define the class of algebras we are talking about. This class of algebras was originally defined by Villadsen [46], but for our purposes we refer to the paper of Toms and Winter [45] in which they expanded upon the work of Villadsen and provided simpler criteria to obtain a larger class of Villadsen algebras that would not be $\mathcal{Z}$-stable.

Definition 2.3.1. [45, Def. 3.2] Let X be a compact Hausdorff space, let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be sequences of natural numbers with $n_1 = 1$, and set $X_i = X^{\times n_i}$. A unital C^*-algebra A is said to be a Villadsen algebra of the first type (a VI algebra) if it can be written as an inductive limit algebra

$$A \cong \lim_{i\to\infty}(M_{m_i} \otimes C(X_i), \phi_i),$$

where each ϕ_i is a VI map.

The inductive system of the above definition is referred to as a *standard decomposition* for A with seed space X.

A VI map is defined as follows.

Definition 2.3.2. [45, Def. 3.1] Let X be a compact Hausdorff space and $n, m, k \in \mathbb{N}$. A unital diagonal $*$-homomorphism

$$\phi\colon\ M_n \otimes C(X) \to M_k \otimes C(X^{\times m})$$

is said to be a Villadsen map of the first type (a VI map) if each eigenvalue map is either a co-ordinate projection or has range equal to a point.

Recall that a $*$-homomorphism $\phi : C(X) \to M_n \otimes C(Y)$ is said to be diagonal if it has the form $f \mapsto \mathrm{diag}(f \circ \lambda_1, \ldots, f \circ \lambda_n)$ where each $\lambda_i : Y \to X$ is continuous. These λ_i are the eigenvalue maps of ϕ. Note also that if a $*$-homomorphism is an amplification of a diagonal map then it is itself a diagonal map; in particular in the above definition note that the given map is an amplification of a diagonal $*$-homomorphism $C(X) \to M_{\frac{k}{n}} \otimes C(X^{\times m})$, and that k must be divisible by n for this to make sense.

Let $N_{i,j}$ be the number of co-ordinate projections from $X^{\times n_j}$ to $X^{\times n_i}$ appearing as eigenvalue maps in $\phi_{i,j}$ and let $M_{i,j}$ be the number of eigenvalue maps in $\phi_{i,j}$, and it can certainly be seen that the ratio of these values must satisfy $0 \leq \frac{N_{i,j}}{M_{i,j}} \leq 1$. Of particular importance here is the ratio $\frac{N_{1,j}}{M_{1,j}}$ since by constructing a VI algebra such that the limit over j of $\frac{N_{1,j}}{M_{1,j}}$ does not vanish, alongside some other hypotheses, one is able to create an entire class of VI algebras that are simple, unital, and nuclear but that are not $\mathcal{Z}$-stable, see [45, Sec. 8].

This relies on the main result of their paper, that can be seen to be an earlier version of the Toms-Winter conjecture above.

Theorem 2.3.3. [45, Thm. 3.4] Let A be a simple VI algebra admitting a standard decomposition with seed space a finite dimensional CW complex, the following are equivalent

1. A is $\mathcal{Z}$-stable.
2. A has finite decomposition rank.
3. A has strict comparison of positive elements.

If A is a VI algebra that is simple with seed space X a finite, but not zero, dimensional CW-complex and the limit over j of $\frac{N_{1,j}}{M_{1,j}}$ does not vanish then this is equivalent to saying $\frac{N_{i,j}}{M_{i,j}} \to 1$, [45, Lemma 5.1]. Thus by [45, Lemma 4.1] this means that the Cuntz semigroup of A does not have the n-comparison property for any n, and in particular A cannot have strict comparison. Thus A is not $\mathcal{Z}$-stable and so certainly does not have finite nuclear dimension by the Toms-Winter conjecture. An alternative way to show that A does not have finite nuclear dimension is to appeal to Robert's result 2.2.12 since if $Cu(A)$ does not have the n-comparison property for any n certainly A cannot have finite nuclear dimension.

Thus when we discuss Villadsen algebras or reference the particular class of Villadsen algebras we are interested in we mean this class where the seed space X is a finite, non-zero, dimensional CW-complex that is simple and the ratio $\frac{N_{1,j}}{M_{1,j}}$ does not vanish. Such algebras are important as a motivating example for us in defining our notion of relative nuclear dimension.

It is an important point to say that the seed space is non-zero dimensional as zero-dimensional CW complexes include the single point space or indeed any finite set X. If one starts with X the single point space then by the definition of VI algebras we would be able to obtain the entire class of UHF algebras, and if X is a finite set then we instead obtain a good range of AF algebras. Certainly these algebras have finite nuclear dimension, so we need to make the distinction that we are dealing with non-zero dimensional CW-complexes.

Chapter 3

Generalised Toeplitz Algebras

In this section we present our work on generalising the work of Brake and Winter [10] that showed the nuclear dimension of the Toeplitz algebra $\mathcal{T}$ is 1. Their work can be generalised to the entire class of algebras that exist as extensions of $C(X)$ for compact metrizable X by the compact operators $\mathcal{K}$, a class of algebras we have elected to name *generalised Toeplitz algebras*

3.1 The Definition and Required Results

Definition 3.1.1. Let X be a compact metrizable space, $\mathcal{H}$ a separable infinite dimensional Hilbert space. A unital C^*-algebra $\mathcal{T}_X$ is said to be a generalised Toeplitz algebra if it exists as an extension:

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{T}_X \xrightarrow{\pi} C(X) \to 0.$$

By proposition 2.2.11 it is clear that for any generalised Toeplitz algebra $\mathcal{T}_X$ its nuclear dimension must satisfy

$$\dim_{\text{nuc}} C(X) \leq \dim_{\text{nuc}} \mathcal{T}_X \leq \dim_{\text{nuc}} C(X) + 1$$

so it is natural to ask if one may apply the reasoning of Brake and Winter [10] to be able to find a specific value for the nuclear dimension of all algebras in this class. This is the content of our main theorem that for all such possible extensions the nuclear dimension is the nicest value it can be.

Theorem 3.1.2. Let $\mathcal{T}_X$ be a generalised Toeplitz algebra for X a compact metrizable space, then $\dim_{\text{nuc}} \mathcal{T}_X = \dim_{\text{nuc}} C(X)$.

We very much follow the same lines of reasoning as in the proof in [10] which is to use some of the inherent *space* in $\mathcal{T}_X$ in order to fit cpc order 0 maps together in such a way that we create a net of cp approximations corresponding to the lower bound for the nuclear dimension.

To give some of the history of this class of algebras, the first to study this particular class of extensions were Brown, Douglas, and Fillmore in [12]. Any extension $0 \to J \to E \to A \to 0$ is naturally associated with a $*$-homomorphism $\tau : A \to \frac{\mathcal{M}(J)}{J}$, where $\mathcal{M}(J)$ is the multiplier algebra of J. This $*$-homomorphism τ is the *Busby invariant* of the extension, see [4, Sec. 15.2]. In the context of generalised Toeplitz algebras this means an extension $0 \to \mathcal{K} \to \mathcal{T}_X \to C(X) \to 0$ corresponds to a unital $*$-homomorphism $\tau : C(X) \to \mathcal{Q}$ where $\mathcal{Q} = \frac{B(\mathcal{H})}{\mathcal{K}}$ is the Calkin Algebra. Brown, Douglas, and Fillmore defined the functor Ext, where the object $\operatorname{Ext}(X)$ is defined as the equivalence classes of the corresponding $*$-homomorphisms under unitary equivalence. For two extensions $\varphi_1, \varphi_2 : C(X) \to \mathcal{Q}$ one can define a binary operation by setting $\varphi_1 + \varphi_2$ as the unital $*$-homomorphism into the Calkin algebra on $\mathcal{H} \oplus \mathcal{H}$. We define this by setting $(\varphi_1 + \varphi_2)(f) = e_{11} \otimes \varphi_1(f) + e_{22} \otimes \varphi_2(f)$ where $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This binary operation makes $\operatorname{Ext}(X)$ into a commutative semigroup, and in fact Brown, Douglas, and Fillmore [12] show that it is an abelian group. The trivial element for this group is the class of *trivial* extensions, these are the extensions that correspond to $*$-monomorphisms $\sigma : C(X) \to B(\mathcal{H})$. The inverse elements of the group are not relevant to our work so we shall not be defining them here, of note however is that Arveson [2] provided a much simpler proof of the existence of the inverses. Particularly the existence of inverses was implied by the ability to lift extensions $\varphi : C(X) \to \mathcal{Q}$ to ucp maps $\phi : C(X) \to B(\mathcal{H})$ that satisfy

1. $\phi(fg) - \phi(f)\phi(g) \in \mathcal{K}$ for all $f, g \in C(X)$,
2. $\phi(f) \in \mathcal{K}$ if and only if $f = 0$.

For such a map ϕ one can define a C^*-algebra $A = \phi(C(X)) + \mathcal{K}$ which is a generalised Toeplitz algebra in the sense of 3.1.1. While the existence of such liftings seems obvious now through the Choi-Effros lifting theorem [17, Thm. 3.10], Arveson's lifting property here was proven roughly two years before Choi-Effros and was a motivating example for their lifting theorem. This lifting of Arveson's then is our preferred way to talk about extensions of $C(X)$, and so we shall need to set some notation.

Definition 3.1.3. Let $\phi : C(X) \to B(\mathcal{H})$ be an extension, that is a ucp map which is a $*$-homomorphism modulo $\mathcal{K}$, then we shall denote by $\mathcal{T}_\phi$ the generalised Toeplitz algebra defined by ϕ.

See also the later paper of Brown, Douglas, and Fillmore [13] for a more complete look on the topic, including weaker notions of equivalence of extensions, utilising partial isometries in place of unitary equivalence. Importantly from this paper is that they make more explicitly the point that the correspondence we have between unital $*$-homomorphisms $\varphi : C(X) \to \mathcal{Q}$ and extensions of $C(X)$ by $\mathcal{K}$ in the short exact sequence sense means that if we have two unitarily equivalent unital $*$-homomorphisms $\varphi_1, \varphi_2 : C(X) \to \mathcal{Q}$ then the corresponding extension algebras are isomorphic, see [13, Rem. 1.6]. In particular what this means in our context is that if we have two extensions $\phi_1, \phi_2 : C(X) \to B(\mathcal{H})$, in the sense of 3.1.3, that are in the same equivalence class then $\mathcal{T}_{\phi_1} \cong \mathcal{T}_{\phi_2}$. Thus if $\phi : C(X) \to B(\mathcal{H})$ is an extension and $\psi : C(X) \to B(\mathcal{H})$ is a trivial extension then we have that $\mathcal{T}_\phi \cong \mathcal{T}_{\phi+\psi}$ since any trivial extension is the trivial element of the group $\mathrm{Ext}(X)$.

Returning then to the question of nuclear dimension for a generalised Toeplitz algebra $\mathcal{T}_\phi$, this can be answered positively quite easily in the case where ϕ is a trivial extension. In this case we can use [30, Prop. 5.2] that states $\mathrm{dr}\mathcal{T}_\phi = \mathrm{dr}C(X)$ in the case that there exists a quasicentral approximate unit for $\mathcal{K}$ in $\mathcal{T}_\phi$ comprised of projections, in other words if $\mathcal{T}_\phi$ is quasidiagonal. This would then necessitate $\dim_{\mathrm{nuc}}\mathcal{T}_\phi = \dim_{\mathrm{nuc}}C(X)$ since $\dim_{\mathrm{nuc}}C(X) = \mathrm{dr}C(X)$, by proposition 2.2.6, and $\dim_{\mathrm{nuc}}\mathcal{T}_\phi \leq \mathrm{dr}\mathcal{T}_\phi$. If ϕ is a trivial extension then it will be unitarily equivalent to a trivial extension that has diagonal image, which would easily commute with any approximate unit for $\mathcal{K}$ formed of projections, so that $\mathcal{T}_\phi$ is quasidiagonal giving a positive answer to our question for the trivial case.

To answer our question in general we take our cue from the proof in [10]. The proof can be seen to rely on two crucial points, the first being able to provide a cpc order 0 lift of an important cpc order 0 map, then being able to define maps relating to this lift that behave well with regards to the nuclear dimension of $C(S^1)$ and of $\mathcal{T}$. The latter point ends up being relatively simple to generalise once one is able to find the lift so it is this first point that can be considered the crux of generalising the result of Brake and Winter to the entirety of this extension class. In their paper, they rely on Lin's theorem on almost commuting normal matrices [31], which allows one to find for a matrix that is *close* to being normal a normal matrix that is *close* to it. This is used by Brake and Winter on the image of the unitary generator of $C(S^1)$ under a certain sequence of maps in order to find their cpc order 0 lift. In order to generalise we make use of another theorem of Lin's that allows us to give an answer to a more general form of this lifting problem.

To state Lin's theorem we shall first need to introduce the notion of *total K-theory* of a C^*-algebra. The total K-theory of A, denoted $\underline{K}(A)$, was first defined by Dadarlat and Loring in [21] to be the direct sum of the K-theory of A and its K-theory with coefficients. That is $\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_*(A, \mathbb{Z}_n)$ where $K_*(A, \mathbb{Z}_n) = K_0(A; \mathbb{Z}_n) \oplus K_1(A; \mathbb{Z}_n)$. The K-theory with coefficients is defined by $K_i(A; \mathbb{Z}_n) = K_i(A \otimes C_n)$ for $i = 0, 1$ where C_n is a commutative algebra such that $K_0(C_n) = \mathbb{Z}_n$ and $K_1(C_n) = 0$. This definition for the K-theory with coefficients is independent of the choice of algebra C_n but when it was introduced by Schochet in [43] the algebra C_n was specifically taken to be the mapping cone of the canonical n degree map $C_0(\mathbb{R}) \to C_0(\mathbb{R})$. Also in this paper he defines the Bockstein operations which allow one to connect these groups with exact sequences. In particular as a consequence one has the following six term exact sequence regarding the K-theory with coefficients,

$$\begin{tikzcd} K_0(A) \arrow[r, "\rho_n^0"] & K_0(A;\mathbb{Z}_n) \arrow[r, "\beta_n^0"] & K_1(A) \arrow[d, "\times n"] \\ K_0(A) \arrow[u, "\times n"] & K_1(A;\mathbb{Z}_n) \arrow[l, "\beta_n^1"] & K_1(A). \arrow[l, "\rho_n^0"] \end{tikzcd}$$

Importantly this diagram informs us that if A is a contractible C^*-algebra then the K-theory with coefficients will similarly be 0 so that the total K-theory of A will vanish.

We also require the following notation; let μ_s be the probability Borel measure induced by a state s on $C(X)$. For all $n \in \mathbb{N}$ we use τ to denote the tracial state on M_n. In particular given any cpc map $\varphi : C(X) \to M_n$ the composition $\tau\varphi$ will be a state on $C(X)$ and $\mu_{\tau\varphi}$ will be its induced probability Borel measure.

Theorem 3.1.4. [33, Theorem 2.10] Let X be a compact metric space, let $\varepsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\eta > 0$ satisfying the following: for any $\sigma > 0$ there exist $\gamma > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{H} \subset C(X)_{\text{s.a.}}$, and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following:

For any unital δ-$\mathcal{G}$-multiplicative cpc linear maps $\varphi, \psi : C(X) \to M_n$ (for some integer $n \geq 1$) for which

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \mu_{\tau\varphi}(O_r) \geq \sigma$$

for all open balls O_r of radius $r \geq \eta$ and

$$|\tau\varphi(a) - \tau\psi(a)| < \gamma \text{ for all } a \in \mathcal{H},$$

where τ is the tracial state of M_n, there is a unitary u in M_n such that

$$\|\varphi(f) - \mathrm{Ad}_u\psi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

We shall now consider what is meant by the symbol $[L]|_{\mathcal{P}}$ for finite $\mathcal{P} \subset \underline{K}(A)$ and $L : A \to B$ a δ-$\mathcal{G}$-multiplicative map. We refer to Lin's paper [32] for the definition. Denote by $\mathbf{P}(A)$ the set of projections in $M_\infty(A^\wedge) \cup \bigcup_{m\geq 1} M_\infty((A \otimes C_m)^\wedge)$ where $A^\wedge$ is the smallest unital algebra containing A. For simplicity also use L to mean the maps $L \otimes \mathrm{id}_{M_k \otimes C_m}$ for all m, k. Then for a given $p \in \mathbf{P}(A)$ it will be the case that $L(p)$ is close to a projection within $\frac{1}{2}$ for suitable δ and $\mathcal{G}$, this shall be denoted by $[L(p)]$ with the understanding that if two such projections are within $\frac{1}{2}$ of $L(p)$ then they are equivalent. This allows us to consider the corresponding map $[L] : \underline{K}(A) \to \underline{K}(B)$, however we need to be careful to check where this is well defined. We can find a finite set $\mathcal{Q}_{\delta,\mathcal{G}} \subset \mathbf{P}(A)$ such that $[L](x)$ is well defined for all $x \in \overline{\mathcal{Q}}_{\delta,\mathcal{G}}$, which is the image of $\mathcal{Q}_{\delta,\mathcal{G}}$ in $\underline{K}(A)$. Here well defined means that for $p_1, p_2 \in \mathcal{Q}_{\delta,\mathcal{G}}$ with $[p_1] = [p_2]$ then $[L(p_1)] = [L(p_2)]$, and that for $p_1, p_2, p_1 \oplus p_2 \in \mathcal{Q}_{\delta,\mathcal{G}}$ we have $[L(p_1 \oplus p_2)] = [L(p_1)] + [L(p_2)]$. We use $[L]|_{\mathcal{P}}$ to denote the usual notion of restricting the map on total K-theory to a subset $\mathcal{P} \subset \underline{K}(A)$, though this is only well defined when $\mathcal{P} \subset \overline{\mathcal{Q}}_{\delta,\mathcal{G}}$. It is this context being used in Lin's theorem 3.1.4.

As noted, for a contractible C^*-algebra A we will have that $\underline{K}(A) = 0$ and it is not too hard to see that for the unitisation $A^\wedge$ we will have that $\underline{K}(A^\wedge) \cong \underline{K}(\mathbb{C})$. So that for a δ-$\mathcal{G}$-multiplicative map $L : A^\wedge \to B$ its image in terms of the total K-theory would depend only on the image of the unit $(0,1)$ in $A^\wedge$ under L. Thus for any two unital δ-$\mathcal{G}$-multiplicative maps $L_1, L_2 : A^\wedge \to B$ we will have $[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$ whenever this is well defined. For our purposes we are going to be applying Lin's theorem 3.1.4 to the cone algebra $C((0,1] \times X)$ for compact Hausdorff metrizable X. More correctly, since Lin's theorem 3.1.4 only applies to maps to compact spaces, we will be working with the unitisation $C((0,1] \times X)^\wedge$ since this corresponds to the continuous functions on the one-point compactification of $(0,1] \times X$. Then since the cone $C((0,1] \times X)$ is a contractible C^*-algebra we know that $[\psi]|_{\mathcal{P}} = [\phi]|_{\mathcal{P}}$ for any two unital δ-$\mathcal{G}$-multiplicative maps $\psi, \phi : C((0,1] \times X)^\wedge \to B$. The consequence of this being that we can ignore the total K-theory conditions of Lin's theorem 3.1.4 since the maps will always agree on the total K-theory in our particular context.

Ignoring the total K-theory condition Lin's theorem 3.1.4 then boils down to two major hypotheses, the tracial condition between the two maps, and a condition on the measures generated by the maps. What follows is a series of lemmas that shall allow us to satisfy these conditions and use Lin's theorem 3.1.4 to create a cpc order 0 lift of a cpc order 0 map. In order to actually do this, rather than being concerned with looking at two given maps, the goal will be to construct a $*$-homomorphism that by Lin's theorem 3.1.4 will be close to a map we care about, and then by the correspondence of maps on the cone this will give us a cpc order 0 map that is a lift of what we started with. Note in particular lemma 3.1.6 shows that it is always possible to construct a $*$-homomorphism that will be tracially close to any map in the desired sense so that the main difficulty of applying Lin's theorem 3.1.4 is showing when the measure condition is satisfied.

For all of the following results we denote by τ the tracial state on M_n for all $n \in \mathbb{N}$.

Lemma 3.1.5. Let $\omega = \sum_{s=1}^{S} \frac{k_s}{n} \delta_{x_s}$ be a convex combination of point mass measures on a compact space X for a given $n \in \mathbb{N}$. Then there exists a $*$-homomorphism $\psi : C(X) \to M_n$ such that $\tau\psi(f) = \int_X f(x)d\omega(x)$ for all $f \in C(X)$.

Proof: Let $e_x : C(X) \to \mathbb{C}$ be the $*$-homomorphism that is evaluation at the point $x \in X$. Since ω is a convex combination of point mass measures we set a $*$-homomorphism ψ to be the diagonal matrix in M_n comprised of e_{x_s} appearing k_s times for all s. ∎

Lemma 3.1.6. Let X be a compact metric space. Given finite $\mathcal{H} \subset C(X)_{s.a.}$ and $\gamma > 0$, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that for all cpc maps $\varphi : C(X) \to M_n$ there exists a $*$-homomorphism $\psi : C(X) \to M_n$ such that

$$|\tau\varphi(a) - \tau\psi(a)| < \gamma \text{ for all } a \in \mathcal{H}.$$

Proof: For each $a \in \mathcal{H}$ we we may find a $\frac{\gamma}{4}$-net $\{\lambda_a^1, \ldots, \lambda_a^s\}$ of the interval $[-\|a\|, \|a\|]$. We then consider the finite collection of tuples $\lambda^j = (\lambda_a^{i_j})_{a\in\mathcal{H}}$ that describes all possible combinations of these λ_a^i. Then for any $n \in \mathbb{N}$ and any cpc map $\varphi : C(X) \to M_n$ there exists a j such that φ is close to λ^j in the sense that $\tau\varphi(a) \approx_{\frac{\gamma}{4}} \lambda_a^{i_j}$ for all $a \in \mathcal{H}$. It may be the case that some λ^j will not have a cpc map close to it in this sense. We shall denote by I the index set of those j for which λ^i does have a cpc map close to it. For each $j \in I$ we arbitrarily pick a cpc map $\varphi_j : C(X) \to M_{n_j}$ which is close to λ^j in this sense. Using the weak density of the convex hull of point mass measures in the space of regular Borel measures we may find, for each $j \in I$, a measure $\omega_j' = \sum_{s=1}^{S_j} \sigma_s \delta_{x_s}$ such that

$$\int_X a(x) d\mu_{\tau\varphi_j}(x) \approx_{\frac{\gamma}{4}} \int_X a(x) d\omega_j'(x)$$

for all $a \in \mathcal{H}$. Let $J = \max_{j\in I} \{S_j\}$ and $H = \max_{a\in\mathcal{H}} \|a\|$, and set N to be the smallest natural number for which $\frac{1}{N} < \frac{\gamma}{4JH}$. Then we claim that for all $n \geq N$ and for all $j \in I$ we may find a convex combination of point mass measures $\omega_{(j,n)} = \sum_{s=1}^{S_j} \frac{k_s}{n} \delta_{x_s}$ such that

$$\int_X a(x) d\omega_{(j,n)}(x) \approx_{\frac{\gamma}{4}} \int_X a(x) d\omega_j'(x)$$

for all $a \in \mathcal{H}$. To see why this is true fix $n \geq N$ and $j \in I$ and consider $s \in \{1, \ldots, S_j\}$. Let q_s be the largest natural number so that $\frac{q_s}{n} \leq \sigma_s$, hence $q_s + 1$ is the smallest natural number such that $\frac{q_s+1}{n} \geq \sigma_s$. Note that σ_s is necessarily within $\frac{\gamma}{4JH}$ of both $\frac{q_s}{n}$ and $\frac{q_s+1}{n}$. If we do this for all $s \in \{1, \ldots, S_j\}$ then we will have natural numbers such that $\sum_{s=1}^{S_j} q_s \leq n$ and $\sum_{s=1}^{S_j} (q_s + 1) \geq n$. For each s we are able to choose either $k_s = q_s$ or $k_s = q_s + 1$ such that $\sum_{s=1}^{S_j} k_s = n$ so that $\omega_{(j,n)} = \sum_{s=1}^{S_j} \frac{k_s}{n} \delta_{x_s}$ will be a convex combination of point mass measures. By construction either choice for k_s for each s will satisfy $\frac{k_s}{n} \approx_{\frac{\gamma}{4JH}} \sigma_s$ so that $\omega_{(j,n)}$ satisfies the required integral condition. Let $\psi_{(j,n)} : C(X) \to M_n$ be the $*$-homomorphism that corresponds to $\omega_{(j,n)}$ as according to lemma 3.1.5. Thus for all $n \geq N$ and all cpc maps $\varphi : C(X) \to M_n$ we can find a $j \in I$ such that φ is within $\frac{\gamma}{4}$ of λ^j and so that $\psi_{(j,n)}$ is a $*$-homomorphism that satisfies

$$\tau\varphi(a) \approx_\gamma \tau\psi_{(j,n)}(a) \text{ for all } a \in \mathcal{H}$$

as required. ∎

Remark 3.1.7. In the above proof, that we are dealing with cpc maps $\varphi : C(X) \to M_n$ has very little impact. Particularly we are more concerned with the state $\tau\varphi$ we obtain by composing with the tracial state τ of M_n than the map φ itself. Thus lemma 3.1.6 can be made more general by replacing cpc maps $\varphi : C(X) \to M_n$ with states $s : C(X) \to \mathbb{C}$. In particular the statement can be changed to read:

Given finite $\mathcal{H} \subset C(X)_{s.a.}$ and $\gamma > 0$, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that for all states $s : C(X) \to \mathbb{C}$ there exists a $*$-homomorphism $\psi : C(X) \to M_n$ such that

$$|s(a) - \tau\psi(a)| < \gamma \text{ for all } a \in \mathcal{H}.$$

The proof would follow the same lines, replacing instances of $\tau\varphi$ with s.

The proof of our main theorem relies heavily on ucp maps into the product of matrix algebras as in the following lemma. This lemma then gives us a condition for which the measure hypothesis of Lin's theorem is satisfied in this particular context.

Lemma 3.1.8. Let X be a compact metrizable space and $\phi : C(X) \to \prod_{\mathbb{N}} M_{k_n}$ a ucp map. If for all positive $f \in C(X)$ we have $\liminf_{n\to\infty} \tau\phi_n(f) > 0$ then for all $\eta > 0$ there exists an $N \in \mathbb{N}$ and $\sigma > 0$ such that for all $n \geq N$ we have

$$\mu_{\tau\phi_n}(B) > \sigma$$

for all open balls B of radius at least η.

Proof: By compactness we may find a finite number of points x_i such that the balls $B(x_i, \frac{\eta}{3})$ of radius $\frac{\eta}{3}$ cover X. Then for each ball B of radius at least η there must be an i such that $B(x_i, \frac{\eta}{3}) \subset B$. For each i we may find a positive function h_i which is 1 on the ball $B(x_i, \frac{\eta}{3} - \frac{\eta}{3^{1000}})$ and 0 outside the ball $B(x_i, \frac{\eta}{3})$. Let $\sigma' = \min_i \liminf_{n\to\infty} \tau\phi_n(h_i) > 0$ and set $\sigma = \frac{\sigma'}{2}$ so that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\inf_{m\geq n} \tau\phi_m(h_i) > \sigma$. Then for any ball B of radius at least η let i be such that $B(x_i, \frac{\eta}{3}) \subset B$ then certainly $\mu_{\tau\phi_n}(B) \geq \tau\phi_n(h_i)$ for all n, hence $\inf_{m\geq N} \mu_{\tau\phi_m}(B) > \sigma$ as required. ∎

Remark 3.1.9. Similar to lemma 3.1.6 the above proof does not rely on the structure of the ucp map $\phi : C(X) \to \prod_{\mathbb{N}} M_{k_n}$ outside of the fact we can compose the component maps with the respective tracial states τ for each M_{k_n}. If we start with a sequence of states $(s_n)_{\mathbb{N}}$ on $C(X)$ we can change the statement of lemma 3.1.8 to read:

If for all positive $f \in C(X)$ we have $\liminf_{n\to\infty} s_n(f) > 0$ then for all $\eta > 0$ there exists an $N \in \mathbb{N}$ and $\sigma > 0$ such that for all $n \geq N$ we have

$$\mu_{s_n}(B) > \sigma$$

for all open balls B of radius at least η. Again the proof is the same, requiring the replacement of instances of $\tau\phi_n$ with s_n.

With these two lemmas we are then able to use Lin's theorem 3.1.4 to get the following result that we will use in the proof of theorem 3.2.1

Theorem 3.1.10. Let Y be a locally compact Hausdorff second countable contractible space and $\phi : C_0(Y) \to \prod_{\mathbb{N}} M_{k_n}$ a cpc lift of a $*$-homomorphism $C_0(Y) \to \frac{\prod_{\mathbb{N}} M_{k_n}}{\bigoplus_{\mathbb{N}} M_{k_n}}$. If for all positive $f \in C_0(Y)$ we have that $\liminf_{n\to\infty} \tau\phi_n(f) > 0$ then there exists a $*$-homomorphism $\psi : C_0(Y) \to \prod_{\mathbb{N}} M_{k_n}$ such that $q\phi = q\psi$.

Proof: Let X be the one point compactification of Y so that $C(X)$ is isomorphic to the unitisation of $C(Y)$, and let $\tilde{\phi}$ be the ucp map obtained from ϕ. Since Y is locally compact, Hausdorff, and second countable it is metrizable by Urysohn's metrization theorem, so $C(Y)$ is separable.

Let $\mathscr{F}_n$ be the first n elements of a countable dense sequence of $C(Y)$ and set $\varepsilon = \frac{1}{n}$, which gives us an $\eta_n > 0$ from Lin's theorem 3.1.4. We obtain an $N'_n \in \mathbb{N}$ and $\sigma_n > 0$ from lemma 3.1.8 such that for all $m \geq N'_n$ we have $\mu_{\tau\tilde{\phi}_m}(\mathcal{O}) > \sigma_n$ for all open balls $\mathcal{O}$ of radius at most η_n. Then Lin's theorem 3.1.4 gives us $\gamma_n > 0$, $\delta_n > 0$, a finite set $G_n \subset C(X)$ and a finite subset $\mathcal{H}_n \subset C(X)_{\text{s.a}}$. By lemma 3.1.6 there is an $N''_n \in \mathbb{N}$ such that for all $m \geq N''_n$ there exists a $*$-homomorphism $\tilde{\psi}_m : C(X) \to M_{k_m}$ such that $\tau\tilde{\phi}_m(a) \approx_{\gamma_n} \tau\tilde{\psi}_m(a)$ for all $a \in \mathcal{H}_n$.

Since ϕ is a lift of a $*$-homomorphism there exists an $N'''_n \in \mathbb{N}$ such that for all $m \geq N'''_n$ we have that ϕ_m is δ_n-G_n-multiplicative, and thus so is $\tilde{\phi}_m$. Set $N_n = \max\{N'_n, N''_n, N'''_n\}$ and for all $N_n \leq m < N_{n+1}$ set $\Psi_m = \tilde{\psi}_m$ where $\tilde{\psi}_m$ is as defined above.

Then by Lin's theorem 3.1.4, for all $N_n \leq m < N_{n+1}$ there exists a unitary $u_m \in M_{k_m}$ such that $\tilde{\phi}_m(f) \approx_{\frac{1}{n}} \mathrm{ad}(u_m)\Psi_m$ for all $f \in \mathscr{F}_n$. Then certainly $\phi_m(f) \approx_{\frac{1}{n}} \mathrm{ad}(u_m)\Psi_m \restriction_Y$ for all $f \in \mathscr{F}_n \subset Y$. By increasing n we may then define a $*$-homomorphism $\Phi : C(Y) \to \prod_{\mathbb{N}} M_{k_m}$ by setting its component maps $\Phi_m = \mathrm{ad}(u_m)\Psi_m \restriction_Y$. Φ is the map required. ∎

For the proof of our main theorem in this section, we want to apply theorem 3.1.10 to $Y \cong (0,1] \times X$ for X a compact metrizable space. Given the correspondence between cpc order 0 maps on A and $*$-homomorphisms on the cone $C((0,1], A)$ this theorem can be seen to be a way to obtain cpc order 0 lifts of a cpc order 0 map in this particular context. We now move onto some results surrounding the liminf hypothesis of theorem 3.1.10.

For an extension $\phi : C(X) \to B(\mathcal{H})$ we are concerned with finding an idempotent quasicentral approximate unit for $\mathcal{K}$ in $\mathcal{T}_\phi$. This is an approximate unit $(h_n)_{\mathbb{N}}$ for $\mathcal{K}$ such that $h_{n+1}h_n = h_n$ for all $n \in \mathbb{N}$, which is the idempotent condition, and $\|h_n\phi(f) - \phi(f)h_n\| \to 0$ for all $f \in C(X)$, which is the quasicentral condition. It is known, due to Arveson [3, Thm. 1], that a quasicentral approximate unit always exists for any ideal I in a C^*-algebra A. Moreover for any approximate unit for I in A there exists a quasicentral approximate unit in its convex hull. For the compact operators $\mathcal{K}$ the idempotent condition is equivalent to saying that h_{n+1} is just the identity matrix for the part that covers h_n. Then we know that an idempotent quasicentral approximate unit for $\mathcal{K}$ in $\mathcal{T}_\phi$ will always exist. Note also that for such an approximate unit we will have that $(h_{n+1}-h_n)^{\frac{1}{2}}B(\mathcal{H})(h_{n+1}-h_n)^{\frac{1}{2}} \cong M_{k_n}$ for some $k_n \in \mathbb{N}$. We shall use τ to refer to the tracial state on $(h_{n+1}-h_n)^{\frac{1}{2}}B(\mathcal{H})(h_{n+1}-h_n)^{\frac{1}{2}}$ under this correspondence.

For the following we denote by $\iota \in C((0,1])$ the canonical inclusion $\mathrm{id}_{(0,1]}$. Thus elements of the form $\iota^a \otimes f$ for $a \in \mathbb{N}$ and $f \in C(X)$ represent the elementary tensors of $C((0,1]) \otimes C(X) \cong C((0,1] \times X)$.

Lemma 3.1.11. Let X be a compact metric space and let μ be a probability measure on $(0,1] \times X$. Then there exists a trivial extension $\phi : C(X) \to B(\mathcal{H})$ and an idempotent quasicentral approximate unit $(h_n)_{\mathbb{N}}$ such that the cpc maps

$$\phi_n : C((0,1] \times X) \to (h_{n+1} - h_n)^{\frac{1}{2}} B(\mathcal{H})(h_{n+1} - h_n)^{\frac{1}{2}}$$

defined by

$$\phi_n(\iota^a \otimes f) := (h_{n+1} - h_n)^{\frac{a}{2}} \phi(f)(h_{n+1} - h_n)^{\frac{a}{2}}$$

satisfy $\liminf_{n\to\infty} \tau\phi_n(f) \geq \frac{1}{2}\int_{(0,1]\times X} f(x)d\mu(x)$ for all positive $f \in C((0,1] \times X)$.

Proof: Let $\mathscr{F}_n$ be the set of the first n elements of a dense sequence in $C((0,1]\times X)$. We may find a convex combination of point mass measures $\omega_n' = \sum_{s=1}^S \lambda_s \delta_{(p_s,x_s)}$ such that

$$\int_{(0,1]\times X} f(x)d\omega_n'(x) \underset{\frac{1}{4n}}{\approx} \int_{(0,1]\times X} f(x)d\mu(x)$$

for all $f \in \mathscr{F}_n$. Let $H = \max_{f\in\mathscr{F}_n} \|f\|$ and so for all $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{1}{4nSH}$ we may find a convex combination of point mass measures $\omega_{n,m}' = \sum_{s=1}^S \frac{k_s}{m}\delta_{(p_s,x_s)}$ such that

$$\int_{(0,1]\times X} f(x)d\omega_{n,m}'(x) \underset{\frac{1}{4n}}{\approx} \int_{(0,1]\times X} f(x)d\omega_n'(x)$$

for all $f \in \mathscr{F}_n$. Let $\{y_1,\dots,y_l\}$ be a finite set of elements which cover X within $\frac{1}{2n}$ and let $\{y_{l_i}\}_{1\leq i\leq I}$ be the subset of this containing those points for which none of the x_s represented in ω_n' are within $\frac{1}{2n}$. Similarly let $\left\{q_{k_j}\right\}_{1\leq j\leq J} \subset \left\{\frac{r}{2n}\right\}_{r=1}^{2n}$ be the set of points that are not within $\frac{1}{2n}$ of any of the points p_s represented in ω_n'. Let $K = \max\{I,J\}$ and consider the set of pairs $\{(q_{k_i}, y_{l_i})\}_{1\leq i\leq K}$ where we set $q_{k_i} = q_{k_J}$ for $J < i \leq I$ if $J < I$ or $y_{l_i} = y_{l_I}$ for $I < i \leq J$ if $I < J$. Then for $1 \leq i \leq K$ we pick a point (p_{s_i}, x_{l_i}) represented in ω_n'. Let m_n be the smallest natural number such that $\frac{1}{m_n} < \min\left\{\frac{1}{4nK}, \frac{1}{4nSH}\right\}$ and $m_n \geq m_{n-1}$. We may then define a new measure which is a convex combination of point mass measures by

$$\omega_n = \sum_{i=1}^K \frac{1}{m_n}\left(\delta_{(q_{k_i},y_{l_i})} - \delta_{(p_{s_i},x_{s_i})}\right) + \omega_{n,m_n}'.$$

Consider the set of pairs $\{(q_{k_i}, y_{l_i})\}_{1\le i\le K} \cup \{(p_s, x_s)\}_{1\le s\le S}$ that are represented in this measure ω_n then relabel them and place them into a multiset $\{(p_{s_t}, x_{s_t})\}_{1\le t\le m_n}$ such that $p_{s_t} \ge p_{s_{t+1}}$ for all t, where a pair (p_s, x_s) is represented in this multiset q_s times where $\frac{q_s}{m_n}$ is the coefficient of $\delta_{(p_s,x_s)}$ in ω_n. We then rewrite ω_n according to this multiset, so we write $\omega_n = \sum_{t=1}^{m_n} \frac{1}{m_n}\delta_{(p_{s_t},x_{s_t})}$. This measure ω_n satisfies

$$\int_{(0,1]\times X} f(x)d\omega_n(x) \approx_{\frac{1}{n}} \int_{(0,1]\times X} f(x)d\mu(x)$$

for all $f \in \mathscr{F}_n$.

If we isolate the points coming from X to obtain $\Omega_n = \sum_{t=1}^{m_n} \frac{1}{m_n}\delta_{x_{s_t}}$ we may use lemma 3.1.5 to obtain a $*$-homomorphism $\Phi_n : C(X) \to M_{m_n}$ corresponding to Ω_n. Then for each $f \in C(X)$ we may define a block diagonal matrix in $B(\mathcal{H})$ by setting the first block to be the matrix to be $\phi_1(f)$, the second block to be $\phi_2(f)$ and so on for increasing n. We may then define a $*$-homomorphism $\phi : C(X) \to B(\mathcal{H})$ by setting each image $\phi(f)$ to be this block diagonal matrix. Note that due to our inclusion of the covering points for X the map ϕ will be a $*$-monomorphism, and thus is a trivial extension.

To define the appropriate approximate unit we set h_n to be the diagonal matrix whose first $\sum_{i=1}^{n-1} m_i$ entries are 1, the next m_n entries are the p_{s_t} that are represented in ω_n in the decreasing order we have given them, and all subsequent entries are 0. By our construction this is a *rough* approximation of the diagonal matrix with entries $\left\{1,\ldots,1,\frac{2n-1}{2n},\ldots,\frac{1}{2n},0\ldots\right\}$. Thus it can be seen that our h_n define an idempotent quasicentral approximate unit as required.

Set $\phi_n = (h_{n+1}-h_n)^{\frac{1}{2}}\phi(\cdot)(h_{n+1}-h_n)^{\frac{1}{2}}$ and note that by construction we have that

$$\tau\phi_n(f) = \frac{m_n}{m_n+m_{n+1}}\sum_{t=1}^{m_n}\frac{1}{m_n}f(((1-p_{s_t}^{(n)}),x_{s_t}^{(n)}))+\frac{m_{n+1}}{m_n+m_{n+1}}\sum_{t=1}^{m_{n+1}}\frac{1}{m_{n+1}}f((p_{s_t}^{(n+1)},x_{s_t}^{(n+1)}))$$

for all $f \in C((0,1]\times X)$. If we assume that f is positive we will have

$$\frac{m_n}{m_n+m_{n+1}}\sum_{t=1}^{m_n}\frac{1}{m_n}f(((1-p_{s_t}^{(n)}),x_{s_t}^{(n)})) \ge 0$$

and since by construction we have $m_{n+1} \ge m_n$ we have $\frac{m_{n+1}}{m_n+m_{n+1}} \ge \frac{1}{2}$. Thus for positive f we have

$$\begin{aligned}\tau\phi_n(f) &\ge \frac{1}{2}\sum_{t=1}^{m_{n+1}}\frac{1}{m_{n+1}}f((p_{s_t}^{(n+1)},x_{s_t}^{(n+1)}))\\ &= \frac{1}{2}\int_{(0,1]\times X} f(x)d\omega_n(x).\end{aligned}$$

Since the ω_n have been defined to approximate the measure μ we can see that

$$\liminf_{n\to\infty}\tau\phi_n(f) \ge \frac{1}{2}\int_{(0,1]\times X} f(x)d\mu(x)$$

for all positive $f \in C((0,1] \times X)$ as required. ∎

Note that if we let μ be a probability measure on $(0,1] \times X$ with full support then this will give us a trivial extension ϕ and idempotent quasicentral approximate unit $(h_n)_{\mathbb{N}}$ such that $\liminf_{n\to\infty} \tau\phi(f) > 0$ for all positive $f \in C((0,1] \times X)$, since if the measure has full support then $\mu(\mathcal{O}) > 0$ for all open balls $O \subset (0,1] \times X$ meaning that $\int_{(0,1]\times X} f(x)d\mu(x) > 0$ for all positive $f \in C((0,1] \times X)$.

Corollary 3.1.12. Let X be a compact metric space, and $\phi : C(X) \to B(\mathcal{H})$ an extension. Then there exists a trivial extension ψ and an idempotent quasicentral approximate unit $(H_n)_{\mathbb{N}}$ for $\mathcal{K}$ in $\mathcal{T}_{\psi+\phi}$ such that the maps

$$(\psi + \phi)_n : C((0,1] \times X) \to (H_{n+1} - H_n)^{\frac{1}{2}} B(\mathcal{H} \oplus \mathcal{H})(H_{n+1} - H_n)^{\frac{1}{2}}$$

defined by

$$(\psi + \phi)_n(\iota^a \otimes g) = (H_{n+1} - H_n)^{\frac{a}{2}} (\psi + \phi)(g)(H_{n+1} - H_n)^{\frac{a}{2}}$$

satisfy $\liminf_{n\to\infty} \tau(\psi + \phi)_n(f) > 0$ for all positive $f \in C((0,1] \times X)$.

Proof: Let $(h'_n)_{\mathbb{N}}$ be an idempotent quasicentral approximate unit for $\mathcal{K}$ in $\mathcal{T}_\phi$ and for $n \in \mathbb{N}$ let

$$\phi_n : C((0,1] \times X) \to (h'_{n+1} - h'_n)^{\frac{1}{2}} B(\mathcal{H})(h'_{n+1} - h'_n)^{\frac{1}{2}}$$

be the cpc maps defined by

$$\phi_n(\iota^a \otimes f) = (h'_{n+1} - h'_n)^{\frac{a}{2}} \phi(f)(h'_{n+1} - h'_n)^{\frac{a}{2}}.$$

Let μ be a probability measure on $(0,1] \times X$ that has full support and let ψ be the trivial extension, $(h_n)_{\mathbb{N}}$ the idempotent quasicentral approximate unit, and

$$\psi_n : C((0,1] \times X) \to (h_{n+1} - h_n)^{\frac{1}{2}} B(\mathcal{H})(h_{n+1} - h_n)^{\frac{1}{2}}$$

the cpc maps obtained from lemma 3.1.11. We then define an idempotent quasicentral approximate unit $(H_n)_{\mathbb{N}}$ for $\mathcal{K}$ in $\mathcal{T}_{\psi+\phi}$ by setting $H_n := h_n \otimes e_{11} + h'_n \otimes e_{22}$ where e_{11} and e_{22} are the standard main diagonal basis vectors in M_2. Recall that we define $(\psi + \phi)(g) = \psi(g) \otimes e_{11} + \phi(g) \otimes e_{22}$ for $g \in C(X)$. We can see that $\tau(\psi + \phi)_n(f) = \frac{1}{2}\tau\psi_n(f) + \frac{1}{2}\tau\phi_n(f)$ for all $f \in C(0,1] \times X)$, and since the maps ϕ_n are all completely positive we have $\tau\phi_n(f) \geq 0$ for all positive f we have that $\liminf_{n\to\infty} \tau(\psi + \phi)_n(f) \geq \frac{1}{2} \liminf_{n\to\infty} \tau\psi_n(f) > 0$ for all positive $f \in C((0,1] \times X)$. ∎

What we wish to do now is to delve a little deeper into the nuclear dimension of $C(X)$ for compact metrizable Hausdorff X and in particular we shall need to define a particular net of approximations for the nuclear dimension that we will be making use of for the main theorem. To begin with, we will need the following lemma which will help us to define the net of approximations we desire.

Lemma 3.1.13. Let X be a compact metrizable space. For $\varepsilon > 0$ there exists a finite cover of X by Borel sets $\{X_i\}$ that are pairwise disjoint with the diameter of each being at most ε and each has a non-empty interior.

This lemma is a paraphrasing of a statement made in the proof of [51, Lemma 2.4].

Let us discuss how this will relate to the nuclear dimension for $C(X)$ for compact metrizable X. Recall that we are able to find a system of n -decomposable cp approximations $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ for $C(X)$ by taking partitions of unity subordinate to *good* n-decomposable finite open covers of X where $n = \dim X$, see remark 2.2.7. In particular recall that a family of subsets $(U_\lambda)_\Lambda$ of X is n-decomposable, [30, Def 1.4], if there exists a decomposition $\Lambda = \bigsqcup_{i=0}^n \Lambda_j$ such that for all $i - 0, \dots, n$ and $\lambda, \lambda' \in \Lambda_i$ we have $U_\lambda \cap U_{\lambda'} = \varnothing$. Recall also that by [30, Prop. 1.6] we have $\dim X = n$ if and only if n is the least integer for which every finite open cover of X has an n-decomposable refinement. Given any n-decomposable finite open cover $(U_{k,j})_{0 \le k \le n, j \in J_k}$ of X, where we have $U_{k,j} \cap U_{k,j'} = \varnothing$ for all k and all $j, j' \in J_k$, we may define an n-decomposable cp map $\varphi : \bigoplus_{k=0}^n \mathbb{C}^{|J_k|} \to C(X)$. If $(h_{k,j})_{0 \le k \le n, j \in J_k}$ is a partition of unity subordinate to this open cover we may define cpc maps $\varphi^{(k)} : \mathbb{C}^{|J_k|} \to C(X)$ by $\varphi^{(k)}((a_j)_{j \in J_k}) = \sum_{j \in J_k} a_j h_{k,j}$. These maps can all be seen to be order 0 since for each k the $h_{k,j}$ correspond to pairwise disjoint subsets, and so their sum $\varphi = \sum_{k=0}^n \varphi^{(k)}$ is n-decomposable. For a collection $\{x_{k,j}\}_{0 \le k \le n, j \in J_k}$ of points of X such that $x_{k,j} \in U_{k,j}$ we may define a cpc map $\psi : C(X) \to \bigoplus_{k=0}^n \mathbb{C}^{|J_k|}$ by $\psi(f) = (f(x_{k,j}))_{0 \le k \le n, j \in J_k}$.

This is possible for any given n-decomposable finite open cover of X, but we need some requirements of the component subsets $U_{k,j}$ to ensure that $(\bigoplus_{k=0}^n \mathbb{C}^{|J_k|}, \psi, \varphi)$ is an n-decomposable cp approximation for a given finite $\mathscr{F} \subset C(X)$ within a given $\varepsilon > 0$. The condition we require is that for all k and $j \in J_k$ we have $|f(x) - f(y)| < \frac{2\varepsilon}{3}$ for all $x, y \in U_{k,j}$ and all $f \in \mathscr{F}$. This is possible by first taking a finite cover of X by open balls $(B_j)_{j=1}^r$ satisfying this condition, then taking an n-decomposable refinement of this cover that we know exists by [30, Prop. 1.6]. Though this need not be the only such n-decomposable finite open cover that would define an approximation for $\mathscr{F}$ within ε. In particular if we set $\eta = \max_j(\operatorname{diam}(B_j))$ then by uniform continuity we know that all subsets $U \subset X$ such that $\operatorname{diam}(U) \le \eta$ will satisfy $|f(x) - f(y)| < \frac{2\varepsilon}{3}$ for all $x, y \in U$ and for all $f \in \mathscr{F}$. Thus for any n-decomposable finite open cover $(U_{k,j})_{0 \le k \le n, j \in J_k}$ of X such that $\max_{k,j}(\operatorname{diam}(U_{k,j})) \le \eta$ we may define an n-decomposable cp approximation (F, ψ, φ) for $\mathscr{F}$ within ε through partitions of unity.

Corollary 3.1.14. Let X be a compact metrizable space and let $n = \dim_{\mathrm{nuc}} C(X)$. Then there exists a sequence of n-decomposable cp approximations $(F_s, \psi_s, \varphi_s)_{s\in\mathbb{N}}$ for $C(X)$ such that the 0^{th} map $\varphi^{(0)} : \mathbb{C}^{x_s(0)} \to C(X)$ such that for each s the collection $(\varphi_s^{(0)}(e_i)^{-1}(0,\infty))_{1\le i\le x_s(0)}$ exists as the interiors of a collection of pairwise disjoint Borel sets $(Y_i^s)_{1\le i\le x_s(0)}$ such that $\bigcup_{i=1}^{x(0)} Y_i^s = X$. Moreover these can be chosen so that $\max_{1\le i\le x_s(0)} \mathrm{diam}(\varphi_s^{(0)}(e_i)^{-1}(0,\infty)) \searrow 0$ as s increases.

Proof: As X is metrizable $C(X)$ is separable, so let $\mathscr{F}_s$ be the first s elements of a dense sequence for $C(X)$ and set $\varepsilon = \frac{1}{s}$. Take a finite open cover $\left(V_j^s\right)_{1\le j\le r_s}$ of X such that $|f(x) - f(y)| < \frac{2\varepsilon}{3}$ for $x, y \in V_j^s$ for all $f \in \mathscr{F}_s$ and $j \in \{1, \dots, r_s\}$, and let $\left(V_{k,j}^s\right)_{0\le k\le n, j\in J_k'}$ be an n-decomposable open refinement of this cover. Set $V_0^s = \bigcup_{j\in J_0'} V_{0,j}^s$ and set $Y^s = X\backslash V_0^s$. Let $\delta_s = \max_{j\in J_0'} \mathrm{diam}(V_{0,j}^s)$. By lemma 3.1.13 we may find a finite cover of Y^s by pairwise disjoint Borel sets $\{Y_i^s\}_{1\le i\le g_s}$ with the diameter of each being at most δ_s and each having nonempty interior. Let $I_i^s = \mathrm{Int}\,(Y_i^s)$ be this interior for each i, and define a collection of sets

$$\left(U_{0,j}^s\right)_{0\le k\le n, j\in J_0} = \left(V_{0,j}^s\right)_{0\le k\le n, j\in J_k'} \cup \left(I_i^s\right)_{1\le i\le g_s}.$$

By construction this will be a collection of pairwise disjoint open sets, and that for all $1 \le i \le g_s$ we will have $|f(x) - f(y)| < \frac{2\varepsilon}{3}$ for all $f \in \mathscr{F}_s$. If we then set $U_{k,j}^s = V_{k,j}^s$ for $1 \le k \le n$ and $j \in J_k'$ then we have an n-decomposable cover $\left(U_{k,j}^s\right)_{0\le k\le n, j\in J_k}$ of X and this cover will allow us to define an n-decomposable cp approximation (F_s, ψ_s, φ_s) for $\mathscr{F}_s$ within $\frac{1}{s}$ by taking a partition of unity subordinate to this cover. Since the diameters of the sets contained in this collection are bounded above by all σ for which $d(x, y) < \sigma$ implies $|f(x) - f(y)| < \frac{2}{3s}$ for all $f \in \mathscr{F}_s$ we necessarily have that the diameters of all the sets in the diameter must decrease to 0 as s increases. In particular $\max_{1\le i\le x_s(0)} \mathrm{diam}(\varphi_s^{(0)}(e_i)^{-1}(0,\infty)) \searrow 0$ as required. ∎

3.2 The Main Theorem

We now have all the pieces required to prove our main theorem.

Theorem 3.2.1. Let X be a compact metric space and $\phi : C(X) \to B(\mathcal{H})$ an extension. Then $\dim_{\mathrm{nuc}} \mathcal{T}_\phi = \dim_{\mathrm{nuc}} C(X)$.

Proof: Let ψ be the trivial extension and $H = (H_n)_{\mathbb{N}}$ the idempotent quasicentral approximate unit for $\mathcal{K}$ in $\mathcal{T}_{\psi+\phi}$ obtained from corollary 3.1.12. Define $\tilde{H} = (H_{n+1})$ to be the same approximate unit as H but with the index shifted by 1. Set $\Phi := \psi + \phi$ and note that $\mathcal{T}_\Phi \cong \mathcal{T}_\phi$ since ψ is a trivial extension. Define

$$A_n = H_n \mathcal{T}_\Phi H_n, B_n = (H_{n+1} - H_n)\mathcal{T}_\Phi(H_{n+1} - H_n), C_n = \overline{(1 - H_{n+1})\mathcal{T}_\Phi(1 - H_{n+1})}$$

and define the corresponding maps

$$\begin{aligned}
\alpha(x) &= (H_n^{\frac{1}{2}} x H_n^{\frac{1}{2}})_{\mathbb{N}} \\
\beta(f) &= ((H_{n+1} - H_n)^{\frac{1}{2}} \Phi(f)(H_{n+1} - H_n)^{\frac{1}{2}})_{\mathbb{N}} \\
\gamma(f) &= ((1 - H_{n+1})^{\frac{1}{2}} \Phi(f)(1 - H_{n+1})^{\frac{1}{2}})_{\mathbb{N}}
\end{aligned}$$

for $x \in \mathcal{T}_\Phi$ and $f \in C(X)$. We consider the following diagram which is a generalisation of the motivating diagram in the proof of the Toeplitz case by Brake and Winter [10].

$$\begin{tikzcd}
 & F_k \arrow[d, dashed, shift left, "\varphi_k"] & \prod_{\mathbb{N}} C_n \arrow[r, "q_C"] \arrow[drr, "\iota_4"] & \frac{\prod_{\mathbb{N}} C_n}{\sum_{\mathbb{N}} C_n} \arrow[drr, "\iota_3"] & & \\
\mathcal{T}_\Phi \arrow[r, shift left, "\pi"] \arrow[rrrrr, bend left=60, "\iota"] \arrow[ddrr, bend right, "\alpha"'] & C(X) \arrow[l, shift left, "\Phi"] \arrow[u, dashed, shift left, "\psi_k"] \arrow[ur, "\gamma"] \arrow[r, dashed, "\varphi_k^{(0)}"] \arrow[dr, "\overline{\beta}"] \arrow[dr, bend right, "\beta"'] & \mathbb{C}^{x(0)} \arrow[d, dashed, "\rho_k"] & & \prod \mathcal{T}_\Phi \arrow[r, "q"] & \frac{\prod \mathcal{T}_\Phi}{\sum \mathcal{T}_\Phi} \\
 & & \prod_{\mathbb{N}} B_n \arrow[r, "q_B"] & \frac{\prod_{\mathbb{N}} B_n}{\sum_{\mathbb{N}} B_n} \arrow[urr, "\iota_2"] & & \\
 & & \prod_{\mathbb{N}} A_n \arrow[uurr, bend right, "\iota_1"'] & & &
\end{tikzcd}$$

The dotted arrows in this diagram represent arrows that will be used to create an *approximate* path through this diagram, and in particular the maps will be coming from a d-decomposable cp approximation (F_k, ψ_k, φ_k) of $C(X)$ where $\dim_{\text{nuc}} C(X) = d$. The maps $\overline{\beta}$ and ρ_k will be defined later in this proof. Here π is the quotient map from the short exact sequence defining the extension, ι is the inclusion map $\mathcal{T}_\Phi \to \frac{\prod_{\mathbb{N}} \mathcal{T}_\Phi}{\sum_{\mathbb{N}} \mathcal{T}_\Phi}$, ι_1 is the inclusion map $\prod_{\mathbb{N}} A_n \to \prod_{\mathbb{N}} \mathcal{T}_\Phi$ and ι_4 the inclusion map $\prod_{\mathbb{N}} C_n \to \prod_{\mathbb{N}} \mathcal{T}_\Phi$. The maps ι_2 and ι_3 are the inclusion maps from $\frac{\prod_{\mathbb{N}} B_n}{\sum_{\mathbb{N}} B_n}$ and $\frac{\prod_{\mathbb{N}} C_n}{\sum_{\mathbb{N}} C_n}$ respectively into $\frac{\prod_{\mathbb{N}} \mathcal{T}_\Phi}{\sum_{\mathbb{N}} \mathcal{T}_\Phi}$. The q, q_B, and q_C are the natural quotient maps into $\frac{\prod_{\mathbb{N}} \mathcal{T}_\Phi}{\sum_{\mathbb{N}} \mathcal{T}_\Phi}$, $\frac{\prod_{\mathbb{N}} B_n}{\sum_{\mathbb{N}} B_n}$, and $\frac{\prod_{\mathbb{N}} C_n}{\sum_{\mathbb{N}} C_n}$ respectively. Importantly from the definitions we have

$$\iota(x) = q\iota_1\alpha(x) + \iota_2 q_B \beta\pi(x) + \iota_3 q_C \gamma\pi(x)$$

for all $x \in \mathcal{T}_\Phi$. Our first goal is to define cpc order 0 maps $\overline{\beta}$ and ρ_k that will allow us to approximate ι using this identity.

Note that in particular $q_B\beta$ is a cpc order 0 map so let $\omega : C((0,1] \times X) \to \frac{\prod_{\mathbb{N}} B_n}{\bigoplus_{\mathbb{N}} B_n}$ be the corresponding $*$-homomorphism that satisfies $\omega(\iota \otimes g) = q_B\beta(g)$. We then set $\Omega : C((0,1] \times X) \to \prod_{\mathbb{N}} B_n$ to be the cpc lift of ω defined by $\Omega_n(\iota^a \otimes g) = (H_{n+1} - H_n)^{\frac{a}{2}}\Phi(g)(H_{n+1} - H_n)^{\frac{a}{2}}$, which can be seen to satisfy $\liminf_{n\to\infty} \tau\Omega_n(f) > 0$ for all positive $f \in C((0,1] \times X)$ since this is the map described in corollary 3.1.12. Thus by theorem 3.1.10 there exists a $*$-homomorphism $\underline{\beta} : C((0,1] \times X) \to \prod_{\mathbb{N}} B_n$ such that $q_B\underline{\beta} = q_B\Omega = \omega$, and so the corresponding cpc order 0 map $\overline{\beta} : C(X) \to \prod_{\mathbb{N}} B_n$ defined by $\overline{\beta}(g) = \underline{\beta}(\iota \otimes g)$ satisfies $q_B\overline{\beta} = q_B\beta$.

Let $d = \dim_{\text{nuc}} C(X)$ and let (F_k, ψ_k, φ_k) be a d-decomposable cp approximation for $\mathscr{F}_k \subset C(X)$, the first k elements of a dense sequence for $C(X)$, within $\frac{1}{k}$ as obtained in the proof of lemma 3.1.14. That is this is an approximation defined by a partition of unity subordinate to an open cover $\left(U^k_{l,j}\right)_{0\le l\le d, j\in J_l}$ that is n-decomposable and such that the $\left(U^k_{0,j}\right)_{j\in J_0}$ are the interiors of pairwise disjoint Borel sets $\left(Y^k_j\right)_{j\in J_0}$ such that $\bigcup_{j\in J_0} Y^k_j = X$ and such that $\max_{j\in J_0} \operatorname{diam}\left(Y^k_j\right) \searrow 0$ as k increases. Denote $F_k = \mathbb{C}^{x(0)} \oplus \cdots \oplus \mathbb{C}^{x(d)}$ and for each $0 \le l \le d$ and for each $j \in J_l$ let $x^k_{lj} \in U^k_{l,j}$ such that $\psi_k(f) = (f(x^k_{l,j}))_{0\le l\le d, j\in J_l}$ for $f \in C(X)$. Let $\chi_{Y^k_j}$ be the characteristic function on the Borel set Y^k_j and so define a cpc order 0 map $\rho_{k,n} : \mathbb{C}^{x(0)} \to B_n$ by

$$\rho_{k,n}((a_j)_{j\in J_0}) = \overline{\beta}_n(\sum_{j=1}^{x(0)} a_j \chi_{Y^k_j})$$

We have implicitly extended $\overline{\beta}_n$ here to the bounded Borel functions. This can be done by considering the corresponding $*$-homomorphism $\underline{\beta} : C((0,1] \times X) \to \prod_{\mathbb{N}} B_n$ and extending this to a $*$-homomorphism on the enveloping von Neumann algebra [22, 12.5.11]. The enveloping von Neumann algebra for any $C(X)$ is the bounded Borel functions on X. Then for all $f \in C(X)$ we have the following

$$\begin{aligned}
&\|\rho_{k,n}\psi^{(0)}_k(f) - \overline{\beta}_n(f)\| \\
&= \|\overline{\beta}_n(\sum_{j=1}^{x(0)} f(x_{0,j})\chi_{Y^k_{0,j}}) - \overline{\beta}_n(f)\| \\
&\xrightarrow{k\to\infty} 0.
\end{aligned}$$

This is relying on the fact that the $Y^k_{0,j}$ cover X and that they are specifically constructed so that $|g(x) - g(y)| < \frac{2}{3k}$ for $x, y \in Y^k_j$ for all $g \in \mathscr{F}_k$ and for all j. With this construction one can see that $\sum_{j=1}^{x(0)} f(x_{0,j})\chi_{Y^k_{0,j}} \to f$ as k increases for all $f \in C(X)$ which gives us our limit above. Note in particular that the convergence here does not rely on n at all. This allows us to define $\rho_k : \mathbb{C}^{x(0)} \to \prod_{\mathbb{N}} B_n$ by $\rho_k(a) = (\rho_{k,n}(a))_{n\in\mathbb{N}}$ so that $\rho_k\psi_k^{(0)}$ converges to $\overline{\beta}$ pointwise and so that the following is true

$$\|\iota(x) - (q\iota_1\alpha(x) + \iota_2 q_B \rho_k \psi_k^{(0)}\pi(x) + \iota_3 q_C \gamma\varphi_k\psi_k\pi(x))\| \xrightarrow{k\to\infty} 0 \tag{3.2.1}$$

for all $x \in \mathcal{T}_\Phi$.

Our next step is to show that $\iota_2 q_B \rho_k + \iota_3 q_C \gamma\varphi_k^{(0)}$ is an order 0 map, though given that each piece of the sum is an order 0 map we only need to show $\iota_2 q_B \rho_k(e_i) \cdot \iota_3 q_C \gamma \varphi_k^{(0)}(e_{i'}) = 0$ for $i \neq i'$ where e_i is the i^{th} canonical generator of $\mathbb{C}^{x(0)}$. We pick a sequence of functions $(d_m)_{\mathbb{N}}$ that are 1 on $U_{0,i}$ and such that $d_m \cdot h_{0,i'}$ uniformly converges to 0. In particular we note that $\chi_{U_{0,i}} \leq d_m$ for all m and so $\rho_k(e_i) \leq \overline{\beta}(d_m)$. Further we can say $q_B\rho_k(e_i) \leq q_B\overline{\beta}(d_m) = q_B\beta(d_m)$ for all m so that

$$\begin{aligned}
&\|\iota_2 q_B \rho_k(e_i) \cdot \iota_3 q_C \gamma\varphi_k^{(0)}(e_{i'})\| \\
&\leq \|\iota_2 q_B \beta(d_m) \cdot \iota_3 q_C \gamma\varphi_k^{(0)}(e_{i'})\| \\
&= \|(\tilde{H} - H)^{\frac{1}{2}}(\iota\Phi(d_m))(\tilde{H} - H)^{\frac{1}{2}} \cdot (1 - \tilde{H})^{\frac{1}{2}}\iota\Phi(h_{0,i'})(1 - \tilde{H})^{\frac{1}{2}}\| \\
&= \|(1 - \tilde{H})(\tilde{H} - H)\iota\Phi(d_m) \cdot \iota\Phi(h_{0,i'})\| \\
&\leq \|\iota\Phi(d_m \cdot h_{0,i'})\| \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}$$

The equality in the second last line is coming from the fact that H and $\tilde{H}$ are quasicentral approximate units for $\mathcal{K}$ in $\mathcal{T}_\Phi$ and thus will commute with the image of Φ into the quotient algebra like this. Thus indeed $\iota_2 q_B \rho_k + \iota_3 q_C \gamma\varphi_k^{(0)}$ is a cpc order 0 map.

We define a cpc order 0 lift $\tilde{\varphi}_k^{(0)} : \mathbb{C}^{x(0)} \to \prod_{\mathbb{N}} \mathcal{T}_\Phi$ of $\iota_2 q_B\rho_k + \iota_3 q_C\gamma\varphi_k^{(0)}$ and cpc order 0 lifts $\tilde{\varphi}_k^{(i)} : \mathbb{C}^{x(i)} \to \prod_{\mathbb{N}} C_n$ of $q_C\gamma\varphi_k^{(i)}$ respectively for $1 \leq i \leq d$, this is doable since, as noted in [30, Remark 2.4], a cpc order 0 map $F \to \frac{A}{J}$ from finite dimensional algebra F is always able to be lifted to a cpc order 0 map $F \to A$, due to the correspondence described by proposition 2.1.3 and the fact that cones over finite dimensional algebras are projective, [34, Thm 10.1.11], [34, Thm. 10.2.1]. Putting this into the context of 3.2.1 we see that we have

$$\lim_{k\to\infty} \|\iota(x) - q(\iota_1\alpha(x) + \tilde{\varphi}_k^{(0)}\psi_k^{(0)}\pi(x) + \sum_{i=1}^{d}(\iota_4\tilde{\varphi}_k^{(i)}\psi_k^{(i)})\pi(x)\| = 0$$

for all $x \in \mathcal{T}_\Phi$ which when we expand the quotient norm as a limit we see that for all $x \in \mathcal{T}_\Phi$

$$\lim_{k\to\infty}\lim_{n\to\infty} \|x - \iota_\alpha(x) + \tilde{\varphi}_k^{(0)}\psi_k^{(0)}\pi(x) + \sum_{i=1}^{d}(\iota_4\tilde{\varphi}_k^{(i)}\psi_k^{(i)})\pi(x)\| = 0.$$

We may then use the separability of $\mathcal{T}_\Phi$ alongside a diagonal argument to find for every k an index n_k such that

$$\lim_{k\to\infty} \|x - \iota_{1,n_k}\alpha_{n_k}(x) + \tilde{\varphi}^{(0)}_{k,n_k}\psi^{(0)}_k\pi(x) + \sum_{i=1}^{d}(\iota_{4,n_k}\tilde{\varphi}^{(i)}_{k,n_k}\psi^{(0)}_k)\pi(x)\| = 0.$$

It is important to note that the cpc map $\iota_{1,n_k} \oplus \iota_{4,n_k}\tilde{\varphi}^{(1)}_{k,n_k} : A_{n_k} \oplus \mathbb{C}^{x(1)} \to \mathcal{T}_\Phi$ is in fact order 0 since the constituent parts are and their images A_{n_k}, C_{n_k} are orthogonal. We now define, for each k,

$$\begin{aligned}
\dot{F}^{(0)}_k &:= \mathbb{C}^{x(0)}, \\
\dot{F}^{(1)}_k &:= A_{n_k} \oplus \mathbb{C}^{x(1)}, \\
\dot{F}^{(i)}_k &:= \mathbb{C}^{x(i)} \text{ for } 2 \le i \le d, \\
\dot{F}_k &:= \bigoplus_{i=0}^{d} \dot{F}^{(i)}_k, \\
\dot{\psi}_k &:= \psi_k\pi + \alpha : \mathcal{T}_\Phi \to \dot{F}_k, \\
\dot{\varphi}^{(0)}_k &:= \tilde{\varphi}^{(0)}_{k,n_k} : \dot{F}^{(0)}_k \to \mathcal{T}_\Phi, \\
\dot{\varphi}^{(1)}_k &:= \iota_{1,n_k} \oplus \iota_{4,n_k}\tilde{\varphi}^{(1)}_{k,n_k} : \dot{F}^{(1)}_k \to \mathcal{T}_\Phi, \\
\dot{\varphi}^{(i)}_k &:= \tilde{\varphi}^{(i)}_k : \dot{F}^{(i)}_k \to \mathcal{T}_\Phi \text{ for } 2 \le i \le d, \\
\dot{\varphi}_k &:= \bigoplus_{i=0}^{d} \dot{\varphi}^{(i)}_k : \dot{F}_k \to \mathcal{T}_\Phi.
\end{aligned}$$

It can then be seen that $(\dot{F}_k, \dot{\psi}_k, \dot{\varphi}_k)_{\mathbb{N}}$ is a sequence of d-decomposable cpc approximations for $\mathcal{T}_\Phi$, so that $\dim_{\mathrm{nuc}}\mathcal{T}_\Phi = \dim_{\mathrm{nuc}}C(X)$. Since $\mathcal{T}_\Phi \cong \mathcal{T}_\phi$ we have that $\dim_{\mathrm{nuc}}\mathcal{T}_\phi = \dim_{\mathrm{nuc}}C(X)$. ∎

Chapter 4

Relative Nuclear Dimension

In this section we present our definition for relative nuclear dimension of C^*-algebras. This is a relatively mild modification of the nuclear dimension introduced with the intention of allowing all nuclear C^*-algebras a notion of finite nuclear dimension relative to *some* interesting algebra. We start by presenting the definition and notable equivalences, and then prove various properties of relative nuclear dimension. Particularly we focus on the case of relative nuclear dimension being 0.

4.1 The Definition

To be able to define the relative nuclear dimension of a C^*-algebra we shall first need to more concretely define the notion of an approximately order 0 map.

Definition 4.1.1. Let A, B be C^*- algebras, $K \subset A$ a compact set of contractions, $\eta > 0$ and $\phi : A \to B$ a cpc map. We say that ϕ is approximately order 0 on K within η, written η-K-order 0, if for $a, b \in K$ such that $\|ab\| < \delta$ with orthogonal elements $a', b' \in A$ within $\frac{\delta}{2}$ of a, b respectively we have $\|\phi(a')\phi(b')\| < \eta$ and thus $\|\phi(a)\phi(b)\| < \eta + \delta$.

And so we present our definition for relative nuclear dimension.

Definition 4.1.2. Let A, B be two C^*-algebras. We say A has nuclear dimension relative to B at most n, written $\mathrm{nd}_B A \leq n$, if for every finite set $\mathscr{F} \subset A$ and $\varepsilon > 0$ there exists a triple $(F \otimes B, \psi, \sim\varphi)$ with F finite dimensional, $\psi : A \to F \otimes B$ completely positive contractive, and $\sim\varphi$ a nonempty set of completely positive maps $\varphi : F \otimes B \to A$ such that

(i) For all $\varphi \in \sim\varphi$ we have $\|\varphi\psi(a) - a\| < \varepsilon$ for all $a \in \mathscr{F}$.

(ii) F decomposes into $n+1$ ideals $F = F^{(0)} \oplus \cdots \oplus F^{(n)}$ such that for any $i = 0, \ldots, n$, for any compact set $K_i \subset F^{(i)} \otimes B$ of contractions and $\eta_i > 0$ there is a $\varphi \in \sim\varphi$ such that the restrictions $\varphi^{(i)} : F^{(i)} \otimes B \to A$ are η_i-K_i-order 0.

Remark 4.1.3. This definition can be presented in a manner closer to that of the definition of nuclear dimension. That is $\mathrm{nd}_B A \leq n$ if there exists a net $(F_\lambda \otimes B, \psi_\lambda, \varphi_\lambda)_\Lambda$ with each F_λ finite dimensional, $\psi_\lambda : A \to F_\lambda \otimes B$ cpc, and $\sim\varphi_\lambda$ a nonempty set of cp maps $\varphi_\lambda : F_\lambda \otimes B \to A$ such that $\varphi_\lambda \psi_\lambda(a) \to a$ uniformly on finite sets of A. This latter point must be true regardless of choice of φ_λ at each step, and the net must satisfy condition (ii) of the above definition for all λ.

It is then reasonable to say $\mathrm{nd}_B A = n$ if n is the least such number satisfying the above definition, and the relative dimension shall be infinite if no such n exists. We shall refer to a triple $(F \otimes B, \psi, \sim\varphi)$ as a relatively n-decomposable cp approximation and a net $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ as a system of relatively n-decomposable cp approximations.

Note that if $\phi : B \to A$ is a cpc order 0 map then certainly it is η-K-order 0 for any η and any compact $K \subset B$. Thus if one is able to find a system $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ for A such that for each λ, $\sim\varphi_\lambda$ contains a map that is n-decomposable in the traditional sense then certainly $\mathrm{nd}_B A \leq n$. Trivially this shows that $\mathrm{nd}_{\mathbb{C}} A \leq \dim_{\mathrm{nuc}} A$ for any C^*-algebra A. It is in fact the case that nuclear dimension relative to $\mathbb{C}$ is equivalent to nuclear dimension, as one might anticipate. The reverse direction that $\dim_{\mathrm{nuc}} A \leq \mathrm{nd}_{\mathbb{C}} A$ is shown later using proposition 4.2.1 but to be able to get to that point we first need to be able to construct an order 0 map from an appropriate collection of approximately order 0 maps.

Recall that for a $C*$-algebra A we denote by A_∞ its sequence algebra. Often this will also be denoted by $A_\infty = \frac{\prod_{\mathbb{N}} A}{\bigoplus_{\mathbb{N}} A}$, that is bounded sequences on A quotiented by those sequences on A that tend to 0.

Lemma 4.1.4. Let A, B be C^*-algebras with B separable such that for every compact set $K \subset B$ of contractions and $\eta > 0$ there is a cpc η-K-order 0 map $\varphi : B \to A$. Then one may find a sequence $(\varphi_n)_{\mathbb{N}}$ of these maps such that $\phi(\cdot) = (\varphi_n(\cdot))_{\mathbb{N}}$ defines a cpc order 0 map $\phi : B \to A_\infty$.

Proof: Assume that for every $K \subset B$ compact and $\eta > 0$ there exists a cpc η-K-order 0 map $B \to A$. Let F_n be the first n elements of a dense sequence of B and let $\varphi_n : B \to A$ be a cpc $\frac{1}{n}$-F_n-order 0 map. This lets us define a cpc map $\varphi : B \to A_\infty$ by setting $\varphi(b) = (\varphi_n(b))_{\mathbb{N}}$. Pick $\varepsilon > 0$, and let $a, b \in B$ be orthogonal. We can find an $N' \in \mathbb{N}$ such that there exists $a' \in F_{N'}$ such that $\|a - a'\| < \frac{\varepsilon}{4\|b\|}$. Then we may find $N'' \in \mathbb{N}$ so that there is an element $b' \in F_{N''}$ such that $\|b - b'\| < \frac{\varepsilon}{4\|a'\|}$. Then let $N \in \mathbb{N}$ be such that $a', b' \in F_N$ and $\frac{1}{N} < \frac{\varepsilon}{2}$ so that φ_N is $\frac{\varepsilon}{2}$-F_N-order 0. Hence for all $n \geq N$ we have

$$\begin{aligned}
\|\varphi_n(a)\varphi_n(b)\| &\leq \|\varphi_n(a)\varphi_n(b) - \varphi_n(a')\varphi_n(b)\| + \|\varphi_n(a')\varphi_n(b) - \varphi_n(a')\varphi_n(b')\| \\
&\quad + \|\varphi_n(a')\varphi_n(b')\| \\
&< \varepsilon
\end{aligned}$$

so that $(\varphi_n(a)\varphi_n(b))_{\mathbb{N}}$ tends to 0. That is $\varphi(a)\varphi(b) = 0$ and hence φ is a cpc order 0 map. ∎

Furthermore if B is nuclear then Choi-Effros [17] quite easily allows us to show the converse of this statement.

Lemma 4.1.5. Let A, B be C^*-algebras with B nuclear and separable. If there is a cpc order 0 map $\phi : B \to A_\infty$ then for any compact set $K \subset B$ of contractions and $\eta > 0$ we may find a cpc η-K-order 0 map $\varphi : B \to A$. Moreover we may take a sequence $(\varphi_n)_{\mathbb{N}}$ of these maps such that $(\varphi_n(b))_{\mathbb{N}} = \phi(b) \in A_\infty$ for all $b \in B$.

Proof: Let $\phi : B \to A_\infty$ be cpc order 0. By Choi-Effros [17] we may find a cpc lift $\Phi : B \to \prod_{\mathbb{N}} A$ of ϕ. Let $K \subset B$ compact and $\eta > 0$ be given. Let $a, b \in K$ be such that $\|ab\| < \delta$ and there are orthogonal a', b' within $\frac{\delta}{2}$ of a, b respectively. Since ϕ is order 0 we have $\pi(\Phi(a')\Phi(b')) = \phi(a')\phi(b') = 0$ where $\pi : \prod_{\mathbb{N}} A \to A_\infty$ is the quotient map. Thus $\Phi_n(a)\Phi_n(b)$ is a sequence that tends to 0 and so there exists some $N_{ab} \in \mathbb{N}$ such that for all $n \geq N_{ab}$ we have $\|\Phi_n(a')\Phi_n(b')\| < \eta$. Let $N = \max N_{ab}$ for all such pairs in K, and set $\varphi = \Phi_N$ which is cpc η-K-order 0. By construction $\Phi = (\varphi_n)_{\mathbb{N}}$ is a sequence such that $(\varphi_n(b))_{\mathbb{N}} = \phi(b) \in A_\infty$ for all $b \in B$. ∎

These lemmas give an equivalence between order 0 maps $B \to A_\infty$ and collections of approximately order 0 maps $B \to A$, at least in the case B is nuclear and separable which is not an unreasonable assumption to make given our context. For such B then one can reformulate the definition of the relative dimension somewhat to be closer to that of nuclear dimension by making use of n-decomposable maps, in the traditional sense, that map into A_∞.

Proposition 4.1.6. Let A, B be C^*-algebras with B nuclear and separable. Then $\mathrm{nd}_B A \leq n$ if and only if there exists a net $(F_\lambda \otimes B, \psi_\lambda, \varphi_\lambda)_\Lambda$ with F_λ finite dimensional algebras, cpc maps $\psi_\lambda : A \to F_\lambda \otimes B$ and completely positive maps $\varphi_\lambda : F_\lambda \otimes B \to A_\infty$ such that

(i) $\varphi_\lambda\psi_\lambda(a) \to \iota(a)$ in norm for all $a \in A$, where $\iota : A \to A_\infty$ is the natural inclusion.

(ii) For each λ, F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that for each i the restriction $\varphi_\lambda^{(i)} : F_\lambda^{(i)} \otimes B \to A_\infty$ of φ_λ is a cpc order 0 map.

Proof: Assume $\mathrm{nd}_B A \leq n$, let $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ be given. Let $(F \otimes B, \psi, \sim\varphi)$ be a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε. For each $i = 0, \ldots, n$ let $F_m^{(i)}$ be the first m elements of a dense sequence of $F^{(i)} \otimes B$ and so pick $\varphi_m \in \sim\varphi$ such that each restriction $\varphi_m^{(i)}$ is $\frac{1}{m}$-$F_m^{(i)}$-order 0. For each i we may then define a cpc order 0 map $\phi^{(i)} : F^{(i)} \otimes B \to A_\infty$ by $\phi^{(i)}(f) = (\varphi_m^{(i)}(f))_{\mathbb{N}}$ by lemma 4.1.4 and note that by construction $\phi := \sum_{i=0}^{n} \phi^{(i)}$ will satisfy $\phi(f) = (\varphi_m(f))_{\mathbb{N}}$ for all $f \in F \otimes B$. Since for each $\varphi \in \sim\varphi$ we have $\|\varphi\psi(a) - a\| < \varepsilon$ for all $a \in \mathscr{F}$ this necessitates that we have $\|\phi\psi(a) - \iota(a)\| < \varepsilon$ for all $a \in \mathscr{F}$.

Conversely let $\mathscr{F}$ and $\varepsilon > 0$ be given and let $(F \otimes B, \psi, \phi)$ be such that F is finite dimensional, $\psi : A \to F \otimes B$ cpc and $\phi : F \otimes B \to A_\infty$ n-decomposable with $\|\phi\psi(a) - \iota(a)\| < \frac{\varepsilon}{2}$ for all $a \in \mathscr{F}$. For each $i = 0, \ldots, n$ we lift the cpc order 0 map $\phi^{(i)} : F^{(i)} \otimes B \to A_\infty$ to a cpc map $\Phi^{(i)} : F^{(i)} \otimes B \to \prod_{\mathbb{N}} A$. Set $\Phi := \sum_{i=0}^{n} \Phi^{(i)}$ and note that this lifts ϕ, so that if $\pi : \prod_{\mathbb{N}} A \to A_\infty$ is the quotient map we have that $\pi\Phi = \phi$. Thus $\|\pi\Phi\psi(a) - \iota(a)\| < \frac{\varepsilon}{2}$ for all $a \in \mathscr{F}$ and so we may find some $N \in \mathbb{N}$ such that for all $m \geq N$ we have $\|\Phi_m\psi(a) - a\| < \varepsilon$. Then for all $K_i \subset F^{(i)} \otimes B$ compact and $\eta_i > 0$ we may find, as in lemma 4.1.5, an $M \geq N$ such that the restrictions $\Phi_M^{(i)}$ are each η_i-K_i-order 0. If we set $\sim \Phi_N$ to be the set of those maps Φ_m for all $m \geq N$ we see that $(F \otimes B, \psi, \sim \Phi_N)$ is a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε. ∎

While we won't be making explicit use of this reformulated definition, the two lemmas that precede it will be very important.

To be able to justify why it should be the case that $\dim_{\text{nuc}} A \leq \text{nd}_{\mathbb{C}} A$ comes down to being able to define an order 0 map $\phi : F \to A$, where F is finite dimensional, given a colection of approximately order 0 maps $F \to A$. If one is able to make that statement then it is not difficult to see that we could make a similar statement that $\dim_{\text{nuc}} A \leq \text{nd}_G A$ for any finite dimensional G since we are still factoring through a finite dimensional algebras $F \otimes G$. In fact then one could generalise this idea further to consider $\text{nd}_B A$ where B has finite nuclear dimension since then if we factor through $F \otimes B$ and B is able to factor through finite dimensional G then A can be factored through $F \otimes G$. All of these cases rely on being able to obtain order 0 maps into A from approximately order 0 maps. This is able to be answered by the stability of order 0 maps on finite dimensional algebras. It is known, by [34, Thm 10.1.11] and [34, Thm. 10.2.1], that cones on finite dimensional algebras are projective, that is any $*$-homomorphism $C((0,1], F) \to \frac{A}{J}$ is liftable to a $*$-homomorphism $C((0,1], F) \to A$ for finite dimensional F and $J \triangleleft A$ an ideal. Combining this with the fact we have a one-to-one correspondence between cpc order 0 maps $B \to C$ and $*$-homomorphisms $C((0,1], B) \to C$ by 2.1.3, easily shows us that any cpc order 0 map $F \to \frac{A}{J}$ is liftable to a cpc order 0 map $F \to A$ for F finite dimensional and $J \triangleleft A$ an ideal. Thus if we start with an appropriate collection of approximately order 0 maps it stands to reason we would be able to find cpc order 0 maps that are *close* in some fashion.

Lemma 4.1.7. Let A, F be C^*-algebras with F finite dimensional. If for all $n \in \mathbb{N}$ we have a cpc map $\varphi_n : F \to A$ that is $\frac{1}{n}$-F_n-order 0 where F_n is the set of the first n elements of a dense sequence of F then there exist cpc order 0 maps $\phi_n : F \to A$ such that $\|\phi_n(f) - \varphi_n(f)\| \to 0$ for all $f \in F$.

Proof: By lemma 4.1.4 we may define a cpc order 0 map $\Phi : F \to A_\infty$ by $\Phi(\cdot) = (\varphi_n(\cdot))_{\mathbb{N}}$. This can be lifted to a cpc order 0 map $\phi : F \to \prod_{\mathbb{N}} A$ by the projectivity of cones on finite dimensional algebras, and since this is a lift we easily have that $\|\phi_n(f) - \varphi_n(f)\| \to 0$ for all $f \in F$. ∎

It is not too difficult to see from this how one might show that indeed $\dim_{\text{nuc}} A \le \text{nd}_G A$ when G is a finite dimensional algebra, we shall not prove it here though since it is an easy corollary of proposition 4.2.1 to come.

4.2 Properties of the Relative Dimension

In this section we present some of the general properties satisfied by the relative nuclear dimension, beginning with a general relationship between relative nuclear dimension and nuclear dimension. We are concerned as well with how well various properties of the nuclear dimension translate to the relative setting, since if the relative definition is supposed to only be a slight modification of nuclear dimension then how well it achieves this can be viewed in some way by how well it generalises properties of the nuclear dimension.

Proposition 4.2.1. Let A and B be C^*-algebras with $\dim_{\text{nuc}} B < \infty$. Suppose also that $\text{nd}_B A < \infty$, then A has finite nuclear dimension. Moreover

$$\dim_{\text{nuc}} A \le (\text{nd}_B A + 1)(\dim_{\text{nuc}} B + 1) - 1.$$

Proof: Let $\text{nd}_B A = m$ and $\dim_{\text{nuc}} B = n$. Let $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ be given. We may pick a relatively m-decomposable cp approximation $(F \otimes B, \psi_A, \sim\!\varphi_A)$ for $\mathscr{F}$ within $\frac{\varepsilon}{3}$. Consider the finite set $\psi_A(\mathscr{F})$ and note that since F is finite dimensional $F \otimes B$ is an algebraic tensor product, thus we may find $M \in \mathbb{N}$ such that for all $a \in \mathscr{F}$ we may write $\psi_A(a) = \sum_{i=1}^M f_{a_i} \otimes b_{a_i}$ for some $f_{a_i} \in F$ and $b_{a_i} \in B$. From this we may define the set $\mathscr{F}_B \subset B$ as the finite set containing the b_{a_i} for all $a \in \mathscr{F}$ and for all i and so obtain an n-decomposable cp approximation (G, ψ_B, φ_B) for $\mathscr{F}_B$ within $\frac{\varepsilon}{3(m+1)M}$. Our intention from here is to create an approximation for A factoring through $F \otimes G$. Consider the ideals $F^{(i)}$ of F for $i = 0, \dots, m$ that come from the relative dimension of A, and the ideals $G^{(j)}$ of G for $j = 0, \dots, n$ coming from the nuclear dimension of B. Then decompose $F \otimes G$ into ideals $(F \otimes G)^{(i,j)} := F^{(i)} \otimes G^{(j)}$ for $i = 0, \dots, m$ and $j = 0, \dots, n$. Denote by $B_{(i,j)}$ the unit ball of $(F \otimes G)^{(i,j)}$ and note that the images of each of these balls $\text{id}_{F^{(i)}} \otimes \varphi_B^{(j)}(B_{(i,j)})$ will be compact in the unit ball of $F^{(i)} \otimes B$. For any $\eta > 0$ we may find a $\varphi_A \in \sim\!\varphi_A$ such that each of the restrictions $\varphi_A^{(i)} : F^{(i)} \otimes B \to A$ are cpc η-$(\text{id}_F \otimes \varphi_B^{(j)})(B_{(i,j)})$-order 0. Necessarily then the composition $\varphi_A^{(i)}(\text{id}_{F^{(i)}} \otimes \varphi_B^{(j)})$ will be a cpc η-$B_{(i,j)}$-order 0 map. In particular for each $d \in \mathbb{N}$ we may find $\varphi_A \in \sim\!\varphi_A$ such that each restriction $\varphi_A^{(i)}(\text{id}_{F^{(i)}} \otimes \varphi_B^{(j)})$ is $\frac{1}{d}$-$B_{(i,j)}$-order 0.

In terms of the composition we shall denote this choice by $(\varphi_A(\mathrm{id}_F \otimes \varphi_B))_d$ and the corresponding restrictions by $(\varphi_A^{(i)}(\mathrm{id}_{F^{(i)}} \otimes \varphi_B^{(j)}))_d$. Thus by lemma 4.1.4 this allows us to define a cpc order 0 map $\phi^{(i,j)} : (F \otimes G)^{(i,j)}$ by $\phi^{(i,j)}(\cdot) = ((\varphi_A^{(i)}(\mathrm{id}_{F^{(i)}} \otimes \varphi_B^{(j)}))_d)_{\mathbb{N}}$. For each i, j we may lift $\phi^{(i,j)}$ to a cpc order 0 map $\Phi^{(i,j)} : (F \otimes G)^{(i,j)} \to \prod_{\mathbb{N}} A$ as in lemma 4.1.7. We may then find $D \in \mathbb{N}$ such that for all $d \geq D$ we have

$$\|\Phi_d^{(i,j)}(f) - (\varphi_A^{(i)}(\mathrm{id}_{F^{(i)}} \otimes \varphi_B^{(j)}))_d(f)\| < \frac{\varepsilon}{3(n+1)(m+1)}$$

for all $f \in (\mathrm{id}_{F^{(i)}} \otimes \psi_B^{(j)})(\psi_A^{(i)})(\mathscr{F})$. Set $\varphi^{(i,j)} = \Phi_D^{(i,j)}$ and define $\varphi := \sum_{i=0}^{m} \sum_{j=0}^{n} \varphi^{(i,j)}$. Set also $\psi = (\mathrm{id}_F \otimes \psi_B)\psi_A$ and let $\psi^{(i,j)} := (\mathrm{id}_{F^{(i)}} \otimes \psi_B^{(j)})\psi_A^{(i)}$ be the respective components.

We then require to show that the composition $\varphi\psi$ approximates the identity map on A. In the following we suppress the notation for neatness but we are working relative to the φ_A that corresponds to $(\varphi_A(\mathrm{id}_F \otimes \varphi_B))_D$. For all $a \in \mathscr{F}$ we have

$$\begin{aligned}
\|\varphi\psi(a) - a\| &\leq \|\varphi\psi(a) - \varphi_A(\mathrm{id}_F \otimes \varphi_B)\psi(a)\| + \|\varphi_A(\mathrm{id}_F \otimes \varphi_B)\psi(a) - a\| \\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n} \|\varphi^{(i,j)}\psi^{(i,j)}(a) - \varphi_A^{(i)}(\mathrm{id}_{F^{(i)}} \otimes \varphi_B^{(i)})\psi^{(i,j)}(a)\| \\
&\quad + \|\varphi_A(\mathrm{id}_F \otimes \varphi_B)\psi(a) - a\| \\
&\leq \frac{\varepsilon}{3} + \|\varphi_A(\mathrm{id}_F \otimes \varphi_B)(\mathrm{id}_F \otimes \psi_B)\psi_A(a) - a\| \\
&\leq \frac{\varepsilon}{3} + \|\varphi_A(\mathrm{id}_F \otimes \varphi_B\psi_B)\psi_A(a) - \varphi_A\psi_A(a)\| + \|\varphi_A\psi_A(a) - a\| \\
&\leq \sum_{k=0}^{M} \|(f_{a_k} \otimes (\varphi_B\psi_B(b_{a_i}) - b_{a_i})\| + \frac{2\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}$$

Thus $(F \otimes G, \psi, \varphi)$ is an $((n+1)(m+1)-1)$-decomposable cp approximation for $\mathscr{F}$ within ε in terms of nuclear dimension. ∎

Corollary 4.2.2. Let A, B be C^*-algebras with B AF. Then $\dim_{\mathrm{nuc}} A \leq \mathrm{nd}_B A$.

Proof: Immediate since B is AF so $\dim_{\mathrm{nuc}} B = 0$. ∎

This shows us that for all C^*-algebras A we have $\dim_{\mathrm{nuc}} A \leq \mathrm{nd}_{\mathbb{C}} A$. It is also quite easy to see $\mathrm{nd}_{\mathbb{C}} A \leq \dim_{\mathrm{nuc}} A$ since all systems for approximations for A in terms of nuclear dimension are automatically systems of relative approximations for A relative to $\mathbb{C}$. Thus we have for all C^*-algebras A that $\mathrm{nd}_{\mathbb{C}} A = A$, though it is notable that it is only relative to $\mathbb{C}$ that we get a general equivalence of the definitions. Even for finite dimensional B we do not get $\mathrm{nd}_B A \leq \dim_{\mathrm{nuc}} A$ in full generality, for example it can be seen that $\mathrm{nd}_{M_2}\mathbb{C} = \infty$ since to obtain a finite relative dimension in this case would necessitate the existence of a non-zero cpc order 0 map $M_2 \to \mathbb{C}_\infty$ when no such map exists.

Since finite nuclear dimension implies nuclearity, as per 2.2.4, we can similarly say that if A has finite nuclear dimension relative to B and B has finite nuclear dimension then A is nuclear. Moreover it is easy to show that the statement A *has finite nuclear dimension relative to some C^*-algebra B* is not by itself enough to imply nuclearity of A. Consider that for any C^*-algebra A it is trivially true that $\mathrm{nd}_A A = 0$ since the triple $(\mathbb{C} \otimes A, \mathrm{id}_A, \mathrm{id}_A)$ will be a relatively 0-decomposable cp approximation for any finite set $\mathscr{F} \subset A$ and for any $\varepsilon > 0$. Thus if A is non-nuclear its finite nuclear dimension relative to itself does not somehow imply nuclearity of A. Though B being of finite nuclear dimension isn't the only way that finite relative dimension can imply nuclearity; if B is nuclear then this will still imply nuclearity of A.

Proposition 4.2.3. Let A and B be C^*-algebras with B nuclear. If $\mathrm{nd}_B A < \infty$ then A is nuclear.

Proof: Assume $\mathrm{nd}_B A = n$ Let $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ be a system of relatively n-decomposable cp approximations for A. By the nuclearity of B we may find finite dimensional G_μ and cpc maps $\Psi_\mu : B \to G_\lambda$, $\Phi_\mu : G_\mu \to B$ such that $\Phi_\mu \Psi_\mu \to \mathrm{id}_B$ uniformly on finite sets of B. For each i the map $\varphi_\lambda^{(i)}(\mathrm{id}_{F_\lambda^{(i)}} \otimes \Phi_\mu)$ is cpc so that the compositions $(\varphi_\lambda(\mathrm{id}_{F_\lambda} \otimes \Phi_\mu))(\mathrm{id}_{F_\lambda} \otimes \Psi_\mu)\psi_\lambda)$ are uniformly bounded by $n+1$ and can be seen to approximate id_A. Then A can be seen to have the completely positive approximation property and thus be nuclear, similar to 2.2.4. ∎

These results quite immediately set some limits on when one can have finite relative nuclear dimension. If A is non-nuclear then it may only have finite dimension relative to algebras that are similarly non-nuclear, and if A is nuclear but does not have finite nuclear dimension, as in the case of the class of Villadsen algebras previously mentioned, then it will only be possible for it to have finite dimension relative to objects that themselves do not have finite nuclear dimension. This would raise the suggestion that to be able to show the Villadsen algebras have finite nuclear dimension relative to something, that something might well have to be another in the class of Villadsen algebras.

Remark 4.2.4. Here are some easy to show relations for the relative nuclear dimension.

(i) Consider an extension $0 \to J \to E \xrightarrow{\pi} A \to 0$ that is split in the sense that there exists a cpc map $\sigma : A \to E$ such that $\pi\sigma = \mathrm{id}_A$, then it can be quite easily seen that $\mathrm{nd}_E A = 0$. To see this note that $(\mathbb{C} \otimes E, \sigma, \pi)$ would be a relatively 0-decomposable cp approximation for any finite set in A for any ε. In particular this can be put into the context of generalised Toeplitz algebras, if $\phi : C(X) \to B(\mathcal{H})$ is an extension it will be a split extension in the sense above so that we have $\mathrm{nd}_{\mathcal{T}_\phi} C(X) = 0$ for any such extension.

(ii) It is also quite easy to see that we have $\mathrm{nd}_B A \otimes B \leq \dim_{\mathrm{nuc}} A$ for any C^*-algebras A, B, where we suppress the norm for the tensor since this statement only has content when $\dim_{\mathrm{nuc}} A < \infty$ thus A is nuclear. So if we assume $\dim_{\mathrm{nuc}} A = n$, let $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ be a system of n-decomposable cp approximations of A. Then for $i = 0, \ldots, n$ the maps $\varphi_\lambda^{(i)} \otimes \mathrm{id}_B : F_\lambda^{(i)} \otimes B \to A \otimes B$ are cpc order 0 by [49, Cor. 3.3] and thus $(F_\lambda \otimes B, \psi_\lambda \otimes \mathrm{id}_B, \varphi_\lambda \otimes \mathrm{id}_B)_\Lambda$ form a system of relatively n-decomposable cp approximations for $A \otimes B$. To see this is not an equality consider the example of $C([0,1]^\infty)$, since $[0,1]^\infty \cong [0,1]^\infty \times [0,1]^\infty$ we can see that $\mathrm{nd}_{C([0,1]^\infty)} C([0,1]^\infty) \otimes C([0,1]^\infty) = \mathrm{nd}_{C([0,1]^\infty)} C([0,1]^\infty) = 0 < \infty = \dim_{\mathrm{nuc}} C([0,1]^\infty)$.

With this remark we can say that for a stable algebra A, that is $A \cong A \otimes \mathcal{K}$, we have $\mathrm{nd}_{\mathcal{K}} A \leq \dim_{\mathrm{nuc}} A$. Combining this with proposition 4.2.1 we can easily see that for a stable algebra A we have $\dim_{\mathrm{nuc}} A = \mathrm{nd}_{\mathcal{K}} A$.

We can also make a statement on $\mathcal{Z}$-stable algebras, those for which $A \cong A \otimes \mathcal{Z}$, that in general for $\mathcal{Z}$-stable algebras A we have $\mathrm{nd}_{\mathcal{Z}} A \leq \dim_{\mathrm{nuc}} A$. Of particular note are those algebras A which fall under the Toms-Winter conjecture, that is simple, separable, unital, nuclear algebras A. In such a case it is known that if A is $\mathcal{Z}$-stable its nuclear dimension is either 0 or 1, depending of course on whether it is AF or not, [16, Cor. C]. If A is AF then immediately we have $\mathrm{nd}_{\mathcal{Z}} A = \dim_{\mathrm{nuc}} A = 0$, but if A is not AF the best we can say is $\mathrm{nd}_{\mathcal{Z}} A \leq \dim_{\mathrm{nuc}} A = 1$. By rearranging proposition 4.2.1 we get that

$$\mathrm{nd}_{\mathcal{Z}} A \geq \frac{\dim_{\mathrm{nuc}} A + 1}{\dim_{\mathrm{nuc}} \mathcal{Z} + 1} - 1,$$

but in this context it merely shows us that $\mathrm{nd}_{\mathcal{Z}} A \geq 0$. So the best we can say for such a $\mathcal{Z}$-stable algebra is that its dimension relative to $\mathcal{Z}$ is either 0 or 1, but it not clear if there should be a general value for the dimension overall. It shall be seen later in this work the meaning of relative dimension 0, which will give at least some insight into when we can say the dimension is 0 or 1 for such a $\mathcal{Z}$-stable algebra.

Otherwise this lower bound on $\mathrm{nd}_B A$ given by proposition 4.2.1 can give us some concrete examples of some non-zero relative dimensions. Consider the torus $\mathbb{T}^2$ and note that $C(\mathbb{T}^2) \cong C(\mathbb{T}) \otimes C(\mathbb{T})$. By proposition 4.2.1 we get

$$\mathrm{nd}_{C(\mathbb{T})} C(\mathbb{T}^2) \geq \frac{\dim_{\mathrm{nuc}} C(\mathbb{T}^2) + 1}{\dim_{\mathrm{nuc}} C(\mathbb{T}) + 1} - 1 = \frac{1}{2}.$$

Since relative dimension must be an integer value we get that $\mathrm{nd}_{C(\mathbb{T})} C(\mathbb{T}^2) \geq 1$. Our remark above shows then that $\mathrm{nd}_{C(\mathbb{T})} C(\mathbb{T}^2) = \mathrm{nd}_{C(\mathbb{T})} C(\mathbb{T}) \otimes C(\mathbb{T}) \leq \dim_{\mathrm{nuc}} C(\mathbb{T}) = 1$.

Hence necessarily $\mathrm{nd}_{C(\mathbb{T})}C(\mathbb{T}^2) = 1$. Similar statements can be made for any locally compact product space $X = Y \times D$. Since $C_0(X) \cong C_0(Y) \otimes C_0(D)$ we can obtain similar estimates. In particular we will have

$$\mathrm{nd}_{C_0(Y)}C_0(X) \geq \frac{\dim_{\mathrm{nuc}}C_0(X) + 1}{\dim_{\mathrm{nuc}}C_0(Y) + 1} - 1$$

and $\mathrm{nd}_{C_0(Y)}C_0(X) \leq \dim_{\mathrm{nuc}}C_0(D)$. So long as D is not the single point space this gives a finite relative dimension that is non-zero.

As a particular sanity check for our definition of relative dimension we would like to say that it satisfies, to some level, the various properties that nuclear dimension satisfies, see proposition 2.2.8, since it should stand that if we are trying to make a small modification to nuclear dimension the same properties, with slight modification themselves should be able to hold. The main difficulty inherent in proving these properties is the approximately order 0 condition, but had we chosen the more traditional notion for n-decomposability some of these properties could not be made to work. For the following properties we attempt to prove them in full generality; however, occasionally we shall need to make restrictions to those algebras they will work for. Particularly we need to make use of algebras being nuclear and separable at points so that we may use Choi-Effros [17]. It is not known if one might be able to prove the affected properties without making such an assumption, however it does not feel particularly unreasonable given that most of the algebras we are interested in dealing with are nuclear and separable.

Proposition 4.2.5. Let A, B, C be C^*-algebras. Then $\mathrm{nd}_C A \oplus B = \max\{\mathrm{nd}_C A, \mathrm{nd}_C B\}$.

Proof: We may assume that dimensions in questions are all finite so let $\mathrm{nd}_C A = n$ and $\mathrm{nd}_C B = m$, we shall also assume $n \geq m$. Let $\mathscr{F} \subset A \oplus B$ finite and $\varepsilon > 0$ be given. Define

$$\mathscr{F}_A := \{a \in A \mid (a, b) \in \mathscr{F} \text{ for some } b \in B\}$$

and similarly define $\mathscr{F}_B \subset B$. We then find a relatively n-decomposable cp approximation $(F \otimes C, \psi_A, \sim\varphi_A)$ for $\mathscr{F}_A$ and a relatively m-decomposable cp approximation $(G \otimes C, \psi_B, \sim\varphi_B)$ for $\mathscr{F}_B$, both within ε. We may then decompose $F \oplus G$ into n ideals using the decompositions coming from these approximations by setting $(F \oplus G)^{(i)} \otimes C = (F^{(i)} \oplus G^{(i)}) \otimes C$ for $i = 0, \ldots, n$ where we set $G^{(i)} \cong 0$ for $i = m+1, \ldots, n$. Note that for all i we have $(F \oplus G)^{(i)} \otimes C \cong (F^{(i)} \otimes C) \oplus (G^{(i)} \otimes C)$. Hence for any compact set $K_i \subset (F \oplus G)^{(i)} \otimes C$ of contractions and $\eta_i > 0$ we are able to define $K_{A_i} := \{a \in F^{(i)} \otimes C \mid (a, b) \in K \text{ for some } b \in G^{(i)} \otimes C\}$, and define $K_{B_i} \subset G^{(i)} \otimes C$ similarly.

We then have that K_{A_i} is a compact set in the unit ball of $F^{(i)} \otimes C$ and K_{B_i} is compact in the unit ball of $G^{(i)} \otimes C$. By the relative dimension of A we may find a $\varphi_A \in \sim\varphi_A$ such that the restriction to each $F^{(i)} \otimes C$ is a cpc η_i-K_{A_i}-order 0 map

$\varphi_A^{(i)} : F^{(i)} \otimes C \to A$ and such that $\varphi_A \psi_A \approx_\varepsilon \mathrm{id}_A$ on $\mathscr{F}_A$. In the same vein we may find a $\varphi_B \in \sim\varphi_B$ such that each restriction is a cpc η_i-K_{B_i}-order 0 map $\varphi_B^{(i)} : G^{(i)} \otimes C \to B$ and such that $\varphi_B \psi_B \approx_\varepsilon \mathrm{id}_B$ on $\mathscr{F}_B$. We may then define a cp map

$$\varphi : (F \oplus G) \otimes C \to A \oplus B$$

by setting $\varphi := \varphi_A \oplus \varphi_B$, and we can see that the restrictions $\varphi^{(i)}$ to the ideals $(F \oplus G)^{(i)} \otimes C$ will satisfy $\varphi^{(i)} = \varphi_A^{(i)} \oplus \varphi_B^{(i)}$. Then for each i we require to show that the restriction $\varphi^{(i)}$ is η_i-K_i-order 0. Note that $K_i \subset K_{A_i} \oplus K_{B_i}$. Then for $(a,b), (a',b') \in K_i$ such that $\|(a,b)\cdot(a',b')\| \le \delta$ and such that we have $(x,y), (x',y') \in (F \oplus G)^{(i)} \otimes C$ orthogonal within $\frac{\delta}{2}$ of these elements respectively we will have

$$\begin{aligned}
\|\varphi^{(i)}(x,y)\varphi^{(i)}(x',y')\| &= \|(\varphi_A^{(i)}(x), \varphi_B^{(i)}(y))(\varphi_A^{(i)}(x'), \varphi_B^{(i)}(y'))\| \\
&= \|(\varphi_A^{(i)}(x)\varphi_A^{(i)}(x'), \varphi_B^{(i)}(y)\varphi_B^{(i)}(y'))\| \\
&= \max\left\{\|\varphi_A^{(i)}(x)\varphi_A^{(i)}(x')\|, \|\varphi_B^{(i)}(y)\varphi_B^{(i)}(y')\|\right\} \\
&< \eta
\end{aligned}$$

as required. Let $\sim\varphi$ be the collection of maps defined by $\varphi = \varphi_A \oplus \varphi_B$ for $\varphi_A \in \sim\varphi_A$ and $\varphi_B \in \sim\varphi_B$. Thus defining the cpc map $\psi := \psi_A \oplus \psi_B$ we have, for all $\varphi \in \sim\varphi$,

$$\begin{aligned}
\|\varphi\psi((a,b)) - (a,b)\| &= \|(\varphi_A\psi_A(a) - a, \varphi_B\psi_B(b) - b)\| \\
&= \max\{\|\varphi_A\psi_A(a) - a\|, \|\varphi_B\psi_B(b) - b\|\} \\
&< \varepsilon
\end{aligned}$$

for all $(a,b) \in \mathscr{F}$. Hence $((F \oplus G) \otimes C, \psi, \sim\varphi)$ is a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε.

To see equality let $\mathscr{F} \subset A$ and $\varepsilon > 0$ be given and assume $\mathrm{nd}_C A \oplus B = n$. Let $\iota_A : A \to A \oplus B$ be the natural inclusion, that is $\iota_A(a) = (a, 0)$ for $a \in A$. Let $(F \otimes C, \psi, \sim\varphi)$ be a relatively n-decomposable cp approximation for $\iota_A(\mathscr{F})$ within ε. Let $\pi_A : A \oplus B \to A$ be the natural projection map, which is a $*$-homomorphism, and note that for any η-K-order 0 $\varphi^{(i)} : F^{(i)} \otimes C \to A \oplus B$ the composition $\pi_A \varphi^{(i)}$ will still be η-K-order 0, since for any orthogonal $f, g \in F^{(i)} \otimes C$ appropriately close to K we will have $\|\pi_A\varphi^{(i)}(f)\pi_A\varphi^{(i)}(g)\| \le \|\varphi^{(i)}(f)\varphi^{(i)}(g)\| < \eta$. Certainly also $\psi\iota_A$ will be cpc. Let $\sim\pi_A\varphi$ be the set of those maps $\pi\varphi$ for all $\varphi \in \sim\varphi$, then $(F \otimes C, \psi\iota_A, \sim\pi\varphi)$ can be seen to be a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε. A similar argument may be made for B also. ∎

Proposition 4.2.6. Let A, B, C, D be C^*-algebras, with C, D nuclear and separable. Then

$$\mathrm{nd}_{C \otimes D} A \otimes B \le (\mathrm{nd}_C A + 1)(\mathrm{nd}_D B + 1) - 1.$$

Note that we may suppress the norm we are using for $A \otimes B$ here since we are assuming C and D are both nuclear. Then since we may assume both $\mathrm{nd}_C A$ and $\mathrm{nd}_D B$ are finite we get that A and B are both nuclear by proposition 4.2.3.

Proof: Assume $\mathrm{nd}_C A = n$ and $\mathrm{nd}_D B = m$ for $n, m \in \mathbb{N}$. Let $\mathscr{F}$ be a finite set in $A \otimes B$ and $\varepsilon > 0$ be given. For each element $x \in \mathscr{F}$ we can find an algebraic tensor $x' \in A \otimes B$ that is within $\frac{\varepsilon}{3(n+1)(m+1)}$ of x, and denote by $\mathscr{F}'$ the finite set of these x'. We may find an $M \in \mathbb{N}$ such that each $x' \in \mathscr{F}'$ may be represented as the sum of M elementary tensors, $x' = \sum_{i=1}^{M} a_i \otimes b_i$, and so define finite sets $\mathscr{F}'_A \subset A$ and $\mathscr{F}'_B \subset B$ as the collection of the $a_i \in A$ and $b_i \in B$ represented in these sums. Let $P = \max\left\{\sup_{a_i \in \mathscr{F}_A} \|a_i\|, \sup_{b_i \in \mathscr{F}_B} \|b_i\|\right\}$. Then we may find a relatively n-decomposable cp approximation $(F \otimes C, \psi_A, \sim\varphi_A)$ for $\mathscr{F}'_A$ within $\frac{\varepsilon}{6MP(m+1)}$ and a relatively m-decomposable cp approximation $(G \otimes D, \psi_B, \sim\varphi_B)$ for $\mathscr{F}'_B$ within $\frac{\varepsilon}{6MP(n+1)}$. Note that $(F \otimes G) \otimes (C \otimes D) \cong (F \otimes C) \otimes (G \otimes D)$, and thus we can decompose the finite dimensional algebras into ideals in the following way:

$$(F \otimes G)^{(i,j)} \otimes (C \otimes D) \coloneqq (F^{(i)} \otimes C) \otimes (G^{(j)} \otimes D).$$

For $i = 0, \ldots, n$ and for $j = 0, \ldots, m$ we may find, by lemma 4.1.4 and by the relative dimension, cpc order 0 maps $\phi_A^{(i)} : F^{(i)} \otimes C \to A_\infty$ and $\phi_B^{(j)} : G^{(j)} \otimes D \to B_\infty$ such that $\phi_A \psi_A \approx \iota_A$ and $\phi_B \psi_B \approx \iota_B$ where ι_A and ι_B are the respective inclusions into the sequence algebras. Then the tensor maps

$$\phi_A^{(i)} \otimes \phi_B^{(j)} : (F \otimes G)^{(i,j)} \otimes (C \otimes D) \to A_\infty \otimes B_\infty$$

will be cpc order 0 for all i, j by 2.1.4. Let $\pi : A_\infty \otimes B_\infty \to (A \otimes B)_\infty$ be the natural $*$-homomorphism, that is the map defined by $\pi((a_n)_\mathbb{N} \otimes (b_n)_\mathbb{N}) = (a_n \otimes b_n)_\mathbb{N}$ and extended to all of $A_\infty \otimes B_\infty$ by [15, Prop. 3.3.7]. Thus the composition $\pi(\phi_A^{(i)} \otimes \phi_B^{(j)})$ is a cpc order 0 map into $(A \otimes B)_\infty$. Then by lemma 4.1.5 we may find, for all compact $K \subset (F \otimes G)^{(i,j)} \otimes (C \otimes D)$ and all $\eta > 0$, some $N \in \mathbb{N}$ such that we have $\varphi^{(i,j)} \coloneqq \pi(\phi_A^{(i)} \otimes \phi_B^{(j)})_N$ is a cpc η-K-order 0 map. We then define a cp map $\varphi : (F \otimes G) \otimes (C \otimes D) \to A \otimes B$ by $\varphi \coloneqq \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi^{(i,j)}$, and so let $\sim\varphi$ be the collection of those φ such that the restrictions $\varphi^{(i,j)}$ are all $\frac{1}{d}$-$K_d^{(i,j)}$-order 0 where $K_d^{(i,j)}$ is the set of the first d elements of a dense sequence of $(F \otimes G)^{(i,j)} \otimes (C \otimes D)$ over $d \in \mathbb{N}$. We thus require to show that such φ give us an approximation for $\mathscr{F}$. Note that for an elementary tensor $f^i \otimes g^j \in (F^{(i)} \otimes C) \otimes (G^{(j)} \otimes D)$ we have $\varphi^{(i,j)}(f^i \otimes g^j) = \phi_A^{(i)}(f^i)_N \otimes \phi_B^{(j)}(g^j)_N = \varphi_A^{(i)}(f^i) \otimes \varphi_B^{(j)}(g^j)$ for appropriate φ_A, φ_B. Recall that for each $x \in \mathscr{F}$ we have chosen an algebraic tensor $x' = \sum_{s=1}^{M} a_s \otimes b_s$, with each a_s, b_s in $\mathscr{F}'_A$ and $\mathscr{F}'_B$ respectively. Define $\psi : A \otimes B \to (F \otimes G) \otimes (C \otimes D)$ by $\psi \coloneqq \psi_A \otimes \psi_B$. We thus have

$$\|\varphi\psi(x) - x\| \le \|\varphi\psi(x) - \varphi\psi(x')\| + \|\varphi\psi(x') - x'\| + \|x' - x\|$$

$$
\begin{aligned}
&< \frac{2\varepsilon}{3} + \sum_{s=1}^{M} \|\varphi_A\psi_A(a_s) \otimes \varphi_B\psi_B(b_s) - a_s \otimes b_s\| \\
&\leq \frac{2\varepsilon}{3} + \sum_{s=1}^{M} (\|\varphi_A\psi_A(a_s) \otimes \varphi_B\psi_B(b_s) - \varphi_A\psi_A(a_s) \otimes b_s\| \\
&\quad + \|\varphi_A\psi_A(a_s) \otimes b_s - a_s \otimes b_s\|) \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

for all $x \in \mathscr{F}$ So $((F \otimes G) \otimes (C \otimes D), \psi, \sim\varphi)$ is a relatively $((n+1)(m+1)-1)$-decomposable cp approximation for $\mathscr{F}$ within ε. ∎

Proposition 4.2.7. Let A, B, C be C^*-algebras with B nuclear and separable. If there exists a surjective $*$-homomorphism $\pi : A \to B$ then $\mathrm{nd}_C B \leq \mathrm{nd}_C A$.

Proof: By Choi-Effros [17] we may find a cpc lift $\sigma : B \to A$ of the identity map on B such that $\pi\sigma = \mathrm{id}_B$. Let finite $\mathscr{F} \subset B$ and ε be given and assume $\mathrm{nd}_C A = n$. Pick a relatively n-decomposable cp approximation $(F \otimes C, \psi, \sim\varphi)$ for $\sigma(\mathscr{F}) \subset A$ within ε. Note that the composition $\psi\sigma$ is certainly cpc. Consider that since π is a $*$-homomorphism then for any η-K-order 0 map $\varphi^{(i)} : F^{(i)} \otimes C \to A$ the composition $\pi\varphi^{(i)}$ will still be η-K-order 0, since for any appropriate orthogonal elements f, g near K we will have $\|\pi\varphi^{(i)}(f)\pi\varphi^{(i)}(g)\| \leq \|\varphi^{(i)}(f)\varphi^{(i)}(g)\| < \eta$. Then for all $\varphi \in \sim\varphi$ we will have

$$
\begin{aligned}
\|\pi\varphi\psi\sigma(b) - b\| &= \|\pi\varphi\psi\sigma(b) - \pi\sigma(b)\| \\
&\leq \|\varphi\psi\sigma(b) - \sigma(b)\| \\
&< \varepsilon
\end{aligned}
$$

for all $b \in \mathscr{F}$. Thus if we denote by $\sim\pi\varphi$ the set of maps $\pi\varphi$ for all $\varphi \in \sim\varphi$ we can see that $(F \otimes C, \psi\sigma, \sim\pi\varphi)$ is a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε. ∎

Proposition 4.2.8. Let A, B be C^*-algebras with A nuclear and separable, and $A = \lim_{\to} A_n$ an inductive limit C*-algebra. Then $\mathrm{nd}_B A \leq \liminf(\mathrm{nd}_B A_n)$.

Proof: Let $\mu_n : A_n \to A$ be the induced map, and note that by 4.2.7 we have that $\mathrm{nd}_B \mu_n(A_n) \leq \mathrm{nd}_B A_n$ for all n. Thus for simplicity we shall assume that A is the closure of the increasing union of the A_n, that is $A = \overline{\bigcup_{\mathbb{N}} A_n}$, rather than the closure of the union of $\mu_n(A_n)$. Let $\liminf_{n\to\infty} \mathrm{nd}_B A_n = M < \infty$. By definition for any δ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ there exists an $m \geq n$ such that $\mathrm{nd}_B A_m < M+\delta$, so that in particular $\mathrm{nd}_B A_m = M$ for small enough δ. We can find some finite set $\mathscr{F}_N \subset A_N$ for $N \in \mathbb{N}$ that is within $\frac{\varepsilon}{12(M+1)}$ of $\mathscr{F}$. For each $n \geq N$ choose a relative approximation $(F_n \otimes B, \psi_n, \sim\varphi_n)$ of A_n for $\mathscr{F}_N$ within $\frac{\varepsilon}{12}$. In particular for all $n \in \mathbb{N}$ there exists $m_n \geq n$ such that $\mathrm{nd}_B A_{m_n} = M$. Let $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ be given. We may find $N \in \mathbb{N}$ and a finite set $\mathscr{F}' \subset A_N$ that is within $\frac{\varepsilon}{4(M+1)}$ of $\mathscr{F}$. Then for each $n \geq N$ let $(F_{m_n} \otimes B, \psi_{m_n}, \sim\varphi_{m_n})$ be a relatively M-decomposable cp approximation of A_{m_n} for $\mathscr{F}' \subset A_{m_n}$ within $\frac{\varepsilon}{4}$. This allows us to define a cpc map

$$\psi : \bigcup_{n\geq N} A_{m_n} \to \frac{\prod_{n\geq N} F_{m_n} \otimes B}{\bigoplus_{n\geq N} F_{m_n} \otimes B}$$

by setting $\psi(a) = (\psi_{m_n}(a))_{\mathbb{N}}$ where we consider $\psi_{m_n}(a) = 0$ for $a \notin A_{m_n}$. This can be continuously extended to a cpc map $\psi' : A \to \frac{\prod_{n\geq N} F_{m_n}\otimes B}{\bigoplus_{n\geq N} F_{m_n}\otimes B}$, and by Choi-Effros [17] we may lift this to a cpc map $\psi'' : A \to \prod_{n\geq N} F_{m_n} \otimes B$. If π is the natural quotient map on $\prod_{n\geq N} F_{m_n} \otimes B$ we have that $\pi\psi'' = \pi'$ and so $\pi\psi'' = \pi$ on $\bigcup_{n\geq N} A_{m_n}$, thus for all $a \in \mathscr{F}$ there exists an $a' \in \mathscr{F}'$ such that $\|\pi\psi''(a) - \psi(a')\| < \frac{\varepsilon}{4(M+1)}$. Thus by definition of the quotient we are able to find a sequence $(x_{m_n})_{n\geq N}$ in $\prod_{n\geq N} F_{m_n} \otimes B$ that tends to 0 such that $\|\psi''(a)_{m_n} - \psi_{m_n}(a') + x_{m_n}\| < \frac{\varepsilon}{4(M+1)}$ so that we have $\|\psi''(a)_{m_n} - \psi_{m_n}(a')\| < \|x_{m_n}\| + \frac{\varepsilon}{4(M+1)}$. Thus there exists $k \geq N$ such that $\|\psi''(a)_{m_k} - \psi_{m_k}(a')\| < \frac{\varepsilon}{2(M+1)}$. Set $\tilde{\psi} := \psi''(\cdot)_{m_k} : A \to F_{m_k} \otimes B$ and take the $\varphi_{m_k} : F_{m_k} \otimes B \to A$ from the relative dimension of A_{m_k}. Thus for all $a \in \mathscr{F}$ we have

$$\begin{aligned}\|\varphi_{m_k}\tilde{\psi}(a) - a\| &\leq \|\varphi_{m_k}\tilde{\psi}(a) - \varphi_{m_k}\psi_{m_k}(a')\| + \|\varphi_{m_k}\psi_{m_k}(a') - a'\| + \|a' - a\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon.\end{aligned}$$

Hence $(F_{m_k} \otimes B, \tilde{\psi}, \sim\varphi_{m_k})$ is a relatively M-decomposable cp approximation for $\mathscr{F}$ within ε. ∎

4.3 Relative Dimension 0

An important question to consider for our relative dimension is the meaning of relative dimension 0. Specifically if $\mathrm{nd}_B A = 0$ can we then make some statement about the structure of A relative to B, in a similar vein to the fact that $\dim_{\mathrm{nuc}} A = 0$ if and only if A is AF. The proof of this latter fact provides some inspiration as to what we might expect such a statement to say, and indeed how we might go around proving it. Certainly we know that if A exists as an inductive limit of the form $(F_n \otimes B, \phi_n)$ for finite dimensional F_n it is easy to see that $\mathrm{nd}_B A = 0$ since we know by 4.2.8 that $\mathrm{nd}_B A \leq \liminf \mathrm{nd}_B F_n \otimes B \leq \liminf \dim_{\mathrm{nuc}} F_n = 0$. To show the other direction is a bit more tricky. If $\dim_{\mathrm{nuc}} A = 0$ then one obtains a system of 0-dimensional approximations $(F_\lambda, \psi_\lambda, \varphi_\lambda)_\Lambda$ such that each of the φ_λ are cpc order 0 maps. Under the correct hypotheses it is known that a cpc order 0 map from a finite dimensional algebra is in fact a $*$-homomorphism, [47, Thm. 4.1.1], and so you are able to show that A is able to be approximated by finite dimensional algebras so is AF in the sense of Bratteli [11]. It is this style of statement we should like to make as a consequence of $\mathrm{nd}_B A = 0$, but since we only get a system $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ of relatively 0-decomposable cp approximations the maps up are cpc approximately order 0. Since the maps are approximately order 0 we won't be able to show that they are in general $*$-homomorphisms, though we can in fact show that they can be assumed to be approximately multiplicative in some sense. Thus it makes sense that we will need some other notion than the usual inductive limit in order to make any statements regarding the structure of A. Particularly we are led to the notion of generalised inductive limit introduced by Blackadar and Kirchberg [5] which is similar to the more usual notion but crucially does not rely on the connecting maps being $*$-homomorphisms, rather only requiring that they are asymptotically so.

First we present the definition of a generalised inductive system.

Definition 4.3.1. [5, Def 2.1.1] A generalised inductive system of C^*-algebras is a sequence (A_n) of C^*-algebras, with coherent maps $\phi_{nm} : A_n \to A_m$ for $n < m$, such that for all $k \in \mathbb{N}$ and all $x, y \in A_k$, $\lambda \in \mathbb{C}$, and all $\varepsilon > 0$, there is an M such that for all $M \leq m < k$,

(1) $\|\phi_{mk}(\phi_{nm}(x) + \phi_{nm}(y)) - (\phi_{nk}(x) + \phi_{nk}(y))\| < \varepsilon$

(2) $\|\phi_{mk}(\lambda\phi_{nm}(x)) - \lambda\phi_{nk}(x)\| < \varepsilon$

(3) $\|\phi_{mk}(\phi_{nm}(x)^*) - \phi_{nk}(x)^*\| < \varepsilon$

(4) $\|\phi_{mk}(\phi_{nm}(x)\phi_{nm}(y)) - \phi_{nk}(x)\phi_{nk}(y)\| < \varepsilon$

(5) $\sup_r \|\phi_{k,r}(x)\| < \infty$.

Remark 4.3.2. The particular notion we are considering here for coherence of maps is an asymptotic version. That is for all $n \in \mathbb{N}$, for all $x \in A_n$ and $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $M \leq m < k$ we have $\|\phi_{nk}(x) - \phi_{mk}\phi_{nm}(x)\| < \varepsilon$.

For our purposes we are only really interested in dealing with generalised systems for which the connecting maps ϕ_{nm} are all cpc. For a collection of cpc maps ϕ_{nm} that satisfy the above asymptotic coherence condition it can be seen that the ϕ_{nm} will trivially satisfy all conditions from the above definition aside from (4)

In order to define the limit algebra A of a generalised system $(A_n, \phi_{n,m})$ one takes the set theoretic limit L of the A_n, which is the disjoint union of the A_n quotiented by the relation of identifying an $x \in A_n$ with $\phi_{n,m}(x)$ for all $m \geq n$. We then use $\phi_n(x)$ to denote this class for $x \in A_n$. Then one defines a pseudometric d on L by setting $d(\phi_n(x), \phi_n(y)) = \sup_{m \in \mathbb{N}} \|\phi_{n,m}(x) - \phi_{n.m}(y)\|$. The limit algebra A then is the completion of L with respect to d, and this is a C^*-algebra, see [5, 2.1.2].

An alternative construction, and one that we make more ready use of, of the limit algebra, [5, 2.1.3], is that it can be embedded in the sequence algebra $\frac{\prod_{\mathbb{N}} A_n}{\bigoplus_{\mathbb{N}} A_n}$ as the closure of the set of elements of the form $\phi_n(x) = (\phi_{nm}(x))_{m \geq n}$ for $x \in A_n$. In general with no assumptions on the connecting maps there's not much that can be said regarding these connecting maps, since each condition of 4.3.1 is only true for a finite set at a time there's no guarantee they force linearity or even boundedness onto the induced map. However for example, [5, Prop. 2.3.1], if all maps ϕ_{nm} are linear and bounded then the induced maps will be bounded through application of Banach-Steinhaus. Necessarily it can be seen that these induced maps will be cpc when the connecting maps are all cpc, but condition (4) of 4.3.1 does not by itself imply that the induced map will be a $*$-homomorphism, for that we would need to make a statement on the multiplicativity of ϕ_{nm} for increasing m.

We are unable to show a precise equivalence between $\mathrm{nd}_B A = 0$ and A being a generalised inductive limit $A \cong (F_n \otimes B, \phi_{nm})$ for F_n finite dimensional with the connecting maps all cpc; however, we are still able to obtain implications in both directions, just with restrictions to their hypotheses that does not allow for an equivalence. We present these results now.

Proposition 4.3.3. Let A, B be C^*-algebras with A nuclear and separable. Assume A exists as a generalised inductive limit $A \cong \lim(F_n \otimes B, \phi_{nm})$ where the F_n are finite dimensional algebras and the connecting maps ϕ_{nm} are all cpc. If for all $n \in \mathbb{N}$ the induced map $\phi_n : F_n \otimes B \to A$ is a $*$-homomorphism, then $\mathrm{nd}_B A = 0$.

Proof: Let $\mathscr{F} \subset A$ and $\varepsilon > 0$ be given. We may find $N \in \mathbb{N}$ such that for all $a \in \mathscr{F}$ there exists $a' \in F_N \otimes B$ such that $\|a - \phi_N(a')\| < \frac{\varepsilon}{4}$, and let $\mathscr{F}' \subset F_N \otimes B$ be the corresponding finite set. By Choi-Effros [17] we may lift the embedding $\iota : A \hookrightarrow \frac{\prod_{\mathbb{N}} F_n \otimes B}{\bigoplus_{\mathbb{N}} F_n \otimes B}$ to a cpc map $\pi : A \to \prod_{\mathbb{N}} F_n \otimes B$. Since π lifts the embedding ι we can also see that $\pi\phi_N$ lifts ϕ_N. Thus we may find some $M' \in \mathbb{N}$ such that for all $m \geq M'$ we have $\|\phi_{nm}(a') - \pi_m(\phi_N(a'))\| < \frac{\varepsilon}{4}$ for all $a' \in \mathscr{F}'$. We may also find, by the coherence of the maps, some $M'' \in \mathbb{N}$ such that for all $M'' \leq m < k$ we have that $\|\phi_{nk}(a') - \phi_{mk}(\phi_{nm}(a')\| < \frac{\varepsilon}{4}$ for all $a' \in \mathscr{F}'$. This implies that for all $m \geq M''$ we have that $\|\phi_N(a') - \phi_m(\phi_{Nm}(a'))\| < \frac{\varepsilon}{4}$ for all $a' \in \mathscr{F}'$ Let $M = \max\{M', M''\}$ so that for all $a \in \mathscr{F}$ we have

$$\begin{aligned}\|\phi_M \pi_M(a) - a\| &\leq \|\phi_M \pi_M(\phi_N(a')) - \phi_N(a')\| + 2\|a - \phi_N(a')\| \\ &\leq \phi_M \pi_M(\phi_N(a')) - \phi_M(\phi_{NM}(a'))\|\end{aligned}$$

$$+\|\phi_M(\phi_{NM}(a')) - \phi_N(a')\| + \frac{\varepsilon}{2}$$
$$< \varepsilon.$$

Then since ϕ_M is a $*$-homomorphism, by assumption, it is certainly cpc order 0, thus $(F_M \otimes B, \pi_M, \phi_M)$ is a relatively 0-decomposable cp approximation for $\mathscr{F}$ within ε. ∎

The hypothesis that the induced maps ϕ_n are $*$-homomorphisms is necessary for the proof in order to find cpc order 0 maps up for our approximations; however, it is unclear what, if any, weakenings of this condition would provide the same conclusion. Certainly we cannot merely assume that the connecting maps are cpc with no additional information as there exist algebras A that are a generalised inductive limit $A \cong \lim(F_n, \phi_{nm})$ where the F_n are finite dimensional and the connecting maps cpc but A is not AF. This is the class of NF algebras, see [5, Sec. 5], that are defined as generalised limits of finite dimensional algebras with cpc connecting maps, and while certainly the AF algebras sit inside this class not all NF algebras are themselves AF. If our assumption for proposition 4.3.3 was merely that the connecting maps are cpc then any NF algebra A would satisfy $\mathrm{nd}_{\mathbb{C}} A = 0$ but since we know $\mathrm{nd}_{\mathbb{C}} A = \dim_{\mathrm{nuc}} A$ this would mean we could find an algebra that isn't AF but $\dim_{\mathrm{nuc}} A = 0$. If $A \cong \lim(F_n, \phi_{nm})$ is an NF algebra with the additional assumption that each of the induced maps ϕ_n are $*$-homomorphisms it can be seen that A is necessarily AF. To see this consider for a finite set $\mathscr{F} \subset A$ and $\varepsilon > 0$ we may find some $N \in \mathbb{N}$ such that for all $a \in \mathscr{F}$ there exists $a' \in F_n$ such $\|a - \phi_N(a')\| < \varepsilon$. Since ϕ_N is a $*$-homomorphism from a finite dimensional F_N satisfying this A immediately satisfies Bratteli's local characterisation of AF algebras [11].

To show the reverse direction we are going to need to make use of an asymptotic intertwining that is defined as follows.

Definition 4.3.4. [5, Def. 2.4.1] Let $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ be generalised inductive systems of C^*-algebras. An asymptotic map from $(A_n, \phi_{m,n})$ to $(B_n, \psi_{m,n})$ is a pair of increasing sequences (r_n) and (s_n) and maps $\alpha_n : A_{s_n} \to B_{r_n}$, such that

(1) for every k, every $x \in A_k$, and $\varepsilon > 0$ there exists an M such that for all $M \leq m < n$,

$$\|\psi_{s_m,s_n}\alpha_m\phi_{k,r_m}(x) - \alpha_n\phi_{k,r_n}(x)\| < \varepsilon.$$

(2) The induced map from $\bigcup \phi_n(A_n)$ to $\bigcup \psi_n(B_n)$ extends to a bounded *-homomorphism α from $A = \lim(A_n, \phi_{m,n})$ to $B = \lim(B_n, \psi_{m,n})$.

An asymptotic intertwining between $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ is a pair of increasing sequences (r_n) and (s_n) and asymptotic maps $\alpha_n : A_{r_n} \to B_{s_n}$ and $\beta_n : B_{s_n} \to A_{r_{n+1}}$ such that for every k, every $x \in A_k, y \in B_k$ and $\varepsilon > 0$ there is an M such that for all $M \leq m < n$,

(3) $\|\beta_n \psi_{s_m,s_n} \alpha_m \phi_{k,r_m}(x) - \phi_{k,r_n+1}(x)\| < \varepsilon$

(4) $\|\alpha_n \phi_{r_{m+1},r_n} \beta_m \psi_{k,s_m}(x) - \psi_{k,s_n}(x)\| < \varepsilon$.

It is noted in [5] that the two $*$-homomorphisms described in condition (2) induced by an asymptotic intertwining will automatically be mutually inverse $*$-isomorphisms.

For our purposes, since we are considering only cases where the connecting maps are cpc, and we shall in fact also be requiring the asymptotic maps are cpc then by [5, Prop 5.1.6] we only need to show that conditions $(1),(3)$, and (4) are satisfied in order to have an asymptotic intertwining.

The basic idea of our proof to show that $\mathrm{nd}_B A = 0$ is to take a system $(F_n, \psi_n, \tilde{\varphi}_n)_{\mathbb{N}}$ of relatively 0-decomposable cp approximations for A, where we shall be assuming A is separable. From such a system we create a generalised inductive system $(F_n \otimes B, \phi_{nm})$ be defining maps $\phi_{nm} = \psi_m \varphi_n$. While there are a few assumptions to be made here so that the coherence of these is satisfied it is notable that we still would require the asymptotic multiplicativity condition to be satisfied for this to actually be a generalised system. The result that cpc order 0 maps from finite dimensional algebras are $*$-homomorphisms in the right setting [47, Thm 4.1.1] has since been improved to a general structure theorem on cpc order 0 maps 2.1.2 that gives us a condition on when any cpc order 0 map can be seen to be a $*$-homomorphism. It is this result that allows us to say that for $\mathrm{nd}_B A = 0$ we may find a system of relatively 0-decomposable cp approximations for A such that the maps up can be taken to be approximately multiplicative. We also take inspiration from 2.2.10 in order to show that the maps down can be taken to be approximately multiplicative. With these results together we are able to create a generalised inductive system $(F_n \otimes B, \psi_m \varphi_n)$ and show that there is an asymptotic intertwining between this and the trivial inductive system on $A \cong \lim(A, \mathrm{id}_A)$.

While 2.2.10 is a result in terms of the decomposition rank it is still relevant to us here. Recall that $\dim_{\mathrm{nuc}} A = 0 \iff \mathrm{dr} A = 0$ so that if we have nuclear dimension 0 we may still use this result to obtain a system of 0-decomposable approximations for A such that the maps down are approximately multiplicative. Similarly here it can be seen that $\mathrm{nd}_B A = 0$ is equivalent to a notion of relative decomposition rank being 0 so it stands to reason we may show the maps down can be taken to be approximately multiplicative. However our result to show this is restricted somewhat, it comes with the caveat that B is a continuous $C(X)$-algebra with simple fibres. Let us explore what we mean by such an object.

$C(X)$-algebras are an object first defined by Kasparov [27] though of course has been elaborated upon over the years, in particular by Blanchard [6]. We refer to Dadarlat [20] as to where we learnt most of the material presented here. Note that while the notion of a $C(X)$-algebra can be defined for any locally compact Hausdorff X we still refer to these objects as $C(X)$-algebras rather than $C_0(X)$-algebras. This is the notational convention in the above sources that we choose to maintain. Let X be a locally compact Hausdorff space, a $C(X)$- algebra is a C^*-algebra A with a structure $*$-homomorphism $\theta : C_0(X) \to ZM(A)$ into the centre of the multiplier algebra on A and such that $C_0(X)A$ is dense in A, where it is understood that we write fa in place of $\theta(f)a$ for $f \in C_0(X)$ and $a \in A$. For a closed subset $Y \subset X$ we denote by $C_0(X,Y)$ the continuous functions vanishing on Y. Then $C_0(X,Y)A$ is a closed two sided ideal of A and we denote by $A(Y)$ the quotient $\frac{A}{C_0(X,Y)A}$. It can be seen that regarding this notation $A = A(X)$, and so we shall denote by $\pi_Y : A(X) \to A(Y)$ the quotient map. For a point $x \in X$ we denote by $A(x)$ the quotient algebra $A(\{x\})$, similarly π_x for the quotient map, and for $a \in A$ the image $\pi_x(a)$ is denoted by $a(x)$. These algebras $A(x)$ for $x \in X$ are what we call the fibres of A. It is known, [20, Lemma 2.1], that for all $a \in A$ the map $x \mapsto \|a(x)\|$ is upper semi-continuous. A continuous $C(X)$-algebra A then is one for which this map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$.

It is the continuous $C(X)$-algebras that we concerned with, since they have a particularly valuable equivalence to a continuous field of C^*-algebras. It is the case, see [6] [8] [36], that A is a continuous $C(X)$-algebra if and only if it is the C^*-algebra of continuous sections of a continuous field of C^*-algebras over X in the sense of Dixmier [22, Def. 10.3.1], the definition for which we shall now go over.

For a topological space X, a continuous field of C^*-algebras over X is a family of C^*-algebras $(A_x)_{x\in X}$ and a $*$-algebra $\Theta \subset \prod_{x\in X} A_x$ of vector fields, functions a defined on X such that $a(x) \in A_x$ for all $x \in X$, such that

1) For every $x \in X$ the set of $a(x)$ for $a \in \Theta$ is dense in A_x,
2) For every $a \in \Theta$ the function $x \mapsto \|a(x)\|$ is continuous,
3) Let $a \in \prod_{x\in X} A_x$ be a vector field. If for every $x \in X$ and every $\varepsilon > 0$ there is $a' \in \Theta$ such that $\|a(y) - a'(y)\| < \varepsilon$ on some neighbourhood of x, then $a \in \Theta$.

For a locally compact space X and a continuous field $\mathscr{A} = (A_x, \Theta)$ over X, take the set $A \subset \Theta$ of elements a such that $\|a(t)\|$ vanishes at infinity on X. A will be a C^*-algebra when equipped with the norm $\|a\| = \sup_{x\in X} \|a(x)\|$. This is the C^*-algebra generated by the continuous field.

If A is a continuous $C(X)$-algebra we may use the fibres of A to define a family of C^*-algebras over X by $A_x = A(x)$. Each $a \in A$ can thus be viewed as a vector field by setting $a(x) = \pi_x(a)$ as above. Then A shall satisfy the definition to be a continuous field algebra. This viewpoint of seeing a continuous $C(X)$-algebra as a continuous field over its fibres in this sense will be particularly useful to us.

It is trivial to see that every C^*-algebra A is a continuous $C(X)$-algebra when X is the one point space, and under this the only fibre will be A itself, $A(x) = A$. Thus any simple A can be trivially viewed as a continuous $C(X)$-algebra with simple fibres, so that all of our following results shall work if B is simple. There are a number of non-simple examples of note to consider also. Trivially for any locally compact Hausdorff space X then $C_0(X)$ is easily a continuous $C(X)$-algebra. For each $x \in X$ we can see the corresponding fibre must satisfy $C_0(X)(x) \cong \mathbb{C}$, and the quotient map π_x is just the evaluation map at x. Thus for $f \in C_0(X)$ $f(x) = \pi_x(f)$ has the typical meaning. That is to say for any locally compact Hausdorff X we have that $C_0(X)$ is a continuous $C(X)$-algebra with simple fibres. This is an easy example of a wider class, see [9, Ex. 2.2.2], of algebras that are $C(X)$-algebras in terms of their primitive ideal space. Let $\mathrm{Prim}(A)$ be the primitive ideal space of a C^*-algebra A which is a T_0-space when given the Jacobson topology. If $\mathrm{Prim}(A)$ is a Hausdorff space then it will be locally compact and for all $a \in A$ the map $K \in \mathrm{Prim}(A) \mapsto \|a + K\|$ is continuous, [22, Cor. 3.3.9]. Then by the Dauns-Hofmann theorem, [39, Cor. 4.4.8], C^*-algebras A with Hausdorff primitive ideal spaces will in fact be continuous $C(\mathrm{Prim}(A))$-algebras with fibres $A(K) = \frac{A}{K}$ for $K \in \mathrm{Prim}(A)$, which in particular are simple. This result had similarly been shown earlier through a result of Fell [24] that showed if $\mathrm{Prim}(A)$ is Hausdorff then A exists as the continuous sections vanishing at infinity over a continuous field of simple C^*-algebras over $\mathrm{Prim}(A)$. That we require continuous $C(X)$-algebras with simple fibres for the following work then can be seen to not be too restrictive a condition. At the very least most examples of note we have considered thus far will satisfy this.

A result that we will need to make use of is how $C(X)$-algebras behave regarding tensor products. The following is a result of Hirshberg, Rørdam, and Winter [25, Prop. 1.6] but exists as a simple generalisation of the same result of Blanchard's [7] that was only for the compact case. Note that we are using $\otimes$ to refer to the minimal tensor product.

Proposition 4.3.5. Let X, Y be locally compact spaces, A a $C(X)$-algebra and B a $C(Y)$-algebra. Suppose A or B are exact. Then $A \otimes B$ is a $C(X \times Y)$-algebra with fibres

$$(A \otimes B)_{(x,y)} \cong A_x \otimes B_y$$

for $x \in X$, $y \in Y$ with structure map μ given by

$$\mu = \mu_A \otimes \mu_B : C_0(X) \otimes C_0(Y) \to \mathcal{M}(A \otimes B).$$

Remark 4.3.6. It is important to note that if A is a continuous $C(X)$-algebra and B is a continuous $C(Y)$-algebra then the tensor product $A \otimes B$ will be a continuous $C(X \times Y)$-algebra. With the isomorphism $(A \otimes B)_{(x,y)} \cong A_x \otimes B_x$ it stands that for the quotient maps we have $\pi_{(x,y)} = \pi_x \otimes \pi_y$. Thus the continuity of the map $(x, y) \mapsto \|f(x, y)\| = \|\pi_{(x,y)}(f)\|$ for $f \in A \otimes B$ is implied by the continuity of the maps $x \mapsto \|a(x)\| = \|\pi_x(a)\|$ and $y \mapsto \|b(y)\| = \|\pi_y(b)\|$ for $a \in A, b \in B$.

Our first step in being able to talk about the meaning of relative dimension 0 is to show the relative version of 2.2.9 that we are able to find a relative system of approximations with the maps down being approximately order 0.

Proposition 4.3.7. Let A, B be C^*-algebras with B a separable continuous $C(X)$-algebra with simple fibres for some locally compact Hausdorff X. If $\mathrm{nd}_B A = n < \infty$ then there exists a system of relatively n-decomposable cp approximations $(F_\lambda, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ such that the ψ_λ are approximately order 0 in the sense the induced map

$$\Psi : A \to \frac{\prod_\lambda (F_\lambda \otimes B)}{\bigoplus_\lambda (F_\lambda \otimes B)}$$

is order 0.

Proof: Let $\mathscr{F} \subset A$ finite and $\varepsilon > 0$ be given. We may assume that $\mathscr{F}$ is comprised of positive normalised elements, and we shall also assume that $\mathscr{F}$ contains at least one pair of elements c, c' such that $\|cc'\| < \frac{\varepsilon^8}{3 \cdot 256(n+1)^4}$ since if we do not the maps in question shall remain unaltered. Pick a relatively n-decomposable cp approximation $(F \otimes B, \psi, \sim\varphi)$ for $\mathscr{F}$ within $\frac{\varepsilon^8}{3 \cdot 256(n+1)^4}$. Let $F \cong M_{r_1} \oplus \cdots \oplus M_{r_s}$ and take ψ_j, φ_j to be the respective components of the maps. As noted each M_{R_j} is a continuous $C(Z)$-algebra, where Z is the single point space, with a single fibre of M_{r_j}. Since M_{r_j} is exact then by proposition 4.3.5 the tensor product $M_{r_j} \otimes B$ is a continuous $C(X)$-algebra with fibres $(M_{r_j} \otimes B)_x \cong M_{r_j} \otimes B_x$ which are in particular simple. We shall thus define for $j \in \{1, \ldots, s\}$ the closed set $W^j_{cc'} \subset X$ as being those $x \in X$ such that $\|\psi_j(c)\psi_j(c')(x)\| \leq \varepsilon$. From this define the closed set

$$G^j := \bigcap_{\left\{\text{pairs } c,c' \in \mathscr{F} \text{ such that } \|cc'\| \leq \frac{\varepsilon^8}{3 \cdot 256(n+1)^4}\right\}} W^j_{cc'}.$$

Define also the closed sets $D^j_{cc'} := \left\{x \in X \mid \|\psi_j(c)\psi_j(c')(x)\| \geq \varepsilon + \frac{\varepsilon}{2^{1000}}\right\}$ and so define

$$NG^j := \bigcup_{\left\{\text{pairs } c,c' \in \mathscr{F} \text{ such that } \|cc'\| \leq \frac{\varepsilon^8}{3 \cdot 256(n+1)^4}\right\}} D^j_{cc'}.$$

We are then able to define a continuous function $\mu_j : X \to [0,1]$ that is 1 on G^j and 0 on NG^j by Urysohn's lemma. Thus we define a new cpc map $\tilde{\psi}_j : A \to M_{r_j} \otimes B$ by setting $\tilde{\psi}_j(a) = \mu_j \psi_j(a)$. This then satisfies

$$\begin{aligned} \|\tilde{\psi}_j(c)\tilde{\psi}_j(c')\| &= \|\mu_j^2 \psi_j(c)\psi_j(c')\| \\ &< \varepsilon + \frac{\varepsilon}{2^{1000}} \end{aligned}$$

for all pairs $c, c' \in \mathscr{F}$ such that $\|cc'\| < \frac{\varepsilon^8}{3 \cdot 256(n+1)^4}$. We shall define $\tilde{\varphi}_j := \varphi_j$ specifically for the purposes of notational neatness. Taking $\tilde{\psi}$ and $\tilde{\varphi}$ as the respective sums of these maps over j we have

$$\|\tilde{\varphi}\tilde{\psi}(a) - a\| \leq \|\tilde{\varphi}\tilde{\psi}(a) - \varphi\psi(a)\| + \|\varphi\psi(a) - a\|$$

$$\begin{aligned}
&\le (n+1)\max_{j\in\{1,\dots,s\}}\|\tilde{\varphi}_j\tilde{\psi}_j(a)-\varphi_j\psi_j(a)\|+\frac{\varepsilon}{2}\\
&=(n+1)\max_{j\in\{1,\dots,s\}}\|\varphi_j((1-\mu_j)\psi_j(a))\|+\frac{\varepsilon}{2}
\end{aligned}$$

for $a\in\mathscr{F}$. So we need to show that

$$\|\varphi_j((1-\mu_j)\psi_j(a)\|<\frac{\varepsilon}{2(n+1)} \tag{4.3.1}$$

for each j for these maps $\tilde{\psi},\tilde{\varphi}$ to appear as part of a relatively n-decomposable cp approximation. Define $\phi_j : M_{r_j}\otimes B\to A_\infty$ to be the cpc order 0 map obtained from the maps φ_j by lemma 4.1.4. Specifically let φ_{j_m} be the map which is $\frac{1}{m}$-F_m-order 0 where F_m is the set of the first m terms of dense sequence of $M_{r_j}\otimes B$. Denote by ϕ the sum of these maps over j. Take $\pi_j : M_{r_j}\otimes B\to A_\infty''$ to be the canonical supporting *-homomorphism for ϕ_j such that there is a positive element $h_j\in A_\infty''$ such that $\phi_j(a)=h_j\pi_j(a)$ for all $a\in M_{r_j}\otimes B$ (see [49, Thm. 2.3]). Then

$$\begin{aligned}
&\|\phi\psi(a)\phi\psi(a')\|\\
&\ge\|\phi_j\tilde{\psi}_j(a)\phi_j\tilde{\psi}_j(a')\|^2 \text{ (by [50, Prop. 3.1])}\\
&=\|h_j^2\pi_j\tilde{\psi}_j(a)\pi_j\tilde{\psi}_j(a')\|^2\\
&\ge\|\pi_j(\tilde{\psi}_j(a)\tilde{\psi}_j(a'))h_j^2\pi_j(\tilde{\psi}_j(a)\tilde{\psi}_j(a'))\|^2\\
&=\|\phi_j(\tilde{\psi}_j(a)\tilde{\psi}_j(a'))\|^4
\end{aligned}$$

for all j and positive normalised elements $a,a'\in A$. What we then get from this is the following

$$\begin{aligned}
\|\phi\psi(a)\phi\psi(a')\|&\le\|\phi\psi(a)\phi\psi(a')-\phi\psi(a)a'\|+\|\phi\psi(a)a'-aa'\|+\|aa'\|\\
&<\frac{\varepsilon^8}{3\cdot256(n+1)^4}+\frac{\varepsilon^8}{3\cdot256(n+1)^4}+\frac{\varepsilon^8}{3\cdot256(n+1)^4}\\
&<\frac{\varepsilon^8}{256(n+1)^4}
\end{aligned}$$

for all $a,a'\in\mathscr{F}$ with $\|aa'\|<\frac{\varepsilon^8}{3\cdot256(n+1)^4}$, and so in particular we have

$$\|\phi_j(\tilde{\psi}_j(a)\tilde{\psi}_j(a'))\|<\frac{\varepsilon^2}{4(n+1)} \tag{4.3.2}$$

for all such a,a'. Since each $\phi_j : M_{r_j}\otimes B\to A_\infty$ is a cpc order 0 map we have a *-homomorphism $\Pi_j : C((0,1],M_{r_j}\otimes B)\to A_\infty$ such that $\phi_j(a)=\Pi_j(\iota\otimes a)$ where $\iota=\mathrm{id}_{(0,1]}$. Since we can view $C_0((0,1])$ as a continuous $C((0,1])$-algebra with all fibres being $\mathbb{C}$ then by proposition 4.3.5 we know that $C_0((0,1],M_{r_j}\otimes B)$ is a continuous

$C((0,1]\times X)$-algebra. We shall denote $C_0((0,1],M_{r_j}\otimes B)=\Gamma$ and note for the fibres we have $\Gamma_{(s,x)}\cong M_{r_j}\otimes B_x$. If we consider the kernel of the map Π_j then for each $(s,j)\in(0,1]\times X$ the corresponding fibre $\Gamma_{(s,x)}$ is either represented fully in the kernel or not at all by simplicity, thus we may find a $Y\subset(0,1]\times X$ such that the kernel of Π_j is those functions that vanish on Y. Then if we define $\Gamma' := \frac{\Gamma}{\ker(\Pi_j)}$ this can be seen to be a continuous $C(Y)$-algebra (see [25, 1.5]), and this gives us a *-homomorphism $\Pi'_j:\Gamma'\to A_\infty$ such that $\Pi'_j(f\restriction_Y)=\Pi_j(f)$, which is injective and thus isometric. Thus we have

$$\begin{aligned}\|\phi_j((1-\mu_j)\psi_j(a)\| &= \|\Pi_j(\iota\otimes(1-\mu_j)\psi_j(a))\|\\ &= \|\Pi'_j((\iota\otimes(1-\mu_j)\psi_j(a))\restriction_Y)\|\\ &= \|(\iota\otimes(1-\mu_j)\psi_j(a))\restriction_Y\|\\ &= \sup_{(s,t)\in Y}\|((1-\mu)\psi_j(a)(t))s\|.\end{aligned}$$

If we assume that $\|\phi_j((1-\mu_j)\psi_j(a))\|>\frac{\varepsilon}{4(n+1)}$, then there is $(s,t)\in Y$ such that $\|((1-\mu_j)\psi_j(a)(t))s\|>\frac{\varepsilon}{4(n+1)}$. Because of how we defined μ_j this would necessitate that $t\in X\smallsetminus G^j$, and since a is of norm at most 1 we have that $s>\frac{\varepsilon}{4(n+1)}$. From this we would have

$$\begin{aligned}\|\phi_j(\tilde\psi_j(a)\tilde\psi_j(a'))\| &= \|\Pi_j(\iota\otimes(\tilde\psi_j(a)\tilde\psi_j(a')))\|\\ &= \|(\iota\otimes(\tilde\psi_j(a)\tilde\psi_j(a')))\restriction_Y\|\\ &\geq \|(\tilde\psi_j(a)\tilde\psi_j(a'))(t)\|s\\ &> \varepsilon\cdot\frac{\varepsilon}{4(n+1)}\end{aligned}$$

for at least one pair $a,a'\in\mathscr{F}$ with $\|aa'\|<\frac{\varepsilon^8}{3\cdot256(n+1)^4}$. This contradicts 4.3.2, and hence $\|\phi_j((1-\mu)\psi_j(a))\|\leq\frac{\varepsilon}{4(n+1)}$ for all $a\in\mathscr{F}$. Given we are working with respect to the sequence algebra A_∞ by this inequality in norm there must exist a sequence (i_m) in A that tends to 0 such that $\|\varphi_{j_m}((1-\mu_j)\psi_j(a)+i_m\|\leq\frac{3\varepsilon}{8(n+1)}$ for all $m\in\mathbb{N}$. Thus there is an $N\in\mathbb{N}$ such that for all $m\geq N$ we have

$$\begin{aligned}\|\varphi_{j_m}((1-\mu_j)\psi_j(a)\| &\leq \|\varphi_{j_m}((1-\mu_j)\psi_j(a)+i_m\|+\|i_m\| &(4.3.3)\\ &< \frac{\varepsilon}{2(n+1)} &(4.3.4)\end{aligned}$$

for all $a\in\mathscr{F}$. This gives us the required inequality 4.3.1 for those maps φ_{j_m} for $m\geq N$. Let $\varphi_m^{(i)}=\sum_{j\in\{1,\dots,s\} s.t. M_{r_j}\subset F^{(i)}}\varphi_{j_m}$ where we split $F=F^{(0)}\oplus\cdots\oplus F^{(n)}$ coming from the approximation we started with. Then necessarily by the separability of $F^{(i)}\otimes B$ for each set $K\subset F^{(i)}\otimes B$ of contractions and η we must be able to find $M\geq N$ such that $\frac{1}{M}<\frac{\eta}{3}$ and that for each $a\in K$ there exists an $a'\in F_M$ that is within $\frac{\eta}{3}$ of a, where F_M

is the set of first M elements of dense sequence in $F^{(i)} \otimes B$. $\varphi_M^{(i)}$ would then necessarily be a η-K-order 0 map, satisfying our final condition for relative dimension. If we denote by $\sim\tilde{\varphi}_N$ the collection of maps $F \otimes B \to A$ defined by all the $\varphi_m^{(i)}$ for $m \geq N$ we can then say that $(F \otimes B, \tilde{\psi}, \sim\tilde{\varphi}_N)$ is a relatively n-decomposable cp approximation for $\mathscr{F}$ within ε and for which the downwards map $\tilde{\psi}$ is ε-$\mathscr{F}$-order 0. The system of these n-decomposable approximations $(F_\lambda \otimes B, \tilde{\psi}_\lambda, \sim\tilde{\varphi}_{N_\lambda})$ indexed over the net $(\mathscr{F}, \varepsilon)$ can then be easily seen to be a system of approximations for which the map induced by the $\tilde{\psi}_\lambda$ is order 0. ∎

We now wish to be able to make similar statements in which we replace approximately order 0 with approximately multiplicative. As noted before thanks to the structure theorem on cpc order 0 maps, theorem 2.1.2, when the nuclear dimension is 0 the maps up are cpc order 0, and can be shown to be $*$-homomorphisms depending on the image of the identity. When the relative dimension is 0 then the maps up will be cpc approximately order 0 and our goal is to show we may assume these are approximately multiplicative. Afterwards we shall need to show that relative dimension 0 implies that the maps down can be taken to be approximately multiplicative in the sense the induced maps are multiplicative. This is based on the idea that $\dim_{\text{nuc}} A = 0 \iff \text{dr} A = 0$ and that one may take the maps for decomposition rank to be approximately mutliplicative, proposition 2.2.10.

Lemma 4.3.8. Let A, B be unital C^*-algebras with B separable. Suppose $\text{nd}_B A = 0$, then there exists a system of relatively 0-decomposable cp approximations $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ such that for each λ, for each finite set $\mathscr{K} \subset F_\lambda \otimes B$ and $\eta > 0$ one can find a φ_λ that is η-$\mathscr{K}$-multiplicative.

Proof: Let $\mathscr{F} \subset A$ a finite set of contractions and $\varepsilon > 0$ be given. Pick a relatively 0-decomposable approximation $(F \otimes B, \psi, \sim\varphi)$ for $\mathscr{F} \cup \{1_A\}$ within $\frac{\varepsilon^2}{9}$. We have $\psi(1_A) \leq 1_B$, so that $\varphi\psi(1_A) \leq \varphi(1_B) \leq 1_A$, and this $\varphi(1_B) \approx_{\frac{\varepsilon^2}{9}} 1_A$. Let $\phi : F \otimes B \to A_\infty$ be the cpc order 0 map obtained from lemma 4.1.4, which will satisfy $\phi\psi(a) \approx_{\frac{\varepsilon^2}{9}} a$ for all $a \in \mathscr{F}$. From this we define a new cpc order 0 map $\Phi : F \otimes B \to A_\infty$ by setting

$$\Phi(\cdot) = \phi(1_B)^{-\frac{1}{2}} \phi(\cdot) \phi(1_B)^{-\frac{1}{2}},$$

which can be seen to satisfy $\Phi\psi(a) \approx_{\frac{\varepsilon}{3}} a$ for all $a \in \mathscr{F}$. Since this is a unital order 0 map it must be a $*$-homomorphism by theorem 2.1.2. Thus for any $\mathscr{K} \subset F \otimes B$ and $\eta > 0$ we may find an $N \in \mathbb{N}$ such that Φ_N is η-$\mathscr{K}$-multiplicative and will satisfy $\Phi_N\psi(a) \approx_\varepsilon a$ for all $a \in \mathscr{F}$. The triple $(F \otimes B, \psi, \sim(\Phi_N))$ is then a relatively 0-decomposable cp approximation for $\mathscr{F}$ within ε. so that we can say $(F \otimes B, \psi, \sim\Phi_N)$ is a 0-decomposable approximation for $\mathscr{F}$ within ε satisfying the required condition. ∎

Proposition 4.3.9. Let A, B be unital C^*-algebras with B a separable $C(X)$-algebra with simple fibres for some locally compact Hausdorff X. Suppose $\mathrm{nd}_B A = 0$ and X is totally disconnected, then there exists a system of relatively 0-decomposable cp approximations $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ such that the map

$$\Psi : A \to \frac{\prod_\lambda (F_\lambda \otimes B)}{\bigoplus_\lambda (F_\lambda \otimes B)}$$

induced by the ψ_λ is multiplicative.

Proof: Let $\mathscr{F} \subset A$ a finite subset of positive contractions and $\varepsilon > 0$ be given. Note that by a consequence of Stinespring's theorem, see [28, Lemma 7.11], [30, Lemma 3.5], we have that for a cpc map $\phi : A \to B$,

$$\|\phi(xy) - \phi(x)\phi(y)\| \leq \|\phi(xx^*) - \phi(x)\phi(x^*)\|^{\frac{1}{2}} \|y\|.$$

Hence to show ε-$\mathscr{F}$-multiplicativity we need only show

$$\|\psi(x^2) - \psi(x)^2\| < \varepsilon^2$$

For all $x \in \mathscr{F}$. Denote the elements of $\mathscr{F}$ by $b_1 \ldots, b_m$ and let $(F' \otimes B, \psi', \sim\varphi')$ be an approximation for $\{b_1, \ldots, b_m, b_1^2, \ldots, b_m^2\}$ within $\frac{\varepsilon^2}{12}$. For each $j \in \{1, \ldots, s\}$ and for each $b_i \in \mathscr{F}$ define the set

$$W_{b_i}^j := \left\{x \in X \mid \|(\psi_j'(b_i^2) - \psi_j'(b_i)^2)(x)\| \leq \varepsilon^2\right\},$$

and so define

$$G^j := \bigcap_{b_i \in \mathscr{F}} W_{b_i}^j.$$

Similarly define closed sets

$$D_{b_i}^j := \left\{x \in X \mid \|\psi_j'(b_i^2) - \psi_j'(b_i)^2\| \geq \varepsilon^2 + \frac{\varepsilon}{2^{1000}}\right\}$$

and thus define

$$NG^j := \bigcup_{b_i \in \mathscr{F}} D_{b_i}^j.$$

As X is locally compact, Hausdorff, and totally disconnected it is therefore totally separated and has a basis of clopen sets. As G^j and NG^j are closed they must be expressible as an intersection of clopen sets. Thus we may find a clopen set $U^j \subset X$ containing G^j that is disjoint with NG^j. In the case where $NG^j = \varnothing$ this is trivially true by setting $U^j = X$. If NG^j is nonempty then such a U^j must exist as otherwise every clopen set containing NG^j must also contain the complement of G^j, contradicting the fact that NG^j is expressible as an intersection of clopen sets. We may then define a continuous function $\mu_j : X \to \{0,1\}$ that is 1 on U^j and 0 otherwise. From this we define a new cpc map $\psi_j : A \to M_{r_j} \otimes B$ by setting $\psi_j(a) = \mu_j \psi'_j(a)$. This satisfies

$$\begin{aligned}\|\psi_j(b_i^2) - \psi_j(b_i)^2\| &= \|\mu_j(\psi'_j(b_i^2) - \psi'_j(b_i)^2)\| \\ &\leq \varepsilon^2 + \frac{\varepsilon}{2^{1000}}\end{aligned}$$

for all $b_i \in \mathscr{F}$. We shall set $\varphi_j = \varphi'_j$ for notational neatness. Taking ψ and φ to be the respective sums over the j we have

$$\|\varphi\psi(b_i) - b_i\| \leq \max_{j\in\{1,\dots,s\}} \|\varphi'_j((1-\mu_j)\psi'_j(b_i)\| + \frac{\varepsilon}{2}$$

for all $b_i \in \mathscr{F}$. Let $\phi_j : F' \otimes B \to A_\infty$ be the cpc order 0 map obtained from the φ_j by lemma 4.1.4. For each $b_i \in \mathscr{F}$ we have the following inequality

$$\begin{aligned}\|\phi_j\psi'_j(b_i^2) - \phi_j(\psi_j(b_i)^2)\| &\leq \|\phi\psi'(b_i^2) - \phi(\psi'(b_i)^2)\| && (4.3.5)\\ &\leq \|\phi\psi'(b_i^2) - b_i^2\| + \|b_i^2 - b_i\phi\psi'(b_i)\| && (4.3.6)\\ &\quad + \|b_i\phi\psi'(b_i) - \phi\psi'(b_i)^2\| && (4.3.7)\\ &< \frac{\varepsilon^2}{4}. && (4.3.8)\end{aligned}$$

We now argue as in proposition 4.3.7. We define the $*$-homomorphism $\Pi_j : C((0,1], M_{r_j} \otimes B) \to A_\infty$ such that $\phi_j(a) = \Pi_j(\iota \otimes a)$ and consider $\Gamma := C((0,1], M_{r_j} \otimes B)$ as a continuous $C((0,1] \times X)$ algebra with simple fibres $\Gamma_{(s,x)} \cong M_{r_j} \otimes B_x$. The kernel of this $*$-homomorphism will then consist of those functions vanishing on some $Y \subset (0,1] \times X$. Then the quotient $\Gamma' := \frac{\Gamma}{\ker(\Pi_j)}$ is a continuous $C(Y)$-algebra and we will have an isometric $*$-homomorphism $\Pi'_j : \Gamma' \to A_\infty$ such that $\Pi'(f \restriction_Y) = \Pi_j(f)$. Thus

$$\begin{aligned}\|\phi_j((1-\mu_j)\psi'_j(b_i))\| &= \|\Pi'_j(\iota \otimes (1-\mu_j)\psi'_j(b_i))\| \\ &= \sup_{(s,t)\in Y} \|(1-\mu_j)\psi_j(b_i)(t)s\|.\end{aligned}$$

If one assumes that $\|\phi_j((1-\mu_j)\psi'_j(b_i)\| > \frac{\varepsilon}{4}$ there must be some $(s,t) \in Y$ such that $\|((1-\mu_j)\psi_j(a)(t))s\| > \frac{\varepsilon}{2}$. This necessitates that $t \in X \backslash G^j$ and $s > \frac{\varepsilon}{4}$ which would mean that for at least one $b_i \in \mathscr{F}$ we would have

$$\begin{aligned}\|\phi_j\psi_j(b_i^2) - \phi_j(\psi_j(b_i)^2)\| &\geq \|(\psi'_j(b_i^2) - \psi'_j(b_i)^2)(t)\|s \\ &> \varepsilon \cdot \frac{\varepsilon}{4},\end{aligned}$$

contradicting 4.3.8.

Recall that the map ϕ_j is defined using the maps φ_{j_m} which are $\frac{1}{m}$-F_m-order 0 for the sets F_m of the first m elements of dense sequence of $M_{r_j} \otimes B$. Since we have

$\|\phi_j((1-\mu_j)\psi_j'(b_j)\| < \frac{\varepsilon}{4}$ then there exists a sequence (i_m) in A such that there exists an $N \in \mathbb{N}$ such that for all $m \geq N$ we have

$$\begin{aligned}\|\varphi_{j_m}((1-\mu_j)\phi_j'(b_i)\| &\leq \|\varphi_{j_m}((1-\mu_j)\psi_j'(b_j)+i_m\| + \|i_m\| \\ &< \frac{\varepsilon}{2}\end{aligned}$$

for all $B_i \in \mathscr{F}$. Let $\sim\varphi_N$ be the collection of maps φ_m defined to be the sum of the φ_{j_m} where $m \geq N$. As in the proof of proposition 4.3.7 it can be seen that this collection of maps are such that we have one which is η-K-order 0 for any compact set $K \subset F' \otimes B$ of contractions and $\eta > 0$. Hence we have that $(F' \otimes B, \psi, \sim\varphi_N)$ is a relatively 0-decomposable cp approximation for $\mathscr{F}$ within ε. The system $(F_\lambda', \psi_\lambda, \sim\varphi_{N_\lambda})$ of 0-decomposable approximations indexed over the $(\mathscr{F}, \varepsilon)$ is easily seen to be a system such that the map Ψ induced by the ψ_λ is multiplicative. ∎

Remark 4.3.10. The preceding result is designed to follow very similar logic to proposition 4.3.7, though comes with the caveat that X, in addition to being a locally compact Hausdorff space, must be totally disconnected. While more restrictive a condition, importantly, it still holds for the case that B is itself simple. It was originally believed that the result would work with the same hypotheses as proposition 4.3.7, except for the requirement that $\mathrm{nd}_B A = 0$, though the proof had a minor mistake which led to finding a counter example.

Let A be a unital simple separable C^*-algebra and set $B = C((0,1], A)$, so that B is a continuous $C((0,1])$-algebra with simple fibers $\Gamma_s \cong A$ for all $s \in (0,1]$. Then $\mathrm{nd}_{C((0,1],A)} A = 0$ which can be seen by taking maps $\psi : A \to C((0,1], A)$ and $\varphi : C((0,1], A) \to A$ defined by $\psi(a) = \mathrm{id}_{(0,1]} \otimes a$ and $\varphi(\mathrm{id}_{(0,1]} \otimes a) = a$. The latter map of which is the $*$-homomorphism that corresponds to the identity map on A by corollary 2.1.3. Then for the maps down to be approximately multiplicative in the correct sense requires us to be able to find a net of maps $\psi_\lambda : A \to F_\lambda \otimes C((0,1], A)$ such that in particular the unit is asymptotically mapped to a projection. This can be seen not to be possible since $C((0,1], A)$ does not contain any projections, and certainly would none of the $F_\lambda \otimes C((0,1], A)$ since any projection can be seen to be unitarily equivalent to a diagonal projection. Since these algebras do not contain any projections they cannot contain any elements that are arbitrarily close to projections due to the stability of projections. Thus our proposition can fail when X is not totally disconnected.

Remark 4.3.11. It is important to note that since the two prior results allow us to obtain approximately multiplicative maps up or down without fundamentally changing the maps down or up, respectively, we should be able to apply both results simultaneously. That is if we have $\mathrm{nd}_B A = 0$ then we may find a system $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$

of relatively 0-decomposable cp approximations for A for which the maps up and down may both be taken to be approximately multiplicative in their respective senses. Moreover if A is separable then we may find a countable system $(F_n \otimes B, \psi_n, \sim\varphi_n)_{\mathbb{N}}$ with the maps up and down are approximately multiplicative.

Theorem 4.3.12. Let A and B be separable unital C^*-algebras with B a continuous $C(X)$-algebra with simple fibres. Suppose $\mathrm{nd}_B A = 0$ and X is totally disconnected, then A exists as the limit of a generalised inductive system $(F_n \otimes B, \phi_{n,m})$ where each of the F_n are finite dimensional algebras.

Proof: By propositions 4.3.8 and 4.3.9 we may find a system $(F_\lambda, \psi_\lambda, \sim\varphi_\lambda)$ of relatively 0-decomposable cp approximations for A such that the maps up and the maps down are both multiplicative in their respective senses. Let $\mathscr{F}_n \subset A$ be the first n elements of a dense sequence and so from the multiplicative system we may find a relatively 0-decomposable cp approximation $(F_1 \otimes B, \psi_1, \sim\varphi_1)$ for $\mathscr{F}_1$ within $\frac{1}{2}$ such that ψ_1 is $\frac{1}{2}$-$\mathscr{F}_1$-multiplicative. Let $K_{1,n}$ be the first n elements of a dense sequence of $F_1 \otimes B$ and so pick $\varphi_{1,1} \in \sim\varphi_1$ that is $\frac{1}{2}$-$K_{1,1}$-order 0. We may then find a relatively 0-decomposable cp approximation $(F_2 \otimes B, \psi_2, \sim\varphi_2)$ for $\mathscr{G}_2 := \mathscr{F}_2 \cup \varphi_{1,1}\psi_1(\mathscr{F}_2) \cup \varphi_{1,1}(K_{1,2})$ within $\frac{1}{2^2}$ such that ψ_2 is $\frac{1}{2^2}$-$\mathscr{G}_2$-multiplicative. Let $K_{2,n}$ be the first n-elements of a dense sequence of $F_2 \otimes B$ and so define a new set $H_{2,2} := K_{2,2} \cup \psi_2\varphi_{1,1}(K_{1,2})$. Pick $\varphi_{2,2} \in \sim\varphi_2$ that is $\frac{1}{2^2}$-$H_{2,2}$-multiplicative.

We continue in this way to obtain for all $n \in \mathbb{N}$ a relatively 0-decomposable cp approximation $(F_n \otimes B, \psi_n, \sim\varphi_n)$ for the finite set

$$\mathscr{G}_n := \mathscr{F}_n \cup \bigcup_{i=1}^{n-1} \varphi_{i,i}\psi_i(\mathscr{F}_n) \cup \bigcup_{i=1}^{n-1} \varphi_{i,i}(K_{i,n}) \cup \bigcup_{1 \le i < j \le n-1} \varphi_{j,j}\psi_j\varphi_{i,i}(K_{i,n})$$

within $\frac{1}{2^n}$ such that that ψ_n is $\frac{1}{2^n}$-$\mathscr{G}_n$-multiplicative where $K_{i,n} \subset F_i \otimes B$ is the set of the first n elements of a dense sequence of $F_i \otimes B$. We also define the set

$$H_{n,n} := K_{n,n} \cup \bigcup_{i=1}^{n-1} \psi_n\varphi_{i,i}(K_{i,n})$$

and so pick $\varphi_{n,n} \in \sim\varphi_n$ that is $\frac{1}{2^n}$-$H_{n,n}$-multiplicative.

For $n, m \in \mathbb{N}, n < m$ we then define a cpc map $\phi_{nm} : F_n \otimes B \to F_m \otimes B$ by $\phi_{nm} := \psi_m\varphi_{n,n}$ and claim that $(F_n \otimes B, \phi_{nm})$ is a generalised inductive system. First note that for any $n \in \mathbb{N}$, any $x \in F_n \otimes B$ and any $\varepsilon > 0$ we may find $n < M \in \mathbb{N}$ such that $\frac{1}{2^M} < \frac{\varepsilon}{3}$ and such that there exists $x' \in K_{n,M}$ such that $\|x - x'\| < \frac{\varepsilon}{3}$, then for all $M \le m < k$ we have

$$\begin{aligned} \|\phi_{nk}(x) - \phi_{mk}\phi_{nm}(x)\| \le{}& \|\phi_{nk}(x) - \phi_{nk}(x')\| \\ &+ \|\phi_{nk}(x') - \phi_{mk}\phi_{nm}(x')\| + \|\phi_{mk}\phi_{nm}(x') - \phi_{mk}\phi_{nm}(x)\| \end{aligned}$$

$$
\begin{aligned}
&< \|\psi_k\varphi_{n,n}(x') - \psi_k\varphi_{m,m}\psi_m\varphi_{n,n}(x')\| + \frac{2\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}
$$

Thus the maps are coherent. Note also that for $n < m$ the maps ϕ_{nm} are $(\frac{1}{2^n} + \frac{1}{2^m})$-$H_{n,n}$-multiplicative, since for any $x, y \in H_{n,n}$ we will have

$$
\begin{aligned}
\|\phi_{nm}(xy) - \phi_{nm}(x)\phi_{nm}(y)\| &\leq \|\psi_m\varphi_{n,n}(xy) - \psi_m(\varphi_{n,n}(x)\varphi_{n,n}(y))\| \\
&\quad + \|\psi_m(\varphi_{n,n}(x)\varphi_{n,n}(y)) - \psi_m\varphi_{n,n}(x)\psi_m\varphi_{n,n}(y)\| \\
&< \frac{1}{2^n} + \frac{1}{2^m}.
\end{aligned}
$$

In particular since $H_{n,n}$ contains all finite sets of the form $\psi_n\varphi_{i,i}(K_{i,n})$ for all $i < n$ we can happily see that in fact ϕ_{nm} is $(\frac{1}{2^n} + \frac{1}{2^m})$-$\phi_{in}(K_{i,n})$-multiplicative. This also allows us to satisfy condition (4) of the definition 4.3.1 for a generalised system. Given $n \in \mathbb{N}$, $x, y \in F_n \otimes B$, and ε we may find x' in the dense sequence for $F_n \otimes B$ such that $\|x - x'\| < \frac{\varepsilon}{7\|y\|}$ and so find y' in the dense sequence of $F_n \otimes B$ such that $\|y - y'\| < \frac{\varepsilon}{7\|x'\|}$. We may then find $M \in \mathbb{N}$ such that $x', y' \in K_{n,M}$, $(\frac{1}{2^M} + \frac{1}{2^{M+1}}) < \frac{\varepsilon}{7}$, and such that for all $M \leq m < k$ we have $\|\phi_{nk}(x') - \phi_{mk}\phi_{nm}(x')\| < \frac{\varepsilon}{7\|y'\|}$ and $\|\phi_{nk}(y') - \phi_{mk}\phi_{nm}(y')\| < \frac{\varepsilon}{7\|x'\|}$.Thus we have for all $M \leq m < k$

$$
\begin{aligned}
&\|\phi_{mk}(\phi_{nm}(x)\phi_{nm}(y)) - \phi_{nk}(x)\phi_{nk}(y)\| \\
&\leq \|\phi_{mk}(\phi_{nm}(x)\phi_{nm}(y)) - \phi_{mk}(\phi_{nm}(x')\phi_{nm}(y))\| \\
&\quad + \|\phi_{mk}(\phi_{nm}(x')\phi_{nm}(y)) - \phi_{mk}(\phi_{nm}(x')\phi_{nm}(y')\| \\
&\quad + \|\phi_{mk}(\phi_{nm}(x')\phi_{nm}(y')) - \phi_{mk}\phi_{nm}(x')\phi_{mk}\phi_{nm}(y')\| \\
&\quad + \|\phi_{mk}\phi_{nm}(x')\phi_{mk}\phi_{nm}(y') - \phi_{nk}(x')\phi_{mk}\phi_{nm}(y')\| \\
&\quad + \|\phi_{nk}(x')\phi_{mk}(y') - \phi_{nk}(x')\phi_{nk}(y')\| \\
&\quad + \|\phi_{nk}(x')\phi_{nk}(y') - \phi_{nk}(x')\phi_{nk}(y)\| \\
&\quad + \|\phi_{nk}(x')\phi_{nk}(y) - \phi_{nk}(x)\phi_{nk}(y)\| \\
&< \varepsilon,
\end{aligned}
$$

so that condition (4) of the definition 4.3.1 is satisfied. The fact that the maps are coherent and all of the maps in question are cpc it is quite easy to see the rest of the estimates for the definition 4.3.1 are satisfied, thus $(F_n \otimes B, \phi_{nm})$ is indeed a generalised inductive system.

Our next goal is to show that there is an asymptotic intertwining between this system $(F_n \otimes B, \phi_{nm})$ and the trivial generalised system for A, $(A_n, \mathrm{id}_{A_{n,m}})$ where each $A_n \cong A$ and $\mathrm{id}_{A_{n,m}}$ is the obvious identity map on A notated this way for consistency. For each $n \in \mathbb{N}$ let $\alpha_n = \psi_n$ and $\beta_n = \varphi_{n,n}$. Since the connecting maps for both generalised systems are *cpc* and these α_n, β_n are cpc also then by [5, Prop. 5.1.6] we only need to satisfy conditions (1), (3), and (4) of definition 4.3.4 in order to see that they form an asymptotic intertwining.

First let $x \in A$, $\varepsilon > 0$ be given. We may find x' in the dense sequence of A such that $\|x - x'\| < \frac{\varepsilon}{3}$ and let $M \in \mathbb{N}$ be such that $x' \in \mathscr{F}_M$ and such that $\frac{1}{2^M} < \frac{\varepsilon}{3}$. Thus for all $M \leq m < k$ we have

$$\begin{aligned}\|\phi_{mk}\alpha_m \mathrm{id}_{A_{nm}}(x) - \alpha_k \mathrm{id}_{A_{n,k}}(x)\| &= \|\psi_k\varphi_{m,m}\psi_m(x) - \psi_k(x)\| \\ &\leq \|\varphi_{m,m}\psi_m(x') - x'\| + 2\|x - x'\| \\ &< \varepsilon.\end{aligned}$$

Thus the (α_n) satisfy condition (1) of 4.3.4.

Next let $n \in \mathbb{N}$, $y \in F_n \otimes B$, and $\varepsilon > 0$ be given. We may find y' in the dense sequence of $F_n \otimes B$ such that $\|y - y'\| < \frac{\varepsilon}{4}$, then let $M \in \mathbb{N}$ be such that $y' \in K_{n,M}$ and $\frac{1}{2^M} \leq \frac{\varepsilon}{4}$. Thus for all $M \leq m < k$ we have

$$\begin{aligned}\|\mathrm{id}_{A_{m,k}}\beta_m\phi_{nm}(y) - \beta_k\phi_{nk}(y)\| &= \|\varphi_{m,m}\psi_m\varphi_{n,n}(y) - \varphi_{k,k}\psi_k\varphi_{n,n}(y)\| \\ &\leq \|\varphi_{m,m}\psi_m\varphi_{n,n}(y') - \varphi_{n,n}(y')\| \\ &\quad + \|\varphi_{k,k}\psi_k\varphi_{n,n}(y') - \varphi_{n,n}(y')\| \\ &\quad + 2\|y - y'\| \\ &< \varepsilon.\end{aligned}$$

Thus the (β_n) satisfy condition (1) of 4.3.4.

Let $x \in A$ and $\varepsilon > 0$ be given. We may find x' in the dense sequence of A such that $\|x - x'\| < \frac{\varepsilon}{4}$ and let $M \in \mathbb{N}$ be such that $x' \in \mathscr{F}_M$ and such that $\frac{1}{2^M} < \frac{\varepsilon}{4}$. Thus for all $M \leq m < k$ we have

$$\begin{aligned}\|\beta_k\phi_{mk}\alpha_m \mathrm{id}_{A_{n,m}}(x) - \mathrm{id}_{A_{n,k}}(x)\| &= \|\varphi_{k,k}\psi_k\varphi_{m,m}\psi_m(x) - x\| \\ &\leq \|\varphi_{k,k}\psi_k\varphi_{m,m}\psi_m(x') - \varphi_{m,m}\psi_m(x')\| \\ &\quad + \|\varphi_{m,m}\psi_m(x') - x'\| \\ &\quad + 2\|x - x'\| \\ &< \varepsilon.\end{aligned}$$

Thus condition (3) of 4.3.4 is satisfied.

Let $n \in \mathbb{N}$, $y \in F_n \otimes B$, and $\varepsilon > 0$ be given. We may find y' in the dense sequence of $F_n \otimes B$ such that $\|y - y'\| < \frac{\varepsilon}{3}$, then let $M \in \mathbb{N}$ be such that $y' \in K_{n,M}$ and $\frac{1}{2^M} \leq \frac{\varepsilon}{3}$. Thus for all $M \leq m < k$ we have

$$\begin{aligned}\|\alpha_k \mathrm{id}_{A_{m,k}}\beta_m\phi_{nm}(y) - \phi_{nk}(y)\| &= \|\psi_k\varphi_{m,m}\psi_m\varphi_{n,n}(y) - \psi_k\varphi_{n,n}(y)\| \\ &\leq \|\varphi_{m,m}\psi_m\varphi_{n,n}(y') - \varphi_{n,n}(y')\| + 2\|y - y'\| \\ &< \varepsilon.\end{aligned}$$

Thus condition (4) of 4.3.4 is satisfied.

Since all the necessary conditions have been satisfied then by [5, Prop. 5.1.6] the (α_n) and (β_n) form an asymptotic intertwining. Thus we have an isomorphism

$$A \cong \lim(F_n \otimes B, \phi_{n,m}).$$

∎

Remark 4.3.13. Unfortunately we do not manage to show that proposition 4.3.3 and theorem 4.3.12 are in fact equivalent. The hypothesis of 4.3.3 that the induced maps are $*$-homomorphisms is equivalent to the statement that for all $n \in \mathbb{N}$, for all $x, y \in F_n \otimes B$, and $\varepsilon > 0$ there exists a $K \in \mathbb{N}$ such that for all $k \geq K$ we have $\|\phi_{nk}(xy) - \phi_{nk}(x)\phi_{nk}(y)\| < \varepsilon$. This is not a condition that our construction of 4.3.12 is able to satisfy, the best we are able to show is that ϕ_{nk} is $(\frac{1}{2^n} + \frac{1}{2^k})$-multiplicative on the set of the first n elements of a dense sequence of $F_n \otimes B$.

An earlier attempt at proving 4.3.12 attempted to remedy this by taking maps $\varphi_{n,k} : F_n \otimes B \to A$ that are $\frac{1}{2^k}$-multiplicative on the set of the first k elements of a dense sequence of $F_n \otimes B$. By defining $\phi_{nk} = \psi_k \varphi_{n,k}$ this would have shown that for any $x, y \in F_n \otimes B$ we would have $\|\phi_{nk}(xy) - \phi_{nk}(x)\phi_{nk}(y)\| \xrightarrow{k\to\infty} 0$. This runs into the problem that coherence relies on showing $\|\varphi_{nm}(x) - \varphi_{nk}(x)\| \to 0$ for increasing $m < k$, for any $x \in F_n \otimes B$, for which there appears to be no obvious reason this should be true.

Thus the system constructed for 4.3.12 is not enough to obtain dimension 0 through 4.3.3.

4.4 On the Comparison Property

In [41] Robert was able to show that if a C^*-algebra has nuclear dimension n then its stabilised Cuntz semigroup $\mathrm{Cu}(A)$ will have n-comparison. This came as a less restrictive version of a similar result by Toms and Winter in terms of the decomposition rank [45, Lemma 6.1] and they were the first to consider this n-comparison property, though with the context of using dimension functions to define the property. The version of the n-comparison property we use for our purposes here, and with which Robert proves his statement, is due to Ortega, Perera, and Rørdam [37]. Let us recall some of the relevant definitions. For $a, b \in M_\infty(A)_+$ we say that a is Cuntz subequivalent to b, written $a \precsim b$, if there exists a sequence $(v_n)_\mathbb{N}$ in $M_\infty(A)$ such that $\|v_n b v_n^* - a\| \to 0$. Then a and b are said to be Cuntz equivalent, written $a \sim b$, if $a \precsim b$ and $b \precsim a$. This gives us an equivalence relation on $M_\infty(A)_+$ and we shall use $[a]$ to denote the equivalence class of $a \in M_\infty(A)_+$. By taking the quotient $W(A) := \frac{M_\infty(A)_+}{\sim}$, equipping this with the binary operation $[a]+[b]=[a\oplus b]$, and equipping it with the partial order $[a]\le[b] \iff a\precsim b$ we get that $W(A)$ is a positively ordered abelian monoid. This object $W(A)$ is the Cuntz semigroup of A. The object $\mathrm{Cu}(A)$ is the stabilised Cuntz semigroup of A, originally introduced by Coward, Elliott, and Ivanescu [19], shown to be order isomorphic to $W(A\otimes\mathcal{K})$. For $a\in M_\infty(A)_+$ and $\varepsilon>0$ we denote by $(a-\varepsilon)_+$ the element of $C^*(a)$ corresponding to the function defined by $f(t)=\max\{0,t-\varepsilon\}$ for $t\in\sigma(a)$.

In order to define the n-comparison property there is another relation we must consider. For elements x,y in an ordered abelian semigroup S we say x is stably dominated by y, written $x\le_s y$ if there exists some $k\in\mathbb{N}$ for which $(k+1)x\le ky$ [37, Def. 2.2]. Then an ordered abelian semigroup W has the n-comparison property if whenever $x,y_0,\dots,y_n\in W$ are such that $x\le_s y_i$ for all i then $x\le\sum_{i=0}^n y_i$ [37, Def 2.8]. It can be seen that such a semigroup W having the 0-comparison property is equivalent to W being almost unperforated. As noted, Toms and Winter in [45] gave their version of the n-comparison property in terms of dimension functions induced by lower semicontinuous 2-quasitraces, see [23, Sec. 4], though this ends up being equivalent to the property as defined in [37] as shown by Robert [41, Lemma 1].

The idea of Robert's proof, see [41, Sec. 2], is to factor through the finite dimensional algebras coming from the nuclear dimension in a particular way. In particular it relies on [50, Prop 3.2] that allows one to pick a system of approximations for the nuclear dimension such that the maps down are asymptotically order 0, and it also relies on the fact that $\mathrm{Cu}(F)$ is unperforated, thus has 0-comparison, for any finite dimensional F. Thus to be able to make a similar statement in terms of our relative dimension largely depends on being able to generalise these ideas to our own setting. We've already shown in proposition 4.3.7 that we are able to generalise Winter and Zacharias' result so we need to consider the comparison property of $\mathrm{Cu}(F\otimes B)$ in relation to that of $\mathrm{Cu}(B)$ for finite dimensional F. Certainly for any $m\in\mathbb{N}$ we have $M_\infty(M_m(B))\cong M_\infty(B)$ thus we have $\mathrm{Cu}(M_m(B))\cong\mathrm{Cu}(B)$ so that if $\mathrm{Cu}(B)$ has the n-comparison property then so will $\mathrm{Cu}(M_m(B))$ for all $m\in\mathbb{N}$. Then for a finite dimensional algebra F we can see that $\mathrm{Cu}(F\otimes B)\cong\mathrm{Cu}(\bigoplus_{k=1}^s M_{m_k}(B))$ which can be seen to have the n-comparison property also. To see why let $(a_k)_{k=1}^s,(b_k^j)_{k=1}^s$ be positive elements in $(\bigoplus_{k=1}^s M_{m_k}(B))\otimes\mathcal{K}\cong\bigoplus_{k=1}^s(M_{m_k}(B)\otimes\mathcal{K})$ such that $\left[(a_k)_{k=1}^s\right]\le_s\left[(b_k^j)_{k=1}^s\right]$ for $j=0,\dots,n$. Then for each k we have $[a_k]\le_s\left[b_k^j\right]$ for $j=0,\dots,n$ and thus $[a_k]\le\sum_{j=0}^n\left[b_k^j\right]$. Then for each k let $(v_{k,r})_{r\in\mathbb{N}}$ be the sequence coming from the definition of the Cuntz subequivalence which gives us a sequence $((v_{k,r})_{k=1}^s)_{r\in\mathbb{N}}$ in $(\bigoplus_{k=1}^s M_{m_k}(B))\otimes\mathcal{K}$ that shows $\left[(a_k)_{k=0}^s\right]\le\sum_{j=0}^n\left[(b_k^j)_{k=0}^s\right]$. This is a simpler version of lemma 4.4.2 to come. Thus if $\mathrm{Cu}(B)$ has the n-comparison property then so does $\mathrm{Cu}(F\otimes B)$ for any finite dimensional F.

We shall still require a number of results given by Robert [41] in order to prove our own version of a dimensional comparison result. The following results and proofs come as generalisations of those present in [41].

Proposition 4.4.1. Let A, B be C^*-algebras with B a separable continuous $C(X)$-algebra with simple fibres. If $\mathrm{nd}_B A = n$ then there exists a net Λ and cpc order 0 maps

$$\Psi^{(i)} : A \to \frac{\prod_\Lambda F_\lambda^{(i)} \otimes B}{\bigoplus_\Lambda F_\lambda^{(i)} \otimes B}$$

and

$$\Phi^{(i)} : \frac{\prod_\Lambda F_\lambda^{(i)} \otimes B}{\bigoplus_\Lambda F_\lambda^{(i)} \otimes B} \to (A_\infty)_\Lambda$$

such that

$$\iota = \sum_{i=0}^{n} \Phi^i \Psi^i \tag{4.4.1}$$

where $\iota : A \to A_\Lambda$ is the diagonal embedding.

Proof: By proposition 4.3.7 we can find a system of n-decomposable cp approximations $(F_\lambda \otimes B, \psi_\lambda, \sim\varphi_\lambda)_\Lambda$ for A such that the induced map $\Psi : A \to \frac{\prod_\Lambda F_\lambda^{(i)} \otimes B}{\bigoplus_\Lambda F_\lambda^{(i)}}$ is cpc order 0, so certainly the component maps $\Psi^{(i)}$ are. By lemma 4.1.4 for each λ and for each i we are able to find a cpc order 0 map $\phi_\lambda^{(i)} : F_\lambda^{(i)} \otimes B \to A_\infty$. The induced maps $\tilde{\Phi}^{(i)} : \prod_\Lambda \to (A_\infty)_\Lambda$ can then be seen to be order 0. That the maps $\Phi^{(i)} : \frac{\prod_\lambda F_\lambda^{(i)} \otimes B}{\bigoplus_\lambda F_\lambda^{(i)} \otimes B} \to (A_\infty)_\Lambda$ are order 0 is due to a fact shown in the proof of lemma 2 of [41] that is if $\tilde{\xi} : C \to D$ is cpc order 0 and $I \trianglelefteq C$ an ideal such that $I \subset \ker(\tilde{\xi})$ then the induced map $\xi : \frac{C}{I} \to D$ is also order 0. This comes from the correspondence between cpc order 0 maps and $*$-homomorphisms on the cone. Then by construction $\iota(a) = \sum_{i=0}^{n} \Phi^{(i)} \Psi^{(i)} a$ for all $a \in A$. ∎

This lemma in particular is a generalisation of Robert's in which he shows the same result for specifically unperforated semigroups.

Lemma 4.4.2. Let A and $(A_\lambda)_\Lambda$ be C^*-algebras.

(i) If $Cu(A)$ has the m-comparison property then so does $Cu(\frac{A}{I})$ for all closed two sided ideals $I \trianglelefteq A$.

(ii) If for each $\lambda \in \Lambda$, $Cu(A_\lambda)$ has the m-comparison property then so do $Cu(\prod_\Lambda A_\lambda)$ and $Cu(\frac{\prod_\lambda A_\Lambda}{\bigoplus_\Lambda A_\lambda})$.

Proof: i) Let $[\tilde{a}], [\tilde{b}_j] \in Cu(\frac{A}{I})$ be such that $[\tilde{a}] \leq_s [\tilde{b}_j]$ for $j = 0, \ldots, m$. Let $[a], [b_j] \in Cu(A)$ be lifts of these elements for $j = 0, \ldots, m$, so that by [18, Prop. 1] we have $[a] \leq_s [b_j] + [i_j]$ for some $i_j \in I \otimes \mathscr{K}$ for $j = 0, \ldots, m$. Since $Cu(A)$ has the m-comparison property we have $[a] \leq \sum_{j=0}^m ([b_j] + [i_j])$. Passing back to the quotient gives us $[\tilde{a}] \leq \sum_{j=0}^m [\tilde{b}_j]$.

ii) Let $(a_\lambda)_\Lambda, (b_{\lambda j})_\Lambda \in (\prod_\Lambda A_\lambda) \otimes \mathscr{K} \subset \prod_\Lambda (A_\lambda \otimes \mathscr{K})$ be positive elements such that $[(a_\lambda)_\Lambda] \leq_s [(b_{\lambda j})_\Lambda]$ for $j = 0, \ldots, m$. Let $\varepsilon > 0$, by [37, Lemma 2.6] there exists some δ such that

$$[((a_\lambda - \varepsilon)_+)_\lambda] \leq_s [((b_{\lambda j} - \delta)_+)_\lambda],$$

for all j. Certainly $[(a_\lambda - \varepsilon)_+] \leq_s [(b_{\lambda j} - \delta)_+]$ for $j = 0, \ldots, m$ and for all λ. To see this consider $(c_\lambda)_\lambda, (d_\lambda)_\lambda \in (\prod_\Lambda A_\lambda) \otimes \mathscr{K}$ such that $[(c_\lambda)_\lambda] \leq [d_\lambda]$. Then there exists a sequence $((v_{\lambda,n})_\lambda)_\mathbb{N}$ such that $\|(v_{\lambda,n})_\lambda (d_\lambda)_\lambda (v_{\lambda,n})_\lambda^* - (c_\lambda)_\lambda\| \to 0$, then we have $(\sup_\lambda \|v_{\lambda,n} d_\lambda v_{\lambda,n}^* - c_\lambda\|) \to 0$. Since each $Cu(A_\lambda)$ has the m-comparison property we have $[(a_\lambda - \varepsilon)_+] \leq \sum_{j=0}^m [(b_{\lambda j} - \delta)_+] = [((\oplus_{j=0}^m b_{\lambda j}) - \delta)_+]$. By the definition of the Cuntz subequivalence we may find $x_\lambda \in A_\lambda \otimes \mathcal{K}$ such that

$$(a_\lambda - 2\varepsilon)_+ = x_\lambda^* x_\lambda \text{ and } x_\lambda x_\lambda^* \in \mathrm{her}(((\oplus_{j=0}^m b_{\lambda j}) - \delta)_+).$$

As $\|x_\lambda\|^2 \leq \|a_\lambda\|$ we have that $(x_\lambda)_\Lambda \in \prod_\lambda (A_\lambda \otimes \mathscr{K})$, we also have $x_\lambda x_\lambda^* \leq \frac{1}{\delta}(\oplus_{j=0}^m b_{\lambda j})$. Thus

$$((a_\lambda - 2\varepsilon)_+)_\Lambda = (x_\lambda^*)_\Lambda (x_\lambda)_\Lambda \text{ and } (x_\lambda)_\Lambda (x_\lambda^*)_\Lambda \in \mathrm{her}(((\oplus_{j=0}^m b_{\lambda j}))_\Lambda).$$

Since $(\prod_\Lambda A_\lambda) \otimes \mathcal{K}$ is a hereditary subalgebra of $\prod_\Lambda (A_\lambda \otimes \mathcal{K})$ and $(x_\lambda)_\Lambda (x_\lambda^*)_\Lambda \in \mathrm{her}(((\oplus_{j=0}^m b_{\lambda j}))_\Lambda)$ we have that $(x_\lambda)_\Lambda \in (\prod_\Lambda A_\lambda) \otimes \mathcal{K}$. This gives us $[((a_\lambda - 2\varepsilon)_+)_\lambda] \leq [((\oplus_{j=0}^m b_{\lambda j}))_\lambda]$, and since ε can be taken to be arbitrarily small we have $[(a_\lambda)_\lambda] \leq \sum_{j=0}^m [(b_{\lambda j})_\lambda]$. ∎

With all these we are now prepared to make our own statement on comparison in the context of relative nuclear dimension.

Theorem 4.4.3. Let A, B be C^*-algebras with B a separable continuous $C(X)$-algebra with simple fibres. If $\mathrm{nd}_B A = n$ and $\mathrm{Cu}(B)$ has the m-comparison property, then $\mathrm{Cu}(A)$ has the $((n+1)(m+1)-1)$-comparison property.

Proof: Assume that we have $[a] \leq_s [b_{ij}]$ for $i = 0, \ldots, n$, $j = 0, \ldots, m$, and let $\Phi^{(i)}$ and $\Psi^{(i)}$ be the cpc order 0 maps defined in proposition 4.4.1. By [49, corollary 3.5] cp order 0 maps preserve Cuntz comparison, and so we have $[\Psi^{(i)}(a)] \leq [\Psi^{(i)}(b_{ij})]$ for $i = 0, \ldots, n$ and $j = 0, \ldots, m$. Since $\mathrm{Cu}(B)$ has m-comparison, so must $Cu(F_\lambda \otimes B)$ for each λ. Thus $[\Psi^{(i)}(a)] \leq \sum_{j=0}^m [\Psi(i)(b_{ij})]$ by lemma 4.4.2. The maps $\Phi^{(i)}$ are order 0 and so will also preserve Cuntz comparison so we have

$$\left[\Phi^{(i)}\Psi^{(i)}(a)\right] \leq \sum_{j=0}^m \left[\Phi^{(i)}\Psi^{(i)}(b_{ij})\right] \leq \sum_{j=0}^m \left[\sum_{i=0}^n \Phi^{(i)}\Psi^{(i)}(b_{ij})\right] = \sum_{j=0}^m \left[\iota(b_{ij})\right].$$

Hence,

$$[\iota(a)] = \left[\sum_{i=0}^{n} \Phi^{(i)}\Psi^{(i)}(a)\right] \leq \sum_{i=0}^{n} \left[\Phi^{(i)}\Psi^{(i)}(a)\right] \leq \sum_{i=0}^{n}\sum_{j=0}^{m} [\iota(b_{ij})] .$$

By [41, Lemma 3] we get $[a] \leq \sum_{i=0}^{n}\sum_{j=0}^{m} [b_{ij}]$. ∎

4.5 Further Considerations

Remark 4.5.1. As the Jiang-Su algebra $\mathcal{Z}$ is simple and separable we can view it as a separable $C(\{x\})$-algebra with simple fibres, where $\{x\}$ is the single point space. As such it satisfies the hypothesis of theorem 4.3.12 to allow us to make some statements surrounding the nuclear dimension relative to $\mathcal{Z}$. Let A be a simple, separable, unital, nuclear, and $\mathcal{Z}$-stable algebra. We have already seen that if A is AF then $\mathrm{nd}_A = 0$ so that any such algebra must exist as a generalised inductive limit $A \cong \lim(F_n \otimes \mathcal{Z}, \phi_{nm})$. However if A is not AF we have only been able to show that $\mathrm{nd}_{\mathcal{Z}}A = 0$ or 1. Trivially we know that $\mathrm{nd}_{\mathcal{Z}}\mathcal{Z} = 0$ and ideally this would be the only non-AF example for A such that $\mathrm{nd}_{\mathcal{Z}}A = 0$. We would like to be able to say that $\mathrm{nd}_{\mathcal{Z}}A = \dim_{\mathrm{nuc}}A$ for all simple, separable, unital, nuclear A such that $A \ncong \mathcal{Z}$. We certainly know that $\mathrm{nd}_{\mathcal{Z}}A = 1$ if A does not exist as a generalised inductive limit of the form $(F_n \otimes \mathcal{Z}, \phi_{nm})$ for F_n finite dimensional, though we do not know if this must be the case when A is not AF or if there are some examples of non-AF $\mathcal{Z}$-stable algebras that can be written as such a generalised inductive limit. Even if there are examples of non-AF A that can be written as such a generalised inductive limit it is not clear that should itself imply $\mathrm{nd}_{\mathcal{Z}}A = 0$ since the condition that the induced maps $\phi_n : F_n \otimes \mathcal{Z} \to A$ are $*$-homomorphisms is not immediate.

Remark 4.5.2. Despite being a motivating example leading to defining our notion of relative nuclear dimension we did not manage to obtain any precise value for the relative nuclear dimension for those Villadsen algebras that are nuclear but with infinite nuclear dimension, outside of obvious trivial examples. We know that if A is such a Villadsen algebra then to have $\mathrm{nd}_B A < \infty$ we must have that B itself has infinite nuclear dimension. Moreover theorem 4.4.3 would imply that B must be such that its Cuntz semigroup does not have the n-comparison property for any n, at least for simple and separable B. As such we were most interested in the case that both A and B are these Villadsen algebras with infinite nuclear dimension. Particularly we had first considered the case that $A \cong \lim_{i\to\infty}(M_{m_i} \otimes C(X^{\times n_i}), \phi_i)$ and $B \cong \lim_{i\to\infty}(M_{m_i} \otimes C(X^{\times n_i}), \psi_i)$, that is the component algebras at each stage being the same but changing in particular the eigenvalue maps at each stage to see what effect that might have on the relative nuclear dimension. In this scenario even we did not find any interesting estimates for the finite relative dimension between these algebras. We do still believe though that it should be possible to show the relative nuclear dimension here must be finite, and further work will hopefully be able to show this.

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