Application of a CSMP Technique and Variational Method to Solve External Flow Problems with Variable Physical Properties

BY

ASHOK KUMAR

Submitted to the Faculty of Science and Engineering of the University of Ottawa in partial fulfillment of the requirements for the degree of

MASTER OF APPLIED SCIENCE

in

Mechanical Engineering

February, 1972

ACKNOWLEDGEMENT

As always, a research work can not be a product of an individual, but is influenced by the writings of admired and respected persons in the field, and by discussions with colleagues. It is never possible to recognize all of these contributers. However, the author would like to thank Dr. S. C. Cheng (Research Director) and Dr. L. G. Birta for their suggestions. Indeed, his interest in CSMP program was aroused by the discussions about the difficulty of division by zero, with Prof. Birta, who spent his many valuable hours trying to simplify my computational problems.

He also wishes to extend his gratitude to all other staff members of the Faculty of Science and Engineering for their interest. Partial financial support provided by the National Research Council, Ottawa, Canada is appreciated.

The much needed encouragement for the completion of the work, from friends is noteworthy. In particular, special thanks go to Preedeep and Vasudev for their help in the preparation of the document. Also, the help of Dev, Raju and Mahendra during the initial stages of the work is duely recorded.

The author is deeply indebted to his family for being a source of constant inspiration during the difficult days of the work.

SYNOPSIS

External boundary layer flow problems with variable physical properties are, in general, difficult to deal with computationally. In this study, wedge flow problems with constant wall temperature or constant wall heat flux are investigated using the variational method and a CSMP (Continous System Modelling Program) package. The physical properties, i.e. the thermal conductivity and the viscosity, are assumed to be linear functions of temperature. In each case a variational formulation based on the concept of local potential is first applied to transform the two fundamental coupled nonlinear partial differential equations into two more tractable ordinary differential equations. The equations are then solved by CSMP technique. The study has yielded information concerning flow performance as well as information on the sensitivity of thermal conductivity and viscosity to the local Nusselt number and the local friction factor. Wherever pcssible a comparison has been made between the results from this study and those obtained from the literature. Close agreement has been obtained in all cases. The results show that the variable properties have substantial influence on the flow performance.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

A	Viscosity coefficient
a	Output signal
$a_1, a_2, --, a_n$	Input signals
B	Conductivity coefficient
C	Constant; see Eq. (A-10)
$c_1, c_2, c_3, ---$	Constants; see Eq. (24)
c_p	Specific heat at constant pressure
E	Local potential
f	Local friction factor
h_x	Heat Transfer coefficient at x
k	Thermal conductivity
L	Logic signal
m	Wedge constant
Nu_x	Local Nusselt number $= \frac{h_x x}{k_\infty}$
p	Pressure
Pr_∞	Prandtl number $= \frac{\mu_\infty c_p}{k_\infty}$
q	Heat flux
Re	Reynolds number; $Re_\infty = \frac{\rho u_\infty \ell}{\mu_\infty}$, $Re_x = \frac{\rho u_\infty x}{\mu_\infty}$

s	Surface
T	Temperature
u, v	Velocity along x and y axes
V	Volume
W	$W = Y^{*^3}$; see Eq. (24 a)
x_0	Unheated starting length
x, y	Cartesian coordinates
Y	(Δ_t/Δ) ratio between thermal and momentum boundary layer thickness
Z	$Z = \Delta^{*^2}$; see Eq. (24 a)

Greek Symbols

ρ	Density
δ	Variation notation
μ	Dynamic viscosity
ν	Kinematic viscosity
Δ	Momentum boundary layer thickness
Δ_t	Thermal boundary layer thickness
θ	Dimensionless temperature variable; $\theta = \dfrac{T-T_\infty}{T_\omega - T_\infty}$
ϕ	Dimensionless temperature variable; $= \dfrac{T-T_\infty}{\dfrac{q_H \ell}{k_\infty}}$
α	Thermal diffusivity

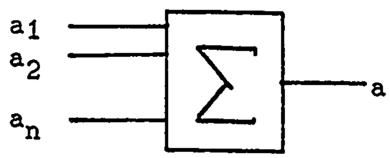
Subscripts

∞	Free stream property
w	Wall property
T	Constant Wall Temperature Case
H	Constant Wall Heat Flux Case

Superscripts

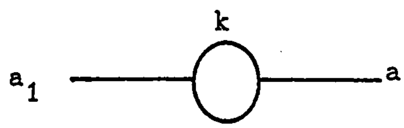
*	Dimensionless quantity
o	Stationary state

PROGRAMMER'S SYMBOLS



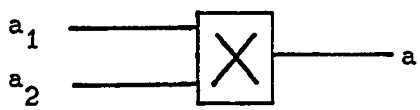
Summer

$a = a_1 + a_2 + \text{---} + a_n$



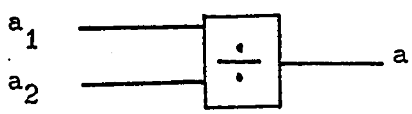
Pot

$a = k\ a_1$



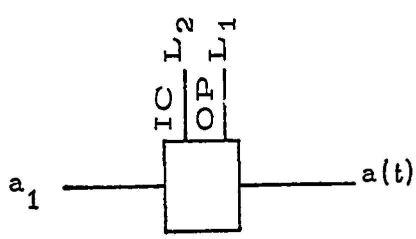
Multiplier

$a = a_1 a_2$



Divider

$a = a_1 / a_2$



Integrator

$a = \text{MODINT}\ (IC, L_1, L_2, a_1)$

$a(t) = \int_0^t a_1\ dt + IC$

for L_1 TRUE, any L_2

$= IC$ for L_1 FALSE,

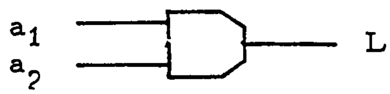
L_2 TRUE

$=$ Last value

for L_1 FALSE, L_2 FALSE

$L_1,\ L_2$ TRUE $\Rightarrow L_1,\ L_2 > 0$

$L_1,\ L_2$ FALSE $\Rightarrow L_1,\ L_2 \leq 0$



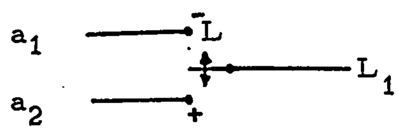
Comparator

L is TRUE (i.e. $L = 1$)

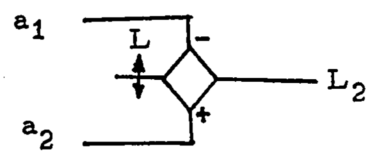
for $a_1 \geqslant a_2$

L is FALSE (i.e. $L = 0$)

otherwise



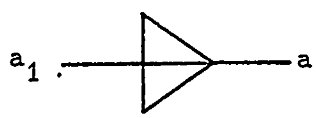
or,



Switch

For L TRUE ($\geqslant 0$), contacts closed to +

For L FALSE (< 0), contacts closed to -

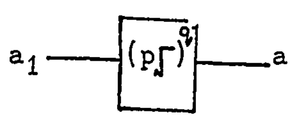


Logical Invertor

$a = \text{NOT}\ (a_1)$

$a = 1 \qquad a_1 \leqslant 0$

$a = 0 \qquad a_1 > 0$



$a = (a_1)^{q/p}$

CHAPTER 1

INTRODUCTION

The solutions of the laminar wedge flow are of importance to many engineering applications. As an example, solutions of this flow can be used, as an approximation for calculating the local heat transfer coefficient, to bodies of arbitrary cross-section such as turbine blades [1], cylinders [2] etc. A better approximation which utilizes the wedge flow solution is presented in reference [3].

For obvious reasons the flow has been studied extensively by many investigators over the last four decades [1 to 15]. Schlicting [13] and Kays [14] summarized the previous work on wedge flow upto 1968. The exact solutions of the laminar boundary layer equations are presented, for the wedge flow with a constant wall temperature under constant property assumption, by Eckert [4]. Levy [5] calculated many cases of constant property flows for Prandtl numbers 0.7, 1.0, 5 and 10. Solutions for porous surface were obtained by Donoughe and Livingood [6] to estimate the effects of transpiration cooling on heat transfer. Reshotko and Cohen [7] further generalized these solutions for Prandtl number 0.7. Approximate solutions for the heat transfer rate with an arbitrary distribution of main stream velocity and wall temperature were obtained by Lighthill [8].

Brown and Donoughe [9] obtained the solution for Prandtl number 0.7 by assuming the fluid properties varying as powers of temperature. The solutions to the compressible laminar boundary layer equations for Prandtl number 0.7 and 1.0 were presented by Soloman [10]. The viscosity was taken to vary linearly with temperature so that $\rho\mu$ = constant. A particular case was dealt in reference [11] by assuming the linear variation of fluid properties. The variable heat flux at the wall case assuming constant properties for m=0 was given by Kays [14] utilizing the step function solution of Klein and Tribus [12]. This paper also presented a summary of analytical results of the general problem of heat convection from surfaces at non-uniform temperature.

Unfortunately, most of the work conducted so far assumed constant properties for cases having Prandtl number greater than one. In this study, the effects of variable viscosity and thermal conductivity on the flow with $Pr_{\infty} > 1$ are investigated by assuming either a constant wall temperature or a constant heat flux at the wall. The problem is solved by the variational method with the concept of local potential and a technique similar to the Kantorovich approach. The concept of local potential was first introduced by Glansdroff and Prigogine [16]. Applications of this formulation are shown in references [17, 18, 19, 11, 15].

In each case, two coupled differential equations are derived by applying the variational formulation to the basic equations of the flow. The System/360 CSMP package which has gained a degree of acceptance for modelling dynamic systems recently, is used to solve these two non-linear, coupled equations. All the aspects regarding the use of S/360 CSMP in obtaining the solution of a problem are described in the literature[20 to 23]. Applications of this package are shown in references[24, 25, 26, 27, 28]. The CSMP program has proved to be a very convenient and effective tool in the study. A combination of the Variational method and the CSMP package to solve this type of problem is a rather new approach. It is hoped that this combined technique can be utilized to solve other engineering problems of similar nature.

CHAPTER 2

BASIC EQUATIONS

The equations of conservation of momentum, energy and mass for a two-dimensional, incompressible boundary layer flow over a wedge neglecting heat dissipation [13] can be expressed as

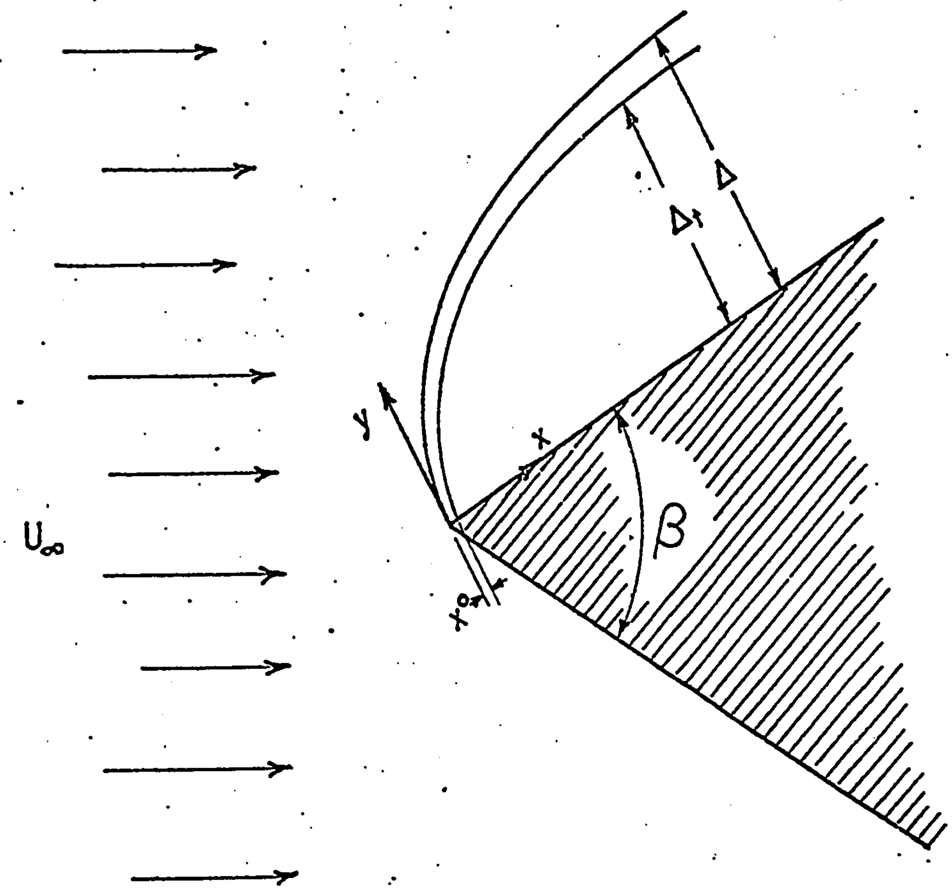
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (1)$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

Here μ and k are functions of T.

Although the problem under study is a steady-state case, one must nevertheless retain the time-dependent character of the equations when forming the local potential.



FLOW PAST A WEDGE ($U_\infty = Cx^m$)

CHAPTER 3

VARIATIONAL FORMULATION OF THE PROBLEM

The closed form solution of simultaneous Eqs. (1), (2) and (3) is, in general, very difficult to obtain because of non-linearities involved. The variational technique can, however, be used to transform these equations into a more tractable form.

In order to construct a local potential for the problem for use in the variational method, a technique used by Glansdorff and Prigogine [16] is followed. Upon multiplying Eq. (1) by $\frac{\partial u}{\partial t}$, Eq. (2) by $\frac{\partial T}{\partial t}$ and Eq. (3) by $-(\frac{\rho}{2})(\frac{\partial V^2}{\partial t})$, summing the results and rearranging the terms, one obtains

$$\begin{aligned}\Psi = -\rho\left(\frac{\partial u}{\partial t}\right)^2 - \rho c_p\left(\frac{\partial T}{\partial t}\right)^2 = {} & \rho u \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial t}\frac{\partial u}{\partial y} \\ & - \frac{\partial u}{\partial t}\left[\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)\right] - \frac{\rho}{2}\frac{\partial V^2}{\partial t} \\ & \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \rho c_p u \frac{\partial T}{\partial x}\frac{\partial T}{\partial t} \\ & + \rho c_p v \frac{\partial T}{\partial y}\frac{\partial T}{\partial t} - \frac{\partial T}{\partial t}\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \\ & + \frac{\partial p}{\partial x}\frac{\partial u}{\partial t} \leq 0 \end{aligned} \qquad (4)$$

Since Eqs. (1), (2) and (3) describe a two-dimensional flow, the value of Ψ will vary in x-y plane but will be independent of z. From the arguments of the variational technique, ϕ is defined in this case by integrating over the area

$$\Phi = \iint_S \psi \, dx \, dy \leq 0 \qquad (5)$$

where s is an area of interest in the x-y plane which is bounded by curve c (the curve which encloses s).

The integrand ψ can be rearranged in the form

$$\begin{aligned}
\psi = & \frac{\partial}{\partial x}\left(\rho u^2 \frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(\rho u v \frac{\partial u}{\partial t}\right) \\
& - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \frac{\partial u}{\partial t}\right) - \rho u \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
& - \rho u^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \rho u v \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) \\
& + \frac{\mu}{2} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\rho}{2} \frac{\partial V^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
& + \rho c_p u \frac{\partial T}{\partial x} \frac{\partial T}{\partial t} + \rho c_p v \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} \\
& - \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial t} \frac{\partial T}{\partial y}\right) + \frac{k}{2} \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right)^2 \\
& + \frac{\partial p}{\partial x} \frac{\partial u}{\partial t} \leq 0 \qquad (6)
\end{aligned}$$

Combining Eqs. (5) and (6) and using Stokes' Theorem, one obtains

$$\begin{aligned}
\Phi = \iint_S \Big[& - \rho u^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - \rho u v \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) \\
& + \frac{\mu}{2} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\rho}{2} \frac{\partial u^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
& - \frac{\rho}{2} \frac{\partial v^2}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
& + \frac{k}{2} \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right)^2 + \rho c_p u \frac{\partial T}{\partial x} \frac{\partial T}{\partial t}
\end{aligned}$$

$$+ \rho c_p v \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} + \frac{\partial p}{\partial x} \frac{\partial u}{\partial t}] \, dx \, dy$$

$$+ \int_c (\rho u^2 \frac{\partial u}{\partial t} dy - \rho u v \frac{\partial u}{\partial t} dx$$

$$+ \mu \frac{\partial u}{\partial y} \frac{\partial u}{\partial t} dx + k \frac{\partial T}{\partial y} \frac{\partial T}{\partial t} dx) \leq 0 \quad (7)$$

Near stationary state, the concept of local potential gives

$$\Phi = \frac{\partial}{\partial t} \iint_S [- \rho u^{0^2} \frac{\partial u}{\partial x} - \rho u^0 v^0 \frac{\partial u}{\partial y} + \frac{\mu^0}{2} (\frac{\partial u}{\partial y})^2$$

$$- \frac{\rho}{2} u^2 (\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}) - \frac{\rho}{2} v^2$$

$$(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}) + \frac{k^0}{2} (\frac{\partial T}{\partial y})^2$$

$$+ \rho c_p^0 u^0 \frac{\partial T^0}{\partial x} T + \rho c_p^0 v^0 \frac{\partial T^0}{\partial y} T$$

$$+ \frac{\partial p}{\partial x} u] \, dx \, dy$$

$$+ \frac{\partial}{\partial t} \int_c (\rho u^{0^2} u \, dy - \rho u^0 v^0 u \, dx$$

$$+ \mu^0 \frac{\partial u^0}{\partial y} u \, dx + k^0 \frac{\partial T^0}{\partial y} T \, dx) \leq 0 \quad (8)$$

Therefore the local potential is

$$E = \iint_S [- \rho u^{0^2} \frac{\partial u}{\partial x} - \rho u^0 v^0 \frac{\partial u}{\partial y} + \frac{\mu^0}{2} (\frac{\partial u}{\partial y})^2$$

$$- \frac{\rho}{2} u^2 (\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y}) - \frac{\rho}{2} v^2 (\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y})$$

$$+ \frac{k^0}{2} (\frac{\partial T}{\partial y})^2 + \rho c_p^0 u^0 \frac{\partial T^0}{\partial x} T$$

$$+ \rho c_p^0 v^0 \frac{\partial T^0}{\partial y} T + \frac{\partial p^0}{\partial x} u] \, dx \, dy$$

$$+ \int_c \left(\rho u^{o^2} u \, dy - \rho u^o v^o u \, dx + \mu^o \frac{\partial u^o}{\partial y} u \, dx + k^o \frac{\partial T^o}{\partial y} T \, dx \right) \qquad (9)$$

with the subsidiary conditions

$$u^o = u$$
$$v^o = v$$
$$T^o = T$$

The line integral portion of the Eq. (9) can be simplified by using the following boundary conditions. In this study the area of interest is a rectangle bounded by the lines $x=0$, $x=\ell$, $y=0$ and $y=\Delta$. Note that the boundary conditions for the problem are

$$u = 0 \qquad y = 0$$
$$v = 0 \qquad y = 0$$
$$u = u_\infty \qquad y = \Delta$$
$$u = u_\infty \qquad x = 0$$

and

$$T = T_w \qquad y = 0$$
$$T = T_\infty \qquad y = \Delta_t$$
$$T = T_\infty \qquad x = 0$$

Therefore, the contribution from line integral is

$$E_{line} = \int_c [\rho u^{o2} u\,dy - \rho u^o v^o u\,dx + \mu^o \frac{\partial u^o}{\partial y} u\,dx + k^o \frac{\partial T^o}{\partial y} T\,dx]$$

$$= \int_0^{\Delta} (\rho u^{o2} u\big|_{x=\ell} - \rho u^{o2} u\big|_{x=0})\,dy$$

$$- \int_0^{\ell} (-\rho u^o v^o u\big|_{y=\Delta} + \rho u^o v^o u\big|_{y=0})\,dx$$

$$+ \int_0^{\ell} (-\mu^o \frac{\partial u^o}{\partial y} u\big|_{y=\Delta} + \mu^o \frac{\partial u^o}{\partial y} u\big|_{y=0})\,dx$$

$$+ \int_0^{\ell} (-k^o \frac{\partial T^o}{\partial y} T\big|_{y=\Delta_t} + k^o \frac{\partial T^o}{\partial y} T\big|_{y=0})\,dx \quad (10)$$

Imposing the boundary conditions, one obtains

$$E_{line} = \int_0^{\Delta} (\rho u^{o2} u\big|_{x=\ell} - \rho u^{o2} u\big|_{x=0})\,dy$$

$$+ \int_0^{\ell} (\rho u^o v^o u\big|_{y=\Delta})\,dx + \int_0^{\ell} (-\mu^o \frac{\partial u^o}{\partial y} u)\big|_{y=\Delta}\,dx$$

$$+ \int_0^{\ell} (-k^o \frac{\partial T^o}{\partial y} T\big|_{y=\Delta_t})\,dx + \int_0^{\ell} (k^o \frac{\partial T^o}{\partial y} T\big|_{y=0})\,dx \quad (11)$$

By using Eqs. (3) and (11), Eq. (9) can be further reduced to

$$E = \iint_S \Big[-\rho u^{o^2} \frac{\partial u}{\partial x} - \rho u^o v^o \frac{\partial u}{\partial y} + \frac{\mu^o}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \rho c_p^o u^o T \frac{\partial T^o}{\partial x} + \rho c_p^o v^o T \frac{\partial T^o}{\partial y} + \frac{k^o}{2}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{\partial p^o}{\partial x} u \Big] dx\, dy$$
$$+ \int_0^{\Delta} \left(\rho u^{o^2} u \Big|_{x=\ell} - \rho u^{o^2} u \Big|_{x=0} \right) dy + \int_0^{\ell} \left(\rho u^o v^o u \Big|_{y=\Delta} \right) dx$$
$$+ \int_0^{\ell} \left(-\mu^o \frac{\partial u^o}{\partial y} u \right)\Big|_{y=\Delta} dx + \int_0^{\ell} \left(-k^o \frac{\partial T^o}{\partial y} T \right)\Big|_{y=\Delta_t} dx$$
$$+ \int_0^{\ell} \left(k^o \frac{\partial T^o}{\partial y} T \right)\Big|_{y=0} dx \qquad (12)$$

or,

$$E = \iint_S F(x, y, u, u_x, u_y, T, T_x, T_y)\, dx dy$$
$$+ \int_0^{\Delta} \left(\rho u^{o^2} u \Big|_{x=\ell} - \rho u^{o^2} u \Big|_{x=0} \right) dy$$
$$+ \int_0^{\ell} \left(\rho u^o v^o u \Big|_{y=\Delta} \right) dx + \int_0^{\ell} \left(-\mu^o \frac{\partial u^o}{\partial y} u \right)\Big|_{y=\Delta} dx$$
$$+ \int_0^{\ell} \left(-k^o \frac{\partial T^o}{\partial y} T \right)\Big|_{y=\Delta_t} dx + \int_0^{\ell} \left(k^o \frac{\partial T^o}{\partial y} T \right)\Big|_{y=0} dx$$

(12 a)

where,
$$F(x, y, u, u_x, u_y, T, T_x, T_y) = -\rho u^{o^2} \frac{\partial u}{\partial x}$$
$$-\rho u^o v^o \frac{\partial u}{\partial y} + \frac{\mu^o}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \rho c_p u^o T \frac{\partial T^o}{\partial x}$$
$$+\rho c_p v^o T \frac{\partial T^o}{\partial y} + \frac{k^o}{2}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{\partial p^o}{\partial x} u \qquad (12\ b)$$

Again it can be seen that the local potential as defined for two-dimensional boundary layer is composed of two parts. One part is an area integral and the other is a line integral.

In order to prove that Eq. (12) is the local potential of the problem, the following operations must be performed. Taking variation of local potential E (Eq. 12 a) with respect to u and T, one obtains

$$\left(\frac{\delta E}{\delta u}\right)_{u^o} = 0 \qquad (13)$$

$$\left(\frac{\delta E}{\delta T}\right)_{T^o} = 0 \qquad (14)$$

Eq. (13) can be rewritten as

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad (15)$$

Substituting Eq. (12 b) to Eq. (15), it follows that

$$\frac{\partial p^o}{\partial x} - \frac{\partial}{\partial x}\left(-\rho u^{o^2}\right) - \frac{\partial}{\partial y}\left(-\rho u^o v^o + \mu^o \frac{\partial u}{\partial y}\right) = 0 \qquad (16)$$

Using the subsidiary conditions

$$u^0 = u$$
$$v^0 = v$$

Eq. (16) becomes

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

or,
$$\rho\, 2u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

Rearranging the terms, the above equation gives

$$\rho u \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

or,
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$$

This is the momentum boundary equation in the x-direction.

Similarly Eq. (14) gives

$$\frac{\partial F}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right) = 0 \qquad (17)$$

or,
$$\rho c_p u^0 \frac{\partial T^0}{\partial x} + \rho c_p v^0 \frac{\partial T^0}{\partial y} - \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}\left(k^0 \frac{\partial T}{\partial y}\right) = 0$$

or,
$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)$$

This is the energy boundary layer equation. Thus it has been shown that Eq. (12) is the local potential of the problem.

In order to proceed, the following velocity and temperature profiles are assumed :

Case (1) Constant Wall temperature (i.e. T_ω=constant)

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \quad 0 \le y \le \Delta \tag{18 a}$$

$$\frac{T - T_\omega}{T_\infty - T_\omega} = \frac{3}{2}\left(\frac{y}{\Delta_t}\right) - \frac{1}{2}\left(\frac{y}{\Delta_t}\right)^3 \quad 0 \le y \le \Delta_t \tag{19 a}$$

These satisfy the boundary conditions

at $y = 0$

$$u = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

$$T = T_\omega$$

$$\frac{\partial^2 T}{\partial y^2} = 0$$

at $y = \Delta$

$$u = u_\infty$$

$$\frac{\partial u}{\partial y} = 0$$

and at $y = \Delta_t$

$$T = T_\infty$$

$$\frac{\partial T}{\partial y} = 0$$

Case (2) Constant Wall Heat Flux (i.e. $q_H = -k_\infty \frac{\partial T}{\partial y} = \text{constant}$)

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \qquad 0 \le y \le \Delta \tag{18 b}$$

$$\frac{T - T_\infty}{\frac{q_H \ell}{k_\infty}} = \frac{1}{3}\left(\frac{\Delta_t}{\ell}\right)\left[1 - 3\left(\frac{y}{\Delta_t}\right) + 3\left(\frac{y}{\Delta_t}\right)^2 - \left(\frac{y}{\Delta_t}\right)^3\right] \tag{19 b}$$

$$0 \le y \le \Delta_t$$

These satisfy the boundary conditions

at $y = 0$

$$u = 0$$
$$\frac{\partial^2 u}{\partial y^2} = 0$$
$$-\frac{\partial T}{\partial y} = \frac{q_H}{k_\infty}$$
$$\frac{\partial^2 T}{\partial y^2} = 0$$

at $y = \Delta$

$$u = u_\infty$$
$$\frac{\partial u}{\partial y} = 0$$

and

at $y = \Delta_t$

$$T = T_\infty$$
$$\frac{\partial T}{\partial y} = 0$$

In this study, only the case $\Delta > \Delta_t$ is considered, where Δ, Δ_t are functions of x to be determined.*

For simplicity, the viscosity and the thermal conductivity are chosen as linear functions of temperature.

Case (1) Constant Wall Temperature

$$\frac{\mu}{\mu_\infty} = 1 + A\theta \qquad (20\text{ a})$$

$$\frac{k}{k_\infty} = 1 + B\theta \qquad (21\text{ a})$$

where, $$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

Case (2) Constant Wall Heat Flux

$$\frac{\mu}{\mu_\infty} = 1 + A\phi \qquad (20\text{ b})$$

$$\frac{k}{k_\infty} = 1 + B\phi \qquad (21\text{ b})$$

where, $$\phi = \frac{T - T_\infty}{\frac{q_H \ell}{k_\infty}}$$

---

* The approach here is similar to that of Kantorovich Method.

In this case the different temperature variable (ϕ instead of θ) is chosen because the wedge temperature at the wall becomes unknown.

These expressions for velocity, temperature, viscosity and conductivity are substituted into Eq. (12). By imposing certain variational arguments the following two coupled equations are obtained. (The rather tedious calculation involved in making this step is given in the Appendices.) Notice that, unlike the conventional method, high order terms of Y^{*4} and Y^{*5} of Eqs. (A-20), (A-21), (B-17) and (B-18) in this study are retained for more accurate calculation.

Case (1) Constant Wall Temperature:

$$ZW'(1 + c_8 W^{2/3}) = c_1 + Z'W(c_2 + c_9 W^{2/3}) + \frac{WZ}{X^*}(c_3 + c_{10} W^{2/3}) \quad (22)$$

$$Z' = c_4 + c_5 W^{1/3} + c_6 W + c_7 \frac{Z}{X^*} + c_{11} W^{5/3} \quad (23)$$

where,

$$c_1 = \frac{64}{3 P_{r_\infty}} \left(\frac{3}{5} + \frac{177}{320} B \right)$$

$$c_2 = -\frac{3}{4}$$

$$c_3 = -\frac{3}{2} m$$

$$
\begin{aligned}
C_4 &= \frac{128}{7} \\
C_5 &= \frac{180}{7} A \\
C_6 &= -\frac{80}{7} A \\
C_7 &= -\frac{52}{7} m \\
C_8 &= -\frac{2}{15} \\
C_9 &= \frac{1}{120} \\
C_{10} &= \frac{m}{10} \\
C_{11} &= \frac{9}{7} A
\end{aligned}
\tag{24}
$$

Here,

$$
\begin{aligned}
x^* &= \frac{x}{\ell Re_\infty} \\
\Delta^* &= \frac{\Delta}{\ell} \\
\Delta_t^* &= \frac{\Delta_t}{\ell} \\
Y^* &= \frac{\Delta_t^*}{\Delta^*} \\
Z &= \Delta^{*2} \\
W &= Y^{*3}
\end{aligned}
\tag{24 a}
$$

the local Nusselt number and the local friction factor can be calculated as follows :

From Fourier's Law :

$$q = - k_{\omega}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= - \frac{3\, k_{\infty}\,(1+B)}{2\,\Delta_t}\,(T_{\infty} - T_{\omega}) \tag{25}$$

also,

$$q = h_x\,(T_{\omega} - T_{\infty}) \tag{26}$$

Eq. (25) and Eq. (26) lead to

$$Nu_x = \frac{3}{2}\,\frac{x}{\Delta_t}\,(1+B)$$

or,

$$\frac{Nu_x}{Re_{\infty}} = \frac{3}{2}\,\frac{x^*}{\Delta_t^*}\,(1+B) \tag{27}$$

By definition

$$f = \frac{\left(\mu \frac{\partial u}{\partial y}\right)_{y=0}}{\frac{\rho u_{\infty}^2}{2}}$$

$$= \frac{3\,\mu_{\infty}\,(1+A)}{\rho\, u_{\infty}\,\Delta}$$

or,

$$f\,Re_{\infty} = \frac{3\,(1+A)}{\Delta^*} \tag{28}$$

Combining Eqs. (27) and (28) one obtains the correlationship between the various flow parameters

$$\frac{Nu_x}{f\, Re_\infty} = \frac{1}{2}\,\frac{(1+B)}{(1+A)}\,\frac{1}{Y} \tag{29}$$

Case (2) Constant Wall Heat Flux Case

Similarly,

$$Z W' (1 + d_9 W^{2/3}) = d_1 + d_2 W^{1/3} Z^{1/2} + Z' W (d_3 + d_{10} W^{2/3}) + \frac{W Z}{x^*} (d_4 + d_{11} W^{2/3}) \tag{30}$$

$$Z' = d_5 + d_6 W^{2/3} Z^{1/2} + d_7 W^{4/3} Z^{1/2} + d_8 \frac{Z}{x^*} + d_{12} W^2 Z^{1/2} \tag{31}$$

where,

$$d_1 = \frac{49}{Pr_\infty}$$

$$d_2 = \frac{2039}{6}\,\frac{B}{Pr_\infty}$$

$$d_3 = -\frac{17}{16}$$

$$d_4 = -\frac{7}{8} m \qquad (31\ a)$$

$$d_5 = \frac{128}{7}$$

$$d_6 = \frac{40}{7} A$$

$$d_7 = -\frac{32}{21} A$$

$$d_8 = -\frac{52}{7} m$$

$$d_9 = -\frac{11}{216}$$

$$d_{10} = \frac{5}{144}$$

$$d_{11} = \frac{m}{36}$$

$$d_{12} = \frac{44}{147} A$$

Local Nusselt number and Local friction factor are

$$q = -k_w \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= -\frac{(T_\infty - T_w)\ 3 k_w}{\Delta_t} \quad (32)$$

also

$$q = h_x (T_w - T_\infty) \quad (33)$$

so

$$\frac{Nu_x}{Re_\infty} = \frac{3 x^*}{\Delta_t^*} \left(1 + B \frac{\Delta_t^*}{3} \right) \quad (34)$$

and

$$f = \frac{\left(\mu \frac{\partial u}{\partial y}\right)_{y=0}}{\frac{\rho u_\infty^2}{2}}$$

$$= \frac{3 \mu_\infty (1 + A)}{\rho u_\infty \Delta} \quad (35)$$

or,

$$f \, Re_{\infty} = \frac{3(1+A)}{\Delta^*} \tag{35}$$

While choosing viscosity coefficients A and conductivity coefficients B, it must be kept in mind that, in general, for incompressible fluids, the thermal conductivity is slightly dependent on temperature and the viscosity is always rapidly decreasing with temperature.

The computer solution is discussed in the next chapter.

CHAPTER 4

COMPUTER SOLUTION OF THE PROBLEMS

Before proceeding to the actual solution of the problems one has to consider the various types of computing facilities available and feasibility of each of them. At present three types of computer hardware are available: the digital computer, the analog computer and a combination of both known as the hybrid computer.

Analog Computer [29, 30, 31, 32]:

This consists of hardware components which are capable of performing certain mathematical operations:

- Integration
- Summation
- Multiplication
- Arbitrary function generator of the form $y = f(x)$
- Certain fixed functions such as $y = Sin(x)$; $y \div Log(x)$
- Compare
- Switches
- Simple logic operations, AND, OR, NOT
- Single bit memories
- Counters, shift registers

Problem solution by the Analog computer is accomplished by analogy; that is, the computer is programmed so that the circuit equations have the same mathematical form as the equations of the problem. In this type of computer, voltages represent various physical quantities, such as velocity, force, temperature, and so on. The magnitudes of the voltages are related to the numerical values of the physical quantities by the use of 'Scale factors'. The analog computer performs its operation in a 'continous process' with respect to time.

The analog simulation combines fast run time and good man-machine interaction but involves the tedious amplitude and time scaling. This can be used advantageously where the number of anticipated runs justifies expensive development costs and skilled staff is available.

Example: Consider the mass - spring - damper system shown in Fig. (1) which is described by the equation

$$M\ddot{x} = -(c\dot{x} + kx)$$

where,

M = mass of the body = 10.2 Kg.

c = damping coefficient = 6.8 N-Sec./M

k = spring constant = 100 N/M

and initial conditions are

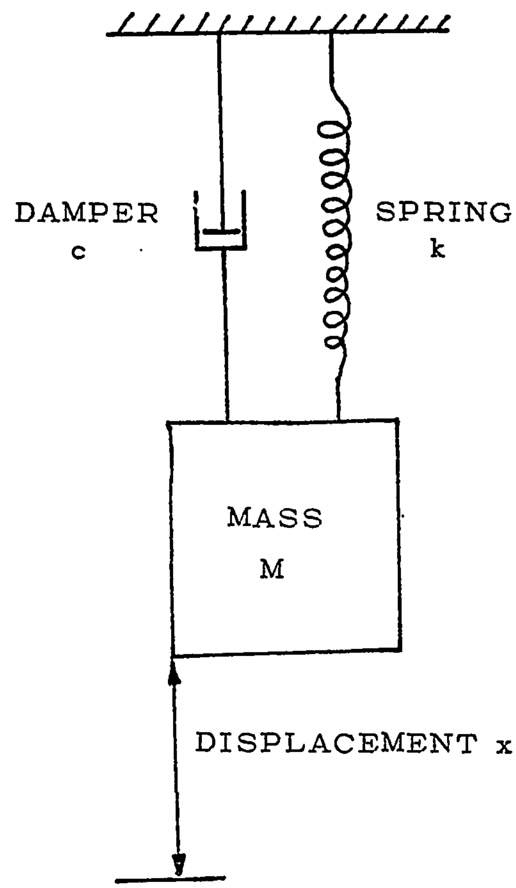


Fig. 1 MODEL OF A MASS - SPRING - DAMPER SYSTEM

Time T = 0 $\dot{x} = 0$, $x = -3$

As this computer is a fixed point device, so introducing the magnitude scaling to the mathematical equation of the model.

$$\frac{\ddot{x}}{\ddot{x}_M} = -\left(\frac{c}{M}\frac{\dot{x}_M}{\ddot{x}_M}\right)\left(\frac{\dot{x}}{\dot{x}_M}\right) - \left(\frac{k}{M}\frac{x_M}{\ddot{x}_M}\right)\left(\frac{x}{x_M}\right)$$

where, subscript 'M' is for maximum value of the parameter. The analog circuit is shown in Fig. (2).

The generated signals for acceleration, velocity, and displacement are continuous because the process, summation, integration etc. , are continuous.

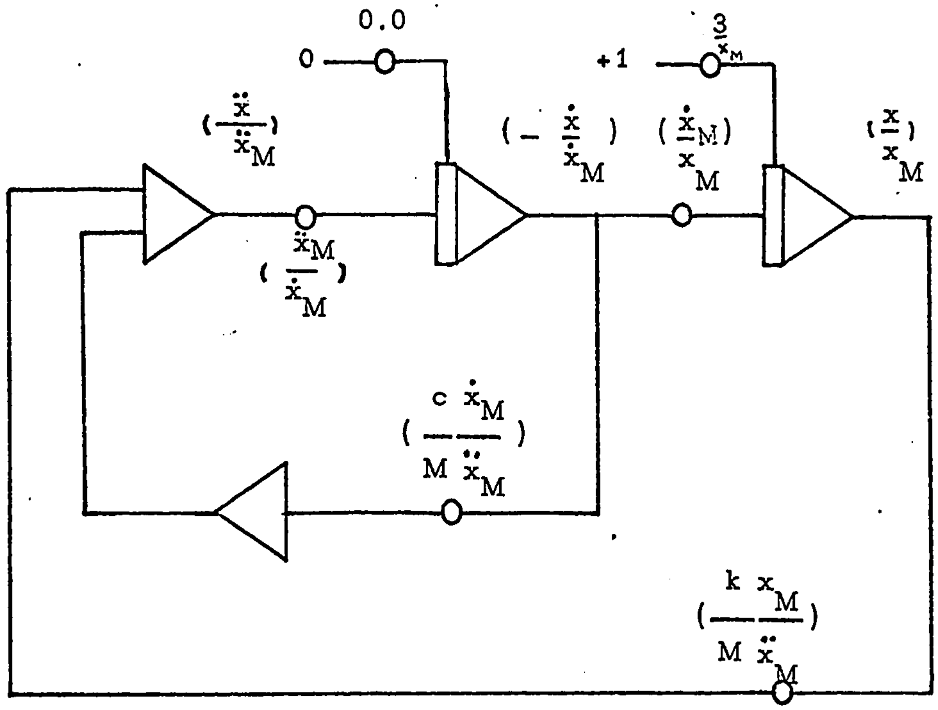


Fig. 2 ANALOG CIRCUIT DIAGRAM FOR THE EXAMPLE

Digital Computer [22, 32, 33]:

This contains simple discrete logical and arithmetic operations that can be executed in sequence to solve the equations. Five basic components are employed: 1. the INPUT unit, which is used to provide data and instructions to the computer; 2. the MEMORY or STORAGE unit, in which data and instructions are stored; 3. the ARITHMETIC - LOGIC unit, which performs the arithmetic operations and provides the "decision - making" ability, or logic, of the computer; 4. the CONTROL unit, which controls the computer operations; and 5. the OUTPUT unit, from which the computer results are obtained.

The digital computer is programmed by defining the sequence of operations necessary to solve the equations of the system, and by coding these operations into one of the languages understood by the computer. In general, the computing languages can be grouped into three main categories - machine (or assembly) languages, procedural languages and problem oriented languages.

For writing the program in the machine language the problem must be broken into a sequence of simple tasks that the hardware of the computer can perform. This makes the programming unnecessarily tedious, although it is only at this level that the full power and flexibility of the computer can be exercised. We will make no attempt to write a machine language program for our example.

The procedural languages, Fortran, Cobol, PL/1 etc., allow the user to define a computational procedure in a form which resembles the equivalent statement of the procedure of the problem. The programming language is considered procedural because the user must exercise the control over the sequencing of these operations in order to secure the maximum approximation of computer operations to the real problem. Using one of these languages, the example would take the form (Fortran):

```
FORTRAN IV G LEVEL  19                    MAIN

                C       SOLUTION OF MASS-SPRING-DAMPER SYSTEM
                C       JAN. 1972
                C       ASHOK KUMAR
                C
0001                    DIMENSION X(101),XDOT(101)
0002                    REAL M,K
                C       SETTING OF CONSTANTS
0003                    M=10.2
0004                    C=6.8
0005                    K=100.0
0006                    DELT=0.1
0007                    J=100
                C       INITIALIZATION
0008                    N=1
0009                    X(1)=-3.0
0010                    XDOT(1)=0.0
```

```
            C
            C      COMPUTATIONAL LOOP
0011        2      X2DOT = (-C*XDOT(N) - K* X(N))/M
0012               XDOT(N+1)=XDOT(N)+DELT*X2DOT
0013               X(N+1)=X(N)+DELT*XDOT(N)
            C      LOOP TERMINATION TESTING
0014               IF(N-J) 20,21,21
            C
            C      CONTINUATION OF THE TIME
0015        20     N=N+1
0016               GO TO 2
            C
            C      STOP AND RESULTS
0017        21     DO 22 I=1,101
0018               R=I-1
0019               T=DELT*R
0020        22     WRITE(3,100) T,X(I),XDOT(I)
0021        100    FORMAT(3F15.5)
0022               STOP
0023               END
```

It is noted that the numerical approximation and a discretization process involving a step - size DELT are introduced during programming. The later is certainly irrelevant as far as the original problem is concerned and is due to the use of the digital computer to solve the equation.

On the other hand the problem oriented languages are designed in such a fashion so that the computer handles the complications which are unrelated to the problem. The languages essentially consists of a collection of predefined procedures which are called into the computer. For using these procedures the the information must be fed into the computer in a prescribed format.

Such languages are tabulated in Table - 1.

The example here is solved by CSMP language. In this language the sequence of procedures are automatically adjusted in order to give a liberty to the user to use the statements as if they are actually occuring in his problem. Thus an attempt has been made towards hiding the digital computer's true nature of being discrete sequential device. CSMP program for the example is shown below:

```
****CONTINUOUS SYSTEM MODELING PROGRAM****
***PROBLEM INPUT STATEMENTS***

*
* CSMP CODING OF THE PROBLEM
* MASS-SPRING-DAMPER SYSTEM
*ASHOK KUMAR
*
CONSTANT M=10.2,C=6.8,K=100.0
INCON    XDOT0=0.0,X0=-3.0
*
DYNAMIC
X2DOT=(-C*XDOT-K*X)/M
XDOT=INTGRL(XDOT0,X2DOT)
X=INTGRL(X0,XDOT)
*
TIMER   OUTDEL=0.1,FINTIM=10.0
*
PRTPLT   X
LABEL    MASS DISPLACEMENT
PRTPLT   XDOT
LABEL    MASS VELOCITY
PRTPLT   X2DOT
LABEL    MASS ACCELERATION
END
STOP
```

TABLE I

EXAMPLES OF PROBLEM - ORIENTED LANGUAGES

#	LANGUAGE NAME	AREA OF APPLICATION
1.	CSMP (Continuous System Modeling Program)	Modeling continuous dynamic systems
2.	GPSS (General Purpose System Simulator)	Modeling discrete dynamic systems
3.	ICES (Integrated Civil Engineering System)	Modeling of various types of Civil engineering design situations
4.	ASKA (Automatic System for Kinematic Analysis)	Modeling structural dynamics
5.	APT (Automatically Programmed Tools)	Preparation of data for numerically controlled machine tools
6.	ADAM (A Generalized Data Management System)	File management and information retrieval

Hybrid Computer [34, 35]:

This computer combines the analog computer and the digital computer with a data and control linkage. Thus hybrid simulation inevitably represent an effort to combine the features of both types of computers. At the present time, virtually all hybrid-computer systems employ so called " stand-alone " analog and digital computers which can function independently as well as in the inter-connected mode. The uses of the resulting unique capabilities of the system are mainly dependent on the nature of the problem under consideration.

Effective utilization of a hybrid computer is made easier by supporting procedural coding languages and packaged procedures which ease the communication problem and activate the detailed manipulations necessary for inter-computer transfer of data. Since the majority of people engaged in research work have familiarity with the Fortran coding language, this language is supported in most systems, with extensions to include certain hybrid-type operations. Since Fortran is a complier language, and the coding used during problem execution is not the coding used to communicate the problem to the computer, run-time dialogue for debugging and interactive running of the digital computer is awkward. This is overcome to some extent in modern hybrid systems by on-line interactive coding languages similar to those used with typewriter terminals used in time shared systems.

Computing Technique:

Each problem in question (represented by Eqs. (22) & (23) and Eqs. (30) & (31)) is a highly nonlinear initial value problem which has certain ill-conditioned features arising from division by zero at the initial portion of the wedge. So the mathematical capability of CSMP package* is used to solve the initial value problem while its logical capability makes it possible to circumvent the difficulties of division by zero occuring in the solution. These matters are, now discussed.

Fig. (3) and Fig. (4) represent the block diagrams for solving the problem represented by Eqs. (22) & (23) and Eqs. (30) & (31) in each case. These diagrams are however only conceptual in nature; the introduction of suitable logic control signals will be dealt later on. Inasmuch as the CSMP package functions in a non-fixed point environment (unlike the analog computer), there is no need for magnitude scaling. Hence the number displayed on the diagrams are actual input data to the machine.

Initial Conditions: The solutions of the problems are interactable because the leading edge of the wedge is a singular point, i. e. at $x^* = 0$, $Y^* = 0/0$ (i.e. $W = 0/0$) and hence undefined. To avoid this difficulty an unheated starting length x^*_0 is assumed. In the unheated region, $0 \leq x^* \leq x^*_0$, $\overset{*}{\Delta}_t = 0$ (i.e. $W = 0$), upon one integration, Eqs.(23)

---

* The computation was performed on IBM 360/65 of Computing Centre, University of Ottawa, Ottawa.

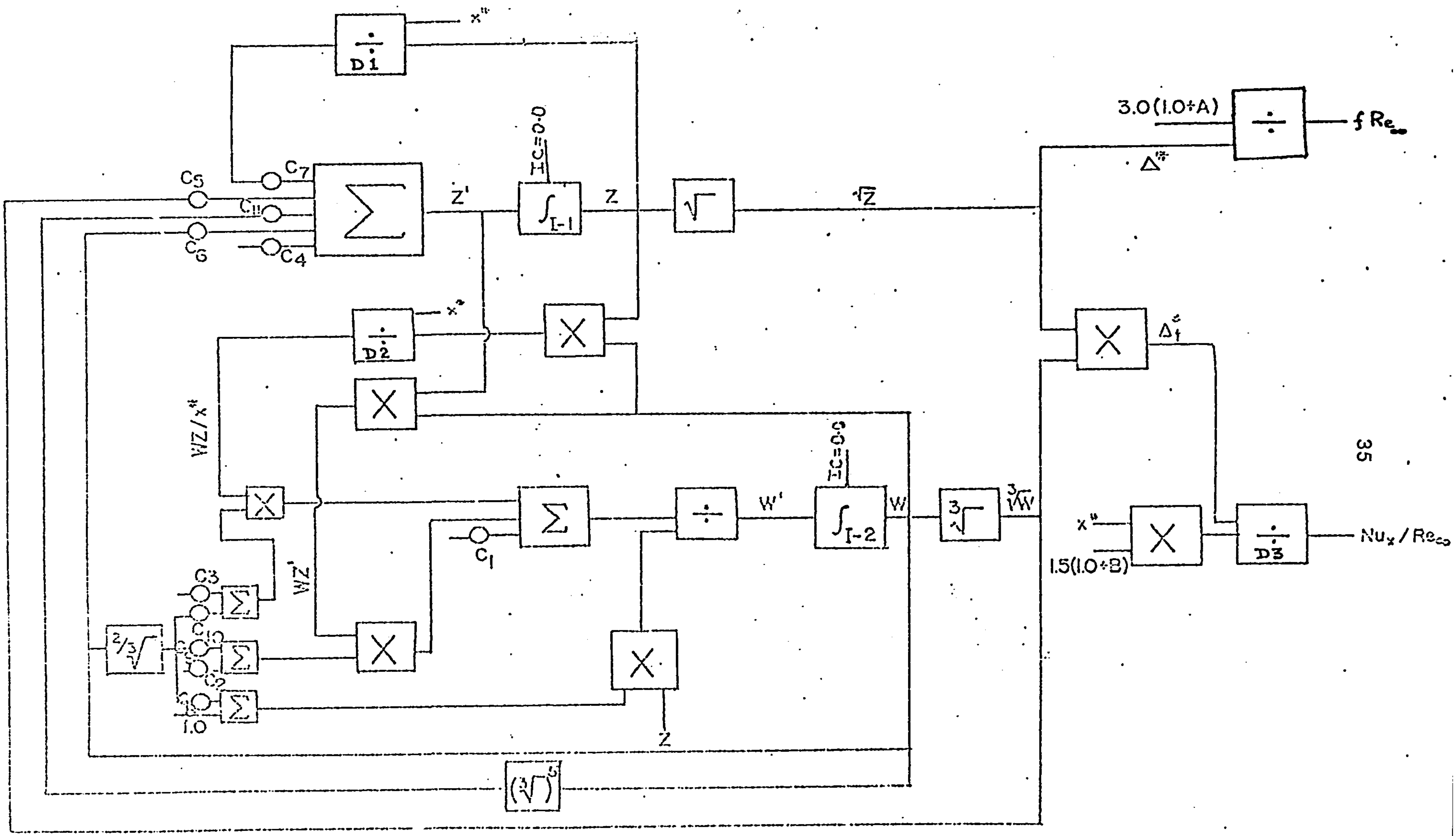


Fig. 3 BLOCK DIAGRAM FOR CONSTANT WALL TEMPERATURE CASE WITHOUT LOGIC

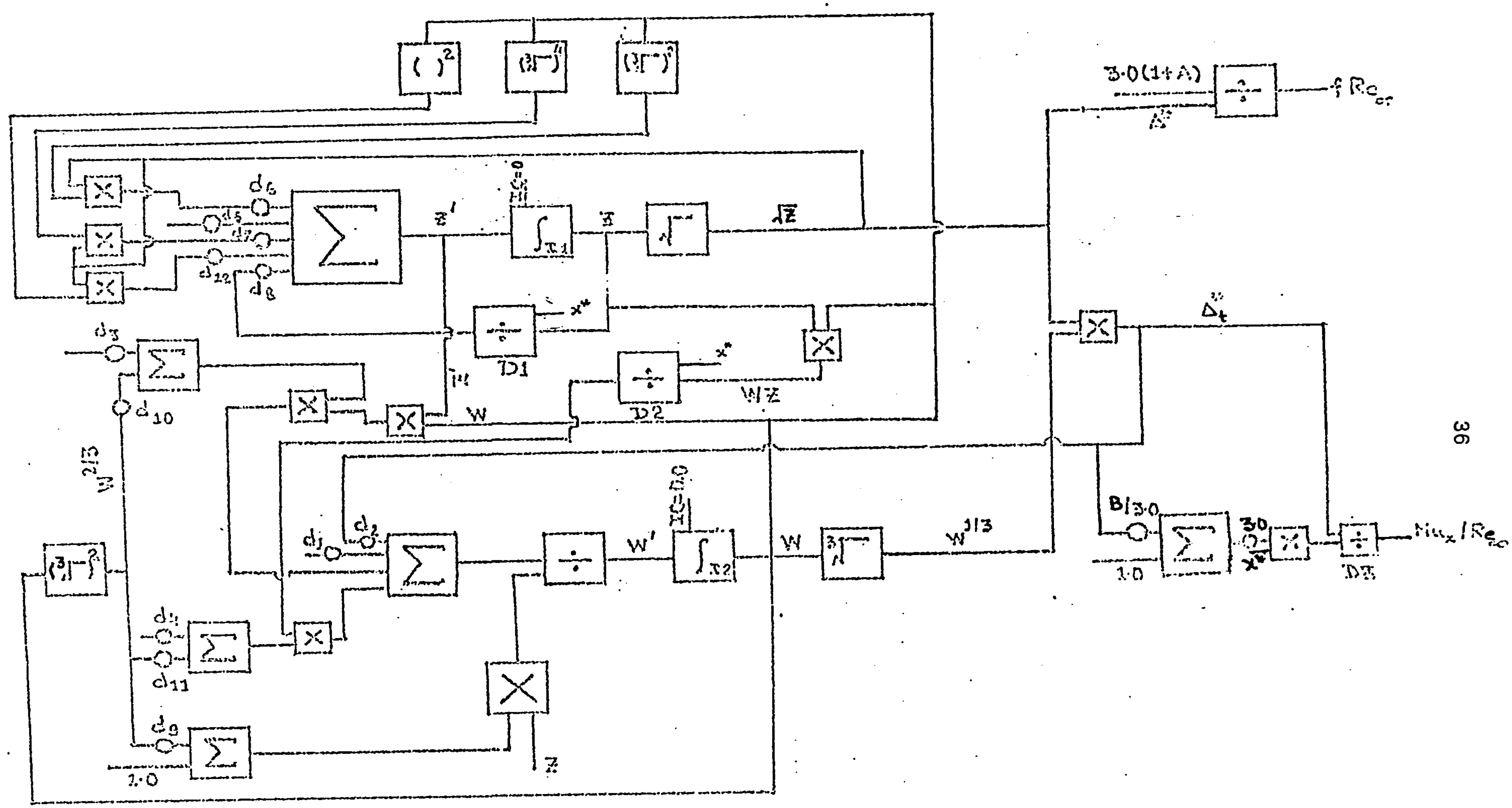


Fig. 4 BLOCK DIAGRAM FOR CONSTANT WALL HEAT FLUX CASE WITHOUT LOGIC

and (31) reduces to $z = (c_4 x^*)/(1-c_7)$ and $z = (d_5 x^*)/(1-d_8)$ respectively. Thus at $x^* = x^*_o$, $z = \frac{c_4 x^*_o}{(1-c_7)}$ & $W = 0$ and $z = \frac{d_5 x^*_o}{(1-d_8)}$ & $W = 0$ for each case.

Integration: CSMP offers a choice of seven numerical integration techniques [20]; the one used was fourth order RKS (Runge-Kutta Subroutine) with a fixed (as opposed to a variable) step size. In order to make sure that step is correct, a number of step sizes are selected to find a stable range for the parameters and then the appropriate step size is chosen.

The above initial conditions if used on the simple INTEGRATORS will not generate the actual results because the computation will start at $x^* = 0$ and not at $x^* = x^*_o$. In order to avoid this difficulty MODE-CONTROLLED INTEGRATORS are used. Use of this functional block will put the integrators in IC mode over a period of $0 \leq x^* \leq x^*_o$ and beyond x^*_o the actual integration procedure will start. The logical signals L_1 and L_2 for this block are generated by a COMPARATOR C1 and a LOGICAL INVERTER IR-1. (Figs. (6) and (7)).

Problem of Division by Zero: The division by zero must be prevented on the DIVIDERS D1, D2 and D3 because the computation starts at $x^* = 0$. This is achieved by using the INPUT SWITCH (RELAY) IS-1 operated by the logical signal L_1. Thus the so produced signal L_3 will prevent the division by zero in the interval of $0 \leq x^* \leq x^*_o$. During computation it has been observed that each divider requires the signal L_3 which is to be generated by seperate Switches and not by one single Switch. The figures like 1.0, 0.001 on the Input Switches IS-1, IS-2 etc., are quite arbitrary and work as a dummy variable during the interval $0 \leq x^* \leq x^*_o$.

Control Statements: The execution run control statement FINISH YO = 1 is used to terminate the run when Y* = 1 (i.e., $\Delta_\xi = \Delta$). The statement RENAME TIME = x is desired to use x as the symbol for the problem independent variable.

CSMP execution sequence and block diagrams upon implementing the logic and modifications are shown in Figs. (5), (6) & (7).

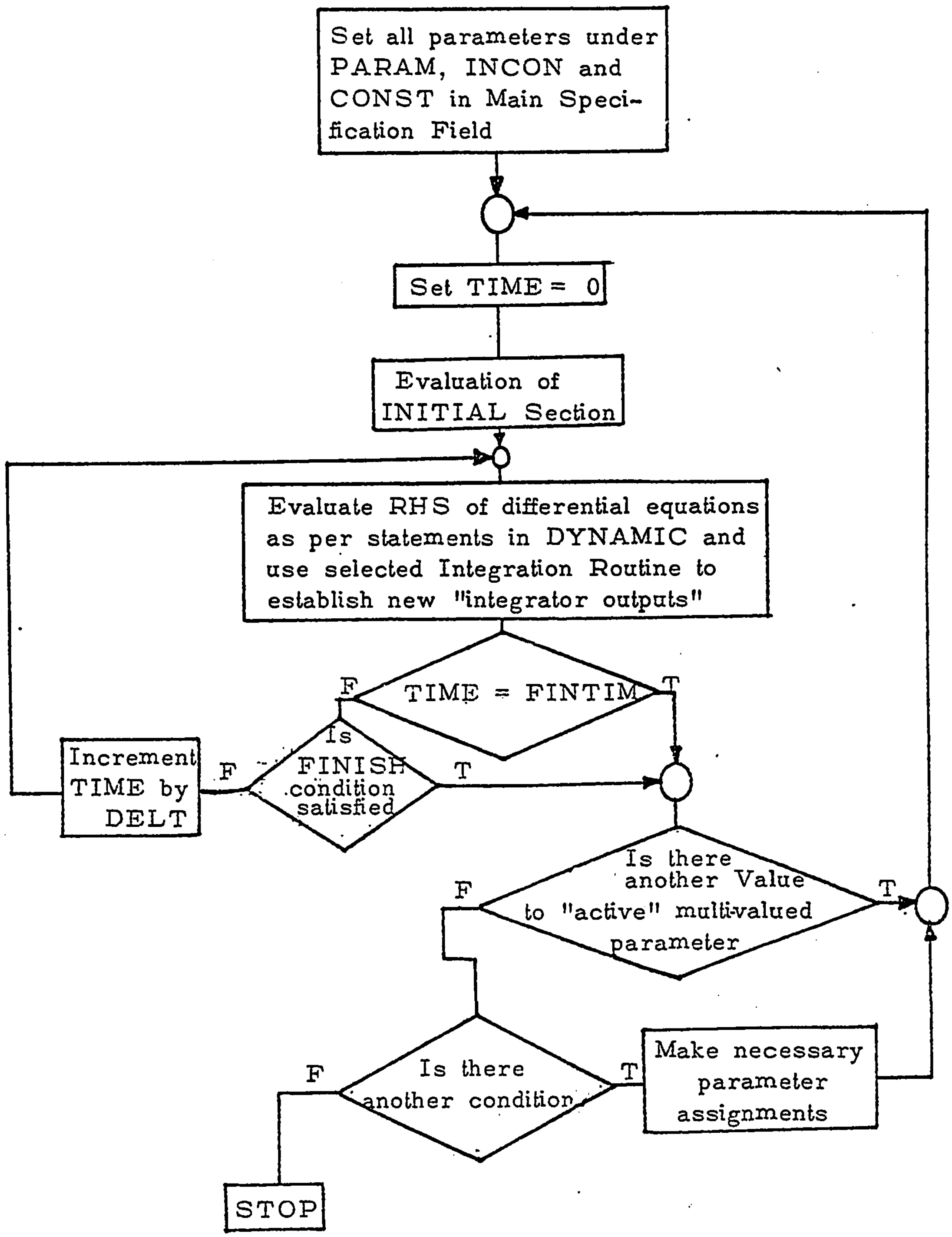


Fig. 5 CSMP EXECUTION SEQUENCE

(similar to - [21] by Birta)

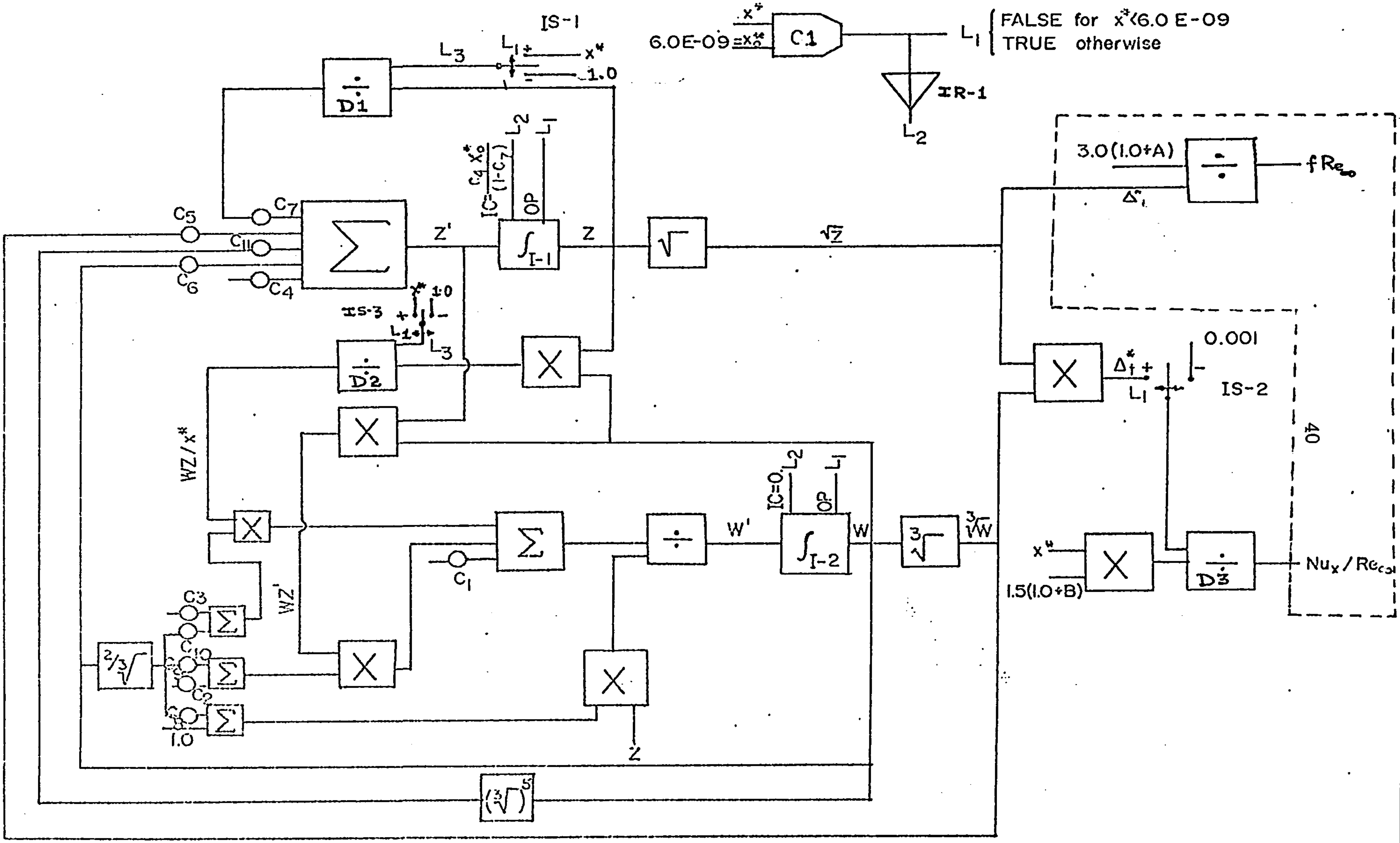


Fig. 6 BLOCK DIAGRAM FOR CONSTANT WALL TEMPERATURE CASE

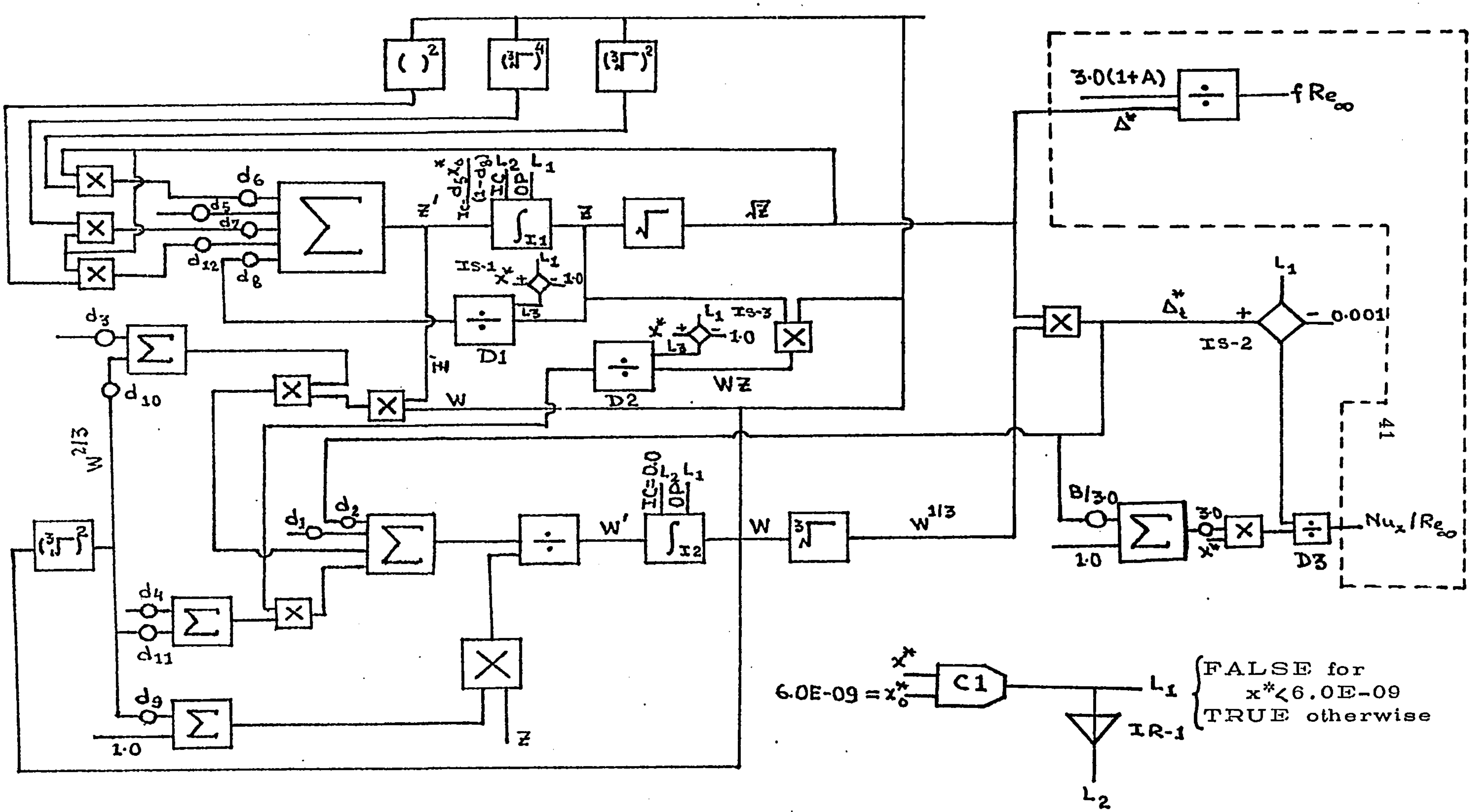


Fig. 7 BLOCK DIAGRAM FOR CONSTANT WALL HEAT FLUX CASE

<u>Logical Circuit for Mapping</u>:

There is implicit in the CSMP package a convenient mechanism for requesting a line-printer plot of system variables (the PRTPLT statement of the program listing). This mechanism, however, does not provide a full degree of flexibility and some special steps were taken in the program to obtain more meaningful output plots. So it is difficult to observe the behaviour of various parameters on the flow performance from the graphs obtained. For the ease, the dotted loops on Figs. (6) and (7) are replaced by the logical circuit shown in Fig. (8). This is a suitable combination of Comparators and Input Switches (Relays). These are logically controlled in such a fashion so as to produce the graphs with the ranges of various parameters as

$$0 \leqslant Nu_x/Re_\infty \leqslant 8.0\text{x}10^{-3}$$

$$0 \leqslant x^* < 2.1\text{x}10^{-3}$$

$$4000 \leqslant f\ Re_\infty \leqslant 0$$

Now it is much easier to study the sensitivity of the variables on the local Nusselt number and the local friction factor. (see Figs. (28) to (75) - given at the end of the next chapter)

(A Study to circumvent these types of difficulties in the solution of other problems is currently under investigation by Birta [23])

Note: The complete program is given on pages 44 to 48.

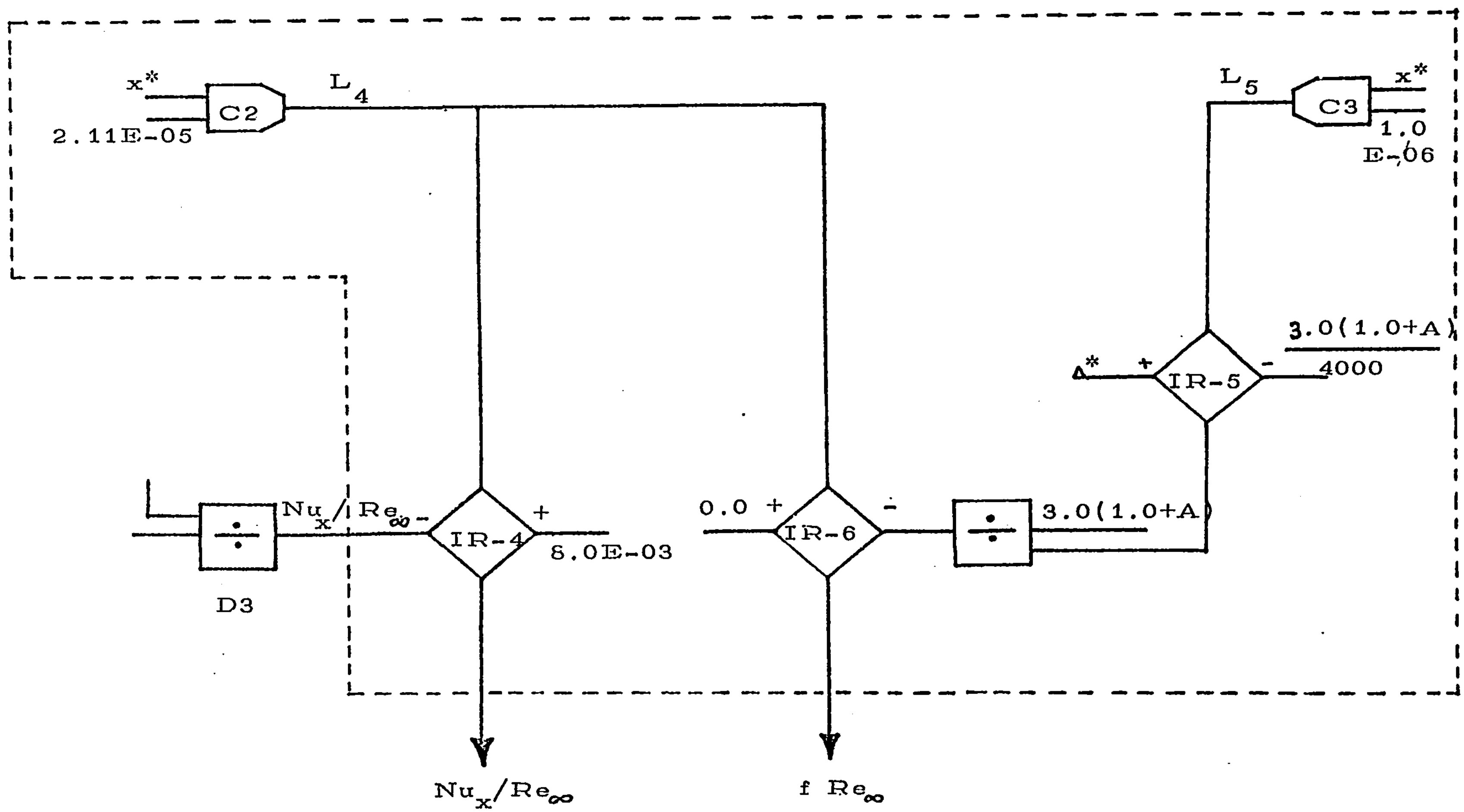


Fig. 8 LOGICAL CIRCUIT USED FOR MAPPING

```
***CONTINUOUS SYSTEM MODELING PROGRAM***

   ***PROBLEM INPUT STATEMENTS***

TITLE FLOW OVER WEDGE
*
*      WM = Wedge Constant (m)
*      XO = Unheated length of Wedge
*      PR = Pr∞
*   CONSTANT WALL TEMP.
*      MAY 1971
*      ASHOK KUMAR
*      ALIGARH U.P. INDIA

RENAME TIME =X
MEMORY      MODINT

*
PARAM    WM=(0.0,0.111,0.222,0.333)
*
INITIAL
XO=6.0E-09
A=1.5
B=-0.2
PR=15.0
*

        C1=64.0*(3.0/5.0+177.0*B/320.0)/3.0/PR
        C2=-3./4.
        C3=-3.0*WM/2.0
        C4=128.0/7.
        C5=180.0*A/7.0
        C6=-80.0*A/7.0
        C7=-52.0*WM/7.0
        C8=-2.0/15.0
        C9=1.0/120.0
        C10= WM/10.0
        C11=9.0*A/7.0

*
DYNAMIC
        ZIN=C4*XO/(1.0-C7)
Q=COMPAR(X, XO    )-0.5
Q1=NOT(Q)-0.5
XX=INSW(Q,1.0    ,X)
       ZD=C4+C5*(W**(1.0/3.0))+C6*W+C7*Z/XX +C11*(W**(5.0/3.0))
Z=MODINT(ZIN,0,Q1,ZD)
```

Program	Actual
Q	L_1
Q1	L_2
XX XN	L_3

```
A0=W**(2.0/3.0)
A1=1.0+C8*A0
A2=C2+C9*A0
A3=C3+C10*A0
XN=INSW(Q, 1.0    ,X)
WD=(C1+A2*W*ZD+A3*W*Z/XN)/(Z*A1)
W=MODINT(0.0,Q,Q1,WD)
*
MBL=Z**0.5
Y0=W**(1.0/3.0)
TBL=Y0*MBL
TB=INSW(Q,0.001,TBL)
Q2=COMPAR(X,2.11E-05)-0.5
NU=INSW(Q2,(1.5*X)/TB*(1.0+B),8.0E-03)

Q3=COMPAR(X,1.0E-06)-0.5
MB=INSW(Q3,3.0*(1.0+A)/4000.,MBL)
FRE=INSW(Q2,3.0*(1.0+A)/MB,0.0)
*
METHOD   RKSFX
*
TIMER     FINTIM=2.2E-05, DELT=6.0E-09, OUTDEL=1.0E-06
*
FINISH     Y0=1.0
*
PRTPLT    NU  (FRE)
*
LABEL     NUSSELT NUMBER/REYNOLDS NUMBER
*
PRTPLT    FRE  (NU)
*
LABEL       FRICTION FACTOR*REYNOLDS NUMBER
*
END
STOP

OUTPUT VARIABLE SEQUENCE
X0       PR       B        C1       C2       C3       C4       A        C5       C6
C7       C8       C9       C10      C11      Q        XX       A0       A1       XN
A3       ZD       Q1       ZIN      770003   Z        A2       WD       770007   W
MBL      Y0       TBL      TB       Q2       NU       Q3       MB       FRE

 OUTPUTS     INPUTS     PARAMS   INTEGS + MEM BLKS   FORTRAN   DATA CDS
 45(500)    48(1400)    4(400)    2+   2=  4(300)    44(600)       10
```

Program	Actual
MBL	Δ^*
TBL	Δ^*_t
Y0	Y^*
Q2	L_4
Q3	L_5
NU	$Nu_x \vert Re_\infty$
FRE	$f Re_\infty$

```
***CONTINUOUS SYSTEM MODELING PROGRAM***

    ***PROBLEM INPUT STATEMENTS***

TITLE FLOW OVER WEDGE
*
*      WM= WEDGE CONSTANT (m)
*      XO=UNHEATED LENGTH OF WEDGE
*      PR=PRANDTL NUMBER (Pr∞)
*
*CONSTANT HEAT FLUX CASE
RENAME TIME =X
MEMORY     MODINT
*      ASHOK KUMAR
*      ALIGARH,U.P.,INDIA

*
PARAM    WM=(0.0,0.111,0.222,0.333)

*
INITIAL
XC=6.0E-09
A=1.0
B=-0.2
PR=15.0
*
*
D1=49./PR
D2=2039./6.*B/PR
D3=-17./16.
D4=-7./8.*WM
D5=128./7.
D6=40./7.*A
D7=-32./21.*A
D8=-52.0/7.0*WM
D9=-11./216.
D10=5./144.
D11=WM/36.0
D12=44.0/147.*A

*
DYNAMIC
ZIN=D5*XO/(1.0-D8)
Q=COMPAR(X, XC      )-0.5
Q1=NOT(Q)-0.5
XX=INSW(Q,1.0,X)
ZD=D5+(D6*(W**(2.0/3.0))+D7*W**(4.0/3.0)+(D12*W*W))*(Z**0.5)+D8*Z/XX
Z=MODINT(ZIN,0,01,ZD)
```

Program	Actual
Q	L_1
Q1	L_2
XX XN	L_3
MBL	Δ^*
TBL	Δ_t^*
YO	Y^*
Q2	L_4
Q3	L_5
NUR	Nu_x/Re_∞
FRE	$f.Re_\infty$

```
*
A0=W**(2.0/3.0)
A1=1.0+D9*A0
A2=D3+D10*A0
A3=D4+D11*A0
XN=INSW(Q,1.0,X)
WA=D1+D2*(W**(1.0/3.0))*(Z**0.5)+A2*ZD*W+A3*W*Z/XN
WD=WA/Z/A1
W=MODINT(0.0,Q,Q1,WD)

*
MBL=Z**0.5
Y0=W**(1.0/3.0)
TBL=Y0*MBL
*
TB=INSW(Q,0.001,TBL)
Q2=COMPAR(X,2.11E-05)-0.5
NUR =3.0*X*(1.0+B/3.0*TBL)/TB
NURE=INSW(Q2,NUR,8.0E-03)
*
Q3=COMPAR(X,1.0E-06)-0.5
MB=INSW(Q3,3.0*(1.0+A)/4000.0,MBL)
FR=3.0*(1.0+A)/MB
FRE=INSW(Q2,FR,0.0)

*
TIMER     FINTIM=2.2E-05,DELT=6.0E-09, OUTDEL=1.0E-06
*
FINISH     Y0=1.0
*
METHOD  RKSFX
*
PRTPLT        NURE (FRE)
*
LABEL     NUSSELT NUMBER/REYNOLDS NUMBER
*
PRTPLT           FRE (NURE)
*
LABEL       FRICTION FACTOR*REYNOLDS NUMBER
*
END
STOP
```

```
OUTPUT VARIABLE SEQUENCE
XO      PR      D1      B       D2      D3      D4      D5      A       D6
D7      D8      D9      D10     D11     D12     Q       XX      A0      A1
XN      A3      ZD      A2      Q1      ZIN     ZZ0003  Z       WA      WD
ZZ0007  W       MBL     Y0      TBL     TB      NUR     Q2      NURE    Q3
MB      FR      FRE

OUTPUTS    INPUTS    PARAMS    INTEGS + MEM BLKS    FORTRAN    DATA CDS
```

A Word about Cost:

Cost of operation for the different computational techniques are varied. None of these techniques can be explicitly said to be the cheapest. However, an idea can be drawn by the studies made by Chubb [27] for two dynamic problems: the first a 5 loop, 5th order non-linear control system, and the other a seat-ejection system with 4th order dynamics and a coordinate transformation. The comparisons are shown in Fig. (9) . The cost analysis used the following datas:

Cost of labour	$ 18/hour
IBM 360/50 Digital Computer	$ 200/hour
AD 256 Analog Computer	$ 18/hour
EAI 680/CDC 1700 Hybrid Computer	$ 80/hour

Conclusions of the study can be summerized as follows:

1. Fortran requires a long time to get the first solution while CSMP greatly reduces this time. Analog or hybrid falls between the two.
2. Time reqired to generate the number of solutions is markedly less for the hybrid or analog as compared to Fortran and CSMP.
3. For dynamic simulation Fortran is not recommanded because of cost considerations.

Thus CSMP is the least expensive when only a few runs

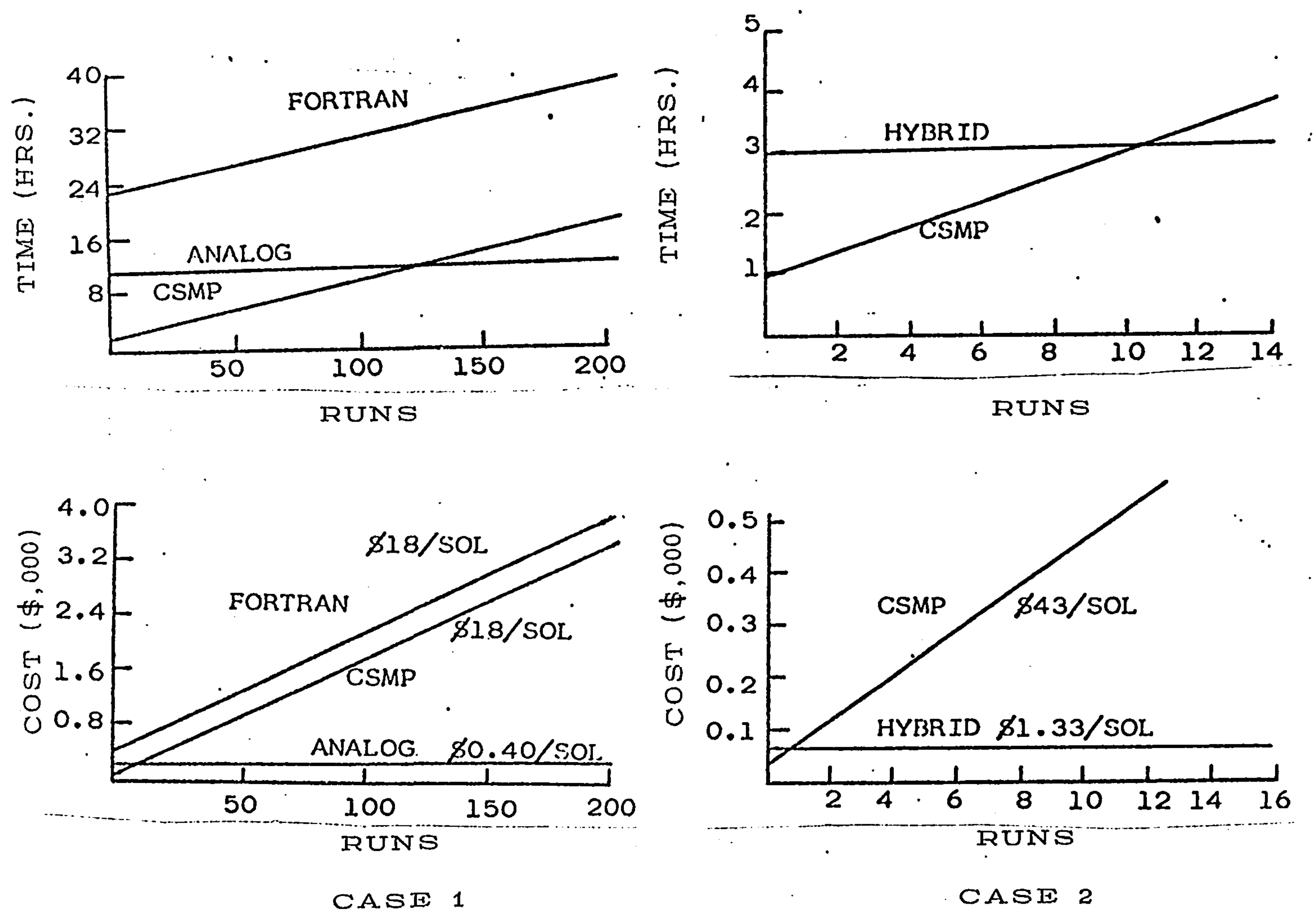


Fig. 9 TIME/COST INFORMATION FOR TYPICAL DYNAMIC SIMULATIONS

(From Chubb [27])

are required and analog or hybrid is advantageous when a large number of runs are needed. Our study is of the same type as of the first mentioned above (solution of coupled differential equations).

CHAPTER 5

RESULTS AND DISCUSSIONS

A number of runs were obtained for the combinations of of various parameters. Some of these results for local Nusselt number and local friction factor using A, B, m and Pr_∞ as parameters are shown in Fig. (10) to (75). A comparision, for constant properties case (i,e. A = 0.0, B = 0.0), between the results from this study and the exact solution [4, 14] is shown in Figs. (10), (12), (14), (15), (18), (19), (20), (21), (25) and (26). For this case, the local Nusselt number is in error by approximately 8% to 10% and the local friction factor is in error by approximately 6% to 8% relative to the exact solution.

For the case of variable properties, there are no suitable solutions available in the literature for comparison. However, the results from this study agree qualitatively with the work for flow through a pipe by Yang [36], flow over a flate plate by Su [11], flow through parallel plates by Schade and McEligot [37] etc. Notice here that the flow properties are evaluated at free stream conditions.

Some remarks from this study can be summarized as follows:

1. Constant Wall Temperature Case:

a) For constant B, m, Pr_∞

The local Nusselt number decreases with increasing values of A while the local friction factor increases with increasing values of A for all x^*. This is due to the fact that positive A indicates the cooling of the liquid in the flow, thus corresponding increase in viscosity near the wall slows down the flow, results in a lower rate of heat transfer relative to the constant properties case.

b) For constant Pr_∞, A, m

The local Nusselt number increases with the increase in B while the effect on the local friction factor is negligible for all x^*.

2. Constant Wall Heat Flux Case:

a) For constant B, m, Pr_∞

The local Nusselt number is insensitive to the change of A while the local friction factor decreases with with decreasing values of A for all x^*. The reason is that the thermal boundary layer thickness for various values of A is approximately equal to the constant properties case. (Also, see Eqs. (34) & (35))

b) For constant Pr_∞, A, m

The effect of B on the local Nusselt number and the local friction factor is negligible for all x^*.

3. Constant Wall Temperature and Constant Wall Heat Flux Cases:

a) For constant A, B, m

The effect on the local friction factor with increasing the Prandtl number is less prominent than on the local Nusselt number for all x^*. The reason for this is that the momentum boundary layer is less affected as compared to the thermal boundary layer with increasing the Prandtl number.

b) For constant A, B, Pr_∞

The local Nusselt number and the local friction factor increase with the increasing value of m as in the case of constant properties for all x^*.

c) For constant A, B, m, Pr_∞

The ratio of boundary layer thicknesses is found to be approximately constant except at the initial portion of the development.

d) The conductance h_x in constant wall heat flux case is higher than at the comparable point on a wedge with constant wall temperature case for constant properties case. The results are shown in Table-2.

e) Computer Work:

(i) By using CSMP technique the higher order terms can be retained without any difficulty for more accurate analysis.

(ii) Attention must be paid while using a Input Switch (Relay) for the like signals required at different functional blocks. (see Chapter 4)

(iii) The graphic technique available on the computer can be modified to plot the results with same Y-axis scale for comparison purpose (shown in Figs. (28) to (75)) by implementing an additional logical circuit as explained in the last chapter.

The results from this study indicate that the variations of thermal conductivity and viscosity to the local Nusselt number and local friction factor are quite significant. Care must be taken in using the constant properties solution.

It is believed that the results can be improved by assuming a higher order velocity profile and a higher order temperature profile in the calculation process. Similarly, more useful conclusions can be drawn by using more realistic expressions of viscosity and thermal conductivity.

The Variational method combined with a CSMP solution technique has been found to be fruitful in the solution of these complex problems. Particularly, it was found that the CSMP program in solving this type of problems is very simple and straightforward.

The procedure utilized in the present study can be used to solve the external and internal heat transfer problems, especially those with nonlinear characteristics. These problems with more general boundary conditions such as wall temperature variations or wall heat flux variations or convection and radiation at wall can now be easily dealt with. Other examples include flows in which all physical properties vary with temperature (linear or nonlinear variation), combined free and forced convection problems, flow over oscillating or rotating surfaces and unsteady heat transfer problems.

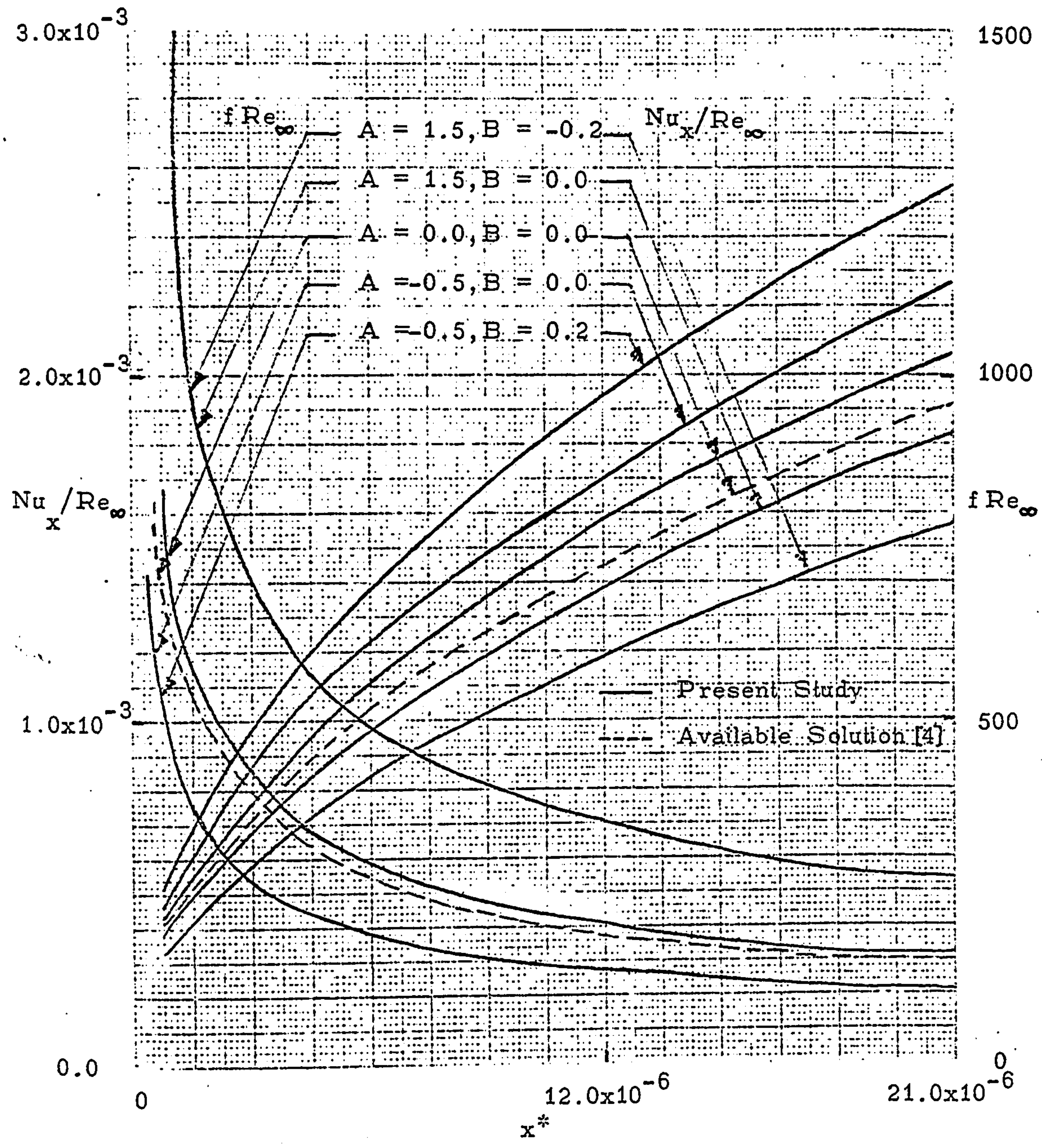


Fig. 10 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $P_{-\infty}=2.0$ CONSTANT WALL TEMPERATURE CASE

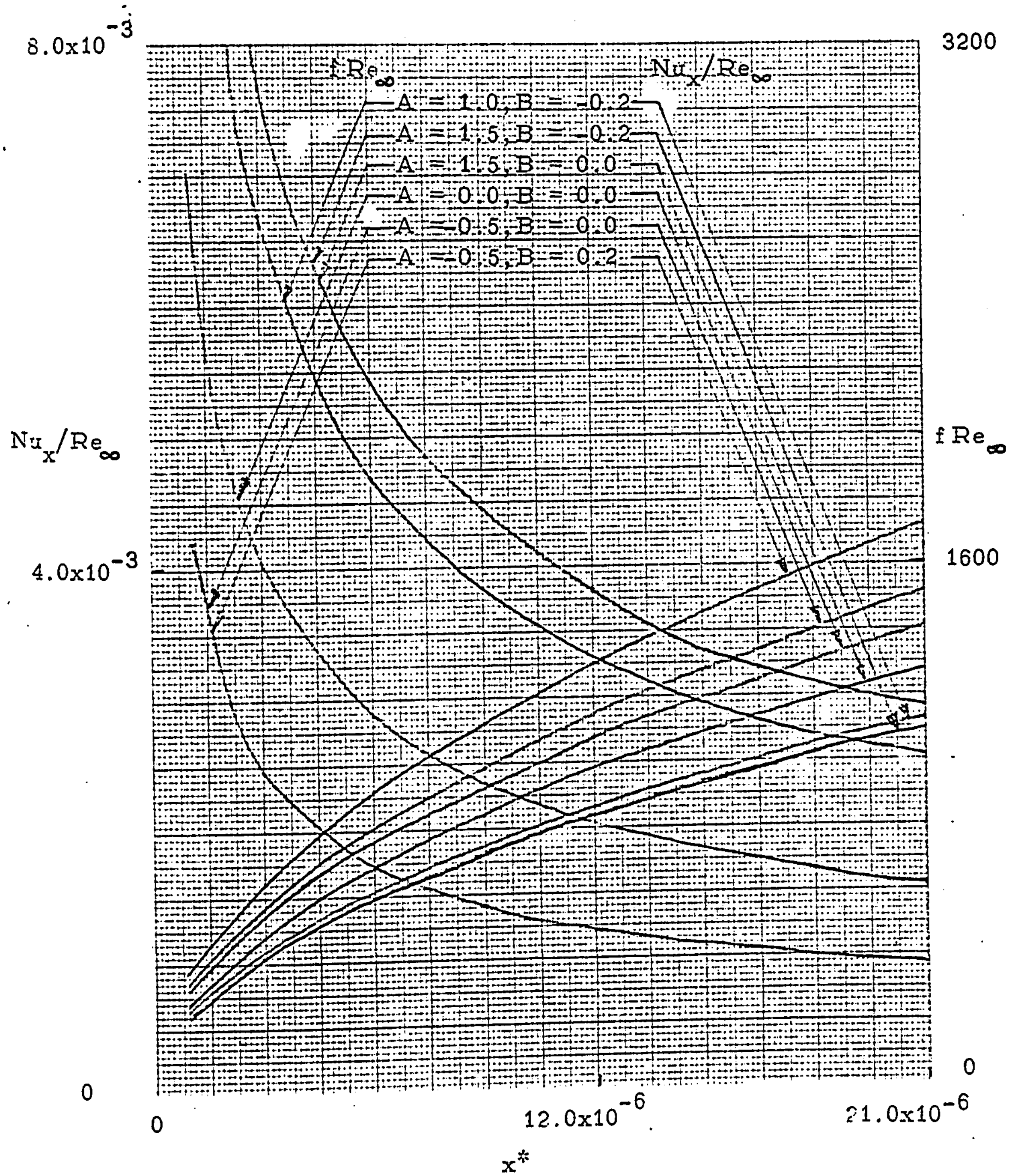


Fig. 11 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 10.0$ CONSTANT WALL TEMPERATURE CASE

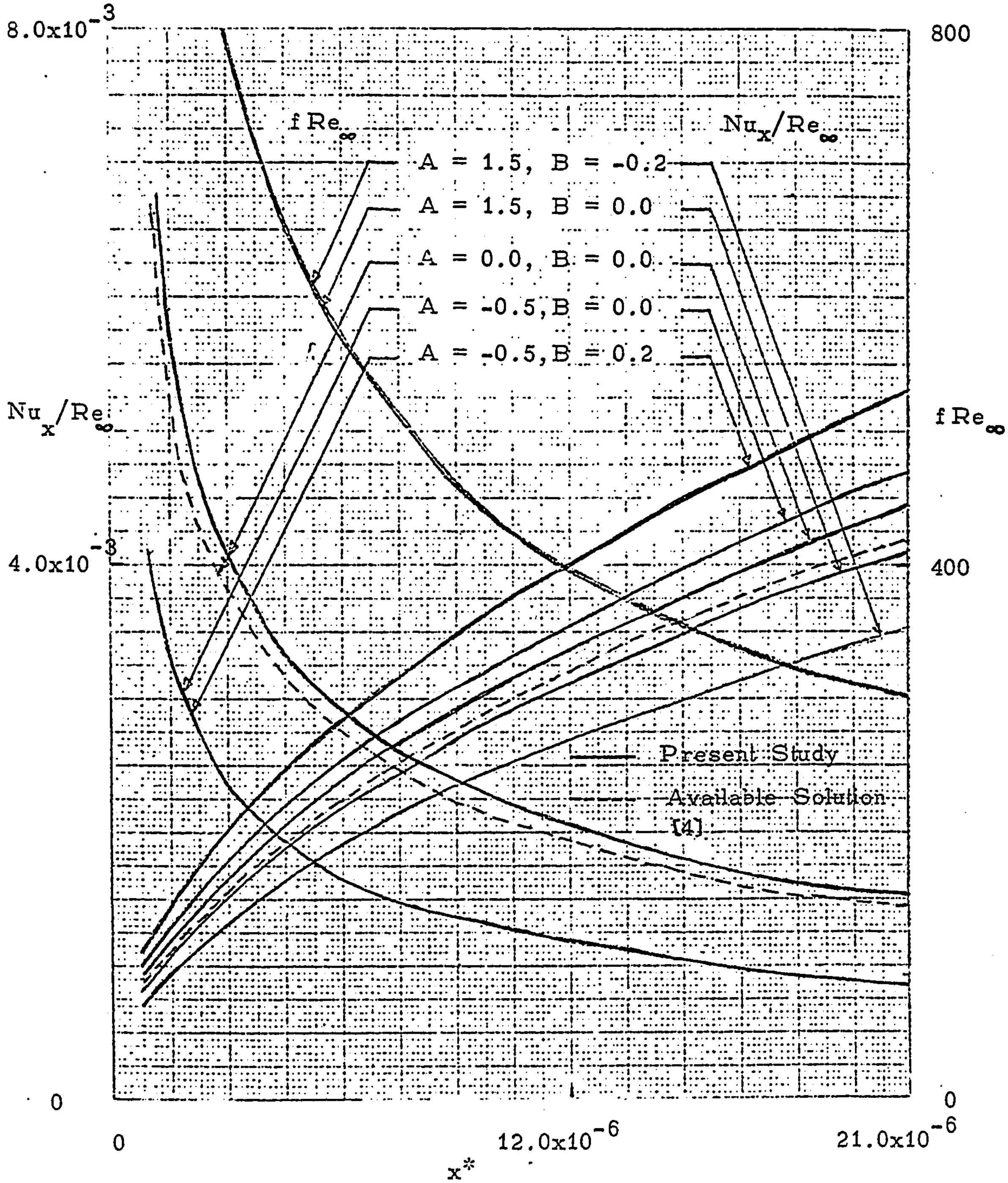


Fig. 12 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 20.0$

CONSTANT WALL TEMPERATURE CASE

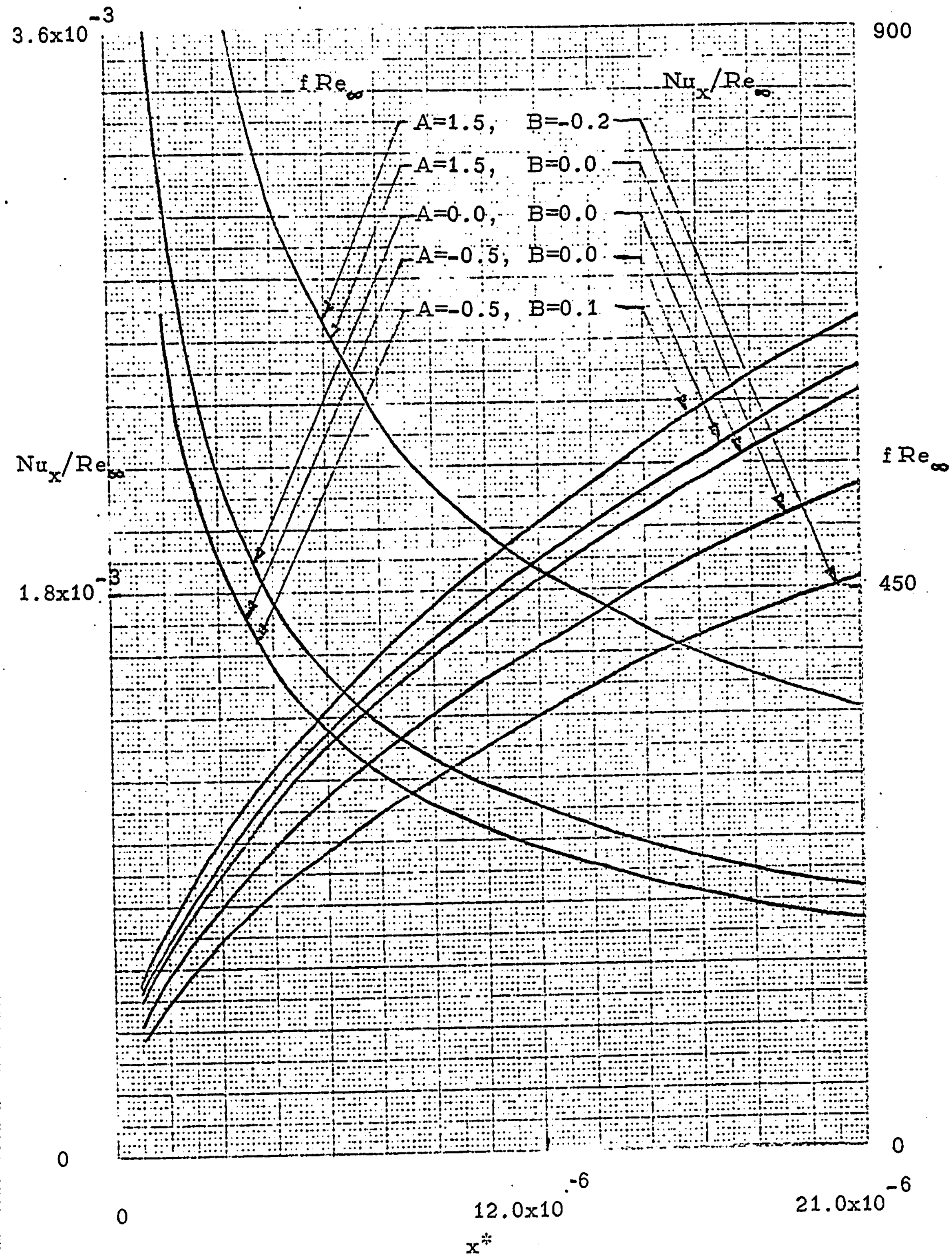


Fig. 13 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 6.111$, $Pr_\infty = 2.0$

CONSTANT WALL TEMPERATURE CASE

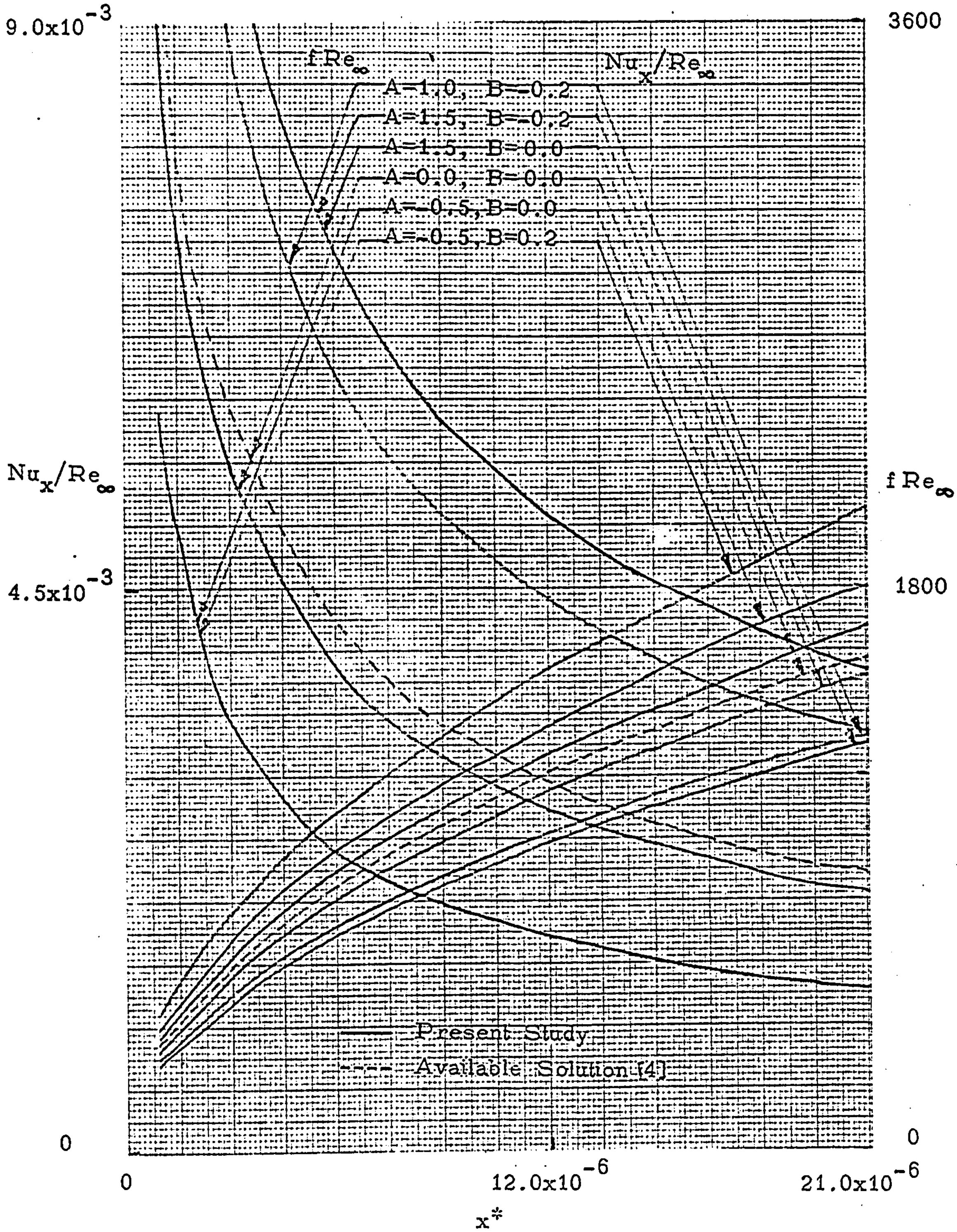


Fig. 14 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.111$, $Pr_\infty = 10.0$

CONSTANT WALL TEMPERATURE CASE

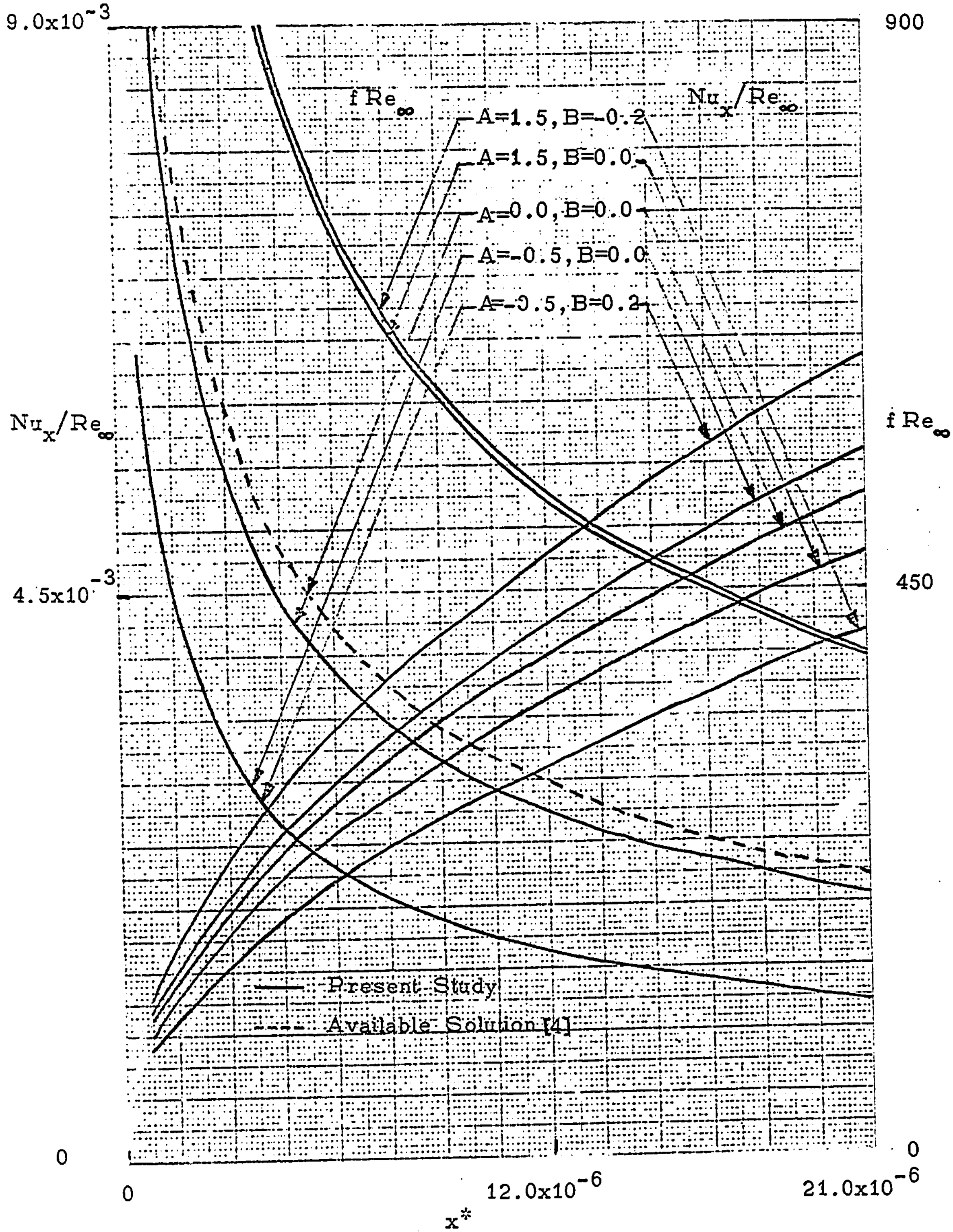


Fig. 15 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.111$, $Pr_{\infty} = 20.0$

CONSTANT WALL TEMPERATURE CASE

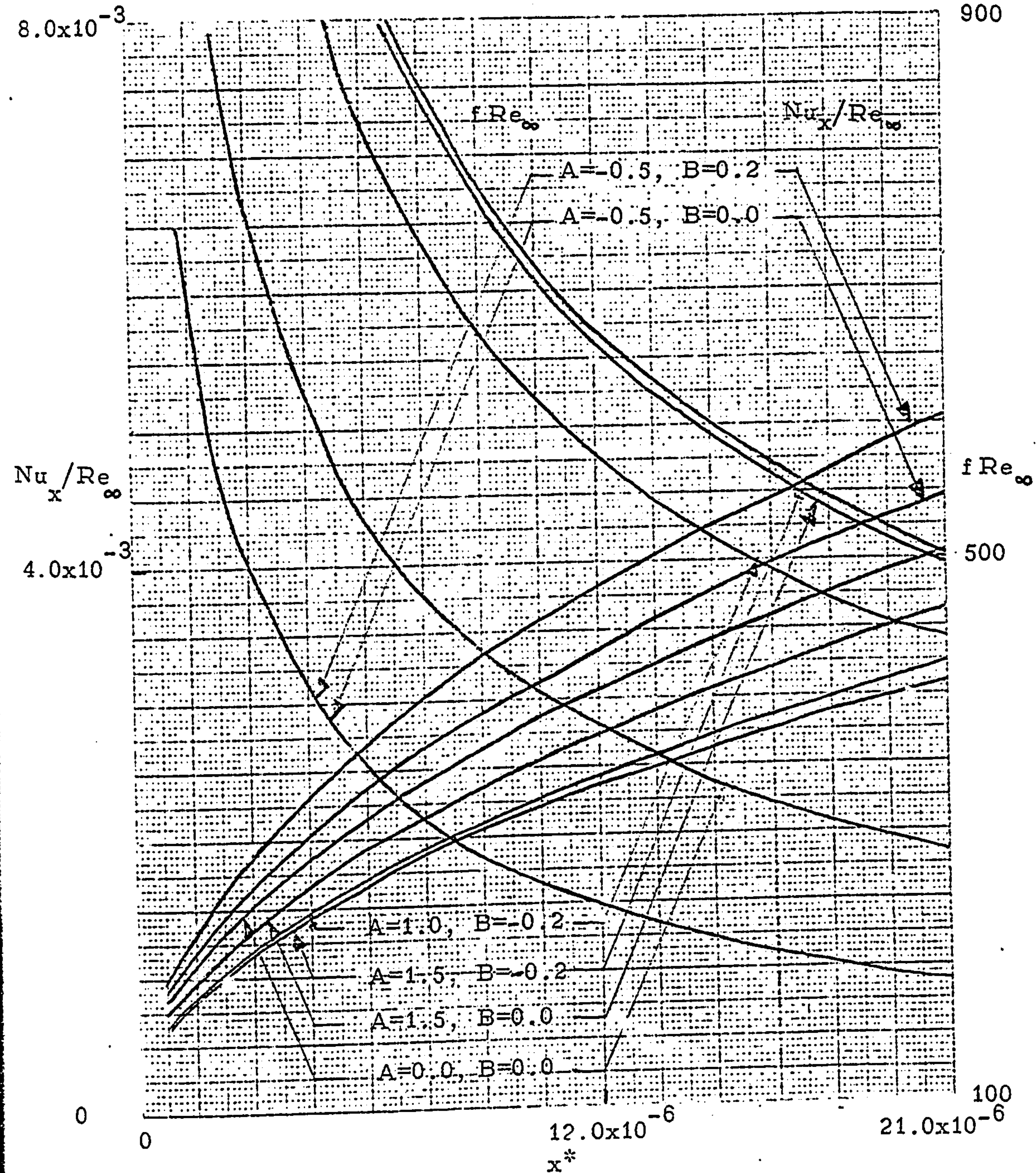


Fig. 16 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.333$, $Pr_\infty = 5.0$

CONSTANT WALL TEMPERATURE CASE

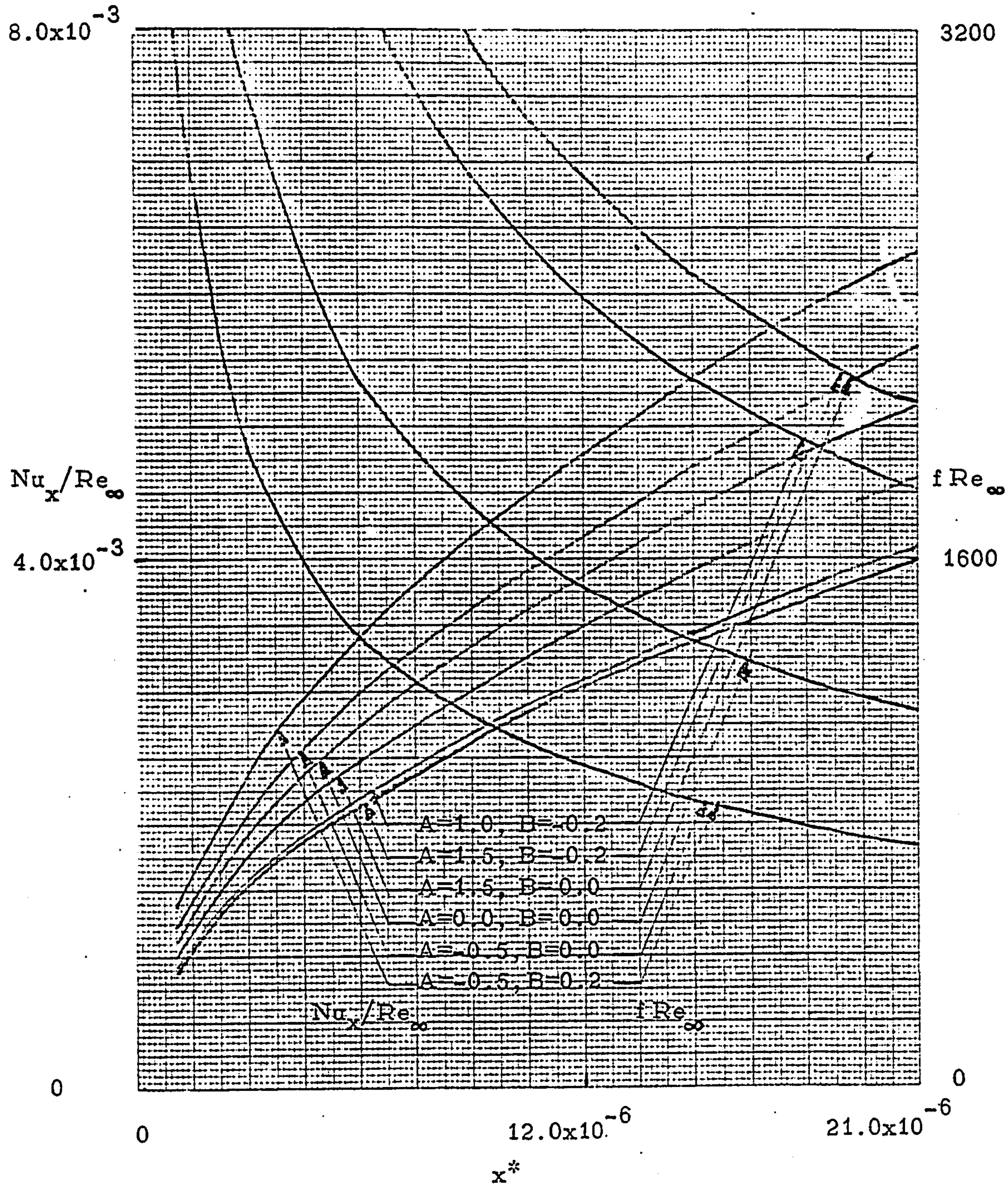


Fig. 17 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.333$, $Pr_{\infty} = 10.0$

CONSTANT WALL TEMPERATURE CASE

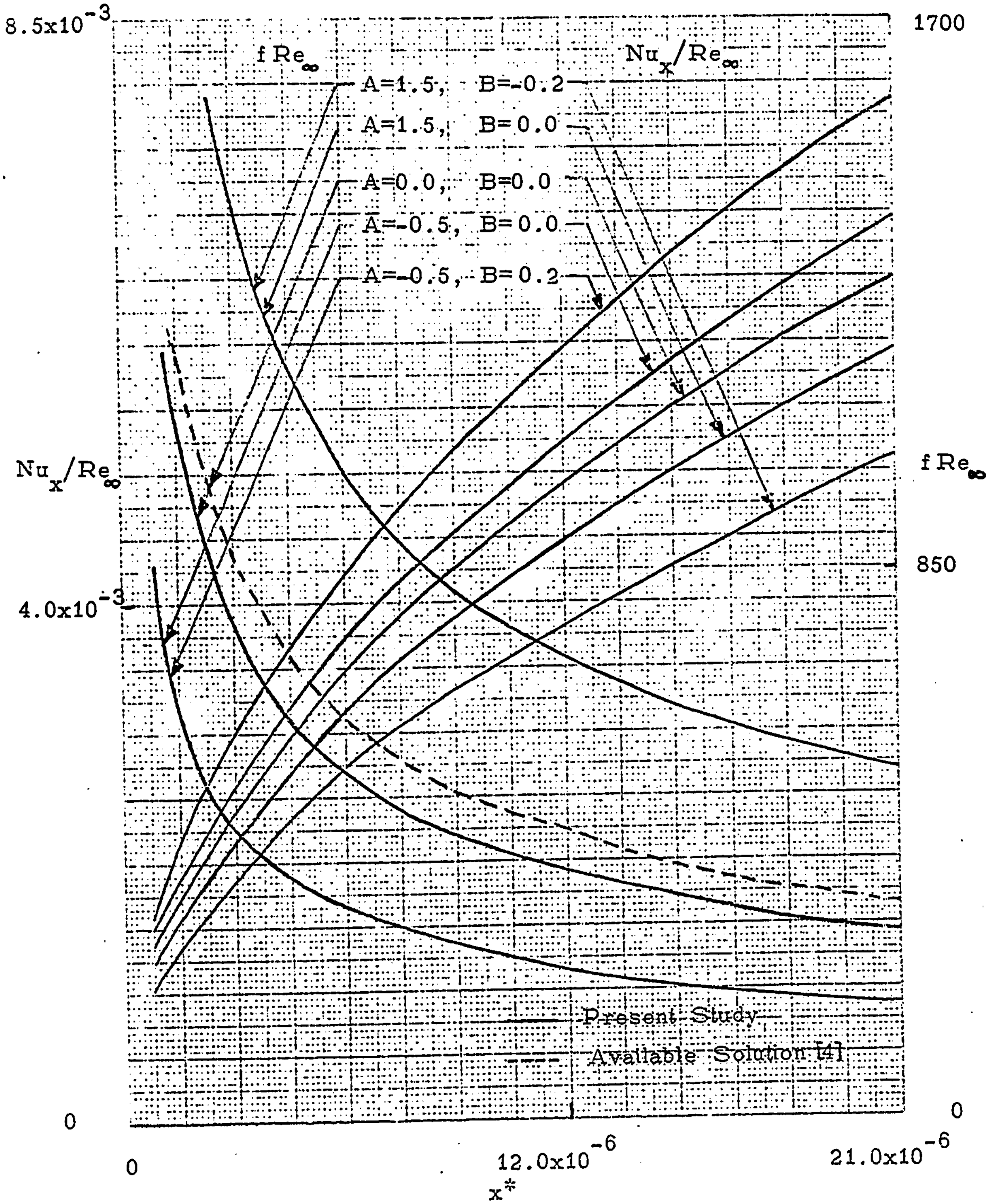


Fig. 18 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.333$, $Pr_\infty = 20.0$ CONSTANT WALL TEMPERATURE CASE

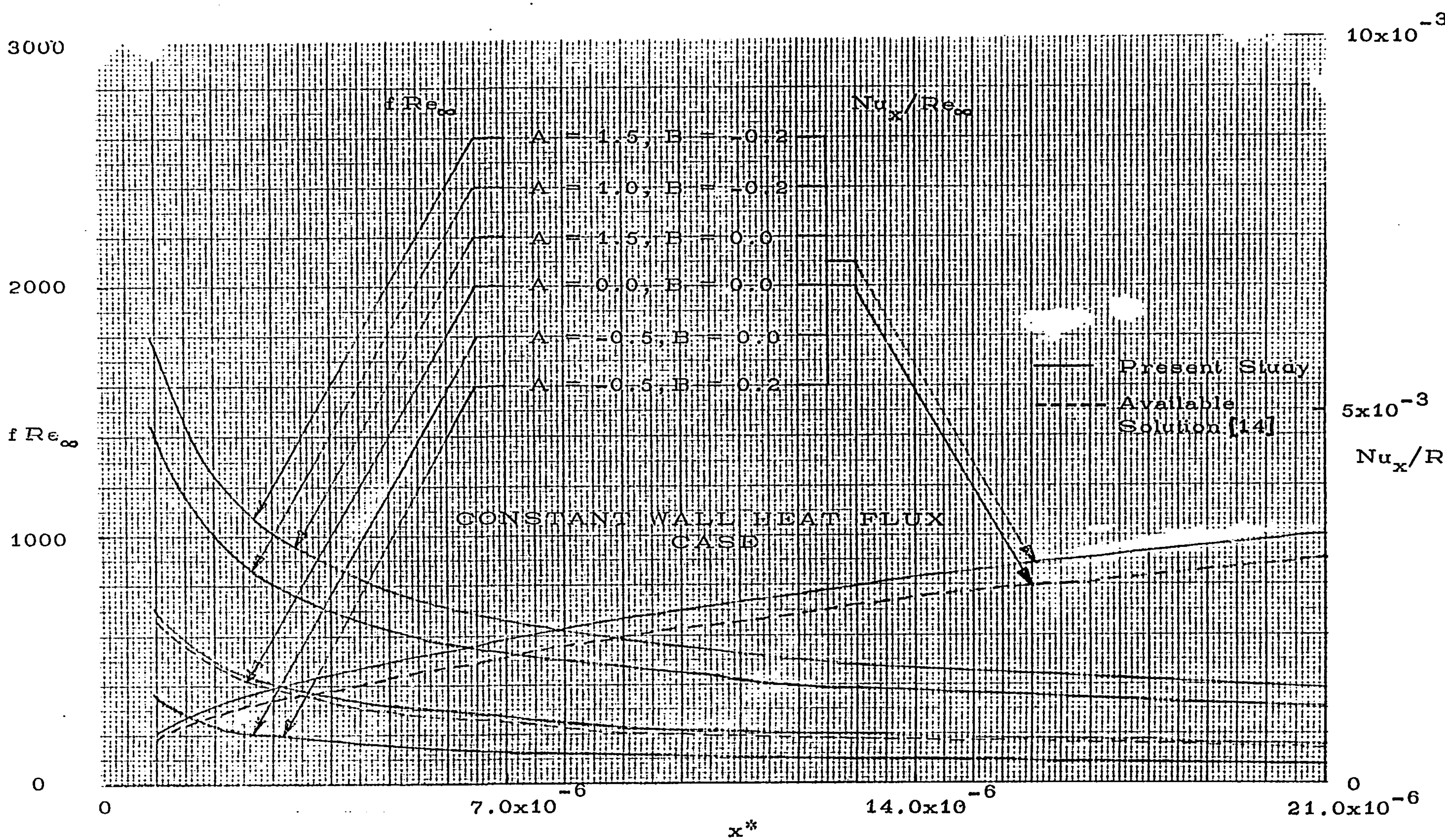


Fig. 19 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 3.0$

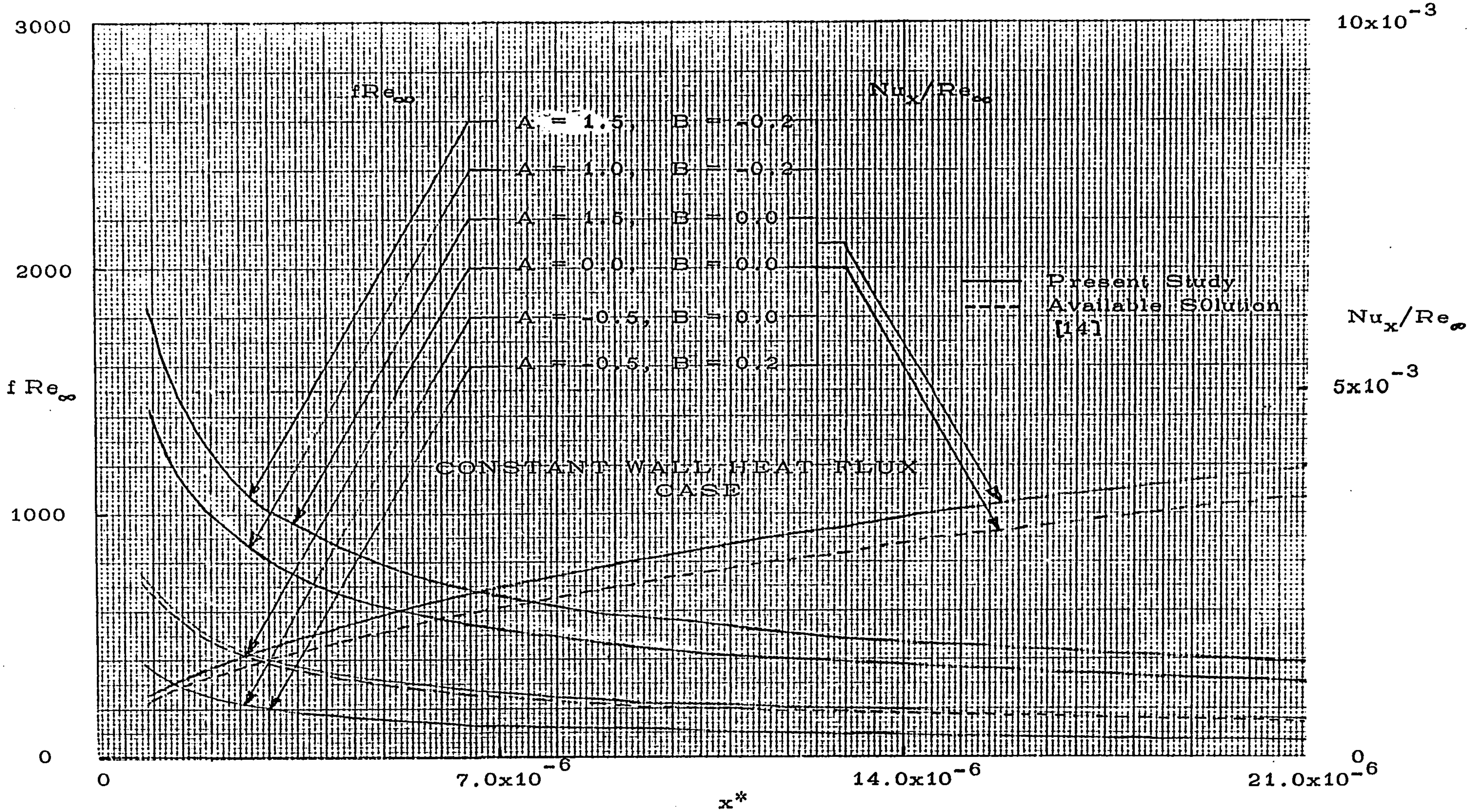


Fig. 20 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 5.0$

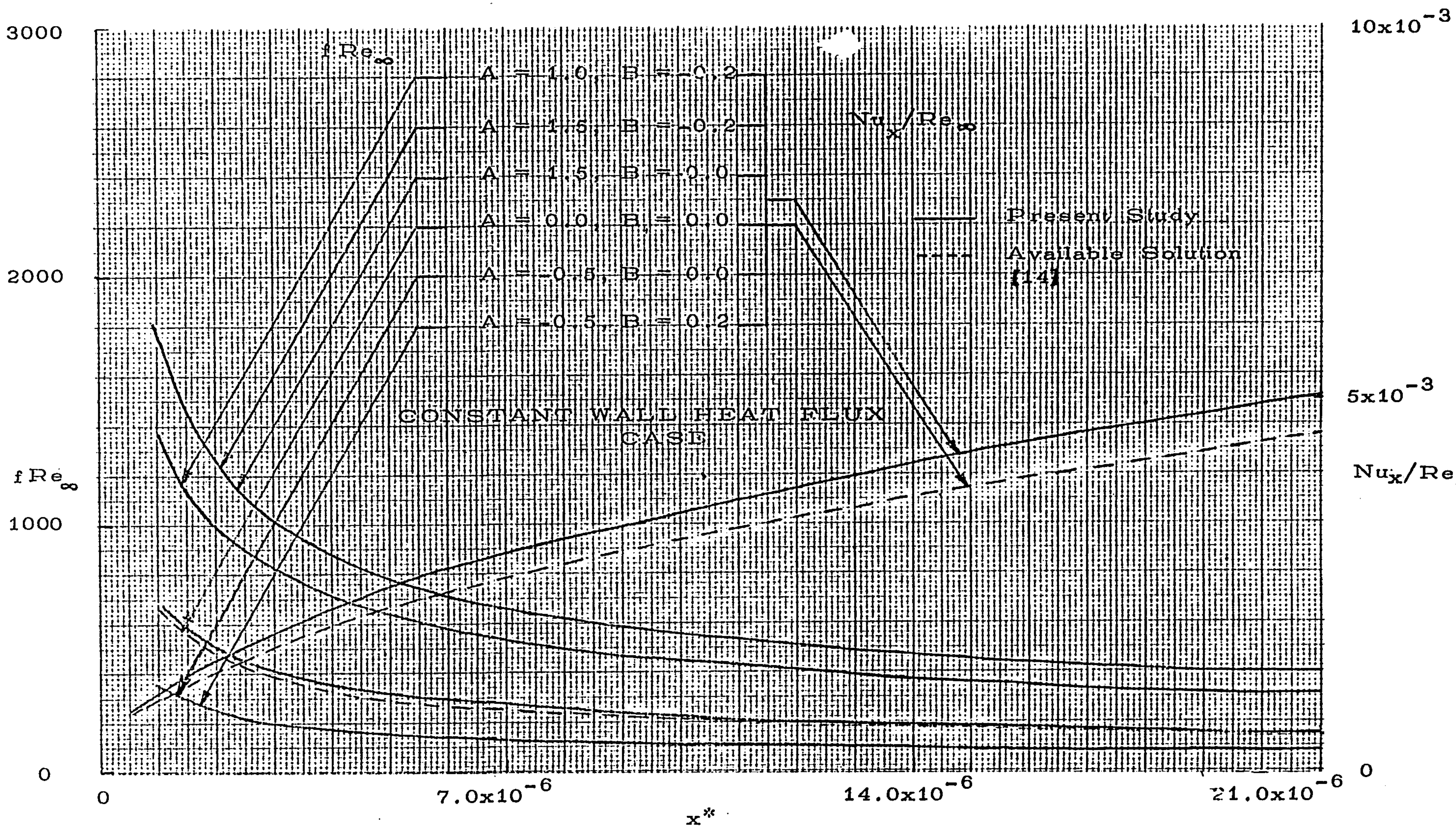


Fig. 21 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 10.0$

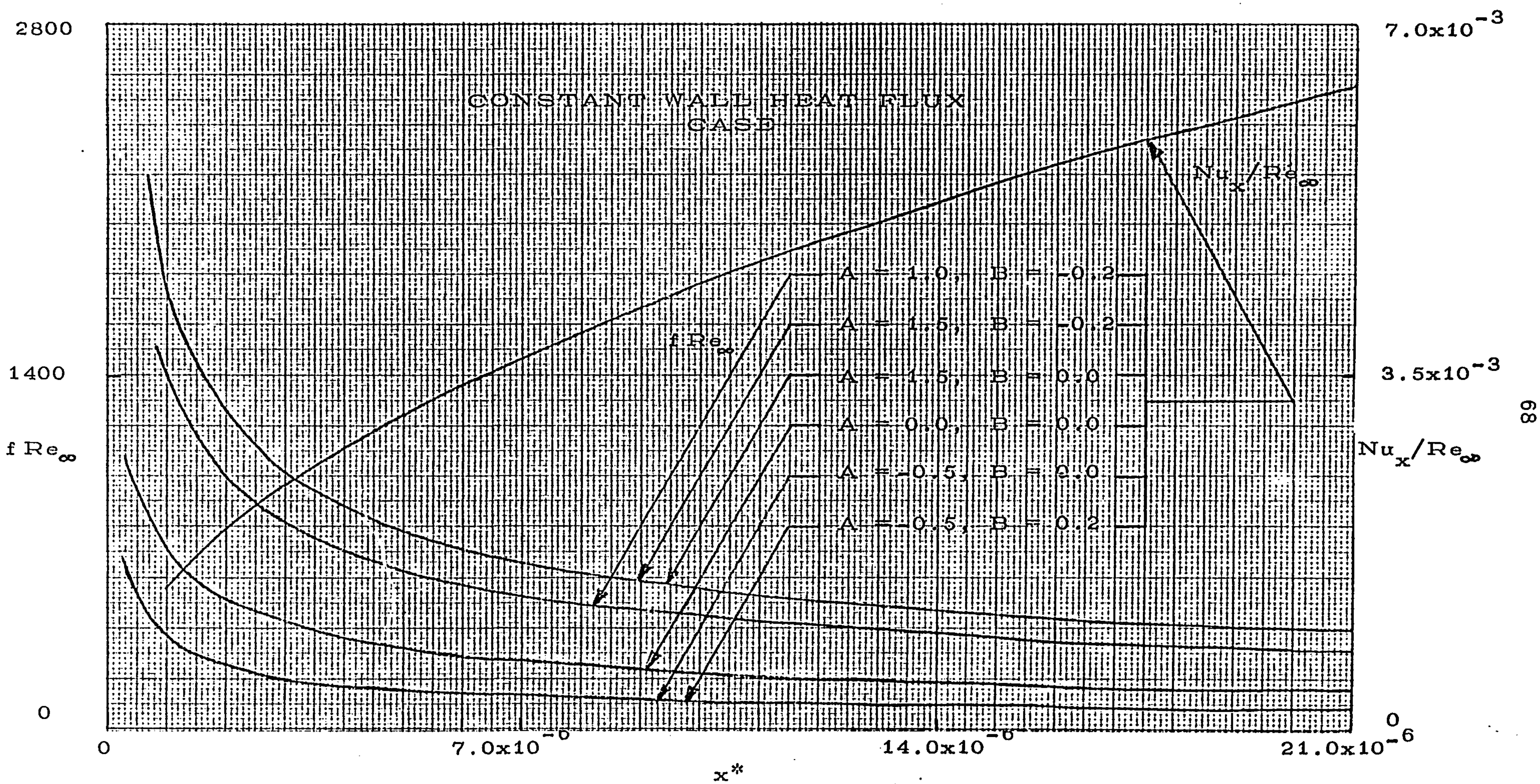


Fig. 22 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.0$, $Pr_\infty = 20.0$

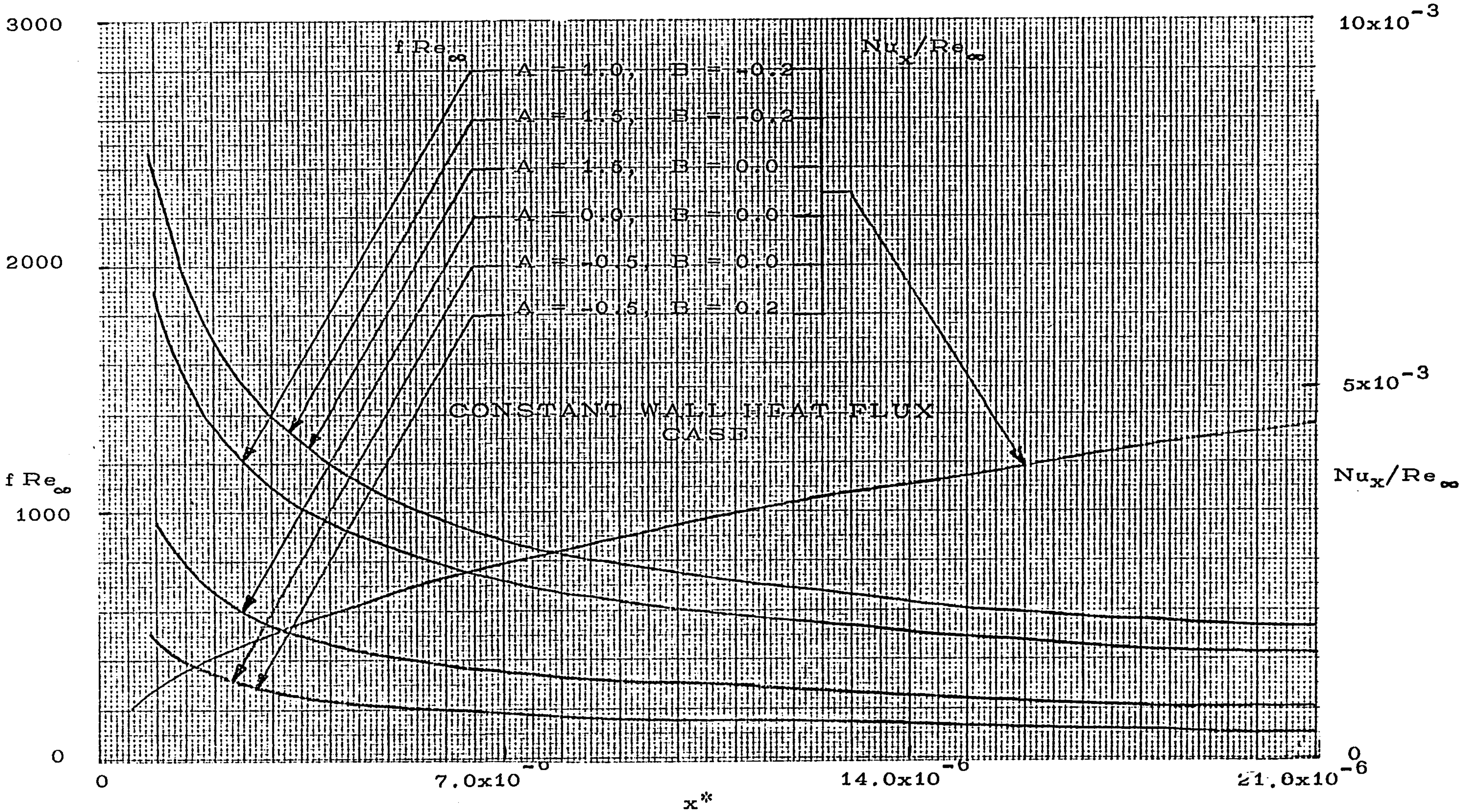


Fig. 23 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.111$, $Pr_\infty = 5.0$

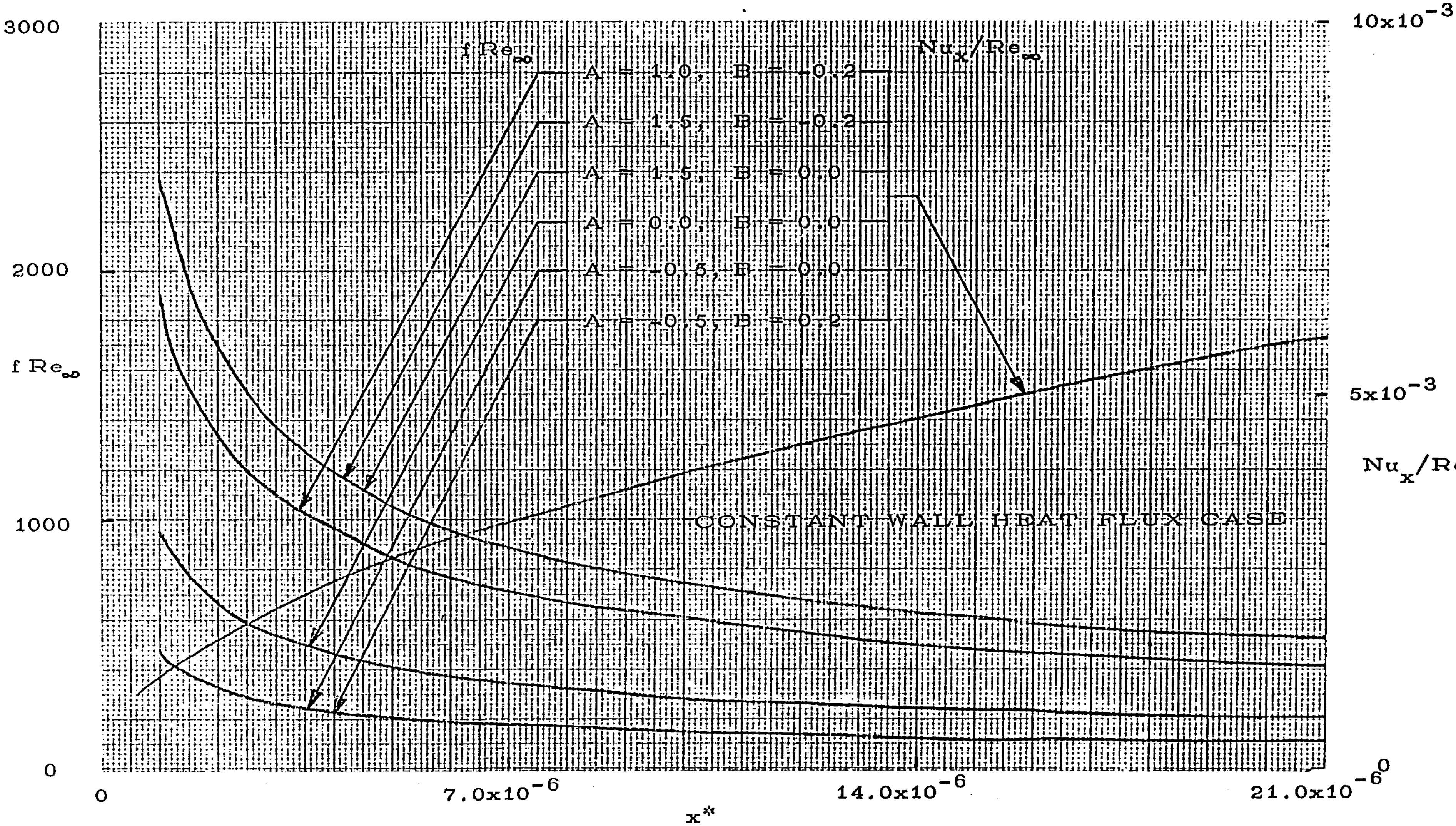


Fig. 24 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.111$, $Pr_\infty = 10.0$

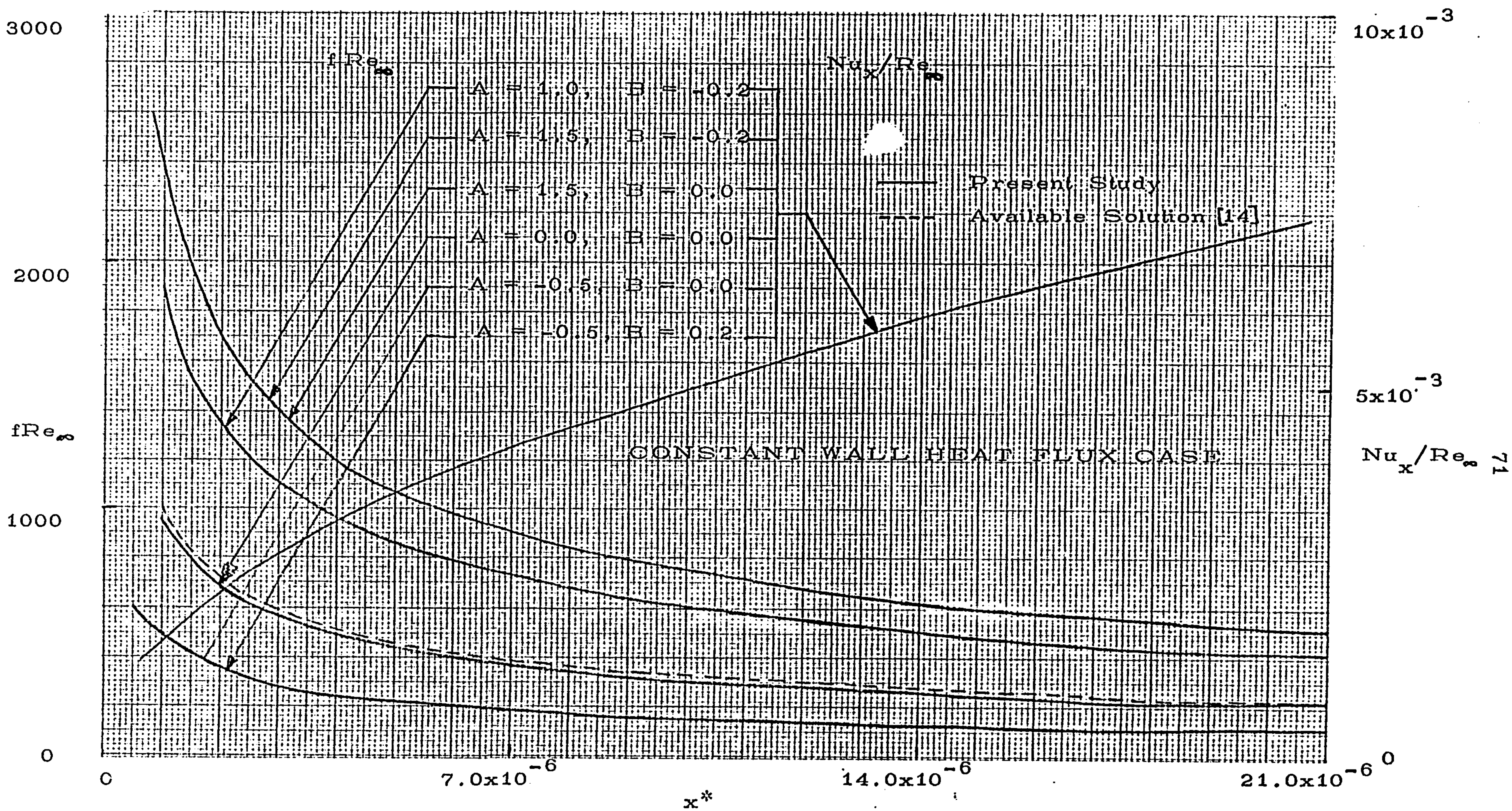


Fig. 25 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.111$, $Pr_\infty = 20.0$

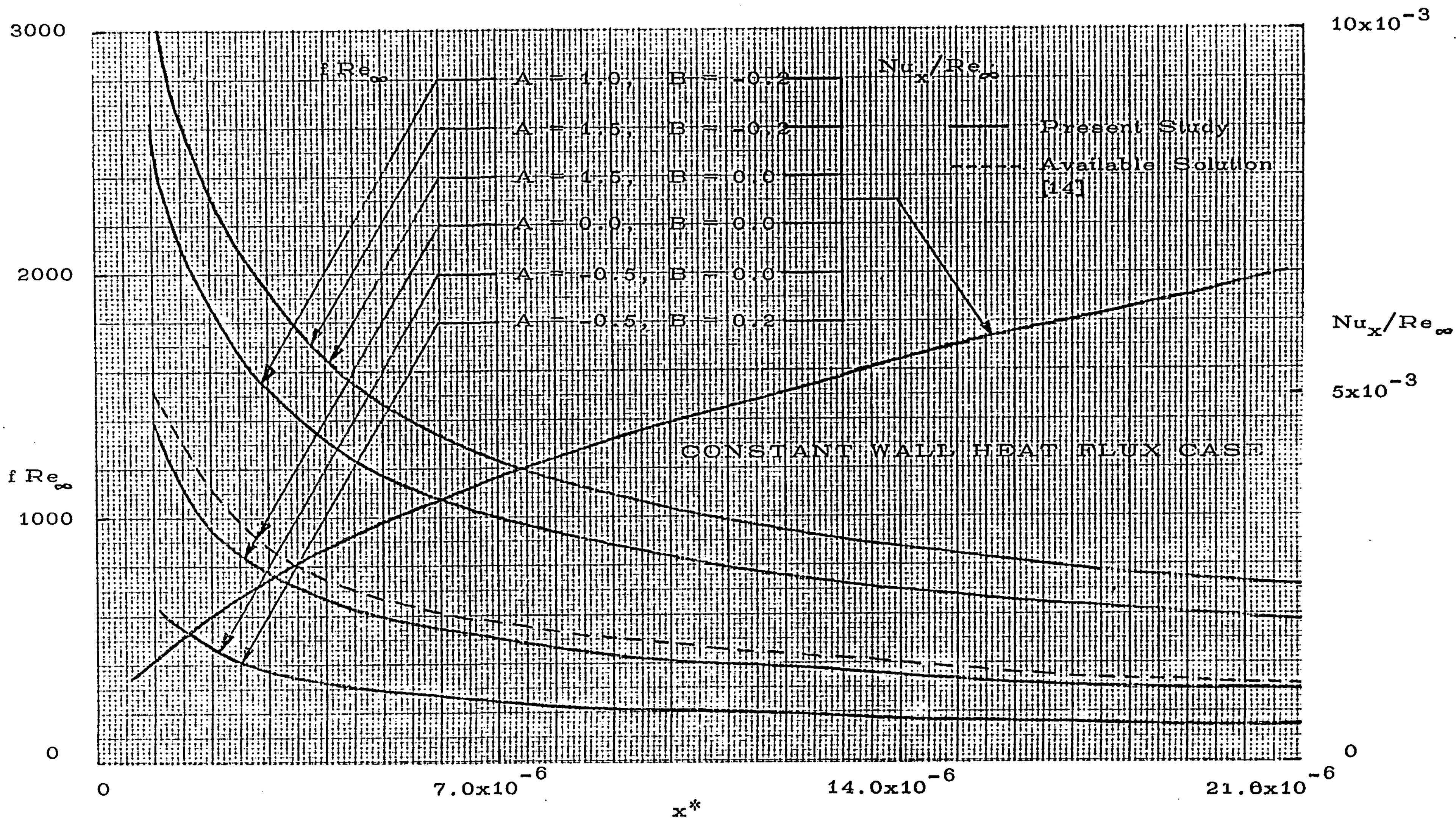


Fig. 26 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.333$, $Pr_\infty = 10.0$

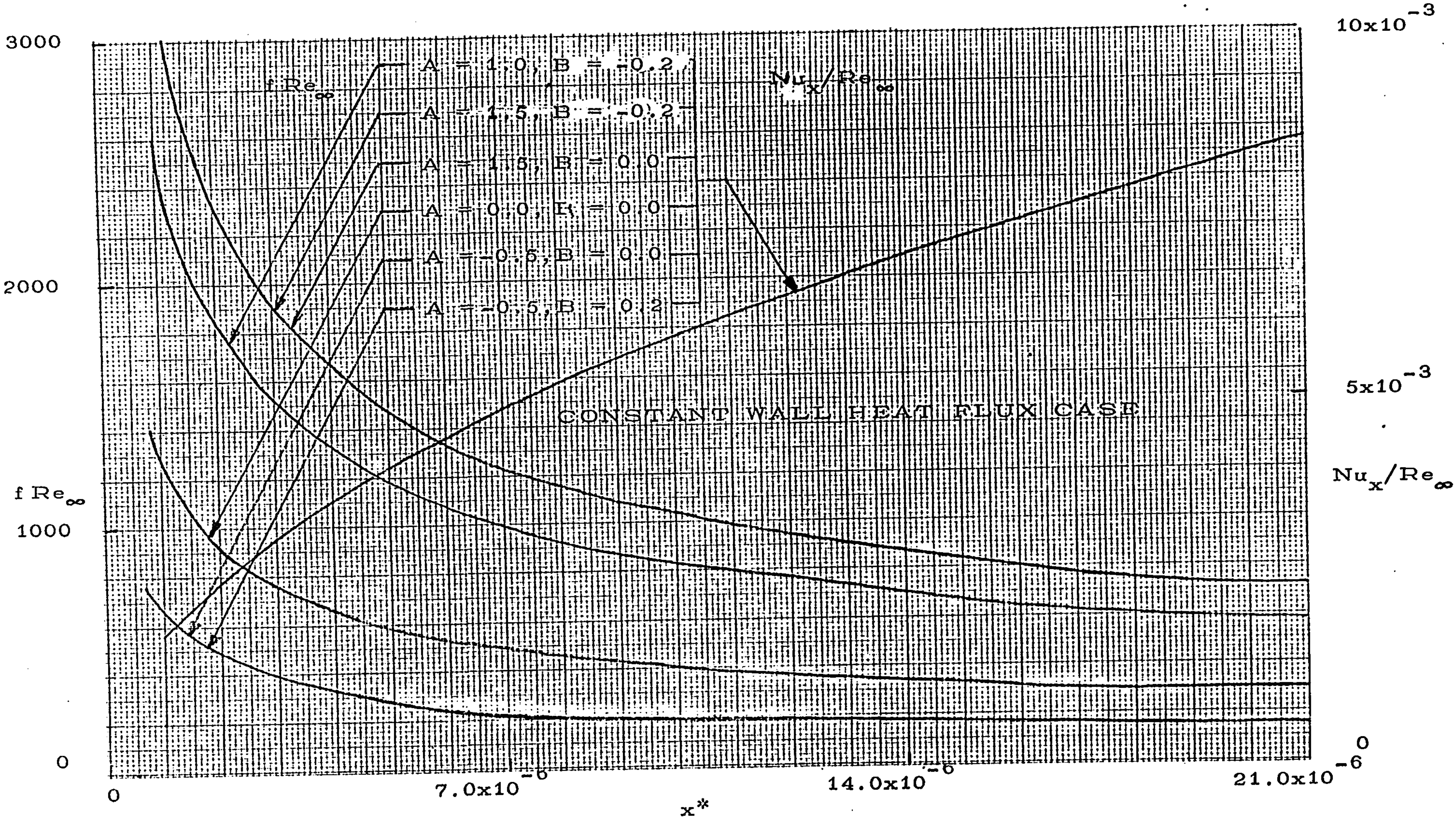


Fig. 27 LOCAL NUSSELT NUMBER AND LOCAL FRICTION FACTOR FOR $m = 0.333$, $Pr_{\infty} = 20.0$

```
NUSSELT NUMBER/REYNOLDS NUMBER                                                       PAGE   1

                              MINIMUM                  NU      VERSUS  X                  MAXIMUM
                                C.0                    WM      = 0.0                   8.0000E-03
    X                NU             I                                                       I
   0.               0.0             +
   1.0000E-06       7.1796E-04      ----+
   2.0000E-06       1.0122E-03      -------+
   3.0000E-06       1.2383E-03      --------+
   4.0000E-06       1.4290E-03      ---------+
   5.0000E-06       1.5971E-03      ----------+
   6.0000E-06       1.7491E-03      -----------+
   7.0000E-06       1.8885E-03      ------------+
   8.0000E-06       2.0191E-03      -------------+
   9.0000E-06       2.1413E-03      --------------+
   1.0000E-05       2.2570E-03      ---------------+
   1.1000E-05       2.3670E-03      ---------------+
   1.2000E-05       2.4721E-03      ----------------+
   1.3000E-05       2.5730E-03      -----------------+
   1.4000E-05       2.6700E-03      -----------------+
   1.5000E-05       2.7636E-03      ------------------+
   1.6000E-05       2.8542E-03      ------------------+
   1.7000E-05       2.9420E-03      -------------------+
   1.8000E-05       3.0272E-03      -------------------+
   1.9000E-05       3.1101E-03      --------------------+
   2.0000E-05       3.1908E-03      --------------------+
   2.1000E-05       3.2696E-03      ---------------------+
   2.2000E-05       8.0000E-03      ----------------------------------------------------------+
```

Fig. 28

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 1.0$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — MAXIMUM 8.0000E-03

NU WM = 0.0

X	NU
0.	0.0
1.0000E-06	7.0273E-04
2.0000E-06	9.9081E-04
3.0000E-06	1.2122E-03
4.0000E-06	1.3989E-03
5.0000E-06	1.5635E-03
6.0000E-06	1.7124E-03
7.0000E-06	1.8493E-03
8.0000E-06	1.9767E-03
9.0000E-06	2.0964E-03
1.0000E-05	2.2097E-03
1.1000E-05	2.3174E-03
1.2000E-05	2.4203E-03
1.3000E-05	2.5190E-03
1.4000E-05	2.6140E-03
1.5000E-05	2.7057E-03
1.6000E-05	2.7944E-03
1.7000E-05	2.8803E-03
1.8000E-05	2.9638E-03
1.9000E-05	3.0449E-03
2.0000E-05	3.1240E-03
2.1000E-05	3.2011E-03
2.2000E-05	8.0000E-03

Fig. 29

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 1.5$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

```
NUSSELT NUMBER/REYNOLDS NUMBER                                                        PAGE    1

                                MINIMUM                 NU     VERSUS X                    MAXIMUM
                                0.0                     WM     = 0.0                       8.0000E-03
      X                NU               I                                                    I
    0.0               0.0               +
    1.0000E-06        8.1766E-04        -----+
    2.0000E-06        1.1529E-03        -------+
    3.0000E-06        1.4105E-03        --------+
    4.0000E-06        1.6278E-03        ----------+
    5.0000E-06        1.8193E-03        -----------+
    6.0000E-06        1.9925E-03        ------------+
    7.0000E-06        2.1518E-03        -------------+
    8.0000E-06        2.3001E-03        --------------+
    9.0000E-06        2.4394E-03        ---------------+
    1.0000E-05        2.5712E-03        ----------------+
    1.1000E-05        2.6966E-03        ----------------+
    1.2000E-05        2.8164E-03        -----------------+
    1.3000E-05        2.9313E-03        ------------------+
    1.4000E-05        3.0418E-03        -------------------+
    1.5000E-05        3.1485E-03        -------------------+
    1.6000E-05        3.2517E-03        --------------------+
    1.7000E-05        3.3517E-03        --------------------+
    1.8000E-05        3.4488E-03        ---------------------+
    1.9000E-05        3.5432E-03        ----------------------+
    2.0000E-05        3.6352E-03        ----------------------+
    2.1000E-05        3.7250E-03        -----------------------+
    2.2000E-05        8.0000E-03        ---------------------------------------------------------+
```

Fig. 30

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 1.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — WM = 0.0 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	8.9155E-04
2.0000E-06	1.2567E-03
3.0000E-06	1.5373E-03
4.0000E-06	1.7740E-03
5.0000E-06	1.9825E-03
6.0000E-06	2.1712E-03
7.0000E-06	2.3446E-03
8.0000E-06	2.5061E-03
9.0000E-06	2.6578E-03
1.0000E-05	2.8012E-03
1.1000E-05	2.9377E-03
1.2000E-05	3.0681E-03
1.3000E-05	3.1932E-03
1.4000E-05	3.3136E-03
1.5000E-05	3.4297E-03
1.6000E-05	3.5421E-03
1.7000E-05	3.6510E-03
1.8000E-05	3.7568E-03
1.9000E-05	3.8596E-03
2.0000E-05	3.9598E-03
2.1000E-05	4.0575E-03
2.2000E-05	8.0000E-03

Fig. 31

LOCAL NUSSELT NUMBER FOR $Pr_\infty = 15.0$, $m = 0.0$, $A = 0.0$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — WM = 0.0 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	9.4269E-04
2.0000E-06	1.3291E-03
3.0000E-06	1.6260E-03
4.0000E-06	1.8763E-03
5.0000E-06	2.0969E-03
6.0000E-06	2.2964E-03
7.0000E-06	2.4799E-03
8.0000E-06	2.6507E-03
9.0000E-06	2.8111E-03
1.0000E-05	2.9629E-03
1.1000E-05	3.1072E-03
1.2000E-05	3.2451E-03
1.3000E-05	3.3774E-03
1.4000E-05	3.5047E-03
1.5000E-05	3.6275E-03
1.6000E-05	3.7463E-03
1.7000E-05	3.8615E-03
1.8000E-05	3.9733E-03
1.9000E-05	4.0821E-03
2.0000E-05	4.1882E-03
2.1000E-05	4.2917E-03
2.2000E-05	8.0000E-03

Fig. 32

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = -0.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 | NU VERSUS X | WM = 0.0 | MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.0727E-03
2.0000E-06	1.5126E-03
3.0000E-06	1.8504E-03
4.0000E-06	2.1353E-03
5.0000E-06	2.3864E-03
6.0000E-06	2.6134E-03
7.0000E-06	2.8222E-03
8.0000E-06	3.0166E-03
9.0000E-06	3.1992E-03
1.0000E-05	3.3719E-03
1.1000E-05	3.5361E-03
1.2000E-05	3.6931E-03
1.3000E-05	3.8437E-03
1.4000E-05	3.9885E-03
1.5000E-05	4.1283E-03
1.6000E-05	4.2635E-03
1.7000E-05	4.3946E-03
1.8000E-05	4.5218E-03
1.9000E-05	4.6456E-03
2.0000E-05	4.7662E-03
2.1000E-05	4.8839E-03
2.2000E-05	8.0000E-03

Fig. 33

LOCAL NUSSELT NUMBER FOR $Pr_\infty = 15.0$, $m = 0.0$, $A = -0.5$, $B = 0.2$

CONSTANT WALL TEMPERATURE CASE

```
NUSSELT NUMBER/REYNOLDS NUMBER                                                PAGE    1
                                MINIMUM          NU      VERSUS X              MAXIMUM
                                  0.0            WM    = 1.1100E-01           8.0000E-03
    X               NU             I                                              I
    0.              0.0            +
 1.0000E-06     8.3980E-04    -----+
 2.0000E-06     1.1860E-03    -------+
 3.0000E-06     1.4518E-03    ---------+
 4.0000E-06     1.6760E-03    ----------+
 5.0000E-06     1.8735E-03    -----------+
 6.0000E-06     2.0522E-03    ------------+
 7.0000E-06     2.2164E-03    -------------+
 8.0000E-06     2.3694E-03    --------------+
 9.0000E-06     2.5130E-03    ---------------+
 1.0000E-05     2.6488E-03    ----------------+
 1.1000E-05     2.7780E-03    -----------------+
 1.2000E-05     2.9015E-03    ------------------+
 1.3000E-05     3.0199E-03    ------------------+
 1.4000E-05     3.1339E-03    -------------------+
 1.5000E-05     3.2438E-03    --------------------+
 1.6000E-05     3.3502E-03    --------------------+
 1.7000E-05     3.4533E-03    ---------------------+
 1.8000E-05     3.5534E-03    ----------------------+
 1.9000E-05     3.6508E-03    ----------------------+
 2.0000E-05     3.7457E-03    -----------------------+
 2.1000E-05     3.8382E-03    -----------------------+
 2.2000E-05     8.0000E-03    -----------------------------------------------------+
```

Fig. 34

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = 1.0$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER — PAGE 1

NU VERSUS X, WM = 1.1100E-01

MINIMUM 0.0 — MAXIMUM 8.0000E-03

X	NU
0.	0.0
1.0000E-06	3.2100E-04
2.0000E-06	1.1595E-03
3.0000E-06	1.4194E-03
4.0000E-06	1.6386E-03
5.0000E-06	1.8318E-03
6.0000E-06	2.0064E-03
7.0000E-06	2.1670E-03
8.0000E-06	2.3166E-03
9.0000E-06	2.4570E-03
1.0000E-05	2.5898E-03
1.1000E-05	2.7161E-03
1.2000E-05	2.8369E-03
1.3000E-05	2.9527E-03
1.4000E-05	3.0641E-03
1.5000E-05	3.1716E-03
1.6000E-05	3.2757E-03
1.7000E-05	3.3765E-03
1.8000E-05	3.4744E-03
1.9000E-05	3.5697E-03
2.0000E-05	3.6624E-03
2.1000E-05	3.7529E-03
2.2000E-05	8.0000E-03

Fig. 35

LOCAL NUSSELT NUMBER FOR $Pr_\infty = 15.0$, $m = 0.111$, $A = 1.5$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

```
   NUSSELT NUMBER/REYNOLDS NUMBER                                                              PAGE    1
                                    MINIMUM                   NU        VERSUS X                     MAXIMUM
                                    0.0                       WM      = 1.1100E-01                  8.0000E-03
       X                NU          1                                                                   I
   0.0              0.0             +
   1.0000E-06       9.5514E-04      -----+
   2.0000E-06       1.3489E-03      ---------+
   3.0000E-06       1.6514E-03      -----------+
   4.0000E-06       1.9064E-03      ------------+
   5.0000E-06       2.1311E-03      ---------------+
   6.0000E-06       2.3343E-03      ----------------+
   7.0000E-06       2.5212E-03      -----------------+
   8.0000E-06       2.6951E-03      ------------------+
   9.0000E-06       2.8585E-03      -------------------+
   1.0000E-05       3.0130E-03      --------------------+
   1.1000E-05       3.1600E-03      ---------------------+
   1.2000E-05       3.3004E-03      ----------------------+
   1.3000E-05       3.4351E-03      -----------------------+
   1.4000E-05       3.5648E-03      ------------------------+
   1.5000E-05       3.6898E-03      -------------------------+
   1.6000E-05       3.8108E-03      -------------------------+
   1.7000E-05       3.9281E-03      --------------------------+
   1.8000E-05       4.0420E-03      ---------------------------+
   1.9000E-05       4.1527E-03      ---------------------------+
   2.0000E-05       4.2606E-03      ----------------------------+
   2.1000E-05       4.3658E-03      -----------------------------+
   2.2000E-05       8.0000E-03      --------------------------------------------------------+
```

Fig. 36

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = 1.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X, WM = .1.1100E-01 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.0482E-03
2.0000E-06	1.4800E-03
3.0000E-06	1.8116E-03
4.0000E-06	2.0913E-03
5.0000E-06	2.3377E-03
6.0000E-06	2.5606E-03
7.0000E-06	2.7655E-03
8.0000E-06	2.9563E-03
9.0000E-06	3.1355E-03
1.0000E-05	3.3050E-03
1.1000E-05	3.4662E-03
1.2000E-05	3.6202E-03
1.3000E-05	3.7680E-03
1.4000E-05	3.9101E-03
1.5000E-05	4.0473E-03
1.6000E-05	4.1800E-03
1.7000E-05	4.3087E-03
1.8000E-05	4.4336E-03
1.9000E-05	4.5551E-03
2.0000E-05	4.6734E-03
2.1000E-05	4.7888E-03
2.2000E-05	8.0000E-03

Fig. 37

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = 0.0$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

NU VERSUS X

WN = 1.1100E-01

MINIMUM 0.0 MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.1181E-03
2.0000E-06	1.5785E-03
3.0000E-06	1.9322E-03
4.0000E-06	2.2304E-03
5.0000E-06	2.4932E-03
6.0000E-06	2.7308E-03
7.0000E-06	2.9493E-03
8.0000E-06	3.1527E-03
9.0000E-06	3.3438E-03
1.0000E-05	3.5245E-03
1.1000E-05	3.6964E-03
1.2000E-05	3.8606E-03
1.3000E-05	4.0182E-03
1.4000E-05	4.1698E-03
1.5000E-05	4.3160E-03
1.6000E-05	4.4575E-03
1.7000E-05	4.5946E-03
1.8000E-05	4.7278E-03
1.9000E-05	4.8573E-03
2.0000E-05	4.9835E-03
2.1000E-05	5.1065E-03
2.2000E-05	8.0000E-03

Fig. 38

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = -0.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — WM = 1.1100E-01 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.2727E-03
2.0000E-06	1.7968E-03
3.0000E-06	2.1992E-03
4.0000E-06	2.5387E-03
5.0000E-06	2.8378E-03
6.0000E-06	3.1082E-03
7.0000E-06	3.3569E-03
8.0000E-06	3.5884E-03
9.0000E-06	3.8059E-03
1.0000E-05	4.0116E-03
1.1000E-05	4.2072E-03
1.2000E-05	4.3942E-03
1.3000E-05	4.5735E-03
1.4000E-05	4.7460E-03
1.5000E-05	4.9125E-03
1.6000E-05	5.0735E-03
1.7000E-05	5.2296E-03
1.8000E-05	5.3812E-03
1.9000E-05	5.5286E-03
2.0000E-05	5.6721E-03
2.1000E-05	5.8122E-03
2.2000E-05	8.0000E-03

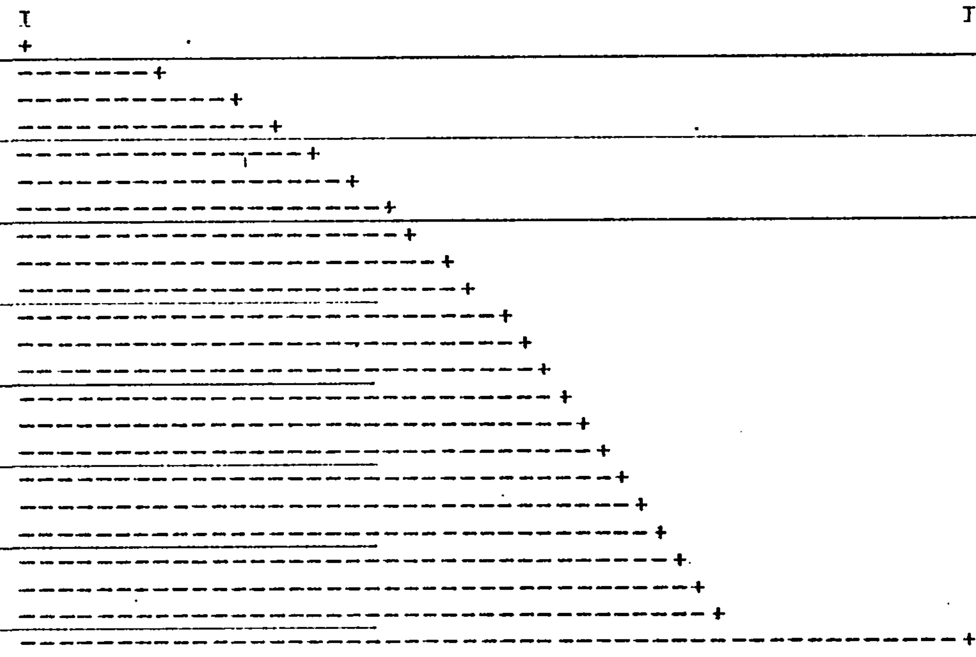


Fig. 39

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = -0.5$, $B = 0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X, WM = 2.2200E-01 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	9.3984E-04
2.0000E-06	1.3293E-03
3.0000E-06	1.6265E-03
4.0000E-06	1.8779E-03
5.0000E-06	2.0994E-03
6.0000E-06	2.2997E-03
7.0000E-06	2.4839E-03
8.0000E-06	2.6554E-03
9.0000E-06	2.8164E-03
1.0000E-05	2.9683E-03
1.1000E-05	3.1136E-03
1.2000E-05	3.2521E-03
1.3000E-05	3.3848E-03
1.4000E-05	3.5126E-03
1.5000E-05	3.6359E-03
1.6000E-05	3.7551E-03
1.7000E-05	3.8707E-03
1.8000E-05	3.9829E-03
1.9000E-05	4.0920E-03
2.0000E-05	4.1983E-03
2.1000E-05	4.3019E-03
2.2000E-05	8.0000E-03

Fig. 40

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.0$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 NU VERSUS X MAXIMUM 8.0000E-03

WM = 2.2200E-01

X	NU
0.	0.0
1.0000E-06	9.1821E-04
2.0000E-06	1.2978E-03
3.0000E-06	1.5891E-03
4.0000E-06	1.8348E-03
5.0000E-06	2.0513E-03
6.0000E-06	2.2470E-03
7.0000E-06	2.4270E-03
8.0000E-06	2.5945E-03
9.0000E-06	2.7518E-03
1.0000E-05	2.9007E-03
1.1000E-05	3.0422E-03
1.2000E-05	3.1775E-03
1.3000E-05	3.3072E-03
1.4000E-05	3.4320E-03
1.5000E-05	3.5525E-03
1.6000E-05	3.6690E-03
1.7000E-05	3.7319E-03
1.8000E-05	3.8915E-03
1.9000E-05	3.9981E-03
2.0000E-05	4.1020E-03
2.1000E-05	4.2034E-03
2.2000E-05	8.0000E-03

Fig. 41

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.5$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — WM = 2.2200E-01 — MAXIMUM 8.0000E-03

Y	NU
0.0	0.0
1.0000E-06	1.0682E-03
2.0000E-06	1.5097E-03
3.0000E-06	1.8486E-03
4.0000E-06	2.1345E-03
5.0000E-06	2.3863E-03
6.0000E-06	2.6140E-03
7.0000E-06	2.8234E-03
8.0000E-06	3.0184E-03
9.0000E-06	3.2015E-03
1.0000E-05	3.3747E-03
1.1000E-05	3.5394E-03
1.2000E-05	3.6967E-03
1.3000E-05	3.8477E-03
1.4000E-05	3.9929E-03
1.5000E-05	4.1331E-03
1.6000E-05	4.2686E-03
1.7000E-05	4.4000E-03
1.8000E-05	4.5276E-03
1.9000E-05	4.6516E-03
2.0000E-05	4.7725E-03
2.1000E-05	4.8904E-03
2.2000E-05	8.0000E-03

Fig. 42

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — MAXIMUM 8.0000E-03

WM = 2.2200E-01

X	NU
0.0	0.0
1.0000E-06	1.1761E-03
2.0000E-06	1.6619E-03
3.0000E-06	2.0349E-03
4.0000E-06	2.3495E-03
5.0000E-06	2.6267E-03
6.0000E-06	2.8773E-03
7.0000E-06	3.1077E-03
8.0000E-06	3.3223E-03
9.0000E-06	3.5237E-03
1.0000E-05	3.7143E-03
1.1000E-05	3.8956E-03
1.2000E-05	4.0688E-03
1.3000E-05	4.2350E-03
1.4000E-05	4.3948E-03
1.5000E-05	4.5491E-03
1.6000E-05	4.6983E-03
1.7000E-05	4.8429E-03
1.8000E-05	4.9833E-03
1.9000E-05	5.1198E-03
2.0000E-05	5.2528E-03
2.1000E-05	5.3826E-03
2.2000E-05	8.0000E-03

Fig. 43

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 0.0$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

NU VERSUS X WM = 2.2200E-01

MINIMUM 0.0 MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.2599E-03
2.0000E-06	1.7802E-03
3.0000E-06	2.1797E-03
4.0000E-06	2.5165E-03
5.0000E-06	2.8134E-03
6.0000E-06	3.0817E-03
7.0000E-06	3.3285E-03
8.0000E-06	3.5582E-03
9.0000E-06	3.7740E-03
1.0000E-05	3.9781E-03
1.1000E-05	4.1722E-03
1.2000E-05	4.3577E-03
1.3000E-05	4.5356E-03
1.4000E-05	4.7068E-03
1.5000E-05	4.8719E-03
1.6000E-05	5.0317E-03
1.7000E-05	5.1866E-03
1.8000E-05	5.3369E-03
1.9000E-05	5.4832E-03
2.0000E-05	5.6256E-03
2.1000E-05	5.7646E-03
2.2000E-05	8.0000E-03

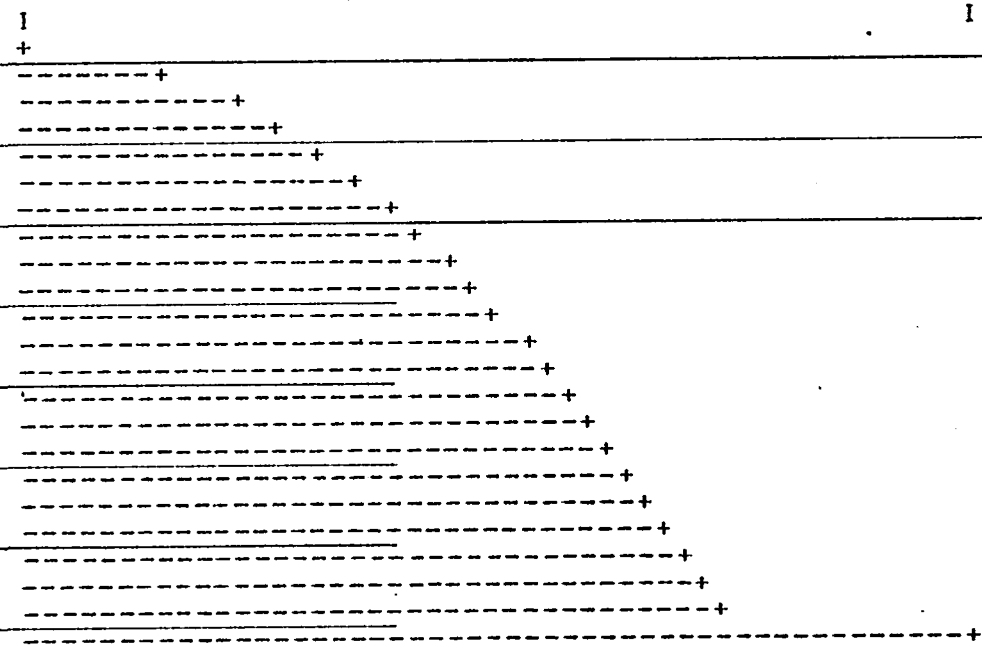


Fig. 44

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = -0.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

NU VERSUS X

MINIMUM 0.0 — WM = 2.2200E-01 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.4340E-03
2.0000E-06	2.0262E-03
3.0000E-06	2.48(8E-03
4.0000E-06	2.8643E-03
5.0000E-06	3.2021E-03
6.0000E-06	3.5076E-03
7.0000E-06	3.7885E-03
8.0000E-06	4.0500E-03
9.0000E-06	4.2956E-03
1.0000E-05	4.5279E-03
1.1000E-05	4.7489E-03
1.2000E-05	4.9600E-03
1.3000E-05	5.1625E-03
1.4000E-05	5.3573E-03
1.5000E-05	5.5453E-03
1.6000E-05	5.7272E-03
1.7000E-05	5.9034E-03
1.8000E-05	6.0745E-03
1.9000E-05	6.2410E-03
2.0000E-05	6.4031E-03
2.1000E-05	6.5612E-03
2.2000E-05	8.0000E-03

Fig. 45

LOCAL NUSSELT NUMBER FOR Pr_{∞} = 15.0, m = 0.222, A = - 0.5, B = 0.2

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER VERSUS X — PAGE 1

MINIMUM 0.0 — NU WM = 3.3300E-03 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.0281E-03
2.0000E-06	1.4536E-03
3.0000E-06	1.7802E-03
4.0000E-06	2.0555E-03
5.0000E-06	2.2981E-03
6.0000E-06	2.5174E-03
7.0000E-06	2.7192E-03
8.0000E-06	2.9069E-03
9.0000E-06	3.0832E-03
1.0000E-05	3.2500E-03
1.1000E-05	3.4086E-03
1.2000E-05	3.5602E-03
1.3000E-05	3.7056E-03
1.4000E-05	3.8455E-03
1.5000E-05	3.9804E-03
1.6000E-05	4.1110E-03
1.7000E-05	4.2375E-03
1.8000E-05	4.3603E-03
1.9000E-05	4.4796E-03
2.0000E-05	4.5962E-03
2.1000E-05	4.7097E-03
2.7900E-05	8.0700E-03

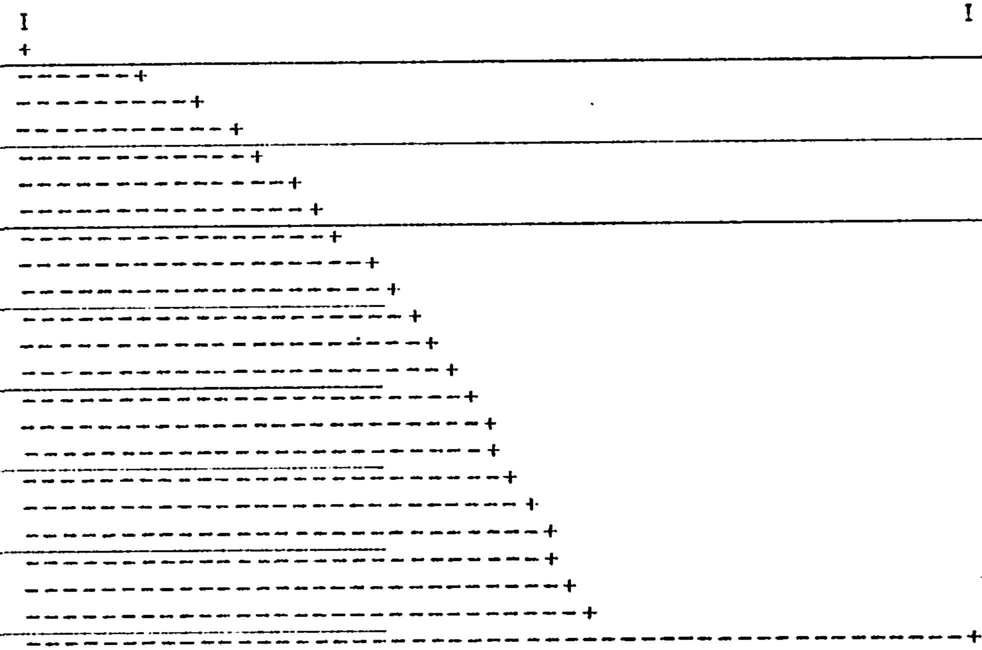


Fig. 46

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = 1.0$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

```
NUSSELT NUMBER/REYNOLDS NUMBER                                                  PAGE    1

                              MINIMUM                NU      VERSUS X                  MAXIMUM
                              0.0                    WM   =  3.3300E-01                8.0000E-03
     X              NU          I                                                        I
     0.0            0.0         +
  1.0000E-06     1.0041E-03   -------+
  2.0000E-06     1.4196E-03   ---------+
  3.0000E-06     1.7386E-03   -----------+
  4.0000E-06     2.0074E-03   -------------+
  5.0000E-06     2.2444E-03   ---------------+
  6.0000E-06     2.4585E-03   ----------------+
  7.0000E-06     2.6555E-03   -----------------+
  8.0000E-06     2.8388E-03   ------------------+
  9.0000E-06     3.0110E-03   -------------------+
  1.0000E-05     3.1739E-03   --------------------+
  1.1000E-05     3.3288E-03   ---------------------+
  1.2000E-05     3.4768E-03   ----------------------+
  1.3000E-05     3.6188E-03   -----------------------+
  1.4000E-05     3.7554E-03   ------------------------+
  1.5000E-05     3.8872E-03   -------------------------+
  1.6000E-05     4.0147E-03   --------------------------+
  1.7000E-05     4.1382E-03   --------------------------+
  1.8000E-05     4.2582E-03   ---------------------------+
  1.9000E-05     4.3749E-03   ----------------------------+
  2.0000E-05     4.4885E-03   -----------------------------+
  2.1000E-05     4.5994E-03   -----------------------------+
  2.2000E-05     8.0000E-03   ----------------------------------------------------+
```

Fig. 47

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = 1.5$, $B = -0.2$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 NU VERSUS X WM = 3.3300E-01 MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.1081E-03
2.0000E-06	1.6514E-03
3.0000E-06	2.0224E-03
4.0000E-06	2.3352E-03
5.0000E-06	2.6108E-03
6.0000E-06	2.8600E-03
7.0000E-06	3.0891E-03
8.0000E-06	3.3024E-03
9.0000E-06	3.5027E-03
1.0000E-05	3.6922E-03
1.1000E-05	3.8724E-03
1.2000E-05	4.0446E-03
1.3000E-05	4.2098E-03
1.4000E-05	4.3687E-03
1.5000E-05	4.5220E-03
1.6000E-05	4.6703E-03
1.7000E-05	4.8140E-03
1.8000E-05	4.9536E-03
1.9000E-05	5.0893E-03
2.0000E-05	5.2216E-03
2.1000E-05	5.3505E-03
2.2000E-05	8.0000E-03

Fig. 48

LOCAL NUSSELT NUMBER FOR $Pr_\infty = 15.0$, $m = 0.333$, $A = 1.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

NU VERSUS X

WM = 3.3300E-01

MINIMUM 0.0 — MAXIMUM 8.0000E-03

X	NU
0.C	0.0
1.0000E-06	1.2885E-03
2.0000E-06	1.8216E-03
3.0000E-06	2.2308E-03
4.0000E-06	2.5758E-03
5.0000E-06	2.8799E-03
6.0000E-06	3.1547E-03
7.0000E-06	3.4075E-03
8.0000E-06	3.6427E-03
9.0000E-06	3.8637E-03
1.0000E-05	4.0727E-03
1.1000E-05	4.2715E-03
1.2000E-05	4.4614E-03
1.3000E-05	4.6436E-03
1.4000E-05	4.8189E-03
1.5000E-05	4.9880E-03
1.6000E-05	5.1516E-03
1.7000E-05	5.3102E-03
1.8000E-05	5.4641E-03
1.9000E-05	5.6138E-03
2.0000E-05	5.7597E-03
2.1000E-05	5.9019E-03
2.2000E-05	8.0000E-03

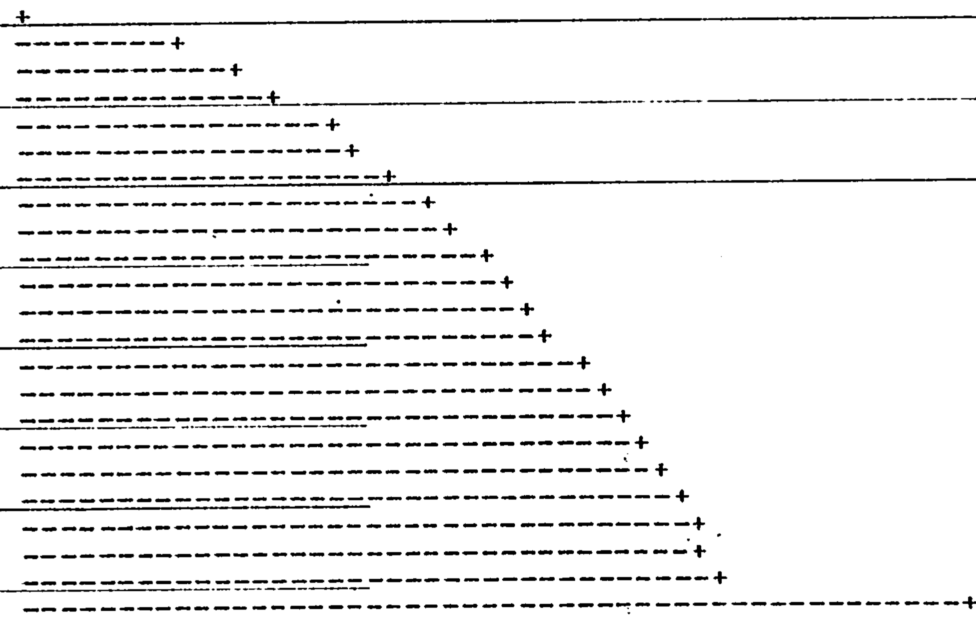


Fig. 49

LOCAL NUSSELT NUMBER FOR $Pr_\infty = 15.0$, $m = 0.333$, $A = 0.0$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER — PAGE 1

NU VERSUS X — WM = 3.3300E-01

MINIMUM 0.0 — MAXIMUM 8.0000E-03

X	NU
0.0	0.0
1.0000E-06	1.3841E-03
2.0000E-06	1.9565E-03
3.0000E-06	2.3959E-03
4.0000E-06	2.7664E-03
5.0000E-06	3.0929E-03
6.0000E-06	3.3881E-03
7.0000E-06	3.6595E-03
8.0000E-06	3.9121E-03
9.0000E-06	4.1494E-03
1.0000E-05	4.3738E-03
1.1000E-05	4.5873E-03
1.2000E-05	4.7913E-03
1.3000E-05	4.9869E-03
1.4000E-05	5.1752E-03
1.5000E-05	5.3568E-03
1.6000E-05	5.5325E-03
1.7000E-05	5.7028E-03
1.8000E-05	5.8681E-03
1.9000E-05	6.0289E-03
2.0000E-05	6.1855E-03
2.1000E-05	6.3383E-03
2.2000E-05	8.0000E-03

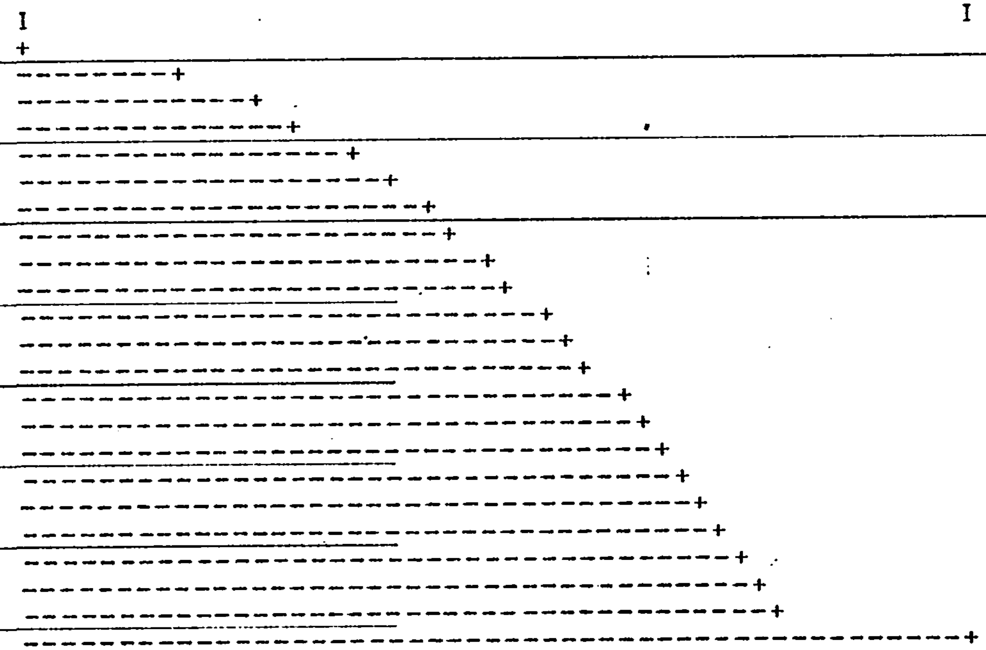


Fig. 50

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = -0.5$, $B = 0.0$

CONSTANT WALL TEMPERATURE CASE

NUSSELT NUMBER/REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — NU VERSUS X — MAXIMUM 8.0000E-03

WM = 3.3300E-01

X	NU
0.0	0.0
1.0000E-06	1.5751E-03
2.0000E-06	2.2265E-03
3.0000E-06	2.7266E-03
4.0000E-06	3.1482E-03
5.0000E-06	3.5197E-03
6.0000E-06	3.8556E-03
7.0000E-06	4.1646E-03
8.0000E-06	4.4521E-03
9.0000E-06	4.7221E-03
1.0000E-05	4.9776E-03
1.1000E-05	5.2205E-03
1.2000E-05	5.4526E-03
1.3000E-05	5.6753E-03
1.4000E-05	5.8895E-03
1.5000E-05	6.0962E-03
1.6000E-05	6.2962E-03
1.7000E-05	6.4899E-03
1.8000E-05	6.6781E-03
1.9000E-05	6.8611E-03
2.0000E-05	7.0393E-03
2.1000E-05	7.2132E-03
2.2000E-05	8.0000E-03

Fig. 51

LOCAL NUSSELT NUMBER FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = -0.5$, $B = 0.2$

CONSTANT WALL TEMPERATURE CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                          PAGE    1
                                     MINIMUM                  FRE     VERSUS X                  MAXIMUM
                                       0.0                    WM      = 0.0                   4.0000E 03
   X                  FRE                I                                                          I
 0.0                4.0000E 03        --------------------------------------------------------------+
 1.0000E-06         1.4036E 03        -----------------+
 2.0000E-06         9.9224E 02        --------------+
 3.0000E-06         8.1007E 02        -----------+
 4.0000E-06         7.0149E 02        ---------+
 5.0000E-06         6.2740E 02        --------+
 6.0000E-06         5.7271E 02        --------+
 7.0000E-06         5.3021E 02        -------+
 8.0000E-06         4.9595E 02        -------+
 9.0000E-06         4.6757E 02        ------+
 1.0000E-05         4.4356E 02        ------+
 1.1000E-05         4.2291E 02        ------+
 1.2000E-05         4.0490E 02        ------+
 1.3000E-05         3.8901E 02        -----+
 1.4000E-05         3.7487E 02        -----+
 1.5000E-05         3.6217E 02        -----+
 1.6000E-05         3.5068E 02        -----+
 1.7000E-05         3.4022E 02        -----+
 1.8000E-05         3.3062E 02        -----+
 1.9000E-05         3.2179E 02        -----+
 2.0000E-05         3.1364E 02        ---+
 2.1000E-05         3.0607E 02        ---+
 2.2000E-05         0.0               +
```

Fig. 52

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 1.0$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER — PAGE 1

FRE VERSUS X — WM = 0.0 — MINIMUM 0.0 — MAXIMUM 4.0000E 03

X	FRE
0.0	4.0000E 03
1.0000E-06	1.7544E 03
2.0000E-06	1.2402E 03
3.0000E-06	1.0125E 03
4.0000E-06	8.7675E 02
5.0000E-06	7.8414E 02
6.0000E-06	7.1578E 02
7.0000E-06	6.6265E 02
8.0000E-06	6.1982E 02
9.0000E-06	5.8435E 02
1.0000E-05	5.5435E 02
1.1000E-05	5.2853E 02
1.2000E-05	5.0601E 02
1.3000E-05	4.8615E 02
1.4000E-05	4.6846E 02
1.5000E-05	4.5257E 02
1.6000E-05	4.3819E 02
1.7000E-05	4.2511E 02
1.8000E-05	4.1313E 02
1.9000E-05	4.0211E 02
2.0000E-05	3.9192E 02
2.1000E-05	3.8248E 02
2.2000E-05	0.0

Fig. 53

LOCAL FRICTION FACTOR FOR $Pr_\infty = 15.0$, $m = 0.0$, $A = 1.5$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 FRE VERSUS X WM = 0.0 MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	--+
1.0000E-06	1.7544E 03	--------------------------+
2.0000E-06	1.2402E 03	-------------------+
3.0000E-06	1.0125E 03	---------------+
4.0000E-06	8.7675E 02	-------------+
5.0000E-06	7.8414E 02	------------+
6.0000E-06	7.1578E 02	-----------+
7.0000E-06	6.6265E 02	-----------+
8.0000E-06	6.1982E 02	---------+
9.0000E-06	5.8435E 02	---------+
1.0000E-05	5.5435E 02	--------+
1.1000E-05	5.2853E 02	--------+
1.2000E-05	5.0601E 02	--------+
1.3000E-05	4.8615E 02	--------+
1.4000E-05	4.6846E 02	-------+
1.5000E-05	4.5257E 02	-------+
1.6000E-05	4.3819E 02	-------+
1.7000E-05	4.2511E 02	-------+
1.8000E-05	4.1313E 02	-------+
1.9000E-05	4.0210E 02	-------+
2.0000E-05	3.9192E 02	-----+
2.1000E-05	3.8247E 02	-----+
2.2000E-05	0.0	+

Fig. 54

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 1.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — FRE VERSUS X — WM = 0.0 — MAXIMUM 4.0000E 03

X	FRE	I
0.0	4.0000E 03	--+
1.0000E-06	7.0191E 02	---------+
2.0000E-06	4.9621E 02	-------+
3.0000E-06	4.0512E 02	------+
4.0000E-06	3.5083E 02	-----+
5.0000E-06	3.1379E 02	----+
6.0000E-06	2.8644E 02	----+
7.0000E-06	2.6519E 02	----+
8.0000E-06	2.4806E 02	----+
9.0000E-06	2.3387E 02	---+
1.0000E-05	2.2187E 02	---+
1.1000E-05	2.1154E 02	---+
1.2000E-05	2.0254E 02	---+
1.3000E-05	1.9459E 02	---+
1.4000E-05	1.8751E 02	---+
1.5000E-05	1.8116E 02	---+
1.6000E-05	1.7541E 02	---+
1.7000E-05	1.7017E 02	---+
1.8000E-05	1.6538E 02	---+
1.9000E-05	1.6097E 02	---+
2.0000E-05	1.5690E 02	--+
2.1000E-05	1.5312E 02	--+
2.2000E-05	0.0	+

Fig. 55

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = 0.0$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                        PAGE    1

                              MINIMUM                       FRE      VERSUS X                    MAXIMUM
                                0.0                         WM       = 0.0                       4.0000E 03
      X                 FRE              I                                                          I
   0.0               4.0000E 03          ----------------------------------------------------------+
   1.0000E-06        3.5098E 02          ----+
   2.0000E-06        2.4813E 02          ---+
   3.0000E-06        2.0258E 02          --+
   4.0000E-06        1.7544E 02          --+
   5.0000E-06        1.5692E 02          -+
   6.0000E-06        1.4324E 02          -+
   7.0000E-06        1.3262E 02          -+
   8.0000E-06        1.2405E 02          -+
   9.0000E-06        1.1696E 02          -+
   1.0000E-05        1.1096E 02          -+
   1.1000E-05        1.0579E 02          -+
   1.2000E-05        1.0129E 02          -+
   1.3000E-05        9.7318E 01          -+
   1.4000E-05        9.3783E 01          -+
   1.5000E-05        9.0609E 01          -+
   1.6000E-05        8.7727E 01          -+
   1.7000E-05        8.5121E 01          -+
   1.8000E-05        8.2727E 01          -+
   1.9000E-05        8.0524E 01          -+
   2.0000E-05        7.8488E 01          +
   2.1000E-05        7.6599E 01          +
   2.2000E-05        0.0                 +
```

Fig. 56

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = -0.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 FRE VERSUS X WM = 0.0 MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	---+
1.0000E-06	3.5098E 02	----+
2.0000E-06	2.4813E 02	---+
3.0000E-06	2.0258E 02	--+
4.0000E-06	1.7544E 02	--+
5.0000E-06	1.5692E 02	-+
6.0000E-06	1.4324E 02	-+
7.0000E-06	1.3262E 02	-+
8.0000E-06	1.2405E 02	-+
9.0000E-06	1.1696E 02	-+
1.0000E-05	1.1096E 02	-+
1.1000E-05	1.0579E 02	-+
1.2000E-05	1.0129E 02	-+
1.3000E-05	9.7318E 01	-+
1.4000E-05	9.3783E 01	-+
1.5000E-05	9.0609E 01	-+
1.6000E-05	8.7737E 01	-+
1.7000E-05	8.5121E 01	-+
1.8000E-05	8.2727E 01	-+
1.9000E-05	8.0524E 01	-+
2.0000E-05	7.8488E 01	+
2.1000E-05	7.6599E 01	+
2.2000E-05	0.0	+

Fig. 57

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.0$, $A = -0.5$, $B = 0.2$

CONSTANT WALL HEAT FLUX CASE

```
 FRICTION FACTOR*REYNOLDS NUMBER                                                          PAGE    1

                                    MINIMUM                      FRE    VERSUS X                          MAXIMUM
                                      0.0                        WM     = 1.1100E-01                   4.0000E 03
        X                FRE            I                                                                  I
    0.0              4.0000E 03   ------------------------------------------------------------------------+
    1.0000E-06       1.8950E 03   ---------------------------------+
    2.0000E-06       1.3399E 03   ------------------------+
    3.0000E-06       1.0940E 03   --------------------+
    4.0000E-06       9.4739E 02   -----------------+
    5.0000E-06       8.4734E 02   ---------------+
    6.0000E-06       7.7349E 02   --------------+
    7.0000E-06       7.1609E 02   -------------+
    8.0000E-06       6.6982E 02   -------------+
    9.0000E-06       6.3150E 02   ------------+
    1.0000E-05       5.9908E 02   ------------+
    1.1000E-05       5.7118E 02   ------------+
    1.2000E-05       5.4685E 02   -----------+
    1.3000E-05       5.2539E 02   -----------+
    1.4000E-05       5.0627E 02   -----------+
    1.5000E-05       4.8909E 02   -----------+
    1.6000E-05       4.7355E 02   ----------+
    1.7000E-05       4.5941E 02   ----------+
    1.8000E-05       4.4645E 02   ----------+
    1.9000E-05       4.3454E 02   ----------+
    2.0000E-05       4.2352E 02   ----------+
    2.1000E-05       4.1331E 02   ----------+
    2.2000E-05       0.0          +
```

Fig. 58

LOCAL FRICTION FACTOR FOR Pr_{∞} = 15.0, m = 0.111, A = 1.0, B = - 0.2

CONSTANT WALL HEAT FLUX CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                     PAGE    1

                              MINIMUM                  FRE    VERSUS X                MAXIMUM
                                0.0                    WM     = 1.1100E-01            4.0000E 03
    X              FRE              I                                                      I
   0.0           4.0000E 03     -----------------------------------------------------+
   1.0000E-06    2.3686E 03     -------------------------------+
   2.0000E-06    1.6747E 03     ----------------------+
   3.0000E-06    1.3673E 03     ------------------+
   4.0000E-06    1.1841E 03     ---------------+
   5.0000E-06    1.0590E 03     --------------+
   6.0000E-06    9.6668E 02     -------------+
   7.0000E-06    8.9493E 02     ------------+
   8.0000E-06    8.3710E 02     -----------+
   9.0000E-06    7.8919E 02     ----------+
   1.0000E-05    7.4867E 02     ----------+
   1.1000E-05    7.1380E 02     ---------+
   1.2000E-05    6.8339E 02     ---------+
   1.3000E-05    6.5656E 02     ---------+
   1.4000E-05    6.3266E 02     --------+
   1.5000E-05    6.1119E 02     --------+
   1.6000E-05    5.9176E 02     --------+
   1.7000E-05    5.7408E 02     --------+
   1.8000E-05    5.5789E 02     -------+
   1.9000E-05    5.4300E 02     -------+
   2.0000E-05    5.2923E 02     -------+
   2.1000E-05    5.1647E 02     -------+
   2.2000E-05    0.0            +
```

Fig. 59

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = 1.5$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — FRE VERSUS X, WM = 1.1100E-01 — MAXIMUM 4.0000E 03

X	FRE
0.0	4.0000E 03
1.0000E-06	2.3686E 03
2.0000E-06	1.6747E 03
3.0000E-06	1.3673E 03
4.0000E-06	1.1841E 03
5.0000E-06	1.0590E 03
6.0000E-06	9.6668E 02
7.0000E-06	8.9493E 02
8.0000E-06	8.3710E 02
9.0000E-06	7.8919E 02
1.0000E-05	7.4867E 02
1.1000E-05	7.1380E 02
1.2000E-05	6.8339E 02
1.3000E-05	6.5656E 02
1.4000E-05	6.3266E 02
1.5000E-05	6.1119E 02
1.6000E-05	5.9176E 02
1.7000E-05	5.7408E 02
1.8000E-05	5.5789E 02
1.9000E-05	5.4299E 02
2.0000E-05	5.2923E 02
2.1000E-05	5.1647E 02
2.2000E-05	0.0

Fig. 60

LOCAL FRICTION FACTOR FOR Pr_{∞} = 15.0, m = 0.111, A = 1.5, B = 0.0

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

FRE VERSUS X

WM = 1.1100E-01

MINIMUM 0.0 MAXIMUM 4.0000E 03

X	FRE
0.0	4.0000E 03
1.0000E-06	9.4765E 02
2.0000E-06	6.7009E 02
3.0000E-06	5.4712E 02
4.0000E-06	4.7382E 02
5.0000E-06	4.2380E 02
6.0000E-06	3.8688E 02
7.0000E-06	3.5818E 02
8.0000E-06	3.3504E 02
9.0000E-06	3.1588E 02
1.0000E-05	2.9967E 02
1.1000E-05	2.8573E 02
1.2000E-05	2.7356E 02
1.3000E-05	2.6283E 02
1.4000E-05	2.5327E 02
1.5000E-05	2.4468E 02
1.6000E-05	2.3691E 02
1.7000E-05	2.2984E 02
1.8000E-05	2.2336E 02
1.9000E-05	2.1741E 02
2.0000E-05	2.1190E 02
2.1000E-05	2.0679E 02
2.2000E-05	0.0

Fig. 61

LOCAL FRICTION FACTOR FOR $Pr_\infty = 15.0$, $m = 0.111$, $A = 0.6$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 FRE VERSUS X WM = 1.1100E-01 MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	--+
1.0000E-06	4.7386E 02	------+
2.0000E-06	3.3509E 02	-----+
3.0000E-06	2.7361E 02	---+
4.0000E-06	2.3695E 02	--+
5.0000E-06	2.1194E 02	--+
6.0000E-06	1.9348E 02	--+
7.0000E-06	1.7913E 02	--+
8.0000E-06	1.6756E 02	--+
9.0000E-06	1.5798E 02	-+
1.0000E-05	1.4988E 02	-+
1.1000E-05	1.4290E 02	-+
1.2000E-05	1.3682E 02	-+
1.3000E-05	1.3145E 02	-+
1.4000E-05	1.2667E 02	-+
1.5000E-05	1.2238E 02	-+
1.6000E-05	1.1850E 02	-+
1.7000E-05	1.1496E 02	-+
1.8000E-05	1.1172E 02	-+
1.9000E-05	1.0874E 02	-+
2.0000E-05	1.0599E 02	-+
2.1000E-05	1.0344E 02	-+
2.2000E-05	0.0	+

Fig. 62

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = -0.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 FRE VERSUS X MAXIMUM 4.0000E 03

WM = 1.1100E-01

X	FRE	I ... I
0.0	4.0000E 03	---+
1.0000E-06	4.7386E 02	------+
2.0000E-06	3.3509E 02	-----+
3.0000E-06	2.7361E 02	---+
4.0000E-06	2.3695E 02	--+
5.0000E-06	2.1194E 02	--+
6.0000E-06	1.9348E 02	--+
7.0000E-06	1.7913E 02	--+
8.0000E-06	1.6756E 02	--+
9.0000E-06	1.5798E 02	-+
1.0000E-05	1.4988E 02	-+
1.1000E-05	1.4290E 02	-+
1.2000E-05	1.3682E 02	-+
1.3000E-05	1.3146E 02	-+
1.4000E-05	1.2667E 02	-+
1.5000E-05	1.2238E 02	-+
1.6000E-05	1.1850E 02	-+
1.7000E-05	1.1496E 02	-+
1.8000E-05	1.1172E 02	-+
1.9000E-05	1.0874E 02	-+
2.0000E-05	1.0599E 02	-+
2.1000E-05	1.0344E 02	-+
2.2000E-05	0.0	+

Fig. 63

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.111$, $A = -0.5$, $B = 0.2$

CONSTANT WALL HEAT FLUX CASE

```
  FRICTION FACTOR*REYNOLDS NUMBER                                                      PAGE    1

                                  MINIMUM                   FRE    VERSUS X                      MAXIMUM
                                  0.0                       WM     = 2.2200E-01                  4.0000E 03
      X                FRE        I                                                                 I
   0.0              4.0000E 03    -------------------------------------------------------------+
   1.0000E-06       2.2834E 03    -----------------------------------+
   2.0000E-06       1.6145E 03    -------------------------+
   3.0000E-06       1.3182E 03    --------------------+
   4.0000E-06       1.1415E 03    -----------------+
   5.0000E-06       1.0210E 03    ---------------+
   6.0000E-06       9.3200E 02    --------------+
   7.0000E-06       8.6283E 02    -------------+
   8.0000E-06       8.0708E 02    -------------+
   9.0000E-06       7.6091E 02    I------------+
   1.0000E-05       7.2184E 02    ------------+
   1.1000E-05       6.8823E 02    -----------+
   1.2000E-05       6.5892E 02    -----------+
   1.3000E-05       6.3305E 02    ---------+
   1.4000E-05       6.1001E 02    ---------+
   1.5000E-05       5.8932E 02    ---------+
   1.6000E-05       5.7059E 02    ---------+
   1.7000E-05       5.5354E 02    --------+
   1.8000E-05       5.3794E 02    --------+
   1.9000E-05       5.2358E 02    --------+
   2.0000E-05       5.1031E 02    --------+
   2.1000E-05       4.9801E 02    --------+
   2.2000E-05       0.0           +
```

Fig. 64

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.0$, $B = - 0.2$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

FRE VERSUS X

WM = 2.2200E-01

MINIMUM 0.0 MAXIMUM 4.0000E 03

X	FRE
0.0	4.0000E 03
1.0000E-06	2.8540E 03
2.0000E-06	2.0179E 03
3.0000E-06	1.6475E 03
4.0000E-06	1.4267E 03
5.0000E-06	1.2760E 03
6.0000E-06	1.1648E 03
7.0000E-06	1.0783E 03
8.0000E-06	1.0086E 03
9.0000E-06	9.5091E 02
1.0000E-05	9.0208E 02
1.1000E-05	8.6006E 02
1.2000E-05	8.2342E 02
1.3000E-05	7.9109E 02
1.4000E-05	7.6229E 02
1.5000E-05	7.3642E 02
1.6000E-05	7.1301E 02
1.7000E-05	6.9171E 02
1.8000E-05	6.7220E 02
1.9000E-05	6.5425E 02
2.0000E-05	6.3767E 02
2.1000E-05	6.2228E 02
2.2000E-05	0.0

Fig. 65

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.5$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 FRE VERSUS X WM = 2.2200E-01 MAXIMUM 4.0000E 03

X	FRE
0.0	4.0000E 03
1.0000E-06	2.8540E 03
2.0000E-06	2.0179E 03
3.0000E-06	1.6475E 03
4.0000E-06	1.4267E 03
5.0000E-06	1.2760E 03
6.0000E-06	1.1648E 03
7.0000E-06	1.0783E 03
8.0000E-06	1.0086E 03
9.0000E-06	9.5051E 02
1.0000E-05	9.0208E 02
1.1000E-05	8.6006E 02
1.2000E-05	8.2342E 02
1.3000E-05	7.9109E 02
1.4000E-05	7.6229E 02
1.5000E-05	7.3642E 02
1.6000E-05	7.1301E 02
1.7000E-05	6.9170E 02
1.8000E-05	6.7219E 02
1.9000E-05	6.5425E 02
2.0000E-05	6.3766E 02
2.1000E-05	6.2228E 02
2.2000E-05	0.0

Fig. 66

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 1.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

```
     FRICTION FACTOR*REYNOLDS NUMBER                                                        PAGE    1
                                   MINIMUM                FRE     VERSUS X                    MAXIMUM
                                   0.0                    WM      = 2.2200E-01.              4.0000E 03
      X               FRE           I                                                          I
     0.0              4.0000E 03    -----------------------------------------------------------+
     1.0000E-06       1.1419E 03    ---------------+
     2.0000E-06       8.0743E 02    ------------+
     3.0000E-06       6.5927E 02    ----------+
     4.0000E-06       5.7095E 02    ---------+
     5.0000E-06       5.1068E 02    --------+
     6.0000E-06       4.6619E 02    ------+
     7.0000E-06       4.3161E 02    ------+
     8.0000E-06       4.0373E 02    ------+
     9.0000E-06       3.8064E 02    -----+
     1.0000E-05       3.6111E 02    -----+
     1.1000E-05       3.4430E 02    -----+
     1.2000E-05       3.2965E 02    -----+
     1.3000E-05       3.1671E 02    ----+
     1.4000E-05       3.0519E 02    ----+
     1.5000E-05       2.9485E 02    ----+
     1.6000E-05       2.8548E 02    ----+
     1.7000E-05       2.7696E 02    ----+
     1.8000E-05       2.6916E 02    ----+
     1.9000E-05       2.6198E 02    ----+
     2.0000E-05       2.5534E 02    ----+
     2.1000E-05       2.4919E 02    ----+
     2.2000E-05       0.0           +
```

Fig. 67

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.222$, $A = 0.0$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                                PAGE    1

                                     MINIMUM                    FRE     VERSUS X                      MAXIMUM
                                     0.0                        WM     = 2.2200E-01                   4.0000E 03
     X                   FRE         I                                                                 I
   0.0                4.0000E 03    --------------------------------------------------------------------+
   1.0000E-06         5.7058E 02    --------+
   2.0000E-06         4.0376E 02    ------+
   3.0000E-06         3.2968E 02    ----+
   4.0000E-06         2.8552E 02    ---+
   5.0000E-06         2.5538E 02    ---+
   6.0000E-06         2.3314E 02    --+
   7.0000E-06         2.1585E 02    --+
   8.0000E-06         2.0191E 02    --+
   9.0000E-06         1.9037E 02    --+
   1.0000E-05         1.8060E 02    --+
   1.1000E-05         1.7220E 02    --+
   1.2000E-05         1.6487E 02    --+
   1.3000E-05         1.5840E 02    -+
   1.4000E-05         1.5264E 02    -+
   1.5000E-05         1.4747E 02    -+
   1.6000E-05         1.4278E 02    -+
   1.7000E-05         1.3852E 02    -+
   1.8000E-05         1.3462E 02    -+
   1.9000E-05         1.3103E 02    -+
   2.0000E-05         1.2772E 02    -+
   2.1000E-05         1.2464E 02    -+
   2.2000E-05         0.0           +
```

Fig. 68

LOCAL FRICTION FACTOR FOR $Pr_\infty = 15.0$, $m = 0.222$, $A = -0.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — FRE VERSUS X — WM = 2.2200E-01 — MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	---+
1.0000E-06	5.7098E 02	--------+
2.0000E-06	4.0376E 02	------+
3.0000E-06	3.2968E 02	-----+
4.0000E-06	2.8552E 02	---+
5.0000E-06	2.5538E 02	---+
6.0000E-06	2.3314E 02	--+
7.0000E-06	2.1585E 02	--+
8.0000E-06	2.0191E 02	--+
9.0000E-06	1.9037E 02	--+
1.0000E-05	1.8060E 02	--+
1.1000E-05	1.7220E 02	--+
1.2000E-05	1.6487E 02	--+
1.3000E-05	1.5840E 02	-+
1.4000E-05	1.5264E 02	-+
1.5000E-05	1.4747E 02	-+
1.6000E-05	1.4278E 02	-+
1.7000E-05	1.3852E 02	-+
1.8000E-05	1.3462E 02	-+
1.9000E-05	1.3103E 02	-+
2.0000E-05	1.2772E 02	-+
2.1000E-05	1.2464E 02	-+
2.2000E-05	0.0	+

Fig. 69

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, m = 0.222, A = - 0.5, B = 0.2

CONSTANT WALL HEAT FLUX CASE

```
     FRICTION FACTOR*REYNOLDS NUMBER                                                    PAGE    1

                                   MINIMUM                        FRE     VERSUS X                     MAXIMUM
                                   0.0                            WM      = 3.3300E-01                 4.0000E 03
      X                 FRE             I                                                                   I
  0.0               4.0000E 03      ------------------------------------------------------------------------+
  1.0000E-06        2.6147E 03      -----------------------------------------------+
  2.0000E-06        1.8488E 03      ----------------------------------+
  3.0000E-06        1.5094E 03      ---------------------------+
  4.0000E-06        1.3071E 03      ------------------------+
  5.0000E-06        1.1691E 03      ---------------------+
  6.0000E-06        1.0672E 03      --------------------+
  7.0000E-06        9.8802E 02      ------------------+
  8.0000E-06        9.2418E 02      -----------------+
  9.0000E-06        8.7130E 02      ---------------+
  1.0000E-05        8.2657E 02      ---------------+
  1.1000E-05        7.8809E 02      --------------+
  1.2000E-05        7.5452E 02      --------------+
  1.3000E-05        7.2490E 02      --------------+
  1.4000E-05        6.9851E 02      ------------+
  1.5000E-05        6.7481E 02      ------------+
  1.6000E-05        6.5337E 02      ------------+
  1.7000E-05        6.3385E 02      -----------+
  1.8000E-05        6.1599E 02      -----------+
  1.9000E-05        5.9955E 02      -----------+
  2.0000E-05        5.8435E 02      -----------+
  2.1000E-05        5.7026E 02      -----------+
  2.2000E-05        0.0             +
```

Fig. 70

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = 1.0$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                    PAGE    1

                              MINIMUM                     FRE      VERSUS  X                MAXIMUM
                                0.0                       WM     = 3.3300E-01              4.0000E 03
     X                FRE           I                                                          I
   0.0              4.0000E 03   ---------------------------------------------------------------+
   1.0000E-06       3.2681E 03   -----------------------------------------------------+
   2.0000E-06       2.3107E 03   ---------------------------------------+
   3.0000E-06       1.8865E 03   ---------------------------------+
   4.0000E-06       1.6327E 03   -----------------------------+
   5.0000E-06       1.4612E 03   ---------------------------+
   6.0000E-06       1.3328E 03   -------------------------+
   7.0000E-06       1.2348E 03   -----------------------+
   8.0000E-06       1.1550E 03   ----------------------+
   9.0000E-06       1.0889E 03   ---------------------+
   1.0000E-05       1.0329E 03   --------------------+
   1.1000E-05       9.8484E 02   --------------------+
   1.2000E-05       9.4289E 02   -------------------+
   1.3000E-05       9.0587E 02   -------------------+
   1.4000E-05       8.7288E 02   ------------------+
   1.5000E-05       8.4326E 02   ------------------+
   1.6000E-05       8.1646E 02   ------------------+
   1.7000E-05       7.9206E 02   -----------------+
   1.8000E-05       7.6972E 02   -----------------+
   1.9000E-05       7.4917E 02   -----------------+
   2.0000E-05       7.3018E 02   -----------------+
   2.1000E-05       7.1256E 02   ----------------+
   2.2000E-05       0.0          +
```

Fig. 71

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = 1.5$, $B = -0.2$

CONSTANT WALL HEAT FLUX CASE

```
FRICTION FACTOR*REYNOLDS NUMBER                                                      PAGE    1

                                  MINIMUM                  FRE     VERSUS X                  MAXIMUM
                                   0.0                     WM      = 3.3300E-01              4.0000E 03
      X                  FRE          I                                                          I
   0.0               4.0000E 03   ----------------------------------------------------------------+
   1.0000E-06        3.2681E 03   ------------------------------------------------+
   2.0000E-06        2.3107E 03   ----------------------------------+
   3.0000E-06        1.8865E 03   ---------------------------+
   4.0000E-06        1.6337E 03   -----------------------+
   5.0000E-06        1.4612E 03   ---------------------+
   6.0000E-06        1.3338E 03   -------------------+
   7.0000E-06        1.2348E 03   ------------------+
   8.0000E-06        1.1551E 03   ----------------+
   9.0000E-06        1.0889E 03   ---------------+
   1.0000E-05        1.0329E 03   ---------------+
   1.1000E-05        9.8484E 02   --------------+
   1.2000E-05        9.4289E 02   -------------+
   1.3000E-05        9.0586E 02   -------------+
   1.4000E-05        8.7288E 02   ------------+
   1.5000E-05        8.4325E 02   ------------+
   1.6000E-05        8.1646E 02   ------------+
   1.7000E-05        7.9206E 02   -----------+
   1.8000E-05        7.6972E 02   -----------+
   1.9000E-05        7.4916E 02   -----------+
   2.0000E-05        7.3017E 02   -----------+
   2.1000E-05        7.1256E 02   ----------+
   2.2000E-05        0.0          +
```

Fig. 72

LOCAL FRICTION FACTOR FOR $Pr_\infty = 15.0$, $m = 0.333$, $A = 1.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 — FRE VERSUS X — WM = 3.3300E-01 — MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	---+
1.0000E-06	1.3076E 03	----------------+
2.0000E-06	9.2458E 02	------------+
3.0000E-06	7.5492E 02	----------+
4.0000E-06	6.5378E 02	---------+
5.0000E-06	5.8477E 02	--------+
6.0000E-06	5.3382E 02	-------+
7.0000E-06	4.9422E 02	-------+
8.0000E-06	4.6230E 02	------+
9.0000E-06	4.3586E 02	------+
1.0000E-05	4.1349E 02	------+
1.1000E-05	3.9425E 02	-----+
1.2000E-05	3.7747E 02	-----+
1.3000E-05	3.6266E 02	-----+
1.4000E-05	3.4947E 02	-----+
1.5000E-05	3.3762E 02	-----+
1.6000E-05	3.2690E 02	-----+
1.7000E-05	3.1714E 02	----+
1.8000E-05	3.0820E 02	----+
1.9000E-05	2.9998E 02	----+
2.0000E-05	2.9239E 02	----+
2.1000E-05	2.8534E 02	----+
2.2000E-05	0.0	+

Fig. 73

LOCAL FRICTION FACTOR FOR Pr_{∞} = 15.0, m = 0.333, A = 0.0, B = 0.0

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

FRE VERSUS X

WM = 3.3300E-01

MINIMUM 0.0 MAXIMUM 4.0000E 03

X	FRE	I ... I
0.0	4.0000E 03	---+
1.0000E-06	6.5383E 02	---------+
2.0000E-06	4.6235E 02	------+
3.0000E-06	3.7751E 02	-----+
4.0000E-06	3.2695E 02	-----+
5.0000E-06	2.9243E 02	----+
6.0000E-06	2.6696E 02	----+
7.0000E-06	2.4716E 02	----+
8.0000E-06	2.3121E 02	---+
9.0000E-06	2.1799E 02	---+
1.0000E-05	2.0681E 02	---+
1.1000E-05	1.9719E 02	---+
1.2000E-05	1.8879E 02	---+
1.3000E-05	1.8139E 02	---+
1.4000E-05	1.7479E 02	---+
1.5000E-05	1.6887E 02	---+
1.6000E-05	1.6351E 02	---+
1.7000E-05	1.5863E 02	--+
1.8000E-05	1.5416E 02	--+
1.9000E-05	1.5005E 02	--+
2.0000E-05	1.4625E 02	--+
2.1000E-05	1.4272E 02	--+
2.2000E-05	0.0	+

Fig. 74

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = -0.5$, $B = 0.0$

CONSTANT WALL HEAT FLUX CASE

FRICTION FACTOR*REYNOLDS NUMBER PAGE 1

MINIMUM 0.0 | FRE VERSUS X | WM = 3.3300E-01 | MAXIMUM 4.0000E 03

X	FRE	I
0.0	4.0000E 03	--+
1.0000E-06	6.5383E 02	---------+
2.0000E-06	4.6235E 02	------+
3.0000E-06	3.7751E 02	-----+
4.0000E-06	3.2695E 02	-----+
5.0000E-06	2.9243E 02	----+
6.0000E-06	2.6696E 02	----+
7.0000E-06	2.4717E 02	----+
8.0000E-06	2.3121E 02	---+
9.0000E-06	2.1799E 02	---+
1.0000E-05	2.0681E 02	---+
1.1000E-05	1.9719E 02	---+
1.2000E-05	1.8879E 02	---+
1.3000E-05	1.8139E 02	---+
1.4000E-05	1.7479E 02	---+
1.5000E-05	1.6887E 02	---+
1.6000E-05	1.6351E 02	---+
1.7000E-05	1.5863E 02	--+
1.8000E-05	1.5416E 02	--+
1.9000E-05	1.5005E 02	--+
2.0000E-05	1.4625E 02	--+
2.1000E-05	1.4272E 02	--+
2.2000E-05	0.0	+

Fig. 75

LOCAL FRICTION FACTOR FOR $Pr_{\infty} = 15.0$, $m = 0.333$, $A = -0.5$, $B = 0.2$

CONSTANT WALL HEAT FLUX CASE

TABLE - 2

CONDUCTANCE h_x RESULTS

#	m	$\left[\frac{h_x(\text{Case 2}) - h_x(\text{Case 1})}{h_x(\text{Case 1})}\right]$ Present Study	Available Solution [14]
1.	0.0	39% to 43%	about 36%
2.	0.111	35% to 39%	-
3.	0.222	33% to 35%	-
4.	0.333	30% to 32%	-

APPENDIX 'A'

DERIVATION OF THE COUPLED EQUATIONS FOR CONSTANT WALL TEMPERATURE

First, one rewrites Eq. (12)

$$
\begin{aligned}
E_T = \iint_S \Big[& -\rho u^{o2} \frac{\partial u}{\partial x} - \rho u^o v^o \frac{\partial u}{\partial y} + \frac{\mu^o}{2}\left(\frac{\partial u}{\partial y}\right)^2 \\
& + \rho c_p^o u^o T \frac{\partial T^o}{\partial x} + \rho c_p^o v^o T \frac{\partial T^o}{\partial y} + \frac{k^o}{2}\left(\frac{\partial T}{\partial y}\right)^2 \\
& + \frac{\partial p^o}{\partial x} u \Big] \, dx\, dy \\
& + \int_0^{\Delta} \left(\rho u^{o2} u \Big|_{x=\ell} - \rho u^{o2} u \Big|_{x=0} \right) dy \\
& + \int_0^{\ell} \left(\rho u^o v^o u \Big|_{y=\Delta} \right) dx + \int_0^{\ell} \left(-\mu^o \frac{\partial u^o}{\partial y} u \right)\Big|_{y=\Delta} dx \\
& + \int_0^{\ell} \left(-k^o \frac{\partial T^o}{\partial y} T \Big|_{y=\Delta_t} \right) dx + \int_0^{\ell} \left(k^o \frac{\partial T^o}{\partial y} T \right)\Big|_{y=0} dx
\end{aligned} \tag{12}
$$

Since

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3$$

it follows that

$$
\begin{aligned}
\frac{\partial u}{\partial x} = & \, u_\infty \left[-\frac{3}{2} \frac{y \Delta'}{\Delta^2} + \frac{3}{2} \frac{y^3 \Delta'}{\Delta^4} \right] \\
& + \frac{\partial u_\infty}{\partial x} \left[\frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \right]
\end{aligned} \tag{A-1}
$$

and
$$\frac{\partial u}{\partial y} = u_\infty \left[\frac{3}{2}\,\frac{1}{\Delta} - \frac{3}{2}\,\frac{y^2}{\Delta^3} \right] \qquad \text{(A-2)}$$

From the continuity equation

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

therefore

$$\begin{aligned} v &= \int_0^y \left(-\frac{\partial u}{\partial x} \right) dy \\ &= u_\infty \left[-\frac{3}{4}\,\frac{y^2 \Delta'}{\Delta^2} - \frac{3}{8}\,\frac{y^4 \Delta'}{\Delta^4} \right] \\ &\quad - \frac{\partial u_\infty}{\partial x} \left[\frac{3}{4}\,\frac{y^2}{\Delta} - \frac{1}{8}\,\frac{y^4}{\Delta^3} \right] \end{aligned} \qquad \text{(A-3)}$$

From Eq. (19a), one obtains

$$T = (T_\infty - T_w) \left[\frac{3}{2} \left(\frac{y}{\Delta_t} \right) - \frac{1}{2} \left(\frac{y}{\Delta_t} \right)^3 \right] + T_w \qquad \text{(A-4)}$$

and so

$$\frac{\partial T}{\partial y} = (T_\infty - T_w) \left[\frac{3}{2}\,\frac{1}{\Delta_t} - \frac{3}{2}\,\frac{y^2}{\Delta_t^3} \right] \qquad \text{(A-5)}$$

Similarly,

$$\frac{\partial T}{\partial x} = (T_\infty - T_w) \left[-\frac{3}{2} \left(\frac{y\,\Delta_t'}{\Delta_t^2} \right) + \frac{3}{2} \left(\frac{y^3 \Delta_t'}{\Delta_t^4} \right) \right] \qquad \text{(A-6)}$$

For wedge flow, the main stream velocity variation is given by

$$u_\infty = c\,x^m \qquad \text{(A-7)}$$

and differentiating gives

$$\frac{du_\infty}{dx} = \frac{m}{x} u_\infty \tag{A-7}$$

Applying Bernoulli's Eq. yields

$$m = \frac{x}{u_\infty}\frac{du_\infty}{dx} = -\frac{x}{\rho u_\infty^2}\frac{dp}{dx} \tag{A-8}$$

Eqs. (20a) and (21a) are

$$\mu = \mu_\infty (1 + A\theta)$$

$$k = k_\infty (1 + B\theta)$$

Substituting all the above expressions into Eq. (12), one obtains

$$E_T = \iint_S \Big[-\rho u_\infty^{o\,2} \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}$$

$$\left\{ u_\infty \left(-\frac{3}{2}\frac{y\Delta'}{\Delta^2} + \frac{3}{2}\frac{y^3\Delta'}{\Delta^4} \right) \right.$$

$$\left. + \frac{\partial u_\infty}{\partial x}\left(\frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\frac{y^3}{\Delta^3} \right) \right\}$$

$$- \rho u_\infty^o \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}$$

$$\left\{ u_\infty^o \left(\frac{3}{4}\frac{y^2\Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8}\frac{y^4\Delta^{o\prime}}{\Delta^{o4}} \right) \right.$$

$$\left. - \frac{\partial u_\infty^o}{\partial x}\left(\frac{3}{4}\frac{y^2}{\Delta^o} - \frac{1}{8}\frac{y^4}{\Delta^{o3}} \right) \right\} \left\{ u_\infty \left(\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2}\frac{y^2}{\Delta^3} \right) \right\}$$

$$+\left\{\frac{\mu_\infty^o - A\mu_\infty^o\left(\frac{3}{2}\frac{y}{\Delta_t^o} - \frac{y^3}{2\Delta_t^{o3}} - 1\right)}{2}\right\}$$

$$\left\{u_\infty^2\left(\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2}\frac{y^2}{\Delta^3}\right)^2\right\}$$

$$+\rho c_p^o u_\infty^o\left\{\frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3\right\}$$

$$\left\{(T_\infty - T_\omega)\left(\frac{3}{2}\frac{y}{\Delta_t} - \frac{1}{2}\frac{y^3}{\Delta_t^3}\right) + T_\omega\right\}$$

$$\left\{(T_\infty^o - T_\omega^o)\left(-\frac{3}{2}\frac{y\,\Delta_t^{o\prime}}{\Delta_t^{o2}} + \frac{3}{2}\frac{y^3\,\Delta_t^{o\prime}}{\Delta_t^{o4}}\right)\right\}$$

$$+\rho c_p^o\left\{u_\infty^o\left(\frac{3}{4}\frac{y^2\Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8}\frac{y^4\Delta^{o\prime}}{\Delta^{o4}}\right)\right.$$

$$\left. - \frac{\partial u_\infty^o}{\partial x}\left(\frac{3}{4}\frac{y^2}{\Delta^o} - \frac{1}{8}\frac{y^4}{\Delta^{o3}}\right)\right\}$$

$$\left\{(T_\infty - T_\omega)\left(\frac{3}{2}\frac{y}{\Delta_t} - \frac{1}{2}\frac{y^3}{\Delta_t^3}\right) + T_\omega\right\}$$

$$\left\{(T_\infty^o - T_\omega^o)\left(\frac{3}{2\Delta_t^o} - \frac{3}{2}\frac{y^2}{\Delta_t^{o3}}\right)\right\}$$

$$+\left\{\frac{k_\infty^o - Bk_\infty^o\left(\frac{3}{2}\frac{y}{\Delta_t^o} - \frac{y^3}{2\Delta_t^{o3}} - 1\right)}{2}\right\}(T_\infty - T_\omega)^2$$

$$\left\{\frac{3}{2\Delta_t} - \frac{3}{2}\frac{y^2}{\Delta_t^3}\right\}^2$$

$$- \rho \frac{u_\infty^{o\,2} m}{x^o} u_\infty \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \Big] dx\,dy$$

$$+ \int_0^\Delta \rho u_\infty^{o\,2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o} \right) - \frac{1}{2} \left(\frac{y}{\Delta^o} \right)^3 \right\}^2$$

$$u_\infty \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \Big|_{x=\ell} dy$$

$$- \int_0^\Delta \rho u_\infty^{o\,2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o} \right) - \frac{1}{2} \left(\frac{y}{\Delta^o} \right)^3 \right\}^2$$

$$u_\infty \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \Big|_{x=0} dy$$

$$+ \int_0^\ell \rho u_\infty^o \left\{ \frac{3}{2} \frac{y}{\Delta^o} - \frac{1}{2} \left(\frac{y}{\Delta^o} \right)^3 \right\}$$

$$\Big\{ u_\infty^o \left(\frac{3}{4} \frac{y^2 \Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8} \frac{y^4 \Delta^{o\prime}}{\Delta^{o4}} \right)$$

$$- \frac{\partial u_\infty^o}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^o} - \frac{1}{8} \frac{y^4}{\Delta^{o3}} \right) \Big\}$$

$$u_\infty \left\{ \frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \left(\frac{y}{\Delta} \right)^3 \right\} \Big|_{y=\Delta} dx$$

$$+ \int_0^\ell - k_\infty^o \left\{ 1 - B \left(\frac{3}{2} \frac{y}{\Delta_t^o} - \frac{y^3}{2\,\Delta_t^{o3}} - 1 \right) \right\}$$

$$\left(T_\infty^o - T_\omega^o \right) \left\{ \frac{3}{2\,\Delta_t^o} - \frac{3}{2\,\Delta_t^o} \left(\frac{y}{\Delta_t^o} \right)^2 \right\}$$

$$\left\{ \left(T_\infty - T_\omega \right) \left(\frac{3}{2} \frac{y}{\Delta_t} - \frac{1}{2} \frac{y^3}{\Delta_t^3} \right) + T_\omega \right\} \Big|_{y=\Delta_t} dx$$

$$+ \int_0^l k_\infty^0 \left\{ 1 - B \left(\frac{3}{2} \frac{y}{\Delta_t^0} - \frac{y^3}{2\Delta_t^{o3}} - 1 \right) \right\}$$

$$(T_\infty^0 - T_\omega^0) \left\{ \frac{3}{2\Delta_t^0} - \frac{3}{2\Delta_t^0} \left(\frac{y}{\Delta_t^0} \right)^3 \right\}$$

$$\left\{ (T_\infty - T_\omega) \left(\frac{3}{2} \frac{y}{\Delta_t} - \frac{1}{2} \frac{y^3}{\Delta_t^3} \right) + T_\omega \right\} \Big|_{y=0} dx \tag{A-9}$$

Taking the variation of E_T (Eq. (A-9)) with respect to Δ_t, one obtains

$$\delta E_T = \int_0^l \int_0^{\Delta_t} \Big[\rho c_p^0 u_\infty^0 \left\{ \frac{3}{2} \left(\frac{y}{\Delta^0} \right) - \frac{1}{2} \left(\frac{y}{\Delta^0} \right)^3 \right\}$$

$$\left\{ (T_\infty - T_\omega) \left(- \frac{3}{2} \frac{y}{\Delta_t^2} + \frac{3}{2} \frac{y^3}{\Delta_t^4} \right) \right\}$$

$$\left\{ (T_\infty^0 - T_\omega^0) \left(- \frac{3}{2} \frac{y \Delta_t^{o\prime}}{\Delta_t^{o2}} + \frac{3}{2} \frac{y^3 \Delta_t^{o\prime}}{\Delta_t^{o4}} \right) \right\}$$

$$+ \rho c_p^0 \left\{ u_\infty^0 \left(\frac{3}{4} \frac{y^2 \Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8} \frac{y^4 \Delta^{o\prime}}{\Delta^{o4}} \right) \right.$$

$$\left. - \frac{\partial u_\infty^0}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^0} - \frac{1}{8} \frac{y^4}{\Delta^{o3}} \right) \right\}$$

$$(T_\infty - T_\omega) \left(- \frac{3}{2} \frac{y}{\Delta_t^2} + \frac{3}{2} \frac{y^3}{\Delta_t^4} \right)$$

$$(T_\infty^0 - T_\omega^0) \left(\frac{3}{2\Delta_t^0} - \frac{3}{2} \frac{y^2}{\Delta_t^{o3}} \right)$$

$$+ \left\{ k_\infty^o - B k_\infty^o \left(\frac{3}{2} \frac{y}{\Delta_t^o} - \frac{y^3}{2 \Delta_t^{o3}} - 1 \right) \right\}$$

$$(T_\infty - T_\omega)^2 \left\{ \left(\frac{3}{2} \cdot \frac{1}{\Delta_t} - \frac{3}{2} \frac{y^2}{\Delta_t^3} \right) \right.$$

$$\left. \left(- \frac{3}{2 \Delta_t^2} - \frac{9}{2} \frac{y^2}{\Delta_t^4} \right) \right\} \delta \Delta_t \, dx \, dy \quad \text{(A-10)}$$

There is no longer any need to distinguish between the varied (Δ_t) and unvaried (Δ_t^o) versions of thermal boundary layer thickness. Dropping the superscript and integrating y from 0 to Δ_t, one obtains

$$\delta E_T = \int_0^l \left[\rho c_p u_\infty (T_\infty - T_\omega)^2 \left\{ \frac{9}{64} \frac{\Delta_t'}{\Delta} - \frac{3}{160} \frac{\Delta_t' \Delta_t^2}{\Delta^3} \right\} \right.$$

$$+ \rho c_p u_\infty (T_\infty - T_\omega)^2 \left\{ \frac{9}{640} \frac{\Delta' \Delta_t^3}{\Delta^4} - \frac{9}{128} \frac{\Delta' \Delta_t}{\Delta^2} \right\}$$

$$- \rho c_p \frac{\partial u_\infty}{\partial x} (T_\infty - T_\omega)^2 \left\{ - \frac{9}{128} \frac{\Delta_t}{\Delta} + \frac{3}{640} \frac{\Delta_t^3}{\Delta^3} \right\}$$

$$\left. + (T_\infty - T_\omega)^2 k_\infty \left\{ - \frac{3}{5 \Delta_t^2} - \frac{177}{320} B \frac{1}{\Delta_t^2} \right\} \right] \delta \Delta_t \, dx$$

(A-11)

Since δE_T must vanish for all $\delta \Delta_t$, therefore

$$\rho c_p u_\infty \left\{ \frac{9}{64} \frac{\Delta_t'}{\Delta} - \frac{3}{160} \frac{\Delta_t' \Delta_t^2}{\Delta^3} + \frac{9}{640} \frac{\Delta' \Delta_t^3}{\Delta^4} \right.$$

$$\left. - \frac{9}{128} \frac{\Delta' \Delta_t}{\Delta^2} \right\}$$

$$- \rho c_p \frac{\partial u_\infty}{\partial x} \left\{ - \frac{9}{128} \frac{\Delta_t}{\Delta} + \frac{3}{640} \frac{\Delta_t^3}{\Delta^3} \right\}$$

$$- \frac{3}{5 \Delta_t^2} k_\infty - \frac{177}{320} \frac{B}{\Delta_t^2} k_\infty = 0 \qquad \text{(A-12)}$$

Rearranging, introducing $Y = \frac{\Delta_t}{\Delta}$ and substituting the expression for $\frac{\partial u_\infty}{\partial x}$, Eq. (A-12) becomes

$$- \frac{3}{5} \alpha - \frac{177}{320} \alpha B + u_\infty \left\{ \frac{9}{128} (2 Y^2 \Delta^2 Y' + \Delta \Delta' Y^3) - \frac{3}{640} (4 Y^4 \Delta^2 Y' + Y^5 \Delta \Delta') \right\}$$

$$- \frac{m}{x} u_\infty \left(- \frac{9}{128} Y^3 \Delta^2 + \frac{3}{640} Y^5 \Delta^2 \right) = 0 \qquad \text{(A-13)}$$

Similarly, taking the variation of E_T(Eq. (A-9)) with respect to Δ gives

$$\delta E_T = \int_0^l \int_0^\Delta \left[- \rho u_\infty^{o\,2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o} \right) - \frac{1}{2} \left(\frac{y}{\Delta^o} \right)^3 \right\}^2 \right.$$

$$u_\infty \left\{ \left(\frac{3 y \Delta'}{\Delta^5} \right) \delta\Delta + \left(- \frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^2}{\Delta^4} \right) \delta\Delta' \right\}$$

$$- \rho u_\infty^{o\,2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o} \right) - \frac{1}{2} \left(\frac{y}{\Delta^o} \right)^3 \right\}^2$$

$$\frac{\partial u_\infty}{\partial x} \left(- \frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^3}{\Delta^4} \right) \delta\Delta$$

$$- \rho u_\infty^0 \left\{ \frac{3}{2} \left(\frac{y}{\Delta^0} \right) - \frac{1}{2} \left(\frac{y}{\Delta^0} \right)^3 \right\}$$

$$\left\{ u_\infty^0 \left(\frac{3}{4} \frac{y^2 \Delta^{0'}}{\Delta^{02}} - \frac{3}{8} \frac{y^4 \Delta^{0'}}{\Delta^{04}} \right) \right.$$

$$\left. - \frac{\partial u_\infty^0}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^0} - \frac{1}{8} \frac{y^4}{\Delta^{03}} \right) \right\}$$

$$\left\{ u_\infty \left(- \frac{3}{2} \frac{1}{\Delta^2} + \frac{9}{2} \frac{y^2}{\Delta^4} \right) \right\} \delta\Delta \Big] dx\, dy$$

$$+ \int_0^l \Big[\int_0^{\Delta_t} \left\{ \mu_\infty^0 - A \mu_\infty^0 \left(\frac{3}{2} \frac{y}{\Delta_t^0} - \frac{y^3}{2 \Delta_t^{03}} - 1 \right) \right\}$$

$$\left\{ u_\infty^2 \left(\frac{3}{2} \frac{1}{\Delta} - \frac{3}{2} \frac{y^2}{\Delta^3} \right) \right.$$

$$\left. \left(- \frac{3}{2} \frac{1}{\Delta^2} + \frac{9}{2} \frac{y^2}{\Delta^4} \right) \right\} \delta\Delta \, dy$$

$$+ \int_{\Delta_t}^{\Delta} \left\{ \mu_\infty^0 \, u_\infty^2 \left(\frac{3}{2} \frac{1}{\Delta} - \frac{3}{2} \frac{y^2}{\Delta^3} \right) \right\} \delta\Delta \, dy \Big] dx$$

$$- \int_0^l \int_0^{\Delta_t} \rho \, \frac{u_\infty^{0\,2} m}{x^0} \, u_\infty \left(- \frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^3}{\Delta^4} \right) \delta\Delta \, dx\, dy$$

$$+ \int_0^{\Delta} \rho u_\infty^{0\,2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^0} \right) - \frac{1}{2} \left(\frac{y}{\Delta^0} \right)^3 \right\}^2$$

$$u_\infty \left(- \frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^3}{\Delta^4} \right) \Big|_{x=l} \delta\Delta \, dy$$

$$- \int_0^{\Delta} \rho\, u_\infty^{o\,2} \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}^2$$

$$u_\infty \left(- \frac{3}{2}\, \frac{y}{\Delta^2} - \frac{1}{2}\, \frac{y^3}{\Delta^4} \right) \Big|_{x=0} \delta\Delta\, dy \qquad \text{(A-14)}$$

As before, there is no longer any need to distinguish between the varied (Δ) and unvaried (Δ^o) versions of the momentum boundary layer thickness. By using the fact $\delta\Delta' = \frac{d(\delta\Delta)}{dx}$, the term involving $\delta\Delta'$ on the right-hand side of Eq. (A-14) can be written as

$$I = \int_0^{\ell} \int_0^{\Delta_t} - \rho\, u_\infty^3 \left\{ \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \right\}^2$$

$$\left\{ - \frac{3}{2}\, \frac{y}{\Delta^2} + \frac{3}{2}\, \frac{y^2}{\Delta^4} \right\} \delta\Delta'\, dx\, dy$$

$$= - \int_0^{\Delta} \int_0^{\ell} \rho\, u_\infty^3 \left\{ \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \right\}^2$$

$$\left\{ - \frac{3}{2}\, \frac{y}{\Delta^2} + \frac{3}{2}\, \frac{y^2}{\Delta^4} \right\} \frac{d(\delta\Delta)}{dx}\, dx\, dy$$

$$= - \int_0^{\Delta} \rho\, u_\infty^3 \left\{ \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \right\}^2$$

$$\left\{ - \frac{3}{2}\, \frac{y}{\Delta^2} + \frac{3}{2}\, \frac{y^2}{\Delta^4} \right\} \Big|_{x=0}^{x=\ell} \delta\Delta\, dy$$

$$+ \int_0^{\ell} \int_0^{\Delta} \frac{d}{dx} \left\{ \rho u_\infty^3 \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right)^3 \left(- \frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^2}{\Delta^4} \right) \right\} \delta\Delta \, dx \, dy \tag{A-15}$$

The first term of this integral cancels out with the last two terms of Eq. (A-14). Finally, integrating y in Eq. (A-14) from 0 to Δ, one obtains

$$\delta E_T = \int_0^{\ell} \Big[\frac{78}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 - \frac{189}{320} \frac{\rho u_\infty^3 m}{x} + \frac{63}{320} \frac{\rho u_\infty^3 m}{x} - \frac{57}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 + \frac{21}{80} \frac{\rho u_\infty^3 m}{x} - \frac{3}{5} \mu_\infty \frac{u_\infty^2}{\Delta^2} - A \mu_\infty u_\infty^2 \left(\frac{27}{32} \frac{\Delta_t}{\Delta^3} - \frac{3}{8} \frac{\Delta_t^3}{\Delta^5} + \frac{27}{320} \frac{\Delta_t^5}{\Delta^7} \right) + \frac{3}{8} \frac{\rho u_\infty^3 m}{x} \Big] \delta\Delta \, dx \tag{A-16}$$

Since δE_T must vanish for all $\delta\Delta$, therefore,

$$\frac{78}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 - \frac{126}{320} \frac{\rho u_\infty^3 m}{x} - \frac{57}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 + \frac{21}{80} \frac{\rho u_\infty^3 m}{x} - \frac{3}{5} \mu_\infty \frac{u_\infty^2}{\Delta^2} - A \mu_\infty u_\infty^2 \left(\frac{27}{32} \frac{\Delta_t}{\Delta^3} - \frac{3}{8} \frac{\Delta_t^3}{\Delta^5} + \frac{27}{320} \frac{\Delta_t^5}{\Delta^7} \right) + \frac{3}{8} \frac{\rho u_\infty^3 m}{x} = 0 \tag{A-17}$$

Introducing the ratio $Y=\frac{\Delta_t}{\Delta}$, Eq. (A-17) becomes

$$\frac{21}{320}\,\rho u_\infty \Delta\Delta' + \frac{78}{320}\,\frac{\rho u_\infty \Delta^2}{x}\,m - \frac{3}{5}\mu_\infty - A\,\mu_\infty\left(\frac{27}{32}Y - \frac{3}{8}Y^3 + \frac{27}{320}Y^5\right) = 0 \tag{A-18}$$

Introducing the dimensionless quantities

$$\begin{aligned} x^* &= \frac{x}{\ell\, Re_\infty} \\ \Delta^* &= \frac{\Delta}{\ell} \\ \Delta_t^* &= \frac{\Delta_t}{\ell} \\ Y^* &= \frac{\Delta_t^*}{\Delta^*} \end{aligned} \tag{A-19}$$

also,

$$Z = \Delta^{*2}$$

$$W = Y^{*3}$$

and rearranging the terms of Eqs. (A-13) and (A-18) one obtains,

$$Z W'\left(1 + c_8 W^{2/3}\right) = c_1 + Z' W\left(c_2 + c_9 W^{2/3}\right) + \frac{WZ}{x^*}\left(c_3 + c_{10} W^{2/3}\right) \tag{A-20}$$

$$Z' = c_4 + c_5 W^{1/3} + c_6 W + c_7 \frac{Z}{x^2} + c_{11} W^{5/3} \qquad \text{(A-21)}$$

where,

$$c_1 = \frac{64}{3 Pr_\infty} \left(\frac{3}{5} + \frac{177}{320} B \right)$$

$$c_2 = -\frac{3}{4}$$

$$c_3 = -\frac{3}{2} m$$

$$c_4 = \frac{128}{7}$$

$$c_5 = \frac{180}{7} A \qquad \text{(A-22)}$$

$$c_6 = -\frac{80}{7} A$$

$$c_7 = -\frac{52}{7} m$$

$$c_8 = -\frac{2}{15}$$

$$c_9 = \frac{1}{120}$$

$$c_{10} = \frac{m}{10}$$

$$c_{11} = \frac{9}{7} A$$

APPENDIX 'B'

DERIVATION OF THE COUPLED EQUATIONS FOR CONSTANT WALL HEAT FLUX

First, one rewrites Eq. (12)

$$E_H = \iint_S \Big[-\rho u^{o2} \frac{\partial u}{\partial x} - \rho u^o v^o \frac{\partial u}{\partial y} + \frac{\mu^o}{2} \Big(\frac{\partial u}{\partial y} \Big)^2$$

$$+ \rho c_p^o u^o T \frac{\partial T^o}{\partial x} + \rho c_p^o v^o T \frac{\partial T^o}{\partial y}$$

$$+ \frac{k^o}{2} \Big(\frac{\partial T}{\partial y} \Big)^2 + \frac{\partial p^o}{\partial x} u \Big] dx\, dy$$

$$+ \int_0^\Delta \Big(\rho u^{o2} u \Big|_{x=\ell} - \rho u^{o2} u \Big|_{x=0} \Big) dy$$

$$+ \int_0^\ell \Big(\rho u^o v^o u \Big|_{y=\Delta} \Big) dx + \int_0^\ell \Big(-\mu^o \frac{\partial u^o}{\partial y} u \Big) \Big|_{y=\Delta} dx$$

$$+ \int_0^\ell \Big(-k^o \frac{\partial T^o}{\partial y} T \Big|_{y=\Delta_t} \Big) dx + \int_0^\ell \Big(k^o \frac{\partial T^o}{\partial y} T \Big|_{y=0} \Big) dx \quad (12)$$

Since

$$\frac{u}{u_\infty} = \frac{3}{2} \Big(\frac{y}{\Delta} \Big) - \frac{1}{2} \Big(\frac{y}{\Delta} \Big)^3$$

it follows that

$$\frac{\partial u}{\partial x} = u_\infty \Big[-\frac{3}{2} \frac{y \Delta'}{\Delta^2} + \frac{3}{2} \frac{y^3 \Delta'}{\Delta^4} \Big] \qquad \text{(B-1)}$$

$$+ \frac{\partial u_\infty}{\partial x} \Big[\frac{3}{2} \Big(\frac{y}{\Delta} \Big) - \frac{1}{2} \Big(\frac{y}{\Delta} \Big)^3 \Big]$$

and

$$\frac{\partial u}{\partial y} = u_\infty \left[\frac{3}{2} \frac{1}{\Delta} - \frac{3}{2} \frac{y^2}{\Delta^3} \right] \tag{B-2}$$

From the continuity equation

$$\frac{\partial v}{\partial y} = - \frac{\partial u}{\partial x}$$

therefore

$$v = \int_0^y \left(- \frac{\partial u}{\partial x} \right) dy$$

$$= u_\infty \left[\frac{3}{4} \frac{y^2 \Delta'}{\Delta^2} - \frac{3}{8} \frac{y^4 \Delta'}{\Delta^4} \right] - \frac{\partial u_\infty}{\partial x} \left[\frac{3}{4} \frac{y^2}{\Delta} - \frac{1}{8} \frac{y^4}{\Delta^3} \right] \tag{B-3}$$

From Eq. (18b), one obtains

$$T = \frac{q_H}{k_\infty} \frac{\Delta_t}{3} \left[1 - \frac{3y}{\Delta_t} + 3\left(\frac{y}{\Delta_t} \right)^2 - \left(\frac{y}{\Delta_t} \right)^3 \right] + T_\infty \tag{B-4}$$

and so

$$\frac{\partial T}{\partial y} = \frac{q_H}{k_\infty} \left[-1 + 2 \frac{y}{\Delta_t} - \frac{y^2}{\Delta_t^2} \right] \tag{B-5}$$

Similarly,

$$\frac{\partial T}{\partial x} = \frac{q_H}{k_\infty} \left[\frac{\Delta_t'}{3} - \frac{y^2 \Delta_t'}{\Delta_t^2} + \frac{2}{3} \frac{y^3 \Delta_t'}{\Delta_t^3} \right] \tag{B-6}$$

For wedge flow, the main stream velocity variation is given by

$$u_\infty = c\, x^m \tag{B-6}$$

and differentiating gives

$$\frac{du_\infty}{dx} = \frac{m}{x} u_\infty \tag{B-7}$$

Applying Bernoulli's Eq. yields

$$m = \frac{x}{u_\infty} \frac{du_\infty}{dx} = - \frac{x}{\rho u_\infty^2} \frac{dp}{dx} \tag{B-8}$$

Eqs. (20b) and (21b) are

$$\mu = \mu_\infty (1 + A\phi)$$
$$k = k_\infty (1 + B\phi)$$

Substituting all the above expressions into Eq. (12), one obtains

$$E_H = \iint_S \Big[- \rho u_\infty^{o\,2} \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}^2$$
$$\left\{ u_\infty \left(- \frac{3}{2} \frac{y\Delta'}{\Delta^2} + \frac{3}{2} \frac{y^3 \Delta'}{\Delta^4} \right) + \frac{\partial u_\infty}{\partial x} \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \right\}$$
$$- \rho u_\infty^o \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}$$
$$\left\{ u_\infty^o \left(\frac{3}{4} \frac{y^2 \Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8} \frac{y^4 \Delta^{o\prime}}{\Delta^{o4}} \right) - \frac{\partial u_\infty^o}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^o} - \frac{1}{8} \frac{y^4}{\Delta^{o3}} \right) \right\}$$
$$\left\{ u_\infty \left(\frac{3}{2} \frac{1}{\Delta} - \frac{3}{2} \frac{y^2}{\Delta^3} \right) \right\}$$

$$+\left\{\frac{\mu_\infty^0 + A\mu_\infty^0 \frac{\Delta_t^0}{3l}\left[1 - 3\left(\frac{y}{\Delta_t^0}\right) + 3\left(\frac{y}{\Delta_t^0}\right)^2 - \left(\frac{y}{\Delta_t^0}\right)^3\right]}{2}\right\}$$

$$\left\{ u_\infty^2 \left(\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2}\frac{y^2}{\Delta^3}\right)^2 \right\}$$

$$+ \rho c_p^0 u_\infty^0 \left\{ \frac{3}{2}\left(\frac{y}{\Delta^0}\right) - \frac{1}{2}\left(\frac{y}{\Delta^0}\right)^3 \right\}$$

$$\left\{ \frac{q_H}{k_\infty}\frac{\Delta_t}{3}\left[1 - \frac{3y}{\Delta_t} + 3\left(\frac{y}{\Delta_t}\right)^2 - \left(\frac{y}{\Delta_t}\right)^3\right] + T_\infty \right\}$$

$$\left\{ \frac{q_H}{k_\infty}\left(\frac{\Delta_t^{0'}}{3} - \frac{y^2\Delta_t^{0'}}{\Delta_t^2} + \frac{2}{3}\frac{y^3\Delta_t^{0'}}{\Delta_t^3}\right) \right\}$$

$$+ \rho c_p^0 \Big\{ u_\infty^0 \left(\frac{3}{4}\frac{y^2\Delta^{0'}}{\Delta^{02}} - \frac{3}{8}\frac{y^4\Delta^{0'}}{\Delta^{04}}\right)$$

$$- \frac{\partial u_\infty^0}{\partial x}\left(\text{[illegible]}\,\frac{y^2}{\Delta^0} - \frac{1}{8}\frac{y^4}{\Delta^{03}}\right)\Big\}$$

$$\left\{ \frac{q_H}{k_\infty}\frac{\Delta_t}{3}\left[1 - 3\left(\frac{y}{\Delta_t}\right) + 3\left(\frac{y}{\Delta_t}\right)^2 - \left(\frac{y}{\Delta_t}\right)^3\right] \right.$$

$$\left. + T_\infty \right\} \left\{ \frac{q_H}{k_\infty^0}\left(-1 + \frac{2y}{\Delta_t^0} - \frac{y^2}{\Delta_t^{02}}\right) \right\}$$

$$+\left[\frac{k_\infty^0 + Bk_\infty^0 \frac{\Delta_t^0}{3l}\left\{1 - 3\left(\frac{y}{\Delta_t^0}\right) + 3\left(\frac{y}{\Delta_t^0}\right)^2 - \left(\frac{y}{\Delta_t^0}\right)^3\right\}}{2}\right]$$

$$\left[\left\{ \frac{q_H}{k_\infty}\left(-1 + \frac{2y}{\Delta_t} - \frac{y^2}{\Delta_t^2}\right) \right\}^2 \right]$$

$$- \rho \frac{u_\infty^{02} m}{x^0} u_\infty \left(\frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\frac{y^3}{\Delta^3}\right)\Big] \, dx\, dy$$

$$+ \int_0^{\Delta} \rho u_\infty^{o^2} \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}^2$$

$$u_\infty \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \Big|_{x=\ell} dy$$

$$- \int_0^{\Delta} \rho u_\infty^{o^2} \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}^2$$

$$u_\infty \left(\frac{3}{2} \frac{y}{\Delta} - \frac{1}{2} \frac{y^3}{\Delta^3} \right) \Big|_{x=0} dy$$

$$+ \int_0^{\ell} \rho u_\infty^o \left\{ \frac{3}{2}\left(\frac{y}{\Delta^o}\right) - \frac{1}{2}\left(\frac{y}{\Delta^o}\right)^3 \right\}$$

$$\left\{ u_\infty^o \left(\frac{3}{4} \frac{y^2 \Delta^{o'}}{\Delta^{o2}} - \frac{3}{8} \frac{y^4 \Delta^{o'}}{\Delta^{o4}} \right) \right.$$

$$\left. - \frac{\partial u_\infty^o}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^o} - \frac{1}{8} \frac{y^4}{\Delta^{o3}} \right) \right\}$$

$$u_\infty \left\{ \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \right\} \Big|_{y=\Delta} dx$$

$$+ \int_0^{\ell} - k_\infty^o \left\{ 1 + B \frac{\Delta_t^o}{3\ell} \left[1 - 3\left(\frac{y}{\Delta_t^o}\right) + 3\left(\frac{y}{\Delta_t^o}\right)^2 \right. \right.$$

$$\left. \left. - \left(\frac{y}{\Delta_t^o}\right)^3 \right] \right\} \left\{ \frac{q_H}{k_\infty^o} \left(-1 + 2 \frac{y}{\Delta_t^o} \right. \right.$$

$$\left. \left. - \frac{y^2}{\Delta_t^{o2}} \right) \right\} \left\{ \frac{q_H}{k_\infty} \frac{\Delta_t}{3} \left(1 - 3\left(\frac{y}{\Delta_t}\right) \right. \right.$$

$$\left. \left. + 3\left(\frac{y}{\Delta_t}\right)^2 - \left(\frac{y}{\Delta_t}\right)^3 \right) + T_\infty \right\} \Big|_{y=\Delta_t} dx$$

$$+ \int_0^{\ell} k_\infty \left\{ 1 + B \frac{\Delta_t^0}{3\ell} \left[1 - 3 \left(\frac{y}{\Delta_t^0} \right) + 3 \left(\frac{y}{\Delta_t^0} \right)^2 - \left(\frac{y}{\Delta_t^0} \right)^3 \right] \right\}$$

$$\left\{ \frac{q_H}{k_\infty^0} \left(-1 + 2 \frac{y}{\Delta_t^0} - \frac{y^2}{\Delta_t^{02}} \right) \right\}$$

$$\left\{ \frac{q_H}{k_\infty} \frac{\Delta_t}{3} \left[1 - 3 \left(\frac{y}{\Delta_t} \right) + 3 \left(\frac{y}{\Delta_t} \right)^2 - \left(\frac{y}{\Delta_t} \right)^3 \right] + T_\infty \right\} \Bigg|_{y=0} dx \qquad \text{(B-9)}$$

Taking the variation of E_H (Eq. (B-9)) with respect to Δ_t one obtains

$$\delta E_H = \int_0^{\ell} \int_0^{\Delta_t} \Big[\rho c_p^0 u_\infty^0 \left\{ \frac{3}{2} \left(\frac{y}{\Delta^0} \right) - \frac{1}{2} \left(\frac{y}{\Delta^0} \right)^3 \right\}$$

$$\left\{ \frac{q_H}{k_\infty} \left(\frac{1}{3} - \frac{y^2}{\Delta_t^2} + \frac{2}{3} \frac{y^3}{\Delta_t^3} \right) \right\}$$

$$\left\{ \frac{q_H}{k_\infty} \left(\frac{\Delta_t^{0'}}{3} - \frac{y^2 \Delta_t^{0'}}{\Delta_t^2} + \frac{2}{3} \frac{y^3 \Delta_t^{0'}}{\Delta_t^3} \right) \right\}$$

$$+ \rho c_p^0 \left\{ u_\infty^0 \left(\frac{3}{4} \frac{y^2 \Delta^{0'}}{\Delta^{02}} - \frac{3}{8} \frac{y^4 \Delta^{0'}}{\Delta^{04}} \right) - \frac{\partial u_\infty^0}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^0} - \frac{1}{8} \frac{y^4}{\Delta^{03}} \right) \right\}$$

$$\left\{ \frac{q_H}{k_\infty} \left(\frac{1}{3} - \frac{y^2}{\Delta_t^2} + \frac{2}{3} \frac{y^3}{\Delta_t^3} \right) \right\}$$

$$\left\{ \frac{q_H}{k_\infty^0} \left(-1 + 2 \frac{y}{\Delta_t^0} - \frac{y^2}{\Delta_t^{02}} \right) \right\}$$

$$+ [k_\infty^0 + B k_\infty^0 \frac{\Delta_t^0}{3\ell} \{ 1 - 3(\frac{y}{\Delta_t^0}) + 3(\frac{y}{\Delta_t^0})^2 - (\frac{y}{\Delta_t^0})^3 \}] \{ \frac{q_H}{k_\infty} (-1 + \frac{2y}{\Delta_t} - \frac{y^2}{\Delta_t^2}) \} \{ \frac{q_H}{k_\infty} (-\frac{2y}{\Delta_t^2} + \frac{2y^2}{\Delta_t^3}) \}] \delta\Delta_t dx dy$$

$$+ \int_0^\ell [\{ k_\infty^0 + B k_\infty^0 \frac{\Delta_t^0}{3\ell} \} (-\frac{q_H}{k_\infty^0}) \{ \frac{q_H}{k_\infty} \frac{1}{3} \}] \delta\Delta_t \, dx \qquad \text{(B-10)}$$

There is no longer any need to distinguish between the varied (Δ_t) and unvaried (Δ_t^0) versions of thermal boundary layer thickness. Dropping the superscript and integrating y from 0 to Δ_t, one finds

$$\delta E_H = \int_0^\ell [\rho c_p u_\infty \frac{q_H^2}{k_\infty^2} \Delta_t' \{ \frac{1}{70} \frac{\Delta_t^2}{\Delta} - (\frac{11}{27} \frac{1}{560}) \frac{\Delta_t^4}{\Delta^3} \} + \rho c_p u_\infty \frac{q_H^2}{k_\infty^2} \{ -\frac{21}{5040} \frac{\Delta' \Delta_t^3}{\Delta^2} + \frac{3}{40} \frac{1}{189} \frac{\Delta' \Delta_t^5}{\Delta^4} \}$$

$$- \rho c_p u_\infty \frac{q_H^2}{k_\infty^2} \frac{m}{x} \{ -\frac{1}{240} \frac{\Delta_t^3}{\Delta} + (\frac{1}{168} \frac{1}{45}) \frac{\Delta_t^5}{\Delta^3} \} + \frac{k_\infty}{10} \frac{q_H^2}{k_\infty^2} + B k_\infty \frac{q_H^2}{k_\infty^2} (-\frac{633}{420} \frac{\Delta_t}{\ell}) - \frac{k_\infty}{3} \frac{q_H^2}{k_\infty^2} - B k_\infty \frac{q_H^2}{k_\infty^2} (\frac{1}{9} \frac{\Delta_t}{\ell})] \delta\Delta_t dx \qquad \text{(B-11)}$$

Since δE_H must vanish for all $\delta\Delta_t$, therefore

$$\rho c_p u_\infty \left\{ \frac{1}{70} \frac{\Delta_t' \Delta_t^2}{\Delta} - \left(\frac{11}{27} \frac{1}{560}\right) \frac{\Delta_t' \Delta_t^4}{\Delta^3} - \frac{21}{5040} \frac{\Delta' \Delta_t^3}{\Delta^2} + \frac{3}{40} \frac{1}{189} \frac{\Delta' \Delta_t^5}{\Delta^4} \right\}$$

$$+ \rho c_p u_\infty \frac{m}{x} \left\{ \frac{1}{240} \frac{\Delta_t^3}{\Delta} - \left(\frac{1}{168} \frac{1}{45}\right) \frac{\Delta_t^5}{\Delta^3} \right\}$$

$$- \frac{7}{30} k_\infty - \frac{2039}{1260} \frac{\Delta_t}{\ell} B k_\infty = 0 \qquad \text{(B-12)}$$

Similarly, taking the variation of E_H (Eq. (B-9)) with respect to Δ gives

$$\delta E_H = \int_0^\ell \int_0^\Delta \Big[- \rho u_\infty^{o2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o}\right) - \frac{1}{2} \left(\frac{y}{\Delta^o}\right)^3 \right\}^2 u_\infty \left\{ \left(\frac{3 y \Delta'}{\Delta^5}\right) \delta\Delta + \left(-\frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^2}{\Delta^4}\right) \delta\Delta' \right\}$$

$$- \rho u_\infty^{o2} \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o}\right) - \frac{1}{2} \left(\frac{y}{\Delta^o}\right)^3 \right\}^2 \frac{\partial u_\infty}{\partial x} \left(-\frac{3}{2} \frac{y}{\Delta^2} + \frac{3}{2} \frac{y^3}{\Delta^4}\right) \delta\Delta$$

$$- \rho u_\infty^o \left\{ \frac{3}{2} \left(\frac{y}{\Delta^o}\right) - \frac{1}{2} \left(\frac{y}{\Delta^o}\right)^3 \right\} \left\{ u_\infty^o \left(\frac{3}{4} \frac{y^2 \Delta^{o\prime}}{\Delta^{o2}} - \frac{3}{8} \frac{y^4 \Delta^{o\prime}}{\Delta^{o4}} \right) - \frac{\partial u_\infty}{\partial x} \left(\frac{3}{4} \frac{y^2}{\Delta^o} - \frac{1}{8} \frac{y^4}{\Delta^{o3}} \right) \right\}$$

$$\left\{ u_\infty \left(-\frac{3}{2} \frac{1}{\Delta^2} + \frac{9}{2} \frac{y^2}{\Delta^4}\right) \right\} \delta\Delta \Big] \, dx\, dy$$

$$+ \int_0^{\ell} \Big[\int_0^{\Delta_t} \Big\{ \mu_\infty^o + A \mu_\infty^o \frac{\Delta_t^o}{3\ell} \Big(1 - 3\Big(\frac{y}{\Delta_t^o}\Big) + 3\Big(\frac{y}{\Delta_t^o}\Big)^2 - \Big(\frac{y}{\Delta_t^o}\Big)^3 \Big) \Big\}$$

$$\Big\{ u_\infty^2 \Big(\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2}\frac{y^2}{\Delta^3} \Big) \Big(-\frac{3}{2}\frac{1}{\Delta^2} + \frac{9}{2}\frac{y^2}{\Delta^4} \Big) \Big\} \delta\Delta\, dy$$

$$+ \int_{\Delta_t}^{\Delta} \Big\{ \mu_\infty^o u_\infty^2 \Big(\frac{3}{2}\frac{1}{\Delta} - \frac{3}{2}\frac{y^2}{\Delta^3} \Big) \Big(-\frac{3}{2}\frac{1}{\Delta^2} + \frac{9}{2}\frac{y^2}{\Delta^4} \Big) \delta\Delta\, dy \Big] dx$$

$$- \int_0^{\ell} \int_0^{\Delta_t} \rho \frac{u_\infty^{o2} m}{x^o} u_\infty \Big(-\frac{3}{2}\frac{y}{\Delta^2} + \frac{3}{2}\frac{y^3}{\Delta^4} \Big) \delta\Delta\, dx\, dy$$

$$+ \int_0^{\Delta} \rho u_\infty^{o2} \Big\{ \frac{3}{2}\Big(\frac{y}{\Delta^o}\Big) - \frac{1}{2}\Big(\frac{y}{\Delta^o}\Big)^3 \Big\}^2 \Big\{ u_\infty \Big(-\frac{3}{2}\frac{y}{\Delta^2} + \frac{3}{2}\frac{y^3}{\Delta^4} \Big) \Big\} \Big|_{x=\ell} \delta\Delta\, dy$$

$$- \int_0^{\Delta} \rho u_\infty^{o2} \Big\{ \frac{3}{2}\Big(\frac{y}{\Delta^o}\Big) - \frac{1}{2}\Big(\frac{y}{\Delta^o}\Big)^3 \Big\}^2 u_\infty \Big(-\frac{3}{2}\frac{y}{\Delta^2} - \frac{1}{2}\frac{y^3}{\Delta^4} \Big) \Big|_{x=0} \delta\Delta\, dy \qquad \text{(B-13)}$$

As before, there is no longer any need to distinguish between the varied (Δ) and unvaried (Δ^0) versions of the momentum boundary layer thickness. By proceeding in the same manner as of Appendix 'A', one obtains

$$\delta E_H = \int_0^{\ell} \Big[\frac{78}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 - \frac{189}{320} \frac{\rho u_\infty^3 m}{x} + \frac{63}{320} \frac{\rho u_\infty^3 m}{x} - \frac{57}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 + \frac{21}{80} \frac{\rho u_\infty^3 m}{x} - \frac{3}{5} \mu_\infty \frac{u_\infty^2}{\Delta^2} - A \mu_\infty u_\infty^2 \frac{\Delta_t}{\ell} \Big(\frac{3}{16} \frac{\Delta_t}{\Delta} - \frac{1}{20} \frac{\Delta_t^3}{\Delta^3} + \frac{11}{1120} \frac{\Delta_t^5}{\Delta^5} \Big) + \frac{3}{8} \frac{\rho u_\infty^3 m}{x} \Big] \delta\Delta \, dx \qquad \text{(B-14)}$$

Since δE_H must vanish for all $\delta\Delta$, therefore

$$\frac{78}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 - \frac{189}{320} \frac{\rho u_\infty^3 m}{x} + \frac{63}{320} \frac{\rho u_\infty^3 m}{x} - \frac{57}{320} \frac{\Delta'}{\Delta} \rho u_\infty^3 + \frac{21}{80} \frac{\rho u_\infty^3 m}{x} - \frac{3}{5} \mu_\infty \frac{u_\infty^2}{\Delta^2} - A \mu_\infty u_\infty^2 \frac{\Delta_t}{\ell} \Big(\frac{3}{16} \frac{\Delta_t}{\Delta} - \frac{1}{20} \frac{\Delta_t^3}{\Delta^3} + \frac{11}{1120} \frac{\Delta_t^5}{\Delta^5} \Big) + \frac{3}{8} \frac{\rho u_\infty^3 m}{x} = 0 \qquad \text{(B-15)}$$

Introducing the dimensionless quantities

$$
\begin{aligned}
x^* &= \frac{x}{\ell \, Re_\infty} \\
\Delta^* &= \frac{\Delta}{\ell} \\
\Delta_t^* &= \frac{\Delta_t}{\ell} \\
Y^* &= \frac{\Delta_t^*}{\Delta^*}
\end{aligned}
\tag{B-16}
$$

also,

$$
\begin{aligned}
z &= \Delta^{*2} \\
W &= Y^{*3}
\end{aligned}
$$

and rearranging the terms of Eqs. (B-12) and (B-15), One obtains

$$
\begin{aligned}
z W' (1 + d_9 W^{2/3}) &= d_1 + d_2 W^{1/3} z^{1/2} \\
&\quad + z' W (d_3 + d_{10} W^{2/3}) \\
&\quad + \frac{W z}{x^*} (d_4 + d_{11} W^{2/3})
\end{aligned}
\tag{B-17}
$$

$$
\begin{aligned}
z' &= d_5 + d_6 W^{2/3} z^{1/2} \\
&\quad + d_7 W^{4/3} z^{1/2} + d_8 \frac{z}{x^*} \\
&\quad + d_{12} W^2 z^{1/2}
\end{aligned}
\tag{B-18}
$$

where,

$$\begin{aligned}
d_1 &= \frac{49}{Pr_\infty} \\
d_2 &= \frac{2039}{6}\ \frac{B}{Pr_\infty} \\
d_3 &= -\frac{17}{16} \\
d_4 &= -\frac{7}{8}\, m \\
d_5 &= \frac{128}{7} \\
d_6 &= \frac{40}{7}\, A \\
d_7 &= -\frac{32}{21}\, A \\
d_8 &= -\frac{52}{7}\, m \\
d_9 &= -\frac{11}{216} \\
d_{10} &= \frac{5}{144} \\
d_{11} &= \frac{m}{36} \\
d_{12} &= \frac{44}{147}\, A
\end{aligned} \qquad \text{(B-19)}$$

REFERENCES

1. W. Byron Brown and Patrick L. Donoughe, Extension of Boundary Layer Heat-Transfer Theory to cobled Turbine Blades. NACA RME 50 f 02, 1950.

2. E. Eckert and O. Drewitz, Calculation of the Temperature Field in the Laminar Boundary Layer of an Unheated Body in a High Speed Flow. R.T.P. Trans. No. 1594, British M.A.P.

3. E.R.G. Eckert and John N.B. Livingood, Method for Calculation of Laminar Heat transfer in Air Flow Around Cylinders of Arbitrary Cross-section (Including large Temperature Differences and Transpiration Cooling). NACA Rep. 1118, 1953 (Supersedes NACA TN 2733)

4. E.R.G. Eckert, Die Berechnung des Warmeubergangs in der laminaren Grenzschicht. VDI-Forschungsh, Vol. 416, pp. 1-24, 1942.

5. Solomon Levy, Heat Transfer to Constant-Property Laminar Boundary Layer Flows with Power-Function Free-Stream Velocity and Wall-Temperature Variation. Journal of the Aeronautical Sciences, May 1952.

6. Patrick L. Donoughe and John N.B. Livingood, Exact Solutions of Laminar Boundary Layer Equations with Constant Property Values for Porous Wall with Variable Temperature. NACA TN 3151, Sept. 1954.

7. Eli Reshotko and clarence B. Cohen, Heat Transfer at the Forward Stagnation Point of Blunt Bodies. NACA TN 3513, July 1955.

8. M.J. Lighthill, Contributions to the theory of Heat Transfer through a Laminar Boundary Layer. Proc. Roy. Soc. (London) Vol. A202, pp. 359, 1950.

9. W.Byron Brown and Patrick L. Donoughe, Tables of Exact Laminar Boundary Layer Solutions when the Wall is Porous and Fluid Properties are Variable. NACA TN 2479, Sept. 1951.

10. Soloman Levy, Effect of Large Temperature Changes (Including viscous heating) upon Laminar Boundary Layers with Variable Free-Stream Velocity. Jour. Aero. Sci., Vol. 21, # 7, pp. 459-474, July 1954.

11. Y.L. Su, Application of a Variational Method to Flow over a Flate Plate in the Entrance Region with Variable Physical

Properties. M.A.Sc Thesis, University of Ottawa, Aug. 1970.

12. John Klein and Myron Tribus, Forced Convection from Non-isothermal Surfaces, ASME Paper No. 53-SA-46, ASME Semi-annual Meeting, 1953.

13. H.Schlicting, Boundary Layer Theory. Chapters VII and X, Sixth Edition, McGraw-Hill, 1968.

14. W.M. Kays, Convective Heat and Mass Transfer. Chapter VII, X and XII, McGraw-Hill, 1966.

15. S.C. Cheng and A. Kumar, A Study of Wedge Flow with Variable Physical Properties by a Variational Technique. Submitted for publication to AIAA Journal.

16. P. Glansdorff and I. Prigogine, On a General Evolution Criterion in Macroscopic Physics. Physica 30, pp. 351-374, 1964.

17. D.F. Hays, An Extended Variational Method Applied to Poiseuille Flow: Temperature dependent viscosity. Int. J. Heat Mass Transfer, Vol. 9, pp. 165-170, 1966.

18. R.J. Donnelly, R. Herman and I. Prigogine, Symposium on Non-equilibrium Thermodynamics: Variational Techniques and

Stability, Part I, University of Chicago Press, 1966.

19. R.S. Schechter, The Variational Method in Engineering. Chapter V, McGraw-Hill, 1967.

20. User's Manual - System-360 Continous System Modeling Program. IBM Application Program No. 360 A-CX-16X, N.Y.

21. L.G. Birta, CSMP Notes of EE 4650 (1970-71). Computer Science Department, University of Ottawa, March 1971.

22. Notes of Seminar on System Modeling and Simulation (June 1971). Analysis Laboratory, National Research Council Ottawa, 1971.

23. L.G. Birta, Extended Plotting Techniques of S/360 CSMP. Computer Science Department Project (in progress), University of Ottawa.

24. Bernard J. Schroer, Expanded Capabilities of CSMP graphics on the IBM 1130 digital computer. Simulation, U.S.A., pp. 205-214, May 1970.

25. H. Tramposch and H.A. Jones Jr., Impact problems efficiently solved with 1130 CSMP. Simulation, U.S.A., pp. 73-79, Feb. 1970.

26. A. Kumar, S.C. Cheng and L.G. Birta, A CSMP Technique to Solve an External Boundary Layer Flow Problem with Constant Wall Heat Flux and Variable Physical Properties. Submitted to 1972 Summer Simulation Conference, U.S.A.

27. B.A. Chubb, Economic Evaluation of the CSMP digital Simulation Language. Simulation, U.S.A., Vol. 14, pp. 101-103, 1970.

28. S.C. Cheng and A. Kumar, A CSMP Approach to Solve the Poiseuille Flow with Variable Physical Properties. Submitted to 1972 Summer Simulation Conference, U.S.A.

29. EAI Handbook of Analog Computation. Publication No. 00 800. 0001-3, Electronic Associates Inc. U.S.A., Dec. 1969.

30. C.L. Johnson, Analog Computer Techniques. McGraw-Hill, 1963.

31. J.J. Blum, Introduction to Analog Computation. Harcourt Brace and World Inc., New York, N.Y., 1969.

32. M.L. James, G.M. Smith and J.C. Wolford, Analog and Digital Computer Methods in Engineering Analysis. International Text Book Co., 1965.

33. W.P. Murrill and C.L. Smith, Fortran IV Programming for Engineers and Scientists. International Text Book Co., Jan. 1969.

34. G.A. Bekey and W.J. Karplus, Hybrid Computation. John Wiley, New York, N.Y. 1968.

35. L.G. Birta, Analog/Hybrid Computer Notes of EE 4650 (1970-71). Computer Science Department, University of Ottawa, 1971.

36. K.T. Yang, Laminar Forced Convection of Liquids in Tubes with Variable Viscosity. ASME(c), J. Heat Transfer, Vol. 84, pp. 353-362, 1962.

37. K.W. Schade and D.M. McEligot, Cartesian Graetz Problems with Air Property Variation. Int. J. Heat Mass Transfer, Vol. 14, pp. 553-666, 1971.