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Locally Nilpotent Derivations and the Cancellation Problem in Affine Algebraic Geometry

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Abstract

Let $\Bbbk$ be a field of characteristic zero and let $R^{[n]}$ denote the polynomial ring in n variables over a ring R for any $n \in \mathbb{N}$, $n > 0$.

We present some basic theory for the study of locally nilpotent derivations as an effective tool in algebraic geometry. Using this tool, we examine the Cancellation Problem in affine algebraic geometry, which asks:

Let A be a $\Bbbk$-algebra such that $A^{[1]} = \Bbbk^{[n+1]}$. Does it follow that $A = \Bbbk^{[n]}$?

This problem is open for $n > 2$. We present the solutions to the cases $n = 1$ and $n = 2$, in the latter case essentially following the algebraic method of Crachiola and Makar-Limanov [9].

We examine a potential counterexample, $R = \Bbbk[X, Y, Z, T]/\langle X + X^2Y + Z^2 + T^3 \rangle$, referred to as Russell's Cubic. We show that while R closely resembles a polynomial ring in 3 variables, we have that $R \neq \Bbbk^{[3]}$, a result due to Makar-Limanov [25]. This is achieved by showing that the Derksen invariant of R is not equal to the Derksen invariant of $\Bbbk^{[3]}$. It is unknown if $R^{[1]}$ is a polynomial ring in 4 variables over $\Bbbk$, nonetheless, we examine some properties of $R^{[1]}$ which highlight its similarities with $\Bbbk^{[4]}$.

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Dedication

To my friends and family, who have supported me through all my endeavors.

Contents

Conventions and Notations

- All rings are assumed to be commutative and associative with identity.
- The letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers (here, $0 \in \mathbb{N}$), the integers, the rational numbers, the real numbers and the complex numbers, respectively.
- The letter $\Bbbk$ denotes a field, $\overline{\Bbbk}$ its algebraic closure.
- If B is a ring, then B^* is its set of units.
- We abbreviate "unique factorization domain" by UFD.
- We abbreviate "principal ideal domain" by PID.
- If $A \subseteq B$ are integral domains, $\mathrm{Frac}(A)$ is the field of fractions of A and $\mathrm{trdeg}_A(B)$ is the transcendence degree of $\mathrm{Frac}(B)$ over $\mathrm{Frac}(A)$.
- If R is a ring and $n \in \mathbb{N}$, then $R^{[n]}$ denotes the polynomial ring in n variables over R.
- If R is a ring and $x \in R$, then we sometimes write R_x for the localized ring $S^{-1}R$, where $S = \{1, x, x^2, \ldots\}$.

Introduction

In this thesis, we study locally nilpotent derivations and use them to explore the Cancellation Problem, which asks:

Let A be a $\Bbbk$-algebra such that $A^{[1]} = \Bbbk^{[n+1]}$. Does it follow that $A = \Bbbk^{[n]}$?

Here, $R^{[n]}$ denotes a polynomial ring in n variables over R. This question is often attributed to Zariski, who posed a similar question in 1949 at the Paris Colloquium on Algebra and the Theory of Numbers (see [32], [40]). While the answer to Zariski's initial question regarding simple transcendental field extensions was negative [4], the Cancellation Problem stated above is still open for $n > 2$ and is of great interest in the field of affine algebraic geometry.

We begin with a concise review of some basic algebraic tools that will be required throughout the thesis. In Chapter 1, we summarize the relevant properties of filtrations, degree functions and graded rings.

In Chapter 2, we discuss a very valuable tool in algebraic geometry: derivations. A derivation is a map D from a ring B to itself which satisfies the following for all elements f and g in B:

(i) $D(f+g) = D(f) + D(g)$

(ii) $D(fg) = fD(g) + gD(f)$.

The set of all derivations of a ring B is denoted $\mathrm{Der}(B)$, and the kernel of a derivation D of B is often referred to as the ring of constants of D and is denoted B^D, or $\ker(D)$.

An element s in B is called a slice of D if $D(s) = 1$.

Of particular importance are locally nilpotent derivations. A derivation D of B is called locally nilpotent if for every element f in B there exists a nonnegative integer n such that $D^n(f) = 0$. The subset of $\mathrm{Der}(B)$ consisting of all locally nilpotent derivations of B is denoted $\mathrm{LND}(B)$.

We will look at some interesting theorems about derivations which will form the basis of a more in-depth discussion on the Cancellation Problem. For example, suppose B is a commutative $\Bbbk$-algebra which is also an integral domain, where $\Bbbk$ is a field of characteristic zero. In 1981, Wright [44] proved the following (see Theorem 2.3.7):

> Let $D \in \mathrm{LND}(B)$ have a slice $s \in B$. Then, $B = B^D[s]$ is a polynomial ring in one variable over the ring of constants of D and $D = \frac{\partial}{\partial s}$.

Thus any locally nilpotent derivation of B which possesses a slice s is essentially the derivative with respect to s.

We review the Makar-Limanov invariant, which was introduced by Leonid Makar-Limanov in the mid-1990s. It is defined as the intersection of the rings of constants of all locally nilpotent derivations of an integral domain B of characteristic zero and is denoted $\mathrm{ML}(B)$. Note that if $B = \Bbbk[X_1, \ldots, X_n]$ is a polynomial ring, then $\mathrm{ML}(B) = \Bbbk$. So if B is a $\Bbbk$-algebra which is also an integral domain and $\mathrm{ML}(B) \neq \Bbbk$, then B is not a polynomial ring over $\Bbbk$. Thus, we can use this ring invariant to distinguish such a B from a polynomial ring.

The Cancellation Problem asks us to determine whether a $\Bbbk$-algebra A is a polynomial ring in n variables over $\Bbbk$ whenever $A^{[1]} = \Bbbk^{[n+1]}$. But this raises the question: what conditions on a ring B are sufficient to ensure that B is a polynomial ring? That is, how can we characterize polynomial rings? This is the topic of discussion in Chapter 3. The problem of characterizing polynomial rings is open for $n > 3$ and is vital to the Cancellation Problem. We present a well-known characterization of $\Bbbk^{[1]}$, and one of $\Bbbk^{[2]}$ introduced by Miyanishi [28].

The Cancellation Problem is open for $n > 2$ and the question has an affirmative answer when $n = 1, 2$. In Chapter 4, we present the solution to the $n = 1$ and $n = 2$ cases. The Cancellation Problem in dimension two over a field of characteristic zero was solved by Fujita [16] and Miyanishi and Sugie [31], while the arbitrary characteristic case was solved by Russell [38]. Their method is quite sophisticated, making heavy use of the theory of open algebraic surfaces. We present the more recent solution due to Crachiola and Makar-Limanov [9], which is more elementary. The solutions to the Cancellation Problem in dimensions one and two each require the characterizations of $\mathbb{k}^{[1]}$ and $\mathbb{k}^{[2]}$ respectively, thereby reiterating the importance of the problem of characterizing polynomial rings.

In Chapter 5, we turn our attention to Russell's Cubic, $R = \mathbb{k}[X, Y, Z, T]/\langle X + X^2Y + Z^2 + T^3\rangle$, where $\mathbb{k}$ is a field of characteristic zero. It originally arose as a potential counterexample to the Linearization Problem [37]. We examine some properties of R which show its resemblance to $\mathbb{k}^{[3]}$, including the fact that R is a unique factorization domain and that $R^* = \mathbb{k}^*$. Makar-Limanov showed that it in fact is not a polynomial ring in 3 variables over a field [25], thereby invalidating it as a counterexample to the Linearization Problem. However, in doing so, it emerges as a potential counterexample to the Cancellation Problem in dimension 3 since it is unknown if $R^{[1]} = \mathbb{k}^{[4]}$.

The Derksen invariant of a ring B, denoted $\mathcal{D}(B)$, which was defined by Derksen in his thesis [13], is the subring of B generated by the union of the rings of constants of all non-zero locally nilpotent derivations of B. It is clear that if $B = \mathbb{k}^{[n]}$ is a polynomial ring over a field, then $\mathcal{D}(B) = B$. Thus, by showing that $\mathcal{D}(R) \neq R$, we can conclude that R is not a polynomial ring. We demonstrate this to be the case by following primarily the proof of Makar-Limanov [26] and by incorporating some of Freudenburg's techniques from [15]. We encountered issues with both proofs and tasked ourselves with resolving them. Thus, in writing our proof we devoted great care to detail and accuracy.

We begin by showing $\mathcal{D}(R) \neq R$ when $\Bbbk$ is an algebraically closed field of characteristic zero. We can then extend the result to an arbitrary field $\Bbbk$ of characteristic zero (see Theorem 5.3.12):

> Let $\Bbbk$ be a field of characteristic zero. We have that $\mathcal{D}(R) \neq R$ and, in particular, $R \neq \Bbbk^{[3]}$.

We conclude by showing the Makar-Limanov invariant of R is $\Bbbk[x]$ (see Corollary 5.3.14).

In the last section, we present some properties of the polynomial ring in one variable over R, $R[w]$, which show that it bears a striking resemblance to $\Bbbk^{[4]}$. It is a unique factorization domain of dimension 4, which is finitely generated as a $\Bbbk$-algebra. It is clear that $\mathcal{D}(R[w]) = R[w]$ and $\mathrm{ML}(R[w]) \subseteq \Bbbk[x]$. Recently [14], Dubouloz proved that the Makar-Limanov invariant of $R[w]$ is in fact $\Bbbk$. Thus neither the Derksen invariant, nor the Makar-Limanov invariant can distinguish $R[w]$ from a polynomial ring in 4 variables over $\Bbbk$. Since it is still unknown if $R[w] = \Bbbk^{[4]}$, Russell's Cubic remains a potential counterexample to the Cancellation Problem in dimension 3.

Chapter 1

Preliminaries

We begin by presenting a sound foundation for the commutative algebra that will be utilized throughout the thesis. Included in this chapter is a basic outline of filtrations, degree functions and graded rings (see Chapter 3, § 2 of [6] for a more comprehensive overview, or Chapter 1 of [15]). We also review polynomial rings and the notion of a coordinate system. While the information presented in this chapter is well established, we want to ensure that all readers have the same fundamentals.

1.1 Filtrations

Let B be a commutative ring and let $(G, +, \leq)$ be a totally ordered abelian group.

Definition 1.1.1 A *G-filtration* of B is a family $\{B_i\}_{i \in G}$ of subsets of B with the following properties:

(i) each B_i is a subgroup of $(B, +)$

(ii) $B_j \subseteq B_i \; \forall \; j \leq i$

(iii) $B = \bigcup_{i \in G} B_i$

(iv) $B_i B_j \subseteq B_{i+j}\ \forall\, i,j \in G$.

If the family $\{B_i\}_{i\in G}$ also satisfies

(v) For all $f \in B \setminus \{0\}$, the set $\{i \in G \mid f \in B_i\}$ has a least element.

(vi) Let $B_{i^-} = \bigcup_{j<i} B_j$. For all $i,j \in G$, if $f \in B_i \setminus B_{i^-}$ and $g \in B_j \setminus B_{j^-}$ then $fg \in B_{i+j} \setminus B_{(i+j)^-}$

then $\{B_i\}_{i\in G}$ is called a *proper G-filtration* of B.

Remark 1.1.2 If B admits a proper G-filtration, then B is an integral domain.

1.2 Degree Functions

The material in this section is well known, and can be found in Chapter 3, §2 of [6]. Let B be a ring and let $(M,+,\leq)$ be a totally ordered commutative monoid.

Definition 1.2.1 A map $\deg : B \to M \cup \{-\infty\}$ is called an *M-valued degree function* if for all $f,g \in B$ it satisfies the following:

(i) $\deg(f) = -\infty \iff f = 0$

(ii) $\deg(fg) = \deg(f) + \deg(g)$

(iii) $\deg(f+g) \leq \max(\deg(f), \deg(g))$

Remark 1.2.2 If B admits a degree function, B must be an integral domain. The important cases in this thesis are $M = \mathbb{N}$ and $M = G$ where G is a totally ordered abelian group.

Remark 1.2.3 Note that if $x,y \in M$ satisfy $2x = 2y$, then $x = y$. Indeed, if $x < y$ then $2x < x+y < 2y$ is a contradiction, and similarly $x > y$ is impossible.

Lemma 1.2.4 *Any degree function* $\deg : B \to M \cup \{-\infty\}$ *satisfies the following for all* $f, g \in B$*:*

(i) $\deg(g) = \deg(-g)$

(ii) *If* $\deg(f) \neq \deg(g)$*, then* $\deg(f+g) = \max(\deg(f), \deg(g))$

Proof:

(i) We have $2 \cdot \deg(g) = \deg(g^2) = \deg((-g)(-g)) = 2 \cdot \deg(-g)$, so by the above remark, $\deg(g) = \deg(-g)$.

(ii) Suppose w.l.o.g. that $\deg(f) > \deg(g)$. Then $\deg(f+g) \leq \deg(f)$. Suppose $\deg(f+g) < \deg(f)$. Then

$$\begin{aligned} \deg(f) &= \deg((f+g) + (-g)) \\ &\leq \max(\deg(f+g), \deg(-g)) \\ &= \max(\deg(f+g), \deg(g)) \\ &< \deg(f) \end{aligned}$$

contradiction. Thus, $\deg(f+g) = \max(\deg(f), \deg(g))$. ∎

For the remainder of this section, assume that B is an integral domain and that G is a totally ordered abelian group.

Remark 1.2.5 Any degree function $\deg : B \to G \cup \{-\infty\}$ induces a proper G-filtration defined by $B_i = \{f \in B | \deg(f) \leq i\}$.

Lemma 1.2.6 *Let* $B = \bigcup_{i \in G} B_i$ *be a proper* G*-filtration. Then* $\deg : B \to G \cup \{-\infty\}$, $f \mapsto \inf\{i \in G | f \in B_i\}$ *for* $f \neq 0$, $0 \mapsto -\infty$ *is a degree function.*

Proof: Let $f, g \in B$. By (v) of Definition 1.1.1 we have that $\deg(f) = -\infty \iff f = 0$. It is clear that if $f = 0$ or $g = 0$ we have $\deg(fg) = \deg(f) + \deg(g)$ and

$\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$. Suppose $f, g \neq 0$ and $\deg(f) = i$, $\deg(g) = j$. Then $f \in B_i \setminus B_{i^-}$ and $g \in B_j \setminus B_{j^-}$, so by Definition 1.1.1 (vi) $fg \in B_{i+j} \setminus B_{(i+j)^-}$ and $\deg(fg) = i + j$. Without loss of generality, suppose $i \leq j$. Then $f \in B_j$, so $f + g \in B_j$ and $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$. ∎

Proposition 1.2.7 (a) *Let* $\deg : B \to G \cup \{-\infty\}$ *be a degree function and let* $\{B_i\}_{i \in G}$ *be the induced filtration as defined in Remark* 1.2.5. *Then the degree function induced by the filtration as defined in Lemma* 1.2.6 *coincides with* deg.

(b) *Let* $\{B_i\}_{i \in G}$ *be a proper* G*-filtration of* B *and let* $\deg : B \to G \cup \{-\infty\}$ *be the induced degree function as defined in Lemma* 1.2.6. *Then the filtration determined by* deg *as in Remark* 1 2.5 *coincides with* $\{B_i\}_{i \in G}$.

Proof: We will prove (a) and leave (b) to the reader. Let $\deg' : B \to G \cup \{-\infty\}$ be the degree function induced by the filtration. Then $\deg(0) = \deg'(0) = -\infty$. Let $f \in B \setminus \{0\}$. Then $f \in B_{\deg(f)} \setminus B_{\deg(f)^-}$, so $\deg'(f) = \deg(f)$. ∎

Remark 1.2.8 Thus the set of G-valued degree functions of B is in bijection with the set of proper G-filtrations of B.

1.3 Graded Rings

The material in this section is well known, see Chapter 2, § 11 of [5].

Definition 1.3.1 Let $(M, +)$ be a commutative monoid. A ring R is a *M-graded ring* if it admits a direct sum decomposition $R = \bigoplus_{i \in M} R_i$ such that each R_i is an additive abelian group and $R_i R_j \subseteq R_{i+j}$ for every $i, j \in M$. We call the family $\{R_i\}_{i \in M}$ a *grading* of R.

Definition 1.3.2 Let R be a M-graded ring. An element $f \in R$ is called *homogeneous* (of degree i) if $f \in B_i$ for some $i \in M$.

Example 1.3.3 Let $B = R[X_1, \ldots, X_n]$ be a polynomial ring. There is a natural $\mathbb{N}$-grading on B defined by taking $B_0 = R$ and B_i to be the subgroup of all R-linear combinations of monomials of degree i.

Remark 1.3.4 Any $\mathbb{N}$-graded ring R can be considered a $\mathbb{Z}$-graded ring by setting $R_i = 0$ for all $i < 0$.

Definition 1.3.5 Let $R = \bigoplus_{i \in M} R_i$ be an M-graded ring, where $(M, +)$ is a commutative monoid. Any $f \in R$ can be written as $f = \sum_{i \in M} f_i$ where $f_i \in R_i$ and $f_i = 0$ for all but finitely many $i \in M$. The f_i are called the *homogeneous components of* f and the expression $f = \sum_{i \in M} f_i$ is the *homogeneous decomposition of* f.

Remark 1.3.6 Let $B = \mathbb{k}[X_1, \ldots, X_n]$ be a polynomial ring in n variables over a field $\mathbb{k}$ of characteristic zero, and let G be a totally ordered abelian group. To define a G-grading on B, it is enough to declare $X_1, \ldots, X_n$ to be homogeneous of degrees $a_1, \ldots, a_n \in G$ respectively. In this case, $B = \bigoplus_{i \in G} B_i$ where B_i is the $\mathbb{k}$-span of monomials $\left\{\lambda X_1^{b_1} \cdots X_n^{b_n} \mid \lambda \in \mathbb{k}^*, \ b_1 a_1 + \ldots + b_n a_n = i\right\}$. Note that one cannot remove the stipulation that the generators are homogeneous, see Remark 1.3.13 below.

Lemma 1.3.7 *Let $B = \bigoplus_{i \in G} B_i$ be a G-graded ring, where G is an abelian group. Then $1 \in B_0$, and B_0 is a subring of B.*

Proof: Let $1 = \sum_{d \in G} g_d$ be the homogeneous decomposition of 1. Let $h \in B_m$ for some $m \in G$. Then $h = 1 \cdot h = \sum_{d \in G} g_d h$ where $g_d h \in B_{m+d}$ for all $d \in G$. By the uniqueness of the homogeneous decomposition of h. we have $h = g_0 h$. So $g_0 h = h$ for all homogeneous elements $h \in B$, and $f = g_0 f$ for all $f \in B$. In particular, $1 = g_0 \cdot 1 = g_0 \in B_0$. ∎

The following is well known, see for instance Proposition 1.1.7 of [2].

Proposition 1.3.8 *An abelian group G can be totally ordered if and only if it is torsion-free.*

Lemma 1.3.9 *Let $B = \bigoplus_{i \in G} B_i$ be a G-graded integral domain, where G is torsion-free. Then*

(i) *If $f, g \in B \backslash \{0\}$ are such that fg is homogeneous, then f and g are homogeneous.*

(ii) *Every unit of B is homogeneous.*

(iii) *If $\Bbbk$ is a field contained in B, then $\Bbbk \subseteq B_0$.*

Proof: By Proposition 1.3.8 we can choose a total order for G.

(i) Let $G_+ = \{g \in G \mid g \geq 0\}$. Define a map $\omega : B \setminus \{0\} \to G_+$, $\omega(f) = \max\{i \in G | f_i \neq 0\} - \min\{j \in G | f_j \neq 0\}$ where $f = \sum_{i \in G} f_i$ is the homogeneous decomposition of f. Notice that f is homogeneous if and only if $\omega(f) = 0$. For all $f, g \in B \setminus \{0\}$, we have $\omega(fg) = \omega(f) + \omega(g)$ since B is an integral domain. So if fg is homogeneous, $0 = \omega(fg) = \omega(f) + \omega(g)$, and we must have $\omega(f) = \omega(g) = 0$.

(ii) If f is a unit of B with inverse f^{-1}, then $ff^{-1} = 1$ is homogeneous by Lemma 1.3.7 so f is homogeneous by (i).

(iii) Suppose $\Bbbk \not\subseteq B_0$. Then we can choose $\lambda \in \Bbbk$. $\lambda \notin B_0$. Since $0 \in B_0$ we know $\lambda \neq 0$. Thus λ is a unit and is homogeneous by (ii). Let $d \in G \setminus \{0\}$ be such that $\lambda \in B_d \setminus \{0\}$. Then $1 + \lambda \in \Bbbk$ is non-zero, since $-1 \in B_0$ by Lemma 1.3.7. So $1 + \lambda \in \Bbbk^*$ but it is not homogeneous, contradicting (ii). ∎

Definition 1.3.10 Let $B = \bigoplus_{i \in G} B_i$ be a ring graded by an abelian group G and denote the grading $\{B_i\}_{i \in G}$ by $\mathcal{G}$. If B is an integral domain and G is totally ordered then we may define a degree function $\deg_{\mathcal{G}} : B \to G \cup \{-\infty\}$ by $\deg_{\mathcal{G}}(f) = \max\{j \in G | f_j \neq 0\}$ (where $0 \neq f = \sum_{j \in G} f_j, f_j \in B_j$) and by $\deg_{\mathcal{G}}(0) = -\infty$. Note that $\deg_{\mathcal{G}}$ satisfies Definition 1.2.1. When no confusion can arise, we will write deg in place of $\deg_{\mathcal{G}}$.

Remark 1.3.11 It follows from Remark 1.2.5 that every G-grading of B induces a proper G-filtration $\{\mathcal{B}_i\}_{i \in G}$ defined by $\mathcal{B}_i = \{f \in B | \deg(f) \leq i\}$, where deg is the degree function induced by the grading. In other words, $\mathcal{B}_i = \bigoplus_{j \leq i} B_j$ for each $i \in G$.

Remark 1.3.12 It should be noted that two distinct G-gradings on a ring B may determine the same degree function. Indeed, let $B = \Bbbk[X]$ be a polynomial ring in one variable over a field $\Bbbk$ of characteristic zero. Define a $\mathbb{Z}$-grading $\mathcal{G}_1$ of B by declaring X to be $\mathcal{G}_1$-homogeneous of degree 1. Note that $B = \Bbbk[X+1]$, and define another $\mathbb{Z}$-grading of B by declaring $X + 1$ to be $\mathcal{G}_2$-homogeneous of degree 1. Explicitly:

$$\mathcal{G}_1 = \{B_i\}_{i \in \mathbb{Z}} \qquad \text{where } B_i = \begin{cases} \{0\}, & \text{if } i < 0; \\ \{\lambda X^i \mid \lambda \in \Bbbk\} & \text{if } i \geq 0 \end{cases}$$

$$\mathcal{G}_2 = \{S_i\}_{i \in \mathbb{Z}} \qquad \text{where } S_i = \begin{cases} \{0\}, & \text{if } i < 0; \\ \{\lambda (X+1)^i \mid \lambda \in \Bbbk\} & \text{if } i \geq 0. \end{cases}$$

Let $\deg_{\mathcal{G}_1}, \deg_{\mathcal{G}_2} : B \to \mathbb{Z} \cup \{-\infty\}$ be as defined in Definition 1.3.10. We know $\deg_{\mathcal{G}_1}(0) = \deg_{\mathcal{G}_2}(0) = -\infty$. Let $p \in \Bbbk[X]$ be non-zero and write $p = a_0 + a_1 X + \cdots + a_n X^n$ where $a_n \neq 0$. Then $\deg_{\mathcal{G}_1}(p) = \max\{j \in \mathbb{Z} \mid a_j \neq 0\} = n$. If we write p as a sum of $\mathcal{G}_2$-homogeneous elements, we see that its highest homogeneous term is $a_n(X+1)^n$. Thus $\deg_{\mathcal{G}_2}(p) = n$. It follows that the distinct gradings $\mathcal{G}_1$, $\mathcal{G}_2$ determine the same degree function on $\Bbbk[X]$.

Remark 1.3.13 Note that two distinct degree functions on a $\Bbbk$-algebra B may agree on the generators of B. Indeed, let $B = \Bbbk[X, Y]$ be a polynomial ring in two variables over a field $\Bbbk$ of characteristic zero. Define a $\mathbb{Z}$-grading $\mathcal{G}_1$ of B by declaring X, Y to be $\mathcal{G}_1$-homogeneous of degrees 1 and 2 respectively. Note that $B = \Bbbk[X, Y - X^2]$, and define another $\mathbb{Z}$-grading $\mathcal{G}_2$ of B by declaring both X and $Y - X^2$ to be $\mathcal{G}_2$-homogeneous of degree 1. Let $\deg_{\mathcal{G}_1}, \deg_{\mathcal{G}_2} : \Bbbk[X, Y] \to \mathbb{Z} \cup \{-\infty\}$ be the degree functions determined by the gradings $\mathcal{G}_1, \mathcal{G}_2$. Then

$$\begin{aligned} \deg_{\mathcal{G}_1}(\lambda) &= 0 = \deg_{\mathcal{G}_2}(\lambda) \qquad \text{for all } \lambda \in \Bbbk^* \\ \deg_{\mathcal{G}_1}(X) &= 1 = \deg_{\mathcal{G}_2}(X) \\ \deg_{\mathcal{G}_1}(Y) &= 2 = \deg_{\mathcal{G}_2}(Y). \end{aligned}$$

However, $\deg_{\mathcal{G}_1} \neq \deg_{\mathcal{G}_2}$ since $\deg_{\mathcal{G}_1}(Y - X^2) = 2$, whereas $\deg_{\mathcal{G}_2}(Y - X^2) = 1$.

Lemma 1.3.14 *Let $(B, \mathcal{G})$ be a G-graded integral domain, and let S be a multiplicative set of B. If each element of S is homogeneous, then there exists a G-grading $\mathcal{G}'$ of $S^{-1}B$ satisfying:*

If $b \in B$ is homogeneous and $s \in S$, then $\frac{b}{s} \in S^{-1}B$ is homogeneous and

$$\deg_{\mathcal{G}'}\left(\frac{b}{s}\right) = \deg_{\mathcal{G}}(b) - \deg_{\mathcal{G}}(s).$$

Proof: Given $x \in S^{-1}B$, let $\mathcal{H}(x)$ be the set of pairs (b, s) such that $b \in B$ is homogeneous, $s \in S$ and $x = \frac{b}{s}$. Note that $\mathcal{H}(x)$ could be empty, but $\mathcal{H}(0) \neq \emptyset$.
Let $x \in S^{-1}B \setminus \{0\}$ be such that $\mathcal{H}(x) \neq \emptyset$. We claim that if $(b, s), (b', s') \in \mathcal{H}(x)$, then $\deg_{\mathcal{G}}(b) - \deg_{\mathcal{G}}(s) = \deg_{\mathcal{G}}(b') - \deg_{\mathcal{G}}(s')$. Since $\frac{b}{s} = x = \frac{b'}{s'}$ and B is an integral domain, we know $s'b = sb' \neq 0$, so $\deg_{\mathcal{G}}(b) - \deg_{\mathcal{G}}(s) = \deg_{\mathcal{G}}(b') - \deg_{\mathcal{G}}(s')$. Thus for such an x, we can define $\deg(x) = \deg_{\mathcal{G}}(b) - \deg_{\mathcal{G}}(s)$. Also define $\deg(0) = -\infty$. So we have a well-defined map

$$\deg : \{x \in S^{-1}B \mid \mathcal{H}(x) \neq \emptyset\} \to G \cup \{-\infty\}$$

satisfying $\deg(x) = -\infty$ if and only if $x = 0$.

For each $g \in G$, define $A_g = \{x \in S^{-1}B \mid \mathcal{H}(x) \neq \emptyset \text{ and } \deg(x) \in \{g, -\infty\}\}$. The reader can verify that $\{A_g\}_{g \in G}$ is a G-grading of $S^{-1}B$. ∎

Definition 1.3.15 Let B be an integral domain, G a totally ordered abelian group and $\{B_i\}_{i \in G}$ a proper G-filtration of B. The *associated graded ring* $\mathrm{Gr}(B)$ is defined as $\mathrm{Gr}(B) = \bigoplus_{i \in G} B_i/B_{i^-}$ where $B_{i^-} = \bigcup_{j<i} B_j$. Let $\deg : B \to G \cup \{-\infty\}$ be the degree function associated with the proper G-filtration as defined in Lemma 1.2.6. We can define the map $\mathrm{gr} : B \to \mathrm{Gr}(B)$ by $\mathrm{gr}(0) = 0$ and $\mathrm{gr}(f) = f + B_{\deg(f)^-} \in B_{\deg(f)}/B_{\deg(f)^-}$. Note that gr is not necessarily a homomorphism, but does preserve multiplication since $\deg(fg) = \deg(f) + \deg(g)$ for all $f, g \in B$.

Remark 1.3.16 Let B be a ring, G a totally ordered abelian group and $\{B_i\}_{i \in G}$ a proper G-filtration of B. Then B and $\mathrm{Gr}(B)$ are integral domains.

Remark 1.3.17 Suppose that $B = \bigoplus_{i \in G} A_i$ is a G-graded integral domain and consider the proper G-filtration $B_i = \bigoplus_{j \leq i} A_j$. Then $\mathrm{Gr}(B) \cong B$.

1.4 Polynomial Rings

Let R be a commutative ring.

Definition 1.4.1 For an R-algebra A, we write $A = R^{[n]}$ to indicate that A is isomorphic (as an R-algebra) to the polynomial ring $R[X_1, \ldots, X_n]$. When this is the case, we often say that A is a *polynomial ring in n indeterminates* over R.

Remark 1.4.2 Note that if $A = R^{[n]}$ and $B = R^{[n]}$ then we do not necessarily have $A = B$.

Proposition 1.4.3 (Universal Property of Polynomial Rings) *Let $B = R[X_1, \ldots, X_n]$ where $0 < n \in \mathbb{N}$, let $\varphi_B : R \to B$ be the inclusion homomorphism, let $\varphi_A : R \to A$ be a ring homomorphism and view A as an R-algebra (via φ_A). For every n-tuple of elements $(a_1, \ldots, a_n) \in A^n$ there exists a unique R-algebra homomorphism $\varphi : B \to A$ such that $\varphi \circ \varphi_B = \varphi_A$ and $\varphi(X_i) = a_i$ for all $1 \leq i \leq n$.*

The following statements are well known, see Proposition 4.29, Corollary 8.21 and Theorem 8.32 in [21]:

Proposition 1.4.4 *Let $B = R^{[n]}$ where $n \in \mathbb{N}$. Then:*

(i) *R is an integral domain if and only if B is an integral domain.*

(ii) *R is a unique factorization domain* (UFD) *if and only if B is a* UFD.

(iii) *R is Noetherian if and only if B is Noetherian.*

(iv) *Let A be a ring over which R is an algebra. Then R is finitely generated as an A-algebra if and only if B is finitely generated as an A-algebra.*

1.5 Coordinates

Let $B = \mathbb{k}^{[n]}$, where $\mathbb{k}$ is a field and $n \in \mathbb{N}$, $n \geq 1$.

Definition 1.5.1 An element $f \in B$ is called a *coordinate* of B if there exist elements $f_2, \ldots, f_n \in B$ such that $B = \mathbb{k}[f, f_2, \ldots, f_n]$. An ordered n-tuple $(f_1, \ldots, f_n) \in B^n$ is called a *coordinate system of B* if $B = \mathbb{k}[f_1, f_2, \ldots, f_n]$.

Definition 1.5.2 Let $\mathrm{Aut}_{\mathbb{k}}(B)$ denote the group of all $\mathbb{k}$-automorphisms of B, i.e. the automorphisms of B as a $\mathbb{k}$-algebra.

Remark 1.5.3 If $(f_1, \ldots, f_n)$ is a coordinate system of B, then the family $(f_1, \ldots, f_n)$ is algrebraically independent over $\Bbbk$. Indeed, if $(f_1, \ldots, f_n)$ were algebraically dependent over $\Bbbk$ it would follow that the transcendence degree of B over $\Bbbk$ is less than n, which is absurd since $B = \Bbbk^{[n]}$. Thus, $(f_1, \ldots, f_n)$ has the universal property of the polynomial ring, that is, given elements $a_1, \ldots, a_n$ of a $\Bbbk$-algebra A there exists a unique $\Bbbk$-homomorphism $\varphi : B \longrightarrow A$ satisfying $\varphi(f_i) = a_i$ for all $i = 1, \ldots, n$.

Proposition 1.5.4 *If $\gamma_1 = (f_1, \ldots, f_n)$ and $\gamma_2 = (g_1, \ldots, g_n)$ are two coordinate systems of B, then there exists a unique $\theta \in \mathrm{Aut}_{\Bbbk}(B)$ such that $\theta(f_i) = g_i$ for all $1 \leq i \leq n$.*

Proof: By Remark 1.5.3, there exists a unique $\Bbbk$-algebra homomorphism $\theta : B \longrightarrow B$ such that $\theta(f_i) = g_i$ for all $1 \leq i \leq n$. This homomorphism is clearly surjective and injectivity follows from Remark 1.5.3 applied to the g_i's. ∎

Chapter 2

Derivations

In this chapter, we outline some basic definitions and properties related to derivations. Derivations, and in particular locally nilpotent derivations, are an effective tool employed in modern algebraic geometry. We discuss at length the properties of derivations, locally nilpotent derivations and their kernels. We then define a slice of a derivation, and we will see in Chapter 4 how the cancellation problem can be reformulated in terms of the kernel of a locally nilpotent derivation with a slice. Also, we cover homogenization of derivations with respect to both gradings and filtrations. Finally, we define the Makar-Limanov invariant of a ring and state some of its elementary properties. This invariant is a principal component that will be used to solve the Cancellation problem in dimension two (see Chapter 4, Section 2) and to show that Russell's Cubic is not $\mathbb{k}^{[3]}$ (see Chapter 5, Section 3).

Most of the material covered in this chapter is well known. Notable references include Freudenburg's book [15], the lecture notes of Daigle [10], [12] and Makar-Limanov [24]. In the case of sections 2.2, 2.5 and 2.6, however, the material is well known in the special case $G = \mathbb{Z}$ but we don't know a suitable reference for the general case.

2.1 Basic Properties

Let B be a commutative ring.

Definition 2.1.1 A map $D : B \to B$ is called a *derivation* if the following conditions are satisfied for all $f, g \in B$:

(i) $D(f+g) = D(f) + D(g)$

(ii) $D(fg) = fD(g) + gD(f)$

The set of all derivations $D : B \to B$ is denoted $\mathrm{Der}(B)$. It is well known that $\mathrm{Der}(B)$ is a B-module.

Remark 2.1.2 Condition (ii) is called Leibniz's law.

Lemma 2.1.3 *Let $D \in \mathrm{Der}(B)$, $f \in B$ and $n \in \mathbb{N}$, $n \geq 1$. Then $D(f^n) = nf^{n-1}D(f)$.*

Lemma 2.1.4 *Let $D \in \mathrm{Der}(B)$, $f, g \in B$, $n \in \mathbb{N}$. $D^n(fg) = \sum_{i+j=n} \binom{n}{i} D^i(f) D^j(g)$.*

We are often working with the kernel of a derivation, i.e. $\ker(D) = \{f \in B \mid D(f) = 0\}$, sometimes called the *ring of constants of* D. We will use the common notation $B^D = \ker(D)$. Note that B^D is a subring of B.

Definition 2.1.5 Let $D \in \mathrm{Der}(B)$. We say that D is *nilpotent* if there exists $n \in \mathbb{N}$ such that $D^n(f) = 0$ for all $f \in B$. If for some $f \in B$ there exists $n \in \mathbb{N}$ such that $D^n(f) = 0$, then D is called *nilpotent at f*. We denote by $\mathrm{Nil}(D)$ the set of all $f \in B$ such that D is nilpotent at f. If D is nilpotent at every $f \in B$ we call D *locally nilpotent*. The set of all locally nilpotent derivations of B is denoted $\mathrm{LND}(B)$.

Let A be a subring of B. The set of all derivations $D \in \mathrm{Der}(B)$ such that $A \subseteq B^D$ is denoted $\mathrm{Der}_A(B)$. The set of all locally nilpotent derivations $D \in \mathrm{LND}(B)$ such that $A \subseteq B^D$ is denoted $\mathrm{LND}_A(B)$. Note that $\mathrm{LND}_A(B) = \mathrm{LND}(B) \cap \mathrm{Der}_A(B)$.

Lemma 2.1.6 *For any $D \in \mathrm{Der}(B)$, $\mathrm{Nil}(D)$ is a subring of B.*

Proof: It is clear that $1 \in \mathrm{Nil}(D)$: $D(1) = D(1 \cdot 1) = 2D(1)$, so $D(1) = 0$. Let $a, b \in \mathrm{Nil}(D)$ and let $n, m \in \mathbb{N} \setminus \{0\}$ be such that $D^n(a) = 0$ and $D^m(b) = 0$. Then $D^{n+m-1}(ab) = \sum_{i+j=n+m-1} \binom{n+m-1}{i} D^i(a) D^j(b)$. For each i, j such that $i+j = n+m-1$, we have either $i \geq n$ and $D^i(a) = 0$ or $j \geq m$ and $D^j(b) = 0$. So $D^{n+m-1}(ab) = 0$ and $ab \in \mathrm{Nil}(D)$. Let $r = \max\{n, m\}$. Then $D^r(a - b) = D^r(a) - D^r(b) = 0$, so $a - b \in \mathrm{Nil}(D)$. It follows that $\mathrm{Nil}(D)$ is a subring of B. ∎

Remark 2.1.7 A derivation $D \in \mathrm{Der}(B)$ is locally nilpotent if and only if $\mathrm{Nil}(D) = B$.

Proposition 2.1.8 *Let $D \in \mathrm{Der}(B)$ and let $I \subseteq B$ be an ideal of B such that $D(I) \subseteq I$. Then the induced derivation D/I is a well-defined derivation of B/I. Furthermore, if $D \in \mathrm{LND}(B)$, then $D/I \in \mathrm{LND}(B/I)$.*

Proof: For any $b \in B$, let $\bar{b}$ denote its congruence class modulo I. Then $D/I : B/I \to B/I$ is defined in the obvious way: $D/I(\bar{b}) = \overline{D(b)}$. If $b' \in B$ is such that $\overline{b'} = \bar{b}$, then $b' = b + a$ for some $a \in I$. So $D/I(\overline{b'}) = \overline{D(b')} = \overline{D(b+a)} = \overline{D(b) + D(a)} = \overline{D(b)}$ since $D(a) \in I$. So D/I is well-defined and clearly a derivation. It is also clear that if $D \in \mathrm{LND}(B)$, then $D/I \in \mathrm{LND}(B/I)$. ∎

Proposition 2.1.9 *Let $D \in \mathrm{Der}_A(B)$ where A is any subring of B and let S be any multiplicatively closed subset of B. Then D induces an A-derivation $S^{-1}D$ of $S^{-1}B$ given by $S^{-1}D(\frac{b}{s}) = \frac{sD(b) - bD(s)}{s^2}$ for any $b \in B$ and $s \in S$.*

Proof: We will check that $S^{-1}D$ is well-defined. Let $b, b' \in B$, $s, s' \in S$ be such that $\frac{b}{s} = \frac{b'}{s'}$, i.e. there exists a $t \in S$ such that $t(bs' - b's) = 0$ in B. To show

$S^{-1}D(\frac{b}{s}) = S^{-1}D(\frac{b'}{s'})$, we must find a $t' \in S$ such that $t'(s'^2(sD(b) - bD(s)) - s^2(s'D(b') - b'D(s'))) = 0$ in B. Since $tbs' = tb's$, we have $D(tbs') = D(tb's)$ and after expanding, $ts'D(b) - tb'D(s) - tsD(b') + tbD(s') = D(t)(b's - bs')$. Take $t' = t^2$. Then

$$\begin{aligned} t'(s'^2(sD(b) - bD(s)) - s^2(s'D(b') - b'D(s'))) &= tss'(ts'D(b) - tb'D(s) \\ &\quad - tsD(b') + tbD(s')) \\ &= tss'D(t)(b's - bs') \\ &= 0 \end{aligned}$$

So $S^{-1}D$ is well-defined. Using the definition, it is easy to check that it is a derivation. If $a \in A$, then $S^{-1}D(\frac{a}{1}) = \frac{D(a)}{1} = 0$, so $S^{-1}D$ is an A-derivation. ∎

Corollary 2.1.10 *Suppose B is an integral domain, $D \in \text{LND}(B)$ and $S \subseteq B^D$ is a multiplicatively closed subset. Then $S^{-1}D \in \text{LND}(S^{-1}B)$ and $S^{-1}B^{(S^{-1}D)} = S^{-1}(B^D)$.*

Proof: Since $S \subseteq B^D$, we have that for any $b \in B$, $s \in S$, $S^{-1}D(\frac{b}{s}) = \frac{D(b)}{s}$ and $S^{-1}D^n(\frac{b}{s}) = \frac{D^n(b)}{s}$ for any $n \in \mathbb{N}$ so $S^{-1}D \in \text{LND}(S^{-1}B)$. Furthermore, $S^{-1}D(\frac{b}{s}) = 0$ if and only if $D(b) = 0$, so $S^{-1}B^{(S^{-1}D)} = S^{-1}(B^D)$. ∎

Remark 2.1.11 In the case $S = \{f^i\}_{i \in \mathbb{N}}$ for some $f \in B$, we write D_f in place of $S^{-1}D$.

2.1.12. Let $\Bbbk$, $\Bbbk'$ be fields of characteristic zero such that $\Bbbk \subseteq \Bbbk'$ is a field extension. Let B be a $\Bbbk$-algebra and let $B' = B \otimes_{\Bbbk} \Bbbk'$. Note that $B \subseteq B'$:
Since B is a flat $\Bbbk$-module, the functor $B \otimes_{\Bbbk} (-)$ is exact. If we apply this functor to the injective $\Bbbk$-linear map $\Bbbk \hookrightarrow \Bbbk'$, we see that the resulting map $B \otimes_{\Bbbk} \Bbbk \to B \otimes_{\Bbbk} \Bbbk' = B'$ is injective. Then, composing this map with the canonical isomorphism $B \to B \otimes_{\Bbbk} \Bbbk$,

$x \mapsto x \otimes 1$, we have a ring homomorphism

$$B \to B'$$
$$x \mapsto x \otimes 1$$

which is injective.

Since any $D \in \mathrm{Der}_{\Bbbk}(B)$ is a $\Bbbk$-linear map, there is a canonical extension of D to a $\Bbbk'$-linear map:

$$D \otimes 1 : \qquad B' \to B'$$
$$b \otimes c \mapsto D(b) \otimes c.$$

A routine verification shows that the $\Bbbk'$-linear map $D \otimes 1$ is a $\Bbbk'$-derivation of B'. Furthermore, we have the following:

Lemma 2.1.13 *Let $B, B', \Bbbk, \Bbbk'$ be as above. Then, $D \in \mathrm{LND}(B) \setminus \{0\}$ if and only if $D \otimes 1 \in \mathrm{LND}(B') \setminus \{0\}$.*

For further properties of the extension of a derivation, see section 5.1 of [33].

Definition 2.1.14 Let B be an integral domain and let A be a subring of B. We say that A is *factorially closed* in B if for all $a, b \in B \setminus \{0\}$ such that $ab \in A$ we have $a, b \in A$. We say that A is *algebraically closed* in B if for all $b \in B$ such that there exists a non-zero polynomial $f \in A[T] \setminus \{0\}$ satisfying $f(b) = 0$ we have $b \in A$.

Remark 2.1.15 Let B be an integral domain and let A be a factorially closed subring of B. It is clear that if $f_1 \cdots f_n \in A$ where $f_1, \ldots, f_n$ are all non-zero, then $f_1, \ldots, f_n \in A$.

Lemma 2.1.16 *Let B be an integral domain of characteristic zero. If $D \in \mathrm{Der}(B)$, then B^D is algebraically closed in B.*

Proof: Let $b \in B$ be algebraic over B^D and let $f(T) \in B^D[T]$ be a non-zero polynomial of minimal T-degree satisfying $f(b) = 0$. Write $f(T) = b_nT^n + \ldots + b_1T + b_0$. Then $f(b) = b_nb^n + \ldots + b_1b + b_0 = 0$ and $0 = D(f(b)) = b_nb^{n-1}D(b) + \ldots + b_1D(b) = D(b)(b_nb^{n-1} + \ldots + b_1)$. Since B is an integral domain, we must have $D(b) = 0$ or $b_nb^{n-1} + \ldots + b_1 = 0$. However, $f(T)$ was chosen to be of minimal T-degree so $b_nb^{n-1} + \ldots + b_1 \neq 0$ and $b \in B^D$. ∎

Lemma 2.1.17 *Let B be an integral domain and let A be a factorially closed subring of B. Then A is algebraically closed in B.*

Proof: Let $b \in B$ be algebraic over A and let $f(T) \in A[T]$ be a non-zero polynomial of minimal T-degree such that $f(b) = 0$. Write $f(T) = a_nT^n + \ldots + a_1T + a_0$. Then $f(b) = a_nb^n + \ldots + a_1b + a_0 = 0$ and $(a_nb^{n-1} + \ldots + a_1)b = -a_0 \in A$. Since $f(T)$ was of minimal T-degree such that $f(b) = 0$, we know $a_nb^{n-1} + \ldots + a_1 \neq 0$. Since A is factorially closed in B, it follows that $b \in A$ and thus A is algebraically closed in B. ∎

Lemma 2.1.18 *Let B be an integral domain and let A be a factorially closed subring of B. Then the following hold:*

(i) $A^* = B^*$.

(ii) *An element of A is irreducible in A if and only if it is irreducible in B.*

(iii) *If B is a* UFD *then so is A.*

Proof:

(i) Let $f \in B^*$. Then $ff^{-1} = 1 \in A$, so $f, f^{-1} \in A$ and $A^* = B^*$.

(ii) Let $f \in A$ be irreducible in A. Suppose $f = bg$ where $b, g \in B$. Then $b, g \in A$ since A is factorial in B. Since f is irreducible in A, either $b \in A^*$ or $g \in A^*$. By (i), $A^* = B^*$ so either $b \in B^*$ or $g \in B^*$.

Let $f \in A$ be irreducible in B. Suppose $f = ag$ where $a, g \in A$. Then $f = ag$ in B, so $a \in B^*$ or $g \in B^*$. By (i), $A^* = B^*$ so either $a \in A^*$ or $g \in A^*$.

(iii) Let $f \in A \backslash A^*$ be non-zero. We first show uniqueness of factorization up to order and associates. Suppose $f = f_1 \cdots f_n = g_1 \cdots g_m$ where $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ are irreducible. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ are irreducible in B by (ii). Since B is a UFD, it follows that $n = m$ and f_i, g_i are associates for $1 \leq i \leq n$ (after re-ordering if necessary).

For existence of factorization, write $f = f_1 \cdots f_n$ as a product of irreducible elements of B. Since B is a UFD, $f_1, \ldots, f_n$ are unique up to order and associates. By Remark 2.1.15, we have $f_1, \ldots, f_n \in A$ and $f_1, \ldots, f_n$ are irreducible by (ii). Since $A^* = B^*$, g_i is an associate of f_i (for some $1 \leq i \leq n$) in A if and only if g_i is an associate of f_i in B. Thus $f = f_1 \cdots f_n$ is a product of irreducible elements of A, unique up to order and associates.

∎

Lemma 2.1.19 *If* $\deg : B \to \mathbb{N} \cup \{-\infty\}$ *is a degree function, then* B *is an integral domain and* $A = \{f \in B \mid \deg(f) \leq 0\}$ *is a factorially closed subring of* B.

Proposition 2.1.20 *Assume that* B *is an integral domain of characteristic zero. Every* $D \in \text{LND}(B)$ *defines a degree function* $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ *given by* $\deg_D(f) = \min\{n \in \mathbb{N} \mid D^{n+1}(f) = 0\}$ *for* $f \in B \setminus \{0\}$ *and* $\deg_D(0) = -\infty$.

Proof: The function is well-defined since for every $f \in B \setminus \{0\}$ there exists a unique minimal $n \in \mathbb{N}$ such that $D^{n+1}(f) = 0$. We must now verify the conditions in Definition 1.2.1. Clearly. $\deg_D(f) = -\infty \iff f = 0$. To show $\deg_D(fg) = \deg_D(f) + \deg_D(g)$, let $f, g \in B \setminus \{0\}$ and let $n = \deg_D(f)$ and $m = \deg_D(g)$. By Lemma 2.1.4, $D^{m+n+1}(fg) = \sum_{i+j=n+m+1} \binom{n+m+1}{i} D^i(f)D^j(g) = 0$ since for any summand either $i \geq n+1$ or $j \geq m+1$. Consider $D^{m+n}(fg) = \sum_{i+j=n+m} \binom{n+m}{i} D^i(f)D^j(g)$. Since $D^{n+1}(f) = D^{m+1}(g) = 0$, we have

$$D^{m+n}(fg) = \sum_{i+j=n+m} \binom{n+m}{i} D^i(f)D^j(g) = \binom{n+m}{n} D^n(f)D^m(g)$$

where $\binom{n+m}{n} \neq 0$ because B has characteristic zero; as B is an integral domain and $D^n(f)$, $D^m(g) \neq 0$ it follows that $D^{m+n}(fg) \neq 0$. Thus $\deg_D(fg) = \deg_D(f) + \deg_D(g)$. It remains to show $\deg_D(f+g) \leq \max(\deg_D(f), \deg_D(g))$. Let $r = \max(\deg_D(f), \deg_D(g))$. If $r = -\infty$, we must have $f = g = 0$, so $\deg_D(f+g) \leq \max(\deg_D(f), \deg_D(g))$. Else, $D^{r+1}(f+g) = D^{r+1}(f) + D^{r+1}(g) = 0$, so $\deg_D(f+g) \leq \max(\deg_D(f), \deg_D(g))$ as required. ∎

Corollary 2.1.21 *Assume that B is an integral domain of characteristic zero. If $D \in \text{LND}(B)$, then B^D is a factorially closed subring of B. In particular, if B is a $\Bbbk$-algebra then $\Bbbk \subseteq B^D$ and $\text{LND}(B) = \text{LND}_{\Bbbk}(B)$.*

Proof: Notice that $B^D = \{f \in B | \deg_D(f) \leq 0\}$, so is factorially closed by Lemma 2.1.19. It follows from 22 2.1.18 that $\Bbbk \subseteq B^D$. ∎

2.2 The Degree of a Derivation

Let B be an integral domain and let G be a totally ordered abelian group.

Definition 2.2.1 Let $\deg : B \to G \cup \{-\infty\}$ be a degree function and $D \in \mathrm{Der}(B)$ a derivation. Consider the non-empty subset

$$U = \{\deg(D(f)) - \deg(f) |\ f \in B \setminus \{0\}\}$$

of the totally ordered set $G \cup \{-\infty\}$. If U has a greatest element, then we say that $\deg(D)$ exists and we define $\deg(D)$ to be that element.
Note that $\deg(D) = -\infty$ if and only if $D = 0$, and that if $\deg(D)$ is defined then $\deg(D(f)) \leq \deg(f) + \deg(D)$ for all $f \in B$.
Also note that if $\deg(D)$ is defined and $D \neq 0$. then there exists an $f \in B \setminus B^D$ such that $\deg(D) = \deg(D(f)) - \deg(f) \in G$.

Definition 2.2.2 Let $\deg : B \to G \cup \{-\infty\}$ be a degree function and let $D \in \mathrm{Der}(B)$. The *defect function* of D is the map

$$\begin{aligned} \mathrm{def}_D : B &\to G \cup \{-\infty\} \\ f &\mapsto \deg(D(f)) - \deg(f) \qquad \text{for } f \neq 0 \\ 0 &\mapsto -\infty \end{aligned}$$

Note that the defect function depends both on the degree function of B and the derivation. When no confusion can arise, we will write def in place of def_D.

Lemma 2.2.3 *Let* $\deg : B \to G \cup \{-\infty\}$ *be a degree function,* $D \in \mathrm{Der}(B)$ *and* $\mathrm{def}_D : B \to G \cup \{-\infty\}$ *be the defect function of* D.

(i) $\mathrm{def}_D(fg) \leq \max\{\mathrm{def}_D(f), \mathrm{def}_D(g)\}$ *for all* $f, g \in B$.

(ii) *If* $f_1, \ldots, f_m \in B$ *are such that* $\deg(\sum_{i=1}^m f_i) = \max_{1 \leq i \leq m} \deg(f_i)$, *then* $\mathrm{def}_D(\sum_{i=1}^m f_i) \leq \max_{1 \leq i \leq m} \mathrm{def}_D(f_i)$.

Proof:

(i) We may assume $f \neq 0$ and $g \neq 0$. Then:

$$\begin{aligned}
\operatorname{def}_D(fg) &= \deg(D(fg)) - \deg(fg) \\
&= \deg(fD(g) + gD(f)) - (\deg(f) + \deg(g)) \\
&\leq \max\{\deg(fD(g)), \deg(gD(f))\} - (\deg(f) + \deg(g)) \\
&\leq \max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} - (\deg(f) + \deg(g))
\end{aligned}$$

In the case that

$$\max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} = \deg(f) + \deg(D(g))$$

we have $\operatorname{def}_D(fg) \leq \deg(D(g)) - \deg(g) = \operatorname{def}_D(g)$.

Otherwise,

$$\max\{\deg(f) + \deg(D(g)), \deg(g) + \deg(D(f))\} = \deg(g) + \deg(D(f)),$$

so $\operatorname{def}_D(fg) \leq \deg(D(f)) - \deg(f) = \operatorname{def}_D(f)$.

Either way, $\operatorname{def}_D(fg) \leq \max\{\operatorname{def}_D(f), \operatorname{def}_D(g)\}$, as desired.

(ii) Observe that

$$\begin{aligned}
\operatorname{def}_D(\sum_{i=1}^{m} f_i) &= \deg(D(\sum_{i=1}^{m} f_i)) - \deg(\sum_{i=1}^{m} f_i) \\
&= \deg(\sum_{i=1}^{m} D(f_i)) - \max\{\deg(f_i) |\ 1 \leq i \leq m\} \\
&\leq \max\{\deg(D(f_i)) |\ 1 \leq i \leq m\} - \max\{\deg(f_i) |\ 1 \leq i \leq m\} \\
&\leq \max\{\deg(D(f_i)) - \deg(f_i) |\ 1 \leq i \leq m\} \\
&\leq \max\{\operatorname{def}_D(f_i) |\ 1 \leq i \leq m\}.
\end{aligned}$$

∎

Lemma 2.2.4 *Let S be a multiplicative set of B and let* $\deg : S^{-1}B \to G \cup \{-\infty\}$ *and* $\deg : B \to G \cup \{-\infty\}$ *be degree functions, were the latter is the restriction of the former. Let $D \in \operatorname{Der}(B)$ and let $S^{-1}D : S^{-1}B \to S^{-1}B$ be the localization of D as defined in Proposition* 2.1.9. *Then* $\deg(D)$ *is defined if and only if* $\deg(S^{-1}D)$ *is defined. Furthermore, if they are defined then* $\deg(D) = \deg(S^{-1}D)$.

Proof: Consider the sets

$$U = \{\deg(D(f)) - \deg(f) \mid f \in B \setminus \{0\}\},$$
$$U' = \left\{\deg(S^{-1}D(f)) - \deg(f) \mid f \in S^{-1}B \setminus \{0\}\right\}.$$

It is clear that $U \subseteq U'$. To prove the lemma, it suffices to show that for each $u' \in U'$ there exists $u \in U$ such that $u' \leq u$. Let $\delta : S^{-1}B \to G \cup \{-\infty\}$ denote the defect function of $S^{-1}D$. Then $\delta(xy) \leq \max\{\delta(x), \delta(y)\}$ for all $x, y \in S^{-1}B$ by Lemma 2.2.3. Let $u' \in U'$. Then $u' = \delta\left(\frac{x}{s}\right)$ for some $x \in B$, $s \in S$. We have $\delta\left(\frac{x}{s}\right) \leq \max\left\{\delta(x), \delta\left(\frac{1}{s}\right)\right\}$, where $\delta(x) \in U$ and

$$\begin{aligned}
\delta\left(\frac{1}{s}\right) &= \deg\left(S^{-1}D\left(\frac{1}{s}\right)\right) - \deg\left(\frac{1}{s}\right) \\
&= \deg\left(\frac{-1}{s^2}D(s)\right) - \deg\left(\frac{1}{s}\right) \\
&= \deg\left(D(s)\right) - \deg(s) \\
&= \delta(s) \in U,
\end{aligned}$$

so there exists $u \in U$ such that $u' \leq u$. ∎

Proposition 2.2.5 *Let B be a G-graded integral domain and a finitely generated* $\Bbbk$*-algebra, where* $\Bbbk$ *is a field and G is a totally ordered abelian group. Let* $\deg : B \to G \cup \{-\infty\}$ *be the corresponding degree function as defined in Definition* 1.3.10. *Then* $\deg(D)$ *is defined for every* $D \in \operatorname{Der}_{\Bbbk}(B)$. *More precisely, if $h_1, \ldots, h_n$ are* *homogeneous elements of B which generate B as a $\Bbbk$-algebra, then*

$$\deg(D) = \max\{\deg(D(h_i)) - \deg(h_i) |\ 1 \leq i \leq n\}.$$

Proof: Let $\operatorname{def} = \operatorname{def}_D : B \to G \cup \{-\infty\}$ be the defect function of D. Let $K = \max\{\operatorname{def}(h_1), \ldots, \operatorname{def}(h_n)\}$. To prove the proposition, we have to show that $\operatorname{def}(f) \leq K$ for all $f \in B$.

From Lemma 2.2.3 (i) we know that if $f_1, \ldots, f_s \in B$ then $\operatorname{def}(f_1 \cdots f_s) \leq \max_{1 \leq i \leq s} \operatorname{def}(f_i)$. In particular, $\operatorname{def}(h_1^{e_1} \cdots h_n^{e_n}) \leq K$ for any $e_1, \ldots, e_n \in \mathbb{N}$.

Claim 1: $\operatorname{def}(\lambda h_1^{e_1} \cdots h_n^{e_n}) \leq K$ for any $\lambda \in \Bbbk^*$ and $e_1, \ldots, e_n \in \mathbb{N}$.

Proof: First notice that since $\lambda \in \Bbbk^*$ we have $\operatorname{def}(\lambda) = -\infty$ since D is a $\Bbbk$-derivation by assumption. Thus $\operatorname{def}(\lambda h_1^{e_1} \cdots h_n^{e_n}) \leq \max\{\operatorname{def}(\lambda), \operatorname{def}(h_1^{e_1} \cdots h_n^{e_n})\} \leq K$.

Claim 2: If $h \in B$ is homogeneous, then $\operatorname{def}(h) \leq K$.

Proof: If $h = 0$ then $\operatorname{def}(h) = -\infty \leq K$, so suppose $h \neq 0$ and h is homogeneous of degree r. We can write $h = \sum_{i=1}^m f_i$ where $f_i = \lambda_i h_1^{e_{i1}} \cdots h_n^{e_{in}}$ and $\sum_{j=1}^n e_{ij}\deg(h_j) = r$ for all $1 \leq i \leq m$. So $\deg(f_i) = \deg(h)$ for all $1 \leq i \leq m$ and $h = \sum_{i=1}^m f_i$ satisfies the condition in Lemma 2.2.3 (ii). Then $\operatorname{def}(h) \leq \max\{\operatorname{def}(f_1), \ldots, \operatorname{def}(f_m)\}$. But $\operatorname{def}(f_i) \leq K$ for all $1 \leq i \leq m$ by claim 1 so $\operatorname{def}(h) \leq K$.

Claim 3: If $f \in B \setminus \{0\}$, then $\operatorname{def}(f) \leq K$.

Proof: Let $f = f_1 + \ldots + f_m$ where f_i are homogeneous elements of B of distinct degrees. Then $\deg(f) = \deg(\sum_{i=1}^m f_i) = \max\{\deg(f_i) |\ 1 \leq i \leq m\}$ so by Lemma 2.2.3 (ii) $\operatorname{def}(f) \leq \max\{\operatorname{def}(f_1), \ldots, \operatorname{def}(f_m)\}$. But by claim 2, $\operatorname{def}(f_i) \leq K$ for all $1 \leq i \leq m$ so $\operatorname{def}(f) \leq K$. ∎

2.3 Slices

Let B be a commutative $\Bbbk$-algebra which is also an integral domain, where $\Bbbk$ is a field of characteristic zero.

Definition 2.3.1 Let $D \in \mathrm{Der}(B)$, $s \in B$. If $D(s) = 1$, then s is called a *slice* of D. If $D(s) \neq 0$ and $D^2(s) = 0$, then s is called a *preslice*.

Remark 2.3.2 Some authors use *local slice* in place of preslice (e.g. [15]).

Proposition 2.3.3 *Every $D \in \mathrm{LND}(B) \setminus \{0\}$ has a preslice.*

Proof: Since $D \neq 0$ there is some $f \in B \setminus \{0\}$ such that $D(f) \neq 0$. As D is locally nilpotent, there exists minimal $n \in \mathbb{N}$, $n \geq 2$ such that $D^n(f) = 0$. Set $s = D^{n-2}(f)$. Then $D(s) = D^{n-1}(f) \neq 0$ and $D^2(s) = D^n(f) = 0$. ∎

Definition 2.3.4 For any $a \in B$, define a map $\mathrm{ev}_a : B[T] \to B$ called *evaluation at a* which maps a polynomial $f(T) \in B[T]$ to $f(a) \in B$. Note that ev_a is a homomorphism of B-algebras.

Definition 2.3.5 For any $D \in \mathrm{LND}(B)$ define a map called the *exponential map associated to D*:

$$\begin{aligned} \xi_D : B &\to B[T] \\ b &\mapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b) T^n \end{aligned}$$

Proposition 2.3.6 *Let $D \in \mathrm{LND}(B)$. The exponential map associated to D is an injective homomorphism of B^D-algebras.*

Proof: If $b \in B^D$, then $D^i(b) = 0$ for all $i \geq 1$ so $\xi_D(b) = b$. In particular, $\xi_D(1) = 1$.

Let $b_1, b_2 \in B$. Then,

$$\begin{aligned} \xi_D(b_1 + b_2) &= \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b_1 + b_2) T^n \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b_1) T^n + \sum_{n \in \mathbb{N}} \frac{1}{n!} D^n(b_2) T^n \end{aligned}$$

$$= \xi_D(b_1) + \xi_D(b_2)$$

so ξ_D preserves addition.

Also,

$$\begin{aligned}
\xi_D(b_1)\xi_D(b_2) &= \left(\sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(b_1)T^i\right)\left(\sum_{j\in\mathbb{N}} \frac{1}{j!} D^j(b_1)T^j\right) \\
&= \sum_{n\in\mathbb{N}} \left(\sum_{i+j=n} \frac{1}{i!j!} D^i(b_1)D^j(b_2)\right) T^n \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n!} D^n(b_1 b_2)T^n \\
&= \xi_D(b_1 b_2)
\end{aligned}$$

where the second last equality follows from Lemma 2.1.4. Thus, ξ_D is an B^D-algebra homomorphism.

Now notice that the map $\mathrm{ev}_0 \circ \xi_D : B \to B$ is the identity map, so ξ_D is injective. ∎

Theorem 2.3.7 (Proposition 2.1 of [44]) *Let $D \in \mathrm{LND}(B)$ have a slice $s \in B$. Then, $B = B^D[s]$ is a polynomial ring in one variable over the ring of constants of D and $D = \frac{\partial}{\partial s}$.*

Proof: Note that B^D is algebraically closed in B by Lemma 2.1.16 and $s \notin B^D$, so s is transcendental over B^D and $B^D[s]$ is a polynomial ring over B^D.

Since $B^D[s] \subseteq B$ it remains to show that $B \subseteq B^D[s]$. Define a map $\xi : B \to B$, $\xi = ev_{-s} \circ \xi_D$. Since both ev_{-s} and ξ_D are homomorphisms of B^D-algebras, so is ξ.

If $x \in B$, then

$$\begin{aligned}
\xi(x) &= ev_{-s}\Big(\sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)T^i\Big) \\
&= \sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)(-s)^i
\end{aligned}$$

and

$$\begin{aligned}
D(\xi(x)) &= D(\sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)(-s)^i) \\
&= \sum_{i\in\mathbb{N}} \frac{1}{i!} D^{i+1}(x)(-s)^i + \sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x) D((-s)^i) \\
&= \sum_{i\in\mathbb{N}} \frac{1}{i!} D^{i+1}(x)(-s)^i + \sum_{i\in\mathbb{N}\setminus\{0\}} \frac{-1}{(i-1)!} D^i(x)(-s)^{i-1} \\
&= 0
\end{aligned}$$

so we have that $\xi(B) \subseteq B^D$. Also, if $a \in B^D$, then $\xi(a) = \sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(a)(-s)^i = a$ so $\xi(B) = B^D$.

To show that for all $x \in B$ we have $x \in B^D[s]$, we proceed by induction on $\deg_D(x)$ (defined in Proposition 2.1.20).
If $\deg_D(x) \leq 0$, then $x \in B^D[s]$ is clear.
Let $x \in B$ be such that $\deg_D(x) > 0$. We can write $x = \xi(x) + (x - \xi(x))$ and note that

$$\begin{aligned}
x - \xi(x) &= x - \sum_{i\in\mathbb{N}} \frac{1}{i!} D^i(x)(-s)^i \\
&= -s \sum_{i\in\mathbb{N}\setminus\{0\}} \frac{1}{i!} D^i(x)(-s)^{i-1}
\end{aligned}$$

so $x - \xi(x) \in sB$. So $x = a + sy$ for some $a \in B^D$ and $y \in B$.
Then $\deg_D(x) = \deg_D(x-a) = \deg_D(sy) = 1 + \deg_D(y)$. It follows that $\deg_D(y) < \deg_D(x)$. By induction $y \in B^D[s]$, so $x \in B^D[s]$. ∎

Corollary 2.3.8 *For any $D \in \text{LND}(B) \setminus \{0\}$, $S^{-1}B = (\text{Frac}(A))^{[1]}$ where $A = B^D$ and $S = A \setminus \{0\}$. In particular, $\text{trdeg}_A(B) = 1$.*

Proof: By Corollary 2.1.10, $S^{-1}D \in \text{LND}(S^{-1}B)$ and $S^{-1}B^{S^{-1}D} = S^{-1}A = \text{Frac}(A)$. Let $s \in B$ be a preslice of D, and write $a = D(s)$. Then $a \in S$, so

$\frac{s}{a} \in S^{-1}B$ and $\frac{s}{a}$ is a slice of $S^{-1}D$ So by Theorem 2 3 7 $B = \mathrm{Frac}(A)^{[1]}$ ∎

2.4 Derivations of Polynomial Rings

For this section, let $B = \Bbbk^{[n]}$ where $n \in \mathbb{N}$ and $n \geq 1$

The following proposition is well known, see for instance Section 3 2 2 of [15] For a generalization to $R^{[n]}$ where R is any commutative ring, see Proposition 1 2 5 of [42]

Proposition 2.4.1 *Let* $B = \Bbbk[X_1, \ldots, X_n]$ *Then* $\mathrm{Der}_{\Bbbk}(B)$ *is a free* B*-module with basis* $\left\{\frac{\partial}{\partial X_i} \mid 1 \leq i \leq n\right\}$ *In particular,* $D = \sum_{i=1}^{n} D(X_i)\frac{\partial}{\partial X_i}$ *for every* $D \in \mathrm{Der}_{\Bbbk}(B)$

Definition 2.4.2 Let $\gamma = (X_1, \ldots, X_n)$ be a coordinate system of B A derivation $D \in \mathrm{Der}_{\Bbbk}(B)$ is called *γ-triangular* if $D(X_i) \in \Bbbk[X_1, \ldots, X_{i-1}]$ for $i > 1$ and $D(X_1) \in \Bbbk$ We call a derivation $D \in \mathrm{Der}_{\Bbbk}(B)$ *triangular* if it is γ-triangular for some coordinate system γ We will denote the set of γ-triangular derivations of B by $\mathrm{Trian}(B, \gamma)$, and the set of all triangular derivations of B by $\mathrm{Trian}(B)$

Lemma 2.4.3 *If* $D \in \mathrm{Trian}(B, \gamma)$ *for some coordinate system* $\gamma = (X_1, \ldots, X_n)$, *then* $D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial X_i}$ *where* $f_i \in \Bbbk[X_1, \ldots, X_{i-1}]$ *for* $i > 1$ *and* $f_1 \in \Bbbk$

Proof: Follows from the definition of triangular and Proposition 2 4 1 ∎

Lemma 2.4.4 *Let* $D \in \mathrm{Trian}(B, \gamma_n)$ *where* $\gamma_n = (X_1, \ldots, X_n)$ *Then for any* $1 \leq m \leq n$, *D restricts to a γ_m-triangular derivation* $D' \colon \Bbbk[X_1, \ldots, X_m] \to \Bbbk[X_1, \ldots, X_m]$ *where* $\gamma_m = (X_1, \ldots, X_m)$

Proof: Define $D' \colon \Bbbk[X_1, \ldots, X_m] \to \Bbbk[X_1, \ldots, X_m]$ by $D'(f) = D(f)$ for all $f \in \Bbbk[X_1, \ldots, X_m]$ Then D' satisfies the conditions in Definition 2 1 1 and for all

$2 \leq i \leq m$ $D'(X_i) = D(X_i) \in \Bbbk[X_1, \ldots, X_{i-1}]$ and $D'(X_1) = D(X_1) \in \Bbbk$. ∎

Proposition 2.4.5 *If D is a triangular derivation of B, then D is locally nilpotent.*

Proof: Suppose D is γ_n-triangular with $\gamma_n = (X_1, \ldots, X_n)$, $n \geq 1$. We prove by induction. If $n = 1$, then $D(X_1) \in \Bbbk$ and D is locally nilpotent. Suppose that any γ_{n-1}-triangular derivation of $\Bbbk[X_1, \ldots, X_{n-1}]$ is locally nilpotent. Let $D : \Bbbk[X_1, \ldots, X_n] \to \Bbbk[X_1, \ldots, X_n]$ be a γ_n-triangular derivation. By Lemma 2.4.4 and the induction hypothesis, $\Bbbk[X_1, \ldots, X_{n-1}] \subseteq \mathrm{Nil}(D)$. Since $\mathrm{Nil}(D)$ is a subring of $\Bbbk[X_1, \ldots, X_n]$ (Lemma 2.1.6) it remains to show $X_n \in \mathrm{Nil}(D)$. But this is clear since $D(X_n) \in \Bbbk[X_1, \ldots, X_{n-1}]$ by definition of γ_n-triangular. ∎

Proposition 2.4.6 *If $n \geq 2$ and $D \in \mathrm{Trian}(B)$, then there exists a coordinate $g \in B$ such that $D(g) = 0$.*

Proof: Choose a coordinate system $\gamma_n = (X_1, \ldots, X_n)$ for B such that D is γ_n-triangular. Then by Lemma 2.4.4, D restricts to a triangular derivation D' of $\Bbbk[X_1, X_2]$ where $D'(f) = D(f)$ for all $f \in \Bbbk[X_1, X_2]$. Write $D' = b\frac{\partial}{\partial X_1} + f_2(X_1)\frac{\partial}{\partial X_2}$ where $b \in \Bbbk$ and $f_2(X_1) \in \Bbbk[X_1]$. If $b = 0$, then $D(X_1) = D'(X_1) = 0$ so D annihilates the coordinate X_1 of $\Bbbk[X_1, \ldots, X_n]$. Suppose $b \in \Bbbk^*$, and $f_2(X_1) = b_n(X_1)^n + \ldots + b_1X_1 + b_0$. Let $f = \frac{b_n}{n+1}(X_1)^{n+1} + \ldots + \frac{b_1}{2}X_1^2 + b_0X_1$, so $D'(f) = bf_2(X_1)$. Set $g = b^{-1}f - X_2$ and notice $D(g) = D'(g) = f_2(X_1) - f_2(X_1) = 0$ and $\Bbbk[X_1, X_2] = \Bbbk[g, X_1]$ so g is a coordinate of $\Bbbk[X_1, X_2]$. It follows that g is a coordinate of B which is annihilated by D. ∎

2.5 Homogeneous Derivations

In this section, fix an integral domain B and a G-grading $\mathcal{G}$ of B where G is a totally ordered abelian group. Let deg denote the degree function induced by $\mathcal{G}$ as defined in Definition 1.3.10.

Definition 2.5.1 Let $D \in \mathrm{Der}(B)$, $D \neq 0$. If there exists $d \in G$ such that $D(B_i) \subseteq B_{i+d}$ for all $i \in G$ we say D is *$\mathcal{G}$-homogeneous of degree d.* When no confusion can arise with the G-grading of B, we will write D is homogeneous. The set of all $\mathcal{G}$-homogeneous derivations on B is denoted $\mathrm{Der}(B, \mathcal{G})$.

Lemma 2.5.2 *If $D \in \mathrm{Der}(B, \mathcal{G})$ and $D \neq 0$ then D is homogeneous of degree* $\deg(D)$, *where* $\deg(D)$ *is defined as in Definition* 2.2.1.

Proof: It is clear that $d \in U = \{\deg(D(f)) - \deg(f) \mid f \in B \setminus \{0\}\}$. We must show that d is the greatest element of U. Let $f \in B \setminus \{0\}$ and write $f = f_{i_1} + \ldots + f_{i_r}$ where $f_{i_s} \in B_{i_s} \setminus \{0\}$ for all $1 \leq s \leq r$ and $i_j < i_k$ for $j < k$. Then $\deg(f) = i_r$. Since D is homogeneous $D(f_{i_s}) \in B_{i_s+d}$, say $D(f_{i_s}) = g_{i_s+d}$ for all $1 \leq s \leq r$. Then $D(f) = D(f_{i_1} + \ldots + f_{i_r}) = D(f_{i_1}) + \ldots + D(f_{i_r}) = g_{i_1+d} + \ldots + g_{i_r+d}$. Since $\deg(f) = i_r$ and $\deg(D(f)) \leq i_r + d$, we must have $\deg(D(f)) - \deg(f) \leq d$ and we are done. ∎

Definition 2.5.3 Let $D \in \mathrm{Der}(B)$ be such that $\deg(D)$ exists. Define the *homogenization of D*, $\widetilde{D} : B \to B$, as follows

If $D = 0$, let $\widetilde{D} = 0$.

If $D \neq 0$ let $d = \deg(D)$ (so $d \in G$), and for all $i \in G$ define

$$\widetilde{D}_i : B_i \to B_{i+d}$$

$$f_i \mapsto p_{i+d}(D(f_i))$$

where $p_j : B \to B_j$ is the canonical projection for all $j \in G$. Now given $f \in B$, write $f = \sum_{i \in G} f_i$ where $f_i \in B_i$ for all $i \in G$, and define

$$\widetilde{D}(f) = \sum_{i \in G} \widetilde{D}_i(f_i).$$

Remark 2.5.4 If $D \in \mathrm{Der}(B)$ is homogeneous, then $D = \widetilde{D}$.

Proposition 2.5.5 *Let $D \in \mathrm{Der}(B)$. If $\deg(D)$ exists then*

(i) $\widetilde{D} \in \mathrm{Der}(B, \mathcal{G})$

(ii) $\widetilde{D} = 0 \iff D = 0$

(iii) $\deg(\widetilde{D}) = \deg(D)$.

Proof: Let $d = \deg(D)$.

(i) We must check that for all $f, g \in B$ $\widetilde{D}(f+g) = \widetilde{D}(f) + \widetilde{D}(g)$ and $\widetilde{D}(fg) = f\widetilde{D}(g) + g\widetilde{D}(f)$. For $f, g \in B$ write $f = \sum_{i \in G} f_i$ and $g = \sum_{i \in G} g_i$ where $f_i, g_i \in B_i$ for all i. Then

$$\begin{aligned}
\widetilde{D}(f+g) &= \widetilde{D}\left(\sum_{i \in G}(f_i + g_i)\right) \\
&= \sum_{i \in G} \widetilde{D}_i(f_i + g_i) \\
&= \sum_{i \in G} p_{i+d}(D(f_i + g_i)) \\
&= \sum_{i \in G} p_{i+d}(D(f_i) + D(g_i)) \\
&= \sum_{i \in G} (p_{i+d}(D(f_i)) + p_{i+d}(D(g_i))) \\
&= \sum_{i \in G} p_{i+d}(D(f_i)) + \sum_{i \in G} p_{i+d}(D(g_i)) \\
&= \widetilde{D}(f) + \widetilde{D}(g)
\end{aligned}$$

And,

$$\begin{aligned}
\widetilde{D}(fg) &= \widetilde{D}\left(\left(\sum_{j\in G} f_j\right)\left(\sum_{k\in G} g_k\right)\right) \\
&= \widetilde{D}\left(\sum_{i\in G}\sum_{j+k=i} f_j g_k\right) \\
&= \sum_{i\in G}\widetilde{D}_i\left(\sum_{j+k=i} f_j g_k\right) \\
&= \sum_{i\in G} p_{i+d}\left(D\left(\sum_{j+k=i} f_j g_k\right)\right) \\
&= \sum_{i\in G} p_{i+d}\left(\sum_{j+k=i} D(f_j g_k)\right) \\
&= \sum_{i\in G} p_{i+d}\left(\sum_{j+k=i} f_j D(g_k) + g_k D(f_j)\right) \\
&= \sum_{i\in G}\sum_{j+k=i} \left(p_{j+k+d}\left(f_j D(g_k)\right) + p_{j+k+d}\left(g_k D(f_j)\right)\right) \\
&= \sum_{i\in G}\sum_{j+k=i} \left(f_j p_{k+d}(D(g_k)) + g_k p_{j+d}(D(f_j))\right) \\
&= \sum_{i\in G}\sum_{j+k=i} f_j p_{k+d}(D(g_k)) + \sum_{i\in G}\sum_{j+k=i} g_k p_{j+d}(D(f_j)) \\
&= \sum_{i\in G}\sum_{j+k=i} f_j \widetilde{D}_k(g_k) + \sum_{i\in G}\sum_{j+k=i} g_k \widetilde{D}_j(f_j) \\
&= f\widetilde{D}(g) + g\widetilde{D}(f)
\end{aligned}$$

So $\widetilde{D} \in \mathrm{Der}(B)$. As $\widetilde{D}(B_i) = \widetilde{D_i}(B_i) \subseteq B_{i+d}$ for all i, $\widetilde{D}$ is homogeneous of degree d.

(ii) If $D = 0$, then $\widetilde{D} = 0$ by definition. Suppose $D \neq 0$. Then $\deg(D) \in G$, so there exists $f \in B \setminus \{0\}$ such that $d = \deg(D(f)) - \deg(f)$. Suppose $\deg(f) = n$ and write $f = f_n + \sum_{i<n} f_i$ where $f_n \neq 0$ and $f_j \in B_j$ for all j. Note that we must have $p_{n+d}(D(f_n)) \neq 0$. Then $\widetilde{D}(f_n) = \widetilde{D}_n(f_n) = p_{n+d}(D(f_n)) \neq 0$, so $\widetilde{D} \neq 0$.

(iii) Note that

$$\deg(\widetilde{D}) = -\infty \iff \widetilde{D} = 0 \iff D = 0 \iff \deg(D) = -\infty$$

So suppose $\deg(D) \in G$. By the proof of (i) we have that $\widetilde{D}$ is homogeneous of degree d. Then by Lemma 2.5.2, $\deg(\widetilde{D}) = d$ so $\deg(\widetilde{D}) = \deg(D)$.

∎

Lemma 2.5.6 *Let* $D \in \mathrm{Der}(B)$ *be such that* $\deg(D)$ *exists. Let* $d = \deg(D)$.

(i) $\widetilde{D}(p_{j+nd}(D^n(f))) = p_{j+(n+1)d}(D^{n+1}(f))$ *for all* $f \in B$, $n \in \mathbb{N}$ *and all* $j \geq \deg(f)$.

(ii) *If* $h \in B_j$ *for some* j, *then for all* $n \in \mathbb{N}$ $\widetilde{D}^n(h) = p_{j+nd}(D^n(h))$.

Proof:

(i) Let $n = 0$ and let $f = \sum_{i \leq j} f_i$. Recall $D^0 = \mathrm{Id}_B$. Then

$$\begin{aligned}\widetilde{D}(p_j(f)) &= \widetilde{D}(f_j)\\ &= \widetilde{D}_j(f_j)\\ &= p_{j+d}(D(f_j))\\ &= p_{j+d}(D(f))\end{aligned}$$

Let $n \in \mathbb{N}$, $g = D^n(f)$ and $i = j + nd$. Then $i \geq \deg(g)$ and

$$\begin{aligned}\widetilde{D}(p_{j+nd}(D^n(f))) &= \widetilde{D}(p_i(g))\\ &= p_{i+d}(D(g))\\ &= p_{j+(n+1)d}(D^{n+1}(f))\end{aligned}$$

(ii) Let $h \in B_j$ for some $j \in G$. We prove by induction on n: If $n = 0$, then $\widetilde{D}^n(h) = h$ and $p_{j+nd}(D^n(h)) = p_j(h) = h$.

Suppose $\widetilde{D}^n(h) = p_{j+nd}(D^n(h))$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\widetilde{D}^{n+1}(h) &= \widetilde{D}(\widetilde{D}^n(h)) \\ &= \widetilde{D}(p_{j+nd}(D^n(h))) \\ &= p_{j+(n+1)d}(D^{n+1}(h))\end{aligned}$$

Thus $\widetilde{D}^n(h) = p_{j+nd}(D^n(h))$ for all $n \in \mathbb{N}$.

∎

Proposition 2.5.7 *Let* $D \in \mathrm{Der}(B)$ *be such that* $\deg(D)$ *exists. If* $D \in \mathrm{LND}(B)$, *then* $\widetilde{D} \in \mathrm{LND}(B)$.

Proof: If $D = 0$, then $\widetilde{D} = 0 \in \mathrm{LND}(B)$. So suppose $\deg(D) \in G$ and $d = \deg(D)$. Let f_j be homogeneous of degree j, $n \in \mathbb{N}$ such that $D^n(f_j) = 0$. By Lemma 2.5.6(ii), $\widetilde{D}^n(f_j) = p_{j+nd}(D^n(f_j)) = 0$ and so $f_j \in \mathrm{Nil}(\widetilde{D})$. Then every homogeneous element of B is contained in $\mathrm{Nil}(\widetilde{D})$, and since $\mathrm{Nil}(\widetilde{D})$ is a subring of B by Lemma 2.1.6 we must have $\mathrm{Nil}(\widetilde{D}) = B$. ∎

Proposition 2.5.8 *Let* $D \in \mathrm{Der}(B)$ *be such that* $\deg(D)$ *exists. If* $f = \sum_{i \leq n} f_i \in B^D$, *then* $f_n \in B^{\widetilde{D}}$.

Proof: Let $d = \deg(D)$. If $d = -\infty$ it is clear, so suppose $d \in G$. Then

$$\begin{aligned}\widetilde{D}(f_n) &= p_{n+d}(D(f_n)) \\ &= p_{n+d}\left(D(f_n) + \sum_{i<n} D(f_i)\right)\end{aligned}$$

$$= p_{n+d}(D(f))$$
$$= 0.$$

∎

Definition 2.5.9 A subring A of B is called a *graded subring* if $A = \bigoplus_{i \in G}(A \cap B_i)$.

Lemma 2.5.10 *If $D \in \mathrm{Der}(B, \mathcal{G})$, then B^D is a graded subring of B, so B^D is generated by homogeneous elements.*

Proof: Let $d \in G$ be such that D is homogeneous of degree d. Suppose $f \in B^D$. Write $f = \sum_{i \in G} f_i$ where $f_i \in B_i$ and only finitely many $f_i \neq 0$. Then $D(f) = \sum_{i \in G} D(f_i) = 0$, where each $D(f_i) \in B_{i+d}$. So we must have $D(f_i) = 0$ for all $i \in G$, and $f_i \in B^D \cap B_i$. ∎

Lemma 2.5.11 *Let B be a finitely generated $\Bbbk$-algebra which is also an integral domain, where $\Bbbk$ is a field of characteristic zero. Let G be a totally ordered abelian group and suppose B is endowed with two G-gradings:*
$\mathcal{G}_1 : B = \bigoplus_{i \in G} B_i$
$\mathcal{G}_2 : B = \bigoplus_{i \in G} S_i$
such that $B = \bigoplus_{(i,j) \in G \times G} B_i \cap S_j$. If we homogenize a non-zero derivation $D \in \mathrm{Der}_{\Bbbk}(B, \mathcal{G}_1)$ with respect to the grading $\mathcal{G}_2$, the resulting derivation $\widetilde{D}$ belongs to $\mathrm{Der}_{\Bbbk}(B, \mathcal{G}_1) \cap \mathrm{Der}_{\Bbbk}(B, \mathcal{G}_2)$.

Proof: By Proposition 2.2.5, $\deg_{\mathcal{G}_1}(D), \deg_{\mathcal{G}_2}(D)$ are defined. Let $\deg_{\mathcal{G}_1}(D) = t, \deg_{\mathcal{G}_2}(D) = d$. Let $\delta = \widetilde{D} \in \mathrm{Der}(B, \mathcal{G}_2)$ be the homogenization of D with respect to $\mathcal{G}_2$. By Proposition 2.5.5, $\deg_{\mathcal{G}_2}(\delta) = d$.
We want to show that $\delta(B_i) \subseteq B_{i+t}$ for all $i \in G$.

Fix an $i \in G$ and let $f \in B_i$. Noting that the assumption on $\mathcal{G}_1$, $\mathcal{G}_2$ implies that $B_i = \bigoplus_{j \in G}(B_i \cap S_j)$, we may write

$$f = \sum_{j \in G} g_j \qquad g_j \in B_i \cap S_j$$

Then

$$\delta(f) = \sum_{j \in G} p_{j+d}(D(g_j))$$

where $p_k : B \longrightarrow S_k$ is the canonical projection.

Since $g_j \in B_i$ and D is $\mathcal{G}_1$-homogeneous of degree t, we have that $D(g_j) \in B_{i+t} = \bigoplus_{k \in G}(B_{i+t} \cap S_k)$.

Write $D(g_j) = \sum_{k \in G} h_k^j$ where $h_k^j \in B_{i+t} \cap S_k$. Then for all $j \in G$

$$p_{j+d}(D(g_j)) = h_{j+d}^j$$

is an element of B_{i+t}. Thus $\delta(f) = \sum_{j \in G} p_{j+d}(D(g_j)) = \sum_{j \in G} h_{j+d}^j \in B_{i+t}$. Therefore, δ is $\mathcal{G}_1$-homogeneous. ∎

2.6 Homogenization of Derivations

In Definition 2.5.3 we defined the homogenization of a derivation with respect to a grading. We now discuss the homogenization of derivations in the more general situation where we are given a degree function.

Let B be an integral domain, G a totally ordered abelian group and $\deg : B \longrightarrow G \cup \{-\infty\}$ a degree function.

Recall from Remark 1.2.5 that the degree function determines a proper G-filtration $\{B_i\}_{i \in G}$ of B. By Definition 1.3.15, this filtration determines an associated G-graded ring $\mathrm{Gr}(B)$, where $\mathrm{Gr}(B)$ is in fact an integral domain by Remark 1.3.16.

Definition 2.6.1 Let $D \in \text{Der}(B)$ be a derivation such that $\deg(D)$ exists, where $\deg(D)$ is as defined in Definition 2.2.1. We define the *associated homogeneous derivation* $\text{gr}(D) : \text{Gr}(B) \to \text{Gr}(B)$ as follows:
If $D = 0$, let $\text{gr}(D) = 0$.
Otherwise, let $t = \deg(D) \in G$ and note that $D(B_i) \subseteq B_{i+t}$ for all $i \in G$. For each $i \in G$, define

$$\begin{aligned} \text{gr}(D) : B_i/B_{i^-} &\to B_{i+t}/B_{(i+t)^-} \\ f + B_{i^-} &\mapsto D(f) + B_{(i+t)^-} \end{aligned}$$

and extend to $\text{Gr}(B)$ linearly.

Lemma 2.6.2 *If $D \in \text{Der}(B)$ is such that $\deg(D)$ exists, then the following hold:*

(i) $\text{gr}(D)$ *is a homogeneous derivation of* $\text{Gr}(B)$

(ii) $\text{gr}(D) = 0$ *if and only if* $D = 0$

(iii) $\text{gr}(B^D) \subseteq \text{Gr}(B)^{\text{gr}(D)}$ *(where* $\text{gr}(B^D)$ *is the image of* B^D *via* $\text{gr} : B \to \text{Gr}(B)$*)*

(iv) *If* $D \in \text{LND}(B)$*, then* $\text{gr}(D) \in \text{LND}(\text{Gr}(B))$*.*

Proof: The proofs are similar to the ones given in 2.5. We prove assertion (iv) as an example. Suppose $D \in \text{LND}(B)$. If $D = 0$, then $\text{gr}(D) = 0$ so we suppose $D \neq 0$. Since $\text{Gr}(B) = \bigoplus_{i \in G} B_i/B_{i^-}$ it is enough to show $\text{gr}(D)$ is locally nilpotent on B_i/B_{i^-} for all $i \in G$. Let $t = \deg(D)$. Let $a \in B \setminus \{0\}$, let $i = \deg(a)$ so $a \in B_i \setminus B_{i^-}$. Then $D(a) \in B_{i+t}$ and $\deg(D(a)) \leq i + t$. If $\deg(D(a)) < i + t$, then $\text{gr}(D)(\text{gr}(a)) = D(a) + B_{(i+t)^-} = 0$. Else, $\deg(D(a)) = i + t$ and $\text{gr}(D)(\text{gr}(a)) = \text{gr}(D(a))$. By iterating, we see that for any $n \in \mathbb{N}$, $\text{gr}(D)^n(\text{gr}(a)) = 0$ or $\text{gr}(D)^n(\text{gr}(a)) = \text{gr}(D^n(a))$. In particular, if $n \in \mathbb{N}$ is such that $D^n(a) = 0$, then $\text{gr}(D)^n(\text{gr}(a)) = 0$. ∎

Remark 2.6.3 Of special note is the case where B is a G-graded integral domain and $\deg : B \to G \cup \{-\infty\}$ is the induced degree function as defined in Definition 1.3.10. By Remark 1.3.17, there exists an isomorphism $\varphi : \mathrm{Gr}(B) \to B$ identifying $\mathrm{Gr}(B)$ and B. Let $D \in \mathrm{Der}(B)$ and consider the homogenization of D, $\widetilde{D}$, as defined in Definition 2.5.3 and the associated homogeneous derivation of D, $\mathrm{gr}(D)$, as defined in Definition 2.6.1. Then $\widetilde{D}$ and $\mathrm{gr}(D)$ are equivalent in the sense that $\varphi^{-1}\widetilde{D}\varphi = \mathrm{gr}(D)$.

2.7 The Makar-Limanov Invariant

Let B be an integral domain of characteristic zero.

Definition 2.7.1 The *Makar-Limanov invariant* of B is $\mathrm{ML}(B) = \bigcap_{D\in \mathrm{LND}(B)} B^D$. It is also called the *ring of absolute constants* and is sometimes denoted $\mathrm{AK}(B)$.

Lemma 2.7.2 *The Makar-Limanov invariant of B has the following properties:*

(i) $\mathrm{ML}(B)$ *is a factorially closed subring of* B.

(ii) $B^* = \mathrm{ML}(B)^*$

(iii) *If* $\mathbb{k}$ *is a field included in* B, *then* $\mathbb{k} \subseteq \mathrm{ML}(B)$.

(iv) *If* B *is a* UFD, *then so is* $\mathrm{ML}(B)$.

Proof: Part (i) follows from the fact that an intersection of factorially closed subrings is again factorially closed. Then parts (ii), (iii) and (iv) follow from Lemma 2.1.18. ∎

Definition 2.7.3 $\mathrm{ML}(B) = B$ if and only if the only locally nilpotent derivation of B is the zero derivation. These integral domains are called **rigid**.

Lemma 2.7.4 *If $B = \Bbbk[X_1, \ldots, X_n]$ is a polynomial ring, then* $\mathrm{ML}(B) = \Bbbk$.

Proof: It is clear that for all $1 \le i \le n$, $D_i = \frac{\partial}{\partial X_i} \in \mathrm{LND}(B)$ and $B^{D_i} = \Bbbk[X_1, \ldots, \hat{X}_i, \ldots, X_n]$. So $\mathrm{ML}(B) \subseteq \bigcap_{1 \le i \le n} B^{D_i} = \Bbbk$. The reverse inclusion follows from Lemma 2.7.2(iii). ∎

The following is well known, see for instance Lemma 2.3 of [7]:

Proposition 2.7.5 *Let $\Bbbk$ be an algebraically closed field of characteristic zero and let B be an integral domain such that $\Bbbk \subseteq B$ and $\mathrm{trdeg}_{\Bbbk}(B) = 1$. If B is not rigid, then $B = \Bbbk^{[1]}$.*

Proof: Choose a non-zero $D \in \mathrm{LND}(B)$. By Lemma 2.1.16, B^D is algebraically closed in B and by Corollary 2.3.8, $\mathrm{trdeg}_{B^D} B = 1$. Since $\Bbbk \subseteq B^D \subsetneq B$ by Corollary 2.1.21, it follows that $B^D = \Bbbk$.
By Proposition 2.3.3, there exists a preslice r of D. Let $D(r) = c$ and note that $c \in \Bbbk^*$. Then $D(c^{-1}r) = c^{-1}D(r) = 1$ and $s = c^{-1}r$ is a slice of D. So by Theorem 2.3.7, $B = B^D[s] = (B^D)^{[1]} = \Bbbk^{[1]}$. ∎

Lemma 2.7.6 *Let B be an integral domain of characteristic zero and let $B[T] = B^{[1]}$. Then* $\mathrm{ML}(B[T]) \subseteq \mathrm{ML}(B)$.

Proof: Since $\frac{\partial}{\partial T} \in \mathrm{LND}(B[T])$ and $B[T]^{\frac{\partial}{\partial T}} = B$ we have $\mathrm{ML}(B[T]) \subseteq B$.
Note that each $D \in \mathrm{LND}(B)$ extends to a $\widetilde{D} \in \mathrm{LND}(B[T])$ by

$$\widetilde{D}\left(\sum_{i=0}^{n} b_i T^i\right) = \sum_{i=0}^{n} D(b_i) T^i$$

and $\widetilde{D}(T) = \widetilde{D}(1 \cdot T) = D(1)T = 0$. For any $b \in \mathrm{ML}(B[T])$ we have $0 = \widetilde{D}(b) = D(b)$, so $b \in \mathrm{ML}(B)$. ∎

Chapter 3

Characterization of Polynomial Rings

The problem of characterizing $\Bbbk^{[n]}$ for $n \in \mathbb{N}$ is important in its own right. In this chapter we examine certain characterizations of polynomial rings in dimensions one and two. The dimension one case is proved using valuation rings of $\Bbbk(x)$. Using locally nilpotent derivations and a characterization of $\Bbbk^{[1]}$, we prove one of the three characterizations of $\Bbbk^{[2]}$ presented by Miyanishi in [28].

The last section briefly looks at characterizations of $\Bbbk^{[n]}$ for $n > 2$.

3.1 Dimension One

Let $\Bbbk$ be a field.

The following is well known, though no suitable reference could be found:

Lemma 3.1.1 *Let A be a* PID. *If B is a ring such that $A \subseteq B \subseteq \text{Frac}A$, then $B = S^{-1}A$ where $S = B^* \cap A$. In particular, if $B^* = A^*$ then $B = A$.*

Proof: Let $\frac{a}{s} \in S^{-1}A$. Then $s \in B^*$ and $a \in B$. So $\frac{a}{s} = s^{-1}a \in B$ and $S^{-1}A \subseteq B$. Let $b \in B$. Since $B \subseteq \text{Frac}A$, we can write $b = \frac{a_1}{a_2}$ where $a_1, a_2 \in A$. Furthermore, we may assume $\gcd(a_1, a_2) = 1$ since A is a PID. It follows that there exist $c_1, c_2 \in A$ such that $c_1a_1 + c_2a_2 = 1$ in A. Thus in Frac$A$ we have $c_1\frac{a_1}{a_2} + c_2 = \frac{1}{a_2}$. Since $\frac{a_1}{a_2}, c_1, c_2 \in B$ we must have $\frac{1}{a_2} \in B$. Consequently, $a_2 \in B^*$ and $a_2 \in S = B^* \cap A$. So $b \in S^{-1}A$ and $B \subseteq S^{-1}A$. ∎

Recall that a valuation ring of a field $\mathbb{K}$ is a subring R of $\mathbb{K}$ with the property that each $x \in \mathbb{K}^*$ satisfies $x \in R$ or $x^{-1} \in R$. Given a field extension $L/\mathbb{K}$, a valuation ring of $L/\mathbb{K}$ is a valuation ring R of L such that $\mathbb{K} \subseteq R \subseteq L$.

The following result is due to Krull [22]. For a proof, see Theorem 6 in Chapter VI, §4 of [46].

Lemma 3.1.2 *Let A be a subring of a field $\Bbbk$. The integral closure $\bar{A}$ of A in $\Bbbk$ is the intersection of all the valuation rings of $\Bbbk$ which contain A.*

For the following, see for instance Theorem 1.2.2 in [41] (note that the definition of a valuation ring used in [41] is slightly different):

Theorem 3.1.3 *The valuation rings of $\Bbbk(x)/\Bbbk$ are $\Bbbk(x)$, $R_{(p(x))} := \left\{\frac{f(x)}{g(x)} | f(x), g(x) \in \Bbbk[x], p(x) \nmid g(x)\right\}$ where $p(x) \in \Bbbk[x]$ is an irreducible polynomial and $R_\infty := \left\{\frac{f(x)}{g(x)} | f(x), g(x) \in \Bbbk[x], \deg f(x) \leq \deg g(x)\right\}$.*

Remark 3.1.4 If $p(x)$ and $q(x)$ are associates in $\Bbbk[x]$, then $R_{(p(x))} = R_{(q(x))}$. Thus, the distinct valuation rings of $\Bbbk(x)/\Bbbk$ are $\Bbbk(x)$, R_∞ and $R_{(p(x))}$ where $p(x)$ is a monic irreducible polynomial.

The following theorem is well known, although a suitable reference could not be found. We reproduce, up to minor changes, a proof provided by Daigle.

Theorem 3.1.5 *Let B be a $\Bbbk$-algebra. Then $B = \Bbbk^{[1]}$ if and only if B is finitely generated as a $\Bbbk$-algebra, B is a* UFD, *$B^* = \Bbbk^*$ and* Frac$B = \Bbbk^{(1)}$.

Proof: The forward implication is clear. For the reverse implication let Frac$B = \Bbbk(x) = \Bbbk^{(1)}$ and let $V = \{R_\infty\} \cup \{R_{(p(x))} \mid p(x) \in \Bbbk[x]$ is monic and irreducible$\}$. By Theorem 3.1.3, $V \cup \{\Bbbk(x)\}$ is the set of all distinct valuation rings of $\Bbbk(x)/\Bbbk$. Let $P = \{R \in V | B \subseteq R\}$ and note that since every UFD is integrally closed in its field of fractions, $B = \bigcap P$ by Lemma 3.1.2.

Suppose $R_\infty \notin P$. Then $P \subseteq V \setminus \{R_\infty\}$ and $B = \bigcap P \supseteq \bigcap(V \setminus \{R_\infty\})$. But $\bigcap(V \setminus \{R_\infty\}) = \Bbbk[x]$ by Lemma 3.1.2. So we have $\Bbbk[x] \subseteq B \subseteq \Bbbk(x)$ and $B^* = \Bbbk^*$ and by Lemma 3.1.1 we get $B = \Bbbk[x]$.

So suppose $R_\infty \in P$. Since $B \neq \Bbbk$, $P \neq V$. Since B is finitely generated as a $\Bbbk$-algebra, and for any non-zero $b \in B$, there are only finitely many valuation rings which do not contain b (see I.3 in [41]) we must have that $V \setminus P$ is finite. So $V \setminus P = \{R_{(p_1(x))}, \ldots, R_{(p_n(x))}\}$ for some $n \in \mathbb{N}, n > 0$, $p_1(x), \ldots, p_n(x)$ distinct monic irreducible polynomials. Let $p(x) = \prod_{i=1}^n p_i(x)$ and $\Bbbk[x]_{p(x)} = \{\frac{f(x)}{p(x)^j} \mid f(x) \in \Bbbk[x],\ j \in \mathbb{N}\}$. Let $\frac{f(x)}{p(x)^j} \in \Bbbk[x]_{p(x)}$. Since $\Bbbk[x]$ is a UFD, for any $R_{(q(x))} \in P$ we have that $q(x) \nmid p(x)^j$ so $\frac{f(x)}{p(x)^j} \in R_{(q(x))}$ and $\Bbbk[x]_{p(x)} \subseteq \bigcap(P \setminus \{R_\infty\})$. Thus $R_\infty \cap \Bbbk[x]_{p(x)} \subseteq \bigcap P = B$.

Since $R_\infty \in P$, to show $B = \bigcap P = R_\infty \cap \Bbbk[x]_{p(x)}$ it remains to show $B \subseteq \Bbbk[x]_{p(x)}$. Let $b \in B$. Then $b = \frac{f(x)}{g(x)}$ where $\deg f(x) \leq \deg g(x)$ and $q(x) \nmid g(x)$ for any monic irreducible $q(x) \in \Bbbk[x]$, $q(x) \neq p_i(x)$ for $1 \leq i \leq n$. Write $g(x) = u \cdot r_1(x) \cdots r_m(x)$ as a product of a unit $u \in \Bbbk^*$ and monic irreducible polynomials $r_1(x), \ldots, r_m(x) \in \Bbbk[x]$. Since $q(x) \nmid g(x)$ for all $q(x)$ such that $R_{(q(x))} \in P$, we must have that $R_{(r_j(x))} \notin P$ for all $1 \leq j \leq m$. Consequently, for each j we have that $r_j(x) = p_i(x)$ for some $1 \leq i \leq n$. Thus we may write $g(x) = u \cdot p_1^{b_1}(x) \cdots p_n^{b_n}(x)$ where $b_1, \ldots, b_n \in \mathbb{N}$. Let $s = \max_{1 \leq i \leq n}\{b_i\}$. Then $b = u^{-1} f(x) \cdot \frac{p_1^{s-b_1}(x) \cdots p_n^{s-b_n}(x)}{p(x)^s} \in \Bbbk[x]_{p(x)}$. So $B \subseteq \Bbbk[x]_{p(x)}$ and $B = \bigcap P = R_\infty \cap \Bbbk[x]_{p(x)}$.

Next we show $n = 1$, so $V \setminus P = \{R_{(p(x))}\}$. Suppose $n > 1$. Let $m_1 = \deg(p_1(x))$, $m_2 = \deg(p_2(x))$. Let $a = \frac{p_1(x)^{m_2}}{p_2(x)^{m_1}}$. We can write $a = \frac{p_1(x)^{m_2}}{p_2(x)^{m_1}} = \frac{p_1(x)^{m_1+m_2}(p_3(x) \ \ p_n(x))^{m_1}}{p(x)^{m_1}}$ so $a \in \Bbbk[x]_{p(x)}$. Furthermore, $\deg(p_1(x)^{m_2}) = m_1 m_2 = \deg(p_2(x)^{m_1})$ so $a \in R_\infty \cap$

$\Bbbk[x]_{p(x)} = B$. For the same reason, $a^{-1} \in B$. Then $a \in B^*$ but $a \notin \Bbbk^*$, contradiction. So $n = 1$ and $p(x)$ is irreducible in $\Bbbk[x]$.

We have:

$$B = R_\infty \cap \Bbbk[x]_{p(x)} = \left\{ \frac{f(x)}{p(x)^j} \mid f(x) \in \Bbbk[x],\ j \in \mathbb{N},\ \deg(f(x)) \leq \deg(p(x)^j) \right\}$$

Let $q = \frac{1}{p(x)}$, $m = \deg(p(x))$. Then q is an irreducible element of B: Suppose $q = ab$ for some $a, b \in B$. Write $a = \frac{f_a(x)}{p(x)^{j_a}}$, $b = \frac{f_b(x)}{p(x)^{j_b}}$ and assume $p(x) \nmid f_a(x)$, $p(x) \nmid f_b(x)$ in $\Bbbk[x]$. If $j_a = 0$, then $\deg(f_a(x)) \leq \deg(p(x)^0) = 0$, so $f_a(x) \in \Bbbk^*$ and a is a unit of B. Similarly, if $j_b = 0$, then b is a unit of B. Suppose $j_a, j_b \geq 1$. Then $q = \frac{1}{p(x)} = \frac{f_a(x)}{p(x)^{j_a}} \cdot \frac{f_b(x)}{p(x)^{j_b}}$, so $p(x)^{j_a+j_b-1} = f_a(x)f_b(x)$ and $j_a + j_b - 1 \geq 1$. Since $p(x)$ is prime in B and $p(x)|f_a(x)f_b(x)$, it follows that $p(x)|f_a(x)$ or $p(x)|f_b(x)$ in $\Bbbk[x]$, a contradiction. Hence q is an irreducible element of B.

Suppose $m > 1$. Then $y = xq \in B$ is such that $q|y^2$ and $q \nmid y$. This yields a contradiction since B is a UFD and thus irreducible elements of B are prime. Then $m = 1$ and $\text{Frac}B = \Bbbk(p(x)) = \Bbbk(q)$.

We have $\Bbbk[q] \subseteq B \subseteq \Bbbk(q)$ and $B^* = \Bbbk^*$, so by Lemma 3.1.1 $B = \Bbbk[q] = \Bbbk^{[1]}$. ∎

The following is well known (see for instance Chapter 2, Lemma 2.8 of [15]) although the proof is outside of the scope of this thesis:

Theorem 3.1.6 *Let $\Bbbk$ be an algebraically closed field and let B be a finitely generated $\Bbbk$-algebra. If B is a* UFD, *$B^* = \Bbbk^*$ and* $\text{trdeg}_{\Bbbk}(B) = 1$, *then $B = \Bbbk^{[1]}$.*

3.2 Dimension Two

The following lemma is a straightforward consequence of Theorem 2.3.1 in [39]:

Lemma 3.2.1 *Let $A \subseteq B$ be integral domains, where B is finitely generated as an A-algebra. Suppose that $S^{-1}B = (S^{-1}A)^{[1]}$ where S is a multiplicative set of A satisfying* *the following condition: each element of S is a product of units of A and of prime elements p of A such that*

(i) *p is a prime element of B*

(ii) $A \cap pB = pA$

(iii) *A/pA is algebraically closed in B/pB.*

Then $B = A^{[1]}$.

For the following. see for instance Lemma 1.39 in [30]·

Proposition 3.2.2 *Let A be an integral domain containing a field $\Bbbk$ such that $\text{trdeg}_{\Bbbk}(A) = 1$. If A is contained in an integral domain which is finitely generated as a $\Bbbk$-algebra, then A is finitely generated as a $\Bbbk$-algebra.*

Theorem 3.2.3 (Theorem 1 in [28]) *Let $\Bbbk$ be an algebraically closed field of characteristic zero and let B be a finitely generated $\Bbbk$-algebra which is also an integral domain such that $\text{trdeg}_{\Bbbk}(B) = 2$. If B is a UFD, $B^* = \Bbbk^*$ and B is not rigid, then $B = \Bbbk^{[2]}$.*

Proof: Since B is not rigid, there exists a $D \in \text{LND}(B) \setminus \{0\}$ Let $A = B^D$ and note that A is factorially closed in B by Corollary 2.1.21. We will show that $B = A^{[1]}$ and $A = \Bbbk^{[1]}$. By Corollary 2.3.8 we know that $\text{trdeg}_{\Bbbk}(A) = 1$, so by Proposition 3.2.2 we have that A is finitely generated as a $\Bbbk$-algebra. Since B is a UFD, $B^* = \Bbbk^*$ and A is factorially closed in B, we have that $A^* = \Bbbk^*$ and A is a UFD by Lemma 2.1.18. It follows from Theorem 3.1.6 that $A = \Bbbk^{[1]}$.

Let $S = A \setminus \{0\}$. From Corollary 2.3.8 we have that $S^{-1}B = (S^{-1}A)^{[1]}$ Now it is enough to show that each prime element $p \in A$ satisfies conditions (i) - (iii) of Lemma 3.2.1.

Part (i) is clear by Lemma 2.1.18, since in a UFD prime is equivalent to irreducible.

It is also clear that $pA \subseteq A \cap pB$. For the reverse inclusion, let $a \in A \cap pB$, $a \neq 0$. Then $a = pb$ for some $b \in B$. Since A is factorially closed in B and $a \in A \setminus \{0\}$, we must have $b \in A$ and therefore $a \in pA$.

Since $\Bbbk$ is algebraically closed and $A = \Bbbk^{[1]}$, we have $A/pA = \Bbbk$ is algebraically closed in B/pB.

Thus $B = A^{[1]}$ by Lemma 3.2.1, and $B = \Bbbk^{[2]}$. ∎

3.3 Higher Dimension

A few characterizations of $\Bbbk^{[3]}$ are known, most notably Miyanishi's characterization in [29]. Here we will only mention the following, which is a corollary of a result due to Kaliman [19]:

Theorem 3.3.1 *Let B be an integral domain and a finitely generated $\mathbb{C}$-algebra, and suppose that f is an element of B satisfying:*

(i) $B/(f-\alpha) \cong \mathbb{C}^{[2]}$ *for infinitely many* $\alpha \in \mathbb{C}$.

Then the condition $B \cong \mathbb{C}^{[3]}$ *is equivalent to*

(ii) $B/(f-\alpha) \cong \mathbb{C}^{[2]}$ *for all* $\alpha \in \mathbb{C}$.

Note that, so far, none of the known characterizations of $\Bbbk^{[3]}$ have made it possible to solve the Cancellation Problem in dimension 3.

Essentially nothing is known about characterizations of $\Bbbk^{[n]}$ when $n > 3$. In 2000, Kaliman and Zaidenberg showed that Miyanishi's characterization of $\Bbbk^{[3]}$ (which appeared in 1984) does not hold for $\Bbbk^{[n]}$ for $n > 3$. In their proof [18], Kaliman and Zaidenberg gave an example of a $\mathbb{C}$-algebra of dimension 4 which satisfies the conditions of Miyanishi's characterization, but is not a polynomial ring in four variables over $\mathbb{C}$.

Chapter 4

The Cancellation Problem in Dimensions One and Two

First, let us state the Cancellation Problem (**CP(n)**) over a field $\Bbbk$ of characteristic zero:

$$\text{Let } A \text{ be a } \Bbbk\text{-algebra such that } A^{[1]} = \Bbbk^{[n+1]}. \text{ Does it follow that } A = \Bbbk^{[n]}? \quad (4.0.1)$$

This problem is open for $n > 2$.

The purpose of this chapter is to present the solution to the Cancellation Problem in dimensions one and two. The Cancellation Problem in dimension one is based on the Generalized Lüroth Theorem and the characterization of the polynomial ring $\Bbbk^{[1]}$ given in Theorem 3.1.6. In 1972, Abyankar, Heinzer and Eakin [1] obtained results which imply as a corollary that question (4.0.1) has an affirmative answer when $n = 1$. The Cancellation Problem in dimension two over a field of characteristic zero was solved by Fujita [16] using results from Miyanishi and Sugie [28], [31]. We present an adaptation of the more elementary method of Crachiola and Makar-Limanov [9]. We first present the case where $\Bbbk$ is algebraically closed (and of characteristic zero), using the characterization of $\Bbbk^{[2]}$ given in Theorem 3.2.3 and properties of the Makar-Limanov invariant; then we use Kambayashi's Theorem [20] to generalize the result to all fields of characteristic zero. Note that the method used by Crachiola and Makar-Limanov [9] is valid for fields of arbitrary characteristic. Originally, the arbitrary characteristic case was solved by Russell [38].

4.1 Formulations of the Cancellation Problem

There are several well-known equivalent statements of the Cancellation Problem, some of which we state below. These equivalent statements are thoroughly discussed in [43], along with some of their generalizations.

Let $\Bbbk$ be a field of characteristic zero.

4.1.1. Slice Problem Let $A = \Bbbk^{[n+1]}$ and let $D \in \text{LND}(A)$ have a slice. Is it true that $A^D = \Bbbk^{[n]}$?

Proposition 4.1.2 *The Cancellation Problem is equivalent to the Slice Problem.*

Proof: Suppose the Cancellation Problem holds. Let $A = \Bbbk^{[n+1]}$ and let $D \in \text{LND}(A)$ have a slice $s \in A$. Then by Theorem 2.3.7, $A = A^D[s] = (A^D)^{[1]}$ and by the Cancellation Problem, $A^D = \Bbbk^{[n]}$. Conversely, suppose the Slice Problem holds. Let A be a $\Bbbk$-algebra such that $A[T] = \Bbbk^{[n+1]}$. The A-derivation $D = \frac{\partial}{\partial T}$ of $A[T]$ is locally nilpotent and has a slice $s = T$. Since $A[T]^D = A$ and the Slice Problem holds, we have that $A = \Bbbk^{[n]}$. ∎

Remark 4.1.3 The geometric equivalent of the Cancellation Problem is: Let $n \geq 1$ and let V be an affine variety over $\Bbbk$ such that $V \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ as algebraic varieties. Does it follow that $V \cong \mathbb{A}^n$ as algebraic varieties?

4.2 The Cancellation Problem in Dimension One

Let $\mathbb{k}$ be a field, and let $\mathbb{k}^{(n)}$ denote the field of fractions of $\mathbb{k}^{[n]}$.

Theorem 4.2.1 (Generalized Lüroth Theorem) *Let $\mathbb{k} \subseteq \mathrm{F} \subseteq \mathrm{K}$ be field extensions where $\mathrm{K} = \mathbb{k}^{(n)}$ and $\mathrm{trdeg}_{\mathbb{k}}\mathrm{F} = 1$. Then $\mathrm{F} = \mathbb{k}(T)$ for some element $T \in \mathrm{K}$ which is transcendental over $\mathbb{k}$.*

For an elegant proof of the Generalized Lüroth Theorem see [34] or Chapter 1 of [30].

Theorem 4.2.2 (CP(1)) *Let B be a $\mathbb{k}$-algebra satisfying $B^{[m]} = \mathbb{k}^{[m+1]}$ for some $m \in \mathbb{N}$. Then $B = \mathbb{k}^{[1]}$.*

Proof: Let $\mathrm{F} = \mathrm{Frac}B$. Then $\mathbb{k} \subseteq \mathrm{F} \subseteq \mathbb{k}^{(m+1)}$ and $\mathrm{trdeg}_{\mathbb{k}}\mathrm{F} = 1$ so by Theorem 4.2.1 we have $\mathrm{F} = \mathbb{k}^{(1)}$. Since $B^{[m]} = \mathbb{k}^{[m+1]}$, it is clear by Proposition 1.4.4 that B is finitely generated as a $\mathbb{k}$-algebra, B is a UFD and $B^* = \mathbb{k}^*$. So by Theorem 3.1.5, $B = \mathbb{k}^{[1]}$. ∎

4.3 The Cancellation Problem in Dimension Two

For the following, see for instance Lemma 5.2 of [10]:

Lemma 4.3.1 *Let B be an integral domain of characteristic zero and let $D \in \mathrm{LND}(B)$. If $D(B) \subseteq aB$ for some $a \in B$, then $a \in B^D$ and $D = aD'$ for some $D' \in \mathrm{LND}(B)$.*

Proof: If $D = 0$, then $D(B) \subseteq aB$ for all $a \in B$. We have $B \subseteq B^D$ and $D = aD$. So suppose $D \neq 0$ and note that we must have $a \neq 0$. Define $D' : B \to B$ to be the unique set map such that $D(x) = aD'(x)$ for all $x \in B$.

D' is a derivation of B: if $f, g \in B$ then

(i) $aD'(f+g) = D(f+g) = D(f) + D(g) = aD'(f) + aD'(g) = a(D'(f) + D'(g))$ so $D'(f+g) = D'(f) + D'(g)$

(ii) $aD'(fg) = D(fg) = fD(g) + D(f)g = faD'(g) + aD'(f)g = a(fD'(g) + D'(f)g)$ so $D'(fg) = fD'(g) + D'(f)g$.

So $D' \in \mathrm{Der}(B)$, and it is clear that $B^D = B^{D'}$. Since $D \in \mathrm{LND}(B) \setminus \{0\}$ there exists a preslice s of D. By definition, $D(s) \in B^D \setminus \{0\}$ and $D(s) = aD'(s)$. But B^D is factorially closed in B (Corollary 2.1.21), so $a \in B^D$.

Claim: $D^n(x) = a^n D'^n(x)$ for all $x \in B$ and all $n \in \mathbb{N}$. We prove by induction. Let $x \in B$. If $n = 0$ or $n = 1$ it is clear. Suppose $D^k(x) = a^k D'^k(x)$ for some $k \in \mathbb{N}$. Then $D^{k+1}(x) = D(a^k D'^k(x)) = aD'(a^k D'^k(x)) = a^{k+1} D'^{k+1}(x)$.

Thus, since D is locally nilpotent, D' must be locally nilpotent. ∎

The following proposition was known to Makar-Limanov in the mid 1990s, around the time of the introduction of the Makar-Limanov invariant. It is mentioned in [17] without proof. For a proof in the case $\mathbb{k} = \mathbb{C}$, see Lemma 2 in [3]. For the general case of a field of any characteristic, see Theorem 3.1 in [9].

Proposition 4.3.2 *Suppose B is an integral domain and a finitely generated algebra over a field of characteristic zero. If* $\mathrm{ML}(B) = B$, *then* $\mathrm{ML}(B[T]) = B$ *where* $B[T] = B^{[1]}$.

Proof: By Lemma 2.7.6 we have $\mathrm{ML}(B[T]) \subseteq B$. Suppose $D \in \mathrm{LND}(B[T]) \setminus \{0\}$. We must show $B \subseteq B[T]^D$.

Give $B[T]$ the standard $\mathbb{Z}$-grading $B[T] = \bigoplus_{i\in\mathbb{Z}} B[T]_i$, where $B[T]_i = 0$ if $i < 0$ and $B[T]_i = BT^i$ for $i \geq 0$. Let $\deg : B[T] \to \mathbb{Z} \cup \{-\infty\}$ be the degree function induced by the grading as defined in Definition 1.3.10. Since $B[T]$ is a $\mathbb{Z}$-graded finitely generated $\mathbb{k}$-algebra, by Proposition 2.2.5 $\deg(D)$ exists and $d = \deg(D) \in \mathbb{Z}$.

Suppose $d \leq 0$. Let $a \in B[T]_0 \setminus \{0\}$. Then $\deg(D(a)) \leq \deg(a) + \deg(D)$ so $\deg(D(a)) \leq 0 + d \leq 0$ and $D(a) \in B[T]_0$. Since $B[T]_0 = B$, $D(B) \subseteq B$ and we can consider the derivation $D' = D|_B \in \text{LND}(B)$. But B is rigid, so $D' = 0$ and $B \subseteq B[T]^D$.

Suppose $d > 0$. Since $\deg(D)$ exists, by Definition 2.5.3 we can consider the homogenization $\widetilde{D}$ of D. By Proposition 2.5.7, $\widetilde{D} \in \text{LND}(B[T])$ and by Proposition 2.5.5 $\widetilde{D}$ is non-zero and homogeneous of degree d. Then for all $i \in \mathbb{N}$, $\widetilde{D}(B[T]_i) \subseteq B[T]_{i+d} = BT^{i+d} \subseteq T^d B[T]$. So by Lemma 4.3.1, $T^d \in B[T]^{\widetilde{D}}$ and $\widetilde{D} = T^d D'$ for some $D' \in \text{LND}(B[T])$. Since $d > 0$ and $B[T]^{\widetilde{D}}$ is factorially closed in $B[T]$ by Corollary 2.1.21, $T \in B[T]^{\widetilde{D}}$. It is clear that D' is homogeneous of degree 0, so $D'(B) \subseteq B$ and $D'|_B \in \text{LND}(B)$. But since B is rigid, $D'|_B = 0$. Then $B \subseteq B[T]^{\widetilde{D}}$ and $T \in B[T]^{\widetilde{D}}$ thus we must have $\widetilde{D} = 0$, a contradiction. Thus we must have $d \leq 0$ and $\text{ML}(B[T]) = B$. ∎

Corollary 4.3.3 *Let B be a $\mathbb{k}$-algebra, where $\mathbb{k}$ is a field of characteristic zero. If $B^{[1]} = \mathbb{k}^{[n+1]}$ for some $n \geq 1$, then B is not rigid.*

Proof: Suppose $B[T] \cong \mathbb{k}[X_1, \ldots, X_{n+1}] = \mathbb{k}^{[n+1]}$. Then

$$\text{ML}(B[T]) = \text{ML}(\mathbb{k}[X_1, \ldots, X_{n+1}]) = \mathbb{k} \neq B$$

and by Proposition 4.3.2 $\text{ML}(B) \neq B$. ∎

We can now present the solution to the Cancellation Problem in dimension two for an algebraically closed field of characteristic zero:

Proposition 4.3.4 *Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. If B is a $\mathbb{k}$-algebra satisfying $B^{[1]} = \mathbb{k}^{[3]}$, then $B = \mathbb{k}^{[2]}$.*

Proof: Since $B^{[1]}$ is a finitely generated $\mathbb{k}$-algebra and a UFD with trivial units, so is B. Clearly. $\text{trdeg}_{\mathbb{k}}(B) = 2$ and by Corollary 4.3.3 B is not rigid. Then by Theorem 3.2.3 $B = \mathbb{k}^{[2]}$. ∎

The following is a well-known result of Kambayashi:

Theorem 4.3.5 (Theorem 3 in [20]) *Let $K/\mathbb{k}$ be a separable field extension. If B is a $\mathbb{k}$-algebra such that $K \otimes_{\mathbb{k}} B = K^{[2]}$, then $B = \mathbb{k}^{[2]}$.*

For the following, see for instance Proposition 1.5 in [23]:

Proposition 4.3.6 *Let A be a commutative ring with identity and let B be an A-algebra. Then $A^{[n]} \otimes_A B = B^{[n]}$.*

Corollary 4.3.7 *Let $\mathbb{k}$ be a field, $\overline{\mathbb{k}}$ its algebraic closure and let B be a $\mathbb{k}$-algebra. Then for all $n \in \mathbb{N}$:*

(i) $\mathbb{k}^{[n]} \otimes_{\mathbb{k}} \overline{\mathbb{k}} = \overline{\mathbb{k}}^{[n]}$

(ii) $B \otimes_{\mathbb{k}} (\overline{\mathbb{k}}^{[n]}) = B^{[n]} \otimes_{\mathbb{k}} \overline{\mathbb{k}}$

Proof:

(i) Apply Proposition 4.3.6 with $A = \mathbb{k}$, $B = \overline{\mathbb{k}}$,

(ii) Apply (i) and Proposition 4.3.6:

$$B \otimes_{\mathbb{k}} (\overline{\mathbb{k}}^{[n]}) = B \otimes_{\mathbb{k}} (\mathbb{k}^{[n]} \otimes_{\mathbb{k}} \overline{\mathbb{k}}) = (B \otimes_{\mathbb{k}} \mathbb{k}^{[n]}) \otimes_{\mathbb{k}} \overline{\mathbb{k}} = B^{[n]} \otimes_{\mathbb{k}} \overline{\mathbb{k}}$$

∎

We can now strengthen Proposition 4.3.4 by dropping the condition that $\mathbb{k}$ is algebraically closed:

Theorem 4.3.8 (CP(2), [16]) *Let $\mathbb{k}$ be a field of characteristic zero. If B is a $\mathbb{k}$-algebra satisfying $B^{[1]} = \mathbb{k}^{[3]}$, then $B = \mathbb{k}^{[2]}$.*

Proof: Let $\overline{\mathbb{k}}$ the algebraic closure of $\mathbb{k}$ and let $\overline{B} = B \otimes_{\mathbb{k}} \overline{\mathbb{k}}$. Since $\overline{B}$ is a $\mathbb{k}$-algebra, $\overline{B}^{[1]} = \overline{B} \otimes_{\mathbb{k}} \mathbb{k}^{[1]}$ by Proposition 4.3.6. Then

$$
\begin{aligned}
\overline{B}^{[1]} &= (B \otimes_{\mathbb{k}} \overline{\mathbb{k}}) \otimes_{\mathbb{k}} \mathbb{k}^{[1]} \\
&= B \otimes_{\mathbb{k}} (\overline{\mathbb{k}} \otimes_{\mathbb{k}} \mathbb{k}^{[1]}) \\
&= B \otimes_{\mathbb{k}} \overline{\mathbb{k}}^{[1]} && \text{by Corollary 4.3.7 (i)} \\
&= B^{[1]} \otimes_{\mathbb{k}} \overline{\mathbb{k}} && \text{by Corollary 4.3.7 (ii)} \\
&= \mathbb{k}^{[3]} \otimes_{\mathbb{k}} \overline{\mathbb{k}} \\
&= \overline{\mathbb{k}}^{[3]} && \text{by Corollary 4.3.7 (i)}
\end{aligned}
$$

Therefore, by Proposition 4.3.4 we have $\overline{B} = \overline{\mathbb{k}}^{[2]}$, so $B \otimes_{\mathbb{k}} \overline{\mathbb{k}} = \overline{\mathbb{k}}^{[2]}$ and by Kambayashi's Theorem 4.3.5 $B = \mathbb{k}^{[2]}$. ∎

Chapter 5

Russell's Cubic

Throughout this chapter, let $\Bbbk$ be a field of characteristic zero, and let

$$R = \Bbbk[X,Y,Z,T]/\langle X + X^2Y + Z^2 + T^3\rangle = \Bbbk[x,y,z,t],$$

where $\Bbbk[X,Y,Z,T]$ is the polynomial ring in 4 variables over $\Bbbk$ and x, y, z, t are the images of X, Y, Z, T via the canonical epimorphism $\Bbbk[X,Y,Z,T] \to R$. The $\Bbbk$-algebra R was first defined in [37] and is known as **Russell's cubic**. Makar-Limanov proved in [25] that $R \neq \Bbbk^{[3]}$ when $\Bbbk = \mathbb{C}$, while the arbitrary characteristic case was proved by Crachiola in [8].

We begin this chapter by presenting some properties of Russell's Cubic that demonstrate its similarity with $\Bbbk^{[3]}$. We review the Derksen invariant of a ring which was first defined by Derksen in his thesis [13]. Like the Makar-Limanov invariant, it is derived from the kernels of locally nilpotent derivations. Using this invariant, we show that $R \neq \Bbbk^{[3]}$ and $\mathrm{ML}(R) \neq \Bbbk$ following mainly the proof presented in [26] with some adjustments for rigour. Whether $R^{[1]}$ is a polynomial ring in four variables over $\Bbbk$ is an open question, so Russell's Cubic is a potential counter-example to the Cancellation Problem in dimension three. We will see that the Derksen and Makar-Limanov invariants do not differentiate $R^{[1]}$ and $\Bbbk^{[4]}$.

5.1 Properties of Russell's Cubic

Definition 5.1.1 Let B be a finitely generated $\mathbb{k}$-algebra, $\mathfrak{p}$ a prime ideal of B The *height* of $\mathfrak{p}$, denoted $\text{ht}(\mathfrak{p})$, is the supremum of all $n \in \mathbb{N}$ such that there exists a chain of distinct prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p}$. The *Krull dimension* of B is defined as $\dim(B) = \sup\{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal of B}\}$.

The following is well known, see for instance Lemma 6.39 in [36]:

Lemma 5.1.2 *Let B be a* UFD.

(i) *If p is an irreducible element of B then the principal ideal $\langle p \rangle$ is a prime ideal of height* 1.

(ii) *Every height* 1 *prime ideal of B is a principal ideal generated by an irreducible element.*

For the following, see for instance Theorem 23 in [27]:

Theorem 5.1.3 *Let B be a finitely generated $\mathbb{k}$-algebra which is also an integral domain. Then* $\dim(B) = \text{trdeg}_{\mathbb{k}}(B)$ *and for any prime ideal $\mathfrak{p}$ of B,* $\dim(B) = \dim(B/\mathfrak{p}) + \text{ht}(\mathfrak{p})$.

Proposition 5.1.4 (i) *R is an integral domain*

(ii) $\dim R = 3$

(iii) $\text{Frac}R = \mathbb{k}^{(3)}$

Proof:

(i) This is because $X+X^2Y+Z^2+T^3$ is irreducible. hence prime since $\mathbb{k}[X,Y,Z,T]$ is a UFD.

(ii) By Lemma 5.1.2 and Theorem 5.1.3, $\dim(R) = \dim(\mathbb{k}[X,Y,Z,T]) - \mathrm{ht}(X + X^2Y + Z^2 + T^3) = 4 - 1 = 3$.

(iii) We know by Theorem 5.1.3 that $\mathrm{trdeg}_{\mathbb{k}}(\mathrm{Frac}R) = \dim(R) = 3$. We have $\mathrm{Frac}R = \mathbb{k}(x,y,z,t)$, but $y = x^{-2}(-x - z^2 - t^3)$ so $\mathrm{Frac}R = \mathbb{k}(x,z,t) = \mathbb{k}^{(3)}$.

∎

Lemma 5.1.5 $R_x = \mathbb{k}[x]_x[z,t]$ *is a localization of a polynomial ring in 3 variables.*

Proof: Note $R_x = \mathbb{k}[x,y,z,t]_x \subseteq \mathbb{k}(x,y,z,t)$. In R_x, we can write $y = \frac{-z^2-t^3-x}{x^2}$ so $R_x = \mathbb{k}[x,z,t]_x$. We know x, z, t are algebraically independent because $\mathrm{trdeg}_{\mathbb{k}}(R) = 3$.
∎

The following is well known, see for instance Theorem 7.53 in [35]:

Lemma 5.1.6 *If B is a* UFD *and $S \subseteq B$ a multiplicatively closed set, then $S^{-1}B$ is a* UFD.

Lemma 5.1.7 *Let B be an integral domain and let $q_1, \dots, q_n \in B$ be prime elements. If $q_1 \cdots q_n \mid xy$ for some $x, y \in B$, then there exist subproducts Q_1, Q_2 of $q_1 \cdots q_n$ such that $Q_1 \mid x$, $Q_2 \mid y$ and $q_1 \cdots q_n = Q_1Q_2$.*

Proof: We prove this by induction. If $n = 1$ it is clear. Let $n \in \mathbb{N}$, $n \geq 2$ and suppose the lemma holds for any prime elements $q_1, \dots, q_{n-1} \in B$. Let $q_n \in B$ be a prime element, and let $x, y \in B$ be such that $q_1 \cdots q_n \mid xy$. Since $q_n \mid xy$, we must have that $q_n \mid x$ or $q_n \mid y$. WLOG, suppose $q_n \mid x$. Then $q_1 \cdots q_{n-1} \mid \frac{x}{q_n}y$ and we can apply the inductive hypothesis: there exist subproducts Q_1', Q_2' of $q_1 \cdots q_{n-1}$ such that $q_1 \cdots q_{n-1} = Q_1'Q_2'$ and $Q_1' \mid \frac{x}{q_n}$, $Q_2' \mid y$. Let $Q_1 = Q_1'q_n$ and $Q_2 = Q_2'$. Then $p_1 \cdots q_n \mid Q_1Q_2$ where $Q_1 \mid x$ and $Q_2 \mid y$ and we are done. ∎

Lemma 5.1.8 (Nagata) *Let B be an integral domain and S a multiplicatively closed subset of B generated by prime elements of B. Then B is a* UFD *if and only if B satisfies the ascending chain condition on principal ideals (ACCP) and $S^{-1}B$ is a* UFD.

Proof: The forward implication follows from Lemma 5.1.6. For the converse, it is enough to show every irreducible element of B is prime. Let $p \in B$ be irreducible. We will first show that p is irreducible or a unit in $S^{-1}B$. Write $p = \frac{a_1}{s_1} \cdot \frac{b_1}{t_1}$ where $a_1, b_1 \in B$, $s_1, t_1 \in S$, so we must show $\frac{a_1}{s_1}$ or $\frac{b_1}{t_1}$ is a unit in $S^{-1}B$. Write $s_1 t_1 = u q_1 \cdots q_r$ where $r \geq 0$, $u \in B^*$ and $q_1, \ldots, q_r \in S$ are prime elements of B. Then $p u q_1 \cdots q_r = a_1 b_1$ in B.

If $r = 0$, then $q_1 \cdots q_r = 1$ and $s_1 t_1 = u$. So $p = u^{-1} a_1 b_1$ and $s_1, t_1 \in B^*$. Since p is irreducible in B, we must have that a_1 or b_1 is a unit. Then either $\frac{a_1}{s_1}$ or $\frac{b_1}{t_1}$ is a unit in B, hence in $S^{-1}B$, and therefore p is irreducible or a unit in $S^{-1}B$.

If $r > 0$, then by Lemma 5.1.7 there exist subproducts Q_1, Q_2 of $q_1 \cdots q_r$ such that $q_1 \cdots q_r = Q_1 Q_2$ and $Q_1 \mid a_1$, $Q_2 \mid b_1$. So $p = u^{-1} \frac{a_1}{Q_1} \frac{b_1}{Q_2}$ where $u \in B^*$. $\frac{a_1}{Q_1}, \frac{b_1}{Q_2} \in B$. Since p is irreducible, we must have that either $\frac{a_1}{Q_1}$ or $\frac{b_1}{Q_2}$ is a unit in B. WLOG, suppose $\frac{a_1}{Q_1}$ is a unit in B. Then $\frac{a_1}{Q_1}$ is a unit in $S^{-1}B$ and it follows that $\frac{a_1}{s_1}$ is a unit in $S^{-1}B$, because $Q_1 \in S$. Thus, p is either irreducible or a unit in $S^{-1}B$.

If p is a unit in $S^{-1}B$, then there exists some $\frac{w}{s} \in S^{-1}B$ ($w \in B$, $s \in S$) such that $p\frac{w}{s} = 1$. So $pw = s$ in B. Write $s = u' q'_1 \cdots q'_n$ where $n \geq 0$, $u' \in B^*$ and $q'_1, \ldots q'_n \in S$ are prime elements of B. So $p w u'^{-1} = q'_1 \cdots q'_n$ in B.

If there exists an i such that $q'_i \mid p$, then q'_i and p are associates and therefore p is a prime element of B.

If not, then $q'_1 \cdots q'_n \mid w$ and p is a unit in B, which is absurd.

So if p is a unit of $S^{-1}B$, it is a prime element of B.

If p is an irreducible element of $S^{-1}B$, then p is prime in $S^{-1}B$ since $S^{-1}B$ is a UFD.

Suppose $p \mid ab$ in B. We must show that $p \mid a$ or $p \mid b$ in B. Since $p \mid ab$ in $S^{-1}B$, it follows that $p \mid a$ or $p \mid b$ in $S^{-1}B$. WLOG, suppose $p \mid a$. Then $a = p\frac{c}{s}$ for some $c \in B$, $s \in S$. Write $s = u'q'_1 \cdots q'_n$ where $n \geq 0$, $u' \in B^*$ and $q'_1, \ldots q'_n \in S$ are prime elements of B. Then $pc = au'q'_1 \cdots q'_n$ in B.

If there exists an i such that $q'_i \mid p$, then q'_i and p are associates and therefore p is a prime element of B.

If not, $s \mid c$ and therefore $\frac{c}{s} \in B$. It follows that $p \mid a$ in B, p is a prime element of B and therefore B is a UFD. ∎

Proposition 5.1.9 *R is a* UFD.

Proof: Since R is finitely generated as a $\Bbbk$-algebra, R satisfies ACCP. By Lemma 5.1.8 it is enough to show that x is a prime element of R and that R_x is a UFD. The latter is clear from Lemma 5.1.5 and Lemma 5.1.6. We will show R/xR is an integral domain. Notice that $R/xR \cong \Bbbk[X,Y,Z,T]/\langle X, X + X^2Y + Z^2 + T^3\rangle$ and $\langle X, X + X^2Y + Z^2 + T^3\rangle = \langle X, Z^2 + T^3\rangle$. Then $R/xR \cong \Bbbk[Y,Z,T]/\langle Z^2 + T^3\rangle$. Since $\Bbbk[Y,Z,T]$ is a UFD and $Z^2 + T^3$ is irreducible (hence prime), $\Bbbk[Y,Z,T]/\langle Z^2 + T^3\rangle$ is an integral domain. It follows that $x \in R$ is prime. ∎

Lemma 5.1.10 *Let $A \subsetneq B$ be integral domains and let $x \in A\setminus\{0\}$. If $xB \cap A = xA$, then $A_x \cap B = A$ (where the last intersection is taken in* Frac(B)*).*

Proof: It is clear that $A \subseteq A_x \cap B$. Let $b \in B \cap A_x$. Write $b = \frac{a}{x^n}$ where $a \in A$ and $n \in \mathbb{N}$ is minimal. Suppose $n > 0$. Then $a = x^n b$ in B and $a \in xB \cap A = xA$. So there exists $a' \in A$ such that $a = xa'$. Then $b = \frac{a}{x^n} = \frac{a'}{x^{n-1}}$ contradicting the assumption that n is minimal. So $n = 0$ and $b \in A$ ∎

In [26], Makar-Limanov defined the following two $\Bbbk$-derivations of $\Bbbk[X,Y,Z,T]$:

$$\begin{aligned} D_1 : \Bbbk[X,Y,Z,T] &\to \Bbbk[X,Y,Z,T] \\ X &\mapsto 0 \\ Y &\mapsto 3T^2 \\ Z &\mapsto 0 \\ T &\mapsto -X^2 \end{aligned} \tag{5.1.1}$$

and

$$\begin{aligned} D_2 : \Bbbk[X,Y,Z,T] &\to \Bbbk[X,Y,Z,T] \\ X &\mapsto 0 \\ Y &\mapsto 2Z \\ Z &\mapsto -X^2 \\ T &\mapsto 0 \end{aligned} \tag{5.1.2}$$

Note that these are γ-triangular where $\gamma = (X,Z,T,Y)$, hence locally nilpotent by Proposition 2.4.5. Since $D_1(X + X^2Y + Z^2 + T^3) = D_2(X + X^2Y + Z^2 + T^3) = 0$, we have induced locally nilpotent derivations $\overline{D}_1$ and $\overline{D}_2$ on R by Proposition 2.1.8. Our goal is to show $\mathrm{ML}(R) = \Bbbk[x]$. We begin with the easy inclusion $\mathrm{ML}(R) \subseteq \Bbbk[x]$ and later (Corollary 5.3.14) we will show the reverse.

Proposition 5.1.11 $\mathrm{ML}(R) \subseteq \Bbbk[x]$

Proof: Since $\mathrm{ML}(R) = \bigcap_{D \in \mathrm{LND}(R)} R^D$, it is enough to find two locally nilpotent derivations $\overline{D}_1$, $\overline{D}_2$ of R such that $R^{\overline{D}_1} \cap R^{\overline{D}_2} \subseteq \Bbbk[x]$. Let $\overline{D}_1$, $\overline{D}_2$ be the derivations on R induced by (5.1.1) and (5.1.2) respectively. We will show $R^{\overline{D}_1} = \Bbbk[x,z]$ and note that the same method can be used to show $R^{\overline{D}_2} = \Bbbk[x,t]$. Let $A_1 = R^{\overline{D}_1}$. Since $x \in A_1$ we have $\overline{D}_{1x} \in \mathrm{LND}(R_x)$ and $R_x^{\overline{D}_{1x}} = (A_1)_x$ by Corollary 2.1.10. Write $A_{1x} = R_x^{\overline{D}_{1x}}$ and write $E = \Bbbk[x]_x$, so $R_x = E[z,t] = E^{[2]}$. Then $E[z] \subseteq A_{1x} \subseteq$

$E[z,t]$. By Corollary 2.3.8, the transcendence degree of $E[z,t]$ over A_{1x} is 1. Since the transcendence degree of $E[z,t]$ over $E[z]$ is also 1 and $E[z]$ is algebraically closed in $E[z,t]$ we must have $A_{1x} = E[z]$. Since the following diagram commutes:

$$\begin{array}{ccc} E[z,t] & \xrightarrow{\overline{D}_{1x}} & E[z,t] \\ \uparrow & & \uparrow \\ R & \xrightarrow{\overline{D}_1} & R \end{array}$$

we have $R^{\overline{D}_1} = R \cap E[z,t]^{\overline{D}_{1x}}$, that is, $A_1 = R \cap A_{1x}$.

Let $S = \Bbbk[x,z]$. If we can show $xR \cap S = xS$, then by Lemma 5.1.10 we have $S = S_x \cap R$. Since $S_x = A_{1x}$, it follows that $A_1 = S$, that is, $R^{\overline{D}_1} = \Bbbk[x,z]$. It is clear that $xS \subseteq xR \cap S$ so suppose $f \in xR \cap S$. Write $f = xr$ for some $r \in R$ and let $G \in \Bbbk[X,Y,Z,T]$ be such that $\pi(G) = r$ where $\pi : \Bbbk[X,Y,Z,T] \twoheadrightarrow R$ is the canonical epimorphism. Also consider $F(X,Z) \in \Bbbk[X,Z]$ such that $\pi(F) = f$. Then $F(X,Z) - XG(X,Y,Z,T) \in \langle X + X^2Y + Z^2 + T^3 \rangle$ and $F(X,Z) \in \langle X, X + X^2Y + Z^2+T^3\rangle = \langle X, Z^2+T^3\rangle$. Write $F(X,Z) = XH + (Z^2+T^3)L$ for $H, L \in \Bbbk[X,Y,Z,T]$. Since the degree of F with respect to T is 0 and $F(0,Z) = (Z^2+T^3)L(0,Y,Z,T)$ we must have $L(0,Y,Z,T) = 0$. But this means $F(0,Z) = 0$, i.e. $X \mid F$ in $\Bbbk[X,Z]$ so $f \in xS$. Thus $xR \cap S = xS$ as required.

Similarly, $R^{\overline{D}_2} = \Bbbk[x,t]$ and finally $\mathrm{ML}(R) \subseteq R^{\overline{D}_1} \cap R^{\overline{D}_1} = \Bbbk[x,z] \cap \Bbbk[x,t] = \Bbbk[x]$, where the intersection is taken in $\Bbbk[x,z,t] = \Bbbk^{[3]}$. ∎

Corollary 5.1.12 $R^* = \Bbbk^*$

Proof: By Lemma 2.7.2, $R^* = \mathrm{ML}(R)^*$ and by Proposition 5.1.11 $\mathrm{ML}(R)^* \subseteq \Bbbk[x]^* = \Bbbk^*$. Since $\Bbbk^* \subseteq R^*$, we have equality. ∎

5.2 The Derksen Invariant

In his Ph.D. thesis [13], Derksen introduced the following ring invariant:

Definition 5.2.1 The *Derksen invariant* of a ring B, denoted $\mathcal{D}(B)$, is the subring of B generated by $\bigcup_{D\in\mathfrak{D}} B^D$ where $\mathfrak{D} = \{D \in \text{LND}(B) | \ D \neq 0\}$.

Remark 5.2.2 If $B = \Bbbk[X_1, \dots, X_n] = \Bbbk^{[n]}$ and $n \geq 2$, then $X_i \in \mathcal{D}(B)$ for all $1 \leq i \leq n$ and $\Bbbk \subseteq \mathcal{D}(B)$, so $\mathcal{D}(B) = B$. If $B = \Bbbk^{[1]}$, then $\mathcal{D}(B) = \Bbbk$.

We continue to consider Russell's Cubic:

$$R = \Bbbk[X,Y,Z,T]/\langle X + X^2Y + Z^2 + T^3\rangle = \Bbbk[x,y,z,t].$$

Our goal is to show $\mathcal{D}(R) = \Bbbk[x,z,t]$. We begin with the easy inclusion $\Bbbk[x,z,t] \subseteq \mathcal{D}(R)$ and later (Theorem 5.3.12) we will show the reverse.

Proposition 5.2.3 $\Bbbk[x,z,t] \subseteq \mathcal{D}(R)$.

Proof: Recall $D_1, D_2 \in \text{LND}(\Bbbk[X,Y,Z,T]) \setminus \{0\}$ as defined in (5.1.1) and (5.1.2):

$$D_1 : \Bbbk[X,Y,Z,T] \to \Bbbk[X,Y,Z,T]$$
$$X \mapsto 0$$
$$Y \mapsto 3T^2$$
$$Z \mapsto 0$$
$$T \mapsto -X^2$$

and

$$D_2 : \Bbbk[X,Y,Z,T] \to \Bbbk[X,Y,Z,T]$$
$$X \mapsto 0$$
$$Y \mapsto 2Z$$

$$Z \mapsto -X^2$$

$$T \mapsto 0$$

These derivations induce non-zero locally nilpotent derivations $\overline{D}_1, \overline{D}_2 \in \text{LND}(R)\setminus\{0\}$. In Proposition 5.1.11 we showed $R^{\overline{D}_1} = \Bbbk[x,z]$ and $R^{\overline{D}_2} = \Bbbk[x,t]$, so we must have $\Bbbk[x,z,t] \subseteq \mathcal{D}(R)$. ∎

5.3 Russell's Cubic is not $\Bbbk^{[3]}$

We show that the Derksen invariant of Russell's Cubic R is not equal to R and in particular, R is not a polynomial ring in three variables over $\Bbbk$. This was first proved by Makar-Limanov in [25] with $\Bbbk = \mathbb{C}$, and he refined his argument in [26]. Other proofs appeared, among which is the one given by Freudenburg in [15] (Theorem 9.6, page 201).

Our approach is based on Makar-Limanov's method in [26], with some ideas borrowed from Freudenburg's proof in [15]. There are some concerns with each of these proofs, as we shall now explain, and writing this section required us to resolve these issues. One of the main results of [26] is Lemma 5, which shows that $R^D \subseteq \mathbb{C}[x,z,t]$ for any $D \in \text{LND}(R) \setminus \{0\}$. Consider the first two sentences of the proof of this lemma:

> Let $G = \mathbb{Z}\times\mathbb{Z}$ which is ordered lexicographically. Define a degree function on R into G by $\deg(x) = (-1,6)$, $\deg(y) = (2,-6)$, $\deg(z) = (0,3)$, and $\deg(t) = (0,2)$.

The problem here is that one cannot define a degree function by giving its values on the generators. Indeed, Remark 1.3.13 gives an example of two $\mathbb{Z}$-valued degree functions on $\mathbb{C}[X,Y]$ which agree on the generators X and Y and map every element of $\mathbb{C}^*$ to zero. In Setup 5.3.2, below, we give the correct procedure for defining the desired degree function on R.

The following sentence in the proof is an unsupported claim:

> The corresponding graded ring $\mathrm{Gr}(R)$ is isomorphic to $\mathbb{C}[x,y,z,t]/(x^2y+z^2+t^3)$.

Our proof of this assertion is given in Lemma 5.3.4 - Corollary 5.3.6, and is an elaboration of the argument given in [15], page 202.

The next line in the proof is the claim (in a different notation) that $\deg(D)$ is defined for every $D \in \mathrm{LND}(R)$. While this is true, the reasoning given in [26] is flawed: on the preceding page, one finds the assertion that if A is a finitely generated $\mathbb{C}$-algebra, then $\deg(D)$ is defined for any $D \in \mathrm{LND}(A)$ and any degree function "deg" on A. This is not correct, as [11] gives an example of a degree function $\deg : \mathbb{C}[X,Y,Z] \to \mathbb{N}\cup\{-\infty\}$ with respect to which $\deg\left(\frac{\partial}{\partial X}\right)$ is not defined. In our proof that $\mathcal{D}(R) \neq R$ (Proposition 5.3.10, below), we use Lemma 2.2.4 and Proposition 2.2.5 to show that $\deg(D)$ is indeed defined for every $D \in \mathrm{LND}(R)$.

There are also some concerns with the proof given in [15], though to a lesser extent. The author begins by defining a $\mathbb{Z}$-valued degree function on $\Bbbk[x]_x[z,t]$ by stipulating its values on x, z, t. The restriction of this degree function to R determines a proper $\mathbb{Z}$-filtration $\{R_\imath\}_{\imath\in\mathbb{Z}}$ on R and an associated graded algebra $\mathrm{Gr}(R) = \bigoplus_{\imath\in\mathbb{Z}} R_\imath/R_{\imath-1}$. He shows $\mathrm{Gr}(R) = \Bbbk[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)]$ and defines a $\mathbb{Z}$-valued degree function on $\mathrm{Gr}(R)$ by assigning values on the generators $\mathrm{gr}(x)$, $\mathrm{gr}(y)$, $\mathrm{gr}(z)$, $\mathrm{gr}(t)$. We have already mentioned that a degree function cannot be defined by simply giving its values on the generators. The author claims this degree function induces a second grading $\mathrm{Gr}(R) = \bigoplus_{\imath\in\mathbb{Z}} S_\imath$. However, as we show in Remark 1.3.12, distinct gradings on a ring S may determine the same degree function on S, so giving a degree function on S does not induce a grading on S.

The question as to whether $\mathrm{gr}(D)$ is defined for every $D \in \mathrm{LND}(R)$ seems to have been overlooked: the author makes use of Principle 15 (page 30) without checking that its hypothesis is satisfied, i.e. if $\deg(D)$ is defined.

Further in the proof, he implicitly uses a critical lemma without reference. We are in the situation where we have a $D \in \text{LND}(R)$, its associated homogeneous derivation $\text{gr}(D) \in \text{LND}(\text{Gr}(R))$ and the homogenization $\widetilde{\text{gr}(D)}$ of $\text{gr}(D)$ with respect to the second $\mathbb{Z}$-grading on $\text{Gr}(R)$. He claims that $\text{Gr}(R)^{\widetilde{\text{gr}(D)}}$ is generated by elements which are homogeneous with respect to both $\mathbb{Z}$-gradings. The missing step involves showing that $\text{Gr}(R) = \bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} (S_i \cap R_j/R_{j-1})$ and applying Lemma 2.5.11.

We begin the proof that $R \neq \Bbbk^{[3]}$ with a useful lemma:

Lemma 5.3.1 (Lemma 9.3 of [15], Lemma 2 of [26]) *Let $m, n \in \mathbb{N}$, $m, n > 1$. Let B be a finitely generated $\Bbbk$-algebra which is also an integral domain, $D \in \text{LND}(B) \setminus \{0\}$. If $D(c_1a^m + c_2b^n) = 0$, where $a, b \in B$, $c_1, c_2 \in B^D \setminus \{0\}$ and $c_1a^m + c_2b^n \neq 0$, then $D(a) = D(b) = 0$.*

Proof: If one of a, b belongs to B^D, say $a \in B^D$ then $0 = D(c_1a^m + c_2b^n) = nc_2b^{n-1}D(b)$, so $D(b) = 0$. Suppose $a, b \notin B^D$. By *Corollary* 2.5.11, $S^{-1}B = \text{Frac}(A)[T] = (\text{Frac}(A))^{[1]}$ where $A = B^D$ and $S = A \setminus \{0\}$. Consider a, b as elements of $\text{Frac}(A)[T]$. Then a and b must be relatively prime non-constant polynomials, otherwise $c_1a^m + c_2b^n \notin \text{Frac}(A)^*$. Since $c_1a^m + c_2b^n \in \text{Frac}(A)$ we must have $0 = \frac{\partial(c_1a^m+c_2b^n)}{\partial T} = mc_1a^{m-1}\frac{\partial a}{\partial T} + nc_2b^{n-1}\frac{\partial b}{\partial T}$. Then $a|\frac{\partial b}{\partial T}$ and $b|\frac{\partial a}{\partial T}$ and $\deg_T a \leq \deg_T b - 1$, $\deg_T b \leq \deg_T a - 1$ where $\deg_T$ is the standard T-degree, contradiction. ∎

Setup 5.3.2 Recall from Lemma 5.1.5 that $R_x = \Bbbk[x]_x[z,t]$ is a localization of a polynomial ring in three variables and note that $\mathcal{B} = \{x^iz^jt^k | i \in \mathbb{Z},\ j,k \in \mathbb{N}\}$ is a basis of R_x as a $\Bbbk$-vector space. Let $G = \mathbb{Z} \times \mathbb{Z}$, where G is totally ordered by the lexicographic order. Define a map

$$\deg : \mathcal{B} \to G$$

$$x \mapsto (-1, 6)$$
$$z \mapsto (0, 3)$$
$$t \mapsto (0, 2)$$
$$x^i z^j t^k \mapsto i\deg(x) + j\deg(z) + k\deg(t)$$

and define a G-grading $R_x = \bigoplus_{g \in G} S_g$ by letting S_g be the $\Bbbk$-span of the monomials $x^i z^j t^k \in \mathcal{B}$ of degree g. Then the grading and the ordering determine a degree function on R_x by Definition 1.3.10:

$$\deg : R_x \to G \cup \{-\infty\} \tag{5.3.1}$$

This degree function restricts to a degree function on R:

$$\deg : R \to G \cup \{-\infty\} \tag{5.3.2}$$

which, by Remark 1.2.5, induces a proper G-filtration $\{R_g\}_{g \in G}$ of R, where

$$R_g = \{f \in R \mid \deg(f) \leq g\}.$$

By Definition 1.3.15, this filtration determines an associated graded ring $\mathrm{Gr}(R) = \bigoplus_{g \in G} R_g / R_{g^-}$ where $R_{g^-} = \bigcup_{h < g} R_h$. Note that since $\{R_g\}_{g \in G}$ is a proper filtration, it follows from Remark 1.3.16 that $\mathrm{Gr}(R)$ is an integral domain.

We will later show that $\mathrm{Gr}(R)$ is generated by the elements $\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z)$ and $\mathrm{gr}(t)$.

First note that there is a natural way to embed $\mathrm{Gr}(R)$ in R_x: Since the degree function (5.3.2) is a restriction of (5.3.1), it follows that for each $g \in G$, $R_g \subseteq S_g$ and $R_g \cap S_{g^-} = R_{g^-}$. Thus we have an injection $R_g / R_{g^-} \hookrightarrow S_g / S_{g^-}$ for each $g \in G$, hence $\mathrm{Gr}(R) \hookrightarrow \mathrm{Gr}(R_x)$.

Recall that since R_x is a graded ring, we can identify $\mathrm{Gr}(R_x)$ and R_x by Remark 1.3.17. Define the embedding $\mu : \mathrm{Gr}(R) \to R_x$ as the composition $\mathrm{Gr}(R) \hookrightarrow \mathrm{Gr}(R_x) \cong R_x$

and consider the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{i} & R_x \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathrm{Gr}(R) & \xrightarrow{\mu} & R_x \end{array} \tag{5.3.3}$$

where $i : R \to R_x$ is the inclusion map and given an element $f \in R_x$, $\theta(f)$ is the highest homogeneous component of f. From (5.3.3) we know the images of $\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z)$ and $\mathrm{gr}(t)$ under μ are:

$$\begin{aligned} \mu : \mathrm{Gr}(R) &\to R_x \\ \mathrm{gr}(x) &\mapsto x \\ \mathrm{gr}(y) &\mapsto \frac{-t^3 - z^2}{x^2} \\ \mathrm{gr}(z) &\mapsto z \\ \mathrm{gr}(t) &\mapsto t. \end{aligned} \tag{5.3.4}$$

Since μ is injective, we must have

$$\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 = 0$$

in $\mathrm{Gr}(R)$. Note that

$$\begin{aligned} \deg(\mathrm{gr}(x)) &= (-1, 6) \\ \deg(\mathrm{gr}(y)) &= (2, -6) \\ \deg(\mathrm{gr}(z)) &= (0, 3) \\ \deg(\mathrm{gr}(t)) &= (0, 2). \end{aligned}$$

We will leave the notations of Setup 5.3.2 in effect until Setup 5.3.13.

Lemma 5.3.3 *Let u, v, w be distinct elements of $\{x, y, z, t\}$. Then the elements $\mathrm{gr}(u), \mathrm{gr}(v), \mathrm{gr}(w)$ of $\mathrm{Gr}(R)$ are algebraically independent over $\Bbbk$. Moreover, consider* *the G-grading of the polynomial ring $\Bbbk[U,V,W]$ obtained by stipulating that the variables U,V,W are homogeneous of degrees* $\deg(u), \deg(v), \deg(w)$ *respectively. Then, for any $P(U,V,W) \in \Bbbk[U,V,W]$, the following hold:*

(i) $\deg(P(u,v,w)) = \deg(P(U,V,W))$,

(ii) $\operatorname{gr}(P(u,v,w)) = \overline{P}(\operatorname{gr}(u), \operatorname{gr}(v), \operatorname{gr}(w))$, *where $\overline{P}$ is the highest homogeneous component of P in the graded ring $\Bbbk[U,V,W]$.*

Proof: Since $\operatorname{Gr}(R) \hookrightarrow R_x \subseteq \Bbbk(x,z,t)$ and $\operatorname{trdeg}_{\Bbbk}(\Bbbk(x,z,t)) = 3$, to show algebraic independence of $\operatorname{gr}(u), \operatorname{gr}(v), \operatorname{gr}(w)$, it is enough to note that the field extension $\Bbbk(x,z,t)/\Bbbk(u,v,w)$ is algebraic when u,v,w are distinct elements of $\{x, \frac{-t^3-z^2}{x^2}, z, t\}$. The assertions (i) and (ii) follow from the algebraic independence of $\operatorname{gr}(u), \operatorname{gr}(v), \operatorname{gr}(w)$. ∎

Lemma 5.3.4 *Any element $r \in R$ can be written in the form*

$$r = p(x,z,t) + y \cdot v(y,z,t) + xy \cdot w(y,z,t).$$

Moreover, the following hold:

(i) *If $p(x,z,t) \neq 0$, then* $\deg(p(x,z,t)) = (i_p, j_p)$ *is such that $i_p \leq 0$.*

(ii) *If $v(y,z,t) \neq 0$, then* $\deg(y \cdot v(y,z,t)) = (i_v, j_v)$ *is such that $i_v > 0$ is even.*

(iii) *If $w(y,z,t) \neq 0$, then* $\deg(xy \cdot w(y,z,t)) = (i_w, j_w)$ *is such that $i_w > 0$ is odd.*

Proof: Let $\mathbb{N}^2$ be well-ordered by the lexicographic order and define a map

$$\delta : \Bbbk[X,Y,Z,T] \to \mathbb{N}^2$$

as follows: Given $F \in \mathbb{k}[X,Y,Z,T]$, write $F = \sum_{i,j} c_{ij} X^i Y^j$ with $c_{ij} \in \mathbb{k}[Z,T]$ and let $S_F = \{(i,j) \in \mathbb{N}^2 \mid c_{ij} \neq 0 \text{ and } i < 2j\}$. Set

$$\delta(F) = \begin{cases} \max S_F, & \text{if } S_F \neq \emptyset; \\ (0,0), & \text{if } S_F = \emptyset. \end{cases}$$

Let $r \in R$. Let $\pi : \mathbb{k}[X,Y,Z,T] \to R$ be the canonical epimorphism and pick $F \in \pi^{-1}(r)$. We claim:

If $(a,b) = \delta(F)$ satisfies $a > 1$, then there exists $F' \in \pi^{-1}(r)$ such that $\delta(F') < \delta(F)$. (5.3.5)

Indeed, suppose that $a > 1$ and write $a = 2n + e$ where $n \in \mathbb{N}$ and $e \in \{0,1\}$. Note that $1 \leq n < b$ because $a < 2b$ by definition of $\delta(F)$. Consider the term $c_{ab}(Z,T)X^aY^b$ of F and evaluate at z,t,x,y, using the abbreviation $c = c_{ab}(z,t)$:

$$\begin{aligned} c_{ab}(z,t)x^ay^b &= cx^ey^{b-n}(x^2y)^n \\ &= cx^ey^{b-n}(-x-z^2-t^3)^n \\ &= c(-1)^n x^e y^{b-n}\left(\sum_{k_1+k_2+k_3=n} \frac{n!}{k_1!k_2!k_3!} x^{k_1} z^{2k_2} t^{3k_3}\right) \\ &= c(-1)^n x^e y^{b-n}\left(\sum_{i=0}^{n} c_i x^i\right) \qquad \text{where } c_i \in \mathbb{k}[t,z] \\ &= \sum_{i=0}^{n} (-1)^n c c_i x^{e+i} y^{b-n} \\ &= G(x,y,z,t) \end{aligned}$$

where $G \in \mathbb{k}[X,Y,Z,T]$ is such that $\delta(G) < (a,b)$. Thus, the polynomial $F' = F - c_{ab}(Z,T)X^aY^b + G$ also belongs to $\pi^{-1}(r)$ and satisfies $\delta(F') < \delta(F)$, proving (5.3.5). It follows that the least element (α,β) of $\{\delta(F) | F \in \pi^{-1}(r)\}$ satisfies $\alpha \leq 1$. Consequently, r can be written as:

$$r = \sum_{i \geq 2j} c_{ij} x^i y^j + \sum_{j \geq 1} c_{0j} y^j + \sum_{j \geq 1} c_{1j} x y^j$$

where $c_{ij} \in \Bbbk[z,t]$ for all i,j.

Note that

$$\begin{aligned}\sum_{i\geq 2j} c_{ij}x^iy^j &= \sum_{i\geq 2j} c_{ij}x^{i-2j}(x^2y)^j \\ &= \sum_{i\geq 2j} c_{ij}x^{i-2j}(-x-z^2-t^3)^j \in \Bbbk[x,z,t].\end{aligned}$$

So $r = p(x,z,t)+y\cdot v(y,z,t)+xy\cdot w(y,z,t)$, where $p(x,z,t) = \sum_{i\geq 2j} c_{ij}x^iy^j$, $v(y,z,t) = \sum_{j\geq 1} c_{0j}y^{j-1}$ and $w(y,z,t) = \sum_{j\geq 1} c_{1j}y^{j-1}$. It remains to show assertions (i) - (iii):

(i) Suppose that $p(x,z,t) \neq 0$ and let $\deg(p(x,z,t)) = (i_p, j_p)$. Note that $(i_p, j_p) = \max_{i=1}^n\{\deg(p_i)\}$ where $p(x,z,t) = \sum_{i=1}^n p_i$ and $p_1,\ldots,p_n$ are monomials in x,z,t. Let $p_m \in \{p_i | 1 \leq i \leq n\}$ be such that $\deg(p_m) = \max_{i=1}^n\{\deg(p_i)\}$ and write $p_m = ax^{k_x}z^{k_z}t^{k_t}$ where $a \in \Bbbk^*$ and $k_x, k_z, k_t \in \mathbb{N}$. Then

$$\begin{aligned}(i_p, j_p) &= k_x\deg(x) + k_z\deg(z) + k_t\deg(t) \\ &= k_x(-1,6) + k_z(0,3) + k_t(0,2) \\ &= (-k_x, 6k_x + 3k_z + 3k_t)\end{aligned}$$

and $i_p = -k_x \leq 0$.

(ii) Suppose that $v(y,z,t) \neq 0$ and let $\deg(y \cdot v(y,z,t)) = (i_v, j_v)$. Note that $(i_v, j_v) = \max_{i=1}^n\{\deg(y \cdot v_i)\}$ where $v(y,z,t) = \sum_{i=1}^n v_i$ and $v_1,\ldots,v_n$ are monomials in y,z,t. Let $v_m \in \{v_i | 1 \leq i \leq n\}$ be such that $\deg(y \cdot v_m) = \max_{i=1}^n\{\deg(y \cdot v_i)\}$ and write $v_m = ay^{k_y}z^{k_z}t^{k_t}$ where $a \in \Bbbk^*$ and $k_y, k_z, k_t \in \mathbb{N}$. Then

$$\begin{aligned}(i_v, j_v) &= (k_y+1)\deg(y) + k_z\deg(z) + k_t\deg(t) \\ &= (k_y+1)(2,-6) + k_z(0,3) + k_t(0,2) \\ &= (2k_y+2, -6k_y + 3k_z + 3k_t - 6)\end{aligned}$$

and $i_v = 2k_y + 2 > 0$ is even.

(iii) Suppose that $w(y,z,t) \neq 0$ and let $\deg(xy \cdot w(y,z,t)) = (i_w, j_w)$. Note that $(i_w, j_w) = \max_{i=1}^{n}\{\deg(xy \cdot w_i)\}$ where $w(y,z,t) = \sum_{i=1}^{n} w_i$ and $w_1, \ldots, w_n$ are monomials in y, z, t. Let $w_m \in \{w_i | 1 \leq i \leq n\}$ be such that $\deg(xy \cdot w_m) = \max_{i=1}^{n}\{\deg(xy \cdot w_i)\}$ and write $w_m = a y^{k_y} z^{k_z} t^{k_t}$ where $a \in \mathbb{k}^*$ and $k_y, k_z, k_t \in \mathbb{N}$. Then

$$\begin{aligned}(i_w, j_w) &= \deg(x) + (k_y+1)\deg(y) + k_z\deg(z) + k_t\deg(t)\\ &= (-1,6) + (k_y+1)(2,-6) + k_z(0,3) + k_t(0,2)\\ &= (2k_y+1, -6k_y + 3k_z + 3k_t)\end{aligned}$$

and $i_w = 2k_y + 1 > 0$ is odd.

∎

Corollary 5.3.5 $\mathrm{Gr}(R) = \mathbb{k}[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)]$.

Proof: It is clear that $\mathbb{k}[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)] \subseteq \mathrm{Gr}(R)$. Let $r \in R$. By Lemma 5.3.4, we can write $r = p(x,z,t) + y \cdot v(y,z,t) + xy \cdot w(y,z,t)$ where $p(x,z,t) \in \mathbb{k}[x,z,t]$ and $v(y,z,t), w(y,z,t) \in \mathbb{k}[y,z,t]$. We also know that the non-zero elements of the set $\{p(x,z,t), y \cdot v(y,z,t), xy \cdot w(y,z,t)\}$ have distinct degrees. In any case, $\mathrm{gr}(r) \in \{\mathrm{gr}(p(x,z,t)), \mathrm{gr}(y \cdot v(y,z,t)), \mathrm{gr}(xy \cdot w(y,z,t))\}$. By Lemma 5.3.3 (ii), $\mathrm{gr}(p(x,z,t)) = \overline{p}(\mathrm{gr}(x), \mathrm{gr}(z), \mathrm{gr}(t))$. $\mathrm{gr}(y \cdot v(y,z,t)) = \mathrm{gr}(y) \cdot \overline{v}(\mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t))$ and $\mathrm{gr}(xy \cdot w(y,z,t)) = \mathrm{gr}(x) \cdot \mathrm{gr}(y) \cdot \overline{w}(\mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t))$, where $\overline{p}, \overline{v}, \overline{w}$ denote the highest deg-homogeneous terms of p, v and w respectively. Consequently, $\mathrm{gr}(r) \in \mathbb{k}[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)]$ and $\mathrm{Gr}(R) = \mathbb{k}[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)]$. ∎

Corollary 5.3.6 *The* $\mathbb{k}$*-algebra homomorphism* $\varphi : \mathbb{k}[X,Y,Z,T] \to \mathrm{Gr}(R)$ *defined* *by*

$$X \mapsto \text{gr}(x), \quad Y \mapsto \text{gr}(y), \quad Z \mapsto \text{gr}(z), \quad T \mapsto \text{gr}(t)$$

is surjective, and $\ker(\varphi) = \langle X^2Y + Z^2 + T^3 \rangle$. *Consequently,*

$$\text{Gr}(R) \cong \Bbbk[X, Y, Z, T]/\langle X^2Y + Z^2 + T^3 \rangle.$$

Proof: Note that $\text{gr}(x)^2\text{gr}(y) + \text{gr}(z)^2 + \text{gr}(t)^3 = 0$ in $\text{Gr}(R)$. Let $\hat{I} = \langle X^2Y + Z^2 + T^3 \rangle$. Then $\varphi(X^2Y + Z^2 + T^3) = \text{gr}(x)^2\text{gr}(y) + \text{gr}(t)^3 + \text{gr}(z)^2 = 0$ and thus $\hat{I} \subseteq \ker\varphi$. By Lemma 5.3.3, $\text{gr}(x), \text{gr}(z)$ and $\text{gr}(t)$ are algebraically independent in $\text{Gr}(R)$. Thus we have $\text{trdeg}_{\Bbbk}(\text{Gr}(R)) \geq 3$, so

$$\text{ht}(\ker\varphi) = \text{trdeg}_{\Bbbk}(\Bbbk[X, Y, Z, T]) - \text{trdeg}_{\Bbbk}(\text{Im}(\varphi)) \leq 1.$$

But $\hat{I} \subseteq \ker\varphi$, $\hat{I}$ is prime and $\text{ht}(\hat{I}) = 1$, so we must have $\hat{I} = \ker\varphi$. It follows from Corollary 5.3.5 that φ is surjective, so $\text{Gr}(R) \cong \Bbbk[X, Y, Z, T]/\langle X^2Y + Z^2 + T^3 \rangle$. ∎

Corollary 5.3.7 *Let* $h \in R \setminus \{0\}$ *and* $\deg(h) = (i, j)$. *Then the following hold.*

(i) $h \notin \Bbbk[x, z, t]$ *if and only if* $i > 0$,

(ii) *if* $h \notin \Bbbk[x, z, t]$, *then* $\text{gr}(h) \notin \Bbbk[\text{gr}(x), \text{gr}(z), \text{gr}(t)]$.

Proof: If $h \in \Bbbk[x, z, t]$ then, by Lemma 5.3.3, $\deg(h) = \deg(M)$ for some monomial M occurring in h Write $M = ax^{k_x}z^{k_z}t^{k_t}$ where $a \in \Bbbk^*$ and k_x, k_z, $k_t \in \mathbb{N}$. Now $\deg(M) = k_x(-1, 6) + k_z(0, 3) + k_t(0, 2) = (-k_x, 6k_x + 3k_z + 2k_t)$. so $\deg(h) = (i, j)$ where $i = -k_x \leq 0$. Thus $i > 0$ implies $h \notin \Bbbk[x, z, t]$.

Conversely, assume that $h \notin \Bbbk[x, z, t]$. By Lemma 5.3.4, we can write

$$h = p(x, z, t) + y \cdot v(y, z, t) + xy \cdot w(y, z, t)$$

where $p(x,z,t) \in \Bbbk[x,z,t]$ and $v(y,z,t), w(y,z,t) \in \Bbbk[y,z,t]$, and where $v(y,z,t) \neq 0$ or $w(y,z,t) \neq 0$. Since $\deg(p(x,z,t)) < \max\{\deg(y \cdot v(y,z,t)), \deg(xy \cdot w(y,z,t))\}$ and $\deg(y \cdot v(y,z,t)) \neq \deg(xy \cdot w(y,z,t))$, we must have $(i,j) = \max\{\deg(y \cdot v(y,z,t)), \deg(xy \cdot w(y,z,t))\}$ and $i > 0$ by Lemma 5.3.4 (ii) and (iii). Note that $\mathrm{gr}(h) \in \{\mathrm{gr}(y \cdot v(y,z,t)), \mathrm{gr}(xy \cdot w(y,z,t))\}$ and $\deg(\mathrm{gr}(h)) = (i,j)$. Since $\deg(\mathrm{gr}(x)) = (-1,6)$, $\deg(\mathrm{gr}(z)) = (0,3)$ and $\deg(\mathrm{gr}(t)) = (0,2)$, any $g \in \Bbbk[\mathrm{gr}(x), \mathrm{gr}(z), \mathrm{gr}(t)] \setminus \{0\}$ where $\deg(g) = (i_g, j_g)$ is such that $i_g \leq 0$. Thus $\mathrm{gr}(h) \notin \Bbbk[\mathrm{gr}(x), \mathrm{gr}(z), \mathrm{gr}(t)]$. ∎

Lemma 5.3.8 *Assume that $\Bbbk$ is algebraically closed. Any non-zero homogeneous element of $\mathrm{Gr}(R)$ can be written as $\lambda \mathrm{gr}(x^a y^b z^c t^d) \prod_{i \in I} (\mathrm{gr}(z)^2 + \mu_i \mathrm{gr}(t)^3)$ for some $\lambda \in \Bbbk^*$, $\mu_i \in \Bbbk \setminus \{0,1\}$, $a,b,c,d \in \mathbb{N}$ and some finite set I.*

Proof: Let $S = \{\mathrm{gr}(x)^i \mathrm{gr}(y)^j \mathrm{gr}(z)^k \mathrm{gr}(t)^l | i,j,k,l \in \mathbb{N}\}$ and note that S is a multiplicative subset of $\mathrm{Gr}(R)$. Since each element of S is homogeneous, it follows from Lemma 1.3.14 that the localization $S^{-1}\mathrm{Gr}(R)$ is a G-graded integral domain.

Consider the subset $S' = \{\mathrm{gr}(x)^i \mathrm{gr}(y)^j \mathrm{gr}(z)^k \mathrm{gr}(t)^l | i,j,k,l \in \mathbb{Z}\}$ of $S^{-1}\mathrm{Gr}(R)$, and the elements $M = \mathrm{gr}(z)^2 \mathrm{gr}(t)^{-3}$ and $N = \mathrm{gr}(x)^2 \mathrm{gr}(y) \mathrm{gr}(z)^{-2}$ of S' and note that $N = -1 - M^{-1}$. It can be verified that $\left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \end{pmatrix} \right\}$ is a basis of the kernel of the $\mathbb{Z}$-linear map $\mathbb{Z}^4 \to \mathbb{Z}^2$, $\begin{pmatrix} i \\ j \\ k \\ l \end{pmatrix} \mapsto \begin{pmatrix} -1 & 2 & 0 & 0 \\ 6 & -6 & 3 & 2 \end{pmatrix} \begin{pmatrix} i \\ j \\ k \\ l \end{pmatrix}$; consequently, any element of degree $(0,0)$ in S' is of the form $M^a N^b$ for some $a, b \in \mathbb{Z}$.

Let $f \in \mathrm{Gr}(R) \setminus \{0\}$ be a homogeneous element. By Corollary 5.3.5, we have that $\mathrm{Gr}(R) = \Bbbk[\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)]$ so we may write $f = \alpha_1 M_1 + \ldots + \alpha_s M_s$ where $\alpha_i \in \Bbbk^*$, $M_i \in S$ for all i and $M_1, \ldots M_s$ are all of the same degree. Then for all i, $\frac{M_i}{M_1} \in S'$ has degree $(0,0)$, so we can write $\frac{M_i}{M_1} = M^{a_i} N^{b_i}$ for some $a_i, b_i \in \mathbb{Z}$; thus $M_i = M_1 M^{a_i} N^{b_i}$. Then

$$f = \alpha_1 M_1 + \alpha_2 M_1 M^{a_2} N^{b_2} + \ldots + \alpha_s M_1 M^{a_s} N^{b_s}$$

$$= M_1 \left(\alpha_1 + \alpha_2 M^{a_2}(-1 - M^{-1})^{b_2} + \ldots + \alpha_s M^{a_s}(-1 - M^{-1})^{b_s}\right)$$

and we can write $f = M_1 M^e P(M)$ where $e \in \mathbb{Z}$ and $P(W) \in \Bbbk[W]$ is a polynomial with non-zero constant term. Then for some $n \in \mathbb{N}$. $\lambda \in \Bbbk^*$ and $\mu_i \in \Bbbk^*$ for all $1 \leq i \leq n$ we have:

$$\begin{aligned} f &= \lambda M_1 M^e \prod_{i=1}^{n} (M + \mu_i) \\ &= \lambda(\mathrm{gr}(t)^{-3n} M_1 M^e) \left(\mathrm{gr}(t)^{3n} \prod_{i=1}^{n} (M + \mu_i) \right) \\ &= \lambda(\mathrm{gr}(t)^{-3n} M_1 M^e) \prod_{i=1}^{n} \left(\mathrm{gr}(z)^2 + \mu_i \mathrm{gr}(t)^3\right). \end{aligned}$$

Now rewrite $\mathrm{gr}(t)^{-3n} M_1 M^e = \mathrm{gr}(x)^a \mathrm{gr}(y)^b \mathrm{gr}(z)^c \mathrm{gr}(t)^d$ where $a, b, c, d \in \mathbb{Z}$. By replacing $\mathrm{gr}(z)^2 + \mathrm{gr}(t)^3$ by $-\mathrm{gr}(x)^2 \mathrm{gr}(y)$ we may assume $\mu_i \neq 1$ for all i. Thus

$$f = \lambda \mathrm{gr}(x)^a \mathrm{gr}(y)^b \mathrm{gr}(z)^c \mathrm{gr}(t)^d \prod_{i=1}^{n} \left(\mathrm{gr}(z)^2 + \mu_i \mathrm{gr}(t)^3\right)$$

where $\lambda \in \Bbbk^*$, $\mu_i \in \Bbbk \setminus \{0, 1\}$ and $a, b, c, d \in \mathbb{Z}$.

It remains to show $a, b, c, d \geq 0$. First, we note that the principal ideals

$$\langle \mathrm{gr}(x) \rangle, \qquad \langle \mathrm{gr}(y) \rangle, \qquad \langle \mathrm{gr}(z) \rangle, \qquad \langle \mathrm{gr}(t) \rangle$$

are prime ideals of $\mathrm{Gr}(R)$. Indeed, $\langle \mathrm{gr}(x) \rangle$ is prime because

$$\mathrm{Gr}(R)/\langle \mathrm{gr}(x) \rangle \cong \Bbbk[X, Y, Z, T]/\langle X, X^2 Y + Z^2 + T^3 \rangle \cong \Bbbk[Y, Z, T]/\langle Z^2 + T^3 \rangle,$$

is an integral domain. The other ideals are prime by a similar argument. Next, we claim:

If $u \in \{\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)\}$ and (5.3.6)
$v \in \{\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t), \mathrm{gr}(z)^2 + \mu \mathrm{gr}(t)^3\} \setminus \{u\}$ where $\mu \in \Bbbk \setminus \{0, 1\}$,
then $v \notin \langle u \rangle$.

We will show that none of $\mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t), \mathrm{gr}(z)^2+\mathrm{gr}(t)^3$ belong to $\langle \mathrm{gr}(x)\rangle$ and leave the other cases to the reader.

By Corollary 5.3.6 we know

$$\mathrm{Gr}(R) \cong \Bbbk[X,Y,Z,T]/\langle X^2Y+Z^2+T^3\rangle.$$

So if one of $\mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t), \mathrm{gr}(z)^2+\mu\mathrm{gr}(t)^3$ belongs to $\langle \mathrm{gr}(x)\rangle$ then one of the polynomials $Y, Z, T, Z^2+\mu T^3$ belongs to the ideal $\langle X, X^2Y+Z^2+T^3\rangle$ of $\Bbbk[X,Y,Z,T]$ and this is not the case. The other cases of (5.3.6) are proved similarly.

Consider the equality:

$$\mathrm{gr}(x)^{|a|}\mathrm{gr}(y)^{|b|}\mathrm{gr}(z)^{|c|}\mathrm{gr}(t)^{|d|}f = \lambda \mathrm{gr}(x)^{a+|a|}\mathrm{gr}(y)^{b+|b|}\mathrm{gr}(z)^{c+|c|}\mathrm{gr}(t)^{d+|d|}\prod_{i=1}^{n}\left(\mathrm{gr}(z)^2+\mu_i\mathrm{gr}(t)^3\right). \quad (5.3.7)$$

If $a<0$ (resp. $b<0, c<0, d<0$) then the left hand side of (5.3.7) belongs to $\langle \mathrm{gr}(x)\rangle$ (resp. $\langle \mathrm{gr}(y)\rangle, \langle \mathrm{gr}(z)\rangle, \langle \mathrm{gr}(t)\rangle$), while the right hand side does not, by (5.3.6). So $a,b,c,d \in \mathbb{N}$ and we are done. ∎

Lemma 5.3.9 *Assume that* $\Bbbk$ *is algebraically closed. If* $D \in \mathrm{LND}(\mathrm{Gr}(R))$ *is non-zero and homogeneous, then* $\mathrm{Gr}(R)^D \in \{\Bbbk[\mathrm{gr}(x),\mathrm{gr}(z)], \Bbbk[\mathrm{gr}(x),\mathrm{gr}(t)]\}$.

Proof: By Corollary 5.3.6, we know $\mathrm{Gr}(R) \cong \Bbbk[X,Y,Z,T]/\langle X^2Y+Z^2+T^3\rangle$, so $\mathrm{trdeg}_{\Bbbk}\mathrm{Gr}(R)^D = 2$. Recall that $\mathrm{Gr}(R)^D$ is generated by homogeneous elements by Lemma 2.5.10. Choose two algebraically independent homogeneous elements $f, g \in \mathrm{Gr}(R)^D$. By Lemma 5.3.8, we can write:

$$f = \lambda_f \mathrm{gr}(x^a y^b z^c t^d)\prod_{i\in I}\left(\mathrm{gr}(z)^2+\mu_i\mathrm{gr}(t)^3\right)$$
$$g = \lambda_g \mathrm{gr}(x^\alpha y^\beta z^\gamma t^\delta)\prod_{j\in J}\left(\mathrm{gr}(z)^2+\nu_j\mathrm{gr}(t)^3\right)$$

for some $\lambda_f, \lambda_g \in \Bbbk^*$, $\mu_i, \nu_j \in \Bbbk \setminus \{0,1\}$, $a,b,c,d,\alpha,\beta,\gamma,\delta \in \mathbb{N}$ and some finite sets I, J.

If $I \neq \emptyset$ then $\mathrm{gr}(z), \mathrm{gr}(t) \in \mathrm{Gr}(R)^D$ by Lemma 5.3.1. Recall:

$$\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 = 0. \tag{5.3.8}$$

So $\mathrm{gr}(x)^2\mathrm{gr}(y) \in \mathrm{Gr}(R)^D$ and since $\mathrm{Gr}(R)^D$ is factorially closed $\mathrm{gr}(x), \mathrm{gr}(y) \in \mathrm{Gr}(R)^D$ contradicting the fact that $\mathrm{trdeg}_{\Bbbk}\mathrm{Gr}(R)^D = 2$. So $I = \emptyset$ and similarly $J = \emptyset$. Hence we can write:

$$f = \lambda_f \mathrm{gr}(x^a y^b z^c t^d)$$
$$g = \lambda_g \mathrm{gr}(x^\alpha y^\beta z^\gamma t^\delta).$$

As any three of $\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t)$ are algebraically independent, we obtain as an immediate consequence:

$$\text{Exactly two of } \mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t) \text{ are elements of } \mathrm{Gr}(R)^D. \tag{5.3.9}$$

Consider the following cases:

- $\{\mathrm{gr}(x), \mathrm{gr}(y)\} \subseteq \mathrm{Gr}(R)^D$
  Then $\mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 \in \mathrm{Gr}(R)^D$ by (5.3.8) and $\mathrm{gr}(z), \mathrm{gr}(t) \in \mathrm{Gr}(R)^D$ by Lemma 5.3.1, contradicting (5.3.9).

- $\{\mathrm{gr}(y), \mathrm{gr}(z)\} \subseteq \mathrm{Gr}(R)^D$
  Then $\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(t)^3 \in \mathrm{Gr}(R)^D$ by (5.3.8) and $\mathrm{gr}(x)\mathrm{gr}(t) \in \mathrm{Gr}(R)^D$ by Lemma 5.3.1, contradicting (5.3.9).

- $\{\mathrm{gr}(y), \mathrm{gr}(t)\} \subseteq \mathrm{Gr}(R)^D$
  Then $\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(z)^2 \in \mathrm{Gr}(R)^D$ by (5.3.8) and $\mathrm{gr}(x), \mathrm{gr}(z) \in \mathrm{Gr}(R)^D$ by Lemma 5.3.1, contradicting (5.3.9).

- $\{\text{gr}(z), \text{gr}(t)\} \subseteq \text{Gr}(R)^D$
  Then $\text{gr}(x)^2\text{gr}(y) \in \text{Gr}(R)^D$ by (5.3.8) and $\text{gr}(x), \text{gr}(y) \in \text{Gr}(R)^D$ since $\text{Gr}(R)^D$ is factorially closed, contradicting (5.3.9).

The only other two possibilities are $\{\text{gr}(x), \text{gr}(z)\} \subseteq \text{Gr}(R)^D$ and $\{\text{gr}(x), \text{gr}(z)\} \subseteq \text{Gr}(R)^D$. Thus $\Bbbk[\text{gr}(x), \text{gr}(z)] \subseteq \text{Gr}(R)^D$ or $\Bbbk[\text{gr}(x), \text{gr}(t)] \subseteq \text{Gr}(R)^D$. To show the reverse inclusions, it is enough to note that any homogeneous element of $\text{Gr}(R)^D$ must be of the form

$$h = \lambda_h \text{gr}(x^{a'} y^{b'} z^{c'} t^{d'}).$$

In the case $\Bbbk[\text{gr}(x), \text{gr}(z)] \subseteq \text{Gr}(R)^D$ we must have $0 = b' = d'$, otherwise $\text{gr}(y)$ or $\text{gr}(t)$ is an element of $\text{Gr}(R)^D$, contradicting (5.3.9). So $h = \lambda_h \text{gr}(x^{a'} z^{c'}) \in \Bbbk[\text{gr}(x), \text{gr}(z)]$ and $\text{Gr}(R)^D = \Bbbk[\text{gr}(x), \text{gr}(z)]$. Similarly, if $\Bbbk[\text{gr}(x), \text{gr}(t)] \subseteq \text{Gr}(R)^D$, then $\text{Gr}(R)^D = \Bbbk[\text{gr}(x), \text{gr}(t)]$.
Thus $\text{Gr}(R)^D \in \{\Bbbk[\text{gr}(x), \text{gr}(z)], \Bbbk[\text{gr}(x), \text{gr}(t)]\}$. ∎

Proposition 5.3.10 *Suppose that $\Bbbk$ is algebraically closed. Then $\mathcal{D}(R) = \Bbbk[x, z, t]$.*

Proof: By Proposition 5.2.3 we have $\Bbbk[x, z, t] \subseteq \mathcal{D}(R)$.
By way of contradiction, suppose that $\mathcal{D}(R) \subseteq \Bbbk[x, z, t]$ is false. Then there exists $D \in \text{LND}(R) \setminus \{0\}$ satisfying $R^D \not\subseteq \Bbbk[x, z, t]$. Choose an $f \in R^D$ such that $f \notin \Bbbk[x, z, t]$. Note that D extends to a $\Bbbk$-derivation D_x of R_x by Proposition 2.1.9. Since R_x is graded, it follows from Proposition 2.2.5 that $\deg(D_x)$ is defined and thus Lemma 2.2.4 implies that $\deg(D)$ is defined. Hence, $\text{gr}(D)$ is defined and $\text{gr}(D) \in \text{LND}(\text{Gr}(R))$ is non-zero and homogeneous. By Lemma 5.3.9, $\text{Gr}(R)^{\text{gr}(D)} \in \{\Bbbk[\text{gr}(x), \text{gr}(z)], \Bbbk[\text{gr}(x), \text{gr}(t)]\}$ and by Lemma 2.6.2(iii) $\text{gr}(f) \in \text{Gr}(R)^{\text{gr}(D)}$. But $f \notin \Bbbk[x, z, t]$ implies that $\text{gr}(f) \notin \Bbbk[\text{gr}(x), \text{gr}(z), \text{gr}(t)]$ by Corollary 5.3.7, leading to a contradiction. Consequently, $\mathcal{D}(R) = \Bbbk[x, z, t]$. ∎

The following is a corollary of Theorem 35 in Chapter III §14 of [45]:

Lemma 5.3.11 *Let $\Bbbk[X_1,\ldots,X_n]$ be a polynomial ring in n variables, let $\overline{\Bbbk}$ be the algebraic closure of $\Bbbk$ and let I be an ideal of $\Bbbk[X_1,\ldots,X_n]$. Write $I = \langle p_1,\ldots,p_m\rangle$ where $p_1,\ldots,p_m \in \Bbbk[X_1,\ldots,X_n]$ and let $\overline{I}$ be the ideal in $\overline{\Bbbk}[X_1,\ldots,X_n]$ generated by $p_1,\ldots,p_m$. Then*

$$\overline{\Bbbk} \otimes_{\Bbbk} \Bbbk[X_1,\ldots,X_n]/I = \overline{\Bbbk}[X_1,\ldots,X_n]/\overline{I}.$$

We can now strengthen Proposition 5.3.10 by no longer assuming $\Bbbk$ is algebraically closed:

Theorem 5.3.12 *$\mathcal{D}(R) = \Bbbk[x,z,t]$. In particular, $R \neq \Bbbk^{[3]}$.*

Proof: By Proposition 5.2.3 we have $\Bbbk[x,z,t] \subseteq \mathcal{D}(R)$.
Let $\overline{\Bbbk}$ be the algebraic closure of $\Bbbk$ and define $\overline{R} = R \otimes_{\Bbbk} \overline{\Bbbk}$. By Lemma 5.3.11, $\overline{R} = \overline{\Bbbk}[X,Y,Z,T]/\langle X + X^2Y + Z^2 + T^3\rangle$. Let $D \in \text{LND}(R) \setminus \{0\}$ and consider $D \otimes 1$, the unique extension of D. By 2.1.13, $D \otimes 1 \in \text{LND}(\overline{R}) \setminus \{0\}$. By way of contradiction, suppose that $R^D \not\subseteq \Bbbk[x,z,t]$. Choose $f \in R^D$ such that $f \notin \Bbbk[x,z,t]$ and let $(i,j) = \deg(f)$. By Corollary 5.3.7 (i) we have that $i > 0$. If we consider f as an element of $\overline{R}$, it follows again from Corollary 5.3.7 (i) that $f \notin \overline{\Bbbk}[x,z,t]$. Since $f \in \overline{R}^{D\otimes 1}$ we have $\overline{R}^{D\otimes 1} \not\subseteq \overline{\Bbbk}[x,z,t]$ contradicting Proposition 5.3.10. So $R^D \subseteq \Bbbk[x,z,t]$ and $\mathcal{D}(R) \subseteq \Bbbk[x,z,t]$. Consequently, $\mathcal{D}(R) = \Bbbk[x,z,t]$ and $R \neq \Bbbk^{[3]}$. ∎

We will now show $\text{ML}(R) = \Bbbk[x]$. This result is due to Makar-Limanov [26], but we follow Freudenburg's proof, Corollary 9.7 in [15].

Setup 5.3.13 Using the same method as in Setup 5.3.2, we define a degree function

$$\deg : R \to \mathbb{Z} \cup \{-\infty\}.$$

First, consider the $\mathbb{Z}$-grading of $R_x = \Bbbk[x]_x[z,t]$ obtained by declaring x, z, t to be homogeneous of degrees $-1, 0, 0$ respectively. This grading determines a degree function $\deg : R_x \to \mathbb{Z} \cup \{-\infty\}$. We then define $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ to be the restriction of $\deg : R_x \to \mathbb{Z} \cup \{-\infty\}$. Let $\mathrm{Gr}(R) = \bigoplus_{g\in\mathbb{Z}} R_g/R_{g^-}$ be the associated graded ring as defined in Definition 1.3.15 and note that $\mathrm{Gr}(R)$ is an integral domain by Remark 1.3.16. It is clear that (5.3.3) and (5.3.4) continue to hold true, and again we have

$$\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 = 0.$$

Note that:

$$\begin{aligned} \deg(\mathrm{gr}(x)) &= -1 \\ \deg(\mathrm{gr}(y)) &= 2 \\ \deg(\mathrm{gr}(z)) &= 0 \\ \deg(\mathrm{gr}(t)) &= 0. \end{aligned}$$

Since $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ is identically zero on $\Bbbk[z,t]$, the map

$$\mathrm{gr}\,|_{\Bbbk[z,t]} \colon \Bbbk[z,t] \to \Bbbk[\mathrm{gr}(z), \mathrm{gr}(t)] \tag{5.3.10}$$

(i.e. the restriction of $\mathrm{gr} : R \to \mathrm{Gr}(R)$) is a $\Bbbk$-algebra isomorphism.

Corollary 5.3.14 ([26]) $\mathrm{ML}(R) = \Bbbk[x]$.

Proof: We already know $\mathrm{ML}(R) \subseteq \Bbbk[x]$ by Proposition 5.1.11.
Let $D \in \mathrm{LND}(R) \setminus \{0\}$ and suppose by way of contradiction that $D(x) \neq 0$. We have $R^D \subseteq \Bbbk[x,z,t]$ by Theorem 5.3.12, and we know that $\mathrm{trdeg}_{\Bbbk} R^D = 2$.
Choose algebraically independent $f, g \in R^D$ and write $f = xf_1(x,z,t) + f_2(z,t)$, $g = xg_1(x,z,t) + g_2(z,t)$. We must have that f_2, g_2 are algebraically independent (so in particular $f_2, g_2 \neq 0$) for otherwise there would exist $H \in \Bbbk[X_1, X_2] \setminus \{0\}$ satisfying $H(f_2, g_2) = 0$; then $x \mid H(f,g)$ and $H(f,g) \in R^D \setminus \{0\}$ would imply $x \in R^D$, a

contradiction.

Consider $\mathrm{gr}(D) \in \mathrm{LND}(\mathrm{Gr}(R)) \setminus \{0\}$. Since $\deg(xf_1(x,z,t)) < \deg(f_2(z,t))$ and $\deg(xg_1(x,z,t)) < \deg(g_2(z,t))$ we must have

$$\begin{aligned} \mathrm{gr}(f) &= \mathrm{gr}(f_2(z,t)) \\ &= f_2(\mathrm{gr}(z), \mathrm{gr}(t)) \qquad \text{by (5.3.10)} \end{aligned}$$

and

$$\begin{aligned} \mathrm{gr}(g) &= \mathrm{gr}(g_2(z,t)) \\ &= g_2(\mathrm{gr}(z), \mathrm{gr}(t)) \qquad \text{by (5.3.10)} \end{aligned}$$

Note that $\mathrm{gr}(f) = f_2(\mathrm{gr}(z), \mathrm{gr}(t))$ and $\mathrm{gr}(g) = g_2(\mathrm{gr}(z), \mathrm{gr}(t))$ are algebraically independent since $f_2, g_2 \in \Bbbk[z,t]$ are algebraically independent and $\mathrm{gr}\mid_{\Bbbk[z,t]} \colon \Bbbk[z,t] \to \Bbbk[\mathrm{gr}(z), \mathrm{gr}(t)]$ is a $\Bbbk$-algebra isomorphism. Consequently, $\Bbbk[\mathrm{gr}(z), \mathrm{gr}(t)]$ is algebraic over $\Bbbk[\mathrm{gr}(f), \mathrm{gr}(g)]$.

We have $\Bbbk[\mathrm{gr}(f), \mathrm{gr}(g)] \subseteq \mathrm{Gr}(R)^{\mathrm{gr}(D)}$ by Lemma 2.6.2 (iii). Since $\Bbbk[\mathrm{gr}(z), \mathrm{gr}(t)]$ is algebraic over $\Bbbk[\mathrm{gr}(f), \mathrm{gr}(g)]$ and $\mathrm{Gr}(R)^{\mathrm{gr}(D)}$ is algebraically closed in $\mathrm{Gr}(R)$, it follows that $\Bbbk[\mathrm{gr}(z), \mathrm{gr}(t)] \subseteq \mathrm{Gr}(R)^{\mathrm{gr}(D)}$.

Now recall that $\mathrm{gr}(x)^2\mathrm{gr}(y) + \mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 = 0$ in $\mathrm{Gr}(R)$, and note that $\mathrm{gr}(z)^2 + \mathrm{gr}(t)^3 \neq 0$. Consequently, $\mathrm{gr}(x)^2\mathrm{gr}(y) \in \mathrm{Gr}(R)^{\mathrm{gr}(D)} \setminus \{0\}$, so $\mathrm{gr}(x), \mathrm{gr}(y), \mathrm{gr}(z), \mathrm{gr}(t) \in \mathrm{Gr}(R)^{\mathrm{gr}(D)}$, which is absurd. ∎

5.4 The Cylinder Over Russell's Cubic

Consider the polynomial ring in one variable over Russell's Cubic:

$$R[w] = \Bbbk[X,Y,Z,T,W]/\langle X + X^2Y + Z^2 + T^3 \rangle = \Bbbk[x,y,z,t,w].$$

The following question is open:

Question 5.4.1 Is $R[w]$ a polynomial ring in four variables over $\mathbb{k}$?

If the answer to Question 5.4.1 is affirmative, then R is a counterexample to the Cancellation Problem in dimension three since $R \neq \mathbb{k}^{[3]}$ by Theorem 5.3.12.

Lemma 5.4.2 *The ring $R[w]$ has the following properties:*

(i) *$R[w]$ is an integral domain.*

(ii) *$R[w]$ is a* UFD.

(iii) *$R[w]$ is finitely generated as a $\mathbb{k}$-algebra.*

(iv) $\dim(R[w]) = 4$.

(v) $\mathrm{Frac}(R[w]) = \mathbb{k}^{(4)}$.

Recall $D_1, D_2 \in \mathrm{LND}(\mathbb{k}[X,Y,Z,T]) \setminus \{0\}$ as defined in (5.1.1) and (5.1.2):

$$\begin{aligned} D_1 : \mathbb{k}[X,Y,Z,T] &\to \mathbb{k}[X,Y,Z,T] \\ X &\mapsto 0 \\ Y &\mapsto 3T^2 \\ Z &\mapsto 0 \\ T &\mapsto -X^2 \end{aligned}$$

and

$$\begin{aligned} D_2 : \mathbb{k}[X,Y,Z,T] &\to \mathbb{k}[X,Y,Z,T] \\ X &\mapsto 0 \\ Y &\mapsto 2Z \\ Z &\mapsto -X^2 \\ T &\mapsto 0 \end{aligned}$$

These derivations induce non-zero locally nilpotent derivations $\overline{D}_1, \overline{D}_2 \in \text{LND}(R)\backslash\{0\}$. Extend $\overline{D}_1, \overline{D}_2$ to non-zero locally nilpotent derivations $\widehat{D}_1, \widehat{D}_2$ of $R[w] = R^{[1]}$ by setting $\widehat{D}_1(w) = \widehat{D}_2(w) = 0$. Then

$$R[w]^{\widehat{D}_1} = \mathbb{k}[x, z, w] \tag{5.4.1}$$

and

$$R[w]^{\widehat{D}_2} = \mathbb{k}[x, t, w]. \tag{5.4.2}$$

We can now show $\mathcal{D}(R[w]) = R[w]$. It follows that the Derksen invariant does not distinguish between $R[w]$ and $\mathbb{k}^{[4]}$.

Lemma 5.4.3 $\mathcal{D}(R[w]) = R[w]$.

Proof: By (5.4.1) we have $\mathbb{k}[x, z, w] \subseteq \mathcal{D}(R[w])$. Let $D = \frac{\partial}{\partial w}$, and note that D is a non-zero locally nilpotent derivation of $R[w]$. It is clear that $R[w]^D = \mathbb{k}[x, y, z, t]$, so $\mathbb{k}[x, y, z, t] \subseteq \mathcal{D}(R[w])$. It follows that $\mathcal{D}(R[w]) = R[w]$. ∎

Remark 5.4.4 By Lemma 2.7.6 and Corollary 5.3.14, the Makar-Limanov invariant $\text{ML}(R[w])$ of $R[w]$ is a subset of $\mathbb{k}[x]$.

Let $\mathbb{k} = \mathbb{C}$. In [14], Dubouloz shows the following:

Theorem 5.4.5 $\text{ML}(R[w]) = \mathbb{C}$.

That is, he shows that there exists a locally nilpotent derivation of $R[w]$ not containing x in its kernel. Consequently, the Makar-Limanov invariant does not differentiate between $R[w]$ and $\mathbb{k}^{[4]}$.

Here is a sketch of Dubouloz' proof in [14]. He shows there exists an isomorphism

$$\varphi : \mathbb{C}[X, Y, Z, T^{\pm 1}, W]/\langle -XY + Z^2 + T\rangle \to \mathbb{C}[X, Y, Z, T^{\pm 1}, W]/\langle X + X^2Y + Z^2 + T^3\rangle$$

which restricts to the identity on $\mathbb{C}[x, t^{\pm 1}]$. The process of finding this isomorphism is quite intricate, but from this point the proof goes as follows. He considers the locally nilpotent derivation

$$\delta = 2iZ\frac{\partial}{\partial X} + iY\frac{\partial}{\partial Z}$$

on $\mathbb{C}[X, Y, Z, T^{\pm 1}, W]$. Note that $\mathbb{C}[T^{\pm 1}] \subseteq \ker(\delta)$. Since $\delta(-XY + Z^2 + T) = 0$, δ induces a locally nilpotent derivation

$$\overline{\delta} \in \text{LND}(\mathbb{C}[X, Y, Z, T^{\pm 1}, W]/\langle -XY + Z^2 + T\rangle).$$

It follows that

$$D = \varphi \circ \overline{\delta} \circ \varphi^{-1} \in \text{LND}(\mathbb{C}[X, Y, Z, T^{\pm 1}, W]/\langle X + X^2Y + Z^2 + T^3\rangle).$$

Since φ fixes $\mathbb{C}[x, t^{\pm 1}]$ and $\overline{\delta}(x) \neq 0$ we have $D(x) \neq 0$ and $\mathbb{C}[t^{\pm 1}] \subseteq \ker(D)$. Since $t \in \ker(D)$, there exists a $k \in \mathbb{N}$ such that $t^k D$ restricts to a locally nilpotent derivation on $\mathbb{C}[X, Y, Z, T, W]/\langle X + X^2Y + Z^2 + T^3\rangle = \mathbb{C}[x, y, z, t, w]$. It remains to observe that $t^k D(x) \neq 0$, thus $\text{ML}(R[w]) = \mathbb{C}$.

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