

The Pagenumber of Ordered Sets

by
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submitted to the School of Graduate Studies and Research
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the requirements for the degree of
Doctor of Philosophy in Mathematics*

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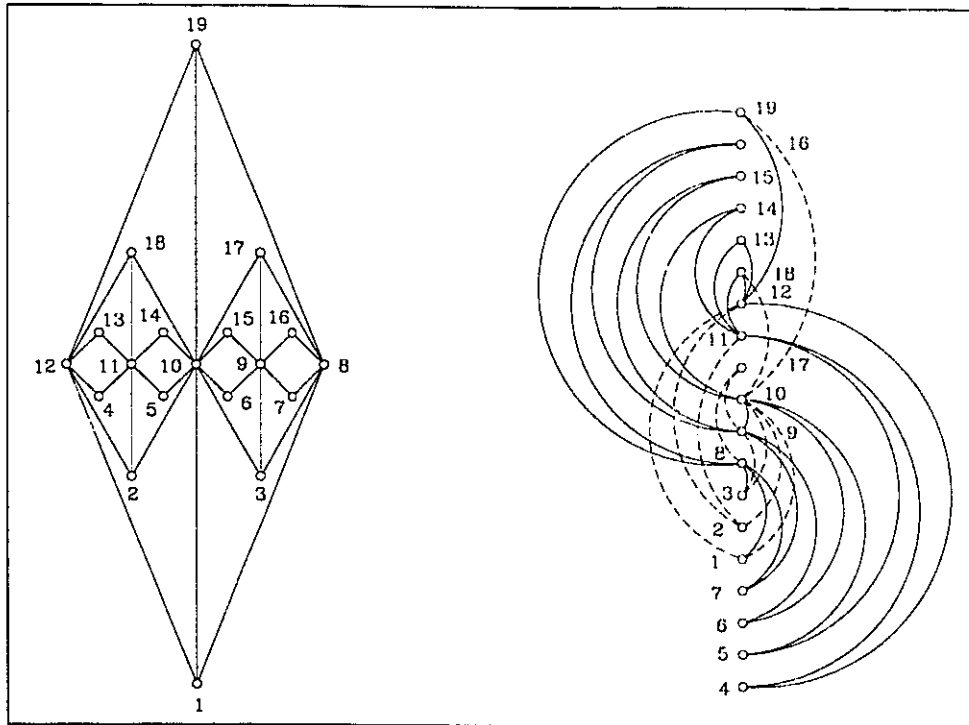
Canada



UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

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By Mohammad Alzohairi



OTTAWA 1996

TO MY MOTHER LATIFAH,
MY WIFE MAHA,
AND MY CHILDREN MALAK,
ABDULAZEEZ AND RIMA

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Abstract

The main result is this

The pagenumber of a series-parallel planar ordered set is at most two.

We present an $O(n^3)$ algorithm to construct a two-page embedding in the case that it is a lattice, where n is the number of the elements of the lattice. One consequence of independent interest, is a characterization of series-parallel *planar* ordered sets.

A k -edge set $\{(a_i, b_i) : 1 \leq i \leq k\}$, in an ordered set P , forms a k -twist in a linear extension L of P , if we have in L $a_1 < a_2 < \dots < a_k < b_1 < b_2 < \dots < b_k$. We give necessary and sufficient conditions for the existence of a linear extension L of an ordered set P such that k edges form a k -twist in L . We also, give lower and upper bounds in some classes of ordered sets. We proved that the problem to determine minimum number of pages required for a fixed linear extension of an ordered set is NP-complete.

We conjecture that the pagenumber of planar ordered sets is unbounded. In contrast, we conjecture that the pagenumber of planar lattices is at most four.

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Introduction

The theory of ordered sets is a modern branch of mathematics. It is based on the simple idea of an order relation; that is a relation which is reflexive, transitive and antisymmetric. Its simplicity makes it a useful tool to model and study a wide variety of problems arising from many fields such as operations research, computer science and social sciences.

A total ordering of the elements of an ordered set P is called a *linear extension* of P , if it is consistent with the ordering of P . The theory of linear extensions is rich and varied. For finite ordered sets, the results are mainly concerned with algorithmic considerations.

A *book embedding* of a graph G consists of an embedding of its nodes along the spine of a book, and an embedding of its edges on pages so that edges embedded on the same page do not intersect. The *pagenumber* of G , $page(G)$, is the minimum number of pages needed, taken over all permutations of the vertices of G .

The pagenumber of an ordered set P , $page(P)$, is the pagenumber of the graph $cov(P)$ taken over only the permutations of the vertices of P which form a linear extension.

The completion of an ordered set is the smallest lattice containing it. The concept of completion plays a central role in proving our main result. We prove first that the page number of a series-parallel planar lattice is at most two and give an $O(n^3)$ algorithm to construct such linear extensions, where n is the number of the elements

of the lattice. Our main result is that the pagenumber number of series-parallel planar ordered sets is at most two.

The idea of the proof of the main result is this. For a series-parallel planar ordered set P , the completion \bar{P} of P , is a series-parallel planar lattice. This implies that it has a two-page linear extension \bar{L} . We use \bar{L} to obtain a two-page linear extension of P .

We also give some lower and upper bounds of the pagenumber of ordered sets and mention important examples. We prove that the problem to determine the minimum number of pages required for a fixed linear extension of an ordered set is NP-complete.

A set of three edges $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$, in an ordered set P , forms a *twist* in a linear extension L of P , if we have in L

$$a_1 < a_2 < a_3 < b_1 < b_2 < b_3$$

We call L twist-free, if it contains no such twist.

It is easy to see that $page(P, L) \geq 3$, if L contains a twist, where $page(P, L)$ denotes the minimum number of pages required for L . We prove that there is a planar ordered set P such that each linear extension of P contains a twist. Also, we give necessary and sufficient conditions for the existence of a linear extension L of an ordered set P such that these three edges form a twist in L . Moreover, we generalize that for k edges, $k \geq 4$.

Chapter 1 is an introduction to ordered sets. We give the basic definitions about ordered sets, its parameters, its operations as well as some basic concepts related to ordered sets. Also, we introduce the class of series-parallel ordered sets, a focal point of this thesis.

Chapter 2 is concerned about lattices — an essential class of ordered sets. We give an idea about the geometry of planar lattices. Finally, we present the completion of an ordered set, the concept which we will use as the bridge in proving our main result in Chapter 6.

In Chapter 3 we give a historical review for the pagewidth for graphs and ordered sets. We give lower and upper bounds of the pagewidth of an ordered set in terms of its combinatorial parameters. Also, we show that finding the minimum number of pages required for a fixed linear extension of an ordered set is NP-complete even if the ordered set is bipartite. We give classes of ordered sets with pagewidth two and a characterization of the class of ordered sets of pagewidth two. Finally, we introduce sequences of ordered sets each of them with unbounded pagewidth. The first one is a sequence of ordered sets of dimension two and the others are sequences of spherical ordered sets and spherical lattices.

In Chapter 4 we indicate the importance of twist-free linear extensions and construct a planar ordered set such that each of its linear extensions contains a twist. For an ordered set P we give necessary and sufficient conditions for the existence of a linear extension L of P such that three edges of P form a twist. Moreover, we generalize that for k edges of P , $k \geq 4$. Finally, we conjecture that the pagewidth of planar ordered sets is unbounded and we give a candidate for such a sequence. In contrast, we conjecture that every planar lattice has a twist-free linear extension. (This implies, if it is true, that the pagewidth of any planar lattice is at most four.)

In Chapter 5 we give a polynomial-time algorithm which embeds any series-parallel planar lattice in two pages.

In Chapter 6 we show that there is a two-page embedding for a series-parallel planar ordered sets. As a consequence, we give a characterization of series-parallel planar ordered sets¹.

¹These results were just announced by M. Alzohairi and I. Rival at Graph Drawing'96. (September 18–20, 1996, Berkeley, California)

Chapter 1

Ordered Sets

1.1 Introduction

This chapter is an introduction to ordered sets. We give the basic definitions about ordered sets, its parameters, its operations as well as some basic concepts related to ordered sets. Also, we introduce the class of series-parallel ordered sets — a focal point of this thesis.

1.2 What¹ is an Ordered Set?

An *ordered set* is a pair consisting of a nonempty set P and a binary relation \leq on P such that the following conditions are satisfied for all a, b and c in P .

- (i) (*Reflexivity*) $a \leq a$.
- (ii) (*Antisymmetry*) If $a \leq b$ and $b \leq a$, then $a = b$.
- (iii) (*Transitivity*) If $a \leq b$ and $b \leq c$, then $a \leq c$.

If a and b are elements of the ordered set P , we also write $b \geq a$ in case $a \leq b$. Also, $a < b$ if $a \leq b$ and $a \neq b$ and we say a is *less than* b . Similarly, $a > b$ if $a \geq b$ and $a \neq b$ and we say a is *greater than* b .

¹For basic terminology we refer the reader to [45] and [13].

On a given set several orderings may be possible. For instance, on the set \mathbb{N} of natural numbers $0, 1, 2, \dots$, either of the following orderings is common

- (a) (*Natural ordering*) For $m, n \in \mathbb{N}$, define $m \leq n$ if $n - m \in \mathbb{N}$.
- (b) (*Divisibility*) For $m, n \in \mathbb{N}$, define $m \leq n$ if $n = q.m$ for some $q \in \mathbb{N}$.

With the exception of the natural ordering in \mathbb{N} , perhaps the most frequently encountered order in all mathematics is the order of *set inclusion* of all subsets of a given set.

- (c) (*Subset ordering*) For subsets S and T of \mathbb{N} define $S \leq T$ if each element of S is an element of T .

For $a \neq b$ in the ordered set P , we say a is *comparable* to b if either $a < b$ or $a > b$. Otherwise, a is *noncomparable* to b , write $a \parallel b$.

An *antichain* is a subset A of an ordered set P such that any two distinct elements of A are noncomparable. A *chain* of P is a subset C of P for which any two elements of C are comparable.

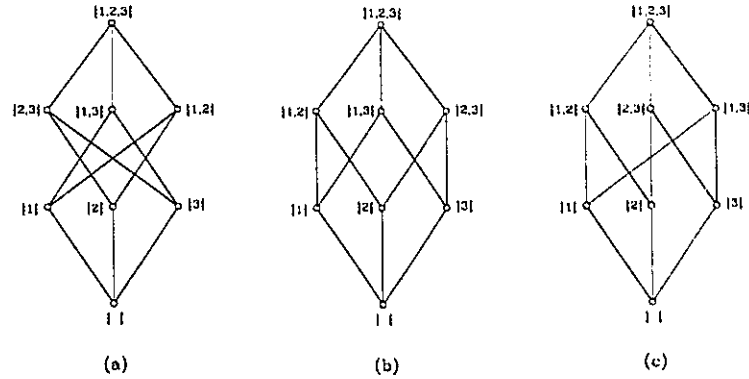
In this thesis every ordered set is finite.

1.3 Upward Drawing

One of the most useful and attractive features of ordered sets is that, in the finite case at least, they can be *drawn*. To describe how to represent ordered sets diagrammatically, we need the idea of covering.

We say a *covers* b (or b is *covered by* a) in the ordered set P , and write $a \succ b$ (or $b \prec a$), if, whenever $a > c \geq b$, then $c = b$. Also, we say a is an *upper cover* of b , or b is a *lower cover* of a , or (a, b) is an *edge* in P .

We say a is a *minimal* (respectively, *maximal*) element of P if a has no lower covers (respectively, a has no upper covers). We denote the set of all minimals (respectively,

Figure 1.1: Upward drawings of 2^3 .

maximals) of P by $\min(P)$ (respectively, $\max(P)$).

The *covering graph* of P , $\text{cov}(P)$, is the graph whose vertices are the elements of P , and the pair $\{a, b\}$ forms an edge in $\text{cov}(P)$ if $a \succ b$ or $a \prec b$.

It is possible (by induction on $|P|$) to draw $\text{cov}(P)$ in the plane such that

- the y -coordinate of a is less than the y -coordinate of b if $a \prec b$, and
- the edge (a, b) is straight and does not pass through any other element of P .

We call such drawing is an *upward drawing* of P . It is common in the theory of ordered sets to represent the vertices of P in the upward drawing by a very small empty circles.

For instance, in Figure 1.1 each of (a),(b),(c) represents an upward drawing of the eight-element ordered set 2^3 , where 2^n denotes all the subsets of any n -element set ordered by inclusion.

An ordered set is *planar* if it has an upward drawing in which no arcs cross. For example, the ordered set on Figure 1.1 is not² planar although its covering graph is planar.

²The point is that every upward drawing of 2^3 has at least some crossing arcs.

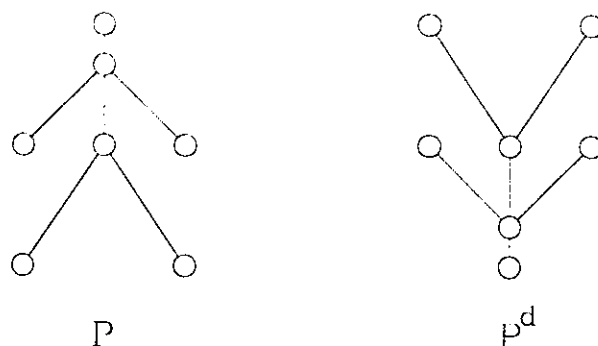


Figure 1.2:

For each integer $i > 0$ we define the i th level of P ,

$$L_i = \min(P - \bigcup_{j=0}^{i-1} L_j)$$

where $L_0 = \min(P)$.

It is clear that there are exactly $\text{height}(P) + 1$ levels for P , where

$$\text{height}(P) = (\text{the size of the longest chain of } P) - 1$$

For instance, the levels for 2^3 in Figure 1.1 are

$$L_0 = \{\{\}\},$$

$$L_1 = \{\{1\}, \{2\}, \{3\}\},$$

$$L_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

$$L_3 = \{\{1, 2, 3\}\}.$$

Given any ordered set P we can form a new ordered set P^d (the *dual* of P) by defining $x \leq y$ to hold in P^d if and only if $y \leq x$ in P . We may obtain an upward drawing for P^d simply by ‘turning upside down’ an upward drawing for P . Figure 1.2 provides a simple illustration.

To each statement about P there corresponds a statement about P^d . For example, we can assert that in P in Figure 1.2, there exists a unique element that covers exactly

three elements in P , while in P^d there exists a unique element covered by three elements. In general, to any given statement Φ about an ordered set P , we obtain the *dual statement* Φ^d by replacing each occurrence of \leq by \geq and vice versa. In fact, if a statement Φ is true for P , then Φ^d is true for P^d . This fact can often be used to give two theorems for the price of one or to reduce work.

1.4 Maps Between Ordered Sets

This section introduces structure-preserving maps between ordered sets. In particular, it provides the machinery for deciding when two ordered sets are essentially the same.

Let P and Q be ordered sets. A map $f : P \rightarrow Q$ is said to be

- (i) *order-preserving* if $x \leq y$ in P implies $f(x) \leq f(y)$ in Q ;
- (ii) *order-embedding* if $x \leq y$ in P if and only if $f(x) \leq f(y)$ in Q ;
- (iii) an *order-isomorphism* if it is an order-embedding of P onto Q .

We say P can be embedded in Q , if there is an order-embedding from P to Q . When there is an order-isomorphism from P to Q , we say P is *isomorphic to* Q and write $P \cong Q$.

Figure 1.3 shows some maps between ordered sets. The map f is not an order-preserving map. Each of g and h is order-preserving, but not order-embedding. The map i is an order-embedding but not an order-isomorphism.

1.5 Linear Extensions

An *extension* Q of an ordered set P is an order defined on the same underlying set (as P) and satisfying $x \leq y$ in Q whenever $x \leq y$ in P . A *linear extension* L of P is an extension which is a chain. For instance, Figure 1.4 illustrates a four-element ordered set together with all of its linear extensions.

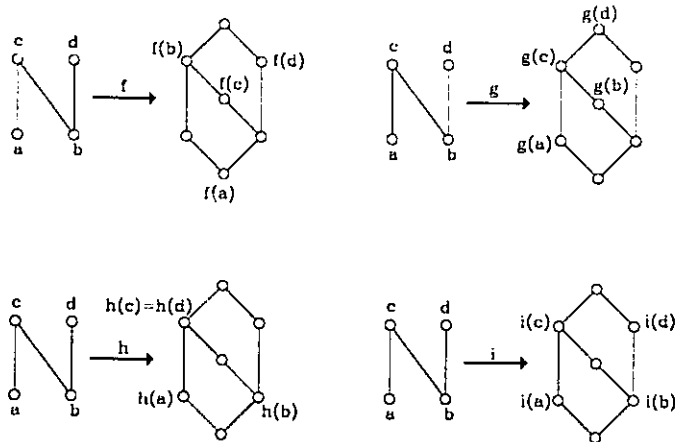


Figure 1.3:

The theory of linear extensions is rich and varied. For finite ordered sets, the results and methods are mainly concerned with algorithmic considerations. The next Theorem is the basic theorem in this subject and it is due to Szpilrajn[48].

Theorem 1.5.1 *Every finite ordered set with elements $b \not\leq a$ has an extension satisfying $a < b$.*

Successive applications of the above theorem on an ordered set produce an extension which is linear.

Actually, a linear extension

$$x_1 < x_2 < \dots < x_n$$

of an n -element ordered set P is constructed according to an algorithm starting with a minimal element x_1 of P , proceeding one element at a time, to delineate the linear ordering $x_1 < x_2 < \dots$ and ending with a maximal element x_n of P . A permutation y_1, y_2, \dots, y_n of the elements of P determines a linear extension of P , if each y_{i+1} is chosen among the minimal elements of the set

$$U_{i+1} = P - \{y_1, y_2, \dots, y_i\}$$

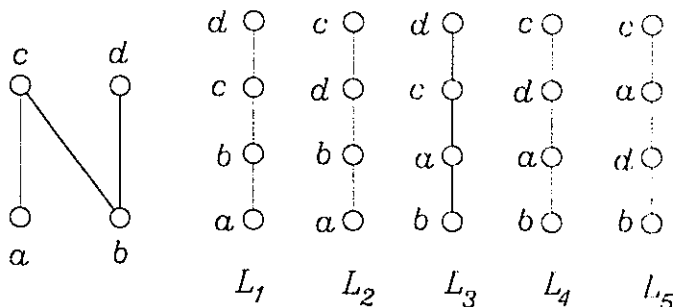


Figure 1.4:

So, $y_1 \in \min(P) = \min(U_1)$ and, for $i = 1, 2, \dots, n - 1, y_{i+1} \in \min(U_{i+1})$.

For some problems a specific algorithm for a constructing a linear extension is useful. For example, a *greedy linear extension* of an ordered set P is a linear extension $x_1 < x_2 < \dots < x_n$ of P such that $x_1 \in \min(P)$ and, for $i \geq 1, x_{i+1} \in \min(P - \{x_1, x_2, \dots, x_i\})$ and, if possible, $x_{i+1} > x_i$. Thus, a greedy linear extension is obtained by following the rule “climb as high as you can”. For instance, in Figure 1.4, L_1, L_2, L_5 are greedy linear extensions. L_3, L_4 are not greedy linear extensions because $a \succ b$ in both, and $a \parallel b$ in P while $d \in \min(P - \{b\})$ and $d > b$ in P .

1.6 Series-Parallel Ordered Sets

The *linear sum* $P \oplus Q$ of the two disjoint ordered sets P, Q is an ordered set on $P \cup Q$, where $a \leq b$ if

1. $a \leq b$ in P , or
2. $a \leq b$ in Q , or
3. $a \in P$ and $b \in Q$.

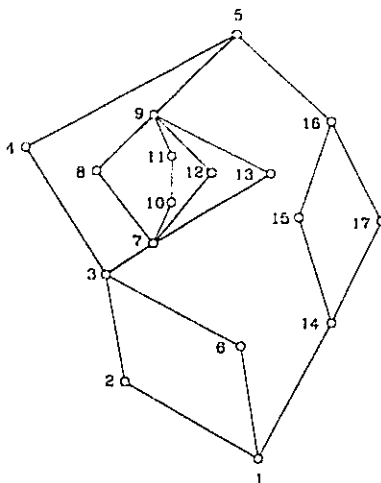


Figure 1.5: A series-parallel planar ordered set.

If we eliminate the third condition of the definition of linear sum, we will have the *disjoint sum* $P + Q$ of P, Q .

An ordered set P is *series-parallel* if P can be constructed from singletons using only the constructions of disjoint sum and linear sum. In other words, P can be decomposed into singletons using only these two operations. For instance, the series-parallel ordered set illustrated in Figure 1.5 can be decomposed into

$$1 \oplus (((2 + 6) \oplus 3 \oplus (4 + (7 \oplus (8 + (10 \oplus 11) + 12 + 13) \oplus 9))) + (14 \oplus (15 + 17) \oplus 16)) \oplus 5$$

With any series-parallel decomposition we may associate a *binary decomposition tree* according to the successive steps \oplus and $+$ in its decomposition into singletons. (This is not unique.) For instance, Figure 1.6 illustrates a binary decomposition tree of the ordered set in Figure 1.5. (The bold face edges correspond to \oplus and the regular ones to $+$.)

A four-element subset $\{a, b, c, d\}$ of an ordered set P forms an **N** if the only comparabilities in P among them are $a < c$, $b < c$ and $b < d$. The next Theorem is well known [51] (cf. [43]).

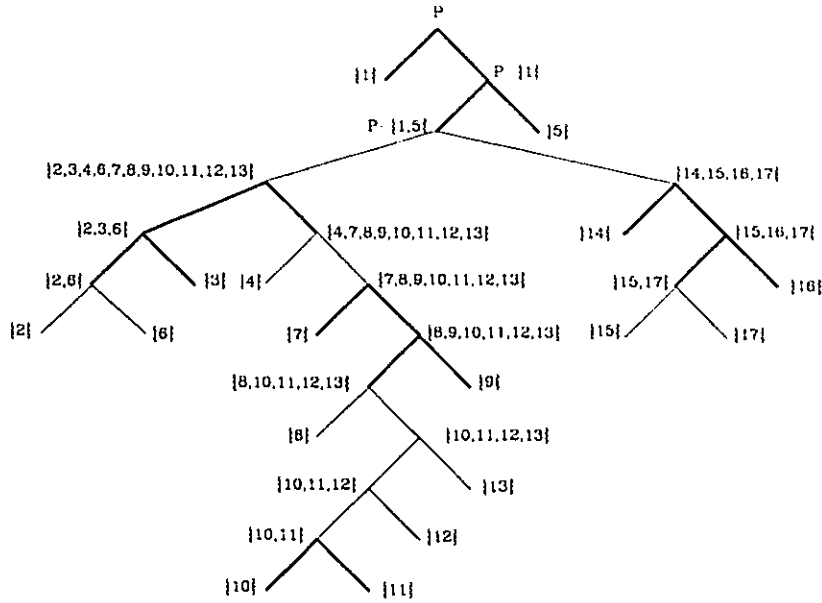


Figure 1.6: A binary tree decomposition of the series-parallel ordered set in Figure 1.5

Theorem 1.6.1 *An ordered set is a series-parallel if and only if it contains no subset N .*

Proof. Let P be a series-parallel ordered set. Suppose that P contains a subset $N = \{a < c > b < d\}$. Consider a binary decomposition tree of P . Let Q be a minimal node in this tree which contains N . We may assume that Q is connected³ since if $Q = Q_1 + Q_2$, then $N \subseteq Q_1$ or $N \subseteq Q_2$ contrary to the minimality of Q . Otherwise, $Q = Q_1 \oplus Q_2$ where both Q_1 and Q_2 are proper subsets of Q .

Since $a \parallel d$, either $\{a, d\} \subseteq Q_1$ or $\{a, d\} \subseteq Q_2$. If $\{a, d\} \subseteq Q_1$, then $b \in Q_1$ because $b < d$. Also, as $c \not\leq d$ and $d \in Q_1$, then $c \in Q_1$. Thus, $N \subseteq Q_1$. Similarly, if $\{a, d\} \subseteq Q_2$, then $N \subseteq Q_2$, contrary to the minimality of Q .

Let P be a finite ordered set which contains no subset N . We show by induction on $|P|$ that P can be decomposed into singletons using $+$ and \oplus .

If P is not connected, then $P = P_1 + P_2$. Since neither P_1 nor P_2 contains N , the result follows by induction.

³An ordered set P is *connected* if $\text{cov}(P)$ is a connected graph.

Let P be connected. Let x be a maximal element in P and let y be a minimal element in P . Then there is a shortest 'zig-zag' $x = z_0 > z_1 < z_2 > \cdots < z_{k-1} > z_k = y, k \geq 1$, connecting x and y . As P contains no subset $N, k = 1$, that is $x > y$. In other words, every maximal element is greater than every minimal element.

For $m \in \min(P)$, let U_m denote the set of all elements $x \geq m$. Set

$$I = \bigcap_{m \in \min(P)} U_m$$

Now, $I \neq \emptyset$; in fact, every maximal element of P belongs to I . Also, $I \cap \min(P) = \emptyset$, otherwise $|\min(P)| = 1$ and $P \cong \min(P) \oplus (P - \min(P))$ so we could apply the induction hypothesis. In particular, $P - I \neq \emptyset$ too.

Finally, we show that

$$P \cong (P - I) \oplus I$$

Suppose there is $x \in P - I$ and $y \in I$ such that $x \not\leq y$. Evidently, $x \notin \min(P)$ so choose a minimal element m satisfying $m \leq x$. Of course, $m \leq y$ too. Moreover, from $x \notin I$, it follows that there is another minimal element m' satisfying $m' \not\leq x$ although $m' < y$. Then, the subset $\{m', y, m, x\}$ forms N . ■

The *direct product* of two ordered sets P and Q , is the ordered set $P \times Q$ on the Cartesian product of the two underlying sets P and Q , such that $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ in P and $y_1 \leq y_2$ in Q . Inductively, we can generalize the direct product to any number of ordered sets⁴.

Informally, the upward drawing of the product $P \times Q$ is drawn by replacing each point of an upward drawing of P by a copy of an upward drawing for Q and connecting corresponding points; this assumes that the points are placed in such a way that the rules for upward drawing are obeyed. Figure 1.7 shows upward drawings for some products.

⁴Notice that the direct product is associative.

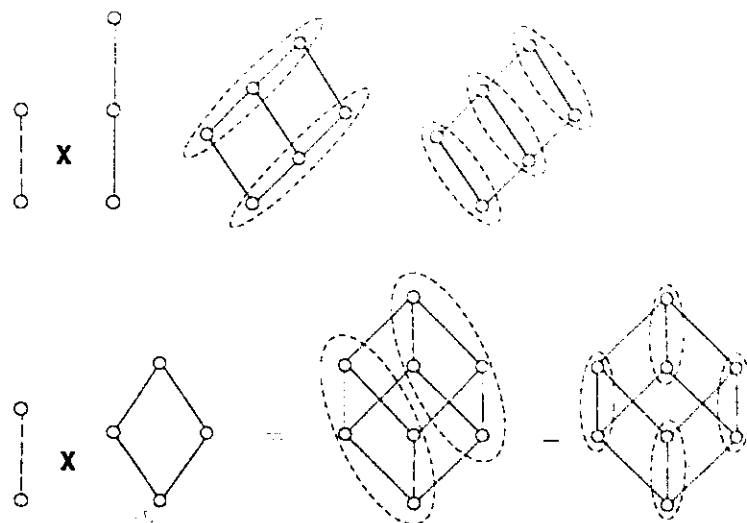


Figure 1.7:

The *order dimension* of an ordered set P is the smallest number m of linear extensions L_1, L_2, \dots, L_m whose intersection is P , that is, $x \leq y$ in P if and only if, for every $i = 1, 2, \dots, m$, $x \leq y$ in L_i . In this case, there is a mapping

$$f : P \longrightarrow L_1 \times L_2 \times \cdots \times L_m$$

according to which $f(x) = (x, x, \dots, x)$, that is, for each $i = 1, 2, \dots, m$, $\text{proj}_i(f(x)) = x$. This is an order-embedding of P into $L_1 \times L_2 \times \cdots \times L_m$. (Indeed, f is injective because for x, y in P , $(x, x, \dots, x) = f(x) = f(y) = (y, y, \dots, y) \Leftrightarrow x = y$. Also, $f(x) = (x, x, \dots, x) \leq (y, y, \dots, y) = f(y)$ if and only if $x \leq y$ in each L_i , $1 \leq i \leq m$.) Conversely, let P be a subset of a direct product $C_1 \times C_2 \times \cdots \times C_m$ of m chains, and, for each $i = 1, 2, \dots, m$, let L_i be the linear extension of P satisfying $x < y$ in L_i if $\text{proj}_i(x) < \text{proj}_i(y)$ in C_i while, if $\text{proj}_i(x) < \text{proj}_i(y)$ in C_i , then $x < y$ in L_i if $x < y$ in P . For instance, the order dimension of the ordered set \mathbf{N} in Figure 1.4 is two because $\mathbf{N} = L_1 \cap L_5$.

The next Lemma is due to **Dushnik and Miller** [15].

Lemma 1.6.1 *The order dimension of a series-parallel ordered set is at most two.*

Proof. Let P be a series-parallel ordered set. We show by induction on $|P|$ that the order dimension of P is at most two.

If $|P| = 1$, then the order dimension of P is one.

Since P is a series-parallel ordered set, there are two nonempty subsets Q and R of P such that either $P = Q + R$ or $P = Q \oplus R$. By the induction hypothesis, there are two linear extensions L_Q, L'_Q of Q and two linear extensions L_R, L'_R of R such that $Q = L_Q \cap L'_Q$ and $R = L_R \cap L'_R$.

Case 1 $P = Q + R$.

Let $L_1 = L_Q \oplus L_R$ and $L_2 = L'_R \oplus L'_Q$. It is clear that L_1 and L_2 are linear extensions of P .

If $x \parallel y$ in P , then either $x \parallel y$ in Q (or R) or $x \in Q$ and $y \in R$.

If $x \parallel y$ in Q (or R), then $x < y$ say in L_Q , so $x < y$ in L_1 , and $y < x$ in L'_Q , so $y < x$ in L_2 . Thus, $x \parallel y$ in $L_1 \cap L_2$.

If $x \in Q$ and $y \in R$, then $x < y$ in L_1 and $y < x$ in L_2 . Therefore, $x \parallel y$ in $L_1 \cap L_2$.

Case 2 $P = Q \oplus R$.

Let $L_1 = L_Q \oplus L_R$ and $L_2 = L'_Q \oplus L'_R$. It is clear that L_1 and L_2 are linear extensions of P .

If $x \parallel y$ in P , then either $x \parallel y$ in Q or $x \parallel y$ in R .

If $x \parallel y$ in Q , then $x < y$ say in L_Q and $y < x$ in L'_Q . Thus, $x < y$ in L_1 and $y < x$ in L_2 . Therefore, $x \parallel y$ in $L_1 \cap L_2$.

Thus, $P = L_1 \cap L_2$. Therefore, the dimension of P is two. ▀

A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. An edge of a graph is said to be *subdivided* when it is deleted

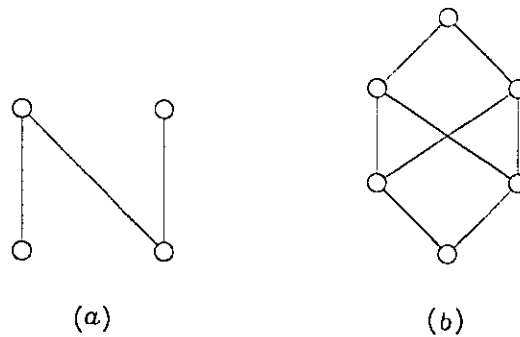


Figure 1.8:

and replaced by a path of length two connecting its ends. A graph is *series-parallel* if it does not contain a subdivision of K_4 .

It seems that series-parallel ordered sets are considerably different from series-parallel graphs. For instance, the ordered set in Figure 1.8(a) is not series-parallel while its covering graph is series parallel. On the other hand, the ordered set in Figure 1.8(b) is series-parallel while its covering graph is not a series-parallel graph.

Chapter 2

Lattices

2.1 Introduction

This chapter is concerned about lattices — an essential class of ordered sets. We will give an idea about the geometry of planar lattices. Finally, we present the completion of an ordered set, the concept which we will use as the bridge in proving our main result in Chapter 6.

2.2 What is a Lattice?

An ordered set P is a *lattice* if every finite subset S has supremum denoted $\sup(S)$, and infimum denoted $\inf(S)$. Thus, $\sup(S)$ is that element a of P satisfying these properties.

- (i) (Upper bound) $s \leq a$ for every $s \in S$.
- (ii) (Least upper bound) If $s \leq b$, for every $s \in S$, then $a \leq b$.

Also, we define dually $\inf(S)$. A simple induction shows that this is equivalent to the existence of the supremum and the infimum for each pair of elements of P .

For instance, the ordered set in Figure 2.1(a) is not a lattice because $\sup(\{a, b\})$ does not exist. On the other hand the two ordered sets in Figure 2.1(b) and in Figure 1.5

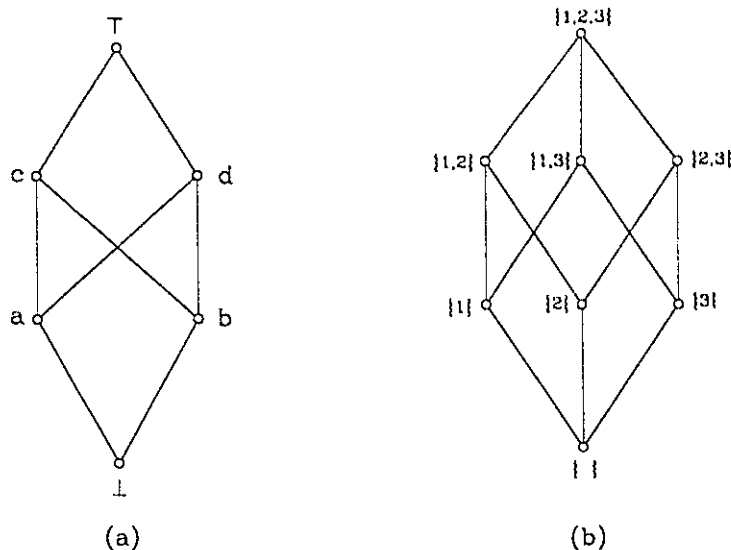


Figure 2.1:

are lattices.

The concept of lattices goes back to **R. Dedekind** (1900) [10].

We say \top is the *top* of an ordered set P if \top is maximal and for every x in P , $\top \geq x$.

Dually, we define the *bottom* \perp of P .

Notice that, in a lattice P , every four-cycle $\{c_1 < c_3 > c_2 < c_4 > c_1\}$ must have a *splitting* element, that is an element x satisfying $c_1, c_2 \leq x \leq c_3, c_4$ (for instance, $x = \sup(\{c_1, c_2\})$ or $x = \inf(\{c_3, c_4\})$). Also, $\sup(P) = \top$ and $\inf(P) = \perp$.

This leads to the following theorem (which is folklore (see [45])), which gives the steps to recognize whether an ordered set P is a lattice or not.

Theorem 2.2.1 *A finite ordered set with \top and \perp is a lattice if and only if every four-cycle has a splitting element.*

Proof. For necessity, suppose that P is an ordered set with \top and \perp , and yet there is $S \subseteq P$ without $\sup(S)$. Then S has at least two maximal elements $a \neq b$.

Let $T = \{x \in P : x > s \text{ for every } s \in S\}$. As $\top \in T$, $T \neq \emptyset$. It is clear that S and T are disjoint. If $|\min T| = 1$, then S has a supremum. Thus, T has at least two

minimal elements c, d . Therefore, P contains four-cycle $\{a, b, c, d\}$ without a splitting element. \blacksquare

Notice that Theorem 2.2.1 is not true for infinite lattices. For instance, take two copies of \mathbb{Q}_1 and \mathbb{Q}_2 of the chain of all rational numbers \mathbb{Q} . Let $a \in \mathbb{Q}_i, b \in \mathbb{Q}_j$. Then $a \leq b$ if $a \leq b$ in \mathbb{Q} . Add a top \top and a bottom \perp . This ordered set is not a lattice yet every four-cycle has a splitting element.

An element a in a lattice is *join irreducible* if it has at most one lower cover. Dually, a is *meet irreducible* if it has at most one upper cover. An element is *doubly irreducible* if it is join and meet irreducible.

Lemma 2.2.1 ([13]) *For every $a \not\leq b$ in a lattice P there is a join irreducible $x \in P$ satisfying $x \leq a$ and $x \not\leq b$. Moreover, for every $a \not\leq b$ in a lattice P there is a meet irreducible $x \in P$ satisfying $x \geq b$ and $x \not\geq a$.*

Proof. Let $S = \{x \in P : x \leq a \text{ and } x \not\leq b\}$. The set S is not empty since it contains a . Let $x \in \min(S)$. We claim that x is join irreducible. If not, then x has at least two lower covers c and d .

By minimality of x , neither c nor d lies in S . Thus, $b > c, d$. As $c \prec x$ and $b > c$, $x \not\prec b$, which implies that $x \parallel b$. Thus, $C = \{c, d, x, b\}$ forms a four-cycle. As $c \prec x$, C has no splitting element, which contradicts Theorem 2.2.1. \blacksquare

2.3 Planar Lattices

Checking whether a finite lattice is planar or not is as easy as checking whether a finite graph is planar or not. Indeed, Platt in [39] showed that a finite lattice is planar if and only if its covering graph, with an additional edge joining its bottom to its top, is itself a planar graph. As graph planarity testing is easy [29], lattice planarity testing is, too. Planarity for lattices is well understood [32]. For ordered sets in general, planarity

testing is hard: Garg and Tamassia [22] proved that checking whether an ordered set is planar or not is NP-complete.

Here are a few elementary terms. Fix a lattice P and fix a planar upward drawing of it. For noncomparable elements $a, b \in P$ such that $a \succ c$ and $b \succ c$, we say a is left of b if any horizontal segment (moving from left to right) which cuts both edges, always cuts the edge (c, a) before the edge (c, b) . For arbitrary $a \parallel b$ in P say that a is left of b , denoted $a \lambda b$, if a' is left of b' , where $a \geq a' \succ \inf(\{a, b\})$ and $b \geq b' \succ \inf(\{a, b\})$. An element a , which does not belong to the maximal chain C is left of C if there is $b \in C$ such that $a \lambda b$. (Of course, all of these ideas are ambidextrous. If a is left of b , then b is right of a , etc.)

Kelly and Rival [32] showed that, if in planar lattice $x \lambda y$, then x is on the left of any maximal chain through y . Moreover, if $x \parallel y$ and x is on the left of some maximal chain through y , then $x \lambda y$.

An immediate consequence of that is, $x \parallel y$ implies that exactly one of $x \lambda y$ or $y \lambda x$.

Once equipped with the equality relation, λ becomes an order relation on the underlying set P , denoted P_λ . (This result is due to J. Zilber¹.)

For example, the ordered set in Figure 2.2(b) is P_λ where P is the planar lattice in Figure 2.2(a).

A complement of an ordered set P , $comp(P)$, is an ordered set on the underlying set P such that x is comparable to y in P if and only if $x \parallel y$ in $comp(P)$. For instance, the ordered set P_λ in Figure 2.2(b) is $comp(P)$ for the lattice P in Figure 2.2(a).

It is clear that each chain in P corresponds to an antichain in $comp(P)$ and vice versa. Moreover, $height(comp(P)) = width(P) - 1$ ($width(P)$ is the maximum size of an antichain in P).

Dushnik and Miller [15] showed that a finite ordered set has a complement if and only if has order dimension two.

¹see [4] page 32, ex. 7(c)

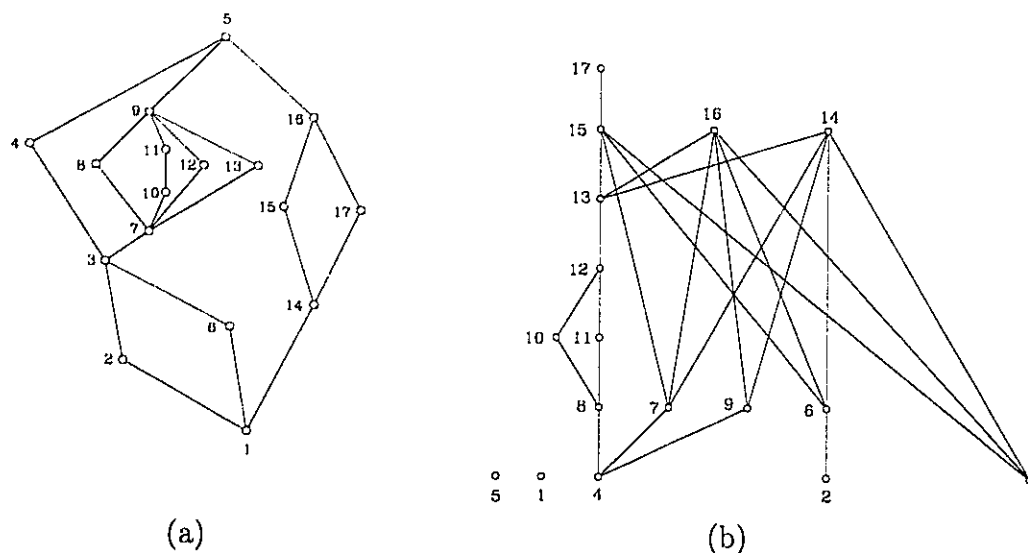


Figure 2.2:

The next theorem (which is due to Baker [1]) gives a characterization for planar lattices.

Lemma 2.3.1 *A lattice is planar if and only if it has order dimension at most two.*

Proof.(cf. Rival [45]) Fix a planar upward drawing of a planar lattice P . The *left greedy* linear extension L_{left} of P is that greedy linear extension whose $(i + 1)$ th element x_{i+1} is the (unique) left-most element belonging to

$$\min(P - \{x_1, \dots, x_i\})$$

from the bottom to the top; the *right greedy* linear extension L_{right} is that greedy linear extension whose $(i + 1)$ th element x_{i+1} belonging to $\min(P - \{x_1, \dots, x_i\})$ Then

$$P = L_{left} \cap L_{right}$$

in the sense that $x \leq y$ in P just if $x \leq y$ in L_{left} and $x \leq y$ in L_{right} . As P is the intersection of these two linear extensions it has order dimension at most two. (In this case, P can be embedded into the direct product $L_{left} \times L_{right}$, according to the

function $f(x) = (x, x)$ for every $x \in P$, that is, P isomorphic to a subset of this direct product.)

Let P be a subset of the direct product $C_1 \times C_2$ of two chains C_1, C_2 . Fix a planar embedding of the planar lattice $C_1 \times C_2$. If P is a cover-preserving subset of $C_1 \times C_2$, then, of course, P is planar. If it is not, take the drawing in which an edge (a, b) of P follows some chains, maximal from a to b in $C_1 \times C_2$. If two such edges cross, say (a, b) and (c, d) , then in $C_1 \times C_2$, they cross at an element which “splits” $\{a, c\}$ from $\{b, d\}$. Thus, $a, c \leq b, d$ and, as P is a lattice, $\sup(\{a, c\}) \leq \inf(\{b, d\})$ in P which, in any case, violates the assumption that $a \prec b$ and $c \prec d$. \square

2.4 The Completion

The *completion* of a finite ordered set P is the smallest lattice into which P can be embedded. The completion is often easy to draw and read. In some cases the completion is used as the bridge linking two problems or classes and uses what is known for one to solve a problem for the other². In this section we show existence and uniqueness of the completion of an ordered set.

In general, there are many ways in which an ordered set can be embedded in a lattice. For instance, any ordered set can be embedded in 2^n (where n is the number of elements of P) by mapping an element x of P to its *down set*, $D(x) = \{y \in P : y \leq x\}$.

For instance, the lattice in Figure 2.3(b) is the completion of the ordered set in Figure 2.3(a). So the six-element ordered set of Figure 2.3(a) can be embedded in a 7-element lattice and also a 46-element lattice 2^6 . (Actually, the completion is the smallest one.)

Let X be a set and $\mathcal{P}(X)$ is the collection of all subsets of X . A map

$$c : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

²See [31]. Also, we will use this idea in Chapter 6.

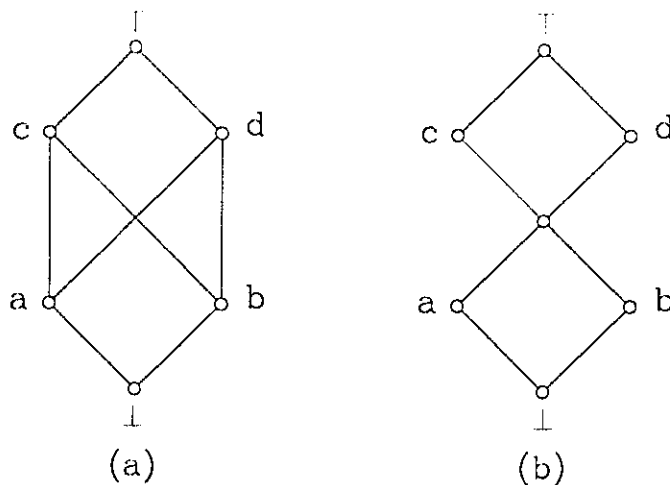


Figure 2.3:

is a *closure operator* [4] on X if, for all $A, B \subseteq X$,

(a) $A \subseteq c(A)$,

(b) if $A \subseteq B$, then $c(A) \subseteq c(B)$,

(c) $c(c(A)) = c(A)$.

A subset A of X is called *closed* if $c(A) = A$. Actually, $c(A)$ is the smallest closed subset of X containing A . Indeed, if $A \subseteq B$ for a closed subset B of X , then $c(A) \subseteq c(B) = B$.

Lemma 2.4.1 (Birkhoff [4, 13]) *For a closure operator c on a set X , the intersection of any family of closed subsets is closed.*

Proof. Let $D = \bigcap_{\alpha \in I} A_\alpha$ for a collection of closed subsets. For every $\alpha \in I$, $D \subseteq A_\alpha$ which implies $c(D) \subseteq c(A_\alpha) = A_\alpha$. Thus, $c(D) \subseteq D$. Therefore, $c(D) = D$. \blacksquare

Lemma 2.4.2 (Birkhoff [4, 13]) *Let c be a closure operator on set X . Then the family*

$$\mathcal{L}_c = \{A \subseteq X : c(A) = A\}$$

of closed subsets ordered by inclusion is a lattice.

Proof. Let $A_\alpha, \alpha \in I$ be a collection of closed subsets of X . From the previous lemma, $\bigcap_{\alpha \in I} A_\alpha$ is closed. Thus

$$\inf(\{A_\alpha : \alpha \in I\}) = \bigcap_{\alpha \in I} A_\alpha$$

Also,

$$\sup(\{A_\alpha : \alpha \in I\}) = c\left(\bigcup_{\alpha \in I} A_\alpha\right)$$

because $c(\bigcup_{\alpha \in I} A_\alpha)$ is the smallest closed subset of X containing $\bigcup_{\alpha \in I} A_\alpha$. Therefore, \mathcal{L}_c is a lattice. \blacksquare

Let P be an ordered set and $A \subseteq P$. Define A “upper” by

$$A^+ = \{x \in P : a \leq x \text{ for every } a \in A\}$$

Dually, we define A “lower”, $A^- = \{x \in P : a \geq x \text{ for every } a \in A\}$.

Lemma 2.4.3 (Birkhoff [4, 13]) *Let A and B be subsets of an ordered set P . Then*

- (i) $A \subseteq (A^+)^-$ and $A \subseteq (A^-)^+$;
- (ii) if $A \subseteq B$, then $A^+ \supseteq B^+$ and $A^- \supseteq B^-$;
- (iii) $A^+ = ((A^+)^-)^+$ and $A^- = ((A^-)^+)^-$.

Proof. We have $a \leq x$ for all $a \in A$ and all $x \in A^+$, which says precisely that $A \subseteq (A^+)^-$. Dually, $A \subseteq (A^-)^+$. Thus, (i) holds.

If $A \subseteq B$, then any element of B^+ is an upper bound of B and so is an upper bound of A and hence belongs to A^+ . Thus (ii) holds.

By (i) we have $A \subseteq (A^+)^-$, whence (ii) yields $A^+ \supseteq ((A^+)^-)^+$. But (i), applied to A^+ , also gives $A^+ \subseteq ((A^+)^-)^+$. Hence, $A^+ = ((A^+)^-)^+$ and, by duality $A^- = ((A^-)^+)^-$, which proves (iii). \blacksquare

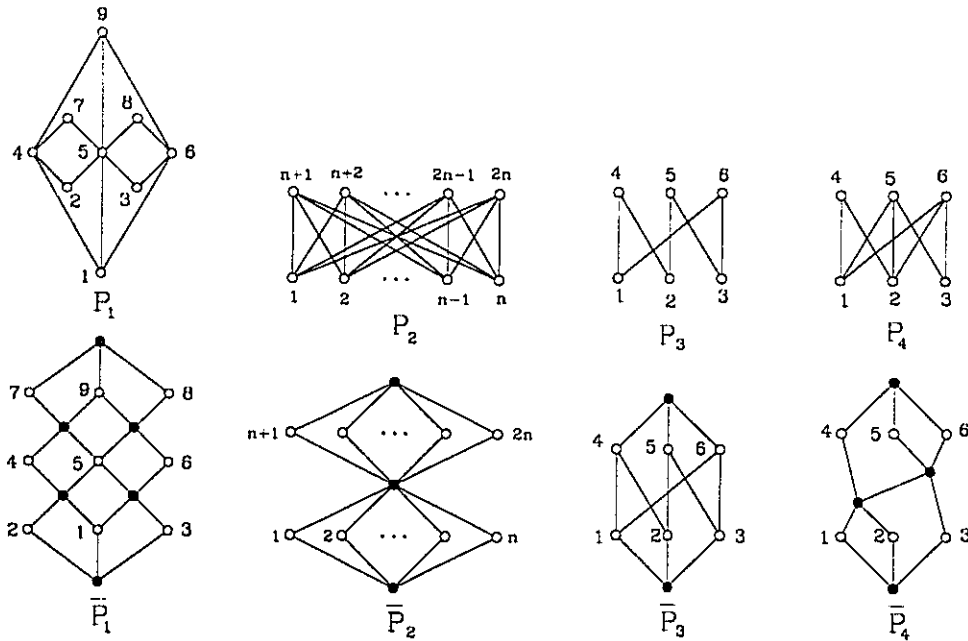


Figure 2.4:

It follows very easily from Lemma 2.4.3 that $c(A) = (A^+)^-$ defines a closure operator on P . By Lemma 2.4.2, the set

$$\bar{P} = \{A \subseteq P : (A^+)^- = A\}$$

ordered by inclusion is a lattice, known as the **Dedekind-MacNeille** [35, 13] completion of P or for short, the completion of P (as we will prove later). Figure 2.4 illustrates the ordered sets P_i and their corresponding completions \bar{P}_i for $i = 1, 2, 3, 4$.

Lemma 2.4.4 (MacNeille [35, 13]) *Let P be an ordered set.*

- (i) *For all $x \in P$, $((D(x))^+)^- = D(x)$ and hence $D(x) \in \bar{P}$.*
- (ii) *If $A \subseteq P$ and $\inf(A)$ exists in P , then $\bigcap_{a \in A} D(a) = D(\inf(A))$*
- (iii) *If $A \subseteq P$ and $\sup(A)$ exists in P , then $(A^+)^- = D(\sup(A))$.*

Proof.

(i) Let $y \in (D(x))^+$; then $z \leq y$ for all $z \in D(x)$ so, in particular, $x \leq y$ as $x \in D(x)$ and hence $y \in U(x)$ ($U(x) = \{y \in P : y \geq x\}$, which is called the *up set* of x in P). Thus, $D(x)^+ \subseteq U(x)$. If $y \in U(x)$, then $y \geq x$ and so, by transitivity, $y \geq z$ for all $z \in D(x)$, that is, $y \in (D(x))^+$. Thus, $U(x) \subseteq (D(x))^+$. Therefore, $(D(x))^+ = U(x)$ and, by duality, $(U(x))^- = D(x)$. Thus, $((D(x))^+)^- = (U(x))^- = D(x)$.

(ii) Let $A \subseteq P$ and assume that $\inf(A)$ exists in P . Note that,

$$\bigcap_{a \in A} D(a) = \{x \in P : (\text{for every } a \text{ in } A) x \leq a\} = A^-$$

Since $\inf(A)$ is a lower bound of A we have $\inf(A) \in A^-$ and hence $D(\inf(A)) \subseteq A^-$. Since $\inf(A)$ is the greatest lower bound of A we have $x \leq \inf(A)$ for all $y \in A^-$ and hence, $A^- \subseteq D(\inf(A))$. Thus, $A^- = D(\inf(A))$.

(iii) Let $A \subseteq P$ and assume that $\sup(A)$ exists in P . Of course, $\sup(A) \in A^+$. Thus, $x \in (A^+)^-$ implies that $x \leq \sup(A)$ and hence $x \in D(\sup(A))$. Consequently, $(A^+)^- \subseteq D(\sup(A))$. Since $\sup(A)$ is the least upper bound of A we have $\sup(A) \leq y$ for all $y \in A^+$ and hence, $\sup(A) \in (A^+)^-$. This gives $D(\sup(A)) \subseteq (A^+)^-$. Hence, $(A^+)^- = D(\sup(A))$, as required. \blacksquare

Theorem 2.4.1 (MacNeille [35, 13]) *Let P be an ordered set and define $\varphi : P \rightarrow \bar{P}$ by $\varphi(x) = D(x)$ for all $x \in P$. Then $P \cong \varphi(P)$.*

Proof. From Lemma 2.4.4 $D(x) \in \bar{P}$ for every $x \in P$. If in P , $D(x) \subseteq D(y)$, then $x \in D(y)$, which implies $x \leq y$. Also, if $x \leq y$ in P and $z \leq x$, then $z \leq y$, i.e. $D(x) \subseteq D(y)$. Thus, φ is an order-embedding map. Therefore, $P \cong \varphi(P)$. \blacksquare

Lemma 2.4.5 (MacNeille [35, 13]) *Let P be an ordered set and $A \subseteq P$. Then in \bar{P}*

1. $\inf(\{D(x) : x \in A^+\}) = (A^+)^-$,

$$2. \sup(\{D(a) : a \in A\}) = (A^+)^-.$$

Proof. For $B \subseteq P$, we have $B^- = \bigcap_{x \in B} D(x)$ because

$y \in B^-$ if and only if (for every $x \in B$) $y \leq x$

if and only if (for every $x \in B$) $y \in D(x)$

if and only if $y \in \bigcap_{x \in B} D(x)$.

With $B = A^+$ this yields, $(A^+)^- = \bigcap_{x \in A^+} D(x)$. From the proof of Lemma 2.4.2, $\inf(\{D(x) : x \in A^+\}) = \bigcap_{x \in A^+} D(x) = (A^+)^-$.

Let $a \in A$; then $a \in (A^+)^-$ as $A \subseteq (A^+)^-$. Hence $D(a) \subseteq (A^+)^-$ as $(A^+)^-$ is the union of down sets. Hence, $(A^+)^-$ is an upper bound in \bar{P} of $\{D(a) : a \in A\}$. If $B \in \bar{P}$ is an upper bound of $\{D(a) : a \in A\}$, then $a \in D(a) \subseteq B$ for all $a \in A$ and hence $A \subseteq B$. Thus, $(A^+)^- \subset (B^+)^- = B$ (two applications of Lemma 2.4.3(ii)). Consequently $(A^+)^-$ is the least upper bound of $\{D(a) : a \in A\}$ in \bar{P} . \square

Lemma 2.4.6 (MacNeille [35, 13]) *Let P be an ordered set. For every $x \in \bar{P}$ $x = \sup(D_P(x)) = \inf(U_P(x))$ where $D_P(x)$, respectively, $U_P(x)$ is the down set, respectively, the upper set of x in P .*

Proof. In \bar{P} every element x is in fact a closed subset of P and every element of P corresponds to a down set for an element of P . Thus, the statement “for every $x \in \bar{P}$ we have $x = \sup(D_P(x)) = \inf(U_P(x))$ ” is equivalent to the statement “for every closed subset A of P we have $A = \sup(\{D(x) : D(x) \subseteq A\}) = \inf(\{U(a) : U(a) \subseteq A\})$ ”.

$$\begin{aligned} & \sup(\{D(x) : D(x) \subseteq A\}) \\ &= \sup(\{D(x) : x \leq a \text{ for every } a \in A\}) \\ &= \sup(\{D(a) : a \in A\}) \\ &= (A^+)^- \text{ (from the second part of Lemma 2.4.5)} \\ &= A \text{ (as } A \in \bar{P}\text{)}. \end{aligned}$$

Also,

$$\begin{aligned}
& \inf(\{D(x) : D(x) \subseteq A\}) \\
&= \inf(\{D(x) : a \in D(x) \text{ for every } a \in A\}) \\
&= \inf(\{D(x) : x \leq a \text{ for every } a \in A\}) \\
&= \inf(\{D(x) : x \in A^+\}) \\
&= (A^+)^- \text{ (from the first part of Lemma 2.4.5)} \\
&= A \text{ (as } A \in \overline{P}\text{)}. \quad \blacksquare
\end{aligned}$$

Theorem 2.4.2 (MacNeille [35, 13]) *Let P an ordered set and \overline{P} its completion. If L is a lattice containing P , then $|\overline{P}| \leq |L|$.*

Proof. Define the function $f : \overline{P} \rightarrow L$ by $f(x) = \sup_L(\{D(x) \cap P\})$, where $D(x)$ is the down set of x taken in \overline{P} and \sup_L is the supremum in L . The map f is well defined because the supremum of any set exists and is unique in L .

If $x \leq y$ in \overline{P} , then $D(x) \subseteq D(y)$. Thus, $f(x) = \sup_L(D(x)) \leq \sup_L(D(y)) = f(y)$, which means that f is an order-preserving map.

It is enough to prove that f is injective. If f is not injective, then there exists $x \neq y$ in \overline{P} such that $f(x) = f(y)$ in L . Let $A = D(x) \cap P$, $A' = U(x) \cap P$, $B = D(y) \cap P$ and $B' = U(y) \cap P$. According to Lemma 2.4.6, $x = \sup(A) = \inf(A')$ and $y = \sup(B) = \inf(B')$ in \overline{P} . Thus, $A \neq B$ and $A' \neq B'$. We have two cases to consider.

Case 1 $A \subset B$ or $B \subset A$. We may assume that $A \subset B$. Thus, $x = \sup(A) \leq$

$\sup(B) = y$ in \overline{P} . Thus, $B' \subset A'$. If there is $a' \in A' - B'$ and $b \in B - A$ such that $a \parallel b$ in P , then as f is order-preserving map we have in L ,

$$b = f(b) \leq f(y) = f(x) \leq f(a') = a'.$$

Thus, $b \leq a'$ in L which implies $b \leq a'$ in P which contradicts our assumption that $b \parallel a'$ in P .

Thus, we have $b \leq a'$ for every $b \in B$ and $a' \in A'$. Hence, every element $a' \in A'$ is an upper bound of B . Thus $a' \geq y$ for every $a' \in A'$. Therefore, $A' - B' = \emptyset$, which contradicts that $B' \subset A'$.

Case 2 $A - B \neq \emptyset$ and $B - A \neq \emptyset$. In this case we claim that $A' - B' \neq \emptyset$ and $B' - A' \neq \emptyset$. Indeed, if $A' \subset B'$, then $x < y$ which implies $A \subset B$ which contradicts our assumption that $B - A \neq \emptyset$.

Let $z \in A - B$. If $z \leq b'$ for every $b' \in B'$, then z will be a lower cover of B' in \overline{P} and thus $z \leq y$ as $y = \inf(B')$. That contradicts that $z \in A - B$. Thus, there is $w \in B' - A'$ such that $z \parallel w$ in \overline{P} . As f is order-preserving map we have in L

$$z = f(z) \leq f(x) = f(y) \leq f(w) = w$$

Therefore, $z \leq w$ in L which implies that $z \leq w$ in P . That contradicts our assumption that $z \parallel w$ in P .

Therefore, f is injective. ■

Lemma 2.4.7 ([13]) *If \overline{P} is the completion of the ordered set P , then the join or meet irreducible elements of \overline{P} are in $P \cup \{\top, \perp\}$.*

Proof. Let $x \in (\overline{P} - (P \cup \{\top, \perp\}))$. Suppose x is join irreducible in \overline{P} , that is x has a unique lower cover y in \overline{P} . Thus $D(x) = \{x\} \cup D(y)$. From Lemma 2.4.6, $x = \sup(D(x) \cap P)$ and $y = \sup(D(y) \cap P)$.

But

$$D(x) \cap P = (\{x\} \cup D(y)) \cap P = D(y) \cap P \text{ as } x \notin P.$$

Thus,

$x = \sup(D(x) \cap P) = \sup(D(y) \cap P) = y$ which contradicts the fact that y is a lower cover of x . ■

Chapter 3

Pagenumber Problem

3.1 Introduction

In this chapter we define the pagenumber problem for graphs and ordered sets and give a historical review. Also, we introduce the best known upper and lower bounds for the pagenumber of an ordered set. A *book embedding* of a graph G consists of an embedding of its nodes along the spine of a book (i.e., a linear ordering of the nodes), and embeddings of its edges on pages so that edges embedded on the same page do not intersect. In a *book embedding* for an ordered set P the vertices of P on the spine form a linear extension. The *pagenumber*¹ ($page(G)$, respectively $page(P)$) in both cases is the minimum number of pages needed (taken over all linear layouts for graphs and all linear extensions for an ordered set). For instance, $page(P) = 2$ for the ordered set illustrated in Figure 3.1, while $page(cov(P)) = 1$. In general, it is clear that $page(cov(P)) \leq page(P)$ for any ordered set P . Notice that, if $page(P) \leq 2$, then P is planar but the converse is not always true; for instance the planar lattice in Figure 3.2 requires three pages. (This example is due to **J. Czyzowicz** [37]).

The pagenumber was first defined for graphs by **Bernhart** and **Kainen** [3], they showed that the one-page graphs are exactly the outerplanar graphs. (A graph is

¹Sometimes it called the *stack* number or the *book thickness*.

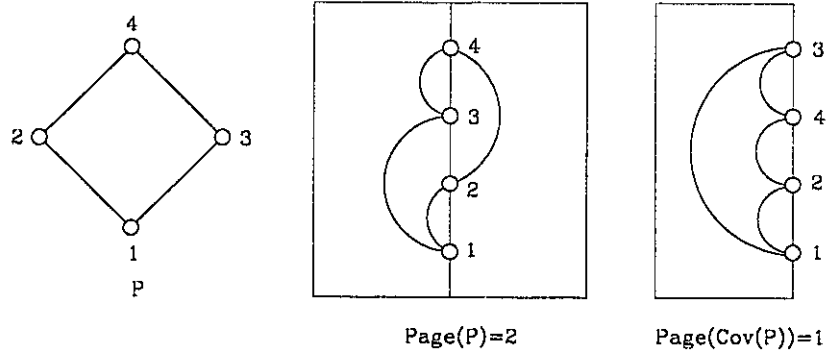


Figure 3.1:

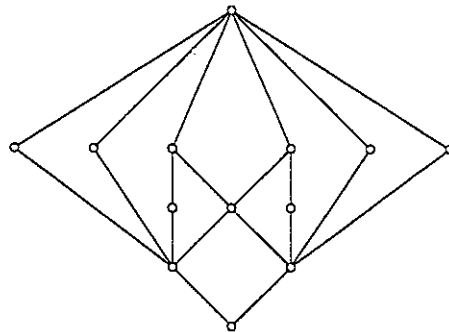


Figure 3.2: Three-page planar lattice

*outerplanar*² if its vertices can be placed on a circle in such a way that its edges are noncrossing chords of the circle.) Also, they conjectured that planar graphs may require an arbitrarily large number of pages. In a series of attempts, it was finally established by **Yannakakis** [53], that $page(G) \leq 4$ for every planar graph G , and this upper bound is achieved. The pagenumber of any planar graph with quadrilateral faces is at most two [19]. For series-parallel graphs the pagenumber is two [8].

The pagenumber for ordered sets has been introduced by **Nowakowski** and **Parker** [37], who show that $page(P) = 1$ if and only if $cov(P)$ is a forest (An ordered set P is a *forest* if $cov(P)$ is a forest (as a graph)). Also, they derive a general lower bound on the page number of ordered sets and upper bounds for special classes of ordered sets. **Hung** [30] shows that there exists a 48-element planar ordered set which requires four pages (see Figure 3.3)³. Moreover, no planar ordered set with page number five is known. **Sysło** [46] provides a lower bound on the page number in terms of its *bump number*. He also shows that $page(P) \leq 2$ if the *jump number* of P is one. Ordered sets with jump number two can have an arbitrarily large page number. For each positive integer n , **Heath** and **Pemmaraju** [38] gave a $6n$ -vertex ordered set P with a planar covering graph such that $page(P) \geq n$ (see Figure 3.4).

Computationally, the book embedding problem for graphs is hard; it is NP-complete to determine whether arbitrary planar graph can be embedded in two pages [8]. Recently, **Heath** and **Pemmaraju** [28] showed that the problem of recognizing whether a directed acyclic graphs (dag) is 6-page is, itself NP-complete⁴. For ordered sets, we show, in Section 3.3, that for a fixed linear extension on the spine, finding the minimum number of pages is NP-complete.

According to **R. Nowakowski** and **A. Parker** ([37]):

“Book embeddings for graphs occur in many contexts (see [7, 8]). In some

²Determining whether a graph is outerplanar can be done in linear time [47].

³This is the smallest known four-page planar ordered set.

⁴In their paper, their representation of a dag need not be an upward drawing.

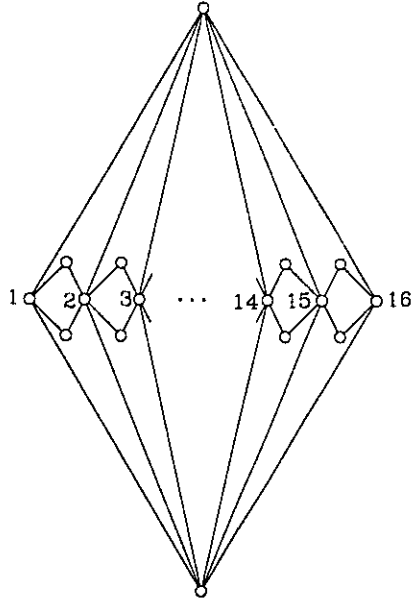


Figure 3.3: Four-page planar ordered set.

instances, the underlying structure may be an ordered set. For example, the vertices of P may represent steps in a calculation and the edges directed into a vertex represent necessary inputs for that calculation to be executed. If the calculation is done on a machine with single processor, then each page represents a stack, the edges indicate the order in which the partial results are entered and removed from the stacks and the pagenumber is the smallest number of stacks required to store data in order for the calculation to be carried out."

The completion of an ordered set is the bridge to proving our main result. There are well-known order-theoretic combinatorial parameters which are invariant with respect to the completion (for instance, the order dimension). The pagenumber of the completion need not equal the pagenumber of the original order. In fact, we show that the pagenumber of an ordered set may be larger than the pagenumber of its completion.

There are quite simple ordered sets with large pagenumber. We introduce three

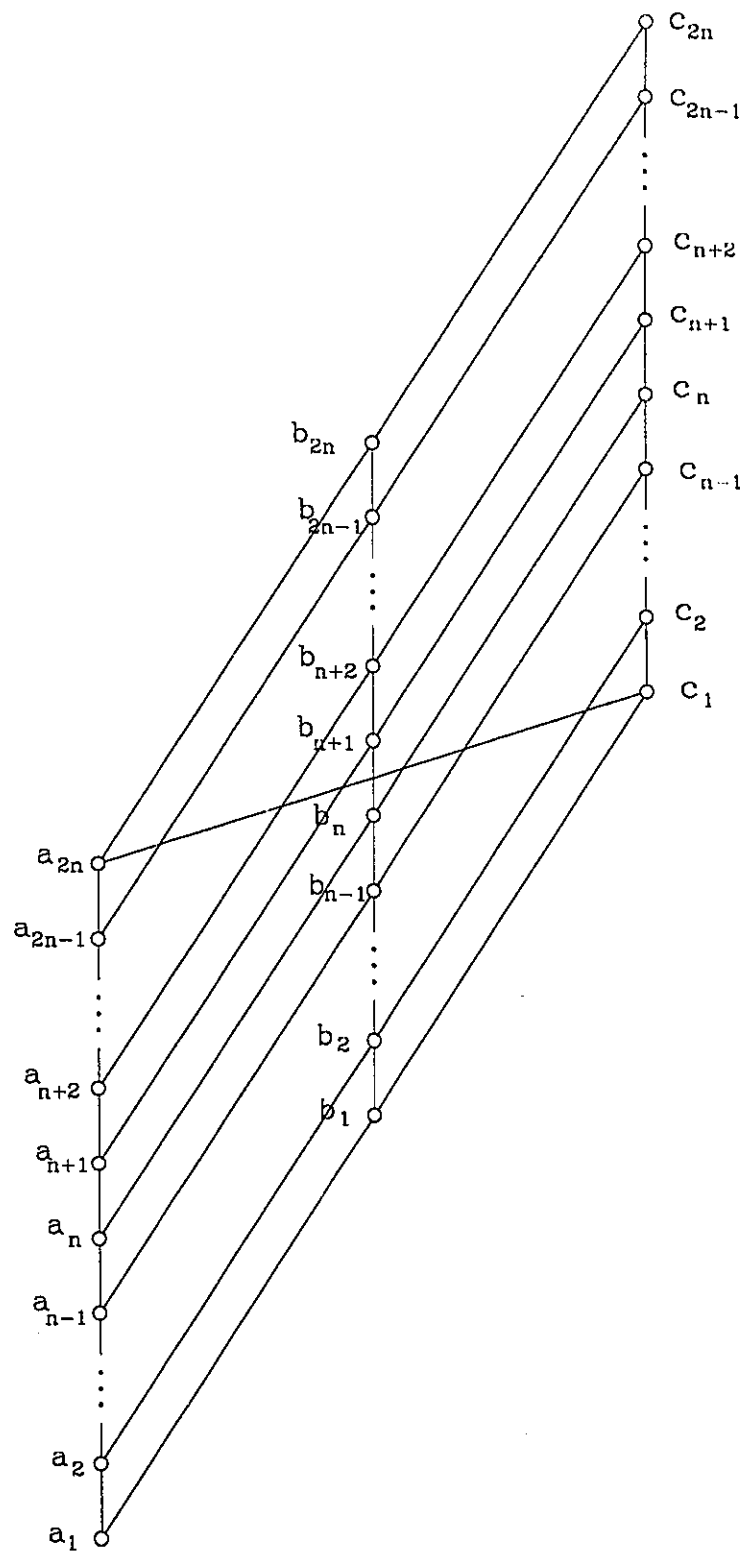


Figure 3.4:

such sequences of ordered sets each with unbounded pagenumber. The first one is a sequence of ordered sets of dimension two and the others are sequences of spherical ordered sets and spherical lattices.

3.2 Lower and Upper Bounds

In this section we give lower and upper bounds for the pagenumber of an ordered set P in terms of combinatorial parameters of P .

For a linear extension L of an ordered set P , let $page(P, L)$ stand for the minimum number of pages required for L .

Lemma 3.2.1 *Let C_1 and C_2 be two nonsingleton disjoint chains in an ordered set P . If $inf(C_1) < inf(C_2) < sup(C_1) < sup(C_2)$ in a linear extension L , then $page(P, L) \geq 2$.*

Proof. Let $C_1 = \{x_1 \prec x_2 \prec \dots \prec x_n\}$ and $C_2 = \{y_1 \prec y_2 \prec \dots \prec y_m\}$, $m, n \geq 2$. Suppose that $x_1 < y_1 < x_n < y_m$ in a linear extension L of P . Let

$$i = \min\{1 \leq k \leq m : x_n < y_k \text{ in } L\}$$

Of course, $i > 1$ because $y_1 < x_n$ in L . Thus $y_1 \leq y_{i-1} < x_n < y_i$ in L . Also, let

$$j = \min\{1 \leq k \leq n : y_{i-1} < x_k \text{ in } L\}$$

Of course, $j > 1$ because $x_1 < y_{i-1}$ in L . Thus $x_{j-1} < y_{i-1} < x_j < y_i$ in L . Hence the two edges (x_{j-1}, x_j) and (y_{i-1}, y_i) can not be drawn in the same page. Therefore, $page(P, L) \geq 2$. ▀

Theorem 3.2.1 *Let x belong to an ordered set P such that x is adjacent to a unique vertex in $cov(P)$ and $P - \{x\}$ is not an antichain. Then $page(P) = page(P - \{x\})$.*

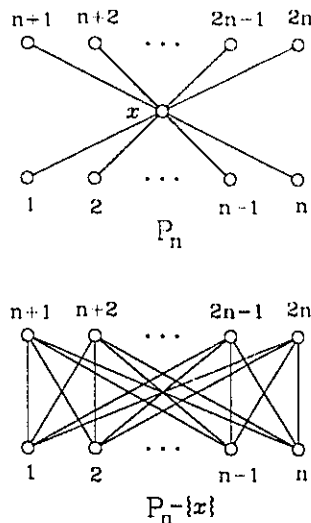


Figure 3.5:

Proof. Since x is adjacent to a unique element in $\text{cov}(P)$, either x is maximal or minimal. Let L be a linear extension of P such that $\text{page}(P, L) = \text{page}(P)$. We may assume that x is maximal and y is the unique lower cover of x in P . We obtain the linear extension L' of $P - \{x\}$ from L by removing x and deleting the edge (y, x) . Thus, $\text{page}(P) \geq \text{page}(P - \{x\})$.

Suppose L' is a linear extension of $P - \{x\}$ such that $\text{page}(P - \{x\}, L') = \text{page}(P - \{x\})$. We obtain the linear extension L of P from L' by adding x right after y (i.e. $x \succ y$ in L). Draw the edge (x, y) in the first page (the first page exists because $P - \{x\}$ is not an antichain). Thus, $\text{page}(P - \{x\}) \geq \text{page}(P)$. \blacksquare

Notice that, in Theorem 3.2.1, the condition that x is adjacent to a unique element in P is necessary. For instance, for each positive integer n , there is an ordered set P_n on n elements such that $\text{page}(P_n - \{x\}) = n \geq 1 = \text{page}(P_n)$ (see Figure 3.5). In contrast, we will see in Section 3.5 an example of an ordered set P such that $\text{page}(P) \geq 3 \geq \text{page}(P - \{x\}) = 2$.

We call the subset C of an ordered set P a *cycle* if C is a cycle in the graph $\text{cov}(P)$.

The next Theorem is due to Nowakowski and Parker [37].

Theorem 3.2.2 *Let P be an ordered set. Then $\text{page}(P) = 1$ if and only if $\text{cov}(P)$ is a forest.*

Proof. Assume first that $\text{cov}(P)$ is a tree. We show, by induction on $|P|$, that $\text{page}(P) = 1$. As $\text{cov}(P)$ is a tree, there is an element x adjacent⁵ to a unique element. Thus, either x is a maximal or a minimal in P ; we may assume it is maximal.

Since $\text{cov}(P - \{x\})$ is a tree, by the induction hypothesis, there is a one-page linear extension L' of $P - \{x\}$. Let y be the unique element adjacent to x in $\text{cov}(P)$.

Obtain L by adding x right above y in L' (i.e. $y \prec x$ in L). It is clear that L is a linear extension of P . Draw the edge (y, x) in the first page, there will be no edge crossing because $x \succ y$ in L .

For the converse, let $\text{cov}(P)$ contain a cycle C . We may assume that C is of a minimum length which implies that C has no chords. Note that the number of maximals equals the number of minimals of C (because by minimality of C , each element of C is adjacent in $\text{cov}(P)$ to exactly two elements of C).

Case 1 C contains a unique maximal and a unique minimal.

Let \perp and \top be, respectively, the minimum and the maximum of C . Let x, y be the only two lower covers of \top in C . Let L be a linear extension of P . We may assume that $x < y$ in L . Thus, the chain C from \perp to y and the edge (x, \top) are two disjoint chains such that $\perp < x < y < \top$ in L . By Lemma 3.2.1, $\text{page}(P, L) \geq 2$.

Case 2 C contains exactly k maximals and exactly k minimals, $k \geq 2$.

Let C contain the minimals $\{a_1, a_2, \dots, a_k\}$ and the maximals $\{b_1, b_2, \dots, b_k\}$, $k \geq 2$ such that

⁵Two vertices a and b are *adjacent* in a graph G if $\{a, b\}$ is an edge in G .

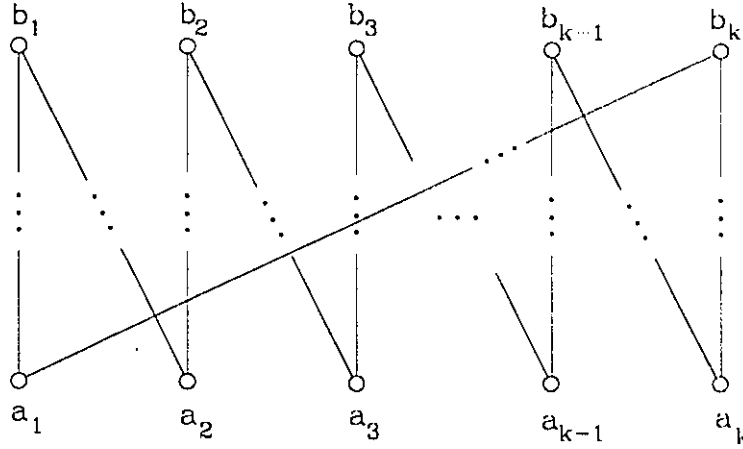


Figure 3.6:

- $a_1 < b_1, b_k$,
- $a_i < b_{i-1}, b_i$ for $2 \leq i \leq k$ (see Figure 3.6).

Let $C_{i,j}$ be the chain from a_i to b_j in C .

Suppose there is a linear extension L of P such that $\text{page}(P, L) = 1$. We may assume that $b_{k-1} < b_k$ in L . As $a_k < b_{k-1}$ in P , then $a_{k-1} < b_{k-1} < b_k$ in L . If $a_{k-1} < a_k$ in L , then $a_{k-1} = \inf(C_{k-1,k-1}) < a_k = \inf(C_k) < b_{k-1} = \sup(C_{k-1,k-1}) < b_k = \sup(C_{k,k})$ in L which implies, according to Lemma 3.2.1, that $\text{page}(P, L) \geq 2$. Thus $a_k < a_{k-1} < b_{k-1} < b_k$ in L .

Also, if $b_{k-2} > b_{k-1}$ in L , then by Lemma 3.2.1, $\text{page}(P, L) \geq 2$.

By a similar argument we finally have

$a_{k-1} < a_{k-2} < \dots < a_1 < b_1 < b_2 < \dots < b_k$ in L . Therefore, $\inf(C_{2,2}) = a_2 < \inf(C_{1,k}) = a_1 < \sup(C_{2,2}) = b_2 < \sup(C_{1,k}) = b_k$ in L which contradicts Lemma 3.2.1.

Therefore, $\text{page}(P, L) \geq 2$.



Let x be an element in an ordered set P . We define the *cone* of x , $\text{cone}(x) = D(x) \cup U(x)$.

Lemma 3.2.2 *Let $a \prec b$ in the ordered set P . Then $\text{cone}(a) = \text{cone}(b)$, if and only if $U(a) = \{a\} \cup U(b)$ and $D(b) = \{b\} \cup D(a)$.*

Proof. We suppose $\text{cone}(a) = \text{cone}(b)$ and show first that $U(a) = \{a\} \cup U(b)$. Since $a < b$, $U(b) \subset U(a)$. Suppose $a < x$ in P . Since $\text{cone}(a) = \text{cone}(b)$, either $x \leq b$ or $b \leq x$. As (a, b) is an edge in P , $x \not\leq b$. Thus, $x \in U(b)$ and hence $U(a) - \{a\} = U(b)$. Therefore, $U(a) = \{a\} \cup U(b)$. Dually, $D(b) = \{b\} \cup D(a)$.

For sufficiency, if $\text{cone}(a) \neq \text{cone}(b)$, then either there exists $x \in P$ such that $x < b$ and $x \not\leq a$ or $x > a$ and $x \not\leq b$ which contradicts the fact that $U(a) = \{a\} \cup U(b)$ and $D(b) = \{b\} \cup D(a)$. ■

Lemma 3.2.3 *Let P be an ordered set such that $a \prec b$. If $\text{cone}(a) = \text{cone}(b)$, then*

$$\text{page}(P) \leq \text{page}(P - \{a\})$$

Proof. Let L be a linear extension of $P - \{a\}$ such that $\text{page}(P - \{a\}, L) = \text{page}(P - \{a\})$. Suppose $x \prec b \prec y$ in L .

Notice that we can get an upward drawing of $P - \{a\}$ from an upward drawing of P just by identifying a and b in P to b . Indeed, if not, then without loss of generality, there are c and d in P such that (c, a) is an edge and $b > d \succ c$. That implies $d \in (D(b) - D(a))$. Thus, $D(b) \neq \{b\} \cup D(a)$ which contradicts Lemma 3.2.2.

To transform L to a linear extension L' of P add a to L such that $x \prec a \prec b \prec y$. Draw the edge (a, b) in the first page. Draw the edge (z, a) for each lower cover z of a in the page where the edge (z, b) was drawn in L . Draw the other edges in the same page drawn for L .

First, the edge (a, b) will not intersect any edge in L' because $a \prec b$ in L' . If the edges (z, a) and (c, d) intersect in the same page, then the edges (z, b) and (c, d) intersect in L in the same page. ■

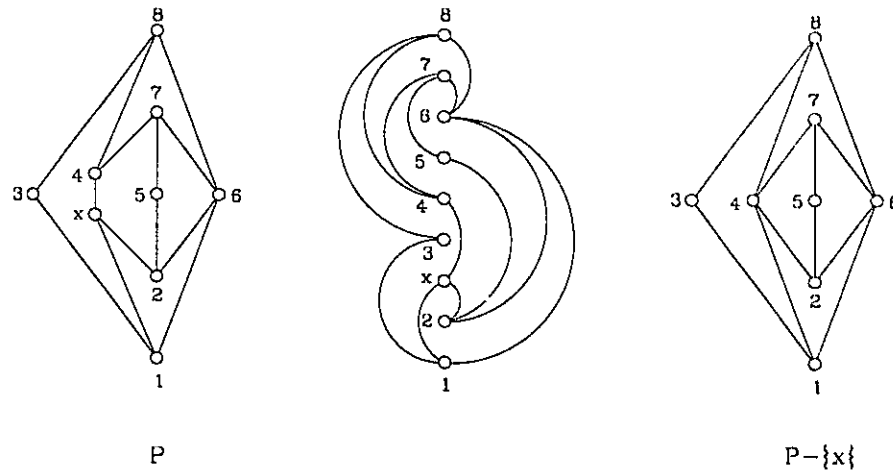


Figure 3.7:

In fact, the inequality in the above lemma may be strict, for instance for the ordered set illustrated in Figure 3.7 $page(P) = 2$ while $page(P - \{x\}) = 3$.

Lemma 3.2.4 *Let P be an ordered set such that $a \prec b \prec c$ is the unique chain from a to c in P . If a is the unique lower cover of b and c is the unique upper cover of b in P , then $page(P) \leq page(P - \{b\})$.*

Proof. As $a \prec b \prec c$ is the unique chain from a to c , so we can obtain an upward drawing of $P - \{b\}$ from an upward drawing of P by deleting b and the two edges $(a, b), (b, c)$ and adding the edge (a, c) .

Let L' be a linear extension of $P - \{b\}$ such that $page(P - \{b\}, L') = page(P - \{b\})$. We obtain L from L' by adding b right below c in L' (i.e $b \prec c$ in L). It is clear that L is a linear extension of P .

Draw the edge (b, c) in the first page and draw the edge (a, b) in the page where the edge (a, c) was drawn in L' . Draw the other edges as drawn in L' .

The edge (b, c) will not intersect any other edge in the first page of P because $b \prec c$ in L . Also, if the edge (a, b) intersects an edge (x, y) in the page where it is drawn, then (a, c) will intersect the edge (x, y) in the same page in L' .

Therefore, $page(P) \leq page(P - \{b\})$. ▀

A subset C of the set of vertices of a graph G is a *vertex cover* in G if for each edge $\{x, y\}$ in G either x or y is in C . The next result, which is due to Sysło [46], uses this vertex covering idea to give an upper bound for the pagenumber.

Lemma 3.2.5 *Let P be an ordered set. Then*

$$page(P) \leq \min\{|C| : C \text{ is a vertex cover in } cov(P)\}$$

Proof. Let L be a linear extension of P and let C be a minimal vertex cover in $cov(P)$. For each vertex x of C draw all edges incident to x in a separate page. Thus, no two edges intersect in the same page. Notice that, all the edges of P have been drawn because C is a vertex cover. ▀

A *k-twist* for L is a set of edges $\{(x_i, y_i) : 1 \leq i \leq k\}$ in P such that we have in L

$$x_1 < x_2 < \dots < x_k < y_1 < y_2 < \dots < y_k$$

A linear extension L of an ordered set P is *k-twist-free* if there is no *k-twist* in L .

In fact, $page(P, L) \geq k$, where k is the size of the maximum twist, because no two edges of the twist can be drawn in the same page. In contrast, the size of the maximum twist in the linear extension may be strictly less than the minimum number of pages required for this linear extension. For example, $page(P, L) = 3$ for the linear extension L of the ordered set P in Figure 3.8(b); in spite of that the maximum twist in L is two.

For the positive integers m, n , the *complete bipartite* ordered set $K_{m,n}$ is an ordered set with m minimals and n maximals such that each maximal covers each minimal. The next Corollary is due to Sysło [46].

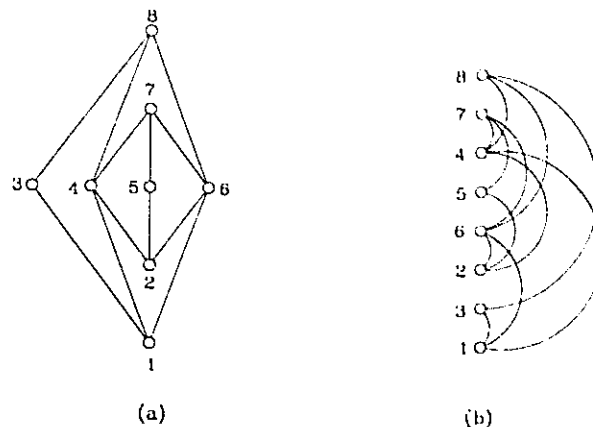


Figure 3.8: (a) The smallest size of a three-page planar ordered set. (b) The layout of the graph $\text{cov}(P)$ with respect to L .

Corollary 3.2.1 *The pagenumber for the complete bipartite⁶ ordered set $K_{m,n}$ is $\min\{m, n\}$.*

Proof. Assume that $\min\{m, n\} = m$. The inequality $\text{page}(K_{m,n}) \leq m$ follows from Lemma 3.2.5. On the other hand, one can see easily that every linear extension of $K_{m,n}$ contains an m -twist, hence $\text{page}(K_{m,n}) \geq m$. \square

Lemma 3.2.6 *Let P be an ordered set containing the set $\{x_i : 1 \leq i \leq n\}$. If the set of upper covers of each x_i equals $\{b\}$ and the set of lower covers equals $\{a\}$, then $\text{page}(P) \leq \text{page}(P - \{x_i : 1 \leq i \leq n\}) + 1$.*

Proof. Let L be an optimal linear extension for $P - \{x_i : 1 \leq i \leq n\}$ such that $y \prec b$ in L . Obtain L' from L by adding the elements $x_1 \prec x_2 \prec \dots \prec x_n$ right below b in L (i.e. in L' we have $y \prec x_1 \prec x_2 \prec \dots \prec x_n \prec b$).

It is clear that L' is a linear extension of P . In L' , for $1 \leq i \leq n$ draw the edge (x_i, b) in the first page and draw the edge (a, x_i) in a new additional page.

Since the vertices $\{x_i : 1 \leq i \leq n\}$ are between b and y in L' , no two edges intersect in the first page. Also, the edges in the new page have the same tail; thus there are no

⁶For the complete bipartite graph $K_{m,n}$, $\text{page}(K_{m,n}) \leq \lceil (m + 2n)/4 \rceil$ [36].

intersecting edges. Therefore, $page(P) \leq page(P - (\{x_i : 1 \leq i \leq n\})) + 1$. \square

Nowakowski and Parker [37] proved that $e(P) \leq 2|P| - 2 - height(P)$ for any planar ordered set P , where $e(P)$ is the number of edges of P . As a consequence of that they showed that

Corollary 3.2.2 *If $page(P) \geq 1$, then*

$$page(P) \geq \begin{cases} \frac{e(P)}{v(P)-1-height(P)/2} & \text{if } height(P) \text{ is even;} \\ \frac{e(P)-2v(P)+2+height(P)}{v(P)-1-(height(P)-1)/2} & \text{if } height(P) \text{ is odd.} \end{cases}$$

Let P be an ordered set on n elements and m edges. Each linear extension of P can be expressed as a linear sum of chains of P , i.e.

$$L = C_0 \oplus C_1 \oplus \cdots \oplus C_k$$

where $C_i (0 \leq i \leq k)$ is a chain in P . The number of breaks (i.e. noncomparabilities) between consecutive elements in L is denoted by $jump(P, L)$ and the *jump number* of P , $jump(P)$, is the minimum such number, that is

$$jump(P) = \min\{jump(P, L) : L \text{ is a linear extension of } P\}$$

The *bump number* of P , $bump(P)$, is

$$bump(P) = \max\{jump(P, L) : L \text{ is a linear extension of } P\}$$

The problem of finding $jump(P)$ is NP-complete for ordered sets[6], whereas the bump number can be calculated in polynomial time[24].

One of the classes of ordered sets for which the jump number is known is the class of series-parallel ordered sets (see [9]). Indeed, $jump(P) = jump(P, L)$ for any greedy linear extension L of a series-parallel ordered set P .

Let us consider a book embedding of P according to L . An edge (x, y) is a *spine edge* if $x \prec y$ in L i.e. $x, y \in C_i$ for $0 \leq i \leq k$. A spine edge can be drawn in any

page. It is clear that for every two chains C_i and C_j , $i < j$, no page contains more than one edge between elements of C_i and C_j . Otherwise, let (u, v) and (x, y) be two edges in P , where $u, x \in C_i$, $v, y \in C_j$ and $u < x$. If $v < y$, then (x, y) is a nonessential⁷ edge and if $y < v$, then the edges (x, y) and (u, v) cross. In general, we can not have a cycle on one page formed by some spine edges and nonspine edges placed on that page. Therefore, if L consists of $k + 1$ chains, then a page in any book embedding of P according to L , may contain at most k nonspine edges of P . The number of spine edges is equal to $n - 1 - k$, hence,

$$\lceil \frac{m - (n - 1 - k)}{k} \rceil \leq \text{page}(P, L)$$

Thus, we obtain the next lemma, which is due to Sysło [46].

Lemma 3.2.7 *Let P be an ordered set. Then*

$$\lceil \frac{m - n + 1}{\text{bump}(P)} \rceil + 1 \leq \text{page}(P)$$

A subset C of an ordered set P is *convex* if $b \in C$ whenever $a \leq b \leq c$ and $a, c \in C$; it is a *cutset* if every path in $\text{cov}(P)$ from any minimal vertex to any maximal vertex contains a vertex of C . A decomposition of

$$P = C_1 \cup C_2 \cup \dots \cup C_k$$

into convex cutsets is called a *layered decomposition* if every vertex of C_i covers only vertices of C_{i-1} and C_i is covered only by vertices of C_i and C_{i+1} . Every ordered set has such a decomposition, in particular one we call the *canonical decomposition*; take C_1 as the minimals of P ; successively take C_i as all the vertices less or equal to the upper covers of C_{i-1} in $P - \bigcup_{j < i} C_j$. The next Theorem and Corollary are due to Nowakowski and Parker [37].

⁷An edge (a, b) is *nonessential* or *not essential* in an ordered set if there is c such that $a < c < b$.

Theorem 3.2.3 *Let $P = C_1 \cup C_2 \cup \dots \cup C_k$ be a layered decomposition of an ordered set P . Then*

$$\text{page}(P) \leq \max\{\text{page}(C_i) : 1 \leq i \leq k\} + 2\max\{|C_j| : 1 \leq j \leq k\}$$

Proof. Take the following linear extension of P . For each i , take an optimal linear extension L_i of C_i . Let

$$L = L_1 \oplus L_2 \oplus \dots \oplus L_k$$

For each i , the n th vertex of C_i taken in this order of L has its upper edges embedded on page $2i - 1$ and its lower edges on page $2i$. Thus,

$$\text{page}(P) \leq \max\{\text{page}(C_i) : 1 \leq i \leq k\} + 2\max\{|C_{2j}| : 1 \leq j \leq k/2\}. \quad \blacksquare$$

In fact, we can modify the above theorem to

$$\text{page}(P) \leq \max\{\text{page}(C_i) : 1 \leq i \leq k\} + 2\min\{\max\{|C_{2j}|\}, \max\{|C_{2j-1}|\} : 1 \leq j \leq k\}$$

because the edges are only between the consecutive layers.

An ordered set P is *graded* if, for edge (x, y) in P such that x belongs to the level L_i , then y belongs to the level L_{i+1} . As a consequence of the previous theorem we have

Corollary 3.2.3 *If P is a graded ordered set and $P = C_1 \cup C_2 \cup \dots \cup C_k$ is a layered decomposition of P , then*

$$\text{page}(P) \leq 2\min\{\max\{|C_{2j}|\}, \max\{|C_{2j-1}|\} : 1 \leq j \leq k\}$$

For two ordered sets P and Q it is clear that $\text{page}(P+Q) = \max\{\text{page}(P), \text{page}(Q)\}$. The next lemma concerns $\text{page}(P \oplus Q)$.

Lemma 3.2.8 *For ordered sets P and Q*

$$\text{page}(P \oplus Q) \leq \max\{\text{page}(P), \text{page}(Q)\} + \min\{|\max P|, |\min Q|\}.$$

Proof. We can obtain an upward drawing of $P \oplus Q$ from an upward drawing of P and upward drawing of Q by drawing Q over P and drawing an edge from each maximal of P to each minimal of Q .

Let L_P and L_Q , respectively, be linear extensions of P and Q , respectively, such that $page(P, L_P) = page(P)$ and $page(Q, L_Q) = page(Q)$. It is clear that $L = L_P \oplus L_Q$ is a linear extension of $P \oplus Q$.

As the edges connecting $maxP$ to $minQ$ forms a complete bipartite ordered set requires, according to Corollary 3.2.1, at least $\min\{|maxP|, |minQ|\}$ pages, then

$$\begin{aligned} page(P \oplus Q) &\leq page(P \oplus Q, L) \\ &\leq \max\{page(P), page(Q)\} + \min\{|maxP|, |minQ|\}. \quad \blacksquare \end{aligned}$$

Nowakowski and Parker [37] showed⁸ that $\frac{n(k-1)}{k} \leq page(\mathbf{k}^n) \leq 2n - 2$ and they asked whether $page(2^n) = n$ and whether it is true that $page(2^n) < page(3^n)$? In general, it is not known whether there is a lower bound of the pagenumber of the ordered set $P \times Q$ greater than $\max\{page(P), page(Q)\}$.

3.3 The Complexity

In this section we show that finding the minimum number of pages required for a fixed linear extension of an ordered set is NP-complete even if it is a bipartite ordered set⁹.

Let P be an ordered set and L be a linear extension of P . A natural question is

What is the complexity of finding $page(P, L)$?

Given a graph $G = (V, E)$ and a permutation σ of the vertices of G , a *layout of G with respect to σ* , is a drawing of G on a page in which vertices are listed on a line

⁸ \mathbf{k}^n is the direct product of n chains each of height $k - 1$.

⁹An ordered set P is *bipartite* if $height(P) = 1$.

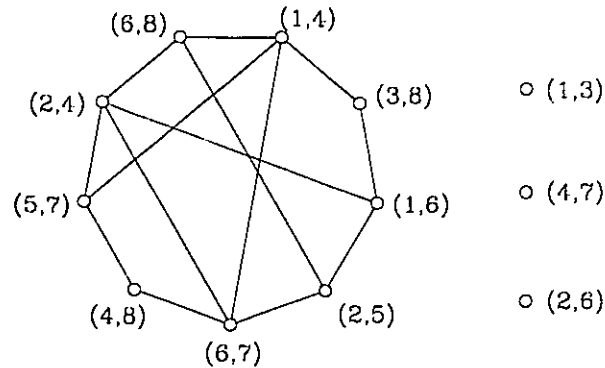


Figure 3.9:

segment at unit intervals according to σ and edges as concave arcs in such a way that two edges intersect only if necessary. Figure 3.8(b) illustrates a layout of the graph $\text{cov}(P)$ of the ordered set P .

The *intersection graph* of the layout graph of G with respect to σ is a new graph whose vertices are the edges of G , and two edges correspond to an edge in the intersection graph if they intersect in the layout graph of G with respect to σ . Figure 3.9 illustrates the intersection graph of the layout graph $\text{cov}(P)$ with respect to L (P and L are in Figure 3.8).

The *chromatic number* of a graph G is the minimum number of colours needed to colour the vertices of G such that no adjacent vertices receive the same colour.

The next theorem (which is due to **Even** and **Itai** [16]) connects the pagenumber problem to the chromatic number problem.

Theorem 3.3.1 *Let P be an ordered set and L a linear extension of P . Then $\text{page}(P, L)$ is equal to the chromatic number of the intersection graph of the layout graph of $\text{cov}(P)$ with respect to L .*

Proof. Suppose that the chromatic number of the intersection graph of the layout of $\text{cov}(P)$ with respect to L is equal to m .

- (i) $page(P, L) \geq m$. Suppose $page(P, L) = n$ and let E_i be the set of edges in the i th page, $1 \leq i \leq n$. Since there are no intersecting edges in any page, there are no two adjacent vertices of the intersection graph of the layout graph of $cov(P)$ with respect to L , in the set E_i . Hence we can color E_i by the color i , for each i . Thus, $m \leq n = page(P, L)$.
- (ii) $page(P, L) \leq m$. Let E_i be the set of edges coloured by the i th color, $1 \leq i \leq t$. We can draw the set of edges E_i in the i th page for each $i = 1, \dots, m$ because in the layout graph of $cov(P)$ with respect to L , no two edges of the set E_i intersect if they are drawn in the same page. Hence, $page(P, L) \leq m$. \square

Corollary 3.3.1 *Let $k \geq 3$ be an integer. The decision question whether, for a given ordered set P and an arbitrary linear extension L of P , $page(P, L) \leq k$, is NP-complete. In fact, this decision question is NP-complete even if P is bipartite.*

Proof. Let P be an ordered set and L be a linear extension of P . Let H be the intersection graph of the layout of $cov(P)$ with respect to L . In H consider each edge (x, y) as an interval in the real line. Thus H is an overlap graph, where a graph K is called an *overlap graph* if its vertices may put into one-to-one correspondence with a collection of intervals on a line such that two vertices are adjacent in K if and only if their corresponding intervals overlap (not just intersect).

It is known that an undirected graph is an overlap graph if and only if it is a circle graph¹⁰, where a *circle graph* is a graph whose vertices are the chords of a cycle in which two vertices form an edge if the two chords are intersect. An example of a circle graph and its representation by a set of chords is given in Figure 3.10.

According to Theorem 3.3.1, the problem of minimizing number of pages that L required is equivalent to colouring circle graphs, hence is NP-complete [21].

¹⁰see [23] page 242.

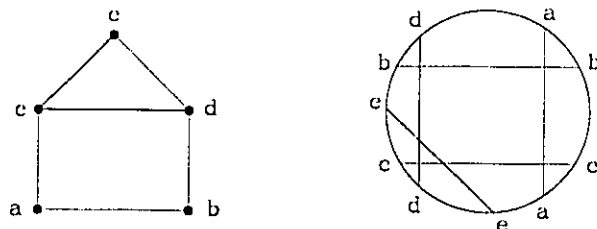


Figure 3.10: A circle graph.

Suppose H is a circle graph. Then H is an overlap graph. Without loss of generality we may assume that the intervals are either open or closed and that no two intervals have a common end point. All of these constructions can be done in polynomial time. Now we want to construct a bipartite ordered set P and linear extension L of P , such that the intersection graph of the layout of $cov(P)$ with respect to L is H . Let the elements of P be endpoints of the intervals, ordered, for each interval I , by $sup(I) \succ inf(I)$. Then order the elements in L as they are ordered in the real line.

The proof is still valid if we restrict it to the class of bipartite ordered sets. ■

3.4 Ordered Sets with Pagenumber Two

In this section we give classes of ordered sets each member of which has pagenumber two. Next, we use the characterization of the graphs of pagenumber two to give a characterization of ordered sets of pagenumber two.

Kelly in [33] considered upward drawing using monotonic edges. A *monotonic* edge for a covering pair $a \prec b$ is any arc from a to b monotonic with respect to y -axis.

Kelly in [33] showed that if an ordered set P has a planar upward drawing in which each edge is a monotonic arc, then there is a planar upward drawing in which each

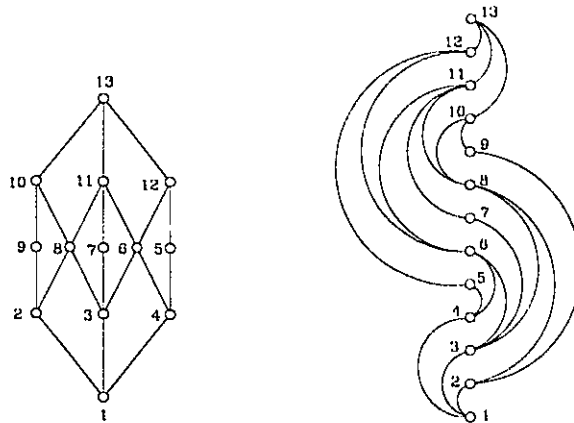


Figure 3.11: A snake linear extension

edge is straight.¹¹

If $\text{page}(P) = 2$ for an ordered set P , then using the Kelly's result, P is planar. In contrast, there are planar ordered sets which require more than two pages. For instance, $\text{page}(P) = 4$ for the planar ordered set illustrated in Figure 3.3.

Figure 3.8(a) illustrates a three page planar ordered set of smallest size (i.e., every planar ordered on seven elements or less has pagenumber two)¹².

The next theorem is due to Nowakowski and Parker [37].

Theorem 3.4.1 *The pagenumber of a planar graded lattice is at most two.*

Proof. Let P be a planar graded lattice. Fix a planar embedding of P . Let $L = L_0 \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_{\text{height}(P)}$ be such that the elements of L_i are ordered according to λ if i is odd and the opposite of the order of λ if i is even. It is clear that L is a linear extension of P . Draw the edges from level L_i to level L_{i+1} in the left page if i is odd and in the right page if i is even (See Figure 3.11). Call the linear extension obtained a *snake* linear extension.

Suppose the two edges (a, b) and (c, d) intersect in the same page. Thus we have

¹¹This result is analogous to the result for planar graphs [18].

¹²The author has enumerated all 7-elements planar ordered sets. Each has pagenumber at most two.

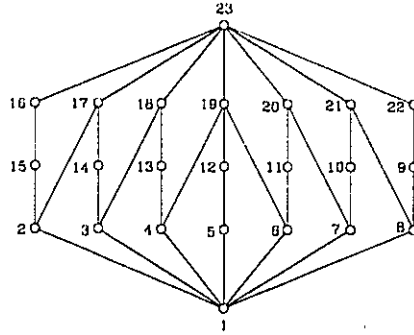


Figure 3.12:

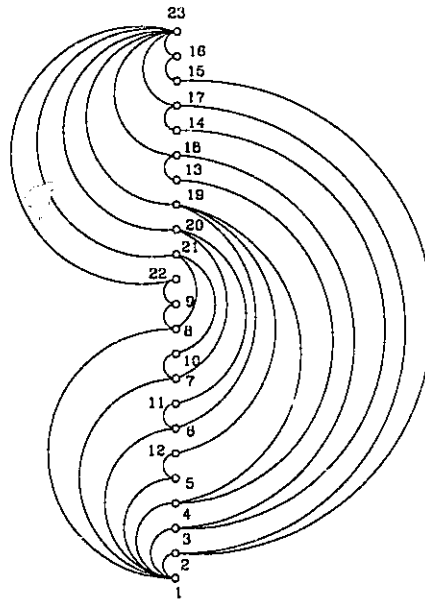


Figure 3.13:

$a < c < b < d$ in L and there is an i such $\{a, c\} \subseteq L_i$ and $\{b, d\} \subseteq L_{i+1}$ which contradicts the fact that we started with a planar embedding of P . ■

Notice that, for non-graded planar lattices, a snake linear extension may be far from optimal. For instance any snake linear extension of the planar lattice P illustrated in Figure 3.12 has a 3-twist, while $page(P) = 2$ (see Figure 3.13). In fact, for each positive integer n we can generalize this example to a planar lattice P_n such that any snake linear extensions has an n -twist and $page(P_n) = 2$.

The next Corollary is due to Hung [46].

Corollary 3.4.1 *Let P be an ordered set. If $\text{jump}(P) = 1$, then $\text{page}(P) \leq 2$.*

Proof. According to Theorem 3.2.1 we may assume that every vertex in P is adjacent to at least to two vertices in $\text{cov}(P)$.

Let L be a linear extension of P with a single jump. Thus, $L = C_1 \oplus C_2$, which means that for any edge (x, y) either x and y are both in C_1 or both in C_2 or $x \in C_1$ and $y \in C_2$.

Let $x \in C_1$ be such that x has no upper covers in C_2 and $y \succ x$ in C_1 . Thus, x has a unique upper cover y in P . Hence $U(x) = \{x\} \cup U(y)$. If $z \leq x$ in P , then, $z \in C_1$ and hence $z \leq y$. Thus, $D(y) = \{y\} \cup D(x)$. According to Lemma 3.2.2, $\text{cone}(x) = \text{cone}(y)$. By Lemma 3.2.3, $\text{page}(P) \leq \text{page}(P - \{x\})$.

Dually, we can show that $\text{page}(P) \leq \text{page}(P - \{x\})$ for $x \in C_2$ such that x has no lower covers in C_1 . By deleting such elements we will end up with a direct product Q of two chains such that $\text{page}(P) \leq \text{page}(Q)$. But Q is a graded lattice; thus by Theorem 3.4.1, $\text{page}(Q) \leq 2$. Thus, $\text{page}(P) \leq 2$. ▀

Notice that, for each positive integer n , there is an ordered set P_n such that $\text{jump}(P_n) = 2$ while $\text{page}(P_n) = n$ (see Figure 3.20)¹³.

Notice that every lattice of width two is planar.

Corollary 3.4.2 *The pagenumber of a lattice of width two is at most two.*

Proof. Let P be a planar lattice of width two. Fix a planar upward drawing of P .

As P_λ is the complement of P , $\text{height}(P_\lambda) = \text{width}(P) - 1 = 1$. Thus, P_λ has only two levels which means that if C_1 is the left boundary chain of P , then $C_2 = P - \{C_1\}$ is a chain too. Figure 3.14(a) illustrates the general structure of P .

¹³We will talk about this sequence in detail in Section 3.7.

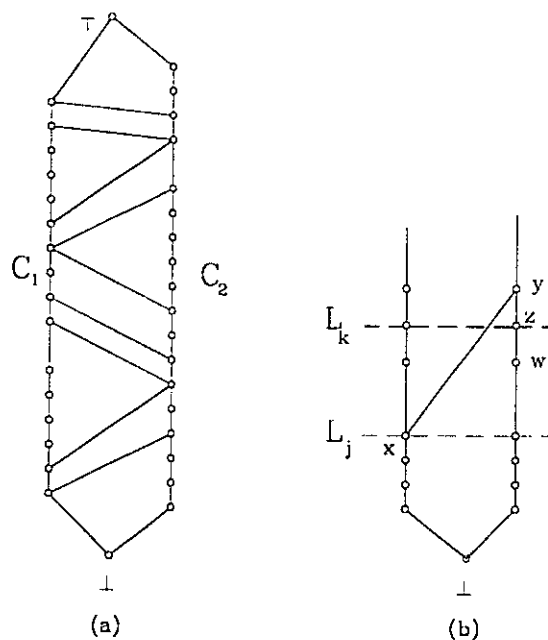


Figure 3.14:

According to Lemma 3.2.3 we may assume that P contains no two elements $a \prec b$ such that $\text{cone}(a) = \text{cone}(b)$. We claim that P is a graded planar lattice (Figure 3.15 shows a two-page embedding of a non-graded planar lattice of width two using Theorem 3.4.1 and Lemma 3.2.3).

If not, suppose $k \geq 2$ is the maximal integer such that P is graded up to the level L_k (i.e., if (x, y) is an edge in P and $y \in L_i, i \leq k$, then $x \in L_{i-1}$).

By maximality of k , there exists an edge (x, y) , $y \in L_{k+1}$ and $x \in L_j, j < k$. Thus, y has two lower covers x and z and hence $x \parallel z$. We may assume that $x \in C_1$ and $z \in C_2$ (see Figure 3.14(b)).

Let $z \succ w$ in C_2 . As $k \geq 2$, $w \neq \perp$. By planarity of P , z does not cover any element of $L_{k-1} \cap C_1$. On the other hand, z does not cover any element of $L_i \cap C_1, i < j$ because P is graded up to the level k . Also, (x, z) is not an edge because $x \parallel z$. Therefore, z

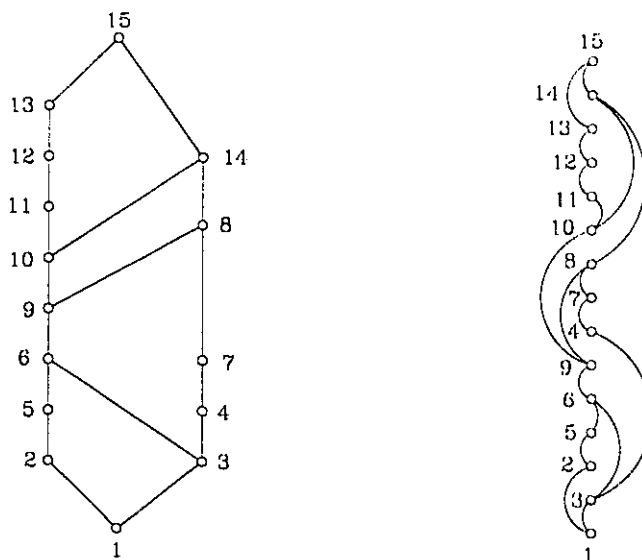


Figure 3.15: Two-page embedding for non-graded planar lattice of width two

has a unique lower cover w .

By planarity of P and since $w \neq \perp$, w has no upper cover in C_1 . Thus, w has a unique upper cover in P .

Therefore, $z \succ w$ in P such that $\text{cone}(z) = \text{cone}(w)$, which contradicts our assumption. □

Nowakowski and Parker in [37] asked for a characterization of the ordered sets with pagenumber two. We will use the next result to describe the ordered sets of pagenumber two. The next theorem, which is due to Bernhart and Kainen in [3], characterizes the two-page graphs. This proof is due to Chung, Leighton and Rosenberg [8].

Theorem 3.4.2 *For a graph G , $\text{page}(G) \leq 2$ if and only if G is a subgraph of a planar Hamiltonian graph.*

Proof. A graph is a subgraph of a planar Hamiltonian graph just if its embeddable in the plane so that

- (i) its vertices lie on a circle,
- (ii) each of its edges lies either totally within the circle or totally outside it,
- (iii) no edges cross in the layout.

Given such “circular” embedding of a subgraph of a planar Hamiltonian graph G , cutting the circle between any two of G ’s vertices yields a planar embedding of G in a line, with each edge lying either totally above the line (i.e., in page 1) or totally below it (i.e., in page 2).

Conversely, given a two-page embedding of the graph G , we view this embedding as placing G in a line with each edge laying totally above the line (page 1) or totally below it (page 2), and with no edges crossing. Pasting together the ends of the line containing G ’s vertices yields a “circular” embedding of G that witnesses that G is subgraph of a planar Hamiltonian graph. ■

Corollary 3.4.3 *The pagenumber of a planar ordered set P is two if and only if $cov(P)$ is a subgraph of a planar Hamiltonian graph and $cov(P)$ has a “circular” embedding and a minimal element x of P such that the permutation starting from x using the order of the circle (in either direction) is a linear extension of P .*

The following corollary, which is due to Chung, Leighton, and Rosenberg [8], is a direct consequence of Wigderson’s [52] result that the problem of deciding whether or not a maximal planar graph is Hamiltonian is NP-complete.

Corollary 3.4.4 *The problem of deciding whether a planar graph can be embedded in two pages is NP-complete.*

3.5 Is $page(\overline{P}) \leq page(P)$?

In many examples we found that the pagenumber of the completion is less than or equal the pagenumber of the original ordered set. For instance in Figure 3.16, $page(P_1) =$

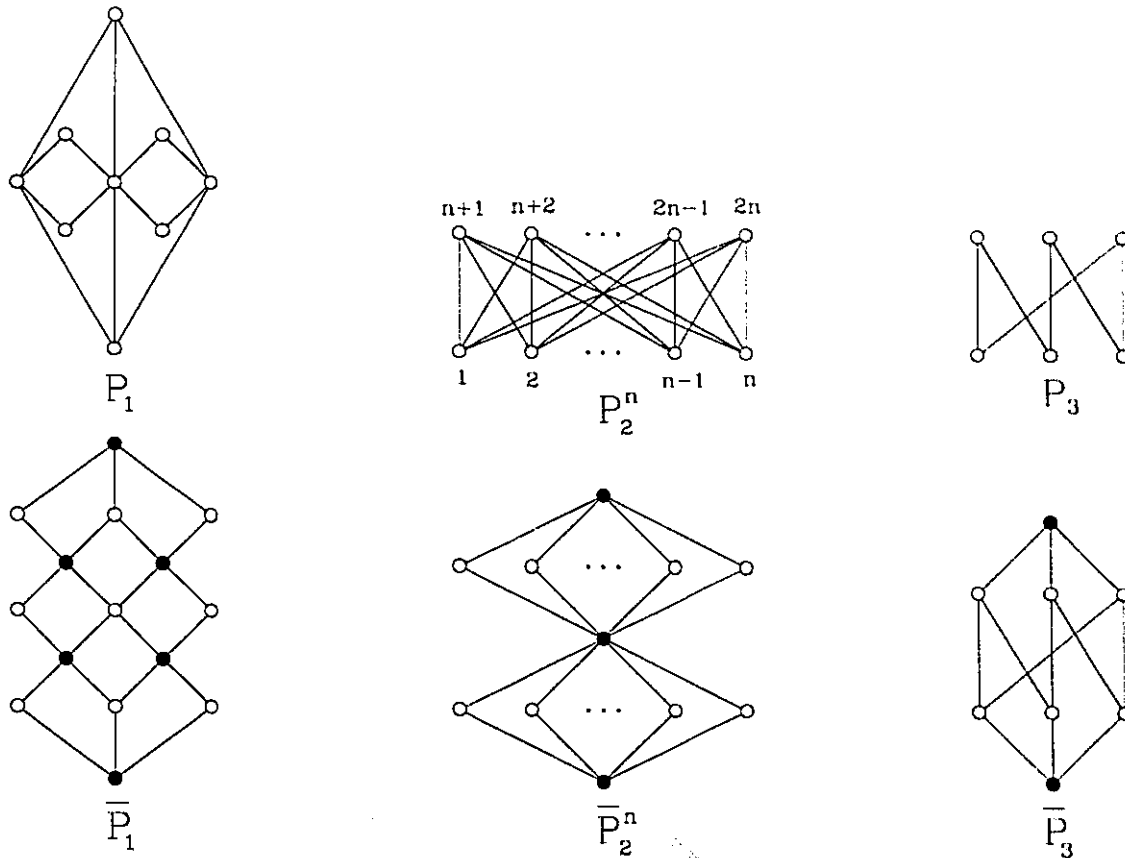


Figure 3.16:

$3 \geq page(\overline{P_1}) = 2$ (the ordered set P_1 is due to Nowakowski and Parker in [37]). In general, for each positive integer n , $page(P_2^n) = n \geq page(\overline{P_2^n}) = 2$.

One may ask

Is the pagenumber of the completion of an ordered set a lower bound of its pagenumber?

The answer is not always yes. For instance, in Figure 3.16, $page(P_3) = 2$ while $page(\overline{P_3}) = 3$. In this case the completion is not planar. It seems the problem arises from the addition of \top and \perp .

On the other hand, the ordered set P in Figure 3.17 needs only two pages, while its completion in Figure 3.18 (without adding top and bottom) is not planar, and

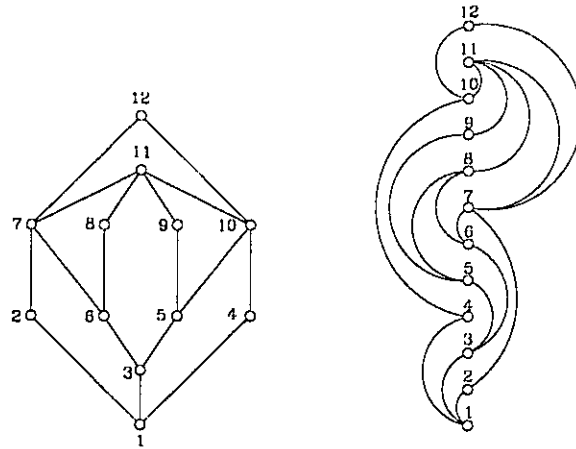


Figure 3.17:

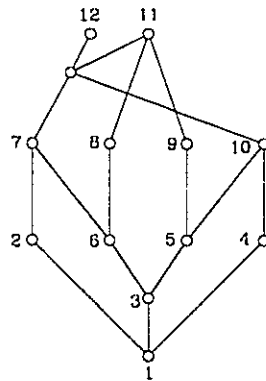


Figure 3.18:

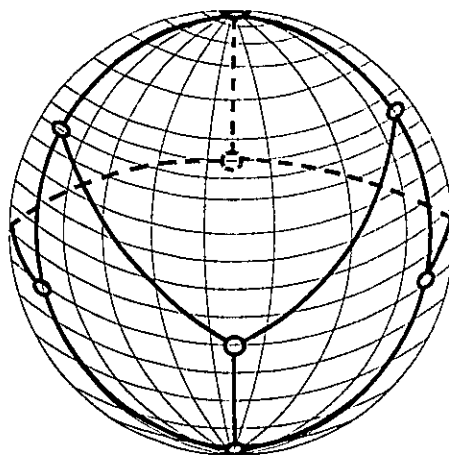


Figure 3.19: 2^3 is a spherical ordered set.

therefore, needs more.

3.6 The Pagenumber and Order Dimension

Is the pagenumber for an ordered set related to the its order dimension? Specifically, is the pagenumber of an ordered set of dimension two bounded? The answer is no.

In this section we construct a sequence of two dimensional ordered sets with unbounded pagenumber.

For each positive integer n , $page(P_2^n) = n$, by Corollary 3.2.1, for the ordered set P_2^n in Figure 3.16 while its order dimension is two. ($P_2^n = L_1 \cap L_2$ where $L_1 = \{1 < 2 < \dots < n - 1 < n < n + 1 < \dots < 2n - 1 < 2n\}$ and $L_2 = \{n < n - 1 < \dots < 2 < 1 < 2n < 2n - 1 < \dots < n - 1 < n\}$).

The question may have a positive answer if we restrict attention to planar ordered sets of dimension two. Planar lattices are such examples.

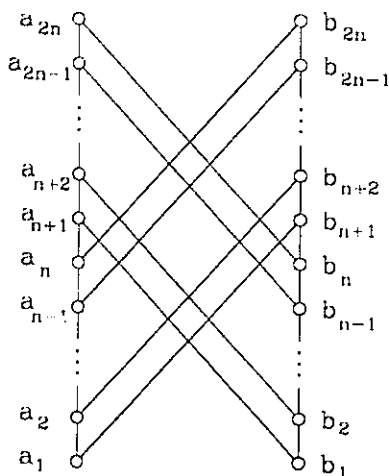


Figure 3.20:

3.7 Spherical Ordered Sets

In this section we give sequences of spherical ordered sets and spherical lattices each of which has unboundedpagenumber. The covering graph for each member of these sequences is planar. The importance of these two examples comes from the fact that a spherical ordered set is “almost” planar.

An ordered set is *spherical* if it has an upward drawing on the surface of the sphere such that all arcs are strictly increasing northward on the sphere, and no pair of arcs cross. (See [44].) Figure 3.19 illustrates a spherical ordered set.

Thepagenumber for general ordered sets is unbounded. Figure 3.20 illustrates a spherical ordered set P_n on $4n$ vertices. Figure 3.21 shows that $\text{cov}(P_n)$ is planar and $\text{page}(\text{cov}(P_n)) \leq 2$. The next theorem shows that $\text{page}(P_n) \geq n$. (The sequences $\{P_n\}$ and $\{L_n\}$ below are due to Kostochka [34]).

Theorem 3.7.1 *For each positive integer n , $\text{page}(P_n) \geq n$ where P_n is the ordered set illustrated in Figure 3.20.*

Proof. Let L be a linear extension of P . Since $a_n \parallel b_{n+1}$ in P_n , either $a_n < b_{n+1}$ or

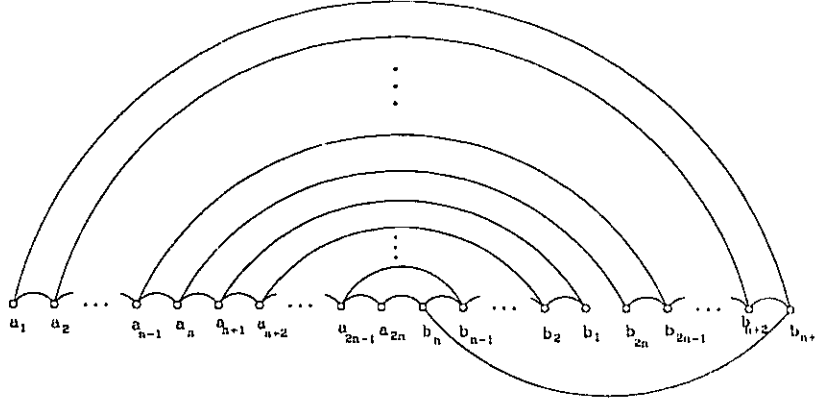


Figure 3.21:

$b_{n+1} < a_n$ in L .

Suppose $a_n < b_{n+1}$ in L . As $a_1 < a_2 < \dots < a_n$ and $b_{n+1} < b_{n+2} < \dots < b_{2n}$ in P , thus we have in L

$$a_1 < a_2 < \dots < a_n < b_{n+1} < b_{n+2} < \dots < b_{2n}$$

which means that L contains the n -twist $\{(a_i, b_{n+i}) : 1 \leq i \leq n\}$. Thus, $\text{page}(P_n, L) \geq n$.

Thus, we may assume $b_{n+1} < a_n$ in L . As $a_n < a_{n+1}$ and $b_n < b_{n+1}$ in P_n , $b_n < a_{n+1}$ in L . Therefore, L contains the n -twist $\{(b_i, a_{n+i}) : 1 \leq i \leq n\}$. Thus, $\text{page}(P_n, L) \geq n$.

Therefore, $\text{page}(P_n) \geq n$. ◻

Theorem 3.7.2 For each positive integer n , $\text{page}(L_n) \geq n$ where L_n is the (spherical) lattice illustrated in Figure 3.22.

Proof. Let L be a linear extension of the lattice L_n . Since $a_n \parallel b_{n+1}$ in L_n , either $a_n < b_{n+1}$ or $b_{n+1} < a_n$ in L .

Suppose $a_n < b_{n+1}$ in L . As $a_1 < a_2 < \dots < a_n$ and $b_{n+1} < b_{n+2} < \dots < b_{2n}$ in L_n , we have in L

$$a_1 < a_2 < \dots < a_n < b_{n+1} < b_{n+2} < \dots < b_{2n},$$

which means that L contains the n -twist $\{(a_i, b_{n+i}) : 1 \leq i \leq n\}$.

Hence, $\text{page}(L_n, L) \geq n$.

Thus, we may assume $b_{n+1} < a_n$ in L . We may assume that $c_{n+1} < b_n$ (Indeed, if not, then by a similar argument to the case $a_n < b_{n+1}$ above, L contains the n -twist $\{(b_i, c_{i+n}) : 1 \leq i \leq n\}$). As $a_{n+1} > a_n$, $b_{n+1} > b_n$ and $c_{n+1} > c_n$ in L_n , we have in L $c_n < c_{n+1} < b_n < b_{n+1} < a_n < a_{n+1}$. As $a_{n+1} < a_{n+2} < \dots < a_{2n}$ and $c_1 < c_2 < \dots < c_n$ in L_n , we have in L

$$c_1 < c_2 < \dots < c_n < a_{n+1} < a_{n+2} < \dots < a_{2n},$$

which means that L contains the n -twist $\{(c_i, a_{i+n}) : 1 \leq i \leq n\}$. Therefore, $\text{page}(L_n) \geq n$. □

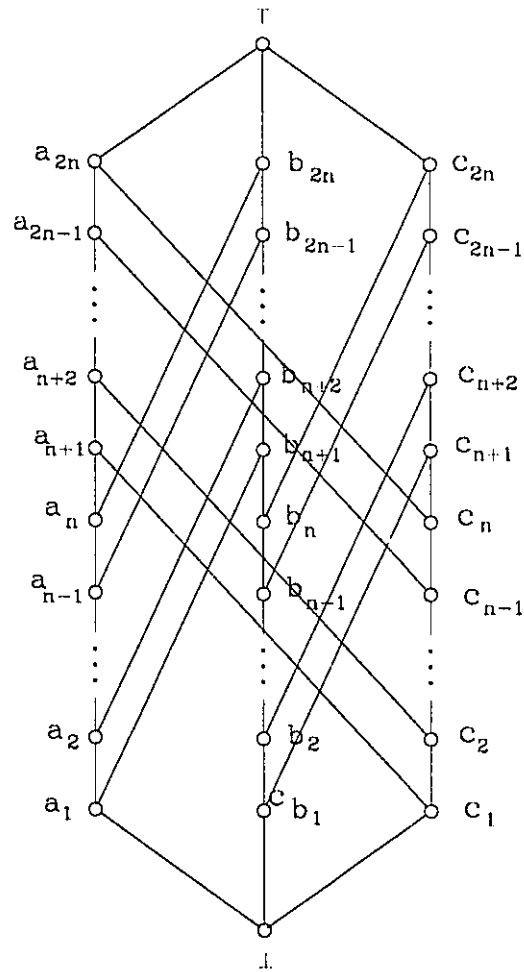


Figure 3.22:

Chapter 4

Twist-Free Linear Extensions

4.1 Introduction

In this chapter, we give necessary and sufficient conditions for the existence of a linear extension L of an ordered set P such that three edges form a 3-twist in L . Moreover, we generalize that for k edges, $k \geq 4$.

At the end, we conjecture that the pagenumber of planar ordered sets is unbounded and we give a candidate for such a sequence. In contrast, we conjecture that the pagenumber of planar lattices is at most four.

4.2 Why Twist-Free?

Recall that a k -twist for L is a set of edges $\{(x_i, y_i): 1 \leq i \leq k\}$ in P such that we have in L

$$x_1 < x_2 < \cdots < x_k < y_1 < y_2 < \cdots < y_k$$

Evidently the only ordered sets with a 1-twist-free linear extension are antichains. Also, according to Theorem 3.2.2, there exists a linear extension L of the ordered set P without 2-twist if and only if $\text{cov}(P)$ is tree. We are primarily concerned with 3-twist-free linear extensions which, for brevity, we call *twist-free*.

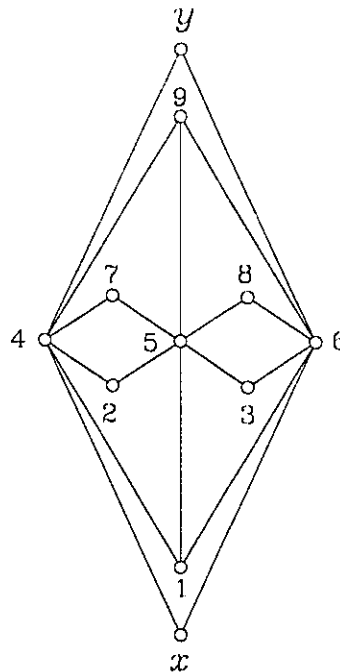


Figure 4.1: Every linear extension has a twist.

In other words, L is twist-free if the intersection graph of the layout of $\text{cov}(P)$ with respect to L contains no triangle.

The importance of twist-free linear extensions comes from this result in graph theory, due to Unger [50], that a circle graph which contains no triangle can be coloured with four colours¹. As a consequence and by Theorem 3.3.1

Theorem 4.2.1 *Let L be a linear extension of an ordered set P . If L is a twist-free, then $\text{page}(P, L) \leq 4$.*

One may ask here whether there exist a linear extension of a planar ordered set P which is twist-free? The next theorem shows that this, in general, not true.

Theorem 4.2.2 *Every linear extension of the ordered set P illustrated in Figure 4.1 contains a twist.*

¹In fact, the result in [50] is more general than that. It states that every circle graph that contains no complete graph K_n can be coloured with $2(n-1)$ colours.

Proof. Suppose there exists a twist-free linear extension L of P . As 4 and 6 are symmetric in the upward drawing of P , then we have three cases to consider $4 < 5 < 6$, $5 < 4 < 6$ and $4 < 6 < 5$ in L .

Case 1 $4 < 5 < 6$ in L .

As $1 < 4$ and $6 < 9$ in P , we have $1 < 4 < 5 < 6 < 9$ in L . Since $6 < 8$ in P , either $9 < 8$ or $6 < 8 < 9$ in L . If $9 < 8$, then L contains the twist $\{(1, 6), (4, 9)(5, 8)\}$. Dually, $1 < 2 < 4$ in L . Thus, we have in L

$$1 < 2 < 4 < 5 < 6 < 8 < 9$$

Since $4, 5 < 7$ in P , either $9 < 7$ or $6 < 7 < 9$ or $5 < 7 < 6$ in L . If $9 < 7$ in L , then L contains the twist $\{(1, 6), (4, 9), (5, 7)\}$. Also, if $6 < 7 < 9$ in L , then L contains the twist $\{(1, 6), (4, 7), (5, 9)\}$. Thus, we have in L

$$1 < 2 < 4 < 5 < 7 < 6 < 8 < 9$$

If $3 < 1$ in L , then L contains the twist $\{(3, 5), (1, 6), (4, 9)\}$. Also, if $1 < 3 < 4$, then L contains the twist $\{(1, 5), (3, 6), (4, 9)\}$. Finally, if $4 < 3 < 5$ in L , then $\{(4, 7), (3, 6), (5, 9)\}$. Thus, this case cannot happen.

Case 2 $5 < 4 < 6$ in L .

Since $1 < 5$ and $6 < 9$ in P , we have $1 < 5 < 4 < 6 < 9$ in L . As $2 < 5$ in P , either $2 < 1$ or $1 < 2 < 4$ in L . If $2 < 1$ in L , then L contains the twist $\{(2, 4), (1, 6), (5, 9)\}$. Thus, we have $1 < 2 < 5 < 4 < 6 < 9$ in L . Also, since $6 < 8$ in P , either $6 < 8 < 9$ or $9 < 8$ in L . If $6 < 8 < 9$ in L , then L contains the twist $\{(1, 6), (5, 8), (4, 9)\}$. Thus, we have in L

$$1 < 2 < 5 < 4 < 6 < 9 < 8$$

Since $3 < 5$ in P , either $1 < 3 < 5$ or $3 < 1$ in L . If $1 < 3 < 5$ in L , then L contains the twist $\{(1, 4), (3, 6), (5, 8)\}$. Thus, we have $3 < 1 < 2 < 5 < 4 < 6 < 9 < 8$ in L .

Since $4 < 7$ in P , either $6 < 7$ or $4 < 7 < 6$ in L . If $6 < 7 < 9$ in L , then L contains the twist $\{(1, 6), (5, 7), (4, 9)\}$ and if $9 < 7 < 8$ in L , then L contains the twist $\{(5, 9), (4, 7), (6, 8)\}$. Also, if $8 < 7$ in L , then L contains the twist $\{(1, 6), (5, 8), (4, 7)\}$. Therefore, we have in L

$$3 < 1 < 2 < 5 < 4 < 7 < 6 < 9 < 8$$

Since $x < 4$ in P , $x < 4$ in L . If $5 < x < 4$ in L , then L contains the twist $\{(5, 7), (x, 6), (4, 9)\}$. Also, if $1 < x < 5$ in L , then L contains the twist $\{(1, 4), (x, 6), (5, 9)\}$. If $x < 1$ in L , then L contains the twist $\{(x, 4), (1, 6), (5, 9)\}$. Hence this case cannot happen.

Case 3 $4 < 6 < 5$ in L .

Since P is self-dual², this case is just a dual of case 2. Thus, also this case can not happen.

That contradicts our assumption. Therefore, every linear extension of P has a twist. ▀

4.3 Subsets with Twist-Free Linear Extensions

In this section we find, for an ordered set P , a subset of P such that P has a twist-free linear extension whenever Q has a twist-free linear extension.

Lemma 4.3.1 *Let a, b be two elements in an ordered set P such that $a \prec b$ and $\text{cone}(a) = \text{cone}(b)$. If $P - \{a\}$ has a twist-free linear extension, then P has a twist-free linear extension.*

²An ordered set P is self-dual if $P \cong P^d$.

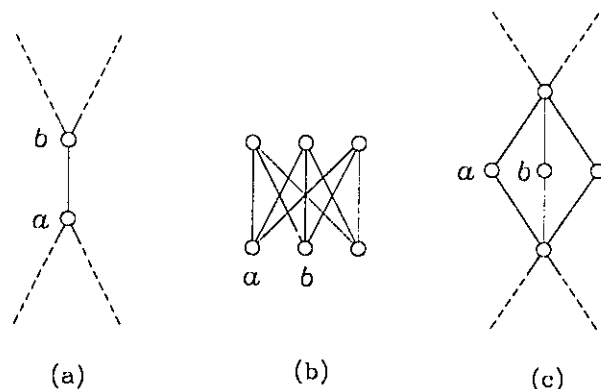


Figure 4.2: (a) $\text{cone}(a) = \text{cone}(b)$, (b) $\text{cone}(a) - \{a\} = \text{cone}(b) - \{b\}$, (c) $\text{cone}(a) - \{a\} = \text{cone}(b) - \{b\}$ and a, b are doubly irreducible.

Proof. Let L' be a twist-free linear extension of $P - \{a\}$. We obtain L from L' by adding a right below b in L' (i.e. $a \prec b$ in L).

Suppose L contains a twist $T = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$. We may assume that $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$ in L . Thus, we have

- (i) $(a, b) \notin T$ because $a \prec b$ in L .
- (ii) $a \in T$ because L' has no twist. We may assume that $b_1 = a$ (because a has a unique upper cover b).
- (iii) $b \in T$ because if not, then $T' = \{(a_1, b), (a_2, b_2), (a_3, b_3)\}$ is a twist in L' . Thus, $b_2 = b$ which contradicts the fact that b has a unique lower cover.

Therefore, L is twist-free linear extension. ▀

What if $a \parallel b$ and $\text{cone}(a) - \{a\} = \text{cone}(b) - \{b\}$? Can the existence of a twist-free linear extension be inferred from its existence for a proper subset? No, for example $\text{cone}(a) - \{a\} = \text{cone}(b) - \{b\}$ for the ordered set P in Figure 4.2(b) and $P - \{a\}$ contains a twist-free linear extension but P does not. However if a and b are doubly irreducible, then it is true (see Figure 4.2(c)).

Theorem 4.3.1 *Let u, v belong to an ordered set P such that the set of the lower covers of v equals the set of the upper covers of u equals $\{a_i : 1 \leq i \leq n\}, n \geq 1$. Suppose each a_i has a unique lower cover u and a unique upper cover v . If $P - \{a_i : 1 \leq i \leq n\}$ has a twist-free linear extension, then P has a twist-free linear extension.*

Proof. The set of the edges of $P - \{a_i : 1 \leq i \leq n\}$ can be obtained from the set of the edges of P by deleting all a_i 's and the $2n$ edges connecting to them and adding the edge (u, v) . Indeed, this is true because the set of the lower covers of v equals the set of the upper covers of u equals $\{a_i : 1 \leq i \leq n\}$.

Let L' be a twist-free linear extension of $P - \{a_i : 1 \leq i \leq n\}$. Obtain the linear extension L of P by adding a permutation of a_i 's right above u in L (i.e. say $u \prec a_1 \prec a_2 \prec \dots \prec a_n$ in L).

Suppose that L contains the twist T . Since L' is twist-free, there is at least one $a_i \in T$. If there are two vertices a_i and $a_j, i < j$, in T , then (a_i, v) and (u, a_j) are in T . Thus, $a_k \notin T$ for every $k \notin \{i, j\}$ because $u < a_k < v$ in L . Therefore, there are two cases to consider.

Case 1 $a_i \in T$ and $a_k \notin T$ for $k \neq i$.

No edge of T intersects the edge (u, a_i) because the only elements between u and a_i in L are a_1, a_2, \dots, a_{i-1} and none of them are in T . Thus, $(a_i, v) \in T$.

Let $T = \{(a_i, v), (x', x), (y', y)\}$. As T is a twist in L , there are three possibilities for the order of the elements of T in L .

- (i) $x' < y' < a_i < x < y < v$.
- (ii) $y' < a_i < x' < y < v < x$.
- (iii) $a_i < x' < y' < v < x < y$.

In every case replacing a_i by u yields a twist in L' .

Case 2 $a_i, a_j \in T, i < j$, and $a_k \notin T$ for $k \neq i, j$.

As $a_i < a_j$ in L and a_i, a_j in T , then (u, a_j) and (a_i, v) are in T . Let (x', x) be the third edge in T . Since $u < a_i < a_j < v$ in L and (u, a_j) and (a_i, v) are in T , there are only three possibilities for where x and x' can go. Thus we must have one of the orderings $x' < u < a_i < x < a_j < v$, $u < x' < a_i < a_j < x < v$ and $u < a_i < x' < a_j < v < x$ in L . For the first case $x = a_k, i < k < j$. Since a_k has a unique lower cover u , $x' = u$, a contradiction. For the second one $u < x' < a_i$ in L and $x' \notin \{a_1, a_2, \dots, a_n\}$ which contradicts the fact that $u \prec a_1 \prec a_2 \prec \dots \prec a_i$ in L . For the last one $x' = a_k, i < k < j$. Since a_k has a unique upper cover v , $x = v$, a contradiction.

Therefore, L is a twist-free linear extension of P . ▀

4.4 Twist and Matching

In this section we give necessary and sufficient conditions for the existence of a linear extension L of an ordered set P such that a set of three edges forms a twist in L .

For a graph G , a subset M of the edges of G is a *matching* if, for each pair of edges of M , there is no common vertex.

Let L be a linear extension of an ordered set P . Suppose L contains the twist $T = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ such that $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$ in L . Evidently, $\{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ is a matching in the graph $\text{cov}(P)$ and no two edges of T lie in the same chain in P (i.e., there is no $a_i > b_j$ in P). In fact, these are sufficient conditions too.

Theorem 4.4.1 *Let P an ordered set. Then $M = \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ is a matching in $\text{cov}(P)$ such that no two edges of M lie in the same chain in P if and only if there is a linear extension L which contains the edges of M as a twist in L .*

Proof. Let $M = \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ be a matching in $\text{cov}(P)$ such that no two edges of M lie in the same chain in P . We may assume that $a_i \prec b_i$ in P , $1 \leq i \leq 3$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. We will construct an extension Q of P such that every linear extension of Q , which is a linear extension of P , contains the twist $T = \{(a_i, b_i) : 1 \leq i \leq 3\}$.

Suppose that $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. We claim that there exists $a_{i_1} \in \min A$ such that $b_{i_1} \in \min B$. If not, then for each $a_{i_1} \in \min A$, $b_{i_1} \notin \min B$ ($\min A = \{a \in A : \text{for every } x \in A, a \leq x\}$).

Suppose that for each $a_{i_1} \in \min A$, $b_{i_1} \notin \min B$ in P . Let $a_{i_1} \in \min A$. Thus, $b_{i_1} \notin \min B$ which implies the existence of $b_{i_2} \in \min B$ such that $b_{i_2} < b_{i_1}$ in P . As $b_{i_2} \in \min B$, $a_{i_2} \notin \min A$. Thus, either $a_{i_2} > a_{i_1}$ or $a_{i_2} > a_{i_3}$.

If $a_{i_2} > a_{i_1}$, then $b_{i_1} > a_{i_2} > a_{i_1}$ which contradicts that (a_{i_1}, b_{i_1}) is an essential edge. Thus, $a_{i_2} > a_{i_3}$. If $a_{i_3} > a_{i_1}$, then the edge (a_{i_1}, b_{i_1}) is not essential. Hence, $a_{i_3} \in \min A$.

As $a_{i_3} \in \min A$, $b_{i_3} \notin \min B$. Thus, $b_{i_3} > b_{i_2}$ which contradicts the fact that the edge (a_{i_3}, b_{i_3}) is an essential edge.

Therefore, there is $a_{i_1} \in \min A$ such that $b_{i_1} \in \min B$.

By an analogous argument, there exists $a_{i_2} \in \min(A - \{a_{i_3}\})$ such that $b_{i_2} \in \min(B - \{b_{i_3}\})$.

Let $Q = P \cup \{a_{i_3} < a_{i_2} < a_{i_3} < b_{i_3} < b_{i_2} < b_{i_1}\}$. In fact, we proved above that Q is an extension of P . ▀

A matching M in a graph G is *perfect* if for every vertex $x \in G$ there is an edge e in M with x as an end point of e . An ordered set P is an *orientation* of a graph G if $\text{cov}(P) \cong G$ (as graphs). Evidently, a graph with a triangle has no orientation. In fact, a drawing of $\text{cov}(P)$ such that every edge is drawn monotonically is an orientation if there are no nonessential edges.

Corollary 4.4.1 *Any triangle-free graph G on 6 vertices with a perfect matching M*

has an orientation P and a linear extension L of P in which the edges of M form a twist in L . Moreover, any such orientation in which no two matched edges lie in the same chain generate a twist.

Proof. Let G be a triangle-free graph on the set $V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ of 6 vertices such that G contains the perfect matching $M = \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$. Either G is bipartite or not.

Case 1 The graph G is bipartite.

A graph G is *bipartite* if the set of vertices V of G equals $V_1 \cup V_2$ such that V_1 and V_2 are two disjoint sets and the set of edges of G is a subset of $V_1 \times V_2$.

We may assume that $V_1 = \{a_1, a_2, a_3\}$. We want now to construct an ordered set P such that $\text{cov}(P) = G$. We will construct P of two levels $L_1 = V_1$ and $L_2 = V_2$. This drawing is an orientation of G because no two vertices of the same level are adjacent and there is no nonessential edge.

Notice that P bipartite implies that no two edges of M are in the same chain in P . Thus, no two edges of the perfect matching M lie in the same chain. Therefore, by Theorem 4.4.1, there exists a linear extension L of P so the edges $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ form a twist.

Case 2 The graph G is not bipartite.

A graph G is bipartite if and only if G has no cycle of odd length. As G contains no triangle, it contains a cycle C of length 5. No non-consecutive vertices in C are adjacent (i.e., no chords in C) because G contains no triangle. Therefore, without loss of generality let $C = \{a_1, b_1, a_2, b_2, b_3\}$ (see Figure 4.3(a)).

Since $\{a_3, b_3\}$ is an edge in G , the set of the edges of G may contain, at most, one extra edge specifically either, $\{b_1, a_3\}$ or $\{a_2, a_3\}$. Construct P on four levels as follows

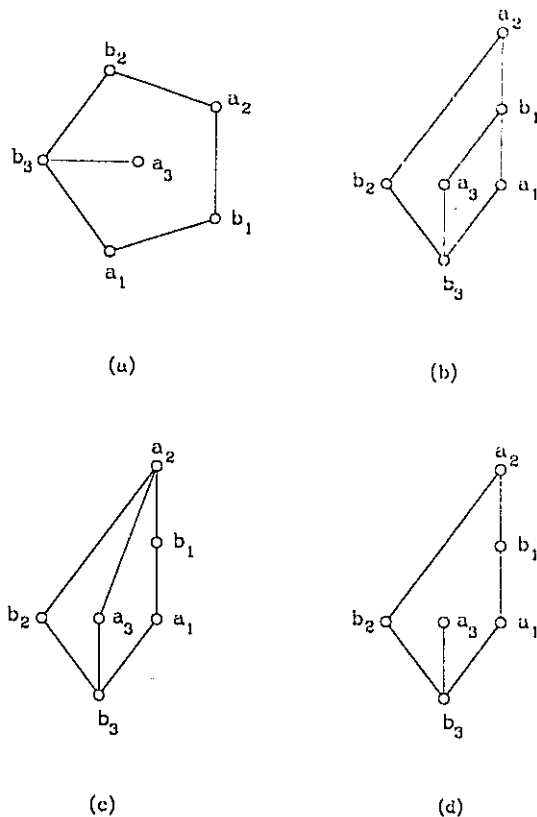


Figure 4.3:

- $L_1 = \{b_3\}$,
- $L_2 = \{b_2, a_3, a_1\}$,
- $L_3 = \{b_1\}$.
- $L_4 = \{a_2\}$.

See an upward drawing of P in Figure 4.3(b) if $\{a_3, b_1\}$ is an edge in $\text{cov}(P)$, in Figure 4.3(c) if $\{a_3, a_2\}$ is an edge in $\text{cov}(P)$ and in Figure 4.3(d) if neither $\{a_3, b_1\}$ nor $\{a_3, a_2\}$ is an edge of $\text{cov}(P)$.

As no two matched edges lie in the same chain, then by Theorem 4.4.1, there exists a linear extension of P such that the edges $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ form a twist. ▣

In the next result, the bold face edges in Figure 4.4 and Figure 4.5 are the edges of the matching M (and edges in P)

Corollary 4.4.2 *Let P be an ordered set such that $\text{cov}(P)$ contains a matching M of size three and no two edges of M lie in the same chain in P . Figure 4.4 and Figure 4.5 illustrate all the possible comparabilities between the six elements. These are classified according to their covering graph.*

We call a set of disjoint edges $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ in an ordered set P a *ladder* if $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ in P .

Lemma 4.4.1 *Let P be an ordered set such that the set of edges $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ forms a ladder in P . Then $a_3 > b_1$ in any twist-free linear extension of P .*

Proof. Let L be a linear extension of P such that $a_3 < b_1$. Thus, we have $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$ in L because $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ in P .

Therefore, L contains the twist $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$. ▀

Lemma 4.4.2 *Let P be an ordered set P . If $\text{cov}(P)$ contains a unique matching M of size 3 in which no two matched edges of M lie in the same chain in P , then P has a twist-free linear extension.*

Proof. Let $M = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ be the unique matching in $\text{cov}(P)$ such that no two edges lie in the same chain in P . We may assume that $a_i < b_i$ in P for $1 \leq i \leq 3$. Thus, there is no $1 \leq i \neq j \leq 3$ such that $a_i > b_j$.

If there is $1 \leq i \neq j \leq 3$ such that $a_i \parallel b_j$ in P , then by Theorem 1.5.1, there is a linear extension L of P such that $a_i > b_j$ in L . Thus, L is twist-free because the two edges (a_i, b_i) and (a_j, b_j) do not intersect.

Suppose that $a_i < b_j$ in P for every $1 \leq i \neq j \leq 3$. Thus, there are c, d such that $b_1 \geq c > a_2$ and $b_2 \geq d > a_1$ in P . Since (a_1, b_1) is an edge in P , $c \not> d$. Also, $d \not> c$

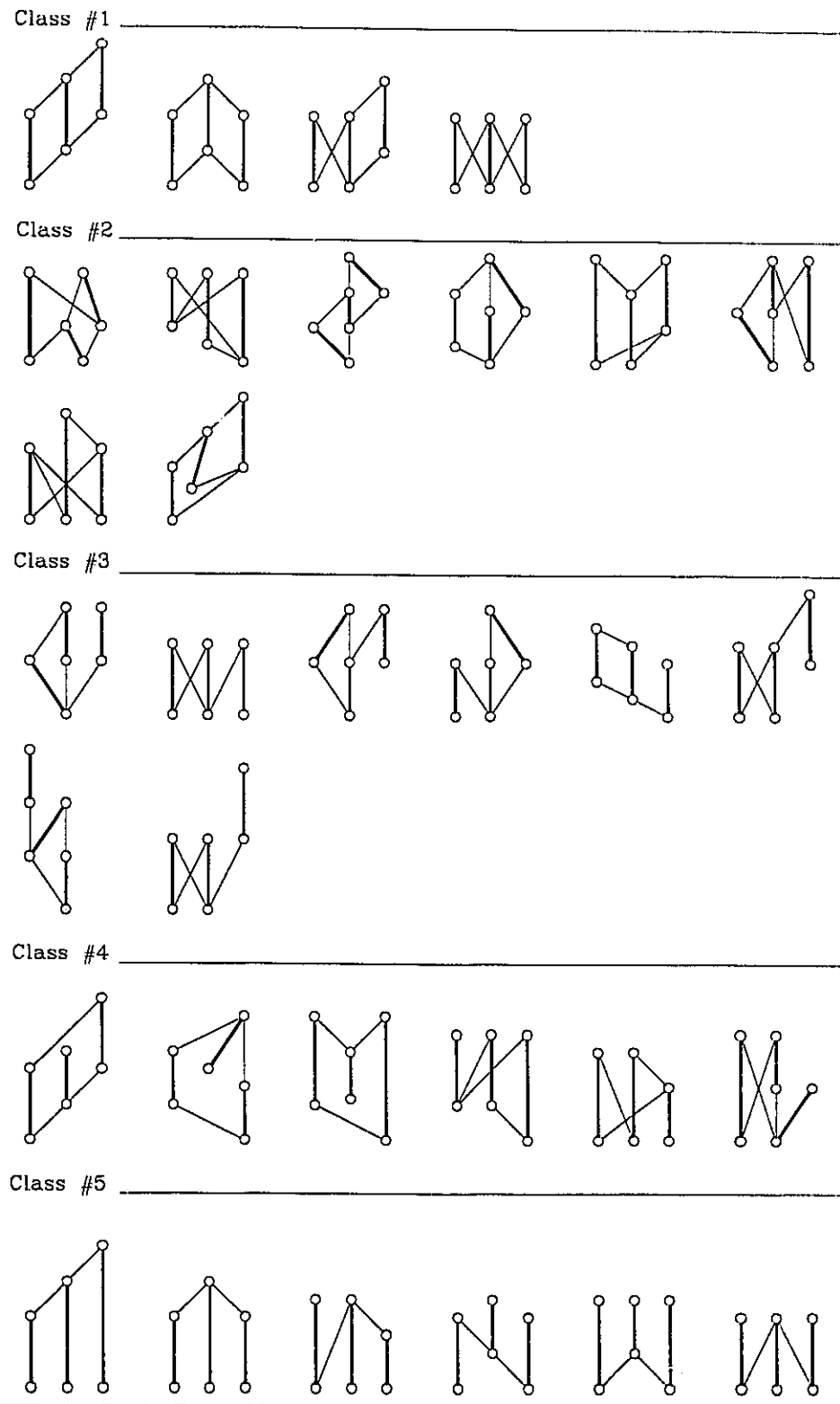


Figure 4.4:

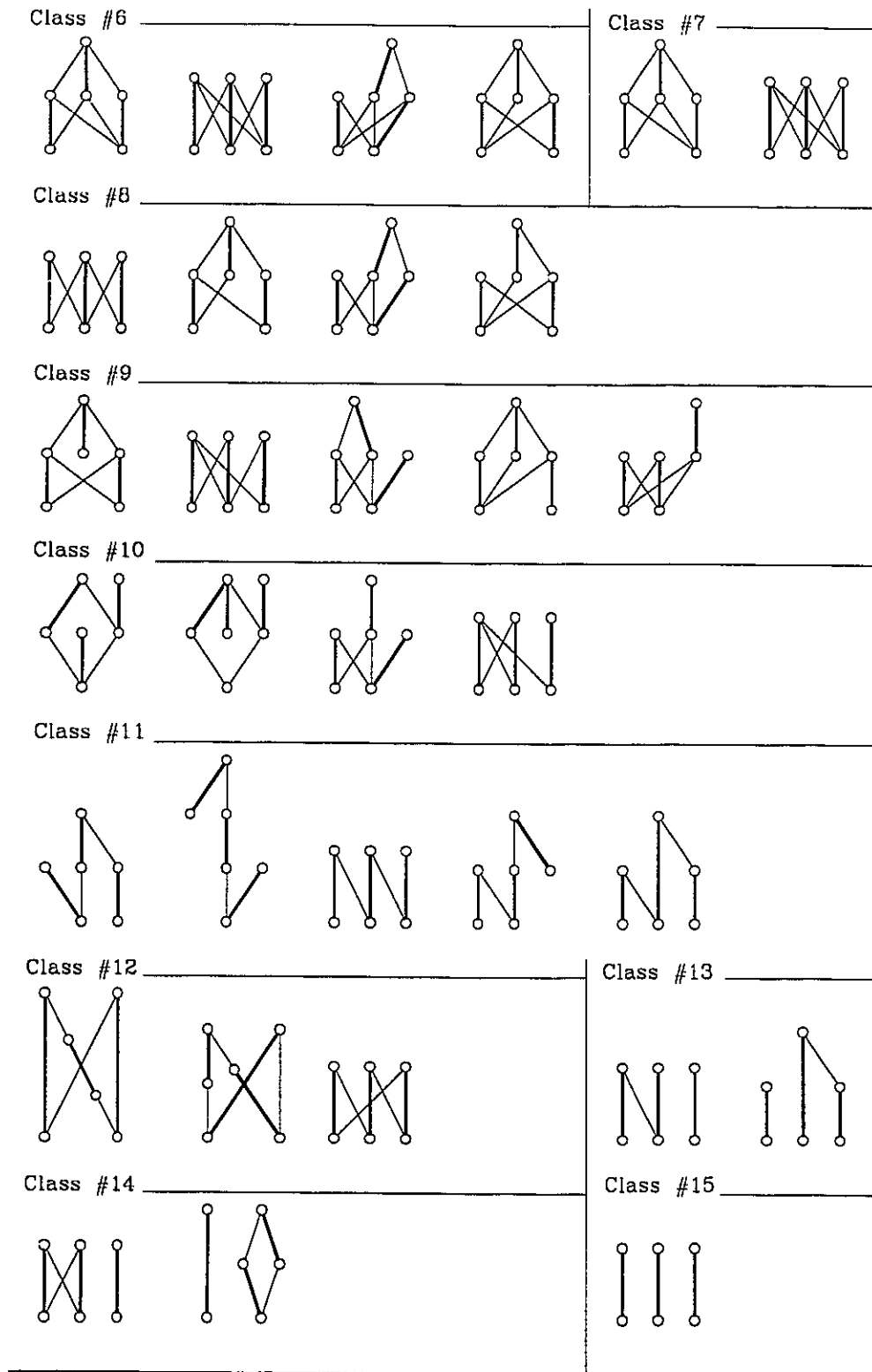


Figure 4.5:

because (a_2, b_2) is an edge in P . Hence, $c \parallel d$. Thus, $M' = \{\{a_1, d\}, \{a_2, c\}, \{a_3, b_3\}\}$ is a matching in $\text{cov}(P)$.

In fact, $a_1 \not\geq c$ as then $b_2 > a_1 > a_2$ and $a_2 \not\geq d$ as then $b_1 > a_2 > a_1$. Also, $a_3 \not\geq c$ because $b_2 > a_3$ and (a_2, b_2) is an edge in P . Similarly $a_3 \not\geq d$. Therefore, no two edges of M' lie in the same chain in P which contradicts the uniqueness of M . ■

4.5 k -Twist-Free Linear Extensions

This section contains a generalization of Theorem 4.4.1. Indeed, we give necessary and sufficient conditions for the existence of a linear extension L of an ordered set such that a set of k edges, $k \geq 4$, is a k -twist in L . However, this generalization has more conditions.

Recall a linear extension L of an ordered set P is k -twist-free if there is no k -twist in L .

Does an ordered set P with a 4-edge matching in $\text{cov}(P)$ with no two edges in the same chain of P have a linear extension L with these 4 edges as a 4-twist?

This generalization of Theorem 4.4.1 is not always true; for example Figure 4.6 illustrates an ordered set P such that $\text{cov}(P)$ has a matching $M = \{\{a_i, b_i\} : 1 \leq i \leq 4\}$ of size four in which no two matched edges lie in the same chain in P , but there is no linear extension L of P such that the matched edges form a 4-twist in L .

Indeed, suppose there exists a linear extension L of P such that the edges $\{(a_i, b_i) : 1 \leq i \leq 4\}$ form a 4-twist in L . Thus, for each $1 \leq i, j \leq 4$, $a_i < b_j$ in L . Also, $a_1 < a_2$ and $a_3 < a_4$ in L because $a_1 < a_2$ and $a_3 < a_4$ in P . Similarly, $b_2 < b_3$ and $b_4 < b_1$ in L .

As the upward drawing of P is symmetric then either $a_1 < a_2 < a_3 < a_4 < b_1 < b_2 < b_3 < b_4$ or $a_1 < a_3 < a_2 < a_4 < b_1 < b_3 < b_2 < b_4$ or $a_1 < a_3 < a_4 < a_2 < a_1 < a_3 < a_4 < a_2$ in L . In the all three cases, L is not linear extension of P because $b_4 < b_1$

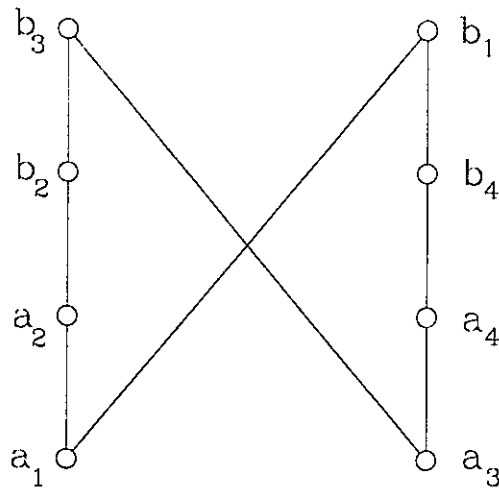


Figure 4.6:

in P .

We call the set of edges $K = \{(a_i, b_i) : 1 \leq i \leq k\}$, $k \geq 4$ in an ordered set P an *alternating cover cycle*³ if

- $a_{2i} > a_{2i+1}$, $1 \leq i < k/2$,
- $b_{2i-1} > b_{2i}$, $1 \leq i \leq k/2$,
- $a_k > a_1$,
- and of course, $a_i < b_i$, $1 \leq i \leq k$ (see Figure 4.7, the **bold face** edges are covering relations in P).

Let us call these comparabilities the *essential relations* in the alternating cover cycle K .

³This should be distinguished from a similar concept defined in [42].

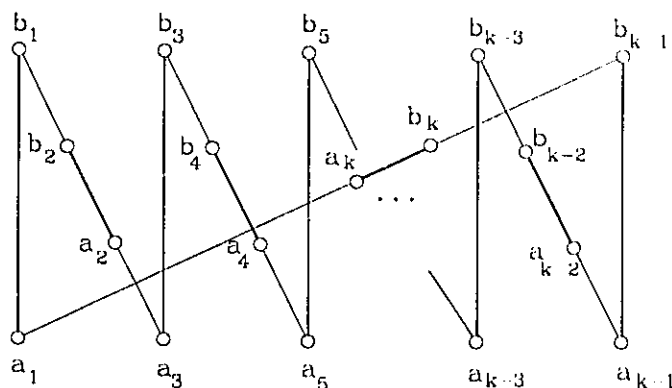


Figure 4.7:

Notice that the number of edges in an alternating cover cycle may be odd (see Figure 4.8(a), an example due to Richter[40]). An alternating cover cycle K is a *minimal alternating cover cycle* if there is no proper subset H of K such that H is an alternating cover cycle. Notice that the number of edges in a minimal alternating cover cycle is even. (Indeed, if $K = \{(a_i, b_i) : 1 \leq i \leq 2k + 1\}$ is a minimal alternating cover cycle, then $k \geq 2$. Thus $K' = \{(a_i, b_i) : 1 \leq i \leq 2k\}$ is an alternating cover cycle (because $a_{2i} > a_{2i+1}$ for $1 \leq i < k$ and $b_{2i-1} > b_{2i}$ for $1 \leq i \leq k$ and $a_{2k} > a_{2k+1} > a_1$) which contradicts the minimality of K .)

The set of edges $\{(a_i, b_i) : 1 \leq i \leq 6\}$ forms a minimal alternating cover cycle in the ordered set P illustrated in Figure 4.8(b) although there are more comparabilities than the essential relations for an alternating cover cycle.

Lemma 4.5.1 *Let $K = \{(a_i, b_i) : 1 \leq i \leq k\}$ be an alternating cover cycle in an ordered set P . If K is a minimal alternating cover cycle, then a nonessential relation $x < y$ in K , x, y are end points of edges in K , implies $x \in \{a_i : 1 \leq i \leq k\}$ and $y \in \{b_i : 1 \leq i \leq k\}$.*

Proof. Let $K = \{(a_i, b_i) : 1 \leq i \leq k\}$ be a minimal alternating cover cycle in an

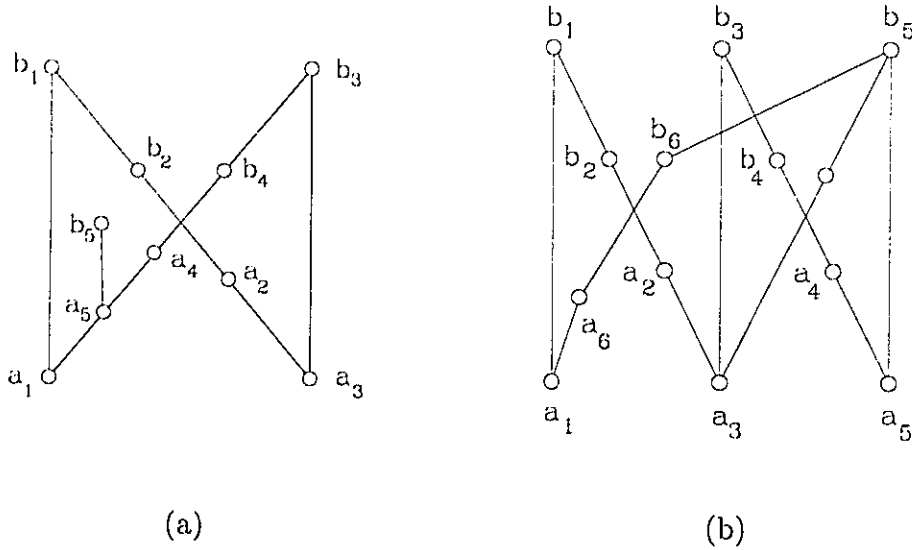


Figure 4.8:

ordered set P (see Figure 4.7). Suppose, for some $m < n$, $b_m > b_n$ in P . There three possibilities; m, n are both even or odd or exactly one of them is even.

Case 1 Both m, n are even.

Notice that it is impossible that $m = 2$ and $n = k$ simultaneously else $b_1 > b_2 > b_k > a_1$, which contradicts that (a_1, b_1) is an essential edge. Let $K' = \{(a_1, b_1), (a_2, b_2), \dots, (a_{m-1}, b_{m-1}), (a_n, b_n), (a_{n+1}, b_{n+1}), \dots, (a_k, b_k)\}$. Hence $\{(a_1, b_1), (a_2, b_2), (a_{k-1}, b_{k-1}), (a_k, b_k)\} \subseteq K'$ which implies $|K'| \geq 4$. Thus, K' is an alternating cover cycle and $K' \subset K$ which contradicts the fact that K is a minimal alternating cover cycle (see Figure 4.9).

Case 2 Both m, n are odd.

We may assume that $b_m = b_1$. Thus, $n \neq k - 1$ because (a_1, b_1) is an edge in P and hence $n \leq k - 3$. Let $K' = \{(a_1, b_1), (a_{n+1}, b_{n+1}), (a_{n+2}, b_{n+1}), \dots, (a_k, b_k)\}$. Thus $\{(a_1, b_1), (a_{k-2}, b_{k-2}), (a_{k-1}, b_{k-1}), (a_k, b_k)\} \subseteq K'$ which implies $|K'| \geq 4$. Thus K' is an alternating cover cycle and $K' \subset K$ which contradicts the fact

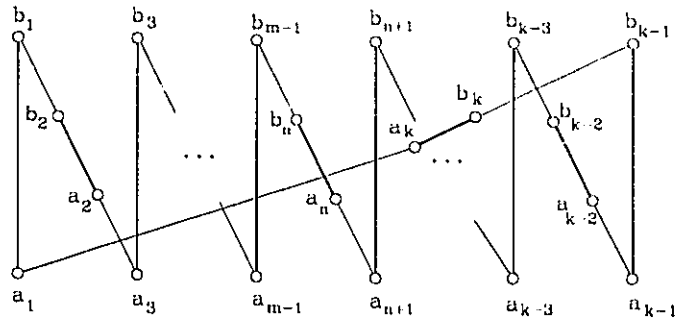


Figure 4.9:

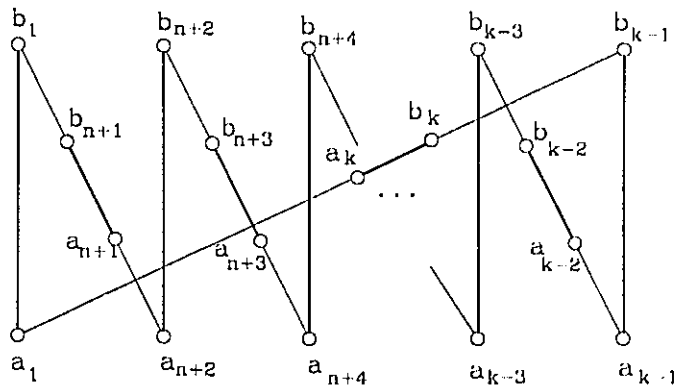


Figure 4.10:

that K is a minimal alternating cover cycle (see Figure 4.10).

Case 3 Exactly one of m, n is even.

If m is odd, we may assume that $m = 1$. Since (a_1, b_1) is an edge, $n \neq k$ and hence $n \leq k - 2$. Let $K' = \{(a_1, b_1), (a_n, b_n), (a_{n+1}, b_{n+1}), \dots, (a_k, b_k)\}$. Thus $\{(a_1, b_1), (a_{k-2}, b_{k-2}), (a_{k-1}, b_{k-1}), (a_k, b_k)\} \subseteq K'$ which implies $|K'| \geq 4$. Thus, K' is an alternating cover cycle and $K' \subset K$ which contradicts the fact that K is a minimal alternating cover cycle.

Also, if m is even, we may assume that $m = 2$. Since (a_1, b_1) is an edge, $n \neq k - 1$ and hence $n \leq k - 3$. Let $K' = \{(a_1, b_1), (a_{n+1}, b_{n+1}), (a_{n+2}, b_{n+2}), \dots, (a_k, b_k)\}$. Hence $\{(a_1, b_1), (a_{k-2}, b_{k-2}), (a_{k-1}, b_{k-1}), (a_k, b_k)\} \subseteq K'$ which implies $|K'| \geq 4$. Thus K' is an alternating cover cycle and $K' \subset K$ which contradicts the fact that K is a minimal alternating cover cycle.

These comparabilities between elements $\{b_i : 1 \leq k\}$ are the essential relations, and, similarly, for the elements of $\{a_i : 1 \leq k\}$. ▀

Theorem 4.5.1 *Let M be a matching in $\text{cov}(P)$ of an ordered set P such that $|M| \geq 4$ and no two edges of M lie in the same chain of P . Then there is a linear extension L of P such that the edges of M form an m -twist in L where $m = |M|$ if and only if there is no minimal alternating cover cycle K of P such that $K \subseteq M$.*

Proof. Let $K = \{(a_i, b_i) : 1 \leq i \leq k\}$ be a subset of edges of M which form a minimal alternating cover cycle in P . Of course, $k \geq 4$. Let $A = \{a_i : 1 \leq i \leq k\}$ and $B = \{b_i : 1 \leq i \leq k\}$. If $a_i \in \min A$, then i odd, so $b_i \notin \min B$ (see Figure 4.7). Thus, there is no linear extension L of P such that $a_{i_1} < a_{i_2} < \dots < a_{i_k} < b_{i_1} < b_{i_2} < \dots < b_{i_k}$ in L . Therefore, there is no linear extension L of P such that the edges of M form a k -twist in L .

Suppose that there is no linear extension L of P such that the edges of M form an m -twist in L . Let $K = \{(a_i, b_i) : 1 \leq i \leq k\}$ be the minimal subset of M such that there exist no linear extension L of P such that K forms a k -twist in L . According to Theorem 4.4.1, $|K| \geq 4$. We want to show that K is an alternating cover cycle. Let $A = \{a_i : 1 \leq i \leq k\}$ and $B = \{b_i : 1 \leq i \leq k\}$.

We claim that if $a_i \in \min A$, then $b_i \notin \min B$. If not, then there is $a_i \in \min A$ such that $b_i \in \min B$. Thus, either there is a linear extension L of P such that $K - \{(a_i, b_i)\}$ forms a $(k - 1)$ -twist in L or not. Let $a_{i_1} = a_i$ and $b_{i_1} = b_i$.

- If there is such linear extension L , then we have $a_{i_2} < a_{i_3} < \dots < a_{i_k} < b_{i_2} < b_{i_3} < \dots < b_{i_k}$ in L . Thus, any linear extension of the extension of P obtained by adding $a_{i_1} < a_{i_2} < a_{i_3} < \dots < a_{i_k} < b_{i_1} < b_{i_2} < b_{i_3} < \dots < b_{i_k}$ will have the k -twist $\{(a_i, b_i) : 1 \leq i \leq k\}$, which contradicts the fact that there is no such linear extension.
- Existence of no such linear extension contradicts the fact that K is a minimal subset of M such that no linear extension, with all of the edges of K , is a k -twist.

Let $a_1 \in \min A$. Since $b_1 \notin \min B$, there exists $b_2 \in \min B$ such that $b_1 > b_2$ in P . As $b_2 \in \min B$, $a_2 \notin \min A$. Thus, there exists $a_3 \in \min A$ such that $a_2 > a_3$. (Evidently $a_3 \neq a_1$ because (a_1, b_1) is an edge in P .)

Using a similar argument leads to the existence of $b_4 \in \min B$ such that $b_3 > b_4$. ($b_4 \neq b_2$) because (a_3, b_3) is an edge in P .)

As $a_4 \notin \min A$, there is $a_5 \in \min A$ such that $a_4 > a_5$ in P . If $k \geq 5$, then by minimality of K , $a_5 \neq a_1$. Also $a_5 \neq a_3$ because (a_3, b_3) is an edge in P .

As $b_5 \notin \min B$, then there is $b_6 \in \min B$ such that $b_5 > b_6$ in P . If $k \geq 6$, then by minimality of K , $b_6 \neq b_2$. Also $b_6 \neq a_4$ because (a_5, b_5) is an edge in P . Using the same argument, we end up with either $a_i \in \min A$ and there is $b_j \in \min B$ such that $b_i > b_j$, $j < i$ if i is odd or $b_i \in \min B$, $b_{i-1} > b_i$ and there is $a_j \in \min A$ such that

$a_i > a_j$, $j < i$ if i is even. By minimality of K , $i = k$ in both cases.

For the first case, $b_k \not\prec b_{k-1}$ and $b_k \not\prec b_{k-2}$ because (a_k, b_k) is an edge in P . Also, by minimality of K , $b_k \not\prec b_j$ for $j < k - 2$. Thus, this case cannot happen and hence k is even.

For the second case, $j \leq k - 4$ because (a_{k-1}, b_{k-1}) is an edge. By minimality of K , $j = 1$. Also, $j \leq k - 4$ implies that $|K| \geq 4$. Thus, K is an alternating cover cycle.

Therefore, P contains a minimal alternating cover cycle. ▀

4.6 Two Conjectures

In this section we give two conjectures. We first conjecture that there is a sequence of unbounded pagenumber planar ordered sets and we give a candidate for such a sequence. In contrast, we conjecture that every planar lattice P has a twist-free linear extension. (If this true, then by Theorem 4.2.1, $\text{page}(P) \leq 4$.) We finally give examples of complex planar lattices with twist-free linear extensions.

Conjecture 1 *For each positive integer n , $\text{page}(P_n) \geq n$ for each ordered set in the following sequence $\{P_n\}$ of planar ordered sets.*

Let P_1 be a two-element chain and P_2 and P_3 be the ordered sets illustrated in Figure 4.11. For $n \geq 3$ we obtain P_n by following the three steps

Step 1 Take two copies of P_{n-1} ,

Step 2 Identify the left most element x on the outer cycle (which is not maximal or minimal in P_{n-1}) on each copy (see Figure 4.12) (P_{n-1} is symmetric),

Let y, z be, respectively, the right most elements of those two copies of P_{n-1} .

Step 3 Add a lower cover and upper cover of the three elements $\{x, y, z\}$ (see Figure 4.12).

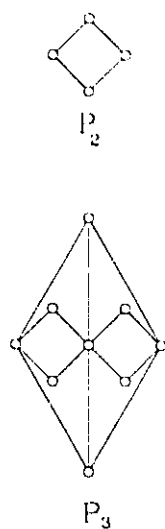


Figure 4.11:

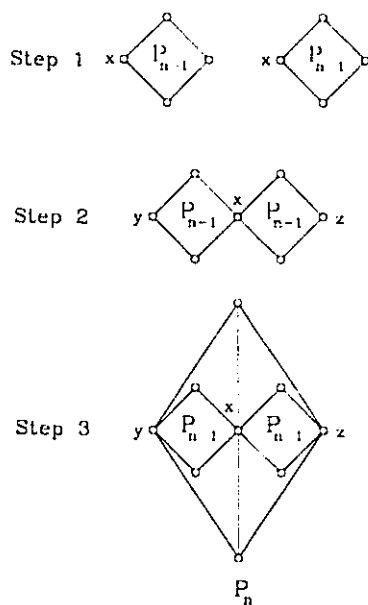


Figure 4.12:

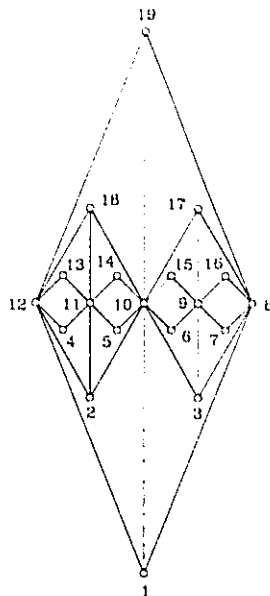


Figure 4.13:

Figure 4.13 illustrates P_4 . In fact P_4 has 17,824,869,000 linear extensions;⁴ each of them, by Theorem 4.2.2, has a twist which implies that $\text{page}(P_4) \geq 3$. In contrast, Figure 4.14 shows that $\text{page}(P_4) \leq 4$.

Conjecture 2 *Each planar lattice has a twist-free linear extension.*

Planar lattices have easier structure than planar ordered sets, in the sense that lattices do not contain four-cycles without splitting element. Each of the following examples illustrated in Figure 4.15 to Figure 4.19 has a twist-free linear extension. One of the twist-free linear extensions is obtained by just following the order $1 \leq 2 \leq \dots \leq |P|$.

⁴This number is obtained by a program counting the number of linear extensions of an ordered set written by K. Ewacha[17].

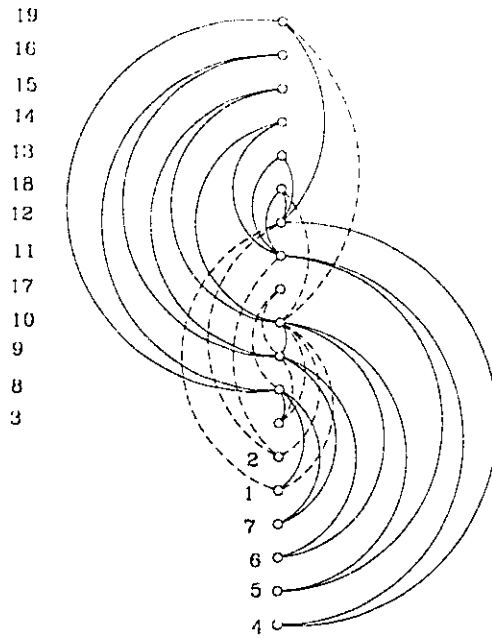


Figure 4.14:

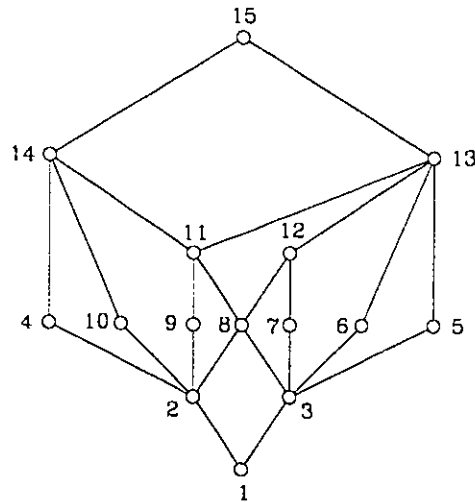


Figure 4.15:

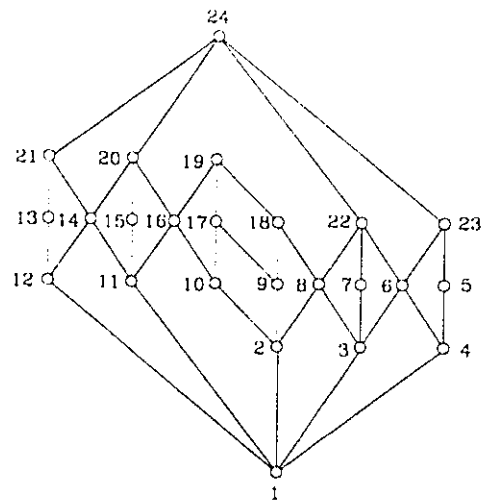


Figure 4.18:

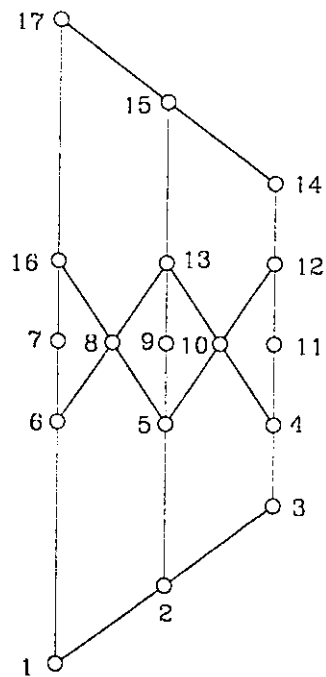


Figure 4.19:

Chapter 5

Series-Parallel Planar Lattices

5.1 Introduction

It is unknown whether there is a constant k such that every planar ordered set can be embedded in at most k pages. It is even unknown for planar lattices.

The purpose of this chapter is to give a polynomial-time algorithm which embeds any series-parallel planar lattice in two pages.

5.2 Two Pages are Enough

In this section we will show the *existence* of a two-page embedding for a series-parallel planar lattice. Let P be a series-parallel planar lattice. Fix a planar embedding of P , and let $C = \{x_1 < x_2 < \dots < x_n\}$ be the left boundary chain. For each $x \in P - C$ define the *interval* $I(x) = (x_i, x_j)$, where

$$i = \max\{1 \leq k \leq n : x > x_k\}$$

$$j = \min\{1 \leq k \leq n : x < x_k\}$$

Of course, $j \geq i + 1$. Notice that, $j > i + 1$ because if $j = i + 1$ then the edge (x_{i+1}, x_i) will not be an essential edge.

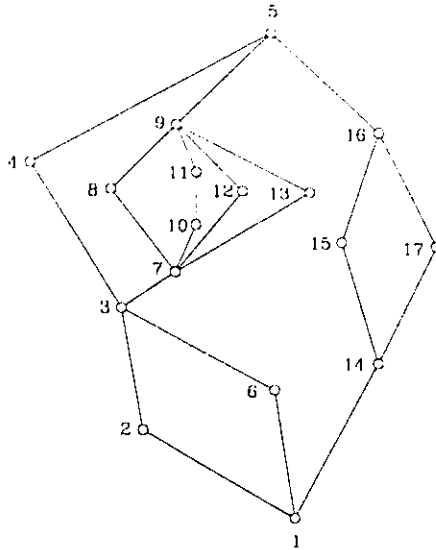


Figure 5.1:

For $y, z \in P - C$, say $y \sim z$ if $I(y) = I(z)$. It is clear that this relation is an equivalence relation on $P - C$; we call the equivalence classes *components*.

Recall, **Kelly and Rival** in [32] define a “component” as any connected subgraph of any open interval of a planar embedding. In fact, our components are unions of such connected subgraphs.

For example, the components of the series-parallel order in Figure 5.1 are:

$C_1 = \{7, 8, 9, 10, 11, 12, 13\}$ which corresponds to the interval $(3,5)$;

$C_2 = \{6\}$ which corresponds to the interval $(1,3)$;

$C_3 = \{14, 15, 16, 17\}$ which corresponds to the interval $(1,5)$.

The ordered set obtained by ordering the intervals by inclusion is shown in Figure 5.2.

For a component A of P , call the element $x_j \in C$, which covers the maximals of A , the *upper connection* of A and the element $x_i \in C$ which is covered by the minimals of A , the *lower connection* of A .

Lemma 5.2.1 *Let P be a series-parallel planar lattice. Then any two intervals, ob-*

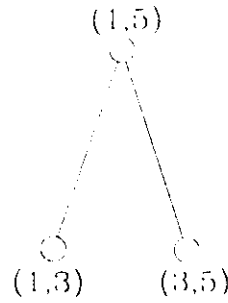


Figure 5.2:

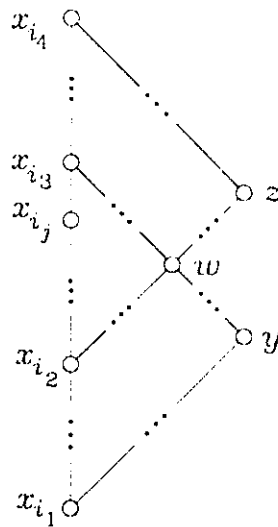


Figure 5.3:

tained according to a planar embedding, are either disjoint or one contains the other. In other words, the set of intervals ordered by inclusion is a forest. Also, there is no edge joining elements from two different components.

Proof. Fix a planar embedding of P , and let $C = \{x_1 < x_2 < \dots < x_n\}$ be the left boundary chain. For each $x \in P - C$ define the interval $I(x) = (x_i, x_j)$.

Suppose there are two intervals $I(y)$ and $I(z)$ such that $I(y) \cap I(z) \neq \emptyset$, $I(y) \not\subseteq I(z)$ and $I(z) \not\subseteq I(y)$. We may assume that $I(y) = (x_{i_1}, x_{i_3})$ and $I(z) = (x_{i_2}, x_{i_4})$ where $x_{i_1} < x_{i_2} < x_{i_3} < x_{i_4}$ in the chain C (see Figure 5.3). Since C is the left boundary,

y and z lie to the right of C . Since P is planar, there is $w \in P - C$ such that $w > x_{i_2}, w > y, w < x_{i_3}$ and $w < z$. Since $x_{i_2} < w < x_{i_3}$, there is $x_{i_j} \in C$ such that $x_{i_3} > x_{i_j} > x_{i_2}$ and $w \parallel x_{i_j}$. Since $I(z) = (x_{i_2}, x_{i_4})$ and $I(y) = (x_{i_1}, x_{i_3})$, $I(w) = (x_{i_2}, x_{i_3}), z \not\asymp x_{i_j}$ and $x_{i_j} \parallel z$.

Therefore, the set $\{x_{i_j}, x_{i_3}, w, z\}$ forms an N which contradicts the fact that P is series-parallel. Hence the set of intervals ordered by inclusion is a forest.

Now we show that there are no edges between two different components. Let a be an element in a component A , $I(a) = (x_i, x_j)$ and let b be an element in a component $B \neq A$, $I(b) = (x_{i'}, x_{j'})$. Suppose that $a \succ b$ in P . There are three cases to consider.

Case 1 $I(a) \subset I(b)$

Thus, $x_{i'} \leq x_i < x_j \leq x_{j'}$. Since $x_{j'} > a > b$ and $I(b) = (x_{i'}, x_{j'})$, $x_{j'} = x_j$ (see Figure 5.4(a)). Suppose $x_{i+1} \succ x_i$ in the left boundary chain.

As $I(a) = (x_i, x_j), a \not\asymp x_{i+1}$. Since $x_{i+1} \succ x_i$ and $a > x_i, x_{i+1} \not\asymp a$. Thus, $x_{i+1} \parallel a$ in P .

Since $I(b) = (x_{i'}, x_{j'}), x_{i+1} \not\asymp b$. Also, $b \not\asymp x_{i+1}$ because $I(a) = (x_i, x_j)$ and $a > b$. Thus $x_{i+1} \parallel b$ in P .

Since $I(b) = (x_{i'}, x_{j'}), b \parallel x_i$. Therefore, the set $\{b, a, x_{i+1}, x_i\}$ forms N, which contradicts the fact that P is series-parallel.

Case 2 $I(b) \subset I(a)$

Thus, $x_i \leq x_{i'} < x_{j'} \leq x_j$. Since $a > b > x_{i'} \geq x_i$ and $I(a) = (x_i, x_j), x_{i'} = x_i$ (see Figure 5.4(b)). Suppose $x_{j'} \succ x_{j'-1}$ in the left boundary chain. By an analogous argument to the one in the Case 1 we can show that $\{x_{j'-1}, x_{j'}, b, a\}$ forms an N, which contradicts the fact that P is series-parallel.

Case 3 $I(a) \cap I(b) = \emptyset$

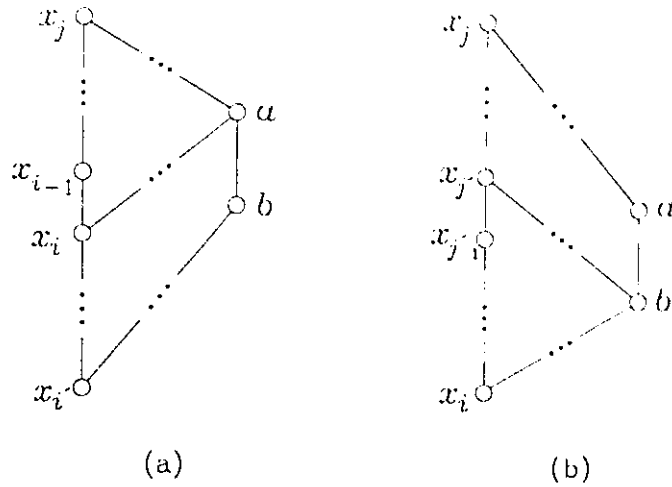


Figure 5.4:

Since $a > b, x_i \geq x_j$. On the other hand, $a > x_i \geq x_j > b$ implies (b, a) is a nonessential edge. Thus, this case cannot happen. \square

Theorem 5.2.1 *If P is a series-parallel planar lattice, then $page(P) \leq 2$.*

Proof. Fix a planar embedding of P , and let $C = \{x_1 < x_2 < \dots < x_n\}$ be the left boundary chain.

If the series-parallel planar lattice P is of height one, then P is a single edge and $page(P) = 1$.

Now we will construct the linear extension of P , component by component, using induction. Our induction hypothesis is

for each series-parallel planar lattice of size less than $|P|$ and height more than one, there is a two-page linear extension in which the edges from the top lie on the left page and the edges to the bottom lie on the right page.

For the basis step, if P is a series-parallel planar lattice of height two, then P is just an antichain with \top and \perp (i.e, $P \cong (\perp \oplus (1 + 1 + \dots + 1) \oplus \top)$) and for any

linear extension of P draw the edges from the top in the left page and the edges to the bottom in the right page.

For the general case, we construct our linear extension step by step:

1. Start from the elements of the left boundary chain. Draw the minimum edge on the right page and all other edges on the left page.
2. Since the set of intervals we obtained according to Lemma 5.2.1, when ordered by inclusion forms a forest, we choose a leaf interval (x_{i-p}, x_i) , $p \geq 2$ and let A be the component corresponding to this interval. Thus, $Q = A \cup \{x_{i-p}, x_i\}$ is a planar series-parallel lattice and $|Q| < |P|$ (because $p \geq 2$). By the induction hypothesis there is a two-page linear extension of Q in which the edges from x_i to the maximals of A lie on the left page and the edges from the minimals of A to x_{i-p} lie on the right page. Thus, we can insert the linear extension of A between x_{i-1} and x_i and still preserve the two-page linear extension.
3. Remove the interval corresponding to the component A from the set of intervals, then go to step 2.

Notice that, if the interval corresponding to B covers the interval corresponding to A (with respect to the inclusion) in the forest of intervals, and B shares the same upper connection x_j , then there exists a place between x_j and the top of A in the linear extension in which we can insert the linear extension of B , such that the edges from the minimals of B to the lower connection of the interval lie on the right page and the edges from x_j to the maximals of B lie on the left page. ■

Recall that the complexity of planarity testing of an n -element lattice is $O(n)$ [39, 29]. For an n -element planar lattice with a fixed planar embedding we need at most $n - 1$ comparisons to find the intervals for an element not on the left boundary. Thus, the the complexity of finding all components is $O(n^2)$. However, we shall prove that

there is an $O(n^3)$ algorithm to draw a series-parallel planar lattice on two pages. We do not presently know whether this is the best possible.

Corollary 5.2.1 *For any maximal chain C in a series-parallel planar lattice P , there is a planar upward drawing of P in which C is the left boundary.*

Proof. Fix a planar upward drawing of P and let C be a maximal chain. Although we define the interval for an element of P with respect to the left boundary chain, we can generalize it to any maximal chain.

For each $x \in P - C$ define the interval for x in C . We can use the same argument in Lemma 5.2.1 to show that the set of the components with respect to C ordered by inclusion forms a forest and there are no edges joining elements in different components. Thus, we can move all the components which lie to the left of C to the right of C in the planar upward drawing of P . We do this, one by one, starting from the component corresponding to the maximal interval in the forest. This has no effect on the planarity of P . ▀

5.3 Structure of Series-Parallel Planar Lattices

In this section we will study the structure of series-parallel planar lattices. We use this as a tool in the construction of a two-page algorithm.

For a series-parallel planar lattice P , fix a planar upward drawing of P , and define the sequence of *peels* of P as follows:

$$L_0 = \{x \in P : x \text{ belongs to the left boundary } \}.$$

Inductively, for $i \geq 1$

$$L_i = \{x \in P - (\bigcup_0^{i-1} L_j) : \text{if } y \text{ lies to the left of } x, \text{ then } y \in \bigcup_0^{i-1} L_j \}.$$

We call any L_i a *peel* of P . Notice that the left boundary of $P - L_0$ may not even exist, in the sense that $P - L_0$ (even with top and bottom) need not be a lattice (here

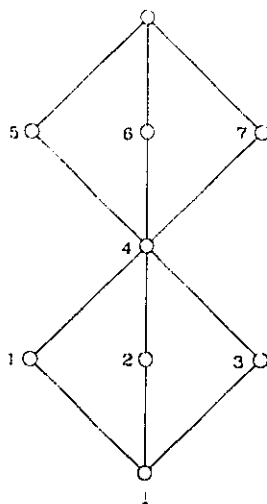


Figure 5.5:

$L_0 = \{\perp, 1, 4, 5, \top\}$ (see Figure 5.5).

Actually, the peels of a planar lattice P are the levels of P_λ , where P_λ is the underlying set P ordered by λ . (Recall $x\lambda y$ in the planar lattice if x lies to the left of y in P .)

If P has $t + 1$ peels $\{L_i : 0 \leq i \leq t\}$, then

$$L_i = \min(P_\lambda - (\bigcup_{j=0}^{i-1} L_j))$$

Of course, t is equal to the height of P_λ . Thus,

$$\text{height}(P_\lambda) = (\text{number of peels of } P) - 1.$$

For example, in the series-parallel ordered set P with respect to the upward drawing shown in Figure 5.1

$$L_0 = \{1, 2, 3, 4, 5\}$$

$$L_1 = \{6, 7, 8, 9\}$$

$$L_2 = \{10, 11\}$$

$$L_3 = \{12\}$$

$$L_4 = \{13\}$$

$$L_5 = \{14, 15, 16\}$$

$$L_6 = \{17\}$$

Lemma 5.3.1 *Let P be a series-parallel planar lattice with peels $\{L_i : i = 0, \dots, t\}$.*

If $0 \leq i \leq t$, then:

1. *for any $x \in L_i, i > 0$, there exists $y \in L_{i-1}$ such that x lies to the right of y ;*
2. *the peel L_i forms a chain;*
3. *the number of peels equals $\text{width}(P)$.*

Proof.

1. Since P_λ is an ordered set and the peels are the levels of P_λ , there is $y \in L_{i-1}$ such that $y \lambda x$.
2. Notice that, P_λ is a complement of P . Since each peel L_i is an antichain in P_λ , L_i forms a chain in P .
3. Since P_λ is the complement of P ,
 $\text{width}(P) = \text{height}(P_\lambda) + 1$. Thus, the number of peels in $P = \text{height}(P_\lambda) + 1 = \text{width}(P)$. ◻

Call a chain C in P *saturated* if all of its covering relations are covering relations in P . Each chain decomposes into its (maximal) saturated chains.

In a series-parallel planar lattice P each peel L_i can be decomposed into maximal saturated chains $C_{i1}, C_{i2}, \dots, C_{in_i}$ called the *clamped chains* for P .

For a clamped chain $C_{ij} \in L_i, i \geq 1$ define:

$$l(C_{ij}) = \{y \in \bigcup_0^{i-1} L_j : \text{inf}(C_{ij}) \succ y\}$$

$$u(C_{ij}) = \{y \in \bigcup_0^{i-1} L_j : \text{sup}(C_{ij}) \prec y\}$$

For example the table below shows the clamped chains in the series-parallel planar lattice in Figure 5.1.

Clamped chain C_{ij}	$l(C_{ij})$	$u(C_{ij})$
$C_0 = \{1, 2, 3, 4, 5\}$	–	–
$C_{11} = \{6\}$	1	3
$C_{12} = \{7, 8, 9\}$	3	5
$C_{21} = \{10, 11\}$	7	9
$C_{31} = \{12\}$	7	9
$C_{41} = \{13\}$	7	9
$C_{51} = \{14, 15, 16\}$	1	5
$C_{61} = \{17\}$	14	16

Lemma 5.3.2 *Let C_{ij} be a clamped chain in a series-parallel planar lattice P .*

1. *Each $l(C_{ij})$ and $u(C_{ij})$ is unique.*
2. *Each $x \in C_{ij} - \{\text{inf}(C_{ij}), \text{sup}(C_{ij})\}$ has neither lower covers nor upper covers in $L_0 \cup L_1 \cup \dots \cup L_{i-1}$. Also, if $\text{inf}(C_{ij}) \neq \text{sup}(C_{ij})$ then $\text{inf}(C_{ij})$ (respectively, $\text{sup}(C_{ij})$) has no upper (respectively, lower) covers in $L_0 \cup L_1 \cup \dots \cup L_{i-1}$.*
3. *If $u(C_{ij}) \in C_{km}$, then $l(C_{ij}) \in C_{km} \cup \{l(C_{km})\}$ and dually if $l(C_{ij}) \in C_{km}$, then $u(C_{ij}) \in C_{km} \cup \{u(C_{km})\}$.*
4. *$\text{inf}(C_{ij})$ (respectively, $\text{sup}(C_{ij})$) has a unique lower (respectively, upper) cover in P .*

Proof.

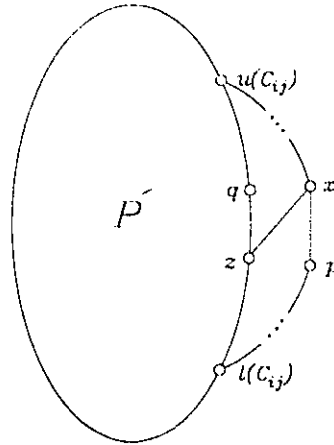


Figure 5.6:

1. Notice, that C_0 is a chain, also $C_0 \cup C_{11}$ has a right chain boundary and inductively $P' = C_0 \cup C_{11} \cup \dots \cup C_{1m_1} \cup \dots \cup C_{i1} \cup \dots \cup C_{ij-1}$ has a right chain boundary. Since P' is planar and contains the top and the bottom of P , there exists a unique element $u(C_{ij})$ in P' which covers $\text{sup}(C_{ij})$. Similarly, $l(C_{ij})$ is unique, too.
2. Suppose that there exists $x \in C_{ij} - \{\text{inf}(C_{ij}), \text{sup}(C_{ij})\}$ such that $x \succ z$ and $z \in L_0 \cup L_1 \cup \dots \cup L_{i-1}$. Planarity of P implies that z belongs to the right boundary chain of $L_0 \cup L_1 \cup \dots \cup L_{i-1}$. Since $x \succ z > l(C_{ij})$, there exists $p \in C_{ij}$ such that $x \succ p \geq \text{inf}(C_{ij})$. Similarly, since $u(C_{ij}) > x \succ z$, there is q which belongs to the right boundary chain $L_0 \cup L_1 \cup \dots \cup L_{i-1}$, such that $u(C_{ij}) > q \succ z$ (see Figure 5.6).

Since q and x are upper covers of z , $q \parallel x$. Also z and p are lower covers of x , $z \parallel p$. By planarity of P , $q \not\parallel p$. Thus, $q \parallel p$.

Therefore, the set $\{q, x, z, p\}$ forms an N , which contradicts the fact that P is series-parallel.

Dually, if $x \in C_{ij} - \{\text{sup}(C_{ij})\}$ has an upper bound, then P is not series-parallel.

Also, we can use a similar argument to show that $\text{inf}(C_{ij})$ (respectively, $\text{sup}(C_{ij})$)

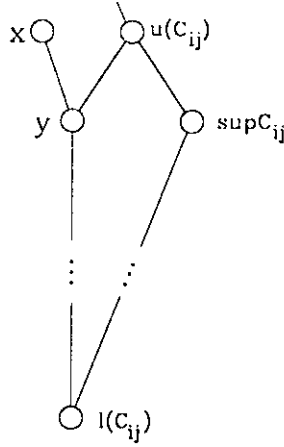


Figure 5.7:

has no lower (respectively, upper) covers in $L_0 \cup L_1 \cup \dots \cup L_{i-1}$ if $\inf(C_{ij}) \neq \sup(C_{ij})$.

3. Suppose that $u(C_{ij}) \in C_{km}$ and $l(C_{ij}) \in C_{np}$, where $C_{km} \neq C_{np}$. Since C_{km} and C_{np} are not both C_0 , we may assume that $C_{km} \neq C_0$.

Let $y = l(C_{km})$. By part 2 of this Lemma, $\sup(C_{ij}) \not\prec y$. If $\sup(C_{ij}) > y$, then there is $s \in L_0 \cup L_1 \cup \dots \cup L_{i-1}$ such that $\sup(C_{ij}) \succ s > y$ which contradicts part 2 of this Lemma. Also, $y \not\prec \sup(C_{ij})$ because $u(C_{ij}) \succ \sup(C_{ij})$. Thus, $y \parallel \sup(C_{ij})$.

Since $y = l(C_{km})$ and $km \neq 0$, there exist $x \succ y$ and x belongs to the right boundary chain of $L_0 \cup L_1 \cup \dots \cup L_{h-1}$ (see Figure 5.7).

By part 2 of this Lemma, $u(C_{ij}) \not\prec x$. Also since $x \succ y$ and $u(C_{ij}) > y$, $x \not\prec u(C_{ij})$. Thus, $x \parallel u(C_{ij})$.

If $x > \sup(C_{ij})$, then by part 2 of this Lemma, $x \succ u(C_{ij})$ contradicts the fact that $x \succ y$. Also, $\sup(C_{ij}) \not\prec x$ because $\sup(C_{ij}) \not\prec y$. Thus, $x \parallel \sup(C_{ij})$.

Therefore, the set $\{x, y, u(C_{ij}), \sup(C_{ij})\}$ forms an N, which contradicts the fact

that P is series-parallel.

4. Suppose that $(y, \inf(C_{ij}))$ is an edge in P and $y \neq l(C_{ij})$. Since $l(C_{ij})$ and y are lower covers of $\inf(C_{ij})$, $y \parallel l(C_{ij})$. Thus, $l(C_{ij})$ lies to the left of y in P . Let C_{km} be the clamped chain containing y . Thus, $y = \sup(C_{km})$ and $\inf(C_{ij}) = u(C_{km})$.

Therefore, according to part 3 of this Lemma, $l(C_{km}) = l(C_{ij})$, which contradicts the fact that $y \parallel l(C_{ij})$. ▀

5.4 Two-Page Algorithm

In this section we will give an $O(n^3)$ two-page algorithm for a series-parallel planar lattice P , where n is the number of elements of P .

- (i) Fix a planar¹ upward drawing for P .
- (ii) List the clamped chains of P in the following order

$$C_0, C_{11}, C_{12}, \dots, C_{1n_1}, C_{21}, C_{22}, \dots, C_{2n_2}, \dots, C_{w1}, C_{w2}, \dots, C_{wn_w}$$

We will process chain by chain according to the above order. (Within a given peel the sequence in which the clamped chains is considered is arbitrary.)

- (iii) Put C_0 on the spine of the book. Draw the bottom edge on the right page and draw all other edges on the left page.
- (iv) Suppose two pages are enough up to $C_{i'j'}$ and C_{ij} is the next in the list from (ii). For C_{ij} put all the elements of C_{ij} right below $u(C_{ij})$. Draw the edge $(\inf(C_{ij}), l(C_{ij}))$ on the right page and draw all C_{ij} edges and the edge $(u(C_{ij}), \sup(C_{ij}))$ on the left page.

We call this algorithm the *two-page algorithm*.

¹There is a linear time algorithm to produce an upward drawing of a planar lattice [2].

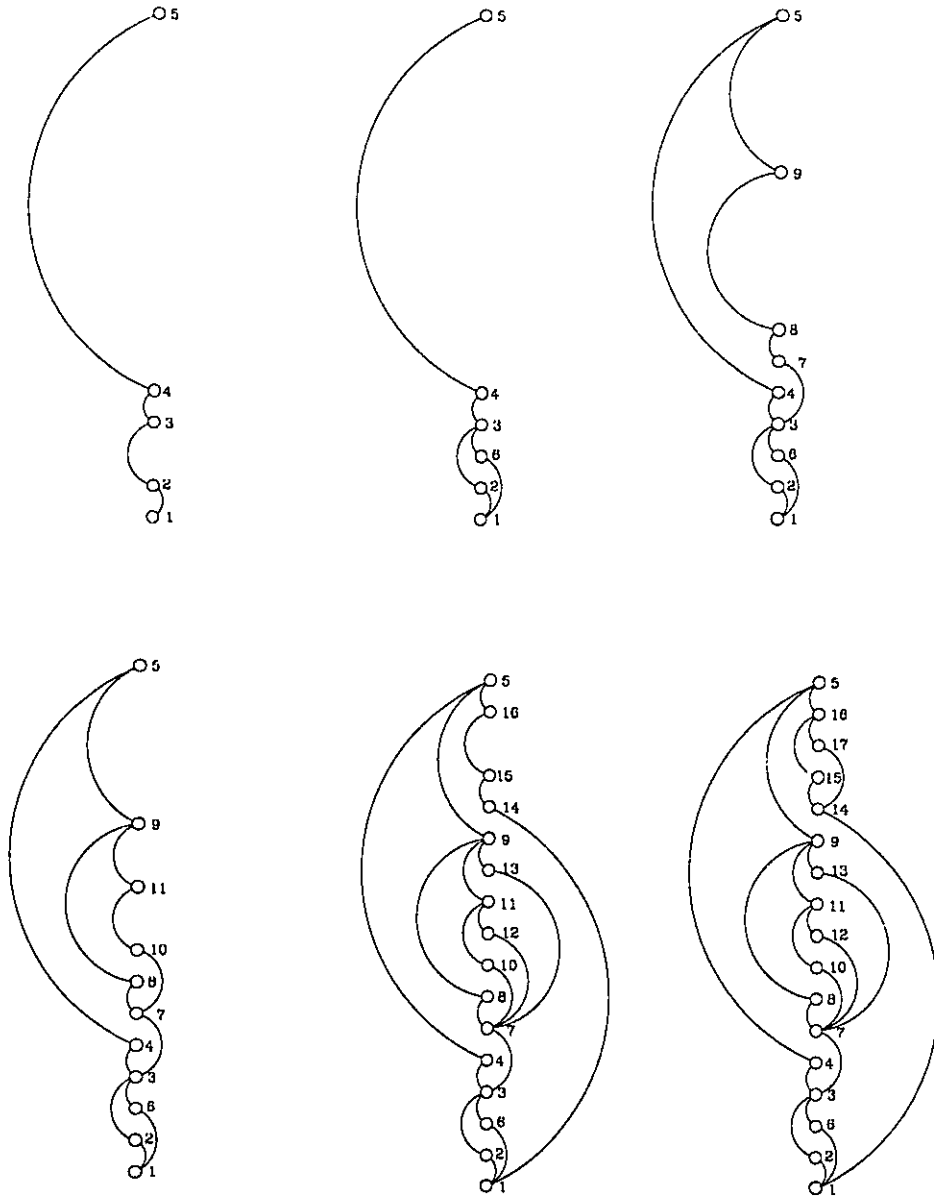


Figure 5.8:

In Figure 5.8 we see the steps of the two-page algorithm applied to the series-parallel planar lattice P in Figure 5.1.

Lemma 5.4.1 *Let P be a series-parallel planar lattice. If L is the permutation obtained by the two-page algorithm, then*

1. L is a linear extension of P ,
2. if $x \parallel y$ in P , and y lies to the left of x , then $y < x$ in L (i.e., L is the left greedy linear extension).

Proof.

1. We will show that after each step in the algorithm we have a linear extension for what we processed.

First C_0 is a chain.

Suppose that the layout we obtained by processing $C_0, C_{11}, C_{12}, \dots, C_{1u_1}, \dots, C_{i1}, C_{i2}, \dots, C_{ij-1}$ is a linear extension for $C_0 \cup C_{11} \cup \dots \cup C_{ij-1}$. We want to prove that adding C_{ij} does not violate the order of P . By part 2 of Lemma 5.3.2, every element in $C_{ij} - \{sup(C_{ij}), inf(C_{ij})\}$ is doubly irreducible in $L_0 \cup L_1 \cup \dots \cup L_{i-1} \cup C_{i1} \cup C_{i2} \cup \dots \cup C_{ij}$ and every comparability of C_{ij} is induced by $sup(C_{ij}) < u(C_{ij})$ and $inf(C_{ij}) > l(C_{ij})$. In the layout we have $u(C_{ij}) > sup(C_{ij}) > inf(C_{ij}) > l(C_{ij})$. Thus, L is a linear extension.

2. Suppose x belongs to the peel L_i and $y \in L_j, i < j$. We will prove that $x < y$ in L by induction on the index of the peel.

Since x lies to the left of $y, i < j$. Let y belong to the clamped chain C_{jp} . As $x \parallel y$, so $x \not> u(C_{jp})$ and $l(C_{jp}) \not> x$.

If $x < u(C_{jp})$ in P , then $x < y < u(C_{jp})$ in L . Else, $x \parallel u(C_{jp})$. Since x lies to the left of $u(C_{jp})$ and the index of the peel of x and $u(C_{jp})$ are less than j , by

induction hypothesis $x < u(C_{jp})$ in L . According to our algorithm x and $u(C_{jp})$ appear in L before y . Therefore, $x < y < u(C_{jp})$ in L . \blacksquare

Corollary 5.4.1 *If P is series-parallel planar lattice, then*

$$\text{jump}(P) = \text{the number of clamped chains} - 1.$$

Proof. Let L be the linear extension obtained by the two-page algorithm for P . According to Lemma 5.4.1, L is a greedy linear extension. Thus, $\text{jump}(P)$ equals the number of jumps in L (cf. page 40).

We will count the number of jumps in L inductively on the number of steps in the two-page algorithm. At the first step when we start with the clamped chain C_0 there is no jump. After adding C_{11} there is only a jump between $\text{inf}(C_{11})$ and the element covered by $u(C_{11})$ in C_0 . For the same reason adding a clamped chain will increase the number of jumps by exactly one. Thus,

$$\begin{aligned} \text{jump}(P, L) &= \text{number of clamped chains} - 1 \\ &= \text{jump}(P). \quad \blacksquare \end{aligned}$$

For example, $\text{jump}(P) = 7$ for the series-parallel planar lattice P in Figure 5.2 and P has eight clamped chains.

Notice that, a greedy (not necessarily left greedy) linear extension of a series-parallel planar lattice may need more than two pages. For example, the linear extension L of the series-parallel planar lattice P in Figure 5.9 needs at least three pages.

Corollary 5.4.2 *The number of clamped chains for a series-parallel planar lattice is independent of the planar embedding.*

Proof. The jump number of an ordered set is independent of its drawing. \blacksquare

Lemma 5.4.2 *If $a < b < c$ in a greedy linear extension L of an ordered set P and $a < c$ in P , then either $a < b$ or $b < c$ in P .*

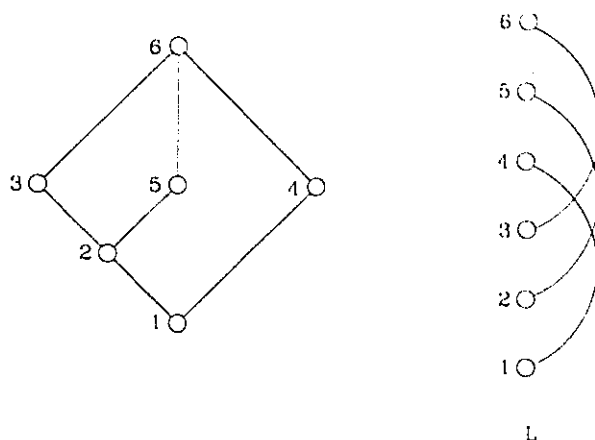


Figure 5.9:

Proof. Suppose that $a \parallel b$ and $b \parallel c$. Thus, there are a' and b' in P such that $c \geq c' > b > a' \geq a$ in L and $c \geq c' \succ a' \geq a$ in P where $a' \parallel b$ and $b \parallel c'$. Therefore, L is not greedy. ▣

Theorem 5.4.1 *The two-page algorithm for an n -element series-parallel planar lattice produces a two-page linear extension in $O(n^3)$ time.*

Proof.

First, from Lemma 5.4.1 the layout is a linear extension. Suppose the algorithm has correctly produced a two-page layout for $C_0, C_{11}, C_{12}, \dots, C_{1n_1}, \dots, C_{i1}, C_{i2}, \dots, C_{ij-1}$. We will show that we can add C_{ij} to the layout and distribute the edges on the two pages such that there are no crossing edges on the same page.

1. Drawing the edge $(sup(C_{ij}), u(C_{ij}))$ on the left page does not create any problem because $u(C_{ij}) \succ sup(C_{ij})$ in the layout.
2. Drawing the edges of C_{ij} on the left page does not create any problem because up to this stage of the algorithm no element not in C_{ij} separates the chain C_{ij} in L .

3. Drawing the edge $(l(C_{ij}), \inf(C_{ij}))$ may create a problem in one of the following two cases.

Case 1 There is an edge (x, y) such that $\inf(C_{ij}) > x > l(C_{ij})$ in the linear extension and the edge (x, y) lies on the right page where $y > u(C_{ij})$ in the linear extension.

If $y \not\prec \sup(C_{ij})$ in P , then $y \parallel \sup(C_{ij})$. Thus, by part 2 of Lemma 5.4.1, y lies to the right of $u(C_{ij})$, which contradicts the fact that we follow a left to right order in processing the clamped chains. Thus, $y > u(C_{ij})$ in P .

Since (x, y) is an edge and $y > u(C_{ij})$, $u(C_{ij}) \not\prec x$. Thus, $x \parallel u(C_{ij})$. Let y belong to the clamped chain C_{hp} . Since (x, y) lies on the right page, $y = \inf(C_{hp})$ and $x = l(C_{hp})$. Since $x \parallel u(C_{ij})$, y has at least two lower covers in P . But $y = \inf(C_{hp})$ which contradicts part 4 of Lemma 5.3.2.

Case 2 There is x and y such $l(C_{ij}) < x < \inf(C_{ij})$ in the linear extension and there is an edge (y, x) on the right page where $y < l(C_{ij})$ in the linear extension.

Since $l(C_{ij}) < x < \inf(C_{ij})$ in L and $l(C_{ij}) < \inf(C_{ij})$ in P , then by Lemma 5.4.2, either $l(C_{ij}) < x$ or $x < \inf(C_{ij})$ in P . Since $\inf(C_{ij})$ has only one lower cover in P , $\inf(C_{ij}) \not\prec x$. Thus, $x > l(C_{ij})$ in P .

Since (y, x) is an edge in P and $x > l(C_{ij})$, $y \parallel l(C_{ij})$. Thus, x has at least two lower covers. Let C_{hp} be the clamped chain containing x . Since the edge (y, x) lies on the right page, $x = \inf(C_{hp})$. That contradicts part 4 of Lemma 5.3.2.

For the complexity, we can find the peel C_0 by checking for each $x \in P$ if there is $y \in P - \{x\}$ such that $y \parallel x$ and y lies to the left of x . Thus, we need at most n^3 comparison operations to obtain the peels of P . To obtain the clamped chains of a certain peel we need first to sort it in $O(n \log n)$ comparisons, then determine the

covering relations in this peel and that can be done in $O(n-1)$ comparisons. Therefore, we can find all clamped chains in $O(n^2 \log n)$ comparisons. For each clamped chain C_{ij} we can find $u(C_{ij})$ by finding the element in $\cup_{k=0}^{i-1} L_k$ which covers $\text{sup}(C_{ij})$ and this can be done in $O(n)$ comparisons. Thus, we can find $u(C_{ij})$ and $l(C_{ij})$ for all clamped chains C_{ij} in $O(n^2)$ comparisons. For the distribution of the edges among the two pages we process each edge just one time; thus, we can decide the page for each edge in $O(n^2)$ comparisons. Therefore, the whole algorithm can be done in $O(n^3)$ comparisons. \blacksquare

Chapter 6

Series-Parallel Planar Ordered Sets

6.1 Introduction

In this chapter we use the fact that there is a two-page embedding for a series-parallel planar lattice to show that there is a two-page embedding for a series-parallel planar ordered set. The idea is to transfer the two-page linear extension of the completion of a series-parallel ordered set to a two-page linear extension of the ordered set itself.

One consequence of this result is a characterization of series-parallel planar ordered sets.

6.2 The Completion

The completion of a planar ordered set P may not be planar even if P contains a top and bottom. For instance, the ordered set P and its completion \bar{P} illustrated in Figure 6.1; \bar{P} is not planar while P is a planar.

Nevertheless, we show that the completion of a series-parallel ordered set is a series-parallel planar lattice.

Lemma 6.2.1 *If the ordered set P has the cycle $\{a \prec c \succ b \prec d \succ a\}$ such that there are u and v in P satisfy $v < a, b$, and $u > c, d$ (see Figure 6.2(a)), then P is not planar.*

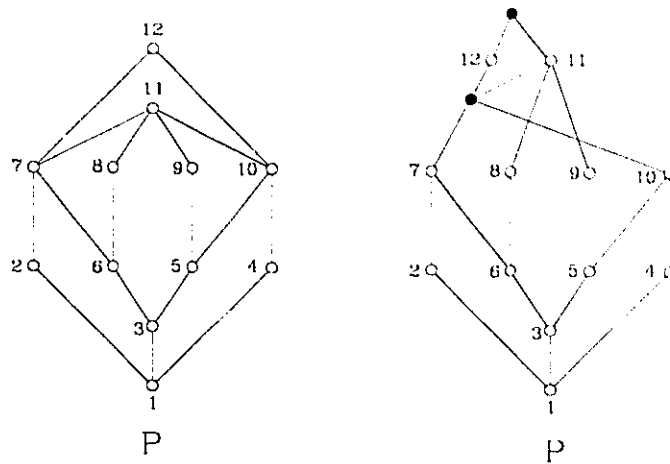


Figure 6.1:

Notice that, although c, d both cover a and b , u need not cover c, d nor v need not be covered by a, b .

Proof. We may assume that v is a maximal element of the set $\{x \in P : x < a, b\}$ and u is a minimal element of the set $\{x \in P : x > c, d\}$.

We can draw the cycle $T = \{v < a < c > b > v\}$ in a planar way in one way up to symmetry as it appears in Figure 6.2(b). Let C and D be chains from c to u and from d to u , respectively.

Since $u > c$, the y -coordinate of u is greater than the y -coordinate of c and u must be outside T . Now, either d is inside or is outside T .

If d is inside T , then, since the y -coordinate of d must be greater than the maximum y -coordinate of a and b , the chain D will intersect either the edge (a, c) or the edge (b, c) which violates planarity.

If d is outside T , then, since $u > d > b$, d must lie to the right of the chain $\{b\} \cup C$. Hence, the chain D or the edge (a, d) will intersect either the chain C or the edge (b, c) , which violates planarity.

Thus, P is not planar. ◻

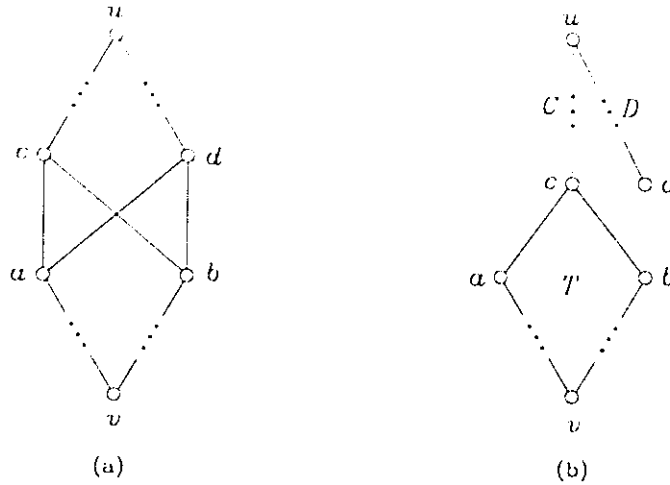


Figure 6.2:

Notice that, the above Lemma is still correct if instead of $\{a < c > b < d > a\}$, we have $\{a < c > b < d > a\}$ such that the chains from a to c , from a to d , from b to c and from b to d are pairwise disjoint, except for common ends from a, b, c, d .

Lemma 6.2.2 *If P is a series-parallel ordered set and \bar{P} its completion, then \bar{P} is a series-parallel planar lattice.*

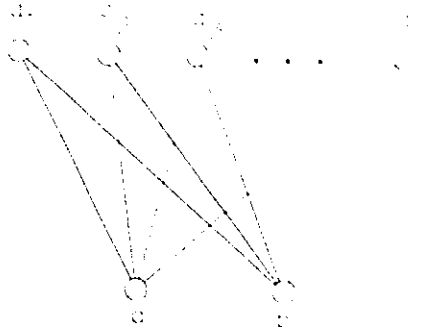
Proof. We first show that \bar{P} is series-parallel. Suppose \bar{P} contains $N = \{a < c > b < d\}$. Since $\{a, d\}$ is an antichain, by Lemma 2.2.1 there exists a join irreducible element a' satisfying $a' \leq a$ and $a' \not\leq d$. Also, $\{a, b\}$ is an antichain, so by Lemma 2.2.1 there exists a join irreducible element b' satisfying $b' \leq b$ and $b' \not\leq a$.

Similarly, $\{c, d\}$ is an antichain, so by Lemma 2.2.1, there is a meet irreducible c' satisfying $c' \geq c$ and $c' \not\geq d$. Also, $\{a', d\}$ is an antichain, so by Lemma 2.2.1 there is a meet irreducible element d' satisfying $d' \geq d$ and $d' \not\geq a'$.

Since $b' \not\leq a$, $b' \not\leq a'$. Also, $b' \not\geq a'$ because $a' \not\leq d$. Thus, $a' \parallel b'$.

Since $d \not\geq c'$, $c' \not\geq d'$. Also, $d' \not\geq c'$ because $d' \not\leq a'$. Thus, $c' \parallel d'$.

Therefore, the set $\{a', b', c', d'\}$ forms an N. By Lemma 2.4.7, $\{a', b', c', d'\} \subseteq P$ which contradicts the fact that P is a series-parallel ordered set.

Figure 6.3: $K_{2,m}$

Now we show \bar{P} is planar. Since P is series-parallel, by the first part of this Lemma, \bar{P} is series-parallel. According to Lemma 1.6.1 the order dimension of \bar{P} is at most two. Therefore, by Lemma 2.3.1, \bar{P} is a planar lattice. \blacksquare

In Lemma 6.2.2 we proved that if the completion of an ordered set P contains N , then P contains N too. Actually, this is just one case of the complete characterization of separable subsets of a finite lattice given by **Duffus** and **Rival** (cf. [14]).

6.3 Neither $K_{2,3}$ nor $K_{3,2}$

In this section we will show that there is a two-page linear extension for any series-parallel ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$ ¹.

Although this result is a special case of the result in the next section, we include it here as motivation and to develop machinery that we shall use in the general case.

A four-cycle $\{a < c > b < d > a\}$ in an ordered set P is a *covering four-cycle* if we have $\{a < c > b < d > a\}$ in P . Notice that a covering four-cycle has no splitting element.

Lemma 6.3.1 *If a series-parallel ordered set P has a four-cycle without a splitting element, then P contains a covering four-cycle $\{a < c > b < d > a\}$.*

¹Here, $K_{m,n}$ is an ordered set (cf. page 111).

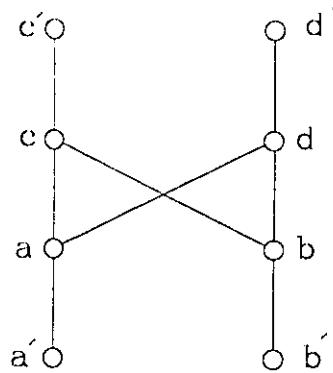


Figure 6.4:

Proof. We say the four-cycle $C = \{a < c > b < d > a\}$ without splitting element is less than the four-cycle $C' = \{a' < c' > b' < d' > a'\}$ without splitting element if $c' \geq c, d' \geq d, a' \leq a$ and $b' \leq b$. We denote this relation by \leq . It is an order relation over all four-cycles without splitting elements. For instance, $\{a, b, c, d\} \leq \{a', b', c', d'\}$ in the eight-element ordered set illustrated in Figure 6.4.

Let $C = \{a < c > b < d > a\}$ be a minimal four-cycle without splitting element. Since there is no splitting element, $a \parallel b$ and $c \parallel d$. Suppose that $c \succ x > a$ in P . We have two cases

Case 1 $d \not\succeq x$.

Since $c \parallel d$, $x \not\succeq d$. Thus, $d \parallel x$. Also, $b \not\succeq x$ because $a \parallel b$. If $x > b$, then $D = \{a < x > b < d > a\}$ is another four-cycle. D has no splitting element because C has none. Also, $D \leq C$ contradicts the minimality of C . Thus, $b \not\succeq x$ and hence $b \parallel x$. Therefore, the set $\{x, c, b, d\}$ forms an N which contradicts the fact that P is a series-parallel ordered set.

Case 2 $d > x$.

Since the four-cycle C has no splitting element, $x \not\succeq b$. Also, since $a \parallel b$, $b \not\succeq x$.

Thus, $x \parallel b$. Therefore, we have another four-cycle $D = \{x < d > b < c > x\}$ without splitting element and $D \leq C$, which contradicts the fact that C is a minimal four-cycle.

Therefore, all the relations in C are covering relations. □

Lemma 6.3.2 *If $C = \{a < c > b < d > a\}$ is a covering four-cycle in a series-parallel ordered set P , then the set of upper covers of a is equal to the set of upper covers of b . Dually, the set of lower covers of c is equal to the set of lower covers of d .*

Proof. Suppose that $x \succ a$. Since x, c and d are upper covers of a , $x \parallel c$, $x \parallel d$ and $c \parallel d$. Also, $a \parallel b$ because a and b are lower covers of c . Since $a \parallel b$, $b \not\succeq x$. If $x \not\succeq b$, then the set $\{x, a, c, b\}$ forms an N which contradicts the fact that P is a series-parallel ordered set. Thus, $x > b$.

We want to show that $x \succ b$: Suppose that $x \succ y > b$. Since $x \parallel c$ and $x \succ y$, $y \not\succeq c$. Also, $c \not\succeq y$ because $c \succ b$ and $y > b$. Thus, $y \parallel c$. Also, $a \parallel y$ because they are lower covers of x . Therefore, the set $\{y, x, a, c\}$ forms an N which contradicts the fact that P is a series-parallel ordered set. Hence, $x \succ b$.

By the same argument we can show the set of lower covers of c is equal to the set of lower covers of d . □

Lemma 6.3.3 *If a series-parallel planar ordered set P contains a covering four-cycle $C = \{a < c > b < d > a\}$ and an edge (t, y) in P such that $b > t$ and $a \not\succeq t$, then $y < c$ and $y < d$ in P .*

Proof. If we prove that $y < c$ then according to Lemma 6.3.2, $y < d$.

Suppose that, $y \not< c$. Since $y \succ t$ and $c > t$, $y \not\succeq c$. Thus, $c \parallel y$.

Since $a \parallel b$ and $b > t$, $t \not\succeq a$. Thus, $a \parallel t$.

If $y > a$, then by Lemma 6.3.2, $y > b$ which contradicts the fact that $b > t$ and $t < y$. Hence, $y \not> a$. Also, $a \not\succeq y$ because $c > a$ and $c \not\succeq y$. Thus, $a \parallel y$.

Therefore, the set $\{y, t, c, a\}$ forms an N which contradicts the fact that P is series-parallel ordered set. Thus, $y < c$. ■

By duality,

Lemma 6.3.4 *If a series-parallel planar ordered set P contains a covering four-cycle $C = \{a \prec c \succ b \prec d \succ a\}$ and an edge (y, t) in P such that $t > c$ and $t \not\succeq d$, then $y > a$ and $y > b$ in P .*

Lemma 6.3.5 *If \bar{L} is a greedy linear extension of the completion \bar{P} of an ordered set P , then the linear order L obtained from \bar{L} by removing the elements of $\bar{P} - P$ is a greedy linear extension of P .*

Proof. If L is not a linear extension of P then there are a and b in P such that $a < b$ in L where $a > b$ in P . This implies $a < b$ in \bar{L} and $a > b$ in \bar{P} . That contradicts the fact that \bar{L} is a linear extension of \bar{P} .

If L is not a greedy linear extension, then there are a, b and c in P such that $a < b < c$ in L where $a < c, a \parallel b$ and $b \parallel c$ in P . This implies $a < b < c$ in \bar{L} where $a < c, a \parallel b$ and $b \parallel c$ in \bar{P} . That contradicts the fact that \bar{L} is a greedy linear extension of \bar{P} . ■

We say the ordered set P contains $K_{m,n}$, $m, n \geq 2$ if it contains a subset $\{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ satisfying $a_i \prec b_j$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Also, we write $K_{m,n} = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ (see Figure 6.3).

Notice that, if P contains $K_{m,n}$, $m, n \geq 2$ then, the sets $\{a_1, a_2, \dots, a_m\}$ and $\{b_1, b_2, \dots, b_n\}$ are antichains.

We say $K_{m,n} = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$ is maximal in P if there is neither $a_{m+1} \neq a_i, 1 \leq i \leq m$ satisfying $a_{m+1} \prec b_j$ for every $1 \leq j \leq n$, nor $b_{n+1} \neq b_j, 1 \leq$

$j \leq n$ satisfying $a_i \prec b_{n+1}$ for every $1 \leq i \leq m$. If a planar ordered set contains $K_{m,n}$, $m, n \geq 2$, then either $m = 2$ or $n = 2$.

Theorem 6.3.1 *If a series-parallel planar ordered set P contains neither $K_{2,3}$ nor $K_{3,2}$, then $\text{page}(P) \leq 2$.*

Proof. Let \overline{P} be the completion of P . By Lemma 6.2.2, \overline{P} is a series-parallel planar lattice. Thus, by Theorem 5.4.1, there exists a two-page linear extension \overline{L} of \overline{P} . We will transfer it to a two-page linear extension L for P .

According to Lemma 6.3.1 and Theorem 2.2.1, if P contains no covering four-cycles, then the two-page linear extension \overline{L} is a two-page linear extension L after the deletion of the top or the bottom, whichever is not in P .

Let $C = \{a \prec c \succ b \prec d \succ a\}$ be a covering four-cycle in P . Since P contains neither $K_{2,3}$ nor $K_{3,2}$, both a and b have exactly two upper covers c and d . Also, c and d have exactly two lower covers a and b . Thus, if $x \in \overline{P}$ is the splitting element of the cycle C in \overline{P} , then x in \overline{P} has exactly two lower covers a and b and exactly two upper covers c and d .

Suppose that in our planar upward drawing of \overline{P} a lies to the left of b and c lies to the left of d . According to the two-page algorithm we should have $a < b < x < c < d$ in \overline{L} and the edges distributed as follows:

- The edges (a, x) , (b, x) and (x, c) lie on the left page (see Figure 6.5).
- The edge (x, d) lies on the right page.

If $\{a, b\}$ has a lower bound in P and $\{c, d\}$ has an upper bound in P , then by Lemma 6.2.1, P is not planar.

To obtain a two-page linear extension L of P from \overline{L}

- Remove the set $\overline{P} - P$ from \overline{L} and all edges connected to them.

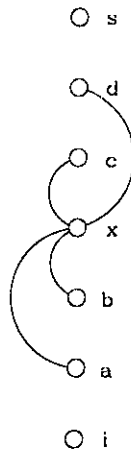


Figure 6.5:

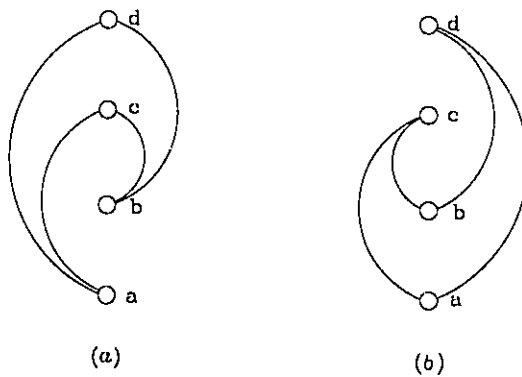


Figure 6.6:

For each covering four-cycle C

- If $\{a, b\}$ has a lower bound in P draw the edges $(a, c), (a, d)$ on the left page and the edges $(b, c), (b, d)$ on the right page (see Figure 6.6(a)).
- If $\{c, d\}$ has an upper bound in P draw the edges $(a, c), (b, c)$ on the left page and the edges $(a, d), (b, d)$ on the right page (see Figure 6.6(b)).

Notice that, by Lemma 6.3.5, L is a greedy linear extension of P .

We will now show that adding the edges of the covering four-cycles C does not create edge crossing on the same page. We shall use the following three facts.

Fact 1 $c \succ b$ in L .

Proof. Suppose there is t in P such that $b < t < c$ in L . Since $b < c$ in P and L is greedy, by Lemma 5.4.2, either $b < t$ or $t < c$ in P . Since b has only two upper covers c, d and $t < c < d$ in L , $t \not\succeq b$ in P . Also, $c \not\succeq t$ in P because c has only two lower covers a, b and $t > b > a$ in L . Thus, we have always $c \succ b$ in L .

Fact 2 If there is t in P such that $a < t < b$ in L , then $t < b$ in P .

Proof. Since $a < t < c$ in L , $a < c$ in P and L is greedy linear extension, then by Lemma 5.4.2, either $t > a$ or $c > t$ in P .

Since the set of the upper covers of a in P is $\{c, d\}$ and $t < c < d$ in L , $t \not\succeq a$ in P . Thus, $t < c$ in P .

Since the set of the lower covers of c in P is $\{a, b\}$ and $t > a$ in L , $t < b$ in P .

Fact 3 If there is t in P such that $c < t < d$ in L , then $t > c$ in P .

Proof. Dual of Fact 2.

The edges of C may intersect one of the edges of P in one of the following cases.

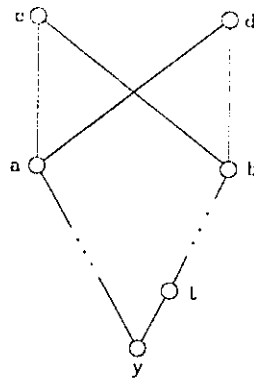


Figure 6.7:

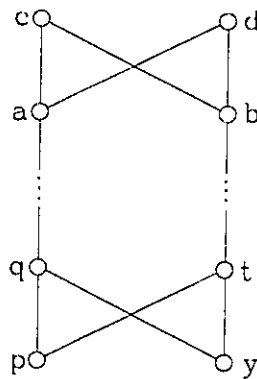


Figure 6.8:

Case 1 There are y and t in P such that $y < a < t < b$ in L and (y, t) is an edge in P .

By Fact 2, $t < b$. Since $y < a < t$ in L and $y < t$ in P , by Lemma 5.4.2, either $y < a$ or $a < t$ in P . Since $a \parallel b$, $t \not> a$ in P . Thus, $a > y$ in P (see Figure 6.7). Therefore, $\{a, b\}$ has a lower bound in P .

If (y, t) is an edge in \bar{P} , then (y, t) lies on the right page because (b, x) was in the left page. Thus, there are crossing edges in \bar{P} .

If (y, t) is not an edge in \bar{P} , then there is a covering four-cycle $D = \{y < q > p < t > y\}$. Since $a > y$ and the set of upper covers of y is $\{p, t\}$, either $a > t$ or

$a > p$ in P . Since $t > a$ in L , $a > p$ in P (see Figure 6.8). Hence, the set $\{p, t\}$ has a lower bound in P . Since $t > p$ in \bar{L} , p lies to the left of t in \bar{P} . Therefore, the edge (y, t) according to our drawing of the edges of D is on the right page and there is no crossing.

Case 2 There are t and y in P such that $a < t < b < c < y < d$.

From Fact 2, $b > t$ and $y > c$ from Fact 3, $y > c > b > t$ in P . Therefore, (t, y) is not an edge in P .

Case 3 There are t and y in P such that $a < t < b < d < y$ in L and (t, y) is an edge in P .

Thus, we have in P :

- covering four-cycle $C = \{a \prec c \succ b \prec d \succ a\}$ and the edge (t, y) .
- $b > t$ (from Fact 1).
- $t \not\prec a, y \not\prec c$ and $y \not\prec d$ because we have $a < t < b < d < y$ in L .

That contradicts Lemma 6.3.3. Thus, this situation cannot happen.

Case 4 (The dual of case 3) There are y and t in P such that $y < a < c < t < d$ in L and (y, t) is an edge in P .

Thus, we have in P :

- covering four-cycle $C = \{a \prec c \succ b \prec d \succ a\}$ and the edge (y, t) .
- $t > c$ (from Fact 3).
- $t \not\prec d, y \not\prec a$ and $y \not\prec b$ because we have $y < a < c < t < d$ in L .

That contradicts Lemma 6.3.4. Thus, this situation cannot happen.

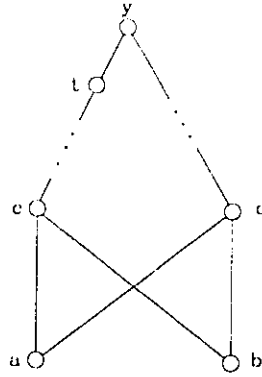


Figure 6.9:

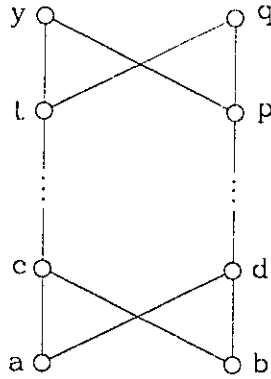


Figure 6.10:

Case 5 There are t and y in P such that (t, y) is an edge in P and $c < t < d < y$ in L .

From Fact 3, $t > c$ in P . Since $t < d < y$ in L and $d \not> t$ in P , by Lemma 5.4.2, $y > d$ in P . Thus, the set $\{c, d\}$ has an upper bound in P (see Figure 6.9).

◁ If the edge $(t, y) \in \overline{P}$, then the edge (t, y) will be in the left page because the edge (x, d) was in the right page. Thus, there is no edge crossing in this case.

If the edge $(t, y) \notin \overline{P}$, then there is a covering four-cycle $D = \{t \prec y \succ p \prec q \succ t\}$. Since $y > d$ and the only lower covers of y are t and p , either $t > d$ or $p > d$ in p . Since $t < d$ in L , $p > q$ in P (see Figure 6.10). Thus, the set $\{p, t\}$ has a lower

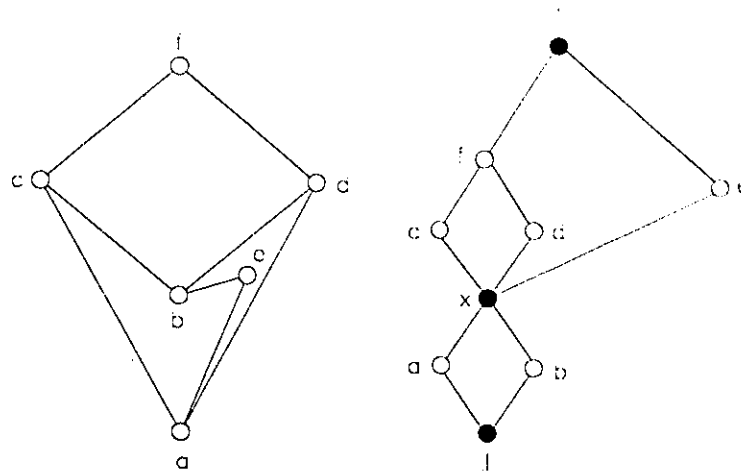


Figure 6.11:

bound in P .

Since $p > t$ in L and $p \parallel t$ in P , t lies to the left of p . Therefore, the edge (t, y) will be in the left page. ▣

6.4 Structure of Series-Parallel Planar Orders

Can we use the same algorithm as used in Theorem 6.3.1, to prove that two pages are enough for any series-parallel planar ordered set?

For example, we consider the series-parallel planar ordered set P and its completion \bar{P} in Figure 6.11. In Figure 6.12 \bar{L} is the two-page linear extension of \bar{P} obtained by the two-page algorithm for series-parallel planar lattice and L is the linear extension obtained from \bar{L} by removing the elements in $\bar{P}-P$.

In Figure 6.13 the intersection graph of the layout graph of $cov(P)$ with respect to L contains a cycle of length five. Thus, according to Theorem 3.3.1 the linear extension L needs at least three pages.

But if we redraw \bar{P} in a different planar embedding as it is in Figure 6.14, then

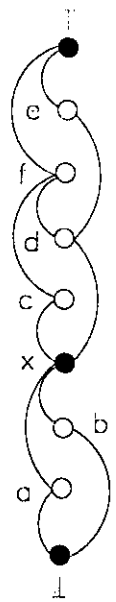


Figure 6.12:

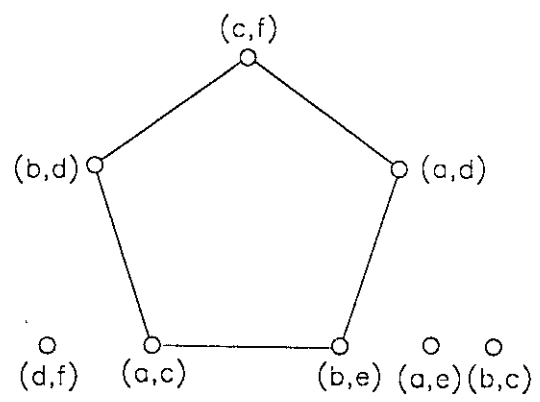


Figure 6.13:

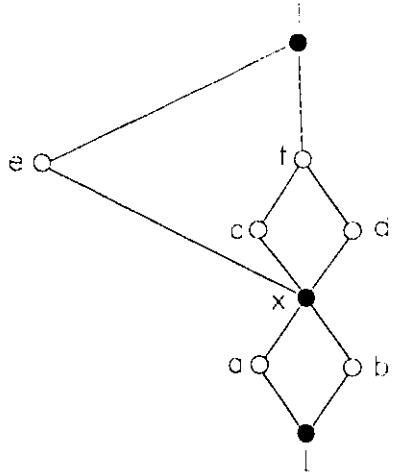


Figure 6.14:

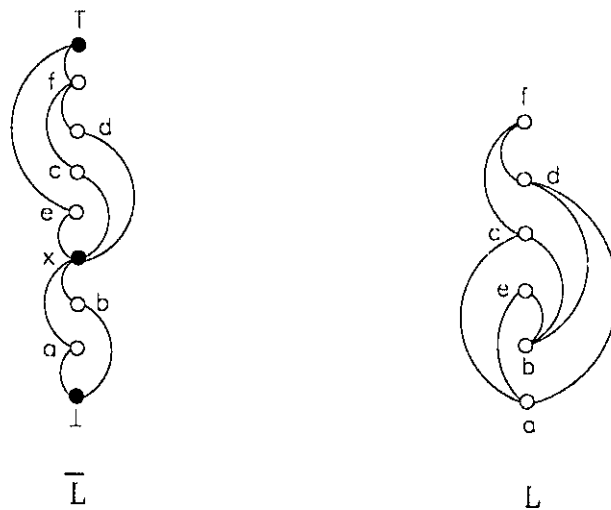


Figure 6.15:

using the two-page algorithm for series-parallel lattices we will obtain the two-page embedding \bar{L} as in Figure 6.15.

In Figure 6.15, we see that the linear extension L of P induced by \bar{L} is a two-page linear extension.

This leads us to the question, whether we can always find a planar embedding of the completion of the series-parallel planar ordered set which can lead finally to a two-page linear extension?

The answer is yes; the details will be in the next section.

Lemma 6.4.1 *If P contains $K_{2,m} = \{a, b, d_1, \dots, d_m\}$, $m \geq 2$, and if P satisfies one of the following conditions, then P is not planar.*

- i) There is an upper bound of some three-element subset of $\{d_1, \dots, d_m\}$.*
- ii) a and b have a common lower bound and some two-element subset of $\{d_1, \dots, d_m\}$ has a common upper bound.*
- iii) There are two different two-element subsets of $\{d_1, \dots, d_m\}$ each of which has an upper bound.*

Proof.

- i) Suppose the set $\{d_1, d_2, d_3\}$ has the upper bound d . We may assume that d is a minimal upper bound. There are three cases to consider.

Case 1 d is minimal upper bound for each two-element subset of $\{d_1, d_2, d_3\}$ (see Figure 6.16)

In this case $\text{cov}(P)$ contains the subset $\{a, b, d_1, d_2, d_3, d\}$ which forms a subdivision of the graph $K_{3,3}$. Thus, $\text{cov}(P)$ is not planar. Therefore, P is not planar.

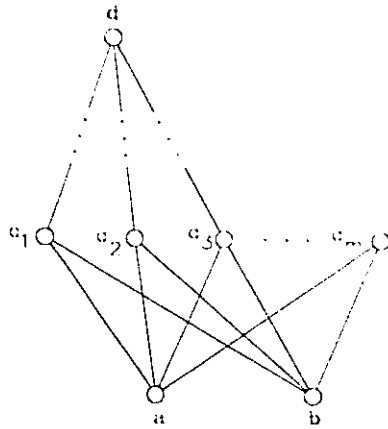


Figure 6.16:

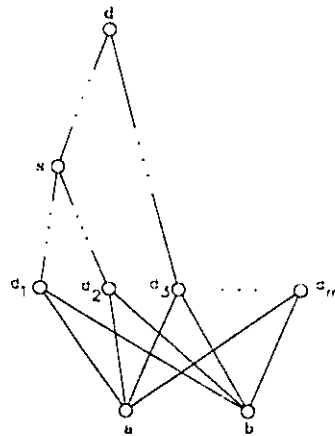


Figure 6.17:

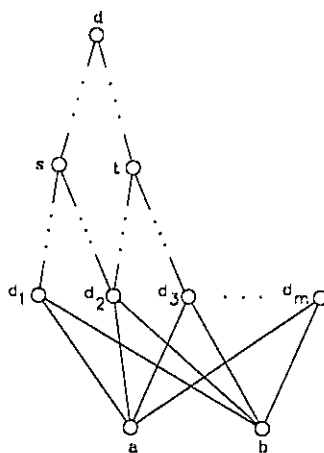


Figure 6.18:

Case 2 d is a minimal upper bound of $\{d_1, d_2, d_3\}$ and s is a minimal upper bound of $\{d_1, d_2\}$ and $s < d$ (See Figure 6.17).

In this case $\text{cov}(P)$ contains the subset $\{a, b, d_1, d_2, d_3, s\}$ which forms a subdivision of the graph $K_{3,3}$. $\text{cov}(P)$ is not planar, thus, P is not planar.

Case 3 d is a minimal upper bound of $\{d_1, d_2, d_3\}$, s and t are minimal upper covers of $\{d_1, d_2\}$ and $\{d_2, d_3\}$, respectively, such that s and t are both less than d (see Figure 6.18).

We can reduce this case to Case 2 by removing the chain from t to d_2 .

- ii) Let the set $\{d_1, d_2\}$ have a minimal upper bound d , and let l be the maximal lower bound of $\{a, b\}$. By Lemma 6.2.1, P is not planar.
- iii) Suppose the subsets $\{d_1, d_2\}$, $\{d_3, d_4\}$ have the minimal upper bounds x, y , respectively, and $x \parallel y$. We will show that there is no planar upward drawing for $\{a, b, d_1, d_2, d_3, d_4, x, y\}$ (see Figure 6.19).

Notice that, for the cycle $R = \{a, d_1, d_2, x\}$ there is (up to symmetry) a unique planar upward drawing. Thus, we may draw R as it is illustrated in Figure 6.20.

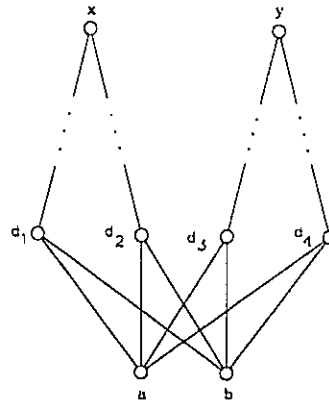


Figure 6.19:

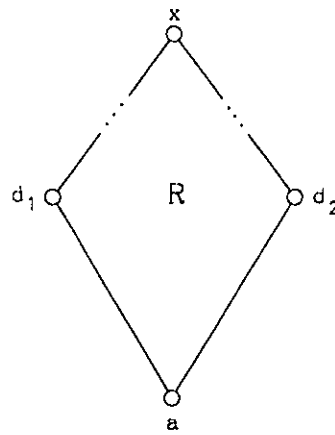


Figure 6.20:

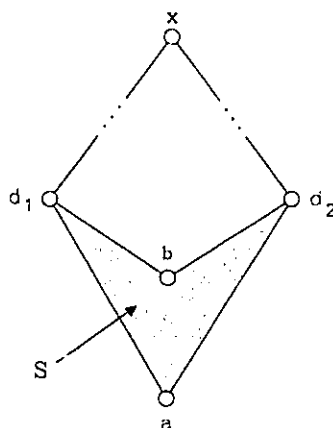


Figure 6.21:

Since b is a lower cover of d_1 and d_2 we can draw it either inside R or outside R . Since a and b are symmetric with respect to d_1 and d_2 we may assume that b is inside R (see Figure 6.20).

Let S be the interior region of the cycle $\{a \prec d_2 \succ b \prec d_1 \succ a\}$. To obtain a planar upward drawing d_3 should lie on S . Hence, we may draw it as it appears in Figure 6.22.

Let T be the interior of the cycle $\{a \prec d_1 \succ b \prec d_3 \succ a\}$. To preserve the planar drawing y should lie on T (see Figure 6.23).

Let C be the chain from y to d_3 . Since $d_4 < y$, d_4 should lie on T . Also, $d_3 \parallel d_4$ and y is a minimal upper bound so d_4 is noncomparable to every element in C . Thus, either d_4 lies to the right or to the left of C .

If d_4 lies to the left of C , then the edge (b, d_4) will intersect the edge (b, d_3) or the chain C . The same problem will happen if d_4 lies to the right of C . Thus, P is not planar.

Now, let us consider the case $d_2 = d_4$ for the two subsets $\{d_1, d_2\}$ and $\{d_3, d_4\}$ such that $\{d_1, d_2\}$ has the upper bound x and $\{d_2, d_3\}$ has the upper bound y .

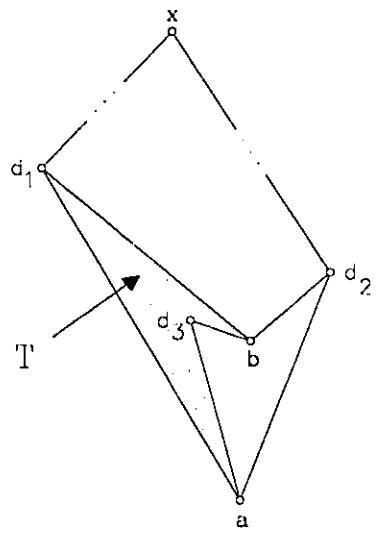


Figure 6.22:

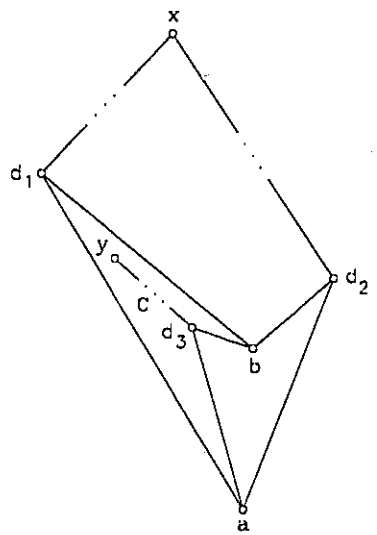


Figure 6.23:

We may assume $x \parallel y$ because if $x > y$, then x is an upper bound of $\{d_1, d_2, d_3\}$ and, by the first part of this lemma, P is not planar.

For this case, we can use a similar argument as used to reach Figure 6.21. If y is inside S then the chain from y to d_2 will intersect the edge (b, d_1) . If y is outside S , then one edge of the chain from y to d_3 will intersect an edge from the cycle $\{a \prec d_2 \succ b \prec d_1 \succ a\}$. Therefore, P is not planar. \square

Dually, we have the following.

Lemma 6.4.2 *If P contains $K_{m,2} = \{d_1, \dots, d_m, a, b\}$, $m \geq 2$, and P satisfies one of the following conditions, then P is not planar.*

- i) There is a lower bound of some three-element subset of $\{d_1, \dots, d_m\}$.*
- ii) a and b have a common upper bound and some two-element subset of $\{d_1, \dots, d_m\}$ has a common lower bound.*
- iii) There are two different two-element subsets of $\{d_1, \dots, d_m\}$ each of which has a lower bound.*

Lemma 6.4.3

- i) If the set $\{a, b, d_1, \dots, d_m\}$ forms a maximal $K_{2,m}$, $m \geq 3$, in a series-parallel planar ordered set P such that the set $\{d_1, d_2\}$ has a minimal upper bound d , then there is a planar upward drawing of the completion lattice \overline{P} of P in which d lies to the right of d_3, d_4, \dots, d_m .*
- ii) If the set $\{d_1, \dots, d_m, a, b\}$ forms a maximal $K_{m,2}$, $m \geq 3$, in a series-parallel planar ordered set P such that the set $\{d_1, d_2\}$ has a minimal lower bound d , then there is a planar upward drawing of the completion lattice \overline{P} of P in which d lies to the left of d_3, d_4, \dots, d_m .*

Proof. By duality, it is enough to prove the first part.

Since P is a series-parallel ordered set, by Lemma 6.2.2, \bar{P} is a series-parallel planar lattice. Since \bar{P} is planar, by the third part of Lemma 6.4.1, $d \parallel d_i$ for $i = 3, \dots, m$.

Fix a planar upward drawing for the lattice \bar{P} and suppose that d_2 lies to right of d_1 in the drawing. Let C be a maximal chain which contains $\{a \prec x \prec d_2 < d\}$, where x is the splitting element of $K_{m,2}$ in \bar{P} .

If there is no $d_j \in \{d_3, d_4, \dots, d_m\}$ which lies to the right of d , then we are done. Else, for each element of $\bar{P} - (C \cup \{y : y \text{ lies to the right of } C\})$ define the interval of it with respect to C as we did in section 5.2 (Lemma 5.2.1). As we have shown there, the set of intervals ordered by inclusion forms a forest and there is no edge between two different components.

Thus, we can move any component which lies to the right of C and contains one element of the set $\{d_3, d_4, \dots, d_m\}$, one component at a time starting with the component with the smallest interval.

Notice that, the interval for $d_1, I(d_1) = (x, y)$ where $y = \sup(\{d_1, d_2\}), y \leq d$ because d is an upper bound in a lattice. ($y = d$ if there is exactly one minimal upper cover of $\{d_1, d_2\}$.)

If $I(d_j) = (x, t), m \geq j \geq 3$, then, $I(d_1) \subset I(d_j)$, else, $d \geq d_j$ contradicts the fact that $d \parallel d_j$ for $i = 3, \dots, m$. Thus, after moving all such components to the left of C we have a new planar upward drawing of \bar{P} such that all $d_j \in \{d_3, d_4, \dots, d_m\}$ lie to the left of d .

Notice that, in the component move we do not change the left relationship of its elements. □

Lemma 6.4.4 *Let P be a series-parallel planar ordered set. Then there is a planar upward drawing of the completion lattice \bar{P} of P in which d lies to the right of d_3, d_4, \dots, d_m and d' lies to the left of e_3, e_4, \dots, e_n for each maximal $K_{2,m} = \{a, b, d_1, \dots, d_m\}, m \geq 3$ and each maximal $K_{n,2} = \{e_1, \dots, e_n, f, g\}, n \geq 3$ such that*

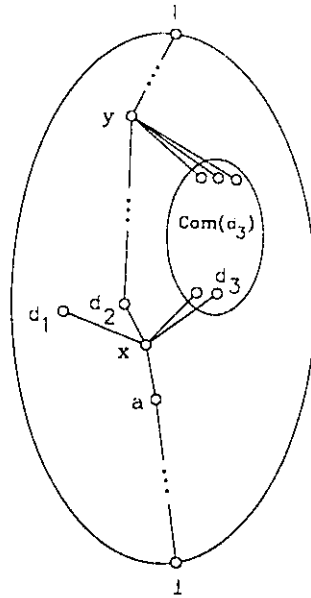


Figure 6.24:

$\{d_1, d_2\}$ has a minimal upper bound d and $\{e_1, e_2\}$ has a maximal lower bound d' ,

Proof. Let $\{a, b, d_1, \dots, d_m\}$ be a maximal $K_{2,m}$ in P such that the set $\{d_1, d_2\}$ has a minimal upper bound d . We may assume that d_i lies to the left of d_j in \bar{P} if $1 \leq i < j \leq m$.

It is enough to show that the component move in the proof of Lemma 6.4.3, neither has any effect on the left relationship of the elements of the maximal $K'_{2,n} = \{a', b', d'_1, \dots, d'_n\}$ nor any effect on the left relationship of the elements of the maximal $K''_{p,2} = \{d''_1, \dots, d''_p, a'', b''\}$, $n, p \geq 2$.

Let x, x' and x'' be the splitting elements of $K_{2,m}, K'_{2,n}$ and $K''_{p,2}$ in \bar{P} , respectively, and suppose that d_3 lies to the right of d in \bar{P} .

Let $Com(d_3)$ be the component containing d_3 in $\bar{P} - (C \cup \{y : y \text{ lies to the left of } C\})$ with respect C , where C is a maximal chain containing $\{a \prec x \prec d_2 \prec d\}$ (see Figure 6.24). As we showed in the proof of Lemma 6.4.3, $I(d_3) = (x, y)$.

If $K'_{2,n} \cap Com(d_3) = \emptyset$ and $K''_{p,2} \cap Com(d_3) = \emptyset$, then the moving of $Com(d_3)$ to the

left of C will have no effect on the left relationship of the elements of $K'_{2,n}$ and $K''_{p,2}$. Moving $Com(d_3)$ may effect this left relation if one of the following four cases happens.

Case 1 $a' \in Com(d_3)$.

Since (a', x') is an edge in \bar{P} , $x' \in Com(d_3) \cup \{y\}$. Since the set of the upper covers of x in \bar{P} is $\{d_1, \dots, d_m\}$, there is $d_j \in Com(d_3), 3 \leq j \leq m$, such that $a' > d_j$.

If $x' = y$, then $x' > d_2$. Since $d'_1 > x'$, $d'_1 > d_2$. Hence, d'_1 is an upper bound of $\{d_2, d_3\}$ in P . Thus, according to the third part of Lemma 6.4.1, P is not planar. Therefore, $x' \in Com(d_3)$.

Since $I(d_3) = (x, y)$ and the set of the upper covers of x is $\{d_1, \dots, d_m\}$, $b' \in Com(d_3)$.

Suppose there is $d'_j \notin Com(d_3), 1 \leq j \leq n$. Since (x', d'_j) is an edge in \bar{P} and $x' \in Com(d_3)$, $d'_j = y$. For any $d'_i \notin Com(d_3), i \neq j$ because $d'_i \parallel d'_j$. Since $d'_i \notin Com(d_3)$ and $d'_i = y$, (x', d'_i) is not an edge in \bar{P} . Thus, $d'_j \in Com(d_3)$ for each $1 \leq j \leq n$.

Therefore, $K'_{2,n} \subset Com(d_3)$ and the moving of $Com(d_3)$ will not effect $K'_{2,n}$.

Case 2 $d'_1 \in Com(d_3)$.

Since d'_1 has a unique lower cover x' and $x' \neq x$, $x' \in Com(d_3)$.

If $d'_i \notin Com(d_3), 2 \leq i \leq n$, then since $x' \in Com(d_3)$ and (x, d_3) is an edge in \bar{P} , $d'_2 = y$. That implies $d'_2 > d'_1$ which contradicts the fact that $d'_1 \parallel d'_2$. Thus, $d'_i \in Com(d_3)$ for each $1 \leq i \leq m$. Also, $a', b' \in Com(d_3)$ because a', b' are the lower covers of x' .

Therefore, $K'_{2,m} \subset Com(d_3)$.

Case 3 $a'' \in Com(d_3)$.

Since (x'', a'') is an edge in \overline{P} and $x'' \neq x$, $x'' \in \text{Com}(d_3)$.

Since (x, x'') is not an edge in \overline{P} , x'' is not a minimal in $\text{Com}(d_3)$. Thus, $d_i'' \in \text{Com}(d_3)$ for each $1 \leq i \leq p$.

If $b'' \notin \text{Com}(d_3)$ and since (x'', b'') is an edge in \overline{P} , $b'' = y$. Thus, $b'' > a''$, which contradicts the fact that $a'' \parallel b''$. Hence, $b'' \in \text{Com}(d_3)$.

Therefore, $K_{p,2}'' \subset \text{Com}(d_3)$.

Case 4 $d_1'' \in \text{Com}(d_3)$.

Since (d_1'', x'') is an edge in \overline{P} , $x'' \in \text{Com}(d_3) \cup \{y\}$. Since the set of the upper covers of x in \overline{P} is $\{d_1, \dots, d_m\}$, there is $d_j \in \text{Com}(d_3)$, $3 \leq j \leq m$, such that $d_1'' > d_j$.

If $x'' = y$, then a'' will be an upper bound of the set $\{d_1, d_2, d_j\}$ in P . Thus, according to Lemma 6.4.1, P is not planar. Therefore, $x'' \in \text{Com}(d_3)$.

Since (x, x'') is not an edge and (d_i'', x'') is an edge in \overline{P} , $d_i'' \in \text{Com}(d_3)$ for each $1 \leq i \leq p$.

If $a'' \notin \text{Com}(d_3)$ and since (x'', a'') is an edge in \overline{P} and $x'' \in \text{Com}(d_3)$ then $a'' = y$. Thus, a'' will be an upper bound of $\{d_1, d_2, d_j\}$ in P . Thus, according to Lemma 6.4.1 P is not planar. Hence, $a'' \in \text{Com}(d_3)$. For the same reason $b'' \in \text{Com}(d_3)$.

Therefore, $K_{p,2}'' \subset \text{Com}(d_3)$. ■

Lemma 6.4.5 *If the ordered set P contains the maximal $K_{2,3} = \{a, b, d_1, d_2, d_3\}$ and the maximal $K_{3,2} = \{d_1', d_2', d_3', a', b'\}$ such that $d_1' > d_1$ and $d_2' > d_2$ (see Figure 6.25), then P is not a series-parallel planar ordered set.*

Proof. Let P be a series-parallel planar ordered set.

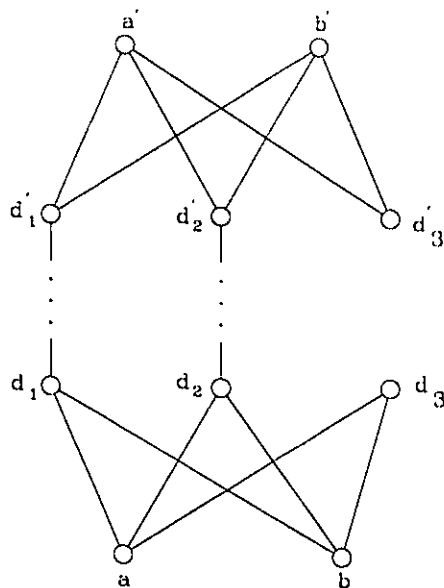


Figure 6.25:

If $d'_3 > d_i$ for some $1 \leq i \leq 3$, then a will be a lower bound of $\{d'_1, d'_2, d'_3\}$. Thus, according to Lemma 6.4.1, $d'_3 \not> d_i$ for any $1 \leq i \leq 3$. If $d_3 > d'_3$, then by Lemma 6.3.2, $d_3 > d'_2$ which contradicts the fact that $d_2 \parallel d_3$. Thus, $d_3 \parallel d'_3$.

If $d'_3 > b$, then b will be a lower bound of the set $\{d'_1, d'_2, d'_3\}$ which contradicts Lemma 6.4.1. Thus, $d'_3 \not> b$. Also, if $b > d'_3$ then according to Lemma 6.3.2 $b > d'_1$ which contradicts that $a \parallel b$. Thus, $d'_3 \parallel b$.

If $b' > d_3$, then b' will be an upper bound of the set $\{d_1, d_2, d_3\}$. Thus, by Lemma 6.4.1, $b' \not> d_3$. Also, $d_3 \not> b'$ because $d_2 \parallel d_3$.

Thus, the set $\{d'_3, b, b', d_3\}$ forms an N which contradicts the fact that P is a series-parallel ordered set. ◻

6.5 Two Pages are Enough

In this section we prove the main results of this chapter. We will first prove that two pages are enough for a series-parallel planar ordered set. Then, as consequence of that, we will give a characterization of series-parallel planar ordered sets.

Theorem 6.5.1 *If P is series-parallel planar ordered set, then $\text{page}(P) \leq 2$.*

Proof. Let \bar{P} be the completion of P . By Lemma 6.2.2, \bar{P} is a series-parallel planar lattice. Fix a planar embedding of \bar{P} satisfying:

- Whenever P contains a maximal $K_{2,m} = \{a, b, d_1, \dots, d_m\}$, $m \geq 3$, such that d is an upper bound of $\{d_{m-1}, d_m\}$, then d lies to the right of $\{d_1, \dots, d_{m-2}\}$.
- Whenever P contains a maximal $K_{m,2} = \{d_1, \dots, d_m, a, b\}$, $m \geq 3$, such that d is a lower bound of $\{d_1, d_2\}$ then d lies to the left of $\{d_3, \dots, d_m\}$.

This is possible according to Lemma 6.4.4.

If P contains either a maximal $K_{2,m}$ or a maximal $K_{m,2}$, $m \geq 2$, we may assume that a lies to the left of b and d_i lies to the left of d_{i+1} for $1 \leq i \leq m-1$, in \bar{P} .

Notice that, if P contains a maximal $K_{2,m}$, then the set of the upper covers of a is $\{d_1, \dots, d_m\}$ which also is the set of the upper covers of b . Also, the set of the lower covers of d_i is $\{a, b\}$ for each $i = 1, \dots, m$.

Similarly, if P contains a maximal $K_{m,2}$, then the set of the upper covers of a is $\{d_1, \dots, d_m\}$ which also is the set of the upper covers of b . Also, the set of the upper covers of d_i is $\{a, b\}$ for each $i = 1, \dots, m$.

Since \bar{P} is a series-parallel parallel planar lattice, by Theorem 5.4.1, there exists a two-page linear extension \bar{L} of \bar{P} . We will transfer it to a two-page linear extension L for P .

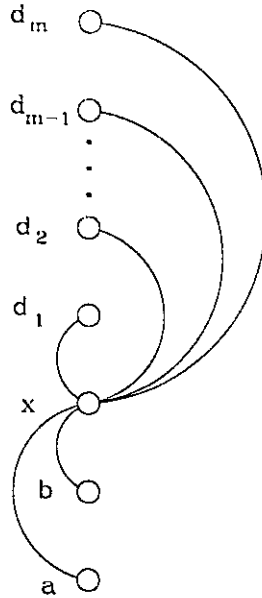


Figure 6.26:

If P contains a maximal $K_{2,m} = \{a, b, d_1, \dots, d_m\}$, $m \geq 2$, such that x is the splitting element of $K_{2,m}$ in \bar{P} , then we have

$$a < b < x < d_1 < d_2 < \dots < d_m \text{ in } \bar{L}$$

with the edges distributed as illustrated in Figure 6.26.

Also, if P has a maximal $K_{m,2} = \{d_1, \dots, d_m, a, b\}$, $m \geq 2$ such that x is the splitting element of $K_{m,2}$ in \bar{P} , then we have

$$d_1 < d_2 < \dots < d_m < x < a < b \text{ in } \bar{L}$$

with the edges distributed as illustrated in Figure 6.27.

Since P is planar, by Lemma 6.4.1 (respectively, Lemma 6.4.2) if $\{a, b\}$ has a lower (respectively, an upper) bound of $K_{2,m}$ (respectively, $K_{m,2}$) in P , then there is no subset of two elements or more of the set $\{d_1, \dots, d_m\}$ which has an upper (respectively, a lower) bound.

To obtain a two-page linear extension L of P from \bar{L}

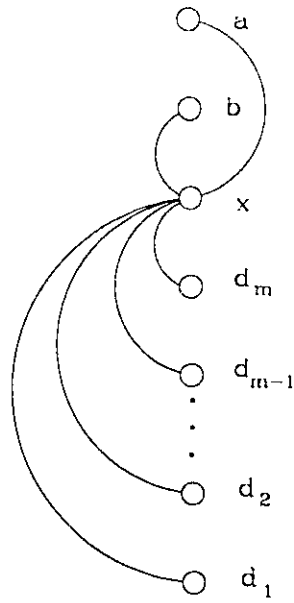


Figure 6.27:

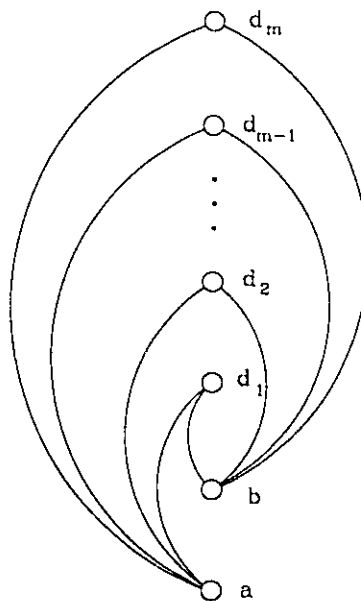


Figure 6.28:

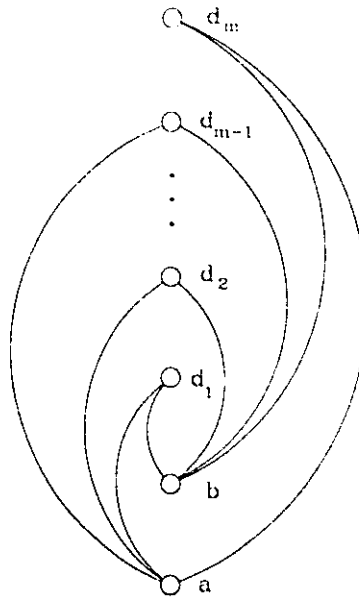


Figure 6.29:

- Remove the set $\bar{P} - P$ from \bar{L} and all edges connected to its vertices.
- For each maximal $K_{2,m} = \{a, b, d_1, \dots, d_m\}$, $m \geq 2$, in P
 - i) If $\{a, b\}$ has a lower bound in P draw the edges (a, d_i) on the left page and the edges (b, d_i) on the right page (see Figure 6.28).
 - ii) If $\{d_{m-1}, d_m\}$ has an upper bound in P draw the edges $\{(b, d_1), (a, d_i) : 1 \leq i \leq m-1\}$ on the left page and draw the edges $\{(a, d_m), (b, d_i) : 2 \leq i \leq m\}$ on the right page for each $1 \leq i \leq m$ (see Figure 6.29).
- For each maximal $K_{m,2} = \{d_1, \dots, d_m, a, b\}$, $m \geq 2$, in P
 - i) If $\{d_1, d_2\}$ has a lower bound in P draw the edges $\{(d_1, b), (d_i, a) : 1 \leq i \leq m-1\}$ on the left page and draw the edges $\{(d_m, a), (d_i, b) : 2 \leq i \leq m\}$ on the right page (see Figure 6.30).

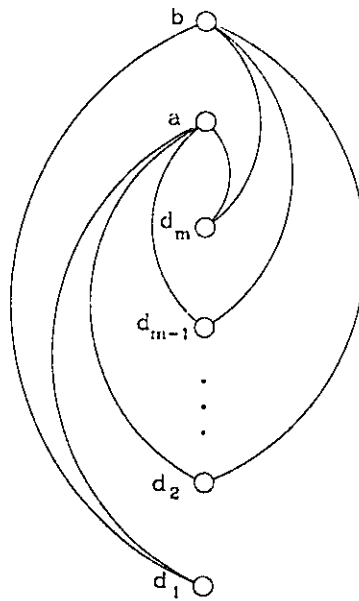


Figure 6.30:

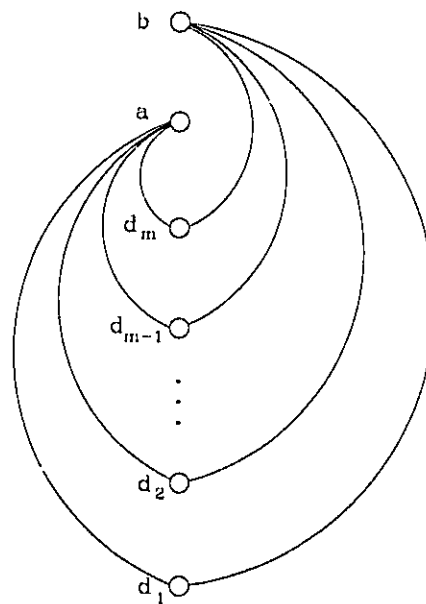


Figure 6.31:

ii) If $\{a, b\}$ has an upper bound in P draw the edges $\{(d_i, a) : 1 \leq i \leq m\}$ on the left page and the edges $\{(d_i, b) : 1 \leq i \leq m\}$ on the right page (see Figure 6.31).

By Lemma 6.3.5, L is greedy linear extension of P . We will show that adding the edges of the maximal $K_{2,m}$'s and $K_{m,2}$'s, $m \geq 2$, will not create a crossing in the same page. We will prove that first for $K_{2,m}$ then for $K_{m,2}$.

Case A P contains a maximal $K_{2,m} = \{a, b, d_1, \dots, d_m\}$, $m \geq 2$.

We shall use these three facts

Fact 1 There is no t in P such that $b < t < d_1$ in L .

Proof: Since $b < d_1$ in P , by Lemma 5.4.2, either $b < t$ or $t < d_1$ in P . If $t < d_1$ and since the set of the lower covers of d_1 is $\{a, b\}$, then either $t < a$ or $t < b$ which contradicts the fact that $t > b > a$ in L . Thus, $t > b$. Since $t > b$ and the set of the upper covers of b is $\{d_1, \dots, d_m\}$, $t > d_i$ in P for some $1 \leq i \leq m$ which contradicts the fact that $t < d_i$ in L for each $1 \leq i \leq m$. Therefore, $b \prec d_1$ in L .

Fact 2 If there is t in P such that $a < t < b$ in L , then $t < b$ in P .

Proof: Since $a < t < d_1$ in L and $a < d_1$ in P , by Lemma 5.4.2, either $a < t$ or $t < d_1$ in P . Since the set of the upper covers of a in P is $\{d_1, \dots, d_m\}$ and $d_i > t$ in L for each $i = 1, \dots, m$, $a \not< t$ in P . Thus, $t < d_1$ in P . Since the set of the lower covers of d_1 is $\{a, b\}$ and $t > a$ in L , $t < b$.

Fact 3 If there is t in P such that $d_i < t < d_{i+1}$ in L , $1 \leq i \leq m-1$, then $d_i < t$.

Proof: Since $b < t < d_{i+1}$ and $b < d_{i+1}$ in P , by Lemma 5.4.2, either $b < t$ or $t < d_i$ in P . Since $d_i < t$ in L , according to Lemma 6.3.2, $t \not< d_{i+1}$ in P . Thus, $t > b$ in P .

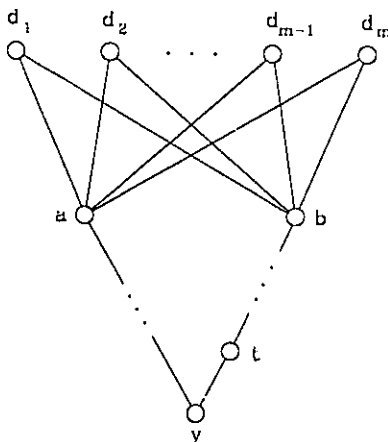


Figure 6.32:

Since the set of the upper covers of b is $\{d_1, \dots, d_m\}$ and $t \not\prec d_{i+1}$ in P , $t > d_j$ in P , $1 \leq j \leq i$. If $d_i \not\prec t$ in P , then we have $d_j < d_i < t$ in L and $d_j < t$ in P where $d_j \parallel d_i$ and $d_i \parallel t$. That contradicts Lemma 5.4.2. Therefore, $d_i < t$.

The edges of the maximal $K_{2,m}$ may intersect one of the edges of P if one of the following cases happen:

Case A.1 There are y and t in P such that $y < a < t < b$ in L and (y, t) is an edge in P .

From Fact 2, $t < b$. Since $y < a < t$ in L and $y < t$ in P , by Lemma 5.4.2, either $y < a$ or $a < t$ in P . If $t > a$ in P , then by Lemma 6.3.2, $t > b$ which contradicts the fact that $b > t$ in L . Thus, $a > y$. Therefore, the set $\{a, b\}$ has a lower bound in P (see Figure 6.32).

If (y, t) is an edge in \bar{P} then (y, t) lies on the right page because (a, x) was on the left page. Thus, there are no edge crossings.

If (y, t) is not an edge in \bar{P} , then P contains a maximal $K_{n,p} = \{y, f_1, \dots, f_{n-1}, g_1, \dots, g_{p-1}, t\}$, $n \geq 2, p \geq 2$.

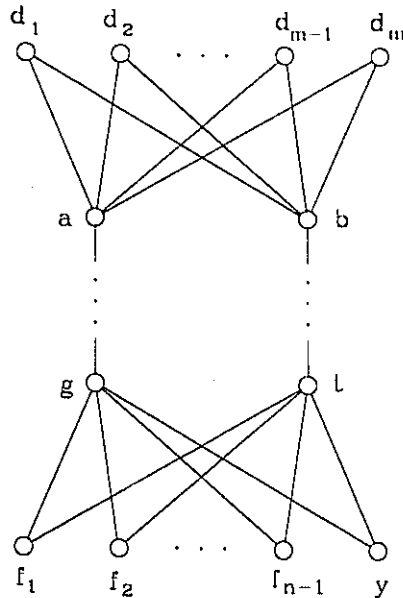


Figure 6.33:

Since P is planar, either $p = 2$ or $n = 2$.

Case A.1.1 $p = 2$.

In this case P contains a maximal $K_{n,2} = \{y, f_1, \dots, f_{n-1}, g, t\}$, $n \geq 2$. Since $a > y$ in P and $a \not> t$, $a > g$ in P (see Figure 6.33). Since a lies to the left of b , g lies to the left of t in \bar{P} . Suppose that f_i lies to the left of f_{i+1} for $1 \leq i \leq n - 2$ and f_{n-1} lies to the left of y in \bar{P} . Thus, we have in L

$$f_1 < \dots < f_{n-1} < y < g < a < t < b < d_1 < \dots < d_m$$

and the edges distributed as illustrated in Figure 6.34.

Thus, the edge (y, t) will be on the right page. Therefore, there are no edge crossings in this case.

Case A.1.2 $n = 2$.

In this case P contains a maximal $K_{p,2} = \{y, f, g_1, \dots, g_{p-1}, t\}$, $p \geq 2$. Since $a > y$ and $a \not> t$, there is $i \in \{1, \dots, p - 1\}$ such that $a > g_i$.

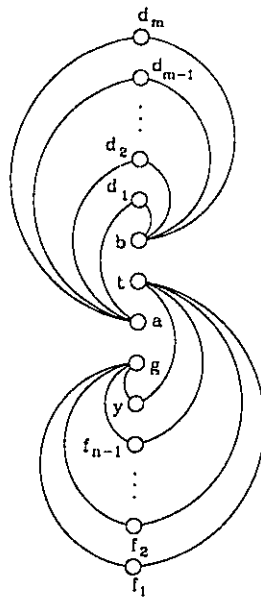


Figure 6.34:

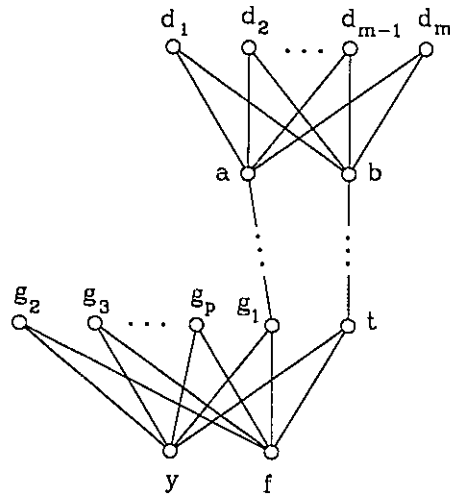


Figure 6.35:

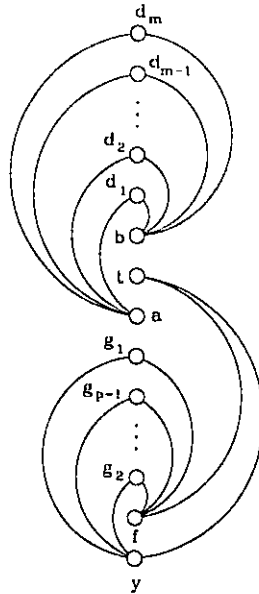


Figure 6.36:

We may assume that $i = 1$ (see Figure 6.35). Thus, the set $\{g, t\}$ has an upper bound d_1 in P . According to Lemma 6.4.1, $g_i \parallel a$ for each $2 \leq i \leq p - 1$. Also, according to Lemma 6.4.4, g_i lies to the left of d_1 for each $2 \leq i \leq p - 1$.

Suppose y lies to the left of f and g_i lies to the left of g_{i+1} if $2 \leq i \leq p - 1$ in \bar{P} . Since a lies to the left of b , so g_1 lies to the left of t in \bar{P} .

Thus, we have in L

$$y < f < g_2 < g_3 < \dots < g_{p-1} < g_1 < a < t < b$$

and the edges are distributed as illustrated in Figure 6.36. Thus, the edge (y, t) will be in the right page. Therefore, there is no edge crossing.

Case A.2 There are t and y in P such that $a < t < b < d_i < y < d_{i+1}$ for $i \in \{1, \dots, m - 1\}$.

From Fact 2, $t < b$ in P and from Fact 3, $y > d_i$ in P . Thus, we have $y > d_i > b > t$ in P . Therefore, there is no edge (t, y) in P .

Case A.3 There are t and y in P such that $a < t < b < d_m < y$ and (t, y) is an edge in P .

From Fact 2, $t < b$. Since $y > d_2 > d_1$ in L , $d_1 \not\prec y$ and $d_2 \not\prec y$ in P . Thus, $a \not\prec t$. Therefore, we have in P

- covering four-cycle $\{a \prec d_1 \succ b \prec d_2 \succ a\}$,
- $t < b$ and (t, y) is an edge in P ,
- $d_1 \not\prec y$, $d_2 \not\prec y$ and $a \not\prec t$ in P .

That contradicts Lemma 6.3.3. Thus, this situation can not happen.

Case A.4 There are y and t in P such that $y < a < d_i < t < d_{i+1}$ in L for $i \in \{1, \dots, m-1\}$ and (y, t) is an edge in P .

From Fact 3, $t > d_i$. Since $y < a < b < d_i < t < d_{i+1}$ in L , then $y \not\prec d_i$, $y \not\prec d_{i+1}$, and $t \not\prec d_{i+1}$ in P . Thus, $y \not\prec a$ and $y \not\prec b$. Therefore, we have in P

- covering four-cycle $\{a \prec d_i \succ b \prec d_{i+1} \succ a\}$,
- $t > d_i$ and (t, y) is an edge in P ,
- $t \not\prec d_{i+1}$, $y \not\prec a$ and $y \not\prec b$ in P .

That contradicts Lemma 6.3.4. Thus, this situation can not happen.

Case A.5 There are y and t in P such that $d_i < y < d_{i+1} < t$ in L for $i \in \{1, \dots, m-1\}$ and (y, t) is an edge in P .

From Fact 3, $y > d_i$ in P . Since $d_i \parallel d_{i+1}$ and $y \not\prec d_{i+1}$ in P , $y \parallel d_{i+1}$.

Since we have $y < d_{i+1} < t$ in L , $y < t$ in P and $y \parallel d_{i+1}$, so by Lemma 5.4.2, $t > d_{i+1}$. Therefore, t is an upper bound of $\{d_i, d_{i+1}\}$. According to Lemma 6.4.1, there is a unique subset of two elements of $\{d_1, \dots, d_m\}$ which has an upper bound in P . Thus, $i = m-1$ and the subset will be $\{d_{m-1}, d_m\}$ (see Figure 6.37).

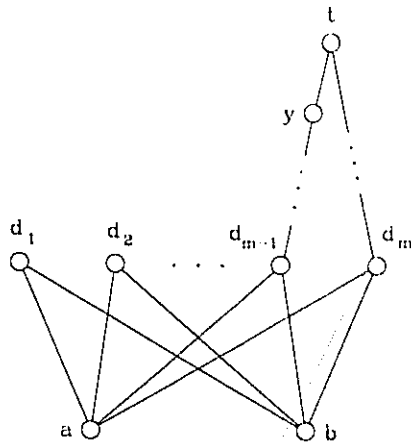


Figure 6.37:

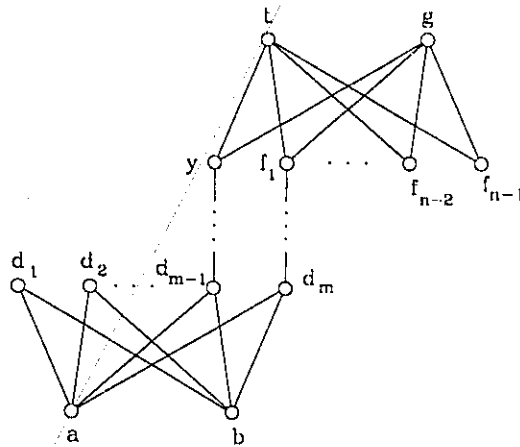


Figure 6.38:

If (y, t) is an edge in \overline{P} , then the edge (y, t) will be on the left page because the edge (x, d_m) was on the right page. Thus, there is no crossing in the same page.

If (y, t) is not an edge in \overline{P} , then there is a maximal $K_{n,p} = \{y, f_1, \dots, f_{n-1}, g_1, \dots, g_{p-1}, t\}$, $n \geq 2$

Since P is planar, so either $p = 2$ or $n = 2$.

Case A.5.1 $p = 2$

In this case P contains a maximal $K_{n,2} = \{y, f_1, \dots, f_{n-1}, g, t\}$, $n \geq 2$.

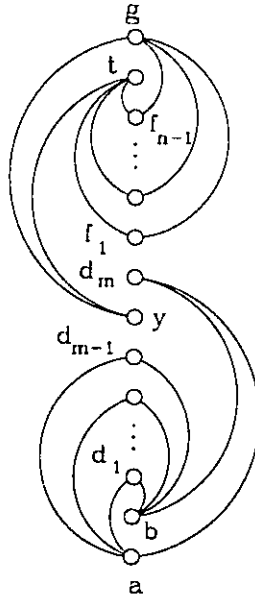


Figure 6.39:

Since $t > d_m$ in P and $y \not> d_m$, there is $i \in \{1, \dots, n-1\}$ such that $f_i > d_m$. We may assume that $i = 1$ (see Figure 6.38).

According to Lemma 6.4.4, b lies to the left of f_i in \overline{P} for each $1 \leq i \leq n-1$. Suppose that f_i lies to left of f_{i+1} if $1 \leq i \leq n-2$ and t lies to the left of g in \overline{P} .

Thus, we have in L

$$a < b < d_1 < \dots < d_{m-1} < y < d_m < f_1 < \dots < f_{n-2} < t < g$$

and the edges are distributed as illustrated in Figure 6.39. Thus, the edge (y, t) lies on the left page. Thus, there are no edge crossings.

Notice that, according to Lemma 6.4.5, $\min\{m, n\} = 2$.

Case A.5.2 $n = 2$

In this case P contains a maximal $K_{p,2} = \{y, f, g_1, \dots, g_{p-1}, t\}$, $p \geq 2$.

Since $t > d_m$ and $y \not> d_m$, $f > d_m$ (see Figure 6.40).

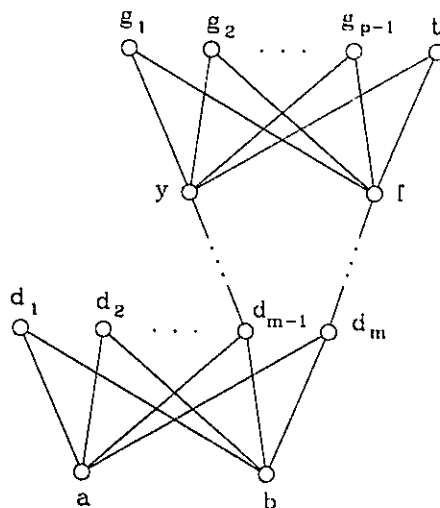


Figure 6.40:

Since d_{m-1} lies to the left of d_m , y lies to the left of f in \overline{P} . Suppose that g_i lies to the left g_{i+1} for $i = 1, \dots, p - 2$ and t lies to the right of g_{p-1} in \overline{P} .

Thus, we have in L

$$a < b < d_1 < \dots < d_{m-1} < y < d_m < f < g_1 < \dots < g_{p-1} < t$$

and the edges are distributed as illustrated in Figure 6.41. Hence, the edge (y, t) lies on the left page. Thus, there are no edge crossings.

Case B P contains a maximal $K_{m,2} = \{d_1, \dots, d_m, a, b\}$, $m \geq 2$. In fact, it is just a dual of case A. ▣

A *simple castle* is a covering four-cycle with the top or bottom. (The top, or bottom, need not be in a cover relation with the covering four-cycle.) (See Figure 6.42) A *castle* is any union of simple castles, which preserves the covering relations of each simple castle. An ordered set P *contains* a castle C if C is a subset of P and P preserves the

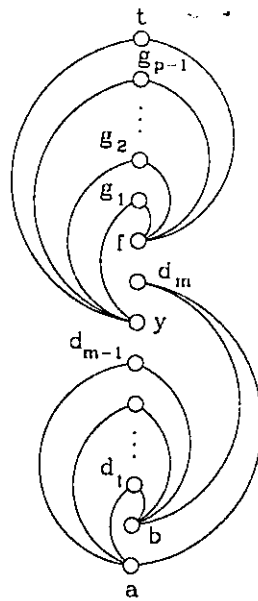


Figure 6.41:

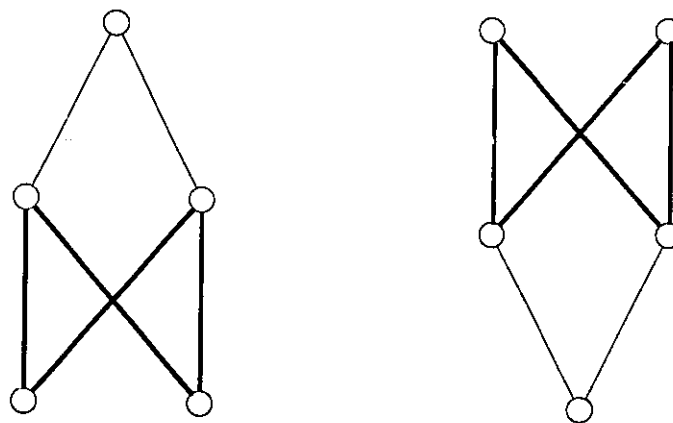


Figure 6.42: Simple castles.

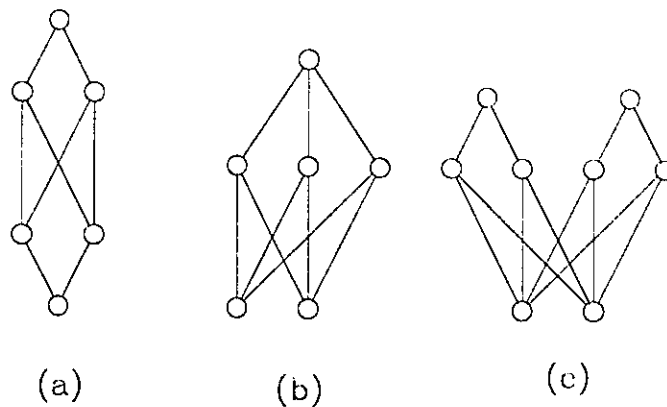


Figure 6.43: Minimal nonplanar castles.

covering relations of its simple castles.(see Figure 6.43)

Theorem 6.5.2 *Let P be a series-parallel ordered set. Then P is planar if and only if P contains no $K_{3,3}$ and P contains no nonplanar castle.*

Proof. Evidently, if P contains a $K_{3,3}$ or a nonplanar castle, then P is not planar.

For sufficiency, suppose P is a series-parallel ordered set that contains neither a $K_{3,3}$ nor a nonplanar castle. Then by Lemma 6.2.2 the completion \bar{P} of P is a series-parallel planar lattice. So by Theorem 5.4.1, \bar{P} has a two-page linear extension \bar{L} .

Notice that the conditions we use in the proof of Lemmas 6.4.1 and 6.4.2 leading up to Theorem 6.5.1 are

- P is a series-parallel,
- P contains no $K_{3,3}$,
- P contains no covering four-cycle with top and bottom(Lemma 6.2.1)(nonplanar castle),

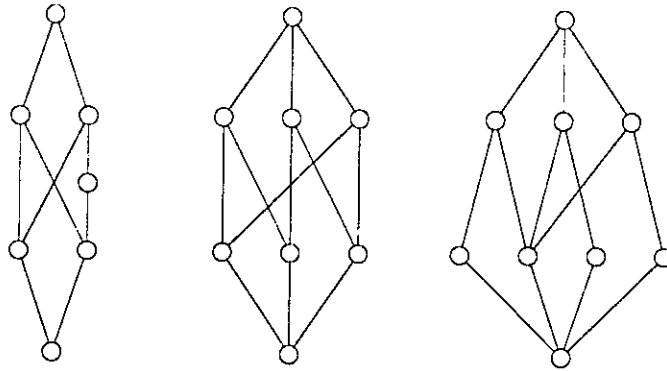


Figure 6.44:

- P contains no $K_{2,3}$ with top or $K_{3,2}$ with bottom (Lemma 6.4.1(i) and Lemma 6.4.2(i)) (nonplanar castle),
- P contains no $K_{2,4} = \{a, b, d_1, d_2, d_3, d_4\}$ such that each of $\{d_1, d_2\}$ and $\{d_3, d_4\}$ has an upper bound (Lemma 6.4.1(iii)) (nonplanar castle),
- Dually, P contains no $K_{4,2} = \{d_1, d_2, d_3, d_4, a, b\}$ such that each of $\{d_1, d_2\}$ and $\{d_3, d_4\}$ has a lower bound (Lemma 6.4.2(iii)) (nonplanar castle).

Since P satisfies the three conditions above, by Theorem 6.5.1, $\text{page}(P) \leq 2$. Therefore, P is planar. ▀

Figure 6.44 illustrates nonplanar ordered sets each of which contains neither $K_{3,3}$ nor a nonplanar castle. In fact, none is series-parallel.

6.6 Open Questions

1. What is the pagenumber for spherical series-parallel ordered sets?

2. What is an upper bound for the pagenumber of (nonplanar) series-parallel ordered set P , depending on the maximal $K_{m,n}$'s in P .
3. For positive integers m, n is there a function $f(m, n)$ such that for any series-parallel ordered set P
 $page(P) \leq \max \{f(m, n) : K_{m,n} \text{ is maximal in } P, m, n \geq 2\}$.

In particular, is there a positive integer k such that

$$f(m, n) \leq \min\{m, n\} + k$$

for every maximal $K_{m,n}$ in P ?

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