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Doctoral Thesis

# Essays on Reciprocity, Institutions, and Political Design


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# Declaration of Authorship

I, Jean Baptiste Tondji, acknowledge the contribution of Dr. Roland Pongou, my supervisor, for the research associated with the first two chapters of this thesis. In both cases, his contribution is equal to my own, although the work in this thesis is in my own words.

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# Abstract

Jean Baptiste Tondji

*Essays on Reciprocity, Institutions, and Political Design*

Chapter 1 of the thesis examines a question of how a voting process can be designed so as to induce rational individuals to display reciprocal and pro-social behaviors. A political procedure, namely, *the reciprocity mechanism*, is proposed to address this issue. The analysis displays that this new mechanism is a modification of the legislative process encountered in democratic countries. The results indicate that, under natural assumptions on voters' preferences, a *stable* policy always exists, and it may be unique if preferences are *single-peaked*. Moreover, any stable policy is Pareto-*efficient*.

It has been argued that the size of the supermajority needed to enact a policy in a decision-making body should depend on the importance of this policy. However, a formal analysis of the relationship between policy importance and the voting rule is still lacking in the literature. Chapter 2 addresses this gap from the perspective of a *preference-blind* political designer. Given the level of *importance* of a policy, the goal is to choose the supermajority rule that guarantees the existence of a stable policy regardless of the extent to which individual political opinions are antagonistic, ensures that all stable policies are efficient, and minimizes status quo bias. Chapter 2 solves this problem. A closed-form relationship between supermajority rule and policy importance is derived. The analysis has practical implications for the optimal design and functioning of political institutions.

The majority rule is widely used to select policies in political institutions. Chapter 3 proves that this rule is not optimal for sufficiently complex policies. To address this issue, natural preference domains are identified for which the majority rule is optimal under a simple sequential procedure. Under this procedure, the majority rule guarantees the existence of a stable policy, ensures that all stable policies are efficient, and minimizes the status quo bias,

no matter the complexity of the policy space. The results imply that this voting rule is not appropriate for certain types of societies, including sufficiently fractionalized societies.

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# General Introduction

The implementation of a public decision in a political economy is often done through several political mechanisms. It is often argued that the pursuit of individual rationality, at the expense of others, may lead to political instability. The failure of an political institution is sometimes characterized by the persistence of inefficient public policies in societies. One factor which drives such a real-life issue is the fact that we, as human beings, are different. We live in an heterogenous society, and individuals have different preferences, tastes, and views about world issues. Public policies must come across a balance between the conflicting interests of heterogeneous voters. At this instance, reaching a collective agreement, upon a social decision, is not always simple. The conflict largely reflects socio-economic factors, deriving from differences in economic variables such as income, age, gender, employment status, geographical residence, occupation, race, to name a few variables. The angular stone of this thesis consists of finding conditions on the structure of the political economy that could lead to the design of *stable*, inclusive, and *efficient* constitutions in societies.

Many studies have proposed different solutions in order to overcome some of the social dilemmas aforementioned towards the well-being of societies. For instance, among the existing contributions, it has been conjectured that the efficiency problem in a public decision can be resolved if agents are induced to display reciprocal or pro-social behavior. To the best of my knowledge, there is no existence of a sequential voting mechanism which induces selfish individuals to adopt such behavior. What could be the properties of such mechanism? Under which conditions, if any, does this mechanism enable policymakers to reach a stable and efficient public policy? I address these questions in the thesis.

The first chapter of this essay provides a deliberative and finite sequential voting process– which I call the *reciprocity mechanism*–, which is a modification of the legislative procedure generally encountered in several democratic countries. The mechanism has desirable features in a political economy. First, it provides incentives for rational individuals to display reciprocal and pro-social behaviors in the allocation of various types of public goods. This feature distinguishes this research from the extant theoretical literature on reciprocity which assumes that individuals have other-regarding preferences including conditional altruism. Second, under wide classes of economy, where an economy is essentially defined by a set of feasible public decisions and a profile of individual preferences over these decisions, the reciprocity mechanism guarantees the existence of stable policies. Third, the mechanism resolves the long-established conflict between individual rationality and efficiency in public goods provision. Basically, any stable policy under the mechanism is Pareto-efficient.

The aggregation of individuals' preferences in a political economy by means of a voting process enables society to implement decisions. Under certain circumstances, this aggregation process may lead to different political and economic failures, and prevent the economy to reach desirable *outcomes*. Many scholars have developed several models that aim at getting the appropriate *voting rule* that could prevent such issues in a decision-making process. However, there is no consensus concerning the properties that the *best voting rule* should satisfy in one hand, and those of the resulting outcomes on another hand. Now, we observe an emergence of decisions rules, each one embedded with interesting and convincing explanations. The second chapter of the thesis displays a rationalization of the multiplicity of supermajority rules in legislative organizations by the *importance level* of public policies in societies. In a fairly and simple model which includes the reciprocity mechanism, the results reveal the minimum supermajority rule required to enact stable and efficient policy, depending on how the corresponding enactment could affect society's life and wellbeing. Ultimately, this finding depends also on the *type* of public policy, and the *preference domain* of legislators over the induced policy space.

The third chapter of the thesis refines the preference domain in which the *majority rule* can be sustained as the optimal rule that aggregates political preference into stable and Pareto-efficient outcomes in decision-making bodies. However, the analysis also reveals that this voting rule is not appropriate for certain types of societies, including sufficiently fractionalized societies. This chapter also displays an application of the model to the problem of public bads. Environmental protection is an important concern and to many, the primary issue of the future of humanity. Many societies face a problem of choosing locations to build public bads. These are goods that produce socially undesirable outcomes in the economy. In several democratic countries, such decisions are made by purposeful and rational political agents legislating in a well-known political mechanism. Since, most citizens want the facility to be located far away from their neighborhoods, how could an agreement be reached among voters? The last chapter of the thesis offers an insight to this issue. It analyzes the existence of stable location of a public bad in a framework under which the policy space is a continuum. The key finding of the study is that, under the reciprocity mechanism, at most two stable locations could be implemented if each individual has single-caved preferences over the policy space.

A key message from each chapter, in this thesis is that political mechanisms that induce rational reciprocal behaviors among citizens could enhance institutions and economies. While each chapter establishes the importance of the theoretical results , it leaves the design of either experimental or empirical validity of such arguments for future research.

# Chapter 1

# A Political Reciprocity Mechanism

## 1.1 Introduction

It is well-known that the pursuit of individual self-interest often leads to the emergence and persistence of inefficient public policies (Samuelson (1954), Falkinger et al. (2000), Acemoglu, Ticchi, and Vindigni (2011)). It has long been conjectured that this social dilemma can be resolved if individuals are induced to display a reciprocal and pro-social behavior (Fehr and Gächter (2000), Axelrod (2006), Falk and Fischbacher (2006)). However, to our knowledge, a formal voting mechanism which induces selfish individuals to adopt such behavior is non-existent. Such a mechanism, by providing an incentive for pro-social actions, would also hopefully be inclusive in the sense of protecting minority rights and interests. It would therefore solve two important problems—policy inefficiency and tyranny of the majority—identified by political scientists as being characteristics of many real-life voting systems. Our goal in this paper is to propose a voting mechanism which addresses these issues. For this reason, we call it the *reciprocity mechanism*.

The reciprocity mechanism, described below, is a sequential procedure that embeds elements of both the one-shot and the infinite horizon political game. A one-shot mechanism (e.g., Gillies (1959), Neumann and Morgenstern (1944)) entails only one stage of voting in which a policy is pitted against the status quo, and the latter is replaced if a (qualified) majority of individuals are in favor of change. In infinite horizon political games (see for e.g., Harsanyi (1974), Baron and Ferejohn (1989), Chwe (1994)), individuals may object and counter-object until they reach a *stable* or an *equilibrium* outcome. This process can also go on forever. It is well known that while the one-shot mechanism always ensures that the chosen policy is Pareto-efficient, it raises two important issues: (1) a lack of political inclusiveness (or the tyranny of the majority); and (2) a general failure of selecting a *stable* policy. On the other hand, the infinite horizon mechanism (some solution concepts for this class of games are surveyed in Fotso, Pongou, and Tchantcho (2017)) may lead to the selection and persistence of inefficient policies. We will show that our reciprocity mechanism resolves all of these issues. Importantly, it reduces the expropriation of decision-power by majorities without the need to resort to a supermajority rule to select policies.[1]

The reciprocity mechanism is a modification of the legislative procedures generally encountered in democratic countries such as the United States, Canada, United Kingdom, and many others. It is described as follows. Consider the problem of reforming a status quo in a legislative institution where a policy is adopted if it is supported by a strict majority. In the legislative chambers encountered in the democracies cited above, the reform process generally entails several stages. In the first stage, a bill is introduced by a legislator or a qualified majority of legislators[2]. In the second stage, the bill is debated, possibly leading to several modifications or amendments. In theory, this stage can last forever. In the final stage, the possibly amended bill is put to a vote. If it passes, it becomes the new law. If it fails, the status quo remains in place.

We show that this traditional decision-making procedure may lead to the selection of an inefficient policy (see section 1.6).

Unlike the classical procedure which can be endless, our mechanism comprises only three stages described as follows and depicted in Figure 1:

- In **stage 1**, a bill $y$ is introduced to replace a status quo $x$. If this introduction is supported by a winning coalition[3] $S$, $y$ is submitted with the possibility of an amendment.
- In **stage 2**, each individual who is not part of the sponsoring coalition $S$ has the right to oppose the proposed policy $y$. If there is no opposition, $y$ is adopted as the new policy and the process ends. If there is any opposition, the sponsoring coalition $S$ is given the opportunity to withdraw the bill $y$. If $S$ chooses to withdraw $y$, a status quo policy

[1] In real-life institutions, the tyranny of the majority is generally resolved by using supermajority rules to make public decisions. To illustrate, suppose that a two-thirds majority rule is used to select policies. Then a minority group that represents exactly one-third of the population would be protected under such a rule if its members all oppose a certain policy. The reciprocity mechanism generally protects minorities under the majority rule.

[2] An appropriate legislative proposal can be introduced by a single legislator in certain systems such as the United States Senate or the Canadian Parliament, whereas in other systems including the European Parliament and the South Korean Parliament, a proposal to be introduced has to be supported by a qualified majority of members.

[3] A winning coalition $S$ has the power to change the status quo policy to a new alternative whatever the preferences of the members outside of $S$, whereas a losing coalition does not have such a veto-right. A formal definition of this concept is provided in the section 1.3.1.

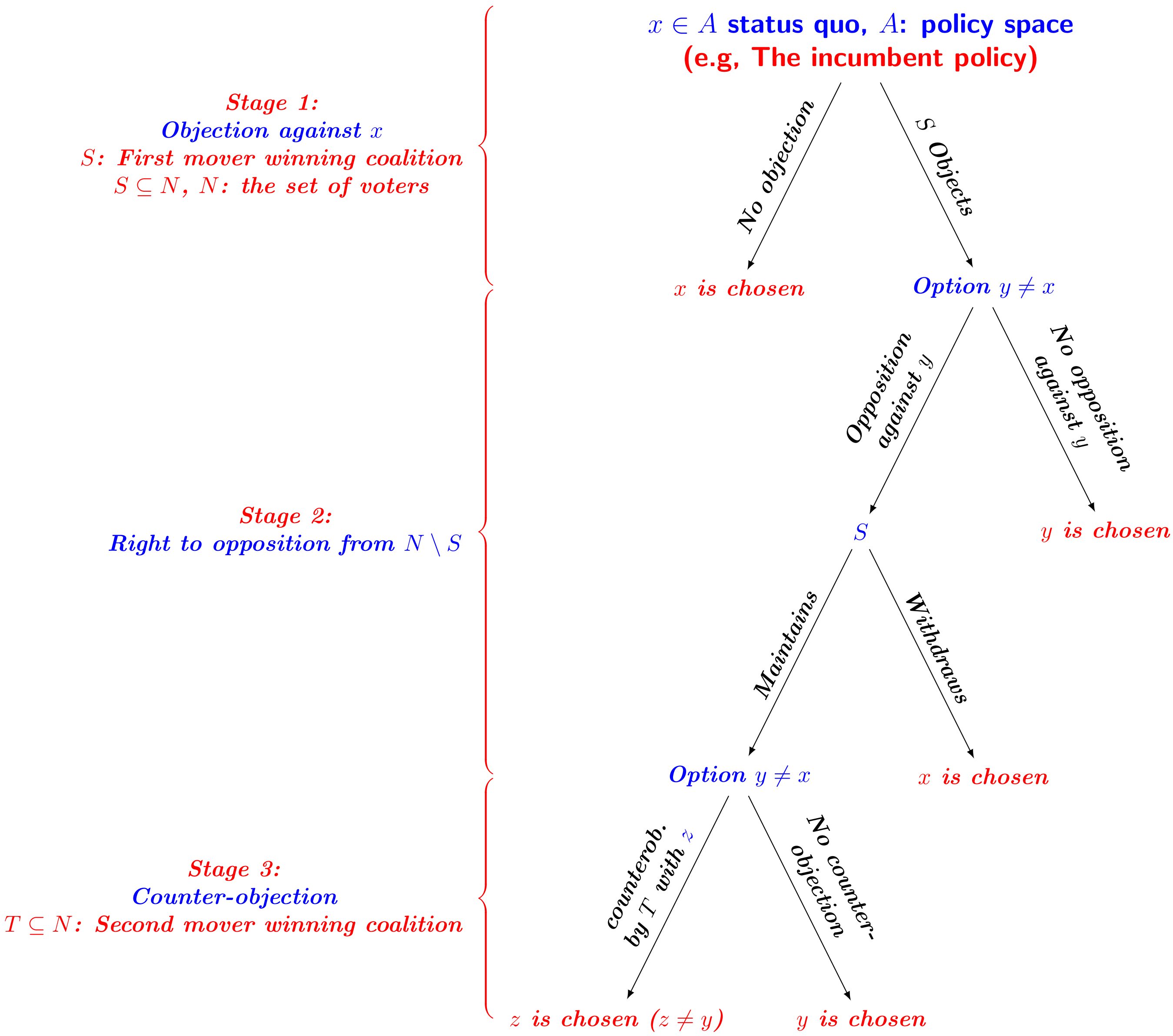



FIGURE 1.1: Reciprocity Mechanism

$x$ remains in place and the process ends. But if $S$ chooses to maintain $y$, the process moves to the final stage.

- In **stage 3**, the opposition has the right to propose an amendment $z$. If it does not, $y$ becomes the new policy and the process ends. If it proposes an amendment $z$, a voting contest is organized between the proposed bill $y$ and $z$; if another winning coalition $T$ supports $z$ against $y$, $z$ becomes the new policy and the process ends; otherwise $y$ is the new policy and the process ends.

We call this political process a reciprocity mechanism because it prevents free-riding by inducing voters to act in a reciprocal way towards one another. In fact, it induces any second mover coalition $T$ to respond positively to any objection by a first mover coalition $S$. The coalition $T$ responds whenever the latter objection favors the interests of its members. Also, the mechanism induces selfish individuals to be farsighted, and then prevents any potential first mover $S$ from taking an action that will hurt the interests of any individual who can successfully retaliate through an amendment (Theorem 1).

In the model considered in this paper, preferences and the bargaining rules are assumed to be common knowledge, all actions (moves) and potential outcomes are observable; so the model involves perfect information. However, the reciprocity mechanism is not an extensive form game of perfect information. There are at least two reasons that sustain the latter argument. First, in the decision-making process described by Figure 2.3, there exists an alternative at each step, whereas in extensive form game, the alternatives are situated at the terminal nodes of the game tree. In the reciprocity mechanism, the payoffs can be collected at either the first, the second or the last stage depending on players' preferences over the policy space. Second, there is no predefined order under which players or coalitions make move in the reciprocity mechanism. For these reasons, the subgame perfect Nash equilibrium is not an appropriate solution concept for the game illustrated by Figure 2.3.

A *stable* or *equilibrium* policy is an alternative such that if proposed as a status quo at the beginning of the voting process, it will be elected. In other words, it is an option that can not be replaced or reformed by another policy if it is proposed as a status quo when the game starts. In section 1.3.4, we introduce the concept of predicted reciprocity set which answers the question about which policies might be enforced from a given status quo. Obviously, an elected outcome under the mechanism is not necessarily a stable policy, and vice versa (see Example 1). The set of all the stable policies, called the *reciprocity set*, is defined in section 2.3 and some of its properties are discussed in sections 1.4 and 1.5. More precisely, this chapter answers the questions of whether this set always contains a policy, and if such a policy is Pareto-*efficient*.

In section 1.4, the conditions under which the reciprocity mechanism lead to a stable policy are provided. Several important classes of economies are examined, where an economy is essentially defined by a set of feasible public decisions and a profile of individual preferences over these decisions. The results cover both discrete and continuous public goods. Examples of such goods are extremely diverse. A decision may consist of selecting a political leader of a country from a finite set of options, choosing the location of a bus stop along a street, deciding on the number of refugees to admit into a country, voting on a fiscal policy, or sharing a given amount of money among a finite set of individuals. Theorem 3 reveals that a stable policy always exists if individuals have strict preferences over the set of possible policies. In particular, if individuals have single-peaked preferences over a left-right political spectrum, there exist at least one equilibrium and at most two equilibria; the equilibrium is unique and corresponds to the median individual's ideal point when the number of individuals is odd (Theorem 4).

Section 1.5 analyzes the welfare implications of stable policies. The key result is that all stable policies are efficient regardless of whether policies are discrete or continuous (Theorem 6). This finding implies that our mechanism resolves the conflict between individual rationality (the pursuit of individual self interests) and efficiency in public decision.

The reciprocity mechanism is compared to other alternative mechanisms studied in the literature, namely the one-shot political games, a modified version of the reciprocity mechanism, and the infinite-horizon sequential political games. These voting environments exhibit different individual's rational behaviors. More details on these comparisons are provided in section 1.6.

In the following section, we relate this work to the literature. Section 2.3 formalizes the mechanism as well as the rational behavior of individuals under this mechanism. It also establishes preliminary results and introduces real-life applications. Section 1.4 studies the existence of stable policies under conditions regarding the structure of preferences and the nature of the policy space. Section 1.5 provides the result concerning the welfare property of the reciprocity mechanism. It reveals that any equilibrium policy is Pareto-efficient. Section 2.5 concludes. For clarity, all proofs are collected in an appendix B.

## 1.2 Related Literature

This study contributes to three different literatures, namely, the literature on preference-based reciprocity, the literature on institutional design, and the literature on game theory and bargaining.

**Preference-based *vs.* institution-induced reciprocity**. Preference-based reciprocity has been extensively studied in the literature. Rabin (1993) argues that "fairness" is the main reason for returning favor for favor and harm for harm. He studies the existence of fairness equilibria in a two-person psychological game. In the latter, Rabin assumes that each individual's subjective expected utility depends on his strategy, his beliefs about the other individual's strategy choice, and his beliefs about the other individual's beliefs about his strategy. Several aspects of this paper have been generalized or tested in laboratory experiments (e.g., Falk and Fischbacher (2006), Bruni, Gilli, and Pelligra (2008)). Experimental studies show that reciprocity promotes cooperation and increases social welfare (e.g., Komorita, Hilty, and Parks (1991), Komorita, Parks, and Hulbert (1992), Fehr and Gächter (2000), Cox (2001)).

This piece of work differs from the aforementioned studies in its scope and primitive assumptions. It provides an overview about how a formal mechanism can be designed so as to induce "selfish" individuals to display reciprocal and pro-social behavior. Therefore, although individuals may only care about their self-interests, they find it rational to take pro-social actions under the reciprocity mechanism. This feature distinguishes our research from the extant theoretical literature on reciprocity which assumes that individuals have other-regarding preferences including conditional altruism (e.g., Rabin (1993), Dufwenberg and Kirchsteiger (2004), Hahn (2009)). In this literature, reciprocity emerges out of other-regarding preferences, whereas in this paper, it is the reciprocity mechanism that provides an incentive for taking reciprocal actions among selfish individuals. Note that in our analysis, preferences can be defined to incorporate other-regarding preferences and fairness considerations (see Example 3 in section 1.3.5). However, this is not needed for individuals in the political model presented in this essay to display reciprocal behavior.

**Institutional design and reciprocity**. This study also contributes to the literature analyzing the different forms of mechanisms that induce people to exhibit reciprocal behavior. In experimental studies, it has been shown that the sanction of non-cooperative behavior is a mechanism capable of inducing cooperation in public goods games (Fehr and Schmidt (1999), Fehr and Gächter (2000)). Another popular mechanism found to induce reciprocity and sustain cooperation, is the repetition of a game (Aumann and Sorin (1989)). Bruni, Gilli, and Pelligra (2008) argue that, in interactive situations where cooperation is the best long-run strategy and defection and opportunism the best one-shot strategy, reciprocity can emerge and sustain cooperation if the interactions are repeated indefinitely. The analysis described in this first essay shares with this literature the assumption that individuals only care about their self-interests. But we differ, in that the mechanism presented by Figure 2.3 is quite static and therefore does not require that a voting game be repeated infinitely in order to induce reciprocity and cooperation among decision-makers. The reciprocity mechanism is more appropriate in a legislative context. It has several merits, including the fact that it minimizes the time-cost of decision-making and guarantees the existence and efficiency of equilibrium outcomes.

**Game theory and bargaining**. The reciprocity mechanism shares some features with certain bargaining frameworks available in the literature (see, e.g., Neumann and Morgenstern (1944), Aumann and Maschler (1964), Rubinstein (1980), Baron and Ferejohn (1989), Zhou (1994), Holzman, Peleg, and Sudhölter (2007) and Pongou, Lambo, and Tchantcho (2008)). Within these frameworks, negotiation often takes the form of a sequence of threats and counter-threats. All individuals can bargain together, with full knowledge of the process, and settle at a stable option that is based on the objections and counter-objections. A new solution concept, precisely the reciprocity set, which identifies stable social policies when people have possibly conflicting views on how society should be run is defined and analyzed. In particular, the reciprocity set displays certain properties of the core (Gillies (1959)). Like the core, it always induces an efficient policy. However, unlike the core, it cannot be empty when preferences are strict. The core is most suited as a solution concept for one-shot games. If the game entails two or more stages, as is the case for several real-life games, the core becomes inappropriate to capture rational behavior (see Mas-Colell (1989)). There exist other solution concepts defined for games which might go on forever. A comparative analysis reviewing the strengths of the reciprocity set to the farsighted stable set (Harsanyi (1974), Ray and Vohra (2015)) and the largest consistent set (Chwe (1994)) is provided. The results indicate that an inefficient policy can persist forever if individuals' rational behavior is captured by these concepts. This is explained by the fact that these concepts were not designed to solve the issue we are solving in this paper.

## 1.3 A Reciprocity Mechanism in a Majoritarian Political Economy

A *majoritarian political economy* is a society $N = \{1, 2, ..., n\}$, populated by a finite-lived number of individuals, and endowed with a constitution $\mathcal{C}$, and a profile of individual preferences $(\succeq_i)_{i \in N}$ over the policy space $A$. This section assumes that each preference relation $\succeq_i$ is reflexive, transitive, and complete. The structure of the policy space depends on the nature of the decision to be made. Certain decisions are discrete in nature (e.g., the choice of a national language), whereas others are continuous (e.g., tax rates). Depending on the nature of the policy space, additional structures will be imposed on preferences, such as continuity in the case of a continuous policy space. The constitution as well as the decision-making mechanism are formally described below.

### 1.3.1 The Constitution

A constitution is a distribution of political decision-making power among the various subgroups of the society. It is formalized as a function $\mathcal{C}$ which maps each subgroup $S$ of the society into either 1 or 0; $\mathcal{C}(S) = 1$ means that the coalition is a "winning coalition", and $\mathcal{C}(S) = 0$ means that $S$ is a losing coalition. A winning coalition $S$ has the power to change the status quo policy to a new alternative whatever the preferences of the members outside $S$, whereas a losing coalition does not have such a veto-right. A simple example of a constitution is the majority rule, which is a rule under which a coalition of agents is winning if and only if it contains more than half of the population. Throughout the paper, we will denote by $P$, the set of winning coalitions.

### 1.3.2 The Decision-Making Process

Consider reforming a given status quo $x \in A$ by way of a deliberative and voting mechanism comprising three stages described as follows:

**Stage 1** (**Objection**): If a winning coalition $S$ proposes that $x$ be replaced by another policy $y$, the pair $(y, S)$ is called an *objection* against $x$. If no objection against $x$ exists, then $x$ remains in place, which ends the process. If an objection $(y, S)$ against $x$ exists, $y$ is recognized as a bill, and the process advances to the second stage.

**Stage 2** (**Right to opposition**): Each individual who is not a member of the sponsoring coalition $S$ has the right to oppose the bill $y$. If there is no opposition, $y$ becomes the new policy, which ends the process. However, if there is any opposition, $S$ has the right to either withdraw or maintain the bill $y$. If $S$ withdraws the bill $y$, the status quo $x$ remains in place and the process ends. However, if $S$ maintains $y$, then the opposing individuals have a right to formulate a counter-objection in the third and final stage.

**Stage 3** (**Counter-objection**): Suppose that $S$ maintains the bill $y$. Each opposing individual has the right to invite other individuals (including opportunists in $S$) to form a winning coalition $T$ (obviously the coalition $T$ is different to the coalition $S$) in order to replace $y$ by another policy $z$. In this case, $z$ is called an amendment of $y$ and $(z, T)$ is a counter-objection to the objection $(y, S)$. If no counter-objection to $(y, S)$ exists, then $y$ becomes the new policy, which ends the process. If a counter-objection $(z, T)$ exists, $z$ is elected as a new policy and the process ends.

Interestingly, in order for the opposition to succeed in the final stage of the above mechanism, it has to form a winning coalition $T$ that supports the amendment $z$ against the initial bill $y$, even by recruiting *opportunists* in the coalition $S$ that initiated the move from $x$ to $y$. This means that there is no agreement that binds the members of the sponsoring coalition $S$. Note that even though some individuals from $S$ would like the latter winning coalition to withdraw its support of the bill $y$ after the opposition at stage 2, the game might still continue. In fact, this scenario could be encountered if the coalition $S$ is not a minimal winning coalition. After the opposition at stage 2, a status quo could be elected if a significant number of individuals decide non-cooperatively to withdraw themselves from $S$, inducing the latter for losing its veto right. An individual joins a coalition in order to achieve his/her main interests. As long as the latter might not be satisfied, he/she may withdraw him/herself from this coalition.

Another difference between the new mechanism and the classical legislative mechanism described in the Introduction is that the final stage of the former opposes $y$ and its amendment $z$, whereas the final stage of the latter opposes the status quo $x$ and the amended bill $z$.

The reciprocity mechanism shares some features of other models proposed in the literature (see, e.g., Aumann and Maschler (1964)). In these models, negotiation may take the form of a finite or infinite sequence of objections and counter-objections. The new mechanism mainly differs in that it offers sponsoring coalitions the possibility to withdraw a bill. Arguably, its design also minimizes the time-cost of decision making, and is therefore appropriate when decisions are subject to a deadline. The mechanism also has three other interesting properties. First, it encourages reciprocal actions among selfish individuals. Second, it always leads to an equilibrium policy under natural assumptions on individuals' preferences. Third, it selects only efficient policies in equilibrium.

### 1.3.3 Rationality and Reciprocity

The formalization of rational behavior of individuals in the voting mechanism described in the above section answers the question of when a winning coalition (group of individuals) will choose to formulate an objection against a status quo under the reciprocity mechanism.

Let $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ be a political economy, $x \in A$ a status quo, and $S \in P$ a winning coalition that wishes to replace $x$ by another policy $y$. The two following conditions should be satisfied:

1. Each member of $S$ should strictly prefer $y$ to $x$ (i.e, $y \succ_S x$).

2. For any winning coalition $T$ different to $S$ that strictly prefers an alternative policy $z$ to $y$ (i.e., $z \succ_T y$), if some member of $T$ strictly prefers a status quo $x$ to the proposed policy $y$ (i.e., $not(y \succeq_T x)$), then each member of $S$ should weakly prefer $z$ over $x$ (i.e., $z \succeq_S x$). More formally, this condition is expressed as follows:

   $[\forall (z, T) \in A \times P,\ S \neq T,\ z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

The first condition is a natural requirement that appears in all two-stage bargaining models of rationality in the literature (see, e.g., Aumann and Maschler (1964) and all the subsequent studies inspired by this paper). This condition is in reality an expression of ambiguity aversion. It expresses the fact that the sponsoring coalition $S$ cannot fully predict the future of the game after introducing the bill $y$. In fact, the game might end at $y$ for several practical reasons. One reason is that even if each member of $S$ knows that there exists another winning coalition $T$ that will be interested in deviating from the proposed policy $y$ to a more preferred policy $z$, and that $z$ is not a worse option for $S$ than $x$, it is not clear that $T$ will be formed. It is generally argued that many exogenous factors that are not under the control of individuals interfere with decision-making in real-life politics (Penn (2009)), and such factors can prevent a coalition to form. It follows that $y$ might end up being selected, implying that $S$ cannot introduce option $y$ without preferring it over the policy $x$.

The second condition says that, if following an objection $(y, S)$ against a status quo $x$, a winning coalition $T$ formulates a counter-objection or an amendment $(z, T)$, then $S$ should weakly prefer $z$ over $x$ provided that certain members of $T$ strictly prefer $x$ to $y$. In fact, it is easy to see that if all the members of $T$ prefer $y$ to $x$ (i.e., $y \succeq_T x$) and some member of $S$ strictly prefers $x$ over $z$, then the coalition $T$ will not oppose the move from $x$ to $y$ and so will not formulate the counter-objection $(z, T)$ against $(y, S)$. Indeed, if it opposes $y$, then $S$ will withdraw its objection, resulting in $x$ remaining in place, which does not benefit the members of $T$ since they all prefer $y$ over $x$. This means that an opposition leading to a counter-objection $(z, T)$ emerges if and only if $T$ prefers $z$ over $y$ (i.e., $z \succ_T y$) and there is at least one member of $T$ who prefers $x$ over $y$ (i.e., $not(y \succeq_T x)$).

The definition of the *reciprocity set* is provided below. It is the set of all stable or equilibrium policies, a stable policy being a policy that cannot be replaced if it is a status quo.

**Definition 1.** Let $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ be a political economy, $M$ the reciprocity mechanism, $S$ be a winning coalition, and $x, y \in A$ be two policies.

1. **Objection**: $(y, S)$ is said to be an objection against $x$ if $y \succ_S x$.
2. **Counter-objection**: let $(y, S)$ be an objection against $x$. A pair $(z, T) \in A \times P$ is said to be a counter-objection against $(y, S)$ if $z \succ_T y$ and $not(y \succeq_T x)$.
3. **Unfriendly counter-objection**: let $(y, S)$ be an objection against $x$ and $(z, T) \in A \times P$ a counter-objection against $(y, S)$. The counter-objection $(z, T)$ is said to be unfriendly if $not(z \succeq_S x)$[4].
4. **Justified objection**: an objection $(y, S)$ against $x$ is said to be justified if there is no unfriendly counter-objection against $(y, S)$.
5. **Reciprocity set**: the reciprocity set of $\mathcal{P}$, denoted $R(\mathcal{P})$, is the set of all the policies in $A$ against which no justified objection exists.

Under the mechanism, all actors bargain and settle on a stable policy which emerges from the sequence of threats and counter-threats. The stability of an outcome, in general, reflects the power or influence of different population coalitions as well as the preferences of individuals. In our context, this power is provided by the constitution $\mathcal{C}$. The reciprocity set derives its name from the fact that reciprocal actions–returning favour for favour and harm for harm–are rational. This behavior is formalized in the first theorem below:

**Theorem 1.** ***(Reciprocal actions are rational)*** *Let $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ be a political economy, and $M$ the reciprocity mechanism. For all winning coalitions $S, T \in P$ and distinct policies $x, y, z \in A$ such that $y \succ_S x$ and $z \succ_T y$, the assertions described hereunder are true:*

[4]The notion of an "unfriendly counter-objection" is inspired by the concept of an "unfriendly amendment" found in the legislative jargon.

1. *If not*$(y \succeq_T x)$*, then* $S$ *will not formulate any objection* $(y, S)$ *against* $x$*, unless* $z \succeq_S x$*. If not*$(y \succeq_T x)$ *and not*$(z \succeq_S x)$*, then* $S$ *will not formulate an objection* $(y, S)$ *against option* $x$*.*

2. *Assume that* $S$ *formulates an objection* $(y, S)$ *against* $x$*. If* $y \succeq_T x$*, then* $T$ *will not formulate an unfriendly counter-objection* $(z, T)$ *against* $(y, S)$ *following an opposition.*

3. *As a corollary of 1. and 2., if* $y \succeq_T x$*, then* $S$ *will formulate an objection* $(y, S)$ *against* $x$*, even if not*$(z \succeq_S x)$*.*

Theorem 1 states that a first coalition mover $S$ cannot rationally take any action that hurts the interests of any potential second mover $T$, unless it is the case that no member of $S$ will be worse off after the subsequent move by $T$; similarly, no second mover $T$ can rationally take an action that hurts the interests of the first mover. Hence, the possible withdrawal action at stage 2 of our mechanism induces individuals to rationally take positive reciprocal actions, which justifies the name proposed to the voting process–*the reciprocity mechanism* .

The next result rationalizes the reciprocity set. First, the definition below introduces a binary relation (dominance relation), denoted $\gtrdot$, over the policy space, which classifies policies from a given preferences' profile.

**Definition 2.** Let $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ be a political economy, and $x$ and $y$ be two policies. $y$ is said to *dominate* $x$, denoted $y \gtrdot x$, if there exists a winning coalition $S \in P$ such that:

a) $y \succ_S x$ and;

b) $[\forall (z, T) \in A \times P,\ S \neq T,\ z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

The result hereunder establishes that any policy belongs to the reciprocity set if and only if it is a maximal element of a binary relation $\gtrdot$. This finding formalizes the rational behavior of individuals under the reciprocity mechanism.

**Theorem 2.** ***(Rationalizing the reciprocity set)*** *Let* $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ *be a political economy,* $M$ *the reciprocity mechanism, and* $x \in A$ *a given policy. Statements 1. and 2. below are equivalent:*

1. $x \in R(\mathcal{P})$*.*

2. *There does not exist a policy* $y \in A$ *such that* $y \gtrdot x$*.*

Let $\mathcal{P} = (N, A, (\succeq_i)_{i\in N})$ be a political economy and $x, y \in A$ two policies such that $y \gg x$. Then, by definition of the binary relation $\gg$, there exists a winning coalition $S$ such that conditions a) and b) in Definition 2 are true. We say that $y$ dominates $x$ via $S$, and denote this by $y \gg_S x$. The binary relation $\gg$ is a very convenient way to identify policies that are stable. It will be especially useful when we study the existence and welfare properties of stable outcomes under the reciprocity mechanism. Throughout this paper, a majoritarian political economy is represented by $\mathcal{P} = (N, A, (\succeq_i)_{i\in N}, M)$, where $M$ is the reciprocity mechanism.

### 1.3.4 The Predicted Reciprocity Set

If a policy, say $x$, is a status quo, which policies will be possibly enforced at the end of the voting process? The following definition introduces a solution concept—the *predicted reciprocity set* which answers this question. It is the set of the policies that can be selected from a given status quo.

Clearly, if $x$ is an equilibrium or a stable policy (that is, $x \in R(\mathcal{P})$), then $x$ will be enforced. But if $x$ is not a stable policy, then it means that an alternative policy $y$ dominates $x$ (that is, $y \gg x$). If there is no opposition or counter-objection against $y$, $y$ will be chosen and so is a possible predicted outcome at $x$. If there is any opposition and a counter-objection against $y$, that may lead to the proposition of amendment $z$, then $z$ will be the predicted outcome at $x$. Indeed, given that $y$ dominates $x$, the latter counter-objection is not unfriendly to the coalition mover supporting the objection. The precise description of the *predicted reciprocity set* is given in the next definition.

**Definition 3.** Let $\mathcal{P}$ be a political economy and $x \in A$ a status quo. The *predicted reciprocity set* at $x$ is defined as: $P(x) = \{x\}$ if $x \in R(\mathcal{P})$; otherwise $P(x) = \{y \in A,\ if\ y \gg x\ and\ \nexists\ z:\ z \succ y\}$, or $P(x) = \{z \in A,\ if\ y \gg x\ and\ z \succ y,\ y \in A\}$.

From this definition, as said above, each stable outcome is its own prediction, but there might exist predicted outcomes that are not stable. The analysis indicates that the predicted reciprocity set from any given policy is non-empty since the voting process is a finite sequential game. The next section illustrates the reciprocity mechanism and the key notions defined so far. The first example below provides the difference between the reciprocity set and the predicted reciprocity set.

**Example 1.** (**Elected vs Stable outcomes**)

Consider a society which consists of a finite set of individuals $N = \{1, 2, 3, 4, 5, 6\}$, and a discrete and finite policy space $A = \{a, b, c, d\}$. Individuals' preferences over the set $A$ are given as follows: $b \succ_1 d \succ_1 c \succ_1 a$; $d \succ_2 b \succ_2 c \succ_2 a$; $c \succ_3 b \succ_3 d \succ_3 a$; $c \succ_4 b \succ_4 a \succ_4 d$; $a \succ_5 d \succ_5 c \succ_5 b$; and $a \succ_6 d \succ_6 c \succ_1 b$.

$$\begin{tikzcd}
a \arrow[r, "S"] \arrow[rr, bend right, "S"'] & b \arrow[r, "T"] & c \arrow[r, "U"] & d
\end{tikzcd}$$

FIGURE 1.2: Elected vs stables outcomes.

Note that $S = \{1, 2, 3, 4\}$, $T = \{3, 4, 5, 6\}$, and $U = \{1, 2, 5, 6\}$. The domination graph among policies based on individuals' preferences over the policy space $A$ is provided by the Figure 1.2 (the arrows indicate the direction of the classical dominance). The predicted reciprocity sets at each status quo are: $P(a) = \{c\}$, $P(b) = \{b\}$, $P(c) = \{d\}$, and $P(d) = \{d\}$. If the policy $a$ is the status quo at the beginning of the voting process, then the policy $c$ will be elected, since policy $c$ dominates $a$. However, the policy $c$ is not a stable outcome, since if $c$ itself is the status quo when the game starts, then the winning coalition $U$ will object against $c$ by proposing $d$. Since there is no other possibility to move from $d$, the latter will be elected. If $b$ is the status quo, it will be elected. If $d$ is the status quo, it will also be elected. The alternatives $b$ and $d$ are the only stable policies in this economy. Plainly, an elected outcome is not necessarily stable.

### 1.3.5 Examples

This part illustrates the reciprocity mechanism through simple examples. These examples cover both discrete and continuous policies. The first example illustrates a situation in which a society that has several ethnolinguistic groups has to choose a language other than the national language(s) to be taught in school and to be used for official communication. The main purpose of the government is to protect linguistic minorities and contribute towards preserving the world's linguistic diversity.

**Example 2. (Preserving a minority language in a country)**

A political economy $\mathcal{P}$ consists of a country that has four ethnic groups. This society has to choose from a set of three languages, the ones to be used both in school and in official communication. Let $N = \{1, 2, 3, 4\}$ be the set of ethnic groups and $L = \{l_1, l_2, l_3\}$ be the set of languages. Each ethnic group has one representative. Language $l_3$ is the most commonly spoken dialect in ethnic groups 1 and 2; $l_2$ is the native language of ethnic group 3; and $l_1$ is a minority language that is mostly spoken in ethnic group 4. It can also be viewed as an indigenous language of the country. Preferences over languages are presented below:

$$\textit{ethnic group } 1 : l_3 \succ_1 l_2 \succ_1 l_1;\ \textit{ethnic group } 2 : l_3 \succ_2 l_2 \succ_2 l_1;$$

$$\textit{ethnic group } 3 : l_2 \succ_3 l_1 \succ_3 l_3;\ \textit{ethnic group } 4 : l_1 \succ_4 l_3 \succ_4 l_2.$$

Figure 1.3 shows the domination relationship among the different languages (e.g., $l_3$ is more popular than $l_2$ as the former is preferred by three ethnic groups).

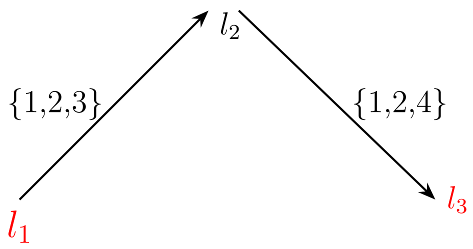



Figure 1.3: Domination relationship among the different languages

The analysis demonstrates that there are two stable languages under the reciprocity mechanism. These languages are $l_1$ and $l_3$. In fact, if $l_3$ is a status quo, it will persist forever, since it is not dominated on the popularity ground. If language $l_2$ is a status quo, the majority coalition $\{1, 2, 4\}$ will object and propose $l_3$, given that each of its members strictly prefers $l_3$ over $l_2$. Even if 3 is opposed to that objection, this individual cannot succeed at forming a majority coalition that will formulate a counter-objection leading to the replacement of $l_3$. It follows that $l_2 \notin R(\mathcal{P})$. If a status quo is $l_1$, no majority coalition will formulate an objection against it. In fact, although the members of $\{1, 2, 3\}$ strictly prefer $l_2$ over $l_1$, they know that if they formulate an objection $(l_2, \{1, 2, 3\})$ against $l_1$, $\{1, 2, 4\}$ will formulate an unfriendly counter-objection $(l_3, \{1, 2, 4\})$ against $(l_2, \{1, 2, 3\})$, causing $\{1, 2, 3\}$ to withdraw its objection. Indeed, the move from $l_1$ to $l_2$ will hurt the interests of ethnic group 4, which is the reason for the counter-objection. The latter is unfriendly because not all members of $\{1, 2, 3\}$ prefer $l_3$ over the status quo $l_1$. Therefore $l_1 \in R(\mathcal{P})$. It follows that $R(\mathcal{P}) = \{l_1, l_3\}$.

Interestingly, note that $l_1$ is a minority language since it is mostly spoken by only one ethnic group, unlike $l_3$. This proves that minority languages can sometimes be preserved under the reciprocity mechanism.

The next example displays that the reciprocity mechanism modifies and extends the traditional ultimatum game to group decision-making.

**Example 3. (Ultimatum game with counter-offer)**

A population of $n$ individuals must share an amount of 100 dollars, with each individual receiving a non-negative portion. A feasible allocation is a vector:

$$x = (x_1, x_2, ..., x_n) \text{ such that } \; x_i \in [0, 100] \text{ and } \sum_{i=1}^{n} x_i \leq 100.$$

The set of feasible allocations is the simplex:

$$A = \{x \in [0, 100]^n : \sum_{i=1}^{n} x_i \leq 100\}.$$

Only groups containing more than half of the population have the right to formulate an objection or a counter-objection.

The decision-making process is described as follows. An arbitrator proposes a status quo allocation $x_0 = (0, 0, ..., 0)$. A majority coalition $S$ may propose an objection $(y, S) \in A \times P$ against $x_0$. If there is no opposition, $y$ is implemented. If there is any opposition, $S$ may withdraw $y$, leading to each individual getting zero. But if $S$ chooses to maintain $y$ following any opposition, another majority coalition $T$ might react by proposing a counter-objection $(z, T)$ against $(y, S)$. In this case, $z$ is implemented and each individual $i$ obtains and consumes his payoff $z_i$.

The traditional ultimatum game is played between two individuals (a proposer and a responder) and, unlike the reciprocity mechanism, it does not allow for further negotiation after the first move.

An interesting question is whether an equilibrium exists. Assume that each individual $i$'s utility function is strictly increasing in his payoff $x_i$, which is a natural assumption. Then, in this framework, an efficient equilibrium always exists, although it might not be unique. Also, if individuals have other-regarding preferences, with each individual $i$'s utility depending on both his payoff $x_i$ and other individuals' payoffs $x_{-i}$, assuming that utilities are continuous (but not necessarily differentiable), and concave, or convex, the result demonstrates that an equilibrium always exists, and that all equilibria are efficient. To illustrate, suppose that there are three individuals who have the following linear utility functions:

$$u_1(x_1, x_2, x_3) = x_1 - \frac{1}{3}(x_2 - x_1) - \frac{1}{3}(x_3 - x_1),$$

$$u_2(x_1, x_2, x_3) = x_2 - \frac{1}{3}(x_1 - x_2) - \frac{1}{3}(x_3 - x_2),$$

$$u_3(x_1, x_2, x_3) = x_3 - \frac{1}{3}(x_1 - x_3) - \frac{1}{3}(x_2 - x_3).$$

Interpreting these utilities functions, individuals have fairness considerations in that fixing the payoff of an individual $i$ and increasing the payoff of another individual decreases $i$'s utility. The analysis proves that (see more details in the appendix A) the set of predicted allocations from the status quo $x_0$ is:

$$P(0,0,0) = \left\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 100 \text{ and } x_i \geq \frac{100}{6};\ i = 1,2,3\right\},$$

and the reciprocity set is:

$$R(\mathcal{P}) = \left\{\begin{array}{l}(x_1, x_2, x_3):\ x_1 + x_2 + x_3 = 100, \text{and } \nexists i \neq j \\ \text{such that } x_i = x_j = 0;\ i, j = 1, 2, 3\end{array}\right\}.$$

Note that the reciprocity set strictly includes the predicted reciprocity set at the status quo $x_0$.

The next example applies to the selection of a fiscal policy.

**Example 4.** (**Fiscal policies**)

A Government should design a specific fiscal policy. Assume that the society consists of $n$ individuals. There are also $k$ categories. The categories might be defined by gender, age group, occupation, income group, marital status, ethnicity, or any combination of these factors. A tax policy is a vector $t = (t_1, ..., t_k) \in [0,1]^k$, which means that a fiscal policy might affect each category differently (e.g., female hygiene products might be taxed less or more than male hygiene products).

The goal is to select a tax policy under the reciprocity mechanism. Each member of society typically belongs to several of these categories and might also be concerned by social justice. This implies that the individual utility derived from a tax policy $t = (t_1, ..., t_k)$ would depend on some or all of the components of $t$. We show that an equilibrium tax policy exists for a wide class of preferences, and that all equilibria are efficient.

The determination of all the equilibrium outcomes under the reciprocity mechanism is possible in certain contexts (e.g., see Examples 2 and 3). However, in certain economies, such as the fiscal policy game of Example 4, the process can be very complicated. The question then arises of whether proving the existence and analyzing the properties of stable policies under very general conditions are feasible. The next section provides a positive answer to that question. It establishes more general results on the existence of equilibrium policies.

## 1.4 Existence of Stable Policies under the Mechanism

The results of this section reveal that stable policies always exist under very natural assumptions on individual preferences. Interestingly, when preferences have certain known structures, there might exist only one or two stable policies. However, uniqueness is not to be expected in general, which might explain why structurally identical societies may have very different policies. Each subsection that follows makes a different assumption on the structure of preferences and policy space, and analyzes the existence of stable policies under this assumption. Generally, the existence of a stable policy depends on three exogenous factors: (1) the structure of preferences; (2) the nature of the political space; and (3) the size of the voting population.

### 1.4.1 Discrete Policies

This section deals with the existence of stable policies when individuals have linear preferences over a finite set of policies. These are reflexive, anti-symmetric, transitive, and complete binary relations over the policy space. The set of such preferences is denoted by $\mathcal{L}$. Preference linearity simply means that individuals have a strict ordering of all the available policies. Example 2 on the choice of a language is an illustration of this class of economies. In this specific domain of preferences, a stable policy always exists.

**Theorem 3.** *Let $\mathcal{P}$ be a political economy where $A$ is a finite policy space ($|A| < \infty$) and preferences are linear. Then, an equilibrium policy always exists.*

Theorem 3 is illustrated by Example 2 on preserving a minority language in a multi-ethnic society. A multiplicity of equilibria can emerge in this environment, as also illustrated by this example. The empirical implications of this finding is that different policies might prevail in structurally identical economies. Indeed, there might not exist any reasonable explanation for why a minority language prevails as the official language in certain countries, but not in others.

### 1.4.2 Continuous Policies

We assume that the policy space $A$ of a political economy $\mathcal{P} = (N, A, (\succeq_i), M)$ is a compact and convex subset of the multidimensional vector space. Without loss of generality, we assume that $A = [0, 1]^k$ where $k$ is a natural number. We assume that policy space $A$ is endowed with the topology of closed convergence (e.g., Hildenbrand (1974)). Similar assumptions have proven useful in the analysis of equilibrium existence in continuous economies (e.g., Arrow and Debreu (1954)). We will also denote by $m$ the Lebesgue measure on the affine manifold spanned by policy space $A$. Given the nature of the policy space, there is a need to put additional structure on preferences. In particular, we assume that preferences are continuous. Preference continuity means that an individual who prefers a policy $x$ over another policy $y$ prefers any alternative policy close enough to $x$ to any alternative policy close enough to $y$. We first examine the existence of a stable policy in a political economy in which individuals have single-peaked preferences. In this class of economies, the analysis of political outcomes are often provided under a majoritarian constitution. We follow the same argument in the next section by assuming that the constitution is the majority rule. The latter has many appealing properties (see, e.g., Dasgupta and Maskin (2008)).

#### Single-Peaked Preferences

A preference relation over a policy space is said to be single-peaked if the policies can be ordered as points on a line; the preference relation has a maximum point; and points farther away from this maximum point are less preferred. To make this definition precise, let us assume that all the policies are ordered by a binary relation denoted $\gg$, and all individuals perceive them as being arranged in this order. An individual $i$ has a single-peaked preference $\succeq_i$, if (1) there exists a policy $a_i$ such that $a \neq a_i$ for any other policy $a \in A$, and (2) for any policies $a, b \in A$, $[a \gg b \gg a_i$ or $a_i \gg b \gg a]$, then $b \succ_i a$.

This class of preferences is very important in real life and has been studied in the literature (see, for e.g., Inada (1964), Moulin (1980), Sprumont (1991), Ehlers, Peters, and Storcken (2002), and a recent survey by Barberà, Berga, and Moreno (2017)). The result illustrates that there exists at least one and at most two stable policies when individuals have single-peaked preferences. In addition, when the number of individuals is odd, there is only one stable policy.

**Theorem 4.** *Let $\mathcal{P}$ be a political economy in which each preference relation $\succeq_i$ is single-peaked over $A$. Then, there exist at least one and at most two equilibria. In addition, if $n$ is odd, there exists only one equilibrium.*

Theorem 4 is similar to the Median Voter Theorem in the framework described in section 2.3. When the number of individuals is odd, the unique equilibrium that exists is the optimum of the median individual. When the number of individuals is even, a median individual may not exist, and in this case, there are two equilibrium that are very close to the center of the political spectrum. The following example on the admission of refugees illustrates the two situations.

**Example 5. (Admitting refugees into a peaceful country)**
How many refugees should the government of a peaceful country admit? We assume that such a decision is made by the legislators of this country. The country derives utility from the number of refugees it admits. The utility can be in terms of the publicity (warm glow) that it receives, or in terms of the skills brought by the refugees. Each member of the country has a different perception of the utility that he/she or the country receives from admitting refugees. We assume that these considerations are reflected in the legislators' utility functions. The net utility received by each legislator $i$ from $x$ refugees being admitted is:

$$v_i(x) = u_i(x) - s_i x,$$

where $u_i$ is an increasing and strictly concave function, and $s_i$ the fraction of the total cost of refugee admission incurred by the constituency of legislator $i$. Under the reciprocity mechanism, there exists at least one equilibrium and at most two equilibria (detail of calculations are relegated to the appendix A).

### Linear Continuous Preferences

As for discrete policies, we analyze the existence of a stable policy under linear individual preferences over a space of continuous policies. Here, owing to the nature of the continuous policy space, we assume that preferences are continuous. The theorem below reveals that, under the reciprocity mechanism, a stable policy always exists.

**Theorem 5.** *Let $\mathcal{P} = (N, A, (\succeq_i), M)$ be a political economy such that each preference relation $\succeq_i$ is linear, continuous, and endowed with the topology of closed convergence. Then, a stable policy exists.*

The results in section 1.4.2 illustrate the different conditions under which a stable policy exists under the reciprocity mechanism when the policy space is continuous. Example 3 on the ultimatum game with counter-offer and Example 4 on the selection of a fiscal policy best illustrate the usefulness of the different findings. The next section studies the efficiency of stable policies.

## 1.5 Welfare under the Reciprocity Mechanism

We analyze the welfare implications of stable policies under the reciprocity mechanism. The result below shows that any equilibrium policy is efficient whatever the nature of individuals' preferences, either linear or weak order preferences. A weak order is a reflexive, transitive, and complete binary relation.

**Theorem 6.** *Let $\mathcal{P} = (N, A, (\succeq_i), M)$ be a political economy where the policy space $A$ is either discrete or continuous, and each individual preference $\succeq_i$ is either a linear or weak order over the policy space $A$. Then, any policy in $R(\mathcal{P})$ is efficient.*

This finding implies that the reciprocity mechanism proposed in this paper resolves the conflict between individual rationality and optimality in public goods provision. By inducing selfish individuals to adopt a reciprocal and pro-social behavior, the mechanism prevents free-riding, which is one reason why it only leads to the selection of efficient policies.

## 1.6 A Comparison with other Mechanisms

We compare our mechanism to some classical mechanisms analyzed in the literature. We begin by modifying our mechanism to remove the incentive for reciprocal actions. This alternative mechanism is formalized as a two-stage game where no policy retraction by a sponsoring coalition is possible. We find that, under this alternative mechanism, an inefficient status quo can persist forever (Proposition 1). This result occurs because, although some individuals might be willing to remove an inefficient status quo, they fear that the new policy might in turn be replaced by a policy that is worse for them compared to the status quo. This fear prevents them from removing a status quo, a situation which is avoided under the reciprocity mechanism. In section 1.6.2, we compare the predictions of the reciprocity mechanism to those of the one-shot mechanism, finding that while the latter always ensures that the chosen policy is efficient, it might fail to select a stable policy (Proposition 2). In section 1.6.3, we consider infinite-horizon political mechanisms. We find that these mechanisms may lead to an inefficient policy persisting forever (Proposition 3).

### 1.6.1 A Modification of the Reciprocity Mechanism

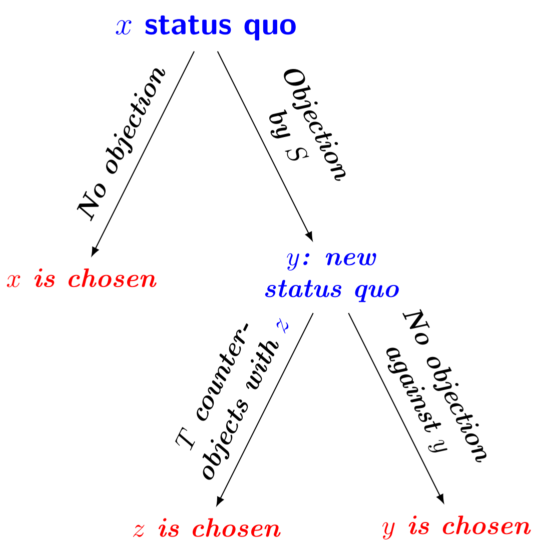



Figure 1.4: Non reciprocity-mechanism

The modified mechanism is described as follows (also see Figure 1.4). Assume that a society wishes to reform a status quo $x \in A$. The decision-making mechanism entails two main stages:

**Stage 1 (Objection)**: If no winning coalition $S$ proposes an objection $(y, S)$ against $x$, then $x$ remains in place, which ends the process. If an objection $(y, S)$ against $x$ exists, $y$ is recognized as a bill, and another coalition might formulate a counter-objection in the second stage.

**Stage 2** (**Counter-objection**): Suppose that an objection $(y, S)$ against $x$ exists. Another winning coalition $T$ might propose a counter-objection or an amendment $(z, T)$ against $(y, S)$. If no counter-objection to $(y, S)$ exists, then $y$ becomes the new policy, which ends the process. If a counter-objection $(z, T)$ exists, $z$ becomes the new policy, which ends the process.

This mechanism differs from the reciprocity mechanism in that it does not allow any sponsoring coalition $S$ to withdraw a bill $y$ once the latter is introduced. This non-retraction clause is found in legislatures. In the Canadian parliamentary system, for instance, there is no disposition that allows a sponsoring coalition to withdraw a bill before the final vote. In other systems, the withdrawal of a bill has to receive the unanimous consent of the members of the legislature to be approved. The mechanism therefore gives any potential second mover $T$ the possibility to free ride on the action of $S$ by deviating to a more preferred policy $z$ from the bill $y$. This deviation still occurs even if the move from $x$ to $y$ initiated by $S$ does not hurt the interests of any member of $T$. If the second move is indeed likely to hurt $S$, $S$ will anticipate this and will refrain from sponsoring the bill. The analysis reveals that this anticipation may result in an inefficient policy persisting forever.

We define below the rational behavior of individuals under this mechanism.

**Definition 4.** Let $\mathcal{P} = (N, A, (\succeq_i)_{i \in N})$ be a political economy, $S$ a winning coalition, and $x, y \in A$ be two policies.

1. **Objection**: $(y, S)$ is said to be an objection against $x$ if $y \succ_S x$.

2. **Counter-objection**: Let $(y, S)$ be an objection against $x$. A pair $(z, T) \in A \times P$ is said to be a counter-objection against $(y, S)$ if $z \succ_T y$.

3. **Unfriendly counter-objection**: Let $(y, S)$ be an objection against $x$ and $(z, T) \in A \times P$ a counter-objection against $(y, S)$. The counter-objection $(z, T)$ is said to be unfriendly if $not(z \succeq_S x)$.

4. **Justified objection**: An objection $(y, S)$ against $x$ is said to be justified if there is no unfriendly counter-objection against $(y, S)$.

5. **Non-reciprocity set**: The non-reciprocity set of the political economy $\mathcal{P}$, denoted by $NR(\mathcal{P})$, is the set of all the policies in $A$ against which no justified objection exists.

The rational behavior under the modified mechanism is defined following the same logic as under the reciprocity mechanism. The main difference resides in the conditions under which a counter-objection is formulated. Under the new mechanism, any winning coalition who wishes to deviate from a sponsored bill to a new policy can formulate a counter-objection. Under the reciprocity mechanism, there exists an additional condition, which is, any winning coalition $T$ that wishes to deviate from a sponsored bill to a more preferred policy can rationally do so only if the interests of some of its members are hurt by the bill. In fact, if the bill does not hurt any member of $T$, any unfriendly counter-objection will cause the first mover $S$ to withdraw its objection. Therefore the unfriendly counter-objection is nullified and the status quo is maintained. This response does not benefit $T$. Under the reciprocity mechanism, it is therefore not rational for $T$ to formulate an unfriendly counter-objection if the objection does not hurt its interests. Whereas under the modified mechanism, this is possible. The result confirms that this absence of incentives for reciprocal actions in the modified mechanism might cause an inefficient policy to persist indefinitely.

**Proposition 1.** *Let $\mathcal{P} = (N, A, (\succeq_i))$ be a political economy under the non-reciprocity mechanism $M'$. Then, a Pareto-dominated policy might be stable.*

The proof is detailed in appendix B.

### 1.6.2 One-Shot Political Games

The most prominent equilibrium concepts for this class of games (Neumann and Morgenstern (1944), Gillies (1959)) are based on the notion of direct dominance. Their definitions are recalled below.

**Definition 5.** Let $\mathcal{P} = (N, A, (\succeq_i))$ be a political economy, $V$ be a subset of $A$, and $x$ and $y$ be two policies.

- $y$ is said to directly dominate $x$, denoted $y >^d x$, if there exist a majority $S$ that strictly prefers $y$ over $x$, that is, $y \succ_S x$.
- The core is the set of all the policies that are not directly dominated.
- $V$ is a vNM stable set if it satisfies the following stability conditions:
    1. (Internal stability): no policy in $V$ is directly dominated by another policy in $V$;

2. (External stability): every policy not in $V$ is directly dominated by some policies in $V$.

It is easy to prove that an inefficient policy cannot belong to the core or a vNM stable set. These equilibrium concepts share this feature with the reciprocity set. However, it is proven below that there exists a political economy for which the reciprocity set is not empty, the core is empty, and a vNM stable set does not exist.

**Proposition 2.** *There exists a political economy $\mathcal{P}_1$ such that the statements below are satisfied:*

1. *the reciprocity set $R(\mathcal{P}_1)$ is non-empty;*
2. *the core is empty; and,*
3. *no subset of the set of policies $A$ satisfies the von Neumann-Morgenstern external and internal stability conditions.*

This analysis makes it possible to compare the properties of the one-shot mechanism and the reciprocity mechanism. As noted above, none of these mechanisms can lead to the selection of an inefficient policy. However, unlike the reciprocity mechanism, the one-shot mechanism may fail to select a stable policy.

### 1.6.3 Infinite Horizon Sequential Political Games

In infinite horizon political games, individuals may bargain everlastingly. In a game, nature randomly chooses a status quo $a$ from the set of policies. If no winning coalition replaces $a$, then $a$ remains forever and the game ends. If a winning coalition $S$ replaces $a$, say by $b$, then $b$ becomes the new status quo, and the process starts over, until a policy is reached that no winning coalition is willing to object to. Once that policy is reached, each individual earns and consumes his payoff and the game ends. Figure 1.5 illustrates this voting process.

The most prominent equilibrium concept in this class of games (Harsanyi (1974), Chwe (1994)) are based on the notion of indirect dominance, a modification of direct dominance, in which first movers anticipate future moves. We recall their definitions below.

**Definition 6.** Let $\mathcal{P} = (N, A, (\succeq_i))$ be a political economy, $K$ be a subset of $A$, and $a$ and $b$ be two policies.

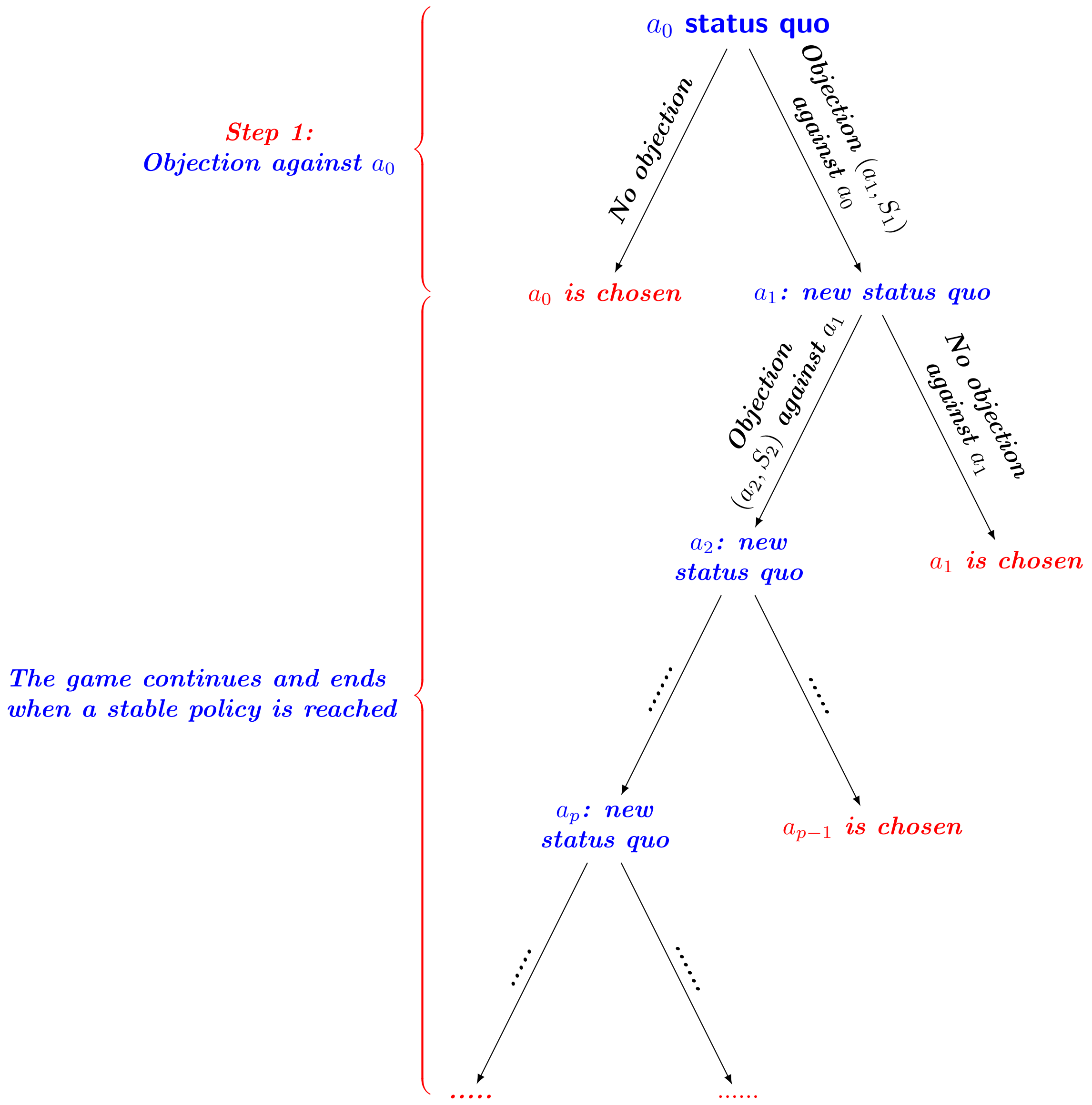



Figure 1.5: Illustration of an infinite horizon sequential political game.

A) Alternative $b$ is said to farsightedly dominate $a$, denoted $b \gg^H a$, if there exists a sequence of policies $a_0,\ a_1,\ ...,\ a_m \in A$ (*where* $a_0 = a$ *and* $a_m = b$) and a sequence of winning coalitions $S_0,\ S_1,\ ..., S_{m-1}$ such that $a_{i+1} >^d a_i$ via $S_{i+1}$ and $b \succ_{S_i} a_i$ for $i = 0, 1, ..., m-1$.

B) $K$ is a farsighted stable set if it satisfies the following conditions:

1. (Internal stability): no policy in $K$ is farsightedly dominated by any other policy in $K$;
2. (External stability): every policy not in $K$ is farsightedly dominated by some policies in $K$.

**Definition 7.** Let $\mathcal{P} = (N, A, (\succeq_i))$ be a political economy, $X$ be a subset of $A$, and $a$ and $b$ be two policies.

A) Alternative $b$ is said to indirectly dominate $a$, denoted $b \gg^C a$, if there exists a sequence of policies $a_0,\ a_1,\ ...,\ a_m \in A$ (*where* $a_0 = a$ *and* $a_m = b$) and a sequence of winning coalitions

$S_0,\ S_1,\ ..., S_{m-1}$ such that $a_{i+1} \longrightarrow_{S_i} a_{i+1}$ and $b \succ_{S_i} a_i$ for $i = 0, 1, ..., m-1$. The relation $a \longrightarrow_S b$ means that if $a$ is a status quo, $S$ can make $b$ the new status quo (Chwe, 1994).

B) $X$ is said to be consistent if:

$$f(X) = \left\{ \begin{array}{l} a \in A : \forall d \in A, S \in P,\ \exists e \in X,\ where \\ e = d\ or\ e \gg^C d\ and\ not(a \succ_S e) \end{array} \right\} = X.$$

C) The largest consistent set is the union of all the consistent sets of $\mathcal{P}$.

These solution concepts formalize the notion that a coalition that moves from a status quo to an alternative policy anticipates the possibility that another coalition might react. A third coalition might in turn react, and so on, without limit. It is therefore important to act in a way that does not lead a coalition to regret its action at the end. What happens in the intermediate stages might not matter, as a coalition simply wants to be better off at the final option relative to a status quo.

The infinite horizon mechanism differs from the reciprocity mechanism in two major respects. First, the reciprocity mechanism is finite and entails only three stages. Second, under the infinite horizon mechanism, a coalition does not have the possibility to revise or withdraw its move, unlike under the reciprocity mechanism. For these reasons, these mechanisms have different equilibrium and welfare properties, as is depicted below.

**Proposition 3.** *There exists a political economy $\mathcal{P}_2$ such that:*

1. *the reciprocity set $R(\mathcal{P}_2)$ is non-empty;*
2. *there is no Harsanyi stable set; and*
3. *the largest consistent set contains a Pareto-dominated policy.*

As already mentioned, Baron and Ferejohn (1989) have also introduced a model of bargaining in a legislature. It is also an infinite horizon game, but unlike the model of Harsanyi and Chwe, it does not accommodate cycles as it is an extensive-form game. Also, in the mechanism of Baron and Ferejohn (1989), there exists a rule that grants veto right to individuals or coalitions. Whereas in the mechanism of Harsanyi and Chwe (and the reciprocity mechanism), no such rule exists as proposals and counter-proposals can be made by any winning coalition. The main differences between these models and the reciprocity mechanism are summarized below:

- The reciprocity mechanism induces selfish individuals to take reciprocal and pro-social actions leading to the selection of efficient policies. It always leads to the selection of an equilibrium policy.

- In contrast, the one-shot mechanism may fail to select a equilibrium policy.

- The infinite horizon mechanism may select an inefficient policy.

## 1.7 Conclusion

In this paper, we propose the *reciprocity mechanism*, which is a political institution that provides incentives for selfish individuals to display reciprocal and pro-social behavior in the allocation of various types of public goods. This new mechanism has desirable properties. In particular, it theoretically resolves the long-established conflict between individual rationality and efficiency in public goods provision.

Our analysis also establishes that the reciprocity mechanism always leads to the selection of a stable policy. This finding covers both discrete and continuous policies. Although a stable policy may be unique under certain natural conditions, multiple equilibria might exist in other environments. We also find that any stable policy is efficient, which means that the reciprocity mechanism prevents political failure. It partly does so by encouraging positive reciprocity and preventing negative reciprocity and free riding.

A comparison of the reciprocity mechanism to other well-known political mechanisms provides that, unlike the former, they lead either to an inefficient policy being selected or fail to select a stable policy.

## 1.8 Appendix A: More Elaboration on Examples

**Example** 3. We derive the predicted reciprocity set at the status quo $x_0 = (0,0,0)$, and the reciprocity set in the ultimatum game with counter-offer. Recall that the utilities functions are:

$$u_1(x_1,x_2,x_3) = x_1 - \frac{1}{3}(x_2 - x_1) - \frac{1}{3}(x_3 - x_1),$$
$$u_2(x_1,x_2,x_3) = x_2 - \frac{1}{3}(x_1 - x_2) - \frac{1}{3}(x_3 - x_2),$$
$$u_3(x_1,x_2,x_3) = x_3 - \frac{1}{3}(x_1 - x_3) - \frac{1}{3}(x_2 - x_3).$$

Assume for instance that individuals 1 and 2 propose a distribution $x = (x_1, x_2, x_3)$ to individual 3, with $x_1 + x_2 + x_3 = 100$. H/Se is indifferent between rejecting $x$ and accepting $x_0 = (0,0,0)$ if $u_3(x) = 0$, which means that $x_3 = \frac{1}{5}(x_1 + x_2)$. Since $x_1 + x_2 + x_3 = 100$, we have $x_1 + x_2 = \frac{500}{6}$ and therefore $x_3 = \frac{100}{6}$. Hence, the utility of individual 3, if he accepts a share $(x_1, x_2, \frac{100}{6})$, with $x_1 + x_2 = \frac{500}{6}$ is zero. Given that utilities are symmetric, the same conclusion applies to each individual. This proves that proposing an allocation which exhausts the total amount of money and gives less than 100/6 to an individual is rejected. Therefore, the set of predicted allocations from the status quo $x_0 = (0,0,0)$ is:

$$P(x_0) = \left\{(x_1,x_2,x_3) : x_1 + x_2 + x_3 = 100 \text{ and } x_i \geq \frac{100}{6};\ i = 1,2,3\right\}.$$

What about the reciprocity set?

- Any allocation $x = (x_1, x_2, x_3)$ such that $x_1+x_2+x_3 < 100$ is dominated. The allocation $A = (x_1 + \frac{100-x_1-x_2-x_3}{3}, x_2 + \frac{100-x_1-x_2-x_3}{3}, x_3 + \frac{100-x_1-x_2-x_3}{3})$ Pareto-dominates $x$, and no one has an incentive to oppose $A$, meaning that $A \gg x$ via $\{1,2,3\}$.

- Any allocation that gives all the money to one individual is dominated. For instance, consider the allocation $x = (100, 0, 0)$. If $x$ is a status quo, then a majority coalition which consists of individuals 2 and 3 will formulate an objection $(y, \{2,3\})$ against $x$, where $y = (34, 33, 33)$. Then, individual 1 will oppose the objection. But individuals 2 and 3 will not withdraw it since they have no incentive to do so. Player 1 can convince individual 2 to formulate a counter-objection $((76, 34, 0), \{1,2\})$ against $(y, \{2,3\})$ without negative ramification. Moreover, any counter-objection $(z, T)$ against $y$ is friendly to $\{2,3\}$, which implies that $y \gg x$.

- Any allocation that gives a strictly positive amount to at least two individuals is stable. Consider, for instance, the allocation $x = (x_1, x_2, 0)$, with $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 = 100$. If $x$ is a status quo, then individual 3 can form a coalition with either individual 1 or individual 2 to initiate an objection against $x$. Let us assume that the majority coalition $\{2, 3\}$ objects against $x$ by proposing the allocation $(0, x_2+a, x_1-a)$, with $a > 0$ and $x_1 > a$. Player 1 will oppose this allocation, causing $\{2, 3\}$ to either withdraw it or maintain it. But $\{2, 3\}$ will withdraw it because otherwise, individual 1 will form a coalition with either individual 2 or individual 3 to formulate a counter-objection. Assume, for instance, that individuals 1 and 3 form a coalition and propose the allocation $(x_1, 0, x_2)$. Then, this will lower the payoff of individual 2. Given that individual 2 is aware of this possibility, he will withdraw from the coalition $\{2, 3\}$, which will invalidate the objection $((0, x_2+a, x_1-a), \{2, 3\})$ against a status quo $x = (x_1, x_2, 0)$. Therefore, there is no objection against a status quo $x = (x_1, x_2, 0)$. Basically, given that $x_1 + x_2 = 100$, it is impossible to formulate an objection that improves all individuals' payoffs. It follows that $x = (x_1, x_2, 0)$ is stable. Given that individuals have symmetric preferences, any allocation that gives a positive amount to at least two individuals is stable. The reciprocity set is therefore:

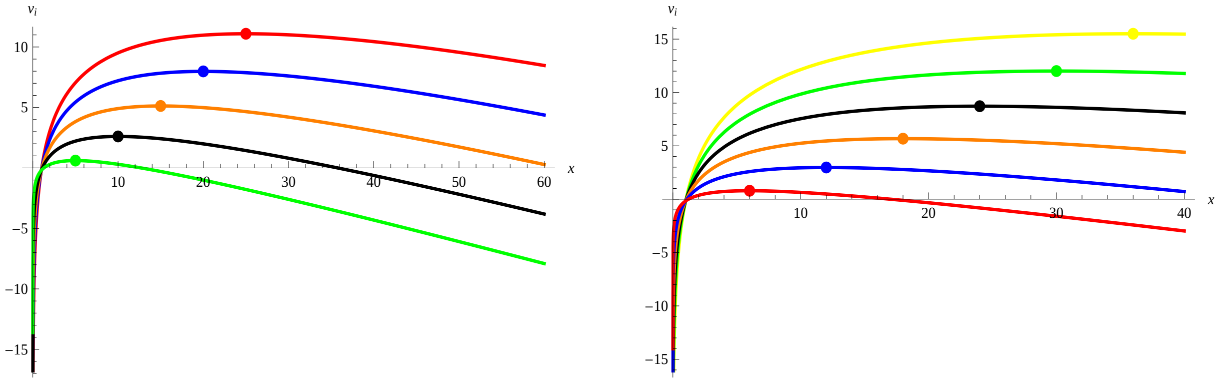
$$R(\mathcal{P}) = \left\{ \begin{array}{l} (x_1, x_2, x_3) : \; x_1 + x_2 + x_3 = 100, \text{and } \nexists i \neq j \\ \text{such that } x_i = x_j = 0; \; i, j = 1, 2, 3 \end{array} \right\}.$$

**Example** 5. To illustrate, assume that there are five legislators who have to decide on the number of refugees to be admitted. The utility function of each legislator $i$ is given by (see Figure 1.6):

$$v_i(x_i) = b_i \ln(x_i) - 1/5x_i.$$

Observe that this function is single-peaked, legislator $i$'s optimum is obtained by solving $v'(x_i) = 0$; this leads to the solution $x_i^* = 5b_i$. Figure 1.6a represents legislator's utility function, assuming $b_1 = 5$, $b_2 = 4$, $b_3 = 3$, $b_4 = 2$ and $b_5 = 1$. Legislators' optima are therefore: $x_1^* = 25$, $x_2^* = 20$, $x_3^* = 15$, $x_4^* = 10$ and $x_5^* = 5$.

We recall that the decision rule is the majority rule. The point $x_3^*$, which is the optimum of the median legislator, will defeat any other legislator's peak in a pairwise majoritarian election under the reciprocity mechanism. Actually, $x_3^* \succ_{1234} x_5^*$, $x_3^* \succ_{123} x_4^*$, $x_3^* \succ_{345} x_2^*$ and




(A) Five legislators (B) Six legislators

Figure 1.6: legislators' utility functions - The bundle on each curve represents legislator $i$'s optimum point.

$x_3^* \succ_{345} x_1^*$. Also, there is no optimum which is preferred by a majority coalition to $x_3^*$. It follows that $x_3^*$ is the only stable optimum: $R(\mathcal{P}) = \{15\}$.

Now, assume that there are six legislators, with legislator $i$'s utility function being $v_i(x) = b_i \ln(x) - 1/6x_i$, where $b_1 = 5$, $b_2 = 4$, $b_3 = 3$, $b_4 = 2$, $b_5 = 1$ and $b_6 = 6$ (see Figure 1.6b). Legislators' optima are: $x_6^* = 36$, $x_1^* = 30$, $x_2^* = 24$, $x_3^* = 18$, $x_4^* = 12$ and $x_5^* = 6$. In this scenario, there is no median legislator. The stable optima are $x_2^*$ and $x_3^*$: $R(\mathcal{P}) = \{24, 18\}$. In fact, in a pairwise majoritarian election, $x_3^*$ defeats $x_5^*$ since $x_3^* \succ_{12346} x_5^*$; $x_3^*$ defeats $x_4^*$ because $x_3^* \succ_{1236} x_5^*$; $x_3^*$ and $x_2^*$ are tied. Also, $x_2^*$ defeats $x_1^*$ because $x_2^* \succ_{2345} x_1^*$; and $x_2^*$ defeats $x_6^*$ since $x_2^* \succ_{2345} x_6^*$. No majority coalition prefers another option to either $x_2^*$ or $x_3^*$. Using Theorem 2, we can conclude that $x_2^* \gtrdot x_1^*$, $x_2^* \gtrdot x_6^*$, $x_3^* \gtrdot x_5^*$ and $x_3^* \gtrdot x_4^*$, which implies that $R(\mathcal{P}) = \{x_2^*, x_3^*\} = \{24, 18\}$. Therefore, either 18 or 24 refugees will be admitted into the country.

## 1.9 Appendix B: Proofs of Results

**Proof of Theorem 1.** Let $\mathcal{P} = (N, A, (\succeq_i)_{i\in N}, M)$ be a political economy, and $S, T$ two winning coalitions, and $x, y, z \in A$ three distinct policies such that $y \succ_S x$ and $z \succ_T y$.

1. Suppose that *not*$(y \succeq_T x)$ and *not*$(z \succeq_S x)$. Assume by contradiction that $S$ formulates an objection $(y, S)$ against $x$. Then, since $z \succ_T y$, a member of $T$ who is not a member of $S$ will oppose the objection. Hence, $S$ will either withdraw or maintain its objection following the prescription in stage 2. Clearly, $S$ will withdraw its objection since, if it does not, a counter-objection $(z, T)$ against $(y, S)$ will be formulated by $T$, leading to $z$ being selected as the new policy. This is a worse outcome for some members of $S$

because $not(z \succeq_S x)$. It follows that $x$ will remain in place, implying that formulating an objection $(y, S)$ against $x$ will not profit $S$. The coalition $S$ will therefore not formulate such an objection, which is a contradiction.

2. Assume that $S$ formulates an objection $(y, S)$ against $x$, and suppose that $y \succeq_T x$. Assume by contradiction that a member of $T$ who is not a member of $S$ opposes the move. Then, $S$ will either withdraw or maintain its objection. Given that the counter-objection $(z, T)$ is a possibility if $S$ does not withdraw $y$ and is an unfriendly counter-objection by definition (since $not(z \succeq_S x)$), $S$ will withdraw its objection, causing $x$ to remain in place. But this does not benefit coalition $T$, since $y \succeq_T x$. Consequently, $T$ will not formulate any unfriendly counter-objection $(z, T)$ against $(y, S)$, a contradiction.

3. Assume that $y \succeq_T x$. If $S$ formulates an objection $(y, S)$ against $x$, there will be no opposition that could lead to an unfriendly counter-objection $(z, T)$ against $(y, S)$. This occurs because such an opposition will lead to $S$ withdrawing its objection and causing the election of $x$. This response will not benefit members of coalition $T$. Then, following the objection, either $y$ will be elected if there is opposition or any opposition and subsequent counter-objection $(z, T)$ will be such that $z \succeq_S x$. In either case, $S$ will benefit, justifying its objection $(y, S)$ against option $x$.

**Proof of Theorem 2**. Let $M(\mathcal{P}, \gtrdot)$ denote the set of all maxima elements of binary relation $\gtrdot$. We want to show that $\mathcal{R}(\mathcal{P}) = M(\mathcal{P}, \gtrdot)$.

1. First, prove that $\mathcal{R}(\mathcal{P}) \subseteq M(\mathcal{P}, \gtrdot)$. Let $x \in A$ be an option such that $x \notin M(\mathcal{P}, \gtrdot)$. Then, there exists a policy $y \in A$ and a winning coalition $S \in P$ such that $y \gtrdot_S x$. It follows that a couple $(y, S)$ is an objection against policy $x$. The objective is to demonstrate that this objection against $x$ is justified. If there is no counter-objection against $(y, S)$, everything is done. Assume that there exists a counter-objection $(z, T)$ against $(y, S)$, thus the assertions described hereunder hold:

   *i)* $z \succ_T y$;

   *ii)* $not(y \succeq_T x)$.

   Given that $y \gtrdot_S x$, then 1.*i*) and 1.*ii*) entail $z \succeq_S x$, and the counter-objection $(z, T)$ against $(y, S)$ is friendly. Therefore, a pair $(y, S)$ is a justified objection against $x$ by definition, meaning that $x \notin R(\mathcal{P})$. We conclude that $R(\mathcal{P}) \subseteq M(\mathcal{P}, \gtrdot)$.

2. Second, prove that $M(\mathcal{P}, \gtrdot) \subseteq \mathcal{R}(\mathcal{P})$. Let $x \in A$ be a policy such that $x \notin R(\mathcal{P})$. Then, there exists a justified objection $(y, S) \in A \times P$ against policy $x$.

   *iii*) If there is no winning coalition $T$ and policy $z$ such that $z \succ_T y$, then $y \succ_S x$ and implication *b*) is satisfied since the right hand side of this implication is false. It follows that $y \gtrdot x$ and $x \notin M(\mathcal{P}, \gtrdot)$.

   *iv*) Assume that there exists $(z, T) \in A \times P$ such that $z \succ_T y$. We have two possibilities described below:

   - If *not*$(y \succeq_T x)$, then $(z, T)$ is a counter-objection against $(y, S)$. Since $(y, S)$ is a justified objection, it follows that any counter-objection against $(y, S)$ is friendly, leading to $z \succeq_S x$. Then, $y \gtrdot x$ and $x \notin M(\mathcal{P}, \gtrdot)$.
   - If $y \succeq_T x$, so *not*$(y \succeq_T x)$ is false, then the implication [*not*$(y \succeq_T x) \Rightarrow z \succeq_S x$] is true. Therefore, $y \gtrdot_S x$ and $x \notin M(\mathcal{P}, \gtrdot)$. We can conclude that $M(\mathcal{P}, \gtrdot) \subseteq R(\mathcal{P})$.

   We conclude the proof that $R(\mathcal{P}) = M(\mathcal{P}, \gtrdot)$.

**Proof of Theorem 3**. For all $x \in A$, define $f(x) = |\{y \in A; y \succ x\}|$ and let $x_0 \in A$ such that $f(x_0) = \min_{x \in A}\{f(x)\}$. Prove that $x_0 \in R(\mathcal{P})$. If this assertion is not true, then there exists $y \in A$ and a winning coalition $S$, such that $y \gtrdot_S x_0$. It follows that:

$$\begin{cases} (\alpha)\ y \succ_S x,\ \ S \in P \\ (\beta)\ \forall (z, T):\ T \neq S,\ \ z \succ_T y, T \in P \text{ and not}(y \succeq_T x) \Rightarrow z \succeq_S x \end{cases} . \tag{1.1}$$

By definition of $x_0$ and given that the preference $\succ$ is asymmetric, there exists $c \in A$ such that $c \succ y$ and *not* $(c \succ x_0)$. Thus, there exists a coalition $T$, with $T \in P$ such that $c \succ_T y$.

- If $y \succ_T x_0$, then we have $c \succ_T y$ and $y \succ_T x_0$, so $c \succ_T x_0$ by transitivity, a contradiction because *not* $(c \succ x_0)$.
- Suppose that *not* $(y \succ_T x_0)$, then according to the assertion $(\beta)$ in equation 1.1, we have $c \succ_S x_0$, a contradiction.

We conclude that $x_0 \in R(\mathcal{P})$.

**Proof of Theorem 4**. Let $\mathcal{P}$ be a political economy in which each preference relation $\succeq_i$ is single-peaked over the policy set $A$. The proof is done by construction. We recall that the constitution is defined here by the majority rule. Let $x \in A$ be a policy and define $S(x)$ as the number of individuals for whom $x$ is the peak. Consider the functions $f$ and $g$ defined on $A$ as follows: for any $q \in A$,

$$f(q) = \sum_{x \leq q} S(x) - \tfrac{n}{2},$$

and

$$g(q) = \sum_{x \geq q} S(x) - \tfrac{n}{2}.$$

Let $A_f = \{p \in A : p \text{ is a peak and } f(p) \geq 0\}$ and $A_g = \{p \in A : p \text{ is a peak and } g(p) \geq 0\}$. Notice that neither $A_f$, nor $A_g$ is empty. In fact, if $\overline{p}$ and $\underline{p}$ are respectively the greatest and the smallest peaks in $A$, then $f(\overline{p}) = n - \frac{n}{2} = \frac{n}{2} > 0$ and $g(\underline{p}) = n - \frac{n}{2} = \frac{n}{2} > 0$, which implies that $\overline{p} \in A_f$ and $\underline{p} \in A_g$, in turn implying that $A_f \neq \emptyset$ and $A_g \neq \emptyset$.

Given that $A_f$ is finite and $f$ is an (strictly) increasing function, there exists a unique peak $q_1^*$ which minimizes $f$ over $A_f$. In addition, for any peak $p < q_1^*$, $f(p) < 0$, which implies that $\sum\limits_{x \leq p} S(x) < \frac{n}{2}$.

Similarly, given that $A_g$ is finite and $g$ is a (strictly) decreasing function, there exists a unique peak $q_2^*$ which minimizes $g$ over $A_g$. In addition, for any peak $p > q_2^*$, $g(p) < 0$, which implies that $\sum\limits_{x \geq p} S(x) < \frac{n}{2}$.

In this proof, it is established that there exists at least one equilibrium and that $q_1^*$ and $q_2^*$ are the only (possible) equilibria. First note the following facts:

**Fact 1:** For any $q \in A$, $f(q) + g(q) = s(q)$. This is derived from the definitions of $f$ and $g$.

**Fact 2:** There is no peak strictly comprised between $q_1^*$ and $q_2^*$. In fact, if $q_1^* = q_2^*$, the claim is proved. Consider that $q_1^* \neq q_2^*$. Two cases will be differentiated: $q_1^* < q_2^*$ and $q_1^* > q_2^*$.

- Suppose $q_1^* < q_2^*$ and by contradiction, that there exists a peak $q^*$ between $q_1^*$ and $q_2^*$. Using the definition of $q_1^*$, we have $f(q^*) > 0$, which implies that $\sum\limits_{x \leq q^*} S(x) - \frac{n}{2} > 0$, and then $\sum\limits_{x \leq q^*} S(x) > \frac{n}{2}$. This means the number of individuals whose peak is weakly on the left of $q^*$ exceeds half of the population. We also know that $g(q_2^*) \geq 0$, which is equivalent to $\sum\limits_{x \geq q_2^*} S(x) - \frac{n}{2} \geq 0$, and $\sum\limits_{x \geq q_2^*} S(x) \geq \frac{n}{2}$. The latter inequality means that the number of individuals whose peak is weakly on the right of $q_2^*$ is at least half of the population. Since $q^* < q_2^*$, there is no individual whose peak is weakly on the left of $q^*$ and weakly on the right of $q_2^*$ at the same time. But, both $\sum\limits_{x \leq q^*} S(x) > \frac{n}{2}$ and $\sum\limits_{x \geq q_2^*} S(x) \geq \frac{n}{2}$ imply that the number of individuals whose peak is weakly on the left of $q^*$ or weakly on the right of $q_2^*$ strictly exceeds the total population $n$, which is a contradiction.

- Suppose that $q_1^* > q_2^*$ and by contradiction, that there exists a peak $q^*$ between $q_1^*$ and $q_2^*$. Using the definition of $q_1^*$ and $q_2^*$, it follows that $f(q^*) < 0$ (since $q^* < q_1^*$) and $g(q^*) < 0$, since $q^* > q_2^*$. This implies that $f(q^*) + g(q^*) < 0$, and $S(q^*) < 0$, because $f(q^*) + g(q^*) = S(q^*)$, which is a contradiction.

**Fact 3:** If $q_1^* \neq q_2^*$, then $q_1^* < q_2^*$. By contradiction, consider that $q_1^* > q_2^*$. It follows from the definition of $q_1^*$ and $q_2^*$ that $f(q_2^*) < 0$ and $g(q_1^*) < 0$. This implies that $\sum\limits_{x \leq q_2^*} S(x) < \frac{n}{2}$ and $\sum\limits_{x \geq q_1^*} S(x) < \frac{n}{2}$, and that $\sum\limits_{x \leq q_2^*} S(x) + \sum\limits_{x \geq q_1^*} S(x) < n$. In other words, the number of individuals whose peak is weakly on the left of $q_2^*$ or weakly on the right of $q_1^*$ is strictly smaller than the size of the entire population $n$. Thus, there exists an individual whose peak is strictly comprised between $q_1^*$ and $q_2^*$. This is a contradiction, because it is proven that there is no peak strictly comprised between $q_1^*$ and $q_2^*$ (see Fact 2).

We prove that $q_1^*$ and $q_2^*$ are the only equilibria. Three cases are covered as described hereunder:

**Case 1**: $n$ is even and $q_1^* = q_2^* = q^*$ (see Figure 1.7). Then, the analysis claims that $q^*$ is the only equilibrium: $R(\mathcal{P}) = \{q^*\}$.

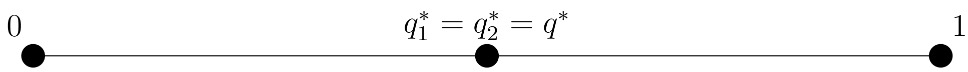



FIGURE 1.7: The voters' size is even and individuals have the same peaks.

Let $p \in A$ be a peak. If $p < q^*$, by definition of $q^*$, $\sum\limits_{x \leq p} S(x) < \frac{n}{2}$ and $\sum\limits_{x \geq q^*} S(x) > \frac{n}{2}$, which implies that $q^*$ beats $p$ in a pairwise majoritarian election. Similarly, if $p > q^*$, we show in the same way that $q^*$ beats $p$ in a pairwise majoritarian election. It follows that $q^*$ beats any other peak in a pairwise majoritarian election. Since there is no other option which beats $q^*$, then the implication b) of Definition 2 is always true, implying that $q^*$ is not dominated in the sense of the domination relation $\gtrdot$. Therefore, $R(\mathcal{P}) = \{q^*\}$.

**Case 2**: $n$ is even and $q_1^* \neq q_2^*$ (see Figure 1.8). In this case,it is shown that, $q_1^*$ and $q_2^*$ are the only equilibria.


FIGURE 1.8: The voters' size is even and the minima peaks $q_1^*$ and $q_2^*$ are different.

First, notice that neither $q_1^*$, nor $q_2^*$ is defeated by any other policy in a pairwise majoritarian election. Also, in a pairwise majoritarian election opposing $q_1^*$ and $q_2^*$, neither will win, since $q_1^* < q_2^*$, and $\sum\limits_{x \leq q_1^*} S(x) + \sum\limits_{x \geq q_2^*} S(x) = n$, each will receive exactly $\frac{n}{2}$ votes. Now, let $p \in A$ be a peak that is distinct from $q_1^*$ and $q_2^*$. It follows from Facts 2 and 3 that, either $p < q_1^*$ or $p > q_2^*$.

1. If $p < q_1^*$, then $q_1^*$ beats $p$ in a pairwise majoritarian election since the number of votes obtained by $q_1^*$ against $p$ is at least equal to $\sum\limits_{x \geq q_1^*} S(x) > \sum\limits_{x \geq q_2^*} S(x) \geq \frac{n}{2}$. By the fact that $q_1^*$ is not defeated by any other policy in the majoritarian sense, $q_1^*$ dominates $p$ in the sense of the binary relation $\gtrdot$.

2. If $p > q_2^*$, then $q_2^*$ beats $p$ in a pairwise majoritarian election since the number of votes obtained by $q_2^*$ against $p$ is at least equal to $\sum\limits_{x \leq q_2^*} S(x) > \sum\limits_{x \leq q_1^*} S(x) \geq \frac{n}{2}$. In addition, given that $q_2^*$ is not dominated by any other policy in the majoritarian sense, it follows that $q_2^*$ dominates $p$ in the sense of $\gtrdot$.

   We conclude that $R(\mathcal{P}) = \{q_1^*, q_2^*\}$.

**Case 3**: $n$ is odd. It is shown that $q_1^* = q_2^*$. Consider the contrary, meaning that $q_1^* \neq q_2^*$. Fact 3 implies that $q_1^* < q_2^*$. And since $n$ is odd and there is no individual whose peak is strictly comprised between $q_1^*$ and $q_2^*$ (Fact 2), the number of individuals whose peak is weakly at the left of $q_1^*$ (i.e., $\sum\limits_{x \leq q_1^*} S(x)$) is different from the number of individuals who peak is weakly at the right of $q_2^*$ (i.e., $\sum\limits_{x \geq q_2^*} S(x)$). Two cases are displayed:
1. Suppose that $\sum\limits_{x \leq q_1^*} S(x) > \sum\limits_{x \geq q_2^*} S(x)$. Recall that by the definition of $q_2^*$, $\sum\limits_{x \geq q_2^*} S(x) \geq \frac{n}{2}$, then, $\sum\limits_{x \leq q_1^*} S(x) > \frac{n}{2}$. This implies that $\sum\limits_{x \leq q_1^*} S(x) + \sum\limits_{x \geq q_2^*} S(x) > n$, which is a contradiction, because $\sum\limits_{x \leq q_1^*} S(x) + \sum\limits_{x \geq q_2^*} S(x) = n$.
2. If $\sum\limits_{x \leq q_1^*} S(x) < \sum\limits_{x \geq q_2^*} S(x)$, then, using the definition of $q_1^*$ ($\sum\limits_{x \geq q_1^*} S(x) \geq \frac{n}{2}$), therefore $\sum\limits_{x \leq q_1^*} S(x) > \frac{n}{2}$. Consequently, $\sum\limits_{x \leq q_1^*} S(x) + \sum\limits_{x \geq q_2^*} S(x) > n$, which is a contradiction, given that $\sum\limits_{x \leq q_1^*} S(x) + \sum\limits_{x \geq q_2^*} S(x) = n$.

That being so, $q_1^* = q_2^*$. Hence, referring to Case 1, as $q_1^* = q_2^* = q^*$, $q^*$ is the unique equilibrium (note that in Case 1, the proof does not necessarily use the assumption that $n$ is even). This concludes that $R(\mathcal{P}) = \{q^*\}$.

**Proof of Theorem 5**. Let $f$ be the function defined over the policy $A$ by $f(x) = \mathfrak{l}(\{y \in A; y \succ x\})$, where $\mathfrak{l}$ is the Lebesgue measure on the manifold spanned by $A$. Given that preferences are continuous, there exists $x_0 \in A$ such that $f(x_0) = \min\limits_{x \in A}\{f(x)\}$. The goal is proving that $x_0 \in R(\mathcal{P})$. By contradiction, there exists $y \in A$ and a winning coalition $S$, such that $y \gtrdot_S x_0$. It follows that:

$$\begin{cases} (\alpha')\ y \succ_S x,\ \ S \in P \\ (\beta')\ \forall (z, T) /\ z \succ_T y, T \in P \text{ and not}(y \succeq_T x) \Rightarrow z \succeq_S x \end{cases}. \tag{1.2}$$

By definition of $x_0$ and given that the preference $\succ$ is asymmetric, there exists $c \in A$ such that $c \succ y$ and *not* $(c \succ x_0)$. Thus, there exists a coalition $T$, with $T \in P$ such that $c \succ_T y$.

- If $y \succ_T x_0$, then we have $c \succ_T y$ and $y \succ_T x_0$, so $c \succ_T x_0$ by transitivity, a contradiction because *not* $(c \succ x_0)$.
- If *not* $(y \succ_T x_0)$, then according to $(\beta')$ in equation 1.2, we have $c \succeq_S x_0$, a contradiction.

We can conclude that $x_0 \in R(\mathcal{P})$.

**Proof of Theorem 6**. Let $x \in A$ be a policy such that $x \in R(\mathcal{P})$. If $x$ is Pareto-dominated, then there exists $y \in A$ such that $y \succ_N x$. It follows that for any $z \in A$ such that $z \succ_T y$ with $T \in P$, we have $y \succ_T x$ because $T \subset N$. Then, not $(y \succeq_T x)$ is always false. Therefore the implication [*not* $(y \succeq_T x) \Rightarrow z \succeq_N x$] is always true i.e $y$ dominates $x$ via $N$ or $(y, N)$ is a justified objection against $x$, which is a contradiction. Thus, $x$ is Pareto-efficient.

**Proof of Proposition 1**. Consider the society $\mathcal{P}_0$ which consists of five individuals in the set $N = \{1, 2, 3, 4, 5\}$, and the policy space $A = \{a, b, c, d, e\}$. The individuals' preferences over these policies are defined hereunder: $c \succ_1 b \succ_1 a \succ_1 e \succ_1 d$, $d \succ_2 c \succ_2 b \succ_2 a \succ_2 e$, $e \succ_3 b \succ_3 a \succ_3 d \succ_3 c$, $e \succ_4 d \succ_4 b \succ_4 a \succ_4 c$, $c \succ_5 d \succ_5 e \succ_5 b \succ_5 a$. Consider $K = \{1, 3, 4\}$, $R = \{2, 4, 5\}$, $L = \{1, 2, 5\}$, and $Q = \{2, 3, 4\}$ the minimal winning coalitions. Figure 1.9

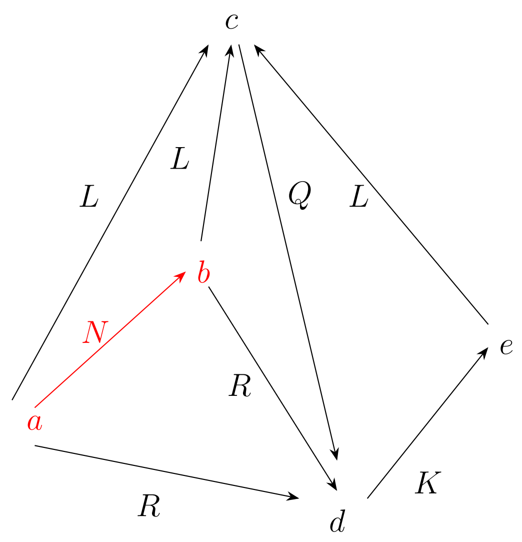



FIGURE 1.9: Majority domination among policies.

provides the graph of the majority domination among the policies (e.g., the arrow from $a$ to $b$ means the grand coalition $N$ strictly prefers policy $b$ to $a$.).

The Pareto-dominated policy $a$ is stable under the modified mechanism $M'$. Although all the individuals prefer $b$ over $a$, no winning coalition will object to $a$ since this will lead to the election of either $d$ (if coalition $L$ objects) or $e$ (if coalition $R$ objects). In either case, some member of the objecting coalition will regret, which is the reason why none of these coalitions will form to object. This implies that $a \in NR(\mathcal{P}_0)$.

Under the modified mechanism some individuals, who might be willing to participate in a coalition in order to remove an inefficient status quo, fear that the interim policy will be replaced by a new policy which is worse for them than a status quo. Under the reciprocity mechanism, coalition $K$, for instance, would be willing to formulate an objection $(b, K)$ against $a$. Once $b$ is introduced, coalition $L$ will not formulate a counter-objection $(L, c)$, knowing that such a counter-objection is unfriendly as it hurts the interest of individuals 3 and 4 in $K$. In fact, if it does, these individuals will withdraw from $K$, destroying $P$ and causing $a$ to remain in place. This response hurts the interest of individuals 2 and 5 in the counter-objecting coalition $L$. These individuals will therefore not participate in coalition $L$ and that coalition will never form to counter-object. Also, coalition $R$ will not formulate a counter-objection $(R, d)$ since individuals 2 and 5 will not participate in that coalition. We note that, in this economy, $NR(\mathcal{P}_0) = \{a, b, c, d, e\}$ and $R(\mathcal{P}_0) = \{b, c, d, e\}$.

**Proof of Proposition 2**. The political economy $\mathcal{P}_0$ considered in this proof is the same that we used in the proof of Proposition 1, except for the mechanism. We consider $\mathcal{P}_1 = \mathcal{P}_0$.

1. As it is proven in the proof of Proposition 1, the set of stable outcomes under the reciprocity mechanism is given by $R(\mathcal{P}_1) = \{b, c, d, e\}$. The reciprocity mechanism expels the Pareto-dominated policy $a$.

2. We can observe in Figure 1.9 that, for any policy $x \in A$, there exists another policy $y \in A$ and a majority coalition $S$ such that all individuals in $S$ strictly prefer $y$ to $x$. This shows that the core is empty.

3. In this political economy, the vNM stable sets do not exist. In this political economy, $c >^d a$, $c >^d b$, $c >^d e$, $b >^d a$, $d >^d a$, $d >^d b$, $d >^d c$ and $e >^d d$. Hence, no subset of the policy set $A$ satisfies von Neumann and Morgenstern external and internal stability conditions.

**Proof of Proposition 3.** Consider the political economy $\mathcal{P}_0$ defined in the proof of Proposition 2. Figure 1.9 above can also be viewed as the graph of "effectiveness relations" (Chwe, 1994) which is equivalent to the graph of domination relationship $>^d$ defined in section 1.6.2. It is straightforward to prove what follows: $c \gg^C a$, $c \gg^C b$, $c \gg^C e$, $b \gg^C a$, $d \gg^C a$, $d \gg^C b$, $d \gg^C c$ and $e \gg^C d$. Note that in this specific political economy, the binary relations $\gg^H$ and $\gg^C$ are equivalent.

1. It is already presented in the previous proof that the reciprocity set is non-empty and contains only efficient policies.

2. Using Figure 1.9 and the farsighted dominance $\gg^H$, we can conclude that there is no subset of policy set $A$ which satisfies internal and external stability conditions, meaning that no Harsanyi stable set exists.

3. Let us determine the largest consistent set. We start with set $X = A$. It turns out that $f(X) = X$ and $A$ is the largest consistent set. For instance, starting with policy $a$, there are three possibilities: either coalition $L$ will move from $a$ to $c$, or coalition $R$ will move to $d$, or all individuals will move to $b$. For each initial move, there might be a next move that reach either $e$, or $d$ or $c$ and some individuals are worse-off. For these reasons, $a$ is stable. We have the same conclusion for each policy that belongs to $A$.

# Chapter 2

# Endogenous Political Rules: A Rationalization Based on Policy Importance

## 2.1 Introduction

> "There are two general rules. First, the more the grave and important the questions discussed the nearer should the opinion that is to prevail approach to unanimity. Second, the more the matter in hand calls for speed, the smaller the prescribed difference in the number of votes may be allowed to become: when an immediate decision has to be reached, a majority of one should suffice" (Rousseau, 1762, n.a).

Rules used to adopt policies in legislative organizations vary widely. For example, the United States House of Representatives may, by a simple majority of vote, impeach a federal official, whereas removal from office requires a two-thirds majority of the Senate. The United Nations Security Council requires a three-fifths supermajority of the members on substantive matters, whereas procedural matters require a simple majority of those present and voting. The legislature of Nebraska can vote to increase property taxes reflecting changes in the Consumer Price Index by a simple majority, while larger increases, up to 5%, require a three-quarters majority. The European Union applies the unanimity rule for particularly "sensitive" issues, and either a simple or qualified majority rule for "technical" matters. The variation in majority requirements within the same legislative institution raises the obvious question of why these requirements differ so much across policy types.

What is the exact relationship between a policy's importance and the voting rule under which such a policy should be adopted? It is argued that the size of the supermajority needed to pass a policy should depend on the importance or complexity of this policy.[1] However, we are not aware of any study that rationalizes voting rules in terms of policy importance. Addressing this gap in the literature has practical implications for the optimal design and functioning of political institutions.

In this paper, we analyze this question from the perspective of a political designer who has to choose a supermajority voting rule in the present for the selection of a policy in the distant future. The political designer consequently has no knowledge of individual political preferences (voting rules are chosen prior to the vote). The political designer only has information about policy importance, the latter being measured by its *social dimensionality*—the number of dimensions or sectors of society's life and wellbeing the policy is likely to affect. For instance, withdrawing from a union (e.g., Brexit) may affect a country in several dimensions including its diplomacy, its ability to attract talented immigrants, its national security, and its economy. The policy importance also consequently represents the different non-correlated dimensions of citizens' preferences that may be affected by a policy (Harris and Sutton (1983), Krijnen, Zeelenberg, and Breugelmans (2015)). Given the social dimensionality of a question (decision) at hand, the political designer's goal is to choose the supermajority rule which:

**(P1)**: ensures that a *stable policy* exists regardless of the extent to which individual preferences, unknown to the political designer, are antagonistic[2];

**(P2)**: leads to all stable policies being *Pareto efficient*; and

**(P3)**: minimizes status quo bias[3].

[1]The introductory quote from Jean-Jacques Rousseau's work is evident of this view. More recent works expressing this basic intuition are Erlenmaier and Gersbach (2001), Barbera and Jackson (2004), Maggi and Morelli (2006), Holden (2005), and Rogoff (2016), among others. Throughout the paper, the words importance and complexity, when they are related to the nature of a decision (policy) are interchangeable.

[2]This requirement is similar to the concept of decisiveness (Dasgupta and Maskin (2008)) or positive responsiveness (May (1952)) in the literature. The existence of a stable policy implies that a permanent decision can be achieved.

[3]This property is equivalent to choose a rule which maximizes neutrality between competing policy alternatives. It is well known that any supermajority rule is biased towards preserving the status quo, and is therefore not neutral between competing policy alternatives. As put by Dasgupta and Maskin (2008, P.2), if "the candidates are, say, various amendments to a nation's constitution, then one might want to give special treatment to the status quo–namely, to no change–and so ensure that constitutional change occurs only with overwhelming support". Minimizing the status quo bias is therefore equivalent to minimizing the size of the supermajority required to change the incumbent policy. It follows that the minimal status quo bias property can also be justified on the ground of cost minimization, especially if the formation of political coalitions is costly as it is generally argued in the political science literature.

The legislative procedure under which decision making takes place is a sequential mechanism, which is a simplification of well-known parliamentary procedures encountered in democratic societies (see section 2.3.3).

The political designer's objective problem is solved both for continuous and for discrete decisions. A continuous problem involves a choice set consisting of a continuum of competing policy alternatives, while a discrete problem involves a finite number of policy alternatives. A closed-form solution is derived, and the latter indicates the exact relationship between policy importance and voting rule (Theorem 7 and Theorem 8 in section 2.4). This relationship is depicted in Figure 2.1 for continuous decisions and in Figure 2.2 for discrete decisions.

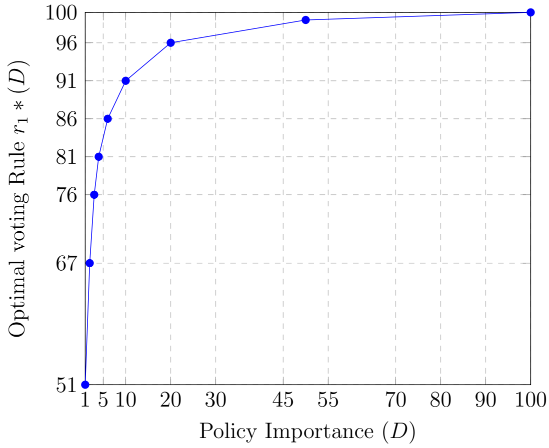



FIGURE 2.1: Optimal supermajority rule as a function of policy importance for continuous decisions.

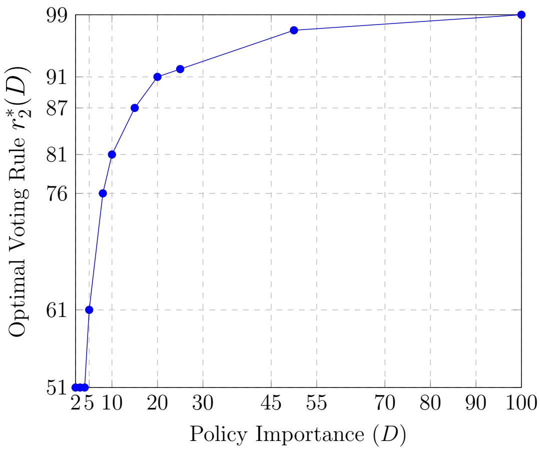



FIGURE 2.2: Optimal supermajority rule as a function of policy importance for discrete decisions.

The optimal supermajority voting rule is an increasing function of policy importance, with a concave shape for continuous decisions. In particular, if the policy importance is sufficiently low, it should be selected using the absolute majority rule. However, policies of sufficiently high importance should be selected using a rule that approaches the unanimity rule. These findings provide an important theoretical foundation to Rousseau's prescriptions.

The results have practical implications for the optimal design of political institutions. If the goal of a society is to reach permanent decisions, then the voting rule should be chosen so as to minimize policy instability. In particular, for societies that have too many factions with competing interests (like multiethnic societies), the results suggest that the optimal supermajority voting rule should be large enough to guarantee that a stable policy always exists. However, in societies characterized by a left-right political spectrum, the majority rule is most appropriate. Importantly, within a society, the social dimensionality of a policy generally depends on its nature. Policy importance might vary depending on whether it concerns education, national defense, international trade, terrorism, health care, human rights, fiscal policy, or land property rights. The findings then suggest that the optimal supermajority requirement should vary depending on the issue. This rationalizes the use of different voting rules to adopt different policies within real-world political institutions.

### 2.1.1 An Illustration: The Brexit Vote

On June 23, 2016, the British people voted on whether to stay in or to leave the European Union. If this vote is considered as a binary decision, then our analysis (Theorem 8) implies that it should be decided using the majority rule, which was the actual rule under which the vote took place. Several scholars however did not regard it as a simple decision, consequently raising some concerns about the Brexit voting process. For instance, Rogoff (2016) argued that the majority rule was not an appropriate voting rule for this decision. He wrote the following in the *Project Syndicate*'s website:

> "This isn't democracy; it is Russian roulette for republics. A decision of **enormous consequence**[4]—far greater even than amending a country's constitution (of course, the United Kingdom lacks a written one)—has been made without any appropriate checks and balances." (Rogoff, 2016, n.a).

If the Brexit vote was not a simple issue, what could its social dimensionality be? Scholars and political observers have argued that the decision to leave the European Union without a well defined plan can significantly impact important sectors of the United Kingdom (see, e.g.,

[4]The emphasis is ours.

Dhingra et al. (2016a), Dhingra et al. (2016b), Rogoff (2016), Stiglitz (2016), Sampson (2017)). The most frequently mentioned sectors are immigration, international trade, national security, and living standards. Assume that each of these four sectors is represented by a continuous variable defined over the interval $[0, 1]$. Then, an alternative is a vector $(x_1, x_2, x_3, x_4)$ with each component taking its values in $[0, 1]$. The analysis (Theorem 7) then suggests that the 80% supermajority rule was the optimal voting rule. This conclusion is reached without any knowledge of individual preferences.

## 2.2 Related Literature

Rousseau (1762) was among the first to suggest that, the size of the supermajority needed to pass a policy, should be related to the social importance of this policy. However, we are not aware of any study that formally analyzes the relationship between policy importance and voting rule in the *absence* of any knowledge about individual political preferences. Rousseau argues that, "The more rapidly the business at hand has to be resolved, the narrower should be the prescribed difference in weighting opinions; in deliberations which have to be concluded straightaway a majority of one should suffice." (Rousseau, 1762, n.a). This assertion is consistent with our findings in the sense that "straightforward" issues, or issues that can be concluded "rapidly", are issues that involve a "small" set of competing alternatives, captured in our framework by the dimension of the policy space (for continuous decisions) and by the number of policy options (for discrete decisions). The authors find that, if social dimensionality is sufficiently small, then the majority rule is the appropriate voting rule. To the best of our knowledge, our work is the first to provide an exact relationship between policy importance and voting rule.

This study also contributes to a small literature on the rationalization of political rules. In their seminal book, *The Calculus of Consent*, Buchanan and Tullock (1962) adopt the rational individualistic approach in collective-choice decision making process to explain the existence of various voting rules in society. Inspired by the works of Buchanan and Tullock, scholars such as Erlenmaier and Gersbach (2001), Barbera and Jackson (2004), Holden (2005), and Maggi and Morelli (2006) derive some rationales concerning the variation of decision rules through different models based on incomplete social contracts.

Erlenmaier and Gersbach (2001) introduce the concept of "flexible" majority rules in public projects to improve the lack of efficiency in democratic processes. The main objective of their study is to propose constitutional rules and principles that lead to the implementation of socially efficient outcomes under uncertainty. By using either a fixed majority rule or a flexible majority rule, they derive some constitutional principles that might be embedded in a well defined incomplete social contract to achieve the first best allocation. Barbera and Jackson (2004) introduce a model in which, "self-stable" constitutions are derived. These are rules that would be immune to change in a society in which they are used to change the extant rules. They demonstrate that these rules include supermajority rules. Maggi and Morelli (2006) determine the optimal size of supermajority in international organizations when there is imperfect enforcement. Holden (2005) analyzes the optimal supermajority requirement determined by multilateral bargaining behind the veil of ignorance, whereby the policy space is unidimensional, and the importance of a decision is assessed by the level of risk aversion.

The research question assessed in this paper significantly differs from the above-mentioned studies in its scope, analyses, and findings. The goal is to analyze the relationship between policy importance and supermajority rule. This relationship is revealed under a usual sequential legislative procedure. The analysis differentiates between discrete and continuous decisions, and ultimately rationalizes the use of different rules to pass policies that differ in their nature and level of importance.

## 2.3 Formalizing the Preference-blind Political Designer's Problem

This section formalizes the political designer's objective problem. It considers that the designer is blind to legislators' preferences. Given policy importance, his or her goal is to determine the supermajority rule that satisfies the properties **(P1)**, **(P2)**, and **(P3)** described in the Introduction. Decision-making follows a well-known sequential procedure, and voters are fully rational. The different notions needed for the formalization of this objective problem is provided below.

### 2.3.1 The Measurement of Decision Importance

In this study, the importance of a policy is assessed by its social dimensionality. A policy is more important when its social dimensionality is larger. In the literature on decision theory, a decision is considered "important" when it involves a complex choice set with many conflicting possibilities (Rousseau (1762), Harris and Sutton (1983), Krehbiel (2004), Krijnen, Zeelenberg, and Breugelmans (2015)). The multiplicity of conflicting alternatives means that the decision is difficult to make. Consistent with this view, Krijnen, Zeelenberg, and Breugelmans (2015) argue that "people assume difficult decisions to be important and important decisions to be difficult". The presence of a large number of feasible options in a decision-making process might also extends the time of reaching an agreement. Krijnen, Zeelenberg, and Breugelmans (2015) argue that "...more important decisions should take more time and effort". In a political context, an important decision is one that is likely to affect several sectors of the society. We will see that the greater the number of a society's sectors that a decision is likely to affect, the larger the decision's choice set.

In formalizing the notion of decision importance, we discriminate between discrete policies and continuous policies. Assume that a certain decision is likely to affect individuals along two distinct and non-correlated dimensions captured by the variables $x_1$ and $x_2$. Consider that these variables are discrete, with each taking values 0 and 1. Then the policy space or the choice set envisioned by the political designer is $\{(0,0),(0,1),(1,0),(1,1)\}$. If instead, the variables $x_1$ and $x_2$ are continuous, with each taking values in the interval $[0,1]$, for example, then the policy space is $[0,1]^2$. More generally, the importance of a decision is measured by the number of competing alternatives when the decision is discrete, and by the dimension of the choice set when the decision is continuous. We assume that the choice set of a continuous decision of dimension $D$ is a compact and convex subset of $\mathbb{R}^D$ (e.g., $[0,1]^D$). A decision that is apparently discrete, such as the decision on whether or not to withdraw from a union, might in actual fact be a continuous decision, especially if each of the social implications of such a decision is captured by a continuous variable. In answering the question of how decision importance determines voting rules, the authors are mostly concerned about the *social dimensionality* of a decision, and not by its mathematical dimension.

To illustrate the notion of decision importance, consider for instance the problem of choosing the official language of a country from the set of its local languages. Following our approach to modeling decision importance, this problem would be more important in a country like Nigeria (with over 250 local languages) than in Niger (with only around 10 local languages). A continuous decision could consist of the problem of choosing the location of the bus stop along a straight road. The political designer may think that this is a unidimensional decision, as each individual would probably care about the distance separating his or her house to the bus stop. We are not assuming that the political designer knows these preferences. In real-life situations, some people might prefer a location that minimizes their distance, whereas others might prefer that the bus stop is not too close to their house, but also not too far away, and still others might prefer the bus stop to be the farthest possible. All these scenarios are possible. The designer simply has to be aware of the dimension of the policy. Such a decision may be viewed as less complex than the decision to join or to withdraw from a union, which might have consequences on national security, immigration, and trade.

It is important to keep in mind that the political designer faced with the problem of choosing the voting rule that is most appropriate for selecting a policy of a given importance cannot anticipate individual preferences over the choice set induced by the importance parameter. This essay only assumes that the designer has enough resources to help her/him identify the main sectors of the society that can be affected by the enactment of a given policy. This information could be obtained by, for instance, gathering the opinions of various experts and specialists (including legislators). Assessing decision importance is one of the major problems addressed in the area of *informational politics*. Krehbiel (2004) argues that "many of the public policies that legislatures address are complex, and legislators differ in their inclinations and abilities to sort through the uncertain consequences of many proposals that fall onto the legislative agenda" (Krehbiel, 2004, P.10). For this reason, consulting different experts in order to gauge the importance of a decision is essential. The analysis will prove that this information suffices for the derivation of the optimal (appropriate) voting rule. That is, a supermajority rule that satisfies the properties of stability, efficiency, and maximal neutrality as outlined in the Introduction.

### 2.3.2 Decision Importance and Induced Legislature

Given the importance level—social dimensionality $D$ of a given decision (e.g., say $D = 2$ if the choice set is $[0, 1]^2$ for a continuous decision), the political designer's goal is to choose the supermajority $r(D)$ such that any legislature voting over the decision using the supermajority rule $r(D)$ selects a stable and efficient policy regardless of the diversity of individual preferences within the legislature.

Formally, a legislature legislating on a decision of importance $D$ is a tuple $\mathcal{L} = (N, D, r(D), (\succeq_i)_{i\in N})$ where:

1. $N = \{1, 2, ..., n\}$ is a finite number of legislators of size $n > 1$. A set $S \subseteq N$, $S \neq \emptyset$, is called a coalition of $N$. $2^N$ denotes the set of all coalitions.

2. $r(D)$ is the supermajority rule used to select a decision of importance $D$, and it is an integer strictly greater than $n/2$. Under the rule $r(D)$, a coalition of voters $S \subseteq N$ is a decisive or winning coalition if it consists of at least $r(D)$ legislators. The condition $r(D) > \frac{n}{2}$ also rules out the possibility to have two non-overlapping winning coalitions under the rule $r(D)$. $P$ denotes the set of winning coalitions in the legislature $\mathcal{L}$. Formally, $P = \{S \in 2^N : |S| \geq r(D)\}$, where $|X|$ denotes the size of a given set $X$.

3. $(\succeq_i)_{i \in N}$ represents the profile of individual preferences, not known to the political designer. Each preference relation $\succeq_i$ is reflexive, complete, and transitive over the choice set or policy space $A$ induced by the decision of importance $D$. $\mathcal{U}$ denotes the set of all such preferences, and $\mathcal{U}^n$ the set of all the preference profiles. If the decision is continuous, we assume in addition that each preference relation is continuous and convex (to be defined in section 2.4.1).

Note that because individual preferences are not known to the political designer, the goal of the latter is to choose the rule $r(D)$, given $D$, to ensure that for any preference profile $(\succeq_i)_{i \in N}$, the legislature $\mathcal{L} = (N, D, r(D), (\succeq_i)_{i \in N})$ selects a *stable* and efficient policy under the sequential mechanism described below.

### 2.3.3 The Political Mechanism

Decisions of any given importance level $D$ are made under a sequential legislative procedure which is a simplification of the deliberative procedure encountered in most legislative bodies in democratic societies. This legislative process (Figure 2.3) was first introduced in Chapter 1. For the sake of clarification to the reader, we briefly recall the description of this finite sequential mechanism as follows:

- *Stage 1 (Objection)*: Let $x$ be the status quo policy in a decision of importance level $D$.[5] Any legislator has the right to seek the replacement of $x$ by proposing a motion $y$. If no motion is submitted, then $x$ remains in place and the game ends. If a motion $y$ is

[5] The status quo is either the incumbent policy or any alternative chosen from the policy space induced by the decision importance $D$.

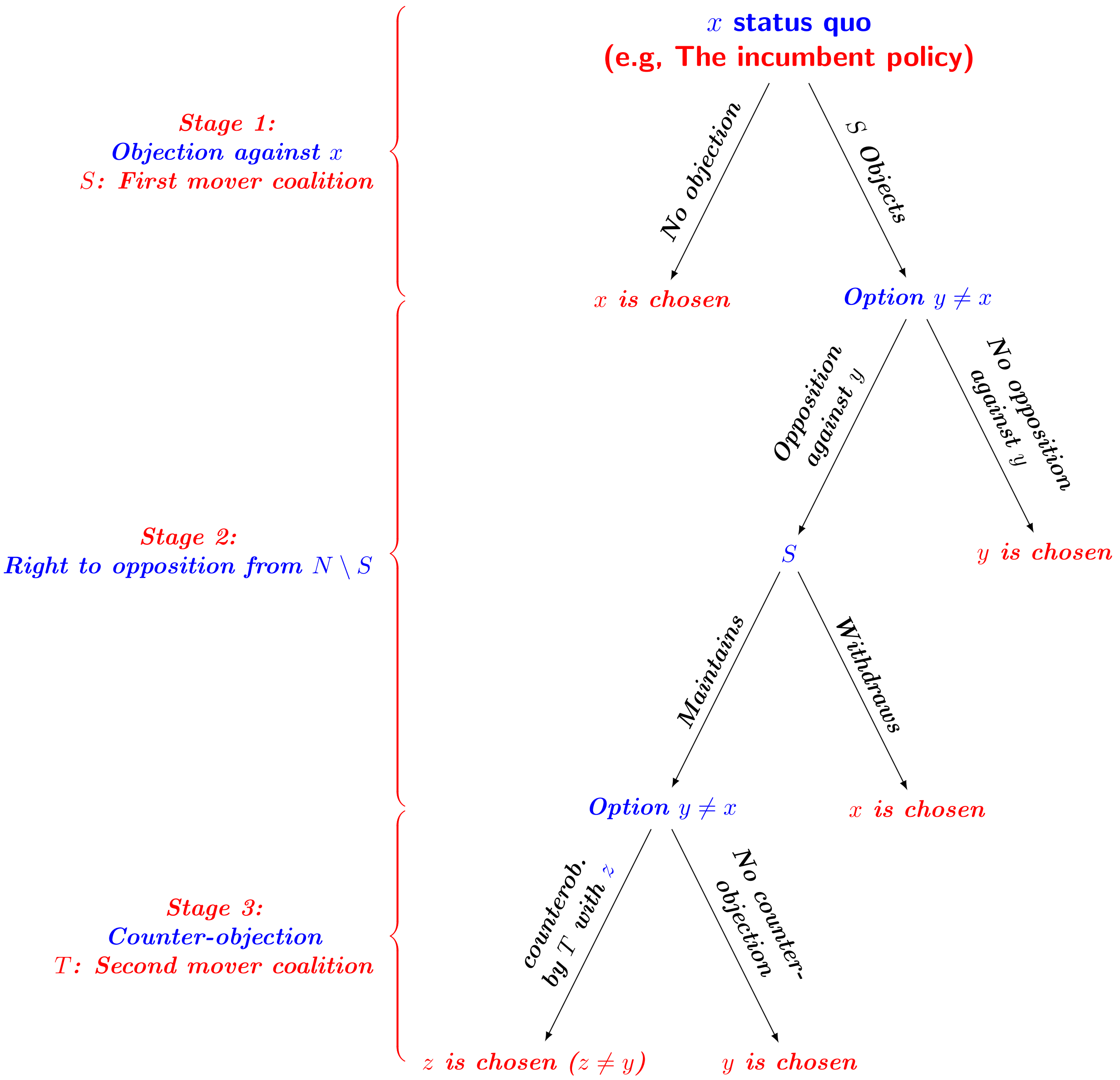



FIGURE 2.3: Political Mechanism

proposed and supported by a coalition (sponsoring coalition) that is winning under the rule $r(D)$, $y$ is submitted with the possibility of an amendment.

- *Stage 2 (Right to opposition)*: Any legislator who is not part of the sponsoring coalition has the right to oppose the bill $y$. If there is no opposition, $y$ is enforced as the new policy and the process ends. If there is any opposition, the sponsoring coalition is given the opportunity to withdraw the bill $y$. If it chooses to withdraw $y$, the status quo policy $x$ remains in place and the process ends. But if it chooses to maintain $y$, the process moves to the third stage.

- *Stage 3 (Counter-objection)*: The opposition has the right to propose an amendment $z$. If it does not propose any amendment, $y$ becomes the new policy and the process ends.

If it proposes an amendment $z$, a vote is organized between $y$ and $z$. If a majority of legislators (under the rule $r(D)$) supports $z$ against $y$, $z$ becomes the new policy and the process ends. Otherwise $y$ is the new policy and the process ends.

A *stable* or *equilibrium* policy is an alternative, such that if it is the status quo, no winning coalition will seek to replace it. We denote by $\mathcal{R}(\mathcal{L})$ the set of stable alternatives in a legislature $\mathcal{L} = (N, D, r(D), (\succeq_i)_{i \in N})$. This solution concept is formally defined in the next section.

### 2.3.4 Equilibrium Concept

This section formalizes the rational behavior of legislators under the mechanism described in the preceding section, and defines the equilibrium concept that captures this behavior.

**Definition 8.** Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, $S$ a winning coalition, and $x,\ y \in A$ two policies, with $A$ being the choice set induced by $D$.

1. **Objection**: $(y, S)$ is said to be an objection against $x$ if $y \succ_S x$ (i.e., each individual in $S$ strictly prefers $y$ to $x$).

2. **Counter-objection**: let $(y, S)$ be an objection against $x$. A pair $(z, T) \in A \times P$ is said to be a counter-objection against $(y, S)$ if $z \succ_T y$ and *not*$(y \succeq_T x)$.

3. **Unfriendly counter-objection**: let $(y, S)$ be an objection against $x$ and $(z, T) \in A \times P$ a counter-objection against $(y, S)$. The counter-objection $(z, T)$ is said to be unfriendly if *not*$(z \succeq_S x)$.

4. **Justified objection**: an objection $(y, S)$ against $x$ is said to be justified if there is no unfriendly counter-objection against $(y, S)$.

5. **Stable policy**: a stable (or equilibrium) policy is an alternative in $A$ against which no justified objection exists. We denote by $R(\mathcal{L})$ the set of all the stable policies in $A$.[6]

The following result states a necessary and sufficient condition for an alternative to be stable.

[6] $R(\mathcal{L})$ is also called the reciprocity set. As we explain below, this is because, under the sequential legislative procedure described above, any second-stage movers cannot rationally free-ride on the action of first-stage movers and first-stage movers cannot take actions that hurt second-stage movers. In other words, this procedure induces legislators to take reciprocal actions.

**Proposition 4.** *Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, and $x \in A$ be an alternative. Statements 1. and 2. below are equivalent:*

1. $x \in \mathcal{R}(\mathcal{L})$.

2. *There does not exist an alternative $y \in A$ such that $y \gtrdot x$, where $y \gtrdot x$ if there exists a winning coalition $S$ such that:*

   *a) $y \succ_S x$ and;*

   *b) $[\forall (z, T) \in A \times P,\ S \neq T,\ z \succ_T y$ and not$(y \succeq_T x)]$ implies $[z \succeq_S x]$.*

In other words, assertions *a)* and *b)* provide a necessary and sufficient condition for an alternative $x$ to be stable. The first assertion *a)* is a basic requirement that appears in many two-stage bargaining models in the literature. Assertion *b)* says that, if following an objection $(y, S)$ against the status quo $x$, a coalition $T$ formulates a counter-objection or an amendment $(z, T)$, then $S$ should weakly prefer $z$ over $x$ only if certain members of $T$ strictly prefer $x$ over $y$. In fact, if all the members of $T$ prefer $y$ over $x$, under the political mechanism presented above, they will not have any incentive to formulate a counter-objection unless this counter-objection does not hurt $S$. This is because doing so will cause $S$ to withdraw the bill $y$, allowing $x$ to remain in place and deteriorating the interests of all the members of $T$ since they all prefer $y$ over $x$. If, on the contrary, some members of $T$ are hurt by the objection $(y, S)$ against the status quo $x$, under our mechanism, these members have the right to oppose this objection and to form $T$ in order to formulate a counter-objection. This occurs only if the objection is not withdrawn. In this case, the members of $S$ should prefer the outcome $z$ of any amendment $(z, T)$ when sponsoring a bill in order to not regret the final outcome of the vote relative to the status quo.

It follows that under the proposed mechanism, a second mover $T$ can not "free-ride" on the action of a first mover $S$ (that is, $T$ has no incentive to react to an action that does not hurt any of its members, unless that reaction guarantees the interests of the members of $S$). Conversely, a first mover $S$ cannot take an action that hurts a member of a potential second mover $T$. It is for this reason that this political mechanism is called a "reciprocity mechanism". Reciprocity in this context, is not driven by individual preferences as is generally assumed in the literature, but by the mechanism that governs the decision-making process. In fact, legislators need not have other-regarding preferences to adopt a reciprocal behavior under our mechanism. Even if they are perfectly selfish, they have an incentive to act in a reciprocal manner.

### 2.3.5 The Political Designer's Objective Problem

This section states the political designer's problem. Given the level of decision importance $D$, the political designer's problem is to find the supermajority rule $r(D)$ such that:

**(P1)**: there exists a stable policy in the legislature $\mathcal{L} = (N, D, r(D), (\succeq_i))$ for any preference profile $(\succeq_i)$—-$\mathcal{R}(\mathcal{L}) \neq \varnothing$ for any $(\succeq_i) \in \mathcal{U}^n$ (or $\mathcal{U}^n_{con}$ for a continuous decision);

**(P2)**: any stable policy is Pareto-efficient—- any $x \in \mathcal{R}(\mathcal{L})$, $x$ is Pareto-efficient;

**(P3)**: $r(D)$ minimizes status quo bias. That is, $r(D)$ minimizes the size of the supermajority needed to change the status quo.

The requirement **(P1)** is also similar to the concept of decisiveness (Dasgupta and Maskin (2008)) or positive responsiveness (May (1952)) in the literature. The existence of a stable policy implies that a permanent decision can be achieved. We add another rationale concerning the third requirement **(P3)** of the expert designer's problem. It is well known that any supermajority rule is biased towards preserving the status quo, and is therefore not neutral between competing policy alternatives. As put by Dasgupta and Maskin (2008, P.2), if "the candidates are, say, various amendments to a nation's constitution, then one might want to give special treatment to the status quo–namely, to no change–and so ensure that constitutional change occurs only with overwhelming support". Minimizing status quo bias or maximizing neutrality between competing alternatives is therefore equivalent to minimizing the size of the supermajority required to change the incumbent policy. Therefore, the minimal status quo bias property can also be justified on the ground of cost minimization. If one admits that the formation of a larger coalition is more costly as is generally argued in the political science literature, this condition is equivalent to minimizing the cost of coalition formation while guaranteeing stability **(P1)** and efficiency **(P2)**.

The preference-blind political designer's objective problem is summarized by the following optimization problem:

$$\begin{aligned} \min \quad & r(D) \\ \text{s.t.} \quad & (\mathbf{P1}) \text{ and } (\mathbf{P2}) \quad \text{are satisfied;} \end{aligned} \tag{2.1}$$

In the next section, this problem is solved for both discrete and continuous policies.

## 2.4 The Solution: Supermajority Rule as a Function of Policy Importance

The solution of problem 2.1 is simplified by the following result according to which any stable alternative under the political mechanism is Pareto-efficient regardless of the nature of the policy space.

**Lemma 1.** *Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature. Any policy in $\mathcal{R}(\mathcal{L})$ is Pareto-efficient.*

The intuition behind this result follows from the fact that the reciprocity mechanism induces legislators to adopt a reciprocal behavior, as explained in the section 2.3.4. No potential second mover will react against a first mover who replaces a status quo $x$ by a policy $y$ which Pareto-dominates $x$. Opposing such a move will cause the sponsoring coalition to withdraw $y$, allowing $x$ to remain in place and inducing the persistence of an alternative that is less preferred by all the legislators.

This lemma is important because it reduces the political designer's objective problem to an optimization problem that only requires a supermajority rule to guarantee that a stable policy exists regardless of the extent to which legislators diverge in their political views. The simplified problem is formulated as follows:

$$\begin{aligned} \min \quad & r(D) \\ \text{s.t.} \quad & \mathcal{R}(\mathcal{L}) \neq \varnothing, \text{ for any } (\succeq_i) \in \mathcal{U}^n \text{ (or } \mathcal{U}^n_{con}) \end{aligned} \tag{2.2}$$

We start by addressing the case of continuous decisions in the next section.

### 2.4.1 Continuous Decisions

We recall that a decision of importance level $D$ is continuous when it involves a continuum of competing policy alternatives forming a compact and convex subset of the multidimensional vector space $\mathbb{R}^D$ (e.g., $[0, 1]^D$). Moreover, $D$ is the number of society's sectors that are likely to be affected by a change in policy. The political designer knows nothing about individual preferences when choosing a rule. He/She only assumes that preferences belong to a large class. More precisely, in addition to being reflexive, transitive, and complete, preferences are continuous and convex. Continuity and convexity are classical assumptions imposed on preferences when the choice set is a continuum. The definitions of these notions are presented below.

**Definition 9.** Let $\succeq$ be a preference relation over a policy space $A$.

1. $\succeq$ is said to be continuous if for any policies $x, y \in A$ such that $x \succ y$, there exists a neighborhood $S(x)$ of $x$ and a neighborhood $S(y)$ of $y$ such that $z \succ t$ for every $z \in S(x)$ and $t \in S(y)$.

2. $\succeq$ is said to be convex if for any policies $x, y \in A$ such that $x \succeq y$ and $x \succeq z$, $x \succeq \lambda y + (1-\lambda)z$ for every $\lambda \in [0, 1]$.

$\mathcal{U}_{conv}$ denotes the set of reflexive, transitive, complete, continuous, and convex preference relations on the policy space $A$. $\mathcal{U}^n_{conv}$ denotes the set of preference profiles corresponding to the set $\mathcal{U}_{conv}$.

To state the main result for this section, the following notation is needed. For a real number $x$, $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to $x$. The exact relationship between policy importance and voting rule is given below.

**Theorem 7.** *Let $D$ be the level of decision importance of a continuous policy in a legislature of size $n$. Then, the optimal solution to problem 2.1 is given by:*

$$r_1^*(D) = \left\lfloor \frac{D}{D+1} n \right\rfloor + 1.$$

The proof of this result is quite involved and is provided in the appendix. First, it is proved that if a voting rule $r$ is such that $r > (\frac{D}{D+1})n$, there always exists a stable policy in a legislature deciding on a policy of importance level $D$ (Lemma 2). This result is quite intuitive. Certainly, if $r$ is the unanimity rule, then a stable policy always exists, since there exists at least one Pareto-efficient policy and any Pareto-efficient policy is stable under unanimity. If $r$ is the majority rule, then a stable policy might not exist if $D$ is sufficiently high. Using an argument that resembles that of the Intermediate Value Theorem, there should be a threshold such that if $r$ is greater than that threshold, a stable alternative always exists. This threshold is $\frac{D}{D+1}n$. The authors also prove that if $r$ is smaller than $\frac{D}{D+1}n$, then there exists a preference profile under which a stable policy does not exist (Lemma 5). Therefore, the optimal rule is $r_1^*(D) = \lfloor \frac{D}{D+1} n \rfloor + 1$. The sought relationship between policy importance and voting rule is shown in Figure 2.1 in the Introduction.

Note that if a policy change is likely to affect only one sector of the society or one dimension of individual preferences, even if we do not know how preferences will be affected, (that is, $D = 1$), the optimal voting rule is the absolute majority rule. For instance, if the decision is to choose a bus stop along a straight road, then the absolute majority rule is the optimal voting rule, especially if the only factor that matters is the distance separating each household from the bus stop, no matter whether some people like to be closer and others like to be farther, and others even have more complex preferences. The same absolute majority rule is the optimal solution in case legislators may have single-peaked preferences (e.g., Moulin (1980)) over the induced policy set considered as a real line (that is the dimension of individuals' preferences is 1). If the social dimensionality of a decision is two, then the optimal rule is the two-thirds majority rule $\lfloor \frac{2n}{3} \rfloor + 1$. For much more important decisions, that is, when $D$ is large enough, the optimal voting rule approaches the unanimity rule. These findings confirm the basic intuition of Rousseau (1762) and are consistent with the variation in majority requirements observed in legislative bodies around the world as noted in the first paragraph of the Introduction. But, what is even more interesting is the derivation of the exact relationship between policy importance and voting rule. Clearly, the size of the supermajority needed to pass a policy is an increasing function of its importance.

### 2.4.2 Discrete Decisions

A decision of importance level $D$ is discrete when it involves $D$ competing policy alternatives. Here, individual preferences over the policy space induced by decision importance are reflexive, transitive, and complete. The main finding below establishes the exact relationship between policy importance and voting rule.

**Theorem 8.** *Let $D$ ($2 \le D < \infty$) be the level of decision importance in a legislature of size $n$. Then, the solution to the political designer's problem 2.1 is given by:*

$$r_2^*(D) = \max\left\{ \left\lfloor \frac{D-2}{D} n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \right\} + 1.$$

As for the proof of Theorem 7, the proof of this result is involved. Due to the discrete property of the policy space, this proof is different from that of Theorem 7, but the logical steps are the same. First, if a voting rule $r$ is such that $r > (\frac{D-2}{D})n$, there always exists a stable policy in a legislature deciding on a policy of importance level $D$ (see Lemma 7 in the appendix). Second, if a voting rule $r$ is smaller or equal than the number $(\frac{D-2}{D})n$, then there exists a preference profile under which a stable policy does not exist (Lemma 8 in the appendix). Consequently, the optimal rule sought by the political designer is $r_2^*(D) = \max\left\{\left\lfloor \frac{D-2}{D}n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right\} + 1$, given that by definition, a supermajority rule is greater than $\frac{n}{2}$. In Theorem 2, there exists at least two competing policy alternatives at stake. Obviously, if the policy's importance induces a policy space with a single alternative, then the latter is elected by unanimity.

The relationship between policy importance and voting rule in the discrete case can be visualized in Figure 2.2 in the Introduction. The latter illustrates that the optimal voting rule for decisions that involve less than five competing options is the absolute majority rule. If the social dimensionality of a decision is 16, then the 87.5% supermajority rule is the most appropriate political rule. The unanimity rule should be used for sufficiently important policies. In general, the size of the supermajority needed to pass a policy weakly increases with the social importance of this policy. Again, these findings are consistent with the prescriptions of Rousseau (1762) regarding the use of the majority and the unanimity rules in political institutions. They also rationalize the fact that different rules are used to pass policies of a different nature in most real-life political institutions.

The findings of this section have practical implications for the design of political institutions. Consider, for instance, the democratic choice of the official language of a country. The analysis implies that such a choice should be made using the absolute majority rule in a country like Rwanda and a rule that approaches the unanimity rule in a country like Cameroon or Nigeria. In general, the result implies that the status quo is more likely to persist in more fractionalized societies.

### 2.4.3 Dropping Neutrality

It is indicated in the previous sections that, the maximization of the neutrality (or the minimization of status quo bias) among policy alternatives together with the properties of Pareto-efficiency and stability lead to the minimum size of majority required in the legislature to amend the status quo. If the neutrality's requirement is relaxed in the formalization of the political designer's objective problem, then for a policy of importance $D$, any supermajority rule of size greater or equal to the threshold $r_1^*(D)$ for continuous policy, and $r_2^*(D)$ for discrete policies will induce legislature to enact stable and efficient outcomes. The shaded areas in the figures (Figure 2.4 for continuous policies, and Figure 2.5 for discrete policy) below display the set of voting rules that the political designer could propose to the legislature depending on the policy importance if neutrality (equality) among political candidates is not a constraint to the design of constitutional institutions.

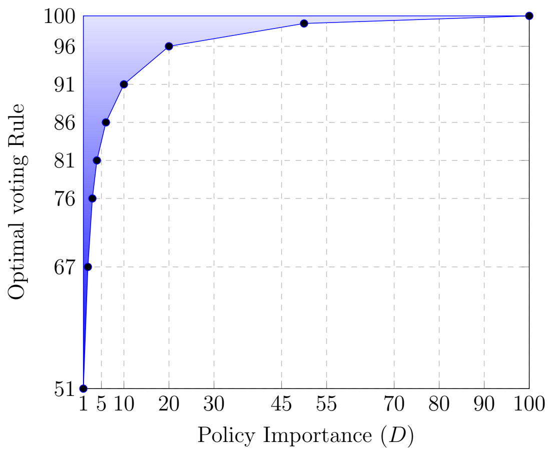



FIGURE 2.4: Set of supermajority rules for continuous decisions.

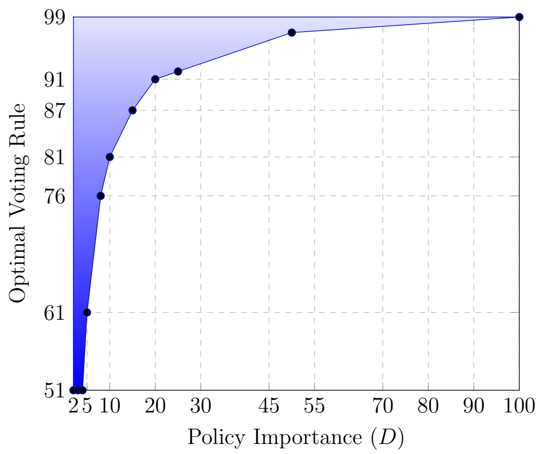



FIGURE 2.5: Set of supermajority rules for discrete decisions.

## 2.5 Conclusion

Voting rules vary widely across policy types within legislative bodies. It is argued that these variations are related to the fact that certain decisions are more important than others. While this intuition is sensible, a formal analysis of how policy importance should determine voting rules in a *preference-blind* environment had not been conducted in the economics or political science literature. This paper addresses this question from the perspective of a political designer who has to choose a voting rule in a legislature that will be used to select policy in a distant future. Consequently, the political designer totally ignores individual preferences.

Given policy importance, her/his goal is to determine the supermajority rule that (1) guarantees the existence of a stable policy regardless of the extent to which individual opinions diverge; (2) ensures that all stable policies are efficient; and (3) minimizes status quo bias. Policies are selected under a simple sequential procedure. A closed-form solution to the designer's objective problem for both continuous and discrete decisions is derived. The analysis shows that the size of the majority needed to adopt policy increases with policy importance, with the functional form varying across continuous and discrete decisions.

The different sectors that are likely to be affected by the enactment of a policy are equally weighted in the measure of the policy importance. If sectors are weighted differently, and the importance of the policy is provided by a single real number, then the results still hold.

The findings have practical implications for the design of political institutions. First, the sequential mechanism that governs policy selection within our framework provides incentives for legislators to adopt reciprocal and pro-social behavior, leading to only efficient policies being adopted. Second, the analysis sheds light on how voting rules should be selected based only on policy importance in order to reach permanent decisions while at the same time minimizing the coordination cost of decision-making.

Our analysis also has testable implications for the functioning of political institutions. It ultimately rationalizes the use of different voting rules to pass policies of different degree of importance within legislative bodies around the world. It also allows the researcher and the political observer to objectively infer the legislator's view about the importance of different policies within a legislature. For example, if one considers the impeachment of a United States President and his removal from office are discrete policies, then our findings imply that, in the legislator's view, the social importance of the former policy is comprised between 2 and 4 whereas that of the latter policy is 6. If one considers those policies to be continuous, the analysis stipulates that, in the legislator's view, the social dimensionality of the former is 1, and that of the latter is 2. The legislator's view might not be consistent with that of an ordinary political observer. A recent real-life example of this sort of discrepancy is the Brexit's aggregation rule decision, which, as noted in the Introduction, Rogoff (2016) thought was a more important decision than the legislator had thought.

Finally, the results have testable implications for the functioning of democratic institutions in ethnically fractionalized societies. If ethnic affiliation determines political preferences, then the findings clearly imply that the size of the majority needed to obtain stable policies should increase with the number of ethnic groups. Because any supermajority rule is biased towards preserving the extant policy, this in turn implies that the status quo is more likely to persist in more fractionalized societies. This could explain why the ruling leaders of such societies tend to stay in power for longer periods.

## 2.6 Appendix: Proofs of Results

**Proof of Proposition 4.** The proof follows the same reasoning as that of Theorem 2 in Chapter 1.

**Proof of Lemma 1.** Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, an $x \in A$ be a policy such that $x \in R(\mathcal{L})$; where $A$ is the policy space. If $x$ is Pareto-dominated, then there exists $y \in A$ such that $y \succ_N x$. It follows that for any $z \in A$ such that $z \succ_T y$ with $T \in P$, we have $y \succ_T x$ because $T \subset N$. Then, not $(y \succeq_T x)$ is always false. Therefore the implication [*not* $(y \succeq_T x) \Rightarrow z \succeq_N x$] is always true i.e $y$ dominates $x$ via $N$ or $(y, N)$ is a justified objection against $x$, which is a contradiction. Thus, $x$ is Pareto-efficient.

**Proof of Theorem 7.** In order to establish this proof, we need to prove some preliminary results. First, the proof illustrates that if a voting rule $r$ is such that $r > (\frac{D}{D+1})n$, there always exists a stable policy in a legislature legislating on a policy of importance level $D$. This result is provided below:

**Lemma 2.** *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature such that $r > (\frac{D}{D+1})n$ and $(\succeq_i) \in \mathcal{U}^n_{conv}$. Then, there exists a stable policy.*

*Proof.* Let $A$ be the policy space induced by the decision of importance $D$ in the legislature. Convex preferences satisfy the following properties: for each legislator $i \in N$, the binary relation $\succeq_i$ has an open graph in $A \times A$ (a binary relation $\succeq_i$ defined on $A$ is a subset of $A \times A$; the binary relation $\succeq_i$ is said to have an open graph if $\succeq_i$ is open in the product topology on $A \times A$, where $A$ is a topological space); and for each legislator $i \in N$ and for each alternative $x \in A$, $x \notin C(A_i(x))$, where $A_i(x) = \{y \in A,\ y \succ_i x\}$, and $C(X)$ denotes the convex hull of $X$. In order to prove Lemma 2, we use the result below.

**Lemma 3.** *(Greenberg, 1979, Lemma 1, P.6) Let $N$ be a finite set with cardinality $n$. Consider the collection $C^s$ of subsets of $N$ of cardinality at least $s$. Then, all intersections of $m + 1$ elements of $C^s$ are non-empty if and only if $s > (\frac{m}{m+1})n$.*

Prove that there exists a stable outcome in the legislature $\mathcal{L}$, whatever the minimum size of majority required $(r)$ is strictly greater than the number $(\frac{D}{D+1})n$. Consider a legislator $i \in N$, and an alternative $x \in A$. Define the sets $A(x, y) = \{i \in N, y \in C_i(x)\}$, and $A_r(x) = \{y \in A, |C(x, y)| \geq r\}$.

**1.** There exists an alternative $x^* \in A$ such that $A_r(x^*)$ is a non-empty set.

**a)** $A_r(x)$ is an open set of $A$ (the topology here is the usual topology in the multidimensional set $\mathbb{R}^D$).

Let $y \in A_r(x)$ be an alternative, then $y \in A_i(x)$ for any legislator $i \in A(x,y)$. Given that legislators' preferences are continuous, for each legislator $i \in A(x,y)$, there exists a neighborhood $A_i(y)$ of alternative $y$ such that option $y' \succ_i x$ for any option $y' \in A_i(y)$. Consider the set $\overline{A} = \bigcap_{i\in A(x,y)} A_i(y)$, and let $\overline{y} \in \overline{A}$ be an alternative. It follows that the option $\overline{y} \in A_i(y)$ for any legislator $i \in A(x,y)$. This means $\overline{y} \succ_i x \ \forall i \in A(x,y)$, and then option $\overline{y} \in A_r(x)$. Therefore, $A_r(x)$ contains an open set $\overline{A}$, which is a neighborhood of alternative $y$, for $y \in A_r(x)$. Hence, the set $A_r(x)$ is an open set of the policy space $A$.

**b)** Let be $C(A_r(x))$ be the convex hull of $A_r(x)$. Demonstrate that, for each alternative $x \in A$, $x \notin C(A_r(x))$. The following result proved to be useful.

**Lemma 4.** *(Nikaido, 1968, Theorem 2.4, P.19) The convex hull $C(X)$ of a set $X$ in $\mathbb{R}^n$ equals the set of all points represented by $\sum_{i=1}^{n+1} \alpha_i x^i$, $\sum_{i=1}^{n+1} \alpha_i = 1$, $\alpha_i \geq 0$ $(i = 1, 2, ..., n+1)$ as the $n+1$ independently range over $X$ and the weight of $\alpha_i$ take on all possible values.*

Given that preferences satisfy the convex property, for each alternative $x \in A$ and for each legislator $i \in N$, $x \notin C(A_i(x))$. Assume by negation that the alternative $\overline{x} \in C(A_r(\overline{x}))$, then following Lemma 4, there exists alternatives $x_1, x_2, ..., x_{D+1}$ and real values $\lambda_1, \lambda_2, ..., \lambda_{D+1}$ with $\sum_{j=1}^{D+1} \lambda_j = 1$, $\lambda_j \geq 0$ such that $\overline{x} = \sum_{j=1}^{D+1} \lambda_j x_j$, and each alternative $x_j \in A_r(\overline{x})$. Alternative $x_j \in A_r(\overline{x})$ implies that $x_j \in A_i(\overline{x})$ for each legislator $i \in A(\overline{x}, x_j)$. Since the number $r > (\frac{D}{D+1})n$, and $|A(\overline{x}, x_j)| \geq r$ for each $j = 1, 2, ..., D+1$ (given that $x_j \in A_r(\overline{x})$), then $|A(\overline{x}, x_j)| > (\frac{D}{D+1})n$. According to Lemma 3, there exists an legislator $i_0 \in N$ such that option $x_j \in A_{i_0}(\overline{x})$ for each $j = 1, 2, ..., D+1$. It follows that the alternative $\overline{x}$ is equal $\sum_{j=1}^{D+1} \lambda_j x_j$, with $x_j \in A_{i_0}(\overline{x})$. Consequently, the option $\overline{x} \in C(A_{i_0}(\overline{x}))$, which is a contradiction by assumption. Thus, for each alternative $x \in A$, $x \notin C(A_r(x))$ and the subset $A_r(x)$ is an open set of the policy space $A$. It follows that there exists an alternative $x^* \in A$ such that the subset $A_r(x^*) = \emptyset$. The alternative $x^*$ is not strictly preferred by another alternative in the policy space $A$.

**2.** The alternative $x^*$ is stable. Assume the contrary, then there exists a policy $y \in A$, and a coalition $C \subseteq N$ such that the alternative $y \succ_i x$ for each legislator $i \in C$. The coalition $C \subseteq A(x,y)$ and the size of $C$, $|C| \geq r$. It follows that the number $|A(x,y)| \geq r$ and the alternative $y \in A_r(x^*)$, which is a contradiction, since the subset $A_r(x^*) = \emptyset$. $\square$

The result below stipulates that if the size of the supermajority needed to pass a policy is less than or equal to the number $(\frac{D}{D+1})n$, then there might exists a preference profile where there is no stable policy.

**Lemma 5.** *If $r \leq (\frac{D}{D+1})n$, there exists a preferences' profile $(\succeq_i) \in \mathcal{U}^n_{conv}$ such that the legislature $\mathcal{L} = (N, D, r, (\succeq_i))$ cannot enact a stable policy.*

*Proof.* Prove that, if the minimum size of the supermajority required $r$ is less or equal than the number $(\frac{D}{D+1})n$, then there exists a profile $(\succeq_i) \in \mathcal{U}^n_{conv}$ which leads the legislature to political cycles, meaning that there exists no stable policy.

Consider the policy space $A$ induced by the decision of importance $D$ in the legislature $\mathcal{L}$. The dimension of $A$ is $D$. It contains a $D$-dimensional simplex $S_D$. Denote by $b^i$, $i = 1, 2, ..., D+1$ the $(D+1)$ affinely independent vertices of $S_D$. For alternatives $x$ and $y$ in the policy space $A$, define the distance from $x$ to $y$, denoted $d(x, y)$ by the real number: $d(x, y) = \|x - y\|$. Also, define the function:

$$\begin{aligned} f_i : \quad A \to & \quad \mathbb{R} \\ x \mapsto & \quad f_i(x) = -d(x, b^i) \end{aligned}$$

$f_i$ is continuous and strictly quasi-concave. The proof differentiates three cases with respect to the size of voters, $n$.

1. $n = D + 1$ and consider that the preference of legislator $i$ is represented by the function $f_i$, $i = 1, 2, ..., D+1$. Legislator $i$'s ideal point is $b^i$ corresponding to a vertex of $S_D$. Consider a policy $x \in A$.

   a) The alternative $x \notin S_D$. Given that $S_D$ is a compact and convex set, there exists an unique alternative $y(x) \in S_D$ close to $x$. Then, $d(y(x), b^i) < d(x, b^i)$ for each $i = 1, 2, ..., D+1$ or $f_i(y(x)) > f_i(x)$ for each $i \in N$ i.e $y(x) \succ_N x$. If there exists an alternative $y' \in A$ such that $y' \succ_T y(x)$, with $T \in P$, then $f_i(y') > f_i(y(x))$ for each $i \in T$. Since, $y(x) \succ_N x$, hence $f_i(y') > f_i(x)$ for each $i \in T$ and $y(x)$ dominates $x$ $(y(x) \gtrdot x)$.

   b) The alternative $x \in S_D$. There exists an $(D-1)$-dimensional face $F$ of $S_D$ such that alternative $x \notin F$. The set $F$ is also a simplex, therefore there exists an unique alternative $y(x) \in F$ close to $x$. It follows that $d(y(x), b^i) < d(x, b^i)$ with

$b^i$, $i = 1, 2, ..., D$ the $D$-vertices affinely independent of $F$. Thus, there exists a coalition $C$ of cardinality $D$ such that $y(x) \succ_C x$. Since the number $D \geq \frac{r}{n-r}$ and $n = D + 1$, then $r \leq D$ and $C$ is a winning coalition. Assume that there exists an alternative $y' \in A$ such that $d(y'^i, b^i) < d(y(x), b^i)$ for $t$ vertices $b^i$ of the policy space $A$ i.e $y'$ is strictly preferred to alternative $y(x)$ by a coalition $T$ with $|T| = t \geq r$.

- If $b^i$, $i = 1, 2, ..., l$ are the vertices of the set $F$, then by definition of $y(x)$, we have $d(y(x), b^i) < d(x, b^i)$ for each $i = 1, 2, ..., t$ i.e $not(y(x) \succeq_T x)$ is false and the implication $[not(y(x) \succeq_T x) \Rightarrow y' \succeq_C x]$ is true, therefore $y(x)$ dominates $x$ $(y(x) \gtrdot x)$.
- Assume that there exists an alternative $h_0 \in \{1, 2, ..., l\}$ such that the vertex $b^{h_0} \notin F$ and $d(y'^{h_0}, b^{h_0}) < d(y(x), b^{h_0})$. If $not(y(x) \succeq_T x)$ is false, then the implication $[not(y(x) \succeq_T x) \Rightarrow y' \succeq_C x]$ is true, so $y(x)$ dominates $x$ via $C$. Assume $not(y(x) \succeq_T x)$ and $not(y' \succeq_C x)$, then there exists a legislator $i_0 \in C$ such that $x \succ_{i_0} y'$ or $d(x, b^{i_0}) < d(y'^{i_0}, b^{i_0})$, $b^{i_0}$ a vertex of $F$. By definition of the alternative $y(x)$, $d(y(x), b^{i_0}) < d(x, b^{i_0})$, then $d(y(x), b^{i_0}) < d(y'^{i_0}, b^{i_0})$, and the legislator $i_0 \notin \{1, 2, ..., t\}$. Given that $F$ contains $D$ vertices, then $h_0 = i_0$, therefore $b^{i_0}$ is not a vertex of $F$, a contradiction. That being said, the alternative $y(x)$ dominates $x$ via $C$.

2. $n < D+1$. Let $F$ be the face of $S_D$ generated by the vertices $b^i$, $i = 1, 2, ..., n$. Consider that the legislators' preferences are represented by the functions $f_i$, $i = 1, 2, ..., n$. $F$ is a simplex of dimension $n-1$, then following **point 1.** of this proof, each alternative which belongs to $F$ can be defeated by a coalition which consists of $n-1$ legislators and each alternative which does not belong to $F$ can be defeated by the coalition $N$. It follows that each alternative can be defeated by a coalition which consists of at least $n-1$ legislators. Since $D \geq \frac{r}{n-r}$ and $n < D+1$, then $r < D$. Furthermore, $\frac{D}{D+1} < 1$ and $r \leq (\frac{D}{D+1})n$; it follows that $r < n$. Since $r$ is an integer, then $r \leq n-1$ meaning that every coalition of cardinality at least $n-1$ is a winning coalition. In the same manner as in **point 1.** of this proof, each alternative is dominated in the sense of the binary relation $\gtrdot$.

3. $n > D+1$. There exists two integers $s$ and $k$, $s \geq 1$ and $0 \leq k < D+1$ such that $n = s(D+1)+k$.

Consider a partition of population $N$ ($|N| = n$) in $(D+1)$ sets where legislators belong to the same set if and only if they have the same preference, and hence the ideal point. Assume that $s+1$ legislators belong to each of the first $k$ sets and $s$ other legislator belong to the remaining $D+1-k$ sets. In each set, the preferences are given by $f_j$, $j = 1, 2, ..., D+1$.

c) If $k = 0$, then $n = s(D+1)$. This means, there are $s$ legislators in each set. According to **point 1.** of this proof, each alternative $x \in A$ is defeated by a coalition of cardinality greater or equal to $n-r$. Since the number $D \geq \frac{r}{n-r}$ and $n = s(D+1)$, then $r \leq Is = n-s$.

Each alternative is dominated. Let $x \in A$ be an alternative. Then, there exists an unique alternative $y(x) \in A$ such that $y(x)$ dominates $x$ via a coalition $C$ with $|C| \geq n-s$. Assume that there exists an alternative $y' \in A$ such that $y'$ is preferred to alternative $y(x)$ by a coalition $T$. It follows that $T$ contains at least $n-s$ legislators.

- If $T = C$, then the option $y' \succ_C y(x) \succ_C x$, so $y' \succ_C x$, and $y(x)$ dominates $x$.
- If $T \neq C$, consider that alternative $y(x) \succeq_T x$. The implication $[not(y(x) \succeq_T x) \Rightarrow y' \succeq_C x]$ is satisfied, then $y(x) \gtrdot_C x$. If $not(y(x) \succeq_T x)$, then there exists $i \in T$ such that $x \succ_i y(x)$. The alternative $y(x)$ is an unique closed point of $x$ and legislator $i$ has $b^i$ as ideal point. It follows that $d(y(x), b^i) < d(x, b^i)$. Therefore, $not(y(x) \succeq_T x)$, then $y(x) \gtrdot_C x$.

d) Assume $k \geq 1$. Each vertex of the simplex corresponds to the ideal point of at most $s+1$ legislators. According to **point 1.** of this proof, each alternative can be defeated by at least $n-(s+1)$ legislators. Since the number $D \geq \frac{r}{n-r}$ and $n = s(D+1)+k$, then $r \leq D(s+\frac{k}{D+1})$ i.e $r < Is+k$ and then $r \leq Is+k-1 = n-(s+1)$, given that $r$ and $Is+k$ are integers. Following the proof in case $k=0$ (**point 3.c**), we can conclude that each alternative is dominated.

It suffices to consider the legislators' preferences profile $(\succeq_i) = (\succeq_1, ..., \succeq_n)$ such that $f_i$ is a cardinal representation of $\succeq_i$, for each legislator $i \in N$. Each preference $\succeq_i$

is represented by a quasi-concave function $f_i$, then $\succeq_i$ is equivalent to ordinal convex preference. We can conclude that $(\succeq_i) \in \mathcal{U}^n_{conv}$ and there is no stable alternative.

Finally, we have all the tools to conclude the proof of this theorem. Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature. Consider $A$ be a compact and convex policy space, subset of Euclidean space $\mathbb{R}^D$, of dimension $D$, and $(\succeq_i) \in \mathcal{U}^n_{conv}$. Lemmas 2 and 5 reveal that there exists a stable outcome if and only if $r(D) > (\frac{D}{D+1})n$. Since $r$ is an integer number, the smallest one which satisfies the latter inequality and is strictly greater than $n/2$ is $\left\lfloor \frac{D}{D+1}n \right\rfloor + 1$. □

**Proof of Theorem 8**. This proof also involves several steps. First, we prove a lemma that illustrates the structure of the voting process in a legislature under which there is no stable policy. Throughout this proof, $A$ denotes the policy space induced by the discrete decision of importance $D$.

**Lemma 6.** *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature with $(\succeq_i) \in \mathcal{U}^n$ such that there is no stable policy. Then, there exists a voting cycle: $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length $p$ $(p \leq D,\ p > 2)$ such that there is no individual $i \in N$ belonging to more than $p-2$ winning coalitions among the set $\mathcal{C} = \{S_1, S_2, ..., S_p\}$.*

*Proof.* Suppose that there exists a legislator $i$ who belongs to $p-1$ winning coalitions among $\mathcal{C} = \{S_1, S_2, ..., S_p\}$. Prove that the legislator $i \notin S_j \cap S_{j+1(modp)}$ for each $j = 1, 2, ..., p$[7]. Without loss of generality, consider that the legislator $i \in S_1 \cap S_p$. By definitions,

$$\{i \in S_1 \text{ and } x_1 \gtrdot_{S_1} x_2\} \Rightarrow \left\{x_1 \succ_i x_2 \text{ and } [not(x_1 \succeq_{S_p} x_2) \Rightarrow x_p \succeq_i x_2]\right\}.$$

Also,

$$\left\{i \in S_p \text{ and } x_p \gtrdot_{S_p} x_1\right\} \Rightarrow \left\{x_p \succ_i x_1 \text{ and } [not(x_p \succeq_{S_{p-1}} x_1) \Rightarrow x_{p-1} \succeq_i x_1]\right\}.$$

**1.** Legislator $i \in S_{p-1} \cup S_{p-2}$, since s/he belongs to $p-1$ winning subgroups among $C = \{S_1, S_2, ..., S_p\}$. The legislator $i \in S_m \in \{S_{p-1}, S_{p-2}\}$. Then,

$$\{i \in S_m \text{ and } x_m \gtrdot_{S_m} x_{m+1}\}.$$

[7] The notation "*mod p*" means "modulo $p$". In our context, $x$ $(modp)$ $y$ if $x - y = kp$, where $k$ is an integer.

This implies that:

$$\left\{x_m \succ_i x_{m+1} \text{ and } [not(x_m \succeq_{S_{m-1}} x_{m+1}) \Rightarrow x_{m-1} \succeq_i x_{m+1}]\right\}.$$

The alternative $x_2 \in \{x_m, x_{m-1}\}$, therefore $x_2 \succeq_i x_{m+1}$. Since the coalition $S_m \in \{S_{p-1}, S_{p-2}\}$, thus the alternative $x_{m+1} \succeq_i x_1$. By transitivity, alternative $x_2 \succeq_i x_1$, which is a contradiction, because the option $x_1 \succ_i x_2$. That being said, $x_2 \notin \{x_m, x_{m-1}\}$.

**2.** Legislator $i$ belongs to coalition $S_m$. Then, as previously, legislator $i$ belongs to a coalition $S_l$ such that $S_l \in \{S_{m-1}, S_{m-2}\}$ and alternative $x_2 \notin \{x_l, x_{l-1}\}$. Certainly,

$$\{i \in S_l \text{ and } x_l \gtrdot_{S_l} x_{l+1}\} \Rightarrow \left\{x_l \succ_i x_{l+1} \text{ and } [not(x_l \succeq_{S_{l-1}} x_{l+1}) \Rightarrow x_{l-1} \succeq_i x_{l+1}]\right\}.$$

If alternative $x_2 \in \{x_l, x_{l-1}\}$, then $x_2 \succeq_i x_{l+1}$ and the option $x_{l+1} \succeq_i x_{m+1} \succeq_i x_1$. Therefore, $x_2 \succeq_i x_1$, a contradiction.

**3.** At each step of this process, some alternatives different than $x_2$ are eliminated. Given that the number of alternatives is finite, there exists an alternative $x_t \in A$ such that $x_2 \in \{x_t, x_{t-1}\}$. Alternative $x_t \gtrdot_{S_t} x_{t+1}$, and the legislator $i \in S_t$. Hence alternative $x_t \succ_i x_{t+1}$, and $[not(x_t \succeq_{S_{t-1}} x_{t+1}) \Rightarrow x_{t-1} \succeq_i x_{t+1}]$. It follows that:

$$x_2 \succeq_i x_{t+1} \succeq_i \ldots \succeq_i x_{l+1} \succeq_i x_{m+1} \succeq_i x_1.$$

By transitivity, the option $x_2 \succeq_i x_1$, which is a contradiction. Consequently, the legislator $i \notin S_j \cap S_{j+1(modp)}$ $\forall j = 1, 2, ..., p$.

**4.** At this point, the legislator $i$ belongs to at most $\frac{p}{2}$ winning coalitions. Therefore, $p - 1 \leq \frac{p}{2}$ (legislator $i$ belongs to $p-1$ winning coalitions). It follows that, $p \leq 2$. This is is absurd, since $p > 2$ by hypothesis. □

The next result indicates that if a voting rule $r$ is such that $r > (\frac{D-2}{D})n$, there exists a stable policy in a legislature deciding on a discrete policy of importance level $D$.

**Lemma 7.** *There exists a stable policy in a legislature $\mathcal{L} = (N, D, r, (\succeq_i))$, $2 < D < \infty$, for any preference profile $(\succeq_i) \in \mathcal{U}^n$ if $r > (\frac{D-2}{D})n$.*

*Proof.* Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, $(\succeq_i) \in \mathcal{U}^n$ be a legislators' preferences profile, and $A$ the policy space of size $D$, with $2 < D < \infty$. The objective is to clarify that, if $r > (\frac{D-2}{D})n$, then there exists an equilibrium outcome.

It is already proved in Lemma 6 that, if there is no stable policy for a legislators' preferences profile $(\succeq_i) \in \mathcal{U}^n$, then there exists a voting cycle $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length $p$ $(p \leq D,\ p > 2)$, such that there is no legislator $i \in N$ belonging to more than $p-2$ winning coalitions among the set $\mathcal{C} = \{S_1, S_2, ..., S_p\}$.

1. First, $n/2 < r < n-1$. Suppose that the set $\mathcal{R}(\mathcal{L})$ is empty for a profile of weak order preferences $(\succeq_i) \in \mathcal{U}^n$. From Lemma 6, there exists a cycle $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length $p$ $(p \leq D,\ p > 2)$. Consider $\overline{s} = \frac{\sum_{j=1}^{p} |S_j|}{n}$ the average number of times that a legislator appears in the winning coalitions. According to Lemma 6, there is no legislator $i \in N$ belonging to more than $p-2$ winning coalitions among $\mathcal{C} = \{S_1, S_2, ..., S_p\}$, thus the number $p-2 \geq \overline{s}$. Let $s_j = |S_j| - r$, $s_j$ is non negative. $|S_j| \geq r$ implies that $S_j$ contains at least $r$ number of legislators. It follows that $\overline{s} = \frac{\sum_{j=1}^{p} s_j + pr}{n}$, and $\overline{s} \leq p-2$. By a simple arrangement, $p \geq \frac{2n}{n-r} + \frac{\sum_{j=1}^{p} s_j}{n-r}$. Since $\frac{\sum_{j=1}^{p} s_j}{n-r}$ is non negative, then $D \geq p \geq \frac{2n}{n-r}$, which is a contradiction.

2. Second, that $r = n-1$. Let $(\succeq_i) \in \mathcal{U}^n$ such that $\mathcal{R}(\mathcal{L})$ is empty. The proof proceeds by induction on the length of the cycle. There are no cycles of length $D$ and demonstrate that, there are no cycles of length $D+1$. In contrast, there is a cycle of length $D+1$ i.e $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_D} x_{D+1} \gtrdot_{S_{D+1}} x_1$.

   **Case 1**: there exists $j \in \{1, 2, ..., D\}$ such that $S_j = S_{j+1}$.

   By using transitivity of $\gtrdot_{S_j}$, we obtain $x_j \gtrdot_{S_j} x_{j+2}$ and thus reducing the length of the cycle to $D$. This contradicts our induction assumption.

   **Case 2**: for each $j \in \{1, 2, ..., D\}$, $S_j \neq S_{j+1}$. Assume without loss of generality that $1 \in S_1 \cap S_{D+1}$. $x_{D+1} \gtrdot_{S_{D+1}} x_1$, then $x_{D+1} \succ_{S_{D+1}} x_1$, and [*not* $(x_{D+1} \succeq_{S_D} x_1) \Rightarrow x_D \succeq_{S_{D+1}} x_1$]. $1 \in S_{D+1}$ and $S_{D+1} \neq S_D$, then $x_{D+1} \succ_1 x_1$ and $x_D \succeq_1 x_1$. Since $S_D \neq S_{D-1}$, then $S_D \cup S_{D-1} = N$, it follows that $1 \in S_D$ or (and) $1 \in S_{D-1}$. If $1 \in S_D$, given that $x_D \gtrdot_{S_D} x_{D+1}$, we have $x_D \succ_1 x_{D+1}$ and $x_{D-1} \succeq_1 x_{D+1}$. It follows that $x_{D-1} \succeq_1 x_{D+1}$ and $x_{D+1} \succ_1 x_1$, then $x_{D-1} \succ_1 x_1$. If $1 \in S_{D-1}$, given that $x_{D-1} \gtrdot_{S_{D-1}} x_D$, we have $x_{D-1} \succ_1 x_D$ and $x_{D-2} \succeq_1 x_D$ because $S_{D-1} \neq S_{D-2}$. It follows that $x_D \succeq_1 x_1$ and $x_{D-1} \succ_1 x_D$, then $x_{D-1} \succ_1 x_1$. Thus in each case, we have $x_{D-1} \succ_1 x_1$.

It has been proven that $x_j \succeq_1 x_1$ for any $j$ greater than or equal to $l$, i.e $x_j \succeq_1 x_1\ \forall j \geq l$. If $1 \in S_l$, since $x_l >_{S_l} x_{l+1}$, then $x_l \succ_1 x_{l+1}$ and $x_{l-1} \succeq_1 x_{l+1}$ given that $S_l \neq S_{l-1}$. Since $x_{l+1} \succeq_1 x_1$, then, with $x_{l-1} \succeq_1 x_{l+1}$, it follows that $x_{l-1} \succeq_1 x_1$. If $1 \notin S_l$, then $1 \in S_{l-1}$, because $S_l \cup S_{l-1} = N$. Given that $x_{l-1} >_{S_{l-1}} x_l$, and $S_{l-1} \neq S_{l-2}$, therefore $x_{l-1} \succ_1 x_l$ and $x_{l-2} \succeq_1 x_l$. $x_l \succeq_1 x_1$, then $x_{l-1} \succ_1 x_l \succeq_1 x_1$. By transitivity, $x_{l-1} \succ_1 x_1$. It has been verified that, in each case $x_{l-1} \succeq_1 x_1$. It follows that $x_2 \succeq_1 x_1$. Since, $1 \in S_1$ and $x_1 >_{S_1} x_2$, then $x_1 \succ_1 x_2$. Hence, $x_1 \succ_1 x_2 \succeq_1 x_1$. By contradiction, $x_1 \succ_1 x_1$, which is a contradiction.

3. Third, $r = n$. Let $(\succeq) \in \mathcal{U}^n$, and $\mathcal{R}(\mathcal{L}) = \emptyset$. Then, there exists a cycle $x_1 >_N x_2 >_N x_3 >_N \ldots >_N x_p >_N x_1$ of length $p$ $(p \leq D,\ p > 2)$. It follows that $x_1 \succ_N x_2 \succ_N x_3 \succ_N \ldots \succ_N x_p \succ_N x_1$. By transitivity, $x_1 \succ_N x_1$, which is a contradiction. Therefore, $\mathcal{R}(\mathcal{L}) \neq \emptyset$.

Summary: if $r > (\frac{D-2}{D})n$, then for each legislators' preferences profile $(\succeq_i) \in \mathcal{U}^n$, there exists a stable outcome. □

Now, the result below reveals that, if a voting rule $r$ is smaller or equal than the number $(\frac{D-2}{D})n$, then there exists a preference profile under which a stable policy does not exist.

**Lemma 8.** *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, $2 < D < \infty$. If $r \leq (\frac{D-2}{D})n$, then there exists a preferences' profile $(\succeq_i) \in \mathcal{U}^n$ which leads to voting cycles, i.e $\mathcal{R}(\mathcal{L}) = \emptyset$.*

Let $\overline{D} = \lceil \frac{2n}{n-r} \rceil$ and $D \geq \overline{D} > 2$ (for a real number $x$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$). The scope of this proof is to construct a preference profile $(\succeq_i) \in \mathcal{U}^n$ such that:

$$x_1 >_{S_1} x_2 >_{S_2} x_3 >_{S_3} \ldots >_{S_{\overline{D}-1}} x_p >_{S_{\overline{D}}} x_1,$$

with $|S_j| \geq r$ for any $j = 1, 2, ..., \overline{D}$. We have to show that:

$$x_l \succ_i x_{l+1} \text{ and } [not(x_l \succeq_{S_{l-1}} x_{l+1}) \Rightarrow x_{l-1} \succeq_{S_l} x_{l+1}]$$

for any $l = 1, 2, ..., \overline{D}$ (for $l = \overline{D}$, $l + 1 = 1$).
We enumerate four cases:

<u>**Case 1**</u>: the integer $n - r$ is even.

We partition the population of $n$ legislators into $\overline{D}$ sets $P_j$, $j = 1, 2, ..., \overline{D}$ where in each set, legislators have the same preferences:

$P_1 : x_1\ x_{\overline{D}}\ x_2\ x_3\ ...\ x_{\overline{D}-1}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_2 : x_2\ x_1\ x_3\ x_4\ ...\ x_{\overline{D}}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_3 : x_3\ x_2\ x_4\ x_5\ ...\ x_1\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_4 : x_4\ x_3\ x_5\ x_6\ ...\ x_2\ (x_{\overline{D}+1}\ ...\ x_D)$

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$P_{\overline{D}-3} : x_{\overline{D}-3}\ x_{\overline{D}-4}\ x_{\overline{D}-2}\ x_{\overline{D}-1}\ ...\ x_{\overline{D}-5}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}-2} : x_{\overline{D}-2}\ x_{\overline{D}-3}\ x_{\overline{D}-1}\ x_{\overline{D}}\ ...\ x_{\overline{D}-4}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}-1} : x_{\overline{D}-1}\ x_{\overline{D}-2}\ x_{\overline{D}}\ x_1\ ...\ x_{\overline{D}-3}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}} : x_{\overline{D}}\ x_{\overline{D}-1}\ x_1\ x_2\ ...\ x_{\overline{D}-2}\ (x_{\overline{D}+1}\ ...\ x_D)$

Consider:

$$p_j = |P_j| = \begin{cases} \frac{n-r}{2} & if \quad j \neq \overline{D} \\ n - \sum_{k=1}^{\overline{D}} p_k = n - \frac{\overline{D}-1}{2}(n-r) & if \quad j = \overline{D} \end{cases}.$$

Given that $N = \bigcup_{j=1}^{\overline{D}} P_j$, consider $S_l = N - (P_{l+1} \cup P_{l+2})$, for any $l = 1, 2, ..., \overline{D}$. (for $l = \overline{D}$, $l+1 = 1$ and $0 < p_{\overline{D}} \leq \frac{n-r}{2}$). Since $p_j = \frac{n-r}{2}$ for $j \neq D$, then $p_{l+1} + p_{l+2} \leq n - r$ and $|S_l| \geq r$.

- For any $i \in S_l$, $x_l \succ_i x_{l+1}$, hence $x_l \succ_{S_l} x_{l+1}$.
- For any $i \in S_l$, $x_{l-1} \succ_i x_{l+1}$ i.e $x_{l-1} \succ_{S_l} x_{l+1}$ and according to the profile, there exists a player $i' \in S_{l-1}$ such that $x_{l+1} \succ_{i'} x_l$ i.e $not(x_l \succeq_{S_{l-1}} x_{l+1})$. It follows that $x_l \gtrdot_{S_l} x_{l+1}$ for any $l = 1, 2, ..., \overline{D}$.

<u>**Case 2**</u>: the integer $n - r$ and $\overline{D}$ are odd.

The same profile of preferences' structure as in **Case 1** are used here, with the exception that the size of each partition $P_j$ changes. Consider:

$$p_j = |P_j| = \begin{cases} \lceil \frac{n-r}{2} \rceil & if \quad j \text{ is even} \\ \lfloor \frac{n-r}{2} \rfloor & if \quad j \text{ is odd and } j \neq \overline{D} \\ n - \frac{\overline{D}-1}{2}(n-r) & if \quad j = \overline{D} \end{cases}.$$

Note that $0 < p_{\overline{D}} \leq \lfloor \frac{n-r}{2} \rfloor$. Using the same reasoning as in **Case 1**, $x_l \succ_{S_l} x_{l+1}$ for any $l = 1, 2, ..., \overline{D}$.

<u>**Case 3**</u>: the integer $n - r$ is odd, $\overline{D}$ is even and $\overline{D} = \frac{2n}{n-r}$.

Consider the same profile as in the previous **Cases** (**1** and **2**) and assume that:

$$p_j = \begin{cases} \lfloor \frac{n-r}{2} \rfloor & if \quad j \ is \ odd \\ \lceil \frac{n-r}{2} \rceil & if \quad j \ is \ even \end{cases}.$$

The reasoning is the same as in the previous **Cases**.

<u>**Case 4**</u>: the integer $n - r$ is odd, $\overline{D}$ is even and $\overline{D} \neq \frac{2n}{n-r}$ (remember that $\overline{D} = \lceil \frac{2n}{n-r} \rceil$) .

The population of $n$ legislators is partitioned in $\overline{D} + 1$ sets described hereunder:

$P_1 : x_1 \quad x_{\overline{D}} \ x_2 \ x_3 \ ... \ x_{\overline{D}-1} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_2 : x_2 \quad x_1 \ x_3 \ x_4 \ ... \ x_{\overline{D}} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_3 : x_3 \quad x_2 \ x_4 \ x_5 \ ... \ x_1 \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_4 : x_4 \quad x_3 \ x_5 \ x_6 \ ... \ x_2 \ (x_{\overline{D}+1} \ ... \ x_D)$

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$P_{\overline{D}-3} : x_{\overline{D}-3} \quad x_{\overline{D}-4} \ x_{\overline{D}-2} \ x_{\overline{D}-1} \ ... \ x_{\overline{D}-5} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}-2} : x_{\overline{D}-2} \quad x_{\overline{D}-3} \ x_{\overline{D}-1} \ x_{\overline{D}} \ ... \ x_{\overline{D}-4} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}-1} : x_{\overline{D}-2} \ x_{\overline{D}} \ x_{\overline{D}-3} \ x_{\overline{D}-1} \ x_1 \ ... \ x_{\overline{D}-4} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}} : \quad x_{\overline{D}-1} \quad x_{\overline{D}-2} \ x_{\overline{D}} \ x_1 \ ... \ x_{\overline{D}-3} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}+1} : x_{\overline{D}} \quad x_{\overline{D}-1} \ x_1 \ x_2 \ ... \ x_{\overline{D}-2} \ (x_{\overline{D}+1} \ ... \ x_D)$

Consider:

$$p_j = |P_j| = \begin{cases} \lfloor \frac{n-r}{2} \rfloor & if \quad j \ is \ odd \ and \ j \notin \{\overline{D}-1, \overline{D}+1\} \\ \lfloor \frac{n-r}{2} \rfloor & if \quad j \in \{\overline{D}-2, \overline{D}\} \\ \lceil \frac{n-r}{2} \rceil & if \quad j \ is \ even \ and \ j < \overline{D}-2 \\ 1 & if \quad j = \overline{D}-1 \\ n - \sum_{i=1}^{\overline{D}} p_i & if \quad j = \overline{D}+1 \end{cases}.$$

Demonstrate that $0 < p_{\overline{D}+1} = n-1-3(\frac{n-r-1}{2})-\frac{\overline{D}-4}{2}(n-r) \leq \lfloor\frac{n-r}{2}\rfloor$. If $n-r = 2x+1$, $x$ an integer, then $\lfloor\frac{n-r}{2}\rfloor = x = \frac{n-r-1}{2}$, and $\lceil\frac{n-r}{2}\rceil = x+1 = \frac{n-r+1}{2}$. Given that $N = \bigcup_{j=1}^{\overline{D}+1} P_j$, consider:

1. $S_l = N - (P_{l+1} \cup P_{l+2})$, for $l = 1, 2, ..., \overline{D}-5$, then $|S_l| \geq r$, and $S_l$ is a wining coalition;
2. For any $l \in \{\overline{D}-4, \overline{D}-3, \overline{D}-2, \overline{D}-1, \overline{D}\}$, $|S_l| \geq r$, since:
   - $S_{\overline{D}-4} = N - (P_{\overline{D}-3} \cup P_{\overline{D}-2} \cup P_{\overline{D}-1})$;
   - $S_{\overline{D}-3} = N - (P_{\overline{D}-2} \cup P_{\overline{D}-1} \cup P_{\overline{D}})$;
   - $S_{\overline{D}-2} = N - (P_{\overline{D}} \cup P_{\overline{D}+1})$;
   - $S_{\overline{D}-1} = N - (P_{\overline{D}-1} \cup P_{\overline{D}+1} \cup P_1)$;
   - $S_{\overline{D}} = N - (P_1 \cup P_2)$;
3. For any $l \in \{1, 2, ..., \overline{D}-5\}$, $x_l >_{S_l} x_{l+1}$ as in **Cases 1 and 2**.
4. According to the new partition: $x_{\overline{D}-4} \succ_{S_{\overline{D}-4}} x_{\overline{D}-3}$, $not(x_{\overline{D}-4} \succeq_{S_{\overline{D}-5}} x_{\overline{D}-3})$, and $x_{\overline{D}-5} \succ_{S_{\overline{D}-4}} x_{\overline{D}-3}$. It follows that $x_{\overline{D}-4} >_{S_{\overline{D}-4}} x_{\overline{D}-3}$. In the same manner: $x_{\overline{D}-3} >_{S_{\overline{D}-3}} x_{\overline{D}-2}$, $x_{\overline{D}-2} >_{S_{\overline{D}-2}} x_{\overline{D}-1}$, $x_{\overline{D}-1} >_{S_{\overline{D}-1}} x_{\overline{D}}$, $x_{\overline{D}} >_{S_{\overline{D}}} x_1$, and a cycle of length $\overline{D}$ occurs.

For each **Case** studied, and for each legislators' preferences in each set $P_j$, $j = 1, 2, ..., \overline{D}$ or $\overline{D}+1$, the alternatives $x_{\overline{D}+1}$, $x_{\overline{D}+2}, ..., x_D$ don't belong to the set $\mathcal{R}(\mathcal{L})$, because they are Pareto-dominated. Moreover, each alternative $x_l$, $l = 1, 2, ..., \overline{D}$ is dominated by only one alternative $x_{l-1}$. Therefore, we have constructed a profile $(\succeq_i) \in \mathcal{U}^n$ such that there is no stable outcome.

Now, we complete the proof of the theorem. Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, with $2 \leq D < \infty$, and $(\succeq_i) \in \mathcal{U}^n$ be a legislators' preferences profile. Lemmas 7 and 8 give necessary and sufficient conditions that a supermajority rule should satisfy in order to fill the constraint in the political designer minimization problem 2.1. They state that if the supermajority size needed to pass a policy of importance level $D$ in a legislature of size $n$ is greater than $(\frac{D-2}{D})n$, then a stable policy exists regardless of the extent to which legislators' preferences are antagonistic. Conversely, if a stable policy of importance level $D$ exists at any preference profile in a legislature of size $n$, then the supermajority size needed to pass a policy should be greater than $(\frac{D-2}{D})n$. Formally, these results illustrate that the set $\mathcal{R}(\mathcal{L})$ is non-empty, for each $(\succeq_i) \in \mathcal{U}^n$ if and only if $r > (\frac{D-2}{D})n$. Since $r$ is an integer number, the smallest one which satisfies the latter inequality is $\left\lfloor \frac{D-2}{D} n \right\rfloor + 1$. Moreover $r > n/2$, then the voting rule: $\max\left\{\left\lfloor \frac{D-2}{D} n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right\} + 1$ is the optimal solution.

# Chapter 3

# A Preference-Based Rationalization of the Majority Rule

## 3.1 Introduction

The *majority rule* is widely used to select policies in decision-making bodies. It is considered as "the most robust"[1] voting rule, because over a wide class of preference domains, it is the only rule which satisfies five classic axioms, namely, Pareto property, anonymity, neutrality, contraction independence—independence of irrelevant alternatives, and generic decisiveness (Dasgupta and Maskin (2008)). In a multi-session bargaining model governed by a closed rule, it is shown that "majority-rule legislature" may adjourn in the first session, thus reducing the cost of a decision-making process (e.g., Buchanan and Tullock (1962), Baron and Ferejohn (1989)) in legislatures. Among other well-known properties of the majority rule, it reduces voting-splitting in an electoral competition and conforms the principle of "one citizen, one vote" in truly democratic societies (e.g., Maskin and Sen (2017a), Maskin and Sen (2017b), and Maskin and Sen (2017c)).

However, it is well-known that the majority rule[2] may lead to inconsistent public decisions, which raises the question of when it should be used (see, for example, Condorcet (1785), Black (1948), May (1952), Sen (1966), Plott (1967), Sen and Pattanaik (1969), Kramer (1973), Grandmont (1978), Shepsle and Weingast (1981) and the references therein, Shepsle and Weingast (1982), among others). In this chapter, I provide a preference-based rationalization of the majority rule under a simple sequential decision-making procedure. I display natural preference domains for which the majority rule can be rationalized in the sense that it always

[1]The quote is from the seminal paper of Dasgupta and Maskin (2008, P.1).

[2]The qualification "majority rule" in this paper has the exact meaning as the "pure majority rule" concept used by some scholars in the literature.

leads to the selection of an equilibrium policy, ensures that all equilibrium policies are efficient, and minimizes status quo bias, no matter the complexity of the policy space. Throughout the chapter, the words "importance" and "complexity", whenever they are related to the nature of a policy (decision) are interchangeable.

Chapter 2 revisits the problem of rationalization of supermajority voting rules. In a tractable political economy defined by a society which consists of a finite set of individuals having preferences over a discrete or continuum set of policy space, the authors derive a closed form relationship between the size of majority required to enact a social policy and the complexity of such policy in a legislature. Among their results, they prove that the majority rule can emerge as the optimal voting rule for the enactment of policies of lower complexity under the *reciprocity mechanism*, if voters in the legislature have preferences that belong to a large preference domain $\mathfrak{D}$. (The description of the reciprocity mechanism is provided in Section 3.2. A preference domain is a subset of preference profiles.) The set $\mathfrak{D}$ includes either convex preferences over a compact and convex policy space or weak order preferences over a discrete and finite policy space.

This chapter refines the large set of individuals' preferences described above. In fact, a preference domain $\mathfrak{D}$ reflects a large view on individuals' socio-economic and political preferences. In some contexts, some of these preferences might be neither feasible nor observable in real-life applications.[3] Finding a small preference domain $\mathfrak{P}$ of $\mathfrak{D}$ (i.e $\mathfrak{P} \subsetneqq \mathfrak{D}$) that could rationalize the use of a well-known decision rule such as the majority rule seems to be of interest. Specifically, what new insight could be drawn from the analysis described above if the preference domain $\mathfrak{D}$ is refined?

The results of my analysis stipulate that the answers to the above objective problem depend on the nature of the induced policy space and the preferences of voters in the council. I note that the majority rule is not optimal for sufficiently complex policies (see Example 3.1). The latter implies that the majority rule is not appropriate for certain types of societies, including sufficiently fractionalized societies. However, if $\mathfrak{P}$ is the set of linear order preferences, the majority rule is the optimal solution independent of the nature of the policy space (Theorem 9). If the policy at stake induces a continuum set of feasible proposals, then the majority rule is the optimal voting rule if each of the following statements is satisfied. 1) $\mathfrak{P}$ is the set of single-caved or single-peaked preferences over a multidimensional policy space (Theorem

[3] Black (1948) stipulates that "ideology" might justify the fact that some preferences may be highly unlikely in an electoral competition.

10). 2) $\mathfrak{P}$ is the set of strictly convex preferences over a continuous and compact policy space (Theorem 12). 3) $\mathfrak{P}$ is the set of continuous preferences that admit concave utility representation over a compact policy space (Theorem 13).

There is a substantial literature investigating the institutionalization of voting rules in legislatures and committees. In this literature, scholars implement legislative games in which the majority rule guarantees the existence of equilibrium outcomes. It is argued that either legislative exchanges (e.g., Tullock (1981)) or institutional arrangements (see, for example, Shepsle and Weingast (1981), Shepsle and Weingast (1982), among others) lead to the stability of majority rule. These studies prove how majority rule in well-defined agenda formation could improve upon the social predictions of multidimensional voting models and legislative choices in real-world legislatures. This study is related to this strand of analyses since the sequential voting procedure—reciprocity mechanism—in the model developed in this chapter can be viewed as a form of structure-induced equilibrium framework capable of solving the instability of the majority rule under natural assumptions of voters' preferences.

In spite of this, to my knowledge, this essay is the first one, after Chapter 2, that rationalizes the majority rule by using the structure of individuals' preferences in a political framework when decisions are made through the reciprocity mechanism. This work departs from the literature analyzing the conditions for the existence of equilibrium of pure majority rule. This literature includes the work, among others, of Black (1948), Arrow (1951), May (1952), Sen (1966), Plott (1967), Sen and Pattanaik (1969), Kramer (1973), Shepsle and Weingast (1981) and the references therein. One reason for the difference between this chapter and the studies cited above is based on the notion of equilibrium set. In fact, the equilibrium concept—*reciprocity set*, first introduced in Chapter 1, which is used in this study, is different from the *majority equilibrium* concept developed in the papers mentioned above. This difference is perceived at least in two dimensions: the structural sequence of exchanges between legislators and the prediction of equilibrium policies.

Section 3.4 displays an application of the framework under which the majority rule is rationalized to a problem of locating a site to build public bads. The latter are public facilities that are profitable for the community but that no one likes to have in his/her close neighborhood. One major problem encountered in the location of a public bad comes from its negative externality. It is therefore rational for each individual to support any public policy which claims to locate such infrastructures far away from his/her land. One casual solution proposed by many political scientists and scholars is voting. Given that voters usually cast ballots according to their preferences and they have antagonistic tastes about public policies, nothing ensures that an assembly will reach an equilibrium location by using a voting mechanism. Finding conditions that guarantee the existence of equilibrium locations of a public bad using a legislative procedure constitute the main objective of Section 3.4. Addressing this problem has important implications for the design of public policies towards protecting both environment and human nature.

The results of the application described in Section 3.4 are related to a few studies in the literature (e.g., see Barberà, Berga, and Moreno (2011), Öztürk, Peters, and Storcken (2013) and Öztürk, Peters, and Storcken (2014), Manjunath (2014)). In these papers, scholars study the existence of a rule to implement the location of a public bad in a given region by voting. A finite set of voters have single-dipped preferences over a finite dimensional location space. They provide conditions on the location space under which the rule (social choice function) satisfies Pareto efficiency, strategy-proofness, and sometimes the unanimous criterion. For example, Manjunath (2014) demonstrates that the majority rule is a Pareto efficient and strategy-proof voting rule. They also characterize the classes of voting rules that satisfy these properties under the same assumption regarding agents' preferences over the set of feasible alternatives. The goal of this chapter is significantly different from this study, in terms of the basic framework to the equilibrium concept. First, the model in which the council selects an equilibrium location of a public bad in the present study is a bargaining and legislative voting procedure. Neither study aforementioned uses this framework. Second, in the proposed setting, members of the council may have a wide range of preferences over a possible multi-dimensional location space. In the decision-making procedure with the majority rule, I am interested in a possible set of individual preferences that guarantees the existence of equilibrium outcomes in the council. The results derive a broad class of agents' preferences that satisfy these specific criteria on the location space. These classes include the case of single-dipped (caved) preferences.

The remaining part of the chapter is organized as follows. Section 3.2 presents some concepts in the formalization of the objective problem, and Section 3.3 provides results. In Section 3.4, I provide a specific application of the analysis to the location of public bads. An appendix contains an outline to the proofs of main theorems and propositions in the text.

## 3.2 The Objective Problem

This section covers some preliminary concepts and formalizes the main objective question of the study. The *social dimensionality* or *complexity* or *importance* of a given policy is the number of dimensions of society's life and wellbeing the policy is likely to affect (for more on these concepts, see, Chapter 2). In this study, the council is labeled by $\mathcal{C} = (N, A(d), (\succeq_i)_{i\in N})$, where $A(d)$ is the policy space induced by a policy of complexity level $d$, $N$ is the set of members or voters, and $(\succeq_i)_{i\in N}$ is the profile of voters' preferences over the set $A(d)$. The policy space, which is either a discrete or a continuum policy space, represents the set of different policies in competition. This information is known by every individual in the council. In case of continuum decision, it is assumed that the policy space is a compact and convex subset of the Euclidean space, $\mathbb{R}^d$, of dimension $d$.

Throughout the chapter, $\mathcal{M}$ denotes the set of *majority coalitions*. Specifically, any subgroup of individuals which consists of more that half of the council's size belongs to $\mathcal{M}$. For an arbitrary set $X$, the label $|X|$ denotes its cardinality. The following section describes the process of selecting policies within the council.

### 3.2.1 The Decision-making Process

The council decides on the choice of a policy under a finite and sequential political mechanism, called the *reciprocity mechanism* introduced in Chapter 1. For clarity, this mechanism is briefly presented below.

Assume that the goal is to reform a given status quo $x \in A(d)$. The latter is either the incumbent policy or any alternative chosen from the policy space. The voting process entails three main stages described as follows:

- *Stage 1 (Objection)*: Any member of the council has the right to demand the replacement of $x$ by proposing a motion $y$. If no motion is submitted, then $x$ remains in place and the vote ends. If a motion $y$ is proposed and supported by a coalition which is winning under the constitution $\mathcal{M}$, then $y$ is submitted with the possibility of an amendment.

- *Stage 2 (Right to opposition)*: Any member who is not part of the sponsoring coalition has the right to oppose the bill $y$. If there is no opposition, $y$ is enforced as the new location and the process ends. If there is an opposition, the sponsoring coalition is given the option to withdraw the bill $y$. If the coalition chooses to withdraw $y$, the status quo

$x$ remains in place and the process ends. But, if this coalition chooses to maintain $y$, the process moves to the third stage.

- *Stage 3 (Counter-objection)*: The opposition has the right to propose an amendment $z$. If it does not propose any amendment, $y$ becomes the new location and the process ends. If the opposition succeeds by proposing an amendment $z$, a vote is organized between $y$ and $z$. If a winning coalition supports $z$ against $y$, $z$ becomes the new location and the process ends. Otherwise $y$ is the new policy and the process ends.

An *equilibrium* policy is an alternative such that if it is the status quo, no majority coalition will seek to replace it. In the decision-making process, a policy, say $y \in A(d)$ *defeats* or *dominates* another policy $x$, labeled by $y \gtrdot x$, if there exists a majority coalition $S \in \mathcal{M}$ such that:

1. $y \succ_S x$ and;
2. $[\forall (z,T) \in A(d) \times \mathcal{M},\ S \neq T,\ z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

The *equilibrium set*, $R(\mathcal{C})$ contains policies that can't be dominated in the council $\mathcal{C}$.

Suppose at the beginning of the bargaining process in the council, a status quo $x \in A(d)$ is at stake. A group of members will gather together in a winning coalition, say $S$ and *object* to the status quo if there exists another alternative, say $y$ which is better than $x$, i.e $y \succ_S x$. After the *objection* $(y, S)$, a member of the remaining group of the council $(N \setminus S)$ may be worse off at $y$, and decides to join a winning coalition $T$ which could propose another policy $z$ to the council, i.e $z \succ_S y$. Thus, a pair $(z, T) \in Z \times \mathcal{M}$ is a *counter-objection* against the objection $(y, S)$ if $z \succ_T y$ and not$(y \succeq_T x)$. Also, the members of $T$ have to consider the fact that the first mover coalition $S$ may withdraw their support for the change of the status quo $x$ in case the new proposition $z$ could hurt some of its members, i.e not$(z \succeq_S x)$. If the latter occurs, then the counter-objection is *unfriendly* against $S$. In the council, a winning coalition $S$ will initiate the first move against the status quo $x$ by proposing a new option $y$ if there is no unfriendly counter-objection against $(y, S)$, i.e $(z \succeq_S x)$; then the objection $(y, S)$ is considered to be *justified*. An equilibrium location is an alternative in $A(d)$ against which no justified objection exists.

Assume the complexity level $d$ of a decision to be made in the council is given, and the decision is made under the reciprocity mechanism with the majority rule. Under which refinement set of individuals' preferences, this rule is optimal in the sense that it always guarantees the existence of an equilibrium policy, ensures that all equilibrium policies are efficient, and minimizes status quo bias?

Following the methodology developed in Chapter 2 on the institutionalization of voting rules, the issue above could be formalized by the optimization problem displayed hereunder:

$$\begin{array}{ll} \min & r(d) \\ \text{s.t.} & \mathcal{R}[N, r(d), (\succeq_i)_{i \in N}] \neq \emptyset, \text{ for any } \succeq_i \in \mathfrak{P} \subset \mathfrak{D} \\ & \forall x \in \mathcal{R}[N, r(d), (\succeq_i)_{i \in N}], \ x \textit{ is } \text{Pareto-efficient}, \end{array} \tag{3.1}$$

where $\mathfrak{P}$ is a proper subset of the preference domain $\mathfrak{D}$, and $r(d)$ is the minimum size of a winning coalition. The set $\mathfrak{D}$ is taken from the objective problem introduced in Chapter 2. The novelty of this present study is the presence of the refinement set $\mathfrak{P}$ that I wish to provide in the next sections. In Chapter 2, it is proven that if the preference domains $\mathfrak{D}$ contains either weak order preferences for policies that induce a discrete set of alternatives, or convex preferences for policies that induce a continuum set of alternatives, then there exists an optimal supermajority rule $r^*(d)$ with respect to $d$, that allows the council to enact an equilibrium and efficient policy. Their analysis rationalizes the wide variation of voting rules across legislative organizations.

The preference domain $\mathfrak{D}$ reflects a large view of individuals' socio-economic and political preferences. In some contexts, some preferences in $\mathfrak{D}$ can be neither feasible nor observable in real-life applications. In this case, other restrictions on individuals' preferences are often made by scholars who study the existence of an equilibrium policy in voting theory (see, for example, Kramer (1973), Grandmont (1978) and the references therein). Identifying a natural preference domains $\mathfrak{P}$ such that the majority rule is the solution of Problem 3.1 is the main purpose of this chapter. This exercise is provided in the next section.

## 3.3 Preference Domains and the Majority Rule

In this section, I provide preference domains under which the majority rule is the solution of the objective problem in a council where decisions are made through the reciprocity mechanism as formalized in the previous section. As mentioned above, the challenge consists of refining the preference domains $\mathfrak{D}$ to a subset $\mathfrak{P}$ under which the majority rule is the optimal solution to Problem 3.1.

The efficiency condition in Problem 3.1 is resolved thanks to the finding derived in Chapter 2.

**Lemma 9.** *Let $\mathcal{C} = (N, A(d), r(d), (\succeq_i))$ be a council, where $r(d)$ is the minimum size of a winning coalition. Any equilibrium policy under the reciprocity mechanism is Pareto-efficient.*

Before displaying the preference domains in which the majority rule is the optimal solution, I introduce below an example where the majority rule does not fulfill all the constraints of the optimality as required in this chapter.

**Example 6.** The setting displays a political structure $\mathcal{C}$ (committee, council, legislature, etc.) which consists of five legislators for whom the task is to make a decision on a policy of complexity level 5. Formally, the set of legislators is labeled by $N = \{1, 2, 3, 4, 5\}$, and the policy space $A = \{x_1, x_2, x_3, x_4, x_5\}$. Legislators have different tastes (preferences) on feasible policies in $A$. They classify policies in the following manner: Legislator 1: $x_1 \succ x_2 \sim x_5 \succ x_3 \succ x_4$, Legislator 2: $x_2 \succ x_1 \sim x_3 \succ x_4 \succ x_5$, Legislator 3: $x_3 \succ x_2 \sim x_4 \succ x_5 \succ x_1$, Legislator 4: $x_4 \succ x_3 \sim x_5 \succ x_1 \succ x_2$, and Legislator 5: $x_5 \succ x_4 \sim x_1 \succ x_2 \succ x_3$. Figure 3.1 illustrates the majority domination among policies at stake in the political structure. Decisions

$$x_1 \xrightarrow{\quad S_1 \quad} x_5 \xrightarrow{\quad S_2 \quad} x_4 \xrightarrow{\quad S_3 \quad} x_3 \xrightarrow{\quad S_4 \quad} x_2 \xrightarrow{\quad S_5 \quad} x_1$$

FIGURE 3.1: Political cycle

are made through the reciprocity mechanism, with the majority rule as the political rule. In Figure 3.1, $S_1 = \{3, 4, 5\}$, $S_2 = \{2, 3, 4\}$, $S_3 = \{1, 2, 3\}$, $S_4 = \{1, 2, 5\}$, and $S_5 = \{1, 4, 5\}$. The graph describes a majoritarian political cycle: $x_1 \succ_{S_5} x_2 \succ_{S_4} x_3 \succ_{S_3} x_4 \succ_{S_2} x_5 \succ_{S_1} x_1$. Under the majority rule, each policy is dominated in the sense of the binary relation $\gtrdot$, meaning that there is no equilibrium outcome. Then, in this political structure, the majority rule is not an appropriate solution to the objective problem described above.

The latter result obtained in the example above is driven either by the complexity of the decision, measured by the social dimensionality of the policy at stake, or the preference profile of individuals in the political structure. Since each policy in the proposals space $A$ is Pareto efficient, and the majority rule is the optimal decision rule which minimizes status quo bias, the only reason that justified the elimination of this rule to the set of solutions to the objective problem is the non-existence of equilibrium alternatives. In the remaining part of this section, I propose several preference domains in which this problem is resolved. I start by analyzing below a political framework, where individual preferences over the induced policy space are reflexive, transitive, antisymmetric, and complete.

### 3.3.1 Linear Order Preferences

**Theorem 9.** *The majority rule is the optimal solution to Problem 3.1 if each individual has a linear order preference over the induced policy space.*

This result covers both discrete and continuous policies. Preference continuity is a natural assumption which means that an individual who prefers a policy $x$ over another option $y$ prefers any alternative close enough to $x$ over any alternative close enough to $y$. This implies that a small perturbation of the policy space does not radically change preferences. This structure of preferences is generally encountered in economic theory.

Formally, let $\succeq$ be a preference relation over a policy space, say $A(d)$. The binary relation $\succeq$ is said to be continuous if for any policies $x, y \in A(d)$, such that $x$ is strictly preferred to $y$ ($x \succ y$), there exists a neighborhood $S(x)$ of $x$ and a neighborhood $S(y)$ of $y$ such that alternative $z$ is strictly preferred to alternative $t$ ($z \succ t$) for every alternatives $z \in S(x)$ and $t \in S(y)$.

In this continuous setting, the policy space is endowed with the topology of closed convergence (see, for example, Hildenbrand (1974)). Topological notions have been useful in the analysis of equilibrium existence in an exchange economy and in game theory (Arrow and Debreu (1954), Debreu (1959) and Debreu (1964), Chichilnisky (1993)). Also, the assumption of a multidimensional policy space covers many problems in welfare economics (e.g., Plott (1967), and Kramer (1973)). In case of continuum policy space, the proof relies on the notions of the Borel set and the Lebesgue measure, denoted here by $\mathfrak{L}$. These concepts have proven to be extremely useful in general equilibrium theory. In the next section, I analyze the solution of objective Problem 3.1 when individual preferences belong to a subset of continuous preferences defined over a one-dimensional policy space.

### 3.3.2 Single-Caved and Single-Peaked Preferences

A preference relation over a policy space is said to be single-peaked if the policies can be ordered as points on a line; the preference relation has a maximum point; and points farther away from this maximum point are less preferred.

To make this definition precise, let us assume that all the policies are ordered by a binary relation denoted $\gg$, and all individuals perceive them as being arranged in this order. An individual $i$ has a single-peaked preference $\succeq_i$, if (1) there exists a policy $a_i$ such that $a \neq a_i$ for any other policy $a \in A(d)$, and (2) for any policies $a, b \in A(d)$, $[a \gg b \gg a_i$ or $a_i \gg b \gg a]$ then $b \succ_i a$. This class of preferences is common in the modelization of people's preferences in real life and has been studied in the literature (see, for example, Inada (1964), Moulin (1980), Sprumont (1991), Ehlers, Peters, and Storcken (2002), and a recent survey by Barberà, Berga, and Moreno (2017)).

Single-caved preferences are the mirrors of single-peaked preferences. A preference relation over a location space is said to be *single-caved* if the locations can be ordered as points on a line, the preference relation has a least preferred point – dip or worse point, and points further away from the least point are more preferred. Formally, assume that all the locations are ordered by a binary relation denoted $\lhd$, and all individuals perceive them as being arranged in this order. An individual $m$ of the council has a *single-caved* preference $\succeq_m$, if for any locations $x$, $y$, and $z$, it holds that: if $(x \lhd y \lhd z)$ or $(z \lhd y \lhd x)$, then $y \succ_m x$ implies that $z \succ_m y$. This class of preferences has been studied in a number of papers (see, e.g., Inada (1964), Ehlers (2002), Barberà, Berga, and Moreno (2011), Öztürk, Peters, and Storcken (2013), Manjunath (2014), Öztürk, Peters, and Storcken (2014), Lahiri, Peters, and Storcken (2016), Yamamura (2016), Peters et al. (2017)).

**Theorem 10.** *The majority rule is the optimal solution of Problem 3.1 if each individual has either a single-caved or single-peaked preference over a one-dimensional policy space.*

The proof of Theorem 10 is constructive. It is shown for both preference domains, single-peaked, and single-caved preferences. The proof also reveals some interesting features. For instance, in the case of single peaked-preference, there exists at least one and at most two equilibrium outcomes. In addition, assuming that the council has an odd number of voters, and that everyone casts the vote, then the political mechanism guarantees the existence of unique equilibrium policy. The latter coincides with the optimum policy of the median voter. When the number of individuals is even, a median individual may not exist, and in this case, there are two equilibrium policies that are very close to the center of the political spectrum (the interval $[0, 1]$). Although single-caved preferences are the mirror-images of single-peaked preferences, the demonstration indicates a different feature concerning the equilibrium set. Formally, if the size of council is odd, there are at most two equilibrium policies. Unlike the case above, there may exist an infinite number of equilibrium policies, if the council consists of an even number of individuals.

### 3.3.3 Multi-Peaked and Multi-Caved Continuous Preferences

In this section, it is assumed that the policy space is a compact and convex subset of the multidimensional vector space $\mathbb{R}^k$(e.g., the segment $[0,1]^2$ is a compact and a convex subset of $\mathbb{R}^2$). Preferences are continuous and endowed with the topology of closed convergence. The definition of the local minimizer of a binary relation is offered below.

**Definition 10.** Let $\succeq$ be a binary relation over a policy space $A(d)$. An alternative $x \in A(d)$ is a local minimizer of $\succeq$ if there exists a neighborhood $S(x)$ of $x$ such that for every $y \in S(x)$, $y \succeq x$.

Denote by $LM(\succeq, A(d))$ the set of all the local minimizers of the binary relation $\succeq$. Note that, if the preference $\succeq$ is a continuous binary relation over a policy space $A(d)$, then $LM(\succeq, A(d))$ is a non-empty Borel subset of $A(d)$. The result of this section guarantees the existence of an equilibrium policy when individuals have both single-caved and multi-caved weak[4] order preferences defined over a continuous location space that is possibly multidimensional. Preferences in this class have one or many local minimizers that form a null set. An interesting subclass of this class of preferences is the set of preferences that have countably many local minimizers. A multi-caved preference is necessarily multi-peaked, where the set of peaks does not need to be countable. It follows that the class of multi-caved preferences includes the subclass of multi-peaked preferences whose local minimizers form a null set. The result indicates that, under this broad class of preferences, the majority rule is the optimal solution to Problem 3.1.

**Theorem 11.** *The majority rule is the optimal solution of Problem 3.1 if for each individual preference $\succeq_i$ defined over a multidimensional policy space $A(d)$ induced by a decision of complexity level $d$, $\succeq_i$ is continuous, and $\mathfrak{L}[LM(\succeq_i, A(d))] = 0$, for $i = 1, 2, ..., n$.*

### 3.3.4 Strictly Convex and Continuous Preferences

Strictly convex preferences have a number of appealing properties and practical implications. A preference relation that is strictly convex and continuous can be represented by a utility

[4] A binary relation $\succeq$ is a weak order if it is reflexive, transitive, and complete.

function that exhibits a diminishing marginal rate of substitution. Let $\succeq$ be a preference relation over a policy space, say $A(d)$. The binary relation $\succeq$ is said to be strictly convex if for any policies $x, y, z \in A(d)$ such that $x \succeq y$ and $x \succeq z$, $x \succ \lambda y + (1 - \lambda)z$ for every $\lambda \in [0, 1[$. Strictly convex preferences have proven useful to the analysis of public goods and externalities (Cherchye, Demuynck, and De Rock (2014)).

**Theorem 12.** *The majority rule is the optimal solution of Problem 3.1 if each individual has continuous and strictly convex preference over a continuous and compact policy space.*

### 3.3.5 Continuous Preferences that Admit a Concave Utility Representation

This part establishes that political institutions could use majority rule to enact political decisions when the individuals' political interests are reflected by ordinal preferences that admit concave utility representation. Owing to their intuitive properties, preferences in this class are probably the most popular in the literature.

**Theorem 13.** *The majority rule is the optimal solution of Problem 3.1 if each individual has continuous preference that admit a concave utility representation over a continuous and compact policy space.*

### 3.3.6 Generic Property of Preference Domains

In what follows, I provide a characterization of the preferences for which the majority rule is optimal when the policy at stake induces a continuum of feasible options. Formally, I prove that the set of continuous and weak orderings' preference profiles for which there exist an equilibrium policy under the reciprocity mechanism with the majority rule is a closed set. This result also introduces a generic property of the equilibrium set under the simple legislative mechanism used in this chapter. Before stating the result, I introduce the following definition:

**Definition 11.** A preference domain is continuous and weak ordered if all the profiles it contains (and thus, all the preferences of all agents at all of its profiles) are continuous and weak orderings.

The definition of continuous and weak ordered preference domain is similar to the preference domain defined by Barberà, Berga, and Moreno (2011). It is a subset of preference profiles in which all the profiles it contains are continuous and weak orderings. But, unlike Barberà, Berga, and Moreno (2011), this chapter analyzes a different problem. The key finding is that, in the case of continuum policies, the preference domain for which the majority rule is the solution to Problem 3.1 is closed in the topology of closed convergence.

**Theorem 14.** *The continuous and weak ordered preference domain in which majority rule is the optimal solution to Problem 3.1 is closed in the topology of closed convergence.*

## 3.4 Application: Equilibrium Locations of a Public Bad

This section addresses the problem of locating a site to build a public bad, i.e., public facilities (e.g., nuclear plant, garbage dumping ground, or any kind of facility with negative externality) that are profitable for the community but no one likes to have in her/his close neighborhood. I solve this problem by delegating this task to a council which consists of a finite set of members for which the sole responsibility is to legislate decisions for the wellbeing of the community. Members of the council make decisions based on the reciprocity mechanism. I assume that members are rational and truthful, and each one may vote according to the preferences of his constituency. The members of the council may represent different states or political parties in the society (e.g., parliament or any legislative body).

Finding an equilibrium location of a public bad has already been of interest to a few scholars in the literature. Theoretical studies first focus on defining and characterizing collective rules, in which each agent has the incentives to reveal his sincere preferences (e.g., Manjunath (2014), and the references therein). Another study departs from this framework by analyzing the choice of locating a public bad from a coalitional standpoint with the equilibrium concepts of strong Nash equilibrium and coalition proof Nash equilibrium (Yamamura (2016)). In the framework used in this chapter, the political mechanism (voting process) under which the decision is made in the council implicitly provides incentives to members to reveal their sincere preferences. The decision-making framework entails three bargaining stages as described above in section 3.2.1, where agents can bargain in the short term in order to reach an outcome which cannot be overruled by a majority if this outcome is submitted to a possible reform or amendment.

### 3.4.1 The Council's Structure

The council consists of either elected or nominated influential citizens in the community whose sole responsibility is to legislate decisions that will improve the quality of individuals' lives in the community. Formally, a *council* is modeled as a triple $\mathcal{C} = (N, Z, (\succeq_m)_{m \in N})$ where $N = \{1, 2, ..., n\}$ is a finite set of individuals or voters, with $n > 1$; $Z$ a set of feasible or conceivable location of the public bad (or the *location space* for short); and $(\succeq_m)_{m \in N}$ a profile of individual preferences over set $Z$. Each preference relation $\succeq_m$ is a weak order i.e reflexive, transitive, and complete. The structure of the location space in the model is a continuum.

### 3.4.2 The Council's Problem

A council is given the task to decide where a garbage dumping grounds or nuclear plant has to be constructed in a community. The location of this public facility depends on the geographical disposition of the community. For example, individuals living in the city could bring their garbage and other recyclable products to the dumpsite or the council could design a policy which includes a weekly collection of garbage from households. These noxious facilities are public bads and most citizens do not want those infrastructures to be built close to their home. Members of the council have weak order preferences over the location space. Specifically, I assume that each member in the council may have ether *single-caved* (*single-dipped*) or multi-caved preference over the location space. For a council's member, the dip represents the worst location of the public bad, for instance, the closest location to where the member lives.

In the council, the decision is taken under the reciprocity mechanism, and voting power is allocated to coalitions by the majority rule. The location space is either a one-dimensional (e.g., $[0, 1]$) or a multi-dimensional (e.g., $[0, 1]^k$, $k < \infty$) set. The main problem is whether there exists an equilibrium location of the dumpsite in the community. The next section solves this interesting problem. One important feature of an equilibrium location is that such an outcome has a higher probability to be implemented in the community, since it cannot be overridden in the same council endowed with the same constitution.

### 3.4.3 Solutions to The Council Problem

Here I provide a solution to the problem faced by the council concerning the location of the dumpsite (for example) usually considered as a public bad. Under reasonable properties on voters' preferences, the council will always select an equilibrium outcome. The results show that in a setting where members of the council have single-caved preferences over a one-dimensional location space, an equilibrium location of a public bad always exists. Moreover, if the size of members is odd, then there exists at most two equilibrium locations, and if the size of members is even, there may exist an infinite number of equilibrium locations.

**Proposition 5.** *Let $\mathcal{C} = (N, Z, (\succeq_m)_{m \in N})$ be a council, and $n = |N|$. If $\succeq_m$ is single-caved for each $m = 1, 2, ..., n$ and $Z = [0, 1]$, then an equilibrium location of a public bad always exists. In addition,*

1. *if $n$ is odd, there are at most two equilibrium locations; and,*

2. *if $n$ is even, there may exist an infinite number of equilibrium locations.*

Next, I extend the dimension of location space to a multidimensional space. The analysis indicates that the existence of an equilibrium location of the public bad is still guaranteed in this context.

**Proposition 6.** *Let $\mathcal{C} = (N, Z, (\succeq_m))$ be a council such that members have multi-caved preferences over a multidimensional location space. Then an equilibrium location of a public bad exists.*

## 3.5 Conclusion

This chapter uses an agenda formation setting to display some possibility theorems for the institutionalization of the majority rule by focussing on the shape of the distribution of individuals' preferences over the set of competing policies. The analyses lead to the identification of natural preference domains for which majority rule emerges as the optimal voting rule in a simple legislative bargaining mechanism in the sense that it breaks political cycles, ensures that all equilibrium policies are Pareto-efficient, and minimizes status quo bias, no matter the complexity of the policy space. A characterization of the preferences for which the majority rule is optimal when the policy at stake induces a continuum of feasible options is also provided. I illustrate the findings to the problem of locating a public bad in the community. I establish that if each council's member has multi-caved preferences over a multidimensional location space, then there always exists an equilibrium location. Moreover, there exist at most two equilibrium locations if the size of members in the council is odd and they have single-caved preferences over a one-dimensional location space.

## 3.6 Mathematical Appendix

The proofs for the various statements given in the text are presented in this section. The theorems and propositions given below follow in the same order as the corresponding descriptions in the text.

**Proof of Theorem 9**. Let $\mathcal{C} = (N, A(d), (\succeq_i)_{i\in N})$ be a council induced by a policy of complexity level $d$. Assume that $\succ_i$ is a linear order over the policy space $A(d)$, for each individual $i \in N$. Members of the council enact decisions under the reciprocity mechanism endowed with the majority rule. To achieve the demonstration, I prove the lemmas below. These results reveal the existence of an equilibrium policy when the decision to be made lead to either a continuum or a discrete space.

**Lemma 10.** *If $A(d)$ is a finite and discrete policy space ($|A(d)| < \infty$) and preferences are linear, then, a policy always exists.*

*Proof.* For each $x \in A(d)$, define $f(x) = |\{y \in A(d); y \succ x\}|$ and let $x_0 \in A(d)$ such that $f(x_0) = \min\limits_{x\in A(d)} \{f(x)\}$. Prove that $x_0 \in R(\mathcal{C})$. If this assertion is not true, then there exists $y \in A(d)$ and a majority $S \subseteq N$, such that $y \gg_S x_0$. It follows that:

$$\begin{cases} (\alpha)\ y \succ_S x,\ |S| > \frac{n}{2} \\ (\beta)\ \forall (z,T):\ T \neq S,\ z \succ_T y, |T| > \frac{n}{2} \text{ and not}(y \succeq_T x) \Rightarrow z \succeq_S x \end{cases} . \tag{3.2}$$

By definition of $x_0$ and given that the preference $\succ$ is asymmetric, there exists $c \in A(d)$ such that $c \succ y$ and *not* $(c \succ x_0)$. Thus, there exists a coalition $T$, with $|T| > \frac{n}{2}$ such that $c \succ_T y$.

- If $y \succ_T x_0$, then I have $c \succ_T y$ and $y \succ_T x_0$, so $c \succ_T x_0$ by transitivity, which is a contradiction, because *not* $(c \succ x_0)$.
- Suppose that *not* $(y \succ_T x_0)$, then according to the assertion $(\beta)$ in equation (3.2), $c \succ_S x_0$, which is a contradiction.

In conclusion, $x_0 \in R(\mathcal{C})$. □

**Lemma 11.** *If $A(d)$ is continuous (assuming compact and convexity properties), and each preference relation $\succeq_i$ is continuous and endowed with the topology of closed convergence, then an equilibrium policy exists.*

*Proof.* Let $f$ be the function defined over the policy $A(d)$ by:

$$f(x) = \mathfrak{L}(\{y \in A(d); y \succ x\}),$$

where $\mathfrak{L}$ is the Lebesgue measure on the manifold spanned by $A(d)$. Given that preferences are continuous, there exists $x_0 \in A(d)$ such that $f(x_0) = \min_{x \in A(d)} \{f(x)\}$. The goal is proving that $x_0 \in R(\mathcal{C})$. By contradiction, there exists $y \in A(d)$ and a winning coalition $S$, such that $y \gtrdot_S x_0$. It follows that:

$$\left\{ \begin{array}{c} (\alpha')\ y \succ_S x,\ |S| > \frac{n}{2} \\ (\beta')\ \forall (z,T)/\ z \succ_T y, |T| > \frac{n}{2} \text{ and not}(y \succeq_T x) \Rightarrow z \succeq_S x \end{array} \right. . \tag{3.3}$$

By definition of $x_0$ and given that the preference $\succ$ is asymmetric, there exists $c \in A(d)$ such that $c \succ y$ and *not* $(c \succ x_0)$. Thus, there exists a coalition $T$, with $|T| > \frac{n}{2}$ such that $c \succ_T y$.

- If $y \succ_T x_0$, then I have $c \succ_T y$ and $y \succ_T x_0$, so $c \succ_T x_0$ by transitivity, which is a contradiction because *not* $(c \succ x_0)$.
- If *not* $(y \succ_T x_0)$, then according to $(\beta')$ in equation (3.3), $c \succeq_S x_0$, which is a contradiction.

Consequently, $x_0 \in R(\mathcal{C})$. □

Given that each equilibrium policy is efficient (see Lemma 9), I conclude the proof.

**Proof of Theorem 10.** Let $\mathcal{C}(d) = (N, A(d), (\succeq_i)_{i \in N})$ be a council induced by a policy of complexity level $d$. Assume that $A(d)$ is a one-dimensional policy space and decisions are made under the majority rule through the reciprocity mechanism. The result below indicates that there exists an equilibrium policy if individual preferences are single-peaked over the policy space.

**Lemma 12.** *If the binary relation $\succeq_i$ is single-peaked for each $i = 1, 2, ..., n$, then there is at least one equilibrium outcome.*

*Proof.* The proof of this result follows the same logic as that of Theorem 4 in Chapter 1. □

The second result proves that single-caved preference domains belong to the set of solutions of the objective problem.

**Lemma 13.** *If $\succeq_i$ is single-caved for each $i = 1, 2, ..., n$, then there is at least one equilibrium outcome.*

*Proof.* Each individual has at most two peak points and, if it is the case, those peak points have the same ranking. The peak locations should coincide with the minimum and maximum elements of $A(d)$.

Let $x \in A(d)$ be a policy, and define the number $S(x)$ as follows: $S(x)$ is a number of individuals whom has $x$ as a peak point. If $x$ is not a peak point, then $S(x)$ takes the value 0. Let $V(q, q')$ denote the set of individuals who vote for $q$ against $q'$ in a pairwise election between $q$ and $q'$. Two cases will be differentiated: each individual has only one peak over the set $A(d)$ or there exists at least one individual who has two peak locations.

**1. Assume $n$ is odd.**

To bring clarity, I begin by proving this first point of theorem for a fixed number of voters. For instance, I consider the case of three voters ($n = 3$). In what follows, $q_m$ is the dip point for individual $m$. Given that individuals have single-caved preferences, they prefer to vote for locations far away from their dip points. For this example, two cases are discussed and the equilibrium set is determined.

**Case a:** $S(q_1) = 1$, $S(q_2) = 0$ and $S(q_3) = 2$ (see Figure 3.2). The pairwise elections

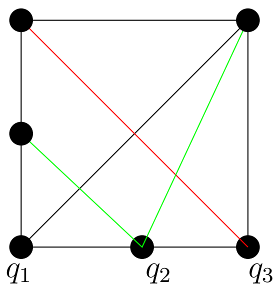


FIGURE 3.2: One individual has only $q_1$ as peak ($S(q_1) = 1$); No one has $q_2$ as peak ($S(q_2) = 0$); and two individuals have $q_3$ as peak ($S(q_3) = 2$).

between locations $q_1$, $q_2$ and $q_3$ give the following results: $q_1 \succ_{23} q_2$ i.e $q_1$ defeats $q_2$ with the support of the majority $\{2, 3\}$; $q_3 \succ_{12} q_1$ or $q_3$ defeats $q_2$ with the support of the coalition $\{1, 2\}$; $q_3 \succ_{12} q_2$ or $q_3$ defeats $q_2$ with the support of the subgroup $\{1, 2\}$. There is no location which defeats option $q_3$. Since $q_3$ defeats $q_1$ and $q_2$, then $q_3 \gtrdot_{12} q_1$, $q_3 \gtrdot_{12} q_2$ and $R(\mathcal{C}) = \{q_3\}$.

**Case b:** $S(q_1) = 2$, $S(q_2) = 0$ and $S(q_3) = 2$ (see Figure 3.3).

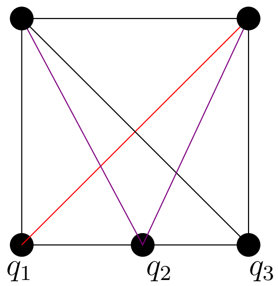


FIGURE 3.3: Two individuals have $q_1$ as peak ($S(q_1) = 2$); No one has $q_2$ as peak ($S(q_2) = 0$); and two individuals have $q_3$ as peak ($S(q_3) = 2$).

As presented previously, the pairwise elections in this case give the following results: $q_1$ defeats $q_2$ with the support of the winning coalition $\{2,3\}$ ($q_1 \succ_{23} q_2$); and $q_3$ defeats $q_2$ with the support of the majority $\{1,2\}$ ($q_3 \succ_{12} q_2$). The pairwise election between $q_1$ and $q_3$ is interesting: $V(q_1, q_3) = \{3\}$ and $V(q_3, q_1) = \{1\}$, then no winner. It follows that the location $q_2$ is dominated by either $q_1$ or $q_3$ and no point defeats either $q_1$ or $q_3$. Hence, $R(\mathcal{C}) = \{q_1, q_3\}$.

This example confirms that the equilibrium set of locations of a public bad may consists of at most two locations if the number of individuals is odd. In what follows, the proof is generalized.

**Assume that $n$ is odd.**

Two different scenarios can be distinguished: each individual has a unique peak point or there exists at least one individual who has two peaks over $A(d)$.

**Case c**: consider the situation where each individual has a unique peak point over $A(d)$. Assume $q_1^*$ and $q_2^*$ to be the two possible peak points (this is possible since $A(d)$ is bounded), then $S(q_1^*) + S(q_2^*) = n$, $S(q_1^*), S(q_2^*) \geq 0$. Without loss of generality, assume that $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$. Consider the following graph (Figure 3.4) described below.

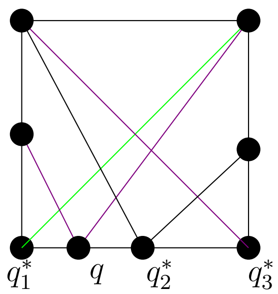


FIGURE 3.4: The size of the Council's individuals is odd, and each individual has a unique peak.

In pairwise election between peaks $q_1^*$ and $q_2^*$, the former wins, because more than half of the council vote for $q_1^*$ against $q_2^*$, since $S(q_1^*) > S(q_2^*)$ and $S(q_1^*) > \frac{n}{2}$. In a pairwise election between $q_1^*$ and any $q \in (q_1^*, q_2^*)$ such that $S(q) = 0$, $q$ loses the election. Indeed, the number of individuals who have $q_1^*$ as dip point vote for $q$. It follows that, at least, $S(q_1^*)$ individuals vote for $q_1^*$ against $q$. Given that $S(q_1^*) > \frac{n}{2}$, then $q_1^*$ dominates all other locations in $A(d)$ and $R(\mathcal{C}) = \{q_1^*\}$.

**Case d**: Assume that there exists at least one individual who has two peaks over $A(d)$. This situation can be illustrated by a graph similar to the one below (Figure 3.5).

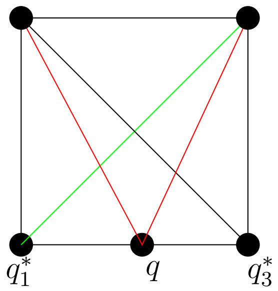


FIGURE 3.5: The size of the Council's individuals is odd, and at least one individual has two peaks.

The individual for whom $q$ is a dip point has two peak points $q_1^*$ and $q_2^*$. Thus s/he is indifferent between those peaks during a pairwise election when they are candidates.

- Let us oppose $q_1^*$ and $q_2^*$ in a pairwise election. Then $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| = n - 1$, because one individual is indifferent between $q_1^*$ and $q_2^*$.

  i) If $S(q_1^*) > S(q_2^*)$, and $S(q_1^*) > \frac{n}{2}$, $q_1^*$ beats $q_2^*$, since $|V(q_1^*, q_2^*)| \geq S(q_1^*)$.

  ii) If $S(q_1^*) = S(q_2^*)$, then $q_1^*$ and $q_2^*$ are tied.

- Let $q$ be any option between $q_1^*$ and $q_2^*$. Using the same previous reasoning, $q$ is beaten by either $q_1^*$ or $q_2^*$ in a pairwise majoritarian election. Consequently, if $S(q_1^*) > S(q_2^*)$ with $S(q_1^*) > \frac{n}{2}$, then $R(\mathcal{C}) = \{q_1^*\}$. Also, if $S(q_1^*) = S(q_2^*)$, then $R(\mathcal{C}) = \{q_1^*, q_2^*\}$.

**2. Assume $n$ is even.**

I start this second point of the proof by looking at a situation where the council consists of four individuals ($n = 4$). The methodology used in the previous cases where the size of the council is odd will be useful in this section.

**Case e**: Each individual has only one peak point. This case can be illustrated in either Figure 3.6 or Figure 3.7.

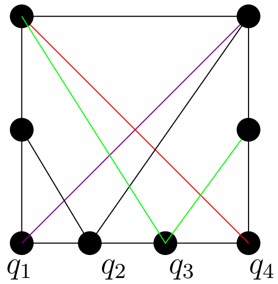


FIGURE 3.6: The size of the Council's individuals is even, and each individual has a unique peak—first scenario.

- The proof begins with the illustration displayed by Figure 3.6.

  The pairwise elections between different locations give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then $q_1$ defeats $q_2$ with the support of majority coalition $\{2, 3, 4\}$. $V(q_1, q_3) = \{2, 3, 4\}$ and $V(q_3, q_1) = \{1\}$, then $q_1$ defeats $q_3$ with the support of majority coalition $\{2, 3, 4\}$. $V(q_1, q_4) = \{3, 4\}$ and $V(q_4, q_1) = \{1, 2\}$, then $q_1$ and $q_4$ are tied. $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then $q_4$ defeats $q_3$ with the support of majority $\{1, 2, 3\}$. As result, the only non defeated locations are the two peak points $q_1$ and $q_4$. Hence, $q_1 \gtrdot_{234} q_2$ and $q_4 \gtrdot_{123} q_3$. Therefore, $R(\mathcal{C}) = \{q_1, q_4\}$.

- I continue the proof with the analysis of Figure 3.7. The pairwise elections between

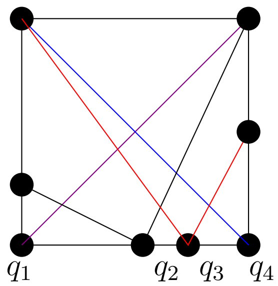


FIGURE 3.7: The size of the Council's individuals is even, and each individual has a unique peak—second scenario.

  different locations give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then $q_1$ defeats $q_2$ with the support of majority coalition $\{2, 3, 4\}$. Voter 2 is indifferent between locations $q_1$ and $q_3$, hence $V(q_1, q_3) = \{3, 4\}$ and $V(q_3, q_1) = \{1\}$, so that $q_1$ and $q_3$ are tied. Also, $V(q_1, q_4) = \{3, 4\}$ and $V(q_4, q_1) = \{1, 2\}$, then $q_1$ and $q_4$ are tied. As proved above, $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then $q_4$ defeats $q_3$ with the support of majority $\{1, 2, 3\}$. At the end, the only non defeated locations are the two peak points $q_1$ and $q_4$ even though $q_1$ and $q_3$ are tied, $q_4 \gtrdot_{123} q_3$. Given that $q_1 \gtrdot_{234} q_2$, then $R(\mathcal{C}) = \{q_1, q_4\}$.

**Case f**: There exists at least one individual who has two peak points. This case can be illustrated in Figure 3.8 (Figures 3.8a and 3.8b).


(A) Figure 8a (B) Figure 8b

FIGURE 3.8: The size of the Council's individuals is even, and at least one individual has two peaks.

The same method (Case e) illustrated above can be applied here. The pairwise elections between different alternatives give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then $q_1$ defeats $q_2$. The only difference is that, from Figure 3.8a, $q_1$ defeats $q_3$, since $V(q_1, q_3) = \{2, 3, 4\}$ and $V(q_3, q_1) = \{1\}$. Whereas in Figure 3.8b, individual 2 is indifferent between candidates $q_1$ and $q_3$, leading to a tie between $q_1$ and $q_3$ (in fact $V(q_1, q_3) = \{3, 4\}$ and $V(q_3, q_1) = \{1\}$). A pairwise election between peaks $q_1$ and $q_4$ leads to a tie, because $V(q_1, q_4) = \{4\}$ and $V(q_4, q_1) = \{1, 2\}$. Finally, $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then $q_4$ defeats $q_3$. In conclusion, $R(\mathcal{C}) = \{q_1, q_4\}$ for the same reasons stated above (Case e).

The proof is illustrated for the situation where there are only four individuals. It has been proven that the equilibrium set consists of only two equilibrium policies. In what follows, I generalize the proof when $n$ is even.

**Assume that $n$ is even.**

Two different scenarios can be identifies: each individual has a unique peak point or there exists at least one individual who has two peaks over $A(d)$.

**Case g**: $n$ is even and each individual has a unique peak point. Consider $q_1^*$ and $q_2^*$ the two possible peak points, then $S(q_1^*) + S(q_2^*) = n$, with $S(q_1^*), S(q_2^*) \geq 0$.

a) Assume that $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$. As previously, $R(\mathcal{C}) = \{q_1^*\}$. This item is similar to the situation already demonstrated, where $n$ is odd (see **Case a** in point 1.).

b) Suppose that $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. In a pairwise majoritarian election between $q_1^*$ and $q_2^*$, there is no winner. Let $q \in A(d)$ be a candidate such that $S(q) = 0$, and oppose this location to the peak point $q_1^*$ in a pairwise election. Members for whom $q_1^*$ is a dip point will vote for $q$ and individuals for whom $q_1^*$ is a peak point will vote for $q_1^*$. Also, individuals for whom $q$ is the dip point will vote for $q_1^*$, although the latter is not their ideal point. Since $S(q_1^*) = \frac{n}{2}$, it follows that $|V(q_1^*, q)| > \frac{n}{2}$, and $q_1^*$ defeats $q$. Therefore all other locations between $q_1^*$ and $q_2^*$ are defeated either by $q_1^*$ or $q_2^*$. Consequently, $q_1^*$ and $q_2^*$ are the only non dominated locations, and $R(\mathcal{C}) = \{q_1^*, q_2^*\}$.

**Case h**: $n$ is even and there exists at least one individual who has two peaks over $A(d)$. The individual for whom $q$ is the dip point has two peak points $q_1^*$ and $q_2^*$, thus she/he is indifferent between those peaks during a pairwise election where they are candidates.

a) Assume $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$. In a pairwise election between $q_1^*$ and $q_2^*$, the result gives: $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| = n - 1$, since one individual is indifferent between $q_1^*$ and $q_2^*$. Since $|V(q_1^*, q_2^*)| \geq S(q_1^*)$ and $S(q_1^*) > \frac{n}{2}$, it follows that $q_1^*$ dominates $q_2^*$. Any other alternative $q$ between $q_1^*$ and $q_2^*$, is dominated. Therefore, $R(\mathcal{C}) = \{q_1^*\}$.

b) Consider $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. A pairwise election between the two peak points $q_1^*$ and $q_2^*$ leads to no winner. Assume a pairwise majoritarian competition between any alternative $q$ such that $S(q) = 0$ and either $q_1^*$ or $q_2^*$. Members who have $q_1^*$ or $q_2^*$ as the dip point will vote for $q$, and voters for whom $q_1^*$ or $q_2^*$ is a peak point vote for $q_1^*$ or $q_2^*$. By hypothesis, $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. It follows that neither $|V(q_1^*, q)|$ nor $|V(q, q_1^*)|$ is greater than half of the council, then there is no winner for the electoral competition. Hence, no candidate is defeated and $R(\mathcal{C}) = A(d)$. □

The results (Lemma 10 and Lemma 13) provide a rationalization of the majority rule in a political framework which includes the classes of single-peaked and single-caved preference domains defined over a one-dimensional policy space. The proof is done, thanks to Lemma 9.

**Proof of Theorem 11.** I start the proof with some notation. For any $x \in A(d)$, I denote $P_C(x) = \{y \in A(d) : y \succ_C x\}$, $C \in \mathcal{M}$ and $P(x) = \{y \in A(d) : y \succ x\}$, and $P(x) = \bigcup_{C \in \mathcal{M}} P_C(x)$. Consider $h$ the mapping from $A(d)$ to the real numbers, and define $h(x) = \mathfrak{L}(P(x))$, for any $x \in A(d)$.

1. $h$ is lower semi-continuous.

   The objective is to prove that $\liminf_{x \to x_0} h(x) \geq h(x_0)$. Consider a sequence $x_k$ which converges to $x_0$ when $k$ tends to infinity and show that $\liminf_{k \to \infty} h(x_k) \geq h(x_0)$. Let $y \in P(x_0)$, then there exists $C \in \mathcal{M}$ such that $y \in P_C(x_0)$. $x_k$ converges to $x_0$ and $P_C$ is continuous, thus for $k$ sufficiently large, $y \in P_C(x_k)$. It follows that $y \in \liminf P_C(x_k)$ for some $C \in \mathcal{M}$, and then $y \in \liminf P(x_k)$, so $P(x) \subseteq \liminf P(x_k)$. Since there exists $k_0$ such that $\mathfrak{l}(\bigcup_{k\geq k_0} P(x_k)) < \infty$, then $\mathfrak{L}(\liminf_{k\to\infty} P(x_k)) \leq \liminf_{k\to\infty} \mathfrak{L}(P(x_k))$, i.e $\mathfrak{L}(\liminf_{k\to\infty} P(x_k)) \leq \liminf_{k\to\infty} h(x_k)$. Since $P(x_0) \subseteq \liminf P(x_k)$, then $\mathfrak{L}(P(x_0)) \leq \mathfrak{L}(\liminf_{k\to\infty} P(x_k))$ and therefore $h(x_0) \leq \lim_{k\to\infty} \inf h(x_k)$.

2. $h$ is lower semi-continuous and $A(d)$ compact, then $h$ has a minima on $A(d)$, i.e there exists $a \in A(d)$ such that $h(a) = \inf_{x\in A(d)} h(x)$.

   Denote by $S = \left\{a \in A(d) : h(a) = \inf_{x\in A(d)} h(x)\right\}$ and $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A(d))]$. Consider the **binary relation** $\gg$ defined as follows: for $x, y \in A(d)$ and $C \subset N$, $y \gg_C x$ if $y \succ_C x$ and $z \succ_C x$ for all $z \in A(d)$ such that $z \succ y$ and $z \notin \overline{LM}$. $y \gg x$ if there exists $C \in \mathcal{M}$ such that $y \gg_C x$. Let $\mathcal{S}(\mathcal{C}) = \{x \in A(d) : not(y \gg x), y \in A(d)\}$.

3. $\mathcal{S}(\mathcal{C})$ is non-empty.

   Assume the contrary and consider an arbitrary location $x_0 \in S$. There exists an alternative location $x_1 \in A(d)$ and $C \in \mathcal{M}$ such that $x_1 \gg_C x_0$ i.e $x_1 \succ_C x_0$ and $z \succ_C x_0$ for any $z \in A(d)$ such that $z \succ x_1$ and $z \notin \overline{LM}$. It follows that $P(x_1) \subseteq P(x_0) \cup (P(x_1) \cap \overline{LM})$. Certainly, if $y \in P(x_1)$, then there exists $C' \in \mathcal{M}$ such that $y \in P_{C'}(x_1)$, so $y \succ_{C'} x_1$. Since the majority rule satisfies the overlapping condition (i.e for any $S, T \in \mathcal{M}$, $S \cap T \neq \emptyset$), there exists a member $i \in C \cap C'$ and $y \succ_i x_1 \succ_i x_0$. By transitivity, $y \succ_i x_0$. Since $P_C$ is continuous, there exists a neighborhood $S(y)$ of $y$ such that $y' \succ_i x_0$ for all $y' \in S(y)$, so $y \in LM(\succeq_i, A(d))$ and $y \in \overline{LM}$. If $C' = C$, then I have $y \succ_C x_1 \succ_C x_0$ i.e $y \succ_C x_0$ or $y \in P(x_0)$. Hence, $P(x_1) \subseteq P(x_0) \cup (P(x_1) \cap \overline{LM})$, therefore $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(P(x_1) \cap \overline{LM}) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(\overline{LM})$. Since $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A(d))]$, then $\mathfrak{L}(\overline{LM}) \leq \sum_{i=1}^{n} \mathfrak{L}[LM(\succeq_i, A(d))]$ and $\mathfrak{L}[LM(\succeq_i, A(d))] = 0$ for all $i = 1, ..., n$ by hypothesis, therefore $\mathfrak{L}(\overline{LM}) = 0$ and $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0))$ i.e $h(x_1) \leq h(x_0)$. Since $h(x_0) = \inf_{x\in A(d)} h(x)$, then the inequality $h(x_1) \leq h(x_0)$ implies that $h(x_1) = h(x_0)$ and $x_1 \in S$. I note two different situations described below:

   **Situation 1**: Assume there exists a neighborhood $S(x_1)$ for which there is no $y \in S(x_1)$ satisfying $y \succ x_1$.

   $x_1 \succ x_0$ and preferences are continuous, then there exists a neighborhood $S'(x_1) \subseteq S(x_1)$ such that $y \succ x_0$ for all $y \in S'(x_1)$. It follows that $S'(x_1) \subseteq P(x_0)$ and $S'(x_1) \subseteq (A(d) - P(x_1))$. Thus, $P(x_1) \subsetneq P(x_0)$ and $h(x_1) < h(x_0)$, which is a contradiction.

   **Situation 2**: For all neighborhood $S(x_1)$ of $x_1$, there exists an alternative $y \in S(x_1)$ such that, $y \succ x_1$.

Let $x_2$ be a location such that $x_2 \gg_{C'} x_1$. As previously, $P(x_2) \subseteq P(x_1) \cup (P(x_1 \cap \overline{LM}))$ and then $h(x_2) = h(x_1)$. We have $x_2 \succ x_1$ and preferences are continuous, thus there exists a neighborhood $S(x_1)$ such that $x_2 \succ a$ for any $a \in S(x_1)$. By assumption, there exists $y \in S(x_1)$ such that $y \succ x_1$. By continuity of preferences, there exists a neighborhood $S(y)$ of $y$ such that $y' \succ x_1$ for any $y' \in S(y)$. Hence for all $y' \in S(y) \cap S(x_1)$, I have $y' \succ x_1$ and $x_2 \succ y'$, so $h(x_1) > h(y')$ and $h(y') > h(x_2)$, then $h(x_1) > h(x_2)$, a contradiction.

Therefore, $\mathcal{S}(\mathcal{C})$ is non-empty.

4. Let $x_0 \in \mathcal{S}(\mathcal{C})$, prove that $x_0 \in R(\mathcal{C})$.

   Assume that $x_0 \notin R(\mathcal{C})$, then there exists $x_1 \in A(d)$ and $C \in \mathcal{M}$ such that $x_1 \gtrdot_C x_0$. Let $z \notin \overline{LM}$ such that $z \succ_T x_1$, for $T \in \mathcal{M}$. I would like to demonstrate that $z \succ_C x_0$. Firstly, $z \succ_T x_1$, and $x_1 \gtrdot_C x_0$. For that reason, $z \succeq_C x_0$.

   **a)** If $z \succ_i x_0$ for any $i \in C$, then $z \succ_C x_0$ and $x_1 \gg x_0$, which is a contradiction.

   **b)** Assume that there exists $i \in C$ such that $z \sim_i x_0$. $z \succ x_1$ and preferences are continuous, then there exists a neighborhood $S(z)$ such that $z' \succ x_i$ for all $z' \in S(z)$. Because $z \notin \overline{LM}$, then $z \notin LM(\succeq_i, A(d))$ for any $i \in N$. Hence, there exists $z' \in S(z)$ such that $z \succ_i z'$. Since, $z \sim_i x_0$, then $x_0 \succ_i z'$ by transitivity. Also, $z' \in S(z)$, therefore $z' \succ x_1$ and by transitivity, $x_0 \succ_i x_0$ if $i \in T$. Otherwise, if $i \notin T$, then, given that $T \cap C \neq \emptyset$, there exists $i' \in T \cap C$ such that $x_0 \succ_{i'} z' \succ_{i'} x_1 \succ_{i'} x_0$ or $x_0 \succ_{i'} x_0$ by transitivity, which is a contradiction and $x_0 \notin \mathcal{S}(\mathcal{C})$. In conclusion, $x_0 \notin R(\mathcal{C})$.

**Proof of Theorem 12.** Let $\mathcal{C} = (N, A(d), (\succeq_i)_{i \in N})$ be a council induced by a policy of complexity level $d$. Assume that the policy space $A(d)$ is a continuous and compact set. Decisions are made in the council under the majority rule in the reciprocity mechanism. I prove that $\mathfrak{L}[LM(\succeq_i, A(d)] = 0$ (recall that $\mathfrak{L}$ represents the Lebesgue measure), for any $i \in N$, which implies that $R[\mathcal{C}] \neq \emptyset$ credited to Theorem 11. For this purpose, I demonstrate that the boundary of $A(d)$, denoted by $\partial A(d)$, contains $LM(\succeq_i, A(d))$. Consider $i \in N$ be an individual, and $x \in A(d)$ be a policy such that $x \in LM(\succeq_i, A(d))$. Assume that $x \notin \partial A(d)$. Then $x$ belongs to the relative interior of $A(d)$. Let $S(x)$ be a neighborhood of $x$ such that $S(x) \subseteq A(d)$ and $y \succeq_i x$ for any $y \in S(x)$. Let $x_1, x_2 \in S(x)$ be two policies such that $x = \lambda x_1 + (1 - \lambda) x_2$ with $\lambda \in (0, 1)$, and $x_1 \neq x_2$. Given that $\succeq_i$ is strongly convex, it follows $x \succ_i \lambda x_1 + (1 - \lambda) x_2$, which means that $x \succ_i x$. This is contradictory. Consequently, $LM(\succeq_i, A(d)) \subseteq \partial A(d)$. Moreover, since $A(d)$ is compact, it follows that $\mathfrak{L}(\partial A(d)) = 0$. This implies that $\mathfrak{L}[(\succeq_i, A(d))] = 0$, because $\mathfrak{L}[LM(\succeq_i, A(d))] \leq \mathfrak{L}[\partial A(d)]$. The proof is done thanks to Theorem 11 and Lemma 9.

**Proof of Theorem 13**. Let $\mathcal{C} = (N, A(d), (\succeq_i)_{i\in N})$ be a council induced by a policy of complexity level $d$. Assume that $A(d)$ is a continuous and compact policy space. For each policy $x \in A(d)$, denote by $P_S(x) = \{y \in A(d) :\succ_S x\}$, $S \in \mathcal{M}$. Also, denote $P(x) = \{y \in A(d) : y \succ x\}$. Then, $P(x) = \bigcup\limits_{S\in\mathcal{M}} P_S(x)$. Let $\overline{LM} = \bigcup\limits_{i=1}^{n}[LM(\succeq_i, A(d))]$. Consider the **binary relation** $\lhd$ defined as follows: for $x$, $y \in A(d)$ and $S \subset N$, $y \lhd_S x$ if $y \succ_S x$ and $z \succ_S x$ for all $z \in A(d)$ such that $z \succ y$ and $z \notin \overline{LM}$. $y \lhd x$ if there exists $S \in \mathcal{M}$ such that $y \lhd_S x$. Consider $\mathcal{S}(\mathcal{C}) = \{x \in A(d)\ not(y \lhd x), y \in A(d)\}$.

The non vacuity of the set $\mathcal{S}(\mathcal{C})$ is justified in the proof of Theorem 11. Let $x_0$ be a maximal element of the binary relation $\lhd$, i.e $x_0 \in \mathcal{S}(\mathcal{C})$.

I prove that $x_0 \in R(\mathcal{C})$. Consider $x \in P(x_0)$. Then there exists a majority coalition $S \in \mathcal{M}$ such that $x \succ_S x_0$. Given that $x_0$ is a maximal element of $\lhd$, there exists $y \in A(d)$ such that $y \neq x$, $y \in P(x)$, $y \notin P(x_0)$ and $y \notin \partial A(d)$. The assertion $[y \notin P(x_0)]$ implies that there exists an individual $i \in S$ such that $x_0 \succeq_i y$. Three possibilities are differentiated.

1. If $x_0 \succ_i y$, then $not(y \succeq_S x_0)$ for $y \succ x$, and therefore $x_0 \in R(\mathcal{C})$.

2. If $x_0 \sim_i y$, then I claim that $y$ is not a local minimizer of $\succeq_i$. Assume the contrary. Given that $y \notin \partial A(d)$, there exists a neighborhood $S(y)$ of $y$ such that $S(y) \subseteq A(d)$ and $z \succeq_i y$ for all $z \in S(y)$. Let $u_i$ be a concave utility function representing $\succeq_i$ and $w \in S(y)$ a policy such that $y = \lambda w + (1-\lambda)z$ with $z \in S(y)$, $\lambda \in [0, 1]$. It follows that $u_i(y) \geq \lambda u_i(w) + (1-\lambda)u_i(z)$. Given that $u_i(w) \geq u_i(y)$, I have $u_i(y) \geq u_i(z)$ for all $z \in S(y)$ and then $y$ is a local maximizer of $\succeq_i$. Since $u_i$ is concave, it follows that $y$ is a global maximizer of $\succeq_i$. By hypothesis, $x_0 \sim_i y$ and $x \succ_i x_0$. By transitivity, $x \succ_i y$, which is a contradiction because $y$ is a global maximizer of $\succeq_i$.

3. If $y$ is not a local minimizer of $\succeq_i$, then there exists $y' \in S(y)$ such that $y \succ_i y'$. Since $z \succeq_i y$ for each $z \in S(y)$, it follows that $y' \succeq_i y \succ_i y'$. That is, $y' \succ_i y'$, which is a contradiction. In conclusion, $x_0 \in R(\mathcal{C})$.

**Proof of Theorem 14**. In this proof, I only have to demonstrate that continuous and weak preference domain for which there exists an equilibrium outcome is closed in the topology of closed convergence thanks to Lemma 9. Let $\mathcal{C} = (N, A(d), (\succeq_m)_{m \in N})$ be a council induced by a policy of complexity level $d$. Assume that $A(d)$ is a continuous and compact policy space. Assume that the sequence $(\succeq_m)_k$ converges to $(\succeq_m)$ in the topology of closed convergence, with $R(\mathcal{C}_k) \neq \emptyset$ for each $k$, with $\mathcal{C}_k = (N, A(d), [(\succeq_m)]_k)$. The goal is to prove that $R(\mathcal{C}) \neq \emptyset$. Consider $x_k \in R(\mathcal{C}_k)$, and assume that $x_k$ converges to $x$ (this is justified because $A(d)$ is compact). It remains to demonstrate that $x \in R(\mathcal{C})$. Assuming the contrary, then there exists $y \in A(d)$ such that $y$ dominates $x$. Thus there exists $C \subseteq N$, $C \in \mathcal{M}$ and $y \succ_m x$ for any $m \in C$. Consider that $not(y \succ_C^k x_k)$, then there exists $m(k) \in C$ such that $x_k \succeq_{m(k)}^k y$. Given that $C$ is finite, there exists $m \in C$ such that $x_k \succeq_{m(k)}^k y$ for many indices $k$. Since $\succeq_m^k \to \succeq_m$ in the topology of closed convergence, and $x_k$ converges to $x$, then $x \succeq_m y$. This is a contradiction. Hence, $y \succ_m x$. It follows that $y \succ_C^k x_k$ for any $k$. Given that $x_k \in R(\mathcal{C}_k)$, there exists $z \in Z$ such that $z \succ_T y$ with $T \subseteq N$, $T \in \mathcal{M}$, $not(y \succeq_T x_k)$ and $not(z \succeq_C x)$. This means, there exists $m(k) \in C$, such that $x_k \succ_{m(k)}^k z$. Since $C$ is finite, there exists $m \in C$ such that $x_k \succ_{m(k)}^k z$, for many indices $k$. Moreover $x_k$ converges to $x$, then $x \succ_m z$, $m \in C$. I just proved that if $y$ dominates $x$ via some winning subgroup $C$, then there exists $z \in A(d)$ such that $z$ dominates $y$ via $T$, a winning coalition with $not(y \succeq_T x)$ and $not(z \succeq_C x)$. That being said, $x \in R(\mathcal{C})$.

**Proof of Proposition 5**. The proof follows the same reasoning as that of Theorem 10.

**Proof of Proposition 6**. The proof follows the same logic as that of Theorem 11.

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