

**Stability of 3D Optical Topologies
and
Reconstruction of Strongly Correlated Bi-photon States**

by

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Declaration of Authorship

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Abstract

Structured light has shown great promises and impacts in various fields such as microscopy, optical trapping, sensing, classical communication, high dimensional quantum information, quantum key distribution (QKD) and quantum metrology [25]. In this Thesis, we will be discussing two projects, in the context of structured light, one about the stability of 3D optical topologies and the other about the reconstruction of strongly correlated bi-photon states.

In the introduction, we will be reviewing some of the basic concepts needed to follow the two projects. Starting from the spatial modes and their characteristics, we discuss about some interferometric approaches to reconstruct the phase and the amplitude of an electromagnetic field, either for a classical laser beam or for photons. Moreover, we briefly explain the spontaneous parametric down conversion (SPDC) process that is commonly used for generating entangled bi-photon states.

In the second chapter, after giving an overview about singular optics and optical topologies, we explore the effect of different types of aberrations on three different 3D optical structures, Trefoil knot, Cinquefoil knot and Hopf Link. In this work we show that these structures are robust under aberrations with the highest strength if the defining aperture of the aberration is big enough. In addition, we discuss about how these aberrations will modify the structure if we decrease the size of the aperture and in which aperture the structures start to break and therefore, we investigate the impact of the strength under such apertures.

In the third chapter, we return to the SPDC process, discuss the thin-crystal approximation, spatial mode correlations of an entangled bi-photon state, and coincidence imaging performed with time stamping cameras. We propose an approach to characterize an unknown two-photon state using interference between the unknown bi-photon state and a reference bi-photon state in addition to coincidence imaging. We show that this approach is faster, scalable and loss-free compared to projective measurements.

Finally, in the conclusion, we put together the results of these two projects and discuss about the future work that can follow up on what is done so far.

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Table of Contents

List of Figures	vii
1 Introduction	1
1.1 Spatial Modes of Light	1
1.1.1 Paraxial Wave Equation and Its Solutions	1
1.1.2 Generation of Spatial Modes	5
1.2 Digital Holography	8
1.2.1 On-Axis Digital Holography	8
1.2.2 Off-Axis Digital Holography	9
1.3 Spontaneous Parametric Down Conversion (SPDC)	9
2 Effect of Aberration on 3D optical topologies	12
2.1 Optical Topologies and Singular Optics	13
2.2 Optical Knots and Links and Their Experimental Realization	15
2.3 Optical Aberrations and Aberrated Knots	18
2.4 Publication: Effect of Aberrations on 3D Optical Topologies	19
2.5 Contributions	29
3 Interferometric imaging of amplitude and phase of spatial biphoton states	30
3.1 SPDC in the Thin-Crystal Approximation	31
3.2 Spatial Correlations in SPDC and Conservation Laws	33

3.3	Coincidence Imaging and Timepix Camera	35
3.4	Publication: Interferometric imaging of amplitude and phase of spatial biphoton states	37
3.5	Contributions	52
4	Conclusion	53
	References	55

List of Figures

1.1	LG and HG modes: a- Examples of $LG_{\ell,p}$ for $\ell \in \{0, \pm 1, \pm 2\}$ and $p \in \{0, 1, 2, 3, 4\}$, b- Examples of $HG_{n,m}$ for $n \in \{0, 1, 2, 3, 4\}$ and $m \in \{0, 1, 2, 3, 4\}$	4
1.2	The hologram for generating $LG_{1,2}$	7
2.1	Tracking the singularities: Intersecting the contours corresponding to the real and imaginary part of the field equal to zero, we find the position of the points in which the amplitude of the field is zero (phase singularities).	14
2.2	Different Zernike functions	17

Chapter 1

Introduction

In this chapter, we will introduce the building blocks and the fundamental physics behind the works presented in the following chapters. In the first section, beginning from the Paraxial Wave Equation (PWE), and its solutions, in particular Gaussian beams, Laguerre Gaussian modes and Hermite Gaussian modes, we talk about the propagation of an arbitrary spatially dependent optical beam and how to generate them in the lab. In the second section, we discuss about the reconstruction of these spatial modes with two different approaches, phase-shifting and off-axis digital holography. Finally, we will be talking about the derivation of the bi-photon SPDC state in the transverse momentum representation. This state will be analyzed more in details in chapter 3 in terms of its spatial mode decomposition.

1.1 Spatial Modes of Light

1.1.1 Paraxial Wave Equation and Its Solutions

The physics of light, as an electromagnetic field, is described by the well-known Maxwell equations. From these equations, the propagation of a monochromatic light signal is given by the Helmholtz equation $\nabla^2 \mathbf{E} = -k^2 \mathbf{E}$, where $\mathbf{E}$ is the light signal and k is the wave-number. Here, we consider solutions that propagate mainly along the z axis, of the following form:

$$\mathbf{E}(\mathbf{r}) = e^{ik_z z} \tilde{\mathbf{E}}(\mathbf{r}), \quad (1.1)$$

where $k_z \approx k = \omega/c$ is the z component of the wave-vector with ω the frequency and c the speed of light in vacuum and $\tilde{\mathbf{E}}(\mathbf{r})$ is the amplitude of the electric field. By applying the slowly varying amplitude approximation or the paraxial approximation, which implies that the beam's amplitude changes slowly upon the propagation, the Helmholtz equation takes the following form [26, 28].

$$\left(\tilde{\nabla}_T + 2ik_z\partial_z\right)\tilde{\mathbf{E}}(\mathbf{r}) = 0, \quad (1.2)$$

where, $\tilde{\nabla}_T = \partial_x^2 + \partial_y^2$. It is important to note that this equation resembles the Schrödinger's equation if we replace time with the propagation direction, z . In many different cases and applications, laser beams are well described by solutions of the paraxial wave equation. Some interesting and useful solution sets (shown by $\mathbf{f}_{a,b}(\mathbf{r})$) are those that are complete and orthonormal:

$$\text{Completeness : } \sum_{a,b} \mathbf{f}_{a,b}(\mathbf{r})\mathbf{f}_{a,b}^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (1.3)$$

$$\text{Orthonormality : } \iint dx dy \mathbf{f}_{a,b}(x, y, z)\mathbf{f}_{a',b'}^*(x, y, z) = \delta_{a,a'}\delta_{b,b'}. \quad (1.4)$$

Any solution of PWE ($\mathbf{U}(\mathbf{r})$), therefore, can be expanded in terms of these orthonormal and complete families of solutions $\mathbf{U}(\mathbf{r}) = \sum_{a,b} c_{a,b}\mathbf{f}_{a,b}(\mathbf{r})$. Gaussian beams, Lagaurre-Gaussian (LG) modes and Hermite-Gaussian (HG) modes are a few well-known solutions of the PWE. In the following, we will briefly examine these modes and their properties.

To begin with, let us consider one of the simplest but very important set of beams, Gaussian beams. A Gaussian beam is considered to be a beam that exhibits a Gaussian intensity profile, and its slowly varying envelope is given by [26]

$$U(r, z) = A_0 \frac{w_0}{w(z)} \exp\left(\frac{-r^2}{w(z)^2}\right) \exp\left(-ik \frac{r^2}{2R(z)} + i \tan\left(\frac{z}{z_R}\right)^{-1}\right), \quad (1.5)$$

where we have used the following definitions of the parameters [26]:

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}. \quad \text{Beam Width} \quad (1.6)$$

$$R(z) = z + \frac{z_R^2}{z}. \quad \text{Wavefront Curvature Radius} \quad (1.7)$$

$$z_R = \frac{\pi w_0^2}{\lambda}. \quad \text{Rayleigh Range} \quad (1.8)$$

Here, w_0 is the beam waist (the width of the beam at $z = 0$), and the Rayleigh range is the distance in which the beam width is less than $\sqrt{2}w_0$. An optical beam is considered “collimated” within this range.

In addition to the Gaussian beams, two different orthonormal complete families of modes are LG modes, and HG modes, defined below. LG modes can be obtained by solving the PWE in cylindrical coordinates r, ϕ, z . The expression for the LG modes is as follows [1].

$$\begin{aligned} \text{LG}_{\ell,p}(r, \phi, z) = & \sqrt{\frac{2p!}{\pi(p+|\ell|)!}} \frac{1}{w(z)} \exp\left(-\frac{r^2}{w^2(z)} - ik\frac{r^2}{2R(z)}\right) \left(\frac{r\sqrt{2}}{w(z)}\right)^{|\ell|} L_p^{|\ell|}\left(\frac{2r^2}{w^2(z)}\right) \\ & \times \exp(-i\ell\phi) \exp\left(i(|\ell| + 2p + 1) \tan^{-1}\left(\frac{z}{z_R}\right)\right), \end{aligned} \quad (1.9)$$

where $L_p^{|\ell|}(\cdot)$ is the generalized Laguerre polynomial, $\ell \in \{0, \pm 1, \pm 2, \dots\}$ is the azimuthal mode index and $p \in \{0, 1, 2, \dots\}$ is the radial mode index. The azimuthal index ℓ corresponds to the Orbital Angular Momentum (in units of $\hbar$) carried by each photon in the beam [1]. The radial index p , instead, gives the number of radial amplitude zeros, in correspondence with which the beam phase experiences a phase jump of π along the radial direction. Therefore, a quantum state labelled by the azimuthal and radial indices ℓ and p , $|\ell, p\rangle$, in the position basis is expressed by the LG mode $\langle r, \phi, z | \ell, p \rangle = \text{LG}_{\ell,p}(r, \phi, z)$. Examples of LG modes are shown in Fig(1.1)a.

Another set of modes are HG modes that can be derived by solving the PWE in the

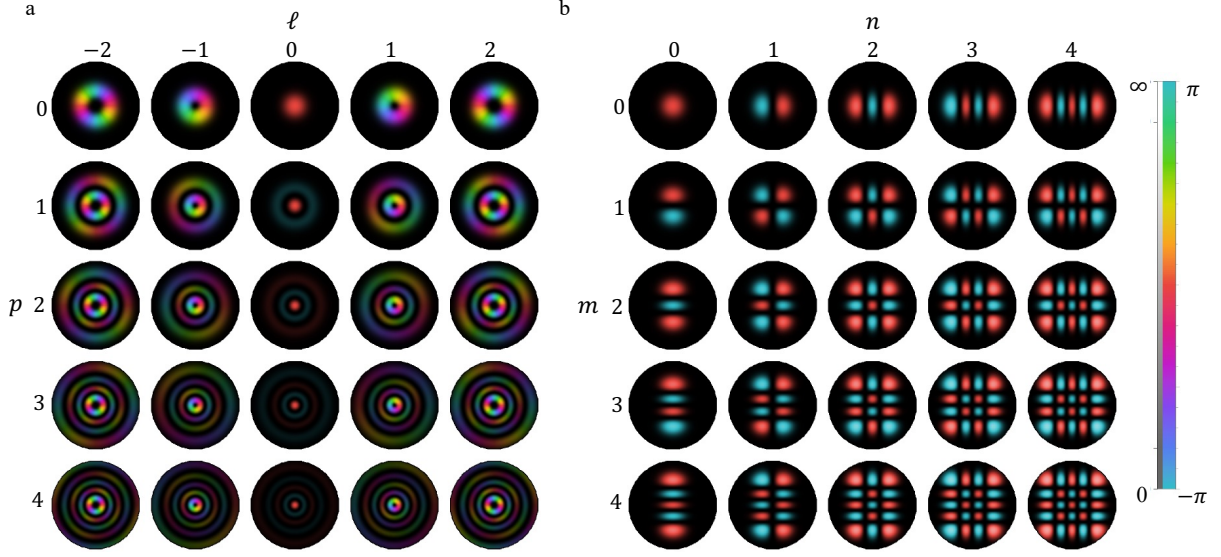


Figure 1.1: LG and HG modes: a- Examples of $\text{LG}_{\ell,p}$ for $\ell \in \{0, \pm 1, \pm 2\}$ and $p \in \{0, 1, 2, 3, 4\}$, b- Examples of $\text{HG}_{n,m}$ for $n \in \{0, 1, 2, 3, 4\}$ and $m \in \{0, 1, 2, 3, 4\}$

Cartesian coordinates. The expression for these modes is as following.

$$\begin{aligned} \text{HG}_{n,m}(x, y, z) = & E_0 \frac{w_0}{w(z)} H_n \left(\frac{\sqrt{2}x}{w(z)} \right) H_m \left(\frac{\sqrt{2}y}{w(z)} \right) \times \\ & \exp \left(-\frac{x^2 + y^2}{w^2(z)} \right) \exp \left(-i \frac{k(x^2 + y^2)}{2R(z)} \right) \times \\ & \exp \left(i(n + m + 1) \tan^{-1} \left(\frac{z}{z_R} \right) \right), \end{aligned} \quad (1.10)$$

where $H_n(\cdot)$ is the Hermite polynomial. Examples of HG modes are shown in Fig(1.1)b.

Any arbitrary solution of the PWE can be expanded in terms of LG or HG modes,

$$\tilde{\mathbf{E}}(\mathbf{r}) = \sum_{\ell,p} a_{\ell,p} \text{LG}_{\ell,p}(r, \phi, z = 0) = \sum_{n,m} b_{n,m} \text{HG}_{n,m}(x, y, z = 0). \quad (1.11)$$

The analytical expression of HG or LG modes for any plane of propagation can be derived. This means if we find the expansion of our arbitrary beam in terms of these modes and use the propagated HG or LG modes, we can find the expression of the beam in any given

plane. For example for $z = z'$ we have,

$$\tilde{\mathbf{E}}(\mathbf{r}') = \sum_{\ell,p} a_{\ell,p} \text{LG}_{\ell,p}(r, \phi, z = z') = \sum_{n,m} b_{n,m} \text{HG}_{n,m}(x, y, z = z'). \quad (1.12)$$

However, in many different cases, in particular, when strong spatial variations of phase and amplitude are present, e.g. in the case of sharp edges, an infinite number of modes is needed for the expansion; therefore, this approach would not be the most efficient way to find the expression of an arbitrary mode upon propagation. For such cases, we will discuss an alternative approach to find the free space propagation. For a beam with a known amplitude and phase in a specific plane one can use the Fresnel propagator [14] as explained in the following.

Starting from the PWE, if we decompose the electromagnetic field in plane waves in the transverse plane and substituting it in the PWE, we have

$$\mathbf{E}(x, y, z) = \int_0^\infty dk_x \int_0^\infty dk_y \left[\exp(i(k_x x + k_y y)) \mathbf{A}(k_x, k_y) \exp\left(-i \frac{(k_x^2 + k_y^2)}{2k} z\right) \right]. \quad (1.13)$$

$$\text{where} \quad \mathbf{A}(k_x, k_y) = \frac{1}{4\pi^2} \int_0^\infty dx \left[\int_0^\infty dy \exp(-i(k_x x + k_y y)) \mathbf{E}(x, y, z = 0) \right]. \quad (1.14)$$

As a result of the above expressions, if we have the expression of an arbitrary beam profile in a given plane, e.g. $z = 0$, we can find the expression of the beam upon propagation at any plane z using Fourier transforms. This approach allows us to find the propagation of different beam profiles numerically using the Fast Fourier Transform (FFT) algorithms available in different programming languages. It is worth mentioning that other methods such as angular spectrum decomposition are also available for the exact same purpose.

1.1.2 Generation of Spatial Modes

So far we have been studying different kinds of spatial modes and their properties and now we are going to look at how to create such beams experimentally. The generation of an arbitrary spatial mode can be challenging because of the difficulties in the manipulation of both the phase and the intensity of the beam. Recently, some liquid-crystal-based devices are developed that can be used for such purposes for example Spatial Light Modulator (SLM) [12, 16] and q-plates [22]. Bellow, we shortly discuss how an SLM manipulates the phase and the amplitude of an incoming beam and generates the desired beam profile [8].

Let us begin by reviewing the physics of anisotropic materials. We know that in such

materials, the dielectric permittivity ϵ is given by a 3×3 symmetric matrix which has three eigenvectors and three eigenvalues. If two of these eigenvalues are equal, the material is called uniaxial. Thus, there will be two eigenvalues (ϵ_o) with multiplicity two and another one (ϵ_e) with multiplicity one. We define the refractive indices along these axes as the ordinary refractive index ($n_o = \sqrt{\epsilon_o}$) and the extraordinary refractive index ($n_e = \sqrt{\epsilon_e}$). This means if an arbitrary polarization state propagates through an anisotropic material, each component of the polarization will gain an additional phase which depends on n_o and n_e . We also know that the effective extraordinary refractive index $n(\gamma)$ is dependent on the angle between the optic axis (the eigenvector of ϵ_e) and the propagation direction [26],

$$\frac{1}{n^2(\gamma)} = \frac{\cos^2(\gamma)}{n_o^2} + \frac{\sin^2(\gamma)}{n_e^2}. \quad (1.15)$$

This means by carefully manipulating the polarization and the incidence angle of the beam we can implement a desired phase acquired by the beam.

In fact, every pixel of an SLM, which consists of nematic liquid crystals, can be considered as an anisotropic material whose orientation can be controlled and changed by applying a specific voltage to the pixel. By applying different voltages to different pixels of the SLM, the orientation of the liquid crystal changes and consequently, the phase of the incoming beam pixel by pixel is modified. This is how we can generate a position-dependent phase structure on the incoming beam. This device is used in both of the discussed projects for generating arbitrary phase patterns for the beams used in the experiments.

Until now, we discussed how one can create the desired phase pattern, but we can also create any arbitrary intensity pattern for the beam using the phase manipulation. Now, if we consider the phase generated by the SLM to be as following [3],

$$t(x, y) = \exp\left(iM(x, y)\text{Mod}\left(F(x, y) + \frac{2\pi x}{\Lambda}, 2\pi\right)\right), \quad (1.16)$$

where Λ is the grating pitch, and $\text{Mod}(\cdot)$ is the modulo function, after selecting the first diffraction order, the output mode will be [3]

$$E(x, y) = -\text{sinc}(\pi M - \pi)e^{i(F + \pi M)}. \quad (1.17)$$

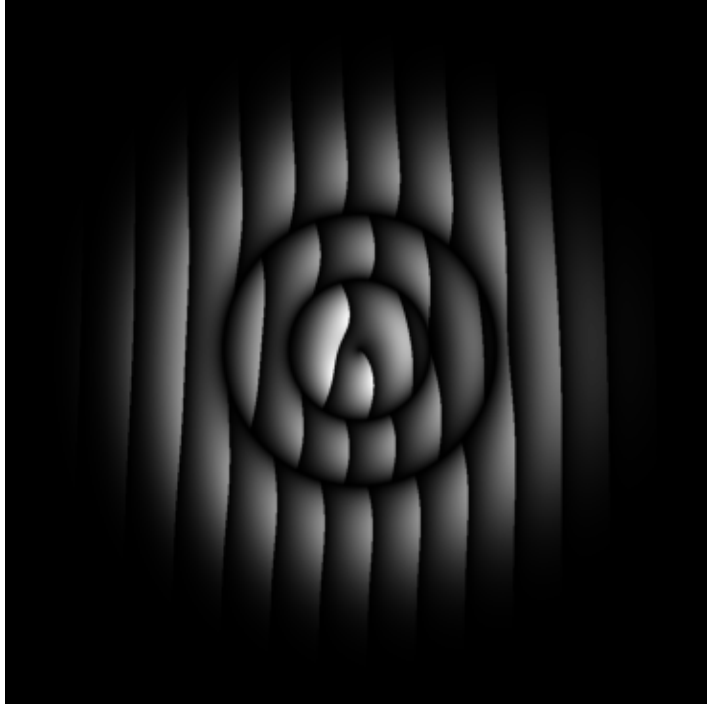


Figure 1.2: The hologram for generating $\text{LG}_{1,2}$

Thus, one can obtain the desired phase and amplitude for the beam using

$$\begin{aligned} M &= 1 - \frac{1}{\pi} \text{sinc}^{-1}(A), \\ F &= \phi - \pi M, \end{aligned} \tag{1.18}$$

where A and ϕ are the amplitude and the phase of the desired beam. We call such patterns displayed on the SLM, holograms. A hologram used to generate an LG mode is shown in Fig(1.2). From this pattern, we can see there are regions of almost uniform phase (dark regions) where the light will be diffracted to the zeroth order and thus filtered out. In this sense, the variation in gray-scale will then be mapped in the intensity variations of the

resulting beam.

1.2 Digital Holography

In the previous section, we discussed about the generation of different phase and amplitude patterns. In this section, we talk about one of the various approaches how to measure or reconstruct the phase and the amplitude of an unknown beam.

The intensity of a beam can be straightforwardly measured using a camera. If there are cameras with high resolution working in the desired wavelength, imaging would be the easiest way to achieve the intensity or the amplitude of the beam. However, the phase is more challenging to reconstruct because light detectors are typically only sensitive to the intensity of the electromagnetic field, and they cannot measure the field itself. To find the phase pattern of the beam, one can use an interferometer to superimpose the unknown beam with a reference beam. Using the information about the reference beam, we can retrieve the information about the phase of the unknown beam. If the two beams are spatially overlapped and have the same propagation direction, the method is called On-Axis or Phase Shifting Digital Holography [32], but if the two beams are tilted with respect to each other, it is called Off-Axis Digital Holography [29].

1.2.1 On-Axis Digital Holography

As mentioned before, if the reference and the unknown beams are co-propagating, after a few lines of simple calculation, we have

$$\{|f_1 + f_2|^2|_{\theta_1=\frac{\pi}{2}} - |f_1 + f_2|^2|_{\theta_1=\frac{3\pi}{2}}\} = 4A_1A_2 \sin \theta_2, \quad (1.19)$$

$$\{|f_1 + f_2|^2|_{\theta_1=0} - |f_1 + f_2|^2|_{\theta_1=\pi}\} = 4A_1A_2 \cos \theta_2, \quad (1.20)$$

where $f_1 = A_1 e^{i\theta_1}$ is the reference plane wave whose phase can be controlled and $f_2 = A_2 e^{i\theta_2}$ is the unknown beam where A_2 and θ_2 are functions of the transverse coordinates. By changing the reference beam's phase and using the interference patterns, we can get,

$$\theta_2 = \tan^{-1} \left(\frac{|f_1 + f_2|^2|_{\theta_1=\frac{\pi}{2}} - |f_1 + f_2|^2|_{\theta_1=\frac{3\pi}{2}}}{|f_1 + f_2|^2|_{\theta_1=0} - |f_1 + f_2|^2|_{\theta_1=\pi}} \right). \quad (1.21)$$

Additionally, for the amplitude A_2 , one can take an intensity photo with a camera and use the square root of the intensity. Putting together the phase and the amplitude, we can reconstruct the field. This approach requires good alignment of the interferometer to get the two beams perfectly superimposed upon propagation. For each plane, five images must be taken, four interference patterns with different phases for the reference beam and one intensity pattern of the unknown beam. We should note that the fifth photo is to avoid the imperfections in the amplitude reconstruction due to phase fluctuations. Now the question that naturally arises is what happens if the reference beam and the unknown beam are not perfectly superimposed and have an angle with respect to each other. Such question leads us to another method that is discussed in the next section.

1.2.2 Off-Axis Digital Holography

In another configuration, in which the reference and the unknown beam do not have the same propagation direction, one still can reconstruct the field with one single measurement but sacrificing spatial resolution. Let us consider f_2 as the unknown beam and $f_1 = A_1 e^{i\theta_1} e^{i(k_{1x}x + k_{1y}y)}$ as the reference beam. From the interference of the two beams, we have

$$|f_1 + f_2|^2 = |f_1|^2 + |f_2|^2 + A_1 e^{i\theta_1} e^{i(k_{1x}x + k_{1y}y)} f_2^* + f_2 A_1 e^{-i\theta_1} e^{-i(k_{1x}x + k_{1y}y)}. \quad (1.22)$$

Now if we take the Fourier transform of the interference pattern, the first two terms on the right-hand side will give us a pattern around the centre in the wave vector space. However, the other two terms will be the Fourier transform of $f_2 A_1 e^{-i\theta_1}$ and $f_2^* A_1 e^{i\theta_1}$ but displaced from the centre by the wave vector of the reference beam (k_{1x}, k_{1y}) . This means if we isolate the region in the wave vector space in which we have the Fourier transform of the unknown beam multiplied by the reference beam by inverse Fourier transforming, we can retrieve the unknown field.

1.3 Spontaneous Parametric Down Conversion (SPDC)

Structured light has many applications in quantum optics, in particular, the two-photon state emerging from nonlinear processes exhibit remarkable high dimensional entanglement in spatial degrees of freedom. SPDC is a well-known and widely used technique for creating entangled photon pairs, which is the result of pumping a nonlinear crystal with a laser beam [21, 31]. This process is the inverse of the second-harmonic generation (SHG) [5], which is when a photon (pump) with a high frequency, ω_3 , is converted to two photons

(signal and idler) with lower frequencies, ω_1 and ω_2 respectively, satisfying $\omega_3 = \omega_1 + \omega_2$. In order to describe the SPDC process, let us begin with the Hamiltonian describing the interaction of the electromagnetic field with a second-order nonlinear crystal [8],

$$\hat{H}(t) = \epsilon_0 \int d^3r \hat{\mathbf{P}}(\mathbf{r}; t) \cdot \hat{\mathbf{E}}(\mathbf{r}; t). \quad (1.23)$$

Here, $\hat{\mathbf{E}}(\mathbf{r}; t)$ is the quantized electric field, and $\hat{\mathbf{P}}(\mathbf{r}; t)$ is the quantized nonlinear polarization which are given by

$$\hat{\mathbf{P}}(\mathbf{r}; t) = \int_0^\infty dt' \int_0^\infty dt'' \chi_{ijk}^{(2)} \hat{\mathbf{E}}_j(\mathbf{r}; t') \hat{\mathbf{E}}_k(\mathbf{r}; t''), \quad (1.24)$$

$$\hat{\mathbf{E}}(\mathbf{r}; t) = \sum_{\mathbf{k}, \sigma} \left(\frac{\hbar \omega}{2\epsilon_0 n^2 V} \right)^{1/2} \{ \mathbf{e}_{\mathbf{k}, \sigma} \hat{a}_{\mathbf{k}, \sigma} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} + \mathbf{e}_{\mathbf{k}, \sigma}^* \hat{a}_{\mathbf{k}, \sigma}^\dagger e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \}, \quad (1.25)$$

where V is the quantization volume, $\hat{a}_{\mathbf{k}, \sigma}$ and $\hat{a}_{\mathbf{k}, \sigma}^\dagger$ are the annihilation and creation operators of a mode with the wave vector $\mathbf{k}$ and polarization $\mathbf{e}_{\mathbf{k}, \sigma}$. Also, ω is the field frequency, and n is the average refractive index of the nonlinear crystal. The quantum state resulted from the interaction with the nonlinear crystal is then given by

$$\begin{aligned} |\psi(t)\rangle &= \exp \left(-\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right) |\psi(0)\rangle \\ &\approx \left(|\psi(0)\rangle - \left(\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right) |\psi(0)\rangle + \dots \right), \end{aligned} \quad (1.26)$$

where we have expanded the exponential by assuming a very small non-linearity for the crystal and that $|\psi(0)\rangle$ is a coherent state which describes the pump laser [21, 27]. The second term of the expansion has two types of contributions such as $(\hat{a}_{\mathbf{k}_1, \sigma_1} \hat{a}_{\mathbf{k}_2, \sigma_2}^\dagger \hat{a}_{\mathbf{k}_3, \sigma_3}^\dagger)$ and $(\hat{a}_{\mathbf{k}_1, \sigma_1}^\dagger \hat{a}_{\mathbf{k}_2, \sigma_2} \hat{a}_{\mathbf{k}_3, \sigma_3})$ which are corresponding to the SPDC process and the SHG process respectively. Since The SPDC process is elastic, the conservation of energy is valid, and therefore we have $\hbar\omega_3 = \hbar\omega_1 + \hbar\omega_2$. The higher terms of the expansion are associated with the creation of more than two-photon pairs with a much lower probability. Thus, the

SPDC state can be written as

$$|\psi\rangle_{\text{SPDC}} = -\frac{i}{\hbar} \sum_{\substack{\mathbf{k}_1, \sigma_1 \\ \mathbf{k}_2, \sigma_2 \\ \mathbf{k}_3, \sigma_3}} \left(\frac{\hbar}{2\epsilon_0 n^2} \right)^{3/2} \sqrt{\omega_1 \omega_2 \omega_3} \chi_{i,j,k}^{(2)} e_{\mathbf{k}_1, \sigma_1}^{(i)} e_{\mathbf{k}_2, \sigma_2}^{(j)} e_{\mathbf{k}_3, \sigma_3}^{(k)} \left[\int_0^t dt' e^{-i(\omega_1 + \omega_2 - \omega_3)t'} \right] \times \\ \left[\int_0^\infty dx \int_0^\infty dy \int_{-\frac{L}{2}}^{\frac{L}{2}} dz e^{i\Delta \mathbf{k} \cdot \mathbf{r}} \right] \hat{a}_{\mathbf{k}_1, \sigma_1}^\dagger \hat{a}_{\mathbf{k}_2, \sigma_2}^\dagger \mathcal{E}_p(\mathbf{k}_3, \sigma_3) |vac\rangle, \quad (1.27)$$

where $\mathcal{E}_p$ is the pump field considered as an input multimode coherent state, L is the length of the crystal and t is the time duration of the process or equivalently the time that the pump photon is travelling through the crystal. The sum over the indices i, j, k is intended, and $\Delta \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$. We should note that Eq. (1.27) is written considering the transverse size of the nonlinear crystal being much bigger than the size of the beam. Finally, by applying the energy conservation and the conservation of the transverse momentum, we can write the state to be

$$|\psi\rangle_{\text{SPDC}} = \sum_{\sigma_1, \sigma_2} \mathcal{C}_{\sigma_1, \sigma_2} \int d^2 q_1 \int d^2 q_2 \int d\omega_1 \int d\omega_2 \times \\ \left[\text{sinc} \left(\frac{\Delta k_z l_z}{2} \right) \mathcal{E}_p(\mathbf{q}_1 + \mathbf{q}_2, \omega_1 + \omega_2) \right] |1_{\mathbf{q}_1, \omega_1, \sigma_1}\rangle |1_{\mathbf{q}_2, \omega_2, \sigma_2}\rangle, \quad (1.28)$$

where we defined $\mathbf{k}_i = (\mathbf{q}_i, \frac{\omega_i}{cn})$. Finally, if we consider a special case of which $\omega_1 = \omega_2 = \omega$ and a Type-I crystal which means $\sigma_1 = \sigma_2$ (and $\sigma_{1,2} \neq \sigma_3$), we get [21, 31]

$$|\psi\rangle_{\text{SPDC}} = \mathcal{N} \int d^2 q_1 \int d^2 q_2 \left[\text{sinc} \left(\frac{\Delta k_z l_z}{2} \right) \mathcal{E}_p(\mathbf{q}_1 + \mathbf{q}_2) \right] |1_{\mathbf{q}_1}\rangle |1_{\mathbf{q}_2}\rangle, \quad (1.29)$$

where $\mathcal{N}$ is a normalization factor, and we have dropped the dependency on the frequencies and the polarization.

Chapter 2

Effect of Aberration on 3D optical topologies

This chapter is based on the following paper:

Dehghan, N., D'Errico, A., Jaouni, T., and Karimi, E. (2023). Effect of Aberrations on 3D optical topologies. arXiv:2302.11531.

In this chapter, we discuss the first project in details. This work is about creating complicated structured beams to generate 3D optical topologies and investigating if these topologies can be modified under phase distortions. We begin with explaining the concept of optical topologies and phase aberrations and then a detailed manuscript about the work is provided.

2.1 Optical Topologies and Singular Optics

When three or more plane waves interfere there are points of destructive interference in which the intensity of the field is zero and, therefore, its amplitude [7]. These points are called phase singularities because the field's phase is undefined where the amplitude is zero [7]. These singularities in the optical field have shown very interesting features, and what concerns our project, their propagation can be manipulated such that their trajectories in free-space form a link or a knot.

The change in the phase along a contour around a phase singularity, for a continuous single-valued field, has to be an integer multiple of 2π [13]. For each of the singularities of the field, one can define a parameter, called the topological charge, describing the strength of the singularity, which is the change of the phase on any chosen contour around the singularity [13], defined as

$$t := \frac{1}{2\pi} \oint_C \nabla \chi \cdot d\mathbf{r} = \frac{1}{2\pi} \oint_C d\chi. \quad (2.1)$$

Here, $\chi(\mathbf{r})$ is the phase of the field defined as $U(\mathbf{r}) = A(\mathbf{r})e^{i\chi(\mathbf{r})}$. Since the change in the phase along the contour has to be an integer multiple of 2π , therefore, a singularity can only have a discretized topological charge. We should note that if two singularities fall into the contour that is chosen, the total topological charge would be the sum of the two individual topological charges [13].

As mentioned before, one can modify the trajectory of these singularities such that upon propagation, two singularities of opposite charge are born and join together and cancel each other out [13]. By different kinds of manipulation, one can create intriguing structures such as knots and links.

For tracking these singularities, considering the fact that the amplitude of the field has to be zero, both the real and the imaginary parts of the field should vanish, and one can find the contours of these functions being zero. The intersection of these contours gives us the position of the singularities [7]. This concept can be understood better from Fig2.1.

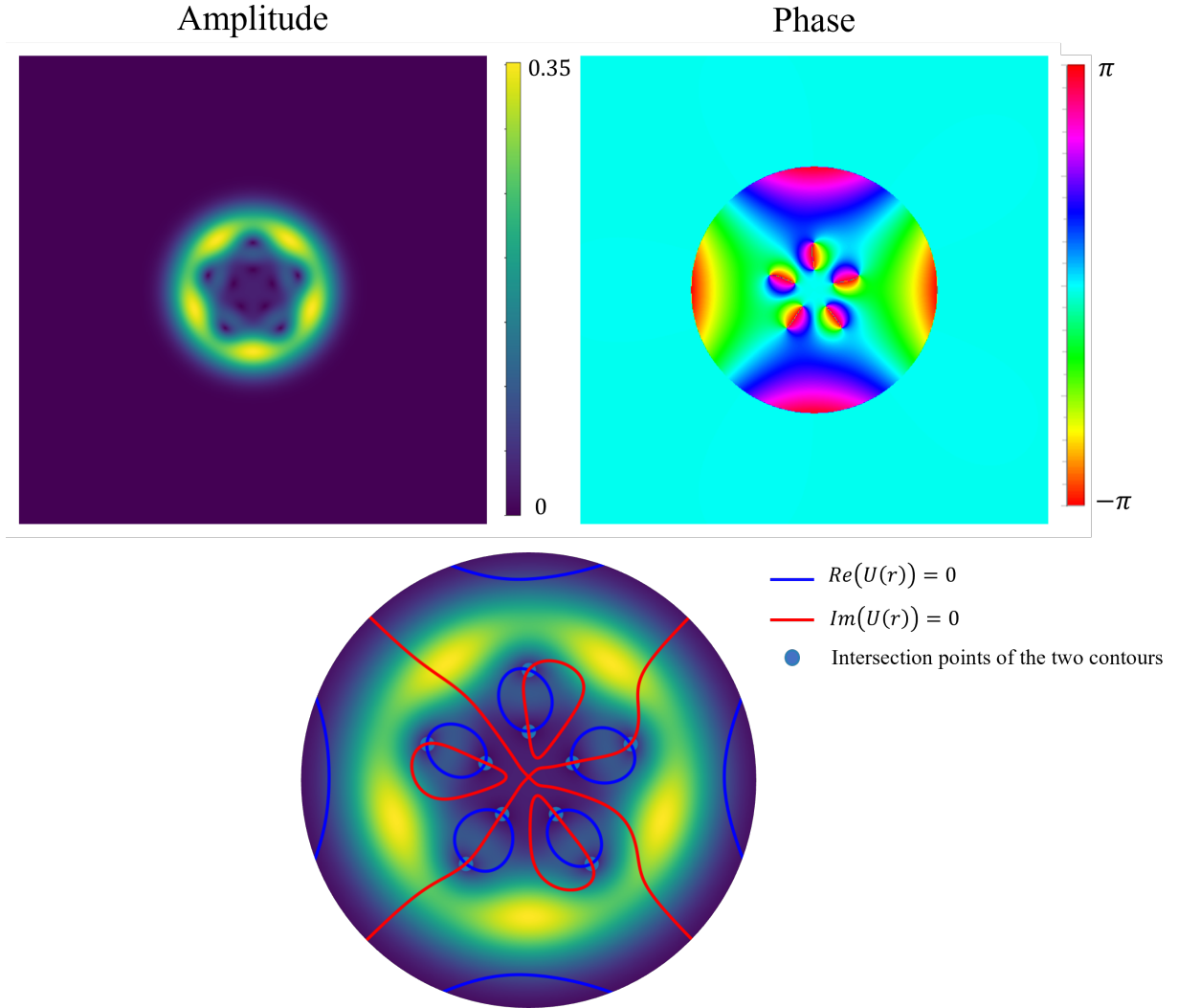


Figure 2.1: Tracking the singularities: Intersecting the contours corresponding to the real and imaginary part of the field equal to zero, we find the position of the points in which the amplitude of the field is zero (phase singularities).

2.2 Optical Knots and Links and Their Experimental Realization

In the previous section, we briefly mentioned that it has been of great interest to modify and combine optical singularities, to create linked or knotted beams. Originally, Sir Michael Berry and Mark Dennis [2] theoretically showed that specific superpositions of optical fields can exhibit knotted or linked phase singularities upon propagation.

The original analysis of knots and links [2] was done using Bessel beams [9]. However, there were technical issues with this set of beams. The ideal Bessel beams are infinitely extended in space, and the dark regions of these beams are so large that it is very difficult to locate the singularity points, especially if they are tracked by finding the intensity zeros. Therefore, an alternative set of spatial modes, LG modes, within the paraxial regime was proposed [19]. Superpositions of LG modes, with different ℓ , p and w were considered and successfully showed to have the characteristics of the knotted and linked beams that were desired.

More recently, it was shown that these optical structures can also be created using specific polynomials multiplied by a Gaussian envelope [17]. Here, we have the polynomials associated with Hopf link, Trefoil and Cinquefoil knots [17],

$$\begin{aligned} \psi_{\text{Hopf link}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (1 - 2(1 + a^2 - b^2)\rho^2 + \rho^4 \\ & - 2(a^2 + b^2)\rho^2 \cos(2\phi) - 4iab\rho^2 \sin(2\phi)), \end{aligned} \quad (2.2)$$

$$\begin{aligned} \psi_{\text{Trefoil}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (-4\rho^3(a^2 - b^2) - 2\rho^3(a - b)^2 e^{-i3\phi} \\ & - 2\rho^3(a + b)^2 e^{i3\phi} + \rho^6 - \rho^4 - \rho^2 + 1), \end{aligned} \quad (2.3)$$

$$\begin{aligned} \psi_{\text{cinquefoil}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (1 + \rho^2 - 2\rho^4 - 2\rho^6 + \rho^8 \rho^{10} \\ & - 16\rho^5(a^2 - b^2) - 8\rho^5(a - b)^2 e^{-i5\phi} \\ & - 8\rho^5(a + b)^2 e^{i5\phi}), \end{aligned} \quad (2.4)$$

where $\rho = \frac{r}{w}$ with w the beam waist, s a parameter that gives the width of the Gaussian envelope and a, b the free parameters that can be changed to modify the shape of the topology. The presence of the Gaussian envelop is guaranteeing the beam to remain paraxial. This approach is more advantageous compared to using the expansions in the LG modes

because these polynomials can be directly deduced from the mathematical description of the desired knot structure. Moreover, the free parameters allow to optimize the optical beam to improve the experimental reconstruction without severely affecting the topology.

In the first experiments, to reconstruct these knotted beams, by saturating the intensity pattern that was imaged by the camera, singularities were spotted as the dark points observed [19]. However, this is not the most accurate way to find the phase singularities, and it is easily affected by many factors in the experimental setup. The main problem is that a dark region, which is detected as a zero because of low exposure or limited dynamical range of the detectors, in reality, can be only a region with a very small intensity without a true phase singularity.

Eventually, to generate these structures in the lab and to reconstruct them, instead of only following the dark areas of the intensity pattern, interferometric approaches were exploited [18] to reconstruct the phase and the amplitude of the knotted or linked beam in every plane.

In our setup, We use a translation stage to scan and reconstruct the phase of the field upon propagation in 80 planes. Creating a knotted field using computer-generated holograms displayed on an SLM, we superimpose the knot with a reference beam, and using phase-shifting digital holography [32] (see section 1.2.1), we are able to retrieve the phase and the amplitude pattern plane by plane (in our case 75-80 planes). Having the optical field in every propagation distance, we can follow the phase singularities and obtain the 3D optical topology.

In particular, for creating the interference, we use a folded-Sagnac interferometer which allows us to send the reference beam and the knotted beam through the same path. The Sagnac configuration ensures that both beams experience the same vibrations in the interferometer allowing for phase stability at the measurement [17]. Additionally, the folding allows us to reduce the incidence angle of the SLM.

Once the optical field is formed in many propagation planes, in each plane, the contours of the real and the imaginary part of the field equal to zero are plotted. The intersection of these sets of contours would give us the position of the phase singularities (see Fig 2.1). Following these singularities upon propagation, we retrieve the skeleton of the singularities.

In order to reconstruct the corresponding continuous curve from the scatter-plot of the phase singularities and thus recognising its topology, we used two alternative approaches, both based on sorting the array of singularity coordinates. The first approach was based on starting from a selected point and finding its nearest neighbour, and proceeding iteratively in this fashion. This method, however, is strongly affected by vibrations in the setup which may cause the algorithm to fail. Another approach, which proved to be more robust, is

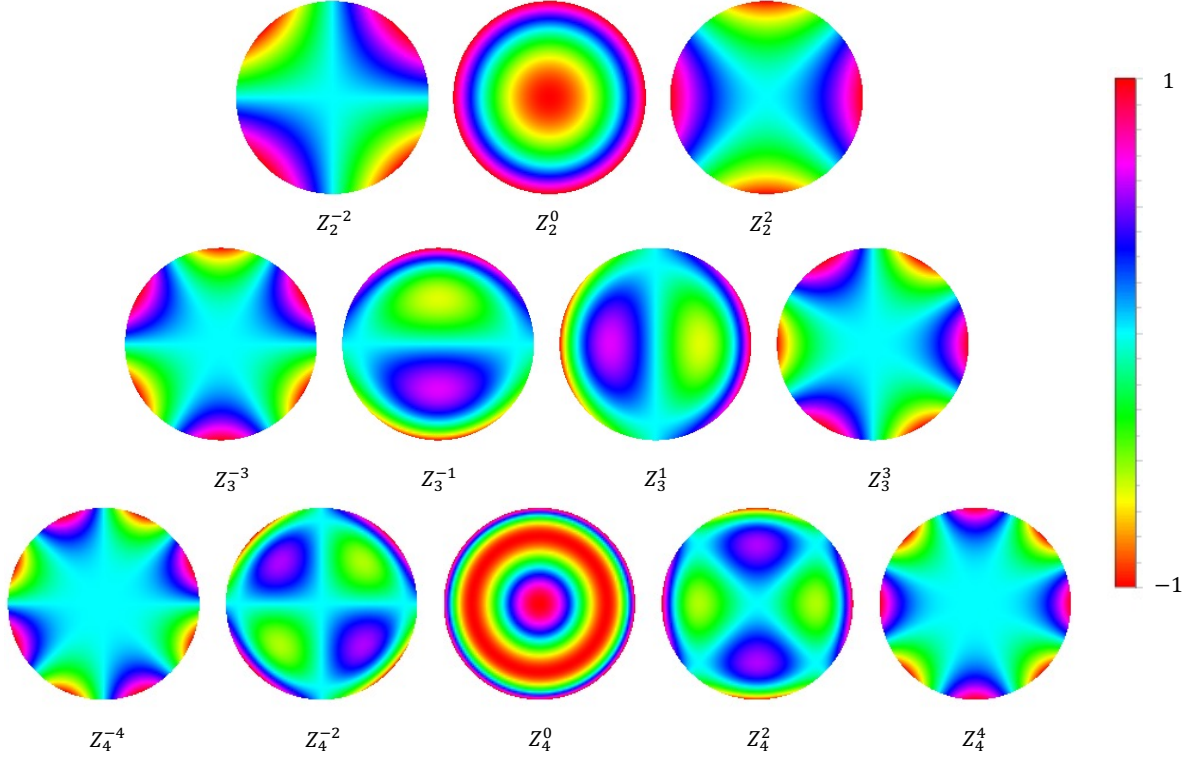


Figure 2.2: Different Zernike functions

sorting the points looking for the shortest tour, that is the closed curve joining all the singularities with the shortest length. This algorithm can converge to a local minima, but this issue can be solved by reiterating the procedure a few times. When these points are properly sorted, they can be joined to form a good approximation of the singular skeleton.

Now that we are familiar with creating and reconstructing knotted or linked beams, we go back to the original question of this project and learn about phase distortions and optical aberrations in the following section.

2.3 Optical Aberrations and Aberrated Knots

Optical Aberrations are known to be the phase distortion on the optical beams applied by different optical elements, especially lenses, and since lenses are circular and have a specific aperture, these phase distortions are considered to be inside a circle. This means if we find an orthogonal complete basis inside a unit circle, we can expand any optical aberration using this basis. A standard choice to describe phase aberration is the use of Zernike polynomials introduced by Zernike, which is an orthogonal complete set of basis inside the unit circle [4]. Zernike functions of different orders are defined as

$$\begin{aligned} Z_n^m(\rho, \phi) &= R_n^m(\rho) \cos(m\phi) \quad \text{for } m > 0, \\ Z_n^{-m}(\rho, \phi) &= R_n^m(\rho) \sin(m\phi) \quad \text{for } m < 0, \end{aligned} \quad (2.5)$$

where

$$R_n^m(\rho) = \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^k (n-k)!}{k! \left(\frac{n+m}{2} - k\right)! \left(\frac{n-m}{2} - k\right)!} \rho^{n-2k}, \quad (2.6)$$

are the radial polynomials, ϕ is the azimuthal angle, and ρ is the radial distance $0 \leq \rho \leq 1$. Therefore, $|Z_n^m(\rho)| \leq 1$. Here we can see examples of different aberrations in Fig 2.2.

An optical field affected by aberration is described by

$$\tilde{U}(x, y) = U(x, y) e^{i\Delta(x, y)} \quad \text{where} \quad \Delta(x, y) = \sum_{n,m} \gamma_{n,m} Z_n^m, \quad (2.7)$$

and $\gamma_{n,m}$ is the strength of the aberration. We define the highest strength of the aberration to be 1 which corresponds to phase variations from $-\pi$ to π . We thus investigate the effect of individual aberrations (different Zernike functions up to the fourth order) on optical knots and links, creating the fields $\psi = \psi_{\text{knot}} e^{i\pi\gamma Z_n^m(\frac{r}{A}, \phi)}$ and exploring the resulting singular skeletons for different ranges of γ and the aperture of the aberration A . We experimentally and theoretically showed that 3D optical topologies, in particular Trefoil knot, Cinquefoil knot, and Hopf link are robust against all types of aberrations up to the 4th order with the highest strength, and with the big enough aperture (order of characteristic size of the beam). Additionally, we found out the aberrations that alter more easily the structure of the singular skeleton, are those with local minima and maxima around the position of the singularities, i.e. Coma (Z_3^{-1} and Z_3^1) that is shown in Fig 2.2. In the following paper, the full analysis of what was described above is provided.

Effect of Aberrations on 3D optical topologies

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Optical knots and links, consisting of trajectories of phase or polarisation singularities, are intriguing nontrivial three-dimensional topologies. They are theoretically predicted and experimentally observed in paraxial [1–6] and non-paraxial regimes [7–10], as well as in random and speckle fields [11]. Framed and nested knots can be employed in security protocols for secret key sharing [12–14], quantum money [15, 16], and topological quantum computation [17]. The topological nature of optical knots suggests that environmental disturbances should not alter their topology; therefore, they may be utilised as a resilient vector of information. Hitherto, the robustness of these nontrivial topologies under typical disturbances encountered in optical experiments has not been investigated. Here, we provide the experimental analysis of the effect of optical phase aberrations on optical knots and links. We demonstrate that Hopf links, trefoil and cinquefoil knots exhibit remarkable robustness under misalignment and phase aberrations. The observed knots are obliterated for high aberration strengths and defining apertures close to the characteristic optical beam size. Our observations recommend employing these photonics topological structures in both classical and quantum information processing in noisy channels where optical modes are strongly affected and not applicable.

Linked or knotted structures can arise from optical fields as trajectories in the three-dimensional space of phase or polarisation singularities [1, 2, 4, 18, 19]. For instance, a scalar paraxial optical field described by the wavefunction $\psi(x, y, z)$, at a given plane z , can exhibit points of zero intensity where the phase is undefined [20]. These singular points are characterised by a topological charge q given by the winding of the field’s phase around the singularity [18–22]: $q = 1/(2\pi) \oint_L \nabla\psi \cdot d\ell$, where L is a closed loop around the singularity and $d\ell$ the infinitesimal arc length. Stable singularities carry the minimal topological charge $q = \pm 1$ and can evolve in space with the constraint of total topological charge conservation [21, 23–25], i.e. singularity pairs, with individual components having opposite q , can annihilate each other or emerge from planes where no charge is present. This dynamics of creation and annihilation of pairs at different planes can result in singularity paths confined in the three-dimensional space and forming a closed trajectory [26]. These curves can be trivial loops, for instance, generated by two charges appearing in one plane and then re-joining each other upon free-space propagation [27]. However, it is now well-known that the wave equation allows for solutions where singularities can track linked or knotted trajectories [1, 26]. These solutions were originally found by

perturbing high-strength vortex line singularities threaded by unstable loop singularities [1]. Other approaches based on numerical optimisation procedures and Laguerre-Gauss mode expansions [3–6] have been subsequently developed. More recently, it has been shown that specific optical knots can be generated by imposing weighted polynomials as boundary conditions on the field amplitude [12, 13, 28]. This result, which still lacks a general proof [28], is particularly intriguing since it may lead to a systematic approach for generating optical knots and links. This may prove extremely useful in implementing secure communications based on optical knots [12–14], quantum money [16], and topological quantum computation [29].

While some kind of singularities, like high-order topological charges and some types of caustics [30], can be structurally unstable, knots and links formed by elementary singularity paths are mathematically stable. Indeed, knots and links are topologically robust objects since they cannot be smoothly deformed into each other. For a knot to change type, a mathematical transformation should reach a point in parameter space where the knot is singular, i.e. self-intersecting. An optical knot is thus expected to keep the same nature under environmental disturbances that smoothly change its phase and amplitude, thus suggesting a potential advantage for environment-resilient transfer of information.

However, in practice, typical disturbances may be strong enough to induce a transition in knot type, thus ruling out the potential of this approach. Here we show, through numerical and experimental investigation, how some types of optical knots recently realised experimentally; namely Hopf link, trefoil, and cinquefoil, are remarkably robust under the action of phase distortions applied on the waist plane. The structures mentioned above can all be generated by an optical field which in the plane $z = 0$ reads: $\psi(\rho, \phi) = \exp(-\rho^2/(2s^2))\text{Poly}(\rho, \exp(i\phi))$, where (ρ, ϕ, z) are cylindrical coordinates, s is a width parameter specifying the size of a Gaussian envelope, and $\text{Poly}(\rho, \exp(i\phi))$ is a Milnor polynomial which specifies the knot type [12, 13, 28] (see Methods for the explicit expressions).

Phase aberrations, which are the most common consequence of environmental disturbances and imperfections in optical setups, can be modelled as an additional phase factor $\delta(\rho, \phi)$ applied on the undistorted field:

$$\tilde{\psi}(\rho, \phi) = \exp(i\delta(\rho, \phi))\psi(\rho, \phi). \quad (1)$$

Monochromatic optical aberrations can be modelled by expanding the phase distortion in Zernike functions:

$$\delta(\rho, \phi) = \pi \sum_{n,m} \gamma_{n,m} Z_n^m(\rho/A, \phi), \quad (2)$$

where $\gamma_{n,m}$ are real numbers which we call the *strength* of the (n, m) -th aberration, and $Z_n^m(\rho/A, \phi) = R_n^m(\rho/A) \cos(m\phi)$

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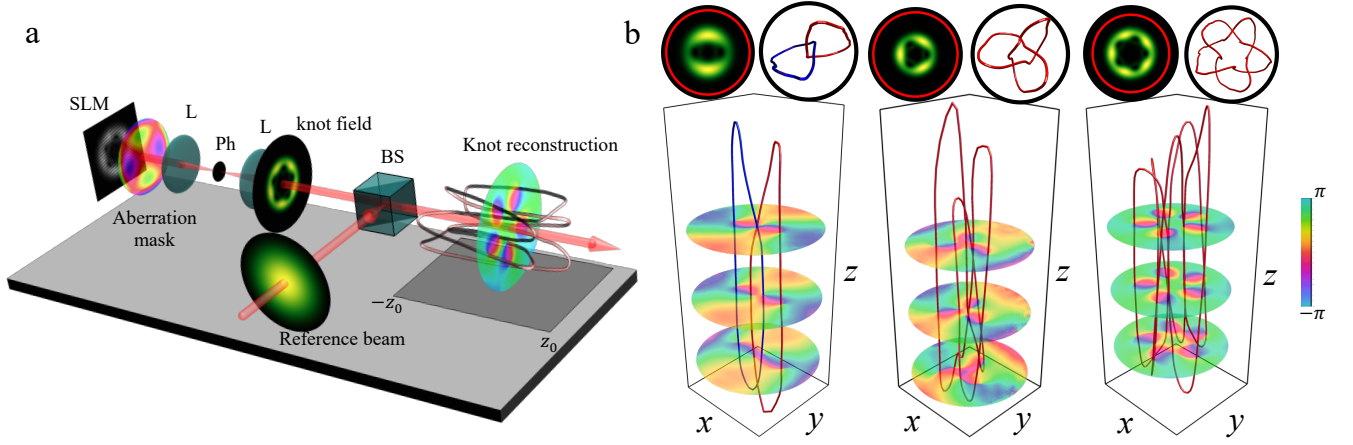


Figure 1. **Experimental scheme and reconstructed aberrated knots.** **a-** Schematics of the experimental setup. An SLM encodes a phase mask which generates the aberrated knot (Eq. (1)) after a 4-f system and selection of the first diffraction order. The resulting field's phase structure is reconstructed by phase-shifting digital holography, interfering with the knot field and a Gaussian reference beam. Recording the interference patterns at different propagation planes allows one to reconstruct the singular skeleton. **b-** Examples of three-dimensional topologies affected by coma with $\gamma = 1$ experimentally reconstructed. From left to right: Hopf link, trefoil, and cinquefoil. The main plots show the singular skeletons and the phase patterns in different propagation planes. Insets show the top view of the singular skeleton (right) and the theoretical intensity patterns in the waist plane with red circles indicating the size of the aperture over which the Zernike polynomials were defined.

for positive m and $Z_n^m(\rho/A, \phi) = R_n^m(\rho/A) \sin(m\phi)$ for negative n , with $R_n^m(\rho/A)$ the radial Zernike polynomial, defined in $0 \leq \rho \leq A$. We recall that $|m| \leq n$ and $|Z_n^m(\rho/A, \phi)| \leq 1$. Here, we mainly investigate the effect of the individual aberrations, up to 4-th order, on optical knots, thus applying phase distortions of the form,

$$\tilde{\psi}(\rho, \phi) = e^{i\pi\gamma Z_n^m(\rho/A, \phi)} \psi(\rho, \phi). \quad (3)$$

In particular, we seek the critical values of the strengths γ and inverse aperture $1/A$ above which the knot topology is altered. It will be shown how knots can survive these aberrations for apertures slightly larger than the characteristic beam size and strengths $\gamma \approx 1$, which are much higher than values encountered in many practical scenarios. We then push towards smaller apertures to determine which aberrations can affect more the topological structure. It is found that aberrations like coma and secondary astigmatism are the most relevant. This is because these aberrations exhibit local maxima and minima in the interior of the aperture, thus introducing wavefront distortion in between the location of the phase singularities and affecting their trajectories.

The robustness of three different 3-dimensional optical topologies (Hopf link, Trefoil and Cinquefoil) under various optical phase aberrations was experimentally investigated.

These structures were generated and detected with an approach similar to Ref. [6]. The principle of the experiment is shown in Fig. 1-a, while details of the setup are reported in the Methods. Optical knots can be obtained from computer-generated holograms displayed on a spatial light modulator (SLM). Exploiting the encoding introduced in Ref. [31], a phase mask is applied on an input Gaussian beam which is transformed in the desired knotted field after selecting the first

diffraction order (this is done with a pinhole placed in the far field of the SLM). In this experiment, the knotted beam interferes collinearly with a reference Gaussian beam. On the translation stage, a CMOS camera is automatically translated to 75 different planes, recording interference patterns that are formed by changing the phase α of the reference beam. The reference phase is controlled using half of the SLM window (see Refs.[6, 12] and Methods). The phase of the structured beam in each plane can be reconstructed by means of phase-shifting digital holography. By tracking the phase singularities upon propagation, the singular skeleton was retrieved. Different aberrations, modelled as Zernike functions [32], were applied individually as additional phases with a specific aperture on the knot hologram.

In agreement with simulations, it was observed that for each topology and aperture A higher than a critical value A_c , the structure survives all different aberrations with $\gamma = 1$. For each example, robustness up to at least $\gamma = 1$ was observed for values of A slightly larger than the characteristic beam intensity radius. Specifically, the topological structures were unaltered for aperture values $A = 8w_0$, $8w_0$, and $4w_0$ for Hopf link, trefoil, and cinquefoil, respectively, where w_0 is the characteristic waist parameter (which however, does not necessarily correspond to the beam size, the latter being also dependent on the parameter s). We point out that simulations predict that trefoil structures should survive high strength aberrations also for $A = 7w_0$, while experimentally, we observe a critical behaviour for Z_4^2 (while other aberrations still give rise to trefoils). We attribute this mismatch to an additional phase perturbation present in the setup. Figure 1-b shows examples of the three structures affected by coma (data for the other aberrations are given in the Supplementary Materials, Fig. S2).

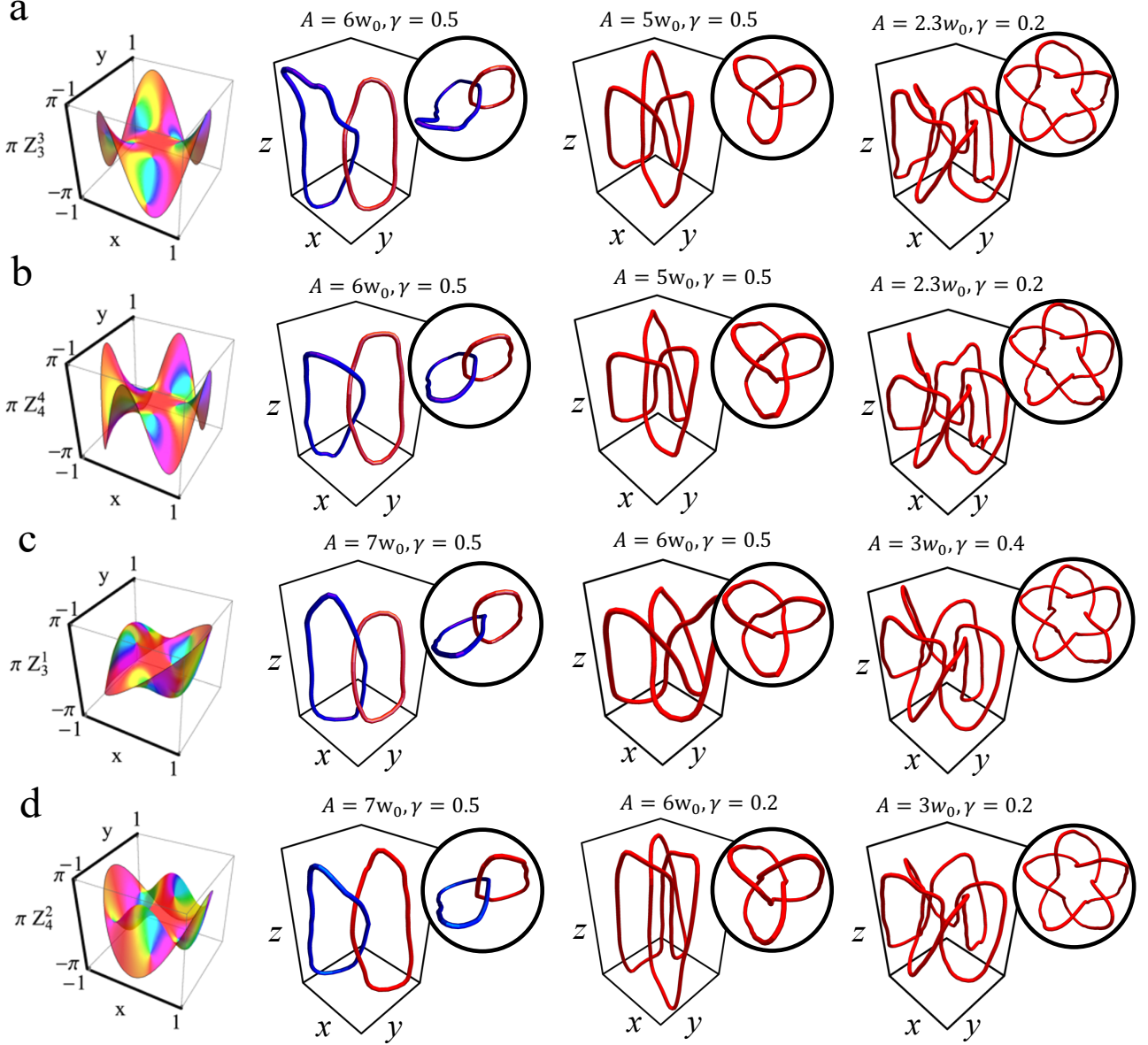


Figure 2. **Knot survival for small apertures.** The figure shows different singular skeletons obtained for aperture values slightly below the critical aperture A_c and for strengths $\gamma < 1$ such that the topology is conserved. The examples shown are for 3-rd and 4-th order aberrations, namely (a) trefoil, (b) quadrifoil, (c) coma, and (d) secondary astigmatism. The first column shows a three-dimensional plot of the aberrations with a texture given by the cinquefoil phase pattern to highlight how the different Zernike polynomials affect the wavefront near the phase singularities.

The main effect of the aberrations is to stretch or compress the individual “lobes” of the curves leaving the overall topology unchanged. Therefore, if the aperture of the phase aberration compared to the beam waist w_0 is bigger than a critical value no aberrations (up to 4-th order) even with $\gamma = 1$ can break the topological structure.

When decreasing A , knots and links can be broken in open trajectories. This effect can be, for instance, caused by singularity pairs created in the far field, which, instead of annihilating

with each other in back-propagation, join with the lines of the singular skeleton. For fixed $\gamma = 1$, we define the critical aperture (A_c) as the A in which the structure breaks up or cannot be fully reconstructed. The value of A_c is different for different aberrations typologies. For instance, in the case of the cinquefoil, A_c for Z_3^1 , Z_4^0 , and Z_4^2 is $\approx 3w_0$ and for Z_3^0 , Z_2^2 , Z_3^3 , and Z_4^4 is $\approx 2.3w_0$. When $A \approx A_c$, the effect of γ can be further investigated. In Fig. 2, experimental examples of links and knots under different aberrations with $A \approx A_c$ and $\gamma < 1$

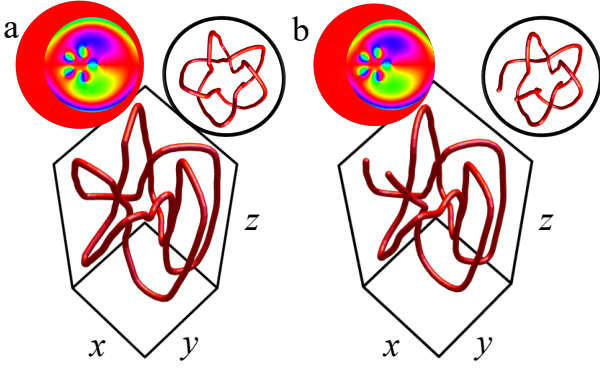


Figure 3. **Effect of aperture displacement.** The singular skeleton of a cinquefoil knot perturbed by a coma aberration with displaced origin (aperture $A = 4w_0$ and $\gamma = 1$). For displacement $\Delta x = w_0$ the knot topology is unaltered (Panel a). For $\Delta x = 1.5w_0$ (b) the singular skeleton opens up due to joining with singularities created in the far field. Insets show the phase pattern in the waist plane (left) and the top view of the singular skeleton (right).

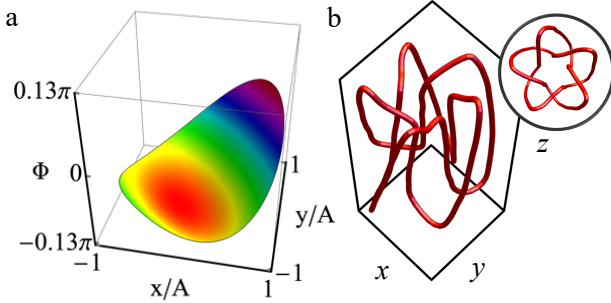


Figure 4. **Cinquefoil knot subject to a superposition of aberrations.** a- Plot of the wavefront distortion applied to a cinquefoil knot. b- Side view and top view (inset) of the reconstructed singular skeleton.

are shown, highlighting how, even if the topology is not preserved under the highest γ , it can still be recovered for smaller strengths. We note that the experimental knots tend to break up for slightly higher apertures than theoretically expected. This is due to either experimental imperfections, which amount to additional aberrations perturbing the beam, or to residual interference with the zeroth order beam diffracted from the SLM. The latter case was analyzed in detail (see Supplementary Materials) but may be less relevant in practical scenarios where aberrations are applied after the spatial filtering.

The topology of the singular skeleton can be more or less sensitive to different Zernike functions. The aberrations with stronger variations close to the singularities affect their evolution more and, therefore, can destroy the topology for larger A compared with aberrations responsible for distortions of the most external parts of the wavefront. This is the reason why to coma (Z_3^1) and secondary astigmatism (Z_4^2) is associated a bigger A_c compared to trefoil (Z_3^3) and quadrafoil (Z_4^4) aberrations. As shown in the first column of Fig. 2, coma and secondary astigmatism have local minima and maxima in

the interior of the defining circle, and so these variations can modify the wave vector distribution around the singularities, thereby altering their trajectory. On the other hand, distortions like Z_3^3 and Z_4^4 , are mostly flat in their central region while being responsible for wavefront distortions close to the boundary of the defining circle. Hence, for large enough apertures, the phase singularities will lie in the central flat region, and the formation of the singular skeleton will be less affected.

Up to this point, we considered the case in which the defining aperture of the Zernike functions is perfectly centred with the unperturbed beam. However, lateral misalignments are a common source of imperfections. Hence, we looked into the effect of a relative displacement between the centre of the beam and the centre of the aberration's defining circle. Figure 3 shows how displaced coma affects the cinquefoil knot. Relatively small displacements leave the singular skeleton topology unaltered, even for $\gamma = 1$. However, for the second displacement ($\Delta x = 1.5w_0$), it can be seen that Z_3^1 breaks the knot. We observed qualitatively similar effects for other aberrations (data are reported in the Supplementary Fig. S3).

So far, we have considered high values of the strength by looking at the effect of individual aberrations. In practical scenarios, one deals with wavefront distortions described by a superposition of Zernike polynomials (as in Eq. (2)). As an example, we consider the effect of the wavefront distortion observed in a previous experiment on underwater high-dimensional quantum key distribution (Ref. [33]). The same distortion was applied to the cinquefoil beam, and, as shown in Fig. 4, the topology survives, which is not surprising since the $\gamma_{n,m}$ are much smaller than 1. We point out that while this wavefront distortion is rather small, it was shown to have a significant effect on the security of OAM-based high-dimensional quantum key distribution. In particular, giving an error rate above the security threshold for dimensions higher than three [33]. This difference in robustness between OAM and knot-based encoding is strictly due to the topological nature of the singular skeletons of structured beams: on the one hand, optical aberrations can abruptly change the decomposition in a given spatial mode basis, thus affecting immediately the fidelity of the transmitted beam and the information encoded within its structure; on the other hand, the change in spatial mode decomposition induced by aberrations does not alter the topological structure associated with a knotted or linked beam, assuming that the strength and/or inverse defining aperture is below a given threshold. Thus, the survival of knot fields under relevant wavefront distortion is a promising example of how these structures can provide a more robust way to encode information. However, we stress that deeper studies on the effects of turbulence must be carried out to certify this advantage. We expect that moderate levels of turbulence, which can be compensated by an adaptive optics system, will not present a serious obstacle to the transmission of either classical or quantum information by means of three-dimensional optical topological structures.

In conclusion, we have demonstrated experimentally how simple optical knots and links are robust under phase aberrations and setup misalignments, thus hinting at their potential advantage in communication protocols. Moreover, we showed how the aberrations exhibiting local minima or maxima in the

interior of the circle defining the Zernike polynomials are those which can more significantly affect the topology of the singular skeleton. These considerations can be useful not only for communication purposes but also in devising setups to generate

knotted fields in more delicate scenarios, e.g. in nanophotonics experiments [10] or in setups for other kinds of structured quantum waves [34, 35], e.g. electrons and neutrons [36].

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Author Contributions. N.D., A.D., and E.K. conceived the idea and devised the experiment. N.D., under the supervision of A.D., performed the experiment and analyzed the data. N.D. and T.J. performed the theoretical simulations. E.K. supervised the project. A.D., N.D., and T.J. prepared the first draft of the manuscript. All authors discussed the results and the final version of the manuscript.

Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Code availability The code used for the data analysis is available from the corresponding author upon reasonable request.

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METHODS

A. Explicit expressions of the knot fields.

In the following, we give expressions for the linked and knotted fields in terms of Milnor polynomials modulated by a Gaussian envelope. We chose a , b and s in such a way as to obtain the cleanest singular skeleton in the not aberrated case. The Hopf-link was generated with a mask imposing, on the SLM's image plane, the field amplitude

$$\psi_{\text{Hopf link}}(\rho, \phi) = e^{-\frac{\rho^2}{2s^2}} (1 - 2(1 + a^2 - b^2)\rho^2 + \rho^4 - 2(a^2 + b^2)\rho^2 \cos(2\phi) - 4iab\rho^2 \sin(2\phi)) \quad (4)$$

with $a = 1$, $b = 1$ and $s = 1.7$. Similarly, the trefoil field was generated from the propagation of the amplitude:

$$\psi_{\text{Trefoil}}(\rho, \phi) = e^{-\frac{\rho^2}{2s^2}} (-4\rho^3(a^2 - b^2) - 2\rho^3(a - b)^2 e^{-i3\phi} - 2\rho^3(a + b)^2 e^{i3\phi} + \rho^6 - \rho^4 - \rho^2 + 1) \quad (5)$$

with $a = 1$, $b = 1$ and $s = 1.2$, while the cinquefoil was obtained from

$$\psi_{\text{cinquefoil}}(\rho, \phi) = e^{-\frac{\rho^2}{2s^2}} (1 + \rho^2 - 2\rho^4 - 2\rho^6 + \rho^8 \rho^{10} - 16\rho^5(a^2 - b^2) - 8\rho^5(a - b)^2 e^{-i5\phi} - 8\rho^5(a + b)^2 e^{i5\phi}) \quad (6)$$

with $a = 0.5$, $b = 0.4$ and $s = 0.65$.

B. Experimental setup

In Fig. 5, we report a more detailed experimental setup. An 808 nm diode laser is coupled to a single-mode fibre, and a Gaussian profile for the beam is generated, which is then sent to a folded-Sagnac interferometer with a Polarising Beam Splitter (PBS) as the input/output port. The vertically polarised beam is converted to a horizontally polarized one by a half-wave plate before reaching the right side of a liquid crystal Spatial Light Modulator (SLM). The SLM replaces the intermediate mirror of the interferometer and displays two holograms on each side, hence applying different transformations on the two beams. On the left side, the knot-generating hologram is applied, while the right side only shifts the phase of the incident beam. Both hologram functions are also superimposed with a blazed grating so that the modulated light is deflected to the first diffraction order. The reference beam is converted to vertical polarization by the same waveplate that acted on the knot hologram's input. In this way, the two beams will both exit the input port of the PBS. The interferometer is aligned to avoid overlap with the input beam and to ensure perfect collinearity of the two output beams. A pinhole placed in the center of the 4- f system selects

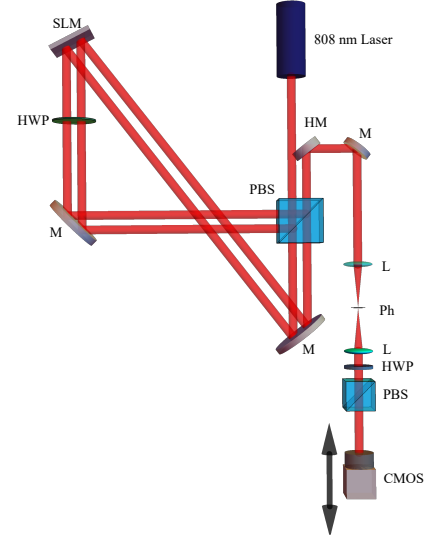


Figure 5. **Experimental setup** SLM: Spatial Light Modulator, M: Mirror, HWP: Half Wave-Plate, PBS: Polarising Beam-Splitter, L: Lens, HM: D-Shaped Mirror, Ph: Pinhole.

the first diffraction order of the SLM. The pinhole aperture and diffraction angle were different depending on the implemented structured beam to minimize the interference with the zeroth order and, at the same time, avoid cutting weak radial contributions necessary for the formation of the desired topology. After the 4- f system the field is $\tilde{\psi}(\rho, \phi, z)\mathbf{e}_H + e^{i\alpha}\psi_{\text{ref}}(\rho, z)\mathbf{e}_V$ with $\alpha = 0, \pi/2, \pi, 3\pi/2$, $\tilde{\psi}(\rho, \phi, z)$ the aberrated knot, $\psi_{\text{ref}}(\rho, z)$ the reference Gaussian beam. Projecting on a diagonal polarization (with a half-wave plate and a PBS) we obtained a scalar superposition between the structured beam and the reference. A CMOS camera placed on a motorized translation stage was used to record the intensity patterns of the resulting superposition for different relative phases, thus allowing for phase reconstruction based on phase-shifting digital holography (see next section).

C. Data analysis.

The experimental reconstruction of the singular skeleton was obtained by measuring, along the propagation direction z , the interference patterns $I_\alpha(z) = \left| \tilde{\psi}(\rho, \phi, z) + e^{i\alpha}\psi_{\text{ref}}(\rho, z) \right|^2$. From these measurements, the knotted field phase $\xi(\rho, \phi, z)$ (assuming a uniform phase for the reference beam) was retrieved as $\xi = \arg[I_0 - I_\pi + i(I_{\pi/2} - I_{3\pi/2})]$. The coordinates (x^*, y^*) of the phase singularities in each plane were extracted finding the intersections of the lines given by the equations $I_0(x, y) - I_\pi(x, y) = 0$ and $I_{\pi/2}(x, y) - I_{3\pi/2}(x, y) = 0$. The solution of these equations at different values of z yields a scatterplot of singular points which are then sorted and joined by a path minimization algorithm to reconstruct the singular skeleton.

Supplementary Information for: Effect of Aberration on 3D optical topologies

Section S1. THEORETICAL SIMULATIONS

To explore the wide parameter space and choose which case to look for experimentally, we simulated the propagation of the aberrated knotted fields using a Fresnel Transfer Function propagator. We define each field over a 2048×2048 array and a simulation window with a half-length of $24w_0$, where w_0 is the simulated beam's dimensionless waist parameter, and a numerical wavelength of $0.0134w_0$. We compute the propagation of the knotted beam given at $z = 0$ within the range $z = \pm z_0$, where $z_0 = \pi w_0^2/\lambda$ refers to the Rayleigh range.

Our experimental results, in some cases, show breakup also in situations where simulations predict the survival of the topological structure. Some discrepancies can be partially explained when considering interference from the zeroth diffraction order of the SLM, which, when applying stronger aberrations, can exhibit a small overlap with the first order. We thus compared the simulations in the ideal case with simulations considering the hologram implementation used in our experiment. We generate a hologram encoding the aberrated knotted field, using the scheme proposed by Ref. [31]. The SLM acts as a phase mask applied onto a Gaussian beam with a large waist (relative to the knotted beam) of $10w_0$. To simulate the effect of the pinhole placed in the far field of the diffracted beam, we applied a Fourier filter encoded as a Heaviside function $\theta(1 - f_r/\sigma)$, where $f_r^2 = f_x^2 + f_y^2$ is the radial component in Fourier space and σ is the pinhole aperture. The value of σ was determined experimentally by imaging the iris aperture when illuminated with a Gaussian beam set with waist w_0 on the SLM plane.

Fig. S1 illustrates an example which demonstrates the effect of simulating the interference of the zeroth order onto the theoretical simulations. When subjected to this aberration at the highest strength and with $A = 7w_0$, the trefoil knot is not observed to survive in the experiment (Fig. S1-c) due to one branch of the singular skeleton opening up. Simulations at $A = 7w_0$ predict, with and without SLM effects included, the existence of the trefoil structure, but with additional singularity lines approaching the knot. Decreasing the value of A to $6.6w_0$ we indeed observe that, if the zeroth diffraction order is taken into account, the knot breaks up as observed in the experiment (Fig. S1-b) while it does not in ideal conditions (Fig. S1-a). The reason we observe a residual discrepancy in the aperture values must be attributed to additional imperfections in the setup, e.g. deformations of the SLM window itself and imperfect centring of the hologram.

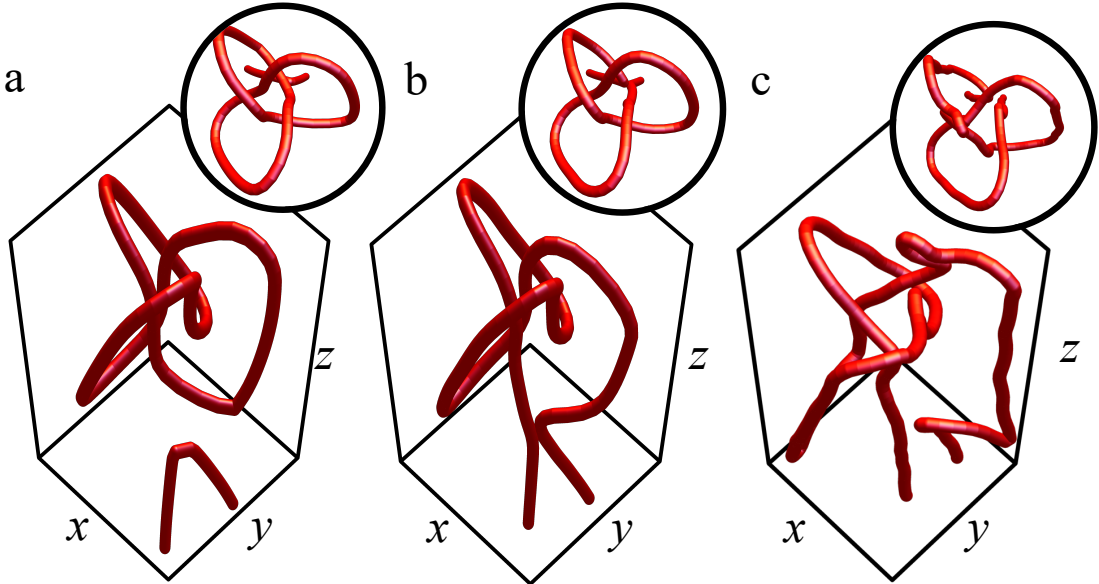


Figure S1. Simulating the effects of diffraction from the SLM. We show the simulation of a trefoil knot affected by secondary astigmatism for aperture $A = 6.6w_0$ without (a) and with (b) the effects of the zeroth diffraction order included. These results partially explain the breakup observed at $A = 7w_0$ in the experiment (panel c).

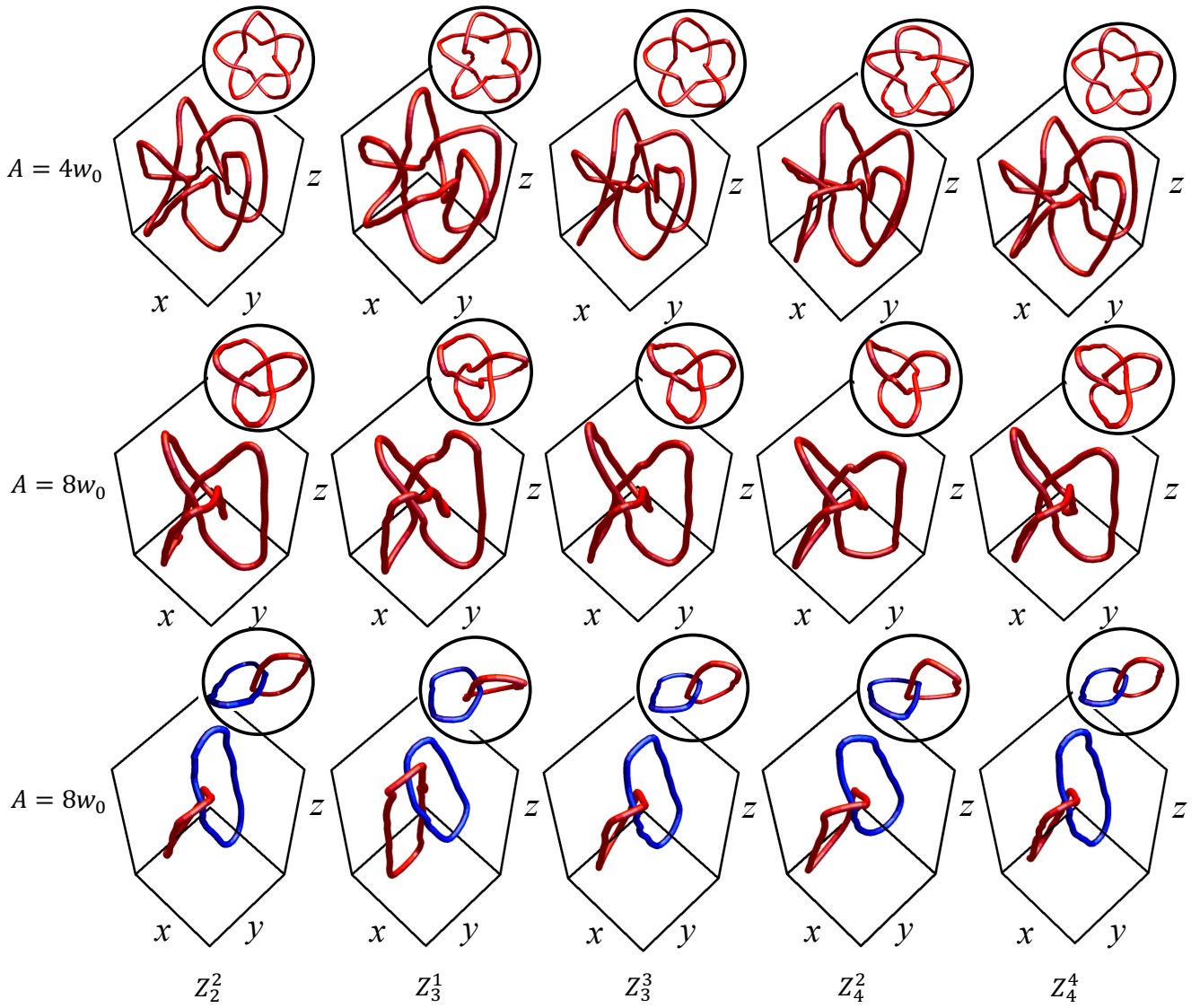


Figure S2. Aberrated Knots and links. Additional examples of topological singular skeleton affected by aberrations with $\gamma = 1$ and aperture A . The rows represent, from top to bottom, cinquefoils, trefoils and Hopf links, with top views as insets. From left to right, the different singular skeletons were affected by, respectively, astigmatism, coma, trefoil, secondary astigmatism, and quadrafoil aberrations.

Section S2. SUPPLEMENTARY DATA.

In Fig. S2, we show how for the apertures $A = 4w_0$ (for cinquefoil) and $A = 8w_0$ (for trefoil and Hopf link) the topology of the singular skeletons is not affected, up to $\gamma = 1$, by aberrations up to order 4.

Figure S3 shows that a displacement of $\Delta x = w_0$ of the window defining the wavefront distortion does not change the topology of cinquefoil knots.

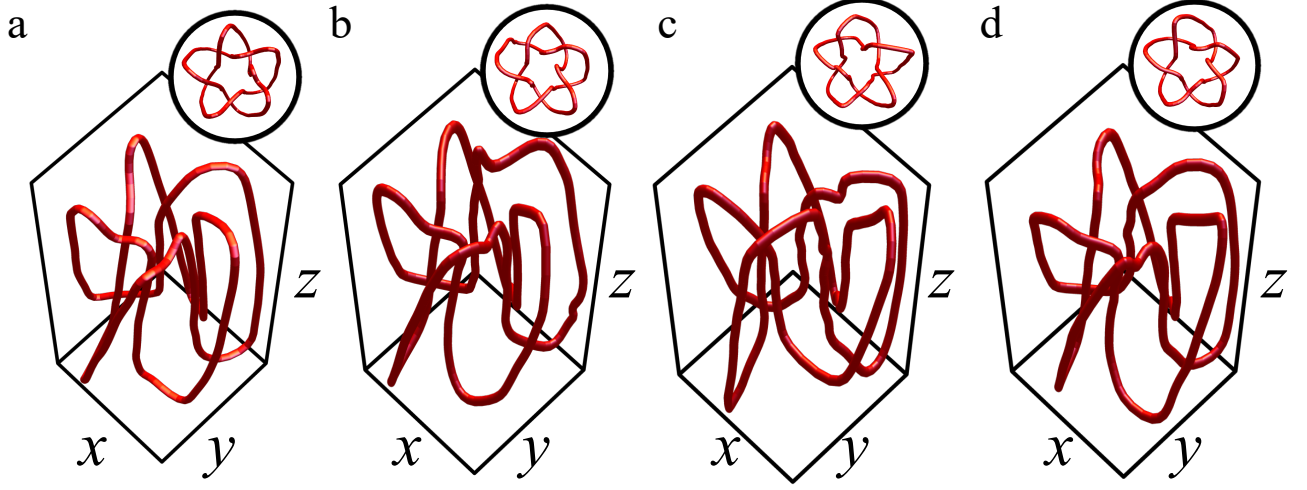


Figure S3. Cinquefoil knots under displaced aberrations. Experimentally reconstructed cinquefoil knots affected by different aberrations displaced by $\Delta x = w_0$ with defining aperture $A = 4w_0$, and strength $\gamma = 1$. The different panels correspond respectively to: a, astigmatism Z_2^2 , b, oblique trefoil Z_3^3 , c, quadrafoil Z_4^4 , and d, secondary astigmatism Z_4^2 .

2.5 Contributions

I, Nazanin Dehghan, together with Alessio D’Errico (A.D.) and Ebrahim Karimi (E.K.), conceived and devised the idea of the experiment. I, under the supervision of A.D., performed the experiment and analyzed the experimental data. I, together with Tareq Jaouni (T.J.), prepared the final version of the theoretical simulation code, did the simulations and analyzed them. I, together with A.D. and T.J., wrote the first draft of the manuscript and prepared the figures. We all discussed the results and the final version of the manuscript.

Chapter 3

Interferometric imaging of amplitude and phase of spatial biphoton states

This chapter is based on the following paper:

Zia, D., **Dehghan, N.**, D’Errico, A., Sciarrino, F., and Karimi, E. (2023). Interferometric imaging of amplitude and phase of spatial biphoton states. *Nature Photonics*, In press

DOI: 10.1038/s41566-023-01272-3

In this chapter, we discuss the second project in details. In this work, We proposed a new approach to reconstructing a bi-photon state. By exploiting well-known methods for reconstructing the phase and the amplitude of a field in the classical domain, we attempt to reconstruct the full bi-photon state using the interference between a reference bi-photon state and the unknown bi-photon state and coincidence imaging. We begin with exploring some important approximations for the theory of SPDC that plays a significant role in this project, the details of our detector for measuring the coincidence distributions and then a detailed manuscript about the work is provided.

3.1 SPDC in the Thin-Crystal Approximation

To begin with, let us continue a bit more on the theory of SPDC and look at the thin crystal approximation. However, it is worth mentioning that the method proposed in this work is not limited to only the bi-photon states in this approximation and can be used for any arbitrary bi-photon state.

In the last section of the introduction (see section 1.3) we showed how the state of the two entangled photons generated by a Type-I degenerate SPDC process can be derived as

$$|\psi\rangle_{\text{SPDC}} = \mathcal{N} \int d^2q_1 \int d^2q_2 \left[\text{sinc}\left(\frac{\Delta k_z l_z}{2}\right) \mathcal{E}_p(\mathbf{q}_1 + \mathbf{q}_2) \right] |1_{\mathbf{q}_1}\rangle |1_{\mathbf{q}_2}\rangle. \quad (3.1)$$

where $\mathcal{N}$ is a normalization factor and $\mathbf{k}_i = (\mathbf{q}_i, \frac{\omega_i}{cn})$. Now in this section, we will look at the state's properties in a well-known approximation called the thin crystal approximation.

One can show [31] that, for typical crystals, we can approximate the phase matching function as $\text{sinc}(\frac{\Delta k_z l_z}{2}) = \text{sinc}(\alpha |\mathbf{q}_1 - \mathbf{q}_2|^2 + \zeta)$ where $\alpha = \frac{Lc}{4\omega_3}$ (L is the length of the crystal) and ζ is the longitudinal mismatch which depends on the crystal orientation. Now let us call this phase matching function $\Phi(\mathbf{q}_1 - \mathbf{q}_2)$ since it's only dependent on the difference of the two transverse wavevectors.

We can expand this state in any orthogonal complete basis. If we have a continuous or a discrete basis, we can write them in terms of the transverse momentum,

$$|\xi\rangle = \int d\mathbf{q} \mathcal{U}(\xi, \mathbf{q}) |\mathbf{q}\rangle \quad (3.2)$$

$$|a, b\rangle = \int d\mathbf{q} f_{a,b}(\mathbf{q}) |\mathbf{q}\rangle \quad (3.3)$$

where $\mathcal{U}$ corresponds to a norm-preserving kernel, i.e. a unitary transformation, and a, b are the discrete indices. Now, let us choose the basis of the transverse position on the crystal plane and rewrite the SPDC state in this space. For the change of basis, we have,

$$|\boldsymbol{\rho}\rangle = \frac{1}{2\pi} \int d\mathbf{q} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} |\mathbf{q}\rangle \implies |\mathbf{q}\rangle = \int d\boldsymbol{\rho} e^{-i\mathbf{q}\cdot\boldsymbol{\rho}} |\boldsymbol{\rho}\rangle \quad (3.4)$$

considering the fact that the transverse position is the Fourier transform of the transverse momentum and vice versa. As a result of substituting these two equations in the SPDC state, we have

$$|\psi\rangle_{\text{SPDC}} = \mathcal{N} \iint \left[\iint \Phi(\mathbf{q}_1 - \mathbf{q}_2) \mathcal{E}_p(\mathbf{q}_1 + \mathbf{q}_2) e^{-i\mathbf{q}_1\cdot\boldsymbol{\rho}_1} e^{-i\mathbf{q}_2\cdot\boldsymbol{\rho}_2} d\mathbf{q}_1 d\mathbf{q}_2 \right] |\boldsymbol{\rho}_1\rangle |\boldsymbol{\rho}_2\rangle d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2. \quad (3.5)$$

Now by a straightforward change of variables, $\mathbf{Q} = \frac{\mathbf{q}_1 + \mathbf{q}_2}{2}$ and $\mathbf{S} = \frac{\mathbf{q}_1 - \mathbf{q}_2}{2}$, we get

$$|\psi\rangle_{\text{SPDC}} = \frac{\mathcal{N}}{4} \iint \left[\iint \Phi(2\mathbf{S}) \mathcal{E}_p(2\mathbf{Q}) e^{-i\mathbf{Q}\cdot(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)} e^{-i\mathbf{S}\cdot(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)} d\mathbf{Q} d\mathbf{S} \right] |\boldsymbol{\rho}_1\rangle |\boldsymbol{\rho}_2\rangle d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2. \quad (3.6)$$

Defining $\mathbf{R} = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}$ and $\boldsymbol{\Delta} = \frac{\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2}{2}$, we have

$$|\psi\rangle_{\text{SPDC}} = \mathcal{N} \iint \tilde{\Phi}(\boldsymbol{\Delta}) \tilde{\mathcal{E}}_p(\mathbf{R}) |\boldsymbol{\rho}_1\rangle |\boldsymbol{\rho}_2\rangle d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2. \quad (3.7)$$

In the thin crystal approximation, as it is mentioned in its name, we consider the crystal to be thin, e.g. $L \approx 1$ mm. Remembering the form of the phase-matching function $\text{sinc}(\alpha|\mathbf{q}_1 - \mathbf{q}_2|^2 + \zeta)$, if we approximate the sinc function with a Gaussian, the width of $\tilde{\Phi}(\boldsymbol{\Delta})$ is given by $\sqrt{\alpha}$ which is of the order of few μm , considering the pump photons to be in the Ultra-Violet (UV) range. This means we can approximate $\tilde{\Phi}(\boldsymbol{\Delta}) \approx \delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$. Thus, we have,

$$|\psi\rangle_{\text{SPDC}} = \mathcal{N} \int \tilde{\mathcal{E}}(\boldsymbol{\rho}) |\boldsymbol{\rho}\rangle |\boldsymbol{\rho}\rangle d\boldsymbol{\rho}. \quad (3.8)$$

This result states that one can neglect the propagation effects inside the crystal and regard the two photons as created always in the same plane and in the same transverse position. Moreover, the two-photon wavefunction is dictated by the pump amplitude and phase distribution on the crystal plane.

As shown in details in the manuscript, the strong position correlations in the thin crystal approximation, allow us to reconstruct the phase and amplitude of the wavefunction directly from the coincidence images captured by the camera.

3.2 Spatial Correlations in SPDC and Conservation Laws

As we discussed in the previous section, we can rewrite the SPDC state in any orthogonal complete basis. Now once we reconstruct the phase and the amplitude of the bi-photon wavefunction, one can be interested in retrieving different kinds of correlations from the state. In this section, we will be looking at this expansion in two orthogonal complete discrete sets of paraxial modes that we introduced in the first chapter (see section 1.3) LG modes and HG modes. The correlations of SPDC states in terms of these modes have been widely investigated [10, 20, 24, 30].

Starting from LG modes, we can write the change of basis as,

$$|\ell, p\rangle = \int d\boldsymbol{\rho} \langle \boldsymbol{\rho} | \ell, p \rangle |\boldsymbol{\rho}\rangle = \int d\boldsymbol{\rho} \text{LG}_{\ell,p}(\boldsymbol{\rho}) |\boldsymbol{\rho}\rangle \implies |\boldsymbol{\rho}\rangle = \sum_{\ell,p} \text{LG}_{\ell,p}^*(\boldsymbol{\rho}) |\ell, p\rangle \quad (3.9)$$

Now rewriting the state 3.8 in $\{|\ell, p\rangle\}$ basis, we have,

$$|\psi\rangle_{SPDC} = \mathcal{N} \int d\boldsymbol{\rho} \tilde{\mathcal{E}}_p(\boldsymbol{\rho}) \left[\sum_{\ell_1, p_1} \text{LG}_{\ell_1, p_1}^*(\boldsymbol{\rho}) |\ell_1, p_1\rangle \right] \otimes \left[\sum_{\ell_2, p_2} \text{LG}_{\ell_2, p_2}^*(\boldsymbol{\rho}) |\ell_2, p_2\rangle \right] \quad (3.10)$$

$$= \mathcal{N} \sum_{\substack{\ell_1, p_1 \\ \ell_2, p_2}} \left[\int d\boldsymbol{\rho} \tilde{\mathcal{E}}_p(\boldsymbol{\rho}) \text{LG}_{\ell_1, p_1}^*(\boldsymbol{\rho}) \text{LG}_{\ell_2, p_2}^*(\boldsymbol{\rho}) \right] |\ell_1, p_1\rangle |\ell_2, p_2\rangle. \quad (3.11)$$

Let us consider the expansion coefficients $\mathcal{C}_{p_1, p_2}^{\ell_1, \ell_2} := \int d\boldsymbol{\rho} \tilde{\mathcal{E}}_p(\boldsymbol{\rho}) \text{LG}_{\ell_1, p_1}^*(\boldsymbol{\rho}) \text{LG}_{\ell_2, p_2}^*(\boldsymbol{\rho})$. If the pump is an LG mode $\text{LG}_{\ell_3, p_3}(\boldsymbol{\rho})$ and if we use the notation of $\text{LG}_{\ell, p} = f_{\ell, p}(r) e^{i\ell\phi}$ we have,

$$\mathcal{C}_{p_1, p_2}^{\ell_1, \ell_2} = \int r dr f_{\ell_3, p_3}(r) f_{\ell_1, p_1}^*(r) f_{\ell_2, p_2}^*(r) \int_0^{2\pi} d\phi e^{i\ell_3\phi} e^{-i\ell_1\phi} e^{-i\ell_2\phi}. \quad (3.12)$$

The second integral has a familiar form of orthogonality of plane waves, and it is equal to $\delta_{\ell_3, \ell_1 + \ell_2}$ which leads to $\ell_3 = \ell_1 + \ell_2$ that implies the OAM conservation. Thus we can write

the final state as,

$$|\psi\rangle_{SPDC} = \sum_{\ell} C_{\ell} |\ell, \ell_3 - \ell\rangle. \quad (3.13)$$

This result has been used to generate and test high-dimensional Bell inequalities [6].

We can go through the same process but this time considering the HG modes as our orthogonal complete basis,

$$|m, n\rangle = \int d\boldsymbol{\rho} \langle \boldsymbol{\rho} | m, n \rangle | \boldsymbol{\rho} \rangle = \int d\boldsymbol{\rho} \text{HG}_{m,n}(\boldsymbol{\rho}) | \boldsymbol{\rho} \rangle \implies | \boldsymbol{\rho} \rangle = \sum_{m,n} \text{HG}_{m,n}^*(\boldsymbol{\rho}) | m, n \rangle, \quad (3.14)$$

leading to,

$$|\psi\rangle_{SPDC} = \sum_{\substack{m_1, n_1 \\ m_2, n_2}} \mathcal{D}_{m_1, m_2}^{n_1, n_2} |m_1, n_1\rangle |m_2, n_2\rangle \quad (3.15)$$

$$\text{where } \mathcal{D}_{m_1, m_2}^{n_1, n_2} = \int \tilde{\mathcal{E}}_p(x, y) \text{HG}_{m_1, n_1}^*(x, y) \text{HG}_{m_2, n_2}^*(x, y) dx dy = I_x \times I_y, \quad (3.16)$$

because for HG modes by looking at their functionality, we see that it is possible to separate the x, y terms and write $\text{HG}_{m,n} = N_{m,n} h_n(\alpha x) h_m(\alpha y) e^{-\frac{x^2}{w^2}} e^{-\frac{y^2}{w^2}}$. We considered the pump to be an HG mode as well so we can write $\tilde{\mathcal{E}}_p(x, y) = F_x(x) F_y(y)$. Consequently,

$$I_x = \int_{-\infty}^{\infty} F_x(x) h_{n_1}(x) h_{n_2}(x) e^{-\frac{x^2}{w^2}} e^{-\frac{x^2}{w^2}} dx = \begin{cases} 0, & \text{if } \pi_3 \neq \pi_1 + \pi_2 \\ 1, & \text{if } \pi_3 = \pi_1 + \pi_2 \end{cases}, \quad (3.17)$$

where π_1, π_2 are the parity of signal and idler, respectively, and π_3 is the parity of the pump field. The same results stands for y . In other words, if the x-dependent part of the pump is odd, there are only two possibilities of either the x-dependent part of the signal being odd and the x-dependent part of the idler even or vice versa. This property of the entangled two-photon state is called parity conservation.

We explore the different spatial correlations we discussed in this chapter using the experimentally reconstructed strongly correlated bi-photon state from the SPDC process later in the paper.

3.3 Coincidence Imaging and Timepix Camera

The Timepix family of cameras have provided us with the possibility of recording the position and the time of arrival of photons with a very high resolution and, therefore, the ability to do coincidence imaging that is used in the following paper. Here, we will be discussing the technology and the techniques behind these cameras in more details.

Timepix cameras use an important technique to store the information very efficiently, called the Data-Driven Readout. This approach is based on the fact that only the data of interest is saved. To implement this technique, the pixels of the camera have a predefined threshold, and only the pixels that are above the threshold perform a time measurement using a free-running 40MHz clock which does not need any external triggering [23].

To detect the photons, a so-called optical detection is performed, which in the case of single photons is performed by an intensifier. The intensifier consists of an off-the-shelf vacuum device with a photo-cathode, a Micro-Channel Plate (MPC) and a scintillator. The photo-cathode can be selected to satisfy the Quantum Efficiency that an experiment requires. In the MPC pores, an avalanche of electrons from the ones emitted by the photo-cathode is produced, with gains up to 10^5 , which allows the sensitivity to single photons. Finally, the scintillators produce light from these electrons, and the light is then sent to the pixels of the camera. Once the threshold of each pixel is passed, it records the time of arrival (TOA) of the photons with a time-resolution of 1.56 ns and the time over threshold of the pixel (TOT) with 25 ns granularity [23]. Therefore, the entire system of the Timepix camera provides us with information about which pixels at which time, with nanosecond resolution, received a photon.

For each detected photon, we can find the spatial distribution of all the time-correlated photons. In other words, we find all the photons that have an arrival time within a few nanoseconds. But, not only now have we the information about when these photons arrived, but also we can determine their position and, therefore, their spatial distribution.

In the case of entangled photons within the thin crystal approximation, we expect them to have sharp spatial correlations in the transverse position if the imaging is done in the image-plane of the crystal. This means if the coincidence imaging is done, we see a sharp line in the plot of x_i vs x_s and y_i and y_s . We should note that by selecting only the photons that have the position correlations or they are detected in the region around the sharp correlations and removing all the rest, we are able to remove a very good amount of background photons that introduce noise to our measurement. Therefore, the coincidence imaging allows us to improve our signal-to-noise ratio significantly. After selecting all the photons that are correlated both in x and y we are able to retrieve the spatial distribution

of the entangled photons.

In the following, we propose a new method to fully reconstruct an unknown two photon-state, using interference between a reference bi-photon state and the desired state and coincidence measurement with a Time-stamping camera. Using this interferometric approach, we reconstruct the strongly correlated two-photon state from the SPDC induced in a thin crystal. Compared to well-known projective measurements that are commonly used, our approach greatly reduces the required measurement time and is scalable in the sense that the accessible space dimensionality is limited only by the resolution and not by the measurement time. The main advantage of this method is the fact that we are able to reconstruct the phase and the amplitude of the bi-photon wavefunction, and therefore we can retrieve the spatial mode correlations, OAM conservation, and parity conservation. It is worth noting that this approach is not limited to the thin crystal approximation and can be generalized to any unknown bi-photon state, even without sharp correlations which is described in more details in the following manuscript.

Interferometric imaging of amplitude and phase of spatial biphoton states

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High-dimensional biphoton states are promising resources for quantum applications, ranging from high-dimensional quantum communications to quantum imaging. A pivotal task is fully characterising these states, which is generally time-consuming and not scalable when projective measurement approaches are adopted. However, new advances in coincidence imaging technologies allow for overcoming these limitations by parallelising multiple measurements. Here, we introduce biphoton digital holography, in analogy to off-axis digital holography, where coincidence imaging of the superposition of an unknown state with a reference one is used to perform quantum state tomography. We apply this approach to single photons emitted by spontaneous parametric down-conversion in a nonlinear crystal when the pump photons possess various quantum states. The proposed reconstruction technique allows for a more efficient (3 order-of-magnitude faster) and reliable (an average fidelity of 87%) characterisation of states in arbitrary spatial modes bases, compared with previously performed experiments. Multi-photon digital holography may pave the route toward efficient and accurate computational ghost imaging and high-dimensional quantum information processing.

I. INTRODUCTION

Photonic qudits are emerging as an essential resource for environment-resilient quantum key distribution, quantum simulation and quantum imaging and metrology [1]. The availability of unbounded photonic degrees of freedom, such as time-bins, temporal modes, orbital angular momentum and radial number [1], allows for encoding large amounts of information in fewer photons than would be required by qubit-based protocols (e.g. when using only polarization). At the same time, the large dimensionality of these states, such as those emerging from the generation of photon pairs, poses an intriguing challenge for what concerns their measurement. The number of projective measurements necessary for a full-state tomography scales quadratically with the dimensionality of the Hilbert space under consideration [2]. This issue can be tackled with adaptive tomographic approaches [3–5] or compressive techniques [6, 7], which are, however, constrained by *a priori* hypotheses on the quantum state under study. Moreover, quantum state tomography via projective measurement becomes challenging when the dimension of the quantum state is not a power of a prime number [8]. Here, we try to tackle the tomographic challenge, in the specific contest of spatially correlated biphoton states, looking for an interferometric approach inspired by digital holography [9–11], familiar in classical optics. We show that the coincidence imaging of the superposition of two biphoton states, one unknown and one used as a reference state, allows retrieving the spatial distribution of phase and amplitude of the unknown biphoton wavefunction. Coincidence imaging can be achieved with modern EMCCD cameras [12, 13], SPAD arrays [14–16] or time stamping cameras [17, 18]. These technologies are commonly exploited in quantum imaging, such as ghost imaging experiments [19] or quantum superresolution [20, 21], as well as for fundamental

applications, such as characterizing two-photon correlations [13, 22], imaging of high-dimensional Hong-Ou-Mandel interference [23–25], and visualization of the violation of Bell inequalities [26]. Holography techniques have been recently proposed in the context of quantum imaging [27–29]; demonstrating the phase-shifting digital holography in a coincidence imaging regime using polarization entanglement [27], and exploiting induced coherence, i.e. the reconstruction of phase objects through digital holography of undetected photons [28].

In this work, we focus on the specific problem of reconstructing the quantum state, in the transverse coordinate basis, of two photons emerging from degenerate Spontaneous Parametric Down Conversion (SPDC). These states are characterised by strong correlations in the transverse position (considered on the plane where the two-photon generation happens), which can be observed in other kinds of photon sources, such as cold atoms [30]. In these sources, the two-photon wavefunction strongly depends on the shape of the pump laser used to induce the down-conversion process [31]. The most used approach to reconstruct the biphoton state emitted by a nonlinear crystal in literature is based on projective techniques [32–34]. This method has drawbacks for what concern measurement times since it needs successive measurements on non-orthogonal bases and for the signal loss due to diffraction. We proposed an imaging-based procedure capable of overcoming both issues mentioned above, while giving the full state reconstruction of the unknown state. The core idea lies in assuming the SPDC state induced by a plane wave as known, and in superimposing this state with the unknown biphoton one. Unless the superposition is achieved directly on the crystal plane, a full analysis of the four dimensional distribution of coincidences is necessary to retrieve the interference between the two wavefunctions. This information can be visualised by observing coincidence images, defined as marginals of the coincidence distribution obtained integrating over the coordinates of one of the two photons. In fact, coincidence images obtained after post selecting specific spatial correlations, allow the retrieval of the phase information also in cases where the state does

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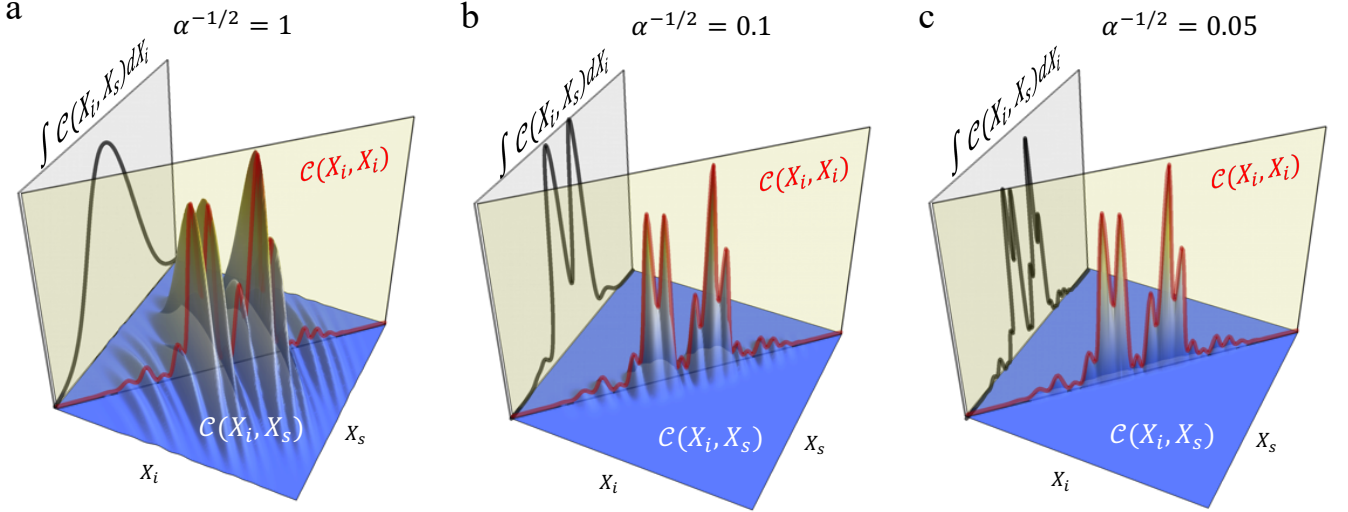


Figure 1. **Biphoton state holographic reconstruction.** Pictorial representations of Eq. (2) (in the simplified scenario of a two-dimensional space (X_i, X_s)) for different scenarios in which the biphoton wavefunctions have variable spatial correlations. To mimic typical SPDC states, we modelled the reference biphoton as a product $\exp(-\alpha(X_i - X_s)^2) \exp(-(X_i + X_s)^2/\sigma^2 + ik(X_i + X_s))$ and the unknown wavefunction as $\exp(-\alpha(X_i - X_s)^2) \exp(-(X_i + X_s)^2/\sigma^2) h_2((X_i + X_s)/\sigma_2)$, where $h_n(x)$ are Hermite polynomials. The parameter α quantifies how narrow are the diagonal correlations. We show, together with the 3D plot of $C(X_i, X_s)$, the marginal correlation $\int C(X_i, X_s) dX_i$ (which corresponds to the coincidence image obtained when no spatial post-selection is performed) and the section $C(X_i, X_i)$ which can be obtained post-selecting on diagonal correlations. We chose $\sigma = 1$, $\sigma_2 = 0.6$, $k = 2\pi/(0.2\sigma)$, and $\alpha = 1$ (panel a), $\alpha = 100$ (b), and $\alpha = 400$ (c). In each panel the correlation width $1/\sqrt{\alpha}$ is reported. We see that, in the strong correlation limit (panel c), interference is also retrieved in the marginal distribution.

not exhibit sharp spatial correlations. We demonstrate this technique for pump beams in different spatial modes, such as Laguerre-Gaussian (LG) and Hermite-Gaussian (HG) modes. We investigate several physical effects from the reconstructed states, such as orbital angular momentum conservation, the generation of high-dimensional Bell states, parity conservation, and radial correlations. Remarkably, we show how, from a simple measurement, one can retrieve information about two-photon states in arbitrary spatial mode bases without the efficiency and alignment issues that affect previously implemented projective characterisation techniques. Indeed, depending on the source brightness and on the required number of detection events, the measurement time can be of the order of tens of seconds, while the previously implemented projective techniques required several hours and were limited to the exploration of a small subspace of spatial modes. As a latter example, we give a proof of principle demonstration of the use of this technique for quantum imaging applications.

II. THEORY BACKGROUND

We start by considering a general scenario where the superposition between two biphoton states is created. We label each state as $|\psi_r\rangle$ and $|\psi_u\rangle$. The state $|\psi_r\rangle$ is considered as known (for example, it can emerge from a source that was previously characterised), while the goal is finding $|\psi_u\rangle$ from measurements performed on the superposition state $|\Psi_{\text{TOT}}\rangle = |\psi_r\rangle + |\psi_u\rangle$. To simplify the analysis, we con-

sider the case in which the photons are frequency-degenerate and with the same polarisation. The biphoton states can be thus decomposed in transverse spatial degrees of freedom (or, equivalently, in the transverse momentum) as:

$$|\psi_\eta\rangle = \iint d^2 X_i d^2 X_s \psi_\eta(\mathbf{X}_i, \mathbf{X}_s) |\mathbf{X}_i\rangle \otimes |\mathbf{X}_s\rangle, \quad (1)$$

where $\eta = r, u$, ψ_η is a complex function of the transverse coordinates of idler and signal photons $(\mathbf{X}_i, \mathbf{X}_s)$ at a given plane. The superposition state is

$$|\Psi_{\text{TOT}}\rangle = \iint d^2 X_i d^2 X_s [\psi_r(\mathbf{X}_i, \mathbf{X}_s) + \psi_u(\mathbf{X}_i, \mathbf{X}_s)] |\mathbf{X}_i\rangle \otimes |\mathbf{X}_s\rangle.$$

If ψ_r is known, one can retrieve information about the phase of ψ_u by coincidence measurements in the transverse position basis. The coincidence count rate corresponding to a simultaneous detection of an idler photon in $\mathbf{X}_1$ and a signal photon in $\mathbf{X}_2$ is proportional to

$$\begin{aligned} \mathcal{C}(\mathbf{X}_1, \mathbf{X}_2) &:= |\langle \mathbf{X}_1, \mathbf{X}_2 | \Psi_{\text{TOT}} \rangle|^2 \\ &= |\psi_r(\mathbf{X}_1, \mathbf{X}_2) + \psi_u(\mathbf{X}_1, \mathbf{X}_2)|^2 \\ &= |\psi_r(\mathbf{X}_1, \mathbf{X}_2)|^2 + |\psi_u(\mathbf{X}_1, \mathbf{X}_2)|^2 \\ &\quad + [\psi_r^*(\mathbf{X}_1, \mathbf{X}_2) \psi_u(\mathbf{X}_1, \mathbf{X}_2) + c.c.]. \end{aligned} \quad (2)$$

The last equality displays an interference term containing the phase difference between the reference and the unknown bipho-

ton wavefunction. We note that Eq. (2) represents an interference pattern in the four-dimensional space $(\mathbf{X}_1, \mathbf{X}_2)$. Figure 1-a gives a pictorial representation of Eq. (2) (here considering only one coordinate per photon) for a general case in which the reference and unknown wavefunction have a large spatial correlation spread. While registering $\mathcal{C}(X_i, X_s)$ is enough to obtain phase information, a further visualization of the experimental interference pattern can be obtained looking at filtered coincidence images. A coincidence image corresponds to the marginal distribution $\int \mathcal{C}(X_i, X_s) dX_a$ ($a = i$ or s) where the interference term is generically washed out. Interference in coincidence images can be, in general, retrieved by looking at sections of the correlation pattern, e.g. extracting coincidence images from the quantity $\mathcal{C}(X_i, X_s)\delta(X_s \pm X_i)$ – Fig. 1 shows the case $X_s = X_i$, given by the red plot. As we will demonstrate in the following section, this operation can be performed by analysing the measurements of a time-stamping camera. As shown below for the case of SPDC filtered coincidence images, besides being useful for data visualization, can allow to isolate different contributions of the biphoton state.

For the interference pattern to have a good contrast, the amplitudes of ψ_r and ψ_u have to be of a similar order. For instance, the phase of an unknown state with strong position correlations would be well resolved if the reference has the same spread of the spatial correlations, while the information hidden in the interference term would be more difficult to retrieve if using a reference state which is spatially uncorrelated or anti-correlated. Interestingly, all these forms of correlations can be observed, in different propagation planes, in the state created in Type I SPDC by a pump beam well approximated by a plane wave shining a thin crystal [31]. In fact, such a state exhibits sharp correlations in the near field (i.e. the image plane of the crystal) and sharp anti-correlations in the far field, while in intermediate propagation planes, one observes wider correlation patterns. Thus, one can, in principle, use this as a reference state for measuring any arbitrary two-photon state. The practical challenge would be then to find the right protocol for creating the superposition between the unknown and the reference state.

In this work, we consider the case in which also the unknown state is generated in an SPDC process. In this case, if the pump beams used for inducing the two SPDC states are in phase, the superposition can be generated either by mixing the biphoton superposition on a beam splitter (see Supplementary Section S4) or directly inducing the two SPDC processes in the same crystal.

The simplest case of the presented scheme arises when both reference and unknown states exhibit sharp position correlations, as is observed in the case of Type I degenerate SPDC from thin crystals. Fig. 1 b-c shows how the transition to a sharp correlation regime allows observing interference in the marginal distribution with no need for post-selection on the spatial correlations. In this widely studied limit which neglects propagation effects in the crystal volume, the two photons are created in the same transverse position which means that, in the transverse coordinates of the crystal's image plane, the state can be written as (see [31] and Supplementary Materials

Section S2)

$$|\Psi\rangle = \mathcal{N} \int d^2\rho \mathcal{E}_p(\rho) |\rho\rangle \otimes |\rho\rangle, \quad (3)$$

where $\mathcal{E}_p(\rho)$ is the pump field on the crystal plane, and $\mathcal{N}$ a normalisation factor. The spatial correlation properties of biphoton states in this approximation are of strong interest for applications in high dimensional quantum entanglement and quantum imaging [32, 34–36].

If the thin crystal approximation holds for both reference and unknown state, Eq. 2 contains relevant contributions only for the case $\mathcal{C}(\mathbf{X}_1, \mathbf{X}_1)$ with $\mathbf{X}_1 = \rho$. These diagonal contributions of the coincidence count rate are given by,

$$\mathcal{C}(\rho, \rho) = |\mathcal{E}_p(\rho) + \mathcal{E}_{\text{ref}}(\rho)|^2. \quad (4)$$

Here, $\mathcal{E}_{\text{ref}}$ is the pump shape used to generate the reference SPDC state. Ideally, $\mathcal{E}_{\text{ref}}$ can be a plane wave or, in practice, a Gaussian beam with a large waist. By controlling the reference pump beam one can map to the two-photon case any interferometric technique that is used in classical optics for amplitude and phase reconstruction. In this work, we experimentally implemented off-axis digital holography, where the reference beam is a Gaussian beam with a tilted wavefront. In off-axis digital holography, with $\mathcal{E}_{\text{ref}}(x, y) = A \exp(-(x^2 + y^2)/w_r^2) \exp(i2\pi(x + y)/\Lambda)$, one has

$$\begin{aligned} |\mathcal{E}_{\text{ref}} + \mathcal{E}_p|^2 &= |\mathcal{E}_{\text{ref}}|^2 + |\mathcal{E}_p|^2 \\ &+ 2Ae^{-\frac{r^2}{w_r^2}} (\mathcal{E}_p e^{-i2\pi\frac{(x+y)}{\Lambda}} + c.c.). \end{aligned}$$

From a spatial Fourier transform one can hence isolate the term proportional to $\mathcal{E}_p$ and reconstruct the amplitude and phase of the unknown field. The proposed scheme can be implemented in two measurement steps: first, the correlations in the crystal image plane are measured to confirm the validity of the thin crystal approximation, second, the coincidences corresponding to Eq. (4) are evaluated and the biphoton state extracted from the resulting interference pattern.

Beyond the thin crystal approximation, one has to reconstruct also the contribution of the phase matching function, which is in general a function of the form $\phi(\mathbf{X}_i - \mathbf{X}_s)$. The phase matching contribution can be retrieved either by analysing the far field SPDC intensity distribution, or, more rigorously, by post-selecting anti-diagonal correlations, i.e. analysing the coincidence image $\mathcal{C}(\mathbf{X}, -\mathbf{X})$.

III. EXPERIMENTAL SETUP AND RESULTS

Following the theoretical description of the previous section, we experimentally implemented a platform in which, through off-axis digital holography, the biphoton state, emitted via SPDC by a Type I crystal of β -Barium Borate (BBO), is reconstructed. In this proof of principle experiment, we generate the unknown and reference SPDC states in the same crystal. A visual scheme of the setup is reported in Fig. 2-a (see Methods and Supplementary Materials Section S5 for

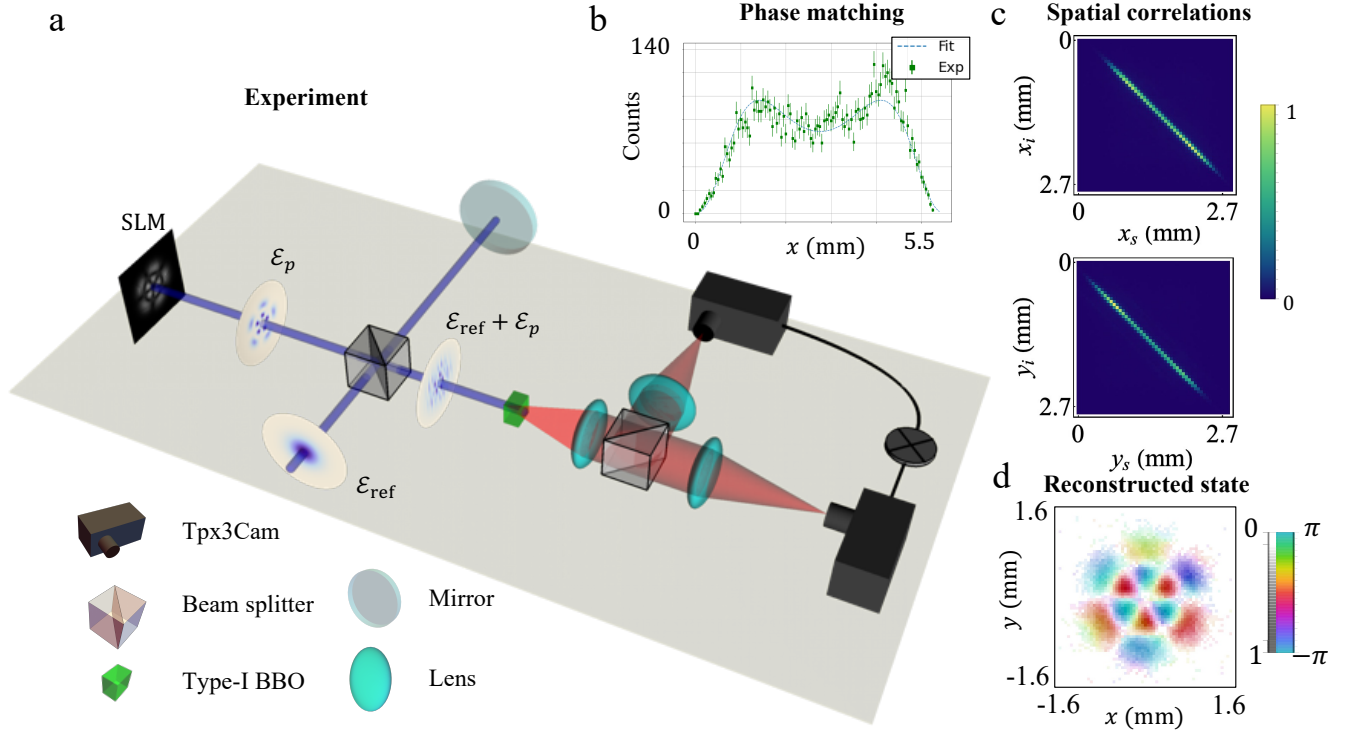


Figure 2. **Experimental setup and state reconstruction:** a) Sketch of the experimental setup: a 405 nm laser in a Gaussian mode ($\mathcal{E}_{ref}$) enters a Michelson interferometer, where an Ultraviolet Spatial Light Modulator (UV-SLM) in one arm is used to shape and generate the unknown pump field ($\mathcal{E}_p$). The interferometer's output is the superposition of the reference and unknown pump field, which is then shined on a 0.5-mm-thick Type-I BBO crystal. Photon pairs are consequently generated and, after being separated through the beam splitter, sent on single photon sensor arrays. The experiment was conducted with one camera and the figure is just for illustration, a more detailed description and figure of the setup can be found in the Supplementary Information. b) By placing the camera in the far field of the crystal and pumping with a large Gaussian beam, we can reconstruct the phase matching function $A \text{sinc}^2(\alpha x^2 - \zeta)$ by direct imaging. The figure shows a scan of the phase matching function with a nonlinear fit yielding $A = 93 \pm 2$ counts, $\alpha = (9.1 \pm 0.2) \times 10^{-6} \text{ mm}^2$ and $\zeta = 0.30 \pm 0.02$. Error bars are standard deviations extracted from the Poissonian counting statistics and assuming as mean values the registered counts (reported as data points). c) Experimental correlations in the x and y coordinates obtained by placing the sensors in the image plane of the crystal. d) Example of reconstructed phase and amplitude of a biphoton state (represented in inverted HSV colours) when pumping the crystal with a superposition of LG modes: $\text{LG}_{1,3} + \text{LG}_{1,-3}$.

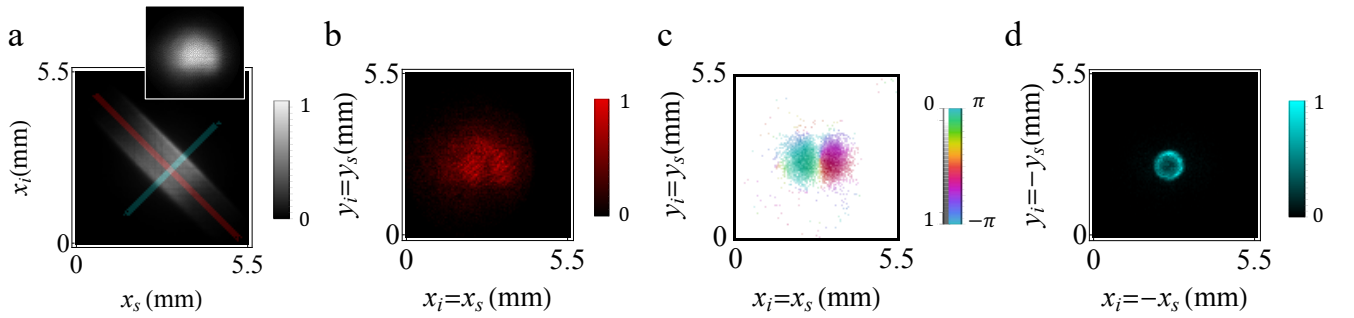


Figure 3. **SPDC state reconstruction at an intermediate plane.** When moving the camera 10 cm away from the crystal image plane, the spatial correlations are broader (as shown in a, for the case of x -correlations). The image obtained after postselecting only on temporal coincidence (shown in the inset) does not exhibit any interference as in the crystal image plane. The red and cyan bands in -a indicate the 2-pixel wide regions selected to analyse coincidence images obtained after selecting spatially correlated and anticorrelated photons, respectively. These images are shown, respectively, in panels b and d. The correlation image (b) shows again the interference unknown and reference pump fields, allowing us to reconstruct the unknown field with off-axis holography (panel c). In (d), the anti-correlation image is shown, displaying the characteristic cone shape of the phase-matching function. This function is not identical to the one obtained in Fig. 2-b, but can be obtained by propagating the amplitude of the latter from the far field to the intermediate plane.

details). For our purpose, we built a Michelson interferometer placing a Spatial Light Modulator (SLM) in one of the

two arms. This allows creating a pump beam in the mode $\mathcal{E}_p + \mathcal{E}_{\text{ref}}$, where the reference mode is a wide Gaussian with a tilted wavefront $\mathcal{E}_{\text{ref}} = \exp(-r^2/w_r^2) \exp(i2\pi(x+y)/\Lambda)$. The mean transverse momentum $2\pi/\Lambda$ is chosen to maximize the spatial resolution of the reconstructed field, and w_r is chosen to be larger than the characteristic waist parameter of $\mathcal{E}_p$, denoted as w_p . The interferometer's output is sent on the BBO crystal, and the two photons' state is recorded using a time-stamping camera (TPX3CAM). The camera is composed of a matrix of 256×256 time-stamping pixels of $55 \mu\text{m}$ size and with ≈ 1 ns time-resolution. We collected data in the crystal's image plane, used to reconstruct the bi-photon state in the thin-crystal approximation. The emitted signal and idler photons were separated by a beam splitter and sent into different regions of the camera sensor, allowing one to check for coincidences between different pixels. In Fig 2 a we represent, for simplicity, the two sensor areas as two independent time stamping cameras. By observing the spatial correlations, we could verify the correctness of the thin-crystal approximation. This is shown by the sharp, ≈ 1 -pixel wide, spatial correlations observed in all the cases under analysis; an example is reported in Fig. 2-c. Data were collected in 1 minute of exposure for each spatial mode under analysis. In particular, we collected both the interference pattern between the two states and a coincidence image of down-converted light induced by $\mathcal{E}_p$ only. The first one is used to retrieve the phase of the state under analysis, while the second one already gives the amplitude of the biphoton field. By exploiting this reconstruction, we were able to fully characterise the biphoton state. An example of the reconstructed phase and amplitude of the biphoton state is in Fig 2-d. Moreover, we also characterise the amplitude of the phase-matching function. To do so, we collect an image of the far field by placing an additional lens in front of the TPX3CAM camera and entering the crystal with a wide Gaussian beam. Fitting the collected data with the function $\text{sinc}^2(\alpha x^2 - \zeta)$, we obtained $\alpha = (9.1 \pm 0.2) \times 10^{-6} \text{ mm}^2$ and $\zeta = 0.30 \pm 0.02$. From α , we retrieve a value of the crystal length $L_{\text{exp}} = 4\omega_p\alpha/c = 0.56 \pm 0.01 \text{ mm}$ (c is the speed of light in the medium), which is in very good agreement with the nominal value $L_{\text{nom}} = 0.5 \text{ mm}$. The phase-matching fit is shown in figure Fig. 2-b. This additional measurement allows, in principle, to evaluate corrections to the quantum state beyond the thin crystal approximation. To show that our approach is not necessarily limited to sharply correlated photon states, we took a measurement moving the camera away from the image plane of the crystal. In this plane, the transverse spatial correlations are broader, and the two-photon wavefunction takes the form $\phi(|\mathbf{X}_i - \mathbf{X}_s|/2)\mathcal{E}((\mathbf{X}_i - \mathbf{X}_s)/2)$ (see Supplementary Materials Section S1 for details) where ϕ is related to the phase matching and $\mathcal{E}$ corresponds to the pump field propagated from the crystal to the measurement plane. As discussed in the previous section and illustrated in Fig. 1-a, in this case, the imaging of the singles or the coincidences, without any post-selection does not show any interference. this is illustrated in inset of Fig. 3 a. However, when we look at the coincidence image after selecting diagonal spatial correlations, we obtain the marginal $\int d^2 X_s \mathcal{C}(\mathbf{X}_i, \mathbf{X}_s) \delta(\mathbf{X}_s - \mathbf{X}_i) = |\mathcal{E}(\mathbf{X}_i)|^2 |\phi(0)|^2$. In our case $\mathcal{E} = \mathcal{E}_p + \mathcal{E}_{\text{ref}}$ so we see the interference be-

tween reference and unknown pump beam (see Fig. 3 b). From this pattern, one can reconstruct the pump field contribution to the SPDC state, as shown in Fig. 3 c. This reconstruction yields only the part of the biphoton state associated with the pump field. However, the phase matching can also be reconstructed from the same measurement by extracting $\int \mathcal{C}(\mathbf{X}_i, \mathbf{X}_s) \delta(\mathbf{X}_i + \mathbf{X}_s) d\mathbf{X}_s = |\mathcal{E}(0)|^2 |\phi(\mathbf{X}_i)|^2$ (see Fig. 3 b). We note that in our setup, the phase matching functions of reference and unknown state are identical, so we only have access to the absolute value of this contribution. It must be noted, however, that a setup where the two SPDC states are created from two independent sources would also allow accessing eventual phase structures in the phase-matching term (see Supplementary Material Section S4). Alternatively, one can extract the phase-matching contribution at different planes and infer its phase by a Gerchberg–Saxton algorithm [37]. We will investigate this non-interferometric approach in a successive work.

IV. RECONSTRUCTION OF SPATIAL MODE CORRELATIONS.

Once the bi-photon state is given, one can extract any desired information about this state, for example, correlations in different degrees of freedom, entanglement, and the decomposition in arbitrary sets of spatial modes.

One of the degrees of freedom of light, which has been extensively studied for high-dimensional quantum applications, is the Orbital Angular Momentum (OAM). Modes in OAM eigenstates are represented by wavefunctions possessing a phase term of the form $\exp(i\ell\phi)$ in position representation. Here, ϕ is the azimuthal angle in cylindrical coordinates, and $\ell \in \mathbb{Z}$ is the OAM value (along the propagation direction) in units of $\hbar$ carried by a photon in such a state. Given a set of OAM carrying modes $\langle r, \phi | p, \ell \rangle := f_p(r) \exp(i\ell\phi)$, where $f_p(r)$ denotes an orthonormal set of radial functions, the probability of detecting the idler photon with OAM ℓ_i and the signal with OAM ℓ_s is (for fixed p_i and p_s)

$$P_{\ell_i, \ell_s}^{p_i, p_s} = \left| \iint \mathcal{E}_p(r, \phi) f_{p_i}^*(r) f_{p_s}^*(r) e^{-i(\ell_i + \ell_s)\phi} r dr d\phi \right|^2. \quad (5)$$

In particular, when the pump beam carries an OAM equal to ℓ_p one has the OAM conservation law:

$$\ell_p = \ell_i + \ell_s, \quad (6)$$

which can be immediately deduced from (5) [31] and was first demonstrated in Ref. [32]. We investigated this relationship for several OAM values by entering the crystal with LG modes [38]. These are a set of modes defined as: $LG_{p, \ell}(r, \phi) := \langle r, \phi | p, \ell \rangle \propto (r/w)^{|\ell|} L_p^{|\ell|}(2r^2/w^2) \exp(-(r/w)^2) \exp(i\ell\phi)$, where $L_p^{|\ell|}(x)$ are associated Laguerre polynomials. They are cylindrically symmetric modes carrying OAM and with minimal divergence in free space [39]. We analyzed the case in which we entered the crystal with states having azimuthal index $\ell \in \{0, 1, 2, 3, 4\}$, the results are reported in Fig. 4. We show

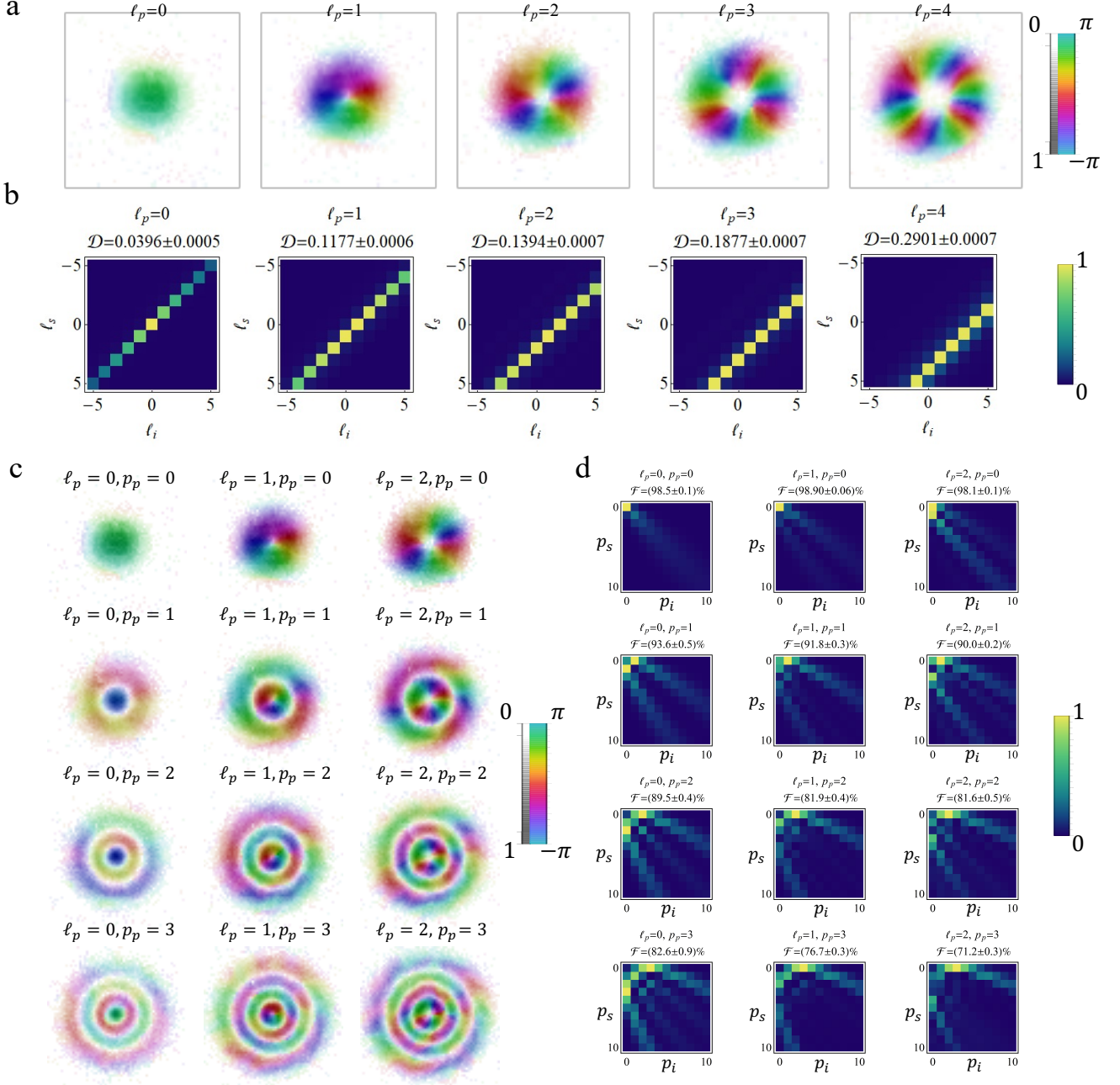


Figure 4. OAM and radial correlations in SPDC biphotons. Reconstructed field of the biphoton state for different OAM-carrying pump beams in $\text{LG}_{p_p=0, \ell_p}$ modes. **(a)** Shows the amplitude and phase of the state for different values of the pump OAM ℓ_p . **(b)** Shows the OAM correlations density plots of generated SPDC photons. It can be seen how increasing the pump OAM, the sum of the OAM values for the idler and signal photons shift in agreement with the conservation law of Eq. (6). **(c)** reconstructed biphoton fields obtained by pumping the crystal with LG modes changing the ℓ and p indexes. **(d)** Shows the correlations in the radial number p . In all the plots, the OAM of signal and idler has been fixed to $\ell_i = 0, \ell_s = \ell_p$. The Fidelities are obtained assuming the theoretical state calculated in the thin crystal approximation. The error analysis is reported in the Methods.

the OAM detection probabilities without specifying the radial function, calculated as $P_{\ell_i, \ell_s} = \sum_{p_i, p_s} P_{\ell_i, \ell_s}^{p_i, p_s}$, where we restricted the sums over the radial indexes for $p_{i,s} = 0, \dots, 10$. It is evident how increasing the OAM carried by the pump causes the OAM correlations to shift in agreement with Eq. (6)

(as also observed in, e.g. [32]). It has been observed that the SPDC state in the OAM basis violates high dimensional Bell inequalities [33, 40]. In Fig. 4-b, we also report the trace distances between the theoretically calculated ($P_{\ell_i, \ell_s}^{\text{th}}$) and reconstructed (P_{ℓ_i, ℓ_s}) probability distributions, defined

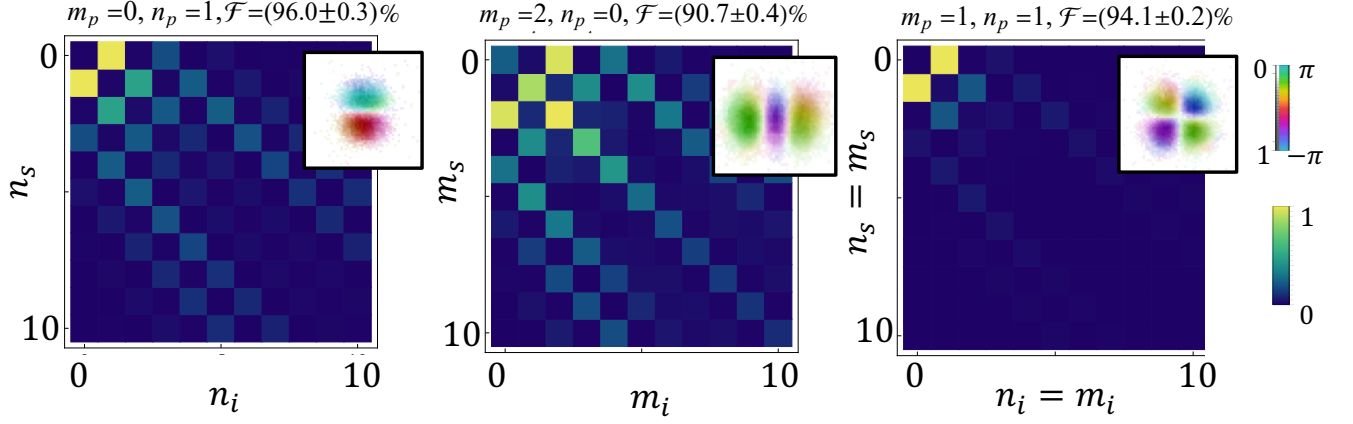


Figure 5. **Hermite-Gauss correlations.** When pumping the crystal with $\mathcal{E}_p(x, y) = \text{HG}_{m,n}(x, y)$ we observe biphoton correlations in the basis of HG modes which highlight the parity conservation of the SPDC process. Insets show the reconstructed biphoton fields from which the correlations have been extracted. Upon each plot, the Fidelity between the retrieved field and the theoretical one in the thin crystal approximation is reported. The error analysis is discussed in the Methods.

as $\mathcal{D} = \sum_{\ell_i, \ell_s} |P_{\ell_i, \ell_s} - P_{\ell_i, \ell_s}^{th}|/2$. The values of $\mathcal{D}$ increase with a higher ℓ_p mainly due to imperfections in the pump preparations (hence the pump beam is better described as a superposition of OAM states). The distributions P_{ℓ_i, ℓ_s}^{th} are shown in the Supplementary Figure S4.

The radial index $p \in \mathbb{N}$ of LG modes corresponds to the number of radial zeros and can be treated as a quantum number [41, 42]. In Fig. 4-c, experimental results of biphoton states for pump beams, prepared as LG modes, are shown. From the reconstructed states, the coefficients of the SPDC state decomposition in LG modes $|\Psi\rangle = \sum_{p_i, \ell_i} C_{p_i, \ell_i}^{p_s, \ell_s} |p_i, \ell_i\rangle \otimes |p_s, \ell_s\rangle$ were extracted. Figure 4-d shows experimental correlations in radial indexes (with OAM indexes of idler and signal fixed as $\ell_i = 0, \ell_s = \ell_p$). When choosing the waist parameter of the decomposition to be equal to the pump waist, the correlations are maximized for $p_{i,s} = p_p, p_{s,i} = 0$. This can be understood from the similarity between the integral expression of $C_{p_i, \ell_i}^{p_s, \ell_s}$ and the orthogonality relationship of LG modes. Within the considered subspaces (where indices $p_{i,s}$ are bounded from 0 to 10 and $\ell_i = 0, \ell_s = \ell_p$), the Fidelity $\mathcal{F} = |\sum_{p_i} C_{p_i, 0}^{*p_s, \ell_p} C_{p_i, 0}^{thp_s, \ell_p}|^2$ was evaluated, where $C_{p_i, 0}^{*p_s, \ell_p}$ are the measured coefficients and $C_{p_i, 0}^{thp_s, \ell_p}$ are the ones expected from the thin crystal approximation, both of the coefficients are obtained after normalizing the state in the reduced subspace.

In this approximation, the spatial modes' parity is also conserved [35]. This effect can be highlighted by considering pump beams in HG modes [38], which, on the crystal plane, read $\text{HG}_{m,n}(x, y) := \langle x, y | m_p, n_p \rangle \propto \exp(-(x^2 + y^2)/w_p^2) h_m(x/w_p) h_n(y/w_p)$, where $h_m(x)$ are Hermite polynomials of order m . HG modes form a complete, orthonormal set with even or odd functions along the x or y directions. Thus, the SPDC state can be decomposed as $|\Psi\rangle = \sum_{m_i, n_i} K_{m_i, n_i}^{m_s, n_s} |m_i, n_i\rangle \otimes |m_s, n_s\rangle$. This basis has been extensively studied [35, 36, 43] and recently considered for biphoton super-resolution measurements [44]. When

studying the SPDC correlations on the basis of HG modes (chosen with the same waist parameter $w = w_p$ of the pump), one has the conservation laws: $n_p = \text{mod}(n_i + n_s, 2)$ and $m_p = \text{mod}(m_i + m_s, 2)$, as can be directly inferred from the parity of the integrands appearing in the expression of $K_{m_i, n_i}^{m_s, n_s}$ (see Methods and Ref. [35] for a detailed proof). Figure 5 shows calculated HG mode correlations for different states (with reconstructed biphoton amplitude and phase shown in the insets). The results show an excellent agreement with the theory; in particular, the parity conservation is evident from the chessboard-like correlation patterns.

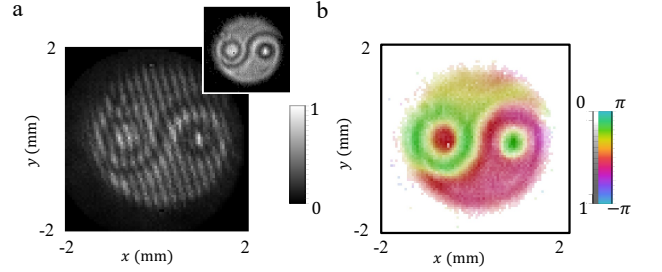


Figure 6. **Image reconstruction.** (a) Coincidence image of interference between a reference SPDC state and a state obtained by a pump beam with the shape of a Ying and Yang symbol (shown in the inset). The inset scale is the same as in the main plot. (b) Reconstructed amplitude and phase structure of the image imprinted on the “unknown” pump.

Finally, figure 6 shows an example of the potential applications of biphoton digital holography. The unknown pump beam can carry information about an image or be scattered by a three-dimensional object. The information about the scatterer is transferred to the SPDC state and can be retrieved through our technique (Fig. 6-b). We show this in the case of off-axis holography, which can present limitations for complex structures due to the limited camera resolution. These limitations

are not related to our proposal and can be improved by employing other approaches, e.g. on-axis phase-shifting digital holography [9].

V. CONCLUSION

In this work, we introduced a novel approach for reconstructing the spatial structure of correlated two photons states. Our proposal exploits the coherent superposition of two SPDC states and the possibility of imaging the amplitude of this superposition with a time-stamping camera. The experimental results showed how, from a single measurement, it is possible to retrieve, in postprocessing, a large amount of information about a two-photon spatial state, e.g. correlations in different degrees of freedom, entanglement and spatial mode decomposition in arbitrary bases. We focused on the simple case of SPDC generated from thin crystals, and, for different pump fields, analysed OAM and parity conservation, high dimensional entangled states, and radial mode correlations. The results show the superiority of this technique, compared to projective techniques (e.g. the ones in refs. [34, 42, 45]), in the context of benchmarking highly correlated quantum states. Indeed, we observe that, if a projective measurement was performed on a 10×10 subspace, as the ones considered in this work for radial modes, the time required to accumulate the necessary statistics on 100^2 projections would have required several days, due to the low count rates associated with the lossy techniques used

for mode projection. In comparison, our approach allows us to obtain the necessary data in a few minutes, independently of the dimensionality of the subspace to be analysed (the latter is only limited by the camera resolution). We also note that, with the time stamping camera, when spatial correlations are present, a further background subtraction is possible thus improving the signal-to-noise ratio of this kind of measurement. We thus achieved a 3 order of magnitude enhancement on the reconstruction time with high Fidelities for the biphoton states, obtaining an average Fidelity equal to 87%. The lowest Fidelity values are due to imperfect pump preparations or an undesired spatially varying phase in the reference beam and not to intrinsic limitations of the technique. Although most of our results are based on a specific kind of two-photon state, we also showed how it is possible to generalise to states where the correlations are not sharp. A key ingredient is to generate reference states with spatial correlations that overlap well with the correlations in the unknown state. Future investigations will be devoted to the generalisation of this approach to arbitrary two and multi-photon states, realising experiments where the reference and unknown states are generated from distant sources. Moreover, we point out that the same technique presented here can be applied to measure biphoton states in the time-frequency degrees of freedom, where several interferometric approaches have been demonstrated [46–48]. Besides the quantum state reconstruction, future investigations will be devoted to the generalisation of our protocol to imaging experiments.

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Author Contributions A.D conceived the idea. D.Z., N.D., and A.D. devised the experimental setup. D.Z., N.D., and A.D. performed the experiment. D.Z. analyzed the data. N.D. analyzed the spatial mode correlations. E.K. and F.S. supervised the project. D.Z., N.D., and A.D. prepared the first version of the manuscript. All authors contributed to the writing of the manuscript.

Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Code availability The code used for the data analysis is available from the corresponding author upon reasonable request.

Ethics declarations Competing Interests. The authors declare no competing interests.

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METHODS

A. Detailed experimental setup.

A Gaussian beam with a wavelength of 405 nm is produced through the second harmonic generation of a 810 nm pulsed Ti:Sa laser (Chameleon Vision II), the latter has a pulse duration of 150 fs and a repetition rate of 80 MHz. The beam is magnified to ≈ 1 cm beam waist and sent in the input of a Michelson interferometer. A reflective liquid crystal spatial light modulator is placed in one arm of the interferometer. At the interferometer output, all the diffraction orders of the SLM, except for the first, are filtered out by a slit placed in the SLM's Fourier plane. The slit allows keeping the reference beam (corresponding to the beam going through the interferometer arm without the SLM) with a different transverse wavevector, thus allowing to perform off-axis digital holography and, at the same time, keeping good interference stability. The phase masks applied on the SLM allows the generation of arbitrary optical fields by means of the technique introduced in Ref. [49]. After filtering, both beams (each with the power of the order of 100 mW) are collimated and sent on the 0.5 mm thick type-I BBO crystal for SPDC generation. The vertically polarized down-converted light is collimated by a lens of focal length $f = 25$ cm, split in two copied by a sequence consisting of a half-wave plate rotated by 22.5° and a polarizing beam splitter (effectively working as an ordinary beam splitter). The two copies are sent on parallel paths (by sending them on another polarizing beam splitter and changing the polarisations in such a way as to maximize the intensity on one output port). The two beams have a lateral shift, that avoids them from being overlapped, and are focused, by means of a $f = 50$ cm lens in the thin crystal approximation configuration and by a $f = 75$ cm lens for the reconstruction of the state in an intermediate plane, on the TPX3CAM sensor. In front of the sensor, a 3 nm bandpass filter is applied to ensure the frequency degeneracy of the analyzed photons. The bandpass filter also ensures a sufficient contrast of the interference fringes. In the far field configuration, used to reconstruct the phase-matching function, an additional $f = 20$ cm lens is placed in front of the camera in a confocal configuration.

B. Data acquisition and analysis.

Data were acquired by collecting SPDC light on the TPX3CAM for 1 minute for each data set. We collected data for cases where both the reference and pump beam were sent on the BBO crystal and cases where only the pump beam was sent on the crystal, after blocking the reference arm. The acquired data files report the timestamp at which counts were detected (for more information see Refs. [17, 24, 25, 50]). In our case, since we shine two copies of the SPDC light on different regions on the camera, we can have counts from these regions detected in the same time window. We considered as coincidences the counts from the two regions with a timestamp difference of 5 ns. From this set of counts, we analyzed the spatial correlations, confirming the validity of the thin crystal

approximation. A weak constant background in the correlation plot is always present (due to dark counts and background light) and can be reduced by removing the counts outside the correlation region. Coincidence images were obtained by plotting the positions of the counts selected as coincidences.

The resulting coincidence images were analyzed using standard off-axis digital holography analysis [9], as described in the main text.

The analysis of the reconstructed states in terms of OAM, HG, and LG modes has been conducted by direct calculations of the expansion coefficients in the respective bases. The errors on the Fidelities have been obtained by repeating the analysis for different state reconstructions, where the original coincidence images were modified pixel by pixel by random amounts within the uncertainty, given by the square root of the coincidences assuming Poissonian statistics. In the main text, we report the average Fidelity and the standard deviation over 20 different realizations.

C. Parity conservation.

The coefficients $K_{m_i, m_i}^{m_s, n_s}$ in the HG expansion of the SPDC state are given by the integral:

$$K_{m_i, m_i}^{m_s, n_s} := (\langle m_i, n_i | \otimes \langle m_s, n_s |) |\Psi\rangle = \mathcal{N} \iint \mathcal{E}(x, y) e^{-2(\frac{x}{w_p})^2} h_{m_i} \left(\frac{x}{w_p} \right) h_{m_s} \left(\frac{x}{w_p} \right) \times e^{-2(\frac{y}{w_p})^2} h_{n_i} \left(\frac{y}{w_p} \right) h_{n_s} \left(\frac{y}{w_p} \right) dx dy.$$

We consider the case in which $\mathcal{E}(x, y) = F_x(x) F_y(y)$ where F_x and F_y are even or odd functions of x and y , respectively (which is the case if the pump is in a HG mode). We have $C_{m_i, m_i}^{m_s, n_s} = I_{F_x} \times I_{F_y}$, where

$$I_{F_\xi} := \sqrt{\mathcal{N}} \int_{-\infty}^{\infty} e^{-2(\frac{\xi}{w_p})^2} F_\xi(\xi) h_{l_i} \left(\frac{\xi}{w_p} \right) h_{l_s} \left(\frac{\xi}{w_p} \right) d\xi$$

with $l = m, n$ for $\xi = x, y$, respectively. The product $h_{l_i} \left(\frac{\xi}{w_p} \right) h_{l_s} \left(\frac{\xi}{w_p} \right)$ is even/odd if $l_i + l_s$ is even/odd. Thus, the integral is zero if the parity of F_ξ is different than the parity of $l_i + l_s$, hence the conservation law mentioned in the main text.

Supplementary Information for: Interferometric imaging of amplitude and phase of spatial biphoton states

Section S1. FREE SPACE PROPAGATION OF THE SPDC BIPHOTON WAVEFUNCTION

We consider the photon pair production in degenerate SPDC. In this process, a second-order nonlinear crystal, pumped by a laser beam with frequency ω_p and with a spatial amplitude $\mathcal{E}_p(x, y, z)$, produces (within a first-order approximation) photon pairs with frequencies ω_i and ω_s such that $\omega_p = \omega_i + \omega_s$, where the subscripts s and i denote the *signal* and *idler* photons, respectively. In the degenerate case, $\omega_i = \omega_s$, the two-photon state, written in the basis of transverse wavevector modes $|\mathbf{q}\rangle := |k_x, k_y\rangle$, is given by [31]:

$$|\Psi\rangle = \mathcal{N} \iint E_p(\mathbf{q}_i + \mathbf{q}_s) \text{sinc}(\alpha|\mathbf{q}_i - \mathbf{q}_s|^2 - \zeta) |\mathbf{q}_i\rangle \otimes |\mathbf{q}_s\rangle d^2\mathbf{q}_i d^2\mathbf{q}_s, \quad (\text{S1})$$

where $\mathcal{N}$ is a normalization factor, $\alpha = Lc/4\omega_p$ (L is the crystal length and c is the speed of light in the medium), E_p is the 2D Fourier transform of the pump mode on the crystal plane, $E_p(\mathbf{q}) = \int \mathcal{E}_p(\boldsymbol{\rho}, z=0) e^{i\boldsymbol{\rho}\cdot\mathbf{q}} d^2\boldsymbol{\rho}$, and ζ is the longitudinal mismatch which depends on the crystal orientation. The contribution of the sinc function is related to the phase matching of the SPDC process $\phi(|\mathbf{q}_i - \mathbf{q}_s|) = \text{sinc}(\alpha|\mathbf{q}_i - \mathbf{q}_s|^2 - \zeta)$. We recall that the transverse momentum coordinates are equivalent, apart from factors depending on the imaging system, to the far field coordinates with respect to the crystal image plane. Thus, if $\mathbf{X}$ are the coordinates on the crystal's Fourier plane, the SPDC, in the $\mathbf{X}$ coordinate basis, takes the form

$$|\Psi\rangle = \mathcal{N} \iint E_p(\mathbf{X}_i + \mathbf{X}_s) \phi(|\mathbf{X}_i - \mathbf{X}_s|) |\mathbf{X}_i\rangle \otimes |\mathbf{X}_s\rangle d^2\mathbf{X}_i d^2\mathbf{X}_s. \quad (\text{S2})$$

The free space evolution of the biphoton state can be evaluated by performing a change of basis from the Fourier plane's coordinates eigenstates, to the coordinates of the plane of interest:

$$|\mathbf{X}'\rangle := \int F(\mathbf{X}', \mathbf{X}; z) |\mathbf{X}\rangle d^2\mathbf{X}, \quad (\text{S3})$$

where F is the free space propagator and z is the distance between the two planes. In the paraxial approximation, F is given by the Fresnel propagator,

$$F(\mathbf{X}', \mathbf{X}; z) \propto \exp(ikz) \exp\left(-i\frac{\pi}{\lambda z}(\mathbf{X}' - \mathbf{X})^2\right). \quad (\text{S4})$$

Performing the change of basis, one gets

$$|\Psi\rangle = \mathcal{N} \iint E'_p(\mathbf{X}'_i + \mathbf{X}'_s, z) \phi'(\mathbf{X}'_i - \mathbf{X}'_s, z) |\mathbf{X}'_i\rangle \otimes |\mathbf{X}'_s\rangle d^2\mathbf{X}'_i d^2\mathbf{X}'_s, \quad (\text{S5})$$

with

$$E'_p(\mathbf{R}', z) := \int F(\mathbf{R}', \mathbf{R}; -z) E_p(\mathbf{R}) d^2\mathbf{R}, \quad (\text{S6})$$

$$\phi'(\mathbf{D}', z) := \int F(\mathbf{D}', \mathbf{D}; -z) \phi(\mathbf{D}) d^2\mathbf{D} \quad (\text{S7})$$

where we defined $\mathbf{R} := (\mathbf{X}_i + \mathbf{X}_s)/2$, $\mathbf{D} := (\mathbf{X}_i - \mathbf{X}_s)/2$, and similarly for $\mathbf{R}'$ and $\mathbf{D}'$. This means that, when postselecting coincidences on correlations (anticorrelations) in a generic plane, one can retrieve the propagated pump field (phase matching amplitude).

Section S2. THIN CRYSTAL APPROXIMATION

In the particular case in which one considers the propagation from the far field to the crystal image plane, the Fresnel transform is replaced by a Fourier transform. The state on the crystal plane is thus

$$|\Psi\rangle = \mathcal{N} \iint \mathcal{E}_p(\boldsymbol{\rho}_i + \boldsymbol{\rho}_s) \Phi(\boldsymbol{\rho}_i - \boldsymbol{\rho}_s) |\boldsymbol{\rho}_i\rangle \otimes |\boldsymbol{\rho}_s\rangle d^2\boldsymbol{\rho}_i d^2\boldsymbol{\rho}_s, \quad (\text{S8})$$

where $\mathcal{E}_p$ and Φ are 2D Fourier transforms of E_p and ϕ , respectively. the thin crystal approximation is performed by noting that the characteristic width of Φ (which can be estimated easily by approximating the phase matching function with a Gaussian [36]) is of the order of $\sqrt{\alpha}$. For a thin crystal, this corresponds to a few wavelengths (in our case $\approx 3 \mu\text{m}$). The thin crystal approximation consists in replacing Φ with a delta function, thus yielding Eq. (3).

Section S3. GENERALISED SCHEME FOR BIPHOTON STATE IMAGING

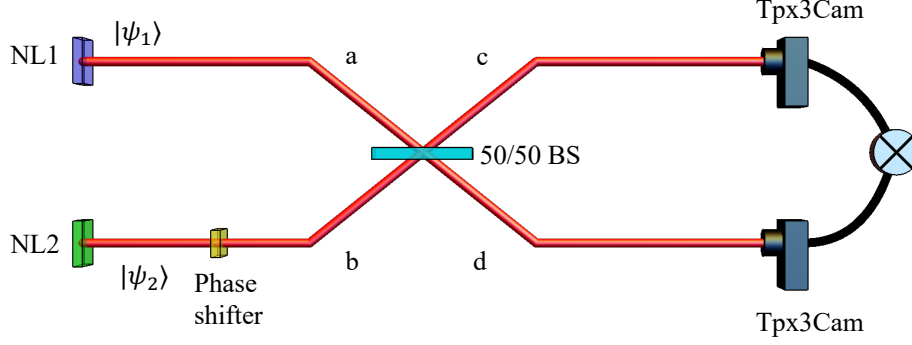


Figure S1. Generalised setup for biphoton state imaging

In our experiment, the unknown and reference state was created in the same crystal. Here we show that, in a more general scenario, where the two states are created from spatially separated sources, our proposed technique can be still implemented when combining the two states on a Beam Splitter and then performing coincidence measurements. This scheme is illustrated in Fig. S1. We denote with $(\hat{a}_{\mathbf{X}}, \hat{a}_{\mathbf{X}}^\dagger)$ the annihilation and creation operator of photons in the mode $|\mathbf{X}\rangle$ entering the port a of a 50/50 beam splitter. Similarly we denote with $(\hat{b}_{\mathbf{X}}, \hat{b}_{\mathbf{X}}^\dagger)$ the operators for photons entering the port b , while $(\hat{c}_{\mathbf{X}}, \hat{c}_{\mathbf{X}}^\dagger)$ and $(\hat{d}_{\mathbf{X}}, \hat{d}_{\mathbf{X}}^\dagger)$ are annihilation and creation operators for the output ports c and d , respectively. The initial state is:

$$|\Psi\rangle = \int [\psi_1(\mathbf{X}, \mathbf{X}') \hat{a}_{\mathbf{X}}^\dagger \hat{a}_{\mathbf{X}'}^\dagger + \psi_2(\mathbf{X}, \mathbf{X}') \hat{b}_{\mathbf{X}}^\dagger \hat{b}_{\mathbf{X}'}^\dagger] d\mathbf{X} d\mathbf{X}' |0\rangle. \quad (\text{S9})$$

The action of the Beam Splitter is $\hat{a}_{\mathbf{X}}^\dagger = (\hat{c}_{\mathbf{X}}^\dagger + i\hat{d}_{\mathbf{X}}^\dagger)/\sqrt{2}$, $\hat{b}_{\mathbf{X}}^\dagger = (i\hat{c}_{\mathbf{X}}^\dagger + \hat{d}_{\mathbf{X}}^\dagger)/\sqrt{2}$. Thus, the state after the beam splitter will be

$$|\Psi'\rangle = \frac{1}{2} \int [\psi_1(\mathbf{X}, \mathbf{X}') (\hat{c}_{\mathbf{X}}^\dagger + i\hat{d}_{\mathbf{X}}^\dagger) (\hat{c}_{\mathbf{X}'}^\dagger + i\hat{d}_{\mathbf{X}'}^\dagger) + \psi_2(\mathbf{X}, \mathbf{X}') (i\hat{c}_{\mathbf{X}}^\dagger + \hat{d}_{\mathbf{X}}^\dagger) (i\hat{c}_{\mathbf{X}'}^\dagger + \hat{d}_{\mathbf{X}'}^\dagger)] d\mathbf{X} d\mathbf{X}' |0\rangle \quad (\text{S10})$$

$$|\Psi'\rangle = \frac{1}{2} \int [(\psi_1(\mathbf{X}, \mathbf{X}') - \psi_2(\mathbf{X}, \mathbf{X}')) (\hat{c}_{\mathbf{X}}^\dagger \hat{c}_{\mathbf{X}'}^\dagger + \hat{d}_{\mathbf{X}}^\dagger \hat{d}_{\mathbf{X}'}^\dagger) + i(\psi_1(\mathbf{X}, \mathbf{X}') + \psi_2(\mathbf{X}, \mathbf{X}')) (\hat{c}_{\mathbf{X}}^\dagger \hat{d}_{\mathbf{X}'}^\dagger + \hat{d}_{\mathbf{X}}^\dagger \hat{c}_{\mathbf{X}'}^\dagger)] d\mathbf{X} d\mathbf{X}' |0\rangle. \quad (\text{S11})$$

When measuring position-dependent coincidences between two different output ports, e.g. the quantity $\mathcal{C}(\mathbf{X}, \mathbf{X}') := \langle \Psi' | \hat{c}_{\mathbf{X}}^\dagger \hat{d}_{\mathbf{X}}^\dagger \hat{d}_{\mathbf{X}'} \hat{c}_{\mathbf{X}'} | \Psi' \rangle$, one obtains the desired interference:

$$\mathcal{C}(\mathbf{X}, \mathbf{X}') = \frac{|\psi_1(\mathbf{X}, \mathbf{X}') + \psi_2(\mathbf{X}, \mathbf{X}')|^2}{4}. \quad (\text{S12})$$

For phase reconstruction applications, one can have a controllable phase-shifter in the reference arm, which adds a phase $\alpha/2$ for each transmitted photon, in such a way to obtain the measurement outputs $\mathcal{C}_\alpha(\mathbf{X}, \mathbf{X}') = |\psi_1(\mathbf{X}, \mathbf{X}') + e^{i\alpha} \psi_2(\mathbf{X}, \mathbf{X}')|^2/4$. In this case, if ψ_2 is known, a full reconstruction of ψ_1 can be obtained following the methods of phase-shifting digital holography, i.e. extracting $[\mathcal{C}_0 - \mathcal{C}_\pi] + i[\mathcal{C}_{\pi/2} + \mathcal{C}_{3\pi/2}] = 4\psi_1\psi_2$.

Section S4. RECONSTRUCTION FIDELITY AND COINCIDENCE COUNTS

We performed a preliminary analysis to quantify how the number of coincidence counts affects the fidelity of the reconstructed state. To do so, we made a discretization of the LG spatial profile simulating the single photons counting and observed the

behaviour of the fidelity. In this case, we simulated the on-axis phase-shifting digital holography where the two interfering fields have the same direction of propagation but a different phase term (see Supplementary Materials [Section S3](#)). We performed the study also adding Poissonian noise to the discretized interference patterns, in order to simulate the experimental conditions. As an example, Fig [S2](#), is reported the behaviour of the fidelity in the function of the coincidence counts of the pixel that detects the highest number of photons for the $LG_{0,3}$ state, both for the case with and without noise.

Similar results are obtained when analyzing different LG modes, for this reason, during the experiment we decided to acquire data for a time that ensured us to obtain at least ≈ 100 counts as a maximum for the interference intensity.

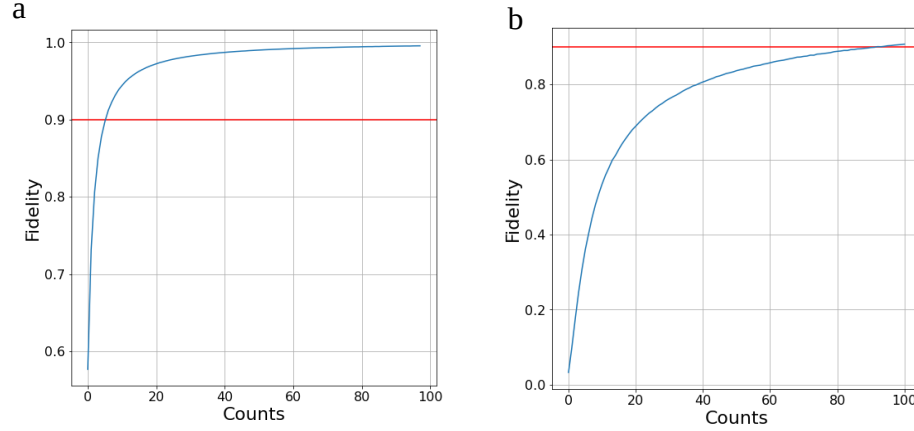


Figure S2. Theoretical fidelity reconstruction behavior. The plots show the behaviour of the fidelity for the reconstruction of the mode $LG_{0,3}$ for different values of the maximum number of counts detected, without noise (panel a) and simulating a Poissonian statistics on the counts (b). The red line indicates a value of the reconstructed fidelity equal to 90%.

Section S5. SUPPLEMENTARY FIGURES

In Fig. [S3](#) we report a more detailed scheme of the setup used in our experiment. In Fig. [S4](#) we show the expected OAM correlations, to be compared with the experimental ones in the main text.

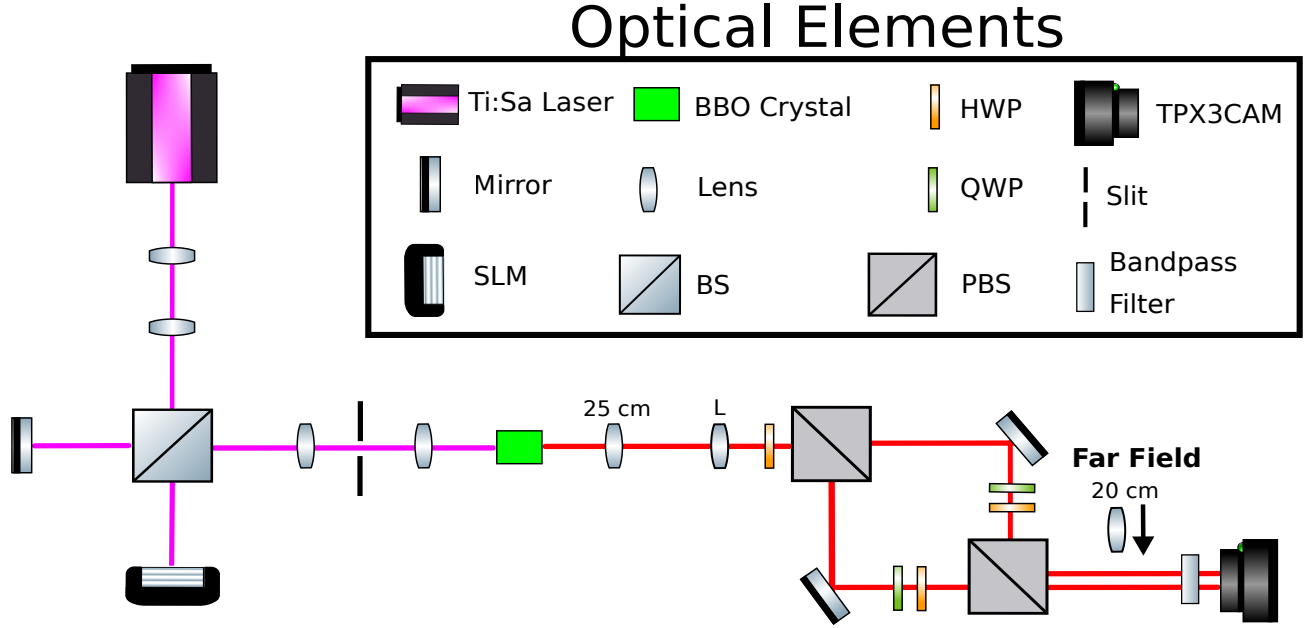


Figure S3. Experimental Setup: The Gaussian beam at 405 nm generated by the second harmonic generation of a 810 nm pulsed Ti:Sa laser (Chameleon Vision II) is sent to a Michelson interferometer. In one of the arms, a reflective UV-SLM is used to shape the field producing the unknown pump beam $\mathcal{E}_p$. This modulation happens in the first diffraction order of the SLM and the state is selected using a couple of lenses and filtering the unwanted orders through a narrow slit. The superposition of $\mathcal{E}_p$ and the reference Gaussian beam $\mathcal{E}_{\text{ref}}$ is then shined on a Type-I BBO crystal to produce the biphoton state. The idler and signal photons having the same polarization are separated using an HWP at 22.5° and a PBS and sent in parallel to different areas of a TPX3CAM sensor. After the BBO crystal, a $4f$ -system is used to image the photons and measure the biphoton wavefunction, in particular, the first lens has a focal length of 25 cm while as the second one we used or a lens of focal length $L = 50$ cm to study the thin crystal approximation case or a lens with $L = 75$ cm for analyzing the reconstruction in an intermediate plane. An additional lens of focal length 20 cm instead is used to image the far field spatial distribution of the state and retrieve the phase-matching function. Finally, a narrow 3 nm bandpass filter is used to ensure the frequency degeneracy of the photons.

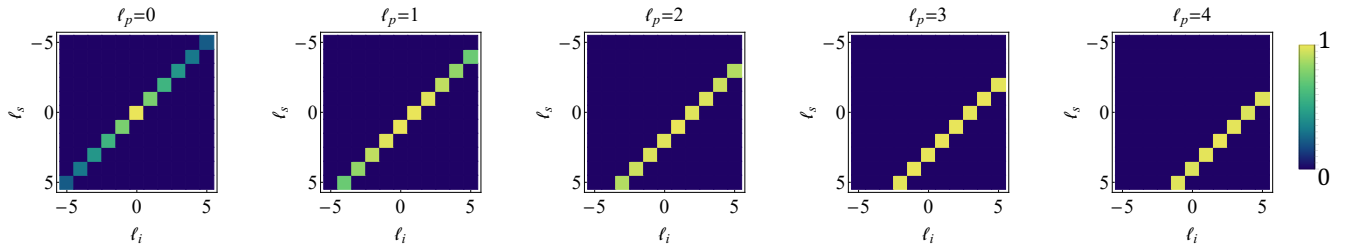


Figure S4. Theoretical OAM correlations. Calculated probabilities P_{ℓ_i, ℓ_s}^{th} used in the main text to evaluate the trace distance $\mathcal{D}$ with the experimental OAM correlations.

3.5 Contributions

I, Nazanin Dehghan, together with Danilo Zia (D.Z.) and Alessio D’Errico (A.D.), devised the experimental setup. I helped D.Z. to perform the experiment and analyze the data. I analyzed the spatial mode correlations and helped A.D. and D.Z. to prepare the first draft of the manuscript. We all discussed the results and contributed to the final version of the manuscript.

Chapter 4

Conclusion

In this thesis, we presented two examples of characterization and investigation of structured light in the classical and quantum optics regimes. The first concerned the reconstruction and characterization of aberrated optical knots and links. The second was the development of a second-order interference technique for reconstructing the spatial mode structure of bi-photon states generated in down-conversion processes.

Until recently, the effect of strong aberrations on 3D topological structures, such as singular skeletons forming knots or links has not been investigated. The complicated structure of these beams may indeed suggest that any imperfection can alter their structure. On the other hand, the topological nature of optical knots and links suggests that they should survive small imperfections but we cannot state how small these imperfections should be. In this thesis, we showed that remarkably these 3D optical topologies can survive very strong aberrations.

In addition to what we showed in [2.4](#), the robustness of 3D optical topologies can be studied under turbulence and complicated superpositions of different Zernike functions. Furthermore, the ability to adjust and compensate for turbulence using adaptive optics systems can be investigated. Also, since now our findings suggest the robustness of these structures, the new communication protocols [[11](#)] that use knots can be practically examined.

Topological structures can be in principle observed also in structured electron beams used in electron microscopes [[15](#)]. In electron microscopy, the alignment and control of aberrations is however more challenging than in optics. Our work has shown a very good robustness for these topological structures under aberration, which then allows us to attempt creating them with electron beams without being worried about the possibility of

creating them with the presence of very strong aberrations in the system.

In the second part of this thesis, we introduced a new interferometric method for the reconstruction of an unknown bi-photon state and retrieving the spatial mode correlations of an SPDC state more time-efficiently compared to projective approaches, loss-free and scalable without any change in the time of the measurement.

In the future, we will be working on applying the approach to spatially separate sources. Furthermore, this approach can be extended to induced coherence setups and be used to discover new quantum phenomena. Also, this technique can be used for developing new imaging technologies.

Hitherto, the topologies considered in the first part of the thesis were only generated in the classical regime. However, nothing can stop one from generating them using correlated bi-photon sources in the quantum regime. This will allow to generate entangled knots, which then can be used for quantum communication protocols as well as topological quantum computation. Therefore, we will apply the approach discussed in the second part of the thesis to reconstruct the topologies from the bi-photon state.

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