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EIGENVECTORS FOR INFINITE MARKOV CHAINS AND DIMENSION GROUPS

By

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Dedication

To my father and mother

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Abstract

This thesis relates the theory of dimension groups to the study of nonnegative infinite matrices. Given a nonnegative matrix $P = (p_{g,h})_{g,h\in\Gamma}$ (Γ countable and infinite), we obtain information concerning the nonnegative eigenvectors of P by studying the associated dimension groups and their trace (state) spaces. For a particular class of countable discrete Markov chains, we exhibit affine homeomophisms between nonnegative eigenvector spaces and certain subspaces of related trace spaces. This thesis also establishes some necessary conditions for the weak ergodicity of sequences of 2×2 real matrices.

Contents

Chapter 1

Introduction

Since its inception by Perron and Frobenius, the theory of nonnegative matrices has developed enormously and is now being used and extended in fields as diverse as probability theory and numerical analysis. Much of the development in this area has been motivated by probabilistic considerations from the theory of Markov chains, that is, in connection with stochastic matrices $P = (p_{ij})$, $i, j \in \{1, 2, \ldots\}$ where $p_{ij} \geq 0$ and

$$\sum_i p_{ij} = 1 \quad \forall j .$$

The study of countable stochastic matrices was initiatied by Kolmogorov in 1936. In this thesis, we extend Handelman's work on nonnegative eigenvectors for Markov chains by employing results from the theory of partially ordered abelian groups.

Let Γ be a countable infinite set and let P be a matrix with rows and columns indexed by Γ such that all entries are nonnegative real numbers. If each column sum $\sum_{g \in \Gamma} p_{g,h}$ equals 1. P determines a Markov process with state space Γ. Namely, the probability of a particle at state g going to state h in one unit of time is $p_{h,g}$. This is the transpose of the probabilists' convention. Conversely, given any Markov process with state space Γ, we can associate an infinite "transition" matrix P which, among other things, has no negative coefficients. One commonly identifies this Markov process with the pair (P, Γ).

In [14], Handelman dealt with Markov processes (P, Γ) where P was column finite. Extending his ideas, we will study Markov processes (P, Γ) where P is permitted to be column infinite. There are some basic differences between the theory of column finite and column infinite Markov processes in relation to order ideals and order units. However, since parallel results exist for the column infinite case, we will adopt the same notation as in [14].

Let R denote the set of all real numbers and consider the real vector space $R\Gamma$ with basis $\{e_g\}_{g\in\Gamma}$ where $e_g : \Gamma \to R$ $(g \in \Gamma)$ is 1 at g and zero elsewhere. In Chapter 2, under some mild additional hypotheses, we study the Choquet boundary of the Markov process represented by $R\Gamma$ and $P : R\Gamma \to R\Gamma$. The first problem is the definition itself. Motivated by [14], we define the Choquet boundary to be the pure trace space of the (real) dimension group associated to P and some initial distribution. In this way, we obtain many results parallel to those in [14]. Chapter 2 also contains many of the definitions and elementary properties used in the rest of the thesis.

Chapter 3 deals with a particular type of interval of eigenvalues for nonnegative infinite matrices. Revisiting a theorem of Pruitt [21], we show that under suitable conditions the set of eigenvalues that admit nonnegative eigenvectors constitutes an interval of the form $[a, \infty)$, some $a > 0$.

Chapter 4 extends some of Handelman's earlier results on Markov chains to our situation [14]. For instance, by imposing an "irreducibility" condition on P, we obtain an affine homeomorphism between the space P_λ of nonnegative eigenvectors (for a fixed eigenvalue λ of P) and a subspace of the trace (state) space of a related dimension group. By letting λ vary, we obtain an analytic one-parameter family of subspaces of the trace space of the said dimension group.

Chapter 5 contains motivational examples which generalize those found in [14]. It is here that we apply some of the results obtained in the earlier chapters.

In Chapter 6, we establish necessary conditions ensuring weak ergodicity for sequences of nonnegative 2×2 matrices. This complements the result of Handelman in [14] in which sufficient conditions for weak ergodicity of such sequences were given.

Terminology and Notation Throughout this thesis, unless otherwise stated,

$$\begin{aligned} N &= \text{the set of all natural numbers,} \\ Z &= \text{the set of all integers,} \\ Z^+ &= \text{the set of all nonnegative integers,} \\ R &= \text{the real number field and} \\ R^+ &= \text{the set of all nonnegative elements of } R. \end{aligned}$$

Chapter 2

Eigenvectors and Dimension Groups

This chapter contains the definitions of $R\Gamma$, P and the dimension group H_Γ. In some order, we will:

- establish correspondences between certain classes of space-time harmonic functions and trace spaces of related dimension groups (For example, adapted from [14], we will prove a correspondence between special global space-time harmonic functions and traces on the dimension group H_Γ.),
- describe the relation between nonzero, nonnegative eigenvectors of P and harmonic functions,
- note the bijective correspondence between space-time cones associated to P and order ideals in the dimension group H_Γ,
- obtain necessary and sufficient conditions for order ideals of the form I_C (C a space-time cone) to have an order unit and necessary and sufficient conditions for H_Γ to have an order unit,

- define Γ_1, C_0 and I_{C_0}.

In this chapter, we also investigate the eigenvector boundary for P and the trace space of H_Γ. We will show that any extremal trace on an order ideal I of H_Γ with order unit corresponds to an eigenvector for the natural endomorphism $\hat{P}$ on I and originates from a harmonic function if the eigenvalue is positive. Under suitable conditions on P, it will be shown that certain pure traces of an order ideal I of H_Γ with order unit extend uniquely to pure traces on H where H is an extension of I. For certain order ideals, we may take $H = H_\Gamma$. (In particular, any faithful pure trace on H_Γ is given by a positive left eigenvector.) Furthermore, we will give conditions on P under which the set of extremal traces of the order ideal I_1 of H_Γ can be written as the union of the sets of extremal traces in E_λ where E_λ is the eigenvector subspace of the trace space of I_1 (see Proposition 2.27 for notation). Similar results continue to hold under weakened hypotheses.

Section 2.3 contains two examples which illustrate some of the concepts and results of Chapter 2. As well, Section 2.3 contains sufficient conditions under which every trace on H_Γ is faithful, sufficient conditions ensuring $I_1 = H_\Gamma$ and sufficient conditions ensuring H_Γ has no proper order ideals with order unit.

Let Γ be a countably infinite set representing the state space for a Markov process. Define $R\Gamma$ to be $l^\infty(\Gamma)$, i.e.,

$$R\Gamma = \{f : \Gamma \to R \text{ such that } |f| \text{ is bounded}\},$$

so that the elements of $R\Gamma$ can be identified with bounded sequences. Let P be a countable transition matrix on Γ such that the row sums are uniformly bounded. That is, $P = (p_{h,g})_{h,g\in\Gamma}$ satisfies:

(1) $p_{h,g} \geq 0 \;\; \forall\, h, g \in \Gamma$,

(2) $s_h = \sum_{g\in\Gamma} p_{h,g} < \infty \;\; \forall\, h \in \Gamma$ and

(3) $\sup\{s_h \mid h \in \Gamma\} = s < \infty$.

Define a linear transformation $P : R\Gamma \to R\Gamma$ by letting

$$(Pv)_h = \sum_{g\in\Gamma} p_{h,g}\, v_g$$

where $v = (v_h)_{h\in\Gamma}$, $v \in R\Gamma$ and $Pv = (Pv)_{h\in\Gamma}$. Given any $v \in R\Gamma$, there exists an $M > 0$ such that $|(Pv)_h| = |\sum_{g\in\Gamma} p_{h,g}\, v_g| \leq \sum_{g\in\Gamma} p_{h,g}\, |v_g| \leq \sum_{g\in\Gamma} p_{h,g} \cdot M \leq M \cdot \sum_{g\in\Gamma} p_{h,g} \leq M \cdot s$ which implies $Pv \in R\Gamma$. Therefore, P is well-defined.

Now, let $(R\Gamma)^+ = \{v \in R\Gamma \mid v_g \geq 0\ \forall g \in \Gamma\}$. Obviously, $(R\Gamma)^+$ is closed under addition, $R\Gamma = (R\Gamma)^+ - (R\Gamma)^+$ and $(R\Gamma)^+ \cap (-(R\Gamma)^+) = \{0\}$. Hence, $R\Gamma$ is a partially ordered abelian group. Furthermore, it is easy to check that $P((R\Gamma)^+) \subseteq (R\Gamma)^+$ so that P is a positive homomorphism. If we define the direct limit

$$H_\Gamma := \varinjlim R\Gamma \xrightarrow{P} R\Gamma \xrightarrow{P} R\Gamma \longrightarrow \ldots = (R\Gamma \times N)/\sim .$$

where $(v, m) \sim (u, n)$ iff $P^{k-m}(v) = P^{k-n}(u)$ for sufficient large k, denote $\overline{(v, m)}$ the class of (v, m). According to [7, p.xvii], H_Γ is a dimension group since $R\Gamma = l^\infty(\Gamma)$ is a lattice ordered abelian group and the direct limit of such groups is always a dimension group. In addition, H_Γ is an ordered vector space over the reals.

2.1 Dimension Groups Associated to Markov Chains

In this section, we adapt and elaborate on some notions from [14].

Definition 2.1 A *space-time cone* C for the Markov process (P, Γ) is a subset of $\Gamma \times N$ satisfying the following two properties:

(a) If $(g, m) \in C$ and there is an edge from g to g' (i.e., $p_{g',g} \neq 0$), then $(g', m + 1) \in C$.

(b) Given $g \in \Gamma$, and $m \in N$, $(g', m+1) \in C$ for every $g' \in \Gamma$ to which there is an edge from g to g', then $(g, m) \in C$.

$\Gamma \times N$ is itself a space-time cone which we will call the *improper* space-time cone. Any subset of $\Gamma \times N$ generates a space-time cone. If the generating set is finite we are naturally inclined to call the resulting space-time cone *finitely generated.*

Lemma 2.2 *The intersection of any family* $\{C_i \mid i \in \mathcal{I}\}$ *of space-time cones of* (P, Γ) *is itself a space-time cone.*

Proof. Let $C = \bigcap_{i \in \mathcal{I}} C_i$ where each $C_i \subseteq \Gamma \times N$ is a space-time cone. Take any $(g, m) \in C$ and any $g' \in \Gamma$ such that there is an edge from g to g'. Applying condition (a) to each $i \in \mathcal{I}$ yields $(g', m+1) \in C_i \ \forall i$. This is precisely condition (a) for C.

To establish condition (b) for C, take $g \in \Gamma$. Suppose there is an m such that, whenever there is a edge from g to g', $(g', m+1) \in C$. Then, for each such g', $(g', m+1) \in C_i \ \forall i$. But each C_i is a space-time cone, thus, condition (b) implies $(g, m) \in C_i \ \forall i \in \mathcal{I}$. Therefore, $(g, m) \in C$ so that C is a space-time cone. ■

Proposition 2.3 *The space-time cone generated by a subset of* $\Gamma \times N$ *is the intesection of all space-time cones which contain the subset.*

Proof. An immediate consequence of the preceding lemma. ■

Definition 2.4 Given a space time cone $C \subseteq \Gamma \times N$, a *space-time harmonic function* is any nonzero function $h : C \to R^+$ satisfying the following condition:

$$h(g, n) = \sum_{f \in \Gamma} h(f, n+1)\, p_{f,g} \quad \forall (g, n) \in C$$

i.e., the values of h are essentially determined by one step in the future.

We will sometimes drop the modifier "space-time" when we talk about cones or harmonic functions. Harmonic functions on the improper cone are called *global harmonic functions.* Our next task is to describe a connection between global harmonic functions and the trace (state) space of the dimension group H_Γ.

Definition 2.5 An *order ideal* of a partially ordered abelian [11, p.6]. group G is any directed convex subgroup of G.

Definition 2.6 An *order unit* in a partially ordered abelian group G is any positive element $u \in G^+$ for which, given any $x \in G$, there is some positive integer n such that $x \leq nu$.

Definition 2.7 A *trace* on H_Γ is any nonzero positive homomorphism from H_Γ to R and the *trace space* of H_Γ, denoted $S(H_\Gamma, R)$, is the set of all traces on H_Γ (when H_Γ has no order unit). If H_Γ has an order unit u, a *(normalized) trace* on (H_Γ, u) is any positive homomorphism from (H_Γ, u) to $(R, 1)$ (i.e., an additive map $s : H_\Gamma \to R$ such that $s(H_\Gamma^+) \subseteq R^+$ and $s(u) = 1$). In such a case, the *(normalized) trace space* of (H_Γ, u), denoted $S(H_\Gamma, u)$, is the set of all (normalized) traces on (H_Γ, u).

We will sometimes express $a : \Gamma \to R$ via the notational device, $a = \prod_{g\in\Gamma} a_g e_g$ where each $e_g : \Gamma \to R$ is as defined in Chapter 1. In this way, we can express any such a as an element of a direct product vector space. If $\exists\, M > 0$ such that $|a_g| < M$ $\forall\, g \in \Gamma$, then $a \in R\Gamma$. The support of such an a, denoted supp(a), is defined to be the set, $\{g \in \Gamma \mid a_g \neq 0\}$. We will often use inner product notation to denote the coefficients of a, that is, $(a, e_g) = a_g$ $(g \in \Gamma)$ where $(\cdot,\cdot)$ can be extended to a bilinear form on $R\Gamma$. We also define

$$\inf(a) \;:=\; \inf\{(a, e_g) \mid g \in \mathrm{supp}(a)\}.$$

Proposition 2.8 *If H_Γ has no order unit, then there is a one to one correspondence between the collection of all global (space-time) harmonic functions h satisfying*

$$\sum_{g\in\Gamma} h(g,n) < \infty, \quad (n = 1, 2, \ldots) \tag{1}$$

and the collection of all traces of H_Γ.

Proof. Let A denote the collection of all global harmonic functions satisfying condition (1) and define a map,

$$\psi : A \to S(H_\Gamma, R)$$

$$h \longmapsto \tau$$

where $\tau : H_\Gamma \to R$ satisfies $\tau(\overline{(e_g, n)}) = h(g,n)$ and is extended linearly. We begin by checking that τ is meaningful. For every g in Γ, $\overline{(e_g, n)} = \overline{(P(e_g), n+1)}$ so that

$$\begin{aligned}
\tau\left(\overline{(P(e_g), n+1)}\right) &= \tau\left(\overline{(\textstyle\sum_{f\in\Gamma} p_{f,g}\, e_f\,,\, n+1)}\right) \\
&= \sum_{f\in\Gamma} p_{f,g}\, \tau\left(\overline{(e_f, n+1)}\right) \\
&= \sum_{f\in\Gamma} \tau\left(\overline{(e_f, n+1)}\right) p_{f,g} \\
&= \sum_{f\in\Gamma} h(f, n+1)\, p_{f,g}.
\end{aligned}$$

As a result, $\tau(\overline{(e_g, n)}) = h(g,n) = \sum_{f\in\Gamma} h(f, n+1)\, p_{f,g} = \tau(\ \overline{(P(e_g), n+1)}\ )$. Given any $a \in H_\Gamma$, we can write $a = \sum_{n\in N} \prod_{g\in\Gamma} a_{g,n}\overline{(e_g, n)}$ where $\exists\, M > 0$ such that $|a_{g,n}| < M$. Thus, $\tau(a) = \sum_{n\in N} \sum_{g\in\Gamma} a_{g,n}\tau(\overline{(e_g, n)})$ so that $|\tau(a)| < \sum_{n\in N}\sum_{g\in\Gamma} M \cdot h(g,n) = M \cdot \sum_{n\in N}\sum_{g\in\Gamma} h(g,n) < \infty$, being a constant multiple of a finite sum of terms of the form $\sum_{g\in\Gamma} h(g,n)$. Therefore, τ is well-defined. Clearly, τ is a homomorphism with $\tau((H_\Gamma)^+) \subseteq R^+$ so that $\tau \in S(H_\Gamma, R)$.

Conversely, we define another map,

$$\phi : S(H_\Gamma, R) \to A,$$

$$\tau \longmapsto h$$

where $h(g,n) = \tau(\overline{(e_g,n)})$. We need to show $h \in A$. Using the identity $\overline{(e_g,n)} = \overline{(P(e_g),n+1)}$, we have $\tau(\overline{(e_g,n)}) = \tau(\overline{(P(e_g),n+1)})$ so that

$$\begin{aligned} h(g,n) &= \tau\left(\overline{(P(e_g),n+1)}\right) \\ &= \sum_{f\in\Gamma} p_{f,g}\,\tau\left(\overline{(e_f,n+1)}\right) \\ &= \sum_{f\in\Gamma} p_{f,g}\,h(f,n+1) \\ &= \sum_{f\in\Gamma} h(f,n+1)\,p_{f,g}. \end{aligned}$$

But $\sum_{g\in\Gamma} h(g,n) = \sum_{g\in\Gamma} \tau(e_g,n) = \tau\left(\overline{(\prod_{g\in\Gamma} e_g,n)}\right) < \infty$. Therefore, $h \in A$.

Given any $h \in A$, $\phi \circ \psi(h) = \phi(\tau)$ for $\tau = \psi(h)$. Thus, $(\phi(\tau))(g,n) = \tau(\overline{(e_g,n)}) = h(g,n)$ which establishes $\phi\circ\psi = id$. Similarly, given any $\tau \in S(H_\Gamma, R)$, $\psi\circ\phi(\tau) = \psi(h)$ for $h = \phi(\tau)$. This implies $(\psi(h))(\overline{(e_g,n)}) = h(g,n) = \tau(\overline{(e_g,n)})$. As a result, $\psi \circ \phi = id$ which completes the proof. ■

Next, we examine the case in which H_Γ has an order unit.

Proposition 2.9 *Suppose H_Γ has an order unit $u = \sum_{n\in N} \prod_{g\in\Gamma} u_{g,n}\,\overline{(e_g,n)}$. If A is the collection of all global (space-time) harmonic functions h satisfying* (1) *and*

$$\sum_{n\in N}\sum_{g\in\Gamma} u_{g,n}\,h(g,n) = 1, \tag{2}$$

then there is a one to one correspondence between A and the trace (state) space $S(H_\Gamma,u)$ of the the dimension group H_Γ.

Proof. Runs parallel to the case in which H_Γ has no order unit. ■

Definition 2.10 A (space-time) harmonic function h is called *spatial* if it originates from a left eigenvector, that is, if there exists a nonnegative function $v : R\Gamma \to R^+$ such that $vP = \lambda v$ for some positive real number λ and $h(g,n) = v(e_g)/\lambda^n$.

In the above definition, note that we do not insist on any boundedness or growth conditions on the entries of v.

Proposition 2.11 *Any nonnegative eigenvector of P corresponding to a nonzero eigenvalue naturally yields a harmonic function on the improper cone.*

Proof. For the purposes of this proof we identify Γ with N. (Recall that Γ is countably infinite.) If v is a nonzero, nonnegative eigenvector of P, then $vP = \lambda v$ which is to say

$$(v_1, v_2, v_3, \cdots)\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdot & \cdot \\ p_{2,1} & p_{2,2} & p_{2,3} & \cdot & \cdot \\ p_{3,1} & p_{3,2} & p_{3,3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \lambda\,(v_1, v_2, v_3, \cdots)$$

where $v = (v_1, v_2, v_3, \cdots)$ and $v_i \geq 0$ for all $i \in \Gamma$. Thus,

$$v_1 p_{1,i} + v_2 p_{2,i} + v_3 p_{3,i} + \cdots = \lambda\, v_i \quad \forall\, i \in \Gamma.$$

implying $\lambda > 0$.

Define the function $h : \Gamma \times N \rightarrow R^+$ by $(g, n) \longmapsto v_g/\lambda^n$. It is obvious that h is a nonzero function and $h(g, n) = v_g/\lambda^n = (1/\lambda^n \cdot 1/\lambda)\ \sum_{f \in \Gamma} v_f\, p_{f,g} = \sum_{f \in \Gamma} (v_f/\lambda^{n+1})\, p_{f,g}$ $= \sum_{f \in \Gamma} h(f, n+1)\, p_{f,g}$. Hence, h is a harmonic function on the improper cone. ∎

Definition 2.12 (a) Given a space-time cone C on (P, Γ) for the dimension group H_Γ, the *n-th component* of C is the set

$$C_n = \{(g, n) \mid (g, n) \in C\}.$$

(b) Given an order ideal I of the dimension group H_Γ, the *n-th component* of I is the subspace

$$I_n = \left\{\overline{(v, n)} \;\middle|\; \overline{(v, n)} \in I\right\}.$$

According to the definition of H_Γ, any component of an order ideal of H_Γ is itself an order ideal.

Proposition 2.13 *Let C be a cone of the dimension group H_Γ and define I_C to be the vector space spanned by*

$$\left\{\overline{(e_g, n)} \;\middle|\; (g, n) \in C\right\} \cup \left\{\prod\nolimits_{(g,n)\in C_n} \overline{(e_g, n)} \;\middle|\; n \in N\right\}.$$

Then,

(1) *I_C is an order ideal, any component of which is an order ideal with order unit.*

(2) *There is a bijective correspondence between space-time cones arising from P and order ideals I of the dimension group H_Γ with the property "any component of I is an order ideal with order unit".*

(3) *I_C has an order unit if and only if $\exists$ a component C_m of C such that*

- *C is generated by C_m and*
- *the element $u_0 = \prod_{(g,m)\in C_m} e_g$ satisfies for all $l > m$, and $u_l = \prod_{(g,l)\in C_l} e_g$, and $\exists\ L_l > 0$ such that $\overline{(u_l, l)} < L_l\overline{(u_0, m)}$.*

(4) *I_C has an order unit if $\exists$ a component C_m of C such that*

- *C is generated by C_m and*
- *the element $u_0 = \prod_{(g,m)\in C_m} e_g$ satisfies $\inf(P^l(u_0)) > 0\ \forall\, l \in N$.*

Proof. (1) By construction,

$$I_C = \left\{\sum_{n\in N} r_n \left(\prod_{(g,n)\in C_n} a_{g,n}\, \overline{(e_g, n)}\right) \;\middle|\; (a_{g,n})_{g\in\Gamma} \in R\Gamma\ \forall n\right\}$$

which coincides with the collection

$$\left\{ \sum_{n \in N} \prod_{(g,n) \in C_n} a_{g,n} \overline{(e_g, n)} \;\middle|\; a_{g,n} \in R \text{ and } \exists\, M > 0 \text{ such that } |a_{g,n}| < M \right\}.$$

Given a, $b \in I_C$, we can write

$$a = \sum_{n \in N} \prod_{(g,n) \in C_n} a_{g,n} \overline{(e_g, n)}$$

where $\exists\, M_1 > 0$ such that $|a_{g,n}| < M_1$ and

$$b = \sum_{n \in N} \prod_{(g,n) \in C_n} b_{g,n} \overline{(e_g, n)}$$

where $\exists\, M_2 > 0$ such that $|b_{g,n}| < M_2$. Thus, given $r_1, r_2 \in R$,

$$\begin{aligned} r_1 \cdot a + r_2 \cdot b &= \sum_{n \in N} \prod_{(g,n) \in C_n} r_1 a_{g,n} \overline{(e_g, n)} + \sum_{n \in N} \prod_{(g,n) \in C_n} r_2 b_{g,n} \overline{(e_g, n)} \\ &= \sum_{n \in N} \prod_{(g,n) \in C_n} (r_1 a_{g,n} + r_2 b_{g,n}) \overline{(e_g, n)} \end{aligned}$$

where $|(r_1 a_{g,n} + r_2 b_{g,n})| \leq (|r_1||a_{g,n}| + |r_2||b_{g,n}|) < [(|r_1| + 1)M_1 + (|r_2| + 1)M_2]$ so that $r_1 \cdot a + r_2 \cdot b \in I_C$. This implies I_C is a subspace of H_Γ. Similarly, using the above notation, if r_1 and r_2 satisfy $0 \leq r_1, r_2 \leq 1$ and $r_1 + r_2 = 1$, then

$$r_1 \cdot a + r_2 \cdot b = \sum_{n \in N} \prod_{(g,n) \in C_n} (r_1 a_{g,n} + r_2 b_{g,n}) \overline{(e_g, n)}$$

where $|(r_1 a_{g,n} + r_2 b_{g,n})| < (M_1 + M_2)$ so that $r_1 \cdot a + r_2 \cdot b \in I_C$. Thus, I_c is a convex subgroup of H_Γ. Furthermore,

$$I_C^+ = \left\{ a = \sum_{n \in N} \prod_{(g,n) \in C_n} a_{g,n} \overline{(e_g, n)} \;\middle|\; \exists M > 0 \text{ such that } M > a_{g,n} > 0 \right\}.$$

From this, it is clear that $I_C = I_C^+ - I_C^+$ and hence, I_C is directed. In total, I_C is an order ideal of H_Γ.

For each $n \in N$, it is obvious that the n-th component of I_C is given by

$$(I_C)_n = \left\{ \prod_{(g,n) \in C_n} a_{g,n} \overline{(e_g, n)} \;\middle|\; a_{g,n} \in R \text{ and } \exists\, M > 0 \text{ such that } |a_{g,n}| < M \right\}.$$

Thus, repeating the above argument, each $(I_C)_n$ is an order ideal. Define

$$u = \prod_{(g,n)\in C_n} \overline{(e_g, n)}.$$

Given $a = \prod_{(g,n)\in C_n} a_{g,n}\, \overline{(e_g, n)} \in (I_C)_n$, there exists $M > 0$ such that $|a_{g,n}| < M$. This implies $a < Mu$, i.e., u is an order unit of $(I_C)_n$.

(2) Let A denote the collection of space-time cones on (P, Γ) and let B denote the collection of all order ideals I of H_Γ for which the statement "every component of I is an order ideal with order unit" is true. From the proof of (1), we have the map $\psi : A \to B$ with $C \longmapsto I_C$. Now, define another map, $\phi : B \to A$ such that $I \longmapsto C$ under ϕ where $C = \{(g, n) \mid \overline{(e_g, n)} \in I\}$. We will verify that this C is a space-time cone. Suppose $(g, m) \in C$ and $g' \in \Gamma$ is such that there is a path from g to g'. Then, $\overline{(e_g, m)} \in I$ and $P(e_g) = \sum_{p_{g',g}\neq 0} p_{g',g}\, e_{g'}$ yields

$$\overline{(e_g, m)} = \overline{(P(e_g), m+1)} = \sum_{p_{g',g}\neq 0} p_{g',g}\, \overline{(e_{g'}, m+1)}.$$

Therefore, since I is an order ideal, $\overline{(e_{g'}, m+1)} \in I$ which implies $(g', m+1) \in C$.

On the other hand, take $g \in \Gamma$ and assume $\exists\, m$ such that for all g' in Γ to which there is an edge from g to g', $(g', m+1) \in C$ (i.e., $p_{g',g} \neq 0$ and $\overline{(e_{g'}, m+1)} \in I$). Then, $\prod_{p_{g',g}\neq 0} \overline{(e_{g'}, m+1)} \in I$ since any component of I is an order ideal with order unit. Furthermore, $\sup\{\sum_{g\in\Gamma} p_{g',g} \mid g' \in \Gamma\} = s < \infty$ implies $|p_{g',g}| < s$ which yields $\prod_{g'\in\Gamma} p_{g',g}\, \overline{(e_{g'}, m+1)} \in I$. However, $\overline{(e_g, m)} = \overline{(P(e_g), m+1)}$ and $P(e_g) = \prod_{g'\in\Gamma} p_{g',g}\, e_{g'}$. Therefore, $\overline{(e_g, m)} \in I$ which implies $(g, m) \in C$. This establishes that C is a cone.

Finally, given a cone C, $\phi \circ \psi(C) = \phi(I_C)$. Both $C \subseteq \phi(I_C)$ and $\phi(I_C) \subseteq C$ are obvious. Thus, $\phi \circ \psi = id$. Similarly, we have $\psi \circ \phi = id$. Hence, ϕ and ψ are mutual inverses.

(3) Suppose there exists a component C_m of the cone C and an element $u_0 = \prod_{(g,m)\in C_m} e_g$ such that C is generated by C_m and $\inf(P^l(u_0)) > 0\ \forall\, l \in N$. By the

construction of I_C. if $a \in (I_C)_n$, then $a = \prod_{(g,n)\in C_n} a_{g,n}\,\overline{(e_g, n)}$ where $\exists\, M > 0$ such that $|a_{g,n}| < M$. Let $\bar{a} = \prod_{(g,n)\in C_n} e_g \in R\Gamma$ and set $\sup\{\sum_{g\in\Gamma} p_{g',g} \mid g' \in \Gamma\} = s < \infty$. There are two cases:

Case (i): $n \leq m$.

By the construction of C,

$$\operatorname{supp}(P^{m-n}(\bar{a})) \subseteq \{g \in \Gamma \mid (g, m) \in C_m\}.$$

Hence, $a < M\,\overline{(\bar{a}, n)} = M\,\overline{(P^{m-n}(\bar{a}), n)} \leq M\, s^{m-n}\overline{(u_0, m)}$.

Case (ii): $n > m$.

By the construction of C, $\operatorname{supp}(P^{n-m}(u_0)) = \operatorname{supp}\bar{a}$. Hence,

$$a \;<\; M\overline{(\bar{a}, n)} \;<\; ML_n\overline{(u_0, m)}.$$

By these two cases and the construction of I_C, $\overline{(u_0, m)}$ is an order unit of I_C.

Conversely, suppose I_C has an order unit u where

$$\begin{aligned} u &= \textstyle\prod_{(g,k_1)\in C_{k_1}} a_{g,k_1}\,\overline{(e_g, k_1)} + \prod_{(g,k_2)\in C_{k_2}} a_{g,k_2}\,\overline{(e_g, k_2)} + \cdots + \prod_{(g,k_t)\in C_{k_t}} a_{g,k_t}\,\overline{(e_g, k_t)} \\ &= \overline{(\textstyle\prod_{(g,k_1)\in C_{k_1}} a_{g,k_1}\, e_g, k_1)} + \overline{(\prod_{(g,k_2)\in C_{k_2}} a_{g,k_2}\, e_g, k_2)} + \cdots + \overline{(\prod_{(g,k_t)\in C_{k_t}} a_{g,k_t}\, e_g, k_t)} \end{aligned}$$

and $\exists\, M > 0$ such that for every $g \in \Gamma$,

$$0 \leq a_{g,k_1}, a_{g,k_2}, \cdots, a_{g,k_t} < M, \quad (0 < k_1 < k_2 < \cdots < k_t).$$

let

$$\begin{aligned} u_1 &= \prod_{(g,k_1)\in C_{k_1}} a_{g,k_1}\, e_g, \\ u_2 &= \prod_{(g,k_2)\in C_{k_2}} a_{g,k_2}\, e_g, \\ &\vdots \end{aligned}$$

$$\begin{aligned} u_t &= \prod_{(g,k_t)\in C_{k_t} a_{g,k_t}} e_g, \\ \bar{u} &= \prod_{(g,k_t+1)\in C_{k_t+1}} e_g. \end{aligned}$$

By the definition of H_Γ, we know that

$$\begin{aligned} u &= \overline{(P^{k_t+1-k_1}(u_1), k_t+1)} + \overline{(P^{k_t+1-k_2}(u_2), k_t+1)} + \cdots + \overline{(P(u_t), k_t+1)} \\ &= \overline{(\sum_{i=1}^{t} P^{k_t+1-k_i}(u_i), k_t+1)}. \end{aligned}$$

From the construction of C, we have $\operatorname{supp}(\sum_{i=1}^{t} P^{k_t+1-k_i}(u_i)) \subsetneq \operatorname{supp} \bar{u}$. Thus, we can write

$$u = \prod_{(g,k_t+1)\in C_{k_t+1}} b_{g,k_t+1}\, \overline{(e_g, k_t+1)}.$$

If $\exists\, M_1 > 0$ such that each $|b_{g,k_t+1}|$ is strictly less than M_1, then $u < M_1\, \overline{(\bar{u}, k_t+1)}$. Since u is order unit of I_C, so is $\overline{(\bar{u}, k_t+1)}$.

Let $\bar{C}$ be the cone generated by C_{k_t+1}. Obviously, $\bar{C} \subseteq C$. According to part (b) of the definition of space-time cone, if $\exists\, m$ such that $(g', m+1) \in \bar{C}$ for all $g' \in \Gamma$ to which there is an edge from $g \in \Gamma$ to g', then $(g, m) \in \bar{C}$. That is, if $P(e_{g'}) < M_2\, \bar{u}$ for some positive real number M_2, then $(g', k_t) \in \bar{C}$. Recursively, if $P^l(e_{g'}) < M_3\, \bar{u}$ for some constant $M_3 > 0$ and some positive integer $l < k_t$, then $(g', k_t - l) \in \bar{C}$. Next, applying part (a) of the definition of space-time cone, if $(g, m) \in \bar{C}$ and there is an edge from g to g', then $(g', m+1) \in \bar{C}$. In other words, if $e_{g'} < M_4\, P(\bar{u})$ for some constant $M_4 > 0$, then $(g', k_t+2) \in \bar{C}$. Recursively, if $e_{g'} < M_5\, P^l(\bar{u})$ for some positive constant M_5 and some positive integer $l > 2$, then $(g', k_t+l+1) \in \bar{C}$.

Now, take any $(g, n) \in C$. If $n \le k_t + 1$, then

$$\overline{(e_g, n)} = \overline{(P^{k_t+1-n}(e_g), k_t+1)} \in I_C$$

so that $\exists\, M_6 > 0$ for which $\overline{(e_g, n)} < M_6\, \overline{(\bar{u}, k_t+1)}$, implying $(g, n) \in \bar{C}$. On the other hand, if $n > k_t + 1$, then

$$\overline{(\bar{u}, k_t+1)} = \overline{(P^{n-k_t-1}(\bar{u}), n)} \in I_C$$

so that $\exists\, M_7 > 0$ for which $\overline{(e_g, n)} < M_7\,\overline{(\bar{u}, k_t + 1)}$, also implying $(g, n) \in \bar{C}$. Therefore, $\bar{C} = C$, i.e., C_{k_t+1} generates C.

Finally, $\overline{(\bar{u}, k_t + 1)}$ is an order unit of I_C, obviously, for all $l > m$, and $u_l = \prod_{(g,l)\in C_l} e_g$, and $\exists\, L_l > 0$ such that $\overline{(u_l, l)} < L_l\overline{(\bar{u}, k_t + 1)}$. From this, we obtain that C_{k_t+1} is the required component of C.

(4) We know for all $n > m$, and $u_n = \prod_{(g,n)\in C_n} e_g$, and $\exists\, L_n = \frac{1}{\inf(P^{n-m}(u_0))}$

$$\overline{(u_n, n)} \;<\; \left[\frac{1}{\inf(P^{n-m}(u_0))}\right] \overline{(P^{n-m}(u_0), n)} \;=\; L_n\overline{(u_0, m)}.$$

According to the above (3), I_C has an order unit. ∎

Corollary 2.14 *The dimension group H_Γ has an order unit if*

$$\inf\left\{\sum\nolimits_{g\in\Gamma} p_{h,g}\,\middle|\, h \in \Gamma\right\} \;>\; 0. \tag{3}$$

Proof. Assume (3) holds. If we define $a = \inf\{\sum_{g\in\Gamma} p_{h,g} \,|\, h \in \Gamma\} > 0$, $b = \prod_{g\in\Gamma} e_g$ and $u = \overline{(b, 1)}$, then $b \in \Gamma$ and

$$\begin{aligned}
u &= \overline{(P(b), 2)} \\
&\geq a\,\overline{(b, 2)} \;=\; a\,\overline{(P(b), 3)} \\
&\geq a^2\,\overline{(b, 3)} \;=\; a^2\,\overline{(P(b), 4)} \\
&\geq a^3\overline{(b, 4)} \\
&\;\vdots \\
&\geq a^{n-2}\,\overline{(b, n-1)} \\
&\geq a^{n-1}\,\overline{(b, n)}.
\end{aligned}$$

Thus, $\forall\, v \in H_\Gamma$, we have

$$v \;=\; \sum_{n\in N}\;\prod_{(g,n)\in C_n} v_{g,n}\,\overline{(e_g, n)}$$

where $\exists M > 0$ such that $|v_{g,n}| < M$. We also know that $C_n = \Gamma \times \{n\}$. Therefore,

$$\begin{aligned} v &= \sum_{n\in N} \prod_{g\in\Gamma} v_{g,n}\,\overline{(e_g, n)} \\ &\le \sum_{n\in N} \prod_{g\in\Gamma} M\,\overline{(e_g, n)} \\ &= M \sum_{n\in N} \overline{(\prod_{g\in\Gamma} e_g, n)} \;=\; M \sum_{n\in N} \overline{(b, n)} \\ &\le \left[M \sum_{n\in N} (a^{n-1})^{-1} \right] u \end{aligned}$$

which implies u is an order unit of H_Γ. ■

Using the notation of the above proposition, we obtain a correspondence between the space of summable harmonic functions on C and the trace space of I_C.

Corollary 2.15 *Fix a space-time cone $C \subseteq \Gamma \times N$ and define A to be the collection of all summable space-time harmonic functions h on C, that is,*

$$A = \left\{ h : C \to R^+ \,\middle|\, \sum_{g\in\Gamma} h(g,n) < \infty \;\forall\, n \in N \right\}.$$

Then, there is a one to one correspondence between A and the trace space $S(I_C, R)$.

Proof. Similar to that of Proposition 2.8. ■

Let Γ_1 be a subset of Γ with the property that for every $g \in \Gamma$, there exist paths from points in Γ_1 to g, that is, $\exists\, h \in \Gamma_1$ and $r_1, r_2, \ldots, r_{s-1}, r_s \in \Gamma$ such that $p_{g,r_s} p_{r_s,r_{s-1}} \cdots p_{r_2,r_1} p_{r_1,h} > 0$. Set

$$u_0 = \prod_{g\in\Gamma_1} e_g$$

and suppose $\inf(P^l(u_0)) > 0 \;\forall\, l \in N$. Let $\Gamma_2, \Gamma_3, \cdots$ be such that $\Gamma_i \bigcap \Gamma_j = \emptyset$ $\forall\, i \ne j$ and $\Gamma = \bigcup_{i\in N} \Gamma_i$. To make things interesting, we normally assume Γ_1 is finite (although there are interesting cases where it is not). If no such finite set exists, we can always start with any finite set, consider the subset of Γ it generates and then work with this subset. Normally, we just assume Γ_1 exists and then define Γ_i $(i \geq 2)$ to be subsets of Γ that satisfy the above conditions. A natural space-time cone C_0 corresponding to such a Γ_1 is the one generated by

$$\{(g, 1) \in \Gamma \times N \mid g \in \Gamma_1\}.$$

The order ideal I_{C_0} it determines is that generated by

$$\left\{\overline{(e_g, 1)} \;\middle|\; g \in \Gamma_1\right\} \cup \left\{\prod_{g \in \Gamma_1} \overline{(e_g, 1)}\right\}.$$

i.e., $I_{C_0} = \{\sum_{(g,m) \in C_0} a_{g,m}\overline{(e_g, m)} \mid \exists M > 0 \text{ such that } |a_{g,m}| < M\}$. Obviously, I_{C_0} is an order ideal of H_Γ with order unit.

Definition 2.16 An *extremal (pure) trace* on the partially ordered abelian group G with order unit u is any trace $\tau \in S(G, u)$ which does not lie in the interior of any line segment with endpoints in $S(G, u)$. In other words, given any convex combination $\tau = t_1\tau_1 + t_2\tau_2$ with $\tau_1, \tau_2 \in S(G, u)$, we must have $t_1 = 1$ or $t_2 = 1$ or $\tau = \tau_1 = \tau_2$. The collection of all extreme (pure) traces in $S(G, u)$ is called the *extremal boundary* of $S(G, u)$ and is denoted by $\partial_e S(G, u)$.

Definition 2.17 The *Choquet boundary* associated to P and Γ_1 is the pure trace space of the order ideal determined by Γ_1.

The element $u = \sum_{(g,m) \in C_0} u_{g,m}\overline{(e_g, m)}$ $(u_{g,m} > 0)$ is an order unit of I_{C_0}. A trace τ on I_{C_0} is normalized if $\tau(u) = 1$. Obviously, any trace on I_{C_0} can be replaced by $\tau/\tau(u)$ which is normalized.

2.2 Eigenvectors and Traces

The dimension group H_Γ admits an obvious order automorphism (an isomorphism of vector spaces for which both it and its inverse are order preserving), $\hat{P} : H_\Gamma \to H_\Gamma$ where $\overline{(v, n)} \longmapsto \overline{(P(v), n)}$. $\hat{P}$ has an inverse $\hat{P}^{-1} : H_\Gamma \to H_\Gamma$ where $\overline{(v, n)} \longmapsto \overline{(v, n+1)}$. Indeed, it is easy to check that

$$\hat{P} \circ \hat{P}^{-1}(\overline{(v, n)}) = \hat{P}(\overline{(v, n+1)}) = \overline{(P(v), n+1)} = \overline{(v, n)}$$

so that $\hat{P} \circ \hat{P}^{-1} = id$. Similarly, $\hat{P}^{-1} \circ \hat{P} = id$.

An order ideal I of H_Γ is said to be $\hat{P}$-*invariant* if $\hat{P}(I) \subseteq I$. Very few order ideals of H_Γ are $\hat{P}$-invariant, however, under reasonable circumstances(see ($\bullet$) below), they are $\hat{P}^{-1}$-invariant.

Definition 2.18 Let p and q be endormorphisms of H_Γ. We write $p \le q$ if the difference $q - p$ is order preserving.

We make the following assumption on our operator P:

$$\exists\, k, K \in N \text{ such that } P^k \le KP^{k+1} \qquad (\bullet)$$

as endmorphisms of $R\Gamma$—in other words, entrywise ($p^{(k)}_{g,h} \le Kp^{(k+1)}_{g,h}$ for all g and h in Γ). This assumption is in force until further notice. For example, this holds if $inf(p_{i,i}) > \epsilon > 0$; then $k = 0$, $K = 1/\epsilon$ will do.

Definition 2.19 A positive endomorphism is called *bounded* if it is bounded above by a multiple of the identity.

Proposition 2.20 *Any bounded positive endomorphism of H_Γ leaves every order ideal of H_Γ stable. In particular, this applies to $\hat{P}^{-1}$ if ($\bullet$) holds.*

Proof. Let I be an order ideal of H_Γ and $\mathcal{F}$ be a bounded positive endomorphism of H_Γ. Then, there exists a positive real number L such that $\mathcal{F} \leq L\mathcal{I}$ where $\mathcal{I}$ denotes the identity homomorphism of H_Γ. Thus, for any a in I^+, $0 \leq \mathcal{F}(a) \leq L\mathcal{I}(a) = La$. Since I is an order ideal, $\mathcal{F}(a) \in I$, and since $I = I^+ - I^+$, it follows that $\mathcal{F}(I) \subseteq I$.

By condition ($\bullet$), for any a in $(R\Gamma)^+$, $P^k(a) \leq KP^{k+1}(a)$. Thus $\forall\, a \in H_\Gamma^+$, $\hat{P}^k(a) \leq K\hat{P}^{k+1}(a)$. Since $\hat{P}^{-1}$ is order preserving, so is $\hat{P}^{-(k+1)}$ and hence,

$$\hat{P}^{-(k+1)}(\hat{P}^k(a)) \leq \hat{P}^{-(k+1)}(K\hat{P}^{k+1}(a)).$$

But $\hat{P}^{-1}(a) \leq Ka = K(\mathcal{I})(a)$ implies $\hat{P}^{-1} \leq K\mathcal{I}$. Therefore, by the first part, $\hat{P}^{-1}$ leaves every order ideal of H_Γ stable. ∎

Now, let I be an order ideal of H_Γ corresponding to some space-time cone and assume that I admits an order unit. I also admits a positive bounded endmorphism, namely the restriction of $\hat{P}^{-1}$ (simply denoted as $\hat{P}^{-1}$). A standard result in dimension groups now applies[9: I.2(c)]: *If I is a partially ordered abelian group with order unit and I admits a bounded positive endomorphism α, then for any extremal trace τ of I which is normalized at the order unit, there exists a nonnegative λ such that* $\tau \circ \alpha = \lambda\tau$. In many cases, $\lambda = 0$. For now, we will concentrate only on those cases in which $\lambda > 0$.

Proposition 2.21 *Let I be an order ideal of H_Γ corresponding to some space-time cone C and assume I has an order unit u. If τ is an extremal trace on I, then $\exists\, \lambda \geq 0$ such that $\tau \circ \hat{P}^{-1} = \lambda\tau$. Moreover, if $\lambda > 0$, then*

(i) *τ is induced by an eigenvector of $\hat{P}$,*

(ii) *τ comes from a harmonic function on C and*

(iii) *τ comes from a sequence of spatial harmonic functions h_i $(i = 1, 2, \ldots)$ and extends to a trace on H_Γ.*

Proof. The existence of $\lambda \geq 0$ follows directly from the italicized result in the above paragraph. For (i), since $\tau \circ \hat{P}^{-1} = \lambda\tau$ and $\lambda > 0$, we have $(1/\lambda)\tau = \tau \circ \hat{P}$ so that τ is induced by an eigenvector of $\hat{P}$. (ii) follows from (i) and Proposition 2.11.

For (iii), since $C = \{(g, n) \mid \overline{(e_g, n)} \in I\}$, we can define for each $i \in N$ a function

$$\begin{aligned} h_i &: C \to R^+, \\ (g, n) &\longmapsto \tau(\overline{(e_g, i)})/\lambda^{-n} = \lambda^n \tau(\overline{(e_g, i)}) \,. \end{aligned}$$

Since $\overline{(P(e_g), i)} = \sum_{f \in \Gamma} \overline{(e_f, i)}\; p_{f,g}$, we have $h_i(g, n) = \lambda^n \,\tau(\overline{(e_g, i)}) = (\lambda^n \cdot \lambda)\, \tau \circ \hat{P}(\overline{(e_g, i)}) = \lambda^{n+1}\, \tau(\overline{(P(e_g), i)}) = \lambda^{n+1}\, \tau(\sum_{f \in \Gamma} \overline{(e_f, i)}\, p_{f,g}) = \sum_{f \in \Gamma} \lambda^{n+1}\, p_{f,g}\, \tau(\overline{(e_f, i)})$ $= \sum_{f \in \Gamma} p_{f,g}\, h_i(f, n+1)$. Therefore, each h_i is a spatial harmonic function on C. ∎

Definition 2.22 A pure (extremal) trace is *faithful* if its kernel contains no nonzero positive elements.

Corollary 2.23 *The harmonic function which corresponds to a faithful pure trace (i.e., which a faithful pure trace comes from) never vanishes on the corresponding cone.*

Proof. An obvious consequence of the definitions. ∎

We now prove several of the main results of this chapter. The following is a detailed proof of [14; 1.1].

Theorem 2.24 *Let P satisfy $(\bullet)$ and let J be an order ideal with order unit in H_Γ corresponding to a space-time cone. Form $H = \sum_{k \in N} \hat{P}^k(J)$. If τ is a pure trace on J such that τ does not vanish identically on $\hat{P}^{-1}(J)$, then*

(a) *$\tau \circ \hat{P}^{-1} = \lambda\, \tau$ for some $\lambda > 0$ and*

(b) *τ extends uniquely to a pure trace $\bar{\tau}$ on H satisfying $\bar{\tau} \circ \hat{P} = \lambda^{-1}\, \bar{\tau}$.*

Proof. (a) By assumption, J has an order unit. As done previously, form $\hat{P}$ and its inverse. By Proposition 2.20, condition (•) guarantees that every order ideal of H_Γ is stable under $\hat{P}^{-1}$.

If τ is a pure trace on J, $\tau \circ \hat{P}^{-1}$ is also a trace and, since $\hat{P}^{-1} \leq K\mathcal{I}$, it follows that $\tau \circ \hat{P}^{-1} \leq K\tau$. From the purity of τ, there exists $\lambda \geq 0$ such that $\tau \circ \hat{P}^{-1} = \lambda\tau$.(cf., [9: I.2(c)]) Since $\tau \circ \hat{P}^{-1}$ is not zero, it must be that $\lambda > 0$.

(b) Now we extend τ to all of H in the obvious way. First note that $\hat{P}^{-1}(J) \subseteq J$ so that $J \subseteq \hat{P}(J)$; the latter is an order ideal in H_Γ. Next, note that $\{\hat{P}^l(J) : l \in N\}$ is an increasing family of order ideals whose union is H. For each positive integer l, we define $\tau_l : \hat{P}^l(J) \to R$ via

$$\tau_l(\hat{P}^l(\overline{(a,m)})) = \lambda^{-l}\tau(\overline{(a,m)})$$

where $\overline{(a,m)} \in J$. To verify that this is a well-defined trace on $\hat{P}^l(J)$, choose $\overline{(b,r)}, \overline{(c,s)} \in J$ and suppose $\hat{P}^l(\overline{(b,r)}) = \hat{P}^l(\overline{(c,s)})$. Since $\hat{P}^l$ is invertible on H_Γ, $\overline{(b,r)} = \overline{(c,s)} \in J$. Hence, $\tau(\overline{(b,r)}) = \tau(\overline{(c,s)})$ and $\tau_l(\hat{P}^l(\overline{(b,r)})) = \lambda^{-l}\tau(\overline{(b,r)}) = \lambda^{-l}\tau(\overline{(c,s)}) = \tau_l(\hat{P}^l(\overline{(c,s)}))$.

Next, we check that $\tau_l|_{\hat{P}^j(J)} = \tau_j$ for any pair $j < l$. Suppose $\overline{(y,k)} = \hat{P}^j(\overline{(x,m)})$ for some $\overline{(x,m)} \in J$ and $\overline{(y,k)} = \hat{P}^l(\overline{(z,m')})$ for some $\overline{(z,m')} \in J$ where $x, y, z \in R\Gamma$ and $k, m, m' \in N$. Then, $\overline{(y,k)} = \hat{P}^j(\overline{(x,m)})$ and since $\hat{P}^j$ is invertible, $\overline{(x,m)} = (\hat{P}^j)^{-1}(\overline{(y,k)}) = \overline{(y,k+j)}$.

By definition, $\exists\ l_1, l_2 \in N$ such that $P^{l_1}(y) = P^{l_2}(x)$ and $l_1 + k + j = m + l_2$. Similarly, $\exists\ l_1', l_2' \in N$ such that $P^{l_1'}(y) = P^{l_2'}(z)$ and $l_1' + k + l = m' + l_2'$. Thus, we have $P^{l_1+l_1'}(y) = P^{l_2+l_1'}(x) = P^{l_1+l_2'}(z)$ and $l - j = m' + l_2' + l_1 - m - l_2 - l_1'$. Therefore,

$$\begin{aligned}\tau_l(\overline{(y,k)}) &= \tau_l(\hat{P}^l(\overline{(z,m')})) = \lambda^{-l}\tau(\overline{(z,m')}) \\ &= \lambda^{-l}\tau(\overline{(P^{l_1+l_2'}(z), l_1 + l_2' + m')})\end{aligned}$$

and

$$\tau_j(\overline{(y,k)}) = \tau_j(\hat{P}^j(\overline{(x,m)})) = \lambda^{-j}\tau(\overline{(x,m)})$$

$$= \lambda^{-j}\,\tau(\overline{(P^{l_2+l_1'}(x), l_2 + l_1' + m)})$$

so that

$$\begin{aligned}
\tau_l(\overline{(y,k)}) &= \lambda^{-l}\,\tau(\overline{(P^{l_1+l_2'}(z), l - j + m + l_2 + l_1')}) \\
&= \lambda^{-l}\,\tau(\overline{(P^{l_2+l_1'}(x), l - j + m + l_2 + l_1')}) \\
&= \lambda^{-l}\,\tau \circ (\hat{P}^{-1})^{l-j}(\overline{(P^{l_2+l_1'}(x), m + l_2 + l_1')}) \\
&= \lambda^{-l} \cdot (\lambda^{l-j}) \cdot \tau(\overline{(P^{l_2+l_1'}(x), l_2 + l_1' + m)}) \\
&= \tau_j(\overline{(y,k)}),
\end{aligned}$$

i.e., $\tau_l|_{\hat{P}^j(J)} = \tau_j$. Since $H = \bigcup_{l\in N} \hat{P}^l(J)$, we can define the map $\tilde{\tau} : H \to R$ by setting $\tilde{\tau}|_{\hat{P}^l(J)} = \tau_l\ \forall\, l \in N$. In total, $\tilde{\tau}$ is a well-defined trace extending τ.

We will now show that $\tilde{\tau}$ satisfies the purity criterion of [8] (see Theorem 3.1 therein) for traces on dimension groups. Choose $\overline{(a,m)}, \overline{(b,n)} \in J^+$. Given any positive integer l, $\hat{P}^l(\overline{(a,m)})$ and $\hat{P}^l(\overline{(b,n)})$ lie in $(\hat{P}^l(J))^+$ and

$$\begin{aligned}
&\min\left\{\tau_l\left(\hat{P}^l(\overline{(a,m)})\right), \tau_l\left(\hat{P}^l(\overline{(b,n)})\right)\right\} \\
&\quad = \lambda^{-l}\min\left\{\tau(\overline{(a,m)}), \tau(\overline{(b,n)})\right\} \\
&\quad = \lambda^{-l}\sup\left\{\tau(\overline{(c,k)}) \,\middle|\, \overline{(c,k)} \in J^+,\ \overline{(c,k)} \le \overline{(a,m)} \text{ and } \overline{(c,k)} \le \overline{(b,n)}\right\} \\
&\quad = \sup\left\{\tau_l(\hat{P}^l(\overline{(c,k)})) \,\middle|\, \hat{P}^l(\overline{(c,k)}) \in (\hat{P}^l(J))^+,\ \hat{P}^l(\overline{(c,k)}) \le \hat{P}^l(\overline{(a,m)})\right. \\
&\qquad\qquad \left.\text{and } \hat{P}^l(\overline{(c,k)}) \le \hat{P}^+(\overline{(b,k)})\right\}.
\end{aligned}$$

Now, for any $l \in N$, $\hat{P}^l(J)$ is an interpolation group and τ_l is a pure trace. Thus, for any $\hat{P}^l(\overline{(a,m)}) \in H$ with $\overline{(a,m)} \in J$,

$$\tilde{\tau} \circ \hat{P}(\hat{P}^l(\overline{(a,m)})) = \tilde{\tau}(\hat{P}^{l+1}(\overline{(a,m)})) = \tau_{l+1}(\hat{P}^{l+1}(\overline{(a,m)})) = \lambda^{-(l+1)}\,\tau(\overline{(a,m)}).$$

On the other hand,

$$\lambda^{-1}\,\tilde{\tau}(\hat{P}^l(\overline{(a,m)})) = \lambda^{-1}\,\tau_l(\hat{P}^l(\overline{(a,m)})) = (\lambda^{-1}\cdot\lambda^{-l})\,\tau(\overline{(a,m)}).$$

Therefore, $\tilde{\tau}$ is a pure trace satisfying $\tilde{\tau} \circ \hat{P} = \lambda^{-1}\tilde{\tau}$.

Finally, to show uniqueness, suppose there is another $\tilde{\tau}'$ which extends τ to H and satisfies $\tilde{\tau}' \circ \hat{P} = \lambda^{-1}\tilde{\tau}'$. Then, $\tilde{\tau}' = \lambda^{-1}\tilde{\tau}' \circ \hat{P}^{-1}$. Furthermore, given any $\hat{P}^l(\overline{(a,m)}) \in H$ with $\overline{(a,m)} \in J$ and $l \in N$, we obtain (recursively),

$$\begin{aligned}
\tilde{\tau}'(\hat{P}^l(\overline{(a,m)})) &= \lambda^{-1}\tilde{\tau}'(\hat{P}^{-1}(\hat{P}^l(\overline{(a,m)}))) \\
&= \lambda^{-1}\tilde{\tau}'(\hat{P}^{l-1}(\overline{(a,m)})) \\
&\vdots \\
&= \lambda^{-l}\tau(\overline{(a,m)})
\end{aligned}$$

so that $\tilde{\tau}(\hat{P}^l(\overline{(a,m)})) = \tau_l(\hat{P}^l(\overline{(a,m)})) = \lambda^{-l}\tau(\overline{(a,m)}) = \tilde{\tau}'(\hat{P}^l(\overline{(a,m)}))$. Therefore, $\tilde{\tau} = \tilde{\tau}'$, i.e., $\tilde{\tau}$ uniquely extends τ. ■

Theorem 2.25 *Suppose Γ and P are defined so that $(\bullet)$ holds and Γ_1 is given. Let I_1 be the order ideal of H_Γ corresponding to the space-time cone C_0 generated by $\{(e_g, 1) \mid g \in \Gamma_1\}$ and let τ be a pure trace of I_1 that does not vanish identically on $\hat{P}^{-1}(I_1)$ (e.g., if τ is faithful). Then,*

(i) *There exists $w : \Gamma \to R^+$ such that the row $v = (w(g))_{g\in\Gamma}$ is a left eigenvector for P with $vP = \lambda^{-1}v$ for some $\lambda > 0$.*

(ii) *τ extends to the pure trace given by $\overline{(a,m)} \longmapsto \lambda^{m-1}(v \cdot a)$.*

(iii) *Any faithful pure trace on H_Γ is given by a positive left eigenvector.*

Proof. We know I_1 has an order unit. We begin by checking that $H_\Gamma = \sum_{k\in N}\hat{P}^k(I_1)$. Take $h \in \Gamma$. If h belongs to the possibly infinite set Γ_1, then $\overline{(e_h, 1)} \in I_1 \subseteq \sum_{k\in N}\hat{P}^k(I_1)$. On the other hand, if $h \notin \Gamma_1$, then $\exists\, g$ in Γ_1 and a path from g to h, that is,

$$\exists\, h, h_2, \ldots, h_{i-3}, h_{i-2} \in \Gamma \text{ such that } p_{h,h_{i-2}}\, p_{h_{i-2},h_{i-3}} \cdots p_{h_2,h_1}\, p_{h_1,g} > 0$$

so that $P^{i-1}(e_g) = p_{h,h_{i-2}} p_{h_{i-2},h_{i-3}} \cdots p_{h_2,h_1} p_{h_1,g} e_h$ + other terms. Since $\hat{P}$ is an order automorphism of H_Γ, so are all of its powers. Hence,

$$\begin{aligned}
\hat{P}^{i-1}\left(\overline{(e_g,1)}\right) &= \overline{(P^{i-1}(e_g),1)} \\
&= \overline{(p_{h,h_{i-2}} p_{h_{i-2},h_{i-3}} \cdots p_{h_2,h_1} p_{h_1,g} e_h + \text{ other terms}, 1)} \\
&= p_{h,h_{i-2}} p_{h_{i-2},h_{i-3}} \cdots p_{h_2,h_1} p_{h_1,g} \overline{(e_h,1)} + \text{ other terms} \\
&\in \hat{P}^{i-1}(I_1).
\end{aligned}$$

Thus, $\overline{(e_h,1)} \in \hat{P}^{i-1}(I_1) \subseteq \sum_{k\in N} \hat{P}^k(I_1)$. Since $\hat{P}^k(I_1) \subseteq \hat{P}^{k+1}(I_1)$ $\forall k$ and $\hat{P}^k(I_1)$ is an order ideal with order unit $\forall k$, $\bigcup_{k\in N} \hat{P}^k(I_1)$ is an order ideal and thus $\sum_{k\in N} \hat{P}^k(I_1)$ $= \bigcup_{k\in N} \hat{P}^k(I_1)$ is an order ideal in H_Γ. Now, take any $g \in \Gamma$ and any m exceeding 1. Since $\hat{P}^{-1} : H_\Gamma \to H_\Gamma$ leaves order ideals stable, so does $(\hat{P}^{-1})^{m-1}$. But $\overline{(e_g,m)} = (\hat{P}^{-1})^{m-1}(\overline{(e_g,1)}) \in \sum_{k\in N} \hat{P}^k(I_1)$. Therefore, $H_\Gamma = \sum_{k\in N} \hat{P}^k(I_1)$.

(i) By Theorem 2.24 and the identity $H_\Gamma = \bigcup_{l\in N} \hat{P}^l(I_1)$, τ extends to a pure trace $\tilde{\tau}$ on H_Γ where

$$\begin{aligned}
&\tilde{\tau} : H_\Gamma \to R, \\
&\hat{P}^l(\overline{(a,m)}) \longmapsto \lambda^{-l}\,\tau(\overline{(a,m)}).
\end{aligned}$$

By this same theorem, $\tilde{\tau}$ satisfies $\tilde{\tau} \circ \hat{P} = \lambda^{-1}\,\tilde{\tau}$ for some $\lambda > 0$. Define

$$\begin{aligned}
&w : \Gamma \to R^+, \\
&g \longmapsto \tilde{\tau}(\overline{(e_g,1)})
\end{aligned}$$

and define the row v whose g-th entry is $w(g)$, that is, $v = (w(g))_{g\in\Gamma}$. To check that v is a left eigenvector (it is obviously nonzero and nonnegative), we calculate

$$\begin{aligned}
v\,P(e_g) &= v \cdot (p_{h,g})_{h\in\Gamma} = \sum_{h\in\Gamma} w(h)\,p_{h,g} = \sum_{h\in\Gamma} \tilde{\tau}(\overline{(e_h,1)})\,p_{h,g} \\
&= \tilde{\tau}\left(\overline{(\Sigma_{h\in\Gamma}\, e_h\, p_{h,g}, 1)}\right) = \tilde{\tau}(\overline{(P(e_g),1)}) = \tilde{\tau} \circ \hat{P}(\overline{(e_g,1)}) \\
&= \lambda^{-1}\,\tilde{\tau}(\overline{(e_g,1)}) = \lambda^{-1}\,w(g) = \lambda^{-1}\,(v \cdot e_g).
\end{aligned}$$

Hence, $v\,P = \lambda^{-1}\,v$.

(ii) Given any $a = \prod_{g\in\Gamma} a_g\, e_g$ in Γ and any $n \in N$, we have

$$\begin{aligned}\tilde{\tau}(\overline{(a,m)}) &= \tilde{\tau} \circ \hat{P}^{-(m-1)}\left(\overline{(\Pi_{g\in\Gamma}\, a_g\, e_g, 1)}\right) \\ &= \lambda^{(m-1)} \sum_{g\in\Gamma} a_g\, \tilde{\tau}(\overline{(e_g, 1)}) \;=\; \lambda^{(m-1)} \sum_{g\in\Gamma} a_g\, v_g \\ &= \lambda^{(m-1)}\,(v \cdot a).\end{aligned}$$

(iii) Any faithful pure trace on H_Γ satisfies the conditions of the present theorem and as such is given by a left eigenvector. ■

The following was noted in [14] if P is column finite.

Theorem 2.26 *Suppose Γ and P are defined so that*

$$\gcd\{k \in N \mid \exists\ m \in N \text{ and } \exists\, \alpha_m > 0 \text{ such that } \alpha_m P^m \leq P^{m+k}\} \;=\; 1. \qquad (\circ)$$

Assume Γ_1 (as described above) exists and let I_1 be the order ideal of H_Γ corresponding to the space-time cone C_0 generated by $\{(e_g, 1) \mid g \in \Gamma_1\}$. If τ is a pure trace of I_1 that does not vanish identically on $\hat{P}^{-r}(I_1)$ $\forall\, r \in N$ (e.g., if τ is faithful), then $\exists$ integers $k_1, k_2, \ldots, k_t$ such that for $l = \sum_{i=1}^{t} k_i$, the following hold:

(i) *τ extends to a pure trace $\tilde{\tau}$ on H_Γ satisfying*

$$\tilde{\tau} \circ \hat{P}^l \;=\; \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}$$

for some $\lambda_1, \cdots, \lambda_t > 0$.

(ii) *There exists $w : \Gamma \to R^+$ such that the row $v = (w(g))_{g\in\Gamma}$ is a left eigenvector for P^l where*

$$vP^l \;=\; \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} v.$$

(iii) *The extended trace $\tilde{\tau}$ in* (i) *is given by*

$$\overline{(a, m\,l + 1)} \;\longmapsto\; \left[\prod_{i=1}^{t} \lambda_i\right]^{m} (v \cdot a)\,, \quad (a \in R\Gamma).$$

In particular, any faithful pure trace on H_Γ is determined by an eigenvector in this fashion.

Proof. By ($\circ$), for each $i \in \{1, 2, \cdots, t\}$, $\exists\, k_i, m_i, \alpha_{m_i} > 0$ such that

$$\gcd(k_1, k_2, \cdots, k_{t-1}, k_t) \;=\; 1$$

and $\alpha_{m_i} P^{m_i} \leq P^{m_i + k_i}$. From this we obtain $\hat{P}^{-k_i} \leq (\alpha_{m_i})^{-1}\mathcal{I}$, that is, $\hat{P}^{-k_i}$ is bounded as an endomorphism of H_Γ. In particular, for any order ideal J of H_Γ, $\hat{P}^{-k_i}(J) \subseteq J$. By the condition on the greatest common divisor, for all sufficiently large n, there exists nonnegative integers $s_1, \cdots, s_t$ such that $n = \sum_{i=1}^{t} s_i k_i$. Let $n_0 - 1$ be the greatest integer not expressible in this form.

Suppose $n \in \mathcal{N}$ is of the form $\sum_{i=1}^{t} s_i k_i$ for some nonnegative integers $s_1, \cdots, s_t$. We will show that τ can be extended to $\hat{P}^n(I_1)$. For each $i = 1, 2, \cdots, t$, $\exists\, \lambda_i > 0$ such that $\tau \circ \hat{P}^{-k_i} = \lambda_i \tau$ and τ does not vanish identically on $\hat{P}^n(I_1)$. We also have $\hat{P}^{-k_i}(I_1) \subseteq I_1$ so that $I_1 \subseteq \hat{P}^{k_i}(I_1)$. Hence, $I_1 \subseteq \hat{P}^{s_i k_i}(I_1)$, $I_1 \subseteq \hat{P}^{\sum_{i=1}^{t} s_i k_i}(I_1) = \hat{P}^n(I_1)$ and $\hat{P}^{-n}(I_1) \subseteq I_1$. Define

$$\begin{aligned} \tau_n : \hat{P}^n(I_1) &\to R, \\ \hat{P}^n(\overline{(a, m)}) &\longmapsto \left(\textstyle\prod_{i=1}^{t} \lambda_i^{-s_i}\right) \tau(\overline{(a, m)}) \end{aligned}$$

where $\overline{(a, m)} \in I_1$. Obviously, τ_n is a trace on $\hat{P}^n(I_1)$.

Next, suppose $\hat{P}^n(\overline{(a, l_1)}) = \hat{P}^m(\overline{(b, l_2)}) \in \hat{P}^n(I_1) \cap \hat{P}^m(I_1)$ where $m = \sum_{i=1}^{t} r_i k_i$ for some nonnegative integers $r_1, \cdots, r_t$ and $\overline{(a, l_1)}, \overline{(b, l_2)} \in I_1$. Then, we have $\hat{P}^{-m}(\overline{(a, l_1)}) = \hat{P}^{-n}(\overline{(b, l_2)}) \in I_1$ and $\tau(\hat{P}^{-m}(\overline{(a, l_1)})) = \tau(\hat{P}^{-n}(\overline{(b, l_2)}))$. Since

$$\begin{aligned} \tau(\hat{P}^{-\sum_{i=1}^{t} r_i k_i}(\overline{(a, l_1)})) &= \left[\textstyle\prod_{i=1}^{t} \lambda_i^{r_i}\right] \tau(\overline{(a, l_1)}) \quad \text{and} \\ \tau(\hat{P}^{-\sum_{i=1}^{t} s_i k_i}(\overline{(b, l_2)})) &= \left[\textstyle\prod_{i=1}^{t} \lambda_i^{s_i}\right] \tau(\overline{(b, l_2)}), \end{aligned}$$

we thus obtain

$$\begin{aligned} \left[\textstyle\prod_{i=1}^{t} \lambda_i^{r_i}\right] \tau(\overline{(a, l_1)}) &= \left[\textstyle\prod_{i=1}^{t} \lambda_i^{s_i}\right] \tau(\overline{(b, l_2)}) \quad \text{and} \\ \left[\textstyle\prod_{i=1}^{t} \lambda_i^{-s_i}\right] \tau(\overline{(a, l_1)}) &= \left[\textstyle\prod_{i=1}^{t} \lambda_i^{-r_i}\right] \tau(\overline{(b, l_2)}). \end{aligned}$$

Hence, $\tau_n(\hat{P}^n(\overline{(a,l_1)})) = \tau_m(\hat{P}^m(\overline{(b,l_2)}))$, i.e., τ_n and τ_m agree on $\hat{P}^n(I_1)\cap\hat{P}^m(I_1)$.

Let $k = \sum_{i=1}^t k_i$. From $\hat{P}^{-k_i}(I_1) \subseteq I_1$, we obtain $I_1 \subseteq \hat{P}^{k_i}(I_1)$ and so $I_1 \subseteq \hat{P}^{\sum_{i=1}^t k_i}(I_1) = \hat{P}^k(I_1)$. Thus, for any $n \geq n_0$, we have $I_1 \subseteq \hat{P}^n(I_1)$ and $\hat{P}^n(I_1) \subseteq \hat{P}^{n+k}(I_1)$. Choose $n_0 \leq j \leq n_0 + k - 1$. Then,

$$\hat{P}^j(I_1) \subseteq \hat{P}^{j+k}(I_1) \subseteq \hat{P}^{j+2k}(I_1) \subseteq \hat{P}^{j+3k}(I_1) \subseteq \cdots .$$

Define

$$H_j := \sum_{q\in Z^+} \hat{P}^{j+qk}(I_1) = \bigcup_{q\in Z^+} \hat{P}^{j+qk}(I_1)$$

and

$$\tilde{\tau}_j : H_j \longrightarrow R,$$
$$\tilde{\tau}_j|_{\hat{P}^{j+qk}(I_1)} = \tau_{j+qk},$$

that is, the restriction of $\tilde{\tau}_j$ to $\hat{P}^{j+qk}(I_1)$ is τ_{j+qk}. As in the proof of Theorem 2.24 (using the criterion of [8: Theorem 3.1] for purity of traces on dimension groups), $\tilde{\tau}_j$ is a pure trace extending τ. Since $\tau_n|_{\hat{P}^n(I_1)\cap\hat{P}^m(I_1)} = \tau_m|_{\hat{P}^n(I_1)\cap\hat{P}^m(I_1)}$ for any $m,n \geq n_0$, we obtain $\tilde{\tau}_{j_1}|_{H_{j_1}\cap H_{j_2}} = \tilde{\tau}_{j_2}|_{H_{j_1}\cap H_{j_2}}$.

Obviously, $\sum_{n\geq n_0} \hat{P}^n(I_1) = \sum_{j=n_0}^{n_0+k-1} H_j$. If we set

$$H := \sum_{n\geq n_0} \hat{P}^n(I_1) = \sum_{j=n_0}^{n_0+k-1} H_j,$$

then H is an order ideal in H_Γ. Therefore, if we define

$$\tilde{\tau} : H \longrightarrow R,$$
$$\textstyle\sum_{j=n_0}^{n_0+k-1} \overline{(a_j,m_j)} \longmapsto \sum_{j=n_0}^{n_0+k-1} \tilde{\tau}_j(\overline{(a_j,m_j)}),$$

the preceding argument implies $\tilde{\tau}$ is a pure trace on H that extends τ.

We now show that $H = H_\Gamma$. Select h in Γ. If $h \in \Gamma_1$, then obviously $\overline{(e_h,1)} \in I_1 \subseteq H$. If $h \notin \Gamma_1$, then $\exists\, g \in \Gamma_1$ and $g_1, g_2, \ldots, g_{u-1}$ in Γ such that $p_{h,g_{u-1}} \cdots p_{g_2,g_1} p_{g_1,g} > 0$. If $u \geq n_0$, then

$$\hat{P}^u(\overline{(e_g,1)}) = \overline{(P^u(e_g),1)}$$

$$\begin{aligned} &= \overline{(p_{h,g_{u-1}}\, p_{g_{u-1},g_{u-2}} \cdots p_{g_2,g_1}\, p_{g_1,g}\, e_h \,+\, \text{other terms},\, 1)} \\ &= p_{h,g_{u-1}}\, p_{g_{u-1},g_{u-2}} \cdots p_{g_2,g_1}\, p_{g_1,g}\, \overline{(e_h, 1)} \,+\, \text{other terms} \\ &\in \hat{P}^u(I_1). \end{aligned}$$

But $\hat{P}^u(I_1) \subseteq \sum_{n \geq n_0} \hat{P}^n(I_1) = H$. thus $\overline{(e_h, 1)} \in H$. On the other hand, suppose $u < n_0$. Select $x \in N$ such that $k_1 x + u \geq n_0$. Since $\alpha_{m_1} \mathcal{I} \leq P^{k_1}$, we have $(\alpha_{m_1})^x \mathcal{I} \leq (P^{k_1})^x$ and thus $(\alpha_{m_1})^x P^u \leq P^{k_1 x + u}$. But as shown above, $\hat{P}^u(I_1) \subseteq \hat{P}^{k_1 x+u}(I_1)$ so that

$$\overline{(e_h, 1)} \in \hat{P}^u(I_1) \subseteq \hat{P}^{k_1 x+u}(I_1) \subseteq \sum_{n \geq n_0} \hat{P}^n(I_1) = H.$$

In total, this establishes $\overline{(e_h, 1)} \in H$ for all h in Γ.

Consider $\overline{(e_h, z)}$. If $z > 1$, $\exists\, y \in N$ such that $k_1 y + z \geq n_0$ and $k_1 y - 1 \geq n_0$. Thus. $(\alpha_{m_1})^y P^z \leq P^{k_1 y + z}$ and $(\alpha_{m_1})^y P^{-1} \leq P^{k_1 y - 1}$ so that $P^{-1}(H) \subseteq P^{k_1 y-1}(H)$ which yields

$$\hat{P}^{-1}(\overline{(e_h, 1)}) = \overline{(e_h, 2)} \in P^{-1}(H) \subseteq P^{k_1 y-1}(H) \subseteq P^{k_1 y-1}\left[\sum\nolimits_{n \geq n_0} \hat{P}^n(I_1)\right] \in H.$$

Therefore, $\overline{(e_h, 2)} \in H$ and by induction, for any positive integer z, $\overline{(e_h, z)} \in H$. From this, we have that H contains the order ideal generated by $\{\overline{(e_h, z)} \mid h \in \Gamma,\, z \in N\}$ which is H_Γ by definition. That is, $H \supseteq H_\Gamma$. Since $H \subseteq H_\Gamma$, this implies $H = H_\Gamma$.

(i) It is obvious that $\bar{\tau}$ is a pure trace on $H = H_\Gamma$. Now we check that

$$\bar{\tau} \circ \hat{P}^{(\Sigma_{i=1}^t k_i)} = \left[\prod\nolimits_{i=1}^t \lambda_i\right]^{-1} \bar{\tau}.$$

Set $k = \sum_{i=1}^t k_i$ and choose $\overline{(a_j, m_j)} \in H_j$ where $n_0 \leq j \leq n_0 + k - 1$. By definition, $\overline{(a_j, m_j)} = \hat{P}^{j+kq}(\overline{(c_j, l_j)})$ for some $\overline{(c_j, l_j)} \in I_1$ where $j = \sum_{i=1}^t s_i k_i$ ($s_i \in Z^+$, $l_j \in N$). We thus calculate

$$\begin{aligned} \bar{\tau} \circ \hat{P}^{(\Sigma_{i=1}^t k_i)}(\overline{(a_j, m_j)}) &= \bar{\tau}_j \circ \hat{P}^{(\Sigma_{i=1}^t k_i)}\left(\hat{P}^{j+kq}(\overline{(c_j, l_j)})\right) \\ &= \bar{\tau}_j\left(\hat{P}^{j+k(q+1)}(\overline{(c_j, l_j)})\right) \end{aligned}$$

$$\begin{aligned}
&= \left[\prod_{i=1}^{t} \lambda_i^{-(s_i+1+q)}\right] \tau(\overline{(c_j, l_j)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}_j \left(\hat{P}^{\Sigma_{i=1}^{t}(s_i+q)k_i}(\overline{(c_j, l_j)})\right) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}_j(\hat{P}^{j+kq}\overline{(c_j, l_j)})) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}_j(\overline{(a_j, m_j)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}(\overline{(a_j, m_j)}).
\end{aligned}$$

By linearity, any element of the form $\sum_{j=n_0}^{n_0+k-1} \overline{(a_j, m_j)}$ (with $\overline{(a_j, m_j)} \in H_j$) lies in H_Γ. This fact allows us to write

$$\begin{aligned}
\tilde{\tau} \circ \hat{P}^{(\Sigma_{i=1}^{t} k_i)} \left(\sum_{j=n_0}^{n_0+k-1} \overline{(a_j, m_j)}\right) &= \sum_{j=n_0}^{n_0+k-1} \tilde{\tau} \circ \hat{P}^{(\Sigma_{i=1}^{t} k_i)} \left(\overline{(a_j, m_j)}\right) \\
&= \sum_{j=n_0}^{n_0+k-1} \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}(\overline{(a_j, m_j)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau} \left(\sum_{j=n_0}^{n_0+k-1} \overline{(a_j, m_j)}\right)
\end{aligned}$$

from which we can conclude $\tilde{\tau} \circ \hat{P}^{(\Sigma_{i=1}^{t} k_i)} = \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}$.

(ii) Define

$$\begin{aligned}
w : \Gamma &\longrightarrow R^+, \\
g &\longmapsto \tilde{\tau}(\overline{(e_g, 1)}).
\end{aligned}$$

Setting $v = (w(g))_{g\in\Gamma}$, we calculate

$$\begin{aligned}
v \cdot P^{\Sigma_{i=1}^{t} k_i} \cdot e_g &= \sum_{h\in\Gamma} \tilde{\tau}(\overline{(e_h, 1)}) \cdot p_{h,g}^{k} \\
&= \sum_{h\in\Gamma} \tilde{\tau} \left(\overline{(p_{h,g}^{k} e_h, 1)}\right) = \tilde{\tau} \left(\overline{(P^k(e_g), 1)}\right) \\
&= \tilde{\tau} \circ \hat{P}^k(\overline{(e_g, 1)}) = \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} \tilde{\tau}(\overline{(e_g, 1)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^{-1} (v \cdot e_g)
\end{aligned}$$

from which we obtain $v\, P^{\Sigma_{i=1}^{t} k_i} = [\prod_{i=1}^{t} \lambda_i]^{-1} v$.

(iii) Take $\overline{(a, m\sum_{i=1}^{t} k_i + 1)} = \overline{(a, mk+1)} \in H_\Gamma$ where $a = \prod_{g\in\Gamma} a_g e_g \in R\Gamma$. The identity $\tilde{\tau} \circ \hat{P}^{-k} = (\prod_{i=1}^{t} \lambda_i)^{-1} \tilde{\tau}$ yields $\tilde{\tau} \circ \hat{P}^{k} = (\prod_{i=1}^{t} \lambda_i) \tilde{\tau}$ so that

$$\tilde{\tau}(\overline{(a, mk+1)}) = \tilde{\tau} \circ \hat{P}^{mk}(\overline{(a, 1)})$$

$$
\begin{aligned}
&= \left[\prod_{i=1}^{t} \lambda_i\right] \tilde{\tau} \circ \hat{P}^{(m-1)k}(\overline{(a,1)}) \\
&\;\vdots \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^m \tilde{\tau}(\overline{(a,1)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^m \tilde{\tau}\left(\overline{(\prod_{g\in\Gamma} a_g\, e_g, 1)}\right) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^m \tilde{\tau}\left(\prod_{g\in\Gamma} a_g\, \overline{(e_g,1)}\right) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^m \sum_{g\in\Gamma} a_g\, \tilde{\tau}(\overline{(e_g,1)}) \\
&= \left[\prod_{i=1}^{t} \lambda_i\right]^m (v \cdot a).
\end{aligned}
$$

Hence, $\tilde{\tau}(\overline{(a, m \cdot \sum_{i=1}^{t} k_i + 1)}) = \left[\prod_{i=1}^{t} \lambda_i\right]^m (v \cdot a)$. ■

Theorems 2.24 and 2.25 can be regarded as corollaries of Theorem 2.26. If t is set to 1 in Theorem 2.26, the other results (except for uniqueness) can be obtained.

Next, we obtain results concerning the Choquet boundary associated to P and Γ_1. Let $S(I_1, u)$ denote the trace space of I_1, where u is an order unit of I_1 and let $\partial_e S(I_1, u)$ denote the set of pure traces of $S(I_1, u)$.

The following extends [14; 1.3] to our situation.

Proposition 2.27 *Suppose that Γ and P are defined so that $(\bullet)$ holds and Γ_1 exists. Let I_1 denote the order ideal of H_Γ corresponding to the space-time cone C_0 generated by $\{\overline{(e_g, 1)} \mid g \in \Gamma_1\}$. Then,*

(i) *For each $0 < \lambda \le \infty$, the set of traces $E_\lambda := \{\tau \in S(I_1, u) \mid \tau \circ \hat{P}^{-1} = \lambda^{-1}\tau\}$ is a closed face in the trace space of (I_1, u) provided E_λ is nonempty.*

(ii) *$\partial_e S(I_1, u) = \bigcup_{0<\lambda\le\infty} \partial_e E_\lambda$ where $\partial_e E_\lambda$ is the set of extremal traces in E_λ.*

(iii) *All faithful pure traces belong to $\bigcup_{0<\lambda<\infty} \partial_e E_\lambda$.*

Proof. (i) We begin by showing that $E_\infty = \{\tau \in S(I_1, u) \mid \tau \circ \hat{P}^{-1} = 0\}$ is a closed face. The inclusion $E_\infty \subseteq S(I_1, u)$ is obvious and the convexity of E_∞ is clear.

Select γ in E_∞ and suppose $\gamma = t_1\,\alpha + t_2\,\beta$ where $\alpha, \beta \in S(I_1, u)$, $0 < t_1, t_2 < 1$ and $t_1 + t_2 = 1$. Since $\gamma \circ \hat{P}^{-1} = t_1\,\alpha \circ \hat{P}^{-1} + t_2\,\beta \circ \hat{P}^{-1}$, we have $\gamma \circ \hat{P}^{-1}(u) = t_1\,\alpha \circ \hat{P}^{-1}(u) + t_2\,\beta \circ \hat{P}^{-1}(u)$. Since $\gamma \circ \hat{P}^{-1}(u) = 0$, $t_1\alpha \circ \hat{P}^{-1}(u) \geq 0$ and $t_2\,\beta \circ \hat{P}^{-1}(u) \geq 0$, it follows that $t_1\alpha \circ \hat{P}^{-1}(u) = 0$ and $t_2\beta \circ \hat{P}^{-1}(u) = 0$. Therefore, $\alpha \circ \hat{P}^{-1}$ and $\beta \circ \hat{P}^{-1}$ are both zero and as such belong to E_∞. Thus, E_∞ is a face of $S(I_1, u)$. Furthermore,

$$E_\infty \;=\; \{\tau \in S(I_1, u) \,|\, \tau \circ \hat{P}^{-1} = 0\} \;=\; \bigcap\nolimits_{x \in I_1} \{\tau \in S(I_1, u) \,|\, \tau(\hat{P}^{-1}(x)) = 0\}$$

implying that E_∞ is a closed subset of $S(I_1, u)$. In total, E_∞ is a closed face in the trace space $S(I_1, u)$.

We now establish this fact for E_λ with $\lambda < \infty$. Assume $0 < \lambda < \infty$ and E_λ is nonempty. Then,

$$\begin{aligned} E_\lambda &= \{\tau \in S(I_1, u) \,|\, \tau \circ (\hat{P}^{-1} - \lambda^{-1} I) = 0\} \\ &= \bigcap\nolimits_{x \in I_1} \{\tau \in S(I_1, u) \,|\, \tau(\hat{P}^{-1}(x) - \lambda^{-1}(x)) = 0\}. \end{aligned}$$

This implies E_λ is closed and therefore compact in $S(I_1, u)$. Once again, the convexity of E_λ is obvious. We will complete the proof that E_λ is a closed face in the trace space $S(I_1, u)$ after we establish part (ii).

(ii) Since E_∞ is a face, $E_\infty \cap \partial_e S(I_1, u) = \partial_e E_\infty$. Suppose τ belongs to $\partial_e S(I_1, u) \setminus E_\infty$ so that τ does not identically vanish on $\hat{P}^{-1}(I_1)$. By Theorem 2.25, $\tau \in E_\lambda$ (some $0 < \lambda < \infty$) and clearly $\tau \in \partial_e E_\lambda$. Hence, $\partial_e S(I_1, u) \subseteq \bigcup_{0<\lambda\leq\infty} \partial_e E_\lambda$.

Conversely, consider

$$E_{\lambda_0} \;=\; \{\tau \in S(I_1, u) \,|\, \tau \circ \hat{P}^{-1} = {\lambda_0}^{-1} \tau\}.$$

Case 1: $0 < \lambda_0 < \infty$.

Choose τ in $\partial_e E_{\lambda_0}$. By standard Choquet theory, there exists a probability measure μ on $\partial_e S(I_1, u)$ such that τ can be represented by the integral

$$\tau(z) \;=\; \int_{\partial_e S(I_1, u)} z \, d\mu \quad \forall\, z \in I,$$

Since $\partial_e S(I_1, u) \subseteq \bigcup_{0<\lambda\leq\infty} \partial_e E_\lambda$,

$$\tau(z) = \int_{E_1} z \, d\mu + \int_{E_2} z \, d\mu + \int_{E_3} z \, d\mu$$

where E_1, E_2, E_3 are Borel subsets defined by

$$\begin{aligned} E_1 &= \partial_e S(I_1, u) \cap (\textstyle\bigcup_{\lambda<\lambda_0} \partial_e E_\lambda), \\ E_2 &= \partial_e S(I_1, u) \cap (\textstyle\bigcup_{\lambda>\lambda_0} \partial_e E_\lambda), \\ E_3 &= \partial_e S(I_1, u) \cap \partial_e E_{\lambda_0}. \end{aligned}$$

If we choose $z \in I^+$, then $\int_{E_i} z \, d\mu$ is nonnegative for $i = 1, 2, 3$. Suppose $\mu(E_1) \neq 0$. Then, for any $\lambda < \lambda_0$,

$$\lambda_0^k \tau(z) \;\geq\; \lambda_0^k \int_{E_1} z \, d\mu \;\geq\; \lambda_0^k \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} z \, d\mu.$$

Evaluating at $z = u$ yields

$$1 \;=\; \tau(u) \;=\; \lambda_0^k \tau \circ \hat{P}^{-k}(u) \;=\; \lambda_0^k \tau(\hat{P}^{-k}(u)) \;\geq\; \lambda_0^k \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^{-k}(u) \, d\mu.$$

But for α in $\partial_e S(I_1, u) \cap \partial_e E_\lambda$, we know that $\alpha(\hat{P}^{-k}(u)) = \lambda^{-k}\alpha(u) = \lambda^{-k}$, so

$$\int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^{-k}(u) \, d\mu \;=\; \lambda^{-k} \mu(\partial_e S(I_1, u) \cap \partial_e E_\lambda).$$

Thus,

$$1 \;\geq\; \frac{\lambda_0^k}{\lambda^k} \mu(\partial_e S(I_1, u) \cap \partial_e E_\lambda).$$

Letting $k \to \infty$ for any fixed $\lambda < \lambda_0$ yields a contradiction. Therefore, $\mu(E_1) = 0$.

Suppose $\mu(E_2) \neq 0$, then for any $\lambda > \lambda_0$,

$$\lambda_0^k \tau(z) \;\geq\; \lambda_0^k \int_{E_2} z \, d\mu \;\geq\; \lambda_0^k \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} z \, d\mu.$$

Evaluating at $z = u$, Theorem 2.25 implies $\tau \circ \hat{P} = \lambda_0 \tau$ so that

$$1 \;=\; \tau(u) \;=\; \lambda_0{}^{-k} \tau \circ \hat{P}^k(u) \;=\; \lambda_0{}^{-k} \tau(\hat{P}^k(u)) \;\geq\; \lambda_0{}^{-k} \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^k(u) \, d\mu.$$

But for α in $\partial_e S(I_1, u) \cap \partial_e E_\lambda$, Theorem 2.25 implies

$$\alpha(\hat{P}^k(u)) = \lambda^k \alpha(u) = \lambda^k$$

so that

$$\int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^k(u)\, d\mu = \lambda^k \mu(\partial_e S(I_1, u) \cap \partial_e E_\lambda).$$

Thus,

$$1 \geq \frac{\lambda^k}{\lambda_0^k} \mu(\partial_e S(I_1, u) \cap \partial_e E_\lambda).$$

Letting $k \to \infty$ for any fixed $\lambda > \lambda_0$ yields a contradiction. Therefore, $\mu(E_2) = 0$.

So far, $\mu(E_1) = \mu(E_2) = 0$ so that

$$\tau(z) = \int_{E_3} z\, d\mu$$

where $E_3 = \partial_e S(I_1, u) \cap \partial_e E_{\lambda_0}$. Now, suppose μ is not point mass. Then, there exist measurable sets U_1, U_2 with positive μ-measure such that $U_1 \bigcap U_2 = \emptyset$, $U_1 \bigcup U_2 = E_3$ and $\mu(U_1) + \mu(U_2) = \mu(E_3)$. We know that $1 = \tau(u) = \int_{E_3} u\, d\mu = \mu(E_3)$. Define $\alpha, \beta : I_1 \to R$ via

$$\begin{aligned} \alpha(z) &= \frac{1}{\mu(U_1)} \int_{U_1} z\, d\mu . \\ \beta(z) &= \frac{1}{\mu(U_2)} \int_{U_2} z\, d\mu . \end{aligned}$$

Obviously, α and β are both positive homomorphisms on I_1. We will show that $\alpha \in E_{\lambda_0}$. First, since $\alpha(u) = \frac{1}{\mu(U_1)} \int_{U_1} u\, d\mu = \frac{1}{\mu(U_1)} \mu(U_1) = 1$, we have $\alpha \in S(I_1, u)$. Furthermore, since $\gamma \circ \hat{P}^{-1}(z) = \frac{1}{\lambda_0} \gamma(z)\ \forall \gamma \in U_1$, we have

$$\alpha \circ \hat{P}^{-1}(z) = \frac{1}{\mu(U_1)} \int_{U_1} \hat{P}^{-1}(z)\, d\mu = \frac{1}{\lambda_0\, \mu(U_1)} \int_{U_1} z\, d\mu = \frac{1}{\lambda_0} \alpha(z).$$

Hence, $\alpha \in E_{\lambda_0}$. Similarly, $\beta \in E_{\lambda_0}$. Furthermore,

$$[\mu(U_1)\, \alpha + \mu(U_2)\, \beta]\, (z) = \mu(U_1)\, \alpha(z) + \mu(U_2)\, \beta(z)$$

$$
\begin{aligned}
&= \int_{U_1} z\,d\mu + \int_{U_2} z\,d\mu \\
&= \int_{U_1 \cup U_2} z\,d\mu \\
&= \int_{E_3} z\,d\mu \\
&= \tau(z).
\end{aligned}
$$

But the identity $\mu(U_1)\,\alpha + \mu(U_2)\,\beta = \tau$ contradicts $\tau \in \partial_e E_{\lambda_0}$. Therefore, μ is supported at a point of E_3.

Case 2: $\lambda_0 = \infty$.

Consider

$$E_\infty = \{\tau \in S(I_1, u) \mid \tau \circ \hat{P}^{-1} = 0\}.$$

Given $\tau \in \partial_e E_\infty$, standard Choquet theory once again yields a probability measure δ on $\partial_e S(I_1, u)$ such that τ can be represented as the corresponding integral using this measure. That is,

$$\tau(z) = \int_{\partial_e S(I_1,u)} z\,d\delta \quad \forall z \in I_1.$$

Since $\partial_e S(I_1, u) \subseteq \bigcup_{0<\lambda\leq+\infty} \partial_e E_\lambda$, we have

$$\tau(z) = \int_{F_1} z\,d\delta + \int_{F_2} z\,d\delta$$

where F_1 and F_2 are Borel subsets given by

$$
\begin{aligned}
F_1 &= \partial_e S(I_1, u) \cap \left(\textstyle\bigcup_{0<\lambda<+\infty} \partial_e E_\lambda\right), \\
F_2 &= \partial_e S(I_1, u) \cap \partial_e E_\infty .
\end{aligned}
$$

If we choose $z \in I^+$, then $\int_{F_i} z\,d\mu$ is nonnegative for $i = 1, 2$. For any $0 < \lambda < \infty$,

$$\tau(z) \geq \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} z\,d\delta.$$

Evaluating at $z = \hat{P}^{-1}(u)$ yields

$$0 = \tau(\hat{P}^{-1}(u)) \geq \int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^{-1}(u)\,d\delta.$$

But for any α in $\partial_e S(I_1, u) \cap \partial_e E_\lambda$, we have $\alpha(\hat{P}^{-1}(u)) = \lambda^{-1}\alpha(u) = \lambda^{-1}$. Thus,

$$\int_{\partial_e S(I_1,u)\cap\partial_e E_\lambda} \hat{P}^{-1}(u)\, d\delta \;=\; \lambda^{-1}\,\delta(\partial_e S(I_1, u) \cap \partial_e E_\lambda)$$

which implies

$$0 \;\geq\; \lambda^{-1}\,\delta(\partial_e S(I_1, u) \cap \partial_e E_\lambda) \;\geq\; 0$$

from which we obtain $\delta(E_1) = 0$ and

$$\tau(z) \;=\; \int_{\partial_e S(I_1,u)\cap\partial_e E_\infty} z\, d\delta.$$

Suppose δ is not point mass. Then, there exist measurable sets V_1, V_2 of positive δ-measure such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = F_2$ and $\delta(V_1) + \delta(V_2) = \delta(F_2)$. Define $\zeta, \eta : I_1 \to R$ via

$$\begin{aligned}
\zeta(z) &= \frac{1}{\delta(V_1)}\int_{V_1} z\, d\delta\,, \\
\eta(z) &= \frac{1}{\delta(V_2)}\int_{V_2} z\, d\delta\,.
\end{aligned}$$

Obviously, ζ and η are positive homomorphisms on I_1. We will show that $\zeta \in E_\infty$. Since $\zeta(u) = \frac{1}{\delta(V_1)}\int_{V_1} u\, d\delta = \frac{1}{\delta(V_1)}\delta(V_1) = 1$, we have $\zeta \in S(I_1, u)$. Also, since $\varepsilon \circ \hat{P}^{-1}(z) = 0$ for every $\varepsilon \in V_1$,

$$\zeta \circ \hat{P}^{-1}(z) \;=\; \zeta(\hat{P}^{-1}(z)) \;=\; \frac{1}{\delta(V_1)}\int_{V_1} \hat{P}^{-1}(z)\, d\delta \;=\; 0.$$

Hence, $\zeta \in E_\infty$ and by a similar argument, $\eta \in E_\infty$. This yields

$$\begin{aligned}
[\delta(V_1)\,\zeta + \delta(V_2)\,\eta](z) &= \delta(V_1)\,\zeta(z) + \delta(V_2)\,\eta(z) \\
&= \int_{V_1} z\, d\delta + \int_{V_2} z\, d\delta \\
&= \int_{V_1\cup V_2} z\, d\delta \\
&= \int_{F_2} z\, d\delta \\
&= \tau(z)
\end{aligned}$$

from which we obtain $\delta(V_1)\,\zeta + \delta(V_2)\,\eta = \tau$. But this contradicts $\tau \in \partial_e E_\infty$, implying that δ is supported at a point of F_2. In light of this fact,

$$\bigcup_{0<\lambda\leq\infty} \partial_e E_\lambda \;\subseteq\; \partial_e S(I_1, u).$$

Therefore, $\partial_e S(I_1, u) = \bigcup_{0<\lambda\leq\infty} \partial_e E_\lambda$, completing the proof of part (ii).

To complete the proof of part (i), in particular, to show that E_λ $(0 < \lambda < \infty)$ is a closed face in the trace space $S(I_1, u)$, note that $S(I_1, u)$ is a simplex [7; 10.5].

By construction, I_1 has an order unit. Thus, $S(I_1, u)$ is a compact, locally convex, Hausdorff space. Hence, $S(I_1, u)$ is a Choquet simplex. We verify the facial criterion of [7] (see Corollary 11.19 therein). When $\lambda < \infty$, $\partial_e E_\lambda$ is a compact subset of $\partial_e S(I_1, u)$ and E_λ is the closure of the convex hull of $\partial_e E_\lambda$. Thus, for the case of $\lambda < \infty$, it follows that E_λ is a closed face of $S(I_1, u)$, completing the proof of part (i).

(iii) As mentioned in Theorem 2.25, a pure faithful trace τ does not vanish identically on $\hat{P}^{-1}(I_1)$. But by part (ii), we know that τ does not belong to $\partial_e E_\infty$ and

$$\tau \;\in\; \partial_e S(I_1, u) \;=\; \bigcup_{0<\lambda\leq\infty} \partial_e E_\lambda.$$

Therefore, the set of faithful pure traces is contained in $\bigcup_{0<\lambda<\infty} \partial_e E_\lambda$. ■

2.3 Examples and Other Results

We start this section with two examples. The first emanates from power series, the second from Laurent power series. The traces are determined and analyzed.

Example 2.28 Take $\Gamma = Z^+$ as the state space of a Markov chain. Here, $R\Gamma$ can be identified with the space of bounded formal power series, that is,

$$R\Gamma \;=\; \left\{ f \in R\{x\} \;\middle|\; f(x) = \textstyle\sum_{i=0}^{\infty} a_i\, x^i \text{ where } \sup_i |a_i| < \infty \right\}.$$

Take $p \in R\Gamma$ where $p(x) = \sum_{i=0}^{\infty}(\frac{x}{2})^i$ and define a linear operator

$$P : R\Gamma \longrightarrow R\Gamma.$$
$$f(x) \longmapsto p(x)f(x).$$

We also can describe the Markov chain as an infinite matrix. Form $M\Gamma$, the real vector space with basis generated by $\{e_g\}_{g\in\Gamma}$ (sometimes thought of as the set of functions $\Gamma \longrightarrow R$ with bounded support). Let P be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2^2} & \frac{1}{2} & 1 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & 0 & \cdot & \cdot \\ \frac{1}{2^4} & \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Clearly, P is a nonnegative infinite matrix and it is easy to check that the set of its row sums is bounded. Thus, $P : M\Gamma \to M\Gamma$ given by

$$e_i \overset{P}{\longmapsto} e_i + \frac{1}{2}e_{i+1} + \frac{1}{2^2}e_{i+2} + \cdots + \frac{1}{2^k}e_{i+k} + \cdots = \prod_{k=0}^{\infty}\frac{1}{2^k}e_{i+k}$$

is well-defined. Regarding $M\Gamma$ as $R\Gamma$, we obtain the dimension group

$$H_\Gamma := \varinjlim R\Gamma \overset{P}{\longrightarrow} R\Gamma \overset{P}{\longrightarrow} R\Gamma \longrightarrow \cdots .$$

Obviously, $id \leq P$ so that P satisfies condition ($\bullet$).

The radius of convergence of $p(x) = \sum_{i=0}^{\infty}\frac{1}{2^i}x^i$ is 2 and the interval of convergence is $(-2, 2)$. We are interested in the nonnegative part of this interval, that is, $[0, 2)$. For $0 \leq r < 2$, define

$$V_r : \Gamma \longrightarrow R,$$
$$i \longmapsto r^i.$$

We calculate

$$
\begin{aligned}
V_r \cdot P &= \begin{pmatrix} 1 & r & r^2 & r^3 & r^4 & \cdot & \cdot \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2^2} & \frac{1}{2} & 1 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & 0 & \cdot & \cdot \\ \frac{1}{2^4} & \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{1}{2^i} r^i & (\sum_{i=0}^{\infty} \frac{1}{2^i} r^i)\, r & (\sum_{i=0}^{\infty} \frac{1}{2^i} r^i)\, r^2 & \cdot & \cdot \end{pmatrix} \\
&= \left[\sum_{i=0}^{\infty} \frac{1}{2^i} r^i\right] \begin{pmatrix} 1 & r & r^2 & r^3 & r^4 & \cdot & \cdot \end{pmatrix} \\
&= p(r)\, V_r .
\end{aligned}
$$

Thus, $p(r) = \sum_{i=0}^{\infty} \frac{1}{2^i} r^i$ is an eigenvalue of P with corresponding eigenvector V_r.

When $r = 0$, the eigenvector $V_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot \end{pmatrix}$ corresponding to the eigenvalue $p(0) = 1$ yields a harmonic function $h_0 : \Gamma \times N \longrightarrow R^+$ given by

$$(g, n) \longmapsto \frac{(V_0)_g}{p(0)^n} = (V_0)_g.$$

That is, $h_0(0, n) = 1$ for all $n \in N$ and $h_0(g, n) = 0$ for all $(g, n) \in (\Gamma \setminus \{0\}) \times N$. It is easy to check that $\sum_{g \in \Gamma} h_0(g, n) = 1 < \infty \ \forall\, n \in N$ so that h_0 induces a trace $\tau_0 : H_\Gamma \longrightarrow R$ defined by

$$\overline{(e_g, n)} \longmapsto h_0(g, n).$$

Clearly, τ_0 is not a faithful trace.

Now, suppose $r \in (0, 1)$. The eigenvector $V_r = \begin{pmatrix} 1 & r & r^2 & r^3 & r^4 & \cdot & \cdot \end{pmatrix}$ corresponding to the eigenvalue $p(r) = \frac{2}{2-r}$ yields a harmonic function $h_r : \Gamma \times N \longrightarrow R^+$ defined by

$$(g, n) \longmapsto \frac{(V_r)_g}{p(r)^n} = \frac{r^g}{(\frac{2}{2-r})^n} = \frac{r^g (2-r)^n}{2^n}.$$

It is easy to check that for any nonnegative integer n,

$$\sum_{g\in\Gamma} h_r(g,n) = \frac{(2-r)^n}{2^n}\sum_{g\in\Gamma} r^g = \frac{(2-r)^n}{2^n}\frac{1}{1-r} < \infty.$$

Thus, h_r induces a trace $\tau_r : H_\Gamma \longrightarrow R$ by means of

$$\overline{(e_g, n)} \longmapsto h_r(g,n) = \frac{r^g(2-r)^n}{2^n}.$$

It is clear that τ_r is faithful.

Definition 2.29 A Markov process (P,Γ) is said to be *irreducible* if for any two states $g, h \in \Gamma$, there exists a path from g to h, that is,

$$\exists\ h_1, h_2, \cdots, h_{k-1}, h_k \in \Gamma \ \text{ such that } \ p_{h,h_k}\, p_{h_k,h_{k-1}} \cdots p_{h_3,h_2}\, p_{h_2,h_1}\, p_{h_1,g} > 0.$$

It is obvious that the Markov process in Example 2.28 is not irreducible. For instance, no path exists from state 2 to state 1. However, we can modify this example so that the resulting process is irreducible.

Example 2.30 Take $\Gamma = Z$ so that $R\Gamma$ can be identified with the space of bounded formal Laurent power series, that is,

$$R\Gamma = \left\{ f \in R\{x\} \,\middle|\, f(x) = \textstyle\sum_{i=-\infty}^{\infty} a_i\, x^i \text{ and } \sup_i |a_i| < \infty \right\}.$$

Set $p(x) = \sum_{i=-\infty}^{\infty} x^i\, 2^{-|i|}$ and define $P : R\Gamma \to R\Gamma$ via

$$f(x) \longmapsto p(x)f(x).$$

The matrix representation of P with the respect to the natural basis is:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \cdot \\ \cdot & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \cdot \\ \cdot & \frac{1}{2^2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2^2} & \cdot \\ \cdot & \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & \frac{1}{2} & \cdot \\ \cdot & \frac{1}{2^4} & \frac{1}{2^3} & \frac{1}{2^2} & \frac{1}{2} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Clearly, P is a nonnegative infinite matrix, the row sums of which are bounded above by 3. As before, we obtain a dimension group via

$$H_\Gamma := \varinjlim R\Gamma \xrightarrow{P} R\Gamma \xrightarrow{P} R\Gamma \longrightarrow \cdots .$$

Note that $id \leq P$ so that P satisfies condition ($\bullet$).

When viewed as a complex function, p converges for all x satisfying $1/2 < |x| < 2$. For each r in $(\frac{1}{2}, 2)$, define a map

$$\begin{aligned} V_r : \Gamma &\to R, \\ i &\longmapsto r^i. \end{aligned}$$

As before, $V_r \cdot P = p(r)\, V_r$ so that V_r is a left eigenvector of P corresponding to the eigenvalue $p(r) = 3r/(2-r)(2r-1)$. This eigenvector yields a harmonic function $h_r : \Gamma \times N \longrightarrow R^+$ by means of

$$(g,n) \longmapsto \frac{(V_r)_g}{p(r)^n} = \frac{r^g}{\left(\frac{3r}{(2-r)(2r-1)}\right)^n} = \frac{r^{g-n}(2-r)^n(2r-1)^n}{3^n}.$$

But

$$\begin{aligned} \sum_{g\in\Gamma} h_r(g,n) &= \frac{(2-r)^n(2r-1)^n}{3^n r^n} \sum_{g\in\Gamma} r^g \\ &= \frac{(2-r)^n(2r-1)^n}{3^n r^n} \left(\sum\nolimits_{i=0}^{\infty} r^i + \sum\nolimits_{i=-\infty}^{-1} r^i\right) \end{aligned}$$

which diverges for any $r \in (\frac{1}{2}, 2)$. Hence,

$$\sum_{g\in\Gamma} h_r(g,n) = \infty$$

so that h_r can not induce a trace on H_Γ.

Proposition 2.31 *Assume (P,Γ) is an irreducible Markov chain satifying ($\bullet$). If*

$$\inf \left\{ \sum\nolimits_{g\in\Gamma} p_{h,g} \,\middle|\, h \in \Gamma \right\} > 0,$$

then every trace on H_Γ is faithful.

Proof. By Corollary 2.14, H_Γ has an order unit, say u. Take a pure trace, $\tau : H_\Gamma \longrightarrow R$. Since $\hat{P}^{-1}$ is a bounded positive endomorphism (in fact, an automorphism), $\exists\, \lambda \geq 0$ such that $\tau \circ \hat{P}^{-1} = \lambda \cdot \tau$. If $\lambda = 0$, then $\tau \circ \hat{P}^{-1} = 0$, whence $\tau = 0$. This leads to a contradiction. Therefore, it must be that $\lambda > 0$.

Suppose there exists v in $(H_\Gamma)^+ \setminus \{0\}$ such that $\tau(v) = 0$. Then, we can write

$$v = \sum_{n \in N} \prod_{(g,n) \in (\Gamma \times N)_n} v_{g,n}\, \overline{(e_g, n)}$$

where $0 \leq v_{g,n} < M \;\forall\, g, n$ for some $M > 0$ and $v_{g,n} > 0$ for some g and n. (Recall that $R\Gamma = l^\infty(\Gamma)$.) Given $n \in N$, since $0 \leq \tau(v_{g,n}\,\overline{(e_g, n)}) \leq \tau(v) = 0$, we have $\tau(\overline{(e_g, n)}) = 0$. Furthermore, $\tau(\overline{(e_g, n+1)}) = \tau \circ \hat{P}^{-1}(\overline{(e_g, n)}) = \lambda \cdot \tau(v_{g,n}\overline{(e_g, n)}) = 0$ so that (recursively) we obtain $\tau(\overline{(e_g, k)}) = 0 \;\forall\, k \in N$ with $k > n$. On the other hand, $\tau = \lambda^{-1}\, \tau \circ \hat{P}^{-1}$ so that

$$\tau(\overline{(e_g, n-1)}) \;=\; \lambda^{-1}\, \tau \circ \hat{P}^{-1}(\overline{(e_g, n-1)}) \;=\; \lambda^{-1}\, \tau(\overline{(e_g, n)}) \;=\; 0$$

which inductively implies $\tau(\overline{(e_g, k)}) = 0 \;\forall\, 1 \leq k < n$. Hence, $\tau(\overline{(e_g, k)}) = 0 \;\forall\, k \in N$.

Since the given Markov chain is irreducible, for any h in Γ, there exists a path from g to h, that is,

$$\exists\, h_1, h_2, \cdots, h_{k-2}, h_{k-1} \in \Gamma \text{ such that } p_{h,h_{k-1}}\, p_{h_{k-1},h_{k-2}} \cdots p_{h_3,h_2}\, p_{h_2,h_1}\, p_{h_1,g} > 0.$$

Moreover,

$$\begin{aligned} \overline{(e_g, n)} &= \overline{(P^k(e_g), n+k)} \\ &= \overline{(p_{h,h_{k-1}}\, p_{h_{k-1},h_{k-2}} \cdots p_{h_2,h_1}\, p_{h_1,g}\, e_h \;+\; \text{other terms},\, n+k)} \end{aligned}$$

and $\tau(\overline{(e_g, n)}) = 0$. Thus, $\tau(\overline{(e_h, n+k)}) = 0$ which, by the argument given in the previous paragraph, implies $\tau(\overline{(e_h, k)}) = 0$. Since H_Γ is the vector space generated by $\{\overline{(e_h, k)} \mid h \in \Gamma,\; k \in N\}$, this implies $\tau = 0$, a contradiction. Therefore, it must be that $\tau(v) \neq 0 \;\forall\, v \in (H_\Gamma)^+ \setminus \{0\}$.

Now, if v is in $H_\Gamma^+ \setminus \{0\}$, then $F = \{\tau \in S(H_\Gamma, u) \mid \tau(v) = 0\}$ is a closed face of $S(H_\Gamma, u)$. Hence, $\partial_e F = F \cap \partial_e S(H_\Gamma, u)$. However, we have just shown that $F \cap \partial_e S(H_\Gamma, u) = \emptyset$, whence $F = \emptyset$. Therefore, all traces on H_Γ are faithful. ■

Definition 2.32 Given (P, Γ) and a subset S of Γ, we write $P^n(S) = \Gamma$ for a positive integer n if for any h in Γ, there exists $g \in S$ and $h_{n-1}, h_{n-2}, \cdots, h_2, h_1 \in \Gamma$ such that

$$p_{h,h_{n-1}}\, p_{h_{n-1},h_{n-2}} \cdots p_{h_2,h_1}\, p_{h_1,g} \;>\; 0.$$

Proposition 2.33 *Suppose Γ_1 is given and (P, Γ) satisfies condition $(\bullet)$. Let I_1 denote the order ideal corresponding to the cone C_0 generated by $\{(g, 1) \mid g \in \Gamma_1\}$. If there exists a postitive integer n such that $P^n(\Gamma_1) = \Gamma$, then $I_1 = H_\Gamma$.*

Proof. Given any $h \in \Gamma$, $\exists\ g \in \Gamma_1$ and $h_{n-1}, h_{n-2}, \ldots, h_2, h_1 \in \Gamma$ such that

$$p_{h,h_{n-1}}\, p_{h_{n-1},h_{n-2}} \cdots p_{h_2,h_1}\, p_{h_1,g} \;>\; 0.$$

Since $\overline{(e_g, 1)} \in I_1$ and

$$\begin{aligned}
\overline{(e_g, 1)} &= \overline{(P^n(e_g), n+1)} \\
&= \overline{(p_{h,h_{n-1}}\, p_{h_{n-1},h_{n-2}} \cdots p_{h_2,h_1}\, p_{h_1,g}\, e_h \;+\; \text{other terms},\ n+1)},
\end{aligned}$$

we have $\overline{(e_h, n+1)} \in I_1$. Note that $u = \prod_{h \in \Gamma} \overline{(e_h, n+1)}$ is an order unit in I_1.

By definition, given any $f \in \Gamma$, $\exists\ M > 0$ such that

$$0 \;\le\; \overline{(e_f, 1)} \;=\; \overline{(P^n(e_f), n+1)} \;\le\; Mu \;\in\; I_1.$$

Thus, $\overline{(e_f, 1)} \in I_1$. Also note that $\hat{P}^{-1}$ leaves I_1 stable. Hence, $\overline{(e_f, 2)} = \hat{P}^{-1}(\overline{(e_f, 1)}) \in I_1$ from which we inductively obtain $\overline{(e_f, l)} \in I_1\ \forall\, l \in N$. Therefore, $\prod_{f \in \Gamma} \overline{(e_f, l)} \in I_1$.

Now, given v in $(H_\Gamma)^+$, we can write

$$v = \sum_{l \in N} \prod_{f \in \Gamma} a_{f,l}\, \overline{(e_f, l)}$$

where $0 \leq a_{g,l} < M_1 \ \forall\, g, l$ for some $M_1 > 0$. This implies

$$0 \ \leq \ v \ \leq \ M_1 \sum_{l \in N} \prod_{f \in \Gamma} \overline{(e_f, l)}\,.$$

Thus, $v \in I_1$, establishing the inclusion $(H_\Gamma)^+ \subseteq I_1$ and implying $H_\Gamma = I_1$. ∎

Corollary 2.34 *Assume (P, Γ) is such that*

(a) *condition $(\bullet)$ is satisfied and*

(b) *for any g in Γ, there exists $n = n(g)$ such that $P^n(g) = \Gamma$.*

Then, H_Γ has no non-trivial order ideals with order unit.

Proof. Let $u_0 = \sum_{m \in N} \prod_{g \in \Gamma} a_{g,m} \overline{(e_g, m)}$ be an order unit of I. Then, there exists m and g such that $a_{g,m} > 0$ and $\overline{(e_g, m)} \in I$. Moreover, by (b), $\exists\, n \in N$ such that $P^n(g) = \Gamma$. Thus, given any $h \in \Gamma$, $\exists\ h_{n-1}, h_{n-2}, \cdots, h_2, h_1 \in \Gamma$ such that

$$p_{h,h_{n-1}}\, p_{h_{n-1},h_{n-2}} \cdots p_{h_2,h_1}\, p_{h_1,g} \ > \ 0.$$

From this, we have

$$\begin{aligned} \overline{(e_g, m)} \ &= \ \overline{(P^{n+m}(e_g), n+m)} \\ &= \ \overline{(p_{h,h_{n-1}}\, p_{h_{n-1},h_{n-2}} \cdots p_{h_2,h_1}\, p_{h_1,g}\, e_h \ + \text{ other terms}\,,\, n+m)} \end{aligned}$$

so that $\overline{(e_g, m)} \in I$ implies $\overline{(e_h, n+m)} \in I$. But I has an order unit u_0. Therefore, $u = \prod_{h \in \Gamma} \overline{(e_h, n+m)} \in I$.

Given any $f \in \Gamma$, it follows by definition that $\exists\, M > 0$ such that

$$0 \ \leq \ \overline{(e_f, 1)} \ = \ \overline{(P^{n+m-1}(e_f), n+m)} \ \leq \ Mu.$$

Since $Mu \in I$, this implies $\overline{(e_f, 1)} \in I$. We also know that $\hat{P}^{-1}$ leaves I stable. Therefore, $\overline{(e_f, 2)} = \hat{P}^{-1}(\overline{(e_f, 1)}) \in I$ so that recursively we obtain $\overline{(e_f, l)} \in I \ \forall\, l \in N$. From this, we have $\prod_{f \in \Gamma} \overline{(e_f, l)} \in I$.

Now, given $v \in (H_\Gamma)^+$, we can write

$$v = \sum_{l \in N} \prod_{f \in \Gamma} a_{f,l} \overline{(e_f, l)}$$

where $0 \leq a_{g,l} < M_1 \ \forall g, l$ for some $M_1 > 0$. Thus,

$$0 \leq v \leq M_1 \sum_{l \in N} \prod_{f \in \Gamma} \overline{(e_f, l)} \in I$$

implying $v \in I$, i.e., $(H_\Gamma)^+ \subseteq I$. Hence, $H_\Gamma \subseteq I$. Since the inclusion $I \subseteq H_\Gamma$ is obvious, we have shown $H_\Gamma = I$. ■

Note that Corollary 2.34 does not yield Proposition 2.33 directly since $P^n(\Gamma_1) = \Gamma$ is weaker than the condition, $P^n(g) = \Gamma \ \forall g \in \Gamma$. At last, we characterize those $f \in R\Gamma$ for which $P^m f \geq 0$.

Proposition 2.35 *Assume (P, Γ) is such that*

(i) *condition $(\bullet)$ is satisfied,*

(ii) *$P^2 \leq MP$ for some positive integer M.*

(iii) *All pure traces in H_Γ are faithful,*

then the following are equivalent:

- *$\exists\, m \in N$ such that $P^m f \geq 0$.*
- *for all left eigenvectors v of P, $v \cdot f > 0$.*

Proof. According to [13] (see 2 there in), H_Γ has an order unit. We know $\overline{(f, 1)} \in H_\Gamma$, so there exists an m in N such that $P^m f \geq 0$ if and only if $\overline{(f, 1)} \in (H_\Gamma)^+$. Furthermore, according to [6] (see I.4 therein), $\overline{(f, 1)} \in (H_\Gamma)^+$ if and only if $\tau(\overline{(f, 1)}) > 0$ for every pure trace τ of H_Γ. By Theorem 2.25, the pure traces are given by left eigenvectors. Therefore, $\tau(\overline{(f, 1)}) > 0$ can be represented by $u \cdot f = \sum_{g \in \Gamma} u(g)\, f(g) > 0$ for some left eigenvector u. That is, for all left eigenvectors v of P, $v \cdot f > 0$. ■

Chapter 3

Minimum Eigenvalue Estimation

This chapter is concerned with eigenvalues for infinite nonnegative matrices. In this chapter, we will simplify Theorem 2 in [21] and give two basic conditions ensuring $v \cdot P = \lambda v$ has a nonnegative, nonzero solution v. Conditions on P under which the set of eigenvalues admitting nonnegative eigenvectors is an interval of the form $[a, \infty)$, some $a > 0$, will be given. Included is an analysis of nonnegative eigenvalues and left nonnegative, nonzero eigenvectors for the irreducible operators described in the previous chapter.

Let $P = (p_{i,j})_{i,j \in \Gamma}$ be a matrix with nonnegative entries where Γ is a countable set. We will sometimes write $P = (p_{i,j})$ to save space. Recall that P is irreducible if for every pair $i, j \in \Gamma$, there is a finite sequence, $k_1, k_2, \cdots, k_n$ in Γ such that $p_{i,k_1} p_{k_1,k_2} \cdots p_{k_n,j} > 0$.

Suppose P is irreducible and all iterates of P exist. We denote the n-th iterate of P by $P^n = (p_{i,j}^{(n)})$ $(n \geq 1)$ where

$$p_{i,j}^{(n)} = \sum_{k \in \Gamma} p_{i,k} \, p_{k,j}^{(n-1)} \quad \forall i, j \in \Gamma.$$

We also take $P^0 = \mathcal{I}$ so that $p_{i,j}^{(0)} = \delta_{i,j} = 1$ when $i = j$ and 0 otherwise. For each i

and j in Γ, define the power series

$$P_{i,j}(x) = \sum_{n=0}^{\infty} p_{i,j}^{(n)} x^n$$

and let $R_{i,j}$ denote its radius of convergence. In [26], it has been shown that $R_{i,j} = \bar{R}$, a value independent of the choice of $i, j \in \Gamma$, and $P_{i,j}(\bar{R})$ either converges for all pairs $i, j \in \Gamma$ or diverges for all such pairs. In light of these result, we introduce the following terminology.

Definition 3.1 The common radius of convergence, $0 \leq \bar{R} < \infty$ of the $P_{i,j}(x)$ is called the *convergence parameter* of the matrix P.

Definition 3.2 An irreducible matrix P is called *$\bar{R}$-transient* if

$$\sum_{n=0}^{\infty} p_{i,j}^{(n)} \bar{R}^n < \infty \quad \forall\, i, j \in \Gamma$$

and is called *$\bar{R}$-recurrent* if

$$\sum_{n=0}^{\infty} p_{i,j}^{(n)} \bar{R}^n \text{ diverges } \quad \forall\, i, j \in \Gamma.$$

We now introduce analogues of the "taboo probabilities" of [5]. Given any $k \in \Gamma$, define the matricies ${}_kP^n = ({}_kp_{i,j}^{(n)})$ $(n \geq 0)$ recursively by

$$\begin{aligned}
{}_kp_{i,j}^{(0)} &= \delta_{i,j}\,(1 - \delta_{j,k})\,, \\
{}_kp_{i,j}^{(1)} &= p_{i,j}\,, \\
{}_kp_{i,j}^{(2)} &= p_{i,j}^{(2)} - {}_kp_{i,k}^{(1)}\, p_{k,j}\,, \\
&\vdots \\
{}_kp_{i,j}^{(n)} &= p_{i,j}^{(n)} - {}_kp_{i,k}^{(n-1)}\, p_{k,j}.
\end{aligned}$$

When $n \geq 2$, we can write

$${}_kp_{i,j}^{(n)} = \sum_{\alpha \neq k} {}_kp_{i,\alpha}^{(n-1)} p_{\alpha,j} \quad \forall\, i, j \in \Gamma.$$

${}_k p_{i,j}^{(n)}$ has a probabilistic interpretation with respect to the Markov process (P, Γ), namely, it is the probability of going from state i to state j in n steps without visiting state k.

Given any $n \geq 0$, it is clear from the above construction that

$${}_k P^n \leq P^n,$$

that is, ${}_k p_{i,j}^{(n)} \leq p_{i,j}^{(n)} \; \forall\, i, j$, and, for this reason,

$${}_k P_{i,j}(x) := \sum_{n=0}^{\infty} {}_k p_{i,j}^{(n)} x^n \leq \sum_{n=0}^{\infty} p_{i,j}^{(n)} x^n = P_{i,j}(x),$$

for all $x \geq 0$.

For the remainder of this chapter, we assume P is irreducible and $\Gamma = Z$.

Theorem 3.3 *([21]; Theorem 2, p.1799) The system $v \cdot P = \lambda\, v$ has a solution with $\lambda > 0$ and v nonzero, nonnegative if and only if one of the following conditions is satisfied:*

(i) *P is $\bar{R}$-recurrent and $\lambda = \bar{R}^{-1}$,*

(ii) *P is $\bar{R}$-transient, $\lambda = \bar{R}^{-1}$ and for each $i = 0, 1, 2, \ldots$, there is an infinite collection K of nonnegative integers such that*

$$\lim_{j \in Z,\, k \in K,\, j,k \to \infty} \frac{\sum_{\alpha=j}^{\infty} {}_i P_{k,\alpha}(\bar{R})\, p_{\alpha,i}}{{}_i P_{k,i}(\bar{R})} = 0,$$

(iii) *$\lambda > \bar{R}^{-1}$ and for each $i = 0, 1, 2, \ldots$, there is an infinite collection K of nonnegative integers such that*

$$\lim_{j \in Z,\, k \in K,\, j,k \to \infty} \frac{\sum_{\alpha=j}^{\infty} {}_i P_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{{}_i P_{k,i}(\lambda^{-1})} = 0.$$

Proof. The argument follows that of the cited result. ∎

For special P, there is a simpler criterion ensuring the existence of nonnegative eigenvectors.

Proposition 3.4 *Suppose*

$$\beta_j := \inf\{p_{i,j} \mid i \in \Gamma\} > 0 \quad \forall j \in \Gamma.$$

If there is an infinite collection K of nonnegative integers such that

$$\lim_{j \in Z,\, k \in K,\, j,k \to \infty} \sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i} = 0 \quad \forall i \geq 0,$$

then the system $v \cdot P = \lambda v$ has a solution with nonnegative v and $\lambda > 0$.

Proof. By definition,

$$\begin{aligned}
{}_iP_{k,i}(\lambda^{-1}) &= \textstyle\sum_{n=0}^{\infty} {}_ip_{k,i}^{(n)}\,(\lambda^{-1})^n \\
&= {}_ip_{k,i}^{(0)} + \textstyle\sum_{n=1}^{\infty} {}_ip_{k,i}^{(n)}\,\lambda^{-n} \\
&= 0 + {}_ip_{k,i}^{(1)}\,\lambda^{-1} + \textstyle\sum_{n=2}^{\infty} {}_ip_{k,i}^{(n)}\,\lambda^{-n} \\
&= p_{k,i}\,\lambda^{-1} + \textstyle\sum_{n=2}^{\infty} {}_ip_{k,i}^{(n)}\,\lambda^{-n} \\
&\geq p_{k,i}\,\lambda^{-1} \\
&\geq \beta_i\,\lambda^{-1}.
\end{aligned}$$

Hence, for each $i \geq 0$,

$$\begin{aligned}
0 &\leq \lim_{j \in Z,\, k \in K,\, j,k \to \infty} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{{}_iP_{k,i}(\lambda^{-1})} \\
&\leq \lim_{j \in Z,\, k \in K,\, j,k \to \infty} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{\beta_i\,\lambda^{-1}} \\
&= (\beta_i\,\lambda^{-1})^{-1} \cdot \lim_{j \in Z,\, k \in K,\, j,k \to \infty} \textstyle\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i} \\
&= (\beta_i\lambda^{-1})^{-1} \cdot 0 \\
&= 0.
\end{aligned}$$

The result now follows by Theorem 3.3. ∎

Corollary 3.5 *Suppose $\beta = \inf\{p_{i,j} \mid i, j \in \Gamma\} > 0$. If there exists an infinite collection K of nonnegative integers such that*

$$\lim_{j\in Z,\, k\in K,\, j,k\to\infty} \sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i} \;=\; 0 \quad \forall\, i \geq 0,$$

then the system $v \cdot P = \lambda\, v$ has a solution with nonnegative v and $\lambda > \bar{R}^{-1}$.

Proof. An immediate consequence of the previous proposition. ∎

In the hope of obtaining more specific results, we further investigate ${}_iP_{k,i}(x)$.

Lemma 3.6 *If ${}_iP_{k,i}(s) < \infty \;\forall\, i, k \in \Gamma$ for some positive real number s, then*

$$\lim_{j\to\infty} \sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(s)\, p_{\alpha,i} \;=\; 0.$$

Proof. By definition,

$$\begin{aligned}
{}_iP_{k,i}(s) &= \textstyle\sum_{n=0}^{\infty} {}_ip_{k,i}^{(n)}\, s^n \\
&= {}_ip_{k,i}^{(0)}\, s^0 \;+\; \textstyle\sum_{n=1}^{\infty} {}_ip_{k,i}^{(n)}\, s^n \\
&= 0 \cdot 1 \;+\; \textstyle\sum_{n=1}^{\infty} \left(\sum_{\alpha\neq i} {}_ip_{k,\alpha}^{(n-1)}\, p_{\alpha,i}\right) s^n \\
&= s \cdot \textstyle\sum_{\alpha\neq i} \left(\sum_{n=1}^{\infty} {}_ip_{k,\alpha}^{(n-1)}\, s^{n-1}\right) p_{\alpha,i} \\
&= s \cdot \textstyle\sum_{\alpha\neq i} \left(\sum_{n=0}^{\infty} {}_ip_{k,\alpha}^{(n)}\, s^{n}\right) p_{\alpha,i} \\
&= s \cdot \textstyle\sum_{\alpha\neq i} {}_iP_{k,\alpha}(s)\, p_{\alpha,i}.
\end{aligned}$$

Therefore, since ${}_iP_{k,i}(s) < \infty$ and $s > 0$, $\sum_{\alpha=1}^{\infty} {}_iP_{k,\alpha}(s)\, p_{\alpha,i} < \infty$ which clearly implies the lemma. ∎

Lemma 3.7 *Fix i and k in Γ and take $\lambda > 0$. If P is $\bar{R}$-recurrent and $\lambda > \bar{R}^{-1}$ (or if P is $\bar{R}$-transient and $\lambda \geq \bar{R}^{-1}$), then*

$$\lim_{j\to\infty} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{{}_iP_{k,i}(\lambda^{-1})} \;=\; 0. \tag{4}$$

Proof. By assumption, $0 < \lambda^{-1} < \bar{R}$ if P is $\bar{R}$-recurrent and $0 < \lambda^{-1} \leq \bar{R}$ if P is $\bar{R}$-transient. In either case, $P_{k,i}(\lambda^{-1}) < \infty$ which, in view of the inequality ${}_iP_{k,i}(\lambda^{-1}) \leq P_{k,i}(\lambda^{-1})$, implies ${}_iP_{k,i}(\lambda^{-1}) < \infty$. Furthermore, since P is irreducible, there exists $n \in N$ such that ${}_ip_{k,i}^{(n)} > 0$. Hence, ${}_ip_{k,i}^{n}\,\lambda^{-n} \leq {}_iP_{k,i}(\lambda^{-1}) < \infty$.

Applying Lemma 3.6, we have

$$\lim_{j\to\infty} \sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i} \;=\; 0.$$

Therefore,

$$\begin{aligned} 0 \;&\leq\; \lim_{j\to\infty} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{{}_iP_{k,i}(\lambda^{-1})} \\ &\leq\; \frac{1}{{}_ip_{k,i}^{(n)}\,\lambda^{-n}} \;\lim_{j\to\infty} \sum\nolimits_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i} \\ &=\; \frac{\lambda^n}{{}_ip_{k,i}^{(n)}} \cdot 0 \;=\; 0 \end{aligned}$$

which confirms (4). ■

Although the limit in (4) exists for all $\lambda > \bar{R}^{-1}$ and $k \in \Gamma$, the convergence need not be uniform in k. To ensure the existence of solutions to $v \cdot P = \lambda\, v$, we will frequently make the following assumption:

For each $\lambda > \bar{R}^{-1}$, the limit in (4) converges to 0 uniformly over all $k \in \Gamma$. (5)

This holds for the examples of section 2.3, but is fairly restrictive.

Definition 3.8 Given a Markov process Q (by our definition, the column sums need not be 1), define

$$\Lambda(Q) \;:=\; \min\{\lambda \in R \,|\, \exists\, v \geq 0, v \neq 0 \text{ such that } v \cdot Q = \lambda\, v\}.$$

Theorem 3.9 *If P is a nonnegative, irreducible, infinite matrix satisfying condition* (5), *then*

(i) $\Lambda(P) = \bar{R}^{-1}$ *where* $\bar{R}$ *denotes the convergence parameter of the matrix* P.

(ii) *If* $\Lambda(P) < \lambda < \infty$, *then there exists a nonnegative, nonzero* $v : \Gamma \to R$ *such that* $v \cdot P = \lambda\, v$.

Proof. (i) If v is any nonnegative left eigenvector for P with corresponding eigenvalue λ, Theorem 3.3 implies $\lambda \geq \bar{R}^{-1}$. Therefore, $\Lambda(P) \geq \bar{R}^{-1}$.

For the opposite inequality, pick $\epsilon > 0$ and let $\lambda = \bar{R}^{-1} + \epsilon$. By (5), the limit in (4) converges uniformly to 0 over all $k \in \Gamma$. Hence, there exists an infinite subset K of nonnegative integers such that

$$\lim_{j \in Z,\, k \in K,\, j,k \to \infty} \frac{\sum_{\alpha=j}^{\infty} {}_iP_{k,\alpha}(\lambda^{-1})\, p_{\alpha,i}}{{}_iP_{k,i}(\lambda^{-1})} = 0 \quad \forall\, i \geq 0.$$

Therefore, by Theorem 3.3, $\exists\, v \geq 0$, $v \neq 0$ such that $v \cdot P = \lambda\, v$, implying

$$\Lambda(P) \;\leq\; \lambda \;= \bar{R}^{-1} + \epsilon.$$

Letting $\epsilon \to 0$, we obtain $\Lambda(P) \leq \bar{R}^{-1}$ which implies (i).

Since no restriction was placed on the size of $\epsilon > 0$ in the proof of part (i), the argument therein also implies (ii) since $\Lambda(P) = \bar{R}^{-1}$. ∎

Corollary 3.10 *Let* P *be a nonnegative, irreducible, infinite matrix and suppose* P *and* P^T *(the transpose of* P*) both satisfy* (5). *Then,*

(i) $\Lambda(P) = \Lambda(P^T) = \bar{R}^{-1}$, *the convergence parameter of the matrix* P.

(ii) *If* $\bar{R}^{-1} < \lambda < \infty$, *then there exists nonnegative, nonzero* $v, u : \Gamma \to R$ *such that* $v \cdot P = \lambda\, v$ *and* $u \cdot P^T = \lambda\, u$.

Proof. This is an obvious consequence of Theorem 3.9 since the convergence parameters of P and P^T are identical. ∎

Example 3.11 Consider the nonnegative, infinite matrix

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & 0 & 1 & 0 & \cdot & \cdot \\ \frac{1}{2^2} & 0 & 0 & 1 & \cdot & \cdot \\ \frac{1}{2^3} & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

We claim that P has only one nonnegative eigenvalue and a unique eigenvector (up to scalar multiple).

First note that P is irreducible— $p_{i,0} = \frac{1}{2^i}$ and $p_{i,i+1} = 1 \ \forall i$. hence, given any $i, j \geq 0$. $p_{i,0}\, p_{0,1}\, p_{1,2} \cdots p_{j-1,j} = \frac{1}{2^i} > 0$. Now, assume $v = (v_i)_{i \geq 0}$ is such that $v \geq 0$. $v \neq 0$ and $v \cdot P = \lambda\, v$ for some $\lambda > 0$. Then,

$$\begin{pmatrix} v_0 & v_1 & v_2 & v_3 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & 0 & 1 & 0 & \cdot & \cdot \\ \frac{1}{2^2} & 0 & 0 & 1 & \cdot & \cdot \\ \frac{1}{2^3} & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \lambda \begin{pmatrix} v_0 & v_1 & v_2 & v_3 & \cdot & \cdot \end{pmatrix}$$

yields the system

$$\begin{cases} v_0 + v_1 \frac{1}{2} + v_2 \frac{1}{2^2} + v_3 \frac{1}{2^3} + \cdots = \lambda\, v_0 \\ v_0 = \lambda\, v_1 \\ v_1 = \lambda\, v_2 \\ v_2 = \lambda\, v_3 \\ \quad \vdots \end{cases}$$

so that

$$v_i = \lambda^{1-i}\, v_0 \quad \forall\, i \geq 1.$$

Since v is nonzero, it must be that $v_i > 0 \ \forall i \geq 0$.

The first equation implies $v_0 = \lambda^{-1}[1 + (2\lambda)^{-1} + (2\lambda)^{-2} + \cdots] \, v_0$. Thus,

$$\lambda = 1 + \frac{1}{2\lambda} + \frac{1}{(2\lambda)^2} + \frac{1}{(2\lambda)^3} + \cdots .$$

Furthermore, $\lambda > 0$ implies $2\lambda > 1$ so that $\frac{1}{1-\frac{1}{2\lambda}} = \lambda$. Solving for λ yields $\lambda = \frac{3}{2}$. Therefore, up to a scalar multiple, the only eigenvector of P is given by $v_i = (2/3)^i$ $(i \geq 0)$ and $\lambda = \frac{3}{2}$ is the only eigenvalue.

Chapter 4

Irreducible Markov Chains

Under the assumption of irreducibility, this chapter establishes an affine homeomorphism between the space P_λ of nonnegative eigenvectors of P for a fixed eigenvalue λ and a subspace of the trace space of the dimension group $DG(\lambda)$ (defined below). Extending the definition given in [14], we define future dimension groups $FDG(\lambda)$ for P which are not necessarily column finite. By imposing additional conditions on P, we obtain a one-parameter family of onto, order preserving homomorphisms $\phi_\lambda : FDG(\lambda) \to DG(\lambda)$, each of which induces a one to one map between the corresponding trace spaces.

4.1 The Affine Homeomorphism Theorem

The set-up is as follows. Let (P, Γ) be an irreducible, infinite Markov process with countably infinite state space Γ. That is, the transition matrix $P = (p_{g,h})_{g,h\in\Gamma}$ (we are using "transition" in a rather loose sense; the column sums need not be 1) is such that $p_{g,h} \geq 0 \;\forall\, h, g$ and, given any states g and h in Γ, there exists a path from g to h, that is,

$$\exists\; h_1, h_2, h_3, \cdots, h_{n-1}, h_n \in \Gamma \text{ such that } p_{h,h_n}\, p_{h_n,h_{n-1}} \cdots p_{h_3,h_2}\, p_{h_2,h_1}\, p_{h_1,g} > 0.$$

Choose an (essentially arbitrary) partition $\{\Gamma_i : i \in N\}$ of the state space Γ where each Γ_i is finite and $\Gamma = \dot{\bigcup}_{i \in N} \Gamma_i$. For each $i \in N$, form $R\Gamma_i$, the real vector space with basis $\{e_g\}_{g \in \Gamma_i}$. Obviously, $R\Gamma_i$ is a finite dimensional vector space. Let B_i denote the transition matrix of Γ restricted to Γ_i so that $B_i = (p_{g,h})_{g,h \in \Gamma_i}$. Note that B_i is a linear transformation from $R\Gamma_i$ to $R\Gamma_i$. For any $i, j \in N$, define $C_{j,i} : R\Gamma_i \to R\Gamma_j$ to be the restriction and compression of P to the corresponding parts of Γ, that is, $C_{j,i} = (p_{g,h})_{(g,h) \in \Gamma_j \times \Gamma_i}$. Thus, with respect to the partition $\{\Gamma_i\}$, P can be written as the block matrix

$$P = \begin{pmatrix} B_1 & C_{1,2} & C_{1,3} & \cdots & C_{1,i} & C_{1,i+1} & \cdots \\ C_{2,1} & B_2 & C_{2,3} & \cdots & C_{2,i} & C_{2,i+1} & \cdots \\ C_{3,1} & C_{3,2} & B_3 & \cdots & C_{3,i} & C_{3,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ C_{i,1} & C_{i,2} & C_{i,3} & \cdots & B_i & C_{i,i+1} & \cdots \\ C_{i+1,1} & C_{i+1,2} & C_{i+3,3} & \cdots & C_{i+1,i} & B_{i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \end{pmatrix}$$

Now, suppose the nonnegative, nonzero row vector $v : \Gamma \to R$ is a left eigenvector of P so that $v \cdot P = \lambda v$ for some $\lambda > 0$. (By Theorem 3.3, it must be that $\lambda \geq \bar{R}^{-1}$.) Let v_i denote the restriction of v on Γ_i, that is, $v_i = v|_{\Gamma_i} : \Gamma_i \to R$ so that we can write

$$v = (\ v_1 \ \ v_2 \ \ v_3 \ \cdot \ \cdot\ ) .$$

From $v \cdot P = \lambda v$ we obtain

$$(\ v_1 \ \ v_2 \ \ v_3 \ \cdot \ \cdot\ ) \cdot P = \lambda (\ v_1 \ \ v_2 \ \ v_3 \ \cdot \ \cdot\ )$$

which yields the system

$$\begin{cases} v_1 B_1 + v_2 C_{2,1} + v_3 C_{3,1} + \cdots + v_i C_{i,1} + v_{i+1} C_{i+1,1} + \cdots = \lambda v_1 \\ v_1 C_{1,2} + v_2 B_2 + v_3 C_{3,2} + \cdots + v_i C_{i,2} + v_{i+1} C_{i+1,2} + \cdots = \lambda v_2 \\ v_1 C_{1,3} + v_2 C_{2,3} + v_3 B_3 + \cdots + v_i C_{i,3} + v_{i+1} C_{i+1,3} + \cdots = \lambda v_3 \\ \vdots \\ v_1 C_{1,i} + v_2 C_{2,i} + v_3 C_{3,i} + \cdots + v_i B_i + v_{i+1} C_{i+1,i} + \cdots = \lambda v_i \\ \vdots \end{cases}$$

If the inverse matrix $(\lambda I - B_1)^{-1}$ exists, then

$$\begin{aligned} v_1 &= v_2 C_{2,1} (\lambda I - B_1)^{-1} + v_3 C_{3,1} (\lambda I - B_1)^{-1} + \\ &\quad v_i C_{i,1} (\lambda I - B_1)^{-1} + v_{i+1} C_{i+1,1} (\lambda I - B_1)^{-1} + \cdots . \end{aligned}$$

For each $k > 1$, define a one-parameter family of matrices, $F_{k,1}^{(\lambda)} : R\Gamma_1 \to R\Gamma_k$ via

$$F_{k,1}^{(\lambda)} = C_{k,1} (\lambda I - B_1)^{-1} .$$

This allows us to write

$$v_1 = v_2 F_{2,1}^{(\lambda)} + v_3 F_{3,1}^{(\lambda)} + \cdots + v_i F_{i,1}^{(\lambda)} + v_{i+1} F_{i+1,1}^{(\lambda)} + \cdots$$

so that

$$\begin{aligned} \lambda v_2 &= v_1 C_{1,2} + v_2 B_2 + v_3 C_{3,2} + \cdots + \\ &\quad v_i C_{i,2} + v_{i+1} C_{i+1,2} + \cdots \\ &= v_2 F_{2,1}^{(\lambda)} C_{1,2} + v_3 F_{3,1}^{(\lambda)} C_{1,2} + \cdots + \\ &\quad v_i F_{i,1}^{(\lambda)} C_{1,2} + v_{i+1} F_{i+1,1}^{(\lambda)} C_{1,2} + \cdots + \\ &\quad v_2 B_2 + v_3 C_{3,2} + \cdots + v_i C_{i,2} + v_{i+1} C_{i+1,2} + \cdots \\ &= v_2 \left(F_{2,1}^{(\lambda)} C_{1,2} + B_2\right) + v_3 \left(F_{3,1}^{(\lambda)} C_{1,2} + C_{3,2}\right) + \cdots + \\ &\quad v_i \left(F_{i,1}^{(\lambda)} C_{1,2} + C_{i,2}\right) + v_{i+1} \left(F_{i+1,1}^{(\lambda)} C_{1,2} + C_{i+1,2}\right) + \cdots . \end{aligned}$$

If we assume the inverse matrix $\left(\lambda I - B_2 - F^{(\lambda)}_{2,1} C_{1,2}\right)^{-1}$ exists, then

$$
\begin{aligned}
v_2 \;=\; & v_3 \left(F^{(\lambda)}_{3,1}\, C_{1,2} \,+\, C_{3,2}\right) \left(\lambda I - B_2 - F^{(\lambda)}_{2,1}\, C_{1,2}\right)^{-1} \,+\, \cdots \,+ \\
& v_i \left(F^{(\lambda)}_{i,1}\, C_{1,2} \,+\, C_{i,2}\right) \left(\lambda I - B_2 - F^{(\lambda)}_{2,1}\, C_{1,2}\right)^{-1} \,+ \\
& v_{i+1} \left(F^{(\lambda)}_{i+1,1}\, C_{1,2} \,+\, C_{i+1,2}\right) \left(\lambda I - B_2 - F^{(\lambda)}_{2,1}\, C_{1,2}\right)^{-1} \,+\, \cdots\,.
\end{aligned}
$$

Defining, for each $k > 2$, a one-parameter family of matrices $F^{(\lambda)}_{k,2} : R\Gamma_2 \to R\Gamma_k$ via

$$
\begin{aligned}
F^{(\lambda)}_{k,2} \;&=\; \left(C_{k,2} \,+\, F^{(\lambda)}_{k,1}\, C_{1,2}\right) \left(\lambda I \,-\, B_2 - F^{(\lambda)}_{2,1}\, C_{1,2}\right)^{-1} \\
&=\; \left(C_{k,2} \,+ \sum_{2>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,2}\right) \cdot \\
&\quad\; \left(\lambda I \,-\, B_2 \,- \sum_{2>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,2}\right)^{-1}.
\end{aligned}
$$

we can rewrite the previous equation as

$$
v_2 \;=\; v_3\, F^{(\lambda)}_{3,2} \,+\, v_4\, F^{(\lambda)}_{4,2} \,+\, \cdots \,+\, v_i\, F^{(\lambda)}_{i,2} \,+\, v_{i+1}\, F^{(\lambda)}_{i+1,2} \,+\, \cdots\,.
$$

For the purpose of an inductive argument, fix $i \in N$. Suppose that for any $l \in N$ satisfying $1 \leq l < i$, the inverse matrix

$$
\left(\lambda I \,-\, B_l \,- \sum_{l>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{l,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,l}\right)^{-1}
$$

exists and furthermore, suppose that for every $k > l$ we have defined a one-parameter family of matrices $F^{(\lambda)}_{k,l} : R\Gamma_l \to R\Gamma_k$ via

$$
\begin{aligned}
F^{(\lambda)}_{k,l} \;&=\; \left(C_{k,l} \,+ \sum_{l>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,l}\right) \cdot \\
&\quad\; \left(\lambda I \,-\, B_l \,- \sum_{l>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{l,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,l}\right)^{-1}
\end{aligned}
$$

so that

$$\begin{cases} v_1 &= v_2\, F_{2,1}^{(\lambda)} + v_3\, F_{3,1}^{(\lambda)} + v_4\, F_{4,1}^{(\lambda)} + \cdots + v_i\, F_{i,1}^{(\lambda)} + v_{i+1}\, F_{i+1,1}^{(\lambda)} + \cdots \\ v_2 &= v_3\, F_{3,2}^{(\lambda)} + v_4\, F_{4,2}^{(\lambda)} + \cdots + v_i\, F_{i,2}^{(\lambda)} + v_{i+1}\, F_{i+1,2}^{(\lambda)} + \cdots \\ &\vdots \\ v_{i-1} &= v_i\, F_{i,i-1}^{(\lambda)} + v_{i+1}\, F_{i+1,i-1}^{(\lambda)} + \cdots . \end{cases}$$

(In the previous paragraph, this was verified for $i = 2$ and $i = 3$. The statement is vacuously true for $i = 1$.) To extend this to the case of $l = i$, note that

$$\begin{aligned} v_1\, C_{1,i} &= v_2\, F_{2,1}^{(\lambda)}\, C_{1,i} + v_3\, F_{3,1}^{(\lambda)}\, C_{1,i} + v_4\, F_{4,1}^{(\lambda)}\, C_{1,i} + \cdots + v_i\, F_{i,1}^{(\lambda)}\, C_{1,i} + \\ &\quad v_{i+1}\, F_{i+1,1}^{(\lambda)}\, C_{1,i} + \cdots \\ &= v_3\, F_{3,2}^{(\lambda)}\, F_{2,1}^{(\lambda)}\, C_{1,i} + v_4\, F_{4,2}^{(\lambda)}\, F_{2,1}^{(\lambda)}\, C_{1,i} + \cdots + v_i\, F_{i,2}^{(\lambda)}\, F_{2,1}^{(\lambda)}\, C_{1,i} + \\ &\quad v_{i+1}\, F_{i+1,2}^{(\lambda)}\, F_{2,1}^{(\lambda)}\, C_{1,i} + \cdots + v_3\, F_{3,1}^{(\lambda)}\, C_{1,i} + v_4\, F_{4,1}^{(\lambda)}\, C_{1,i} + \cdots + \\ &\quad v_i\, F_{i,1}^{(\lambda)}\, C_{1,i} + v_{i+1}\, F_{i+1,1}^{(\lambda)}\, C_{1,i} + \cdots \\ &= v_3 \left(F_{3,2}^{(\lambda)}\, F_{2,1}^{(\lambda)} + F_{3,1}^{(\lambda)}\right) C_{1,i} + v_4 \left(F_{4,2}^{(\lambda)}\, F_{2,1}^{(\lambda)} + F_{4,1}^{(\lambda)}\right) C_{1,i} + \cdots + \\ &\quad v_i \left(F_{i,2}^{(\lambda)}\, F_{2,1}^{(\lambda)} + F_{i,1}^{(\lambda)}\right) C_{1,i} + v_{i+1} \left(F_{i+1,2}^{(\lambda)}\, F_{2,1}^{(\lambda)} + F_{i+1,1}^{(\lambda)}\right) C_{1,i} + \cdots \\ &\;\vdots \\ &= v_i \left(\sum_{i>j_1>j_2>\cdots>j_p>1} F_{i,j_1}^{(\lambda)}\, F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,1}^{(\lambda)} \right) C_{1,i} + \\ &\quad v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>1} F_{i+1,j_1}^{(\lambda)}\, F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,1}^{(\lambda)} \right) C_{1,i} + \cdots . \end{aligned}$$

Similarly,

$$\begin{aligned} v_2\, C_{2,i} &= v_i \left(\sum_{i>j_1>j_2>\cdots>j_p>2} F_{i,j_1}^{(\lambda)}\, F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,2}^{(\lambda)} \right) C_{2,i} + \\ &\quad v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>2} F_{i+1,j_1}^{(\lambda)}\, F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,2}^{(\lambda)} \right) C_{2,i} + \cdots \end{aligned}$$

and inductively,

$$v_{i-1}\, C_{i-1,i} = v_i\, F_{i,i-1}^{(\lambda)}\, C_{i-1,i} + v_{i+1}\, F_{i+1,i-1}^{(\lambda)}\, C_{i-1,i} + \cdots .$$

Combining these identities, we can thus write

$$\begin{aligned}
\lambda v_i &= v_1 C_{1,i} + v_2 C_{2,i} + v_3 C_{3,i} + \cdots + v_i B_i + v_{i+1} C_{i+1,i} + \cdots \\
&= v_i \left(\sum_{i>j_1>j_2>\cdots>j_p>1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,1} \right) C_{1,i} + \\
&\quad v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>1} F^{(\lambda)}_{i+1,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,1} \right) C_{1,i} + \cdots + \\
&\quad v_i \left(\sum_{i>j_1>j_2>\cdots>j_p>2} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,2} \right) C_{2,i} + \\
&\quad v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>2} F^{(\lambda)}_{i+1,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,2} \right) C_{2,i} + \cdots + \\
&\quad v_i F^{(\lambda)}_{i,i-1} C_{i-1,i} + v_{i+1} F^{(\lambda)}_{i+1,i-1} C_{i-1,i} + \cdots + v_i B_i + v_{i+1} C_{i+1,i} + \cdots \\
&= v_i \left(B_i + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right) + \\
&\quad v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} + C_{i+1,i} \right) + \cdots .
\end{aligned}$$

Continuing with the induction argument (namely, establishing the case of $l = i$), suppose the inverse matrix

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right)^{-1}$$

exists and define a one-parameter family of matrices $F^{(\lambda)}_{k,i} : R\Gamma_i \to R\Gamma_k$ via

$$\begin{aligned}
F^{(\lambda)}_{k,i} &= \left(C_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right) \cdot \\
&\quad \left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right)^{-1}
\end{aligned}$$

Then, by the previous identity,

$$v_i = v_{i+1} \left(\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} + C_{i+1,i} \right) \cdot$$

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1} +$$

$$v_{i+2}\left(\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} + C_{i+2,i}\right) \cdot$$

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1} + \cdots$$

which implies

$$v_i = v_{i+1} F^{(\lambda)}_{i+1,i} + v_{i+2} F^{(\lambda)}_{i+2,i} + v_{i+3} F^{(\lambda)}_{i+3,i} + \cdots . \tag{6}$$

This completes the induction, proving that the $F^{(\lambda)}_{k,i}$ $(k > i)$ satisfy (6) for each $i \in N$ (provided they exist). It remains to be shown that each $F^{(\lambda)}_{k,i}$ is well-defined.

Lemma 4.1 *If $v = (v_j)$ is a nonnegative, nonzero row vector satisfying $v \cdot P = \lambda v$ for some $\lambda > 0$, then $v_j > 0 \; \forall j$.*

Proof. By assumption, $v_i > 0$ for some i. Thus, by irreducibility, given any $j \in \Gamma$, we have a path from j to i, that is, $p_{i,j_n} p_{j_n,j_{n-1}} \cdots p_{j_2,j_1} p_{j_1,j} > 0$ for suitable indicies $j_1, \cdots, j_k$. But $v \cdot P^{n+1} = \lambda^{n+1} v$. Therefore,

$$0 < v_i p_{i,j_n} p_{j_n,j_{n-1}} \cdots p_{j_2,j_1} p_{j_1,j} \leq \lambda^{n+1} v_j$$

from which the lemma follows. ∎

Lemma 4.2 *Using the above notation, the matrix*

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1}$$

is defined for every $i \in N$ and the matrix

$$F^{(\lambda)}_{k,i} = \left(C_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right) \cdot$$

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1}$$

is nonnegative for every $k > i \geq 1$.

Proof. Let P, v, λ and $\{\Gamma_i\}$ be as introduced earlier in this section. Define

$$\Delta = \operatorname{diag}(v)$$

and set

$$P' = \Delta P \Delta^{-1} \lambda^{-1}.$$

(By Lemma 4.1, Δ^{-1} is well-defined.) Then,

$$v \Delta^{-1}(g) = 1$$

for all $g \in \Gamma$ and

$$v \Delta^{-1} \Delta P \Delta^{-1} = v P \Delta^{-1} = \lambda v \Delta^{-1}.$$

(This is a standard trick called an H-transform.) Obviously, the matrix P' is non-negative, irreducible and satisfies

$$(\mathbf{1} \;\; \mathbf{1} \;\; \mathbf{1} \;\; \cdot \;\; \cdot) \cdot P' = (\mathbf{1} \;\; \mathbf{1} \;\; \mathbf{1} \;\; \cdot \;\; \cdot).$$

i.e., all column sums of P' are 1. For this reason, P' can be regarded as the transition matrix of an irreducible, stationary Markov chain with state space $\Gamma = \bigcup_{i \in N} \Gamma_i$ (see p.7 of [5]).

For each $i \in N$, define $\Delta_i : R\Gamma_i \to R\Gamma_i$ via

$$\Delta_i = \Delta|_{R\Gamma_i}.$$

We obtain a linear transformation $B_i' : R\Gamma_i \to R\Gamma_i$ when we define

$$B_i' := \Delta_i B_i^{-1} \Delta_i^{-1} \lambda^{-1}.$$

Similarly, for each pair $i \neq j$ in Γ, define $C_{j,i}' : R\Gamma_i \to R\Gamma_j$ by taking

$$C_{j,i}' := \Delta_j C_{j,i} \Delta_i^{-1} \lambda^{-1}.$$

In terms of the said Markov chain, B_i' represents the relative transition matrix from Γ_i to Γ_i and $C_{j,i}'$ represents the relative transition matrix from Γ_i to Γ_j.

For every $k > 1$, define a linear transformation $F_{k,1}^{(1)} : R\Gamma_1 \to R\Gamma_k$ by taking

$$F_{k,1}^{(1)} := \Delta_k C_{k,1} \Delta_1^{-1} \lambda^{-1} \left(I - \Delta_1 B_1 \Delta_1^{-1} \lambda^{-1} \right)^{-1}.$$

We need to show that each $F_{k,1}^{(1)}$ exists. To this end, we consider how a particle in Γ_1 may jump to Γ_k in any number of steps without passing through the other Γ_i.
This can be divided into infinitely many disjoint events: one jump from Γ_1 to Γ_k, staying one time in Γ_1 and then jumping, staying two times in Γ_i and then jumping and so on. The transition matricies for these probabilities are given by $C_{k,1}'$, $C_{k,1}'B_1'$, $C_{k,1}'(B_1')^2, \ldots$ (respectively) and the sum of these matrices yields the transition matrix for the process,

$$C_{k,1}' \sum_{j=0}^{\infty} (B_1')^j.$$

An elementary irreducibility argument for finite matrices yields that the spectral radius of B_1' is less than 1. Therefore, $\sum_{j=0}^{\infty}(B_1')^j$ converges to $(I - B_1')^{-1}$. Of course, this is nonnegative. Thus, $F_{k,1}^{(1)} = C_{k,1}' (I - B_1')^{-1}$ is well-defined and nonnegative. Furthermore,

$$\begin{aligned}
\Delta_k^{-1} F_{k,1}^{(1)} \Delta_1 &= \Delta_k^{-1} C_{k,1}' (I - B_1')^{-1} \Delta_1 \\
&= \Delta_k^{-1} C_{k,1}' \Delta_1 \left(I - \Delta_1^{-1} B_1' \Delta_1 \right)^{-1} \\
&= \lambda^{-1} C_{k,1} \left(I - B_1 \lambda^{-1} \right)^{-1} \\
&= F_{k,1}^{(\lambda)}
\end{aligned}$$

so that each $F_{k,1}^{(\lambda)}$ is well-defined and nonnegative. Similarly,

$$\begin{aligned}
\lambda^{-1} \Delta_1^{-1} (I - B_1')^{-1} \Delta_1 &= \lambda^{-1} \Delta_1^{-1} \left(I - \Delta_1 B_1 \Delta_1^{-1} \lambda^{-1} \right)^{-1} \Delta_1 \\
&= (\lambda I - B_1)^{-1},
\end{aligned}$$

implying $(\lambda I - B_1)^{-1}$ is well-defined.

Next, define linear transformations $F^{(1)}_{k,2} : R\Gamma_2 \to R\Gamma_k$ via

$$\begin{aligned} F^{(1)}_{k,2} &= \left(\frac{\Delta_k C_{k,2}\Delta_2^{-1}}{\lambda} + F^{(1)}_{k,1}\frac{\Delta_1 C_{1,2}\Delta_2^{-1}}{\lambda}\right)\left(I - \frac{\Delta_2 B_2 \Delta_2^{-1}}{\lambda} - F^{(1)}_{2,1}\frac{\Delta_1 C_{1,2}\Delta_2^{-1}}{\lambda}\right)^{-1} \\ &= \left(C'_{k,2} + F^{(1)}_{k,1} C'_{1,2}\right)\left(I - B'_2 - F^{(1)}_{2,1} C'_{1,2}\right)^{-1} \end{aligned}$$

for every $k > 2$. To show that $F^{(1)}_{k,2}$ exists, we consider the process which begins in Γ_2 and enters Γ_k after some number of steps in such a way that at any intermediate step, we either jump from Γ_2 to Γ_1, jump from Γ_1 to Γ_2 or stay in Γ_1 or Γ_2 and, at the last step, jump from either Γ_1 or Γ_2 into Γ_k. In other words, we jump from Γ_2 into Γ_k without entering any Γ_i for which $i \neq 1$ or 2.

This can be divided into infinitely many disjoint events. The events and corresponding transition maticies are as follows: never staying in Γ_2,

$$C'_{k,2} + F^{(1)}_{k,1} C'_{1,2},$$

staying one time in Γ_2 just before jumping to Γ_k,

$$\left(C'_{k,2} + F^{(1)}_{k,1} C'_{1,2}\right)\left(B'_2 + F^{(1)}_{2,1} C'_{1,2}\right)$$

and, for any $j > 2$, staying j times in Γ_2 before jumping to Γ_k,

$$\left(C'_{k,2} + F^{(1)}_{k,1} C'_{1,2}\right)\left(B'_2 + F^{(1)}_{2,1} C'_{1,2}\right)^j.$$

Summing these matrices yields the transition matrix of this process,

$$\left(C'_{k,2} + F^{(1)}_{k,1} C'_{1,2}\right)\sum_{j=0}^{\infty}\left(B'_2 + F^{(1)}_{2,1} C'_{1,2}\right)^j.$$

Note that $B'_2 + F^{(1)}_{2,1}C'_{1,2}$ has column sums strictly less than 1 since it represents the part of the transition matrix associated to the Markov chain from Γ_2 to Γ_2 passing through any states in Γ. Therefore, the above matrix series converges with

$$\sum_{j=0}^{\infty}\left(B'_2 + F^{(1)}_{2,1} C'_{1,2}\right)^j = \left(I - B'_2 - F^{(1)}_{2,1} C'_{1,2}\right)^{-1}.$$

Because of this,

$$F^{(1)}_{k,2} = \left(C'_{k,2} + F^{(1)}_{k,1}C'_{1,2}\right)\left(I - B'_2 - F^{(1)}_{2,1}C'_{1,2}\right)^{-1}$$

is well-defined and nonnegative. But

$$\begin{aligned}\Delta_k^{-1}F^{(1)}_{k,2}\Delta_2 &= \Delta_k^{-1}\left(C'_{k,2} + F^{(1)}_{k,1}C'_{1,2}\right)\left(I - B'_2 - F^{(1)}_{2,1}C'_{1,2}\right)^{-1}\Delta_2 \\ &= \left(C_{k,2}\lambda^{-1} + F^{(\lambda)}_{k,1}C_{1,2}\lambda^{-1}\right)\left(I - B_2\lambda^{-1} - F^{(\lambda)}_{2,1}C_{1,2}\lambda^{-1}\right)^{-1} \\ &= F^{(\lambda)}_{k,2}.\end{aligned}$$

Hence, $F^{(\lambda)}_{k,2}$ is also well-defined and nonnegative. Similarly,

$$\begin{aligned}&\lambda^{-1}\Delta_2^{-1}\left(I - B'_2 - F^{(1)}_{2,1}C'_{1,2}\right)^{-1}\Delta_2 \\ &= \lambda^{-1}\Delta_2^{-1}\left(I - \Delta_2 B_2\Delta_2^{-1}\lambda^{-1} - F^{(1)}_{2,1}\Delta_1 C_{1,2}\Delta_2^{-1}\lambda^{-1}\right)^{-1}\Delta_2 \\ &= \left(\lambda I - B_2 - F^{(\lambda)}_{2,1}C_{1,2}\right)^{-1},\end{aligned}$$

implying $\left(\lambda I - B_2 - F^{(\lambda)}_{2,1}C_{1,2}\right)^{-1}$ is well-defined.

In general, given any $k > i \geq 3$, define $F^{(1)}_{k,i} : R\Gamma_i \to R\Gamma_k$ via

$$\begin{aligned}F^{(1)}_{k,i} &= \left(\frac{\Delta_k C_{k,i}\Delta_i^{-1}}{\lambda} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1}F^{(1)}_{j_1,j_2}\cdots F^{(1)}_{j_p,j}\frac{\Delta_j C_{j,i}\Delta_i^{-1}}{\lambda}\right)\cdot \\ &= \left(I - \frac{m_iB_im_i^{-1}}{\lambda} - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1}F^{(1)}_{j_1,j_2}\cdots F^{(1)}_{j_p,j}\frac{\Delta_j C_{j,i}\Delta_i^{-1}}{\lambda}\right)^{-1}\end{aligned}$$

To show that such $F^{(1)}_{k,2}$ exist, consider the process which begins in Γ_i and eventually ends at Γ_k in such a way that at any intermediate step, we either jump to or remain in one of $\Gamma_1, \Gamma_2, \ldots, \Gamma_i$ and, at the last step, we jump to Γ_k.

Again, this can be divided into infinitely many disjoint events: never staying in Γ_i,

$$\left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1}F^{(1)}_{j_1,j_2}\cdots F^{(1)}_{j_p,j}C'_{j,i}\right)$$

staying one time in Γ_i just before jumping to Γ_k,

$$\left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right) \cdot$$

$$\left(B'_i + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right)$$

and, given any $j > 2$, staying in Γ_i j times before jumping to Γ_k,

$$\left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right) \cdot$$

$$\left(B'_i + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right)^j$$

The sum of these matrices yields the transition matrix for the process.

$$\left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right) \cdot$$

$$\sum_{j=0}^{\infty} \left(B'_i + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right)^j$$

Since $B'_i + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}$ has column sums strictly less than 1 (being the part of the transition matrix associated to the Markov chain from Γ_i to Γ_i passing through any states in Γ), the above matrix series converges to

$$\left(I - B'_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right)^{-1}.$$

Therefore,

$$\begin{aligned} F^{(1)}_{k,i} &= \left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right) \cdot \\ &\quad \left(I - B'_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i}\right) \end{aligned}$$

is well-defined nonnegative. But

$$\begin{aligned}
\Delta_k^{-1} F^{(1)}_{k,i} \Delta_i &= \Delta_k^{-1} \left(C'_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{k,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i} \right) \cdot \\
&\quad \left(I - B'_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i} \right) \Delta_i \\
&= \left(\frac{C_{k,i}}{\lambda} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} \frac{C_{j,i}}{\lambda} \right) \cdot \\
&\quad \left(I - \frac{B_i}{\lambda} - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} \frac{C_{j,i}}{\lambda} \right)^{-1} \\
&= F^{(\lambda)}_{k,i} .
\end{aligned}$$

Hence, $F^{(\lambda)}_{k,i}$ is also well-defined and nonnegative. Similarly,

$$\begin{aligned}
&\lambda^{-1}\Delta_i^{-1} \left(I - B'_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} C'_{j,i} \right)^{-1} \Delta_i \\
&= \frac{1}{\lambda} \Delta_i^{-1} \left(I - \frac{\Delta_i B_i \Delta_i^{-1}}{\lambda} - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(1)}_{i,j_1} F^{(1)}_{j_1,j_2} \cdots F^{(1)}_{j_p,j} \frac{\Delta_j C_{j,i} \Delta_i^{-1}}{\lambda} \right)^{-1} \Delta_i \\
&= \left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right)^{-1}
\end{aligned}$$

so that

$$\left(\lambda I - B_i - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i} \right)^{-1}$$

is well-defined. ∎

Lemma 4.3 *Using the above notation, if we set*

$$H = \begin{pmatrix} F^{(\lambda)}_{2,1} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F^{(\lambda)}_{3,1} & F^{(\lambda)}_{3,2} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F^{(\lambda)}_{4,1} & F^{(\lambda)}_{4,2} & F^{(\lambda)}_{4,3} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ F^{(\lambda)}_{i+1,1} & F^{(\lambda)}_{i+1,2} & F^{(\lambda)}_{i+1,3} & \cdot\ \cdot & F^{(\lambda)}_{i+1,i} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

then

$$(\ v_2 \quad v_3 \quad v_4 \quad \cdot\ \cdot\ ) \cdot H \ = \ (\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ )$$

if and only if

$$(\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ ) \cdot P \ = \ \lambda(\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ ).$$

Proof. Suppose

$$(\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ ) \cdot P \ = \ \lambda(\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ ).$$

As verified in the course of defining the $F^{(\lambda)}_{k,i}$, we have

$$\begin{cases} v_1 = v_2 F^{(\lambda)}_{2,1} + v_3 F^{(\lambda)}_{3,1} + \cdots \\ v_2 = v_3 F^{(\lambda)}_{3,2} + v_4 F^{(\lambda)}_{4,2} + \cdots \\ \quad\vdots \\ v_i = v_{i+1} F^{(\lambda)}_{i+1,i} + v_{i+2} F^{(\lambda)}_{i+2,i} + \cdots \\ \quad\vdots \end{cases}$$

Thus,

$$(\ v_2 \quad v_3 \quad v_4 \quad \cdot\ \cdot\ ) \cdot H \ = \ (\ v_1 \quad v_2 \quad v_3 \quad \cdot\ \cdot\ ).$$

Conversely, suppose

$$(\; v_2 \quad v_3 \quad v_4 \quad \cdot \quad \cdot \;) \cdot H \;=\; (\; v_1 \quad v_2 \quad v_3 \quad \cdot \quad \cdot \;).$$

Then, given any $i \in \mathcal{N}$, we can write

$$v_i \;=\; v_{i+1}\, F^{(\lambda)}_{i+1,i} \;+\; v_{i+2}\, F^{(\lambda)}_{i+2,i} \;+\; \cdots$$

where for each $k > i \geq 1$,

$$\begin{aligned} F^{(\lambda)}_{k,i} \;=\; & \Big(C_{k,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \cdot \\ & \Big(\lambda I \;-\; B_i \;-\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big)^{-1} \end{aligned}$$

By substitution,

$$\begin{aligned} & v_i \Big(\lambda I \;-\; B_i \;-\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \\ & \quad = \; v_{i+1}\, \Big(C_{i+1,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+ \\ & \qquad\;\; v_{i+2}\, \Big(C_{i+2,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+\; \cdots \end{aligned}$$

from which we obtain

$$\begin{aligned} \lambda\, v_i \;=\; & v_i\, B_i \;+\; v_{i+1}\, C_{i+1,i} \;+\; v_{i+2}\, C_{i+2,i} \;+\; \cdots \;+ \\ & v_i\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+ \\ & v_{i+1}\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+ \\ & v_{i+2}\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+\; \cdots . \end{aligned}$$

However,

$$\begin{aligned} & v_i\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+ \\ & v_{i+1}\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+ \\ & v_{i+2}\, \Big(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\Big) \;+\; \cdots \end{aligned}$$

is equal to

$$
\begin{aligned}
& v_i \left(F^{(\lambda)}_{i,i-1}\, C_{i-1,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1} \left(F^{(\lambda)}_{i+1,i-1}\, C_{i-1,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+2} \left(F^{(\lambda)}_{i+2,i-1}\, C_{i-1,i} \;+\; \textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+\; \cdots
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& \left(v_i F^{(\lambda)}_{i,i-1} \;+\; v_{i+1}\, F^{(\lambda)}_{i+1,i-1} \;+\; v_{i+2}\, F^{(\lambda)}_{i+2,i-1} \;+\; \cdots\right) C_{i-1,i} \;+ \\
& v_i \left(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1} \left(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+2} \left(\textstyle\sum_{i>j_1>j_2>\cdots>j_p>j\geq 1,\, i-1>j} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+\; \cdots
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& v_{i-1}\, C_{i-1,i} \;+\; v_i\, F^{(\lambda)}_{i,i-1} \left(\textstyle\sum_{i-1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i-1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_i \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1}\, F^{(\lambda)}_{i+1,i-1} \left(\textstyle\sum_{i-1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i-1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1} \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+2}\, F^{(\lambda)}_{i+2,i-1} \left(\textstyle\sum_{i-1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i-1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+2} \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+\; \cdots
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& v_{i-1}\, C_{i-1,i} \;+\; \left(v_i\, F^{(\lambda)}_{i,i-1} \;+\; v_{i+1}\, F^{(\lambda)}_{i+1,i-1} \;+\; v_{i+2}\, F^{(\lambda)}_{i+2,i-1}\right) \cdot \\
& \left(\textstyle\sum_{i-1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i-1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_i \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1} \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+2} \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+2,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+\; \cdots
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& v_{i-1}\, C_{i-1,i} \;+\; v_{i-1} \left(\textstyle\sum_{i-1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i-1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_i \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+ \\
& v_{i+1} \left(\textstyle\sum_{i-1>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{i+1,j_1}\, F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i}\right) \;+\cdots\,.
\end{aligned}
$$

Continuing by induction, this becomes equal to

$$v_{i-1}\,C_{i-1,i} \;+\; v_{i-2}\,C_{i-2,i} \;+\; \cdots \;+\; v_3\,C_{3,i} \;+\; v_3\left(\textstyle\sum_{3>j_1>j\geq 1} F^{(\lambda)}_{3,j_1}\,F^{(\lambda)}_{j_1,j}\,C_{j,i}\right) \;+$$
$$v_4\left(\textstyle\sum_{3>j_1>j\geq 1} F^{(\lambda)}_{4,j_1}\,F^{(\lambda)}_{j_1,j}\,C_{j,i}\right) \;+\; v_5\left(\textstyle\sum_{3>j_1>j\geq 1} F^{(\lambda)}_{5,j_1}\,F^{(\lambda)}_{j_1,j}\,C_{j,i}\right) \;+\; \cdots$$

which equals

$$\begin{aligned}
&v_{i-1}\,C_{i-1,i} \;+\; v_{i-2}\,C_{i-2,i} \;+\; \cdots \;+\; v_3\,C_{3,i} \;+\\
&v_3\,F^{(\lambda)}_{3,2}\,C_{2,i} \;+\; v_3\,F^{(\lambda)}_{3,2}\,F^{(\lambda)}_{2,1}C_{1,i} \;+\; v_3\,F^{(\lambda)}_{3,1}\,C_{1,i} \;+\\
&v_4\,F^{(\lambda)}_{4,2}\,C_{2,i} \;+\; v_4\,F^{(\lambda)}_{4,2}\,F^{(\lambda)}_{2,1}\,C_{1,i} \;+\; v_4\,F^{(\lambda)}_{4,1}\,C_{1,i} \;+\\
&v_5F^{(\lambda)}_{5,2}\,C_{2,i} \;+\; v_5\,F^{(\lambda)}_{5,2}\,F^{(\lambda)}_{2,1}\,C_{1,i} \;+\; v_5\,F^{(\lambda)}_{5,1}\,C_{1,i} \;+\; \cdots
\end{aligned}$$

which equals

$$\begin{aligned}
&v_{i-1}\,C_{i-1,i} \;+\; v_{i-2}\,C_{i-2,i} \;+\; \cdots \;+\; v_3\,C_{3,i} \;+\\
&\left(v_3\,F^{(\lambda)}_{3,2} \;+\; v_4\,F^{(\lambda)}_{4,2} \;+\; v_5\,F^{(\lambda)}_{5,2} \;+\; \cdots\right) C_{2,i} \;+\\
&\left(v_3\,F^{(\lambda)}_{3,2} \;+\; v_4\,F^{(\lambda)}_{4,2} \;+\; v_5\,F^{(\lambda)}_{5,2} \;+\; \cdots\right) F^{(\lambda)}_{2,1}\,C_{1,i} \;+\\
&v_3\,F^{(\lambda)}_{3,1}\,C_{1,i} \;+ v_4\,F^{(\lambda)}_{4,1}\,C_{1,i} \;+\; v_5\,F^{(\lambda)}_{5,1}\,C_{1,i} \;+\; \cdots
\end{aligned}$$

which equals

$$\begin{aligned}
&v_{i-1}\,C_{i-1,i} \;+\; v_{i-2}\,C_{i-2,i} \;+\; \cdots \;+\; v_3\,C_{3,i} \;+\; v_2\,C_{2,i} \;+\\
&v_2\,F^{(\lambda)}_{2,1}\,C_{1,i} \;+\; v_3\,F^{(\lambda)}_{3,1}\,C_{1,i} \;+\; v_4\,F^{(\lambda)}_{4,1}\,C_{1,i} \;+\; \cdots
\end{aligned}$$

which in turn equals

$$v_{i-1}\,C_{i-1,i} \;+\; v_{i-2}\,C_{i-2,i} \;+\; \cdots \;+\; v_3\,C_{3,i} \;+\; v_2\,C_{2,i} \;+\; v_1\,C_{1,i}.$$

Hence, using the previous identity for $\lambda\,v_i$,

$$\lambda\,v_i \;=\; v_1\,C_{1,i} \;+\; v_2\,C_{2,i} \;+\; \cdots \;+\; v_{i-1}\,C_{i-1,i} \;+\; v_i\,B_i \;+\; v_{i+1}\,C_{i+1,i} \;+\; \cdots\,.$$

Since $i \in N$ was arbitrarily chosen, we have thus shown

$$(\;v_1 \quad v_2 \quad v_3 \quad \cdot \quad \cdot\;)\cdot P \;=\; \lambda(\;v_1 \quad v_2 \quad v_3 \quad \cdot \quad \cdot\;)$$

which completes the proof. ■

We now begin our construction of dimension groups associated to nonnegative eigenvalues. Recall that for each $j \in \mathbb{N}$, $R\Gamma_j$ is a finite dimensional vector space over R with basis $\{e_g\}_{g\in\Gamma_j}$. Defining

$$R\Gamma_j^+ \;:=\; \{a \in R\Gamma_j \mid a_g \geq 0 \;\forall\; g \in \Gamma_j\}$$

makes $R\Gamma_j$ an ordered vector space. For each i, define the infinite cartesian product,

$$\mathcal{U}_i \;:=\; R\Gamma_i \times R\Gamma_{i+1} \times R\Gamma_{i+2} \times \cdots \;=\; \prod_{j=i}^{\infty} R\Gamma_j.$$

We make each $\mathcal{U}_i$ an ordered vector space by defining

$$\mathcal{U}_i^+ \;:=\; \{u_i \in \mathcal{U}_i \mid u_{i,j} \in R\Gamma_j^+ \;\forall\; j \geq i,\; j \in \mathbb{N}\}$$

Define a linear transformation, $H_1 : \mathcal{U}_1 \to \mathcal{U}_2$ via

$$H_1 \;=\; \begin{pmatrix} F_{2,1}^{(\lambda)} & 0 & 0 & \cdot\;\cdot & 0 & \cdot\;\cdot \\ F_{3,1}^{(\lambda)} & F_{3,2}^{(\lambda)} & 0 & \cdot\;\cdot & 0 & \cdot\;\cdot \\ F_{4,1}^{(\lambda)} & F_{4,2}^{(\lambda)} & F_{4,3}^{(\lambda)} & \cdot\;\cdot & 0 & \cdot\;\cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot \\ F_{j+1,1}^{(\lambda)} & F_{j+1,2}^{(\lambda)} & F_{j+1,3}^{(\lambda)} & \cdot\;\cdot & F_{j+1,j}^{(\lambda)} & \cdot\;\cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot \end{pmatrix}$$

(This is precisely the matrix H in Lemma 4.3.) Since $F_{j+1,s}^{(\lambda)}$ is a nonnegative matrix for any $j \in N$ and $j+1 > s \geq 1$, so is H_1. From

$$H_1 \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ \vdots \\ u_{1,j} \\ \vdots \end{pmatrix} = \begin{pmatrix} F_{2,1}^{(\lambda)}(u_{1,1}) \\ F_{3,1}^{(\lambda)}(u_{1,1}) + F_{3,2}^{(\lambda)}(u_{1,2}) \\ F_{4,1}^{(\lambda)}(u_{1,1}) + F_{4,2}^{(\lambda)}(u_{1,2}) + F_{4,3}^{(\lambda)}(u_{1,3}) \\ \vdots \\ F_{j+1,1}^{(\lambda)}(u_{1,1}) + F_{j+1,2}^{(\lambda)}(u_{1,2}) + \cdots + F_{j+1,j}^{(\lambda)}(u_{1,j}) \\ \vdots \end{pmatrix}$$

it is clear that H_1 is a well-defined positive homomorphism. Similarly, for each $i > 1$, define a linear transformation $H_i : \mathcal{U}_i \to \mathcal{U}_{i+1}$ via

$$H_i = \begin{pmatrix} F_{i+1,i}^{(\lambda)} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F_{i+2,i}^{(\lambda)} & F_{i+2,i+1}^{(\lambda)} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F_{i+3,i}^{(\lambda)} & F_{i+3,i+1}^{(\lambda)} & F_{i+3,i+2}^{(\lambda)} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ F_{j+1,i}^{(\lambda)} & F_{j+1,i+1}^{(\lambda)} & F_{j+1,i+2}^{(\lambda)} & \cdot\ \cdot & F_{j+1,j}^{(\lambda)} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

so that

$$H_i \begin{pmatrix} u_{i,i} \\ u_{i,i+1} \\ u_{i,i+2} \\ \vdots \\ u_{i,j} \\ \vdots \end{pmatrix} = \begin{pmatrix} F_{i+1,i}^{(\lambda)}(u_{i,i}) \\ F_{i+2,i}^{(\lambda)}(u_{i,i}) + F_{i+2,i+1}^{(\lambda)}(u_{i,i+1}) \\ F_{i+3,i}^{(\lambda)}(u_{i,i}) + F_{i+3,i+1}^{(\lambda)}(u_{i,i+1}) + F_{i+3,i+2}^{(\lambda)}(u_{i,i+2}) \\ \vdots \\ F_{j+1,i}^{(\lambda)}(u_{i,i}) + F_{j+1,i+1}^{(\lambda)}(u_{i,i+1}) + \cdots + F_{j+1,j}^{(\lambda)}(u_{i,j}) \\ \vdots \end{pmatrix}.$$

Once again, it is clear that H_i is a well-defined positive homomorphism.

Lemma 4.4 *If*

$$(\ v_2 \quad v_3 \quad v_4 \quad \cdot \quad \cdot \ ) \cdot H_1 \ = \ (\ v_1 \quad v_2 \quad v_3 \quad \cdot \quad \cdot \ ). \tag{7}$$

then for any $i > 1$, we have

$$(\ v_{i+1} \quad v_{i+2} \quad v_{i+3} \quad \cdot \quad \cdot \ ) \cdot H_i \ = \ (\ v_i \quad v_{i+1} \quad v_{i+2} \quad \cdot \quad \cdot \ ). \tag{8}$$

Proof. By (7), for every $j \in \mathcal{N}$,

$$v_j \ = \ v_{j+1} F^{(\lambda)}_{j+1,j} \ + \ v_{j+2} F^{(\lambda)}_{j+2,j} \ + \ \cdots .$$

In particular, given any $i \in \mathcal{N}$, this holds for every $j \geq i$, implying (8). ■

Now, define

$$\mathcal{V}_1 \ := \ R\Gamma_1 \bigoplus R\Gamma_2 \bigoplus R\Gamma_3 \bigoplus \cdots \ = \ \bigoplus_{j=0}^{\infty} R\Gamma_j \ \subseteq \ \mathcal{U}_1.$$

Equipped with the relative ordering inherited from $\mathcal{U}_1$, $\mathcal{V}_1$ becomes an ordered vector space. Given any $i > 1$, define

$$\mathcal{V}_i \ := \ \left(R\Gamma_i \bigoplus R\Gamma_{i+1} \bigoplus R\Gamma_{i+2} \bigoplus \cdots\right) \ \bigoplus \ H_{i-1}(\mathcal{V}_{i-1}).$$

By a simple induction, $\mathcal{V}_{i-i} \subseteq \mathcal{U}_{i-1} \ \forall\, i \geq 2$. Hence,

$$H_{i-1}(\mathcal{V}_{i-1}) \ \subseteq \ H_{i-1}(\mathcal{U}_{i-1}) \ \subseteq \mathcal{U}_i \quad \forall\, i \geq 2$$

(Note that $H_i(\mathcal{V}_i) \subseteq \mathcal{V}_{i+1}$.) Assume the relative ordering of $\mathcal{U}_i$ on $\mathcal{V}_i$ and denote $H_i|_{\mathcal{V}_i} : \mathcal{V}_i \to \mathcal{V}_{i+1}$ simply as H_i. Each H_i is a well-defined order preserving transformation from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$.

Define the direct limit

$$DG(\lambda) \ := \ \varinjlim \ \mathcal{V}_1 \xrightarrow{H_1} \mathcal{V}_2 \xrightarrow{H_2} \mathcal{V}_3 \xrightarrow{H_3} \cdots$$

equipped with the direct limit ordering—declare $\overline{(v,m)} \in DG(\lambda)$ to be nonnegative if and only if there exists a positive integer l such that

$$H_{m+l-1} \circ H_{m+l-2} \circ \cdots \circ H_{m+2} \circ H_{m+1} \circ H_m(v) \in \mathcal{V}^+_{m+l}.$$

Since each $\mathcal{V}_i$ is a dimension group (in fact a lattice ordered vector space), $DG(\lambda)$ is also a dimension group. The following is the infinite analogue of [14; 3.2]; the extra complications here arise from differing contexts (e.g., $DG(\lambda)$ is different here).

Theorem 4.5 *For each $\lambda > 0$, there is an affine homeomorphism ϕ between the space P_λ of nonnegative, nonzero left eigenvectors of P corresponding to λ and*

$$\tilde{S} = \left\{\tau \in S(DG(\lambda), R) \,\middle|\, \tau(\overline{(u_i,l)}) = \tau(\overline{(u_i,n)})\ \forall n \geq l,\ u_i \in R\Gamma_i\right\}$$

where $S(DG(\lambda), R)$ denotes the trace space of $DG(\lambda)$. In particular, given an eigenvector $v = (v_i = v|_{\Gamma_i})_{i\in N}$, we set $\phi(v) = \tau_v$ where $\tau_v : DG(\lambda) \to R$ is the trace defined by

$$\tau_v(\overline{(u_i,l)}) = v_i \cdot u_i \quad \forall i \in N.$$

(Here, $v_i \cdot u_i$ denotes $\sum_{g\in\Gamma_i} v_i(g)\, u_i(g)$, an inner product.)

Proof. Let P_λ denote the space of nonzero, nonnegative left eigenvectors of P corresponding to λ, that is,

$$P_\lambda = \{v : \Gamma \to R \,|\, v \geq 0,\ v \neq 0 \text{ and } v \cdot P = \lambda v\}.$$

Given $v \in P_\lambda$, we have

$$(\, v_1 \;\; v_2 \;\; v_3 \;\; \cdot \;\; \cdot \,) \cdot P = \lambda(\, v_1 \;\; v_2 \;\; v_3 \;\; \cdot \;\; \cdot \,)$$

so that Lemma 4.3 implies

$$(\, v_2 \;\; v_3 \;\; v_4 \;\; \cdot \;\; \cdot \,) \cdot H = (\, v_1 \;\; v_2 \;\; v_3 \;\; \cdot \;\; \cdot \,)$$

and thus, by Lemma 4.4,

$$\begin{pmatrix} v_{i+1} & v_{i+2} & v_{i+3} & \cdot & \cdot \end{pmatrix} \cdot H_i = \begin{pmatrix} v_i & v_{i+1} & v_{i+2} & \cdot & \cdot \end{pmatrix}$$

for every $i \in N$. Define the map $\phi : P_\lambda \to \tilde{S}$ by $v \mapsto \tau_v$ where

$$\tau_v(\overline{(u_j, l)}) = v_j \cdot u_j = \tau_v(\overline{(u_j, n)})$$

for $u_j \in R\Gamma_j$ and $n \geq l$. We need to show τ_v is a trace of $DG(\lambda)$.

First, we check that $\tau_v(u) < \infty \ \forall\, u \in DG(\lambda)$. There are several cases.

(i) $u = \overline{(\sum_{j\geq 1} u_j, 1)}$ where $\sum_{j\geq 1} u_j \in \mathcal{V}_1$ and $u_j \in R\Gamma_j \ \forall\, j \geq 1$. In this case,

$$\tau_v(u) = \sum_{j\geq 1} v_j \cdot u_j < \infty$$

since $v_j \cdot u_j < \infty$ and $u_j = 0$ for all but finitely many j.

(ii) $u = \overline{(\sum_{j\geq 2} u_j, 2)} + \overline{(H_1(w_1), 2)}$ where $\sum_{j\geq 2} u_j \in \mathcal{V}_2$ and $u_j \in R\Gamma_j \ \forall\, j \geq 2$ and $w_1 \in \mathcal{V}_1$. In this case,

$$\begin{aligned} \tau_v(u) &= \sum_{j\geq 2} v_j \cdot u_j + \begin{pmatrix} v_2 & v_3 & v_4 & \cdot & \cdot \end{pmatrix} \cdot H_1(w_1) \\ &= \sum_{j\geq 2} v_j \cdot u_j + \begin{pmatrix} v_1 & v_2 & v_3 & \cdot & \cdot \end{pmatrix} \cdot w_1 \\ &= \sum_{j\geq 2} v_j \cdot u_j + \tau_v(w_1) \end{aligned}$$

which is finite since $\tau_v(w_1) < \infty$ by (i) and $u_j = 0$ for all but finitely many j.

(iii) Suppose $\tau_v(u) < \infty$ whenever $u = \overline{(u', i)}$ with $u' \in \mathcal{V}_i$. Then, if $u = \overline{(\sum_{j\geq i+1} u_j, i+1)} + \overline{(H_i(w_i), i+1)}$ where $\sum_{j\geq i+1} u_j \in \mathcal{V}_{i+1}$, $u_j \in R\Gamma_j$ and $w_i \in \mathcal{V}_i$, we have

$$\begin{aligned} \tau_v(u) &= \sum_{j\geq i+1} v_j \cdot u_j + \begin{pmatrix} v_{i+1} & v_{i+2} & v_{i+3} & \cdot & \cdot \end{pmatrix} \cdot H_i(w_i) \\ &= \sum_{j\geq i+1} v_j \cdot u_j + \begin{pmatrix} v_i & v_{i+1} & v_{i+2} & \cdot & \cdot \end{pmatrix} \cdot w_i \\ &= \sum_{j\geq i+1} v_j \cdot u_j + \tau_v(w_i) \end{aligned}$$

which is finite since $\tau_v(w_i)$ is finite by assumption and $u_j = 0$, all but finitely many j.

(iv) Finally, given any u in $DG(\lambda)$, we can write $u = \overline{(w,n)}$ where $w \in \mathcal{V}_n$. By the induction in case (iii), we know that $\tau_v(u) = \tau_v(\overline{(w,n)}) < \infty$.

Next, we show that τ_v is well-defined on $DG(\lambda)$. For this purpose, select $u = \overline{(u_1,m)} = \overline{(u_2,n)}$ where $m,n \in \mathcal{N}$, $u_1 \in \mathcal{V}_m$ and $u_2 \in \mathcal{V}_n$. By definition, $\exists\, l_1, l_2 \in \mathcal{N}$ such that $n' = l_1 + m = l_2 + n$ and

$$H_{n'-1}\,H_{n'-2}\cdots H_{m+1}\,H_m\,(u_1) \;=\; H_{n'-1}\,H_{n'-2}\cdots H_{n+1}\,H_n\,(u_2) \;=\; u_3$$

for some $u_3 \in \mathcal{V}_{n'}$. Thus,

$$\begin{aligned}
\tau_v(\overline{(u_3,n')}) &= (\; v_{n'} \quad v_{n'+1} \quad v_{n'+2} \quad \cdot \quad \cdot\;) \cdot u_3 \\
&= (\; v_{n'} \quad v_{n'+1} \quad v_{n'+2} \quad \cdot \quad \cdot\;) \cdot H_{n'-1}\,H_{n'-2}\cdots H_{m+1}\,H_m(u_1) \\
&= (\; v_{n'-1} \quad v_{n'} \quad v_{n'+1} \quad \cdot \quad \cdot\;) \cdot H_{n'-2}\cdots H_{m+1}\,H_m(u_1) \\
&= (\; v_{n'-2} \quad v_{n'-1} \quad v_{n'} \quad \cdot \quad \cdot\;) \cdot H_{n'-3}\cdots H_{m+1}\,H_m(u_1) \\
&\;\;\vdots \\
&= (\; v_{m+2} \quad v_{m+3} \quad v_{m+4} \quad \cdot \quad \cdot\;) \cdot H_{m+1}\,H_m(u_1) \\
&= (\; v_{m+1} \quad v_{m+2} \quad v_{m+3} \quad \cdot \quad \cdot\;) \cdot H_m(u_1) \\
&= (\; v_m \quad v_{m+1} \quad v_{m+2} \quad \cdot \quad \cdot\;) \cdot u_1 \\
&= \tau_v(\overline{(u_1,m)}).
\end{aligned}$$

Similarly, $\tau_v(\overline{(u_3,n')}) = \tau_v(\overline{(u_2,n)})$ so that $\tau_v(\overline{(u_2,n)}) = \tau_v(\overline{(u_1,m)})$, i.e., τ_v is well-defined. Therefore, since τ_v is linear and maps positive elements of $DG(\lambda)$ into R^+, τ_v constitutes a trace on $DG(\lambda)$.

To check that $\phi(t_1 v + t_2 \bar{v}) = t_1\phi(v) + t_2\phi(\bar{v})$ for any given $v, \bar{v} \in P_\lambda$ and any given t_1, $t_2 > 0$ with $t_1 + t_2 = 1$, note that

$$\begin{aligned}
(t_1\,v \;+\; t_2\,\bar{v}) \cdot P &= t_1\,v \cdot P \;+\; t_2\,\bar{v} \cdot P \\
&= t_1\,\lambda\,v \;+\; t_2\,\lambda\,\bar{v}
\end{aligned}$$

$$= \lambda(t_1 v + t_2 \bar{v}).$$

Hence, $t_1 v + t_2 \bar{v} \in P_\lambda$. Given any $u_j \in R\Gamma_j$ and any $l \in N$, we can thus calculate

$$\begin{aligned}
\phi(t_1 v + t_2 \bar{v})(\overline{(u_j, l)}) &= (t_1 v_j + t_2 \bar{v}_j) \cdot u_j \\
&= t_1 v_j \cdot u_j + t_2 \bar{v}_j \cdot u_j \\
&= t_1 \phi(v)(\overline{(u_j, l)}) + t_2 \phi(\bar{v})(\overline{(u_j, l)}) \\
&= (t_1 \phi(v) + t_2 \phi(\bar{v}))(\overline{(u_j, l)}).
\end{aligned}$$

Furthermore, if $DG(\lambda)$ possesses an order unit $\bar{u}$ for which $\phi(v)(\bar{u}) = 1$ and $\phi(\bar{v})(\bar{u}) = 1$, it is easy to check that $\phi(t_1 v + t_2 \bar{v})(\bar{u}) = 1$. Therefore, ϕ is an affine linear map.

Conversely, we construct a map $\varphi : \tilde{S} \to P_\lambda$ as follows. Given, τ in $\tilde{S}$, define $v = (v_i)$ by setting

$$v_i(g) = \tau(\overline{(e_{i,g}, 1)}) \quad \forall g \in \Gamma_i$$

for every $i \in N$ and then setting $\varphi(\tau) = v$. Given any $i \geq 1$,

$$\begin{aligned}
v_i(g) &= \tau(\overline{(e_{i,g}, 1)}) \\
&= \tau(\overline{(H_1 \cdot e_{i,g}, 2)}) \\
&= \tau(\overline{(F^{(\lambda)}_{i+1,i} \cdot e_{i,g} + F^{(\lambda)}_{i+2,i} \cdot e_{i,g} + F^{(\lambda)}_{i+3,i} \cdot e_{i,g} + \cdots, 2)}) \\
&= \tau(\overline{(F^{(\lambda)}_{i+1,i} \cdot e_{i,g}, 2)}) + \tau(\overline{(F^{(\lambda)}_{i+2,i} \cdot e_{i,g}, 2)}) + \tau(\overline{(F^{(\lambda)}_{i+3,i} \cdot e_{i,g}, 2)}) + \cdots \\
&= \textstyle\sum_{j=i+1}^{\infty} \tau(\overline{(F^{(\lambda)}_{j,i} \cdot e_{i,g}, 2)}).
\end{aligned}$$

Furthermore, given any pair $j > i$, if we write the entries of $F^{(\lambda)}_{j,i}$ as $(f_{h,g})_{(h,g) \in \Gamma_j \times \Gamma_i}$,

$$\begin{aligned}
\tau(\overline{(F^{(\lambda)}_{j,i} \cdot e_{i,g}, 2)}) &= \tau(\overline{(\textstyle\sum_{h \in \Gamma_i} e_{j,h} f_{h,g}, 2)}) \\
&= \textstyle\sum_{h \in \Gamma_i} \tau(\overline{(e_{j,h}, 2)}) f_{h,g} \\
&= \textstyle\sum_{h \in \Gamma_i} \tau(\overline{(e_{j,h}, 1)}) f_{h,g} \\
&= \textstyle\sum_{h \in \Gamma_i} v_j(h) f_{h,g} \\
&= v_j \cdot F^{(\lambda)}_{j,i} \cdot e_{i,g}.
\end{aligned}$$

Hence, we have

$$\begin{aligned} v_i(g) &= \sum_{j=i+1}^{\infty} v_j \cdot F_{j,i}^{(\lambda)} \cdot e_{i,g} \\ &= \left(\sum_{j=i+1}^{\infty} v_j \cdot F_{j,i}^{(\lambda)}\right) \cdot e_{i,g} \\ &= \left(\sum_{j=i+1}^{\infty} v_j \cdot F_{j,i}^{(\lambda)}\right)(g) \end{aligned}$$

which implies

$$v_i \;=\; v_{i+1}\, F_{i+1,i}^{(\lambda)} \;+\; v_{i+2}\, F_{i+2,i}^{(\lambda)} \;+\; \cdots \quad \forall\, i \geq 1$$

so that

$$(\; v_2 \;\; v_3 \;\; v_4 \;\; \cdot \;\; \cdot \;) \cdot H_1 \;=\; (\; v_1 \;\; v_2 \;\; v_3 \;\; \cdot \;\; \cdot \;)\,.$$

Therefore, by Lemma 4.3, $v \cdot P = \lambda\, v$. But $\tau \neq 0$ implies $v \neq 0$. Thus, $v \in P_\lambda$ which implies φ is well-defined.

Given any $\alpha, \beta \in \tilde{S}$ and any $t_1, t_2 \geq 0$ for which $t_1 + t_2 = 1$, it is routine to check that $t_1\, \alpha + t_2\, \beta \in \tilde{S}$ and

$$\varphi(t_1\, \alpha \,+\, t_2\, \beta) \;=\; t_1\, \varphi(\alpha) \,+\, t_2\, \varphi(\beta)\,,$$

i.e., φ is an affine linear map. Moreover, it is straightforward to show that φ is continuous with respect to the topologies of pointwise convergence (weak topologies) on the respective spaces.

Now, take any $v \in P_\lambda$. If we set

$$u \;=\; \varphi \circ \phi(v) \;=\; \varphi(\phi(v))\,,$$

then for any $i \in N$ and $g \in \Gamma_i$,

$$\begin{aligned} u_i(g) &= \phi(v)(\overline{(e_{i,g}, 1)}) \\ &= \tau_v(\overline{(e_{i,g}, 1)}) \\ &= v_i \cdot e_{i,g} \\ &= v_i(g)\,. \end{aligned}$$

Hence, $u = v$ which implies $\varphi \circ \phi(v) = v$, that is, $\varphi \circ \phi = id$. On the other hand, given any $\tau \in \tilde{S}$, $u_i \in R\Gamma_i$ and $l \in \mathbb{N}$,

$$\begin{aligned}
(\phi \circ \varphi(\tau))\,(\overline{(u_i, l)}) &= (\phi(\varphi(\tau)))\,(\overline{(u_i, l)}) \\
&= (\varphi(\tau))_i \cdot u_i \\
&= \textstyle\sum_{g\in\Gamma_i} (\varphi(\tau))_i(g)\, u_i(g) \\
&= \textstyle\sum_{g\in\Gamma_i} \tau(\overline{(e_{i,g}, 1)})\, u_i(g) \\
&= \textstyle\sum_{g\in\Gamma_i} \tau(\,\overline{(e_{i,g} u_i(g), 1)}\,) \\
&= \tau(\,\overline{(\textstyle\sum_{g\in\Gamma_i} e_{i,g}\, u_i(g), 1)}\,) \\
&= \tau(\overline{(u_i, 1)}) \\
&= \tau(\overline{(u_i, l)})\,.
\end{aligned}$$

This implies $\phi \circ \varphi = id$. Therefore, since φ is a continuous bijective affine map, it is necessarily an affine homeomorphism which completes the proof. ∎

4.2 Other Results

We now define "future dimension groups" for P which are not necessarily column finite. This is done in analogy to [14] (column finite case). The future dimension group was motivated by a desire to simplify computations of eigenvectors and eigenvalues.

For any $k > i \geq 1$, define $D_{k,i}^{(\lambda)}$ and $G_i^{(\lambda)}$ via the recursive relations,

$$D_{k,1}^{(\lambda)} = C_{k,1}\,, \qquad G_1^{(\lambda)} = \left(I - \frac{B_1}{\lambda}\right)^{-1},$$

$$\begin{aligned}
D_{k,i}^{(\lambda)} &= C_{k,i} + \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} \frac{D_{k,j_1}^{(\lambda)}\, G_{j_1}^{(\lambda)}\, D_{j_1,j_2}^{(\lambda)}\, G_{j_2}^{(\lambda)} \cdots D_{j_p,j}^{(\lambda)}\, G_j^{(\lambda)}}{\lambda^{p+1}}\, C_{j,i}\,, \\
G_i^{(\lambda)} &= \left(I - \frac{B_i}{\lambda} - \sum_{i>j_1>j_2>\cdots>j_p>j\geq 1} \frac{D_{i,j_1}^{(\lambda)}\, G_{j_1}^{(\lambda)}\, D_{j_1,j_2}^{(\lambda)}\, G_{j_2}^{(\lambda)} \cdots D_{j_p,j}^{(\lambda)}\, G_j^{(\lambda)}}{\lambda^{p+2}}\, C_{j,i}\right)^{-1}.
\end{aligned}$$

Lemma 4.6 *For any $k > i \geq 1$, $D_{k,i}^{(\lambda)}$ and $G_i^{(\lambda)}$ are well-defined and*

$$\lambda F_{k,i}^{(\lambda)} = D_{k,i}^{(\lambda)} G_i^{(\lambda)}.$$

Proof. Our proof is by induction. The case of $i = 1$ is obvious. Fix $n \in N$ and assume that for every $1 \leq i \leq n-1$, $\lambda F_{k,i}^{(\lambda)} = D_{k,i}^{(\lambda)} G_i^{(\lambda)}$ holds and both $D_{k,i}^{(\lambda)}$ and $G_i^{(\lambda)}$ are well-defined. Then, given any $k > n$, we can write

$$\begin{aligned}
D_{k,n}^{(\lambda)} &= \left(C_{k,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} \frac{D_{k,j_1}^{(\lambda)} G_{j_1}^{(\lambda)} D_{j_1,j_2}^{(\lambda)} G_{j_2}^{(\lambda)} \cdots D_{j_p,j}^{(\lambda)} G_j^{(\lambda)}}{\lambda^{p+1}} C_{j,n} \right) \\
&= \left(C_{k,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} \frac{\lambda F_{k,j_1}^{(\lambda)} \lambda F_{j_1,j_2}^{(\lambda)} \cdots \lambda F_{j_p,j}^{(\lambda)}}{\lambda^{p+1}} C_{j,n} \right) \\
&= \left(C_{k,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F_{k,j_1}^{(\lambda)} F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,j}^{(\lambda)} C_{j,n} \right)
\end{aligned}$$

and

$$\begin{aligned}
G_n^{(\lambda)} &= \left(I - \frac{B_n}{\lambda} - \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} \frac{D_{n,j_1}^{(\lambda)} G_{j_1}^{(\lambda)} D_{j_1,j_2}^{(\lambda)} G_{j_2}^{(\lambda)} \cdots D_{j_p,j}^{(\lambda)} G_j^{(\lambda)}}{\lambda^{p+2}} C_{j,i} \right)^{-1} \\
&= \left(I - \frac{B_n}{\lambda} - \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} \frac{\lambda F_{n,j_1}^{(\lambda)} \lambda F_{j_1,j_2}^{(\lambda)} \cdots \lambda F_{j_p,j}^{(\lambda)}}{\lambda^{p+2}} C_{j,i} \right)^{-1} \\
&= \left(I - \frac{B_n}{\lambda} - \frac{1}{\lambda} \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F_{n,j_1}^{(\lambda)} F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,j}^{(\lambda)} C_{j,i} \right)^{-1}.
\end{aligned}$$

Therefore, by Lemma 4.2, both $D_{k,n}^{(\lambda)}$ and $G_n^{(\lambda)}$ are well-defined. Moreover,

$$\begin{aligned}
D_{k,n}^{(\lambda)} G_n^{(\lambda)} &= \left(C_{k,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F_{k,j_1}^{(\lambda)} F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,j}^{(\lambda)} C_{j,n} \right) \\
&= \left(I - \frac{B_n}{\lambda} - \frac{1}{\lambda} \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F_{n,j_1}^{(\lambda)} F_{j_1,j_2}^{(\lambda)} \cdots F_{j_p,j}^{(\lambda)} C_{j,i} \right)^{-1} \\
&= \lambda F_{k,n}^{(\lambda)}
\end{aligned}$$

which completes the induction. ∎

We now construct the future dimension group $FDG(\lambda)$. Given any positive integer i, define the linear transformation $K_i : \mathcal{U}_i \to \mathcal{U}_{i+1}$ by

$$K_i = \begin{pmatrix} D^{(\lambda)}_{i+1,i} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ D^{(\lambda)}_{i+2,i} & D^{(\lambda)}_{i+2,i+1} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ D^{(\lambda)}_{i+3,i} & D^{(\lambda)}_{i+3,i+1} & D^{(\lambda)}_{i+3,i+2} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ D^{(\lambda)}_{j+1,i} & D^{(\lambda)}_{j+1,i+1} & D^{(\lambda)}_{j+1,i+2} & \cdot\ \cdot & D^{(\lambda)}_{j+1,j} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

acting on the column $u_i = (u_i(g))^T$. By an argument similar to that found in Section 4.1, each K_i is a well-defined positive homomorphism.

Define (as in the construction of the dimension group with H, etc.)

$$\mathcal{W}_1 = R\Gamma_1 \bigoplus R\Gamma_2 \bigoplus R\Gamma_3 \bigoplus \cdots \subseteq \mathcal{U}_1$$

and let $\mathcal{W}_1$ inherit the relative ordering from $\mathcal{U}_1$. Given any $i \geq 2$, define

$$\mathcal{W}_i = (R\Gamma_i \bigoplus R\Gamma_{i+1} \bigoplus R\Gamma_{i+2} \bigoplus \cdots) \bigoplus K_{i-1}(\mathcal{W}_{i-1}) \subseteq \mathcal{U}_i,$$

$\mathcal{W}_i$ inheriting the relative ordering from $\mathcal{U}_i$. Analogous to the situtión in Section 4.1, each $K_i : \mathcal{W}_i \to \mathcal{W}_{i+1}$ is an order preserving transformation. We define the *future dimension group* to be the ordered vector space given by the direct limit

$$FDG(\lambda) := \varinjlim \mathcal{W}_1 \xrightarrow{K_1} \mathcal{W}_2 \xrightarrow{K_2} \mathcal{W}_3 \xrightarrow{K_3} \cdots .$$

This is obviously a dimension group (see Section 3 in [14]).

Theorem 4.7 *Suppose each Γ_i can be identified with a finite set in such a way that the corresponding matrices B_i, $C_{j,i}$ all commute with each other and for any $i \in$*

$\mathcal{N}$, $G_i^{(\lambda)} = G^{(\lambda)}$. Then, there is a one-parameter family of onto order preserving homomorphisms, $\phi_\lambda : FDG(\lambda) \to DG(\lambda)$ (λ any positive eigenvalue) each of which induces a one to one map between the respective trace spaces.

Proof. Commutativity of the matrices B_i, $C_{j,i}$ guarantees that $D_{k,i}^{(\lambda)}$, $G_{k,i}^{(\lambda)} = G_i^{(\lambda)}$ all commute with each other. We know that for any $i \in \mathcal{N}$,

$$H_i = \begin{pmatrix} F_{i+1,i}^{(\lambda)} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F_{i+2,i}^{(\lambda)} & F_{i+2,i+1}^{(\lambda)} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ F_{i+3,i}^{(\lambda)} & F_{i+3,i+1}^{(\lambda)} & F_{i+3,i+2}^{(\lambda)} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ F_{j+1,i}^{(\lambda)} & F_{j+1,i+1}^{(\lambda)} & F_{j+1,i+2}^{(\lambda)} & \cdot\ \cdot & F_{j+1,j}^{(\lambda)} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

$$= \lambda^{-1} K_i \begin{pmatrix} G_i^{(\lambda)} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ 0 & G_{i+1}^{(\lambda)} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ 0 & 0 & G_{i+2}^{(\lambda)} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & G_j^{(\lambda)} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

But by hypothesis,

$$K_i G_i^{(\lambda)} = G_i^{(\lambda)} K_i = G^{(\lambda)} K_i.$$

Therefore, $H_i = \lambda^{-1} G^{(\lambda)} K_i$.

Define a map $\phi_\lambda : FDG(\lambda) \to DG(\lambda)$ via the following diagram:

$$\begin{array}{ccccccccc}
\mathcal{W}_1 & \xrightarrow{K_1} & \mathcal{W}_2 & \xrightarrow{K_2} & \mathcal{W}_3 & \xrightarrow{K_3} & \mathcal{W}_4 & \xrightarrow{K_4} \cdots & FDG(\lambda) \\
=\downarrow & & \lambda^{-1}G^{(\lambda)}\downarrow & & \lambda^{-2}(G^{(\lambda)})^2\downarrow & & \lambda^{-3}(G^{(\lambda)})^3\downarrow & & \phi_\lambda\downarrow \\
\mathcal{V}_1 & \xrightarrow{H_1} & \mathcal{V}_2 & \xrightarrow{H_2} & \mathcal{V}_3 & \xrightarrow{H_3} & \mathcal{V}_4 & \xrightarrow{H_4} \cdots & DG(\lambda)
\end{array}$$

Since $H_i = \lambda^{-i}G_i^{(\lambda)}K_i = \lambda^{-i}G^{(\lambda)}K_i$, this diagram commutes and hence ϕ_λ makes sense. Since all the matrices are nonnegative, ϕ_λ is order preserving. The restriction map on traces, $\tau \longmapsto \tau \circ \phi_\lambda$ is an affine map between the respective trace spaces. Since $G^{(\lambda)}$ is invertible, the range of ϕ_λ is all of $DG(\lambda)$ implying that τ is uniquely determined by its restriction, i.e.. this map is one to one on the trace spaces. ∎

Theorem 4.8 *Suppose P is a nonnegative irreducible matrix which can be written in the form*

$$P = \begin{pmatrix}
B_1 & C_{1,2} & C_{1,3} & \cdots & C_{1,i} & C_{1,i+1} & \cdots \\
C_{2,1} & B_2 & C_{2,3} & \cdots & C_{2,i} & C_{2,i+1} & \cdots \\
0 & C_{3,2} & B_3 & \cdots & C_{3,i} & C_{3,i+1} & \cdots \\
0 & 0 & C_{4,3} & \cdots & C_{4,i} & C_{4,i+1} & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & 0 & \cdots & B_i & C_{i,i+1} & \cdots \\
0 & 0 & 0 & \cdots & C_{i+1,i} & B_{i+1} & \cdots \\
0 & 0 & 0 & \cdots & 0 & C_{i+2,i+1} & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots
\end{pmatrix}$$

where $C_{j,i} = 0$ whenever $j > i+1$. Given an eigenvalue $\lambda > 0$ of P, if we define

$$DG'(\lambda) := \varinjlim R\Gamma_1 \xrightarrow{F_{2,1}^{(\lambda)}} R\Gamma_2 \xrightarrow{F_{3,2}^{(\lambda)}} R\Gamma_3 \xrightarrow{F_{4,3}^{(\lambda)}} \cdots,$$

then there is an affine homeomorphism between the space P_λ of nonzero, nonnegative left eigenvectors of P corresponding to λ and the trace space of $DG'(\lambda)$ given by assigning $v = (v_i = v|_{\Gamma_i})_{i \in N}$ to $\tau_v : DG'(\lambda) \to R$ where $\tau_v(\overline{(u_i, i)}) = v_i \cdot u_i \;\; \forall\, i \in N$. (Here, $v_i \cdot u_i$ denotes $\sum_{g \in \Gamma_i} v_i(g)\, u_i(g)$, an inner product.)

Proof. We begin by verifying that $F^{(\lambda)}_{k,i} = 0$ whenever $k \geq i + 2$. This will be achieved via a simple induction. In the course of this induction, we will obtain a simple form for $F^{(\lambda)}_{i+1,i}$ $(i \in N)$. Recall that for any $k > i \geq 1$,

$$\begin{aligned} F^{(\lambda)}_{k,i} \;=\; & \left(C_{k,i} + \sum_{i > j_1 > j_2 > \cdots > j_p > j \geq 1} F^{(\lambda)}_{k,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i} \right) \cdot \\ & \left(\lambda I - B_i - \sum_{i > j_1 > j_2 > \cdots > j_p > j \geq 1} F^{(\lambda)}_{i,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j}\, C_{j,i} \right)^{-1} \end{aligned}$$

When $i = 1$, we know that $F^{(\lambda)}_{2,1} = C_{2,1}(\lambda I - B_1)^{-1}$. Furthermore, by assumption, $C_{k,1} = 0 \;\; \forall\, k \geq 3$. Thus,

$$F^{(\lambda)}_{k,1} \;=\; C_{k,1}\,(\lambda I - B_1)^{-1} \;=\; 0 \cdot (\lambda I - B_1)^{-1} \;=\; 0$$

for all $k \geq 3$. When $i = 2$, $F^{(\lambda)}_{3,1} = 0$ implies

$$\begin{aligned} F^{(\lambda)}_{3,2} &= \left(C_{3,2} + F^{(\lambda)}_{3,1} C_{1,3} \right) \cdot \left(\lambda I - B_2 - F^{(\lambda)}_{2,1} C_{1,2} \right)^{-1} \\ &= C_{3,2} \left(\lambda I - B_2 - F^{(\lambda)}_{2,1} C_{1,2} \right)^{-1}. \end{aligned}$$

But $C_{k,2} = 0$ and $F^{(\lambda)}_{k,1} = 0 \;\; \forall\, k \geq 4$. Hence,

$$\begin{aligned} F^{(\lambda)}_{k,2} &= \left(C_{k,2} + F^{(\lambda)}_{k,1} C_{1,3} \right) \cdot \left(\lambda I - B_2 - F^{(\lambda)}_{2,1} C_{1,2} \right)^{-1} \\ &= (0 + 0 \cdot C_{1,3}) \cdot \left(\lambda I - B_2 - F^{(\lambda)}_{2,1} C_{1,2} \right)^{-1} \\ &= 0 \end{aligned}$$

for all $k \geq 4$.

Now, take $n \in \mathcal{N}$ and assume that for every $1 \leq i \leq n-1$, $F^{(\lambda)}_{k,i} = 0 \ \forall k \geq i+2$. Then, for $i = n$ we have

$$\begin{aligned}
F^{(\lambda)}_{n+1,n} &= \left(C_{n+1,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{n+1,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right) \cdot \\
&\quad \left(\lambda I - B_n - \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{n,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1} \\
&= C_{n+1,n}\left(\lambda I - B_n - \sum_{n>j\geq 1} F^{(\lambda)}_{n,n-1} F^{(\lambda)}_{n-1,n-2} \cdots F^{(\lambda)}_{j+1,j} C_{j,i}\right)^{-1}
\end{aligned}$$

since $F^{(\lambda)}_{n+1,j_1} = 0$ and, for every $i \leq n-1$, $F^{(\lambda)}_{k,i} = 0$ whenever $k \geq i+2$. Furthermore, since $C_{k,n} = 0$ and $F^{(\lambda)}_{k,j_1} = 0$ for every $k \geq n+2$ and $j_1 < n$,

$$\begin{aligned}
F^{(\lambda)}_{k,n} &= \left(C_{k,n} + \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{k,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right) \cdot \\
&\quad \left(\lambda I - B_n - \sum_{n>j_1>j_2>\cdots>j_p>j\geq 1} F^{(\lambda)}_{n,j_1} F^{(\lambda)}_{j_1,j_2} \cdots F^{(\lambda)}_{j_p,j} C_{j,i}\right)^{-1} \\
&= 0
\end{aligned}$$

for every $k \geq n+2$ which completes the induction. From this, we can write

$$H = H_1 = \begin{pmatrix}
F^{(\lambda)}_{2,1} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\
0 & F^{(\lambda)}_{3,2} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\
0 & 0 & F^{(\lambda)}_{4,3} & \cdot\ \cdot & 0 & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\
0 & 0 & 0 & \cdot\ \cdot & F^{(\lambda)}_{j+1,j} & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot
\end{pmatrix}$$

and, given any $i \geq 2$, we can write

$$H_i = \begin{pmatrix} F^{(\lambda)}_{i+1,i} & 0 & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ 0 & F^{(\lambda)}_{i+2,i+1} & 0 & \cdot\ \cdot & 0 & \cdot\ \cdot \\ 0 & 0 & F^{(\lambda)}_{i+3,i+2} & \cdot\ \cdot & 0 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & F^{(\lambda)}_{j+1,j} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot\ \cdot \end{pmatrix}$$

so that

$$\mathcal{V}_i = \left(R\Gamma_i \bigoplus R\Gamma_{i+1} \bigoplus R\Gamma_{i+2} \bigoplus \cdots\right) \bigoplus H_{i-1}(\mathcal{V}_{i-1}) = \bigoplus_{j=i}^{\infty} R\Gamma_i$$

for every $i \geq 2$.

Recall the dimension group

$$DG(\lambda) = \varinjlim \mathcal{V}_1 \xrightarrow{H_1} \mathcal{V}_2 \xrightarrow{H_2} \mathcal{V}_3 \xrightarrow{H_3} \cdots$$

and the space

$$\tilde{S} = \left\{\tau \in S(DG(\lambda), R) \,\middle|\, \tau(\overline{(u_i, l)}) = \tau(\overline{(u_i, n)})\ \forall n \geq l,\ u_i \in R\Gamma_i\right\}.$$

According to the affine homeomorphism theorem, there exists an affine homeomorphism between $\tilde{S}$ and the space

$$P_\lambda = \{v : \Gamma \to R \,|\, v \geq 0,\ v \neq 0 \text{ and } v \cdot P = \lambda v\}.$$

Denote the trace space of $DG'(\lambda)$ by $S(DG'(\lambda), R)$ and define a map

$$\phi : S(DG'(\lambda), R) \to \tilde{S}$$

as follows. Given $\tau \in S(DG'(\lambda), R)$, set

$$\phi(\tau)(\overline{(u_i, l)}) = \tau(\overline{(u_i, i)})$$

for any $l \leq i$ and $u_i \in R\Gamma_i$ where $\overline{(u_i, l)} \in DG(\lambda)$ and $\overline{(u_i, i)} \in DG'(\lambda)$. Indeed, it is easy to check that

$$\begin{aligned} \phi(\tau)(\,\overline{(H_l(u_i), l+1)}\,) &= \phi(\tau)(\,\overline{(F^{(\lambda)}_{i+1,i}(u_i), l+1)}\,) \\ &= \tau(\,\overline{(F^{(\lambda)}_{i+1,i}(u_i), i+1)}\,) \\ &= \tau(\overline{(u_i, i)}) \\ &= \phi(\tau)(\overline{(u_i, l)}) \end{aligned}$$

so that $\phi(\tau)$ is a state, i.e., $\phi(\tau) \in \tilde{S}$.

Similarly, define a map $\varphi : \tilde{S} \to S(DG'(\lambda), R)$ as follows. For any $l \leq i$ and $u_i \in R\Gamma_i$ where $\overline{(u_i, l)} \in DG(\lambda)$ and $\overline{(u_i, i)} \in DG'(\lambda)$, set

$$\varphi(\theta)(\overline{(u_i, i)}) = \theta(\overline{(u_i, l)}).$$

It is obvious that

$$\begin{aligned} \varphi(\theta)(\,\overline{(F^{(\lambda)}_{i+1,i}(u_i), i+1)}\,) &= \theta(\,\overline{(F^{(\lambda)}_{i+1,i}(u_i), l+1)}\,) \\ &= \theta(\,\overline{(H_l(u_i), l+1)}\,) \\ &= \theta(\overline{(u_i, l)}) \\ &= \varphi(\theta)(\overline{(u_i, i)})\,. \end{aligned}$$

Hence, $\varphi(\theta) \in S(DG'(\lambda), R)$.

It is clear that both φ and ϕ are affine homeomorphisms and $\varphi \circ \phi = \phi \circ \varphi = id$. Therefore, $S(DG'(\lambda), R)$ is affine homeomorphic to $\tilde{S}$ which, as mentioned above, implies $S(DG'(\lambda), R)$ is affine homeomorphic to P_λ. ■

Chapter 5

Motivational Examples

In this chapter, we present several processes to which our earlier results can be applied. The examples we will consider are motivated by (but are distinct from) those given in [14]. The central difference here is that our P are column infinite whereas [14] only considers P which are column finite.

Example 5.1 (*Laurent power series with nonnegative coefficients.* [10, 11, 13, 14]) Let $Q(x) = \sum_{w \in Z^n} r_w \, x^w$ $(x = (x_1, x_2, \ldots, x_n) \in R^n)$ be a Laurent power series with nonnegative coefficients, that is, $r_w \geq 0 \ \forall \, w \in Z^n$. ($x^w$ is defined to be $\prod_{i=1}^{n} x_i^{w_i}$.) This describes a walk on the lattice $\Gamma = Z^n$ where $P : R\Gamma \to R\Gamma$ is defined via

$$e_v \longmapsto \prod_{w \in Z^n} r_w \, e_{v+w}.$$

(Here, $R\Gamma$ is the direct product over R with basis generated by $\{e_w\}_{w \in \Gamma = Z^n}$.) If $Q(1, 1, \ldots, 1) = 1$, this simply corresponds to a (infinitely supported) random walk on Z^n. There is no harm in assuming this here since, in this case, r_w is precisely the probability of going from a lattice point v to $v+w$. That is, $p_{v+w,v} = r_w$ so that

$$
\begin{array}{ccc}
 & v\text{-th} & \\
P \;=\; & \begin{pmatrix}
\cdots & \cdot & \cdots \\
\cdots & \cdot & \cdots \\
\cdots & r_{w_1} & \cdots \\
\cdots & r_{w_2} & \cdots \\
\cdots & \cdot & \cdots \\
\cdots & \cdot & \cdots \\
\cdots & r_{w_n} & \cdots \\
\cdots & \cdot & \cdots \\
\cdots & \cdot & \cdots
\end{pmatrix} &
\begin{array}{l}
\\ \\ (v+w_1)\text{-th} \\ (v+w_2)\text{-th} \\ \\ \\ (v+w_n)\text{-th} \\ \\ \\
\end{array}
\end{array}
$$

Let Log Q denote $\{w \in Z^n \mid r_w \neq 0\}$ and let $\langle \text{Log } Q\rangle$ denote the semigroup generated by Log Q.

Proposition 5.2 $Z^n = \langle \text{Log } Q\rangle$ *if and only if* P *is irreducible.*

Proof. Assume $Z^n = \langle \text{Log}\, Q\rangle$. Then, given any $w, v \in Z^n$, $w-v \in Z^n = \langle \text{Log}\, Q\rangle$. Hence, $\exists\ v_1, v_2, \ldots, v_{m-1}, v_m \in$ Log Q such that $w - v = v_1 + v_2 + \cdots + v_{m-1} + v_m$ which implies $w = v + v_1 + v_2 + \cdots + v_{m-2} + v_{m-1} + v_m$. Therefore,

$$p_{w,\, v+v_1+v_2+\cdots+v_{m-2}+v_{m-1}}\, p_{v+v_1+v_2+\cdots+v_{m-1},\, v+v_1+v_2+\cdots+v_{m-2}} \cdots p_{v+v_1+v_2,\, v+v_1}\, p_{v+v_1,\, v}$$

equals

$$r_{v_m}\, r_{v_{m-1}}\, r_{v_{m-2}} \cdots r_{v_2}\, r_{v_1}$$

which is positive, implying P is irreducible.

Conversely, assume P is irreducible. Then, given any $w \in Z^n$, there exists a path from 0 to w, that is, $\exists\ w_1, w_2, \ldots, w_{m-1}, w_m \in Z^n$ such that

$$p_{w,w_m}\, p_{w_m,w_{m-1}}\, p_{w_{m-1},w_{m-2}} \cdots p_{w_2,w_1}\, p_{w_1,0}$$

$$
\begin{aligned}
&= r_{w-w_m} r_{w_m-w_{m-1}} r_{w_{m-1}-w_{m-2}} \cdots r_{w_2-w_1} r_{w_1} \\
&> 0 .
\end{aligned}
$$

This implies $w - w_m, w_m - w_{m-1}, \ldots, w_2 - w_1, w_1 \in \text{Log } Q$ so that

$$
\begin{aligned}
w &= w - w_m + w_m - w_{m-1} + w_{m-1} - w_{m-2} + \cdots + w_2 - w_1 + w_1 \\
&= (w - w_m) + (w_m - w_{m-1}) + (w_{m-1} - w_{m-2}) + \cdots + (w_2 - w_1) + w_1 \\
&\in \langle \text{Log } Q \rangle .
\end{aligned}
$$

Therefore, since w was arbitrary, $Z^n \subseteq \langle \text{Log } Q \rangle$. Since $\langle \text{Log } Q \rangle \subseteq Z^n$, we have thus shown $Z^n = \langle \text{Log } Q \rangle$. ∎

Lemma 5.3 *Let P and Q be as defined above and let $(R^n)^{++}$ be the set of all $r \in R^n$ with the property that every component of r is positive. Given any $r \in (R^n)^{++}$ with*

$$
Q(r) = \sum_{w \in Z^n} r_w r^w < \infty . \tag{9}
$$

if we define the row vector $V_r : \Gamma = Z^n \to R$ by

$$
w \longmapsto r^w .
$$

then V_r is a left nonnegative nonzero eigenvector of P for the eigenvalue $Q(r)$.

Proof. Take $r \in (R^n)^{++}$ satisfying (9). Then, given any $w_0 \in \Gamma$,

$$
\begin{aligned}
V_r \cdot P \cdot e_{w_0} &= V_r \cdot (P \cdot e_{w_0}) \\
&= V_r \cdot \prod\nolimits_{w \in Z^n} r_w \, e_{w+w_0} \\
&= \sum\nolimits_{w \in Z^n} r_w \, r^{w+w_0} \\
&= \left(\sum\nolimits_{w \in Z^n} r_w \, r^w \right) r^{w_0} \\
&= Q(r) \, V_r \cdot e_{w_0} .
\end{aligned}
$$

Thus, $V_r \cdot P = Q(r) \, V_r$. Since $r \in (R^n)^{++}$, V_r is a nonnegative nonzero row vector. That is, V_r is a left nonnegative nonzero eigenvector of P for the eigenvalue $Q(r)$. ∎

Theorem 5.4 *Let $\Omega \subseteq R^n$ denote the domain of convergence of Q. If we define*

$$\Lambda_{\min}(Q) := \inf\left\{Q(r) \;\middle|\; r \in \Omega \cap (R^n)^{++}\right\},$$

$$\Lambda_{\max}(Q) := \sup\left\{Q(r) \;\middle|\; r \in \Omega \cap (R^n)^{++}\right\}.$$

then

(i) *For any $\lambda \in (\Lambda_{\min}(Q), \Lambda_{\max}(Q))$, there exists a nonnegative nonzero eigenvector V_r of P for the eigenvalue λ.*

(ii) *If $\langle \mathrm{Log}\ Q \rangle = Z^n$, then $\bar{R}^{-1} \leq \Lambda_{\min}(Q)$ where $\bar{R}$ denotes the convergence parameter of the matrix P (see Definition 3.1).*

Proof. (i) Ω being a connected domain implies $\Omega \cap (R^n)^{++}$ is a connected domain so that the continuity of $Q : \Omega \to R$ implies $Q(\Omega \cap (R^n)^{++})$ is connected. Thus, given any $\lambda \in (\Lambda_{\min}(Q), \Lambda_{\max}(Q))$, the connectedness of $Q(\Omega \cap (R^n)^{++})$ implies the existence of $r \in \Omega \cap (R^n)^{++}$ such that $Q(r) = \lambda$. By Lemma 5.3, $V_r : \Gamma = Z^n \to R$ defined by $w \longmapsto r^w$ is a nonnegative nonzero left eigenvector of P for eigenvalue $Q(r) = \lambda$.

(ii) By Proposition 5.2, $\langle \mathrm{Log}\ Q \rangle = Z^n$ implies P is irreducible. Thus, Theorem 3.3 implies $\bar{R}^{-1} \leq \Lambda_{\min}(Q)$. ∎

We can now characterize those Laurent power series f for which $Q^m f$ has nonnegative coefficients for all sufficiently large m.

Proposition 5.5 *Let P and Q be as defined above and let f be a Laurent power series, say*

$$f(x) = \sum_{g \in Z^n} l_g x^g \quad (x = (x_1, \ldots x_n) \in R^n)$$

with coefficients $l_g \in R$. If

(i) *$Q \leq KQ^2$ for some positive integer K,*

(ii) $Q(1, 1, \ldots, 1) < \infty$ *and*

(iii) $Z^n = \langle \text{Log } Q \rangle$,

then the following are equivalent:

- $\exists\, m \in N$ *such that* $Q^m f$ *has no negative coefficients,*
- $f(r) > 0 \;\; \forall\, r \in \Omega \cap (R^n)^{++}$.

Proof. In view of the correspondence between P and Q, $Q \le KQ^2$ implies $P \le KP^2$. If $P = (p_{g,h})_{g,h \in \Gamma}$, then $p_{g,h} = r_{g-h}$ so that, given any $g \in \Gamma$,

$$\begin{aligned} \sum\nolimits_{h \in \Gamma} p_{g,h} &= \sum\nolimits_{h \in \Gamma} r_{g-h} \\ &= \sum\nolimits_{w \in Z^n} r_w \\ &= Q(1, 1, \ldots, 1)\,. \end{aligned}$$

Since $g \in \Gamma$ was arbitrary, this implies

$$\inf \left\{ \sum\nolimits_{h \in \Gamma} p_{g,h} \,\middle|\, g \in \Gamma \right\} \;=\; Q(1, 1, \ldots, 1) \;>\; 0.$$

Therefore, by Corollary 2.14, H_Γ has an order unit. Furthermore, since (iii) implies P is irreducible (Proposition 5.2), all pure traces of H_Γ are faithful (Proposition 2.31).

If we define the column vector $v : Z^n = \Gamma \to R$ by

$$g \longmapsto l_g,$$

then $Q^m f$ has no negative coefficients if and only if $P^m(v) \ge 0$, that is, $\overline{(v, 1)} \in (H_\Gamma)^+$. Furthermore, according to [6] (see I.4 therein), $\overline{(v, 1)} \in (H_\Gamma)^+$ if and only if $\tau(\overline{(v, 1)}) > 0$ for every pure trace τ of H_Γ. By Theorem 4.5, the traces are given by left eigenvectors and the pure traces are given by extremal left eigenvectors. Therefore, $\tau(\overline{(v, 1)}) > 0$ can be represented by $U \cdot v = \sum_{g \in \Gamma} U(g)\, v(g) > 0$ for some extremal left eigenvector U. But by [9] (see I.3 therein), all extremal eigenvectors $U : \Gamma \to R$ are of the form

$$g \longmapsto r^g$$

(some $r \in \Omega \cap (R^n)^{++}$). Hence,

$$U \cdot v = \sum_{g \in \Gamma} U(g)\, v(g) = \sum_{g \in \Gamma} r^g\, l_g = f(r) > 0$$

which completes the proof. ∎

Example 5.6 (*Matrix-valued random walks over Laurent power series.* [14, 15, 20]) Generalizing Example 5.1, we take an $m \times m$ matrix

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdot & \cdot & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdot & \cdot & a_{2,m} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdot & \cdot & a_{3,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdot & \cdot & a_{m,m} \end{pmatrix}$$

of Laurent power series, each with nonnegative coefficients. That is, for every pair $1 \leq i, j \leq m$, $a_{i,j}(x) = \sum_{w \in Z^n} l_w^{i,j}\, x^w$ with $l_w^{i,j} \geq 0\ \forall w$ and $x = (x_1, \ldots, x_d) \in R^n$. As before, we define $\operatorname{Log} a_{i,j} := \{w \in Z^n \mid l_w^{i,j} \neq 0\}$.

This M describes a process P with Γ being the union of m disjoint copies of Z^n. We can think of these copies as being stacked one on top of the other so that

$$\Gamma = \left.\begin{pmatrix} Z^n \\ Z^n \\ Z^n \\ \vdots \\ Z^n \end{pmatrix}\right\} m\,.$$

A point γ in Γ can then be written as a pair (v, j) where $v \in Z^n$ and $j \in \{1, 2, \dots, m\}$. This process sends a particle at (v, j) to $(v + w, i)$ with weight determined by the coefficients of the Laurent power series $a_{i,j}$. That is, $P : R\Gamma \to R\Gamma$ sends $e_\gamma = e_{(v,j)}$ to

$$P(e_{(v,j)}) = \sum_{i=1}^{m} \prod_{w \in \operatorname{Log} a_{i,j}} l_w^{i,j} e_{(v+w,i)}.$$

We recapture the previous class of examples by taking $m = 1$.

A matrix is said to be *primitive* if all sufficiently large powers have no zero entries. Given any r in $(R^n)^{++}$ with $a_{i,j}(r) < \infty$ $\forall i, j$, the real matrix $M(r)$ (obtained by evaluating each entry at r) is primitive and as such admits a left Perron eigenvector. If we let $V(r)$ denote this eigenvector and let $\beta(r)$ denote the corresponding eigenvalue, we can then write

$$\begin{aligned} \beta(r) V(r) &= V(r) \cdot M(r) \\ &= \left(\sum_{l=1}^{m} V_l(r)\, a_{l,1}(r) \quad \sum_{l=1}^{m} V_l(r)\, a_{l,2}(r) \quad \cdots \quad \sum_{l=1}^{m} V_l(r)\, a_{l,m}(r) \right) \end{aligned}$$

where $V(r) = (V_1(r) \quad V_2(r) \quad \cdots \quad V_m(r))$.

If we define Ω' by

$$\Omega' = \left\{ r \in (R^n)^{++} \,\middle|\, a_{i,j}(r) < \infty \ \forall i, j \in \{1, 2, \dots, m\} \right\},$$

then Ω' is open and nonempty (without branch points—a consequence of primitivity and the Perron-Frobenius theorem). The above mentioned $V(r)$ can be normalized in such a way that each coordinate of $r \longmapsto V(r)$ is analytic on Ω'. Moreover, $r \longmapsto \beta(r)$ is analytic on Ω'.

Proposition 5.7 *Given $r \in \Omega'$, using the above notation, the row vector $f : \Gamma \to R$ defined via*

$$(u, j) \longmapsto V_j(r)\, r^u$$

is a nonnegative nonzero left eigenvector of P for the eigenvalue $\beta(r)$.

Proof. By the above construction, f is clearly nonzero and nonnegative. Furthermore, given any $(u, j) \in \Gamma$,

$$\begin{aligned} f \cdot P \cdot e_{(u,j)} &= f \cdot (P \cdot e_{(u,j)}) \\ &= f \cdot \sum_{i=1}^{m} \prod_{w \in \operatorname{Log} a_{i,j}} l_w^{i,j}\, e_{(u+w,i)} \\ &= \sum_{i=1}^{m} V_i(r) \left(\sum_{w \in \operatorname{Log} a_{i,j}} r^{u+w}\, l_w^{i,j} \right) \\ &= r^u \cdot \sum_{i=1}^{m} V_i(r) \left(\sum_{w \in \operatorname{Log} a_{i,j}} r^w\, l_w^{i,j} \right) \\ &= r^u \cdot \sum_{i=1}^{m} V_i(r)\, a_{i,j}(r) \\ &= r^u \cdot \beta(r) \cdot V_j(r) \\ &= \beta(r)\, V_j(r)\, r^u \\ &= \beta(r)\, f \cdot e_{(u,j)} \,. \end{aligned}$$

Therefore, $f \cdot P = \beta(r)\, f$, i.e., g is a nonnegative nonzero left eigenvector of P for the eigenvalue $\beta(r)$. ∎

One can also obtain examples of processes based on infinite trees or infinite directed graphs. Recall that a *tree* is any connected graph without cycles whereas a *directed graph* is a graph whose edges are ordered pairs. We briefly discuss the former.

Example 5.8 (*Random walks on infinite trees.* [14])
Let G be an infinite tree with (countably infinite) vertex set $V(G)$. One possible definition of the *adjacency matrix* $A = A(G) = (a_{i,j})_{i,j \in V(G)}$ is

$$a_{i,j} = \begin{cases} 1 & \text{if there exists a edge from } j \text{ to } i \\ 0 & \text{otherwise}\,. \end{cases}$$

An other definition would include multiplicities. Among other things, $A(G)$ is an infinite nonnegative matrix.

Now, take $\Gamma = V(G)$ as the "state space" for a Markov process in which $a_{i,j}$ represents the probability of jumping from state j to state i. In this way, $P = A(G)$ can be viewed as the transition matrix of an infinite process. If P is irreducible, define

$$\Lambda(G) = \inf\{\lambda \in R \mid \exists v \geq 0, v \neq 0 \text{ such that } v \cdot P = \lambda v\}.$$

The *Poisson boundary* of the tree G is then defined to be

$$B(G) = \{v \mid v \geq 0, v \neq 0 \text{ and } v \cdot P = \Lambda(P)\, v\}.$$

A typical question in this situation is: "When does there exist a unique nonnegative nonzero eigenvector in $B(G)$ corresponding to $\Lambda(G)$?" This is not known at present.

Before presenting the next class of examples, we need to set some basic terminology. Let X be a discrete group and $\{\mu_0, \mu_1, \mu_2, \ldots\}$ a family of probability measures on X with μ_0 a positive infinitely supported measure. (μ_0 is termed an *initial distribution* on X.) Denote the convolution of measures μ and γ by $\mu * \gamma$.

Definition 5.9 The *space-time cone* determined by the above $\mu_0, \mu_1, \mu_2, \ldots$ is

$$\mathcal{C} = \{(x, n) \in X \times N \mid \mu_n * \mu_{n-1} * \mu_{n-2} * \cdots * \mu_2 * \mu_1 * \mu_0(\{x\}) > 0\}.$$

A *space-time harmonic function* h on $\mathcal{C}$ is any nonzero nonnegative function $h : \mathcal{C} \to R$ satisfying

$$h(x, n) = \sum_{y \in X} h(x + y, n + 1)\, \mu_{n+1}(\{y\}) \quad \forall\, (x, n) \in \mathcal{C}.$$

Example 5.10 (*Infinite diffuse measures on R^n.* [12, 14])

The previous Laurent power series problem can be reiterated in terms of repeated convolutions of an infinite distribution. Let $X = Z^d$ and, as in Example 5.1, define $Q(x) = \sum_{w \in Z^d} r_w\, x^w$ where for all $w \in Z^d, r_w \geq 0$. Define an infinite diffuse measure μ_Q on X via

$$w \longmapsto r_w$$

and, given a Laurent power series $f(x) = \sum_{w \in Z^d} b_w x^w$, define $\mu_f : X \to R$ by

$$w \longmapsto b_w.$$

The convolution $\mu_Q * \mu_f : X \to R$ is then defined by

$$w \longmapsto \sum r_u b_v$$

where the above sum ranges over all $u, v \in X$ for which $u + v = w$.

Given $m \in N$, let $(\mu)^m$ denote the m-fold convolution of μ.

Proposition 5.11 *Using the above notation, $Q^m f \geq 0$ for some $m \in N$ if and only if $(\mu_Q)^m * \mu_f \geq 0$.*

Proof. Obvious. ■

Let $M(X)$ denote the linear space of diffuse measures on X. According to Definition 5.9, the space-time cone determined by μ_Q is the set

$$\mathcal{C}_Q = \{(x, n) \in X \times N \mid (\mu_Q)^{n+1}(\{x\}) > 0\}.$$

If we define a linear transformation $P : M(X) \to M(X)$ by

$$\mu \longmapsto \mu_Q * \mu$$

(so that P can be written as a nonnegative infinite matrix) and take $\Gamma = X$, then (P, Γ) is an infinite Markov process.

Theorem 5.12 *Any nonzero, nonnegative eigenvector of P yields a harmonic function on the space-time cone $\mathcal{C}_Q$.*

Proof. Let v be a nonzero nonnegative eigenvector of P. Then, $v \cdot P = \lambda\, v$ for some $\lambda > 0$. Define the nonnegative function $h : \mathcal{C}_Q \rightarrow R^+$ by

$$(w, n) \longmapsto v_w/\lambda^n .$$

Clearly, h is a nonzero function.

Given any $w \in X$, let δ_w denote the measure on X which is 1 on w and 0 otherwise. Since $v \cdot P = \lambda\, v$,

$$v \cdot P \cdot \delta_w \;=\; \lambda\, v \cdot \delta_w \quad \forall\, w \in X.$$

Thus, given any $w \in X$,

$$\lambda\, v_w \;=\; v \cdot (\mu_Q * \delta_w) \;=\; v \cdot \sum_{u \in X} r_u\, \delta_{u+w} \;=\; \sum_{u \in X} v_{u+w}\, r_u$$

which implies

$$\frac{v_w}{\lambda^n} \;=\; \sum_{u \in X} \frac{v_{u+w}}{\lambda^{n+1}}\, r_u \,.$$

Therefore,

$$h(w, n) \;=\; \sum_{u \in X} h(u + w, n + 1)\, \mu_Q(\{u\})$$

on $\mathcal{C}_Q$, i.e. h is a harmonic function on the space-time cone $\mathcal{C}_Q$. ∎

Example 5.13 (*Multiplication by power series.* [13, 14])
Let (P, Γ) be an infinite Markov process with $\Gamma = Z^+$. Assume $P = (p_{i,j})_{i,j \in \Gamma}$ is such that there exists $M > 0$ for which

$$|\liminf_{i \to \infty} p_{i,j}\,| \;<\; M \quad \forall\, j \in \Gamma\,.$$

We will only consider a specific case of this situation.

Take a power series $Q(z) = \sum_{i=0}^{\infty} a_i\, z^i$ $(z \in R)$ with $a_i \geq 0$ $\forall\, i \in Z^+$ such that $\liminf_{i \to \infty} a_i < M$. Let $R\Gamma$ denote the direct product over R with basis generated by $\{e_i\}_{i \in \Gamma = Z^+}$ and define $P : R\Gamma \rightarrow R\Gamma$ by setting

$$e_0 \longmapsto \prod_{i=0}^{\infty} a_i\, e_i$$

and, for any $i \in \Gamma \setminus \{0\}$,

$$e_i \longmapsto \sum_{i=0}^{\infty} a_i \, e_{i+1}.$$

We can write P in the following form:

$$P = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdot & \cdot \\ a_1 & a_0 & 0 & 0 & \cdot & \cdot \\ a_2 & a_1 & a_0 & 0 & \cdot & \cdot \\ a_3 & a_2 & a_1 & a_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Proposition 5.14 *Let R' denote the radius of convergence for Q. Given any $0 < r < R'$, if we define $V_r : \Gamma \to R$ via $i \longmapsto r^i$, then V_r is a nonnegative nonzero eigenvector of P for the eigenvalue $Q(r)$.*

Proof. The proof is similar to that of Lemma 5.3. ∎

Chapter 6

Weak Ergodicity for 2×2 Matrix Sequences

In [14], Handelman studied sufficient conditions ensuring a unique nonnegative eigenvector for each eigenvalue of a Markov chain (P, Γ) with P nonnegative, infinite and column finite. One such condition was "weak ergodicity". In [14], Handelman investigated the weak ergodicity of sequences of 2×2 matrices in great detail. In the present chapter, we give necessary conditions for weak ergodicity of such sequences.

Definition 6.1 A sequence of nonnegative matrices $\{A_i\}_{i \in N}$ (usually assumed to be strictly positive—although this is not necessary) is called (*backwardly*) *weakly ergodic* if for every $k \in N$, the products $A_{M+k} A_{M+k-1} \cdots A_{k+1}$ have the property that, as $M \to \infty$, the angles between the columns tend to zero. (This is "projective convergence"—each column represents a point on the unit sphere in the appropriate Euclidean space.)

Theorem 6.2 *Let* $\{a_i\}$, $\{b_i\}$ *and* $\{c_i\}$ *be sequences of nonnegative real numbers with*

$b_i > 0\ \forall i \in N$. If the sequence $\{A_i\}$ with

$$A_i = \begin{pmatrix} 1 & c_i \\ a_i & b_i \end{pmatrix}$$

is backwardly weakly ergodic, then at least one of

$$\sum_{i=1}^{\infty} \frac{a_i}{\prod_{j=1}^{i} b_j} \quad \text{and} \quad \sum_{i=1}^{\infty} c_i \left(\prod_{j=1}^{i-1} b_j\right) \tag{10}$$

must diverge.

Proof. We will make use of the usual $o(\,)$-notation, if k goes to infinity, then $o(1)$ goes to 0. If both series in (10) converge, then

$$\sum_{i=k+1}^{\infty} \frac{a_i}{\prod_{j=1}^{i} b_j} = o(1) \quad \text{and} \quad \sum_{i=k+1}^{\infty} c_i \left(\prod_{j=1}^{i-1} b_j\right) = o(1)$$

so that

$$a_i \le o(1) \prod_{j=1}^{i} b_j \quad \text{and} \quad c_i \le o(1) \left(\prod_{j=1}^{i-1} b_j\right)^{-1}$$

whenever $i \ge k+1$. Set

$$\begin{aligned} A^{(M,k)} &= \begin{pmatrix} 1 & c_{M+k} \\ a_{M+k} & b_{M+k} \end{pmatrix} \begin{pmatrix} 1 & c_{M+k-1} \\ a_{M+k-1} & b_{M+k-1} \end{pmatrix} \cdots \begin{pmatrix} 1 & c_{k+2} \\ a_{k+2} & b_{k+2} \end{pmatrix} \begin{pmatrix} 1 & c_{k+1} \\ a_{k+1} & b_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} R_M^{1,1} & S_M^{1,2} \\ T_M^{2,1} & U_M^{2,2} \end{pmatrix}. \end{aligned}$$

By direct calculation, we obtain

$$\begin{aligned} R_M^{1,1} &= \sum d_{1,i_{M+k}}^{(M+k)} d_{i_{M+k},i_{M+k-1}}^{(M+k-1)} \cdots d_{i_{k+3},i_{k+2}}^{(k+2)} d_{i_{k+2},1}^{(k+1)}, \\ S_M^{1,2} &= \sum d_{1,i_{M+k}}^{(M+k)} d_{i_{M+k},i_{M+k-1}}^{(M+k-1)} \cdots d_{i_{k+3},i_{k+2}}^{(k+2)} d_{i_{k+2},2}^{(k+1)}, \\ T_M^{2,1} &= \sum d_{2,i_{M+k}}^{(M+k)} d_{i_{M+k},i_{M+k-1}}^{(M+k-1)} \cdots d_{i_{k+3},i_{k+2}}^{(k+2)} d_{i_{k+2},1}^{(k+1)}, \\ U_M^{2,2} &= \sum d_{2,i_{M+k}}^{(M+k)} d_{i_{M+k},i_{M+k-1}}^{(M+k-1)} \cdots d_{i_{k+3},i_{k+2}}^{(k+2)} d_{i_{k+2},2}^{(k+1)} \end{aligned}$$

where each sum ranges over all ordered sets $\{i_{M+k}, i_{M+k-1}, \cdots, i_{k+3}, i_{k+2}\}$ where, given any $j \in N$, $i_j \in \{1, 2\}$ and given any $l \in N$,

$$d^l_{1,1} = 1\,,\; d^l_{1,2} = c_l\,,\; d^l_{2,1} = a_l \text{ and } d^l_{2,2} = b_l\,.$$

Obviously,

$$\begin{aligned}
R^{1,1}_{M+1} &= R^{1,1}_{M} + c_{M+k+1} \cdot T^{2,1}_{M}\,, \\
S^{1,2}_{M+1} &= S^{1,2}_{M} + c_{M+k+1} \cdot U^{2,2}_{M}\,, \\
T^{2,1}_{M+1} &= b_{M+k+1} \cdot T^{2,1}_{M} + a_{M+k+1} \cdot R^{1,1}_{M}\,, \\
U^{2,2}_{M+1} &= b_{M+k+1} \cdot U^{2,2}_{M} + a_{M+k+1} \cdot S^{1,2}_{M}\,.
\end{aligned}$$

By induction, we will establish the following inequalities for every $M \in N$:

$$(*)\left\{\begin{array}{rcccl}
1 & \leq & R^{1,1}_M & \leq & 1 + o(1)\,, \\
 & & S^{1,2}_M & \leq & o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right)\,, \\
 & & T^{2,1}_M & \leq & o\left(\prod_{j=1}^{k+M} b_j\right)\,, \\
\prod_{j=k+1}^{k+M} b_j & \leq & U^{2,2}_M & \leq & \prod_{j=k+1}^{k+M} b_j + o\left(\prod_{j=k+1}^{k+M} b_j\right)\,.
\end{array}\right.$$

When $M = 1$, we have

$$A^{(1,k)} = \begin{pmatrix} 1 & c_{k+1} \\ a_{k+1} & b_{k+1} \end{pmatrix}$$

so that

$$\begin{array}{rcccccccl}
1 & \leq & R^{1,1}_1 & = & 1 & \leq & 1 + o(1)\,, & & \\
 & & S^{1,2}_1 & = & c_{k+1} & \leq & o(1) \cdot \prod_{j=1}^{k+1-1} b_j & = & o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right)\,, \\
 & & T^{2,1}_1 & = & a_{k+1} & \leq & o(1) \cdot \prod_{j=1}^{k+1} b_j & = & o\left(\prod_{j=1}^{k+1} b_j\right)\,, \\
b_{k+1} & \leq & U^{2,2}_1 & = & b_{k+1} & \leq & b_{k+1} + o(b_{k+1}) & &
\end{array}$$

Thus, $(*)$ is valid for $M = 1$.

Now, take $n \in \mathbb{N}$ and suppose $(*)$ is true for $M = n$. Then,

$$\begin{aligned} 1 \leq R_{n+1}^{1,1} &= R_n^{1,1} + c_{n+k+1} \cdot T_n^{2,1} \\ &\leq 1 + o(1) + o(1) \cdot \left(\prod_{j=1}^{n+k+1-1} b_j\right)^{-1} \cdot o\left(\prod_{j=1}^{k+n} b_j\right) \\ &= 1 + o(1) + o(1) \cdot o(1) \\ &= 1 + o(1) \,. \end{aligned}$$

$$\begin{aligned} S_{n+1}^{1,2} &= S_n^{1,2} + c_{n+k+1} \cdot U_n^{2,2} \\ &\leq o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) + o(1)\left(\prod_{j=1}^{n+k+1-1} b_j\right)^{-1} \cdot \left[\prod_{j=k+1}^{k+n} b_j + o\left(\prod_{j=k+1}^{k+n} b_j\right)\right] \\ &= o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) + o(1)\left(\prod_{j=1}^{k} b_j\right) + o(1) \cdot o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) \\ &= o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) \,. \end{aligned}$$

$$\begin{aligned} T_{n+1}^{2,1} &= b_{n+k+1} \cdot T_n^{2,1} + a_{n+k+1} \cdot R_n^{1,1} \\ &\leq b_{n+k+1} \cdot o\left(\prod_{j=1}^{k+n} b_j\right) + o(1)\left(\prod_{j=1}^{n+k+1} b_j\right) \cdot [1 + o(1)] \\ &= o\left(\prod_{j=1}^{k+n+1} b_j\right) + o(1)\left(\prod_{j=1}^{n+k+1} b_j\right) \cdot [1 + o(1)] \\ &= o\left(\prod_{j=1}^{k+n+1} b_j\right) \end{aligned}$$

and

$$\begin{aligned} \prod_{j=k+1}^{k+n+1} b_j &\leq U_{n+1}^{2,2} \\ &= b_{n+k+1} \cdot U_n^{2,2} + a_{n+k+1} \cdot S_n^{1,2} \\ &\leq b_{n+k+1} \cdot \left[\prod_{j=k+1}^{k+n} b_j + o\left(\prod_{j=k+1}^{k+n} b_j\right)\right] + \\ &\quad o(1)\left(\prod_{j=1}^{n+k+1} b_j\right) \cdot o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) \\ &= \prod_{j=k+1}^{k+n+1} b_j + o\left(\prod_{j=k+1}^{k+n+1} b_j\right) + o(1) \cdot o\left(\prod_{j=k+1}^{n+k+1} b_j\right) \\ &= \prod_{j=k+1}^{k+n+1} b_j + o\left(\prod_{j=k+1}^{k+n+1} b_j\right) . \end{aligned}$$

Therefore, $(*)$ is true for $M = n + 1$, completing the induction.

Given any $M \in N$, $(*)$ implies

$$\prod_{j=k+1}^{k+M} b_j \;\leq\; R_M^{1,1} \cdot U_M^{2,2}$$

and

$$S_M^{1,2} \cdot T_M^{2,1} \;\leq\; o\left(\left(\prod_{j=1}^{k} b_j\right)^{-1}\right) \cdot o\left(\prod_{j=1}^{k+M} b_j\right) \;=\; o\left(\prod_{j=k+1}^{k+M} b_j\right).$$

Thus, there exists a sufficiently large k so that

$$S_M^{1,2} \cdot T_M^{2,1} \;\leq\; \frac{1}{2} \prod_{j=k+1}^{k+M} b_j \;<\; \prod_{j=k+1}^{k+M} b_j \;\leq\; R_M^{1,1} \cdot U_M^{2,2}$$

from which we obtain

$$\frac{R_M^{1,1}}{S_M^{1,2}} \bigg/ \frac{T_M^{2,1}}{U_M^{2,2}} \geq 2 > 1 .$$

But this implies $\{A_i\}$ is not backwardly weakly ergodic, a contradiction. Therefore, at least one of

$$\sum_{i=1}^{\infty} \frac{a_i}{\prod_{j=1}^{i} b_j} \quad \text{and} \quad \sum_{i=1}^{\infty} c_i \left(\prod_{j=1}^{i-1} b_j\right)$$

must diverge. ■

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