

APPROXIMATELY INNER AUTOMORPHISMS OF VON NEUMANN FACTORS

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Abstract

Through von Neumann's reduction theory, the classification of injective von Neumann algebras acting on separable Hilbert spaces translates into the classification of injective factors. In his proof of the uniqueness of the injective type II_1 factor, Connes showed an alternate characterization of the approximately inner automorphisms of type II_1 factors. Moreover, he conjectured that this characterization could be extended to all types of factors acting on separable Hilbert spaces. In this thesis, we present a general toolbox containing the basic notions needed to study von Neumann algebras, before describing our work concerning Connes' conjecture in the case of type III_λ factors. We have obtained partial results towards the proof of a modified version of this conjecture.

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Introduction

Von Neumann algebras originated in the work of von Neumann in the 1930s, in particular in his search for a natural mathematical framework for quantum mechanics. While C^* -algebras are the noncommutative analogues of topology, von Neumann algebras can be thought of as the noncommutative version of measure and integration theory. In the context of theoretical physics, the von Neumann algebras of interest are often the ones acting on separable Hilbert spaces¹.

Reduction theory, a discovery due to von Neumann himself, greatly simplifies the study of von Neumann algebras acting on separable Hilbert spaces. It states that any such algebra is a direct integral of von Neumann factors, which are defined as von Neumann algebras with a trivial center. It can be shown that numerous properties pass from the algebra to the factors appearing in its decomposition, and vice versa.

One such property is injectivity. To be precise, a “separable” von Neumann algebra is injective if and only if almost all the factors in its integral decomposition are injective. Therefore, the study and classification of injective “separable” von Neumann algebras can be reduced to those of injective “separable” factors. The complete classification of injective factors was attained during the latter half of the 20th century. This achievement is mainly due to the work of Alain Connes, and was completed in 1987 when Uffe Haagerup solved the last remaining case (see Sec. 2.2.5).

Von Neumann factors are divided into the following different types: I_n ($n \in \mathbb{N}$), I_∞ , II_1 , II_∞ , III_1 , III_λ ($0 < \lambda < 1$) and III_0 . The aforementioned classification states that there exists a unique “separable” injective factor for all these types but the III_0 one, for which an uncountable infinity of “separable” injective factors exists (these are in bijective correspondence with non-transitive ergodic flows).

In an intermediate step of his proof of the uniqueness of the “separable” injective II_1 factor, Connes showed an alternate characterization of approximately inner automorphisms of “separable” II_1 factors [Con76]. Moreover, in [Con85], he conjectured that this characterization could be extended to all types of “separable” factors. In [GM93], the authors developed an alternative proof to this characterization for the

¹In this work, we will designate as “separable” the von Neumann algebras that can be represented faithfully on a separable Hilbert space. Note that such algebras are not necessarily separable as Banach spaces, which is why we will always use quotation marks.

type II_1 case, and were able to use their technique to prove Connes conjecture in the II_∞ case.

The objective of this thesis was to adapt the techniques of [GM93] to the case of type III_λ factors satisfying Araki's property L'_λ . Indeed, in 2018, [Mar20] found a counterexample to Connes' conjecture in the case of type III_λ factors, but the author showed that the conjecture holds if the factor is required to have the L'_λ property. This led us to seek an analogue of the proof of [GM93], in the case of a III_λ factor with property L'_λ . As we shall see in Chap. 3, we obtained partial results towards this proof, but technical points still need to be addressed in order to obtain a complete proof.

In Chap. 1, we present a "toolbox" containing all the basic results needed to study von Neumann algebras. In Chap. 2, we discuss the structure of type III factors. To do so, we introduce the Tomita-Takesaki theorem and the group of modular automorphisms, along with Connes' S and T invariants. This allows us to finish an exhaustive list of examples of factors of each type, which is initiated in Sec. 1.9. Finally, we present in Chap. 3 our results related to Connes' conjecture for a factor of type III_λ .

Chapter 1

von Neumann algebras

In this chapter, we cover the basic toolbox one needs to study von Neumann algebras. We summarize the types of operator algebras (1.1), and we discuss the various useful topologies (1.2) and linear functionals (1.3) on von Neumann algebras. We then get more to the heart of the subject with projections (1.4), weights (1.5), the GNS construction (1.6), completely positive maps (1.7) and injectivity (1.8). After that is the longest section of this chapter, dealing with factors (1.9), where we discuss the structure of such objects and start an exhaustive list of examples which will be completed in Chap. 2. We finish this chapter with discrete crossed products (1.10), a construction that allow us to build a whole new range of von Neumann algebras from existing ones. The main references that were used to write this chapter are [Bry13, Ped89, Put19, Sun87].

1.1 Operator algebras

Before delving into the subject of von Neumann algebras, we define more general types of operator algebras.

Definition 1.1.1. *A complex Banach algebra A is a complex Banach space endowed with an associative bilinear multiplication for which the norm is submultiplicative, i.e. $\|xy\| \leq \|x\| \cdot \|y\|, \forall x, y \in A$.*

In this work, we will solely be focusing on Banach algebras that have more restrictions on their structure, namely C^* -algebras .

Definition 1.1.2. A C^* -algebra A is a complex Banach algebra endowed with an involution

$$\begin{aligned} * : A &\rightarrow A \\ a &\mapsto a^* \end{aligned}$$

which is an antilinear operation that satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$, $\forall x, y \in A$. Furthermore, it is required to satisfy the so-called C^* -identity $\|x\|^2 = \|x^*x\|$, $\forall x \in A$.

A C^* -algebra may or may not have a multiplicative unit. In the case where it is non-unital, it can always be embedded in a unital C^* -algebra called the unitization of the original algebra.

Perhaps surprisingly, the C^* -identity turns out to be a very strong structural requirement on the algebra. For example, if A is unital, we can introduce the spectrum of an element $x \in A$

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid 1 - \lambda x \text{ is not invertible}\},$$

which is always a non-empty compact subset of \mathbb{C} . It can then be shown that

$$\|x\|^2 = \sup\{|\lambda| \mid \lambda \in \sigma(x^*x)\},$$

an equality that originates from the C^* -identity and which shows that the latter strongly ties the algebraic and analytic structures of the algebra, since the norm becomes linked to the existence of multiplicative inverses. Note that these concepts are easily extended to non-unital C^* -algebras through their unitization. Since the right-hand side of the last equation solely depends on the algebraic structure, this equality also shows that the norm of a C^* -algebra is unique, in the sense that the algebra cannot be a C^* -algebra under two different norms.

There are two basic examples of C^* -algebras that arise quite naturally, and in fact they are much more general than one could think at first glance. To construct a commutative example, we consider a locally compact Hausdorff space X ; recall that a complex function f on X is said to *vanish at infinity* if for every $\epsilon > 0$, there exists a compact set $K \subseteq X$ s.t. $|f(x)| < \epsilon$ for all $x \in X \setminus K$. The algebra of continuous complex functions vanishing at infinity on X , denoted $C_0(X)$, forms a commutative C^* -algebra with the usual pointwise operations (with the adjoint given by the pointwise complex conjugation) and the supremum norm. This algebra is unital if and only if X is compact. This example gains all its generality through the following theorem.

Theorem 1.1.3. (*Gelfand-Naimark Theorem*) For any commutative C^* -algebra A , there exists a locally compact Hausdorff space X such that A is isometrically isomorphic to $C_0(X)$.

For a non-commutative example, let \mathcal{H} be a Hilbert space with inner product $(\xi, \eta) \mapsto (\xi|\eta)$, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of continuous linear operators on \mathcal{H} . For any $x \in \mathcal{B}(\mathcal{H})$, there exists a unique $x^* \in \mathcal{B}(\mathcal{H})$ such that

$$(x\xi|\eta) = (\xi|x^*\eta), \forall \xi, \eta \in \mathcal{H}$$

This correspondence defines an involution on $\mathcal{B}(\mathcal{H})$, and with the operator norm

$$\|x\| = \sup\{\|x\xi\| \mid \xi \in \mathcal{H}, \|\xi\| = 1\}$$

$\mathcal{B}(\mathcal{H})$ is a C^* -algebra. Of course, it is non-commutative if $\dim(\mathcal{H}) \geq 2$. It is easy to check that any $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ which is norm closed is also a C^* -algebra. The correspondence is actually much stronger: as will be discussed in Sec. 1.6, every C^* -algebra can be represented as a subalgebra of $\mathcal{B}(\mathcal{H})$ for a given Hilbert space \mathcal{H} . Therefore, even though C^* -algebras are defined abstractly, one can always think of a C^* -algebra as a subalgebra of $\mathcal{B}(\mathcal{H})$.

We must define a few basic terms central to the study of C^* -algebras. For a C^* -algebra A , we say that an element $x \in A$ is

- *self-adjoint* if $x = x^*$.
- *normal* if $xx^* = x^*x$.
- *positive* if $x = y^*y$ for some $y \in A$.

The set of positive elements in A is denoted A_+ . Finally, a map $f : A \rightarrow B$ between two C^* -algebras is called positive if it preserves positive elements.

There are many beautiful and important facts that we could discuss about C^* -algebras and that will be used throughout this work. However, for brevity, we only mention one of the most important ones here, namely a first form of the spectral theorem. In the following statement, $C^*(x)$ denotes the norm closure in the algebra A of the set of polynomials composed of $\mathbb{1}_A$, x and x^* . Moreover, $\chi_{\sigma(x)}$ is the characteristic function of $\sigma(x)$ and $f : \sigma(x) \rightarrow \mathbb{C}$ is defined as $f(z) = z$.

Theorem 1.1.4. (*Spectral theorem - Continuous functional calculus*) *Let A be a unital C^* -algebra and x be a normal element of A . There is a unique isometric algebra isomorphism ϕ from $C(\sigma(x))$ onto $C^*(x)$ such that $\phi(\chi_{\sigma(x)}) = \mathbb{1}_A$ and $\phi(f) = x$.*

This theorem is extremely useful as it allows one to translate problems in the C^* -algebra in terms of continuous functions, which are easy to deal with. We note that it can be slightly modified to treat the case of non-unital algebras. This theorem allows one to prove a multitude of results, including the following one.

Proposition 1.1.5. *Let A, B be C^* -algebras and $\pi : A \rightarrow B$ be a $*$ -homomorphism. Then, $\|\pi\| \leq 1$, and π is an isometry iff it is injective. Furthermore, $\pi(A)$ is a C^* -subalgebra of B .*

Proof: Prop. 5.2 and Prop. 5.3 of [Tak02]. ■

We finish this brief discussion of C^* -algebras by mentioning the following elegant result, which fully classifies finite dimensional C^* -algebras. Note that in this work \mathbb{M}_n will be used to denote $n \times n$ matrices.

Theorem 1.1.6. *Let A be a finite dimensional C^* -algebra. Then, we can find a list of positive integers n_1, \dots, n_k such that $A \cong \bigoplus_{i=1}^k \mathbb{M}_{n_i}(\mathbb{C})$.*

Proof: Thm. I.11.2 of [Tak02]. ■

Finite dimensional C^* -algebras are therefore all unital and semisimple, and it will be clear by the end of this section that this theorem also implies that they are von Neumann algebras.

Finally, we introduce the operator algebras which will be the focus of this work, namely von Neumann algebras. Let \mathcal{H} be a Hilbert space. For a subset S of $\mathcal{B}(\mathcal{H})$, we can consider its commutant $S' = \{y \in \mathcal{B}(\mathcal{H}) \mid xy = yx, \forall x \in S\}$, composed of the operators that commute with every element of S . It is easy to check that for any $S \subset \mathcal{B}(\mathcal{H})$ we have $S \subset S''$ and $S' = S'''$. Therefore, if we denote the result of taking the commutant n times by $S^{(n)}$, we have that $S^{(n)}$ is the same for all odd values of n , while the process stabilizes only at $n = 2$ for even values. Von Neumann algebras are precisely the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that are equal to their bicommutant.

Definition 1.1.7. *A von Neumann algebra M on a Hilbert space \mathcal{H} is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ such that $M = M''$.*

From this definition, one can check that the commutant of a $*$ -invariant set and the intersection of a finite number of von Neumann algebras are both von Neumann algebras. As the commutant of a set is norm closed, von Neumann algebras are always C^* -algebras, but the latter form a much bigger class. In opposition to the above definition of C^* -algebras, this definition is inherently concrete, i.e. the algebra is already represented on a Hilbert space \mathcal{H} . It is important to mention that von Neumann algebras also have an abstract definition, but this will be discussed in Sec. 1.3. Nevertheless, it will always be assumed that our von Neumann algebras are represented on a given Hilbert space \mathcal{H} .

The simplest example of a von Neumann algebra is $\mathcal{B}(\mathcal{H})$, for a given Hilbert space. Indeed, $\mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbb{1}_{\mathcal{H}}$, as, on a Banach space, a linear operator that commutes with all other such operators must be a complex multiple of the identity, a result that can be proven by using the Hahn-Banach extension theorem. Therefore, $\mathcal{B}(\mathcal{H})'' = \mathcal{B}(\mathcal{H})$, which shows that this is a von Neumann algebra, non-commutative for $\dim \mathcal{H} \geq 2$.

For a commutative example, let λ be the Lebesgue measure on \mathbb{R} and consider the map

$$\begin{aligned} L^\infty([0, 1], \lambda) &\rightarrow \mathcal{B}(L^2([0, 1], \lambda)) \\ f &\mapsto m_f \end{aligned}$$

where m_f is the operator corresponding to the multiplication by f . This is an isometric algebra homomorphism, and the image of $L^\infty([0, 1], \lambda)$ under this map is a von Neumann algebra on $L^2([0, 1], \lambda)$. As in the case of commutative C^* -algebras, this result is representative of the general case.

Theorem 1.1.8. *Let M be a commutative von Neumann algebra on a Hilbert space. Then there exists a locally compact space X and a positive measure μ on X such that M is $*$ -isomorphic to $L^\infty(X, \mu)$.*

Proof: Thm. 1 of I.7.3 in [Dix81]. ■

One might compare this theorem to Thm. 1.1.3. Indeed, as mentioned before, a von Neumann algebra M is always a C^* -algebra. Hence, if M is commutative, we have by the previous theorem that $M \cong L^\infty(X_1, \mu)$ for some locally compact space X_1 and we have by Thm. 1.1.3 that $M \cong C_0(X_2)$ for some locally compact space X_2 . In general, X_1 is not isomorphic to X_2 . In fact, the locally compact spaces associated to commutative von Neumann algebras through the Gelfand-Naimark theorem are called hyperstonean spaces, which are extremely disconnected spaces (see Chap. III.1 of [Tak02]).

The center of a von Neumann algebra M , denoted $Z(M)$, is defined as the operators in M that commute with all the other elements of the algebra. In terms of the ambient Hilbert space, we thus have $Z(M) = M \cap M'$. This equality shows that $Z(M)$ is also a von Neumann algebra. As von Neumann algebras are unital, we always have $\mathbb{C}\mathbb{1}_M \subset Z(M)$. A von Neumann algebra will be said to be a *factor* when we have precisely $Z(M) = \mathbb{C}\mathbb{1}_M$, in other words, when the center of the algebra is as small as possible. Therefore, factors are the contrast to commutative von Neumann algebras. The simplest examples of factors are the algebras $\mathcal{B}(\mathcal{H})$, for a given Hilbert space, since $\mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbb{1}_{\mathcal{H}}$. We will elaborate on the structure and the importance of factors in Sec. 1.9.

Finally, we note the following two constructions.

Proposition 1.1.9. *Let $(M_i)_{i \in I}$ be a family of von Neumann algebras, each acting on a Hilbert space \mathcal{H}_i . We can define the direct sum of this family as $\bigoplus_I M_i = \{(T_i)_{i \in I} \mid T_i \in M_i \text{ with } \sup \|T_i\| < \infty\}$. This is a von Neumann algebra, with the norm $\|(T_i)_{i \in I}\| = \sup \|T_i\|$ and the obvious representation on $\mathcal{H} = \bigoplus_I \mathcal{H}_i = \{(\xi_i)_{i \in I} \mid \xi_i \in \mathcal{H}_i \text{ with } \sum_I \|\xi_i\|^2 < \infty\}$, given, for $T = (T_i)_{i \in I}$ and $\xi = (\xi_i)_{i \in I} \in \mathcal{H}$, by $(T\xi)_i = T_i\xi_i$.*

Proof: Prop. 0.11 of [Bry13]. ■

Proposition 1.1.10. *Let M, N be von Neumann algebras acting respectively on the Hilbert spaces \mathcal{H}, \mathcal{K} , and let $\mathcal{H} \otimes \mathcal{K}$ denote the Hilbert completion of the algebraic tensor product $\mathcal{H} \odot \mathcal{K}$. For $S \in M$ and $T \in N$, there is a unique linear operator $S \otimes T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $(S \otimes T)(x \otimes y) = Sx \otimes Ty$ for $x \in \mathcal{H}$ and $y \in \mathcal{K}$. The set*

$$M \otimes N = \{S \otimes T \mid S \in M, T \in N\}'' \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$$

is then a von Neumann algebra, called the von Neumann algebra tensor product of M and N .

Proof: See Chap. 1 of [Bry13] for the details, more precisely Prop. 1.19 and Def. 1.4. ■

Note that the norm in this algebra satisfies $\|S \otimes T\| = \|S\| \cdot \|T\|$ for $S \in M$ and $T \in N$. Furthermore, it is important to note that $M \otimes N$ depends up to isomorphism only on M and N , and not on the Hilbert spaces on which the algebras are represented (see III.1.5.4 of [Bla06]). In this work, this is the tensor product which will be implicitly used for von Neumann algebras.

Before going any further, one must discuss the topological description of von Neumann algebras.

1.2 Locally convex topologies on $\mathcal{B}(\mathcal{H})$

Given a separating family \mathcal{F} of seminorms on a vector space V (if $x \neq y$ in V , $\exists p \in \mathcal{F}$ with $p(x - y) \neq 0$), one can obtain a topology on the latter, under which it is a locally convex topological vector space. Indeed, for each pair in $(p, y) \in \mathcal{F} \times V$, we can associate the function $x \mapsto p(x - y)$; the set of all such functions separates points in V . Thus, the initial topology (weakest topology that makes all these functions continuous) induced by this set on V is a Hausdorff topology. For $x \in V$, sets of the form

$$\{y \in V \mid p_k(y - x) < \epsilon, 1 \leq k \leq n\}$$

for $\epsilon > 0$ and p_1, \dots, p_n a finite number of elements of \mathcal{F} form then a basis of neighbourhood at x , and it can be checked that this makes V a locally convex topological vector space. We will refer to this topology as the *weak topology induced by \mathcal{F} on V* . Working with such a topology is made much easier through the following result:

Proposition 1.2.1. *Let \mathcal{F} be a separating family of semi-norms on V . Then $x_n \rightarrow x$ in the weak topology induced by \mathcal{F} iff $p(x_n - x) \rightarrow 0, \forall p \in \mathcal{F}$.*

We are now ready to list the topologies on $\mathcal{B}(\mathcal{H})$ that, with the usual norm topology, are paramount to the study of von Neumann algebras.

1. The **weak operator topology** (WOT) is induced by the family of seminorms $x \mapsto |(x\xi|\eta)|$ for $\xi, \eta \in \mathcal{H}$. We have that $x_n \xrightarrow{\text{WOT}} x$ iff $(x_n\xi|\eta) \rightarrow (x\xi|\eta)$ for all $\xi, \eta \in \mathcal{H}$.
2. The **strong operator topology** (SOT) is induced by the family of seminorms $x \mapsto \|x\xi\|$ for $\xi \in \mathcal{H}$. We have that $x_n \xrightarrow{\text{SOT}} x$ iff $x_n\xi \rightarrow x\xi$ for all $\xi \in \mathcal{H}$.
3. The **ultraweak topology** (UWOT, or σ -weak topology) is induced by the family of seminorms $x \mapsto |\sum_{n=1}^{\infty} (x\xi_n|\eta_n)|$ for square summable sequences (ξ_n) and (η_n) in \mathcal{H} .
4. The **ultrastrong topology** (USOT, or σ -strong topology) is induced by the family of seminorms $x \mapsto (\sum_{n=1}^{\infty} \|x\xi_n\|^2)^{1/2}$ for square summable sequences (ξ_n) in \mathcal{H} .

Note that, in all these topologies, multiplication is always continuous in one variable, but is not in general jointly continuous (it however is jointly continuous on bounded subsets of $\mathcal{B}(\mathcal{H})$ for the strong and ultrastrong topologies). Through the convergence conditions, we have the following result:

Proposition 1.2.2. *Let $S \subset \mathcal{B}(\mathcal{H})$. We have*

$$\overline{S}^{\|\cdot\|} \subseteq \overline{S}^{\text{UST}} \subseteq \overline{S}^{\text{SOT}} \subseteq \overline{S}^{\text{WOT}} \quad \overline{S}^{\|\cdot\|} \subseteq \overline{S}^{\text{UST}} \subseteq \overline{S}^{\text{UWT}} \subseteq \overline{S}^{\text{WOT}}$$

Simple calculations with the same convergence conditions also show that the weak and ultraweak (or respectively the strong and ultrastrong) topologies coincide on bounded subsets of $\mathcal{B}(\mathcal{H})$. It turns out that these topologies are not interesting when \mathcal{H} is finite dimensional, as they all coincide with the norm topology on $\mathcal{B}(\mathcal{H})$ since the locally convex topology on a finite dimensional topological vector space is unique. However, if \mathcal{H} is infinite dimensional, all these topologies differ. Indeed, a simple calculation shows that the adjoint operation is continuous in the weak and ultraweak topologies; however, it is not the case in the strong and ultrastrong topologies. To prove this, let $\mathcal{H} = \ell^2(\mathbb{N})$ and let $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$ define the shift operator. Then S^n converges to 0 strongly, but $(S^*)^n$ does not (the same proof applies for the ultrastrong topology as the sequence lies in the unit ball, a bounded set). For the proof that the weak and ultraweak (resp. strong and ultrastrong) topologies differ in infinite dimensions, see Prop. 1.5.7.

Although the ultraweak and ultrastrong topologies might seem less “natural” and more complicated than their simpler counterparts, we will see in the next section that they are in fact intrinsic topologies for von Neumann algebras on \mathcal{H} , in the sense that their properties are preserved under $*$ -isomorphisms. On the contrary, the weak and strong topologies are space-dependent, i.e. they depend on the Hilbert space on which a given von Neumann algebra is represented.

These topologies take all their importance in the following theorem due to von Neumann, which constitutes the cornerstone of von Neumann algebras.

Theorem 1.2.3. (*Bicommutant theorem*) *For a unital $*$ -subalgebra M of $\mathcal{B}(\mathcal{H})$,*

$$M'' = \overline{M}^{SOT} = \overline{M}^{WOT} = \overline{M}^{UWT} = \overline{M}^{UST}$$

Proof: Thm. 4.6.7 and Prop. 4.6.15 of [Ped89]. ■

Through this theorem, it is seen that a von Neumann algebra could equivalently be defined to be a unital $*$ -subalgebra M of $\mathcal{B}(\mathcal{H})$ which is closed in any of these topologies.

1.3 Linear functionals and normality

Linear functionals are a powerful tool to learn more about the vector space itself on which they act. In general, if V is a Banach space, one can look at its (topological) dual space $V^* = \{\phi : V \rightarrow \mathbb{C} \mid \phi \text{ is linear and bounded in norm}\}$, which is itself a Banach space. In the case of C^* -algebra A , the dual has slightly more structure.

Definition 1.3.1. *Let A be a C^* -algebra and $\phi \in A^*$.*

- *If $\phi(x^*x) \geq 0$ for all $x \in A$, we say that ϕ is positive.*
- *If A is unital, ϕ is positive and $\phi(\mathbb{1}_A) = 1$, we say that ϕ is a state.*
- *If ϕ is positive and $\phi(xy) = \phi(yx)$ for all $x, y \in A$, we say that ϕ is a trace.*
- *If ϕ is positive and $\phi(x^*x) = 0$ implies $x = 0$, we say that ϕ is faithful.*

Note that in this work we reserve the word *trace* for what some authors call a finite trace. In Sec. 1.5, we will define an infinite trace as a *tracial weight*. By lack of a better place to mention it, we include here the definition of the 2-norm associated with a trace.

Definition 1.3.2. *If τ is a faithful trace on a C^* -algebra A , the sesquilinear form $(a|b) = \tau(b^*a)$ is an inner product and we call the induced norm $\|x\|_2 = \tau(x^*x)^{1/2}$ the 2-norm.*

The dual of a von Neumann algebra has even more structure, which originates from the existence of suprema of increasing nets of self-adjoint operators within the algebra.

Proposition 1.3.3. *Let (x_i) be a uniformly bounded increasing net ($(\|x_i\|)$ is a bounded net and $i \leq j \implies x_i \leq x_j$) of self-adjoint operators in a von Neumann algebra M . Then, there exists $x \in M$ such that $x_i \leq x$ for all i and $x = \lim x_i$ in all four topologies introduced in Sec. 1.2. This limit further has the property that if $y \in M$ is such that $x_i \leq y$ for all i , then $x \leq y$. For this reason, we write $x = \sup x_i$.*

Proof: P.16-17 of [Sun87]. ■

This result allows us to define the concept of a normal linear functional.

Definition 1.3.4. *Let M be a von Neumann algebra and $\phi \in M^*$. We say that ϕ is normal if for any uniformly bounded increasing net of self-adjoint operators (x_i) in M , the net $\phi(x_i)$ converges to $\phi(\sup x_i)$.*

We note that, through a similar proof as for Prop. 1.2.2, a linear functional continuous in any of the topologies defined in Sec. 1.2 is automatically norm continuous and therefore bounded. As the following theorem shows, normal linear functionals are at the heart of the topological properties of the algebra.

Theorem 1.3.5. *Let M be a von Neumann algebra on \mathcal{H} and $\phi \in M^*$. The following are equivalent:*

- ϕ is normal.
- ϕ is ultraweakly continuous.
- ϕ is ultrastrongly continuous.
- there exist square summable sequences (ξ_n) and (η_n) in \mathcal{H} such that $\phi(x) = \sum(x\xi_n|\eta_n)$ for all $x \in M$.

Proof: Thm. 3.6.4 of [Ped18], and Prop. 4.6.11 + Remark 4.6.16 of [Ped89]. ■

As an aside, note that a similar result holds for weakly and strongly continuous linear functionals.

Proposition 1.3.6. *Let M be a von Neumann algebra on \mathcal{H} and $\phi \in M^*$. The following are equivalent:*

- ϕ is weakly continuous.
- ϕ is strongly continuous.

- there exist finite sequences (ξ_n) and (η_n) in \mathcal{H} such that $\phi(x) = \sum_{n=1}^N (x\xi_n|\eta_n)$ for all $x \in M$.

Proof: Prop. 4.6.4 of [Ped89]. ■

We will denote the set of normal linear functionals on M by M_* , a notation that gains all its significance through the following fundamental theorem.

Theorem 1.3.7. *Let M be a von Neumann algebra. Then M_* is a Banach space and M is isometrically isomorphic to $(M_*)^*$.*

Proof: In [Ped89], Thm. 4.6.17 with Prop. 4.6.11. ■

It turns out that a space with this property is unique: if M is a von Neumann algebra and V is a Banach space such that $V^* = M$, then V is isometrically isomorphic to M_* (See [Sak98], Cor.1.13.3). For this reason, we call M_* the *predual* of M . Note that, along with Thm. 1.3.5, this shows that the ultraweak topology is the same as the weak-* topology that M inherits as the dual space of M_* . However, the main takeaway of this theorem is that a von Neumann algebra is always the dual space of a Banach space. This result is actually fundamental: Sakai showed that the converse is also true, namely that a C^* -algebra which is the dual of a Banach space always possesses a representation on a Hilbert space under which it is a von Neumann algebra as per Def. 1.1.7 (see Sakai, Thm. 1.16.7). Therefore, one can give an abstract definition of the concept, in the sense that it does not depend on a particular representation.

Definition 1.3.8. *A von Neumann algebra is a C^* -algebra which is isometrically isomorphic to the dual space of a Banach space.*

The notion of normality can actually be extended to the case of positive linear maps between von Neumann algebras. Doing so strengthens the importance of this concept in the structure of the algebras.

Definition 1.3.9. *Let M, N be von Neumann algebras and $f : M \rightarrow N$ a bounded positive linear map. We say that f is normal if for any uniformly bounded increasing net of self-adjoint operators (x_i) in M , we have $f(\sup x_i) = \sup f(x_i)$.*

Proposition 1.3.10. *Let M, N be von Neumann algebras and $f : M \rightarrow N$ a bounded positive linear map. The following are equivalent:*

1. f is normal.
2. for all $\phi \in N_*$, we have $\phi \circ f \in M_*$.
3. f is ultraweakly-to-ultraweakly continuous.

Furthermore, if we assume that f is completely positive (see Sec. 1.7), we have that f is normal iff it is ultrastrongly-to-ultrastrongly continuous.

Proof: For the first three equivalences, see Prop. 2.46 and Cor. 2.47 of [Bry13], and for the last statement, see III.2.2.2 of [Bla06]. Note that, for the last equivalence, the hypothesis of f being completely positive was not specified in [Bry13], but it seems necessary. ■

Proposition 1.3.11. *Let M, N be von Neumann algebras and $\pi : M \rightarrow N$ be a $*$ -isomorphism. Then π is an ultraweakly-to-ultraweakly (and an ultrastrongly-to-ultrastrongly) homeomorphism.*

Proof: This follows from the previous result by checking that π and π^{-1} are both normal and completely positive linear maps, since they are $*$ -isomorphisms. ■

This last result fails to be true in the case of the weak and strong topologies. Indeed, let M be the von Neumann algebra $\mathcal{B}(\mathcal{H})$ for \mathcal{H} countably infinite dimensional, and $\Phi : M \rightarrow \mathcal{B}(\oplus_{\mathbb{N}} \mathcal{H})$ defined by $\Phi(x)((\xi_n)_{n \in \mathbb{N}}) = (x\xi_n)_{n \in \mathbb{N}}$. The map Φ is an injective $*$ -homomorphism and $\Phi(M)$ is a von Neumann algebra, so it is an isomorphism between M and $\Phi(M)$. It is easy to check that it is an ultraweakly-to-weakly and ultrastrongly-to-strongly homeomorphism. Therefore, it is impossible that it is either a weakly-to-weakly or strongly-to-strongly homeomorphism, because that would imply that the weak and ultraweak topologies are the same on M (similarly for the strong and ultrastrong topologies), which we know to be false (see Prop. 1.5.7 for a proof).

Therefore, the ultraweak and ultrastrong topologies are intrinsic to the algebra, while the weak and strong topologies are space-dependent. Consequently, normality of maps between von Neumann algebras will often be required, as it ensures the preservation of intrinsic topological properties of the algebras.

Finally, we introduce the notion of “separable” von Neumann algebras.

Proposition 1.3.12. *Let M be a von Neumann algebra. Consider the following conditions.*

1. M can be represented faithfully on a separable Hilbert space.
2. M_* is separable as a Banach space, i.e. it possesses a dense sequence.
3. There exists a faithful normal state ϕ on M .
4. If $(p_i)_{i \in I}$ is a list of nonzero mutually orthogonal projections in M (see next section), then I is countable.

We have $1 \implies 2 \implies 3 \iff 4$.

Proof: 1 \implies 2: Suppose that M is represented on a separable Hilbert space \mathcal{H} . Let $(\xi_n)_{n \in \mathbb{N}}$ be a dense sequence in \mathcal{H} . We claim that the set

$$\mathcal{S} = \left\{ \sum_{n \in \mathbb{N}} (\cdot \xi_{k_n} | \xi_{\ell_n}) \mid (k_n)_{n \in \mathbb{N}} \text{ and } (\ell_n)_{n \in \mathbb{N}} \text{ s.t. } \sum_{n \in \mathbb{N}} \|\xi_{k_n}\|^2, \sum_{n \in \mathbb{N}} \|\xi_{\ell_n}\|^2 < \infty \right\} \subset M_*$$

is countable and dense in M_* . It is clear that it is countable, since it is in bijection with a subset of \mathbb{N}^3 . Let $\psi = \sum_{n \in \mathbb{N}} (\cdot \eta_n | \kappa_n) \in M_*$ (see Thm. 1.3.5) and $\epsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$, we can find a $k_n \in \mathbb{N}$ and $\ell_n \in \mathbb{N}$ such that $\|\eta_n - \xi_{k_n}\| < \epsilon/n$ and $\|\kappa_n - \xi_{\ell_n}\| < \epsilon/n$. It is simple to check that $\phi = \sum_{n \in \mathbb{N}} (\cdot \xi_{k_n} | \xi_{\ell_n}) \in \mathcal{S}$. For any $x \in M$ with $\|x\| = 1$, we then have

$$\begin{aligned} \|\psi - \phi\| &\leq |\psi(x) - \phi(x)| = \left| \sum_{n \in \mathbb{N}} (x \eta_n | \kappa_n) - \sum_{n \in \mathbb{N}} (x \xi_{k_n} | \xi_{\ell_n}) \right| \\ &\leq \left| \sum_{n \in \mathbb{N}} (x(\eta_n - \xi_{k_n}) | \kappa_n) \right| + \left| \sum_{n \in \mathbb{N}} (x \xi_{k_n} | \kappa_n - \xi_{\ell_n}) \right| \\ &\leq \left(\sum_{n \in \mathbb{N}} \|\eta_n - \xi_{k_n}\|^2 \right)^{1/2} \left(\sum_{n \in \mathbb{N}} \|\kappa_n\|^2 \right)^{1/2} \\ &\quad + \left(\sum_{n \in \mathbb{N}} \|\xi_{k_n}\|^2 \right)^{1/2} \left(\sum_{n \in \mathbb{N}} \|\kappa_n - \xi_{\ell_n}\|^2 \right)^{1/2} \\ &\leq \epsilon \frac{\pi}{\sqrt{6}} \left(\left(\sum_{n \in \mathbb{N}} \|\kappa_n\|^2 \right)^{1/2} + \left(\sum_{n \in \mathbb{N}} \|\xi_{k_n}\|^2 \right)^{1/2} \right) \end{aligned}$$

Since ψ and ϵ were arbitrary, we conclude that \mathcal{S} is dense in M_* .

2 \implies 3: If M_* is a separable Banach space, $(M_*)^+$ is also separable. Let $(\phi_n)_{n \in \mathbb{N}}$ be a dense sequence in $(M_*)^+$. We define the normal state

$$\phi(x) = \frac{6}{\pi^2} \sum_{n \in \mathbb{N}} \frac{1}{n^2 \|\phi_n\|} \phi_n(x)$$

for $x \in M$. Suppose that ϕ is not faithful, i.e. that there exists $y \in M$ such that $\phi(y) = 0$ and $\|y\| \neq 0$; we then have $\phi_n(x) = 0 \forall n \in \mathbb{N}$. Let $\psi \in (M_*)^+$ and $\epsilon > 0$ be arbitrary. We choose $N \in \mathbb{N}$ such that $\|\psi - \phi_N\| \leq \epsilon \|y\| > 0$. We have

$$|\psi(y)| \leq \|\psi - \phi_N\| \|y\| + |\phi_N(y)| \leq \epsilon$$

Hence, $\psi(y) = 0$ for all $\psi \in (M_*)^+$. We conclude that we must have $y = 0$, which is a contradiction. Therefore, ϕ is faithful.

3 \iff 4: Prop. II.3.19 in [Tak02]. ■

A von Neumann algebra satisfying the last two conditions is usually called σ -finite or *countably decomposable*. We will however mostly consider the class of algebras satisfying the first condition.

Definition 1.3.13. *We say that a von Neumann algebra is “separable” if it can be represented faithfully on a separable Hilbert space.*

It is important to note that a “separable” von Neumann algebra is not necessarily separable as a Banach space. In fact, a von Neumann algebra is separable as a Banach space if and only if it is finite dimensional (Thm. 1.1.6 proves the “if” part; for the “only if” part, see Ex. III.3.1 of [Tak02], or Cor. 1.3.17 of [Ole12] for a complete proof).

1.4 Projections in a von Neumann algebra

In a C^* -algebra, an element p is called an *orthogonal projection* if $p = p^2 = p^*$. In this work, we will simply call such an operator a projection. A crucial aspect that differentiates von Neumann algebras from other C^* -algebras is the abundance of projections in the former. This originates from the fact that the isomorphism of Thm. 1.1.4 can be extended as a homomorphism to the class of Borel functions of the spectrum of a normal element.

In the following, the von Neumann algebra generated by a normal element $x \in M \subseteq \mathcal{B}(\mathcal{H})$ is $W^*(x) = \{x\}'' = \overline{C^*(x)}^{\text{UWT}}$. As was seen in Sec. 1.3, the last equality makes this definition independent of the representation of M . Additionally, we let $\mathfrak{B}_b(\sigma(x))$ denote the bounded Borel functions on $\sigma(x)$.

Theorem 1.4.1. *(Spectral theorem - Borel Functional Calculus) Let M be a von Neumann algebra and $x \in M$ be a normal operator. There exists a norm decreasing $*$ -homomorphism ψ_x from $\mathfrak{B}_b(\sigma(x))$ into $W^*(x)$ which extends the isomorphism of Thm. 1.1.4. Furthermore, if (f_n) is a bounded increasing sequence of real functions in $\mathfrak{B}_b(\sigma(x))$, then $\psi_x(\sup f_n) = \sup \psi_x(f_n)$.*

Proof: Thm. 4.5.4 of [Ped89]. ■

Since for any Borel subset B of $\sigma(x)$, the characteristic function χ_B is a projection in $\mathfrak{B}_b(\sigma(x))$, ψ_x induces a map from the Borel subsets of $\sigma(x)$, denoted $\mathcal{B}(\sigma(x))$, to the projections of M , denoted $\mathcal{P}(M)$:

$$\begin{aligned} \mathcal{B}(\sigma(x)) &\rightarrow \mathcal{P}(M) \\ B &\mapsto \psi_x(\chi_B) \end{aligned}$$

This suggests an abundance of projections in M , which is in fact confirmed by the next result.

Proposition 1.4.2. *A von Neumann algebra is the norm closure of the set of linear combinations of its projections.*

Proof: Corollary 0.4.9 of [Sun87] and the discussion below it. ■

To illustrate how this property fails in the general case of C^* -algebras, one can look at the case of $A = C([0, 1])$, the continuous complex functions on $[0, 1]$. It is easy to check that the only projections in A are 0 and $\mathbb{1}_A$. Therefore, the norm closed subspace generated by the projections is merely $\mathbb{C}\mathbb{1}_A$.

This result also suggest that one way to learn about the structure of a von Neumann algebra is to study in depth its projections. We recall the following notions about projections in a C^* -algebra A .

Definition 1.4.3. *For two projections $p, q \in A$, we say that $p \leq q$ if one of the following equivalent conditions is satisfied.*

1. $q - p$ is a positive operator.
2. $qp = p$
3. $pq = p$
4. If $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , $\text{Im}(p) \subset \text{Im}(q)$

We say that p and q are mutually orthogonal if one of the following equivalent conditions is satisfied.

1. $pq = 0$
2. $qp = 0$
3. If $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , $p(\mathcal{H}) \perp q(\mathcal{H})$

Now if $\{p_\lambda \mid \lambda \in \Lambda\}$ is a family of projections in a von Neumann algebra M , we define $\sup_\Lambda p_\lambda$ and $\inf_\Lambda p_\lambda$ as the projections on the subspaces $[\cup_\Lambda \text{Im}(p_\lambda)]$ and $\cap_\Lambda \text{Im}(p_\lambda)$, respectively, where $[S]$ denotes the closed subspace generated by S . As their names suggest, $\sup_\Lambda p_\lambda$ is the smallest projection that majorizes this family, and the converse statement holds for $\inf_\Lambda p_\lambda$. The following result shows that $\mathcal{P}(M)$ always forms a lattice.

Proposition 1.4.4. *For any family $\{p_\lambda \mid \lambda \in \Lambda\}$ of projections in M , $\sup_\Lambda p_\lambda$ and $\inf_\Lambda p_\lambda$ are contained in M .*

Proof: Prop. V.1.1 of [Tak02]. ■

For a given projection $p \in \mathcal{P}(M)$ we can consider the *reduced algebra* or *corner algebra* defined aso $pMp = \{pxp \mid x \in M\}$.

Proposition 1.4.5. *Let M be a von Neumann algebra acting on \mathcal{H} and $p \in \mathcal{P}(M)$. Then, pMp is a von Neumann algebra acting on $p\mathcal{H}$, with identity $\mathbb{1}_{pMp} = p$. Furthermore, if M is a factor, then pMp is also a factor.*

Proof: Prop. 3.10 of [Tak02]. Note that the last assertion should be written $(M')_e = (M_e)'$. ■

For a projection $p \in \mathcal{P}(M)$, we define the *central cover* of p to be the projection $c(p) = \inf\{q \in \mathcal{P}(M) \cap Z(M) \mid p \leq q\}$. For us, this concept will gain its importance through the following results.

Lemma 1.4.6. *Let M be a von Neumann algebra and $p, q \in \mathcal{P}(M)$. The following are equivalent:*

1. $pxq = 0 \forall x \in M$
2. $c(p)c(q) = 0$

Proof: Lemma 0.4.16 of [Sun87]. ■

In a C^* -algebra, an element u is called a partial isometry if u^*u is a projection. It is simple to check through the C^* -condition that u is a partial isometry if and only if u^* is.

Proposition 1.4.7. *Let M be a factor and $p, q \in \mathcal{P}(M)$ be nonzero projections. There exists a nonzero partial isometry $u \in M$ such that $u^*u \leq p$ and $uu^* \leq q$.*

Proof: In a factor, we always have $\mathcal{P}(M) \cap Z(M) = \{0, \mathbb{1}_M\}$. Therefore, $c(p) = c(q) = \mathbb{1}_M$ and the previous lemma shows that there exists $x \in M$ for which $pxq \neq 0$. For an operator $x \in M$, we recall that we have the polar decomposition $x = u|x|$ for a partial isometry u and a positive operator $|x|$, both in M . If we let $pxq = u|pxq|$ be the polar decomposition, u fulfills the role of the desired partial isometry. ■

It turns out that a fruitful way to study the projections in a von Neumann algebra is to classify them according to their relative size. This is done by defining an appropriate equivalence relation on $\mathcal{P}(M)$ and an order relation on the set of equivalence classes.

Definition 1.4.8. *Two projections $p, q \in M$ are said to be Murray-von Neumann equivalent (relative to M) if there exists a partial isometry $v \in M$ such that $p = vqv^*$ and $q = v^*pv$. Equivalently, there exists a partial isometry $v \in M$ such that $p = vv^*$ and $q = v^*v$. We will then write $p \sim q$ and the equivalence class of p will be denoted $[p]$.*

It is important to note that this equivalence relation depends solely on M ; if M is acting on a given Hilbert space \mathcal{H} , two projections could be equivalent in $\mathcal{B}(\mathcal{H})$ while not being equivalent in M . This equivalence relation behaves nicely on sums of projections.

Proposition 1.4.9. *If $(p_n)_{n \geq 1}, (q_n)_{n \geq 1} \subseteq \mathcal{P}(M)$ are such that $p_n \sim q_n$ for all n , while $p_n \perp p_m$ and $q_n \perp q_m$ for $n \neq m$, then $\sum^{USOT} p_n \sim \sum^{USOT} q_n$.*

Proof: Proposition 1.1.7 of [Sun87] proves this result for the strong operator topology. Since these two sums of projections are bounded in norm, we automatically get the result in the ultrastrong topology. ■

At first sight, one might ask why the definition of the equivalence is not given as $p \sim q$ iff there exists a unitary $u \in M$ such that $p = uqu^*$ and $q = u^*pu$. The reason for choosing partial isometries instead is due to the fact that unitary equivalence is too strong for our purpose.

To illustrate this, we take $\mathcal{H} = \ell^2(\mathbb{N})$, $M = \mathcal{B}(\ell^2(\mathbb{N}))$, $p = \mathbb{1}_M$ and q as the projection that cuts out the first coordinate. Let ξ_i denote the i -th vector in the canonical basis of $\ell^2(\mathbb{N})$. For any unitary $u \in M$, we have $u^*pu = u^*\mathbb{1}_M u = u^*u = \mathbb{1}_M = p \neq q$, so these two projections are not unitary equivalent. However, we clearly have $\text{Im}(p) \cong \text{Im}(q)$ as subspaces, so these two projections should have the same “size”. On the other hand, if we define $u : \mathcal{H} \rightarrow \mathcal{H}$ by $u(\xi_i) = \xi_{i+1}$, we have $u^*(\xi_i) = \xi_{i-1}$ for $i \geq 1$ and $u^*(\xi_0) = 0$, and hence $p = u^*u$ while $q = uu^*$. Therefore, the Murray-von Neumann equivalence respects the intuitive idea of when two projections are “of the same size”.

On these equivalence classes, the following order relation is well-defined: if $[p]$ and $[q]$ are Murray-von Neumann equivalence classes, we will say that $[p] \leq [q]$ if there exists \tilde{p} such that $p \sim \tilde{p} \leq q$. In general, this order is only partial. The simplest example is to take $M = \mathbb{C} \oplus \mathbb{C}$, with $p = 1 \oplus 0$ and $q = 0 \oplus 1$. However, the order will be total in the case where M is factor.

Proposition 1.4.10. *If M is a factor and $p, q \in \mathcal{P}(M)$, either $[p] \leq [q]$ or $[q] \leq [p]$.*

Proof: See [Dix81], Part III, Chap. 1, 2. A theorem on comparability, Cor. 1. ■

For the rest of this section, we will therefore restrict our scope to the case where M is a factor. The first use of the equivalence relation comes in terms of characterizing what we mean by an infinite projection.

Definition 1.4.11. *A projection $p \in \mathcal{P}(M)$ is said to be finite (relative to M) if $q \in \mathcal{P}(M)$ and $p \sim q \leq p$ implies that $q = p$. Otherwise, we say that p is infinite.*

It is simple to check that finiteness is preserved under the equivalence relation. This definition mirrors that of an infinite set, where the latter is defined as a set in bijection with one of its own proper subset. Furthermore, we have the following results, which are completely analogous to the notion of infinity in set theory.

Proposition 1.4.12. *Let $p, q \in \mathcal{P}(M)$.*

1. *If $[p] \leq [q]$ and q is finite, p is finite.*

2. The supremum of finitely many finite projections is finite.
3. If q is infinite, then $[p] \leq [q]$. Consequently, any two infinite projections are equivalent.

Proof: p.22-27 of [Sun87]. ■

The last statement follows from the antisymmetry of the order relation on the equivalence classes.

Proposition 1.4.13. *Let $p, q \in \mathcal{P}(M)$, with $p \neq 0$. Then, there exists $e \in \mathcal{P}(M)$ and a family $(p_i)_{i \in I}$ of mutually orthogonal projections in M such that $q = (\sum_{i \in I}^{SOT} p_i) + e$, $[e] \leq [p]$ and $p_i \sim p$ for all $i \in I$. Furthermore, if I is an infinite set, there exists another such decomposition where $e = 0$.*

Proof: If $[q] \leq [p]$, this is trivial by taking $I = \emptyset$ and $e = p$. Therefore, we can suppose $[p] \leq [q]$. Let $p' \in \mathcal{P}(M)$ be such that $p' \sim p$ and $p' \leq q$. Consider Corollary 1 in [Dix81], Part III, Chap. 1., 2. A theorem on comparability. Applying this corollary to the factor qMq , with q as the only element of the family (E_i) , yields the result we seek. ■

1.5 Weights

When integrals are introduced in measure theory, the matter of infinite integrals and infinite measures follows quite naturally. These generalize the concept of the finite integral. In a completely analogous fashion, we will introduce in this section weights, which are an extension to the notion of positive linear functionals. Weights will prove essential to the study of von Neumann factors, as we shall see in the later sections.

Definition 1.5.1. *A weight on a von Neumann algebra M is a function $\phi : M_+ \rightarrow [0, \infty]$ such that $\phi(\lambda x + y) = \lambda\phi(x) + \phi(y)$ for $x, y \in M_+$ and $\lambda \in [0, \infty)$ (with the operations executed in the extended real numbers, for example $0 \cdot \infty = 0$). We will say that ϕ is*

- *faithful if $x \in M_+$ and $x \neq 0$ imply $\phi(x) > 0$.*
- *normal, if $\phi(\sup x_i) = \sup \phi(x_i)$ whenever (x_i) is a monotone increasing net in M_+ .*
- *tracial, if $\phi(x^*x) = \phi(xx^*)$ for all $x \in M$.*

A weight will be called finite if $\phi(x) < \infty$ for all $x \in M_+$, and the following proposition shows that these weights are non other than the positive linear functionals.

Proposition 1.5.2. *Let ϕ be a weight on a von Neumann algebra M . The following are equivalent:*

1. ϕ is finite.
2. $\phi(1) < \infty$.
3. there exists a unique positive linear functional $\tilde{\phi}$ on M such that $\phi = \tilde{\phi}|_{M_+}$.

Proof: 1) \implies 2): Obvious.

2) \implies 3): Uniqueness follows easily from the fact that any element of M is a sum of four positive operators. We show the existence of such an extension. For any element $x \in M_+$, we have $x \leq \|x\|1_M$ and so $\phi(1) < \infty$ implies that $\phi(x)$ is finite. Since any self-adjoint element $y \in M$ is the difference of two positive operators $y_+, y_- \in M_+$, we can then define $\tilde{\phi}$ on y by $\tilde{\phi}(y) = \phi(y_+) - \phi(y_-)$. This is well-defined: if $y_+ - y_- = y'_+ - y'_-$ for $y_+, y_-, y'_+, y'_- \in M_+$, we have

$$\begin{aligned} \phi(y_+) + \phi(y'_-) &= \phi(y_+ + y'_-) = \phi(y'_+ + y_-) = \phi(y'_+) + \phi(y_-) \\ &\implies \phi(y_+) - \phi(y_-) = \phi(y'_+) - \phi(y'_-) \\ &\implies \tilde{\phi}(y_+ - y_-) = \tilde{\phi}(y'_+ - y'_-) \end{aligned}$$

A simple calculation shows that $\tilde{\phi}$ is real linear on M_{sa} , the set of self-adjoint elements of M . Now for any $z \in M$, the real part $z_r = (z + z^*)/2$ and imaginary part $z_i = (z - z^*)/(2i)$ are two self-adjoint operators such that $z = z_r + iz_i$. As this decomposition is unique, $\tilde{\phi}(z) = \tilde{\phi}(z_r) + i\tilde{\phi}(z_i)$ is well-defined. With the fact that $\tilde{\phi}$ is real linear on M_{sa} , a simple verification shows that this defines a linear functional on M . It is then obvious that $\phi = \tilde{\phi}|_{M_+}$ and that $\tilde{\phi}$ is positive.

3) \implies 1): For any $x \in M_+$, we have $\phi(x) = \tilde{\phi}(x) < \infty$. ■

Definition 1.5.3. *Let ϕ be a weight on a von Neumann algebra M . We define the sets*

- $\mathcal{D}_\phi = \{x \in M_+ \mid \phi(x) < \infty\}$
- $\mathcal{N}_\phi = \{x \in M \mid \phi(x^*x) < \infty\}$
- $\mathcal{M}_\phi = \{\sum_{i=1}^n x_i^* y_i \mid x_i, y_i \in \mathcal{N}_\phi, n \in \mathbb{N}\}$

Proposition 1.5.4. *These subsets have the following properties.*

- a) \mathcal{D}_ϕ is a hereditary positive convex cone; i.e. $x, y \in \mathcal{D}_\phi$ and $\lambda \in [0, \infty) \implies \lambda x + y \in \mathcal{D}_\phi$, and $x \in \mathcal{D}_\phi, z \in M_+, z \leq x \implies z \in \mathcal{D}_\phi$. Note that for $x, z \in M_+$, we always have $z \leq x \implies \phi(z) \leq \phi(x)$.

- b) \mathcal{N}_ϕ is a left ideal in M , i.e. $M\mathcal{N}_\phi \subseteq \mathcal{N}_\phi$.
- c) \mathcal{M}_ϕ is a self-adjoint subalgebra of M .
- d) $\mathcal{D}_\phi = \mathcal{M}_{\phi_+} = \mathcal{M}_\phi \cap M_+$.
- e) $x, z \in \mathcal{N}_\phi, y \in M \implies x^*yz \in \mathcal{M}_\phi$.
- f) there is a unique linear functional $\tilde{\phi}$ on \mathcal{M}_ϕ such that $\tilde{\phi}|_{\mathcal{M}_{\phi_+}} = \phi|_{\mathcal{M}_{\phi_+}}$.

Proof: Prop. 2.4.5 of [Sun87]. ■

Of course, not all weights carry a lot of information. For example, the mapping $\phi : M_+ \rightarrow [0, \infty]$, defined as $\phi(x) = \infty$ for nonzero x and $\phi(0) = 0$, is a faithful tracial weight on any von Neumann algebra M . However, the following so-called *semifinite* property allows us to characterize when a weight takes finite values on “enough” elements.

Proposition 1.5.5. *For a weight ϕ on a von Neumann algebra M , the following conditions are equivalent:*

1. \mathcal{M}_ϕ is ultraweakly dense in M .
2. $\mathbb{1}_M = \sup\{p \in \mathcal{P}(M) \mid \phi(p) < \infty\}$.
3. there is a monotone increasing net (x_i) in \mathcal{D}_ϕ such that $\|x_i\| < 1$ and $\sup x_i = \mathbb{1}_M$.

In addition, if ϕ is tracial, we can add the following two conditions to the list of equivalences.

4. for all nonzero $x \in M_+$, there exists $y \in M_+$ such that $0 < y \leq x$ and $\phi(y) < \infty$.
5. for all nonzero $p \in \mathcal{P}(M)$, there exists $q \in \mathcal{P}(M)$ such that $0 < q \leq p$ and $\phi(q) < \infty$.

Proof: For the equivalence of 1), 2) and 3), see [Sun87] p.56-57. If ϕ is tracial, we have:

3) \implies 4): Let $x \in M_+$ be nonzero (hence $x = b^2$ for some $b \in M_+$) and let (x_i) be a net as in 3). Then $(bx_i b)$ is a monotone increasing net with $\sup(bx_i b) = x$. Therefore, there must exist some I such that $bx_I b \neq 0$. We then have $0 < bx_I b \leq x$ and $\phi(bx_I b) = \phi(x_I^{1/2} b^2 x_I^{1/2}) \leq \|b\|^2 \phi(x_I) < \infty$, since ϕ is tracial and $x_I^{1/2} b^2 x_I^{1/2} \leq \|b\|^2 x_I$.

4) \implies 5): Let $p \in \mathcal{P}(M)$ be nonzero. We know from 4) that there exists $y \in M_+$ such that $0 < y \leq p$ and $\phi(y) < \infty$. Using the Borel functional calculus, consider the projections $q_n = \psi_y(\chi_{[1/n, \infty)})$. As $(\chi_{[1/n, \infty)})$ is a monotone increasing net with limit

$\chi_{(0,\infty)}$, we have $\psi_y(\chi_{(0,\infty)}) = \sup q_n$. Since $y = y\psi_y(\chi_{(0,\infty)})$, we then know that there exists N such that q_N is nonzero. But $q_N \leq Ny$, so Prop. 1.5.4 a) yields $\phi(q_N) < \infty$. Combining $q_N \leq Ny$ with $y \leq p$, we obtain $q_N \leq p$, which finishes the proof.

5) \implies 2): Let $e = \sup\{p \in \mathcal{P}(M) \mid \phi(p) < \infty\}$. Suppose that $p = \mathbb{1}_M - e \neq 0$. As p is a nonzero projection, we know from 5) that there exists $q \in \mathcal{P}(M)$ such that $0 < q \leq p$ and $\phi(q) < \infty$. But then $q \leq e$ and $q \leq \mathbb{1}_M - e$, so $q = 0$, which is a contradiction.

Note that the last two implications do not require the fact that ϕ is tracial. \blacksquare

Definition 1.5.6. *Let ϕ be a weight on a von Neumann algebra M . If ϕ satisfies the equivalent conditions of the preceding proposition, we say that it is semifinite.*

With all these tools in hand, we can now look at concrete examples of weights. The easiest example is the motivating concept, the Lebesgue integral. If $M = L^\infty(\mathbb{R}, \lambda)$ with λ the Lebesgue measure, then $\phi(f) = \int f d\lambda$ is a faithful normal semifinite tracial weight (for semifiniteness, see 2. of Def. 1.5.5). Then, $\mathcal{D}_\phi = L^\infty(\mathbb{R}, \lambda)_+ \cap L^1(\mathbb{R}, \lambda)$, $\mathcal{N}_\phi = L^\infty(\mathbb{R}, \lambda) \cap L^2(\mathbb{R}, \lambda)$ and $\mathcal{M}_\phi = L^\infty(\mathbb{R}, \lambda) \cap L^1(\mathbb{R}, \lambda)$. From Prop. 1.5.4, we know that ϕ can be extended uniquely to a linear functional on $L^\infty(\mathbb{R}, \lambda) \cap L^1(\mathbb{R}, \lambda)$; this is simply the Lebesgue integral on integrable functions.

The classical example of weight in the noncommutative case is the trace on the separable Hilbert space, a generalization of the usual trace on matrices. Let $\mathcal{H} = \ell^2(\mathbb{N})$ with the canonical orthonormal basis (ξ_n) and $M = \mathcal{B}(\mathcal{H})$. Then, we define the trace $\text{Tr} : M_+ \rightarrow [0, \infty]$ by $\text{Tr}(T) = \sum_{\mathbb{N}} (T\xi_n | \xi_n)$. This is a faithful normal semifinite tracial weight. We have

- $\mathcal{N}_{\text{Tr}} = \mathcal{B}^2(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{Tr}(T^*T) < \infty\}$, the *Hilbert-Schmidt* operators.
- $\mathcal{M}_{\text{Tr}} = \mathcal{B}^1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{Tr}(|T|) < \infty\}$, the *trace-class* operators.

This example actually leads to many important facts: \mathcal{N}_{Tr} is a Hilbert space with inner product $(S|T) = \text{Tr}(T^*S)$, and \mathcal{M}_{Tr} is a Banach space with norm $\|T\| = \text{Tr}(|T|)$. Furthermore, they are both self-adjoint two-sided ideals of $\mathcal{B}(\mathcal{H})$. We also have $\mathcal{B}_f(\mathcal{H}) \subseteq \mathcal{B}^1(\mathcal{H}) \subseteq \mathcal{B}^2(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$, where $\mathcal{B}_f(\mathcal{H})$ are the finite rank operators and $\mathcal{B}_0(\mathcal{H})$ are the compact operators. Finally, it turns out that $\mathcal{B}^1(\mathcal{H})$ is precisely the predual of $\mathcal{B}(\mathcal{H})$. For a complete treatment of the trace on $\mathcal{B}(\mathcal{H})$, see Sec. 3.4 of [Ped89].

We finish this section by finally proving the following result which was stated in Sec. 1.2.

Proposition 1.5.7. *Let \mathcal{H} be an infinite dimensional separable Hilbert space. Then the weak and ultraweak (resp. strong and ultrastrong) topologies differ on $\mathcal{B}(\mathcal{H})$.*

Proof: We identify \mathcal{H} with $\ell^2(\mathbb{N})$ and denote the canonical orthonormal basis by $(\xi_n)_{n \geq 1}$. The operator T defined by $T\xi_n = (1/n)\xi_n$ is bounded and trace-class, and so by Prop. 4.6.11 of [Ped89], $\phi(S) = \text{Tr}(ST)$ (for $S \in \mathcal{B}(\mathcal{H})$) is an ultraweakly functional on $\mathcal{B}(\mathcal{H})$. Assume that it is also weakly continuous, so that $\phi(S) = \sum (Sx_n | y_n)$ for two finite sequences (x_n) and (y_n) in \mathcal{H} . Then, the proof in [Ped89] of Prop. 4.6.11 (i) \Rightarrow (iii) shows that $\phi(S) = \text{Tr}(SU)$ for a finite rank operator U . It follows that $\text{Tr}(S(T - U)) = 0$ for all $S \in \mathcal{B}(\mathcal{H})$; since $\mathcal{B}^2(\mathcal{H})$ is a Hilbert space, we have $T = U$. As it is clear that T is not of finite rank, this is a contradiction and we conclude that ϕ cannot be weakly continuous. Therefore, the weak and ultraweak topologies do not share the same continuous functionals, so we conclude that they must differ. As the strong and weak (resp. ultrastrong and ultraweak) topologies possess the same set of continuous functionals, we also conclude that the strong and ultrastrong topologies differ. ■

1.6 GNS construction and standard form

We have already alluded to the notion of representations in this work, without giving a clear definition of the concept. We do so here.

We say that a vector η is *cyclic* for a subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$ if $[A\eta]$, the closed subspace generated by elements of the form $x\eta$ for $x \in A$, is equal to \mathcal{H} . We say that η is *separating* for the same subalgebra if $x\eta = 0$ for $x \in A$ implies $x = 0$. These two notions are linked.

Proposition 1.6.1. *Let M be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. A vector $\eta \in \mathcal{H}$ is cyclic for M iff it is separating for M' .*

Proof: Prop. 2.22 of [Bry13]. ■

We are now ready to discuss representations.

Definition 1.6.2. *A representation (π, \mathcal{H}) of a C^* -algebra A on a Hilbert space \mathcal{H} is a $*$ -homomorphism $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$. We say that π is faithful if it is injective, and that π is cyclic if $\pi(A)$ possess a cyclic vector $\xi \in \mathcal{H}$.*

By Prop. 1.1.5, the image of such a representation is always a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$. For a C^* -algebra A , the following result gives a range of representations of A .

Theorem 1.6.3. *(GNS construction) Let A be a unital C^* -algebra and let ϕ be a state on A .*

1. $N_\phi = \{a \in A \mid \phi(a^*a) = 0\}$ is a closed left ideal in A .
2. The sesquilinear form $(a + N_\phi, b + N_\phi) = \phi(b^*a)$ defines an inner product on the vector space A/N_ϕ . We denote the Hilbert space completion of the latter by \mathcal{H}_ϕ .
3. For $a \in A$, the formula $\pi_\phi(a)(b + N_\phi)$ defines by extension a bounded linear operator $\pi_\phi(a)$ on \mathcal{H}_ϕ . This defines a cyclic representation $(\pi_\phi, \mathcal{H}_\phi)$ of A , with cyclic vector $\xi_\phi = \mathbb{1}_A + N_\phi$.

Proof: Thm. 1.12.4 of [Put19]. ■

The triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ is called the Gelfand-Naimark-Segal (GNS) representation of A associated to the state ϕ . If A is a von Neumann algebra, the GNS representation actually gives us more.

Proposition 1.6.4. *Let M be a von Neumann algebra, and ϕ be a faithful normal state on M . Then $\pi_\phi(M)$ is a von Neumann algebra on \mathcal{H} and π_ϕ is a $*$ -isomorphism from M to $\pi_\phi(M)$.*

Proof: Thm. 2.2.1. of [Sun87]. ■

It is in fact through the GNS construction that one proves the fundamental fact that we can always think of a C^* -algebra as a norm closed $*$ -subalgebra of the bounded linear operators on a given Hilbert space.

Theorem 1.6.5. *Every C^* -algebra admits a faithful representation on a Hilbert space.*

Proof: Thm. 1.12.8. of [Put19]. ■

We close this section by presenting the *standard form* of a von Neumann algebra, a result that will be central in our work.

Definition 1.6.6. *Let C be a subset of a complex vector space V . We say that*

- C is a cone if for all $\alpha \geq 0$ and $v \in C$, we have $\alpha v \in C$.
- C is a complex cone if it is a cone and for all $u, v \in C$, we have $u + v \in C$.

If V is a Hilbert space, we define the dual cone of a subset $B \subseteq V$ as $B^* = \{u \in V \mid (u|v) \geq 0 \text{ for all } v \in B\}$. This is always a convex cone. We say that a cone $C \subseteq V$ is a self-dual cone if $C = C^*$.

Theorem 1.6.7. (Standard Form) *Any von Neumann algebra M can be represented faithfully on a Hilbert space \mathcal{H} , such that there exists a conjugate linear isometric involution $J : \mathcal{H} \rightarrow \mathcal{H}$, and a self-dual cone P in \mathcal{H} with the following properties:*

1. $JMJ = M'$

2. $JcJ = c^*, \forall c \in Z(M)$
3. $J\xi = \xi, \forall \xi \in P$
4. $aa^t(P) \subseteq P, \forall a \in M$ where $a^t = JaJ$.

Proof: Thm. 1.6 of [Haa75]. ■

When a representation of a von Neumann algebra satisfies these properties, we say that the algebra is in standard form, or that (M, \mathcal{H}, J, P) is a standard form. This standard form also comes with the following properties.

Theorem 1.6.8. *Let (M, \mathcal{H}, J, P) be a standard form.*

- P is closed and $\text{span}(P) = \mathcal{H}$
- For every $*$ -isomorphism θ of M , there exists a unique unitary $u_\theta \in \mathcal{B}(\mathcal{H})$ such that
 1. $\theta(x) = u_\theta x u_\theta^*, \forall x \in M$.
 2. $J = u_\theta J u_\theta^*$
 3. $u_\theta(P) = P$
- For every positive $\phi \in M_*$, there exists a unique $\xi_\phi \in P$ such that $\phi(x) = (x\xi_\phi | \xi_\phi)$ for all $x \in M$. Therefore, $P = \{\xi_\phi | \phi \in M_*^+\}$. We have that for positive $\phi, \psi \in M_*$,

$$\|\xi_\phi - \xi_\psi\|^2 \leq \|\phi - \psi\| \leq \|\xi_\phi - \xi_\psi\| \|\xi_\phi + \xi_\psi\|$$

Proof: For the first point, see p.311 of [SZ79]. For the second and third points, see Thm. 3.2. and Lemma 2.10 of [Haa75]. ■

Proposition 1.6.9. *If (M, \mathcal{H}, J, P) is a standard form, the weak and ultraweak (resp. strong and ultrastrong) notions of convergence on M are the same. Therefore, the respective topologies also coincide.*

Proof: We already know that ultraweak convergence implies weak convergence, so we need to show the reverse implication. Let (x_α) be a net in M that converges weakly to $x \in M$, and $\phi \in M_*^+$. From the previous theorem, we know that there exists $\xi_\phi \in P$ such that $\phi(y) = (y\xi_\phi | \xi_\phi)$ for all $y \in M$. Therefore, we immediately get that $(\phi(x_\alpha))$ converges to $\phi(x)$. As $\phi \in M_*^+$ was arbitrary and as any ultraweak functional is a linear combination of positive ultraweak functionals, this shows that (x_α) converges ultraweakly to x .

We already know that ultrastrong convergence implies strong convergence, so we need to show the reverse implication. Let (x_α) be a net in M that converges strongly to $x \in M$, and (ξ_n) be any square summable sequence in \mathcal{H} . We then have

$$\sum_{i=1}^{\infty} \|(x - x_\alpha)\xi_n\|^2 = \sum_{i=1}^{\infty} ((x - x_\alpha)^*(x - x_\alpha)\xi_n | \xi_n)$$

The term on the right represents a positive ultraweakly continuous functional applied to $(x - x_\alpha)^*(x - x_\alpha)$, so we know that there exists a vector $\xi \in P$ such that

$$\sum_{i=1}^{\infty} \|(x - x_\alpha)\xi_n\|^2 = ((x - x_\alpha)^*(x - x_\alpha)\xi | \xi) = \|(x - x_\alpha)\xi\|^2 \rightarrow 0$$

where we have applied the fact that (x_α) converges strongly to x . Hence, this convergence is also ultrastrong. \blacksquare

1.7 Completely positive maps

In the category of C^* -algebras, $*$ -homomorphisms are the maps that preserve the structure. However, requiring a map to be as such is very restrictive. For example, a $*$ -homomorphism is necessarily a contraction, and being injective forces it to be an isometry. A slightly weaker type of map is a completely positive map (CP map). Such a map does not preserve the multiplication, but is still very close to a $*$ -homomorphism, as Stinespring's theorem will show.

Recall that an element x in a C^* -algebra A is called positive if $x = y^*y$ for some $y \in A$, and a map $f : A \rightarrow B$ between two C^* -algebras is called positive if it preserves positive elements. For example, any $*$ -homomorphism is positive. A CP map is an intermediate between a positive linear map and a $*$ -homomorphism. Therefore, we first list results about positive maps, as these will apply to CP maps.

Proposition 1.7.1. *If A, B are C^* -algebras and $f : A \rightarrow B$ is a positive linear map, then f is bounded. Furthermore, for any $a \in A$, we have $f(a^*) = f(a)^*$.*

Proof: Prop. 3.7 and 3.9 of [Bry13]. \blacksquare

Let A be a C^* -algebra, $n \in \mathbb{N}$ be fixed and $\mathbb{M}_n(A)$ be the $n \times n$ matrices with entries in A . With the usual matrix addition and multiplication, and the involution given by taking the transpose of the matrix along with the involution of A to each entry, $\mathbb{M}_n(A)$ is a $*$ -algebra. As we know from Sec. 1.6, A necessarily has a faithful representation π on a Hilbert space \mathcal{H} . We can then consider $\mathcal{H}^n = \bigoplus_{k=1}^n \mathcal{H}$; it is simple to check that

$\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$, where we apply the usual matrix multiplication on a vector in \mathcal{H}^n . It turns out, applying π entry-wise gives us an injective $*$ -homomorphism ϕ from $\mathbb{M}_n(A)$ into $\mathcal{B}(\mathcal{H}^n)$. The norm on the latter can then be pulled back on $\mathbb{M}_n(A)$, and this norm makes $\mathbb{M}_n(A)$ a C^* -algebra. We are now ready to define CP maps.

Definition 1.7.2. *Let A, B be C^* -algebras. For $n \in \mathbb{N}$, a linear map $\Phi : A \rightarrow B$ is said to be n -positive if the induced map $\Phi_n : \mathbb{M}_n(A) \rightarrow \mathbb{M}_n(B)$, given by applying Φ to each entry, is positive. Furthermore, Φ is said to be completely positive if it is n -positive for all $n \in \mathbb{N}$.*

Therefore, a CP map is positive, and so it is bounded and preserves the adjoint operation. To further study CP maps, it is useful to develop criteria to evaluate the positivity of the elements and maps of matrix algebras. We will denote an element $a \in \mathbb{M}_n(A)$ by its entries in the form $a = (a_{ij})$.

Proposition 1.7.3. *Let A be a C^* -algebra. An element $a = (a_{ij}) \in \mathbb{M}_n(A)$ is positive iff it is a finite sum of matrices of the form $a_{ij} = a_i^* a_j$ for a_1, \dots, a_n in A . Additionally, $a = (a_{ij})$ is positive iff for all b_1, \dots, b_n in A , $\sum_{i,j} b_i^* a_{ij} b_j$ is positive in A .*

Proof: Lemmas 3.1 and 3.2 of [Tak02]. ■

Putting all this together, we obtain a computational way of checking if a map is completely positive.

Corollary 1.7.4. *Let A, B be C^* -algebras and $\phi : A \rightarrow B$ a linear map. ϕ is n -positive iff for all $x_1, \dots, x_n \in A$ and $y_1, \dots, y_n \in B$, $\sum_{i,j} y_i^* \phi(x_i^* x_j) y_j$ is positive in B .*

Proof: Follows directly from Prop. 1.7.3. ■

Using this criterion, it becomes easy to prove the following fact.

Corollary 1.7.5. *Let A, B be C^* -algebras, with B abelian. Then, any positive linear map $\phi : A \rightarrow B$ is completely positive.*

Proof: Cor. 3.5 of [Tak02]. ■

In particular, this shows that any positive linear functional on a C^* -algebra is a CP map. The next result is very useful.

Proposition 1.7.6. *If A, B are C^* -algebras, A is unital and $\phi : A \rightarrow B$ is a linear map which is 2-positive, then $\|\phi\| = \|\phi(\mathbb{1}_A)\|$.*

Proof: Prop. 3.9 of [Bry13]. ■

We are now ready to state the main theorem of this section, which establishes the link between CP maps and $*$ -homomorphisms.

Theorem 1.7.7. (*Stinespring's Theorem*) Let A be a C^* -algebra, \mathcal{H} be a Hilbert space and $\phi : A \rightarrow \mathcal{B}(\mathcal{H})$ be a linear map. The following are equivalent:

1. ϕ is completely positive.
2. There exists a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$ and a representation $\pi : A \rightarrow \mathcal{K}$ such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$.

If A is unital, π can be chosen to be unital. Furthermore, if A is a von Neumann algebra and ϕ is normal, π can be chosen to be normal.

Proof: Thm. 3.16 of [Bry13]. ■

Conceptually, this theorem is crucial; it shows that a CP map is merely a truncation of a $*$ -homomorphism, and it fully classifies all such maps. Computationally, it can be used as an instrument of proof, as the following example shows.

Corollary 1.7.8. Let A, B be C^* -algebras and $\phi : A \rightarrow B$ be a CP map. For all $a \in A$, we have $\phi(a)^*\phi(a) \leq \|\phi\|\phi(a^*a)$.

Proof: By representing B faithfully on a Hilbert space \mathcal{H} , we can apply the previous theorem and work as if $B \subseteq \mathcal{B}(\mathcal{H})$. Let V and π denote the respective bounded operator and representation from Thm. 1.7.7. We have

$$\begin{aligned} \phi(a)^*\phi(a) &= (V^*\pi(a)V)^*V^*\pi(a)V \\ &= V^*\pi(a^*)VV^*\pi(a)V \\ &\leq V^*\pi(a^*)(\|VV^*\|\mathbb{1}_{\mathcal{K}})\pi(a)V \\ &= \|VV^*\|V^*\pi(a^*a)V \\ &\leq \|\phi\|\phi(a^*a) \end{aligned}$$

where we have used the inequality $\|VV^*\| = \|V^*V\| \leq \|\phi\|$. In the case where A is unital, this inequality follows easily from Prop. 1.7.6 and Thm. 1.7.7. In the case where A is non-unital, we refer back to the notation in the proof of the non-unital case of Stinespring's theorem by [Bry13], at the top of p. 75. Through the unital case, we have $\|V^*V\| = \|\theta \circ \phi^{**}\|$. However, $\|\theta\| = 1$ as it is a $*$ -homomorphism and $\|\phi^{**}\| = \|\phi\|$ as per Prop. 3.13 of the same text, so we obtain $\|V^*V\| \leq \|\phi\|$. ■

To conclude this section, we define the following concept, introduced by Mingo in [Min89], that will be essential in Chap. 3.

Definition 1.7.9. Let M be a von Neumann algebra and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A such that $\sum_n^{UWOT} a_n^*a_n$ converges, or, equivalently, such that the sequence of partial sums is bounded (see Prop. 1.3.3). Then, the map $\Phi : M \rightarrow M$ defined by $\Phi(x) = \sum_n^{UWOT} a_n^*xa_n$ is completely positive (see Cor. 1.7.4). Any map of this form is called an inner CP map.

1.8 Injectivity

As in many mathematical theories, one of the goals in the study of von Neumann algebras would be the classification of all such objects. The classification of the subclass of injective “separable” von Neumann algebras, completed during the latter half of the 20th century, is up to now the most important accomplishment towards this objective. Using reduction theory (see the introduction of the next section), it is enough to classify the injective “separable” factors. The classification of these factors is stated in the next chapter. In this section, we introduce the concept of injectivity and the range of properties equivalent to it.

Definition 1.8.1. *A C^* -algebra A is called injective if it possesses the following property. Whenever B and \tilde{B} are C^* -algebras such that there exists a CP map $\phi : B \rightarrow A$ and $B \subseteq \tilde{B}$, then there exists a CP map $\tilde{\phi} : \tilde{B} \rightarrow A$ that extends ϕ .*

It is easy to check that this property is preserved under isomorphisms, and so this defines an intrinsic algebraic property. We are actually used to working with injective C^* -algebras, as the following example shows.

Proposition 1.8.2. *For any Hilbert space \mathcal{H} , the C^* -algebra $\mathcal{B}(\mathcal{H})$ is injective.*

Proof: This is essentially Arveson’s extension theorem [Arv69]. See Prop. 4.2 of [Bry13] for an efficient proof using Stinespring’s theorem. ■

The beauty of this property is in its functorial nature, which allows to show that it is preserved under multiple operations on the algebras. To name a few, the direct sum of von Neumann algebras is injective iff each summand is, any corner algebra of an injective von Neumann algebra is injective, and the tensor product of two von Neumann algebras is injective iff each algebra is. For the proofs and more examples, see Sec. 4.2 of [Bry13].

In counterpart, while being an extremely stable property, proving that a certain algebra is injective can be difficult. In the case of a von Neumann algebra, injectivity is actually equivalent to a whole range of properties. It is not as obvious that these are stable under the different constructions, because they are either space dependent or formulated in terms of the internal structure of the algebra. However, for these reasons, it is often easier to show that a given algebra satisfies one of these properties.

Therefore, the strength of having this range of equivalent properties is that in practice some of them are much easier to show for a given algebra, and we know through the equivalence with injectivity that they are all stable under multiple constructions. In what follows, we list some of these properties and state properly the equivalence. To introduce the first of these properties, we need the following vocabulary.

Definition 1.8.3.

- Let V, W be vector spaces with $V \subseteq W$. A map $E : W \rightarrow V$ is called a *projection* if it is surjective and $E(v) = v$ for all $v \in V$.
- Let A, B be C^* -algebras such that $B \subseteq A$. A *conditional expectation* is a contractive and completely positive projection $E : A \rightarrow B$ such that $E(b_1 a b_2) = b_1 E(a) b_2$ for all $a \in A$, $b_1, b_2 \in B$.

The following beautiful theorem by Tomiyama shows that a conditional expectation simply needs to be a contractive projection.

Theorem 1.8.4. *Let A, B be C^* -algebras with $B \subseteq A$ and $E : A \rightarrow B$ be a projection. The following are equivalent:*

1. E is a conditional expectation.
2. E is contractive and completely positive.
3. E is contractive.

Proof: Thm. 4.5 of [Bry13]. ■

This allows us to define the second property, introduced by Hakeda and Tomiyama.

Definition 1.8.5. *Let \mathcal{H} be a Hilbert space and $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We say that A has *property E* if there exists a projection $E : \mathcal{B}(\mathcal{H}) \rightarrow M$ of norm 1.*

Obviously, the statement of this first property is inherently in terms of the space on which the algebra is represented. We now introduce the final two properties, which explicitly deal with the internal structure of the algebra. The first was introduced by Effros and Lance, while the second one is an idea that was already present in the pioneering work of Murray and von Neumann.

Definition 1.8.6.

- A von Neumann algebra M is *semidiscrete* if the identity mapping $id_M : M \rightarrow M$ can be approximated pointwise ultraweakly by normal, completely positive maps of finite rank fixing $\mathbb{1}_M$. In other words, if there is a net (ϕ_α) of normal, complete positive maps of finite rank such that, for all α , $\phi_\alpha(\mathbb{1}_M) = \mathbb{1}_M$ and $\phi_\alpha(x) \rightarrow x$ ultraweakly for all $x \in M$.
- A von Neumann algebra M is *approximately finite dimensional (AFD)* if for any $x_1, \dots, x_n \in M$ and any $*$ -ultrastrong neighbourhood V of 0 in A , we can find a finite dimensional $*$ -subalgebra N of M and $y_1, \dots, y_n \in N$ such that $x_i - y_i \in V$. Note that the term “hyperfinite” is sometimes used instead of AFD, but some authors reserve it to describe finite AFD algebras.

The $*$ -ultrastrong topology is induced by the family of seminorms

$$x \mapsto \left(\sum_{n=1}^{\infty} \|x\xi_n\|^2 + \|x^*\xi_n\|^2 \right)^{1/2}$$

as in Sec. 1.2, and also satisfies the bicommutant theorem. This definition focuses on the local aspect of hyperfiniteness; on the other hand, the following formulation focuses on the global aspect of this property.

Proposition 1.8.7. *Let M be a von Neumann algebra. If there exists a net of finite dimensional $*$ -subalgebras (M_α) , directed under inclusion, such that $M = \overline{\cup_\alpha M_\alpha}^{UWOT}$ (closure in the ultraweak topology), then M is AFD.*

We see that in general, the global property implies the local one. We note that in the case of “separable” algebras von Neumann algebras, these two definitions are equivalent.

As we already announced, these properties are unified by the following paramount theorem.

Theorem 1.8.8. *Let M be a von Neumann algebra. The following are equivalent:*

- a) M is injective.
- b) M is semidiscrete.
- c) M is AFD.
- d) If $M \subseteq \mathcal{B}(\mathcal{H})$, M satisfies the property E.

Proof: For a) \iff b), see Thm. 5.30 in [Bry13]. For a) \iff d), see Cor. 4.3 in [Bry13]. For a) \iff c), see [Ell78]. ■

Historically, this theorem was the culmination of the work of many authors. It is important to note that the crucial step, first achieved by Connes in his proof of the uniqueness of the injective “separable” type II_1 factor, was to show that an injective “separable” von Neumann factor is AFD [Con76]. This completed the chain of equivalence of these properties. See Chap. 5.7 of [Con94] for a more detailed exposition of the historical context behind this theorem.

1.9 von Neumann Factors

Factors were defined in Sec. 1.1, and will be the central concept of this work. Factors gain their importance through the reduction theory of von Neumann algebras, developed by von Neumann himself, which we outline here (based on 5.1.β of [Con94]). For a separable Hilbert space \mathcal{H} , the space \mathcal{F} of all factors in $\mathcal{B}(\mathcal{H})$ can be given a Borel structure. It then makes sense to talk about a Borel function $t \mapsto M(t)$ from a standard Borel space X to \mathcal{F} . If X is in addition equipped with a probability measure μ , we can define the von Neumann algebra $\int_X M(t)d\mu$, called the *direct integral* of the family $(M(t))_{t \in X}$ with respect to the measure μ , which acts on the Hilbert space $L^2(X, \mu, \mathcal{H})$. Intuitively, this von Neumann algebra can be thought of as a “continuous” direct sum of the factors $M(t)$. This leads to the central theorem in reduction theory.

Theorem 1.9.1. *Let M be a von Neumann algebra acting on a separable Hilbert space. Then M is isomorphic to a direct integral of factors $\int_X M(t)d\mu$.*

This decomposition finds its usefulness in the fact that some properties of the algebra M are linked to those of the factors in the family $(M(t))_{t \in X}$. In many cases, the study and classification of “separable” von Neumann algebras can then be reduced to the study of factors. Factors can therefore be thought of as the “building blocks” of this class of algebras.

For example, for a von Neumann algebra acting on a separable Hilbert space, M is injective if and only if $M(t)$ is injective almost everywhere (see Prop. IV.2.2.12 of [Bla06]). Classifying “separable” injective factors is then sufficient to characterize the class of “separable” injective von Neumann algebras. We state this classification in Sec. 2.2.5.

In this section, we introduce the fundamental tools to study and classify factors, and we construct examples of type I and II factors, while leaving the examples of type III factors for the next chapter.

The first tool to classify factors is the *dimension function* of Murray and von Neumann.

Theorem 1.9.2. *Let M be a factor. There exists a function $D : \mathcal{P}(M) \rightarrow [0, \infty]$ such that, for any $p, q \in \mathcal{P}(M)$,*

1. $[p] = [q] \iff D(p) = D(q)$
2. $pq = 0 \implies D(p + q) = D(p) + D(q)$
3. p is finite $\iff D(p) < \infty$

where the operations are in the extended positive real numbers. This function is uniquely determined up to a positive constant multiple.

Proof: P. 27-32 of [Sun87]. ■

This dimension function possesses the following measure-like properties.

Proposition 1.9.3. *Let M be a factor and D be a dimension function. If $p, q \in \mathcal{P}(M)$, we have*

1. $[p] < [q] \iff D(p) < D(q)$
2. If (p_n) is a sequence of mutually orthogonal projections, $D(\sum_{SOT} p_n) = \sum D(p_n)$

Proof: Prop. 1.3.11 of [Sun87]. ■

The importance of the dimension function becomes clear when one analyses the possible sets that it can have as its range. Indeed, without even having an explicit factor at hand, one can show that the possible ranges of D can be listed neatly. This gives us the type classification of factors.

Theorem 1.9.4. *Let M be a factor and D be a dimension function. Then, M can be classified according to a type, which is determined by $\text{Im}(D)$ as follows:*

1. Type I_n , with $n \in \mathbb{N}^*$, if $\text{Im}(D) = \{0, \alpha, 2\alpha, \dots, n\alpha\}$ for $\alpha \in]0, \infty[$
2. Type I_∞ , if $\text{Im}(D) = \{0, \alpha, 2\alpha, \dots, \infty\}$ for $\alpha \in]0, \infty[$
3. Type II_1 , if $\text{Im}(D) = [0, \alpha]$ for $\alpha \in]0, \infty[$
4. Type II_∞ , if $\text{Im}(D) = [0, \infty]$
5. Type III , if $\text{Im}(D) = \{0, \infty\}$

Proof: Prop. 1.3.14 of [Sun87]. ■

Before covering examples of each type of factor, we must first examine the following important tools.

Proposition 1.9.5. *Let M be a von Neumann factor.*

- (a) *If τ is a faithful trace on M , then $D = \tau|_{\mathcal{P}(M)}$ is a dimension function for M .*
- (b) *If τ is a faithful semifinite tracial weight on M , then $D = \tau|_{\mathcal{P}(M)}$ is a dimension function for M .*

Proof: We first note that, in Thm. 1.9.2, 1), 2) and 3)(\Rightarrow) already imply 3)(\Leftarrow). Indeed, only assuming the restricted set of assumption, we can show the converse to 3)(\Leftarrow). Let $p \in \mathcal{P}(M)$ be infinite. Then, by Cor. 1.2.4 of [Sun87], we can decompose p as $p = q + (p - q)$, where $[p] = [q] = [p - q]$, so $D(p) = D(q) + D(p - q) = 2D(p)$. Therefore, if $D(p) < \infty$, we have $D(p) = 0$ and, by 1), $[p] = [0]$. But this is a contradiction since 0 is a finite projection; hence, $D(p) = \infty$.

(a) When τ is a trace, it is a positive linear functional, so we already know that $D = \tau|_{\mathcal{P}(M)}$ has the right codomain. Note that it only takes finite values. It remains to show that D satisfies 1), 2) and 3)(\Rightarrow) from Thm. 1.9.2.

1) Let $p, q \in \mathcal{P}(M)$. If $[p] = [q]$, we know that there exists a partial isometry $u \in M$ such that $p = uu^*$ and $q = u^*u$. Therefore, $D(p) = D(q)$ through the trace property of τ . On the converse, if $D(p) = D(q)$, we know from Prop. 1.4.10 that we can assume $[p] \leq [q]$. Let $p \sim e \leq q$. We then have

$$\begin{aligned} 0 &= D(q) - D(p) = D(q) - D(e) = \tau(q - e) \\ &\implies q - e = 0 \implies q = e \implies q \sim p \implies [q] = [p] \end{aligned}$$

where we have use the fact that $q - e \geq 0$ and that τ is faithful.

- 2) This is automatically satisfied by the linearity of τ .
3)(\Rightarrow) is trivial as τ only takes finite values.

(b) In the case where τ is a faithful semifinite tracial weight, we must slightly adapt the preceding proof, as τ may now take infinite values.

1)(\Rightarrow) The proof is the same as above for $[p] = [q] \implies D(p) = D(q)$.

3)(\Rightarrow) Assume that $p \in \mathcal{P}(M)$ is finite; as τ is semifinite, we know from Def. 1.5.5 that there exists $p_0 \in \mathcal{P}(M)$ such that $0 < p_0 \leq p$ and $\tau(p_0) < \infty$. By Prop. 1.4.13, there exists a family of mutually orthogonal projections $(p_\alpha)_{\alpha \in A}$ such that $p_\alpha \sim p_0$ for each α , along with a projection $q \leq p$, st. $p = \sum_A^{SOT} p_\alpha + q$. By Prop. 1.4.9, we know that if A is infinite, $\sum_A^{SOT} p_\alpha$ is an infinite projection and in turn p is then infinite through Prop. 1.4.12. Therefore, A must be finite and so $D(p) = \sum_A D(p_\alpha) + D(q) \leq (|A| + 1)D(p_0) < \infty$.

1)(\Leftarrow) The proof also remains the same for $D(p) = D(q) < \infty$. It must however be changed in the infinite case, as the subtraction is then not defined. But if $D(p) = D(q) = \infty$, we know from 3)(\Rightarrow) that p and q are infinite, and so $[p] = [q]$ by Prop. 1.4.12.

- 2) This is automatically satisfied by the linearity of τ . ■

This result is extremely useful in practice: if we can find a faithful trace or a semifinite tracial weight on a factor M , we can simply study the image of its restriction to projections to identify the type of factor we are dealing with. The examples that

follow will illustrate this. Before covering those however, we state a crucial result about finite factors.

Corollary 1.9.6. *A von Neumann factor M admits a normal faithful trace τ iff $\mathbb{1}_M$ is finite, i.e. if M is of type I_n or II_1 . Moreover, if we fix $\tau(\mathbb{1}_M) = 1$, this trace is unique.*

Proof: If M admits a normal faithful trace, $\mathbb{1}_M$ is finite by the previous proposition. For the existence of the trace when M is finite, see Thm. 2.4 of [Tak02].

If τ and $\tilde{\tau}$ are two faithful traces on M normalized as above, $D = \tau|_{\mathcal{P}(M)}$ and $\tilde{D} = \tilde{\tau}|_{\mathcal{P}(M)}$ are both dimension functions for M that take the same value at $\mathbb{1}_M$. Hence, by Thm. 1.9.2, we have $\tilde{\tau}|_{\mathcal{P}(M)} = \tau|_{\mathcal{P}(M)}$. By Prop. 1.4.2, the linear span of $\mathcal{P}(M)$ is norm dense in M ; hence, since both traces are bounded linear functionals, we must have $\tilde{\tau} = \tau$ on M . ■

In what follows, we will give examples of factors for each types I and II. Type III factors will be explored in details in Chapter 2.

Type I: It is a simple exercise to check that for a factor of the form $\mathcal{B}(\mathcal{H})$, the function $D(p) = \dim(\text{Im}(p))$ is a dimension function, and therefore $\mathcal{B}(\mathcal{H})$ is of type I, with the index being equal to the dimension of \mathcal{H} . These actually exhaust all the examples of type I.

Theorem 1.9.7. *Let M be a factor. If M is of type I_n , then $M \cong \mathbb{M}_n(\mathbb{C})$. If M is of type I_∞ , then $M \cong \mathcal{B}(\mathcal{H})$, for \mathcal{H} an infinite Hilbert space.*

Proof: Cor. 1.28 of [Tak02]. ■

Factors of type II and III are more exotic, in the sense that they are fundamentally different from both type I categories. Indeed, we see that in the case of type III, all the nonzero projections are infinite, while type II factors possess a continuous range of finite projections. This means that, if M is a factor of type II and $p \in \mathcal{P}(M)$ is a finite projection, we can always find $q \in \mathcal{P}(M)$ st. $0 \neq q < p$. Since this process can be repeated an infinite numbers of times, we note that this also implies that if $M \subset \mathcal{B}(\mathcal{H})$, the finite nonzero projections of M are infinite relative to $\mathcal{B}(\mathcal{H})$. As could be expected, constructing examples of type II and III factors requires more work than for factors of type I.

Type II₁: To give examples of factors of type II_1 , we will introduce a few notions, starting with the concept of an ICC group, i.e. a group that satisfies the *infinite conjugacy class property*. A group has the ICC property if all of its conjugacy classes, except the trivial one, are infinite. One can check that S_∞ , the group of finite permutations on \mathbb{N} , and \mathbb{F}_2 , the free group on two generators, are ICC groups.

Additionally, we introduce the *group von Neumann algebra* of a discrete countable group G . Let G be a discrete countable group; the space of square summable complex

sequences on G , denoted $\ell^2(G)$, is a Hilbert space under the usual component-wise operations. This space has a canonical orthonormal basis, i.e. the set $\{\xi_g \mid g \in G\}$ where $\xi_g(h) = \delta_{gh}$. The map $\lambda : G \rightarrow \mathcal{U}(\mathcal{H})$, defined on the basis by $(\lambda(k))(\xi_g) \equiv \lambda_k(\xi_g) = \xi_{kg}$, is a unitary representation of G on $\ell^2(G)$, called the *left regular representation of G* . Therefore, the set $\{\lambda_g \mid g \in G\}$ is $*$ -invariant, and it follows from Sec. 1.1 that $vN(G) = \{\lambda_g \mid g \in G\}''$ is a von Neumann algebra,

Definition 1.9.8. *The von Neumann algebra $vN(G)$ is called the von Neumann algebra of the group G .*

A clever application of the bicommutant theorem also shows that $vN(G)$ is the closure in any of the four topologies introduced in Sec. 1.2 of the set of finite linear combinations of elements of $\{\lambda_g \mid g \in G\}$. The representation λ has a counterpart, namely the right regular representation $\rho : G \rightarrow \mathcal{U}(\mathcal{H})$ defined by $(\rho(k))(\xi_g) \equiv \rho_k(\xi_g) = \xi_{gk^{-1}}$. It is easy to check that $\{\rho_g \mid g \in G\} \subseteq vN(G)'$.

We are now ready to show that, for a countable discrete ICC group G , $M = vN(G)$ is a type II_1 factor. Let ε denote the identity element of G ; it is immediate that ξ_ε is cyclic for both M and its commutant, and so it is also separating for M . Let $T \in M \cap M'$ and let $c_g = (T\xi_\varepsilon | \xi_g)$. A simple calculation shows that for any $h, g \in G$, we have $c_{hgh^{-1}} = c_g$. Let $h \neq \varepsilon$ and \mathcal{C}_h denote its conjugacy class. Since

$$|c_h|^2 \sum_{g \in \mathcal{C}_h} 1 = \sum_{g \in \mathcal{C}_h} |c_g|^2 \leq \sum_{g \in G} |c_g|^2 = \|T\xi_\varepsilon\|^2 < \infty$$

and since \mathcal{C} is infinite, we conclude that $c_h = 0$. As h was arbitrary, we must have $T\xi_\varepsilon = c_\varepsilon \xi_\varepsilon$, so $(T - c_\varepsilon \mathbb{1}_M)\xi_\varepsilon = 0$. As ξ_ε is separating for M , we must have $T = c_\varepsilon \mathbb{1}_M$, so M is a factor.

To show that it is of type II_1 , we define $\tau : M \rightarrow \mathbb{C}$ by $\tau(x) = (x\xi_\varepsilon | \xi_\varepsilon)$. It is straightforward to show that this is a faithful trace. Therefore, by Prop. 1.9.5, $D = \tau|_{\mathcal{P}(M)}$ is a dimension function. Since $D(\mathbb{1}_M) = 1$, $\text{Im}(D) \subseteq [0, 1]$ and we know from Thm. 1.9.4 that M is either of type I_n or II_1 . However, the set $\{\lambda_g \mid g \in G\} \subseteq M$ is linearly independent and infinite. As we have seen that a factor of type I_n is necessarily isomorphic to $\mathbb{M}_n(\mathbb{C})$, we conclude that M is of type II_1 .

Therefore, $vN(S_\infty)$ and $vN(\mathbb{F}_2)$ are both examples of factors of type II_1 . Of course, the next question to ask is whether these two factors are isomorphic. As each ‘‘separable’’ factor of type I_n was unique, we could expect it to be true; it is however not so, as the following property will show.

Definition 1.9.9. *A type II_1 factor M is said to have property Γ if for all $x_1, \dots, x_n \in M$ and $\epsilon > 0$, there exists a unitary $u \in M$ such that $\tau(u) = 0$ and $\|x_j u - u x_j\|_2 \leq \epsilon$ (see Def. 1.3.2) for all j .*

By the uniqueness of the trace, it is clear that property Γ is an invariant of the algebra (i.e. it is preserved under $*$ -isomorphisms).

Proposition 1.9.10. *The factor $vN(S_\infty)$ has the property Γ while $vN(\mathbb{F}_2)$ does not. Therefore, these two factors are non isomorphic.*

Proof: $\mathbf{M} = \mathbf{vN}(S_\infty)$: Let $x_1, \dots, x_n \in M$ and $\epsilon > 0$ be arbitrary, and set $M_0 = \text{Span}\{\lambda_g \mid g \in S_\infty\}$. We already know that M_0 is strongly dense in M , and since $\|x\|_2 = \tau(x^*x)^{1/2} = (x\xi_\epsilon|x\xi_\epsilon)^{1/2} = \|x\xi_\epsilon\|$, it follows that for $1 \leq j \leq n$ we can find $y_j \in M_0$ such that $\|x_j - y_j\|_2 < \epsilon/2$. Each y_j is a finite linear combination of elements of the type λ_g for $g \in S_\infty$. Therefore, there exists a finite set $\mathcal{S} \subseteq S_\infty$ such that $y_1, \dots, y_n \in \text{Span}\{\lambda_g \mid g \in \mathcal{S}\}$. As this set is finite and each of its elements are finite permutations of \mathbb{N} , we know that there exists $N \in \mathbb{N}$ such that $g(n) = n$ for all $g \in \mathcal{S}$ and $n \geq N$. Then, if h is the permutation that exchanges N and $N+1$ while leaving the rest of \mathbb{N} unchanged, it is clear that the unitary $u = \lambda_h$ commutes with each y_j . For any j , we then have

$$\begin{aligned} \|x_j u - u x_j\|_2 &= \|(x_j - y_j)u + y_j u - u(x_j - y_j) - u y_j\|_2 \\ &\leq \|(x_j - y_j)u\|_2 + \|y_j u - u y_j\|_2 + \|u(x_j - y_j)\|_2 \\ &= 2\|(x_j - y_j)\|_2 < \epsilon \end{aligned}$$

As it is clear that $\tau(u) = 0$, we have that $vN(S_\infty)$ satisfies the property Γ .

$\mathbf{M} = \mathbf{vN}(\mathbb{F}_2)$: Assume that M has the property Γ ; we will arrive at a contradiction. Let a, b denote the generators of \mathbb{F}_2 . Let $x_1 = \lambda_a$, $x_2 = \lambda_b$, $\epsilon = 1/20$ and $u \in vN(\mathbb{F}_2)$ be a unitary that fulfils the property Γ for this set of conditions. By Prop. V.7.6 of [Tak02], we know that $u = \sum_{g \in \mathbb{F}_2}^{USOT} \alpha_g \lambda_g$ for some sequence of complex numbers $(\alpha_g)_{g \in G}$. Note that

$$0 = \tau(u) = (u\xi_\epsilon|\xi_\epsilon) = \sum_{g \in \mathbb{F}_2} \alpha_g (\lambda_g \xi_\epsilon | \xi_\epsilon) = \alpha_\epsilon$$

and

$$1 = \|u\xi_\epsilon\|^2 = (u\xi_\epsilon|u\xi_\epsilon) = \sum_{h \in \mathbb{F}_2} \sum_{g \in \mathbb{F}_2} \alpha_g^* \alpha_h (\lambda_h \xi_\epsilon | \lambda_g \xi_\epsilon) = \sum_{g \in \mathbb{F}_2} |\alpha_g|^2$$

For $h \in \mathbb{F}_2$ and $U \subseteq \mathbb{F}_2$ arbitrary we then have

$$\begin{aligned} \|u - \lambda_h^* u \lambda_h\|_2 &= \|(u - \lambda_h^* u \lambda_h)\xi_\epsilon\| = \left\| \sum_{g \in \mathbb{F}_2} (\alpha_g \lambda_g - \alpha_g \lambda_h^* \lambda_g \lambda_h) \xi_\epsilon \right\| \\ &= \left\| \sum_{k \in \mathbb{F}_2} (\alpha_k - \alpha_{hkh^{-1}}) \lambda_k \xi_\epsilon \right\| = \left(\sum_{k \in \mathbb{F}_2} |\alpha_k - \alpha_{hkh^{-1}}|^2 \right)^{1/2} \end{aligned}$$

We now define a finite measure ν on \mathbb{F}_2 by $\nu(S) = \sum_{k \in S} |\alpha_k|^2$. We have

$$|\nu(S) - \nu(h^{-1}Sh)| = (\nu(S)^{1/2} + \nu(h^{-1}Sh)^{1/2}) \cdot |\nu(S)^{1/2} - \nu(h^{-1}Sh)^{1/2}|$$

$$\begin{aligned}
&\leq |\nu(\mathbb{F}_2)^{1/2} + \nu(\mathbb{F}_2)^{1/2}| \cdot |\nu(S)^{1/2} - \nu(h^{-1}Sh)^{1/2}| \leq 2|\nu(S)^{1/2} - \nu(h^{-1}Sh)^{1/2}| \\
&= 2 \left| \left(\sum_{k \in S} |\alpha_{k, \mathfrak{B}}|^2 \right)^{1/2} - \left(\sum_{k \in S} |\alpha_{hkh^{-1}}|^2 \right)^{1/2} \right| \\
&\leq 2 \left(\sum_{k \in S} |\alpha_k - \alpha_{hkh^{-1}}|^2 \right)^{1/2} \leq 2 \left(\sum_{k \in \mathbb{F}_2} |\alpha_k - \alpha_{hkh^{-1}}|^2 \right)^{1/2} \leq 2\epsilon
\end{aligned}$$

Now let $F \subseteq \mathbb{F}_2$ be the subset composed of words that start either with a or a^{-1} in their irreducible form. We have that $F \cup (aFa^{-1}) = \mathbb{F}_2 \setminus \{\varepsilon\}$ and that the three sets F , bFb^{-1} and $b^{-1}Fb$ are all disjoint. This gives us respectively

$$\begin{aligned}
\nu(F) + \nu(aFa^{-1}) &\geq \nu(\mathbb{F}_2) - \nu(\{\varepsilon\}) \geq 1 \quad (I) \\
\nu(F) + \nu(bFb^{-1}) + \nu(b^{-1}Fb) &\leq \nu(\mathbb{F}_2) \leq 1 \quad (II)
\end{aligned}$$

But applying the inequality $|\nu(S) - \nu(h^{-1}Sh)| \leq 2\epsilon$ gives us

$$\begin{aligned}
\nu(aFa^{-1}) - \nu(F) &\leq 2\epsilon \quad (III) \\
\nu(F) - \nu(bFb^{-1}) &\leq 2\epsilon \quad (IV) \\
\nu(F) - \nu(b^{-1}Fb) &\leq 2\epsilon \quad (V).
\end{aligned}$$

Subtracting (I) – (III) yields

$$2\nu(F) \geq 1 - 2\epsilon \quad (VI)$$

and combining (II), (IV), (V) gives

$$\nu(F) + (\nu(F) - 2\epsilon) + (\nu(F) - 2\epsilon) \leq 1 \implies \nu(F) \leq (1/3)(1 + 4\epsilon) \quad (VII).$$

Finally, combining (VI) and (VII) yields

$$(1/2)(1 - 2\epsilon) \leq (1/3)(1 + 4\epsilon).$$

We finally get $(1/14) \leq \epsilon$, which is a contradiction because $\epsilon = 1/20$. Hence, $vN(\mathbb{F}_2)$ does not have the property Γ . \blacksquare

Therefore, $vN(S_\infty)$ and $vN(\mathbb{F}_2)$ are non-isomorphic, so we have constructed two different examples of type II_1 factors. Note that $vN(S_\infty) = \overline{\cup_n \text{span}\{\lambda_g \mid g \in S_n\}}^{UWOT}$ with the S_n seen as subsets of S_∞ ; this implies that $vN(S_\infty)$ is AFD, or equivalently that it is an injective von Neumann factor. Furthermore, we will see in Sec. 2.2.5 that $vN(\mathbb{F}_2)$ cannot therefore be injective, as the injective II_1 factor is unique up to isomorphism.

Type II_∞ : We carry on our task to list examples of all types of factors by constructing factors of type II_∞ . For a Hilbert space \mathcal{H} , we let $\tilde{\mathcal{H}} = \bigoplus_{\mathbb{N}} \mathcal{H}_n$, where $\mathcal{H}_n = \mathcal{H}$ for all n . For a vector $\eta \in \tilde{\mathcal{H}}$, we will write $\eta(j)$ as the j -th component of η . For a vector $\xi \in \mathcal{H}$, we will denote by $\xi^{(j)}$ the vector in $\tilde{\mathcal{H}}$ which satisfies $\xi^{(j)}(i) = \delta_{ij}\xi$. The following proposition shows that we can think of operators in $\mathcal{B}(\tilde{\mathcal{H}})$ as infinite matrices with entries in $\mathcal{B}(\mathcal{H})$ and obeying the usual matrix operations.

Proposition 1.9.11. *For an operator $x \in \mathcal{B}(\tilde{\mathcal{H}})$, the following is true.*

- For each $i, j \in \mathbb{N}$, there exists a unique operator $x(i, j) \in \mathcal{B}(\mathcal{H})$ such that for all $\xi, \eta \in \mathcal{H}$, we have $(x\xi^{(j)}|\eta^{(i)}) = (x(i, j)\xi|\eta)$.
- For $y \in \mathcal{B}(\tilde{\mathcal{H}})$, $x = y$ iff $x(i, j) = y(i, j)$ for all $i, j \in \mathbb{N}$.
- For $\xi \in \tilde{\mathcal{H}}$ and $j \in \mathbb{N}$, $(x\xi)(j) = \sum_i x(j, i)\xi(i)$, where the series converges in norm in \mathcal{H} .
- For $i, j \in \mathbb{N}$, $x^*(i, j) = (x(j, i))^*$.
- For $y \in \mathcal{B}(\tilde{\mathcal{H}})$ and $i, j \in \mathbb{N}$, $(xy)(i, j) = \sum_k^{USOT} x(i, k)y(k, j)$.

Now if $M \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, we define $\tilde{M} = \{x \in \mathcal{B}(\tilde{\mathcal{H}}) \mid x(i, j) \in M, \forall i, j \in \mathbb{N}\}$. With the results of the previous proposition, it is simple to check that $\tilde{M}' = \{x \in \mathcal{B}(\tilde{\mathcal{H}}) \mid x(i, j) = \delta_{ij}x'$ for some $x' \in M', \forall i, j \in \mathbb{N}\}$ and that \tilde{M} is a von Neumann algebra through the bicommutant theorem. Furthermore, this shows that \tilde{M} is a factor if M is. In fact, it is not too complicated to check that $\tilde{M} \cong M \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$.

Now let G be a countable discrete ICC group, $\mathcal{H} = \ell^2(G)$ and $M = vN(G)$. We will use the same notation as defined in the type II_1 example. We claim that \tilde{M} is a type II_∞ factor. To prove so, we define the faithful semifinite tracial weight $\tilde{\tau} : \tilde{M}_+ \rightarrow [0, \infty]$ by $\tilde{\tau}(x) = \sum_n (x(n, n)\xi_\varepsilon|\xi_\varepsilon)$. Note that this is well defined as if x is positive, $x(n, n)$ is positive for all n . To prove that it is faithful, let $x = b^*b$ for $b \in \tilde{M}$ be such that $\tilde{\tau}(x) = 0$. We then have

$$0 = \sum_n (x(n, n)\xi_\varepsilon|\xi_\varepsilon) = \sum_n \left(\left(\sum_k^{USOT} (b(k, n))^*b(k, n) \right) \xi_\varepsilon \middle| \xi_\varepsilon \right) = \sum_n \sum_k \|b(k, n)\xi_\varepsilon\|^2$$

As ε is separating for M , this imposes $b(k, n) = 0$ for all $k, n \in \mathbb{N}$ and so $x = 0$. Using similar tricks, one shows that $\tilde{\tau}$ inherits traciality from the trace $\tau(x) = (x\xi_\varepsilon|\xi_\varepsilon)$ on M . Finally, we can infer that $\tilde{\tau}$ is semifinite through the existence of the family of projections $\{\mathbb{1}_M^{(j)} \mid j \in \mathbb{N}\}$ which implies 2) of Def. 1.5.5. Therefore, we know from Prop. 1.9.5 that $\tilde{\tau}$ induces a dimension function D on $\mathcal{P}(\tilde{M})$.

We first note that $D(\mathbb{1}_{\tilde{M}}) = \infty$. In addition, as M is a factor of type II_1 with $\tau(\mathbb{1}_M) = 1$, we know that for all $\alpha \in [0, 1]$, there exists $e_\alpha \in \mathcal{P}(M)$ with $\tau(e_\alpha) = \alpha$.

As $e_\alpha^{(1)} \in \mathcal{P}(\tilde{M})$ and $\tilde{\tau}(e_\alpha^{(1)}) = \alpha$, we have that $[0, 1] \subseteq \text{Im}(D)$. Combining these two facts, we deduce from Thm. 1.9.4 that \tilde{M} is of type II_∞ .

As mentioned before, we have $\tilde{M} \cong M \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$. Therefore, this shows that $vN(S_\infty) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$ and $vN(\mathbb{F}_2) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$ are two examples of factors of type II_∞ . Since a tensor product of von Neumann algebras is injective if and only if each summand is (see Prop. 4.12 of [Bry13]), we know that $vN(S_\infty) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$ is injective while $vN(\mathbb{F}_2) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$ is not. Hence, these are two non isomorphic examples of type II_∞ factors.

Actually, this way of constructing a type II_∞ factor represents the general case. Indeed, all type II_∞ factors arise from a type II_1 in this way.

Proposition 1.9.12. *If N is a type II_∞ factor acting on a separable Hilbert space \mathcal{K} , there exists a type II_1 factor M acting on a Hilbert space \mathcal{H} and a unitary $u : \tilde{\mathcal{H}} \rightarrow \mathcal{K}$ such that $u\tilde{M}u^* = N$. Therefore $N \cong \tilde{M}$.*

Proof: Let $p \in N$ be a nonzero finite projection. By Prop. 1.4.13, we know that there exists a family of mutually orthogonal projections $(p_n)_{n \in \mathbb{N}} \subset N$ such that $\mathbb{1}_N = \sum_n^{USOT} p_n$ and $p_n \sim p$ for all n (note: the summation index has to be infinite, since $\mathbb{1}_N$ is infinite and p is finite, and has to be countable since \mathcal{K} is separable). Let $u_n \in N$ be such that $u_n^* u_n = p$ and $u_n u_n^* = p_n$. We let $\mathcal{H} = p\mathcal{K}$ and $M = pNp$. By Prop. 1.4.5, M is a factor acting on \mathcal{H} . Since $\mathbb{1}_M = p$ is finite in N , it must also be finite in M . Furthermore, since N is a type II_∞ factor, we know that we can find a strictly decreasing infinite sequence of projections that are smaller than p (which implies that they lie in M). Combining these two facts, we know that M is a type II_1 factor.

If we now regard u_n as an operator from $p\mathcal{K} = \mathcal{H}$ to \mathcal{K} and u_n^* as an operator from \mathcal{K} to \mathcal{H} , we can define $u : \tilde{\mathcal{H}} \rightarrow \mathcal{K}$ by

$$u\xi = \sum_n u_n \xi(n).$$

This is then a unitary with inverse $u^* : \mathcal{K} \rightarrow \tilde{\mathcal{H}}$ defined by

$$(u^*\eta)(n) = u_n^* \eta.$$

It is then easy to check that $\tilde{M} = u^* N u$. ■

1.10 Discrete crossed products

We start off this section by a definition.

Definition 1.10.1. *An action α of a countable discrete group G on a von Neumann algebra M is a homomorphism*

$$\begin{aligned}\alpha : G &\rightarrow \text{Aut}(M) \\ g &\mapsto \alpha_g\end{aligned}$$

Throughout this section, M will be a von Neumann algebra acting on a Hilbert \mathcal{H} , and G will be a countable group acting on M . We denote the unit of G by ε . Recall the construction of $\tilde{\mathcal{H}}$ and the corresponding matrix element representation from the examples of type II_∞ factors in Sec. 1.9. Let $\tilde{\mathcal{H}}$ be the analogous construction where the indexing set has been replaced by G instead of \mathbb{N} . We define the *crossed product of M by G* as

$$M \rtimes_\alpha G = \{x \in \mathcal{B}(\tilde{\mathcal{H}}) \mid x(g, h) \in M \text{ and } \alpha_{h^{-1}}(x(gh^{-1}, \varepsilon)) \quad \forall g, h \in G\}$$

One can check that $M \rtimes_\alpha G$ is an ultraweakly closed $*$ -subalgebra of $\mathcal{B}(\tilde{\mathcal{H}})$ containing the identity, and so it is a von Neumann algebra. We define the representation $\pi : M \rightarrow M \rtimes_\alpha G$ as $\pi(x)(g, h) = \delta_{gh} \alpha_{h^{-1}}(x)$. This is a normal isomorphism, and so $\pi(M)$ is a von Neumann subalgebra of $M \rtimes_\alpha G$. We also define the unitary representation $\lambda : G \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$ by $\lambda_k(g, h) = \delta_{g, kh}$. One can check that $\lambda(G) \subseteq M \rtimes_\alpha G$ and that $\lambda_k \pi(x) \lambda_k^* = \pi(\alpha_k(x))$ for all $k \in G$ and $x \in M$.

Let us now see when the crossed product will be a factor.

Definition 1.10.2.

- α is said to be *free* if for all $t \neq \varepsilon$, the automorphism α_t is free, i.e. $xy = \alpha_t(y)x$ for all $y \in M$ implies that $x = 0$.
- α is said to be *ergodic* if its fixed point algebra is trivial, i.e. $M^\alpha = \{x \in M \mid \alpha_g(x) = x \quad \forall g \in G\} = \mathbb{C}\mathbb{1}_M$.

Note that we always have $Z(M) \subseteq M^\alpha$. Therefore, by restriction of α , we can consider the action $\alpha|_{Z(M)}$ of G on $Z(M)$.

Proposition 1.10.3. *Suppose that α is free. Then $M \rtimes_\alpha G$ is a factor if and only if $\alpha|_{Z(M)}$ is ergodic.*

Proof: Prop. 4.1.15 in [Sun87]. ■

We now briefly introduce the group measure space construction, an application of the crossed product which will be used to state the complete classification of “separable” injective factors. For this, suppose that (X, \mathfrak{F}, μ) is a standard Borel measure space.

Definition 1.10.4. *An automorphism of (X, \mathfrak{F}, μ) is a bijection T on X st.*

1. *For all $E \in \mathfrak{F}$ we have $T(E), T^{-1}(E) \in \mathfrak{F}$.*
2. *If $E \in \mathfrak{F}$, then $\mu(E) = 0$ if and only if $\mu(T^{-1}(E)) = 0$.*

Given an automorphism T of (X, \mathfrak{F}, μ) , we can then define an action α_T of \mathbb{Z} on $M = L^\infty(X, \mu)$ by $(\alpha_T(n))(f) = f \circ T^{-n}$. The group measure space construction associated with the space (X, \mathfrak{F}, μ) and the automorphism T is then the crossed product $L^\infty(X, \mu) \rtimes_{\alpha_T} \mathbb{Z}$.

Chapter 2

Type III von Neumann factors

In Sec. 1.9, we introduced factors and started to give an exhaustive list of examples of each type, but we left out type III factors. In this chapter, we cover the tools needed to study their structure. We start by introducing the group of modular automorphisms associated with a faithful normal state ϕ on a von Neumann algebra M , and the algebraic invariants $S(M)$ and $T(M)$ that can be defined using this group (2.1). These invariants give a subclassification of type III factors, and in (2.2) we give examples of factors for each of these subtypes. We conclude this section by stating the complete classification of “separable” injective factors.

We recall that we call a von Neumann algebra “separable” if it can be represented faithfully on a separable Hilbert space.

2.1 Group of modular automorphisms

Since a type III factor is infinite, i.e. its unit is an infinite projection, we know from Prop. 1.9.5 that it cannot admit a faithful trace (or faithful semifinite tracial weight), because the latter would induce a nontrivial dimension function on the factor. However, a “separable” type III factor M always admits several faithful normal states, see for example Prop. 1.3.12.

This absence of trace can actually be used to learn more about the structure of these factors. Let ϕ be a faithful normal state on a type III factor M and let $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ be the associated GNS representation. Through the polarization identity

for a sesquilinear form, it is easy to check that if we had

$$\|\pi_\phi(x^*)\xi_\phi\|^2 = \phi(xx^*) = \phi(x^*x) = \|\pi_\phi(x)\xi_\phi\|^2$$

for all $x \in M$, ϕ would be a trace. It follows then that the operation $\pi_\phi(x)\xi_\phi \mapsto \pi_\phi(x^*)\xi_\phi$, defined on a dense subspace of \mathcal{H}_ϕ , is not isometric, and it is the lack of symmetry of this operation which will give us information on the structure of M . In general, it also turns out that this operator might not even be bounded. While it is not in our goal here to introduce the theory of unbounded operators, keep in mind that the operators we will be dealing with here, namely $S_0, S_\phi, F_0, F_\phi, J_\phi$ and Δ_ϕ can be unbounded in this construction. See Chap. 9 of [SZ79] for the related terminology and more details on the theory of unbounded operators.

In what follows, M is a von Neumann algebra, ϕ is a faithful normal state on M and $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ is the associated GNS representation.

Proposition 2.1.1. *Let S_0 be the antilinear operator with domain $\pi_\phi(M)\xi_\phi$ defined as $S_0(\pi_\phi(x)\xi_\phi) = \pi_\phi(x^*)\xi_\phi$, and let F_0 be the antilinear operator with domain $\pi_\phi(M)\xi_\phi$ defined as $F_0(x'\xi_\phi) = (x')^*\xi_\phi$. These are densely defined closable operators, and their closure S_ϕ and F_ϕ satisfy $S_\phi = F_\phi^* = F_0^*$ and $F_\phi = S_\phi^* = S_0^*$.*

Proof: Prop. 2.3.1 of [Sun87]. ■

The two operators S_ϕ and F_ϕ are actually tools that allow us to define two more useful operators that will be at the center of our work.

Proposition 2.1.2. *Let $S_\phi = J_\phi\Delta_\phi^{1/2}$ be the polar decomposition of the closed operator S_ϕ . We then have that*

1. J_ϕ is a self-adjoint antiunitary operator and Δ_ϕ is an injective positive self-adjoint operator, invertible as an unbounded operator.
2. the polar decomposition of F_ϕ is $F_\phi = J_\phi\Delta_\phi^{-1/2}$, and we have $F_\phi S_\phi = \Delta_\phi$ and $S_\phi F_\phi = \Delta_\phi^{-1}$.
3. if f is any finite valued Borel function (possibly unbounded) on $[0, \infty)$, then $J_\phi f(\Delta_\phi) J_\phi = \bar{f}(\Delta_\phi^{-1})$.

Proof: Prop. 2.3.2 of [Sun87]. ■

This brings us to the main theorem of this section.

Theorem 2.1.3. *(Tomita-Takesaki theorem) With the same notation as in the preceding propositions, we have*

1. $\Delta_\phi^{it} M \Delta_\phi^{-it} = M$ for all $t \in \mathbb{R}$.

$$2. J_\phi M J_\phi = M'.$$

Proof: Thm. 2.3.3 of [Sun87]. ■

This theorem allows us to introduce the object which is the main takeaway from this section.

Corollary 2.1.4. *The formula $\sigma_t^\phi(x) = \Delta_\phi^{it} x \Delta_\phi^{-it}$ for $t \in \mathbb{R}$ defines a σ -strongly* continuous one parameter group of automorphisms of M . It is called the group of modular automorphisms of M associated with ϕ .*

Proof: Exercise 2.3.4 of [Sun87] and the discussion that follows. ■

The group of modular automorphisms satisfies the following simple but important identity.

Proposition 2.1.5. *We have $\phi \circ \sigma_t^\phi = \phi$, for all $t \in \mathbb{R}$.*

Proof: Discussion preceding Cor. 2.5.12 in [Sun87]. ■

The group of modular automorphisms can also be constructed for a weight. This requires more work than the case of a state; we omit the details here but allow ourselves to assume the existence of the group of modular automorphisms of a weight. The first use of the power of the group of modular automorphisms is in defining algebraic invariants of the algebra M .

Let ϕ be any faithful normal semifinite weight on M . In [Con73], Connes introduced the set

$$T(M) = \{t \in \mathbb{R} \mid \sigma_t^\phi \text{ is an inner automorphism}\}$$

and showed that it does not depend on the choice of faithful normal semifinite weight by his famous 2×2 matrix trick (see Thm. 3.1.1 of [Sun87]). Hence, it is a well-defined algebraic invariant of M . Moreover, it is easy to see that $T(M)$ is a subgroup of the additive group \mathbb{R} . The following result will be useful to compute the set $T(M)$ later in this chapter.

Proposition 2.1.6. *If M_1 and M_2 are von Neumann algebras, we have $T(M_1 \otimes M_2) = T(M_1) \cap T(M_2)$.*

Proof: Thm. 1.3.4 of [Con73]. ■

We also note the following alternate definition of the set $T(M)$.

Proposition 2.1.7. *Let M be a “separable” von Neumann algebra. Then*

$$T(M) = \{t \in \mathbb{R} \mid \exists \text{ a faithful normal state } \phi \text{ on } M \text{ such that } \sigma_t^\phi = id_M\}$$

Proof: Remark 1.3.3 in [Con73]. ■

This will be useful later in this work, when we suppose that we have a periodic state on M , i.e. a state that satisfies this condition.

Now, let M_ϕ denote the centralizer of M with respect to ϕ , namely

$$M_\phi = \{x \in M \mid \sigma_t^\phi(x) = x \quad \forall t \in \mathbb{R}\}$$

For every nonzero projection $e \in M_\phi$, we can define a faithful normal weight on the corner algebra eMe by

$$\phi_e(x) = \phi(x) \quad \forall x \in eMe, x \geq 0$$

We then define the set

$$S(M) = \bigcap_{e \neq 0} \text{Spectrum } \Delta_{\phi_e}$$

This set was also introduced by Connes in [Con73], and he showed that it is a well defined algebraic invariant of M .

Proposition 2.1.8. *If M is a factor, $S(M) \cap \mathbb{R}_+^*$ is a closed subgroup of \mathbb{R}_+^* .*

Proof: Cor. 3.2.3 of [Con73]. ■

As a first example, let M be a factor of type I or II. If M is finite, we know from Cor. 1.9 that M admits a normal faithful finite trace. If M is of type I_∞ , we know that M admits the usual faithful normal trace defined on $\mathcal{B}(\mathcal{H})$. Finally, if M is of type II_∞ , M inherits a faithful normal tracial weight from its “ II_1 block” through the construction of Prop. 1.9. Hence, we can assume that M admits either a faithful normal trace or a faithful normal semifinite tracial weight ϕ . In this case, the associated operators S_0 and S are bounded and isometric. Hence $\Delta_\phi = \text{id}_M$ and $\sigma_t^\phi = \text{id}_M$ for all $t \in \mathbb{R}$. Therefore, we automatically have $T(M) = \mathbb{R}$. In the case of $S(M)$, we have that ϕ_e is tracial for all projections e . Hence $\Delta_{\phi_e} = \text{id}_{eMe}$ and $\text{Spectrum } \Delta_{\phi_e} = \{1\}$ and so $S(M) = \{1\}$. We see that these invariants are trivial in the case of factors of type I and II. It is in the case of a factor of type III that they can be used to uncover the inner structure of the factor.

If M is a factor of type I or II, we found that $T(M) = \mathbb{R}$. This is actually an equivalence in the separable case.

Proposition 2.1.9. *Let M be a “separable” factor. It is of type III if and only if $T(M) \neq \mathbb{R}$.*

Proof: Thm. 1.3.4 of [Con73]. ■

Moreover, we have a similar result for $S(M)$.

Proposition 2.1.10. *If M is a factor, $0 \in S(M)$ if and only if M is of type III.*

Proof: Lemma 3.1.2 of [Con73]. ■

In the case of a type III factor, since $S(M) \cap \mathbb{R}_+^*$ is a closed subgroup of \mathbb{R}_+^* , it is then easy to see that the only possibilities for the set $S(M)$ are

- a) $S(M) = \{0, 1\}$
- b) $S(M) = \{0\} \cup \lambda^{\mathbb{Z}}$, for $0 < \lambda < 1$.
- c) $S(M) = [0, \infty)$

This further refines the classification of type III factors into subclasses. We say that the factor is of type III_0 , III_λ and III_1 if the set $S(M) \cap \mathbb{R}_+^*$ is respectively of the type a), b) or c). As we will see in the next section, all these types are realized.

For all factors except the type III_0 factors, it turns out that the invariants S and T are actually linked.

Proposition 2.1.11. *Let M be a factor such that $S(M) \neq \{0, 1\}$. Then, $T(M)$ is the orthogonal of $S(M) \cap \mathbb{R}_+^*$ with respect to the duality*

$$\begin{aligned} T(M) \times S(M) \cap \mathbb{R}_+^* &\rightarrow \mathbb{R} \\ (t, \lambda) &\mapsto \lambda^{it} \end{aligned}$$

i.e. $T(M) = \{t \in \mathbb{R} \mid \lambda^{it} = 1 \quad \forall \lambda \in S(M) \cap \mathbb{R}_+^*\}$.

Proof: Thm. 3.4.1 of [Con73]. ■

For an overview of the invariants S and T , see 5.5 of [Con94].

2.2 Examples of type III factors

In this section, we continue the work that was started in Sec. 1.9, namely to give examples of factors of all possible types. We discuss each subclass of type III factors.

2.2.1 ITPFI factors

To give examples of type III factors, we introduce the construction of infinite tensor products of finite type I factors (ITPFI factors). For this, suppose we have an infinite family $(M_n, \psi_n)_{n \geq 1}$ of matrix algebras M_n with faithful states ψ_n .

Now let $A_n = \otimes_{k=1}^n M_k$ and $A = \cup_{n=1}^{\infty} A_n$. With the natural injections $i_n : x \mapsto x \otimes \mathbb{1}_{M_{n+1}}$ from A_n to A_{n+1} , the addition, multiplication and involution are naturally defined on A . Furthermore, one can check that these injections are isometric, so the norm is well defined on A . This allows us to define the C*-algebra \tilde{A} as the norm completion of A .

Now, we define on A_n the linear functionals $\phi_n = \otimes_{i=1}^n \psi_i$. It can be checked that each of the ϕ_n are faithful states. Since $\phi_{n+1} \circ i_n = \phi_n$, this family defines a linear functional ϕ on A satisfying $\phi|_{A_n} = \phi_n$. Because each ϕ_n is a state, ϕ is bounded with norm 1. Therefore, ϕ extends to a state $\tilde{\phi}$ on \tilde{A} . Let (\mathcal{H}, π) be the GNS representation associated with the pair $(\tilde{A}, \tilde{\phi})$, and consider the C*-algebra $\pi(\tilde{A})$. We define the von Neumann algebra $\otimes_{n \geq 1} (M_n, \psi_n) = \pi(\tilde{A})''$, the bicommutant of $\pi(\tilde{A})$ in $\mathcal{B}(\mathcal{H})$.

This algebra is a factor (see Prop. III.3.1.3 of [Bla06]). Note that, as $\pi(A) = \cup_{i=n}^{\infty} \pi(A_n)$ is norm dense in $\pi(\tilde{A})$ and $\pi(\tilde{A})$ is ultraweakly dense in $\otimes_{n \geq 1} (M_n, \psi_n)$, we have that $\pi(A)$ is ultraweakly dense in $\otimes_{n \geq 1} (M_n, \psi_n)$. Therefore, $\otimes_{n \geq 1} (M_n, \psi_n)$ is AFD, which we know by Thm. 1.8.8 to be equivalent to being injective. Finally, it is evident that this factor acts on a separable Hilbert space.

To compute the invariant $S(M)$ of an ITPFI factor $M = \otimes_{n \geq 1} (M_n, \psi_n)$, it is useful to introduce an additional invariant defined by Araki and Woods, the *asymptotic ratio set* $r_{\infty}(M)$.

For a faithful state ψ on $\mathbb{M}_n(\mathbb{C})$, we can always find a positive matrix $h \in \mathbb{M}_n(\mathbb{C})$ such that $\psi(x) = \text{Tr}(hx)$. Indeed, if we let e_{ij} be the matrix elements (1 in the i -th row and the j -th column, 0 elsewhere), we can write any $x \in \mathbb{M}_n(\mathbb{C})$ as $x = \sum x_{ij} e_{ij}$ for coefficients $x_{ij} \in \mathbb{C}$. We then have $\psi(x) = \sum x_{ij} \psi(e_{ij})$, which allows us to define the matrix $h \in \mathbb{M}_n(\mathbb{C})$ as $h_{ij} = \psi(e_{ji})$. For any $b_1, \dots, b_n \in \mathbb{C}$, we have

$$\sum_{ij} b_i^* h_{ij} b_j = \sum_{ij} h_{ij} b_i^* b_j = \sum_{ij} \psi(e_{ji}) b_i^* b_j = \psi\left(\sum_{ij} b_j^* b_i e_{ij}\right) \geq 0$$

The last inequality comes from the fact that ψ is a positive functional and that $\sum_{ij} b_j^* b_i e_{ij}$ is positive in $\mathbb{M}_n(\mathbb{C})$ by Prop. 1.7.3. By the second part of the same proposition, this implies that h is positive in $\mathbb{M}_n(\mathbb{C})$. Finally, we have for any $x \in \mathbb{M}_n(\mathbb{C})$

$$\text{Tr}(hx) = \sum_i (hx)_{ii} = \sum_{ij} h_{ij} x_{ji} = \sum_{ij} \psi(e_{ji}) x_{ji} = \psi(x)$$

As ψ is a faithful state, the eigenvalues of h are then all strictly positive and sum up to 1. If α_i for $1 \leq i \leq n$ denote the eigenvalues of h , we define the ratio set of

ϕ as $r(\phi) = \{\alpha_i/\alpha_j \mid 1 \leq i, j \leq n\}$. It is straightforward to check that, if $(h_i)_{i=1}^k$ are the matrices representing the faithful states $(\psi_i)_{i=1}^k$ on the matrix algebras $(M_i)_{i=1}^k$, then $\otimes_{i=1}^k h_i$ represents $\otimes_{i=1}^k \psi_i$ on $\otimes_{i=1}^k M_i$, hence, it is easy to check that $r(\otimes_{i=1}^k \psi_i) = \{\lambda_1 \dots \lambda_k \mid \lambda_i \in r(\phi_i)\}$. This allows us to define the asymptotic ratio set $r_\infty(M)$ of the ITPFI factor $M = \otimes_{n \geq 1} (M_n, \psi_n)$, as the set of values $x \geq 0$ such that for all $m \in \mathbb{N}$ and $\epsilon > 0$, there exists an $\ell \in \mathbb{N}$ and a β in the ratio set of $\psi_{m+1} \otimes \dots \otimes \psi_{m+\ell}$ with $|x - \beta| < \epsilon$. This is an algebraic invariant of M , which is specific to ITPFI factors in this current definition (see Def. III.3.1.11 in [Bla06]).

2.2.2 Type III_λ factors, $0 < \lambda < 1$

Fix $0 < \lambda < 1$. We will construct a type III_λ factor as an ITPFI factor. For this, we choose the family $M_n = \mathbb{M}_2(\mathbb{C})$ and $\psi_n = \phi_\lambda$ defined on M_n as

$$\phi_\lambda(x) = \text{tr} \left(\begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix} x \right), \text{ for all } x \in M_n,$$

for all n . Then, $R_\lambda = \otimes_{i \geq 1} (M_2(\mathbb{C}), \phi_\lambda)$ is an injective factor.

The asymptotic ratio set of an ITPFI factor is actually linked to the family R_λ .

Proposition 2.2.1. *Let $0 < \lambda < 1$ and $M = \otimes_{n \geq 1} (M_n, \psi_n)$ be an ITPFI factor. Then $\lambda \in r_\infty(M)$ iff $M \otimes R_\lambda \cong M$.*

Proof: Cor. 5.5 and Lemma 5.8 of [AW69]. ■

Note that this property allows one to generalize the concept of $r_\infty(M)$ for an arbitrary factor M , by defining it as the set of positive values² λ such that $M \otimes R_\lambda \cong M$. This result also allows us to link $r_\infty(M)$ with $S(M)$.

Proposition 2.2.2. *Let $0 < \lambda < 1$ and M be a von Neumann factor. Then $M \otimes R_\lambda \cong M$ implies $\lambda \in S(M)$. If M is an ITPFI factor, the converse is also true.*

Proof: Thm. 3.6.1 of [Con73]. Note that Connes uses the generalized definition of $r_\infty(M)$. ■

We are now ready to compute the invariants of each member of the family R_λ . In this case, we have $r(\phi_\lambda) = \{1, \lambda, \lambda^{-1}\}$. Hence, $r(\phi_n) = \{\lambda^k \mid -n \leq k \leq n\}$. It is then easy to check that $\lambda^{\mathbb{Z}} \subseteq r_\infty(R_\lambda)$. Therefore, by the two previous propositions, we immediately see that $S(R_\lambda) = \{0\} \cup \lambda^{\mathbb{Z}}$. Hence, for each $\lambda \in (0, 1)$, R_λ is an injective factor of type III_λ . For future reference, note that Prop. 2.1.11 then implies that $T(R_\lambda) = \{2\pi n / (\ln \lambda) \mid n \in \mathbb{Z}\}$.

²Of course, under this simple formulation, the definition is unclear for $\lambda \notin (0, 1)$. See Sec. 6 of [AW69] for the precise general definition.

2.2.3 Type III₀ factor

To construct examples of III₀ factors, we will also use the ITPFI construction. Let $(L_k)_{k \geq 1}$ be a sequence of strictly positive integers, and $(\lambda_k)_{k \geq 1}$ be a sequence of numbers such that $0 < \lambda_k < 1$. Then, we define $M(L_k, \lambda_k) = \otimes_{k \geq 1} (\otimes_{i=1}^{L_k} \mathbb{M}_2(\mathbb{C}), \otimes_{i=1}^{L_k} \phi_{\lambda_k})$, where the states ϕ_{λ_k} are defined as in Sec. 2.2.2. As seen in Sec. 2.2.1, $M(L_k, \lambda_k)$ is an injective factor. Its type is determined by the sequences $(L_k)_{k \geq 1}$ and $(\lambda_k)_{k \geq 1}$. We will consider the special case $\lambda_k = e^{-r_k}$, where r_k is an increasing sequence of positive real numbers.

Proposition 2.2.3.

- a) If $\lambda_k = e^{-r_k} \rightarrow 0$ and $\sum_{k \geq 1} L_k e^{-r_k} = \infty$, $M(L_k, \lambda_k)$ is of type III.
- b) If $\sum_{k \geq 1} L_k e^{-r_k} (r_k/r_{k+1})^2 < \infty$, $T(M(L_k, \lambda_k))$ is uncountable.
- c) If the two above conditions are satisfied, $M(L_k, \lambda_k)$ is of type III₀.

Proof: For a), we note that we can consider $M(L_k, \lambda_k)$ as an infinite tensor product of type I₂ factors, by repeating L_k times in the sequence the factor $(\mathbb{M}_2(\mathbb{C}), \phi_{\lambda_k})$, for each k . We can then more easily apply point (iii) of Thm. III.3.1.12 of [Bla06]. We choose $\alpha_1^{(i)} = 1/(1 + e^{-r_{k_i}})$. Since $\lambda_k \rightarrow 0$, we know that $1/\lambda_k$ is bounded from below by a constant greater than 1, and so we can choose $C = \inf_k \{|1/\lambda_k - 1|^2\} > 0$. We then get

$$\begin{aligned} \sum_{i,n} \alpha_n^{(i)} \inf\{|\alpha_1^{(i)}/\alpha_n^{(i)} - 1|^2, C\} &= \sum_{i,n} \alpha_n^{(i)} \inf\{|1/\lambda_k - 1|^2, C\} \\ &= \sum_{k \geq 1} L_k \frac{e^{-r_k}}{1 + e^{-r_k}} C \\ &\geq \frac{C}{2} \sum_{k \geq 1} L_k e^{-r_k} \end{aligned}$$

which proves a). For b), see Prop. 1.3 of [GS85]. To obtain c), note that from Prop. 2.1.11 we know that if the invariant $T(M)$ of a type III factor M is uncountable, M is of type III₀. ■

Let us now construct a concrete example of a type III₀ factor. Let $L_k = 3^{k^k}$ and $r_k = k^k \ln 3$, for $k \geq 1$. We then have $\lambda_k = e^{-r_k} \rightarrow 0$ and

$$\begin{aligned} \sum_{k \geq 1} L_k e^{-r_k} &= \sum_{k \geq 1} 1 = \infty \\ \sum_{k \geq 1} L_k e^{-r_k} (r_k/r_{k+1})^2 &= \sum_{k \geq 1} \left(\frac{k^k}{(k+1)^{k+1}} \right)^2 \leq \sum_{k \geq 1} \frac{1}{(k+1)^2} < \infty \end{aligned}$$

Therefore, by the previous proposition, $M(L_k = 3^{k^k}, \lambda_k = e^{-k^k \ln 3})$ is an injective type III₀ factor.

2.2.4 Type III₁ factor

To construct a type III₁ factor, let R_ν and R_μ be type III_λ factors constructed as in Sec. 2.2.2, with $0 < \nu, \mu < 1$ and $\ln \nu / \ln \mu$ irrational. Define $R_\infty = R_\nu \otimes R_\mu$. R_∞ is then a factor, as a tensor product of two factors is a factor (see for example Thm. IV.5.9 and Cor. IV.5.10 of [Tak02]). We have

$$T(R_\infty) = T(R_\nu) \cap T(R_\mu) = \{2\pi n / (\ln \nu) \mid n \in \mathbb{Z}\} \cap \{2\pi n / (\ln \mu) \mid n \in \mathbb{Z}\} = \{0\}$$

Hence, $T(R_\infty) \neq \mathbb{R}$, so R_∞ must be a factor of type III.

We have $R_\infty \otimes R_\mu \cong R_\nu \otimes (R_\mu \otimes R_\mu) \cong R_\nu \otimes R_\mu = R_\infty$. By Prop. 2.2.2, this implies that $\mu \in S(R_\infty)$. By the same reason, we have $\nu \in S(R_\infty)$. Since $0 < \nu, \mu < 1$, this implies that $S(R_\infty) \neq \{0, 1\}$. Therefore, since $T(R_\infty) = \{0\}$, we have by Prop. 2.1.11 that $S(M) = [0, \infty)$. Hence, we conclude that R_∞ is a type III₁ factor.

We note that it is injective and “separable” as both components of the product satisfy these properties.

2.2.5 Classification of injective factors

We are now ready to state the complete classification of “separable” injective factors. This classification is remarkably simple, as the “separable” injective factor of a given type is unique up to isomorphism for all but the III₀ type. It is no coincidence that the examples we have constructed so far are examples of each type of injective factors.

Theorem 2.2.4. *Let M be a “separable” injective factor.*

- If M is of type I_n , then $M \cong \mathbb{M}_n(\mathbb{C})$.
- If M is of type I_∞ , then $M \cong \mathcal{B}(\ell^2(\mathbb{N}))$.
- If M is of type II_1 , then $M \cong vN(S_\infty)$.
- If M is of type II_∞ , then $M \cong vN(S_\infty) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$.
- If M is of type III_0 , there exists a measure space (X, \mathfrak{F}, μ) and an automorphism T of this space such that $M \cong L^\infty(X, \mu) \rtimes_{\alpha_T} \mathbb{Z}$. Such factors are in one-to-one correspondance with non-transitive ergodic flows.
- If M is of type III_λ , $0 < \lambda < 1$, then $M \cong R_\lambda$.
- If M is of type III_1 , then $M \cong R_\infty$.

The case of type I factors was known since the original works of von Neumann. On the other hand, the classification of the two other types is historically linked. The first major breakthrough, the discrete decomposition of type III_λ factors, was discovered by Connes in 1973. It states that any III_λ factor can be expressed as a discrete crossed product of a II_∞ factor by an automorphism which scales the tracial weight. Moreover, in this decomposition, the latter factor inherits injectivity from the former. Connes also extended the discrete decomposition to the case of a type III_0 factor. In particular, this decomposition allowed him to show that any injective type III_0 factor can be expressed as the discrete cross product of an abelian von Neumann algebra by a single automorphism. Results by Connes, Takesaki and Krieger then imply that injective III_0 factors are classified by non-transitive ergodic flows.

In 1976, Connes proceeded to prove the equivalence of injectivity and the AFD property, which allowed him to prove the uniqueness of the injective type II_1 factor. This immediately implied the uniqueness of the injective type II_∞ factor through Prop. 1.9. Combined with his classification of the automorphisms of the injective II_∞ factor given a year before, this implied for each $0 < \lambda < 1$ the uniqueness of the III_λ factors through the discrete decomposition. Connes was also able to reduce the question of the uniqueness of the injective factor of type III_1 to what is called the bicentralizer problem. This problem was finally solved by Haagerup in 1987, which completed the classification of injective factors. A summary of this classification is given in Sec. 5.9 of [Con94], and the references to the original works can be found there.

It is important to note that the III_0 case is the only one which is not associated with a unique “separable” injective factor. While $S(M)$ and $T(M)$ are complete invariants for the other cases of type III factors, the *flow of weights* is the complete invariant for the type III_0 case.

On a side note, since we proved that $vN(S_\infty)$ and $vN(\mathbb{F}_2)$ are non isomorphic “separable” factors of type II_1 , $vN(\mathbb{F}_2)$ cannot be injective.

Chapter 3

Approximately inner automorphisms of factors

We are now ready to introduce the notions central to the main body of our work. We recall that we call a von Neumann algebra “separable” if it can be represented faithfully on a separable Hilbert space. On the space $\mathcal{B}(M)$ of continuous linear maps from M to itself, we will consider the following Hausdorff topologies:

1. the p -topology, or point ultraweak topology, induced by the family of seminorms $\psi \mapsto |\phi \circ \psi(x)|$ for $\psi \in \mathcal{B}(M)$, $\phi \in M_*$ and $x \in M$.
2. the u -topology, induced by the family of seminorms $\psi \mapsto \|\phi \circ \psi\|$ for $\psi \in \mathcal{B}(M)$ and $\phi \in M_*$.

It can be checked that $\text{Aut}(M)$ is a topological group under the u -topology (see 2.23, [Str81]). Recall that $\theta \in \text{Aut}(M)$ is called *inner* if there exists a unitary $u \in M$ such that $\theta(x) = u x u^* = \text{Ad } u(x)$.

Definition 3.0.1. *Let M be a von Neumann algebra. We say that an automorphism θ is approximately inner if there exists a net of inner automorphisms $(\text{Ad } u_\alpha)$ which converges to θ in the u -topology. If the predual of M is a separable Banach space, we can take this net to be a sequence.*

Definition 3.0.2. [Min90] *Let M be a von Neumann algebra. A CP map is called an approximately inner CP map if it can be approximated in the p -topology by a net of inner CP maps of the form $\Psi(x) = \sum_{k \in \mathbb{N}}^{UWOT} a_k^* x a_k$, for a sequence $(a_k) \subset M$ for which $\sum_{k \in \mathbb{N}}^{UWOT} a_k^* a_k \leq \mathbb{1}_M$.*

As we have seen before, an automorphism is completely positive. In this chapter, we explore the conditions under which these two notions of approximate innerness are equivalent for an automorphism of a von Neumann factor.

One direction is fairly simple.

Theorem 3.2.2. *Let M be a “separable” factor and $\theta \in \text{Aut}(M)$. If θ is approximately inner as an automorphism, then θ is approximately inner as a CP map.*

This result is already known (see for example [GM93]), and we include the proof in Sec. 3.2. The converse is however more complex. Indeed, an automorphism θ of a factor M induces an automorphism $\text{mod}(\theta)$ of the flow of weights of M [CT77], a concept which we briefly mentioned in the previous chapter. It turns out that if θ is approximately inner, $\text{mod}(\theta)$ is trivial. Therefore, this condition is necessary for the converse to be true. This led Connes to formulate the following conjecture:

Conjecture A [Con85] Let M be a “separable” factor and $\theta \in \text{Aut}(M)$. Then θ is an approximately inner automorphism if and only if it is approximately inner as a CP map and $\text{mod}(\theta) = \text{id}$.

Note that Connes actually formulated his conjecture in terms of the boundedness of a map induced by θ on $C^*(M, M')$, the C^* -algebra generated by M and M' when M is in standard form. We show the equivalence of the two formulations in Sec. 3.1.1. This conjecture was verified for the class of “separable” injective type III factors in [KST92]. Moreover, it was verified for “separable” type I and II factors in [GM93]. However, the conjecture fails in the non-injective type III case. As shown in [Mar20], for the type III_λ and III_1 cases, there are examples of automorphisms with trivial mod which are approximately inner as CP maps, but are not approximately inner automorphisms.

While the conjecture was proven to be false in its current state, the next natural question is to ask whether additional conditions on M could be added for it to be true. In Sec. 3.3-3.4, we set out to explore this question for the case of a “separable” factor of type III_λ . Precisely, we aim to obtain a result of the form:

(*) *Let M be a “separable” factor of type III_λ which satisfies a set of conditions S and let $\theta \in \text{Aut}(M)$. Then θ is an approximately inner automorphism if and only if it is approximately inner as a CP map and $\text{mod}(\theta) = \text{id}$.*

In [Mar20], the author showed that this result holds when M has Araki’s property L'_λ , i.e. $M \cong M \otimes R_\lambda$, where R_λ is the injective type III_λ factor. To study this question, we adapt the proof of [GM93], which shows Conjecture A for the type II_1 case. Their proof can be broken up into two results.

Proposition 3.0.3. *Let M be a “separable” von Neumann algebra and $\theta \in \text{Aut}(M)$ be approximately inner as a CP map. Then θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M$.*

This result is proven in Lemma 2 of [GM93]. Section §2 of [GM93] is then devoted to proving the following result.

Proposition 3.0.4. *Let M be a “separable” finite von Neumann algebra and $\theta \in \text{Aut}(M)$, such that θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M$. Then θ is an approximately inner automorphism.*

Our goal is to adapt this method to the case of a “separable” type III_λ factor M with an automorphism θ such that $\text{mod}(\theta) = \text{id}_M$. The main difficulty resides in the fact that M does not admit any trace, whose existence was used extensively in the construction of [GM93]. However, the fact that $\text{mod}(\theta) = \text{id}$ allows us to suppose the existence of a faithful normal state ϕ such that $\phi \circ \theta = \phi$ and $\sigma_T^\phi = \text{id}_M$ for $T = -2\pi/\ln \lambda$ (see Appendix C). For such a state, we can consider its centralizer $M_\phi = \{x \in M \mid \sigma_t^\phi(x) = x \quad \forall t \in \mathbb{R}\}$, which is a finite von Neumann algebra. The idea is then to replace the trace in the proof of [GM93] by the state ϕ . Replacing the trace by a state adds numerous additional complexities. In Sec. 3.3, we adapt the proof of Prop. 3.0.4 to obtain the following analogous result.

Theorem 3.3.16. *Let M be a “separable” von Neumann algebra, $\theta \in \text{Aut}(M)$ and ϕ a faithful normal state such that $\phi \circ \theta = \phi$. Suppose that θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M_\phi$. Then θ is an approximately inner automorphism.*

As is apparent, the only difference is that the elements a_j are required to lie in M_ϕ . The techniques in this section are based on unpublished work by Giordano and Mingo following the publication of [GM93]. Their notes included a series of lemmas with sketches of proofs. Our work consisted in filling these gaps, adding the necessary additional lemmas and resulted in the series of results presented in Sec. 3.3, which leads to the theorem stated above.

The next logical step is to seek a result analogous to Prop. 3.0.3 where the elements a_j can be chosen in M_ϕ . Of course, since Conjecture A was shown not to hold in the case of a type III_λ factor, such a result cannot be obtained without imposing additional conditions on M . We describe qualitatively in Sec. 3.4 the approach we have taken to try to prove such a result when M has property L'_λ , and the shortcomings of this calculation. Indeed, it was not conclusive and we therefore do not have a full proof of a result of the form (*).

Before moving on to the heart of this chapter, we present in Sec. 3.1 technical results that will prove useful in this chapter. The first two subsections include well

known results, namely an equivalent definition for a map to be an approximately inner CP map and a list of algebraic identities. We then finish the section with new results related to the approximation of elements of M by elements of M_ϕ , and we apply these results to show that, under certain conditions, we can choose the coefficients of inner maps in M_ϕ .

3.1 Technical results

In this section, we list technical results related to the topic of this chapter. The content of the first two subsections is well known and is listed here for completeness. However, the content of the last two subsections is new and consists of partial results that we have obtained in our attempts to generalize Prop. 3.0.3.

3.1.1 Equivalent description of an approximately inner CP map

In this subsection, we describe an alternate definition of an approximately inner CP map. This is actually the original concept which appears in Conjecture A, while the definition of an approximately inner CP map that we have stated earlier is due to [Min90].

Let M be a “separable” factor. We represent M in standard form (M, \mathcal{H}, J, P) on a Hilbert space \mathcal{H} . We define $\mathbb{C}(M, M')$ to be the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ generated by M and M' , i.e. elements of the form $\sum_{i=1}^n x_i y_i$ for $x_1, \dots, x_n \in M$ and $y_1, \dots, y_n \in M'$. We denote by $C^*(M, M')$ the C^* -algebra generated by M and M' in $\mathcal{B}(\mathcal{H})$, which corresponds to the norm closure of $\mathbb{C}(M, M')$. For $\theta \in \text{Aut}(M)$, we define $\Theta \in \text{Aut}(\mathbb{C}(M, M'))$ by $\Theta(\sum_{i=1}^n x_i y_i) = \sum_{i=1}^n \theta(x_i) y_i$. That this map is well defined is a simple consequence of the following result of Murray and von Neumann.

Proposition 3.1.1. *Let N be a von Neumann factor acting on \mathcal{H} , and let $x_i, \dots, x_n \in N$ and $y_1, \dots, y_n \in N'$. Then $\sum_{i=1}^n x_i y_i = 0$ iff there exists a matrix $(\lambda_{ij}) \in \mathbb{M}_n(\mathbb{C})$ such that $\sum_{i=1}^n \lambda_{ij} x_i = 0$ and $\sum_{i=1}^n \lambda_{ij} y_j = y_i$.*

Proof: Lemma 5.22 of [Bry13]. ■

Now let $M \odot M'$ denote the algebraic tensor product of M and M' , which is a $*$ -algebra (see [Bry13] for a thorough coverage of tensor products of $*$ -algebras).

Definition 3.1.2.

- A binormal representation of $M \odot M'$ is a $*$ -homomorphism $\pi : M \odot M' \rightarrow \mathcal{B}(\mathcal{K})$ for a Hilbert space \mathcal{K} such that the mappings $x \in M \mapsto \pi(x \odot \mathbb{1}_{M'})$ and $y \in M' \mapsto \pi(\mathbb{1}_M \odot y)$ are normal representations.

- The binormal norm on $M \odot M'$ is given by

$$\|x\|_{bin} = \sup\{\|\pi(x)\| \mid \pi \text{ is a binormal representation of } M \odot M'\}.$$

- The C^* -algebra $M \otimes_{bin} M'$ is the completion of $M \odot M'$ under the norm $\|\cdot\|_{bin}$.

We note that binormal representations are equivalent to correspondences of von Neumann algebras [Min89]. It is immediate that any binormal representation of $M \odot M'$ on \mathcal{K} is bounded with respect to $\|\cdot\|_{bin}$, and so it extends to a representation of $M \otimes_{bin} M'$ on \mathcal{K} . The two binormal representations which are of use to us are given by

$$\begin{aligned} \tilde{\pi}_{id} : M \odot M' &\rightarrow \mathcal{B}(\mathcal{H}) \\ \sum_{i=1}^k x_i \odot y_i &\mapsto \sum_{i=1}^k x_i y_i \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}_\theta : M \odot M' &\rightarrow \mathcal{B}(\mathcal{H}) \\ \sum_{i=1}^k x_i \odot y_i &\mapsto \sum_{i=1}^k \theta(x_i) y_i \end{aligned}$$

We will denote by π_{id} and π_θ the corresponding representations of $M \otimes_{bin} M'$. The following concept will link these notions to approximately inner CP maps, see for example [Min90].

Definition 3.1.3. *Given two representations π_1 and π_2 of a C^* -algebra A , we say that π_1 is weakly contained in π_2 if $\ker(\pi_2) \subseteq \ker(\pi_1)$. This is equivalent to saying that there is a $*$ -homomorphism $\tilde{\pi} : \pi_2(A) \rightarrow \pi_1(A)$ such that $\pi_1 = \tilde{\pi} \circ \pi_2$. We denote this by $\pi_1 \subseteq_w \pi_2$.*

Proposition 3.1.4. *Let M be a factor represented in standard form. The map $\Theta \in \text{Aut}(\mathbb{C}(M, M'))$ is bounded iff $\pi_\theta \subseteq_w \pi_{id}$.*

Proof: If Θ is bounded, it extends to a $*$ -homomorphism on $C^*(M, M')$. Then, $\Theta \circ \pi_{id}$ is defined on $M \otimes_{bin} M'$. We have $\pi_\theta = \Theta \circ \pi_{id}$ on $M \odot M'$ and so, by density of the latter, this equality stands on $M \otimes_{bin} M'$. Therefore, $\pi_\theta \subseteq_w \pi_{id}$.

Now assume that $\pi_\theta \subseteq_w \pi_{id}$, and hence that there is a $*$ -homomorphism $\alpha : \pi_{id}(M \otimes_{bin} M') \rightarrow \pi_\theta(M \otimes_{bin} M')$. Since $\pi_\theta(M \otimes_{bin} M')$ and $\pi_{id}(M \otimes_{bin} M')$ are C^* -subalgebras of $\mathcal{B}(\mathcal{H})$, we know that α is bounded. We have

$$\Theta\left(\sum_{i=1}^k x_i y_i\right) = \pi_\theta\left(\sum_{i=1}^k x_i \odot y_i\right) = \alpha \circ \pi_{id}\left(\sum_{i=1}^k x_i \odot y_i\right) = \alpha\left(\sum_{i=1}^k x_i y_i\right)$$

Therefore, $\Theta = \alpha$ on $\mathbb{C}(M, M')$ and so it is bounded. \blacksquare

Proposition 3.1.5. *Let M be a “separable” factor. For $\theta \in \text{Aut}(M)$, we have $\pi_\theta \subseteq_w \pi_{\text{id}}$ iff θ is approximately inner as a CP map.*

Proof: Since M is “separable”, Thm. 2.6 of [ADH90] applies, and so we know that $\pi_\theta \subseteq_w \pi_{\text{id}}$ iff θ can be approximately factored by the identity mapping (as defined in [ADH90]). It is straightforward to check that θ can be approximately factored by the identity mapping iff θ is approximately inner as a CP map. \blacksquare

Combining these result, we obtain this alternative condition for an automorphism to be approximately inner as a CP map.

Corollary 3.1.6. *Let M be a “separable” von Neumann factor and (M, \mathcal{H}, J, P) be a standard form. An automorphism θ of M is approximately inner as a CP map if and only if the induced automorphism $\Theta \in \text{Aut}(\mathbb{C}(M, M'))$ is bounded.*

3.1.2 Some technical results

Let M be a von Neumann algebra. The following results hold and will be used freely in this chapter. Note that for $y \in M$ and $\psi \in M^*$, the commutator $[y, \psi] \in M^*$ is defined as $[y, \psi](x) = \psi(yx) - \psi(xy)$, for all $x \in M$. Moreover, for $\psi \in M_+^*$, we define on M the semi-norm $\|x\|_\psi = \psi(x^*x)^{1/2}$. Such a semi-norm satisfies a Cauchy-Schwarz inequality: for $\psi \in M_+^*$ and $x, y \in M$, we have $|\psi(x^*y)| \leq \|x\|_\psi \cdot \|y\|_\psi$. Moreover, it is a norm if and only if ψ is faithful. We will also use the semi-norm $\|x\|_\psi^\# = \psi(x^*x + xx^*)^{1/2}$; it is easy to see that we have $\|x\|_\psi \leq \|x\|_\psi^\#$ and $\|x^*\|_\psi \leq \|x\|_\psi^\#$.

Proposition 3.1.7. *For $u \in \mathcal{U}(M)$ and $\psi \in M_+^*$, $\|\psi \circ \text{Ad}_u - \psi\| = \|\psi \circ \text{Ad}_{u^*} - \psi\| = \|[u, \psi]\|$. Moreover, for any $y \in M$, we have $\|[y, \psi]\| \leq 2\|y - \mathbb{1}_M\|_\psi^\# \|\psi\|^{1/2}$.*

Proof:

$$\begin{aligned}
\|[y, \psi]\| &= \sup_{\|x\| \leq 1} |\psi(yx) - \psi(xy)| = \sup_{\|x\| \leq 1} |\psi(yx - x) + \psi(x - xy)| \\
&= \sup_{\|x\| \leq 1} |\psi((y - \mathbb{1}_M)x) + \psi(x(\mathbb{1}_M - y))| \\
&\leq \sup_{\|x\| \leq 1} |\psi((y - \mathbb{1}_M)x)| + \sup_{\|x\| \leq 1} |\psi(x(\mathbb{1}_M - y))| \\
&\leq \sup_{\|x\| \leq 1} \|y^* - \mathbb{1}_M\|_\psi \|x\|_\psi + \sup_{\|x\| \leq 1} \|y - \mathbb{1}_M\|_\psi \|x^*\|_\psi \\
&= 2\|y - \mathbb{1}_M\|_\psi^\# \|\psi\|^{1/2}
\end{aligned}$$

Proposition 3.1.8. *For ϕ a state on M and $a, x \in M$, we have $\|ax\|_\phi \leq \|a\| \cdot \|x\|_\phi$*

Proof:

$$\|ax\|_\phi = \phi(x^*a^*ax)^{1/2} \leq \phi(x^*\|a^*a\|\mathbb{1}_Mx)^{1/2} = \|a\| \cdot \|x\|_\phi$$

Proposition 3.1.9. *For ϕ a faithful normal state on M , $a \in M_\phi$ and $x \in M$, we have*

$$\|xa\|_\phi \leq \|a\| \cdot \|x\|_\phi$$

Proof: Consider the linear functional $\psi : M_\phi \rightarrow \mathbb{C}$ defined by $\psi(y) = \phi(x^*xy)$. For $b \in M_\phi$, we have

$$\psi(b^*b) = \phi(x^*xb^*b) = \phi((xb^*)^*xb^*) \geq 0$$

where we have used 1. of Prop. 3.1.11. This functional is positive and so

$$\begin{aligned} \|xa\|_\phi &= \phi(a^*x^*xa)^{1/2} = \phi(x^*xaa^*)^{1/2} = |\psi(aa^*)|^{1/2} \\ &\leq \|\psi\|^{1/2} \|aa^*\|^{1/2} = |\psi(\mathbb{1}_M)|^{1/2} \|a\| = \|a\| \|x\|_\phi \end{aligned}$$

Proposition 3.1.10. *Let M be a “separable” von Neumann algebra and ϕ a faithful normal state on M . Then we can find a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $[0, \phi] = \{\psi \in (M_*)^+ \mid \psi \leq \phi\}$ such that $\text{span}_{n \in \mathbb{N}} \psi_n$ is dense in M_* .*

Proof: By the proof of Thm. 10.25 of [SZ79], the set $S = \{\psi \in (M_*)^+ \mid \exists \lambda > 0 \text{ such that } \psi \leq \lambda\phi\}$ is dense in $(M_*)^+$. Since M_* is separable, S is separable. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence dense in S , with the associated $(\lambda_n)_{n \in \mathbb{N}}$ such that $\psi_n \leq \lambda_n\phi$. This sequence is also dense in $(M_*)^+$. Since each element in M_* is a linear combination of four positive normal forms, it is straightforward to verify that $((1/\lambda_n)\psi_n)_{n \in \mathbb{N}}$ is the sequence we are looking for. ■

3.1.3 Approximation by elements of M_ϕ

In this subsection, we develop methods to approximate elements of M by elements of M_ϕ . We first note the following facts about M_ϕ .

Proposition 3.1.11. *Let M be a von Neumann algebra and ϕ a faithful normal state on M . Then,*

1. $M_\phi = \{x \in M \mid \phi(xy) = \phi(yx) \quad \forall y \in M\}$
2. *There exists a normal conditional expectation $E_\phi : M \rightarrow M_\phi$ such that $\phi \circ E_\phi = \phi$. This conditional expectation is uniquely determined by ϕ .*
3. *It ϕ is periodic, i.e. there exists $T > 0$ such that $\sigma_T^\phi = id_M$, then $E_\phi(x) = (1/T) \int_0^T \sigma_t^\phi(x) dt$. The integral is defined as in Lemma 13.1.1 of [Str81], by taking $\mathcal{X} = M$ and $\mathcal{X}_* = M_*$.*

Proof: For 1., see Thm. VIII.2.6 of [Tak03]. For 2., see Thm. IX.4.2 of [Tak03]. To prove 3., note that the integral of [Str81] satisfies $\|\int x(t) dt\| \leq \int \|x(t)\| dt$ and apply Thm. 1.8.4. \blacksquare

The following three results allow the approximation of elements “close” to M_ϕ by elements of M_ϕ of the same nature. The proofs are based on the techniques used by Connes in [Con75].

Lemma 3.1.12. *Let M be a von Neumann algebra, ϕ a state on M , $N \subseteq M$ a finite von Neumann factor which is the image of a conditional expectation $E : M \rightarrow N$, and $u \in \mathcal{U}(M)$. If $v = E(u)$, there exists $\tilde{w} \in \mathcal{U}(N)$ such that*

$$\max\{\|u - \tilde{w}\|_\phi, \|u^* - \tilde{w}^*\|_\phi\} \leq 6(\|u - v\|_\phi^\# + \|\phi \circ Ad_u - \phi\|^{1/2})$$

Proof: Let $v = w|v|$ and $v^* = w^*|v^*|$ be the polar decompositions of v and v^* , respectively. Since $\mathbb{1}_M \leq (|v| + \mathbb{1}_M)^*(|v| + \mathbb{1}_M)$, we have

$$\begin{aligned} \||v| - \mathbb{1}_M\|_\phi^2 &= \phi((|v| - \mathbb{1}_M)^*(|v| - \mathbb{1}_M)) \leq \phi((|v| - \mathbb{1}_M)^*(|v| + \mathbb{1}_M)^*(|v| + \mathbb{1}_M)(|v| - \mathbb{1}_M)) \\ &= \||v| + \mathbb{1}_M\|_\phi \||v| - \mathbb{1}_M\|_\phi^2 \end{aligned}$$

Hence

$$\begin{aligned} \||v| - \mathbb{1}_M\|_\phi &\leq \||v|^2 - \mathbb{1}_M\|_\phi = \|v^*v - \mathbb{1}_M\|_\phi \leq \|v^*(v - u)\|_\phi + \|(v^* - u^*)u\|_\phi \\ &\leq \|v - u\|_\phi + \|v^* - u^*\|_{\phi \circ Ad_{u^*}} \\ &\leq \|v - u\|_\phi + \|v^* - u^*\|_\phi + \|\phi \circ Ad_{u^*} - \phi\|^{1/2} \cdot \|v^* - u^*\| \\ &\leq \|v - u\|_\phi + \|v^* - u^*\|_\phi + 2\|\phi \circ Ad_{u^*} - \phi\|^{1/2} \end{aligned}$$

Similarly

$$\begin{aligned}
\| |v^*| - \mathbb{1}_M \|_\phi &\leq \| |v^*|^2 - \mathbb{1}_M \|_\phi = \| v v^* - \mathbb{1}_M \|_\phi \leq \| v(v^* - u^*) \|_\phi + \| (v - u)u^* \|_\phi \\
&\leq \| v^* - u^* \|_\phi + \| v - u \|_{\phi \circ Ad_u} \\
&\leq \| v - u \|_\phi + \| v^* - u^* \|_\phi + \| \phi \circ Ad_u - \phi \|^{1/2} \cdot \| v - u \| \\
&\leq \| v - u \|_\phi + \| v^* - u^* \|_\phi + 2 \| \phi \circ Ad_u - \phi \|^{1/2}
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\| u - w \|_\phi &= \| u - w|v \|_\phi + \| w(|v| - \mathbb{1}_M) \|_\phi \\
&\leq \| u - v \|_\phi + \| |v| - \mathbb{1}_M \|_\phi
\end{aligned}$$

and

$$\begin{aligned}
\| u^* - w^* \|_\phi &= \| u - w^*|v^* \|_\phi + \| w^*(|v^*| - \mathbb{1}_M) \|_\phi \\
&\leq \| u^* - v^* \|_\phi + \| |v^*| - \mathbb{1}_M \|_\phi
\end{aligned}$$

Define $g_1 = w^*w$ and $g_2 = ww^*$. We then have

$$\begin{aligned}
\| \mathbb{1}_M - g_1 \|_\phi &= \| (\mathbb{1}_M - g_1)u^* \|_{\phi \circ Ad_{u^*}} = \| (\mathbb{1}_M - g_1)(u^* - w^*) \|_{\phi \circ Ad_{u^*}} \leq \| u^* - w^* \|_{\phi \circ Ad_{u^*}} \\
\| \mathbb{1}_M - g_2 \|_\phi &= \| (\mathbb{1}_M - g_2)u \|_{\phi \circ Ad_u} = \| (\mathbb{1}_M - g_2)(u - w) \|_{\phi \circ Ad_u} \leq \| u - w \|_{\phi \circ Ad_u}
\end{aligned}$$

Note that, since $g_1 \sim g_2$ in N , which is a finite factor, we have $\mathbb{1}_M - g_1 \sim \mathbb{1}_M - g_2$. Therefore, there exists a partial isometry w_0 in N such that $\mathbb{1}_M - g_1 = w_0^*w_0$ and $\mathbb{1}_M - g_2 = w_0w_0^*$. Then $\tilde{w} = w + w_0 \in \mathcal{U}(N)$ satisfies

$$\begin{aligned}
\| u - \tilde{w} \|_\phi &\leq \| u - w \|_\phi + \| w_0 \|_\phi = \| u - w \|_\phi + \| \mathbb{1}_M - g_1 \|_\phi \leq \| u - w \|_\phi + \| u^* - w^* \|_{\phi \circ Ad_{u^*}} \\
&\leq \| u - w \|_\phi + \| u^* - w^* \|_\phi + 2 \| \phi \circ Ad_{u^*} - \phi \|^{1/2} \\
&\leq \| u - v \|_\phi + \| |v| - \mathbb{1}_M \|_\phi + \| u^* - v^* \|_\phi + \| |v^*| - \mathbb{1}_M \|_\phi + 2 \| \phi \circ Ad_{u^*} - \phi \|^{1/2} \\
&\leq 3 \| u - v \|_\phi + 3 \| u^* - v^* \|_\phi + 4 \| \phi \circ Ad_{u^*} - \phi \|^{1/2} + 2 \| \phi \circ Ad_u - \phi \|^{1/2} \\
&\leq 6 (\| u - v \|_\phi^\# + \| \phi \circ Ad_u - \phi \|^{1/2})
\end{aligned}$$

and

$$\begin{aligned}
\| u^* - \tilde{w}^* \|_\phi &\leq \| u^* - w^* \|_\phi + \| w_0^* \|_\phi = \| u^* - w^* \|_\phi + \| \mathbb{1}_M - g_2 \|_\phi \leq \| u^* - w^* \|_\phi + \| u - w \|_{\phi \circ Ad_u} \\
&\leq \| u^* - w^* \|_\phi + \| u - w \|_\phi + 2 \| \phi \circ Ad_u - \phi \|^{1/2} \\
&\leq \| u^* - v^* \|_\phi + \| |v^*| - \mathbb{1}_M \|_\phi + \| u - v \|_\phi + \| |v| - \mathbb{1}_M \|_\phi + 2 \| \phi \circ Ad_u - \phi \|^{1/2} \\
&\leq 3 \| u - v \|_\phi + 3 \| u^* - v^* \|_\phi + 4 \| \phi \circ Ad_u - \phi \|^{1/2} + 2 \| \phi \circ Ad_{u^*} - \phi \|^{1/2} \\
&\leq 6 (\| u - v \|_\phi^\# + \| \phi \circ Ad_u - \phi \|^{1/2})
\end{aligned}$$

■

Lemma 3.1.13. *Let M be a von Neumann algebra, ϕ a faithful normal state on M , E_ϕ the conditional expectation from M to M_ϕ , and $p \in \mathcal{P}(M)$. If $v = E_\phi(p)$ and $\|p - v\|_\phi < 1/3$, there exists $c \in \mathcal{P}(M_\phi)$ such that $\|p - c\|_\phi \leq \|p - v\|_\phi^{1/2}(\|p - v\|_\phi^{1/2} + 2\sqrt{3})$.*

Proof: See Appendix B. ■

Lemma 3.1.14. *Let M be a von Neumann algebra, ϕ a faithful normal state on M , E_ϕ the conditional expectation from M to M_ϕ , and $u \in M$ be a partial isometry. If $v = E_\phi(u)$ and $\|u - v\|_\phi^\# < 1/10$, there exists a partial isometry $w \in M_\phi$ such that $\|w - u\|_\phi^\# \leq \|u - v\|_\phi^\# + 6\sqrt{10}(\|u - v\|_\phi^\#)^{1/2}$.*

Proof: See Appendix B. ■

3.1.4 Choosing the coefficients of inner maps in M_ϕ

We now use the results of the previous subsection to prove that, when an automorphism θ can be approximated in the u -topology by a sequence of inner maps, the coefficients of these maps can be chosen in M_ϕ .

These are intermediate results that were obtained through our attempts to show that when an automorphism can be approximated by a sequence of inner CP maps, the coefficients of these maps can be chosen in M_ϕ (see Sec. 3.4).

Proposition 3.1.15. *Let M be a “separable” type III_λ factor, ϕ a faithful normal state on M with $\sigma_T^\phi = id_M$ for $T = -2\pi/\ln \lambda$, and $\alpha \in \overline{Int}(M)$ such that $\phi \circ \alpha = \phi$. Then, there exists a sequence $(w_n)_{n \geq 1} \subseteq \mathcal{U}(M_\phi)$ such that $Ad_{w_n} \rightarrow \alpha$ in the u -topology on $Aut(M)$.*

Proof: Let (u_n) be a sequence in $\mathcal{U}(M)$ such that $Ad_{u_n} \rightarrow \alpha$ in the u -topology on $Aut(M)$. By Thm. 4.2.6 of [Con73], 1 is an isolated point of $sp\Delta_\phi$, and so by Prop. 2.3 of [Con74], we have $\|u_n - E_\phi(u_n)\|_\phi^\# \rightarrow 0$. Moreover, we know by Section 1 of [Haa89] that M_ϕ is a type II_1 factor. Let $w_n \in \mathcal{U}(M_\phi)$ be the unitary of Lemma 3.1.12 corresponding to u_n . We then have

$$\max\{\|u_n - w_n\|_\phi, \|u_n^* - w_n^*\|_\phi\} \leq 6(\|u_n - E(u_n)\|_\phi^\# + \|\phi \circ Ad_{u_n} - \phi\|^{1/2}) \rightarrow 0$$

where we have used the fact that $\phi = \phi \circ \alpha$. By Prop. 3.1.10, let (ψ_k) be a sequence in $[0, \phi]$ such that $\text{span}(\psi_k)$ is dense in M_* . We have

$$\|\psi_k \circ \alpha - \psi_k \circ Ad_{w_n}\| \leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{w_n}\|$$

$$\begin{aligned}
&\leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{w_n}\| \\
&\leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + \|\psi_k \circ Ad_{u_n w_n^*} - \psi_k\| \\
&\leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + 2\|\psi_k\|^{1/2}\|u_n w_n^* - \mathbb{1}_M\|_{\psi_k}^{\#} \\
&\leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + 2\|\phi\|^{1/2}\|u_n(w_n^* - u_n^*)\|_{\phi}^{\#} \\
&\leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + 2\|u_n\| \cdot \|u_n^* - w_n^*\|_{\phi}^{\#} \\
&= \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + 2\|u_n^* - w_n^*\|_{\phi}^{\#} \rightarrow 0
\end{aligned}$$

Since $\text{span}(\psi_k)$ is dense in M_* , this finishes the proof. \blacksquare

Proposition 3.1.16. *Let M be a “separable” type III_λ factor, ϕ a faithful normal state on M with $\sigma_T^\phi = id_M$ for $T = -2\pi/\ln \lambda$ and $\alpha \in \text{Aut}(M)$ such that $\phi \circ \alpha = \phi$. Suppose that there exists a sequence $(u_n)_{n \geq 1}$ of partial isometries of M such that $Ad_{u_n} \rightarrow \alpha$ in the u -topology on $CP(M)$, and that $\|u_n^* u_n - \mathbb{1}_M\|_{\phi}$ and $\|u_n u_n^* - \mathbb{1}_M\|_{\phi}$ both converge to 0. Then, we can find a sequence $(w_n)_{n \geq 1} \subseteq M_{\phi}$ of partial isometries such that $Ad_{w_n} \rightarrow \alpha$ in the same topology.*

Proof: See Appendix B. \blacksquare

Proposition 3.1.17. *Let M be a “separable” type III_λ factor, ϕ a faithful normal state on M with $\sigma_T^\phi = id$ for $T = -2\pi/\ln \lambda$ and $\alpha \in \text{Aut}(M)$ such that $\phi \circ \alpha = \phi$. Suppose that there exists a sequence $(a_n)_{n \geq 1}$ in M such that $Ad_{a_n} \rightarrow \alpha$ in the u -topology on $CP(M)$, and that $\|a_n^* a_n - \mathbb{1}_M\|_{\phi}$, $\|a_n a_n^* - \mathbb{1}_M\|_{\phi}$ both converge to 0. Finally, suppose that $\|a_n\| \leq C$ for some $C \in \mathbb{R}$. Then, we can find a sequence $(b_n)_{n \geq 1} \subseteq M_{\phi}$ such that $Ad_{b_n} \rightarrow \alpha$ in the same topology.*

Proof: See Appendix B. \blacksquare

3.2 An approximately inner automorphism is approximately inner as a CP map

In this section, we show that if M is any “separable” factor, then any approximately inner automorphism of M is an approximately inner CP map.

Let (M, \mathcal{H}, J, P) be a “separable” factor in standard form. Recall that for any $\theta \in \text{Aut}(M)$ there exists a unique unitary $u_{\theta} \in P$ such that $\theta = Ad_{u_{\theta}}|_M$ (see Thm. 1.6.8). The following result is due to Haagerup (Prop. 3.5 in [Haa75]).

Proposition 3.2.1. *If (θ_n) is a sequence of automorphisms of M converging to $\theta \in \text{Aut}(M)$ in the u -topology, then u_{θ_n} converges strongly to u_θ .*

Proof: For $\phi \in M_*^+$, $\vartheta \in \text{Aut}(M)$ and $x \in M$, we have

$$(xu_\vartheta\xi_\phi|u_\vartheta\xi_\phi) = (u_\vartheta^*xu_\vartheta\xi_\phi|\xi_\phi) = (\vartheta^{-1}(x)\xi_\phi|\xi_\phi) = \phi \circ \vartheta^{-1}(x)$$

and therefore, by the uniqueness of the vector representing a normal form in Thm. 1.6.8, we have $u_\vartheta\xi_\phi = \xi_{\phi \circ \vartheta^{-1}}$. Hence, for all $\phi \in M_*^+$, we have by Thm. 1.6.8

$$\|u_{\theta_n}\xi_\phi - u_\theta\xi_\phi\|^2 = \|\xi_{\phi \circ \theta_n^{-1}} - \xi_{\phi \circ \theta^{-1}}\|^2 \leq \|\phi \circ \theta_n^{-1} - \phi \circ \theta^{-1}\| \rightarrow 0$$

where we have used the fact that the inverse operation on $\text{Aut}(M)$ is continuous in the u -topology. As $\text{Span}\{\xi_\phi \mid \phi \in M_*^+\} = \mathcal{H}$, this shows the desired result. \blacksquare

We now ready to prove the eponymous result of this section.

Theorem 3.2.2. *Let M be a “separable” factor and $\theta \in \text{Aut}(M)$. If θ is approximately inner as an automorphism, then θ is approximately inner as a CP map.*

Proof: Let $\theta_n = \text{Ad } u_n$ be a sequence of inner automorphisms that converges to θ in the u -topology. We know by the previous proposition that u_{θ_n} converges strongly to u_θ . We then know that for any $x \in M$,

$$\theta_n(x) = u_n x u_n^* = u_{\theta_n} x u_{\theta_n}^* \xrightarrow{\text{SOT}} u_\theta x u_\theta^* = \theta(x)$$

as the adjoint operation is strongly continuous on unitaries and multiplication is jointly strongly continuous on bounded sequences. Using this convergence condition, it becomes easy to show that Θ is bounded on $\mathbb{C}(M, M')$. Indeed, for any $\xi \in \mathcal{H}$, $x_i, \dots, x_n \in M$ and $y_1, \dots, y_n \in M'$, we have

$$\begin{aligned} \|\Theta(\sum_{i=1}^k x_i y_i)\xi\| &= \|(\sum_{i=1}^k \theta(x_i) y_i)\xi\| \\ &= \lim_n \|\sum_{i=1}^k u_n x_i u_n^* y_i \xi\| \\ &= \lim_n \|u_n (\sum_{i=1}^k x_i y_i) u_n^* \xi\| \quad (u_n^* \in M, y_i \in M') \\ &\leq \lim_n \|\sum_{i=1}^k x_i y_i\| \|u_n^* \xi\| \end{aligned}$$

$$= \left\| \sum_{i=1}^k x_i y_i \right\| \|\xi\|$$

Therefore, $\|\Theta\| \leq 1$. By Cor. 3.1.6, we then know that θ is approximately inner as a CP map. \blacksquare

3.3 Adaptation of Prop. 3.0.4

We introduce an additional topology on $CP(M)$ for a von Neumann algebra M . Consider the family of functions from $CP(M)$ to M defined by $\psi \mapsto \psi(x)$ for $x \in M$. This family separates points and so, if we equip M with the ultrastrong topology, the associated initial topology on $CP(M)$ is Hausdorff and is called the *point ultrastrong topology*. Convergence in this topology can be summarized in the following way: a net (ψ_α) in $CP(M)$ converges to $\psi \in CP(M)$ if for all $x \in M$, $(\psi_\alpha(x))$ converges ultrastrongly to $\psi(x)$ in M . Note that the point ultraweak topology introduced earlier could also have been defined in a similar way. As is common in the field of Banach spaces, these two topologies coincide on convex subsets, a fact that we will use amply in the proof.

Lemma 3.3.1. *Let M be a von Neumann algebra and F be a convex subset of $CP(M)$. Then $\overline{F}^{\text{pt-}UWOT} = \overline{F}^{\text{pt-}USOT}$.*

Proof: As in Prop. 1.2.2, we immediately get $\overline{F}^{\text{pt-}USOT} \subseteq \overline{F}^{\text{pt-}UWOT}$. To show the reverse inclusion, we represent M in standard form (M, \mathcal{H}, J, P) . Prop. 1.6.9 then implies that we only need to show $\overline{F}^{\text{pt-}WOT} \subseteq \overline{F}^{\text{pt-}SOT}$. Let $\Psi \in \overline{F}^{\text{pt-}WOT}$ be arbitrary. Let $x_1, \dots, x_n \in M$, ξ_1, \dots, ξ_n and $\epsilon > 0$ be given; if we can show that there exists $\psi \in F$ such that $\|(\Psi(x_i) - \psi(x_i))\xi_i\| < \epsilon$, this will imply that $\Psi \in \overline{F}^{\text{pt-}SOT}$. Define $x = (x_1, \dots, x_n)$ in the von Neumann algebra $\oplus^n M$ (acting on the Hilbert space $\oplus^n \mathcal{H}$) and $\mathfrak{F} = \{(\psi(x_1), \dots, \psi(x_n)) \mid \psi \in F\} \subseteq \oplus^n M$. As \mathfrak{F} is a convex set in $\oplus^n M$, we have by Prop. 1.3.6 and Thm. A.7 of [Bry13] that $\overline{\mathfrak{F}}^{WOT} = \overline{\mathfrak{F}}^{SOT}$. Since $\Psi \in \overline{F}^{\text{pt-}WOT}$, it is easy to check that $(\Psi(x_1), \dots, \Psi(x_n)) \in \overline{\mathfrak{F}}^{WOT}$. Hence, $(\Psi(x_1), \dots, \Psi(x_n)) \in \overline{\mathfrak{F}}^{SOT}$ and so we can find $\psi \in F$ such that $\|((\Psi - \psi)x_1, \dots, (\Psi - \psi)x_n)(\xi_1, \dots, \xi_n)\| < \epsilon$, which in turn implies $\|((\Psi - \psi)x_i)\xi_i\| < \epsilon$ for all i . \blacksquare

In this section, we let M be a “separable” von Neumann algebra and $\theta \in \text{Aut}(M)$. We suppose that ϕ is a faithful normal state such that $\phi \circ \theta = \phi$, and that θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M_\phi$.

We represent M in standard form (M, \mathcal{H}, J, P) . With left action $(x, \xi) \mapsto x\xi$ and right action $(\xi, x) \mapsto Jx^*J\xi$ for $x \in M$ and $\xi \in \mathcal{H}$, the Hilbert space \mathcal{H} is then a normal M -bimodule. Namely, these maps are bilinear and satisfy

$$x(\xi y) = (x\xi)y$$

for $x, y \in M$ and $\xi \in \mathcal{H}$. Moreover, the mapping $x \mapsto L_x$ from M to $\mathcal{B}(\mathcal{H})$ (the operator defined by left multiplication by x) is a normal unital homomorphism and the mapping $x \mapsto R_x$ (the operator defined by right multiplication by x) is a normal unital antihomomorphism.

The goal of this section is to prove Thm. 3.3.16. The above hypotheses will remain true throughout this section, which is structured in a series of lemmas and corollaries leading to the main theorem. The results between Lemma 3.3.2 and Cor. 3.3.10 allow us to suppose that the maps $x \mapsto \sum_{j=1}^p a_j x a_j^*$ can be chosen such that $x \mapsto \sum_{j=1}^p a_j a_j^* \leq \mathbb{1}_M$ and $x \mapsto \sum_{j=1}^p a_j^* a_j \leq \mathbb{1}_M$. We then apply Haagerup's probability techniques in Lemma 3.3.13, a method which he introduced in [Haa85, Haa89]. Thm. 3.3.16 is finally proven in two steps through Thm. 3.3.14 and Cor. 3.3.15 with the help of ultraproducts of Hilbert spaces (see Appendix D). We recall that the work presented in this section is based on unpublished notes by Giordano and Mingo following the publication of [GM93].

From the hypotheses listed above, we immediately get:

Lemma 3.3.2. θ^{-1} is a point ultrastrong limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_i \in M_\phi$.

Proof: Apply Lemma 3.3.1. ■

Lemma 3.3.3. θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_i \in M_\phi$ and $\sum_{i=1}^p a_i a_i^* \leq \mathbb{1}_M$.

Proof: Let $\phi_1, \dots, \phi_m \in (M_*^+)_1$, $x_1, \dots, x_n \in M_1$ (where the subscript indicates the unit ball) and $0 < \epsilon < 1$. By Lemma 3.3.2, we can choose $b_1, \dots, b_p \in M_\phi$ such that

$$\begin{aligned} \|\theta^{-1}(\mathbb{1}_M) - \sum_{j=1}^p b_j b_j^*\|_{\phi_k} &< (\epsilon/6)^2 \text{ and} \\ |\phi_k(\theta^{-1}(x_i) - \sum_{j=1}^p b_j x_i b_j^*)| &< \epsilon/2. \end{aligned}$$

We define $b = \mathbb{1}_M + (\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M)_+$, which is in M_ϕ as it is a von Neumann algebra. We have $b \geq \mathbb{1}_M$ and therefore $(b^{1/2} - \mathbb{1}_M)^2 \leq b - \mathbb{1}_M$. Now let $a_j = b^{-1/2} b_j \in M_\phi$

and $\Phi(x) = \sum_{j=1}^p a_j x a_j^*$. Note that

$$\begin{aligned} b - \left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right)_- &= \mathbb{1}_M + \left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right)_+ - \left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right)_- \\ &= \mathbb{1}_M + \left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right) = \sum_{j=1}^p b_j b_j^* \end{aligned}$$

Hence,

$$\begin{aligned} \Phi(\mathbb{1}_M) &= b^{-1/2} \sum_{j=1}^p b_j b_j^* b^{-1/2} \\ &= b^{-1/2} \left(b - \left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right)_- \right) b^{-1/2} \leq \mathbb{1}_M, \end{aligned}$$

We also have

$$\begin{aligned} \|b^{1/2} - \mathbb{1}_M\|_{\phi_k}^2 &= \phi_k((b^{1/2} - \mathbb{1}_M)^2) \leq \phi_k(b - \mathbb{1}_M) \\ &= \phi_k\left(\left(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M\right)e_+\right) \\ &\leq \left\| \sum_{j=1}^p b_j b_j^* - \mathbb{1}_M \right\|_{\phi_k} < (\epsilon/6)^2, \end{aligned}$$

where e_+ is the support projection of $(\sum_{j=1}^p b_j b_j^* - \mathbb{1}_M)_+$. In the last line, we have used the Cauchy-Schwarz inequality for the sesquilinear form $(x, y) \mapsto \phi_k(y^*x)$. We then have

$$\begin{aligned} & \left| \phi_k\left(\Phi(x_i) - \sum_{j=1}^p b_j x_i b_j^*\right) \right| \\ &= \left| \phi_k\left((\mathbb{1}_M - b^{1/2})\Phi(x_i) + b^{1/2}\Phi(x_i)(\mathbb{1}_M - b^{1/2})\right) \right| \\ &\leq \|\mathbb{1}_M - b^{1/2}\|_{\phi_k} (\|\Phi(x_i)\|_{\phi_k} + \|\Phi(x_i)^* b^{1/2}\|_{\phi_k}) \\ &\leq \|\mathbb{1}_M - b^{1/2}\|_{\phi_k} (\|\phi_k\|^{1/2} \cdot \|\Phi(x_i)\| + \sqrt{\phi_k(b^{1/2}\Phi(x_i)\Phi(x_i)^*b^{1/2})}) \\ &\leq \|\mathbb{1}_M - b^{1/2}\|_{\phi_k} (\|\Phi\| \cdot \|x_i\| + \sqrt{\phi_k(b^{1/2}\|\Phi(x_i)\|^2 b^{1/2})}) \\ &\leq \|\mathbb{1}_M - b^{1/2}\|_{\phi_k} (1 + \|b^{1/2}\|_{\phi_k}) \\ &\leq \|\mathbb{1}_M - b^{1/2}\|_{\phi_k} (1 + \|\mathbb{1}_M\|_{\phi_k} + \|b^{1/2} - \mathbb{1}_M\|_{\phi_k}) \end{aligned}$$

$$\leq \epsilon/6(2 + \epsilon/6) = \epsilon/3 + \epsilon^2/36 < \epsilon/2$$

And so we finally get

$$\begin{aligned} & |\phi_k(\theta^{-1}(x_i) - \Phi(x_i))| \\ & \leq |\phi_k(\theta^{-1}(x_i) - \sum_{j=1}^p b_j x_i b_j^*)| + |\phi_k(\sum_{j=1}^p b_j x_i b_j^* - \Phi(x_i))| \\ & < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

■

Recall (see for example Prop. 2.5.3 of [Bry13]) that if $N = \bigoplus_{i=1}^n M$, N is a von Neumann algebra with predual $N_* = \bigoplus_{i=1}^n M_*$. If $\varphi = (\varphi_1, \dots, \varphi_n) \in N_*$, then $\|\varphi\| = \sum \|\varphi_i\|$, and for all $\bar{x} = (x_1, \dots, x_n) \in N$, $\phi(\bar{x}) = \sum \varphi_i(x_i)$.

Corollary 3.3.4. *Given $x_1, \dots, x_n \in M$ and $\epsilon > 0$, there exists a_1, \dots, a_k in M_ϕ such that $\sum a_i a_i^* \leq \mathbb{1}_M$ and, for all $y \in M$ and $1 \leq i \leq n$,*

$$|\phi((\theta(x_i) - \sum_k a_k^* x_i a_k) y)| < \epsilon \|y\|$$

Proof: Let $x_1, \dots, x_n \in M$ and $\epsilon > 0$ be fixed. For each $a \in M$, we can define $\phi_a \in M_*$ by $\phi_a(x) = \phi(ax)$. Moreover, we fix $N = \bigoplus_{i=1}^n M$ and $N_* = \bigoplus_{i=1}^n M_*$. Now define

$$\begin{aligned} F &= \{\Phi \in CP(M) \mid \exists p \in \mathbb{N} \text{ and } a_i \in M_\phi, 1 \leq i \leq p, \text{ such that} \\ & \quad \Phi(x) = \sum_{i=1}^p a_i^* x a_i, \text{ for } x \in M, \text{ and } \sum_{i=1}^p a_i a_i^* \leq \mathbb{1}_M\} \end{aligned}$$

and

$$\tilde{F}(\tilde{x}) = \{(\phi_{\Phi(x_1)}, \dots, \phi_{\Phi(x_n)}) \mid \Phi \in F\} \subseteq N$$

Since θ leaves ϕ invariant, we know that for a given map $\Phi \in F$ and $x, y \in M$, we have

$$\phi((\theta(x) - \sum a_i^* x a_i) y) = \phi(x(\theta^{-1}(y) - \sum a_i y a_i^*))$$

Now let $\tilde{z}_1, \dots, \tilde{z}_m \in N$ be arbitrary, with $\tilde{z}_i = (z_{i,1}, \dots, z_{i,n})$ where $z_{i,j} \in M$. By the previous lemma, we can find $b_1, \dots, b_p \in M_\phi$ with $\sum_{k=1}^p b_k b_k^* \leq \mathbb{1}_M$ such that

$$|\phi(x_j(\theta^{-1}(z_{i,j}) - \sum_{k=1}^p b_k z_{i,j} b_k^*))| < \epsilon/n$$

Denoting $S(x) = \sum_{j=1}^p b_k^* x b_k$ for $x \in M$, we then have

$$\begin{aligned}
|\tilde{z}_i((\phi_{\theta(x_1)}, \dots, \phi_{\theta(x_n)}) - (\phi_{S(x_1)}, \dots, \phi_{S(x_n)}))| &= \left| \sum_{j=1}^n (\phi_{\theta(x_j)} - \phi_{S(x_j)})(z_{i,j}) \right| \\
&= \left| \sum_{j=1}^n \phi\left(\theta(x_j) - \sum_{k=1}^p b_k^* x_j b_k\right) z_{i,j} \right| \\
&= \left| \sum_{j=1}^n \phi\left(x_j(\theta^{-1}(z_{i,j}) - \sum_{k=1}^p b_k z_{i,j} b_k^*)\right) \right| \\
&\leq \sum_{j=1}^n \left| \phi\left(x_j(\theta^{-1}(z_{i,j}) - \sum_{k=1}^p b_k z_{i,j} b_k^*)\right) \right| < \epsilon
\end{aligned}$$

Therefore, $(\phi_{\theta(x_1)}, \dots, \phi_{\theta(x_n)})$ is in the closure of $\tilde{F}(\tilde{x})$ in the weak topology σ induced by N on N_* . It is easy to check that $\tilde{F}(\tilde{x})$ is a convex set, and so by Thm. V.1.4 of [Con90], its closure in the topology σ is the same as its norm closure. Hence, $(\phi_{\theta(x_1)}, \dots, \phi_{\theta(x_n)})$ is in the norm closure of $\tilde{F}(\tilde{x})$, which means that we can find $\Phi \in F$ such that $\|(\phi_{\theta(x_1)}, \dots, \phi_{\theta(x_n)}) - (\phi_{\Phi(x_1)}, \dots, \phi_{\Phi(x_n)})\| < \epsilon$. We denote by $a_1, \dots, a_p \in M_\phi$ the elements corresponding to Φ . Now, let $y \in M$ be arbitrary. For a given i , define \tilde{y}_i as the vector in N with y in its i -th coordinate and zero elsewhere. We then have

$$|\phi((\theta(x_i) - \sum_k a_k^* x_i a_k)y)| = |\tilde{y}_i(\phi_{\theta(x_1)}, \dots, \phi_{\theta(x_n)}) - (\phi_{\Phi(x_1)}, \dots, \phi_{\Phi(x_n)})| \leq \|y\| \epsilon$$

■

Definition 3.3.5. Let σ_t^ϕ be the group of modular automorphisms associated to ϕ . We say that an element $x \in M$ is entire if the function $t \in \mathbb{R} \mapsto \sigma_t^\phi(x)$ can be extended to an M -valued entire function over \mathbb{C} , in the sense that $\alpha \in \mathbb{C} \mapsto \omega(\sigma_\alpha^\phi(x))$ is entire for all $\omega \in M_*$. We define M_α^ϕ to be the set of entire elements of M .

Proposition 3.3.6. M_α^ϕ is a σ -weakly dense $*$ -subalgebra of M . For any $x, y \in M_\alpha^\phi$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned}
\sigma_\alpha^\phi(xy) &= \sigma_\alpha^\phi(x)\sigma_\alpha^\phi(y) \\
\sigma_{\alpha+\beta}^\phi(x) &= \sigma_\alpha^\phi(\sigma_\beta^\phi(x)) \\
\sigma_\alpha^\phi(x) &= \sigma_\alpha^\phi(x^*)^*
\end{aligned}$$

Moreover, for any $x \in M$ and $\gamma > 0$, $x_\gamma = \sqrt{\gamma/\pi} \int_{\mathbb{R}} e^{-\gamma t^2} \sigma_t^\phi(x) dt$ is entire. The integral is defined as in Lemma 13.1.1 of [Str81], by taking $\mathcal{X} = M$ and $\mathcal{X}_* = M_*$. Finally, for any $x \in M$ and $y \in M_a^\phi$, we have

$$\phi(xy) = \phi(\sigma_i^\phi(y)x) \quad \text{and} \quad \phi(x\sigma_{-i}^\phi(y)) = \phi(yx)$$

Proof: Lemma VIII.2.3 and Exercise VIII.2.2 in [Tak03]. ■

Lemma 3.3.7. *If $x_0 = \mathbb{1}_M$, $x_1, \dots, x_m \in M_1$, $y_1, \dots, y_n \in (M_a^\phi)_1$ and $\epsilon > 0$, there exists $a_1, \dots, a_p \in M_\phi$ such that $\sum_{k=1}^p a_k^* a_k \leq \mathbb{1}_M$, $\sum_{k=1}^p a_k a_k^* \leq \mathbb{1}_M$ and*

$$|\phi(y_j(\theta(x_i) - \sum a_k^* x_i a_k))| < \epsilon$$

Proof: By Cor. 3.3.4, we can choose $b_1, \dots, b_p \in M_\phi$ such that $\sum b_k b_k^* \leq \mathbb{1}_M$ and for all $y \in M$

$$|\phi((\theta(x_i) - \sum_k b_k^* x_i b_k)y)| < \epsilon \|y\|.$$

We define $b = \mathbb{1}_M + (\sum b_k^* b_k - \mathbb{1}_M)_+$ and $a_k = b_k b^{-1/2}$. Then

$$\sum b_k^* b_k = \mathbb{1}_M + (\sum b_k^* b_k - \mathbb{1}_M)_+ - (\sum b_k^* b_k - \mathbb{1}_M)_- = b - (\sum b_k^* b_k - \mathbb{1}_M)_-$$

and so

$$\sum a_k^* a_k = b^{-1/2} \sum b_k^* b_k b^{-1/2} = b^{-1/2} (b - (\sum b_k^* b_k - \mathbb{1}_M)_-) b^{-1/2} \leq \mathbb{1}_M$$

Finally,

$$\sum a_k a_k^* = \sum b_k b^{-1} b_k^* \leq \sum b_k b_k^* \leq \mathbb{1}_M$$

Moreover,

$$\begin{aligned} \|b^{1/2} - \mathbb{1}_M\|_\phi^2 &= \phi((b^{1/2} - \mathbb{1}_M)^2) \leq \phi(b - \mathbb{1}_M) \\ &= \phi((\sum b_k^* b_k - \mathbb{1}_M)_+) = \phi((\sum b_k^* b_k - \mathbb{1}_M)e_+) \leq \epsilon \end{aligned}$$

where e_+ is the support projection of $(\sum b_k^* b_k - \mathbb{1}_M)_+$. Defining $c_i = \sum a_k^* x_i a_k$ and $d_i = \sum b_k^* x_i b_k$, we then have

$$\begin{aligned} &|\phi(y_j(\theta(x_i) - \sum a_k^* x_i a_k))| \\ &\leq |\phi(y_j(\theta(x_i) - d_i))| + |\phi(y_j(d_i - c_i))| \end{aligned}$$

$$\begin{aligned}
&\leq |\phi((\theta(x_i) - d_i)\sigma_{-i}^\phi(y_j))| + |\phi(y_j(b^{1/2}c_i b^{1/2} - c_i))| \\
&\leq \epsilon + |\phi(y_j b^{1/2} c_i (b^{1/2} - \mathbb{1}_M))| + |\phi(y_j (b^{1/2} - \mathbb{1}_M) c_i)| \\
&\leq \epsilon + \|b^{1/2} y_j^*\|_\phi \|c_i (b^{1/2} - \mathbb{1}_M)\|_\phi + |\phi((b^{1/2} - \mathbb{1}_M) c_i \sigma_{-i}^\phi(y_j))| \\
&\leq \epsilon + |\phi(y_j b y_j^*)|^{1/2} \cdot \|c_i\| \cdot \|b^{1/2} - \mathbb{1}_M\|_\phi + \|b^{1/2} - \mathbb{1}_M\|_\phi \|c_i \sigma_{-i}^\phi(y_j)\|_\phi \\
&\leq \epsilon + |\phi(b y_j^* \sigma_{-i}^\phi(y_j))|^{1/2} \cdot \|b^{1/2} - \mathbb{1}_M\|_\phi + \|b^{1/2} - \mathbb{1}_M\|_\phi \|c_i\| \cdot \|\sigma_{-i}^\phi(y_j)\| \\
&\leq \epsilon + |\phi(b^{1/2} y_j^* \sigma_{-i}^\phi(y_j) b^{1/2})|^{1/2} \sqrt{\epsilon} + \sqrt{\epsilon} \\
&\leq \epsilon + \|b^{1/2}\|_\phi^{1/2} \cdot \|y_j^* \sigma_{-i}^\phi(y_j) b^{1/2}\|_\phi^{1/2} \sqrt{\epsilon} + \sqrt{\epsilon} \\
&\leq \epsilon + \|y_j^* \sigma_{-i}^\phi(y_j)\|^{1/2} \cdot \|b^{1/2}\|_\phi \sqrt{\epsilon} + \sqrt{\epsilon} \\
&\leq \epsilon + \|b^{1/2}\|_\phi \sqrt{\epsilon} + \sqrt{\epsilon} \\
&\leq \epsilon + (1 + \|b^{1/2} - \mathbb{1}_M\|_\phi) \sqrt{\epsilon} + \sqrt{\epsilon} \\
&\leq \epsilon + (1 + \sqrt{\epsilon}) \sqrt{\epsilon} + \sqrt{\epsilon}
\end{aligned}$$

With an appropriate relabeling of ϵ , the result is obtained. ■

Lemma 3.3.8. *For any $y \in M$ and $\epsilon > 0$, we can find $\tilde{y} \in M_a^\phi$ such that $|\phi((y - \tilde{y})c)| \leq \epsilon$ for all $c \in M_1$.*

Proof: Observe that, for $x \in M$, we have

$$\begin{aligned}
\|x - \sigma_t^\phi(x)\|_\phi^2 &= \phi(x^*x) - \phi(\sigma_t^\phi(x^*)x) - \phi(x^* \sigma_t^\phi(x)) + \phi(x^*x) \\
&= \phi(x^*(x - \sigma_t^\phi(x))) + \phi((x^* - \sigma_t^\phi(x^*))x)
\end{aligned}$$

Since $t \rightarrow \sigma_t^\phi(y^*)$ is ultraweakly continuous and $\sigma_0^\phi(y^*) = y^*$, the preceding calculation shows that we can find $\delta > 0$ such that $|t| \leq \delta$ implies $\|y^* - \sigma_t^\phi(y^*)\|_\phi \leq \epsilon$. Next, note that we can find $\gamma \in \mathbb{R}$ large enough such that

$$\sqrt{\frac{\gamma}{\pi}} \int_{|t|>\delta} e^{-\gamma t^2} dt < \epsilon/\|y\|$$

and that the integral over the whole real line has a value of 1. We now define

$$\tilde{y} = \sqrt{\frac{\gamma}{\pi}} \int e^{-\gamma t^2} \sigma_t^\phi(y) dt$$

By Prop. 3.3.6, we have $\tilde{y} \in M_a^\phi$. Now, for any $c \in M_1$, we have

$$|\phi((y - \tilde{y})c)| = \left| \phi \left(\sqrt{\frac{\gamma}{\pi}} \int e^{-\gamma t^2} (y - \sigma_t^\phi(y)) c dt \right) \right|$$

$$\begin{aligned}
&\leq \sqrt{\frac{\gamma}{\pi}} \int e^{-\gamma t^2} |\phi((y - \sigma_t^\phi(y))c)| dt \\
&= \sqrt{\frac{\gamma}{\pi}} \int_{|t| < \delta} e^{-\gamma t^2} |\phi((y - \sigma_t^\phi(y))c)| dt + \sqrt{\frac{\gamma}{\pi}} \int_{|t| > \delta} e^{-\gamma t^2} |\phi((y - \sigma_t^\phi(y))c)| dt \\
&\leq \sup_{|t| < \delta} \|y^* - \sigma_t^\phi(y^*)\|_\phi \sqrt{\frac{\gamma}{\pi}} \int_{|t| < \delta} e^{-\gamma t^2} dt + \sqrt{\frac{\gamma}{\pi}} \int_{|t| > \delta} e^{-\gamma t^2} \|y - \sigma_t^\phi(y)\| \cdot \|c\| dt \\
&\leq \epsilon \sqrt{\frac{\gamma}{\pi}} \int e^{-\gamma t^2} dt + 2\|y\| \sqrt{\frac{\gamma}{\pi}} \int_{|t| > \delta} e^{-\gamma t^2} dt \\
&\leq \epsilon + 2\epsilon = 3\epsilon
\end{aligned}$$

■

Corollary 3.3.9. *If $x_1, \dots, x_n \in M_1$, $\psi_1, \dots, \psi_m \in M_*$ and $\epsilon > 0$, there exists $a_1, \dots, a_p \in M_\phi$ such that $\sum_{k=1}^p a_k^* a_k \leq \mathbb{1}_M$, $\sum_{k=1}^p a_k a_k^* \leq \mathbb{1}_M$ and*

$$|\psi_j(\theta(x_i) - \sum a_k^* x_i a_k)| < \epsilon$$

Proof: For each $a \in M$, we can define $\phi_a \in M_*$ by $\phi_a(x) = \phi(ax)$. Since ϕ is faithful and M is the dual of M_* , $\{\phi_a \mid a \in M\}$ is dense in M_* (see Cor. 2.3.5 of [Ped89]). Choose $y_1, \dots, y_m \in M$ such that $\|\psi_j - \phi_{y_j}\| \leq \epsilon$. For each y_j , let \tilde{y}_j be the corresponding element of M_a^ϕ as per the previous lemma. By Lemma 3.3.7, we can find $a_1, \dots, a_p \in M_\phi$ such that $\sum_{k=1}^p a_k^* a_k \leq \mathbb{1}_M$, $\sum_{k=1}^p a_k a_k^* \leq \mathbb{1}_M$ and

$$|\phi(\tilde{y}_j(\theta(x_i) - \sum a_k^* x_i a_k))| < \epsilon, \text{ for } 1 \leq i \leq n.$$

We then have

$$\begin{aligned}
|\psi_j(\theta(x_i) - \sum a_k^* x_i a_k)| &\leq \|\psi_j - \phi_{y_j}\| \|\theta(x_i) - \sum a_k^* x_i a_k\| + |\phi_{y_j}(\theta(x_i) - \sum a_k^* x_i a_k)| \\
&\leq 2\epsilon + |\phi((y_j - \tilde{y}_j)(\theta(x_i) - \sum a_k^* x_i a_k))| + |\phi(\tilde{y}_j(\theta(x_i) - \sum a_k^* x_i a_k))| \\
&\leq 2\epsilon + 2\epsilon + \epsilon = 5\epsilon
\end{aligned}$$

■

Corollary 3.3.10. *θ is a point ultrastrong limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j^* x a_j$ with $a_j \in M_\phi$ s.t. $\sum_{j=1}^p a_j a_j^* \leq \mathbb{1}_M$ and $\sum_{j=1}^p a_j^* a_j \leq \mathbb{1}_M$.*

Proof: The preceding corollary showed that θ is a point ultraweak limit of a net of elements of this type of completely positive maps. Combining this with Lemma 3.3.1, we have the desired result. \blacksquare

Lemma 3.3.11. *Let $u_1, \dots, u_m \in M$ be unitaries and $\epsilon > 0$. Then there exists $a_1, \dots, a_p \in M_\phi$ such that*

$$\begin{aligned} \sum_{k=1}^p a_k^* a_k &\leq \mathbb{1}_M, & \sum_{k=1}^p a_k a_k^* &\leq \mathbb{1}_M \\ \left\| \sum_{k=1}^p a_k^* a_k - \mathbb{1}_M \right\|_\phi &< \epsilon, & \sum_{k=1}^p \|a_k \theta(u_i) - u_i a_k\|_\phi^2 &< \epsilon \\ & & \sum_{k=1}^p \|a_k^* u_i - \theta(u_i) a_k^*\|_\phi^2 &< \epsilon \end{aligned}$$

Proof: Let us set $u_0 = \mathbb{1}_M$. By Cor. 3.3.10, we can choose $a_1, \dots, a_n \in M_\phi$ such that

$$\begin{aligned} \sum_{k=1}^p a_k^* a_k &\leq \mathbb{1}_M, & \sum_{k=1}^p a_k a_k^* &\leq \mathbb{1}_M \\ \left\| \sum_{k=1}^p a_k^* u_i a_k - \theta(u_i) \right\|_\phi &< \epsilon/2 \end{aligned}$$

for $0 \leq i \leq m$. We then have

$$\begin{aligned} \sum_{k=1}^p \|a_k \theta(u_i) - u_i a_k\|_\phi^2 &= \sum_{k=1}^p \phi((a_k \theta(u_i) - u_i a_k)^* (a_k \theta(u_i) - u_i a_k)) \\ &= \sum_{k=1}^p \phi(\theta(u_i)^* a_k^* a_k \theta(u_i) + a_k^* a_k - a_k^* u_i^* a_k \theta(u_i) - \theta(u_i)^* a_k^* u_i a_k) \end{aligned}$$

As $\sum_{k=1}^p \phi(\theta(u_i)^* a_k^* a_k \theta(u_i)) \leq \phi(\theta(u_i)^* \theta(u_i))$ and $\phi(\sum_{k=1}^p a_k^* a_k) \leq \phi(\mathbb{1}_M) = \phi(\theta(u_i)^* \theta(u_i))$, we have

$$\begin{aligned} \sum_{k=1}^p \|a_k \theta(u_i) - u_i a_k\|_\phi^2 &\leq \phi(\theta(u_i)^* \theta(u_i)) - \left(\sum_{k=1}^p a_k^* u_i^* a_k \right) \theta(u_i) + \theta(u_i)^* \theta(u_i) - \theta(u_i)^* \left(\sum_{k=1}^p a_k^* u_i a_k \right) \\ &\leq \phi((\theta(u_i)^* - \sum_{k=1}^p a_k^* u_i^* a_k) \theta(u_i)) + \phi(\theta(u_i)^* (\theta(u_i) - \sum_{k=1}^p a_k^* u_i a_k)) \end{aligned}$$

$$\begin{aligned}
&\leq \|\theta(u_i) - \sum_{k=1}^p a_k^* u_i a_k\|_\phi \cdot \|\theta(u_i)\|_\phi + \|\theta(u_i)\|_\phi \cdot \|\theta(u_i) - \sum_{k=1}^p a_k^* u_i a_k\|_\phi \\
&\leq 2\|\theta(u_i) - \sum_{k=1}^p a_k^* u_i a_k\|_\phi < \epsilon
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{k=1}^p \|a_k^* u_i - \theta(u_i) a_k^*\|_\phi^2 &= \sum_{k=1}^p \phi((a_k^* u_i - \theta(u_i) a_k^*)^* (a_k^* u_i - \theta(u_i) a_k^*)) \\
&= \sum_{k=1}^p \phi(u_i^* a_k a_k^* u_i + a_k a_k^* - u_i^* a_k \theta(u_i) a_k^* - a_k \theta(u_i)^* a_k^* u_i)
\end{aligned}$$

Note that

$$\sum_{k=1}^p \phi(u_i^* a_k^* a_k u_i) \leq \phi(u_i^* u_i) = \phi(\theta(u_i)^* \theta(u_i))$$

and

$$\phi\left(\sum_{k=1}^p a_k a_k^*\right) \leq \phi(\mathbb{1}_M) = \phi(\theta(u_i)^* \theta(u_i))$$

Therefore, using the fact that $a_k \in M_\phi$, we have

$$\begin{aligned}
\sum_{k=1}^p \|a_k^* u_i - \theta(u_i) a_k^*\|_\phi^2 &\leq \phi(\theta(u_i)^* \theta(u_i)) - \left(\sum_{k=1}^p a_k^* u_i^* a_k\right) \theta(u_i) + \theta(u_i^*) \theta(u_i) - \theta(u_i)^* \left(\sum_{k=1}^p a_k^* u_i a_k\right) \\
&< \epsilon
\end{aligned}$$

as before. Finally,

$$\left\| \sum_{k=1}^p a_k^* a_k - \mathbb{1}_M \right\|_\phi = \left\| \sum_{k=1}^p a_k^* u_0 a_k - \theta(u_0) \right\|_\phi < \epsilon/2$$

■

Lemma 3.3.12. *For $x \in M_\phi$, we have $\xi_\phi x = x \xi_\phi$, where ξ_ϕ is the vector lying in the cone P and representing ϕ , i.e. $\phi(x) = (x \xi_\phi | \xi_\phi)$ for all $x \in M$.*

Proof: For any $t \in \mathbb{R}$, we have

$$\Delta^{it} x \xi_\phi = \Delta^{it} x \Delta^{-it} \Delta^{it} \xi_\phi = \sigma_t^\phi(x) \xi_\phi = x \xi_\phi$$

since $\Delta^{it} \xi_\phi = \xi_\phi$ (see 10.24 in [SZ79]). By Corollary 9.15 of [SZ79], we then have $\Delta^\alpha x \xi_\phi = x \xi_\phi$ for all $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 0$. Finally, we have

$$\xi_\phi x = J x^* J \xi_\phi = J x^* \xi_\phi = J S(x \xi_\phi) = \Delta^{1/2} x \xi_\phi = x \xi_\phi$$

■

Lemma 3.3.13. *For unitaries, $u_1, \dots, u_n \in M$, $\epsilon > 0$ and an integer $r < 1/\epsilon$, there are $c_1, \dots, c_r \in M_\phi$ such that*

$$\begin{aligned} \|c_i\| &< \sqrt{r} \\ \left\| \sum_{i=1}^r c_i^* c_i - \mathbb{1}_M \right\|_\phi^2 &< 24/r \\ \left\| \sum_{i=1}^r c_i c_i^* - \mathbb{1}_M \right\|_\phi^2 &< 24/r \\ \sum_{i=1}^r \|c_i \theta(u_j) - u_j c_i\|_\phi^2 &< 8n\epsilon \\ \sum_{i=1}^r \|c_i^* u_j - \theta(u_j) c_i^*\|_\phi^2 &< 8n\epsilon \end{aligned}$$

Proof: Using Lemma 3.3.11, we choose $a_1, \dots, a_m \in M_\phi$ such that

$$\begin{aligned} \sum_{j=1}^m a_j^* a_j &\leq \mathbb{1}_M, & \sum_{j=1}^m a_j a_j^* &\leq \mathbb{1}_M \\ \left\| \sum_{j=1}^m a_k^* a_k - \mathbb{1}_M \right\|_\phi &< \epsilon \\ \sum_{j=1}^m \|a_j \theta(u_i) - u_i a_j\|_\phi^2 &< \epsilon \\ \sum_{j=1}^m \|a_j^* u_i - \theta(u_i) a_j^*\|_\phi^2 &< \epsilon \end{aligned}$$

Note that $\phi(\mathbb{1}_M - \sum_{k=1}^m a_k^* a_k) \leq \|\sum_{k=1}^m a_k^* a_k - \mathbb{1}_M\|_\phi < \epsilon$, and so $\phi(\sum_{k=1}^m a_k^* a_k) > 1 - \epsilon > 1 - 1/r$. What follows is based on Lemma 4.3 of [Haa85]. We let $\Omega =$

$\{(s_1, \dots, s_m) \mid s_i \in \mathbb{C}, |s_i| = 1\}$ (the complex m -torus), s_i be the i -th coordinate function on Ω , $d\omega$ be the normalized Haar measure on Ω . We set $\Omega^r = \Omega \times \dots \times \Omega$, the Cartesian product of r copies of Ω , and consider the product measure $d\bar{\omega}$. We define the function $a(\omega) = \sum_{i=1}^m s_i(\omega)a_i$ for $\omega \in \Omega$. Since ϕ is a trace on M_ϕ , one can follow the calculations of Lemma 4.3 of [Haa85] to obtain

$$\int_{\Omega^r} \left\| \frac{1}{r} \sum_{i=1}^r a(\omega_i)^* a(\omega_i) - \mathbb{1}_M \right\|_\phi^2 d\bar{\omega} < 3/r$$

$$\int_{\Omega^r} \left\| \frac{1}{r} \sum_{i=1}^r a(\omega_i) a(\omega_i)^* - \mathbb{1}_M \right\|_\phi^2 d\bar{\omega} < 3/r$$

Since $d\bar{\omega}$ is a product measure, we also have

$$\int_{\Omega^r} \frac{1}{r} \sum_{i=1}^r \|a(\omega_i)\theta(u_k) - u_k a(\omega_i)\|_\phi^2 d\bar{\omega} = \int_{\Omega} \|a(\omega)\theta(u_k) - u_k a(\omega)\|_\phi^2 d\omega < \epsilon$$

$$\int_{\Omega^r} \frac{1}{r} \sum_{i=1}^r \|a(\omega_i)^* u_k - \theta(u_k) a(\omega_i)^*\|_\phi^2 d\bar{\omega} = \int_{\Omega} \|a(\omega)^* u_k - \theta(u_k) a(\omega)^*\|_\phi^2 d\omega < \epsilon$$

We can now define

$$E = \{\bar{\omega} \in \Omega^r \mid \|1/r \sum_{i=1}^r a(\omega_i)^* a(\omega_i) - \mathbb{1}_M\|_\phi^2 \geq 12/r\}$$

$$F = \{\bar{\omega} \in \Omega^r \mid \|1/r \sum_{i=1}^r a(\omega_i) a(\omega_i)^* - \mathbb{1}_M\|_\phi^2 \geq 12/r\}$$

$$G_k = \{\bar{\omega} \in \Omega^r \mid 1/r \sum_{i=1}^r \|a(\omega_i)\theta(u_k) - u_k a(\omega_i)\|_\phi^2 \geq 4n\epsilon\}$$

$$H_k = \{\bar{\omega} \in \Omega^r \mid 1/r \sum_{i=1}^r \|a(\omega_i)^* u_k - \theta(u_k) a(\omega_i)^*\|_\phi^2 \geq 4n\epsilon\}$$

The preceding calculations then show that

$$m(E) < 1/4$$

$$m(F) < 1/4$$

$$m(G_k) < 1/(4n)$$

$$m(H_k) < 1/(4n)$$

where m denotes the measure $d\bar{\omega}$ on Ω^r . Therefore, the measure of the set $E \cup F \cup_{i=1}^n (G_i \cup H_i)$ is less than 1, and so we can find a $\tilde{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^r$ such that

$$\|1/r \sum_{i=1}^r a(\omega_i)^* a(\omega_i) - \mathbb{1}_M\|_\phi^2 < 12/r$$

$$\begin{aligned} \left\| \frac{1}{r} \sum_{i=1}^r a(\omega_i) a(\omega_i)^* - \mathbb{1}_M \right\|_\phi^2 &< 12/r \\ \frac{1}{r} \sum_{i=1}^r \|a(\omega_i) \theta(u_k) - u_k a(\omega_i)\|_\phi^2 &< 4n\epsilon \\ \frac{1}{r} \sum_{i=1}^r \|a(\omega_i)^* u_k - \theta(u_k) a(\omega_i)^*\|_\phi^2 &< 4n\epsilon \end{aligned}$$

for all k . Therefore, defining $b_i = (1/\sqrt{r})a(\omega_i) \in M_\phi$, we have

$$\begin{aligned} \left\| \sum_{i=1}^r b_i^* b_i - \mathbb{1}_M \right\|_\phi^2 &< 12/r \\ \left\| \sum_{i=1}^r b_i b_i^* - \mathbb{1}_M \right\|_\phi^2 &< 12/r \\ \sum_{i=1}^r \|b_i \theta(u_k) - u_k b_i\|_\phi^2 &< 4n\epsilon \\ \sum_{i=1}^r \|b_i^* u_k - \theta(u_k) b_i^*\|_\phi^2 &< 4n\epsilon \end{aligned}$$

for all k .

We define $c_i = g(b_i b_i^*) b_i$, where the function g is defined as

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq r \\ \sqrt{r/t} & \text{if } r < t \end{cases}$$

Note that with the function $h(t) = tg(t)^2$, we have $c_i^* c_i = h(b_i^* b_i)$, and so $\|c_i\| \leq \sqrt{r}$. Since $b_i, c_i \in M_\phi$ and ϕ is a trace on M_ϕ , one can follow the calculations in Lemma 4.4 of [Haa85] to show that (see Appendix A for a clarification)

$$\begin{aligned} \left\| \sum_{i=1}^r c_i^* c_i - \mathbb{1}_M \right\|_\phi^2 &\leq 2 \left\| \sum_{i=1}^r b_i^* b_i - \mathbb{1}_M \right\|_\phi^2 < 24/r \\ \left\| \sum_{i=1}^r c_i c_i^* - \mathbb{1}_M \right\|_\phi^2 &\leq 2 \left\| \sum_{i=1}^r b_i b_i^* - \mathbb{1}_M \right\|_\phi^2 < 24/r \end{aligned}$$

Let

$$f(t) = \begin{cases} -\sqrt{r} & \text{if } t < -\sqrt{r} \\ t & \text{if } -\sqrt{r} \leq t \leq \sqrt{r} \\ \sqrt{r} & \text{if } \sqrt{r} < t \end{cases}$$

Note that this function is a contraction, i.e. $|f(s) - f(t)| \leq |s - t|$ for $s, t \in \mathbb{R}$. In $\mathbb{M}_2(M)$, we define the elements

$$h_i = \begin{pmatrix} 0 & b_i^* \\ b_i & 0 \end{pmatrix} \quad s_k = \begin{pmatrix} \theta(u_k) & 0 \\ 0 & u_k \end{pmatrix}$$

and the faithful state

$$\phi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \phi(a + d)/2$$

As in Lemma 4.4 of [Haa85], we then have

$$f(h_i) = \begin{pmatrix} 0 & c_i^* \\ c_i & 0 \end{pmatrix}$$

By direct computation, one can check that

$$\begin{aligned} 2\|h_i s_k - s_k h_i\|_{\phi_2}^2 &= \|b_i \theta(u_k) - u_k b_i\|_{\phi}^2 + \|b_i^* u_k - \theta(u_k) b_i^*\|_{\phi}^2 \\ 2\|f(h_i) s_k - s_k f(h_i)\|_{\phi_2}^2 &= \|c_i \theta(u_k) - u_k c_i\|_{\phi}^2 + \|c_i^* u_k - \theta(u_k) c_i^*\|_{\phi}^2 \end{aligned}$$

Recall that M is represented in standard form on a Hilbert space \mathcal{H} . Consider the Hilbert space $\mathbb{M}_2(\mathcal{H})$ with the obvious inner product inherited from \mathcal{H}^4 . For each i , let L_{h_i} (resp. R_{h_i}) be the operator on $\mathbb{M}_2(\mathcal{H})$ defined by left (resp. right) multiplication and left (resp. right) action by h_i . These are two commuting self adjoint operators, and so we can apply the multivariable version of the continuous functional calculus (see 4.4.2 in [Ped89]) to obtain a representation $\pi : \mathbb{C}(\sigma(L_{h_i}) \times \sigma(R_{h_i})) \rightarrow \mathcal{B}(\mathbb{M}_2(\mathcal{H}))$ which satisfies $k(s)g(t) \mapsto k(L_{h_i})g(R_{h_i})$ for $k \in C(\sigma(L_{h_i}))$ and $g \in C(\sigma(R_{h_i}))$. One can check that we actually have $k(L_{h_i}) = L_{g(h_i)}$ and $g(R_{h_i}) = R_{g(h_i)}$ for $k \in C(\sigma(L_{h_i}))$ and $g \in C(\sigma(R_{h_i}))$. Since the previously defined function f is contractive, i.e. $|f(s) - f(t)| \leq |s - t|$ for $s, t \in \mathbb{R}$, we get $(L_{f(h_i)} - R_{f(h_i)})^2 \leq (L_{h_i} - R_{h_i})^2$.

Now note that

$$\xi_{\phi_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{\phi} & 0 \\ 0 & \xi_{\phi} \end{pmatrix} \in \mathbb{M}_2(\mathcal{H})$$

is a cyclic and separating vector for $\mathbb{M}_2(M)$ realizing ϕ_2 . Moreover, by Lemma 3.3.12,

$$s_k \xi_{\phi_2} h_i = s_k \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{\phi} & 0 \\ 0 & \xi_{\phi} \end{pmatrix} \begin{pmatrix} 0 & b_i^* \\ b_i & 0 \end{pmatrix} = s_k \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \xi_{\phi} b_i^* \\ \xi_{\phi} b_i & 0 \end{pmatrix} = s_k \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & b_i^* \xi_{\phi} \\ b_i \xi_{\phi} & 0 \end{pmatrix} = s_k h_i \xi_{\phi_2}$$

Similarly, $s_k \xi_{\phi_2} f(h_i) = s_k f(h_i) \xi_{\phi_2}$. Therefore,

$$\|f(h_i) s_k - s_k f(h_i)\|_{\phi_2}^2 = \|f(h_i) s_k \xi_{\phi_2} - s_k f(h_i) \xi_{\phi_2}\|_{\phi_2}^2$$

$$\begin{aligned}
&= \|f(h_i)s_k\xi_{\phi_2} - s_k\xi_{\phi_2}f(h_i)\|^2 \\
&\leq \|h_ik_s\xi_{\phi_2} - s_k\xi_{\phi_2}h_i\|^2 \\
&\leq \|h_ik_s\xi_{\phi_2} - s_kh_i\xi_{\phi_2}\|^2 \\
&\leq \|h_ik_s - s_kh_i\|_{\phi_2}^2
\end{aligned}$$

for all i . Hence, for all k ,

$$\begin{aligned}
\sum_{i=1}^r \|c_i\theta(u_k) - u_kc_i\|_{\phi}^2 &\leq \sum_{i=1}^r 2\|f(h_i)s_k - s_kf(h_i)\|_{\phi_2}^2 \\
&\leq \sum_{i=1}^r 2\|h_ik_s - s_kh_i\|_{\phi_2}^2 \\
&= \sum_{i=1}^r (\|b_i\theta(u_k) - u_kb_i\|_{\phi}^2 + \|b_i^*u_k - \theta(u_k)b_i^*\|_{\phi}^2) \\
&\leq 8n\epsilon
\end{aligned}$$

Similarly, one gets

$$\sum_{i=1}^r \|c_i^*u_k - \theta(u_k)c_i^*\|_{\phi}^2 \leq 8n\epsilon$$

for all k , which finishes the proof. ■

Theorem 3.3.14. *For every $\epsilon > 0$ and unitaries $u_1, \dots, u_n \in M$, there is a unitary $w \in M_{\phi}$ such that*

$$\|\theta(u_k) - w^*u_kw\|_{\phi} = \|w\theta(u_k) - u_kw\|_{\phi} < \epsilon$$

Proof: We first prove the result for $n = 1$. Let F be a free ultrafilter on \mathbb{N} (see Appendix D), let \mathcal{H}_F be the ultraproduct of the constant sequence $\mathcal{H}_m = \mathcal{H}$ along F , and let $\mathcal{M} = \oplus_{m \in \mathbb{N}} M_{\phi}$. We define $\eta = (u\xi_{\phi}, u\xi_{\phi}, \dots) \in \mathcal{H}_F$ and $\xi = (\theta(u)\xi_{\phi}, \theta(u)\xi_{\phi}, \dots) \in \mathcal{H}_F$, and let r be an arbitrary positive integer. We choose for each $m \in \mathbb{N}$ an ϵ_m s.t. $\epsilon_m < 1/m$ and $\epsilon_m < 1/r$. From the previous lemma, we also choose for each $m \in \mathbb{N}$ a list $c_1^{(m)}, \dots, c_r^{(m)} \in M_{\phi}$, such that $\|c_i^{(m)}\| < \sqrt{r}$ and

$$\begin{aligned}
\left\| \sum_{i=1}^r c_i^{(m)*} c_i^{(m)} - \mathbb{1}_M \right\|_{\phi}^2 &< 24/r \\
\left\| \sum_{i=1}^r c_i^{(m)} c_i^{(m)*} - \mathbb{1}_M \right\|_{\phi}^2 &< 24/r
\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^r \|c_i^{(m)}\theta(u) - uc_i^{(m)}\|_\phi^2 &< 8\epsilon_m \\ \sum_{i=1}^r \|c_i^{(m)*}u - \theta(u)c_i^{(m)*}\|_\phi^2 &< 8\epsilon_m\end{aligned}$$

We then have

$$\begin{aligned}\left(\sum_{i=1}^r c_i^{(m)*}c_i^{(m)} - \mathbb{1}_M\right)\theta(u) &= \sum_{i=1}^r c_i^{(m)*}(c_i^{(m)}\theta(u) - uc_i^{(m)}) + \sum_{i=1}^r (c_i^{(m)*}u - \theta(u)c_i^{(m)*})c_i^{(m)} \\ &\quad + \theta(u)\left(\sum_{i=1}^r c_i^{(m)*}c_i^{(m)} - \mathbb{1}_M\right)\end{aligned}$$

Hence

$$\begin{aligned}\left\|\left(\sum_{i=1}^r c_i^{(m)*}c_i^{(m)} - \mathbb{1}_M\right)\theta(u)\right\|_\phi &\leq \sum_{i=1}^r \|c_i^{(m)}\| \left(\|c_i^{(m)}\theta(u) - uc_i^{(m)}\|_\phi + \|c_i^{(m)*}u - \theta(u)c_i^{(m)*}\|_\phi\right) \\ &\quad + \left\|\sum_{i=1}^r c_i^{(m)*}c_i^{(m)} - \mathbb{1}_M\right\|_\phi \\ &\leq \sqrt{r} \sqrt{\sum_{i=1}^r 1} \left(\sqrt{\sum_{i=1}^r \|c_i^{(m)}\theta(u) - uc_i^{(m)}\|_\phi^2} \right. \\ &\quad \left. + \sqrt{\sum_{i=1}^r \|c_i^{(m)*}u - \theta(u)c_i^{(m)*}\|_\phi^2} \right) + \sqrt{\frac{24}{r}} \\ &\leq 2r\sqrt{8\epsilon_m} + \sqrt{\frac{24}{r}}\end{aligned}$$

Similarly,

$$\left\|\left(\sum_{i=1}^r c_i^{(m)}c_i^{(m)*} - \mathbb{1}_M\right)u\right\|_\phi \leq 2r\sqrt{8\epsilon_m} + \sqrt{\frac{24}{r}}$$

Now let $c_i = (c_i^{(1)}, c_i^{(2)}, \dots) \in \mathcal{M}$. We then have

$$\left\|\left(\sum_{i=1}^r c_i c_i^* - \mathbb{1}_M\right)\eta\right\|^2 = \lim_{m \rightarrow F} \left\|\left(\sum_{i=1}^r c_i^{(m)}c_i^{(m)*} - \mathbb{1}_M\right)u\right\|_\phi^2 \leq \lim_{m \rightarrow F} \left(2r\sqrt{8\epsilon_m} + \sqrt{\frac{24}{r}}\right)^2 = \frac{24}{r}$$

$$\begin{aligned}
\left\| \left(\sum_{i=1}^r c_i^* c_i - \mathbb{1}_M \right) \xi \right\|^2 &= \lim_{m \rightarrow F} \left\| \left(\sum_{i=1}^r c_i^{(m)*} c_i^{(m)} - \mathbb{1}_M \right) \theta(u) \right\|_\phi^2 \leq \lim_{m \rightarrow F} \left(2r\sqrt{8\epsilon_m} + \sqrt{\frac{24}{r}} \right)^2 = \frac{24}{r} \\
\sum_{i=1}^r \|c_i \xi - \eta c_i\|^2 &= \lim_{m \rightarrow F} \sum_{i=1}^r \|c_i^{(m)} \theta(u) - u c_i^{(m)}\|_\phi^2 \leq \lim_{m \rightarrow F} 8\epsilon_m = 0 \\
\sum_{i=1}^r \|c_i^* \eta - \xi c_i^*\|^2 &= \lim_{m \rightarrow F} \sum_{i=1}^r \|c_i^{(m)*} u - \theta(u) c_i^{(m)*}\|_\phi^2 \leq \lim_{m \rightarrow F} 8\epsilon_m = 0
\end{aligned}$$

Note that we have implicitly used Lemma 3.3.12 in the last two equations. Therefore, $c_i \xi = \eta c_i$ and $c_i^* \eta = \xi c_i^*$, for $1 \leq i \leq r$. As this construction is valid for all $r \in \mathbb{N}$, we know by Lemma 2.6 of [Haa89] that there exists a unitary $\tilde{w} \in \mathcal{M}$ such that $\|\tilde{w}\xi - \eta\tilde{w}\| < \epsilon$. Since $\mathcal{M} = \bigoplus_{m \in \mathbb{N}} M_\phi$, this unitary has to be of the form $\tilde{w} = (\tilde{w}_m)_{m \in \mathbb{N}}$ with, $\tilde{w}_m \in M_\phi$. We then have $\lim_F \|\tilde{w}_m \theta(u) \xi_\phi - u \xi_\phi \tilde{w}_m\| = \|\tilde{w}\xi - \eta\tilde{w}\| < \epsilon$. Therefore, we can find a unitary $w \in M_\phi$ such that $\|w\theta(u) - uw\|_\phi = \|w\theta(u)\xi_\phi - u\xi_\phi w\| < \epsilon$. As $\|\theta(u) - w^*uw\|_\phi = \|w\theta(u) - uw\|_\phi$, this finishes the proof for $n = 1$.

For the case $n \geq 2$, we replace \mathcal{H} by $\mathcal{H}^{(n)} = \mathcal{H} \oplus \mathcal{H} \oplus \dots \oplus \mathcal{H}$, η by $u_1 \xi_\phi \oplus u_2 \xi_\phi \oplus \dots \oplus u_n \xi_\phi$ and similarly for ξ . The proof is then completely analogous with component wise left and right actions. \blacksquare

Corollary 3.3.15. *The map θ is an approximately inner automorphism.*

Proof: Since every element in M is a linear combination of four unitaries, the previous result immediately implies that for every $\epsilon > 0$ and $x_1, \dots, x_n \in M$, there is a unitary $w \in M_\phi$ such that

$$\|\theta(x_i) - wx_i w^*\|_\phi < \epsilon$$

Now let $x_1, \dots, x_n \in M$, $\phi_1, \dots, \phi_m \in M_*$ and $\epsilon > 0$ be arbitrary. By Prop. 3.1.10, let (ψ_k) be a sequence in $[0, \phi]$ such that $\text{span}(\psi_k)$ is dense in M_* . For $1 \leq j \leq m$, we can choose $\alpha_1^{(j)}, \dots, \alpha_{q_j}^{(j)} \in \mathbb{C}$ and $\psi_{k_1^{(j)}}, \dots, \psi_{k_{q_j}^{(j)}}$ such that $\|\phi_j - \sum_\ell \alpha_\ell^{(j)} \psi_{k_\ell^{(j)}}\| < \epsilon / (2 \max_i \{\|x_i\|\})$. Finally, we can choose a unitary $w \in M_\phi$ such that

$$\|\theta(x_i) - wx_i w^*\|_\phi \leq \epsilon / \max_j \left\{ \sum_\ell |\alpha_\ell^{(j)}| \right\}$$

We then have

$$\begin{aligned}
|\phi_j(\theta(x_i) - Ad_w(x_i))| &\leq |(\phi_j - \sum_\ell \alpha_\ell^{(j)} \psi_{k_\ell^{(j)}})(\theta(x_i) - Ad_w(x_i))| \\
&\quad + \left| \sum_\ell \alpha_\ell^{(j)} \psi_{k_\ell^{(j)}}(\theta(x_i) - Ad_w(x_i)) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|\phi_j - \sum_{\ell} \alpha_{\ell}^{(j)} \psi_{k_{\ell}^{(j)}}\| 2\|x_i\| + \sum_{\ell} |\alpha_{\ell}^{(j)}| \cdot |\psi_{k_{\ell}^{(j)}}(\theta(x_i) - Ad_w(x_i))| \\
&\leq \epsilon + \sum_{\ell} |\alpha_{\ell}^{(j)}| \cdot \|\theta(x_i) - Ad_w(x_i)\|_{\psi_{\ell}^{(j)}} \\
&\leq \epsilon + \|\theta(x_i) - Ad_w(x_i)\|_{\phi} \sum_{\ell} |\alpha_{\ell}^{(j)}| \leq 2\epsilon
\end{aligned}$$

where we have used in the last line the fact that $\psi_k \leq \phi$ implies $\|x\|_{\psi_k} \leq \|x\|_{\phi}$ for all $x \in M$. This shows that θ is in the closure in the p -topology of the set $\{Ad_w \mid w \in M_{\phi}\}$. By Cor. 2.25 of [Str81], we know that the p -topology and the u -topology coincide on the set of automorphisms that leave ϕ invariant. Therefore, we conclude that θ is in the closure in the u -topology of the set $\{Ad_w \mid w \in M_{\phi}\}$, i.e. it is an approximately inner automorphism. \blacksquare

Therefore, the result we have proven in this section is the following:

Theorem 3.3.16. *Let M be a “separable” von Neumann algebra, $\theta \in Aut(M)$ and ϕ a faithful normal state such that $\phi \circ \theta = \phi$. Suppose that θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M_{\phi}$. Then θ is an approximately inner automorphism.*

3.4 Adaptation of Prop. 3.0.3

In this section, we discuss the adaptation of Prop. 3.0.3 in the case of a type III_{λ} factor, so that it can be applied to prove a result of the form (*) (see the introduction to Chap. 3). The idea is to modify it in the following way:

(**) *Let M be a “separable” von Neumann factor of type III_{λ} , $\theta \in Aut(M)$ and ϕ a faithful normal state such that $\phi \circ \theta = \phi$. Suppose that θ is approximately inner as a CP map and that we impose a set of conditions S on M . Then θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M_{\phi}$.*

Combined with Thm. 3.3.16, this would imply (*). Of course, since it has been shown that Conjecture A does not hold for a factor of type III_{λ} , we know that the set of conditions S must be nonempty.

As it has been shown in [Mar20] that (*) stands if $S = \{M \text{ has property } L'_{\lambda}\}$, the first objective would be to obtain (**) with the same condition on M . Although we have not been able to show this result, we briefly describe qualitatively here the attempt that has been made in that way, and point out how this attempt fails for now.

In the following, we suppose that M is a “separable” von Neumann factor of type III_λ satisfying the property L'_λ . We let $\theta \in \text{Aut}(M)$ be approximately inner as a CP map. As explained in Appendix C, we suppose that there is a faithful normal state ϕ on M with $\phi \circ \theta = \phi$ and $\sigma_T^\phi = \text{id}_M$ for $T = -2\pi/\ln \lambda$. Let $\chi = e^{-2\pi/T}$. We use a decomposition due to [Tak73]. For $n \in \mathbb{Z}$, we introduce the eigenspaces

$$M_n = \{x \in M \mid \sigma_t^\phi(x) = \chi^{int} x \quad \forall t \in \mathbb{R}\}$$

Note that Prop. 3.0.3 still applies in this case. Therefore, we know that θ^{-1} is a point ultraweak limit of a net of inner completely positive maps of the form $x \mapsto \sum_{j=1}^p a_j x a_j^*$ with $a_j \in M$. The goal is to be able to project these elements in M_ϕ .

The first step is to find them in the eigenspaces M_n . We let \mathfrak{M} denote the $*$ -algebra generated by $\cup_{n \in \mathbb{Z}} M_n$. We will consider $\overline{\mathfrak{M}}$, the norm closure of \mathfrak{M} , which is a C^* -algebra. It is possible to prove a result of this form:

Let $x_1, \dots, x_m \in \overline{\mathfrak{M}}$ and $\epsilon > 0$ be arbitrary. Then, we can find $a_i \in \cup_{n \in \mathbb{Z}} M_n$ such that $\sum_{i=1}^k a_i a_i^ \leq \mathbb{1}_M$ and $\|\theta^{-1}(x_j) - \sum_{i=1}^k a_i x_j a_i^*\|_\phi < \epsilon$.*

Although $\overline{\mathfrak{M}}$ is ultraweakly dense in M , we were not able to prove that this result holds for arbitrary $x_1, \dots, x_m \in M$. If that would be possible, the next step would be to use this result, the fact that $M \cong M \otimes R_\lambda$ and methods developed by [Gol78] applied to both M and R_λ , to show that the coefficients a_i can be chosen in M_ϕ . We studied such a proof; unfortunately, the same problem arises at the end of this calculation, namely, we are only able to get a result of the following form:

There exists an ultraweakly dense $$ -subalgebra N of M such that, for $x_1, \dots, x_m \in N$ and $\epsilon > 0$ arbitrary, we can find $a_i \in M_\phi$ with $\sum_{i=1}^k a_i a_i^* \leq \mathbb{1}_M$ and $\|\theta^{-1}(x_j) - \sum_{i=1}^k a_i x_j a_i^*\|_\phi < \epsilon$.*

Therefore, to this point, our work in finding a version of Prop. 3.0.3 which we could pair with Thm. 3.3.16 in order to prove a result of the form $(*)$ has not been conclusive.

Appendix A

Clarification in the proof of Lemma 4.4 of [Haa85]

Let N be a type II₁ factor. In the proof of Lemma 4.4 of [Haa85], Haagerup considers a continuous function f defined on \mathbb{R}^+ such that

$$\begin{aligned} 0 \leq t - f(t) &\leq \frac{1}{2r}(t-1)^2 & \text{for } t \geq 1 \\ t - f(t) &= 0 & \text{for } 0 \leq t \leq 1 \end{aligned}$$

For an element $b \in N$, Haagerup claims that

$$0 \leq b^*b - f(b^*b) \leq \frac{1}{2r}((b^*b - \mathbb{1}_N)_+)^2$$

We clarify here how to obtain this last inequality. Note that from the first two inequalities, we have

$$0 \leq t - f(t) \leq \frac{1}{2r}((t-1)\chi(t))^2 \quad \text{for } t \geq 0$$

where $\chi(t)$ is the characteristic function of the set $\{t \geq 1\}$. The functions $e_+(t) = (t-1)\chi(t)$ and $e_-(t) = -(t-1)(1-\chi(t))$ are continuous, and so we can apply the continuous functional calculus with respect to b^*b to obtain $e_+(b^*b)$ and $e_-(b^*b)$. These are positive operators which lie in $C^*(b^*b) = C^*(b^*b - \mathbb{1}_N)$ and satisfy $e_+(b^*b) - e_-(b^*b) = b^*b - \mathbb{1}_N$ with $e_+(b^*b)e_-(b^*b) = 0$. Since $C^*(b^*b - \mathbb{1}_N)$ is isomorphic to the algebra of continuous functions on the spectrum of $b^*b - \mathbb{1}_N$, and since the positive part of a real function is unique, we conclude that $e_+(b^*b) = (b^*b - \mathbb{1}_N)_+$, which proves Haagerup's statement.

Appendix B

Proofs of the approximations stated in Sec. 3.1.3

We include here the proofs of Lemmas 3.1.13 and 3.1.14, and Prop. 3.1.16 and 3.1.17. This Appendix uses extensively the definitions and technical results stated in Sec. 3.1.2. Note that the proofs of Lemmas 3.1.13 and 3.1.14 are adaptations of Connes' technical lemmas in Sec. 1 of [Con75].

Lemma 3.1.13. *Let M be a von Neumann algebra, ϕ a faithful normal state on M , E_ϕ the conditional expectation from M to M_ϕ , and $p \in \mathcal{P}(M)$. If $v = E_\phi(p)$ and $\|p - v\|_\phi < 1/3$, there exists $c \in \mathcal{P}(M_\phi)$ such that $\|p - c\|_\phi \leq \|p - v\|_\phi^{1/2}(\|p - v\|_\phi^{1/2} + 2\sqrt{3})$.*

Proof: We have

$$\|v^2 - v\|_\phi = \|v^2 - pv\|_\phi + \|pv - v\|_\phi \leq \|(v - p)v\|_\phi + \|p(v - p)\|_\phi + \|p - v\|_\phi \leq 3\|p - v\|_\phi$$

Since $\|p - v\|_\phi < 1/3$, this implies that we can apply Lemma 1.1.5 of [Con75] to find $c \in \mathcal{P}(M_\phi)$ s.t. $\|v - c\|_\phi \leq 2\sqrt{3}\|p - v\|_\phi^{1/2}$. Then,

$$\|p - c\|_\phi \leq \|p - v\|_\phi + \|v - c\|_\phi \leq \|p - v\|_\phi^{1/2}(\|p - v\|_\phi^{1/2} + 2\sqrt{3}).$$

■

Lemma 3.1.14. *Let M be a von Neumann algebra, ϕ a faithful normal state on M , E_ϕ the conditional expectation from M to M_ϕ , and $u \in M$ be a partial isometry. If $v = E_\phi(u)$ and $\|u - v\|_\phi^\# < 1/10$, there exists a partial isometry $w \in M_\phi$ such that $\|w - u\|_\phi^\# \leq \|u - v\|_\phi^\# + 6\sqrt{10}(\|u - v\|_\phi^\#)^{1/2}$.*

Proof: We define $p = u^*u \in \mathcal{P}(M)$ and $a = v^*v \in M_\phi$. We have

$$\|p - a\|_\phi \leq \|u^*(u - v)\|_\phi + \|(u^* - v^*)v\|_\phi \leq \|u - v\|_\phi + \|u^* - v^*\|_\phi \leq 2\|u - v\|_\phi^\#$$

Moreover,

$$\begin{aligned} \|a^2 - p^2\|_\phi &= \|(a + p)(a - p) + ap - pa\|_\phi \\ &\leq 2\|a - p\|_\phi + \|a(p - a)\|_\phi + \|(a - p)a\|_\phi \leq 4\|a - p\|_\phi \end{aligned}$$

and so

$$\|a^2 - a\|_\phi \leq \|a^2 - p^2\|_\phi + \|p - a\|_\phi \leq 5\|a - p\|_\phi \leq 10\|u - v\|_\phi^\#$$

We then apply Lemma 1.1.5 of [Con75] to find a spectral projection $g \in M_\phi$ of a s.t. $\|a - g\|_\phi < 2\sqrt{10}(\|u - v\|_\phi^\#)^{1/2}$ and $\|a^{1/2} - g\|_\phi < 3\sqrt{10}(\|u - v\|_\phi^\#)^{1/2}$. We define $w = kg \in M_\phi$, where $v = ka^{1/2}$ is the polar decomposition of v . Note that k^*k is the support projection of a and so we must have $g \leq k^*k$. Hence,

$$ww^*w = kggk^*kg = kgg = w$$

and so w is a partial isometry. We have

$$\begin{aligned} \|w - v\|_\phi^\# &\leq \|w - v\|_\phi + \|w^* - v^*\|_\phi \\ &\leq \|k(g - a^{1/2})\|_\phi + \|(g - a^{1/2})k^*\|_\phi \\ &\leq 2\|g - a^{1/2}\|_\phi \leq 6\sqrt{10}(\|u - v\|_\phi^\#)^{1/2} \end{aligned}$$

and so

$$\begin{aligned} \|w - u\|_\phi^\# &\leq \|w - v\|_\phi^\# + \|v - u\|_\phi^\# \\ &\leq \|u - v\|_\phi^\# + 6\sqrt{10}(\|u - v\|_\phi^\#)^{1/2} \end{aligned}$$

■

Proposition 3.1.16. *Let M be a separable type III_λ factor, ϕ a faithful normal state on M with $\sigma_T^\phi = id_M$ for $T = -2\pi/\ln \lambda$ and $\alpha \in Aut(M)$ s.t. $\phi \circ \alpha = \phi$. Suppose that there exists a sequence $(u_n)_{n \geq 1}$ of partial isometries in M s.t. $Ad_{u_n} \rightarrow \alpha$ in the u -topology on $CP(M)$, and that $\|u_n^*u_n - \mathbb{1}_M\|_\phi$ and $\|u_nu_n^* - \mathbb{1}_M\|_\phi$ both converge to 0. Then, we can find a sequence $(w_n)_{n \geq 1} \subseteq M_\phi$ of partial isometries s.t. $Ad_{w_n} \rightarrow \alpha$ in the same topology.*

Proof: First, note that for a positive functional $\psi \in M^*$ and $a \in M$, we have for any $x \in M$ with $\|x\| \leq 1$

$$\begin{aligned} |[a, \psi](x)| &\leq |\psi(ax - xa) + \psi(ax(aa^* - \mathbb{1}_M))| + |\psi(ax(aa^* - \mathbb{1}_M))| \\ &\leq |\psi(a(xa)a^* - xa)| + \|a\| \cdot \|\psi\|^{1/2} \cdot \|aa^* - \mathbb{1}_M\|_\psi \\ &\leq \|a\|(\|\psi \circ Ad_a - \psi\| + \|\psi\|^{1/2} \cdot \|aa^* - \mathbb{1}_M\|_\psi) \\ &\implies \|[a, \psi]\| \leq \|a\|(\|\psi \circ Ad_a - \psi\| + \|\psi\|^{1/2} \cdot \|aa^* - \mathbb{1}_M\|_\psi) \end{aligned}$$

and

$$\begin{aligned} \|\psi(axa^* - x)\| &\leq |\psi(a(xa^*) - (xa^*)a)| + |\psi(x(a^*a - \mathbb{1}_M))| \\ &\leq \|a\| \cdot \|[a, \psi]\| + \|\psi\|^{1/2} \cdot \|a^*a - \mathbb{1}_M\|_\psi \\ &\implies \|\psi \circ Ad_a - \psi\| \leq \|a\| \cdot \|[a, \psi]\| + \|\psi\|^{1/2} \cdot \|a^*a - \mathbb{1}_M\|_\psi \end{aligned}$$

By the first inequality, we see that we must have $\|[u_n, \phi]\| \rightarrow 0$. By Thm. 4.2.6 of [Con73], 1 is an isolated point of $\text{sp}\Delta_\phi$. Therefore, by Prop. 2.3 of [Con74], we have $\|u_n - E_\phi(u_n)\|_\phi^\# \rightarrow 0$. Let $w_n \in M_\phi$ be the partial isometry of Lemma 3.1.14 corresponding to u_n . We then have $\|u_n - w_n\|_\phi^\# \rightarrow 0$. By Prop. 3.1.10, let (ψ_k) be a sequence in $[0, \phi]$ s.t. $\text{span}(\psi_k)$ is dense in M_* . We have

$$\begin{aligned} \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{w_n}\| &\leq \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{u_n} \circ Ad_{u_n^* w_n}\| \\ &\quad + \|\psi_k \circ Ad_{u_n} \circ Ad_{u_n^* w_n} - \psi_k \circ Ad_{w_n}\| \end{aligned}$$

we break this into steps. Applying the second inequality obtained above and Prop. 3.1.7, we have

$$\begin{aligned} \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{u_n} \circ Ad_{u_n^* w_n}\| &\leq \|[u_n^* w_n, \psi_k \circ Ad_{u_n}]\| + \|w_n^* u_n u_n^* w_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{u_n}} \\ &\leq 2\|\psi_k \circ Ad_{u_n}\|^{1/2} \|u_n^* w_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{u_n}}^\# + \|w_n^* u_n u_n^* w_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{u_n}} \\ &\leq 2\|u_n^* w_n - \mathbb{1}_M\|_{\phi \circ Ad_{u_n}}^\# + \|w_n^* u_n u_n^* w_n - \mathbb{1}_M\|_{\phi \circ Ad_{u_n}} \\ &\leq 2(4\|\phi \circ Ad_{u_n} - \phi\|^{1/2} + \|u_n^* w_n - \mathbb{1}_M\|_\phi^\#) \\ &\quad + 2\|\phi \circ Ad_{u_n} - \phi\|^{1/2} + \|w_n^* u_n u_n^* w_n - \mathbb{1}_M\|_\phi \\ &\leq 10\|\phi \circ Ad_{u_n} - \phi\|^{1/2} + 2\|u_n^* w_n - \mathbb{1}_M\|_\phi^\# + \|w_n^* u_n u_n^* w_n - \mathbb{1}_M\|_\phi \end{aligned}$$

for which we have

$$\begin{aligned} \|u_n^* w_n - \mathbb{1}_M\|_\phi^\# &\leq \|(u_n^* - w_n^*)w_n\|_\phi^\# + \|w_n^* w_n - \mathbb{1}_M\|_\phi^\# \\ &\leq \|(u_n^* - w_n^*)w_n\|_\phi + \|w_n^*(u_n - w_n)\|_\phi + 2\|w_n^* w_n - \mathbb{1}_M\|_\phi \\ &\leq \|u_n^* - w_n^*\|_\phi + \|u_n - w_n\|_\phi \end{aligned}$$

$$\begin{aligned}
& + 2(\|(w_n^* - u_n^*)w_n\|_\phi + \|u_n^*(w_n - u_n)\|_\phi + \|u_n^*u_n - \mathbb{1}_M\|_\phi) \\
& \leq 3\|u_n^* - w_n^*\|_\phi + 3\|u_n - w_n\|_\phi + 2\|u_n^*u_n - \mathbb{1}_M\|_\phi \rightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
\|w_n^*u_nu_n^*w_n - \mathbb{1}_M\|_\phi & \leq \|w_n^*(u_nu_n^* - \mathbb{1}_M)w_n\|_\phi + \|(w_n^* - u_n^*)w_n\|_\phi \\
& \quad + \|u_n^*(w_n - u_n)\|_\phi + \|u_n^*u_n - \mathbb{1}_M\|_\phi \\
& \leq \|u_nu_n^* - \mathbb{1}_M\|_\phi + \|w_n^* - u_n^*\|_\phi \\
& \quad + \|w_n - u_n\|_\phi + \|u_n^*u_n - \mathbb{1}_M\|_\phi \rightarrow 0
\end{aligned}$$

For the second step, we have similarly

$$\begin{aligned}
\|\psi_k \circ Ad_{u_n} \circ Ad_{u_n^*} - \psi_k \circ Ad_{w_n}\| & \leq \|\psi_k \circ Ad_{u_nu_n^*} - \psi_k\| \\
& \leq \|[u_nu_n^*, \psi_k]\| + \|u_nu_n^*u_nu_n^* - \mathbb{1}_M\|_{\psi_k} \\
& \leq 2\|\psi_k\|^{1/2}\|u_nu_n^* - \mathbb{1}_M\|_{\psi_k}^\# + \|u_nu_n^*u_nu_n^* - \mathbb{1}_M\|_{\psi_k} \\
& \leq 2\|u_nu_n^* - \mathbb{1}_M\|_\phi^\# + \|u_nu_n^*u_nu_n^* - \mathbb{1}_M\|_\phi \\
& \leq 2\sqrt{2}\|u_nu_n^* - \mathbb{1}_M\|_\phi + \|u_nu_n^*u_nu_n^* - \mathbb{1}_M\|_\phi
\end{aligned}$$

for which we have

$$\begin{aligned}
\|u_nu_n^*u_nu_n^* - \mathbb{1}_M\|_\phi & \leq \|u_nu_n^*(u_nu_n^* - \mathbb{1}_M)\|_\phi + \|u_nu_n^* - \mathbb{1}_M\|_\phi \\
& \leq 2\|u_nu_n^* - \mathbb{1}_M\|_\phi \rightarrow 0
\end{aligned}$$

Hence, we have $\|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{w_n}\| \rightarrow 0$ which in turn implies

$$\|\psi_k \circ \alpha - \psi_k \circ Ad_{w_n}\| \leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{u_n}\| + \|\psi_k \circ Ad_{u_n} - \psi_k \circ Ad_{w_n}\| \rightarrow 0$$

Since this is valid for any k and since $\text{span}(\psi_k)$ is dense in M_* , this finishes the proof. \blacksquare

Proposition 3.1.17. *Let M be a separable type III_λ factor, ϕ a faithful normal state on M with $\sigma_T^\phi = id$ for $T = -2\pi/\ln \lambda$ and $\alpha \in \text{Aut}(M)$ s.t. $\phi \circ \alpha = \phi$. Suppose that there exists a sequence $(a_n)_{n \geq 1}$ in M s.t. $Ad_{a_n} \rightarrow \alpha$ in the u -topology on $CP(M)$, and that $\|a_n^*a_n - \mathbb{1}_M\|_\phi, \|a_nu_n^* - \mathbb{1}_M\|_\phi$ both converge to 0. Finally, suppose that $\|a_n\| \leq C$ for some $C \in \mathbb{R}$. Then, we can find a sequence $(b_n)_{n \geq 1} \subseteq M_\phi$ s.t. $Ad_{b_n} \rightarrow \alpha$ in the same topology.*

Proof: As before, note that for a positive functional $\psi \in M^*$ and $a \in M$, we have that

$$\|[a, \psi]\| \leq \|a\|(\|\psi \circ Ad_a - \psi\| + \|\psi\|^{1/2} \cdot \|aa^* - \mathbb{1}_M\|_\psi)$$

and

$$\|\psi \circ Ad_a - \psi\| \leq \|a\| \cdot \|[a, \psi]\| + \|\psi\|^{1/2} \cdot \|a^*a - \mathbb{1}_M\|_\psi$$

By the first inequality, we see that we must have $\|[a_n, \phi]\| \rightarrow 0$. By Thm. 4.2.6 of [Con73], 1 is an isolated point of $\text{sp}\Delta_\phi$. Therefore, by Prop. 2.3 of [Con74], we have $\|a_n - E_\phi(a_n)\|_\phi^\# \rightarrow 0$. We define $b_n = E_\phi(a_n)$. By Prop. 3.1.10, let (ψ_k) be a sequence in $[0, \phi]$ s.t. $\text{span}(\psi_k)$ is dense in M_* . We have

$$\begin{aligned} \|\psi_k \circ Ad_{a_n} - \psi_k \circ Ad_{b_n}\| &\leq \|\psi_k \circ Ad_{a_n} - \psi_k \circ Ad_{a_n} \circ Ad_{a_n^*b_n}\| \\ &\quad + \|\psi_k \circ Ad_{a_n} \circ Ad_{a_n^*b_n} - \psi_k \circ Ad_{b_n}\| \end{aligned}$$

we break this into steps:

$$\begin{aligned} &\|\psi_k \circ Ad_{a_n} - \psi_k \circ Ad_{a_n} \circ Ad_{a_n^*b_n}\| \\ &\leq C^2 \|[a_n^*b_n, \psi_k \circ Ad_{a_n}]\| + \|\psi_k \circ Ad_{a_n}\|^{1/2} \cdot \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{a_n}} \\ &\leq 2C^2 \|\psi_k \circ Ad_{a_n}\|^{1/2} \|a_n^*b_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{a_n}}^\# + C \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_{\psi_k \circ Ad_{a_n}} \\ &\leq 2C^3 \|a_n^*b_n - \mathbb{1}_M\|_{\phi \circ Ad_{a_n}}^\# + C \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_{\phi \circ Ad_{a_n}} \\ &\leq 2C^3 (2(C^2 + 1) \|\phi \circ Ad_{a_n} - \phi\|^{1/2} + \|a_n^*b_n - \mathbb{1}_M\|_\phi^\#) \\ &\quad + C(C^4 + 1) \|\phi \circ Ad_{a_n} - \phi\|^{1/2} + C \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_\phi \\ &\leq (5C^5 + 4C^3 + C) \|\phi \circ Ad_{a_n} - \phi \circ \alpha\|^{1/2} \\ &\quad + 2C^3 \|a_n^*b_n - \mathbb{1}_M\|_\phi^\# + C \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_\phi \end{aligned}$$

for which we have

$$\begin{aligned} \|a_n^*b_n - \mathbb{1}_M\|_\phi^\# &\leq \|(a_n^* - b_n^*)b_n\|_\phi^\# + \|b_n^*b_n - \mathbb{1}_M\|_\phi^\# \\ &\leq \|(a_n^* - b_n^*)b_n\|_\phi + \|b_n^*(a_n - b_n)\|_\phi + 2\|b_n^*b_n - \mathbb{1}_M\|_\phi \\ &\leq C \|a_n^* - b_n^*\|_\phi + C \|a_n - b_n\|_\phi \\ &\quad + 2(\|(b_n^* - a_n^*)b_n\|_\phi + \|a_n^*(b_n - a_n)\|_\phi + \|a_n^*a_n - \mathbb{1}_M\|_\phi) \\ &\leq 3C \|a_n^* - b_n^*\|_\phi + 3C \|a_n - b_n\|_\phi + 2\|a_n^*a_n - \mathbb{1}_M\|_\phi \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \|b_n^*a_n a_n^*b_n - \mathbb{1}_M\|_\phi &\leq \|b_n^*(a_n a_n^* - \mathbb{1}_M)b_n\|_\phi + \|(b_n^* - a_n^*)b_n\|_\phi \\ &\quad + \|a_n^*(b_n - a_n)\|_\phi + \|a_n^*a_n - \mathbb{1}_M\|_\phi \\ &\leq C^2 \|a_n a_n^* - \mathbb{1}_M\|_\phi + C \|b_n^* - a_n^*\|_\phi \\ &\quad + C \|b_n - a_n\|_\phi + \|a_n^*a_n - \mathbb{1}_M\|_\phi \rightarrow 0 \end{aligned}$$

For the second step, we have

$$\begin{aligned}
\|\psi_k \circ Ad_{a_n} \circ Ad_{a_n^* b_n} - \psi_k \circ Ad_{b_n}\| &\leq C^2 \|\psi_k \circ Ad_{a_n a_n^*} - \psi_k\| \\
&\leq C^4 \|[a_n a_n^*, \psi_k]\| + C^2 \|a_n a_n^* a_n a_n^* - \mathbb{1}_M\|_{\psi_k} \\
&\leq 2C^4 \|\psi_k\|^{1/2} \|a_n a_n^* - \mathbb{1}_M\|_{\psi_k}^{\#} + C^2 \|a_n a_n^* a_n a_n^* - \mathbb{1}_M\|_{\psi_k} \\
&\leq 2\sqrt{2}C^4 \|a_n a_n^* - \mathbb{1}_M\|_{\phi} + C^2 \|a_n a_n^* a_n a_n^* - \mathbb{1}_M\|_{\phi}
\end{aligned}$$

for which we have

$$\begin{aligned}
\|a_n a_n^* a_n a_n^* - \mathbb{1}_M\|_{\phi} &\leq \|a_n a_n^* (a_n a_n^* - \mathbb{1}_M)\|_{\phi} + \|a_n a_n^* - \mathbb{1}_M\|_{\phi} \\
&\leq (C^2 + 1) \|a_n a_n^* - \mathbb{1}_M\|_{\phi} \rightarrow 0
\end{aligned}$$

Hence, we have $\|\psi_k \circ Ad_{a_n} - \psi_k \circ Ad_{b_n}\| \rightarrow 0$ which in turn implies

$$\|\psi_k \circ \alpha - \psi_k \circ Ad_{b_n}\| \leq \|\psi_k \circ \alpha - \psi_k \circ Ad_{a_n}\| + \|\psi_k \circ Ad_{a_n} - \psi_k \circ Ad_{b_n}\| \rightarrow 0$$

Since $\text{span}(\psi_k)$ is dense in M_* , this finishes the proof. ■

Appendix C

Choice of an invariant periodic state

In this section, we show that, in the case of an automorphism θ of a “separable” type III_λ factor, the condition $\text{mod}(\theta) = \text{id}$ allows us to show that there exists an invariant periodic state.

We will say that two automorphisms $\theta_1, \theta_2 \in \text{Aut}(M)$ are *inner conjugate* if there exists a unitary $u \in M$ with $\theta_1 = \text{Ad}_u \circ \theta_2$. Note that in this case, θ_1 is an approximately inner automorphism if and only if θ_2 is. Similarly, θ_1 is approximately inner as a CP map if and only if θ_2 is. Therefore, Prop. C.0.3 shows that in our study of Conjecture A for a “separable” factor of type III_λ , we can replace the condition $\text{mod}(\theta) = \text{id}$ by the existence of a normal faithful state ϕ on M such that $\phi \circ \theta = \phi$ and $\sigma_T^\phi = \text{id}_M$ for $T = -2\pi/\ln \lambda$.

Definition C.0.1. *Let M be a type III_λ factor and let $T = -2\pi/\ln \lambda$. A generalized trace on M is a normal faithful strictly semifinite weight (see Def. 3.1.5 of [Con73]) ϕ on M s.t. $\phi(\mathbb{1}_M) = \infty$ and $\sigma_T^\phi = \text{id}_M$.*

Definition C.0.2. *Let M be a von Neumann algebra. We will say that two weights ϕ_1 and ϕ_2 on M are equivalent if there exists a unitary $u \in M$ such that $\phi_1(x) = \phi_2(uxu^*)$ for all $x \in M_+$. This will be denoted $\phi_1 \simeq \phi_2$.*

Proposition C.0.3. *Let M be a “separable” type III_λ factor and $\theta \in \text{Aut}(M)$ be such that $\text{mod}(\theta) = \text{id}$. Then we can find $\tilde{\theta} \in \text{Aut}(M)$ and a faithful normal state $\tilde{\psi}$ on M such that*

- *there exists a unitary $\tilde{w} \in M$ with $\theta = \text{Ad}_{\tilde{w}} \circ \tilde{\theta}$*
- *$\tilde{\psi} \circ \tilde{\theta} = \tilde{\psi}$*
- *$\sigma_T^{\tilde{\psi}} = \text{id}_M$ for $T = -2\pi/\ln \lambda$*

Proof: By 30.10 of [Str81], there exists a generalized trace ϕ on M such that $\phi \circ \theta^{-1} \simeq \mu\phi$ for some $\mu \in S(M) \cup \mathbb{R}_*^+$. By Thm. 4.3.2 of [Con73], we then know that $\phi \simeq \phi \circ \theta^{-1}$. We let $u \in M$ be the unitary such that $\phi = \phi \circ \theta^{-1} \circ \text{Ad}_u$ and define $\theta' = \text{Ad}_{u^*} \circ \theta \in \text{Aut}(M)$.

Now by Thm. 4.2.6 of [Con73], M_ϕ is a factor and, by 3.1.4 of [Con73], $\phi|_{M_\phi}$ is a normal semifinite faithful tracial weight on M_ϕ . Then, by Prop. 1.9.5, M_ϕ is a II_1 factor with ϕ as a dimension function.

We choose a nonzero finite projection $p \in \mathcal{P}(M_\phi)$. Since $\phi = \phi \circ \theta'$, we know that p and $\theta'(p)$ are equivalent in $\mathcal{P}(M_\phi)$. Similarly, $\mathbb{1}_M - p$ and $\theta'(\mathbb{1}_M - p)$ are equivalent in $\mathcal{P}(M_\phi)$. Let v_1 and v_2 be partial isometries in M_ϕ s.t. $v_1 p v_1^* = \theta'(p)$ and $v_2(\mathbb{1}_M - p)v_2^* = \theta'(\mathbb{1}_M - p)$. Then, $v = v_1 + v_2$ is a unitary such that $v p v^* = \theta'(p)$, and we define $\theta'' = \text{Ad}_{v^*} \circ \theta'$. On the corner algebra pMp , we define the faithful normal state ψ by $\psi(x) = (1/\phi(p))\phi(pxp)$. We have $\psi \circ \theta'' = \psi$. Moreover, $\sigma_t^\psi(x) = \sigma_t^\phi(x)$ for all $x \in pMp$ and $t \in \mathbb{R}$ (this follows from the fact that σ_t^ϕ satisfies the KMS condition with respect to ψ , see Thm. VIII.1.2 of [Tak03]). Therefore, $\sigma_T^\psi = \text{id}_{pMp}$.

Since M is a factor of type III, we know that p is equivalent to $\mathbb{1}_M$ in M . Hence, let $w \in M$ be a partial isometry such that $w^*w = \mathbb{1}_M$ and $ww^* = p$. Then $w^*\theta''(w)$ is a unitary, since

$$\begin{aligned} (w^*\theta''(w))(\theta''(w^*)w) &= w^*\theta''(ww^*)w = w^*\theta''(p)w = w^*v^*\theta'(p)vw = w^*pw = \mathbb{1}_M \\ (\theta''(w^*)w)(w^*\theta''(w)) &= v^*\theta'(w^*)v p v^*\theta'(w)v = v^*\theta'(w^*pw)v = v^*\theta'(\mathbb{1}_M)v = v^*v = \mathbb{1}_M \end{aligned}$$

Let us denote by $\tilde{\theta}$ the automorphism $\text{Ad}_{w^*} \circ \theta'' \circ \text{Ad}_w$ and by $\tilde{\psi}$ the faithful normal state $\psi \circ \text{Ad}_w$ on M . By construction, $\tilde{\psi} \circ \tilde{\theta} = \tilde{\psi}$. Moreover, $\sigma_t^{\tilde{\psi}}(x) = w^*\sigma_t^\psi(wxw^*)w$ for all $x \in M$ and $t \in \mathbb{R}$ (as before, this can be checked through the fact that the right hand side of this equation is a one parameter group of automorphisms satisfying the KMS condition with respect to $\tilde{\psi}$). Therefore, $\sigma_T^{\tilde{\psi}} = \text{id}_M$. \blacksquare

Appendix D

Ultrafilters and ultraproducts

This appendix is mainly based of [SW99]. We start by establishing the necessary terminology.

Definition D.0.1. *Let P be a partially ordered set (poset) and $F \subseteq P$ be a non-empty subset. We say that F is a filter if*

- $\forall x, y \in F, \exists z \in F$ such that $z \leq x$ and $z \leq y$.
- $\forall x \in F, \forall y \in P, x \leq y$ implies $y \in F$.

Definition D.0.2. *Let P be a partially ordered set and $F \subseteq P$ be a filter. We say that*

- F is **proper** if $F \subsetneq P$.
- F is an **ultrafilter** if it is a maximal proper filter, i.e. $F \subseteq G \subsetneq P$ for a filter G implies $F = G$.
- F is a **principal filter** if it contains a least element; equivalently, if there exists $y \in P$ such that $F = \{x \in P \mid x \geq y\}$.
- F is **free** if it is not principal.

Moreover, if $G \subseteq P$ is another filter and $G \subseteq F$, we say that F is **finer** than G .

For an arbitrary set S , its power set $\mathcal{P}(S)$ with the order given by inclusion is a poset. For a filter $F \subseteq \mathcal{P}(S)$, Def.(D.0.1) is equivalent to the following

Definition D.0.3. *Let S be a set and $F \subseteq \mathcal{P}(S)$ be non-empty. We say that F is a filter on S if*

- $A, B \in F$ implies $A \cap B \in F$.

- $A \in F$ and $B \in \mathcal{P}(S)$ with $A \subseteq B$ implies $B \in F$.

It follows that F is a proper filter iff $\emptyset \notin F$. Similarly, the definition of an ultrafilter can be rephrased in this case.

Definition D.0.4. *Let S be a set and $F \subseteq \mathcal{P}(S)$ be a proper filter on S . We say that F is an ultrafilter on S if for all $A \subseteq S$, either $A \in F$ or $X \setminus A \in F$.*

Through a simple “Zornification”, one can show that if X is a non-empty set, it always possesses an ultrafilter. Finally, we end this long list of definitions with the following concept.

Definition D.0.5. *Let S be a set and $\mathfrak{F} \subseteq \mathcal{P}(S)$ be non-empty. We say that \mathfrak{F} is a filter base if for all $A, B \in \mathfrak{F}$, there exists $C \in \mathfrak{F}$ such that $C \subseteq A \cap B$. The collection of subsets of S containing a member of \mathfrak{F} then forms a filter, the filter generated by \mathfrak{F} .*

An important example of filter is the Fréchet filter \mathcal{F} (or cofinite filter) on an arbitrary set X , defined as the collection of subsets of X with a finite complement.

Proposition D.0.6. *Let S be a set and $F \subseteq \mathcal{P}(S)$ be an ultrafilter on S .*

- F is principal iff there exists $x \in S$ such that $F = \{A \in \mathcal{P}(S) \mid x \in A\}$.
- F is principal iff it contains a finite set.
- If S is infinite, F is free iff it contains the Fréchet filter on S .

Another natural example of a filter is a concept well-known in topology: the neighbourhood filter $\mathcal{N}(x)$ of a point x in a topological space X , defined as the collection of sets containing an open set in which lies x . It turns out that filters are actually a tool at the heart of topology. Indeed, one can reformulate the usual notions of this field in terms of filters. For us however, we will mostly need the following notion of convergence.

A filter F on a topological space X is said to converge to $x \in X$ if F is finer than the neighbourhood filter of x . If Y is a set, $F \subseteq \mathcal{P}(Y)$ is a filter and $f : Y \rightarrow X$ is a function, it can be checked that $f(F)$ is a filter base on X . We say that f converges to $x \in X$ along F if the filter generated by $f(F)$ converges to x . In the case where $Y = \mathbb{N}$, we will write this result as $\lim_F y_n = x$.

This notion of convergence is actually a generalization of the usual ones defined in elementary topology. For example, if we take $f : \mathbb{N} \rightarrow X$ with F as the Fréchet filter on \mathbb{N} , it can be checked that convergence of f along F to a point x is the same as the usual notion of convergence of the sequence defined by F , i.e. $\lim_{n \rightarrow \infty} x_n = x$ iff $\lim_F x_n = x$.

As could be expected, the basic results of topology will also be true for filters. For example, if X, Y are topological spaces, $f : S \rightarrow X$ converges to $x \in X$ along a filter F on S , and $g : X \rightarrow Y$ is a continuous function, then $g \circ f$ converges to $g(x)$ along F . On the opposite, new exotic results are also true in the case of filters. For example, let F be an ultrafilter on S , X be a compact topological space, and $f : S \rightarrow X$ be a map. Then f converges to a point $x \in X$ along F .

We now construct the object which will be of main use to us. Let $(\mathcal{H}_n)_{n \geq 1}$ be a sequence of Hilbert spaces and F be a free ultrafilter on \mathbb{N} . Let

$$\mathcal{H}_\infty = \{(\xi_n)_{n \geq 1} \mid \xi_n \in \mathcal{H}_n, \sup \|\xi_n\| < \infty\}$$

which is a Banach space under the norm $\|\xi\| = \sup \|\xi_n\|$, and let

$$H_0 = \{\xi \in \mathcal{H}_\infty \mid \lim_{\omega} \|\xi_n\| = 0\}$$

It can be checked that H_0 is a closed subspace of \mathcal{H}_∞ , and so the quotient space $\mathcal{H}_F = \mathcal{H}_\infty / H_0$ is a Banach space, called the ultraproduct of the Hilbert spaces $(\mathcal{H}_n)_{n \geq 1}$ along F . The construction of this space through the quotient map naturally induces the norm $\|\xi\| = \inf\{\sup \|\xi_n - \eta_n\| \mid (\eta_n) \in H_0\}$, but it can be shown to be equal to the much simpler norm $\|\xi\| = \lim_F \|\xi_n\|$. It is then easy to see that \mathcal{H}_F is actually a Hilbert space with the inner product $(\xi|\eta) = \lim_F (\xi_n|\eta_n)$.

Finally, if $M = \oplus M_n$ is the direct sum in the von Neumann algebra sense, where each von Neumann algebra M_n admits a unital representation π_n on \mathcal{H}_n , then M admits a unital representation π on \mathcal{H}_F , defined naturally by

$$\pi((x_n)_{n \in \mathbb{N}})((\xi_n)_{n \in \mathbb{N}}) = (\pi_n(x_n)(\xi_n))_{n \in \mathbb{N}}$$

We then have

$$\|\pi((x_n)_{n \in \mathbb{N}})\| = \lim_F \|\pi_n(x_n)\|$$

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