

SYNTHESIS OF LINEAR SYSTEMS
THROUGH THE EIGENVALUE APPROACH

by

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ABSTRACT

The quadratic performance index is considered in this thesis as a generalized criterion for the design of linear multivariable systems. A method is developed in this thesis to design high order multivariable linear systems when the dominant eigenvalues are specified.

In Chapter I, the problem of system synthesis is stated and developed. In Chapter II, asymptotic properties of the optimal system are given and relevant details are investigated. The concept of 'lower order' control is introduced and developed in Chapter III. The original system is contracted to a system of lower order. The contracted system is optimally controlled and this control is used to generate the control for the original higher order system. This control is termed the 'lower order' control. It is shown that the eigenvalues of the optimally controlled contracted system, and the original system controlled by the 'lower order' control are identical. The cost of this 'lower order' control is shown to be the solution of a linear matrix equation. When the elements of the state-weighting matrix are generated in a specific way, it is shown that this cost can be found without solving the linear matrix equation. Two methods are given for finding the contraction matrix. A procedure for design is given and tests are incorporated in it to ensure the stability of the resulting system. Three examples are worked out to demonstrate the implementation of the theory of 'lower order' control. In Chapter IV, a special case is considered and the properties of the control-weighting matrix are investigated. The possibility of other applications is given and the various aspects of the theory are discussed and

conclusion are drawn at the end of this final chapter.

In Appendix A, a method is given to solve the linear matrix equation. In Appendix B, a method is developed to obtain the eigenvalues of a generalized matrix and the Vandermonde matrix is shown to be useful in obtaining the contraction matrix under certain conditions. Important proofs and tests are given in Appendix C to validate mathematically the theory of 'lower order' control. In Appendix D, details of savings made in Computer time resulting from the use of 'lower order' control are given. The proof of a zero quadratic form is given in Appendix E.

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LIST OF SYMBOLS

Given system of dimension 'n':Matrices

A	-	(n x n)	system matrix
B	-	(n x r)	input matrix
H	-	(l x n)	output matrix
Q_n	-	(n x n)	state weighting matrix
Q, Q_l	-	(l x l)	state weighting matrix
R	-	(r x r)	control weighting matrix
S	-	(l x l)	penalty function matrix for the finite time problem
N	-	(n x n)	gain matrix also known as cost matrix
K_n	-	(r x n)	feedback matrix
A_{opt}, A_o	-	(n x n)	optimum system matrix
F_c	-	(2n x 2n)	canonical system matrix
$\gamma(s)$	-		adjoint matrix

Vectors

x	-	(n x 1)	system state vector
u	-	(r x 1)	control vector
y	-	(l x 1)	output vector
x_o	-	(n x 1)	system state vector at the initial time
p	-	(n x 1)	costate or the adjoint vector

Contracted system of dimension 'm'

C	-	(m x n)	contraction matrix
F	-	(m x m)	contracted system matrix
G	-	(m x r)	C. S. input matrix
P	-	(m x m)	C. S. output matrix
Q_m	-	(m x m)	C. S. state weighting matrix
U	-	(m x m)	penalty function matrix for the finite time problem of the C. S.
M	-	(m x m)	gain matrix also known as the cost matrix
K_m, K	-	(r x m)	feedback matrix
F_o	-	(m x m)	optimum C. S. matrix

Vectors

z	-	(m x 1)	C. S. state vector
w	-	(m x 1)	C. S. output vector
z_o	-	(m x 1)	initial condition C. S. state vector

Scalar Variables

- s - Laplace variable
 t - running time
 J - performance index or cost of the control
 \hat{J} - cost of the optimum control of the given system
 J_n - cost of the optimum control of the C. S.
 J_m - cost of the 'lower order control' of the given system
 H - the Hamiltonian function
 $\bar{D}(s)$ - $\det [sI - A]$
 $\bar{D}(-s)$ - $(-1)^n \det [sI + A]$
 $D(s)$ - $\det [sI - A_0]$
 $Q_i(s)$ - denominator polynomial obtained from expanding $\det [sI_{2n} - F_c]$
 $P_i(s)$ - numerator polynomial obtained from expanding $\det [sI_{2n} - F_c]$
 $\gamma_{ij}(s)$ - an element of the matrix $\gamma(s)$

Constants

- t_0 - initial time
 T, t_f - final time in regulator problem
 ω_n - natural frequency
 σ - a parameter used to define the diagonal matrix Q
 a_i, b_i - polynomial coefficients
 λ_i, μ_i - eigenvalues or the variables of characteristic polynomials

Abbreviations

- $[]^*$ - optimal quantity
 $[]'$ - transpose of a matrix
 $[]^{-1}$ - inverse of a matrix
 $[]\dot{}$ - time derivative
 $[\hat{}]$ - hat matrix notation used in infinite-time Regulator problem
 \langle , \rangle - inner or dot product of vectors
 adj - adjoint of a matrix
 $\det []$ - determinant of a matrix
 $\text{tr} []$ - trace or spur of a matrix
 $Q = \text{dia} [q_{11}, q_{22}, q_{33}]$ - Q is diagonal matrix with element q_{11}, q_{22}, q_{33}
 D - $\frac{d}{dt}$, the time derivative

D^k - $\frac{d^k}{dt^k}$, the k^{th} order time derivative

$\mathcal{L}(D)$ - the time derivative operator

General

$[0]_n$ - null matrix of the indicated dimension

I_n, I_m - identity matrix of the proper dimension

U, V - capital letters are used for matrices

s, x, y, z - small letters are used for vector or scalar variables

n, l, r, m - constants indicating the relevant dimensions

$x^T A x$, $\langle x, A x \rangle$ - both the symbols are used for quadratic form

\mathcal{L}^{-1} - inverse Laplace transform

CHAPTER I

INTRODUCTION

In recent years, a great deal of interest has developed in the use of optimal control theory as a basis for designing control systems. This interest has arisen because application of the theory of optimization results in improved system performance for a wide class of systems. In aerospace systems of today problems of minimum time, minimum fuel and minimum energy are encountered in navigation and guidance^{(1),(2),(3)} and come under this classification.

1.1 General Considerations :

The essential elements of a typical control problem or the problem of system design, based on optimization theory are :

1. Identification of the dynamical system to be controlled and obtaining a valid mathematical model of the system.
2. Specifying the output of the system.
3. A set of admissible inputs or controls^{(4),(5)} and
4. A design index, which is termed as the performance functional or cost functional and which measures the effectiveness of a given control action.

If the mathematical model of the system is known, then the specified output can be attained by using different admissible inputs which will depend on the desired output and the initial condition of the system. Thus the designer will have to seek a measure of performance which will allow him to choose the control which will extremize the performance functional chosen. This performance functional must contain terms which are relevant to important performance specifications, for instance minimizing the required energy. The performance functional is extremized⁽⁶⁾ to obtain the optimal control. The resulting control system is optimal

only with respect to the design index selected; and there is no guarantee that a given optimal system will provide a good or even acceptable performance from another point of view, for instance the decay of the transient response. This seeming contradiction is due to the fact that no method is known for formulating a design index which gives rise to satisfactory system behaviour as required by other system specifications.

Thus from an engineering point of view, a unified design procedure is needed and this would include determining the parameters of the design index. The development of such a procedure will not only reduce the time and cost required to carry out the design of complex systems but will also lead to automation of system design for many classes of problems.

It may be asked if some specific considerations can be used in selecting this performance functional. One specific consideration is that the resulting optimal system should be stable.^{(7),(8)} However such an approach is based on indices which have generalized mathematical forms and the resulting optimal system, although stable, does not necessarily have rapidly decaying transient response. The definition of a reliable performance index that represents most of the design requirements may or may not be possible. If it is possible, the solution for the optimal control function will be a specific solution to a particular problem and will be numerical in nature. To obtain a closed form analytical solution for the control with a view to evaluate different performance indices, it will be necessary to use suitable performance indices. However, many of the design requirements will not be accommodated by such performance indices. Thus, the selection of a performance index is a compromise between a comprehensive, reliable criterion and one that is mathematically tractable.

In the literature^{(9), (10), (14)} the quadratic type of performance index is considered as a suitable criterion for designing optimal linear multi-variable systems. This quadratic type of performance index consists of quadratic forms and their integrals and is known as the quadratic criterion. The advantages of using the quadratic criterion are:

1. By specifying the numerical elements of the matrices used in the quadratic forms, the quadratic criterion can be fully defined.
2. The use of the quadratic criterion makes it possible to obtain closed form analytical solutions for the control functions. The closed form solution enables the designer to readily evaluate the dynamical behaviour of the resulting optimal system.
3. The optimal systems which result from quadratic criteria exhibit properties like overshoot, damping ratio and natural frequency. Thus the designer is in a position to develop a conceptual bridge between the classical frequency domain approach and the time domain approach of the optimization theory to the problem of system design.
4. If the original system is completely controllable and observable, and if the matrices used in the quadratic criterion satisfy certain mathematical conditions⁽⁹⁾, then the resulting optimal system is always stable.

It must be stressed that the entire design process of an optimal system is contained in the definition of the performance index. In linear systems, when the elements of the matrices of the quadratic criterion are specified, the optimal feedback gains can be obtained by solving a set of non-linear equations by a computer. However when the system order 'n' is high, the solution of the non-linear equations becomes a time-consuming and involved task.

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1.2 Background of this Research :

In this section the general nature of the problem and the previous research done will be examined.

It is assumed that the plant or system can be described by a set of first order time-invariant linear differential equations written in the following standard vector differential equation form.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{H} \mathbf{x}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}\tag{1-1}$$

Here \mathbf{A} is the plant or system matrix, \mathbf{B} is the input matrix and \mathbf{H} is the output matrix. The vectors and matrices have the following dimensions:

- \mathbf{x} - state vector - $(n \times 1)$
- \mathbf{u} - control vector - $(r \times 1)$
- \mathbf{y} - output vector - $(l \times 1)$
- \mathbf{A} - system matrix - $(n \times n)$
- \mathbf{B} - input matrix - $(n \times r)$
- \mathbf{H} - output matrix - $(l \times n)$

The function of a state regulator is to keep the state near the origin by expending minimum control energy. When the system given by eqn (1-1) is required to perform the function of an infinite-time regulator, it will be necessary to minimize the following performance functional under the constraints of eqn. (1-1).

$$J = \frac{1}{2} \int_0^{\infty} [\mathbf{y}'(t) \mathbf{Q} \mathbf{y}(t) + \mathbf{u}'(t) \mathbf{R} \mathbf{u}(t)] dt\tag{1-2}$$

For the finite-time state regulator, the performance functional defined below is minimized under the constraint of eqn. (1-1).

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$$J = 1/2 y'(T) S y(T) + 1/2 \int_0^T [y'(t)Q y(t) + u'(t)R u(t)] dt \quad (1-3)$$

Here the various matrices are as follows:

- Q - state weighting matrix - ($l \times l$)
- R - control weighting matrix - ($r \times r$)
- S - terminal cost matrix - ($l \times l$)

The resulting optimal system depends on the nature of the Q, R and S matrices. By assuming the R and S matrices, many rationales for obtaining the Q matrix have been investigated and are reported in the literature. Kalman⁽¹¹⁾ has dealt with single input case using frequency domain techniques. Transfer function techniques⁽¹²⁾ have produced some results. The choice of the elements of this Q matrix may be done by trial and error⁽¹³⁾ or by root-locus⁽¹⁴⁾ techniques. The Q matrix may also be chosen on the basis of sensitivity^{(15), (16), (17)} constraints. When the input vector which is the control function, is not weighted the state weighting matrix Q is chosen to minimize the Integral Squared Error⁽¹⁸⁾. A related problem of pole assignment of multi-input linear systems has been investigated. (8), (19), (20), (21) Dynamic systems of upto two degrees of freedom have also been designed.⁽²²⁾

1.3 General Observations :

When controlling high order linear systems, quite often it is required to control only some of the eigenvalues which are dominant and which need to be changed. It is these dominant eigenvalues which largely determine the time-domain behaviour of these systems. Hence the problem of specifying only m dominant eigenvalues from the n eigenvalues of an n^{th} order system is quite realistic. Thus, if m of the eigenvalues which would be designed are the dominant eigenvalues the process of design would achieve the objective of controlling the

time-domain behaviour of the system. The control which can achieve this objective must also minimize the cost functional. This cost functional is taken to be the quadratic cost functional; this quadratic cost functional is completely specified once the numerical elements of R , S and Q matrices are specified.

The very idea of specifying the eigenvalues means getting to know the time-response of the system. If the time-response of the system, at least part of which springs from the dominant eigenvalues, is known beforehand, then the optimal design will have the advantage of giving the designer a better feel for what he is designing.

In the past, several authors^{(11)...,(18)} have designed systems which were optimum in some specific mathematical sense. Some authors^{(11),(22)} have derived the Q matrix for low order systems to obtain a specific time response. However when the order of the system is large, such derivation of the Q matrix becomes impracticable. The designer has to assume arbitrary R and Q matrices which are positive definite. The resulting optimal system has arbitrary eigenvalues. Thus the time response of the optimal system becomes arbitrary and the designer thereby loses the feel for the resulting optimal system. However, in this thesis a method has been found to specify the dominant eigenvalues of a high order linear system and also to obtain the precise numbers for the elements of the state weighting matrix Q .

1.4 Statement of the Problem :

It is desired to establish a correlation between the time-response criteria of a system and the elements of the weighting matrix Q which some times are called weighting factors. This can enable the designer to make an initial selection of these weighting factors. If the initial selection gives rise to a system that has a sluggish transient response, then the Q matrix will have to be altered. For high order systems, if the

correlation between the dominant eigenvalues and the elements of the Q matrix is known, then the alterations in the Q matrix can be done rationally and without any arbitrariness.

1.4-1 The Problem Considered Is :

'It is required to minimize the cost functional given by eqn. (1-2) under the constraints given by eqn. (1-1) with respect to $u(t)$. Furthermore, this is to be achieved by choosing the elements of the state-weighting matrix Q in such a way that the resulting optimal system will have a certain number, 'm' ($m < n$, n-large) of the eigenvalues as specified by the designer'.

An explicit solution for the elements of the Q matrix to shift all the 'n' eigenvalues of a system of n^{th} order requires the solution of $n(n+1)/2$ non-linear equations. When n is large, this becomes quite complicated and the solutions, if obtainable, become very time-consuming. In many cases of multi-input systems, only m dominant eigenvalues need to be shifted. In this thesis it will be shown that in a given system, if the behaviour of the state-trajectory of an optimal regulator is to be specified partially by keeping the elements of the R matrix unaltered and selecting only the elements of the Q matrix, then it is possible to achieve this objective by the method of 'lower order' control. The term 'lower order' control is used in this thesis to signify that the n^{th} order system is controlled by a control which is derived from a system of lower order m, $m < n$.

It is believed that for high order systems, this scheme will enable the procedure of optimal design to be used for design itself, rather than for just obtaining comparison models.

1.5 Development of the Problem :

The solution to the finite-time regulator problem is found by minimizing eqn. (1-3) under the constraints of the time-invariant system

given by eqn. (1-1) and it is known^{(10),(23)} to be

$$u^*(t) = -K^*(t)x(t) \quad (1-4)$$

where $K^*(t)$ is the time-varying optimal feedback matrix and is given by

$$K^*(t) = R^{-1}B'N^*(t)$$

where $N^*(t)$ satisfies the matrix Riccati differential equation

$$-\dot{N}^*(t) = A'N^*(t) + N^*(t)A - N^*(t)BR^{-1}B'N^*(t) + H'QH$$

$$\text{with } N^*(T) = H'SH \quad (1-5)$$

In the infinite-time regulator problem specified by eqns. (1-1) and (1-2) it has been proved⁽¹⁰⁾ that

$$\lim_{T \rightarrow \infty} N^*(t) = \hat{N} \quad (1-6)$$

Here \hat{N} is the time-invariant matrix, and the solution to the infinite-time regulator problem is

$$u^*(t) = -\hat{K}x(t) \quad (1-7)$$

where

$$\hat{K} = R^{-1}B'\hat{N} \quad (1-8)$$

The matrix \hat{N} can be obtained by solving matrix differential equation (1-5) as a steady-state matrix algebraic equation :

$$0 = A'\hat{N} + \hat{N}A - \hat{N}BR^{-1}B'\hat{N} + H'QH \quad (1-9)$$

Then substituting eqns. (1-7) and (1-8) in eqn. (1-1), the optimal system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu^*(t) \\ &= [A - BR^{-1}B'\hat{N}]x(t) \\ &= A_{\text{opt}}x(t) \end{aligned} \quad (1-10)$$

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It has been proved^{(7), (23)} that the stability of the optimal system given by eqn. (1-10) is ensured when R is a positive definite matrix and so is the Q matrix. In the output regulator problem $H'QH$ in eqn. (1-9) will be a positive semi-definite matrix and the resulting system will be stable as long as the original system is completely controllable and observable.

The R matrix is associated with control vector 'u' and thus will have significant influence on the optimal control and the cost. The choice of the R matrix is closely linked with the economic factors governing the control of the system. It can be thought of as a normalized 'value' matrix. The R matrix will be made positive definite by assuming it to be an r-dimensional identity matrix I_r . An attempt will be made to establish some of the relationships concerning the R matrix elements.

However in this research the specific interest is to specify the dominant eigenvalues and to obtain the Q matrix. The Q matrix is associated with the state vector of the system. The choice of the q_{ij} elements will affect the behaviour of the state-vector, the eigenvalues and the time response of the optimal system. Assuming that the R matrix is fixed, how should the q_{ij} elements be chosen to make the optimal system have the specified time response or state-trajectory ?

1.6 Some Remarks on the Q Matrix :

It will be interesting to find how the Q matrix can be obtained. From eqn. (1-10) it can be seen that if A_{opt} is completely specified⁽²⁰⁾ as A_o , then

$$A_o = A - BR^{-1}B'N^{\wedge}$$

Thus $T_n^{\wedge} N = A - A_o$ (1-11)

where $T_n \triangleq BR^{-1}B'$ (1-12)

Rewriting eqn. (1-9)

$$H'QH = -A'N - NA + NT_n N$$

Using eqn. (1-11)

$$H'QH = -A'N - NA + N[A - A_o] \quad (1-13)$$

After collecting terms

$$A'N + NA_o + H'QH = 0 \quad (1-14)$$

Thus eqns. (1-11) and (1-14) will have to be satisfied simultaneously.

In general T_n may not have an inverse. If however T_n has an inverse then

$$N = [T_n]^{-1} [A - A_o] \quad (1-15)$$

and

$$H'QH = -A'[T_n]^{-1} [A - A_o] - [T_n]^{-1} [A - A_o]A_o \quad (1-16)$$

As indicated in Appendix A, eqn. (1-14) can be solved under the conditions stated there. However this solution of eqn. (1-14) will also have to satisfy eqn. (1-11). It is quite reasonable to suspect that the matrix A_o chosen arbitrarily will give rise to an N which may not satisfy both the eqns. (1-11) and (1-14) simultaneously. If however T_n has an inverse, N can be obtained uniquely from eqn. (1-15) and the Q matrix can be constructed from it as given by eqn. (1-14). When T_n does not have an inverse however, the system given by eqn. (1-1) can be contracted (24), (25), (26), (27), (28) to a system of lower order m , $m < n$.

When the system is thus contracted, eqns. similar to (1-15) and (1-16) can be developed and the 'lower order' control can be obtained.

1.7 Outline of the Thesis :

Characteristic equations and asymptotic properties (14), (29) of optimal

systems are reviewed in Chapter II. A canonical form based on the state and costate vector approach is given and characteristic equations are expanded to obtain the asymptotic properties. These asymptotic properties allow understanding of the behaviour of the optimal systems when the elements of the Q matrix are very large and also show how the behaviour of the system is affected by the input matrix B and the system matrix A .

The concept of 'lower order' control is developed in Chapter III. The method of contraction^{(25), (27), (28)} is utilized and further developed to obtain the lower order control. First the infinite-time regulator system is contracted and the specified eigenvalues are designed in it with the 'lower order' control. The original order system is controlled by utilizing the 'lower order' control, and it is shown that the lower order control transfers the specified eigenvalues to the higher order system. The question of stability of the higher order system resulting from the 'lower order' control is considered. The cost of the 'lower order' control and Q matrix are derived. A similar development is carried out for the finite-time regulator problem. Three numerical examples are worked out by using the design procedure. In the numerical examples, different types of system matrices are chosen to demonstrate the implementation of the proposed theory of lower order control.

In the fourth and final Chapter, a special case of finite-time regulator problem is considered. The possibility of applying the proposed scheme of design in other types of problems is considered. The effect of variations in the elements of R matrix is considered in brief. The positive semi-definiteness of the Q matrix and a possible method of solution of obtaining this Q matrix in general are discussed. The lower order control is reviewed and the contraction matrix C is discussed. Finally useful conclusions are drawn and the advantages of the 'lower order'

control theory which is developed in this thesis, are given.

In Appendix A, details of the method of solving a linear matrix equation are given. In Appendix B a method is given for obtaining the eigenvalues of a matrix; the modal transformation matrix which is useful in obtaining the contraction matrix C is also derived in Appendix B. In Appendix C two definitions are given and important proofs along with tests are given to validate the mathematical aspects of the theory of 'lower order' control. Details of savings made in Computer time by using the 'lower order' control are given in Appendix D. Two propositions are stated and proved to obtain the conditions regarding the zero quadratic form in Appendix E.

CHAPTER II

STRUCTURE OF OPTIMAL SYSTEMS

The asymptotic properties of optimal systems give the behaviour of these systems when the elements of the Q matrix are very large, and here they are referred to as the structural properties. Characteristic equations and asymptotic properties of optimal systems are known.^{(14), (29)} The structural properties are included in this Chapter and pertinent details are developed because it is felt that these properties are quite relevant to understanding the problem considered in Chapter I subsection 1.4-1, and also in developing its nomenclature.

In section 2.1, the canonical form with the state and costate vector approach^{(30), (31), (32)} is developed. This canonical form consists of a $2n$ dimensional system and the Q matrix appears in the canonical system matrix F_c in an explicit way. Hence this canonical form is useful for deriving the asymptotic properties of the optimal systems.

The canonical system matrix F_c is expanded to obtain the characteristic equation in section 2.2. Several equivalent forms⁽¹⁴⁾ of this characteristic equation are written and expanded in polynomial form to simplify the resulting expressions.

In section 2.3 the polynomial expansion of the canonical system is given when the original second order system has two inputs and a diagonal Q matrix. The details of these polynomials are given.

In section 2.4, brief description is given of the rules for finding the asymptotes of root-locus diagrams. A table and a figure is given to show the properties of the resulting Butterworth functions upto order 5.

In section 2.5, asymptotic properties are derived for a single input general system which is in the phase-variable form. A relation for the

order of the Butterworth function is also given.

In section 2.6 asymptotic properties of a third order system with three inputs are given. Many detailed observations are made regarding the order of the resulting Butterworth functions.

In section 2.7 asymptotic properties of a system similar to one in section 2.6 are given. However the difference is in the Q matrix. In this section the Q matrix is taken to be σI instead of the general diagonal matrix.

In this chapter the known^{(14), (29)} asymptotic properties of optimal systems are reviewed, however sections 2.5, 2.6 and parts of 2.3 and 2.7 have been developed for further improving the understanding of these asymptotic properties.

2.1 Canonical Form of the Optimal Systems:

If the system is given by

$$\begin{aligned}\dot{x}(t) &= A x(t) + Bu(t) \\ y(t) &= H x(t) \\ x(t_0) &= x_0\end{aligned}\tag{2-1}$$

and it is required to find the optimal control $u^*(t)$ such that the cost functional given below is minimized.

$$J = 1/2 x'(t_f) S x(t_f) + 1/2 \int_{t_0}^{t_f} [x'(t)H'QH x(t) + u'(t)R u(t)] dt\tag{2-2}$$

The Hamiltonian H for the system given by eqn. (2-1) and the cost functional given by eqn. (2-2) is

$$H = 1/2 \langle x, H'QHx \rangle + 1/2 \langle u, Ru \rangle + \langle x, A'p \rangle + \langle u, B'p \rangle\tag{2-3}$$

Here p is the costate vector.

From the relation

$$\dot{x}(t) = \frac{\partial H}{\partial p(t)}$$

and

$$\dot{p}(t) = \frac{-\partial H}{\partial x(t)}$$

the canonical system of equations for the state vector $x(t)$ and costate vector $p(t)$ become

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -H'QH & -A' \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = F_c \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \quad (2-4)$$

It has been shown^{(30), (31)} that

$$u^*(t) = -R^{-1} B' p(t) \quad (2-5)$$

If a linear relationship is assumed⁽³²⁾ between $p(t)$ and $x(t)$

$$p(t) = N^*(t) x(t)$$

then eqn. (2-5) becomes

$$u^*(t) = -R^{-1} B' N^*(t) x(t) \quad (2-6)$$

where $N^*(t)$ is the solution of the following eqn.

$$-\dot{N}^*(t) = N^*(t) A + A' N^*(t) - N^*(t) B R^{-1} B' N^*(t) + H' Q H \quad (2-7)$$

with $N^*(t_f) = S$

When $t_f = \infty$, the terminal cost matrix S is assumed to be zero and eqn. (2-6) and (2-7) become respectively

$$u^*(t) = -R^{-1} B' \hat{N} x(t) \quad (2-8)$$

$$0 = \hat{N} A + A' \hat{N} - \hat{N} B R^{-1} B' \hat{N} + H' Q H \quad (2-9)$$

and $\hat{N}(t_f = \infty) = 0$.

\hat{N} is the solution of the algebraic eqn. (2-9) which is the steady state form of the differential eqn. (2-7). Thus the infinite-time output regulator with the optimal control has the following system equation

$$\dot{x}(t) = [A - BR^{-1} \hat{B}' \hat{N}] x(t) \quad (2-10)$$

Analogous expression can be written for the case when $t_f \neq \infty$ and $S \neq 0$.

2.2 Equivalent Forms of the Characteristic Equation :

The characteristic equation of the optimal system defined by eqn. (2-10) can be written as

$$\det [sI_n - (A - BR^{-1} \hat{B}' \hat{N})] = \prod_{i=1}^{(n-2m)} (s+a_i) \cdot \prod_{k=1}^m [s+(a_k + j\omega_k)][s+(a_k - j\omega_k)] = 0 \quad (2-11)$$

However the expansion of eqn. (2-11) cannot have the various q_{ij} 's in a simple functional relationship with the a's and ω 's. The characteristic equation derived from the canonical form is more useful since the matrix Q appears in an explicit way. Thus the characteristic equation of eqn. (2-4) is

$$\det [sI_{2n} - F_c] = \det \begin{bmatrix} (sI_n - A) & (BR^{-1} B') \\ (H'QH) & (sI_n + A') \end{bmatrix} = 0 \quad (2-12)$$

If $[sI_{2n} - F_c]$ is premultiplied by the transformation matrix T, it is quasi-diagonalized. This transformation T is given below

$$T = \begin{bmatrix} (sI_n - A)^{-1} & 0 \\ -H'QH(sI_n - A)^{-1} & I_n \end{bmatrix} \quad (2-13)$$

Then using the theory of quasi-diagonalized partitioned matrices⁽³³⁾

and taking determinants and simplifying

$$\det \begin{bmatrix} (sI-A) & BR^{-1}B' \\ H'QH & (sI+A') \end{bmatrix} = \det (sI-A) \cdot \det (sI+A') \cdot \det [I_n - (sI+A')^{-1}H'QH(sI-A)^{-1}BR^{-1}B'] \quad (2-14)$$

Whenever the subscript of I is dropped, it is understood to be n.

Using the determinant identity⁽³⁴⁾

$$\det [AB' \pm \lambda I_n] = (\pm \lambda)^{n-r} \cdot \det [B'A \pm \lambda I_r]$$

with $r \leq n$, $A - n \times r$, $B' - r \times n$,

eqn.(2-14) can be rewritten as

$$\det \begin{bmatrix} (sI-A) & BR^{-1}B' \\ H'QH & (sI+A') \end{bmatrix} = \det (sI-A) \cdot \det (sI+A') \cdot \det [I_r + R^{-1}B'(-sI-A')^{-1}H'QH(sI-A)^{-1}B] \quad (2-15)$$

with $r \leq l \leq n$

or

$$\det \begin{bmatrix} (sI-A) & BR^{-1}B' \\ H'QH & (sI+A') \end{bmatrix} = \det (sI-A) \cdot \det (sI+A') \cdot \det (I_l + H(sI-A)^{-1}BR^{-1}B'(-sI-A')^{-1}H'Q) \quad (2-16)$$

with $l \leq r \leq n$

By spectral factoring Letov⁽³⁵⁾, (36), (37) has shown that

$$\begin{aligned} \det [sI_{2n} - F_c] &= D(s) \cdot D(-s) \\ &= \prod_{i=1}^{(n-m)} (s+a_i)(s-a_i) \cdot \prod_{k=1}^m [s+(a_k+j\omega_k)][s+(a_k-j\omega_k)] \\ &\quad \cdot [s-(a_k-j\omega_k)][s-(a_k+j\omega_k)] \end{aligned} \quad (2-17)$$

It is assumed that the system described by A, B and H matrices is completely controllable and observable as defined in Appendix C. This implies that the optimal system $A_o = [A - BR^{-1}B'N]$ is stable and thus the eigenvalues of F_c with negative real parts are the eigenvalues of A_o .

2.3 Polynomial Expansion of Characteristic Equation :

The canonical characteristic equation (2-14) can be rewritten as

$$\det [sI_{2n} - F_c] = \det [sI - A] \cdot \det [(sI + A') - H'QH(sI - A)^{-1} BR^{-1} B']$$

$$= \bar{D}(s) \cdot \det [(sI + A') - H'QH \frac{\gamma(s)}{\bar{D}(s)} BR^{-1} B'] \quad (2-18)$$

$$= D(s) D(-s) \quad (2-19)$$

Here

$$\bar{D}(s) = \det [sI - A]$$

$$\gamma(s) = \text{Adjoint } [sI - A]$$

The square bracket determinant in eqn. (2-18) can be expanded into $\sum_{r=0}^n C_r = 2^n$ determinants and eqn. (2-19) can be expressed as

$$D(s) D(-s) = \bar{D}(s) \bar{D}(-s) + (-1)^n \sum_{i=1}^{(2n-1)} k_i P_i(s) P_i(-s) \quad (2-20)$$

Here

$$\bar{D}(-s) = \det [-sI - A] = \det [-sI - A'] = (-1)^n \det [sI + A']$$

For a typical second order system with

$$A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

and

$$R = I = H,$$

$$\det [sI_{2n} - F_c] = D(s) D(-s)$$

and

$$D(s) D(-s) = \bar{D}(s) \bar{D}(-s) + q_{11} P_1(s) P_1(-s) + q_{22} P_2(s) P_2(-s) + q_{11} q_{22} P_3(s) P_3(-s) \quad (2-21)$$

Here

$$\begin{aligned} \bar{D}(s) &= [(s+a_{11})(s+a_{22}) - a_{12}a_{21}] \\ P_1(s)P_1(-s) &= [-\bar{b}_{11}(s+a_{22})(s-a_{22}) + a_{12}^2 \bar{b}_{22} - 2a_{12} \bar{b}_{21} a_{22}] \\ P_2(s)P_2(-s) &= [-\bar{b}_{22}(s+a_{11})(s-a_{11}) + a_{21}^2 \bar{b}_{11} - 2a_{11} \bar{b}_{12} a_{21}] \\ P_3(s)P_3(-s) &= [\bar{b}_{11} \bar{b}_{22} - \bar{b}_{12} \bar{b}_{21}] = \det [BR^{-1}B'] \end{aligned}$$

and \bar{b}_{ij} are the various elements of the matrix $[-BR^{-1}B']$.

2.4 Asymptotic Properties of a Class of Optimal Systems :

In the preceding section, it is seen that the characteristic equation of $[sI_{2n} - F_c]$ can be expressed as a polynomial in the elements of Q . The asymptotic properties of optimal systems can be found by observing the properties of these polynomials by varying the elements of Q . If a single diagonal element q_{ii} of the Q matrix is varied and all others are held constant, then because of the symmetrical properties of these polynomials, the determinant $[sI_{2n} - F_c]$ can be expressed as

$$\det [sI_{2n} - F_c] = D(s) D(-s) = 0 = 1 + \frac{kq_{ii} P(s)P(-s)}{Q(s) Q(-s)} \quad (2-22)$$

Here k is a constant, $P(s) P(-s)$ is the polynomial coefficient of q_{ii} and the denominator polynomial $Q(s) Q(-s)$ is that portion of the expansion of the above determinant which is not a function of q_{ii} . If all the diagonal elements of the Q matrix are zero except q_{ii} , then $Q(s)$ becomes $\bar{D}(s)$ as defined before.

For changes in the elements of Q that result in expansion of the characteristic determinant of the form given by eqn. (2-22), the rules for finding the asymptotes of a root-locus diagram may be applied. If the degrees of the polynomials $Q(s)$ and $P(s)$ are n and m respectively, then $2m$ of the $2n$ roots of $Q(s)Q(-s)$ will terminate on the zeros of $P(s)P(-s)$ for large values of q_{ii} . The remaining $2(n-m)$ roots of eqn. (2-22) can be approximated by :

$$\det [sI_{2n} - F_c] \approx k \prod_{i=1}^{(n-m)} \left(\frac{s}{t_i} + 1 \right) \left(\frac{-s}{t_i} + 1 \right) \quad (2-23)$$

Here k is a constant and t_i are the values of the roots. For large values of q_{ii} , $\left| \frac{s}{t_i} \right| \gg 1$ and the roots of eqn. (2-22) will lie on a circle about the origin. Since the roots of $\det [sI_{2n} - F_c] = 0$ that contain negative real parts are the eigenvalues of the optimal system, the determinant $D(s)$ of the optimal system is a function whose roots lie on a semi-circle in the L.H. Plane. Functions of this type are called Butterworth functions. When the optimal system determinant $D(s)$ approaches a Butterworth function, it is characterized for any order system by : (1) a single natural frequency ω_n and (2) a frequency response which is always 0.707 of its maximum value, at ω_n . Some properties of Butterworth functions are given on the next page.

Order of the Butterworth Function = (n-m)	θ is the Angle of Asymptotes	D(s) for large q_{ii}
1	0°	$s + \omega_n$
2	45°	$s^2 + \sqrt{2} \omega_n s + \omega_n^2$
3	$0^\circ, 60^\circ$	$(s^2 + \omega_n s + \omega_n^2)(s + \omega_n)$
4	$22.5^\circ, 67.5^\circ$	$(s^2 + 1.85\omega_n s + \omega_n^2)(s^2 + 0.762\omega_n s + \omega_n^2)$
5	$0^\circ, 36^\circ, 72^\circ$	$(s^2 + 1.62\omega_n s + \omega_n^2)(s^2 + 0.69\omega_n s + \omega_n^2)(s + \omega_n)$

Table 2.1

Here θ is defined by :

$$\theta = \frac{(2c+1) 180^\circ}{2(n-m)}, \quad c = 0, 1, 2, \dots ; (n-m) \text{ even.}$$

$$\theta = \frac{(2c+1) 360^\circ}{2(n-m)}, \quad c = 0, 1, 2, \dots ; (n-m) \text{ odd.}$$

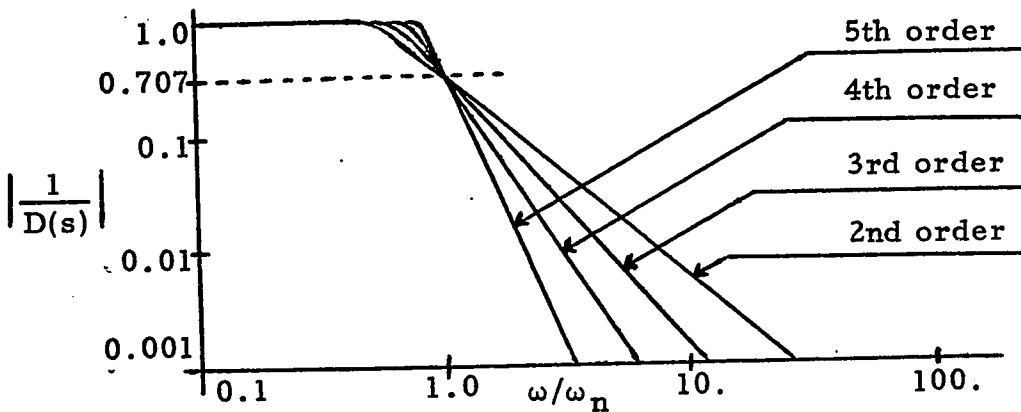


Fig. 2.1

Any multiple-input multiple-output system whose determinant can be expressed in the form given by eqn. (2-22) will have the properties of Butterworth function for large values of any of the q_{ii} elements of the Q matrix.

2.5 Asymptotic Properties of Single Input Systems:

It is known^{(38),(39),(40)} that any single input system which is completely controllable can be transformed into the phase-variable form with the system matrix A having the companion matrix form and the input matrix B having all entries zero except b_{n1} .

Consider the following single input generalized system in the phase-variable form.

$$A = \begin{bmatrix} 0 & 1 & \dots & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \dots & -a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_{n1} \end{bmatrix} \quad Q = \begin{bmatrix} q_{11} \\ \vdots \\ q_{nn} \end{bmatrix}$$

with $H = I$ and $R = r$ (2-24)

Then

$$\det [sI_{2n} - F_c] = \det \begin{bmatrix} (sI-A) & BR^{-1}B' \\ H'QH & (sI+A') \end{bmatrix} = 0$$

can be expanded as

$$1 + (-1)^{n+1} \frac{b_{n1}^2}{r} [q_{nn} s^{2(n-1)} - q_{n-1,n-1} s^{2(n-2)} + (-1)^{2n-k} q_{n-k,n-k} s^{2(n-k-1)} + \dots] = 0$$

$$\bar{D}(s) \bar{D}(-s) \tag{2-25}$$

where

$$\bar{D}(s) = s^n + a_{nn} s^{n-1} + \dots + a_{n1}$$

When $n = 2$, eqn. (2-25) will become

$$1 - \frac{b_{21}^2}{r} \frac{[q_{22} s^2 - q_{11}]}{(s^2 + a_{22} s + a_{21})(s^2 - a_{22} s + a_{21})} = 0 \tag{2-26}$$

When $n = 3$, eqn. (2-25) will become

$$1 + \frac{b_{31/r}^2 [q_{33}s^4 - q_{22}s^2 + q_{11}]}{\bar{D}(s)\bar{D}(-s)} = 0$$

where $\bar{D}(s) = s^3 + a_{33}s^2 + a_{32}s + a_{31}$ (2-27)

If any one particular q_{ii} is varied, it can be said that the order of the Butterworth function that results will depend on the order of $\bar{D}(s)$ and the polynomial coefficient of q_{ii} that is made very large. In general $\bar{D}(s)$ is an n^{th} order polynomial and the power of the 's' term which is the coefficient of q_{11} is $2(i-1)$. Thus when any particular q_{ii} is made very large

Butterworth Function ORDER = $p = n - (i-1)$ (2-28)

If all the q_{ii} elements are set equal to σ and σ is made very large, then the single input system in the phase-variable form will have the properties of first order Butterworth function.

2.6 Asymptotic Properties of Multi-Input Systems :

If the system is given by the equation $\dot{x}(t) = A x(t) + B u(t)$

with

$$A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad R = I = H$$

and Q the diagonal matrix given by

$$Q = \text{dia} [q_{11}, q_{22}, q_{33}] \quad (2-29)$$

then the expansion of the characteristic determinant of the matrix F_c will result in $2^3 = 8$ polynomials in 's' given below.

$$\det [sI_{2n} - F_c] = \bar{D}(s) \bar{D}(-s)$$

$$-q_{11} \cdot \det \begin{bmatrix} [\gamma_{11}(s)\bar{b}_{11} + \gamma_{12}(s)\bar{b}_{21} + \gamma_{13}(s)\bar{b}_{31}] & [\gamma_{11}\bar{b}_{12} + \gamma_{12}\bar{b}_{22} + \gamma_{13}\bar{b}_{32}] & [\gamma_{11}\bar{b}_{13} + \gamma_{12}\bar{b}_{23} + \gamma_{13}\bar{b}_{33}] \\ a_{12} & s + a_{22} & a_{32} \\ a_{13} & a_{23} & s + a_{33} \end{bmatrix}$$

$$-q_{22} \cdot \det \begin{bmatrix} s + a_{11} & a_{21} & a_{31} \\ [\gamma_{21}\bar{b}_{11} + \gamma_{22}\bar{b}_{21} + \gamma_{23}\bar{b}_{31}] & [\gamma_{21}\bar{b}_{12} + \gamma_{22}\bar{b}_{22} + \gamma_{23}\bar{b}_{32}] & [\gamma_{21}\bar{b}_{13} + \gamma_{22}\bar{b}_{23} + \gamma_{23}\bar{b}_{33}] \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$-q_{33} \cdot \det \begin{bmatrix} s + a_{11} & a_{21} & a_{31} \\ a_{12} & s + a_{22} & a_{32} \\ [\gamma_{31}\bar{b}_{11} + \gamma_{32}\bar{b}_{21} + \gamma_{33}\bar{b}_{31}] & [\gamma_{31}\bar{b}_{12} + \gamma_{32}\bar{b}_{22} + \gamma_{33}\bar{b}_{32}] & [\gamma_{31}\bar{b}_{13} + \gamma_{32}\bar{b}_{23} + \gamma_{33}\bar{b}_{33}] \end{bmatrix}$$

$$+ q_{11}q_{22}P_4(s)P_4(-s) + q_{11}q_{33}P_5(s)P_5(-s) + q_{22}q_{33}P_6(s)P_6(-s)$$

$$+ q_{11}q_{22}q_{33} \det [BR^{-1}B'] \quad (2-30)$$

Here all the γ 's are functions of 's' but for convenience the 's' in the brackets is dropped, and $\gamma_{ij}(s)$ are the elements of the adjoint matrix and \bar{b}_{ij} refer to the general elements of the $BR^{-1}B'$ matrix.

The general expansion of the determinant⁽⁴¹⁾ $[sI_{2n} - F_c]$ is done by

substituting $i = 1$ to n rows of $\frac{\gamma(s)BR^{-1}B'}{\bar{D}(s)}$ into $(sI + A')$ as defined in eqns. (2-18) and (2-19) and forming the successive determinants.

When any i^{th} row is substituted into $(sI + A')$, and the determinant is taken, the resulting expansion is the coefficient of the q_{ii} element. When any two rows are substituted for the i^{th} and $(i+1)^{\text{th}}$ row, the resulting expansion is the coefficient of $q_{ii} \cdot q_{i+1,i+1}$ and so on.

Consider the case when q_{11} is varied. Eqn. (2-30) can be written as

$$\det [sI_{2n} - F_c] = 1 - \frac{k_1 q_{11} P(s) P(-s)}{Q(s) Q(-s)} \quad (2-31)$$

$Q(s) Q(-s)$ is the combination of $\bar{D}(s) \bar{D}(-s)$ and the polynomial coefficients of q_{22} and q_{33} in eqn. (2-30). $P(s) P(-s)$ is the sum of the polynomial coefficients of q_{11} , $q_{11} q_{22}$, $q_{11} q_{33}$ in eqn. (2-30) and k_1 is a constant..

Rewriting eqn. (2-18)

$$\det [sI_{2n} - F_c] = \bar{D}(s) \cdot \det [(sI + A') - H'QH \frac{\gamma(s)}{\bar{D}(s)} BR^{-1}B'] \quad (2-31a)$$

when $H = I$ and Q is a diagonal matrix, $H'QH \frac{\gamma(s)}{\bar{D}(s)} BR^{-1}B'$ matrix is such that any i^{th} row is multiplied by q_{ii} . As seen from eqn. (2-30), for $i = 1$, q_{11} can be brought outside the determinant. The first row of $\frac{\gamma(s)}{\bar{D}(s)} BR^{-1}B'$ has been substituted. Now $\gamma(s)$, the adjoint matrix has

polynomial elements of degree $(n-1)$. $\bar{D}(s)$ is a polynomial of degree n .

The remaining $(n-1)$ rows of the matrix $(sI + A')$ have polynomial elements of degree 1 and are in the diagonal position. Thus when the first row of $(sI + A')$ in eqn. (2-31a) has been substituted by the first row of $\frac{\gamma(s)BR^{-1}B'}{\bar{D}(s)}$,

the resulting expansion will give the polynomial coefficients of q_{11} .

The degree of these polynomial coefficients can be expressed as follows:

$$\bar{D}(s) \cdot q_{11} \det \begin{bmatrix} \text{1st row of } \frac{\gamma(s) BR^{-1} B'}{\bar{D}(s)} \\ \text{Other rows of } (sI + A') \end{bmatrix}$$

and will be given by

$$\begin{array}{ccc} \downarrow & + & \downarrow & + & \downarrow \\ n & & [\{ (n-1) - n \} & & 1 (n-1)] = (2n-2) \quad (2-32) \\ \text{degree of} & & \text{degree of 1 row of} & & \text{degree of remaining} \\ \bar{D}(s) & & \frac{\gamma(s) BR^{-1} B'}{\bar{D}(s)} & & (n-1) \text{ rows of 1st degree} \\ & & & & \text{elements} \end{array}$$

Thus when any one row is substituted in eqn. (2-31a), the resulting expansion has a polynomial of degree (2n-2). In general if k rows are substituted at a time, then the resulting expansion is a polynomial and its degree is

$$\begin{array}{ccc} \downarrow & + & \downarrow & + & \downarrow \\ n & & [k \{ (n-1) - n \} & & 1(n-k)] = 2(n-k) \\ \bar{D}(s) & & k \text{ rows of } \frac{\gamma(s) BR^{-1} B'}{\bar{D}(s)} & & \text{remaining } (n-k) \text{ rows} \\ & & & & \text{of 1st degree elements} \end{array} \quad (2-33)$$

[Degree of the Expanded Polynomial] = 2(n-k)
with k rows substituted

Thus the polynomial coefficients of $q_{11}q_{22}$ or $q_{11}q_{33}$ will have degree of $2(3-2) = 2$, since $n = 3$ and $k = 2$. Thus the degree of $P(s) P(-s)$ in eqn. (2-31) is the same as that of the polynomial which results by substituting only one row as explained in eqn. (2-32). Thus the degree of $P(s) P(-s)$ is

$$\text{Degree} = 2n - 2 = 2 \times 3 - 2 = 4$$

It may be mentioned that in expansion given by eqn. (2-30)

$$\gamma_{11}(s) = s^2 + (a_{22} + a_{33})s + a_{22}a_{33} - a_{32}a_{23}$$

$$\gamma_{12}(s) = -a_{12}s - (a_{12}a_{33} - a_{32}a_{13})$$

$$\gamma_{13}(s) = -a_{13}s + (a_{12}a_{23} - a_{13}a_{22})$$

Other elements of $\gamma(s)$ can be readily found also. The above results for the degree of the polynomial coefficient of q_{11} hold, only because $\bar{b}_{11} \neq 0$. Similarly it can be said that the coefficient of any q_{ii} will be a polynomial of degree $(2n-2)$ if the i^{th} row of B contains at least one non-zero element so that $\bar{b}_{ii} \neq 0$.

Since $P(s)P(-s)$ is a polynomial of degree 4 and $Q(s)Q(-s)$ is a polynomial of degree 6, a 1st order Butterworth function results for large values of q_{11} when the other elements of the Q matrix are held constant. Similarly if any other diagonal element q_{ii} of Q is made very large while the other elements are held constant, a 1st order Butterworth function results provided the i^{th} row of the B matrix contains at least one non-zero element.

If the first row of the B matrix contains all zero elements, then the first row and first column of $BR^{-1}B'$ will be zero. The polynomial coefficients of q_{11} in expansion (2-30) will be of degree 2, provided that a_{12} and a_{13} are non-zero. It can also be shown that coefficients of $q_{11}q_{22}$ and $q_{11}q_{33}$ are of degree 2 unless a_{11} is zero. Thus if the first row of B is zero and the first row of A is non-zero, then a second order Butterworth function is obtained. The results of this section can be summarized as follows:

1. The general effect of more inputs or elements in the B matrix is to decrease the order of the Butterworth function that describes the

asymptotic properties of $[sI - (A - BR^{-1} \hat{B}'N)]$.

2. If at least one element in the i^{th} row of the B matrix is non-zero, then the system determinant $[sI - (A - BR^{-1} \hat{B}'N)]$ approaches a first order Butterworth function as q_{ii} becomes large.

3. If all the elements of the i^{th} row of the B matrix are zero, then the characteristic equation described by the system determinant approaches a second order Butterworth function as q_{ii} becomes large. This presupposes that the i^{th} row of the matrix A is non-zero.

4. From the general trend seen in the expansion of $P(s)P(-s)$, it may be said that the degree of $P(s)P(-s)$ becomes smaller as more rows of B matrix are zero, and also as more rows of the A matrix are zero. Thus for a large value of q_{ii} and more than one row of the B matrix being zero, Butterworth functions of order 3 through n may be obtained. This presupposes that the matrix A has more numerous zero elements.

2.7 Asymptotic Properties of Optimal Systems with $Q = \sigma I$:

When the Q matrix is defined as $Q = \sigma I$, the asymptotic properties of the system could be investigated with the help of a multi-input multi-output system of order 3.

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with

$$A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad \text{and } Q = \sigma I$$

$$H = I = R$$

(2-34)

Then $\det [sI_{2n} - F_c] = 0$ can be expanded as

$$1 + \frac{\sigma k_1 P_1(s)P_1(-s) + \sigma^2 k_2 P_2(s)P_2(-s) + \sigma^3 k_3 P_3(s)P_3(-s)}{\bar{D}(s) \bar{D}(-s)} = 0 \quad (2-35)$$

where

$$\bar{D}(s) = [(s+a_{11})(s+a_{22})(s+a_{33}) - a_{32}a_{23}(s+a_{11}) - a_{12}a_{21}(s+a_{33}) - a_{13}a_{31}(s+a_{22}) + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32}]$$

$$k_1 = \bar{b}_{11} + \bar{b}_{22} + \bar{b}_{33}$$

$$P_1(s) = [s^2 + bs + c] \quad \text{where } b \text{ and } c \text{ are constants.}$$

$$k_2 = \bar{b}_{11}\bar{b}_{22} + \bar{b}_{11}\bar{b}_{33} + \bar{b}_{22}\bar{b}_{33}$$

$$P_2(s) = (s+a) \quad \text{where } a \text{ is a constant.}$$

$$k_3 = \bar{b}_{11}(b_{22}b_{33} - \bar{b}_{32}\bar{b}_{23}) - \bar{b}_{12}(\bar{b}_{21}\bar{b}_{33} - \bar{b}_{31}\bar{b}_{23}) + \bar{b}_{13}(\bar{b}_{21}\bar{b}_{32} - \bar{b}_{31}\bar{b}_{22})$$

$$= \det [BR^{-1} B']$$

$$P_3(s) = 1$$

In general the n^{th} order system can be expanded as

$$1 + \frac{\sigma k_1 P_1(s)P_1(-s) + \dots + \sigma^i k_i P_i(s)P_i(-s) + \dots + \sigma^n k_n P_n(s)P_n(-s)}{\bar{D}(s) \bar{D}(-s)} = 0 \quad (2-36)$$

The degree of $P_i(s)$ is $(n-i)$ and that of $P_i(s)P_i(-s)$ is $2(n-i)$

where n is the order of the system. When σ is very large, eqn.

(2-36) can be expressed as

$$1 + \frac{\sigma k_1 s^{2(n-1)} + \sigma^2 k_2 s^{2(n-2)} + \dots + \sigma^n k_n s^0}{s^{2n} + a_1 s^{2(n-1)} + \dots + a_0 s^0} = 0 \quad (3-37)$$

The ratio of the polynomials in eqn. (2-37) has two more poles than zeros. Although the optimal system for large σ may be considered to approach a first order system, it might not have the properties of a Butterworth function. This is so because the zeros of the numerator polynomial are not fixed but vary with σ .

The specific properties of eqn. (2-37) depend on the various values of the constants k_i . These constants in general depend on the elements of the B matrix. They are found to be

$$k_1 = \bar{b}_{11} + \bar{b}_{22} + \dots + \bar{b}_{nn} = \text{sum of the diagonal elements}$$

$$= \text{trace} [BR^{-1}B']$$

$$k_2 = (\bar{b}_{11} \times \bar{b}_{22}) + (\bar{b}_{11} \times \bar{b}_{33}) + \dots + (\bar{b}_k \times \bar{b}_{k+1}) + \dots + (\bar{b}_{n-1} \times \bar{b}_n)$$

$$= \text{sum of all the diagonal elements of } BR^{-1}B' \text{ multiplied two at a time.}$$

$$k_j = (\bar{b}_{11} \times \bar{b}_{12} \times \dots \times \bar{b}_{jj}) + (\bar{b}_{22} \times \bar{b}_{33} \times \dots \times \bar{b}_{j+1}) + \dots$$

$$= \text{sum of the diagonal elements of } BR^{-1}B' \text{ multiplied 'j' at a time.}$$

and

$$k_n = \det [BR^{-1}B']$$

Here the input matrix B is a rectangular matrix of dimension (n x r) and $r \leq n$. If $r < n$, then $BR^{-1}B'$ is singular and $k_n = 0$. In some systems, it may happen that the various constants k_i are sufficiently small so that most of the σ_i terms can be ignored. Thus it may be possible to reduce the expansion in eqn. (2-37) to a form similar to the one in eqn. (2-22) and it could be written as

$$1 + \frac{\sigma k P(s) P(-s)}{Q(s) Q(-s)} = 0 \quad (2-38)$$

In this case the asymptotic properties may be found by the root-locus technique. However if $\sigma \rightarrow 0$, the expansion given by eqn. (2-37) will not be valid. It is known that for $Q = 0$, the optimal control function will be a null function and the resulting system will continue to be an n^{th} order system.

Thus it can be said in general that the structure of the input matrix greatly influences the asymptotic behaviour of the optimal system. This behaviour can be altered if the input matrix is altered. However the asymptotic behaviour of the system has very little to say about the behaviour of the optimal system when the elements of the Q matrix are finite and not very large. But the details of the structure of optimal system given in this chapter can enable the designer to develop a significant insight in synthesizing systems.

CHAPTER III
LOWER ORDER CONTROL
AND
SOME NUMERICAL EXAMPLES

It will be shown in this Chapter that the 'lower order' control transfers the specified eigenvalues from the lower order system to the higher order system. The Q matrix for the original system is obtained during the process of design as required by the problem considered in section 1.4 of the first chapter.

Here the term 'lower order' control is used to indicate that this 'lower order' control will first control a system of lower order. This lower order system is obtained by reducing the order of the original system or in other words contracting the order of the original system. The order of the contracted system will be determined by the number of inputs to be used. The specified eigenvalues are designed into the contracted system by the lower order control. By utilizing the lower order control in the original order system, it is shown that the lower order control transfers the specified eigenvalues to the higher order system.

In the first six sections, the infinite time regulator is considered. In section 3.1, the infinite time system is contracted to obtain the lower order system. In the process of contraction, the dominant eigenvalues of the original system are retained in the lower order system. In the past, the concept of contraction was used for simplification of dynamical analysis⁽²⁶⁾ of systems and for obtaining less complicated control functions⁽²⁵⁾ without specifying the eigenvalues which determine the time response of the system. Contraction was also used to control⁽²⁷⁾ weakly coupled⁽²⁸⁾ dynamical systems. This control was implemented without specifying the eigenvalues and the stability of the resulting system could not be guaranteed.⁽²⁷⁾ However throughout this thesis, the process of

contraction is developed for designing generalized multi-input systems which have the specified eigenvalues and which are assuredly stable.

In section 3.2, the eigenvalues of the lower order system are specified by specifying the optimal lower order system matrix. The Q matrix for the lower order system is obtained by satisfying condition for the rank of the matrix T_m which has to have an inverse. This condition implies that the order of the contracted system is equal to or less than the number of inputs. If the Q matrix for the higher order system is derived from the Q matrix for the lower order system, then it is shown in this section that the equivalence between the expanded equation (3-19) and eqn. (3-22) becomes an equality when additional conditions of uniqueness are satisfied.

In section 3.3, the concept of 'lower order' control is defined mathematically and in section 3.4 it is shown that this lower order control does transfer the specified eigenvalues from the lower order system to the resulting higher order system.

In section 3.5, it is observed that the resulting higher order system will be asymptotically stable if the conditions of stabilizability and detectability are satisfied. These conditions also ensure the uniqueness of the solution to eqn. (3-19).

In section 3.6, the cost of the 'lower order' control is derived. It is found to be the solution of a linear matrix equation. A method is given in Appendix A to solve such a linear matrix equation.

The finite time output regulator problem is considered in section 3.7 and development similar to one given above is carried out in brief. In section 3.8 the contraction transformation is considered in general first. Then two methods are given to obtain this contraction transformation matrix which is referred to as just contraction matrix. The first method is

based on the Vandermonde transformation matrix and it is particularly useful when the eigenvalues of the original system are distinct and real. The second method is quite general and the contraction matrix can be obtained notwithstanding the nature of the eigenvalues of the original system.

In section 3.9 , a digital computer oriented algorithm is given as a procedure for design. This procedure for design incorporates the tests for stabilizability and detectability. This procedure for design is utilized in designing the three systems given in the numerical examples. In section 3.10, a third order system with three stable, distinct and real eigenvalues and having the companion matrix form is designed by specifying two eigenvalues. In section 3.11, the system considered is third order and has one pair of complex conjugate eigenvalues and one real eigenvalue. This pair of complex conjugate eigenvalues is shifted. In section 3.12, the system considered is also a third order system but it has two unstable eigenvalues which are shifted to a specified position and stabilized. In all the three examples, the time response of the systems before and after applying the 'lower order' control is given.

3.1 Contraction of the Infinite Time System:

Since the 'lower order' control is obtained from the system of lower order, a method of lowering or contracting the order of the system is given below. This process of contraction is developed below for obtaining the 'lower order' control. Let the n^{th} order system be given by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A} \mathbf{x}(t) + \mathbf{B} u(t) \\ \mathbf{y}(t) &= \mathbf{H} \mathbf{x}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}\tag{3-1}$$

and let

$$\mathbf{J}_n = 1/2 \int_0^{\infty} [\mathbf{y}'(t) \mathbf{Q} \mathbf{y}(t) + u'(t) \mathbf{R} u(t)] dt\tag{3-2}$$

be the cost functional which is to be minimized. Here the various vectors are of the following dimensions :

$$\begin{aligned} \mathbf{x} & - (n \times 1) \\ \mathbf{y} & - (l \times 1) \\ \mathbf{u} & - (r \times 1) \end{aligned}$$

The system matrices A, B, H and the performance matrices Q and R are of the proper dimensions and conform with the vectors.

The cost functional can be rewritten as follows :

$$J_n = \frac{1}{2} \int_0^{\infty} [\mathbf{x}'(t) H' Q H \mathbf{x}(t) + \mathbf{u}'(t) R \mathbf{u}(t)] dt \quad (3-3)$$

The control vector \mathbf{u} has r components. If an m dimensional vector \mathbf{z} is defined with $m \leq r$, then

$$\mathbf{z}(t) = C \mathbf{x}(t) \quad (3-4)$$

where C is the $(m \times n)$ contraction matrix with

$$m \leq r \leq n .$$

This C matrix is a constant matrix.

Then

$$\dot{\mathbf{z}}(t) = C \dot{\mathbf{x}}(t)$$

Substituting eqn. (3-1) into the above equation

$$\dot{\mathbf{z}}(t) = CA \mathbf{x}(t) + CB \mathbf{u}(t) \quad (3-5)$$

Let

$$\dot{\mathbf{z}}(t) = F \mathbf{z}(t) + G \mathbf{u}(t) \quad (3-6)$$

Using eqn. (3-4), eqn. (3-6) can be rewritten as

$$\dot{\mathbf{z}}(t) = FC \mathbf{x}(t) + G \mathbf{u}(t) \quad (3-7)$$

Comparing eqns. (3-5) and (3-7), it can be seen that

$$\begin{aligned} FC & = CA \\ G & = CB \end{aligned} \quad (3-8)$$

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For the m^{th} order system given by eqn. (3-6) let the cost functional be defined as

$$J_m = 1/2 \int_0^{\infty} [w'(t) Q_m w(t) + u'(t) R u(t)] dt \quad (3-9)$$

Here

$$w(t) = P z(t)$$

$$z(0) = z_0$$

The various new matrices have the following dimensions :

$$F \quad - \quad (m \times m) \qquad P \quad - \quad (m \times m)$$

$$G \quad - \quad (m \times r) \qquad Q_m \quad - \quad (m \times m)$$

With eqn. (3-6) as the constraint eqn. (3-9) can be minimized^{(23), (32)} with the optimal control $u^*(t)$

$$u^*(t) = - \hat{K}_m z(t) \quad (3-10)$$

where

$$\hat{K}_m = R^{-1} G' \hat{M} \quad (3-11)$$

Here \hat{M} is the steady-state, symmetric matrix solution of the matrix Riccati equation given below

$$0 = \hat{M} F + F' \hat{M} - \hat{M} G R^{-1} G' \hat{M} + P' Q_m P \quad (3-12)$$

The optimal cost J_m for the m^{th} order system will be

$$\hat{J}_m = 1/2 \langle z_0, \hat{M} z_0 \rangle \quad (3-13)$$

The optimum m^{th} order system equation will be given by

$$\dot{z}(t) = F z(t) - G R^{-1} G' \hat{M} z(t)$$

Thus with F_0 defined as below

$$F_0 \triangleq [F - G R^{-1} G' \hat{M}] \quad (3-14)$$

$$\dot{z}(t) = F_0 z(t) \quad (3-15)$$

Now for this m^{th} order system, the optimal system matrix F_o is completely specified ; thus

$$\hat{M} = [T_m]^{-1} [F - F_o] \quad (3-16)$$

where T_m is defined below.

$$T_m \triangleq [GR^{-1}G']$$

The above development is comparable to the analysis in section 1.6 .

3.2 Expansion of the m^{th} Order System:

In this section it is proposed that by expanding the m^{th} order system in a specific way, the state-weighting matrix $H'Q_l H$ can be obtained from the Q_m matrix.

When F_o is completely specified, \hat{M} can be expressed as

$$\hat{M} = [T_m]^{-1} [F - F_o]$$

Here

$$T_m = GR^{-1}G' \text{ can have an inverse, since}$$

$\text{Rank } [GR^{-1}G'] \leq \min [m, r, n] \leq m$. See Appendix C for the specific conditions which are proved there.

Rewriting eqn. (3-12)

$$P' Q_m P = -\hat{M} F - F' \hat{M} + \hat{M} GR^{-1} G' \hat{M}$$

and substituting eqn. (3-16) into it

$$P' Q_m P = -[T_m]^{-1} [F - F_o] F_o - F' [T_m]^{-1} [F - F_o] \quad (3-17)$$

If eqn. (3-12) is premultiplied by C' and postmultiplied by C , then

$$0 = C' \hat{M} F C + C' F' \hat{M} C - C' \hat{M} GR^{-1} G' \hat{M} C + C' P' Q_m P C \quad (3-18)$$

Using eqn. (3-8), eqn (3-18) can be rewritten as

$$0 = (C' \hat{M} C)A + A'(C' \hat{M} C) - (C' \hat{M} C)BR^{-1}B'(C' \hat{M} C) + C'P'Q_m PC \quad (3-19)$$

If an identical development is carried through for minimizing the cost functional given by eqn. (3-2) under the constraints of eqn. (3-1) for the n^{th} order system, the n^{th} order optimal system will be given by the optimal control

$$u^*(t) = -\hat{K}_n x(t) \quad (3-20)$$

where

$$\hat{K}_n = R^{-1} B' \hat{N} \quad (3-21)$$

Here \hat{N} is a symmetric matrix and satisfies the following eqn.

$$0 = \hat{N}A + A'\hat{N} - \hat{N}BR^{-1}B'\hat{N} + H'Q_l H \quad (3-22)$$

Also the optimum n^{th} order system is

$$\dot{x}(t) = [A - B\hat{K}_n] x(t) \quad (3-23)$$

and

$$\hat{J}_n = 1/2 \langle x_0, \hat{N} x_0 \rangle \quad (3-24)$$

If eqns. (3-19) and (3-22) are compared, it will be noticed that \hat{N} is equivalent to $(C' \hat{M} C)$. If $H'Q_l H$ is equated to $C'P'Q_m PC$, then the non-linear algebraic eqn. (3-22), will have a certain solution \hat{N}_1 given by

$$\hat{N}_1 = C' \hat{M} C \quad (3-25)$$

with

$$H'Q_l H = C'P'Q_m PC \quad (3-26)$$

In Appendix C it is shown that \hat{N} is a unique solution when certain conditions are satisfied and that it satisfies the relation

$$\hat{N} = \hat{N}_1 = C' \hat{M} C \quad (3-27)$$

If eqn. (3-22) is solved as a matrix Riccati differential equation by backward integration and with the proper end condition, and if $H'Q_l H$ is the one given by eqn. (3-26), which is positive semi-definite, it is observed from numerous examples of different systems that \hat{N} is positive semi-definite.

Thus when F_o is completely specified, $P'Q_m P$ is found from eqn. (3-17) and $H'Q_l H$ is obtained from eqn. (3-26). If H is an $(n \times n)$ matrix and if its inverse exists, then the Q matrix will be an $(n \times n)$ matrix Q_n and can be expressed as

$$Q_n = [H']^{-1} [C'P'Q_m PC] [H]^{-1}$$

3.3 Lower Order Control :

The functional given by eqn. (3-2) will be minimized under the constraints of eqn. (3-1) if $u^*(t) = -\hat{K}_n x(t)$ given by eqn. (3.20) is used as the optimal control. However for this eqn. (3-22) will have to be solved.

On the other hand if $u^*(t) = -\hat{K}_m z(t)$ given by eqn. (3-10) is used, then eqn. (3-12) will have to be solved. When m is much smaller than n , eqn. (3-12) will be considerably easier to solve.

If

$$u(t) = -\hat{K}_m z(t) = \hat{K}_m C x(t) \quad (3-28)$$

is used for the n^{th} order system, it will be derived from the lower order system. This will be termed as the Lower Order Control and for convenience will be denoted by

$$u(t)_{\text{lower order}} = -KC x(t) \quad (3-29)$$

This lower order control gives rise to the following n^{th} order system

$$\begin{aligned} \dot{x}(t) &= A x(t) + B [-KC x(t)] \\ &= [A - BKC] x(t) \end{aligned} \quad (3-30)$$

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3.4 Eigenvalues of the System :

The lower order control $u(t) = -KCx(t)$ will give rise to the n^{th} order system

$$\dot{x}(t) = [A - BKC]x(t) \quad (3-30)$$

If this n^{th} order system $[A - BKC]$ has eigenvalues μ_i and eigenvectors x_i , certain properties will be established in this section to show that m of these eigenvalues are the same as those of F_o .

Since

$$[A - BKC]x_i = \mu_i x_i ; 1 \leq i \leq n \quad (3-31)$$

Thus

$$\begin{aligned} C[A - BKC]x_i &= C\mu_i x_i \\ [CA - CBKC]x_i &= \mu_i Cx_i \end{aligned} \quad (3-32)$$

Using eqns. (3-8) and (3-11), eqn. (3-32) can be rewritten as

$$[F - GR^{-1}G' \hat{M}]Cx_i = \mu_i Cx_i \quad (3-33)$$

By using eqn. (3-14), eqn. (3-33) can be written as

$$\begin{aligned} [F_o]Cx_i &= \mu_i (Cx_i) \\ Cx_i &\neq 0 ; 1 \leq i \leq m \end{aligned} \quad (3-34)$$

This shows that when $Cx_i \neq 0$, F_o has the same m eigenvalues as $[A - BKC]$; however both of these matrices have different eigenvectors.

The same can be proved the other way round. If F_o has eigenvalues λ_i and certain eigenvectors, then F_o' has the same eigenvalues with the same or different eigenvectors.

Knowing that $K = R^{-1}G' \hat{M}$

$$[F - GR^{-1}G' \hat{M}]' z_i = \lambda_i z_i ; 1 \leq i \leq m \quad (3-35)$$

Premultiplying by C' and using eqn. (3-8)

$$[A'C' - C'MGR^{-1}G'] z_i = C'\lambda_i z_i$$

$$[A' - C'K'B']C'z_i = \lambda_i (C'z_i)$$

Hence

$$[A - BKC]' C' z_i = \lambda_i (C' z_i) \quad ; \quad 1 \leq i \leq m \quad (3-36)$$

Thus when $C'z_i \neq 0$, $[A - BKC]$ has the same eigenvalues as F_o .

Thus it is seen that the lower order control transfers all the designed 'm' eigenvalues of F_o to the $[A - BKC]$ matrix.

3.5 Comments on Stability :

In this section the important question of the stability of the $[A - BKC]$ matrix will be considered.

In section 3.4 it was shown that the n^{th} order system matrix $[A - BKC]$ has m of its n eigenvalues identically the same as the m eigenvalues of the m^{th} order system $[F - GR^{-1}G'M]$. These m eigenvalues are stable since the matrix F_o is designed to be stable. Thus these m eigenvalues of $[A - BKC]$ are definitely known to be stable. The nature of the other $(n-m)$ eigenvalues can be predicted if the system is checked for stabilizability and Detectability. Stabilizability and detectability are defined in Appendix D.

Since

$$\begin{aligned} [A - BKC] &= [A - BR^{-1}G'MC] \\ &= [A - BR^{-1}B'(C'MC)] \end{aligned} \quad (3-37)$$

As shown in section 3.2, $N_1 = C'MC$ is one solution. As shown in Appendix C⁽⁴²⁾, this solution will be unique with $N_1 = N$ and resulting matrix $[A - BR^{-1}B'N]$ will be asymptotically stable if and only if the system is stabilizable and detectable.

In the design procedure, the stabilizability and detectability of the system is checked to ensure that

$$\begin{aligned} [A - BKC] &= [A - BR^{-1}B'(C'MC)] \\ &= [A - BR^{-1}B'N] \end{aligned} \quad (3-38)$$

is asymptotically stable. What is important is that $\hat{N} = C' \hat{M} C$ will be the relation which will enable \hat{N} to be found conveniently from \hat{M} .

3.6 Cost of the Lower Order Control:

In this section the cost of implementing the lower order control will be investigated.

The cost is given by the cost functional given by eqn. (3-3)

$$J_n = 1/2 \int_0^{\infty} [x'(t) H' Q H x(t) + u'(t) R u(t)] dt \quad (3-3)$$

The lower order control is given by eqn. (3-29) and is

$$u(t) = -K C x(t) \quad (3-29)$$

Thus J_n can be rewritten as

$$\begin{aligned} J_n &= 1/2 \int_0^{\infty} [x'(t) H' Q H x(t) + \{-K C x(t)\}' R \{K C x(t)\}] dt \\ &= 1/2 \int_0^{\infty} x'(t) [H' Q H + C' K' R K C] x(t) dt \end{aligned} \quad (3-39)$$

The resulting n^{th} order system is

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ &= [A - B K C] x(t) \end{aligned} \quad (3-40)$$

In section 3.5, it is mentioned that this resulting system will be asymptotically stable, if it is stabilizable and detectable. This test for stabilizability and detectability is incorporated into the synthesis algorithm given in section 3.9.

Thus

$$x(t) = e^{(A - B K C) t} x(0) \quad (3-41)$$

$$\text{and } x(\infty) = 0 \quad (3-42)$$

Define the Lyapunov function $V(x)$ such that

$$V[x(t)] = 1/2 x'(t) N x(t) \quad (3-43)$$

$$\dot{V} [\mathbf{x}(t)] = \frac{dV}{dt} = 1/2 [\dot{\mathbf{x}}'(t) N \mathbf{x}(t) + \mathbf{x}'(t) N \dot{\mathbf{x}}(t)] \quad (3-44)$$

Substituting eqn. (3-40) in eqn. (3-44)

$$\frac{dV}{dt} = 1/2 \mathbf{x}'(t) [(A-BKC)' N + N(A-BKC)] \mathbf{x}(t) \quad (3-45)$$

Introduce

$$(A-BKC)' N + N(A-BKC) = - [H' Q_l H + C' K' R K C] \quad (3-46)$$

Then

$$\int_0^{\infty} \frac{dV}{dt} \cdot dt = V(\infty) - V(0) = -1/2 \int_0^{\infty} \mathbf{x}'(t) [H' Q_l H + C' K' R K C] \mathbf{x}(t) dt$$

Since the resulting system is asymptotically stable

$$V [\mathbf{x}(\infty)] = 0 = V(\infty)$$

Hence

$$V(0) = 1/2 \int_0^{\infty} \mathbf{x}'(t) [H' Q_l H + C' K' R K C] \mathbf{x}(t) dt \quad (3-47)$$

Thus

$$V(0) = J_n = 1/2 \mathbf{x}'(0) N \mathbf{x}(0) \quad (3-48)$$

where N satisfies eqn. (3-46) which can be rewritten as

$$(A-BKC)' N + N(A-BKC) + C' K' R K C + H' Q_l H = 0 \quad (3-49)$$

Since initial condition $\mathbf{x}(0)$ is a constant, N can be explicitly written as below by using eqns. (3-41), (3-47) and (3-48).

$$N = \int_0^{\infty} e^{(A-BKC)' t} [H' Q_l H + C' K' R K C] e^{(A-BKC)t} dt \quad (3-50)$$

Evaluating the integral in eqn. (3-50) is complicated because of matrix exponents. A much easier way is to solve for N in the equation given by (3-49). This equation is linear in N and can be solved for N by the algorithm given in Appendix A. It may be mentioned that eqn. (3-49) is linear in N as contrasted with the non-linear eqn. (3-22) in N.

If $H'Q_l H = C'P'Q_m PC$, then eqn. (3-49) can be written as

$$(A-BKC)'N + N(A-BKC) + C'K'RKC + C'P'Q_m PC = 0 \quad (3-51)$$

$N = C' \hat{M} C = \hat{N}$ is a solution of the above equation if

$$H'Q_l H = C'P'Q_m PC .$$

This can be verified by simplifying the following and substituting.

$$[A-BR^{-1}B'C'\hat{M}C]N + N[A-BR^{-1}B'C'\hat{M}C] + C'\hat{M}CBR^{-1}B'C'\hat{M}C + C'P'Q_m PC = 0 \quad (3-51a)$$

3.7 Finite Time Output Regulator Problem:

In this section the concept of lower order control will be extended to the finite time output regulator problem.

If the n^{th} order system is given by eqn. (3-1) and the cost functional J_n given below is to be minimized

$$J_n = 1/2 y'(T)S y(T) + 1/2 \int_0^T [y'(t)Q_l y(t) + u'(t)Ru(t)] dt \quad (3-52)$$

then development can be carried out as in eqns. (3-4) to (3-8). If the cost functional J_m is defined as

$$J_m = 1/2 w'(T)U w(T) + 1/2 \int_0^T [w'(t)Q_m w(t) + u'(t)Ru(t)] dt \quad (3-53)$$

where

$$w(t) = Pz(t) \quad \text{and} \quad P - (m \times m) \text{ matrix.}$$

Other matrices and vectors have the appropriate dimensions.

Using the transversality condition for the costate vector p ,

$$p(T) = \frac{\partial}{\partial z(T)} [1/2 \langle z(T), P'UP z(T) \rangle] \quad (3-54)$$

and carrying the analysis through, the optimum control will be given by

$$\begin{aligned} u^*(t) &= -R^{-1}G' M^*(t) z(t) \\ &= -K_m^* z(t) \end{aligned} \quad (3-55)$$

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where $M^*(t)$ satisfies the following equation.

$$-\dot{M}^*(t) = M^*(t)F + F'M^*(t) - M^*(t)GR^{-1}G'M^*(t) + P'Q_m P$$

$$\text{with } M^*(T) = P'UP \quad (3-56)$$

Equation (3-56) can be solved backward in time with a negative increment in time.

If the lower order control given by eqn. (3-55) is used for the n^{th} order system, then

$$\dot{x}(t) = [A - BK_m^* C] x(t) \quad (3-57)$$

The same control will give an m^{th} order system given by

$$\begin{aligned} \dot{z}(t) &= [F - GR^{-1}G' \hat{M}(t)] z(t) \\ &= F_0(t) z(t) \end{aligned} \quad (3-58)$$

If the finite time regulator problem described by (3-1) and (3-52) is controlled without contraction, then

$$\dot{x}(t) = [A - BK_n^*] x(t) \quad (3-59)$$

where

$$K_n^* = R^{-1} B' N^*(t)$$

and $N^*(t)$ satisfies the following equation

$$-\dot{N}^*(t) = N^*(t)A + A'N^*(t) - N^*(t)BR^{-1}B'N^*(t) + H'Q_\ell H \quad (3-60)$$

$$\text{with } N^*(T) = H'SH$$

Eqn. (3-60) is integrated backward in time. If eqn. (3-60) is premultiplied by C' and postmultiplied by C , then the equivalence between $N^*(t)$ and $C'M^*(t)C$ and the correspondence between $H'Q_\ell H$ and $C'P'Q_m PC$ will be noticed. As before, if $H'Q_\ell H$ were equated to $C'P'Q_m PC$, then the equivalence will be even more significant.

In the case of finite time problem, the concept of eigenvalues is rather inappropriate since the feedback matrices K_m^* or K_n^* are time varying. This means that the eigenvalues of both the m^{th} order and the n^{th} order systems will be changing with time. If eqns (3-57) and (3-58) are compared and an analysis similar to the one made in section 3.4 is carried through, it will be noticed that although the feedback gains are time-varying, the eigenvalues of the two systems will be identical at any given corresponding point in time.

If a Lyapunov function $V(x)$ is defined as

$$V[x(t)] = 1/2 \langle x(t), N(t) x(t) \rangle \quad (3-61)$$

then with

$$\begin{aligned} u(t) &= -R^{-1} G' M^*(t) C x(t) \\ &= -KC x(t) \end{aligned}$$

the cost functional given by eqn. (3-52) can be shown to be

$$J_n = 1/2 \langle x(t), N(t) x(t) \rangle \quad (3-62)$$

where $N(t)$ satisfies the linearized matrix differential equation

$$-\dot{N}(t) = (A-BKC)' N(t) + N(t) (A-BKC) + C' K' R K C + H' Q H$$

$$\text{with } N(T) = H' S H \quad (3-63)$$

It may be mentioned that, without contraction, J_n^* would be given by

$$J_n^* = 1/2 \langle x(t), N^*(t) x(t) \rangle$$

where $N^*(t)$ is the solution of the non-linear differential equation (3-60).

3.8 Contraction Matrix C:

The contraction matrix C plays an important role in implementing the lower order control. In this section the contraction transformation⁽²⁵⁾ will be investigated and two methods will be proposed for obtaining the contraction matrix C .

3.8.1 Contraction Transformation :

The lower order system with dimension m is derived from the higher n dimensional system through contraction transformation.

Consider the n^{th} order system

$$\dot{x} = Ax + Bu \quad (3-64)$$

If

$$y = T^{-1}x \quad (3-65)$$

$$\Lambda = T^{-1}AT \quad (3-66)$$

Then the above co-ordinate transformation will result in

$$\begin{aligned} \dot{y} &= T^{-1}\dot{x} \\ &= T^{-1}Ax + T^{-1}Bu \\ &= T^{-1}ATy + T^{-1}Bu \end{aligned} \quad (3-67)$$

If T^{-1} is partitioned row-wise

$$T^{-1} = \begin{bmatrix} C & \uparrow \\ \dots & \downarrow \\ D & \uparrow \\ & \downarrow \end{bmatrix} \begin{matrix} m \\ n-m \end{matrix} \quad (3-68)$$

and z is the vector with first m components of y , then

$$z = Cx$$

is the contraction transformation. If all the natural frequencies are distinct, then the first m natural frequencies are retained by

$$\dot{z} = Fz + Gu$$

where

$$FC = CA$$

and $G = CB$

The following relation can be established

$$\begin{aligned} FCC' &= CAC' \\ F &= CAC'(CC')^{-1} \end{aligned} \quad (3-69)$$

The matrix F determined by eqn. (3-69) will satisfy the relation $FC = CA$ when the following condition is met.

$$CA = CAC' (CC')^{-1} C \tag{3-70}$$

Since C is $(m \times n)$ matrix, CC' is $(m \times m)$ and the CC' matrix can be assumed to have an inverse. Two methods of obtaining this C Matrix will be described below.

3.8.2 1st Method :

If the n^{th} order system matrix A has λ_i as the eigenvalues and y_i ; as the eigenvectors, then

$$[A]y_i = \lambda_i y_i \quad ; \quad 1 \leq i \leq n$$

Thus

$$CAy_i = C\lambda_i y_i \tag{3-71}$$

Since $FC = CA$, eqn. (3-71) becomes

$$[F] Cy_i = \lambda_i Cy_i \quad ; \quad 1 \leq i \leq m \tag{3-72}$$

with $Cy_i \neq 0$

Thus for the m eigenvalues of F , eqn. (3-72) will hold.

When λ_i takes on other $(n-m)$ eigenvalues, the following relation will hold.

$$Cy_i = 0 \quad ; \quad (m+1) \leq i \leq n$$

Since y_i are the eigenvectors of the n^{th} order matrix, it is known that n independent eigenvectors can be found and

$$T = \begin{bmatrix} y_{1,1} & y_{2,1} & \dots & y_{n,1} \\ y_{1,2} & y_{2,2} & & y_{n,2} \\ \vdots & \vdots & & \vdots \\ y_{1,n} & y_{2,n} & & y_{n,n} \end{bmatrix}$$

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is the transformation matrix more commonly known as the modal matrix. When the eigenvalues are distinct and the system matrix A is in the companion matrix form, the matrix given below becomes the modal matrix. This matrix is known as the Vandermonde matrix and in Appendix B it is shown that it is in fact the modal matrix. A method for obtaining eigenvalues is also given there.

$$T = \begin{bmatrix} \lambda_1^0 & \lambda_2^0 & \dots & \lambda_n^0 \\ \lambda_1^1 & \lambda_2^1 & & \lambda_n^1 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix} \quad (3-73)$$

This modal matrix will transform the system matrix into a diagonal matrix Λ since the eigenvalues are distinct.

Thus with distinct eigenvalues

$$\Lambda = T^{-1} A T$$

If the eigenvalues are repeated, the transformation matrix T_r cannot be found as easily as in eqn. (3-73) and a Jordan canonical form will be the result of transformation. Thus with repeated eigenvalues

$$J = T_r^{-1} A T_r$$

The condition for obtaining m eigenvalues from the A matrix into the F matrix is

$$C y_i \neq 0 \quad ; \quad 1 \leq i \leq m$$

and can be satisfied if

$$C = [T^{-1}] \quad (3-74)$$

(mxn)

This is the same as saying that the contraction matrix C is formed from the m appropriate rows of the inverse of the modal transformation matrix to obtain the m dominant eigenvalues in the contracted system matrix F .

The method proposed in eqn. (3-74) will do an effective job when all the eigenvalues are non-zero, distinct and real. Since the contraction matrix C has to be real, it will be difficult to obtain if the eigenvalues are complex. Repeated eigenvalues create the same kind of problem in obtaining the various matrices. The problem is even worse if complex eigenvalues are repeated.

3.8.3 2nd Method :

It is seen from eqn. (3-72) that m eigenvalues of F are identical to m of the eigenvalues of the A matrix. The matrix equation $FC = CA$ has only one known matrix A and two unknown matrices F and C . Since F can have the same eigenvalues as A , F will be assumed to have these m dominant eigenvalues and also a standard companion matrix form. Since the companion matrix form is always real, and

$$\begin{aligned} FC &= CA \\ G &= CB \end{aligned}$$

C will be a companion matrix of real elements and so will be G . Thus the m^{th} order matrix Riccati differential equation will have all the matrices real. A simple example will demonstrate the method.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix} \quad (3-75)$$

Then the eigenvalues are given by

$$(\lambda + 1)(\lambda^2 + 4\lambda + 5) = 0$$

The matrix A has one real eigenvalue and one pair of complex eigenvalues. Let F be the 2nd order matrix with the complex pair of eigenvalues retained. Thus F will have $(\lambda^2 + 4\lambda + 5 = 0)$ as the characteristic equation ; F will be assumed to have the companion matrix form :

$$F = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \quad (3-76)$$

If $FC = CA$ is solved with

$$C_{13} = a$$

$$C_{22} = b, \quad \text{then matrix C will be}$$

$$C = \begin{bmatrix} (b+9a) & (b+10a) & (a) \\ (-5a) & (b) & (5a+b) \end{bmatrix} \quad (3-77)$$

Here 'a' and 'b' are any real numbers and thus C will be a non-unique matrix. However since C is a rectangular transformation matrix, it is quite reasonable that it is non-unique. For simplicity, it will be assumed that $a = b$; then

$$C = a \begin{bmatrix} 10 & 11 & 1 \\ -5 & 1 & 6 \end{bmatrix}$$

Constant 'a' can have any value and it can very well be taken to be unity. This method is quite general and yields a C matrix having elements which are real numbers.

3.9 Procedure for Design :

A concise scheme for designing multi-variable linear systems will be given here. Rather than putting it in a flow chart form, the procedure will be written down as a descriptive algorithm useful for a typical digital computer. All the steps are to be followed sequentially unless specified otherwise.

- STEP 1: DEFINE THE DIMENSIONS n, r, l, m .
- STEP 2: READ IN VARIOUS MATRICES A, B, H, R, S, F_0 . WRITE DOWN THESE MATRICES.
- STEP 3: FIND THE EIGENVALUES OF THE SYSTEM MATRIX A . WRITE DOWN THESE EIGENVALUES.
- STEP 4: ARE ANY EIGENVALUES COMPLEX OR ANY EIGENVALUES REPEATED? IF THEY ARE, GO TO STEP 6. IF THEY ARE NON-REPEATED REAL EIGENVALUES, GO TO STEP 5.
- STEP 5: FORM THE MODAL TRANSFORMATION MATRIX T . OBTAIN INVERSE T^{-1} . GENERATE THE CONTRACTION MATRIX C AS DESCRIBED IN 1st METHOD IN SECTION 3.8.2. WRITE THESE MATRICES. GO TO STEP 7.
- STEP 6: FORM THE F MATRIX IN THE COMPANION MATRIX FORM. OBTAIN THE CONTRACTION MATRIX C AS DESCRIBED IN 2nd METHOD IN SECTION 3.8.3. WRITE THESE MATRICES.
- STEP 7: OBTAIN THE CONTRACTED SYSTEM THROUGH THE CONTRACTION MATRIX C . WRITE CONTRACTED SYSTEM MATRICES F AND G .
- STEP 8: OBTAIN THE MATRIX T_m . $T_m = GR^{-1}G'$. CHECK T_m FOR AN INVERSE BY CHECKING $\text{RANK}[G] = m$. IF $\text{RANK}[G]$ IS LESS THAN m , GO BACK TO STEP 4. FIND ANOTHER C ; IF $\text{RANK}[G]$ IS m , THEN GO TO STEP 9.
- STEP 9: OBTAIN m^{th} ORDER STATE-WEIGHTING MATRIX Q_m FROM THE RELATION;
$$Q_m = - [T_m]^{-1} [F - F_0] F_0 - F' [T_m]^{-1} [F - F_0].$$
 WRITE MATRIX Q_m .

- STEP 10: FORM THE n^{th} ORDER MATRIX Q_n . $Q_n = C' Q_m C$. FROM THE Q_n MATRIX CHECK FOR STABILIZABILITY AND DETECTABILITY OF THE n^{th} ORDER SYSTEM. IF THE SYSTEM IS STABILIZABLE AND DETECTABLE, GO TO STEP 11. IF NOT, GO TO STEP 2 AND READ IN ANOTHER F_o .
- STEP 11: INTEGRATE THE m^{th} ORDER MATRIX RICCATI DIFFERENTIAL EQUATION BACKWARDS. DOES THE SOLUTION OF THE RICCATI EQUATION CONVERGE? IF IT DOES CONVERGE GO TO STEP 12. IF IT DOES NOT, GO TO STEP 2 AND READ IN ANOTHER F_o . REPEAT TILL THE SOLUTION OF THE RICCATI EQUATION IS CONVERGENT.
- STEP 12: WRITE THE SOLUTION OF THE MATRIX RICCATI EQUATION $M(t)$ ONCE IN EVERY THOUSAND STEPS OR ANY OTHER REASONABLE INTERVAL. WHEN THE S MATRIX IS ZERO \hat{M} IS THE STEADY-STATE SOLUTION. WRITE THIS MATRIX DOWN.
- STEP 13: OBTAIN THE FEED-BACK GAIN MATRIX K FROM THE RELATION $K = R^{-1} G' \hat{M}$. GENERATE THE LOWER ORDER CONTROL FROM THE RELATION $u(t)_{\text{lower order}} = -KC x(t)$.
- STEP 14: LOWER ORDER CONTROL GIVES RISE TO SYSTEM MATRIX Z . $Z = [A - BKC]$. ALSO FORM THE MATRIX $C' \hat{M} C$.
- STEP 15: INTEGRATE THE n^{th} ORDER MATRIX RICCATI DIFFERENTIAL EQUATION BACKWARDS WITH THE Q_n MATRIX GIVEN ABOVE. COMPARE THE SOLUTION MATRIX \hat{N} WITH THE MATRIX $C' \hat{M} C$ FROM STEP 14.
- STEP 16: FORM THE MATRIX $C' K' R K C$. OBTAIN MATRIX Y . HERE, $Y = - [C' K' R K C + Q_n]$. WRITE VARIOUS MATRICES.
- STEP 17: FORM THE LOWER ORDER CONTROL EQUATION $NZ + Z' N = Y$. SOLVE THIS EQUATION FOR MATRIX N . COMPARE THIS N WITH \hat{N} FROM STEP 15 AND $C' \hat{M} C$ FROM STEP 14. SINCE $Q_n = C' \hat{M} C$, IT CAN BE VERIFIED THAT $N = \hat{N} = C' \hat{M} C$. WRITE THE VARIOUS MATRICES. STEPS 15, 16, AND 17 ARE USED FOR DOUBLE-CHECKING ONLY.

STEP 18: DESIGN OF SPECIFIED m EIGENVALUES IS COMPLETE. IF ANOTHER SYSTEM IS TO BE DESIGNED, GO TO STEP 1 AND READ IN ANOTHER DATA SET. COMPLETE THE DESIGN ALGORITHM. END OF DESIGN PROCEDURE.

In the three concluding sections specific numerical examples will be worked out to demonstrate the implementation of the theory of lower order control developed in this chapter.

3.10 1st Numerical Example :

In this example, a third order system is considered. The system matrix is in the companion matrix form and has three stable, distinct and real eigenvalues. The system design is carried out by contracting the system to a system of second order. This second order system retains two dominant eigenvalues of the original third order system. These two dominant eigenvalues are shifted to the prescribed position ; then they are designed into the third order system by using the lower order control. This is achieved essentially by the process of expansion.

Let the n^{th} order system be

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{3-78}$$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -43 & -14 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

It is required to minimize

$$J_n = \frac{1}{2} \int_0^{\infty} [x'(t) Q_n x(t) + u'(t) R u(t)] dt$$

with $R = I$ (3-79)

The eigenvalues of the system are all real and distinct.

$$\lambda_1 = -1.0$$

$$\lambda_2 = -3.0$$

$$\lambda_3 = -10.0$$

Let it be required to design a system with the design eigenvalues

$$\lambda_{D.1} = -2.86$$

$$\lambda_{D.2} = -24.25$$

and F_o given by

$$F_o = \begin{bmatrix} -22.55 & 1.432 \\ 23.47 & -4.56 \end{bmatrix} \quad (3-80)$$

Going through the design procedure given in section 3.9, the following matrices are obtained.

Contraction matrix C will be given by

$$C = \begin{bmatrix} 1.666 & 0.722 & 0.0555 \\ -0.714 & -0.785 & -0.0714 \end{bmatrix} \quad (3-81)$$

The m^{th} order system matrices will be

$$F = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & -3.0 \end{bmatrix} \quad (3-82)$$

and

$$G = \begin{bmatrix} 0.0555 & 0.722 \\ -0.0714 & -0.785 \end{bmatrix} \quad (3-83)$$

The matrix F has (-1.) and (-3.) as its eigenvalues which are the two dominant eigenvalues of the matrix A.

The matrices F and G are obtained from the relations

$$FC = CA \quad \text{and} \quad G = CB.$$

Then

$$\begin{aligned} [T_m]^{-1} &= [GR^{-1}G']^{-1} \\ &= \begin{bmatrix} 9.88 \times 10^3 & 9.07 \times 10^3 \\ 9.07 \times 10^3 & 8.33 \times 10^3 \end{bmatrix} \end{aligned}$$

The Q_m matrix will be obtained from the relation

$$Q_m = -[T_m]^{-1} [F - F_0] F_0 - F' [T_m]^{-1} [F - F_0]$$

and

$$Q_m = \begin{bmatrix} 9.01 \times 10^2 & -4.62 \\ -4.62 & 1.04 \times 10^2 \end{bmatrix} \quad (3-84)$$

Matrix \hat{M} will be

$$\hat{M} = \begin{bmatrix} 5.23 \times 10^1 & 1.22 \times 10^1 \\ 1.22 \times 10^1 & 1.55 \times 10^1 \end{bmatrix} \quad (3-85)$$

The matrix Q_n will be given by

$$\begin{aligned} Q_n &= C' Q_m C \\ &= \begin{bmatrix} 2.66 \times 10^3 & 1.22 \times 10^3 & 9.61 \\ 1.22 \times 10^3 & 5.87 \times 10^2 & 4.64 \times 10^1 \\ 9.61 & 4.64 \times 10^1 & 3.68 \end{bmatrix} \quad (3-86) \end{aligned}$$

The feedback which is derived from the gain matrix K and KC are

$$K = \hat{K}_m = R^{-1} G' \hat{M}$$

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$$= \begin{bmatrix} 2.05 & -4.28 \times 10^{-1} \\ 28.33 & -3.41 \end{bmatrix} \quad (3-87)$$

$$KC = \begin{bmatrix} 3.69 & 1.80 & 0.14 \\ 51.33 & 22.97 & 1.804 \end{bmatrix} \quad (3-88)$$

The lower order control will give rise to the system matrix $[A-BKC]$ where

$$[A-BKC] = \begin{bmatrix} 0 & 1 & 0 \\ -51.33 & -22.97 & -0.804 \\ -33.69 & -44.80 & -14.14 \end{bmatrix} \quad (3-89)$$

The eigenvalues of the matrix $[A-BKC]$ are found to be :

$$\lambda_{D.1} = -2.86$$

$$\lambda_{D.2} = -24.25$$

and

$$\lambda_3 = -10.0$$

The cost of the lower order control N can be obtained by solving

$$[A-BKC]^T N + N[A-BKC] + C^T K^T R K C + Q_n = 0$$

and with $R = I$,

$$N = \begin{bmatrix} 1.24 \times 10^2 & 4.93 \times 10^1 & 3.69 \\ 4.93 \times 10^1 & 2.29 \times 10^1 & 1.804 \\ 3.69 & 1.804 & 1.43 \times 10^{-1} \end{bmatrix} \quad (3-90)$$

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\hat{N} can be solved with the \hat{Q}_n matrix given by eqn. (3-86) and solving the n^{th} order matrix Riccati equation. However it can be ascertained that $\hat{N} = C' M C$.

If, $R = 10I$, then

$$\hat{N} = \begin{bmatrix} 3.39 \times 10^2 & 1.43 \times 10^2 & 1.09 \times 10^{-1} \\ 1.43 \times 10^2 & 6.58 \times 10^1 & 5.14 \\ 1.09 \times 10^1 & 5.14 & 4.04 \times 10^{-1} \end{bmatrix} \quad (3-91)$$

3.10.1 Unit Step Response :

The concept of unit step response is widely used in the case of single input systems. In the case of multi-input systems it can be generalized as follows.

With

$$\dot{x}(t) = A x(t) + B u(t)$$

Taking Laplace transforms on both sides.

$$s X(s) - x(0_+) = A X(s) + B U(s)$$

$$x(t) = \mathcal{L}^{-1} \{ [sI - A]^{-1} [BU(s) + x(0_+)] \}$$

For the unit step response the initial condition will be assumed to be zero and

$$x(t) = \mathcal{L}^{-1} \{ [sI - A]^{-1} [BU(s)] \} \quad (3-92)$$

Equation (4-15) can be written as a convolution integral

$$x(t) = \int_0^t \phi(t-\lambda) B u(\lambda) d\lambda \quad (3-93)$$

where the transition matrix is known to be

$$\phi(t) = \mathcal{L}^{-1} [sI - A]^{-1} \quad (3-94)$$

It is generally more convenient to compute the inverse Laplace transform given by eqn. (3-92) by the residue theory than evaluating the convolution integral. For understanding the time response, only the first component $x_1(t)$ of the vector $x(t)$ will be considered. The other components could be worked out similarly. In a two-dimensional control, the control vector $u(t)$ can be taken to be

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For a higher dimensional control, any other combination of 1's may be taken. However to make the results meaningful and manageable, in the case of this two-dimensional control $u(t)$ will be assumed as

$$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following diagram will indicate the procedure followed to obtain the unit step response.

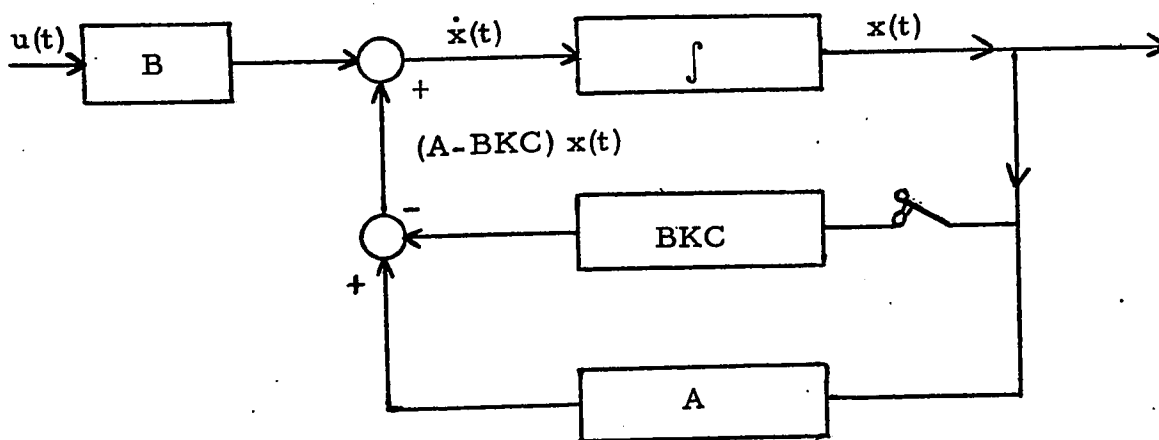


Fig. 3-1

Since the state regulator has no outside control vector and the feedback is strictly state feedback, the above block diagram with the switch in

the closed position will be the regulator with the lower order control

$$\dot{x}(t) = [A-BKC] x(t) + B u(t)$$

And with the switch open, it will be the system

$$\dot{x}(t) = A x(t) + B u(t)$$

This scheme will ensure that the input matrix B will be the same for both the controlled and the uncontrolled system. For the unit step response it will be assumed that the instant the switch is closed, the resulting regulator matrix becomes [A-BKC]. Then the transition matrix of the state regulator will be :

$$\phi(t) = \mathcal{L}^{-1} [sI - (A-BKC)]^{-1} \quad (3-95)$$

Here the inverse matrix is given by

$$[sI - (A-BKC)]^{-1} = \frac{1}{D(s)} \begin{bmatrix} [(s+26.02)(s+11.09)] & [(s+14.14)] & [-(0.804)] \\ [-(51.33s+699.)] & [s(s+14.14)] & [-(0.804s)] \\ [-(33.7s-1516.)] & [-(44.80s+33.7)] & [(s+20.43)(s+2.54)] \end{bmatrix}$$

where

$$D(s) = (s+10)(s+2.86)(s+24.25) \quad (3-96)$$

Evaluating the various terms of the transition matrix $\phi(t)$, they are

$$\begin{aligned} \phi_{11}(t) &= 1.25e^{-2.86t} - 1.72 \times 10^{-1} e^{-10t} - 7.8 \times 10^{-2} e^{-24.25t} \\ \phi_{12}(t) &= 7.4 \times 10^{-2} e^{-2.86t} - 4.06 \times 10^{-2} e^{-10t} - 3.34 \times 10^{-2} e^{-24.25t} \\ \phi_{13}(t) &= -5.28 \times 10^{-3} e^{-2.86t} + 7.92 \times 10^{-3} e^{-10t} - 2.66 \times 10^{-3} e^{-24.25t} \\ \phi_{21}(t) &= -3.64e^{-2.86t} + 1.84e^{-10t} + 1.8 e^{-24.25t} \\ \phi_{22}(t) &= -2.12 \times 10^{-1} e^{-2.86t} + 4.06 \times 10^{-1} e^{-10t} + 8.06 \times 10^{-1} e^{-24.25t} \\ \phi_{23}(t) &= 1.52 \times 10^{-2} e^{-2.86t} - 7.94 \times 10^{-2} e^{-10t} + 6.42 \times 10^{-2} e^{-24.25t} \\ \phi_{31}(t) &= 10.55 e^{-2.86t} - 18.3 e^{-10t} + 7.75 e^{-24.25t} \\ \phi_{32}(t) &= 6.2 \times 10^{-1} e^{-2.86t} - 4.07 e^{-10t} + 3.45 e^{-24.25t} \\ \phi_{33}(t) &= -3.8 \times 10^{-2} e^{-2.86t} + 7.7 \times 10^{-1} e^{-10t} + 2.68 \times 10^{-1} e^{-24.25t} \end{aligned}$$

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It can be verified that

$$\phi(t=0) = I$$

and $\phi(t=\infty) = 0$

Thus

$$x(t) = \int_0^t \phi(t-\lambda) B u(\lambda) d\lambda$$

Since the matrix B is (n x r) and the vector u is (r x 1), the product Bu will be a vector (n x 1). Define $[bu]_{j1}$ as the elements of the product Bu.

Then

$$x_1(t) = \int_0^t \sum_{j=1}^{n=3} \phi_{1j}(t-\lambda) [bu]_{j1}(\lambda) d\lambda \quad (3-97)$$

The same can be obtained from

$$x(t) = \mathcal{L}^{-1} \{ [sI - (A - BKC)]^{-1} [B U(s)] \}$$

Thus

$$x_1(t) = \mathcal{L}^{-1} \frac{-0.804}{s \cdot D(s)}$$

where D(s) is defined in eqn. (3-96). From the residue theory

$$x_1(t) = \sum_{s=-r_1, \dots, -r_k, \dots, -r_n} (s+r_k) \left\{ \frac{-0.804}{s \cdot D(s)} \right\} e^{st}$$

where the various r's are the simple poles. Evaluating this

$$x_1(t) = -1.15 \times 10^{-3} + 1.84 \times 10^{-3} e^{-2.86t} - 7.92 \times 10^{-4} e^{-10t} + 1.095 \times 10^{-4} e^{-24.25t} \quad (3-98)$$

This result is exactly the same as would be obtained from the convolution integral. This output $x_1(t)$ will be within 1/2 % of the steady state in less than 2.05 seconds.

With the switch open, the regulator will not have the internal state feedback and

$$x(t) = \mathcal{L}^{-1} \{ [sI - A]^{-1} [B u(s)] \}$$

Here

$$[sI - A]^{-1} = \frac{1}{D_o(s)} \begin{bmatrix} [(s^2 + 14s + 43)] & [(s + 14)] & [(1)] \\ [-(30)] & [s(s + 14)] & [(s)] \\ [-(30s)] & [-(43s + 40)] & [(s^2)] \end{bmatrix}$$

where $D_o(s) = (s+1)(s+3)(s+10)$

Thus

$$x_1(t) = \mathcal{L}^{-1} \frac{1}{s \cdot D_o(s)}$$

Evaluating the inverse Laplace transform by the residue theory

$$x_1(t) = 3.33 \times 10^{-2} - 5.56 \times 10^{-2} \times 10^{-2} e^{-t} + 2.38 \times 10^{-2} e^{-3t} - 1.59 \times 10^{-3} e^{-10t} \quad (3-99)$$

This output will be within 1/2 % of the steady state in less than 5.8 seconds. Thus it can be appreciated that the regulator with the lower order control and the designed eigenvalues, reaches within 1/2 % of the steady state much faster.

If

$$u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{then}$$

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1} \frac{(s+14)+1}{s \cdot D_o(s)} \\ &= \mathcal{L}^{-1} \frac{(s+14)}{s \cdot D_o(s)} + \mathcal{L}^{-1} \frac{1}{s \cdot D_o(s)} \end{aligned}$$

This is nothing else but the sum of the responses for

$$u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For a case where $u(t)$ is of higher dimension, it will be noticed that the total response will be the sum of the individual responses with the appropriate component of $u(t)$ taken to be unity and the other components taken to be zero. This is once again a reassertion of the superposition principle of linear systems.

3.11 2nd Numerical Example :

In this example, the third order system has one real eigenvalue and one pair of complex conjugate eigenvalues. The contraction gives rise to the second order system which retains this pair of complex eigenvalues. These are shifted into desired positions and incorporated back into the third order system by using the lower order control.

The n th order system is given by

$$\dot{x}(t) = A x(t) + B u(t)$$

$$x(t_0) = x_0$$

Here

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -11 & -5 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3-100)$$

It is required to minimize

$$J_n = 1/2 \int_0^{\infty} [x'(t) Q_n x(t) + u'(t) R u(t)] dt$$

with $R = I$ (3-101)

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One eigenvalue is real and the other two are a complex conjugate pair.

$$\lambda_1 = -3.$$

$$\lambda_{2,3} = -1. \pm j 2.$$

It is required to design a system with the design eigenvalues

$$\lambda_{D,1,2} = -5.724 \pm j 2.9$$

with F_o given by

$$F_o = \begin{bmatrix} -9.3071 & 3.2882 \\ -6.4211 & -2.1410 \end{bmatrix} \quad (3-102)$$

Going through the design procedure, the following results are obtained.

Contraction matrix C is given by

$$C = \begin{bmatrix} 12.0 & 7.0 & 1.0 \\ -15.0 & 1.0 & 2.0 \end{bmatrix} \quad (3-103)$$

The m^{th} order system matrices will be given by

$$F = \begin{bmatrix} 0 & 1 \\ -5.0 & -2.0 \end{bmatrix} \quad (3-104)$$

$$G = \begin{bmatrix} 1.0 & 7.0 \\ 2.0 & 1.0 \end{bmatrix} \quad (3-105)$$

The matrix F has the complex pair as its eigenvalues.

$$\begin{aligned}
 [T_m]^{-1} &= [G R^{-1} G']^{-1} \\
 &= \begin{bmatrix} 2.9586 \times 10^{-2} & -5.3254 \times 10^{-2} \\ -5.3254 \times 10^{-2} & 2.9586 \times 10^{-2} \end{bmatrix}
 \end{aligned}$$

The matrix Q_m will be given by

$$Q_m = \begin{bmatrix} 9.9951 \times 10^{-1} & 3.0121 \times 10^{-4} \\ 3.0121 \times 10^{-4} & 9.9951 \times 10^{-1} \end{bmatrix} \quad (3-106)$$

Steady state m^{th} order matrix will be

$$\hat{M} = \begin{bmatrix} 1.9964 \times 10^{-1} & -7.5187 \times 10^{-2} \\ -7.5187 \times 10^{-2} & 1.6356 \times 10^{-1} \end{bmatrix} \quad (3-107)$$

$$Q_n = \begin{bmatrix} 3.6878 \times 10^2 & 6.8933 \times 10^1 & -1.7998 \times 10^1 \\ 6.8933 \times 10^1 & 4.998 \times 10^1 & 9.0008 \\ -1.7998 \times 10^1 & 9.0008 & 5.0001 \end{bmatrix} \quad (3-108)$$

Matrix $K = \hat{K}_m$
 $= R^{-1} G' \hat{M}$ is given by

$$K = \begin{bmatrix} 4.9266 \times 10^{-2} & 2.5193 \times 10^{-1} \\ 1.3222 & -3.6274 \times 10^{-1} \end{bmatrix} \quad (3-109)$$

and matrix KC will be

$$KC = \begin{bmatrix} -3.1877 & 5.9679 \times 10^{-1} & 5.5312 \times 10^{-1} \\ 2.1309 \times 10^1 & 8.8933 & 5.9676 \times 10^{-1} \end{bmatrix} \quad (3-110)$$

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The lower order control will give rise to the system matrix [A-BKC] where

$$[A-BKC] = \begin{bmatrix} 0 & 1 & 0 \\ -21.309 & -8.8933 & 0.40321 \\ -11.812 & -11.597 & -5.5531 \end{bmatrix} \quad (3-111)$$

The eigenvalues of the matrix [A-BKC] are found to be

$$\lambda_{D,1,2} = -5.724 \pm j 2.9$$

$$\lambda_3 = -3.$$

The cost of the lower order control is N and can be obtained by solving the equation

$$[A-BKC]^T N + N[A-BKC] + C^T K^T R K C + Q_n = 0$$

and with R = I

$$N = \begin{bmatrix} 9.2631 \times 10^1 & 2.1313 \times 10^1 & -3.1879 \\ 2.1313 \times 10^1 & 8.8949 & 5.9683 \times 10^{-1} \\ -3.1879 & 5.9683 \times 10^{-1} & 5.5315 \times 10^{-1} \end{bmatrix} \quad (3-112)$$

$\hat{N} = C^T \hat{M} C$ can be verified by solving for \hat{N} from the n^{th} order matrix Riccati equation with Q_n given by eqn. (3-108).

If R = 10I, then

$$N = \begin{bmatrix} 1.6814 \times 10^2 & 5.859 \times 10^1 & 8.4831 \times 10^{-1} \\ 5.859 \times 10^1 & 2.9941 \times 10^1 & 3.4702 \\ 8.4831 \times 10^{-1} & 3.4702 & 1.0625 \end{bmatrix} \quad (3-113)$$

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3.11.1 Unit Step Response :

Using the concept developed in sub-section 3.10.1

and with $u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the unit step response can be written as

$$x(t) = \mathcal{L}^{-1} \{ [sI - (A - BKC)]^{-1} [B U(s)] \}$$

and

$$[sI - (A - BKC)]^{-1} = \frac{1}{D(s)} \begin{bmatrix} [(s^2 + 14.44s + 54.06)] [(s + 5.55)] [(0.403)] \\ [- (21.3s + 122.76)] [s(s + 5.55)] [(0.403s)] \\ [(-11.81s + 142)] [- (11.59s + 11.81)] [(s^2 + 8.89s + 21.3)] \end{bmatrix}$$

where

$$D(s) = (s+3)(s+5.724 + j2.9)(s+5.724 - j2.9) \quad (3-114)$$

Then

$$x_1(t) = \mathcal{L}^{-1} \frac{0.403}{s \cdot D(s)} \quad (3-115)$$

Evaluating the inverse Laplace transform by residue theory

$$x_1(t) = 3.27 \times 10^{-3} - 8.5 \times 10^{-3} e^{-3t} + 5.23 \times 10^{-3} e^{-5.724t} \sin(2.9t + \theta)$$

$$\text{where } \theta = \tan^{-1} 3.4 \quad (3-116)$$

The unit step response of the original system with no feedback will be given by

$$x(t) = \mathcal{L}^{-1} \{ [sI - A]^{-1} [B U(s)] \}$$

Here

$$[sI - A]^{-1} = \frac{1}{D_0(s)} \begin{bmatrix} [(s^2 + 5s + 11)] [(s+5)] [(1)] \\ [- (15)] [s(s+5)] [(s)] \\ [-(15s)] [- (11s+15)] [(s^2)] \end{bmatrix}$$

where

$$D_0(s) = (s+3)(s+1+j2)(s+1-j2) \quad (3-117)$$

Using

$$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1(t) = \mathcal{L}^{-1} \frac{1}{s.D_0(s)} \quad (3-118)$$

Evaluating $x_1(t)$ from eqn. (4-41), it is found to be

$$x_1(t) = 6.667 \times 10^{-2} - 4.17 \times 10^{-2} e^{-3t} - 7.9 \times 10^{-2} e^{-t} \sin(2t + \theta)$$

where

$$\theta = \tan^{-1} 0.333 \quad (3-119)$$

Comparing eqns. (3-116) and (3-119) it will be seen that eqn. (3-116) will reach the narrow band around the steady state much quicker than eqn. (3-119).

3.12 3rd Numerical Example :

In this example, the system matrix has a general form and three distinct and real eigenvalues. One eigenvalue is stable and the other two are unstable. The contracted system is made to retain these two unstable eigenvalues. The second order system is stabilized and these two stable eigenvalues are incorporated into the third order system by using the lower order control.

The n^{th} order system is given by

$$\dot{x}(t) = A x(t) + B u(t)$$

$$x(t_0) = x_0$$

and

$$A = \begin{bmatrix} 2.0 & -2.0 & 3.0 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 3.0 & -1.0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3-120)$$

It is required to minimize

$$J_n = 1/2 \int_0^{\infty} [x'(t) Q_n x(t) + u'(t) R u(t)] dt$$

with $R = I$ (3-121)

This generalized system matrix A has three real and distinct eigenvalues. One is a stable eigenvalue and the other two are unstable.

$$\lambda_1 = -2.0$$

$$\lambda_2 = +3.0$$

$$\lambda_3 = +1.0$$

It is required to design the 2nd order system with the design eigenvalues of

$$\lambda_{D.1} = -30.50$$

$$\lambda_{D.2} = -2.20$$

and the F_o matrix is given by

$$F_o = \begin{bmatrix} -5.045 & -8.481 \\ -8.382 & -27.701 \end{bmatrix} \quad (3-122)$$

The design procedure yields the following matrices :

Contraction matrix C is given by

$$C = \begin{bmatrix} 1.0 & 0.530 & 0.824 \\ 3.350 & 1.0 & 2.710 \end{bmatrix} \quad (3-123)$$

The m^{th} order system matrices will be

$$F = \begin{bmatrix} 0 & 1.0 \\ -3. & +4. \end{bmatrix} \quad (3-124)$$

$$G = \begin{bmatrix} 0.824 & 0.53 \\ 2.71 & 1.0 \end{bmatrix} \quad (3-125)$$

The matrix F will have (+3) and (+1) as the two dominant, unstable eigenvalues.

$$[T_m]^{-1} = \begin{bmatrix} 2.225 \times 10^1 & -7.369 \\ -7.369 & 2.560 \end{bmatrix}$$

The matrix Q_m will be given by

$$Q_m = \begin{bmatrix} 1.075 \times 10^2 & 9.896 \\ 9.896 & 9.191 \times 10^1 \end{bmatrix} \quad (3-126)$$

Steady state m^{th} order matrix \hat{M} will be

$$\hat{M} = \begin{bmatrix} 7.18 \times 10^1 & -2.273 \times 10^1 \\ -2.273 \times 10^1 & 1.119 \times 10^1 \end{bmatrix} \quad (3-127)$$

The matrix Q_n will be given by

$$Q_n = \begin{bmatrix} 1.205 \times 10^3 & 3.924 \times 10^2 & 9.772 \times 10^2 \\ 3.924 \times 10^2 & 1.326 \times 10^2 & 3.184 \times 10^2 \\ 9.772 \times 10^2 & 3.184 \times 10^2 & 7.922 \times 10^2 \end{bmatrix} \quad (3-128)$$

Feedback matrices K and KC will be given by

$$K = \begin{bmatrix} -3.3 & 11.6 \\ 15.37 & -0.86 \end{bmatrix} \quad (3-129)$$

$$KC = \begin{bmatrix} 3.64 \times 10^1 & 1.031 \times 10^1 & 2.944 \times 10^1 \\ 1.246 & 7.267 & 1.031 \times 10^1 \end{bmatrix} \quad (3-130)$$

The system matrix resulting from the lower order control will be [A-BKC] and will be given by

$$[A-BKC] = \begin{bmatrix} 2.0 & -2.0 & 3.0 \\ -11.463 & -6.267 & -9.312 \\ -35.44 & -7.312 & -30.442 \end{bmatrix} \quad (3-131)$$

The eigenvalues of the matrix [A-BKC] will be

$$\lambda_{D.1} = -30.50$$

$$\lambda_{D.2} = -2.20$$

$$\lambda_3 = -2.0$$

The lower order control will have the cost given by the matrix N and with R = I

$$N = \begin{bmatrix} 4.518 \times 10^1 & 1.243 \times 10^1 & 3.646 \times 10^1 \\ 1.243 \times 10^1 & 7.288 & 1.030 \times 10^1 \\ 3.646 \times 10^1 & 1.030 \times 10^1 & 2.943 \times 10^1 \end{bmatrix} \quad (3-132)$$

$\hat{N} = C^T M C$ can be verified by solving for \hat{N} from the n^{th} order matrix Riccati equation with Q_n obtained from eqn. (4-51).

If $R = 10I$, then

$$N = \begin{bmatrix} 1.818 \times 10^2 & 4.166 \times 10^1 & 1.460 \times 10^2 \\ 4.166 \times 10^1 & 3.533 \times 10^1 & 3.530 \times 10^1 \\ 1.460 \times 10^2 & 3.530 \times 10^1 & 1.174 \times 10^2 \end{bmatrix} \quad (3-133)$$

3.12.1 Unit Step Response :

Making use of the previous development, the unit step response of the system with

$$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{can be written as}$$

$$x(t) = \mathcal{L}^{-1} \{ [sI - (A-BKC)]^{-1} [B U(s)] \}$$

where

$$[sI - (A-BKC)]^{-1} = \frac{1}{D(s)} \begin{bmatrix} [(s+32.95)(s+3.75)] & [-(2s+82.82)] & [(3s+37.42)] \\ [-(11.46s+18.)] & [(1.67)(s+26.77)] & [-(9.31s+15.76)] \\ [-(35.44s+138.75)] & [-(7.31s-85.5)] & [(s+8.43)(s-4.16)] \end{bmatrix}$$

$$\text{where } D(s) = (s+2.0)(s+30.50)(s+2.20) \quad (3-134)$$

Then

$$x_1(t) = \mathcal{L}^{-1} \frac{(3s+37.42)}{s \cdot D(s)}$$

Evaluating the inverse transform, $x_1(t)$ is found to be

$$x_1(t) = 0.281 - 2.753e^{-2t} + 2.474e^{-2.2t} - 0.0022e^{-30.50t} \quad (3-135)$$

The unit step response of the original system without any feedback will be given by

$$x(t) = \mathcal{L}^{-1} \{ [sI - A]^{-1} [B U(s)] \}$$

where $[sI - A]^{-1}$ is given below.

$$[sI - A]^{-1} = \frac{1}{D_o(s)} \begin{bmatrix} [(s^2 - 4)] & [-(2s-7)] & [(3s-5)] \\ [(s+2)] & [(s^2 - s - 5)] & [(s+1)] \\ [(s+2)] & [(3s-8)] & [(s-3)] \end{bmatrix} \quad (3-136)$$

$$\text{where } D_o(s) = (s+2)(s-1)(s-3).$$

Using $u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x_1(t) = \mathcal{L}^{-1} \frac{(3s-5)}{s \cdot D_o(s)} \quad (3-137)$$

Evaluating the inverse transform

$$x_1(t) = -0.834 + 0.367 e^{-2t} + 0.333 e^{+t} + 0.134 e^{+3t} \quad (3-138)$$

Comparing equations (3-135) and (3-138), it will be seen that an unstable system has been made stable.

It will be seen from the examples that $u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

gave a unit step response which exhibited all the system modes.

If however some of the system modes were going to be suppressed with

$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ would turn out more suitable to excite all the system modes.

It will also be observed that the gain matrices K and KC which enable the lower order control to be implemented, have elements which are of reasonable magnitude. This indicates that generating such gain matrices will be practicable. If the magnitude of these elements is excessively large, then the implementation of such feed-back matrices will not be possible and alternate designs will have to be specified and the feasibility of using these gain matrices will have to be ascertained.

In Appendix D, these three examples are solved by choosing both the weighting matrices R and Q_n arbitrarily. This results in systems which have arbitrary eigenvalues. There, a comparison is made of the Computer time required for the numerical examples which are worked out in this Chapter and in Appendix D. Some other relevant properties are also given in that Appendix.

The function of the infinite-time regulator is to drive all the initial states to the origin in infinite time. The output of such a regulator which is designed here is given by the expression

$$x(t) = \mathcal{L}^{-1} \{ [sI - (A - BKC)]^{-1} [x(0_+)] \} \quad (3-139)$$

The system modes will depend upon the eigenvalues of the matrix $[A - BKC]$. The design algorithm ensures that all the eigenvalues of the matrix $[A - BKC]$ will be asymptotically stable. Further the algorithm enables the eigenvalues to be moved farther to the left in the Left Half of the Complex Plane or $\text{Re}[\lambda]$ to be made reasonably negative.

Thus from an engineering point of view a regulator, which is asymptotically stable and which drives all the initial states to the origin in infinite time, will be a good regulator if it also drives all its initial states into a small region around the origin (the ϵ - neighbourhood) in a finite time. The examples worked out in this Chapter demonstrate these considerations.

When systems are designed with the aid of the synthesis algorithm given, they are made stable and their dominant eigenvalues are made to have $\text{Re}[\lambda]$ which are more negative. This enables the unit step response of the system to reach within an acceptable narrow band around the steady state quicker. This in turn implies that the transient response of the regulator decays more rapidly or the initial states of the regulator reach some specified ϵ -neighbourhood around the origin faster.

CHAPTER IV

A SPECIAL CASE, OTHER APPLICATIONS, DISCUSSIONS AND CONCLUSIONS

The theory of 'lower order' control developed in Chapter III will be scrutinized in this concluding chapter. In section 4.1 a special case of the finite time problem will be considered. In Chapter III, section 3.7 it was shown that the 'lower order' control gives rise to identical eigenvalues in the lower order and higher order systems at any given corresponding point in time. However these eigenvalues will change with time. In section 4.1 it is shown that the finite-time regulator problem is equivalent to the infinite-time regulator problem under the condition which is stated in that section. If this condition is satisfied the lower order control for the finite-time problem can be found as in the infinite-time problem as developed in sections 3.1 through 3.6.

In section 4.2 it is shown that a class of tracking problem can be put in the form of an output regulator problem. In such cases, the lower order control synthesis could be used.

In section 4.3, the effect of variations of the elements of the R matrix are considered. In the scheme of lower order control the Q_n or Q_l matrix is derived in such a way that it depends on the R matrix. This is evident from eqns. (3-17) and (3-26) and this aspect is investigated in section 4.3.

In section 4.4, the positive semi-definiteness of the Q matrix and an iterative method of solving for this Q matrix is discussed. The necessity of using the lower order control is also discussed. The method of obtaining the contraction matrix C is also reviewed.

In section 4.5, conclusions are drawn from the 'lower order' control theory which is developed in this thesis. The advantages of using the lower order control are enumerated and the thesis is concluded.

4.1 A Special Case of the Finite-Time Problem :

In this section it is shown that when condition given by eqn. (4-4) is satisfied, the finite-time regulator problem and the infinite-time regulator problem are equivalent.

If the system is given by eqn. (3-1) and it is required to minimize the cost functional given by eqn. (3-52), then the minimum cost is

$$J_n^* = 1/2 \langle x(t), N^*(t) x(t) \rangle \quad (4-1)$$

where $N^*(t)$ is the time-varying matrix solution of eqn. (3-60) with

$$N^*(T) = H' S H$$

and
$$u^*(t) = -BR^{-1}B' N^*(t) x(t) \quad (4-2)$$

It will be shown that the cost J_n will be minimized and will be given by

$$J_n^* = 1/2 \langle x(t), H' S H x(t) \rangle \quad (4-3)$$

if $H' S H$ satisfies the following equation :

$$A'(H'SH) + (H'SH)A - (H'SH) B R^{-1} B'(H'SH) + H' Q_l H = 0 \quad (4-4)$$

(32)(43)(44)

When eqn. (4-3) is the minimum cost, the following Hamilton Jacobi

Partial Differential equation will be satisfied.

$$\begin{aligned} & \frac{\partial J_n^* [x(t), t]}{\partial t} + \min u(t) \{ 1/2 [\langle x(t), H' Q_l H x(t) \rangle + \langle u(t), R u(t) \rangle] \\ & + [\langle A x(t), \frac{\partial J_n^* [x(t), t]}{\partial x(t)} \rangle + \langle B u(t), \frac{\partial J_n^* [x(t), t]}{\partial x(t)} \rangle] \} = 0 \end{aligned} \quad (4-5)$$

Since H and S are constant matrices

$$\frac{\partial J_n^*}{\partial t} = 0 \quad (4-6)$$

The expression inside the braces is minimized when.

$$R u(t) + B' \frac{\partial J_n^* [x(t), t]}{\partial x(t)} = 0$$

$$\therefore u^*(t) = -R^{-1} B' \frac{\partial J_n^* [x(t), t]}{\partial x(t)} \quad (4-7)$$

When the cost is given by eqn. (4-65)

$$\frac{\partial J_n^* [x(t), t]}{\partial x(t)} = H' S H x(t) \quad (4-8)$$

Thus

$$u^*(t) = -R^{-1} B' H' S H x(t) \quad (4-9)$$

Using eqns. (4-8) and (4-6) eqn. (4-5) can be written as

$$\left\{ \frac{1}{2} \left[\langle x(t), H' Q H x(t) \rangle + \langle -R^{-1} B' H' S H x(t), -B' H' S H x(t) \rangle \right] \right. \\ \left. + \left[\langle A x(t), H' S H x(t) \rangle + \langle -R^{-1} B' H' S H x(t), H' S H x(t) \rangle \right] \right\} = 0 \quad (4-10)$$

Simplifying eqn. (4-10), the following equation can be written

$$\frac{1}{2} \langle x(t), [H' Q H - (H' S H) B R^{-1} B' (H' S H) + A' (H' S H) + (H' S H) A] x(t) \rangle = 0 \quad (4-10a)$$

for all $x(t)$.

Thus from the above quadratic form, the following equation can be written.

See Appendix E.

$$A' (H' S H) + (H' S H) A - (H' S H) B R^{-1} B' (H' S H) + H' Q H = 0 \quad (4-11)$$

This shows that if eqn. (4-11) is satisfied, eqn. (4-3) will give the cost.

Comparing eqns. (4-1) and (4-3), it is evident that the two costs are identical when $t = T$. Since the minimum costs are the same and they are

achieved by unique controls,

$$u^*(t) = -R^{-1} B' N^*(t) x(t)$$

and

$$u^*(t) = -R^{-1} B' H' S H x(t)$$

are the same and

$$N^*(t) = H' S H \quad (4-12)$$

Thus $N^*(t)$ is a constant matrix and the following algebraic equation will hold.

$$0 = A' N^* + N^* A - N^* B R^{-1} B' N^* + H' Q H \quad (4-13)$$

Equation (4-13) is the typical equation of an infinite-time problem and so is equation (4-11). As a matter of fact when eqn. (4-11) is satisfied, the finite-time and infinite-time problems are one and the same.

If it is a state regulator problem rather than the output state regulator problem, then eqn. (4-11) can be written as

$$A'S + SA - SBR^{-1} B'S + Q_n = 0 \quad (4-14)$$

4.2 Other Applications :

In this section a special class of tracking problem is discussed. Certain types of tracking problems⁽⁴⁵⁾ can be put in forms equivalent to output regulator problems and the synthesis algorithm could then be used. Here a scalar case is considered to indicate the possibilities.

Let the system be given by

$$\dot{x}(t) = A x(t) + B u(t)$$

$$x(t_0) = x_0$$

$$y(t) = H x(t)$$

and let the desired output be $z(t)$. It is required to keep $y(t)$ as near $z(t)$

as possible without expending excessive control energy. Then if

$$e(t) = z(t) - y(t),$$

the requirement is the same as minimizing the cost functional

$$J = 1/2 e'(T) S e(T) + 1/2 \int_0^T [e'(t)Q e(t) + u'(t)R u(t)] dt \quad (4-15)$$

A simple scalar example for infinite-time, will bring out the underlying principle.

Let

$$D = \frac{d}{dt}$$

and

$$\mathcal{L}(D) = [D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0] \quad (4-16)$$

If the scalar output $y(t)$ and scalar input $u(t)$ are related by

$$\mathcal{L}(D)y(t) = u(t) \quad (4-17)$$

If the error $e(t)$ is defined as $e(t) = z(t) - y(t)$, where $z(t)$ is the desired output and if J is given by :

$$J = 1/2 \int_0^\infty [q e^2(t) + r u^2(t)] dt \quad (4-18)$$

Applying the linear operator $\mathcal{L}(D)$ to $e(t)$

$$\mathcal{L}(D) e(t) = \mathcal{L}(D) [z(t) - y(t)] \quad (4-19)$$

If $z(t)$ is differentiable 'n' times and if

$$\mathcal{L}(D) z(t) = [D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0]z(t) = 0 \quad (4-20)$$

From eqns. (4-17) and (4-20), eqn. (4-19) can be rewritten as

$$\mathcal{L}(D) e(t) = -u(t) \quad (4-21)$$

Eqn. (4-21) is the input-output relationship for a linear time-invariant system with $-u(t)$ as input and $e(t)$ as output.

$$\text{If } x_i(t) = D^{i-1} e(t) \quad ; \quad i = 1, 2, \dots, n \quad (4-22)$$

Then the system can be put in the form of a state regulator problem and optimal control will be a linear combination of the first (n-1) derivatives, which in effect is an n-state feedback. The theory proposed could then be used in this case. Eqn. (4-20) is not very likely to be satisfied ordinarily. However, whenever it is satisfied, other equations will follow.

4.3 Variations of the R Matrix :

In the performance functional used throughout the thesis, it is not only the state weighting matrix 'Q' which is at the disposal of the designer but also the 'R' matrix which weighs inputs or controls. In this section, this R matrix will be considered in a qualitative way to obtain some insight.

In Chapter III it was shown that with a specified F_o , $P'Q_m P$ was given by

$$P'Q_m P = -[GR^{-1}G']^{-1}[F-F_o]F_o - F'[GR^{-1}G']^{-1}[F-F_o] \quad (4-23)$$

and the following equation was satisfied by \hat{M} :

$$0 = F'\hat{M} + \hat{M}F - MGR^{-1}G'\hat{M} + P'Q_m P \quad (4-24)$$

The lower order control is given by

$$\begin{aligned} u(t) &= -\hat{K}_m C x(t) \\ &= -KC x(t) \end{aligned} \quad (4-25)$$

where

$$\hat{K}_m = K = R^{-1}G'\hat{M} \quad (4-26)$$

The cost of the lower order control is given by the matrix N which satisfies the following linear eqn.

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$$[A - BKC]'N + N[A - BKC] + C'K'RKC + H'Q_l H = 0 \quad (4-27)$$

Equations (4-23) through (4-27) are all developed systematically in Chapter III.

For convenience R matrix will be considered to have a diagonal form and identical elements. Thus $R \hat{=} a \cdot I_r$ where 'a' is some constant. Then $R^{-1} = \frac{1}{a} I_r$. The existence of R^{-1} is an absolute necessity throughout the development of the theory. Thus at the outset $a = 0$ can be ruled out as inadmissible. When $a = 1$, all the previous examples will have to be solved completely. Two separate cases will be considered qualitatively.

4.3.1 Case I - 'a' Very Large :

When 'a' is very large R^{-1} will be almost a zero matrix and so also the matrix $[GR^{-1}G']$. But the matrix $[GR^{-1}G']^{-1}$ will have elements of large magnitude. Thus eqn. (4-24) can be approximated as

$$0 = F'M + MF + P'Q_m P \quad (4-28)$$

The lower order control given by

$$u(t) = -R^{-1} G' M C x(t) \quad (4-29)$$

will be very small. This seems reasonable; if the control is going to be very heavily weighted in the cost functional, the control will have to be small to minimize the cost. Eqn. (4-27) can be rewritten as

$$[A - BR^{-1}G'MC]'N + N[A - BR^{-1}G'MC] + C'MGR^{-1}R R^{-1}G'MC + H'Q_l H = 0$$

and can be approximated as

$$A'N + NA + H'Q_l H = 0 \quad (4-30)$$

If 'a' is very large eqns. (4-28) and (4-30) are both seen to be linear. These two equations may be solved by the method which is given in Appendix A. However as eqn. (4-29) shows, this gives rise to a very small control.

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4.3.2 Case II - 'a' Very Small :

When 'a' is very small, R^{-1} will be a matrix with elements of large magnitude and so also the matrix $[GR^{-1}G']$. Naturally the matrix $[GR^{-1}G']^{-1}$ will have very small elements. It must be remembered that all other matrices are assumed to have elements of similar order - not very large or not very small. $P'Q_m P$ will be a matrix of very small elements and eqn. (4-24) can be approximated as

$$0 = F' \hat{M} + M F - \frac{1}{a} \hat{M} G G' \hat{M} + P' Q_m P \quad (4-31)$$

and the control will be given by

$$u(t) = - \frac{1}{a} G' \hat{M} C x(t) \quad (4-32)$$

Since 'a' is small, the control will be quite large for a given $P'Q_m P$ or Q_n . This makes sense from the point of view of minimizing the cost functional. The cost of this lower order control will be found by approximating eqn. (4-27)

$$\left[A - \frac{1}{a} B G' \hat{M} C \right]' N + N \left[A - \frac{1}{a} B G' \hat{M} C \right] + \frac{1}{a} C' \hat{M} G G' \hat{M} C = 0 \quad (4-33)$$

since $H' Q_t H = C' P' Q_m P C$ will be a matrix of small elements.

From Case I and Case II, two facts stand out. When 'a' is very large with respect to unity, the control is heavily weighted. This gives rise to a very large state-weighting matrix $P'Q_m P$. This also gives rise to a small control.

When 'a' is small, the control is lightly weighted and the matrix $P'Q_m P$ will also be quite small. The control will be quite large. However the cost of the performance functional will be kept finite since $P'Q_m P$ will be small.

The cost of the performance functional is dependent on the value of 'a' and on the elements of the state-weighting matrix Q. The state-weighting matrix Q is itself dependent on 'a' in the scheme developed.

Hence 'a' is assumed to be unity in this thesis unless otherwise specified.

4.4 Discussions :

Many numerical examples for synthesizing the plants upto 6th order were considered. However, only third order systems were given in the thesis because they illustrate the scheme of synthesis quite adequately.

Computer programs for synthesizing systems were developed from the design procedure given in Section 3.9. Programs for synthesizing systems of order higher than six could be developed from the same design procedure.

Throughout the thesis a quadratic performance index was considered. For the infinite-time problem, the typical index was of the form

$$J = 1/2 \int_0^{t=\infty} [x'(t) Q_n x(t) + u'(t) R u(t)] dt$$

R was assumed to be positive definite so that R^{-1} existed and the synthesis was possible by the scheme developed in this thesis. Since the matrix Q_n was obtained from the matrix Q_m by the process of expansion, involving the contraction matrix, the matrix Q_n turns out to be positive semi-definite. If this Q_n is to be used in the conventional nth order matrix Riccati equation, there are two ways to ensure that the resulting optimum system is asymptotically stable and that it drives all the initial states to the origin.

First way is to add a small positive definite matrix ρI_n to Q_n . Thus the positive semi-definite matrix Q_n can be written as :

$$Q_n \geq 0$$

The positive-definite matrix ρI_n with $\rho > 0$ but small enough to have very little practical significance, can be added to Q_n to obtain Q'_n which will be positive definite.

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Thus

$$\rho I_n > 0$$

Hence

$$Q'_n = Q_n + \rho I_n > 0$$

If this Q'_n were to be used, the resulting optimum system will be stable. However since ' ρ ' is arbitrary, it will shift the designed eigenvalues arbitrarily too. Hence the second method given below is preferred.

In the second method, the n^{th} order system is checked for Stabilizability and Detectability with the Q_n matrix which in turn is generated from the Q_m matrix. In Appendix D, the rationale for the Stabilizability and Detectability tests is clearly given ; if the n^{th} order system is Stabilizable and Detectable with the Q_n matrix, the resulting optimum system will always be stable. Besides, the designed eigenvalues will not shift arbitrarily as in the first method.

In section 1.5, Chapter I it was mentioned that if A_{opt} is completely specified as A_o , then the following two equations

$$\hat{T}_n N = A - A_o$$

and

$$\hat{A}'N + NA_o + H'QH = 0$$

would have to be solved simultaneously. The second equation can be solved by the method given in Appendix A. The solution for \hat{N} may not satisfy the first equation for an arbitrary $H'QH$. An attempt was made to develop an algorithm to solve these two equations simultaneously by iteration, starting with an arbitrary $H'QH$ and specified A_o . The attempt did not prove to be encouraging.

It may be mentioned that a single input state-regulator which is completely controllable can always be transformed^{(38), (39), (40)} into the

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phase-variable form. For these systems both the matrices A and A_0 can be in the companion matrix form and the above two equations can then be solved in a relatively easy way. In more general cases, when the dimension of the systems is high, even the specification of the structure of A_0 is a significant problem.

Since a satisfactory method to solve the two above equations simultaneously was not found, the method of 'lower order' control proved very useful for system synthesis. This lower order control is obtained by contraction of the system order. The concept of contraction is used in such a way that the contracted matrix F of order m retains m of the dominant eigenvalues of the original n^{th} order system matrix A . The lower order system is controlled to have the specified stable eigenvalues in the matrix F_0 . This gives rise to the Q_m matrix and \hat{M} matrix. The validity of this Q_m matrix is checked by solving the m^{th} order matrix Riccati equation for \hat{M} which has to be convergent. The control which satisfies these conditions is the 'lower order' control. This lower order control is used to control the n^{th} order system through the process of expansion. This process of expansion can be thought of as the reverse of contraction and it is the contraction matrix which achieves it. This lower order control results in the n^{th} order matrix $[A - BKC]$.

For instance, a system matrix A has two eigenvalues which are dominant. They may be unstable and in the Right Half Plane or they may be in the Left Half Plane and very close to the imaginary axis. These two dominant eigenvalues can be retained in the F matrix and then moved well into the LHP in the F_0 matrix. By using the 'lower order' control for the original system, the resulting matrix $[A - BKC]$ will be seen to have these two eigenvalues well into the LHP just as the F_0 matrix.

The design procedure makes use of the concept of contraction. The contraction matrix C is used not only to lower the order of the original system but also to expand the 'lower order' control back to the

original system. Two methods of obtaining this C matrix are discussed. The first method may be used when the eigenvalues of the matrix A are real and non-repeated. The modal transformation matrix T is generated and the appropriate m rows of the $[T]^{-1}$ matrix are selected to obtain the (mxn) dimensional C matrix. This C matrix gives rise to the m^{th} order system matrix F which retains the m dominant eigenvalues of the n^{th} order system. The second method is quite general. It assumes the F matrix to have the required dominant eigenvalues and also to have the companion matrix form. The resulting (mxn) simultaneous and homogeneous linear algebraic equations are solved by assuming one element each from the m rows of the C matrix.

4.5 Conclusions :

In multi-input system of high order, there are r ($r > 1$) inputs to the system. If m, $m \leq r \leq n$, eigenvalues are specified, then the design of the n^{th} order system can be achieved by the 'lower order' control. This lower order control is obtained by contracting the n^{th} order system to a system of lower order m. This m^{th} order system is controlled to obtain the specified eigenvalues as described in the previous section. The resulting m^{th} order control or the 'lower order' control is used to control the original n^{th} order system through the process of expansion. This results in the n^{th} order matrix $[A-BKC]$. In section 3.4 it is shown that this method of synthesis carries the m specified eigenvalues of the m^{th} order matrix F_o back to the expanded n^{th} order matrix $[A-BKC]$.

As explained in section 3.5, the other (n-m) eigenvalues of the n^{th} order matrix $[A-BKC]$ will be assuredly stable if the n^{th} order system is checked for stabilizability and detectability with the Q_n matrix which is derived from the Q_m matrix. This test for stabilizability and

detectability is incorporated into the design procedure given in section 3.9 to ensure that the system is stabilizable and detectable.

In section 3.2 it is mentioned that the order m of the contracted system is such that $m \leq r \leq n$. This requirement is proved in Appendix C, section C-2. It is interesting to know what happens when $r = 1$. When $r = 1$, or in case of single input systems the order m of the contracted system will be such that $m = 1$. In other words, the lower order system will be of order 1. Thus for single input systems, lower order control will enable only $m=1$ eigenvalue to be specified for the F_o matrix and thus designed into the $[A-BKC]$ matrix. The process of lower order control can be repeated over and over again to move as many real eigenvalues as necessary with the proviso that only one can be moved in one operation. However with $r = 1$, complex eigenvalues cannot be moved. As mentioned in section 4.4, when for a single input system the matrices A and A_o can be put in the companion matrix form, the matrix Q can be obtained relatively easily. This is achieved by specifying the A_o matrix in the companion matrix form with the coefficients of the characteristic polynomial. Thus when $r = 1$, the 'lower order' control scheme need not be used.

However when $r \geq 2$, the scheme of synthesis is extremely useful since it can conveniently move all types of eigenvalues. In high order multiple-input systems, invariably $r \geq 2$ and thus $m \geq 2$. In such cases, the scheme developed is advantageous since the required number of eigenvalues $m \geq 2$ can be designed and at the same time the Q_n matrix can be generated. In a synthesis problem there is no pressing reason why only one input ($r = 1$) should be used. However, if it is not necessary to specify that $r = 1$, then the synthesis procedure developed in this thesis can be used in a very wide class of synthesis of multivariable systems.

The numerical examples worked out in sections 3.10, 3.11 and 3.12 demonstrate the procedure of implementing the theory of 'lower order' control. The three examples were chosen with an idea to show that it is possible to design high order multi-input linear systems by the theory developed notwithstanding the structure of the system matrix A. The contraction matrix C can always be obtained no matter whether the eigenvalues of A are real, complex, stable or unstable. This was also verified for repeated eigenvalues although a sample example was not included in Chapter III. In the examples given in the thesis, m of the eigenvalues of the matrix [A-BKC] are the same as the m specified eigenvalues of the F_0 matrix. The other (n-m) eigenvalues of the [A-BKC] matrix were observed to be the same as the original matrix A. The same phenomenon was observed in the case of a large number of other examples, many of them of higher order which were worked out but not included in this thesis.

In the scheme developed, the Q_n or Q_l matrix is derived from the Q_m matrix. This is shown in section 3.2. The Q_m matrix has rank m as it is positive definite. If the contraction matrix C also has rank m then the matrix Q_l or Q_n can also have rank m. From the various numerical examples worked out, it may be observed that in this method of 'lower order' control, m of the eigenvalues of the matrix [A-BKC] are the same as those of F_0 if the rank of Q_l or Q_n is m. The other (n-m) eigenvalues are the same as those of the original A matrix.

When the original system is checked for stabilizability and detectability with either the Q_l or Q_n matrix, it has been shown in Appendix C, section C-3 that N^{\wedge} is unique and that it is identically equal to the matrix $C^{\wedge} M C$. In other words when the 'lower order' control scheme of synthesis is followed, only the mth order matrix Riccati equation need

be solved ; the n^{th} order matrix Riccati equation does not have to be solved. Thus the greatest advantage is in terms of saving Computer time. The details of the savings in the computer time are given in Appendix D. When n is large and m is of the order of 2, the saving in time can be quite significant. For instance when $n = 6$ and $m = 2$, the 'lower order' control is achieved in 1 min 10 sec CPU time ; whereas by assuming the Q_n matrix arbitrarily and solving the 6^{th} order matrix Riccati equation, the design requires about 9 minutes and results in arbitrary eigenvalues. If n is even larger, the savings in Computer time as well as the core-memory requirement can be very significant. The approximate relation of proportionality given in Appendix D has not been verified for very large values of n . However it may be said that the savings in Computer time and memory requirements can become extremely significant with very large n .

The cost of the 'lower order' control can be found by solving the linear matrix equation for N which is derived in section 3.6. A method is given in Appendix A to solve a generalized linear matrix equation. This can be useful in solving for N to obtain the cost of the lower order control. It may be possible in the future to use this generalized method for developing an iterative algorithm for solving for the Q matrix with a specified A_o matrix. In section 3.6 it is also shown that the N matrix and the N matrix are identical when a specific condition is satisfied.

The synthesis scheme is developed for both the infinite-time regulator and the finite-time regulator problem. A digital computer oriented design algorithm is included in section 3.9 to facilitate the use of the method of synthesis developed.

The advantages of the 'lower order' control synthesis developed in this thesis can be restated in brief as follows:

1. The 'lower order' control makes it possible to design the specified eigenvalues in a high order multiple-input system and thus control its time response.
2. The lower order control necessitates the solution of matrix Riccati equation of m^{th} order instead of n^{th} order. This gives rise to considerable savings in computer time and requirement of memory locations when designing high order systems.
3. The lower order control is obtained through the process of contraction. It is always possible to obtain this contraction matrix.
4. The Q_n or Q_l matrix is obtained from the Q_m matrix by the process of expansion. When the conditions of stabilizability and detectability are satisfied, the lower order control always gives rise to a stable system.
5. The cost of the lower order control can be found by solving a linear matrix equation.

Thus it can be said that the lower order control achieves the objectives of the problem considered in section 1.4-1 .

APPENDIX A

SOLUTION OF A LINEAR MATRIX EQUATION

In section 1.5 it was mentioned that the matrix equation given there, can be solved under certain conditions. In this Appendix the conditions for the solution ⁽⁴⁶⁾ ⁽⁴⁷⁾ of the general form of the equation will be given. Certain forms of a particular solution will also be given

A.1 The General Equation:

A linear matrix equation of the following general form is required to be solved.

$$TU + VT = S \quad (A-1)$$

All the matrices in eqn. (A-1) are real and eqn. (A-1) is solved for the unknown matrix T. Here T and S matrices are of dimension (m x n) and U and V are square matrices of dimension (n x n) and (m x m) respectively. The solution of eqn. (A-1) is known ⁽⁴⁷⁾ and the following details are given for completeness.

A.1.1 Recursive Relation:

To obtain a closed form solution to eqn. (A-1), recursive relations are given below. In these relations it will be noticed that the unknown matrix T retains its original first power and this property is utilized in subsection A.1.2.

$$\text{Let } S_0 = 0 \quad (A-2)$$

$$S_1 = S = TU + VT \quad (A-3)$$

$$\begin{aligned} \text{then } S_2 &= VS_1 - S_1U + VS_0U \\ &= V(TU + VT) - (TU + VT)U + VS_0U \\ &= V^2T - TU^2 \end{aligned} \quad (A-4)$$

$$\begin{aligned} S_3 &= VS_2 - S_2U + VS_1U \\ &= V^3T + TU^3 \end{aligned} \quad (A-5)$$

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$$\begin{aligned} S_4 &= VS_3 - S_3U + VS_2U \\ &= V^4T - TU^4 \end{aligned} \quad (A-6)$$

In general

$$S_k = V^k T - (-1)^k T U^k \quad (A-7)$$

The general S_k term is generated from the recursive relation

$$S_k = VS_{k-1} - S_{k-1}U + VS_{k-2}U \quad (A-8)$$

A. 1.2 Characteristic Matrix Polynomials:

Since V and U are $(m \times m)$ and $(n \times n)$ square matrices respectively, characteristic equations can be written as follows:

$$\det[\lambda I_m - V] = \lambda^m + a_1 \lambda^{m-1} + a_2 \lambda^{m-2} + \dots + a_m = 0$$

or

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_m) = 0 \quad (A-9)$$

and

$$\det[\mu I_n - U] = \mu^n + b_1 \mu^{n-1} + b_2 \mu^{n-2} + \dots + b_n$$

or

$$(\mu - \mu_1)(\mu - \mu_2) \dots (\mu - \mu_n) = 0 \quad (A-10)$$

Now the Cayley-Hamilton Theorem states that these characteristic equations are satisfied by the matrices themselves.

Thus

$$V^m + a_1 V^{m-1} + a_2 V^{m-2} + \dots + a_m I_m = 0$$

and

$$U^n + b_1 U^{n-1} + b_2 U^{n-2} + \dots + b_n I_n = 0 \quad (A-11)$$

Using eqns (A-11) and (A-7), the following eqn. can be written

$$\begin{aligned} S_n - b_1 S_{n-1} + \dots + (-1)^{n-1} b_{n-1} S_1 &= V^n T - (-1)^n T U^n \\ - b_1 [V^{n-1} T - (-1)^{n-1} T U^{n-1}] + \dots + (-1)^{n-1} b_{n-1} [VT + TU] \end{aligned} \quad (A-12)$$

The R.H.S. of eqn (A-12) can be simplified as follows :

$$\begin{aligned} \text{R.H.S.} &= V^n T - b_1 V^{n-1} T + \dots + (-1)^{n-1} b_{n-1} V T \\ &- (-1)^n T U^n + (-1)^{n-1} b_1 T U^{n-1} + \dots + (-1)^{n-1} b_{n-1} T U \quad (\text{A-13}) \end{aligned}$$

Now

$$\begin{aligned} &[-(-1)^n T U^n + (-1)^{n-1} b_1 T U^{n-1} + \dots + (-1)^{n-1} b_{n-1} T U] \\ &= (-1)^n T [-U^n - b_1 U^{n-1} - \dots - b_{n-1} U] \\ &= (-1)^n T [b_n I_n] \\ &= (-1)^n T b_n \quad (\text{A-14}) \end{aligned}$$

The last step is evident from eqn. (A-11). Also for any k

$$\frac{(-1)^{k-1}}{(-1)^k} = -1 \quad \text{where } k \text{ is an integer.}$$

From eqns. (A-14) and (A-13), eqn (A-12) can be rewritten as

$$\begin{aligned} &[S_n - b_1 S_{n-1} + \dots + (-1)^{n-1} b_{n-1} S_1] \\ &= [V^n - b_1 V^{n-1} + \dots + (-1)^{n-1} b_{n-1} V + (-1)^n b_n I_n] T \quad (\text{A-15}) \end{aligned}$$

Therefore

$$T = G^{-1} [S_n - b_1 S_{n-1} + \dots + (-1)^{n-1} b_{n-1} S_1] \quad (\text{A-16})$$

where

$$G \triangleq [V^n - b_1 V^{n-1} + \dots + (-1)^{n-1} b_{n-1} V + (-1)^n b_n I_n] \quad (\text{A-17})$$

An alternate solution can be obtained as follows. Consider the following polynomial :

$$\begin{aligned} &[S_m + a_1 S_{m-1} + \dots + a_{m-1} S_1] = V^m T - (-1)^m T U^m \\ &+ a_1 [V^{m-1} T - (-1)^{m-1} T U^{m-1}] + \dots + a_{m-1} [V T + T U] \quad (\text{A-18}) \end{aligned}$$

The R.H.S. of eqn. (A-18) can be simplified as follows:

$$\begin{aligned} \text{R.H.S.} &= [V^m + a_1 V^{m-1} + \dots + a_{m-1} V] T \\ &+ T [(-1)^m U^m - (-1)^{m-1} a_1 U^{m-1} - (-1)^{m-(m-1)} a_{m-1} U] \end{aligned}$$

Using eqn. (A-11)

$$\text{R.H.S.} = -a_m I_m T - (-1)^m T [U^m - a_1 U^{m-1} + \dots + (-1)^{m-1} a_{m-1} U] \quad (\text{A-19})$$

for m, which is an odd or even integer. This is so because for an integer k $(-1)^k = (-1)^{-k}$.

Eqn.(A-19) can be further simplified.

$$\begin{aligned} \text{R.H.S.} &= (-1)^{m-1} \left[\frac{(-1)}{(-1)^{m-1}} a_m I_m T + T \{U^m - a_1 U^{m-1} + \dots + (-1)^{m-1} a_{m-1} U\} \right] \\ &= (-1)^{m-1} T (H) \end{aligned} \quad (\text{A-20})$$

where

$$H = \Delta [U^m - a_1 U^{m-1} + \dots + (-1)^{m-1} a_{m-1} U + (-1)^m a_m I_m] \quad (\text{A-21})$$

A. 1.3 Closed-form Solutions:

The closed form solutions for T in eqn. (A-1) are given below.

$$T = (-1)^m [S_m + a_1 S_{m-1} + \dots + a_{m-1} S_1] H^{-1} \quad (\text{A-22})$$

or

$$T = G^{-1} [S_n - b_1 S_{n-1} + \dots + (-1)^{n-1} b_{n-1} S_1] \quad (\text{A-23})$$

The matrices H and G have been defined by eqns. (A-21) and (A-17) respectively. The various coefficients a_i s and b_j s are defined as follows: ⁽⁴⁸⁾⁽⁴⁹⁾

$$a_i = -\frac{1}{i} [a_{i-1} \cdot \text{tr}(V) + a_{i-2} \cdot \text{tr}(V^2) + \dots + a_0 \cdot \text{tr}(V^i)] \quad (\text{A-24})$$

$$1 \leq i \leq m$$

and

$$b_j = -\frac{1}{j} [b_{j-1} \cdot \text{tr}(U) + b_{j-2} \cdot \text{tr}(U^2) + \dots + b_0 \cdot \text{tr}(U^j)] \quad (\text{A-25})$$

$$1 \leq j \leq n$$

where $a_0 = 1 = b_0$, and tr indicates the process of taking the trace

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of a square matrix.

A.1.4 Necessary and Sufficient Condition for Solution:

From eqns. (A-22) and (A-23), it is seen that T will have a unique solution if H and G respectively have an inverse. Hence in this subsection the conditions for the existence of the inverse of H and G will be established.

If the coefficients of the k^{th} power term of an n^{th} degree polynomial are multiplied by $(-1)^{n-k}$, then the roots of the polynomial change sign. In other words if

$$\sum_{k=0}^n a_k x^k = a_0 + a_1 x + \dots + a_n x^n$$

has the roots $-\mu_i$, then

$$\sum_{k=0}^n a_k x^k = \prod_{i=1}^n (x + \mu_i) \tag{A-26}$$

then
$$\sum_{k=0}^n (-1)^{n-k} a_k x^k = \prod_{i=1}^n (x - \mu_i) \tag{A-27}$$

which means that

$$\sum_{k=0}^n (-1)^{n-k} a_k x^k = (-1)^n a_0 + (-1)^{n-1} a_1 x + \dots + a_n x^n$$

will have the roots $+\mu_i$.

From eqn. (A-11) it is known that

$$V^m + a_1 V^{m-1} + \dots + a_m I_m = 0$$

This can be factored into

$$(V - \lambda_1 I_m) \cdot (V - \lambda_2 I_m) \cdot \dots \cdot (V - \lambda_m I_m) = 0 \tag{A-28}$$

Thus

$$H = U^m - a_1 U^{m-1} + \dots + (-1)^m a_m I_n$$

can be factored into

$$H = (U + \lambda_1 I_n) \cdot (U + \lambda_2 I_n) \cdot \dots \cdot (U + \lambda_m I_n) \tag{A-29}$$

Similarly since

$$U^n + b_1 U^{n-1} + \dots + b_n I_n = 0$$

Can be factored into

$$(U - \mu_1 I_n) \cdot (U - \mu_2 I_n) \dots (U - \mu_n I_n) = 0 \quad (A-30)$$

$$G = V^n - b_1 V^{n-1} + \dots + (-1)^n b_n I_m$$

can be factored into

$$G = (V + \mu_1 I_m) (V + \mu_2 I_m) \dots (V + \mu_n I_m) \quad (A-31)$$

To be able to obtain a closed form solution for T, H and G must have an inverse⁽⁵⁰⁾. The necessary and sufficient condition for H and G to have an inverse is that their determinants must be non-zero.

Since the determinant of a product of matrices is the product of the constituent determinants,

$$\det [H] = \prod_{i=1}^m \det [U + \lambda_i I_n] \neq 0 \quad (A-32)$$

and

$$\det [G] = \prod_{j=1}^n \det [V + \mu_j I_m] \neq 0 \quad (A-33)$$

From eqns. (A-32) and (A-33) it can be seen that U cannot have $(-\lambda_i)$ and V cannot have $(-\mu_j)$ as their respective eigenvalues. Since the eigenvalues of U are (μ_i) and those of V are (λ_j) , the following condition is necessary and sufficient for H and G to have an inverse.

$$\mu_j \neq -\lambda_i$$

or

$$\lambda_i + \mu_j \neq 0, \quad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix} \quad (A-34)$$

Hence the solution to be possible, none of the eigenvalues of V must be the negative of any of the eigenvalues of U and vice-versa.

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A.2 A Particular Case of the Matrix Equation :

In section 3.6, the cost of the lower order control was given by linear matrix equation of the following form.

$$TU + U'T = S \quad (A-35)$$

This equation is the same as

$$TU + VT = S$$

$$V = U'$$

Thus when solving eqn. (A-35), the various a_i s and b_i s will be identical. This is so because U and U' have the same characteristic polynomials and the same eigenvalues.

Since

$$V = U'$$

$$m = n$$

and $a_i = b_i$

From eqn. (A-17) G can be rewritten as

$$G = [V^n - a_1 V^{n-1} + \dots + (-1)^n a_n I_n] \quad (A-36)$$

From eqn. (A-11) it is known that

$$0 = [V^n + a_1 V^{n-1} + \dots + a_n I_n] \quad (A-37)$$

Adding eqn. (A-37) to (A-36)

$$G = 2 [V^n + a_2 V^{n-2} + a_4 V^{n-4} + \dots +] \quad (A-38)$$

Subtracting eqn. (A-37) from (A-36)

$$G = -2 [a_1 V^{n-1} + a_3 V^{n-3} + a_5 V^{n-5} + \dots +] \quad (A-39)$$

For an n^{th} order system there are $(n+1)$ terms in the characteristic equation. Thus when n is an odd integer both the expressions given by (A-38) and (A-39) will have the same number of terms. However when n is even, expression (A-39) will always have one term less. Thus expression (A-39) will have a slight advantage over (A-38), and as such

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will be used as a standard form.

Similarly H can be written as

$$H = [U^n - a_1 U^{n-1} + \dots + (-1)^n a_n I_n]$$

and the standard form used will be

$$H = -2[a_1 U^{n-1} + a_3 U^{n-3} + a_5 U^{n-5} + \dots] \quad (A-40)$$

The various a_i s have been defined in eqn. (A-24). Thus the equation

$$TU + U'T = S$$

can be solved in all cases when the following condition is satisfied.

Since U and U' have identical eigenvalues, eqn. (A-34) will take the form

$$\lambda_i + \lambda_j \neq 0, \quad \begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq n \end{matrix} \quad (A-41)$$

If U denotes an asymptotically stable matrix, then the real part of the eigenvalue, $\text{Re}[\lambda]$, will have the same negative sign and eqn. (A-41) will always be satisfied.

If $\lambda_i = 0$, eqn. (A-41) cannot be satisfied. Thus when U or U' matrix is singular, a closed form solution will not be possible.

When the matrix U has eigenvalues of the type $(\lambda - \lambda_i)(\lambda + \lambda_i)$ eqn (A-41) cannot be satisfied and closed form solution will not be possible. Obviously such a U matrix will not be stable.

A.3 System Algorithms:

Algorithms will be explicitly written for a 3rd order, 6th order and n^{th} order systems.

A.3.1 3rd Order System:

$$TU + VT = S$$

with $a_0 = 1$, $n = 3$ and $V = U'$

$$G = -2[a_1 V^2 + a_3 I_3] \quad (A-42)$$

$$H = -2 [a_1 U^2 + a_3 I_3] \quad (A-43)$$

$$\begin{aligned} S_3 &= VS_2 - S_2U + VS_1U \\ &= V^2S - VSU + SU^2 \end{aligned} \quad (A-44)$$

$$T = G^{-1} [S_3 - a_1 S_2 + a_2 S_1] \quad (A-45)$$

or

$$T = (-1)^{3-1} [S_3 + a_1 S_2 + a_2 S_1] H^{-1} \quad (A-46)$$

Here

$$a_3 = -1/3 [a_2 \cdot \text{tr}(U) + a_1 \cdot \text{tr}(U^2) + \text{tr}(U^3)] \quad (A-47)$$

S_0, S_1 and S_2 are defined in eqns. (A-2), (A-3) and (A-4).

Also a_1, a_2 are defined in eqn. (A-24).

A.3.2 6th Order System:

$$TU + VT = S$$

with $a_0 = 1, n = 6$ and $V = U'$

$$G = -2 [a_1 V^5 + a_3 V^3 + a_5 V] \quad (A-48)$$

$$H = -2 [a_1 U^5 + a_3 U^3 + a_5 U] \quad (A-49)$$

$$\begin{aligned} S_6 &= VS_5 - S_5U + VS_4U \\ &= V^5S - V^4SU + V^3SU^2 - V^2SU^3 + VSU^4 - SU^5 \end{aligned} \quad (A-50)$$

Then

$$T = G^{-1} [S_6 - a_1 S_5 + a_2 S_4 - a_3 S_3 + a_4 S_2 - a_5 S_1] \quad (A-51)$$

or

$$T = (-1)^{6-1} [S_6 + a_1 S_5 + a_2 S_4 + a_3 S_3 + a_4 S_2 + a_5 S_1] H^{-1} \quad (A-52)$$

The various a_i s and S_i s are previously defined.

A.3.3 nth Order System :

The generalized formulae for T, G and H are given by eqns. (A-22) or (A-23) and (A-39) and (A-40). The a_i s are previously defined.

$$S_n = V S_{n-1} - S_{n-1} U + V S_{n-2} U \quad (A-53)$$

or

$$S_n = \sum_{k=0}^n (-1)^k V^{(n-k-1)} S U^k \quad (A-54)$$

A.4 A 2nd Order Worked Example :

$$TU + VT = S$$

with $V = U'$

$$U = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad V = U' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$a_1 = -\text{tr}(U) = -5$$

$$a_2 = -1/2 [a_1 \cdot \text{tr}(U) + \text{tr}(U^2)] = -2$$

$$G = -2 [a_1 V] = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix} = H$$

Thus

$$G^{-1} = \frac{-1}{200} \begin{bmatrix} 40 & -20 \\ -30 & 10 \end{bmatrix}$$

$$H^{-1} = \frac{-1}{200} \begin{bmatrix} 40 & -30 \\ -20 & 10 \end{bmatrix}$$

Hence

$$\begin{aligned} T &= (-1)^{2-1} [S_2 + a_1 S_1] H^{-1} \\ &= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix} \end{aligned}$$

Similarly

$$T = G^{-1} [S_2 - a_1 S_1]$$

gives

$$T = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

By expanding the system in the conventional manner, the following is obtained.

$$\begin{bmatrix} t_{11} \\ t_{12} \\ t_{22} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 2 \\ 0 & 6 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \quad \text{as before.}$$

A.5 Remarks:

The major advantage of this method which is given here lies in the fact that no system expansion has to be done. When n is large, the expansion is an awkward process and the dimension of the expanded system is $\frac{n(n+1)}{2}$ which is quite large. Thus the inversion of the expanded system matrix can be a serious problem.

Computer programs were written to mechanize these algorithms. It must be emphasized that the algorithm will yield a closed form solution as long as $\lambda_i + \lambda_j \neq 0$. This condition will always be satisfied as long as U or U' matrices are asymptotically stable, non-singular and when they do not have eigenvalues which are of the type $(\lambda - \lambda_i)(\lambda + \lambda_i)$.

APPENDIX B

MATRIX EIGENVALUES BY SUCCESSIVE SIMILARITY TRANSFORMATIONS AND MODAL TRANSFORMATION MATRIX FOR CONTRACTION

B-1 Matrix Eigenvalues:

The design procedure given in section 3.9 of Chapter III, makes use of eigenvalues of various matrices. In this section a method will be given to find the eigenvalues of a generalized matrix.

If a matrix can be transformed into one of the canonical forms^{(33), (49)}, the coefficients of the characteristic polynomial can be readily determined. When the characteristic equation is known the eigenvalues can be found by Newton's method^{(51), (52), (53)}. In this section, the generalized matrix will be transformed into the Frobenius normal form.

If the matrix A is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad (B-1)$$

Then the matrix A may be put into the Frobenius normal form by similarity transformation given by

$$P = T^{-1} A T .$$

Since similar matrices have identical eigenvalues, the characteristic equation of the A matrix can be written as follows:

$$\det [A - \lambda I] = \det [P - \lambda I]$$

$$= \det \begin{bmatrix} P_1 - \lambda & P_2 & \dots & P_{n-1} & P_n \\ 1 & -\lambda & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -\lambda \end{bmatrix} \quad (B-2)$$

then the determinant can be readily expanded as

$$\det [A - \lambda I] = (-1)^n [\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \dots - p_n] \quad (B-3)$$

Since similar matrices have identical characteristic polynomials, the Frobenius normal form can be obtained if the given matrix A can be reduced to the matrix P by successive similarity transformations.

Here the matrix P is :

$$P = \begin{bmatrix} P_1 & P_2 & \dots & P_{n-1} & P_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (B-4)$$

It will be shown that (n-1) successive similarity transformations will transform matrix A into matrix P.

To change $[a_{n1} \ a_{n2} \ \dots \ a_{n,n-1} \ a_{n,n}]$ into the row $[0 \ 0 \ \dots \ 1 \ 0]$, divide all the elements of the row by $a_{n,n-1}$; then subtract the (n-1)th column multiplied by $a_{n1}, a_{n2} \ \dots \ a_{nn}$ respectively from all the rest of the columns. These are elementary transformations⁽³³⁾ on columns and are nothing but post-multiplication of matrix A by T_{n-1} . Here it is assumed that $a_{n,n-1} \neq 0$.

Thus

$$T_{n-1} = \begin{bmatrix} 1 & 0 & \text{----} & 0 & 0 \\ 0 & 1 & \text{----} & 0 & 0 \\ \text{-----} & & & & \\ \frac{-a_{n1}}{a_{n,n-1}} & \frac{-a_{n2}}{a_{n,n-1}} & \text{----} & \frac{1}{a_{n,n-1}} & \frac{-a_{nn}}{a_{n,n-1}} \\ 0 & 0 & \text{---} & 0 & 1 \end{bmatrix} \quad (B-5)$$

The inverse of T_{n-1} can be readily found and is given by

$$T_{n-1}^{-1} = \begin{bmatrix} 1 & 0 & \text{----} & 0 & 0 \\ 0 & 1 & \text{----} & 0 & 0 \\ \text{-----} & & & & \\ a_{n1} & a_{n2} & \text{----} & a_{n,n-1} & a_{n,n} \\ 0 & 0 & \text{----} & 0 & 1 \end{bmatrix} \quad (B-6)$$

The elements of $B = AT_{n-1}$ can be found as follows:

$$AT_{n-1} = \begin{bmatrix} a_{11} & a_{12} & \text{----} & a_{1,n-1} & a_{1n} \\ \text{-----} & & & & \\ a_{n-1,1} & a_{n-1,2} & \text{----} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \text{---} & a_{n,n-1} & a_{nn} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & \text{----} & & \\ \text{-----} & & & & \\ t_{n-1,1} & t_{n-1,2} & \text{----} & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & \text{---} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1,n-1} & b_{1,n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (\text{B-7})$$

Here

$$t_{n-1,j} = \frac{-a_{nj}}{a_{n,n-1}} \quad ; \quad j \neq n-1$$

$$t_{n-1,n-1} = \frac{1}{a_{n,n-1}} \quad (\text{B-8})$$

and

$$b_{ik} = a_{ik} + a_{i,n-1} \cdot t_{n-1,k} \quad , \quad i \leq n-1, k \neq n-1$$

$$b_{i,n-1} = a_{i,n-1} t_{n-1,n-1} \quad (\text{B-9})$$

The similarity transformation matrix $C = T_{n-1}^{-1} A T_{n-1}$ is found as

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n,n-1} & a_{n,n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} b_{11} & b_{12} & b_{1,n-1} & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{1,n-1} & b_{1n} \\ \sum_{k=1}^n a_{nk} b_{k1} & \sum_{k=1}^n a_{nk} b_{k2} & \sum_{k=1}^n a_{nk} b_{k,n-1} & \sum_{k=1}^n a_{nk} b_{kn} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (B-10)$$

Thus the premultiplication by T_{n-1}^{-1} changes only the (n-1)th row of the B matrix. The elements of this row are found with only one accumulation. The process can be continued for one more step if $c_{n-1,n-2} \neq 0$. The entire process of (n-1) similarity transformations by means of the matrices $T_{n-1}, T_{n-2}, \dots, T_1$ can be fitted into a computational scheme so that $T = T_{n-1}, T_{n-2}, \dots, T_1$.

After several steps the C matrix has the following form :

$$C = \begin{bmatrix} c_{11} & & & & c_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{k1} & c_{k2} & \dots & c_{kk} & \dots & c_{kn} \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 \end{bmatrix} \quad (B-11)$$

If it turn out that $c_{k,k-1} = 0$, there are two possible cases. If $c_{k,j}$ ($j < k-1$) is non-zero, the j^{th} and $(k-1)^{\text{th}}$ columns are interchanged and at the same time the rows in similar positions are interchanged too. This transformation is achieved by pre and post-multiplication by a matrix S of the form :

$$S = \begin{bmatrix} 1 & & & & & \\ & 0 & \dots & 1 & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \dots & 1 \\ & & & & & & & 1 \end{bmatrix} \quad \begin{matrix} (j) \\ \\ \\ \\ (k-1) \end{matrix} \quad \text{(B-12)}$$

This matrix has the property: $S^2 = I$ or $S = S^{-1}$. Thus the multiplication on the left and right generates a similarity transformation. After this transformation one can proceed as usual.

If $c_{k,j} = 0$ ($j \leq k-1$), the situation is even simpler. The matrix C will have the form

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1,k-1} & c_{1,k} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{k-1,1} & c_{k-1,2} & \dots & c_{k-1,k-1} & c_{k-1,k} & \dots & c_{k-1,n} \\ 0 & 0 & \dots & 0 & c_{k,k} & \dots & c_{k,n} \\ 0 & 0 & \dots & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 1 & & 0 \end{bmatrix} \quad \text{(B-13)}$$

$$= \begin{bmatrix} C_1 & D \\ 0 & C_2 \end{bmatrix}$$

Here the dimensions of the C_1 and C_2 matrices are :

- C_1 - $[(k-1), (k-1)]$
- C_2 - $[(n-k+1), (n-k+1)]$

and D is a rectangular matrix of appropriate dimension.

Then

$$\det [C - \lambda I] = \det [C_1 - \lambda I] \cdot \det [C_2 - \lambda I] \quad (B-14)$$

Since C_2 is in the Frobenius form $\det [C_2 - \lambda I]$ can be found right away. The matrix C_1 can be transformed to the Frobenius form in the same way matrix C_2 was transformed. Once the characteristic equation is determined, the eigenvalues can be readily obtained.

The eigenvectors of the generalized matrix A can be found from the total transformation matrix as shown below.

Since A and P are related by similarity transformation, they have identical eigenvalues. If x is the eigenvector of P associated with eigenvalue λ , then

$$Px = \lambda x \quad (B-15)$$

If x_1, x_2, \dots, x_n are the n components of this eigenvector then using the Frobenius form for P given by eqn. (B-4), eqn. (B-15) is expanded and the following equations will result.

$$\begin{aligned} p_1 x_1 + p_2 x_2 + \dots + p_{n-1} x_{n-1} + p_n x_n &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ x_2 &= \lambda x_3 \\ &\vdots \\ x_{n-1} &= \lambda x_n \end{aligned} \quad (B-16)$$

If x_n is assumed to be equal to 1, then

$$\begin{aligned} x_n &= 1 \\ x_{n-1} &= \lambda \\ &\vdots \\ x_2 &= \lambda^{n-2} \\ x_1 &= \lambda^{n-1} \end{aligned}$$

and

$$p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \dots + p_{n-1} \lambda + p_n = \lambda^n \quad (B-17)$$

When the n components of x satisfy the above relations, the last relation given in eqn. (B-17) is identical to the characteristic equation (B-3). Thus when the components of x satisfy the relations given by eqn. (B-17), x is the eigenvector of the matrix P associated with the eigenvalue λ . The remaining eigenvectors of P can be found similarly. The relation between the eigenvectors of A and those of P can be found as follows :

$$\begin{aligned} Px &= \lambda x \\ \therefore T^{-1} A Tx &= \lambda x \end{aligned} \tag{B-18}$$

$$\text{If } Tx = y \tag{B-19}$$

$$\text{then } T^{-1} Ay = \lambda x \tag{B-19a}$$

Premultiplying eqn. (B-19a) by T

$$Ay = \lambda Tx = \lambda y \tag{B-20}$$

Eqn. (B-20) shows that y is the eigenvector of A . Thus when x , the eigenvector of P is found by the scheme of eqn. (B-17), the eigenvector of the original matrix A can be readily obtained from eqn. (B-19). This is a significant advantage of the above method.

B.2 Modal Transformation Matrix :

In subsection 3.8-2, it was mentioned that if the system matrix A is in the companion matrix form and has distinct eigenvalues, then the Vandermonde matrix becomes the modal transformation matrix.

This will be proved here.

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \dots & \dots & \dots & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \tag{B-21}$$

It will be shown that if

$$T = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \quad (B-22)$$

then

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix} = T^{-1} A T \quad (B-23)$$

The matrix T is known as the Vandermonde matrix.

Eqn. (B-23) is applicable when λ 's are distinct. To prove eqn. (B-23), it can be shown that

$$T^{-1} A T e_j = \lambda_j e_j \quad ; \quad j = 1, \dots, n \quad (B-24)$$

where e_j is the natural basis of R_n .

Now

$$T e_1 = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{n-1} \end{bmatrix} \quad (B-25)$$

and

$$ATe_1 = \begin{bmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ -a_0 \lambda_1^0 - a_1 \lambda_1^1 - a_2 \lambda_1^2 - \dots - a_{n-1} \lambda_1^{n-1} \end{bmatrix} \quad (B-26)$$

Now λ_1 is a solution of the characteristic equation

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

Hence

$$-a_0 \lambda_1^0 - a_1 \lambda_1^1 - a_2 \lambda_1^2 - \dots - a_{n-1} \lambda_1^{n-1} = \lambda_1^n \quad (B-27)$$

Therefore

$$ATe_1 = \begin{bmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^n \end{bmatrix} = \lambda_1 \begin{bmatrix} \lambda_1^0 \\ \lambda_1^1 \\ \vdots \\ \lambda_1^{n-1} \end{bmatrix} = \lambda_1 Te_1 \quad (B-28)$$

Thus

$$\begin{aligned} T^{-1} A T e_1 &= T^{-1} \lambda_1 T e_1 \\ &= \lambda_1 e_1 \end{aligned} \quad (B-29)$$

Similarly

$$T^{-1} A T e_j = \lambda_j e_j \quad ; \quad j = 2, 3, \dots, n \quad (B-30)$$

Hence

$$T^{-1} A T = \Lambda \quad (B-31)$$

When all the λ 's are real and distinct, T^{-1} , the inverse of the transformation matrix is useful in obtaining the contraction matrix C.

APPENDIX C

DEFINITION OF CONTROLLABILITY AND OBSERVABILITY, CONDITIONS FOR THE EXISTENCE OF THE INVERSE OF T_m AND THE UNIQUENESS OF \hat{N}

C.1 Definition :

In section of Chapter II, the terms controllability and observability were mentioned. In this section these terms will be formally defined.

When the linear time-invariant system given below is :

$$\begin{aligned}\dot{x}(t) &= A x(t) + B u(t) \\ y(t) &= H x(t)\end{aligned}\tag{C-1}$$

with

A - (nxn) - system matrix	x - (nx1) - state vector
B - (nxr) - input matrix	y - (lx1) - output vector
H - (lxn) - output matrix	u - (rx1) - control vector

the controllability and observability are defined as follows.

C.1.1 Controllability :

$$\text{If Rank } [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] = n \tag{C-2}$$

then the system given by eqn. (C-1) is said to be controllable. This means that the $[(n) \times (n \cdot r)]$ matrix given in eqn. (C-2) must contain n linearly independent column vectors.

C.1.2 Observability :

$$\text{If Rank } [H' \quad A'H' \quad (A')^2 H' \quad \dots \quad (A')^{n-1} H'] = n \tag{C-3}$$

then the system given by eqn. (C-1) is said to be observable. In other words the $[(n) \times (n \cdot l)]$ matrix in eqn. (C-3) must contain n linearly independent column vectors.

C.2 Condition for the Existence of Inverse of T_m :

In chapter III, section 3.2 on system expansion, the theory was based on the fact that T_m has an inverse. The condition for the existence of this inverse will be established here.

By definition

$$T_m = GR^{-1}G'$$

Since R is a diagonal matrix with all the elements often being unity, for practical purposes T_m can be defined as :

$$T_m \triangleq GG' \tag{C-4}$$

The matrix G is an $(m \times r)$ rectangular matrix with $m \leq r$.

Thus for T_m to have an inverse, $\det [GG']$ must be non-zero.

Hence the following proposition will be stated and proved.

C.2.1 Proposition 1 :

A necessary and sufficient condition for $\det [GG'] \neq 0$ is
 $\text{Rank } [G] = m$

Proof: Assume that $\det [GG'] \neq 0$ but $\text{Rank } [G] < m$.

The condition that $\text{Rank } [G] < m$ implies that there exists a non-zero $(m \times 1)$ vector x , such that

$$x'G = 0 \quad ; \quad x \neq 0 \tag{C-5}$$

Hence

$$x'GG' = 0 \tag{C-6}$$

Transposing eqn. (C-6)

$$GG'x = 0 \tag{C-7}$$

Eqn. (C-7) is true for any non-zero x , if and only if

$$\det [GG'] = 0 \tag{C-8}$$

However eqn. (C-8) contradicts the assumption that $\det [GG'] \neq 0$.

Hence

$$\text{Rank } [G] < m \quad (\text{C-9})$$

Since the matrix G is an $(m \times r)$ matrix with $m \leq r$

$$\text{Rank } [G] \neq m \quad (\text{C-10})$$

Therefore

$$\text{Rank } [G] = m \quad (\text{C-11})$$

Conversely assume that

$$\text{Rank } [G] = m \text{ but } \det [GG'] = 0.$$

Now $\det [GG'] = 0$ implies that there exists a non-zero $(m \times 1)$ vector x such that

$$GG'x = 0 \quad (\text{C-12})$$

Premultiplying eqn. (C-12) by x'

$$x'GG'x = 0 = \|G'x\|^2 \quad (\text{C-13})$$

Hence

$$G'x = 0$$

Therefore

$$x'G = 0 \quad (\text{C-14})$$

Eqn. (C-14) implies that $\text{Rank } [G] < m$, which contradicts the assumption. This completes the proof of the proposition.

C.2.2 Ranks of the Various Matrices :

Now

$$G = CB \quad (\text{C-15})$$

Where C - $(m \times n)$ matrix

B - $(n \times r)$ matrix $m \leq r \leq n$

By the product rule⁽³³⁾ of matrices,

$$\text{Rank } [G] = \text{Rank } [CB] \leq \min [r_C, r_B] \quad (\text{C-16})$$

Here

$$r_C \triangleq \text{Rank } [C]$$

$$r_B \triangleq \text{Rank } [B]$$

As shown by Proposition 1

$$\text{Rank } [G] = m, \text{ hence}$$

$$m \leq \min [r_C, r_B] \quad (C-17)$$

Since the matrix C is (m x n) and $m < n$, its rank r_C can at the most be m. The matrix B is (n x r) and $m \leq r \leq n$, its rank r_B can at the most be r. However to satisfy expression (C-17)

$$r_C = \text{Rank } [C] = m \quad (C-18)$$

$$r_B = \text{Rank } [B] \geq m \quad (C-19)$$

Condition (C-18) is satisfied by ensuring that the contraction matrix C has all of its m row-vectors linearly independent. Condition (C-19) implies that at least m of the rows or columns of the B matrix are linearly independent. Ordinarily matrix B has all of its r column vectors ($r \geq m$) linearly independent.

To ensure that GG' has an inverse, step 8 is used in the design algorithm given in section 3.9, to check that the determinant of GG' is non-zero. This is achieved by checking that $\text{Rank } [G] = m$.

C-3 Conditions for the Uniqueness of \hat{N} :

In section 3.2 it was mentioned that the solution of the algebraic matrix Riccati equation is unique. In this sections conditions will be given which make this assertion valid.

If the system is given by

$$\dot{x} = Ax + Bu$$

$$x(0) = x_0$$

$$y = Hx$$

(C-20)

and it is required to minimize the performance index

$$J = 1/2 \int_0^{\infty} [x'H'Q_x Hx + u'Ru] dt \quad (C-21)$$

then it is well known that the optimal control law is $u = -R^{-1}B'Px$ and this gives rise to the closed loop system matrix $A_0 = A - BR^{-1}B'P$. Here the matrix P is obtained from the algebraic matrix Riccati equation given below.

$$PA + A'P - PBR^{-1}B'P + H'Q_l H = 0$$

For convenience

$$BR^{-1}B' = UU' \text{ with Rank } [U] = \text{Rank } [BR^{-1}B'] \quad (C-22)$$

and

$$H'Q_l H = V'V \text{ with Rank } [V] = \text{Rank } [H'Q_l H] \quad (C-23)$$

Thus the algebraic matrix Riccati equation becomes

$$PA + A'P - PUU'P + V'V = 0 \quad (C-24)$$

A brief recapitulation of the major result in the literature on this equation is given below.

1. If (A, U) is completely controllable⁽¹⁰⁾⁽⁵⁴⁾, then there exists a solution P to eqn. (C-24), such that P is positive semi-definite i.e. $P \geq 0$.
2. If (A, U) is completely controllable and (V, A) ⁽¹¹⁾⁽⁵⁴⁾ is completely observable, then there exists a unique solution P for eqn. (C-24) such that P is positive definite i.e. $P > 0$.
These conditions have been further weakened⁽⁵⁵⁾ as follows:
3. If (A, U) is stabilizable and (V, A) is completely observable, then there exists a unique solution P to eqn. (C-24) such that $P > 0$ and it yields a stable closed loop system.
4. If (A, U) is stabilizable and (V, A) is detectable, then there exists a solution $P \geq 0$ to eqn. (C-24) such that it yields an asymptotically stable closed-loop system.

Here the terms stabilizable and detectable, which are used in the above two results are defined after stating Proposition 2. The above four results are sufficiency conditions only. In the literature certain

general forms of the solution P have been found⁽⁵⁶⁾ and some necessary conditions have also been proved⁽⁵⁷⁾.

However a necessary and sufficient condition has been obtained by Dr. Kucera⁽⁴²⁾ for the existence of a unique positive semi-definite solution to eqn. (C-24). The author is aware of these results through private communication with Dr. Kucera and the important result is given below in Proposition 2.

C.3.1 Proposition 2 :

Equation (C-24) possesses a Unique solution $P \geq 0$ and the closed loop matrix $A_o = A - UU'P$ is asymptotically stable if and only if (A, U) is stabilizable and (V, A) is detectable.

The definitions of stabilizability and detectability are given below⁽⁴²⁾⁽⁵⁵⁾.

C.3.2 Stabilizability :

The pair (A, U) is stabilizable if and only if every eigenvalue of (A, U) which is not asymptotically stable is controllable. In other words there exists a real matrix D such that the matrix $(A + UD)$ is asymptotically stable.

C.3.3 Detectability :

The pair (V, A) is detectable if and only if every eigenvalue of (V, A) which is not asymptotically stable is observable. In other words the pair (A', V') is stabilizable.

From the above definitions, the following quantitative checks can be formulated.

C.3.4 Test for Stabilizability :

If λ is any eigenvalue of the matrix A with $\text{Re} [\lambda] \geq 0$, then from the definition of stabilizability given above λ must be a controllable eigenvalue of the pair (A, U) . For such a λ controllability can be checked as follows.

If there exists a row vector $w \neq 0$, such that

$$\begin{aligned} wA &= \lambda w \\ wU &\neq 0 \end{aligned} \tag{C-25}$$

then λ is controllable. With $\text{Re} [\lambda] \geq 0$, test given by eqn. (C-25) implies stabilizability.

C.3.5 Test for Detectability :

If λ is any eigenvalue of the matrix A with $\text{Re} [\lambda] \geq 0$, then from the definition of detectability given above λ must be an observable eigenvalue of the pair (V, A) . For this λ , the observability can be checked as follows.

If there exists a column vector $z \neq 0$, such that

$$\begin{aligned} Az &= \lambda z \\ Vz &\neq 0 \end{aligned} \tag{C-26}$$

then λ is observable. With $\text{Re} [\lambda] \geq 0$, test given by eqn. (C-26) implies detectability.

Check given by conditions (C-25) and (C-26) need be performed only when the eigenvalues λ of A are such that $\text{Re} [\lambda] \geq 0$. When $\text{Re} [\lambda] \geq 0$, these two tests can be readily performed on these eigenvalues of matrix A since the eigenvalues and the eigenvectors of the A matrix are found in a routine manner.

C.3.6 Uniqueness of the \hat{N} Matrix :

In the design algorithm given in section 3.9, it is ascertained that (A, U) is stabilizable and (V, A) is detectable.

With

$$UU' = BR^{-1}B' \tag{C-27}$$

and

$$V'V = H'Q_t H = C'P'Q_m PC \tag{C-28}$$

or

$$V'V = Q_n = C'Q_m C \tag{C-29}$$

$$\hat{N} = C' \hat{M} C \quad (C-30)$$

is one positive semi-definite solution of eqn. (C-24).

Since the system is checked for stabilizability and detectability,

\hat{N} is Unique by Proposition 2. Also

$$\begin{aligned} A_o &= A - BR^{-1}B' \hat{N} \\ &= A - UU' \hat{N} \end{aligned} \quad (C-31)$$

is asymptotically stable.

Hence

$$\begin{aligned} A - BR^{-1}B' \hat{N} &= A - BR^{-1}B' C' \hat{M} C \\ &= A - BKC \end{aligned} \quad (C-32)$$

is also asymptotically stable. Here

$$K \triangleq R^{-1} G' M \hat{M} .$$

APPENDIX D

NUMERICAL EXAMPLES WITH ARBITRARY R AND Q_n MATRICES AND TIME REQUIREMENTS

D.1 Numerical Examples with Arbitrary Matrices:

In chapter III, three numerical examples were considered. The Q_m matrix was obtained by contraction. The Q_n matrix was generated from the Q_m matrix and the R matrix was arbitrarily assumed.

In this section both R and Q_n matrices will be arbitrarily assumed. Thus the various matrices will be obtained by solving the third order matrix Riccati differential equation. The resulting eigenvalues of the optimal systems will also be arbitrary.

D.1.1 1st Numerical Example :

The system is given by eqn. (3-78) and it is required to minimize the cost-functional given by eqn. (3-79). In this sub-section the problem will be solved by arbitrarily assuming $R = I_2$ and $Q_n = I_3$. The following matrices will be obtained by solving the third order matrix Riccati eqn.

$$\hat{N} = \begin{bmatrix} 1.798 & 1.508 & -2.019 \times 10^{-3} \\ 1.508 & 1.633 & 5.215 \times 10^{-3} \\ -2.019 \times 10^{-3} & 5.215 \times 10^{-3} & 3.603 \times 10^{-3} \end{bmatrix} \quad (D-1)$$

$$\hat{K}_n = \begin{bmatrix} -2.019 \times 10^{-3} & 5.215 \times 10^{-3} & 3.603 \times 10^{-3} \\ 1.508 & 1.633 & 5.215 \times 10^{-3} \end{bmatrix} \quad (D-2)$$

where

$$\hat{K}_n = \Delta R^{-1} B' \hat{N}$$

Then

$$A_o = \begin{bmatrix} 0 & 1 & 0 \\ -1.508 & -1.633 & 0.944 \\ -29.998 & -43.005 & -14.036 \end{bmatrix} \quad (D-3)$$

where

$$A_o = [A - BK_n^{\wedge}]$$

All the matrices have been defined before. The characteristic equation is :

$$\det [\lambda I - A_o] = \lambda^3 + 15.699\lambda^2 + 66.558\lambda + 44.5 = 0 \quad (D-4)$$

The eigenvalues are:

$$\lambda_1 = -0.82$$

$$\lambda_2 = -8.37$$

$$\lambda_3 = -6.47$$

D.1.2 2nd Numerical Example :

The problem is defined by eqns. (3-100) and (3-101).

Here the problem will be solved by arbitrarily assuming $R = I_2$ and $Q_n = I_3$.

The following matrices will be obtained by solving the third order matrix Riccati eqn.

$$\hat{N} = \begin{bmatrix} 2.9156 & 1.5196 & -4.3735 \times 10^{-2} \\ 1.5196 & 1.655 & 5.8938 \times 10^{-2} \\ -4.3735 \times 10^{-2} & 5.8938 \times 10^{-2} & 1.1022 \times 10^{-1} \end{bmatrix} \quad (D-5)$$

$$\hat{K}_n = \begin{bmatrix} -4.3735 \times 10^{-2} & 5.8938 \times 10^{-2} & 1.1022 \times 10^{-1} \\ 1.5196 & 1.655 & 5.8938 \times 10^{-2} \end{bmatrix} \quad (D-6)$$

$$A_o = \begin{bmatrix} 0 & 1 & 0 \\ -1.5196 & -1.655 & 0.94106 \\ -14.956 & -11.059 & -5.1102 \end{bmatrix} \quad (D-7)$$

The characteristic equation is :

$$\det [\lambda I - A_o] = \lambda^3 + 5.76\lambda^2 + 20.36\lambda + 21.82 = 0 \quad (D-8)$$

The eigenvalues are:

$$\begin{aligned} \lambda_1 &= -1.56 \\ \lambda_{2,3} &= -2.1 \pm j 3.066 \end{aligned}$$

D.1.3 3rd Numerical Example :

The problem is defined by eqns. (3-120) and (3-121).

Here the problem is solved by arbitrarily assuming $R = I_2$ and $Q_n = I_3$.

The following matrices will be obtained by solving the third order matrix Riccati equation.

$$\hat{N} = \begin{bmatrix} 1.058 \times 10 & 3.63 \times 10^{-1} & 7.701 \\ 3.63 \times 10^{-1} & 3.116 & 7.48 \times 10^{-1} \\ 7.701 & 7.48 \times 10^{-1} & 6.011 \end{bmatrix} \quad (D-9)$$

$$\hat{K}_n = \begin{bmatrix} 7.701 & 7.48 \times 10^{-1} & 6.011 \\ 3.63 \times 10^{-1} & 3.116 & 7.48 \times 10^{-1} \end{bmatrix} \quad (D-10)$$

$$A_o = \begin{bmatrix} 2.0 & -2.0 & 3.0 \\ 6.36 \times 10^{-1} & -2.116 & 2.51 \times 10^{-1} \\ -6.701 & 2.251 & -7.011 \end{bmatrix} \quad (D-11)$$

The characteristic equation is :

$$\det [\lambda I - A_0] = 0 = \lambda^3 + 7.127\lambda^2 + 17.406\lambda + 13.160 \quad (D-12)$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &= -1.4 \\ \lambda_{2,3} &= -2.863 \pm j 1.1 \end{aligned}$$

D.2 Time Requirements:

The method of system design by contraction is described in the Procedure for Design in section 3.9. In this algorithm steps 15, 16 and 17 are used for double-checking only. Step 15 is used to obtain the \hat{N} matrix by integration of the n^{th} order matrix Riccati differential equation. This \hat{N} matrix is then compared with the $C^1\hat{M}C$ matrix to verify that they are identical.

Out of the steps 15, 16 and 17, step 15 is the most time consuming since it is in this step that the integration of the n^{th} order equation is carried out. And integration is always a very time consuming process. Steps 16 and 17 are not essential in the scheme of design by contraction. However step 16 and step 17 require a very short time compared to step 15 where the integration is carried out. A comparison is made below among the three schemes for the time required to complete each of these schemes.

The time requirement is observed from the three examples considered in the thesis and many others which were not included in this thesis. The relation of proportionality between the time required and scheme used is only approximate. The time considered here is strictly the Central Processing Unit (CPU) time. The total time required is always more than CPU time and varies with Input/Output and Peripheral devices used. As defined before, n is the order of the original system and m is the order of the contracted system. The CPU time was on an IBM-360/65 installation. The following table summarizes the observed results.

$m = 2$ $n = 3$	Contraction Method with step 15	Contraction Method without step 15.	Method of Appendix D with Arbitrary R and Q_n Matrices.
Approximate Relation of Proportionality	$m^2 + n^2$	m^2	n^2
CPU Time	1 min 40 sec.	25 sec.	1 min 5 sec.

m = dimension of the contracted system
 n = dimension of the original system

Table D-1

The results above give a fair idea of the time saved in designing high order systems by the method of contraction without using step 15.

In all the examples included in the thesis, the dimension of the contracted lower order system was two. This was possible because the input vector was at least of dimension two. It was observed from many other examples, that when possible $m = 2$ was an adequate dimension so as to be able to move either real or complex-conjugate pairs of poles to the desired location. Also the Q_m and Q_n matrices could be obtained relatively easily. Relocating more than two poles could be achieved by successive application of the design procedure.

APPENDIX E

PROOF OF A ZERO QUADRATIC FORM

E.1 Zero Quadratic Form :

In section 4.1 it is required to prove the zero quadratic form. Here it will be shown that if the quadratic form given below is zero for all x , then for a symmetric matrix S

$$\langle x, Sx \rangle = x'Sx = 0 \quad (\text{E-1})$$

then the above equation (E-1) implies that

$$S = 0 \quad (\text{E-2})$$

The implication given by eqn. (E-2) by first stating and then proving two propositions.

E.2 Proposition 1 :

A necessary and sufficient condition that a linear transformation A on an inner product space be zero is that

$$\langle Ax, y \rangle = 0 \quad \text{for all } x \text{ and } y.$$

Proof :

Assume that

$$\langle Ax, y \rangle = 0 \quad \text{for all } x \text{ and } y \quad (\text{E-3})$$

If this assumption holds true for all x and y , then it also holds true for

$$y = Ax \quad (\text{E-4})$$

Hence eqn. (E-3) becomes

$$\langle Ax, Ax \rangle = 0 = \|Ax\|^2 \quad (\text{E-5})$$

Hence

$$Ax = 0 \quad \text{for all } x \quad (\text{E-6})$$

Expanding eqn. (E-6) and solving for an (nx1) vector x given by

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

, then

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \quad (\text{E-7})$$

Eqn. (E-7) implies that

$$a_{11} = a_{21} = \dots, = a_{n1} = 0 \quad (\text{E-8})$$

Similarly if x is chosen as

$$x = e_i \quad ; \quad i = 2, 3, \dots, n \quad (\text{E-9})$$

where e_i is the natural basis of R_n , then

$$Ae_i = 0 \quad ; \quad i = 2, 3, \dots, n \quad (\text{E-10})$$

implies

$$a_{1i} = a_{2i} = \dots, = a_{ni} = 0 \quad (\text{E-11})$$

for $i = 2, 3, \dots, n$

Eqn. (E-11) shows that all the elements of the matrix A have to be identically equal to zero for all x and y in eqn. (E-3).

Hence

$$A = 0 \quad (\text{E-12})$$

Conversely if A is assumed to be zero, then

$$A = 0$$

Hence

$$Ax = 0 \quad \text{for all } x \quad (\text{E-13})$$

Therefore,

$$y'Ax = \langle Ax, y \rangle = 0 \quad \text{for all } x \text{ and } y. \quad (\text{E-14})$$

This completes the proof of Proposition 1.

E.3 Proposition 2 :

A necessary and sufficient condition that a symmetric linear transformation A on an inner product space be zero is $\langle Ax, x \rangle = 0$ for all x .

Proof :

To prove this proposition, the following identity will be first verified.

$$\langle Ax, y \rangle + \langle Ay, x \rangle = \langle A(x+y), (x+y) \rangle - \langle Ax, x \rangle - \langle Ay, y \rangle \quad (\text{E-15})$$

$$\begin{aligned} \text{R.H.S.} &= (x+y)' A (x+y) - x'Ax - y'Ay \\ &= x'Ax + x'Ay + y'Ax + y'Ay - x'Ax - y'Ay \\ &= x' Ay + y'Ax \end{aligned} \quad (\text{E-16})$$

Thus

$$\begin{aligned} \text{R.H.S.} &= \langle Ax, x \rangle + \langle Ay, x \rangle \\ &= \text{L.H.S.} \end{aligned} \quad (\text{E-17})$$

Assume that

$$\langle Ax, x \rangle = 0 \quad \text{for any } x \quad (\text{E-18})$$

Then for $y = x$ eqn. (E-18) must hold. Thus

$$\langle Ay, y \rangle = 0 \quad (\text{E-19})$$

Now if

$$\langle A(x+y), (x+y) \rangle = 0 \quad (\text{E-20})$$

Then by eqns. (E-16) and (E-15)

$$x'Ay + y'Ax = 0 \quad (E-21)$$

Transposing the quadratic form,

$$\begin{aligned} x'Ay + y'Ax &= y'A'x + y'Ax \\ &= y'(A' + A)x \end{aligned} \quad (E-22)$$

Hence from eqn. (E-21)

$$y'(A' + A)x = 0 \quad (E-23)$$

Since matrix A is symmetric, $A = A'$ and eqn. (E-23) becomes

$$2y'Ax = 0 \quad (E-24)$$

Hence

$$y'Ax = 0 \quad \text{for all } x \text{ and } y \quad (E-25)$$

From Proposition 1, it is known that, eqn. (E-25) implies

$$A = 0 \quad (E-26)$$

Conversely if A is assumed to be zero, then

$$A = 0$$

Hence

$$Ax = 0 \quad \text{for all } x \quad (E-27)$$

and

$$x'Ax = 0 \quad \text{for all } x \quad (E-28)$$

This completes the proof of Proposition 2. This proposition can be extended to all linear self-adjoint transformations. However, here it is proved for symmetric transformation only.

E.4 Proof of the Quadratic Form :

From Propositions 1 and 2 it can be seen that

$$\langle x, Sx \rangle = 0 \quad \text{for all } x$$

implies that $S = 0$

when S is a symmetric matrix.

Thus eqn. (4-10a) in Chapter IV can be written as

$$\langle x(t), [A'(H'SH) + (H'SH)A - (H'SH)BR^{-1}B'(H'SH) + H'Q_{\ell}H]x(t) \rangle = 0 \quad (\text{E-29})$$

Since eqn. (E-29) is true for all $x(t)$,

$$A'(H'SH) + (H'SH)A - (H'SH)BR^{-1}B'(H'SH) + H'Q_{\ell}H = 0 \quad (\text{E-30})$$

It can be verified that the L.H.S. of eqn. (E-30) is a symmetric matrix.

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