MODERN OPTIMAL CONTROL OF A MICRO-ECONOMIC SYSTEM

by

Guy J. Jutras

A thesis submitted to
the School of Graduate Studies
of the University of Ottawa
in partial fulfillment of the requirements
for the degree of Master of Applied Science

Department of Electrical Engineering
Faculty of Science and Engineering
University of Ottawa
Ottawa, Ontario
January 1977

UMI Number: EC52260

INFORMATION TO USERS

TABLE OF CONTENTS

Table of Contents cont'd....

ABSTRACT

This thesis presents a microeconomic model for a manufacturing firm, including typical control variables. Constraints are established for the controls and a realistic cost criterion derived. The resulting optimization problem is solved by the use of Pontryagin's Maximum Principle which yields the required form for the controls. Numerical results are obtained through the solution of a two point boundary value problem (T.P.B.V.P.).

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his supervisor Dr. N. U. Ahmed for initiating the problem and for his continued assistance, good advice and encouragement without which this thesis could not have been possible.

The author also wishes to thank Dr. E. Apel of the Economics Department at the University of Ottawa for useful discussions and advice, and to thank his fellow students at the University for their friendship and discussions.

The financial assistance of the National Research Council Postgraduate Scholarship is gratefully acknowledged.

INTRODUCTION

A problem which is often considered in economics is that of a manufacturing firm in a competitive industry. Often a number of algebraic "balance" equations are optimized through a linear programming method; models for purposes of planning or prediction have been constructed from linear difference equations with the right hand side (r.h.s.) coefficients generated through a statistical analysis of the past behavior of the equation variables. These models describe a linear extrapolation of the past behavior but do not attempt to represent the dynamics of the system. A reason why attention has traditionally been given to these models is that the classical analytical solution of this type of problem is well known.

However it is easy to see that in a model meant to represent the dynamics of the actual system, the differential (or difference) equations used will in general not be linear; it is the author's intention in this work to demonstrate the solution of a microeconomic model containing non-linear differential equations.

In the first chapter a brief review is made of the theoretical background for the Pontryagin's Maximum

Principle. The theorems presented are used in the solution of the problem defined in later chapters, but more general and powerful theorems also apply which are not mentioned, for the sake of brevity.

In the second chapter, the microeconomic model, composed of five differential equations, is introduced; of course this model is highly aggregated and represents many simplifying assumptions. The equations for capital, labour and inventory stock are quite simple and flow directly from traditional economic concepts. The equations describing the production rate and sales rate are explained in chapter two; these are based on an assumed production function and demand curve. Many other functions are possible, but our purpose in this work is not to reccomend economic theories but rather to demonstrate the use of modern optimal control theory in economic work through the solution of a specific problem.

In chapter three is develloped the "cost" criterion to be maximized in the optimal control problem. This criterion represents the economic goal to be reached; again many possible goals could be stated but a specific one was chosen to represent profitability. Also solved is a second problem where it is wished to reach a certain economic state in a minmum amount of time.

Then, for both problems the form of the solution is derived using Pontryagin's Maximum Principle.

In chapter four is given an example of how an actual problem might be modelled. Using arbitrary initial assumptions, the parameters in our microeconomic model are estimated. The cost criterion is develloped and the numerical solution is obtained, giving the optimum control for our microeconomic system. Results are presented graphically

Finally, in a last chapter, the results are discussed and other conclusions and comments are presented.

CHAPTER 1

A REVIEW OF
PONTRYAGIN'S MAXIMUM PRINCIPLE

§ 1.1 BASIC ASSUMPTIONS

It is assumed that the control system is described by a system of differential equations of the form:

$$\frac{d\,x_i(t)}{dt} = f_i(x(t),u(t)) \qquad ,i = 1,2,3,\ldots,n \tag{1}$$

In this case, the right-hand side (r.h.s.) does not depend explicitly on t. Setting these equations in a vector form, we get

$$\frac{d\,x(t)}{dt} = f(x(t),u(t)) \tag{2}$$

where $x(t) \equiv \{x_1(t),x_2(t),\ldots,x_n(t)\}$ is the state vector,

$f(x(t),u(t)) \equiv \{f_1(x(t),u(t),f_2(x(t),u(t)),\ldots$
$\ldots f_n(x(t),u(t))\}$ is

the vector of r.h.s. equations, and

$u(t) \equiv \{u_0(t),u_1(t),\ldots,u_m(t)\}$ is the control vector.

We shall go on to define a criterion to be maximized and to derive the control policy u(t) appropriate to this end.

We assume that the functions $f_i(x(t),u(t))$ are defined for $x(t) \varepsilon E^n$ and for $u(t) \varepsilon U$, $U \varepsilon E^m$.

U is the region of admissible controls which is usually a closed bounded set in E^m.

Also it is assumed that all

$$\frac{\partial}{\partial x_j(t)} f_i(x(t),u(t)) \qquad ,i = 1,2,\ldots,n, \quad j = 1,2,\ldots,n.$$

are continuous on $E^n \times U$.

§ 1.2 THE OPTIMAL CONTROL PROBLEM

Let U be the set of all piecewise continuous functions defined for the interval $I \equiv [t',t'']$ to E^m such that $u(t) \varepsilon U$ for $t \varepsilon I$. We consider U as the class of admissible controls.

If the functional to be minimized is $J \equiv \int_{t'}^{t''} f_0(x(t),u(t))\, dt$, where $f_0(x(t),u(t))$ satisfies all the conditions previously required of the functions $f_i(x(t),u(t))$, $i = 1,2, \ldots ,n$, then we define a new phase coordinate $x_0(t)$ such that $\frac{d\, x_0(t)}{dt} = f_0(x(t),u(t))$.

By adjoining the new coordinate, we obtain the (n + 1) dimensional phase space E^{n+1}. Then equation (2) takes the form

$$\frac{d\, \tilde{x}(t)}{dt} = \tilde{f}(x(t),u(t)) \quad \text{where} \tag{3}$$

$\tilde{f}(x(t),u(t)) \equiv \{f_0(x(t),u(t)), f_1(x(t),u(t)), \ldots ,f_n(x(t),u(t))\}$ and $\tilde{x}(t) \equiv \{x_0(t),x_1(t), \ldots ,x_n(t)\}$ are the expanded vectors.

§ 1.3 STATEMENT OF THE FUNDAMENTAL PROBLEM

Let us define $x' \equiv x(t')$ and $x'' \equiv x(t'')$ where t' and t" are the initial and final times respectively. Let us consider, in the n+1 dimensional phase space E^{n+1}, the point $\tilde{x}' \equiv (0,x')$ (the cost functional at the initial time is of course zero) and the line $\Pi \equiv \{(x_0,x''): x_0 \varepsilon E\}$. The line Π is parallel to the x_0 axis, passing through the point (0,x"). The minimization consists in finding that admissible control u(t),having the property that its solution $\tilde{x}(t)$ of equation (3) starting from the point $\tilde{x}'$ intersects Π, which intersects the line Π with the smallest possible coordinate x_0 (i.e. smallest value for the cost functional at final time t"). The solution to a problem of this type is given by Pontryagin's Maximum Principle. The results are as follows.

§ 1.4 THE MAXIMUM PRINCIPLE

Let us consider the following functional $H(x(t),\Psi(t),u(t))$, which we can call the Hamiltonian, defined as such:

$$H(x(t),\Psi(t),u(t)) \equiv (\Psi(t) \cdot \tilde{f}(x(t),u(t)))$$

$$= \sum_{i=0}^{n} \Psi_i(t) \cdot f_i(x(t),u(t)) \tag{4}$$

where

$$\Psi(t) \equiv \{\Psi_0(t),\Psi_1(t), \ldots ,\Psi_n(t)\} \tag{5}$$

satisfies the differential equations

$$\frac{d\Psi_i}{dt}(t) = -\sum_{k=0}^{n} \Psi_k(t) \cdot \frac{\partial}{\partial x_i(t)} f_k(x(t),u(t)) \tag{6}$$

From the definition of the Hamiltonian we can write equations (3) and (6) as

$$\frac{d\tilde{x}_i}{dt}(t) = \frac{\partial}{\partial \Psi_i(t)} H(x(t),\Psi(t),u(t)) \tag{7}$$

$$i = 0,1, \ldots ,n$$

$$\frac{d\Psi_i}{dt}(t) = -\frac{\partial}{\partial x_i(t)} H(x(t),\Psi(t),u(t)) \tag{8}$$

$$i = 0,1, \ldots ,n$$

We now present Pontryagin's Maximum Principle in the following theorem:

THEOREM 1 (Ref.{1}, th. 1, pp.19)

Let $u(t)$, $t \in (t', t'')$, be an admissible control such that the corresponding trajectory $\tilde{x}(t)$ (see(7)) which begins at the point $\tilde{x}'$ at the time t' passes, at some time t'', through a point on the line Π. In order that $u(t)$ and $\tilde{x}(t)$ be optimal it is necessary that there exist a nonzero continuous vector function $\Psi(t) \equiv \{\Psi_0(t), \Psi_1(t), \ldots, \Psi_n(t)\}$ corresponding to the functions $u(t)$ and $\tilde{x}(t)$ (see(8)), such that:

1-/ For every t, $t' \leqslant t \leqslant t''$, the function $H(x(t), \Psi(t), u(t))$ of the variable $u(t) \in U$ attains its maximum at the point $u(t) = u^*(t)$:

$$H(x(t), \Psi(t), u(t)) = H(x(t), \Psi(t), u^*(t)) \equiv M(x(t), \Psi(t)); \qquad (9)$$

2-/ At the terminal time t'' the relations

$$\Psi_0(t'') \leqslant 0, \; M(x(t''), \Psi(t'')) = 0 \qquad (10)$$

are satisfied. Furthermore, it turns out that if $\Psi(t)$, $\tilde{x}(t)$ and $u(t)$ satisfy system (7), (8), and condition 1-/, the timefunctions $\Psi_0(t)$ and $M(x(t), \Psi(t))$ are constant. Thus, (10) may be verified at any time t, $t' \leqslant t \leqslant t''$, and not just at t''.

§ 1.5 THE TIME OPTIMAL CASE

In this case we wish to minimize the time required to reach the specified final conditions. Thus we set $f_0(x(t),u(t)) = 1$ and we can consider the Hamiltonian

$$H(x(t),\Psi(t),u(t)) = \Psi_0 + \sum_{i=1}^{n} \Psi_i(t) \cdot f_i(x(t),u(t))$$

where $\Psi(t)$ is now an n-dimensional vector,

$\Psi(t) \equiv \{\Psi_1(t),\Psi_2(t), \ \ldots \ ,\Psi_n(t)\}$

Here we consider an n-dimensional phase space. Since $\Psi_0(t)$ is constant the Hamiltonian system becomes:

$$\frac{d\, x_i(t)}{dt} = \frac{\partial H(x(t),\Psi(t),u(t))}{\partial \Psi_i(t)} \qquad i = 1,2, \ \ldots \ ,n \tag{11}$$

$$\frac{d\, \Psi_i(t)}{dt} = -\frac{\partial H(x(t),\Psi(t),u(t))}{\partial x_i(t)} \qquad i = 1,2, \ \ldots \ ,n \tag{12}$$

The necessary condition for time optimality is given below.

<u>THEOREM 2</u> (Ref.{1}, th. 2, pp. 20)

Let $u(t)$, $t' \leqslant t \leqslant t''$, be an admissible control which transfers the phase point from x' to x'', and let $x(t)$ be the corresponding trajectory (see(11)), so that $x(t') = x'$, $x(t'') = x''$. In order that $u(t)$ and $x(t)$ be time optimal it is necessary that there exist a nonzero, continuous vector function $\Psi(t) \equiv \{\Psi_1(t),\Psi_2(t), \ \ldots \ ,\Psi_n(t)\}$ corresponding to $u(t)$ and $x(t)$ (see(12)) such that:

1-/ For all t, $t' \leqslant t \leqslant t''$, the function $H(x(t),\Psi(t),u(t))$ of the variable $u(t) \in U$ attains its maximum at the point

$u(t) = u^*(t)$:

$$H(x(t),\Psi(t),u(t)) = H(x(t),\Psi(t),u^*(t)) \equiv M(x(t),\Psi(t)) \; ; \qquad (13)$$

2-/ At the terminal time the relation

$$M(x(t''),\Psi(t'')) \geqslant 0 \qquad (14)$$

is satisfied. Furthermore, it turns out that if $\Psi(t)$, $x(t)$, and $u(t)$ satisfy system (11), (12), and condition 1-/, the time function $M(x(t),\Psi(t))$ is constant. Thus, (14) may be verified at any time t, $t' \leqslant t \leqslant t''$, and not just at $t = t''$

§ 1.6 TRANSVERSALITY CONDITIONS

Let us now consider a modification of the control problem stated previously whereby it is no longer required that the final phase coordinate $x(t'')$ equal x'', a specific point, but merely that the point $x(t'')$ fall on a manifold M. We can consider this manifold M as a set of many possible final endpoints. Taking an individual point, we can define an optimal control problem for it as previously stated (i.e. with a fixed endpoint)and obtain the corresponding solution and the corresponding value for the cost functional Therefore we require additional relationships in the new type of problem, which will determine which of the many possible final endpoints on M is optimal with respect to the others.

These additional relationships are known as the transversality conditions.

THEOREM 3

Let $u(t)$, $t' \leqslant t \leqslant t''$, be an admissible control which transfers the phase point from the position $x(t') = x'$ to the position $x(t'') \in M$, and let $\tilde{x}(t)$ be the corresponding trajectory (starting at the point $\tilde{x}' \equiv (0,x')$). In order that $u(t)$ and $\tilde{x}(t)$ yield the solution of the optimal problem with a variable final endpoint, it is necessary that there exist a nonzero continuous vector function $\Psi(t)$, which satisfies the conditions of theorem 1, and in addition, the transversality condition at the endpoint on S of the trajectory $\tilde{x}(t)$.

The formulation of the transversality condition is as such :

Let $x'' \in M$ be a certain point, and let T be a plane tangent to M at the point x''. The plane T is in X, and has the dimension of the manifold M. Furthermore, let $u(t)$, $\tilde{x}(t)$, $t' \leqslant t \leqslant t''$, be the solution of the optimal control problem with the fixed final endpoint x''. Finally, let $\Psi(t)$ be the vector whose existence is asserted in theorem 3. We shall say that the vector $\Psi(t)$ satisfies the transversality condition at the right-hand endpoint of the trajectory $\tilde{x}(t)$, (i.e. at the point $\tilde{x}(t")$ if the vector $\Psi(t") = \{\Psi_1(t"), \Psi_2(t"), \ldots, \Psi_n(t")\}$ is orthogonal to T. In other words, the transversality condition signifies that $(\Psi(t")\cdot\Theta) = 0$ for every vector $\Theta \equiv \{\Theta_1, \Theta_2, \ldots, \Theta_n\}$ belonging (or parallel) to T.

We will now consider a specific application. Consider the case where all the desired final values for $x(t")$ are specified except for one coordinate $x_i(t")$ which is left free such that $x_i(t") \in E'$. Then, application of the transversality condition in this case yields a requirement that $\Psi_i(t") = 0$. More generally, when several (or all) of the phase coordinates $x_i(t")$ at the final time are left free, the transversality condition requires that the corresponding variables $\Psi_i(t)$ at the final time t" equal zero ($\Psi_i(t") = 0$).

CHAPTER 2

THE MICROECONOMIC SYSTEM

§ 2.1 INTRODUCTION

In this chapter we shall present a micro-economic model for a firm in a competitive industry producing a good which is not perfectly substitutable. Five dynamic behavioral equations represent the "facts of life" for this firm without containing any implied management policy. Such policies will be executed through the 3 control variables which appear exogenously to the basic model but will be obtained by solving the optimal control problem.

§ 2.2 THE VARIABLES

$x_1(t)$: Capital Stock

$x_2(t)$: Labour Stock

$x_3(t)$: Inventory Stock

$x_4(t)$: Accumulated Production

$x_5(t)$: Accumulated Sales

$u_1(t)$: Gross Capital Investment

$u_2(t)$: Labour Turnover Rate

$u_3(t)$: Sales (asking) Price

$p(t)$: Externally determined market-average price

$r(t)$: Externally determined interest rate

§ 2.3 CAPITAL EQUATION

Consider the equation

$$\frac{d\, x_1(t)}{dt} = -\alpha_1 x_1(t) + u_1(t)x_1(t) \qquad , \alpha_1 > 0$$

The first term on the right hand side represents a depreciation at a proportion of α_1 of the total capital per unit time. $u_1(t)$ is the proportionate gross capital investment rate with similar units to that of α_1. We assume that gross capital investment is limited to a certain rate ($u_1(t) \leqslant u_1 max$) and that gross capital disinvestment is only possible through depreciation ($u_1(t) \geqslant 0$).

§ 2.4 LABOUR EQUATION

Consider the equation

$$\frac{d\, x_2(t)}{dt} = u_2(t)x_2(t)$$

$u_2(t)$ is the proportional hiring ($u_2 > 0$) and firing ($u_2 < 0$) rate. Both rates are considered limited by external factors (e.g. logistics, labour,unions, etc.) so that $u_2 min \leqslant u_2 \leqslant u_2 max$ where $u_2 min < 0$ and $u_2 max > 0$

§ 2.5 PRODUCTION EQUATION

Three basic factors of production are materials, capital and labour. Let us assume that materials are freely available at a certain price and will not restrict production; therefore the production function will depend only on the other factors. Let us consider that for the most efficient type of production there exists an ideal "capital utilized" to "labour utilized" ratio; once we have attained this condition increasing only one of the factors should result in a decreasing marginal productivity.

For the purpose of this work we make the harshest assumption: increasing the amount of one of the factors above the amount corresponding to the ideal ratio has no effect on production; the extra amount is useless. One possible production function is then:

$$\frac{d\,x_4(t)}{dt} = \alpha_2 (\, x_1(t) \,\Lambda\, \lambda x_2(t)\,)^{\alpha_3}$$

$$\alpha_2 > 0$$

$$\alpha_3 > 1.$$

$$\lambda > 0$$

where $\frac{d\,x_4(t)}{dt}$ is the production rate,

where λ is the ideal $\frac{x_1}{x_2}$ ratio

and the operator Λ is defined as such:

$$a \,\Lambda\, b \equiv \begin{cases} a \text{ if } a \leqslant b \\ b \text{ if } b < a \end{cases}$$

The term $(x_1(t) \,\Lambda\, \lambda x_2(t))$ is raised to the power α_3 where $\alpha_3 > 1$. This implies a production function with increasing returns to scale, consistent with the possible increased efficiency of a larger operation. α_2 is a scaling constant. As will be shown later, a proper choice of scaling for $x_1(t)$ and $x_2(t)$ will allow us to set $\lambda = 1$.

Of course many other forms of production function are possible: this one was chosen in part to illustrate the solution of a problem with discontinuities

§ 2.6 SALES EQUATION

$u_3(t)$ is that control variable representing the firm's sales price. Our equation for the sales rate should reflect the fact that the lower the price (compared to p(t), the market-average price) the greater will be the sales rate; furthermore, the lower the price, the greater will be the marginal rate of increase in the sales rate. One possible form for the equation is then:

$$\frac{d\, x_5(t)}{dt} = \alpha_4 \left(\frac{p(t)}{u_3(t)} - \alpha_5 \right) \qquad \begin{matrix} \alpha_4 > 0 \\ \alpha_5 > 0 \end{matrix} \qquad u_3(t) > 0$$

which yields a demand curve as shown in fig. 1

We will assume a limited range for $u_3(t)$ such that $u_3\text{min} \leqslant u_3(t) \leqslant u_3\text{max}$. α_4 and α_5 are parameters which determine the shape of the curve; α_5 should be chosen so that negative sales are impossible.

It should be noted that it is possible for the sales rate to exceed the production rate: in this case,

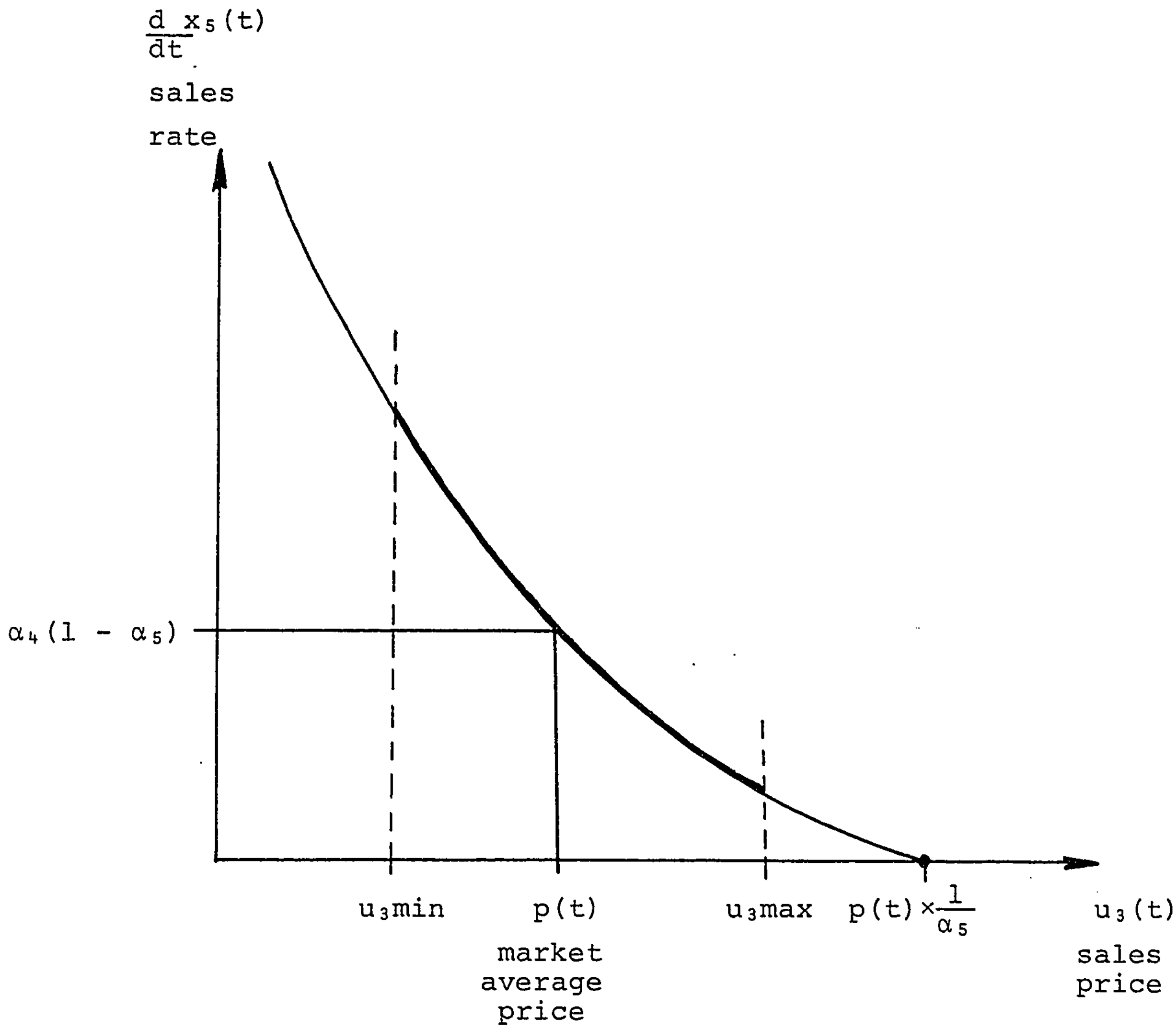

Fig. 1 ASSUMED DEMAND CURVE
FOR SALES RATE EQUATION

the demand is met by depleting the inventory stock. We shall assume that a negative inventory, corresponding to undelivered orders, is acceptable on a short term basis.

§ 2.7 INVENTORY EQUATION

Rate of change in the inventory is the production rate minus the sales rate.

$$\frac{d\,x_3(t)}{dt} = \frac{d\,x_4(t)}{dt} - \frac{d\,x_5(t)}{dt}$$

§ 2.8 THE FINAL MODEL

$$\frac{d\,x_1(t)}{dt} = (u_1(t) - \alpha_1)x_1(t) \qquad \equiv f_1$$

$$\frac{d\,x_2(t)}{dt} = u_2(t)\cdot x_2(t) \qquad \equiv f_2$$

$$\frac{d\,x_3(t)}{dt} = \alpha_2(x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3} - \alpha_4\left(\frac{p(t)}{u_3(t)} - \alpha_5\right) \qquad \equiv f_3$$

$$\frac{d\,x_4(t)}{dt} = \alpha_2(x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3} \qquad \equiv f_4$$

$$\frac{d\,x_5(t)}{dt} = \alpha_4\left(\frac{p(t)}{u_3(t)} - \alpha_5\right) \qquad \equiv f_5$$

CHAPTER 3

FORMULATION OF THE OPTIMAL CONTROL PROBLEM

§ 3.1 THE FIXED TIME OPTIMAL CONTROL PROBLEM

The nature of any optimal control problem is determined by the control objective (i.e. cost functional). For a business firm a very obvious objective is to maximize net profit over a fixed time period. Let us now consider what are the flows of income and expenditure for our firm.

§ 3.2 INCOME RATE

We assume that the only income is from sales; hence it is equal to sales rate multiplied by the sales price:

$$\beta_0 u_3(t) \frac{d\ x_5(t)}{dt}$$

β_0 is a scaling constant for monetary units.

§ 3.3 EXPENDITURE RATES

We assume that the cost of maintainance for capital is accounted for in the depreciation rate, and hence in the gross capital investment $u_1(t) \cdot x_1(t)$.

The cost of labour is proportional to the amount employed: $\beta_1 x_2(t)$.

We assume the cost of maintaining the inventory is proportional to the inventory stock: $\beta_2 x_3(t)$.

We assume the cost of materials is proportional to the amount used which is reflected in the output rate:

$$\beta_3 \frac{d\ x_4(t)}{dt}$$

We shall neglect the cost of distribution and sales.

Therefore the net rate of income at any time t is:

$$i(t) = \beta_0 u_3(t) \frac{d}{dt} x_5(t) - \beta_1 x_2(t) - \beta_2 x_3(t) - \beta_3 \frac{d}{dt} x_4(t) - u_1(t) x_1(t)$$

We might therefore express the functional to be maximized as

$$\int_{t'}^{t''} i(t)\, dt$$

But if we consider that our goal is to have accumulated the maximum amount of net profit at time t", then we must give an extra weighting to those profits realized earlier on since they are collecting interest (we assume) over the time period. Introducing the appropriate weighting factor, the income functional now becomes :

$$\int_{t'}^{t''} e^{r(t''-t)}\, i(t)\, dt$$

where e is the number 2.718...

$$= \int_{t'}^{t''} e^{r(t''-t)} \left\{ \beta_0 u_3(t) \frac{d}{dt} x_5(t) - \beta_1 x_2(t) - \beta_2 x_3(t) - \beta_3 \frac{d}{dt} x_4(t) - u_1(t) x_1(t) \right\} dt$$

In chapter 1 are presented theorems dealing with the minimization of a certain functional; since minimization in this case meant to make as negative (or less positive) as possible, maximization of a functional can be acheived by changing the sign of the functional to be maximized and applying the minimization techniques developed earlier.

In solving a numerical problem we shall specify the initial conditions: $x_1(t') = x_1'$, $x_2(t') = x_2'$,

$x_3(t') = x_3'$, $x_4(t') = x_4'$, $x_5(t') = x_5'$.

The terminal conditions are specified only for some of the phase coordinates, as consistent with practical considerations of the problem being solved. In our case we shall specify terminal values for the amount of capital, labour and inventory stock. So, the final state is expressed as being on the manifold

$$M \equiv \{ x_1(t'') = x_1'',\ x_2(t'') = x_2'',\ x_3(t'') = x_3'',\ x_4(t'') > 0,\ x_5(t'') > 0 \}$$

§ 3.4 FORMULATION OF THE PROBLEM

The control problem can now be formally stated. Subject to the model of the system given previously (including control constraints), find the admissible controls which will transfer the system from the specified initial state to the desired final state (on manifold M) at the final time t" while maximizing the cost functional (maximizing profit).

As stated previously, the argument of the functional was

$$e^{r(t''-t)} \left(\beta_0 u_3(t) \frac{d}{dt} x_5(t) - \beta_1 x_2(t) - \beta_2 x_3(t) - \beta_3 \frac{d}{dt} x_4(t) - u_1(t) x_1(t) \right)$$

But it was found during numerical computations that the maximum (i.e. most positive) value for the functional was obtained by attributing large negative values to the variables $x_1(t)$ and $x_2(t)$, capital stock and labour stock. Of course negative values for these variables are physically unrealistic. Methods exist for solving an optimal control problem containing constraints on the state variables; however, it was judged simpler to modify the cost functional so as to penalize negative values not consistent with the practical considerations of the problem being solved. The resulting argument for the income functional is then

$$e^{r(t''-t)} \left(\beta_0 u_3(t) \frac{d}{dt} x_5(t) - \beta_1 |x_2(t)| - \beta_2 x_3(t) - \beta_3 \left| \frac{d}{dt} x_4(t) \right| - u_1(t) |x_1(t)| \right)$$

The absolute value terms ensure that negative values of $x_1(t)$ and $x_2(t)$ no longer make the functional more positive.

The Hamiltonian H therefore becomes:

$$H = e^{r(t''-t)} \{ \beta_0 u_3(t) \frac{d\,x_5}{dt}(t) - \beta_1 |x_2(t)| - \beta_2 x_3(t) - \beta_3 \left| \frac{d\,x_4}{dt}(t) \right| - u_1(t)|x_1(t)| \}$$

$$+ \Psi_1(t)\{ u_1(t)x_1(t) - \alpha_1 x_1(t) \}$$

$$+ \Psi_2(t)\{ u_2(t)x_2(t) \}$$

$$+ \Psi_3(t)\{ \alpha_2(x_1(t) \wedge \lambda x_2(t))^{\alpha_3} - \alpha_4(\frac{p(t)}{u_3(t)} - \alpha_5) \}$$

$$+ \Psi_4(t)\{ \alpha_2(x_1(t) \wedge \lambda x_2(t))^{\alpha_3} \}$$

$$+ \Psi_5(t)\{ \alpha_4(\frac{p(t)}{u_3(t)} - \alpha_5) \}$$

Maximizing the Hamiltonian H with respect to the controls, we obtain the form of the optimal controls.

$$u_1^*(t) = u_1 min + (u_1 max - u_1 min) S\{ \Psi_1(t)x_1(t) - e^{r(t''-t)}|x_1(t)| \}$$

where the function S{x} is defined as such:

$$S\{x\} \equiv \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

$$u_2^*(t) = u_2 min + (u_2 max - u_2 min) S\{ \Psi_2(t)x_2(t) \}$$

$$u_3^*(t) = \begin{cases} u_3 min & \text{if } \omega \geq 0 \\ \sqrt{\frac{\omega}{\psi}} \text{ or the nearest possible value such that } u_3 min \leq u_3(t) \leq u_3 max & \text{if } \omega < 0 \\ \text{where } \quad \omega \equiv \alpha_4 p(t)\cdot(\Psi_5(t) - \Psi_3(t)) & \\ \text{and } \quad \psi \equiv - \beta_0 \alpha_4 \alpha_5 e^{r(t''-t)} & \end{cases}$$

The adjoint equations are:

$$\frac{d\ \Psi_1(t)}{dt} = \{\ \beta_3 e^{r(t''-t)} \operatorname{sgn}(\alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}) - \Psi_3(t) \ldots$$

$$- \Psi_4(t)\ \} \times \frac{\partial}{\partial x_1(t)} \{\ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}\ \}$$

$$+\ e^{r(t''-t)} u_1(t) \operatorname{sgn}(x_1(t)) - \Psi_1(t)(u_1(t) - \alpha_1)$$

$$\frac{d\ \Psi_2(t)}{dt} = \{\ \beta_3 e^{r(t''-t)} \operatorname{sgn}(\alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}) - \Psi_3(t) \ldots$$

$$- \Psi_4(t)\ \} \times \frac{\partial}{\partial x_2(t)} \{\ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}\ \}$$

$$+\ e^{r(t''-t)} \beta_1 \operatorname{sgn}(x_2(t)) - \Psi_2(t) u_2(t)$$

$$\frac{d\ \Psi_3(t)}{dt} = \beta_2 e^{r(t''-t)}$$

$$\frac{d\ \Psi_4(t)}{dt} = 0$$

$$\frac{d\ \Psi_5(t)}{dt} = 0$$

where

$$\frac{\partial}{\partial x_1(t)} \{\ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}\ \}$$

$$= \begin{cases} \alpha_2 \alpha_3 (x_1(t))^{\alpha_3 - 1} & \text{if } x_1(t) < \lambda x_2(t) \\ 0 & \text{if } x_1(t) > \lambda x_2(t) \end{cases}$$

$$\frac{\partial}{\partial x_2(t)} \{\ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3}\ \} \equiv$$

$$= \begin{cases} \alpha_2 \alpha_3 \lambda (x_2(t))^{\alpha_3 - 1} & \text{if } x_1(t) > \lambda x_2(t) \\ 0 & \text{if } x_1(t) < \lambda x_2(t) \end{cases}$$

and where outside the switching boundaries

$$\frac{\partial}{\partial x_1(t)} \operatorname{sgn}(\alpha_2 (x_1(t) \wedge \lambda x_2(t))^{\alpha_3}) = 0 ,$$

$$\frac{\partial}{\partial x_2(t)} \operatorname{sgn}(\alpha_2 (x_1(t) \wedge \lambda x_2(t))^{\alpha_3}) = 0 ,$$

$$\frac{\partial}{\partial x_1(t)} \operatorname{sgn}(x_1(t)) = 0 \quad \text{and}$$

$$\frac{\partial}{\partial x_2(t)} \operatorname{sgn}(x_2(t)) = 0 .$$

It should be noted that the terms $x_1(t)$, $x_2(t)$ and $\alpha_2(x_1(t) \wedge \lambda x_2(t))^{\alpha_3}$ will initially be set to positive values. If the values of $x_1(t)$ and $x_2(t)$ remain positive throughtout the duration of the solution period ($t' < t < t''$) then the switching boundaries are never encountered.

By applying the transversality conditions we obtain the end conditions for the adjoint variables:

$$\Psi_4(t'') = \Psi_5(t'') = 0 ,$$

$\Psi_1(t'')$, $\Psi_2(t'')$ and $\Psi_3(t'')$ are left free.

We now have a system of 10 differential equations describing the dynamics of the state and costate variables. We also know the required form of the control variables, for optimizing the cost functional, in terms of the costate variables. To know the values of the control variables consistent with the optimal control, we must know the trajectories of the costate variables during the solution time period. The equations must be integrated forward in time toghether, but we only have initial values specified for the state variables; since there are terminal conditions to be satisfied for some of the state and costate variables the resulting problem is called a two-point boundary value problem. Once this two-point boundary value problem has been solved, the optimal controls over the time period are known, and of course the trajectories of of the state variables.

§ 3.5 THE TIME OPTIMAL PROBLEM

Another type of problem encountered in practice is called the time optimal problem.

Consider the manufacturing firm as described previously by the state equations. It may be desired to reach a certain goal within the shortest possible time. For example, such a goal might be a certain value of gained capital, labour and inventory stock representing a desired expansion of the firm within the shortest possible time. The dynamics of the controlled system have not changed; hence the only change appears in the cost functional. Since we wish to minimize the transition time, the functional to be minimized is

$$\int_{t'}^{t''} 1 \, dt$$

where the value of t" is left free and is the factor to be minimized.

The terminal state which must be reached at time t" will be specified as being on the manifold

$$M \equiv \{ x_1(t'') = x_1'',\ x_2(t'') = x_2'',\ x_3(t'') = x_3'',\ x_4(t'') > 0,\ x_5(t'') > 0 \}$$

where we have specified the final values of capital stock and inventory stock. The Hamiltonian function H is now

$$\begin{aligned} H = & -1 + \Psi_1(t)\ (u_1(t)x_1(t) - \alpha_1 x_1(t)) \\ & + \Psi_2(t)\ u_2(t)\ x_2(t) \\ & + \Psi_3(t)\ \{ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3} - \alpha_4 (\frac{p(t)}{u_3(t)} - \alpha_5) \} \\ & + \Psi_4(t)\ \{ \alpha_2 (x_1(t)\ \Lambda\ \lambda x_2(t))^{\alpha_3} \} \\ & + \Psi_5(t)\ \{ \alpha_4 (\frac{p(t)}{u_3(t)} - \alpha_5) \} \end{aligned}$$

Maximizing the Hamiltonian with respect to the control variables, we obtain the required form for the controls:

$$u_1(t) = u_1\min + (u_1\max - u_1\min)\ S\{ \Psi_1(t)x_1(t) \}$$

$$u_2(t) = u_2\min + (u_2\max - u_2\min)\ S\{ \Psi_2(t)x_2(t) \}$$

$$u_3(t) = u_3\min + (u_3\max - u_3\min)\ S\{(\Psi_3(t) - \Psi_5(t))\alpha_4 p(t)\}$$

The adjoint equations are

$$\frac{d\,\Psi_1(t)}{dt} = \{ -\Psi_3(t) - \Psi_4(t) \} \frac{\partial}{\partial x_1(t)} (\alpha_2 (x_1(t) \wedge \lambda x_2(t))^{\alpha_3})$$

$$- \Psi_1(t)(u_1(t) - \alpha_1)$$

$$\frac{d\,\Psi_2(t)}{dt} = \{ -\Psi_3(t) - \Psi_4(t) \} \frac{\partial}{\partial x_2(t)} (\alpha_2 (x_1(t) \wedge \lambda x_2(t))^{\alpha_3})$$

$$- \Psi_2(t) u_2(t)$$

$$\frac{d\,\Psi_3(t)}{dt} = 0$$

$$\frac{d\,\Psi_4(t)}{dt} = 0$$

$$\frac{d\,\Psi_5(t)}{dt} = 0$$

By applying the transversality conditions we obtain the end conditions for the adjoint variables:

$$\Psi_4(t'') = \Psi_5(t'') = 0$$

$\Psi_1(t'')$, $\Psi_2(t'')$ and $\Psi_3(t'')$ are left free.

Again, as in the fixed time problem, we have a system of 10 differential equations with some of the initial conditions and some of the final conditions specified. We will solve this two-point boundary value problem to obtain the optimizing trajectories of the control variables, and hence the trajectories of the state variables in the optimal case.

CHAPTER 4

EXAMPLES OF MODELLING AND NUMERICAL SOLUTIONS

§ 4.1 MODELLING EXAMPLE

Let us consider the example of a manufacturing firm with 10 employees and capital ressources valued at $200,000.00 which is producing 50,000 items per month; the optimal capital-labour ratio was attained. ratio is in effect. Materials cost is 50¢ per item; the cost of keeping an inventory is 2¢ per item per month; the market-average price is $1.15 per unit and the interest rate for all purposes is 12% (1% per month). There is an initial inventory of 50,000 items.

We must now decide on the scaling of the variables in our model; our basic unit of time is 1 month and other basic units are chosen as 50,000 items = 1 item unit and $200,000.00 = 1 capital unit. If we set 10 employees = 1 labour unit then the optimal capital-labour ratio, λ, is equal to 1. The monthly depreciation rate for capital is 1% therefore $\alpha_1 = .01$.

Because of the scaling chosen, the initial values of $x_1(t)$ and $x_2(t)$ are equal to 1. Since the initial production rate is 1 item unit (50,000 items) per month, and we have the equation

$$\frac{d\, x_4(t)}{dt} = \alpha_2 (x_1(t) \wedge \lambda x_2(t))^{\alpha_3} ,$$

then it is required that $\alpha_2 = 1$ and α_3 is left free. We shall arbitrarily choose $\alpha_3 = 1.5$.

If we decide that the demand curve for the product is such that sales drop to nothing when the sales price is twice the market-average price, this implies that $\alpha_5 = .5$

If the curve is such that the sales rate is 1 unit per month when the sales price is set equal to the market-average, this implies $\alpha_4 = 2$.

Now we will establish the parameters of the cost functional. Since sales of 1 item unit at a unit sales price (\$1.00 / item) produce a revenue of .25 money units where we have set 1 money unit $\equiv$ \$200,000.00 . Hence we set $\beta_0 = .25$.

One labour unit implies a payroll of \$10,000.00 per month, hence $\beta_1 = .05$.

Keeping 1 item unit in inventory costs \$1,000.00 per month, so $\beta_2 = .005$.

Producing 1 item unit costs \$25,000.00 in materials, so $\beta_3 = .125$.

Since the units of capital and the money units are equal (1 unit = \$200,000.00), no scaling constant is required for the $u_1(t)x_1(t)$ term.

§ 4.2 NUMERICAL EXAMPLES

We shall give an example of the solution of both the fixed time and time optimal problem. For the model specified in Chapter 2 we shall set the parameter values at $\alpha_1 = .01$

$\alpha_2 = 1.0$

$\alpha_3 = 1.5$

$\alpha_4 = 2.0$

$\alpha_5 = .5$ as determined previously.

We shall set $u_1 min = 0$ $u_1 max = .04$ (4% increase per month)

$u_2 min = -.1$ $u_2 max = .05$

$u_3 min = .5$ $u_3 max = 2.$

Numerically, it is easy to find the solution of a two point boundary value problem (TPBVP) if the equations involved are linear; one assumes certain values for the unknown initial conditions, integrates the equations forward in time,and by analysing the final conditions acheived, it is possible to calculate the values of the unknown initial conditions (IC's) which will yield the desired final conditions. If the calculation of a solution proves impossible, this constitutes a proof that no solution exists.

Solution of a TPBVP with non-linear differential equations is more difficult. Many methods now used are shooting methods whereby the initial conditions are guessed, the equations integrated forward and the resulting final conditions, through an analysis algorithm, are used to formulate new guesses for the initial conditions; hopefully, the method converges after a certain number of iterations. The analysis algorithm can be based on one of various linearizing assumptions or steepest descent methods, educated guessing, or another method. There are also many other types of solution methods such as integrating from guessed conditions at both the initial and final conditions, the continuation method, or embedding techniques. In general it is very difficult (usually impossible) to determine whether there exists a solution, many solutions, or no solution. For iterative methods it is usually impossible to prove whether the method will converge or if the point of solution is a global (rather than local) extremum; in fact the convergence and speed of convergence may depend not only on the form of the problem in question, but also on the actual numbers involved.

Since one of our system equations is discontinuous on the right hand side, the relation between the initial and the final conditions may be discontinuous; in attempting the numerical solution it was found to be so. A great deal of time was spent attempting to solve the problem with various shooting methods; the mechanized methods failed to converge and required large amounts of computing time so finally the solution was obtained by an "educated guessing" method; existing automatic methods could have been "optimized" to give the solution with proper initial guesses, but the method would then be specific to the problem and of no special value for other problems.

§ 4.3 THE FIXED TIME PROBLEM

The values of the parameters in the cost functional are:

$\beta_0 = .25 \qquad \beta_1 = .05$

$\beta_2 = .005 \qquad \beta_3 = .125$

The initial values of the state variables are:

$x_1(t') = 1. \qquad x_2(t') = 1.$

$x_3(t') = 1. \qquad x_4(t') = 0.$

$x_5(t') \quad 0.$

where we have initially chosen unit values for capital, labour and inventory stock.

We must now decide on terminal conditions for the state variables. But since we have not formally studied the attainability of endpoints with the given control constraints we can not be certain that arbitrarily chosen values for the endpoints would be attainable; for example, it is easy to see that a negative value for $x_1(t'')$ (an illogical choice anyway) is unattainable.

One means of obtaining proper values for the endpoints is to arbitrarily assume fixed values (or other arbitrary trajectories) for the control variables which are in the region of admissible controls, substitute in the state equations and integrate forward in time from the given initial conditions. The resulting final values are known to be attainable with the given control constraints. In the worst case, this final point may be uniquely attainable, and in this case the optimal control as found by the solution of the Maximum Principle may simply mimic the behaviour of the previous arbitrarily chosen controls. Otherwise, solving the optimal control problem may yield a different control strategy which arrives at the same final endpoint but for which the value of the cost functional is more optimal.

For our example we chose to set

$u_1(t) = .01$

$u_2(t) = 0.$

$u_3(t) = 1.15$

The resulting final conditions were:

$x_1(t'') = 1.$

$x_2(t'') = 1.82113$

$x_3(t'') = 1.$

The cost functional was calculated for later comparison and was

$J(u) = 1.012727$

So, for our TPBVP the specified terminal conditions are

$x_1(t'') = 1. \qquad x_2(t'') = 1.82113 \qquad x_3(t'') = 1.$

$\Psi_4(t'') = 0 \qquad \Psi_5(t'') = 0$

The problem will be solved over a time period of 12 months, so we set $t' = 0$ and $t'' = 12$.

The TPBVB was solved numerically and the results are as such:

Optimizing initial conditions

$\Psi_1(t')$	$\Psi_2(t')$	$\Psi_3(t')$	$\Psi_4(t')$	$\Psi_5(t')$
1.00380	.02512	.11980	0.	0.

Final conditions

	$x_1(t'')$	$x_2(t'')$	$x_3(t'')$	$\Psi_4(t'')$	$\Psi_5(t'')$
desired	1.	1.821129	1.	0.	0.
actual	1.002119	1.821129	1.000088	0.	0.

Value of the cost functional: 1.0405

The trajectories of the state variables and the control variables are given in fig. 2 and fig. 3..

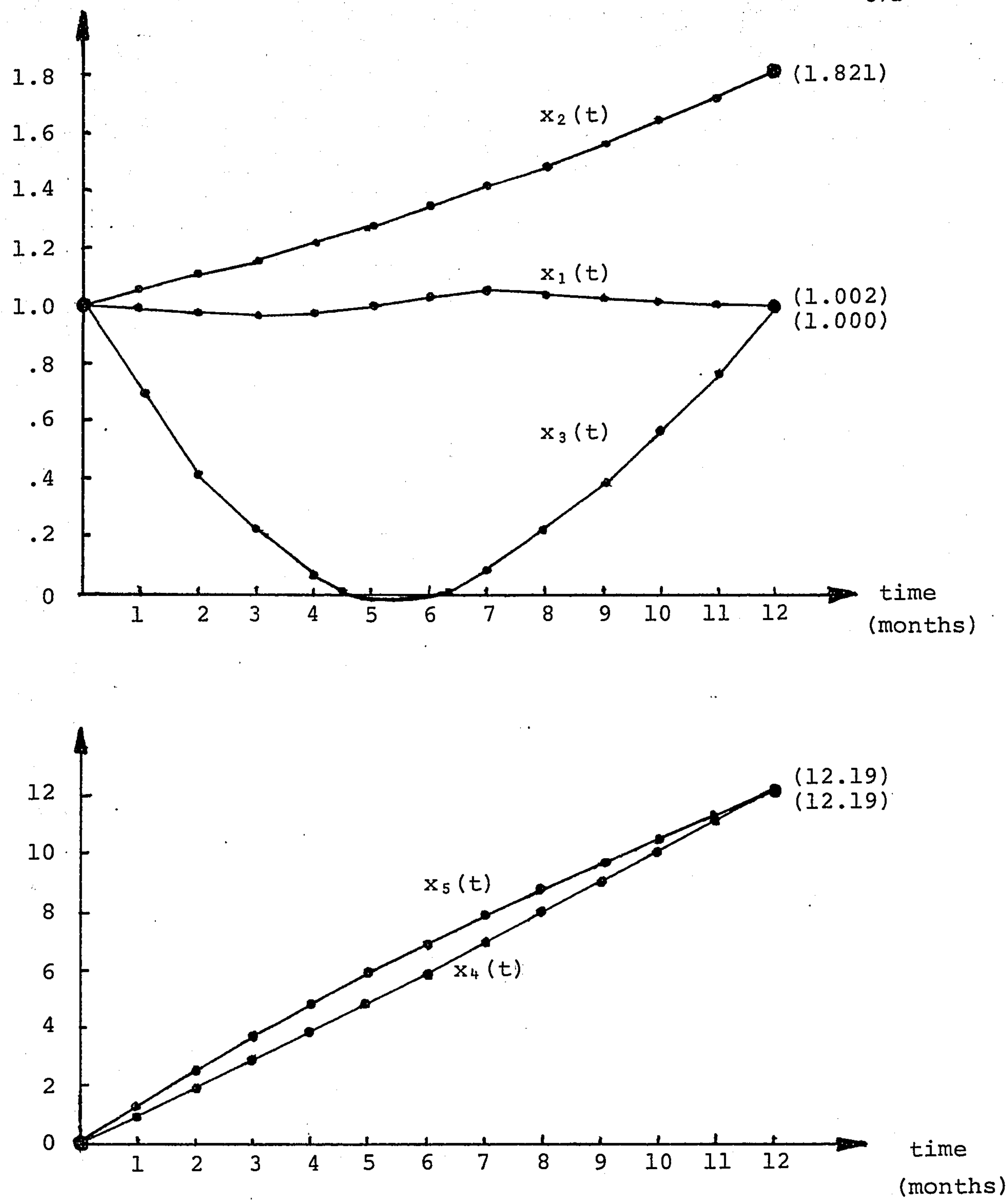

Fig. 2 THE FIXED TIME PROBLEM
STATE VARIABLES

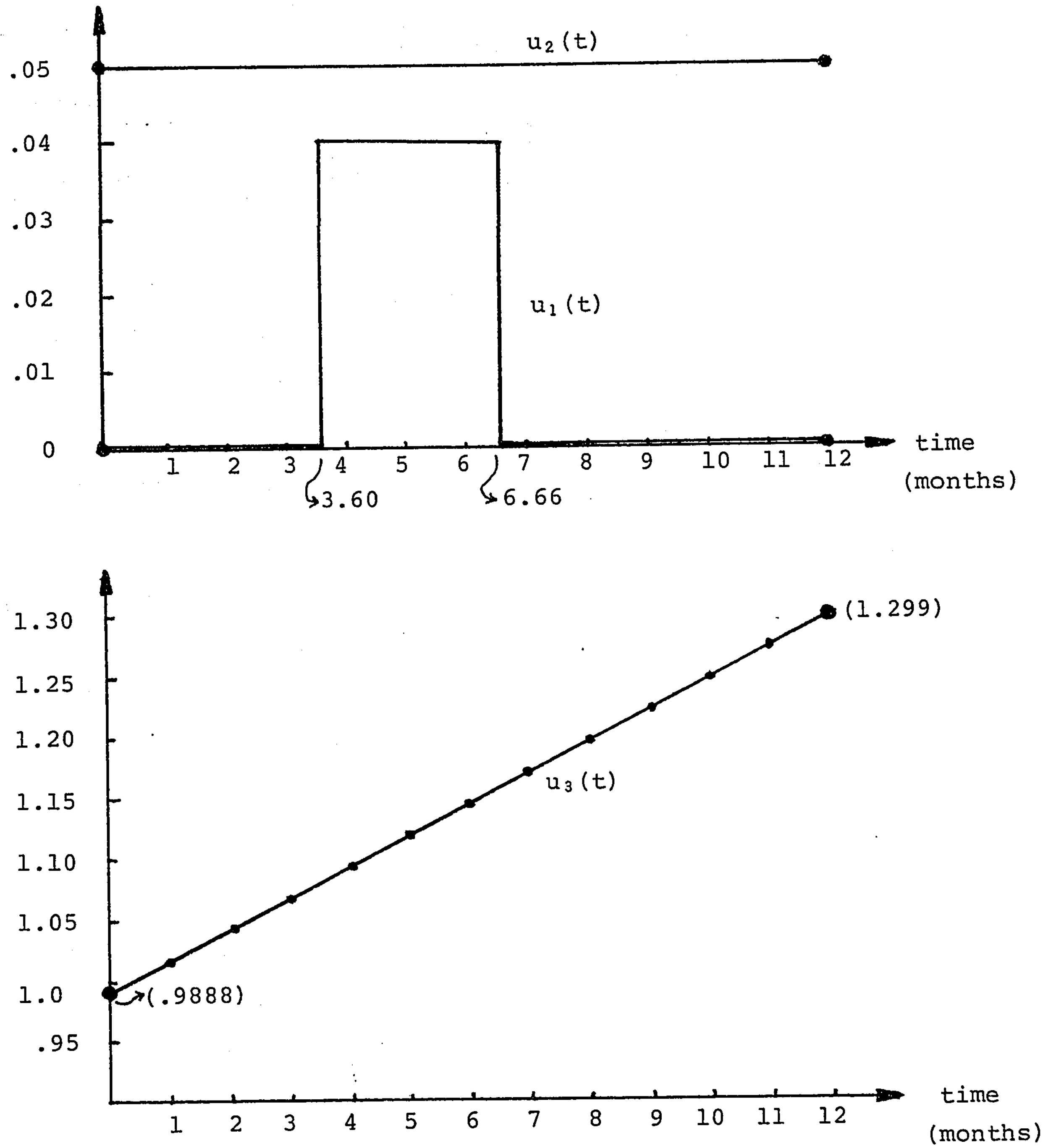

Fig. 3 THE FIXED TIME PROBLEM
CONTROL VARIABLES

§ 4.4 THE TIME OPTIMAL PROBLEM

The initial value of the state variables are the same as in the fixed time case.

It would have been possible to choose the same final values as in the fixed time problem, however since the final values for capital and labour stock are equal to their initial values this is an unrealistic goal; in practice, a firm would likely aim for a certain expansion of capital, so we have arbitrarily chosen $x_1(t")$ and $x_2(t")$ equal to 1.5 instead of 1 . We require that $x_3(t") = -.6701$; the reason for this value will be given later.

We set $t' = 0$ and t" is the factor to be optimized.

In attempting the numerical solution, a problem was soon encountered. Because of the nature of the problem (particularly the discontinuity) the control $u_3(t)$ described by any solution of the TPBVP can only assume one of the two limiting values u_3min or u_3max and cannot change value during the course of the solution. If some numerical solutions are attempted it is seen possible to select almost any negative value for $\Psi_3(t')$ and find values of $\Psi_1(t')$ and $\Psi_2(t')$ such that the final conditions are satisfied; the final value of $x_3(t")$ is obtained because of the specific value of $u_3$min. In general, because $u_3(t)$ has the value of either $u_3$min or $u_3$max, the possible values of $x_3(t")$ has an upper and a lower bound, but cannot reach intermediate values; since there is a singular arc in the solution, this problem is called a singular problem. Although this type of problem can be solved analytically (see ref. 4), for the sake of obtaining the form of the optimal controls we can also arrive at a numerical solution. For example, if we specify a problem with a goal of $x_3(t") = 1.$, a numerical solution can be obtained by assuming that the S(x) function in the term

$$u_3(t) = u_3min + (u_3max - u_3min)\ S\{(\Psi_3(t)-\Psi_5(t))\alpha_4 p(t)\}$$

is not quite discontinuous, but has a certain slope at the switching point. Then we can select a value for $\Psi_3(t')$ (which will not change) such that $u_3(t)$ has an intermediate value between $u_3$min and $u_3$max; corresponding to this value it is possible to find the required values of $\Psi_1(t')$ and $\Psi_2(t')$ such that $x_1(t") = x_2(t") = 1.5$ as required. This process can be repeated until the value of $u_3(t)$ is found which also yields the desired value for $x_3(t")$ (e.g. 1.) as well as the values of $x_1(t")$ and $x_2(t")$.

We see that the only difference in the controls (which is what we wish to obtain in solving the TPBVP) between this solution and a solution where $u_3(t)$ assumes one of the limiting values, is that $u_3(t)$ has a different constant value. So we can now express the solutions to a range of problems by specifying the same form for $u_1(t)$ and $u_2(t)$ and an appropriate value for $u_3(t)$ chosen so that $x_3(t")$ meets its objective; this is possible because in our system of state equations, $x_1(t)$ and $x_2(t)$ are completely independent of the variables $x_3(t)$, $x_4(t)$, $x_5(t)$ and $u_3(t)$. It is obvious that the solution is the time-optimal one since the variable $x_1(t)$ is the limiting factor and climbs at its maximum rate.

For the numerical example of the time-optimal problem we have set

$u_1 min = 0.$ $\quad$ $u_1 max = .04$

$u_2 min = -.05$ $\quad$ $u_2 max = .1$

$u_3 min = 1.$ $\quad$ $u_3 max = 2.$

The desired final conditions are

$x_1(t") = 1.5$ $\quad$ $x_2(t") = 1.5$ $\quad$ $x_3(t") = -.6701$

$\Psi_4(t") = 0.$ $\quad$ $\Psi_5(t") = 0.$

The value of $x_3(t")$ was chosen for simplicity since it represents a solution for one of the limiting values of $u_3(t)$.

The TPBVP was solved numerically and the results are as such:

<u>Optimizing initial conditions</u>

$\Psi_1(t')$	$\Psi_2(t')$	$\Psi_3(t')$	$\Psi_4(t')$	$\Psi_5(t')$
1.23580	-2.60300	-1.10400	0.	0.

<u>Final conditions</u>

	$x_1(t'')$	$x_2(t'')$	$x_3(t'')$	$\Psi_4(t'')$	$\Psi_5(t'')$
desired	1.5	1.5	-.6701	0.	0.
actual	1.49929	1.50015	-.67013	0.	0.

Optimal final time: 13.5155 months.

The trajectories of the state variables and the control variables are given in fig. 4 and fig. 5 .

Another concept illustrated in this example is that while the solution given by the Pontryagin's Maximum Principle is optimal, it is not unique. Figure 6 illustrates one of infinitely many alternative solutions meeting the same final conditions at the same, optimal time.

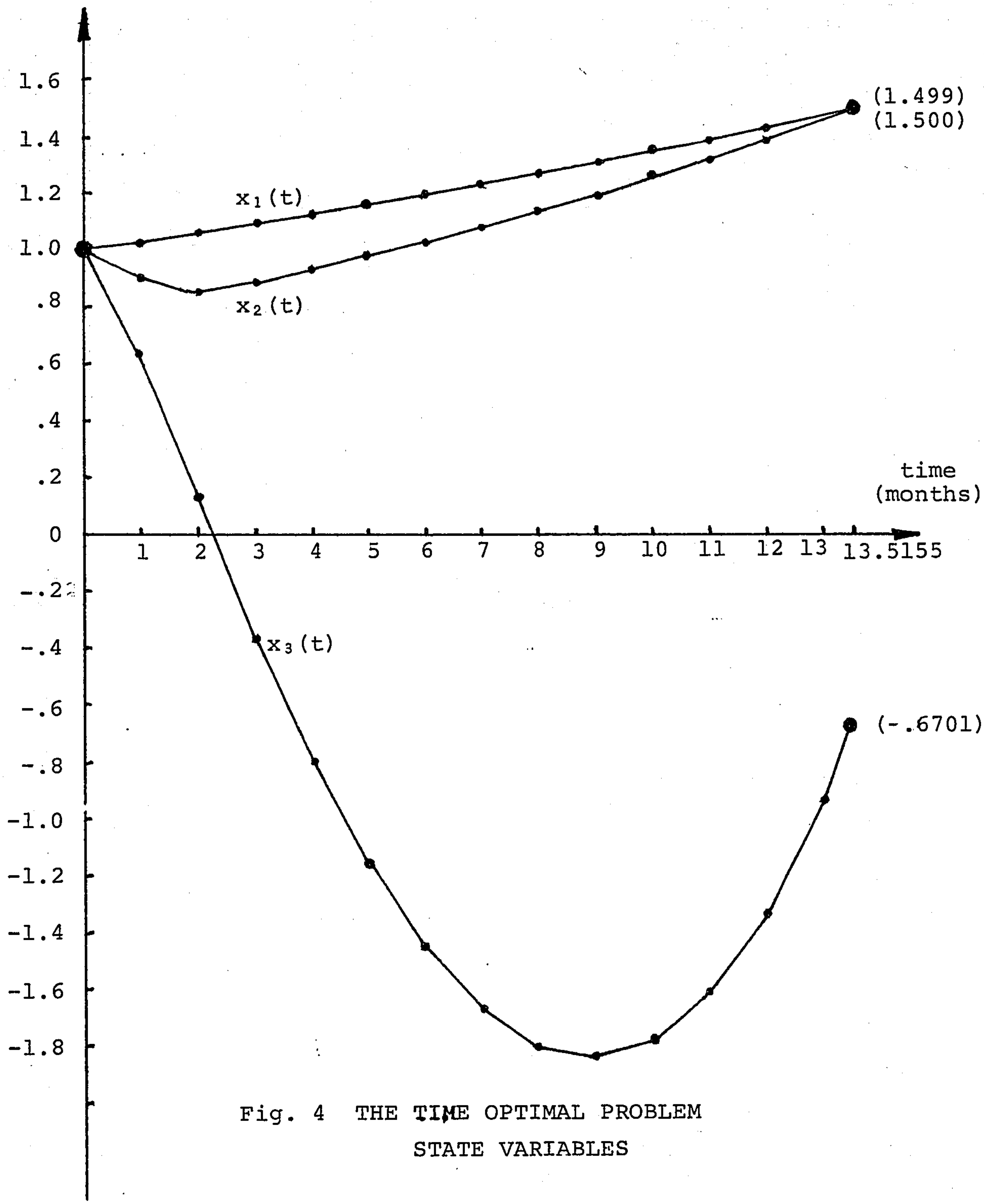

Fig. 4 THE TIME OPTIMAL PROBLEM
STATE VARIABLES

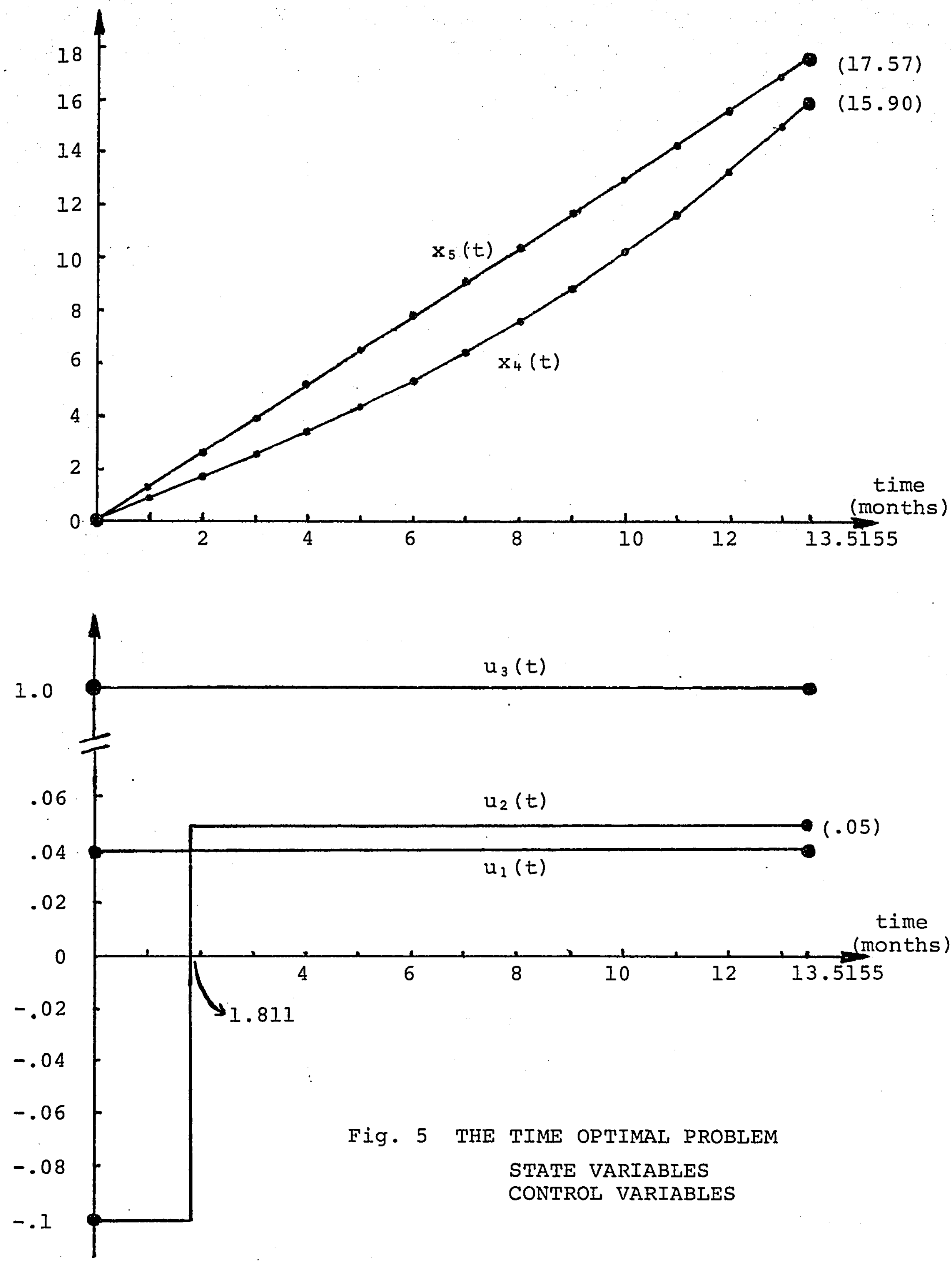

Fig. 5 THE TIME OPTIMAL PROBLEM
STATE VARIABLES
CONTROL VARIABLES

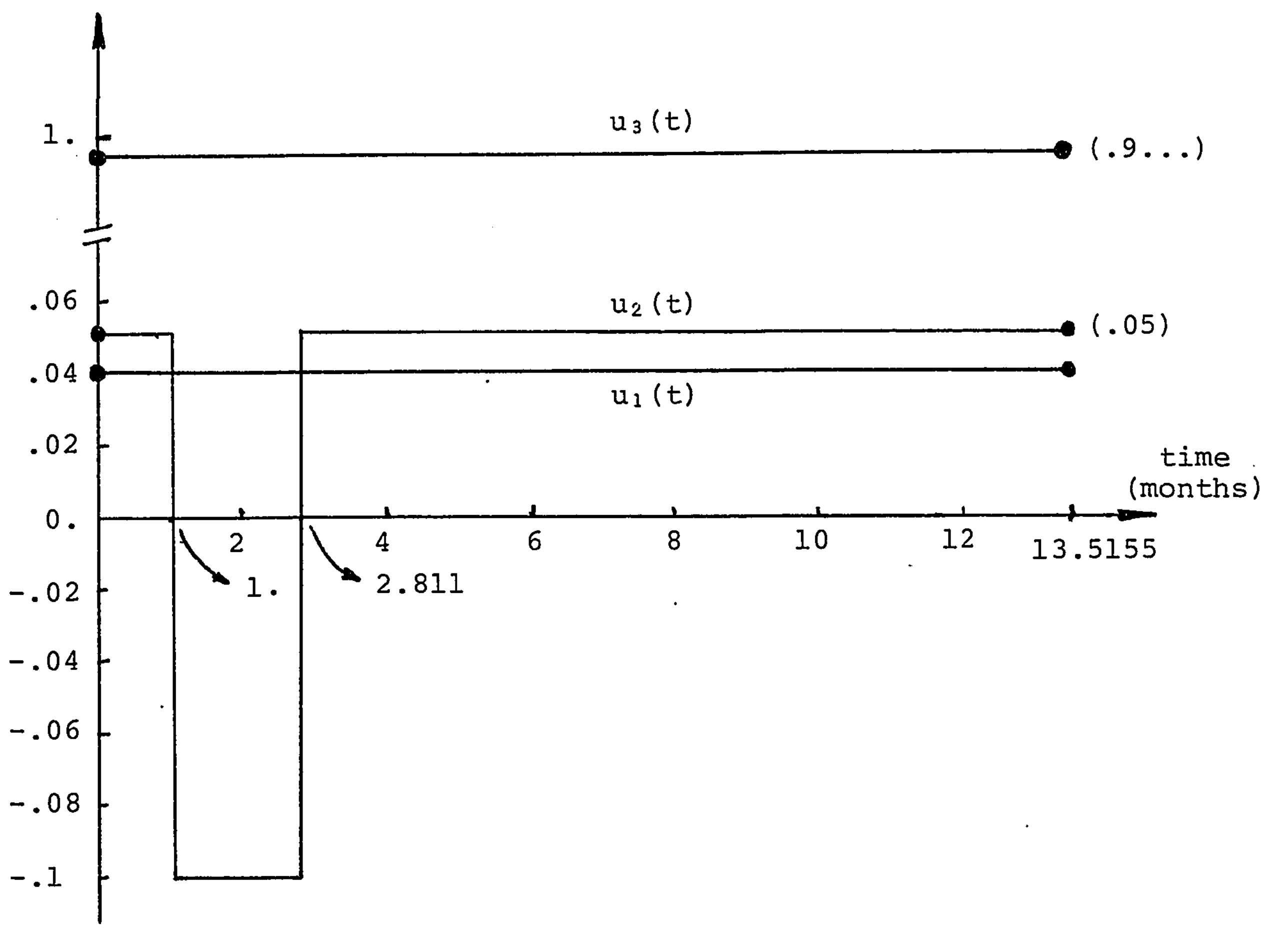

Fig. 6 CONTROL VARIABLES FOR
AN ALTERNATIVE SOLUTION OF
THE TIME OPTIMAL PROBLEM

DISCUSSIONS AND CONCLUSIONS

In this paper it has been attempted to demonstrate the solution of an economic system using modern optimal control theory. It is beleived by the author that most models presently usedin economic planning are, by their nature, of limited value.

Many presently used models employ equations with right hand sides composed of a linear combination of some system variables weighted by coefficients; these coefficients are evaluated by fitting the equations to past data by a statistical analysis. Obviously, no attempt is made to actually reproduce the behavior of the system and so the predictive value of such a model is limited. In an attempt to improve the accuracy of some models, a greater number of variables are used by disaggregating some of the factors (e.g. instead of a variable for consumer consumption, introduce variables for several types of consumption). Depending on the purpose of the model, this may be necessary, but a proliferation of system variables will not guarantee greater accuracy if the structure of the model is not realistic. Predictive models have been formulated with dozens, even hundreds of state variables; these require an impressive amount of computational effort to calculate the parameters (and an index of statistical confidence for each one) yet cannot predict the yearly change (annual models) in the major economic factors with any impressive accuracy.

It is the author's belief that no consistent predictions can be done until the predictive models used simulate the dynamics of the actual system. In general, economic behaviour is controlled by a large number of factors which are complex in nature and often unmeasurable; this is the major obstacle for the devellopment of such models.

But economists have often cited the "difficulty" of obtaining numerical solutions as a reason for avoiding non-linear or even non-static equations. By demonstrating the solution of an "inelegant" system, we wish to point out that a wide variety of problems can be solved by the use of Modern Optimal Control Theory and so, modelling validity can become the principal concern of the economist. The model presented is not meant as a contribution to economic theory but rather a meaningful example; consequently it was shown how a numerical solution can be obtained in certain situations through reasonable assumptions and approximations while an analytical solution would require more involved and rigorous techniques.Again, the economist need not unduly fear mathematical complexities if a numerical solution is sought; however caution must be exercised.

The theorems presented in Chapter 1, although sufficient for the needs of this thesis, are not the most general ones existing in Modern Control Theory. Many more types of problems can be solved with more advanced theorems and progress is continuing in this field.

It should be noted that the conditions set on page 25 concerning the switching boundaries of the partial derivatives are respected in both examples. (i.e. the values of $x_1(t)$ and $x_2(t)$ are always greater than zero)).

It was not possible to choose a "real-life" case for the modelling example because of the simplicity of our model and the difficulty of obtaining data which could be formatted and aggregated to the desired degree.

As noted in the text, the solutions of the TPBVPs were obtained by an iterative method of guessing the unknown initial conditions and observing the resulting final conditions. Even a simple analysis of the system equations and interdependencies (which would be difficult to program into a general mechanized method) speeds up the iterative process considerably. Since the behaviour of the variables was quite smooth, numerical integration was accomplished through a rectangular integration scheme with 1000 steps.

As noted previously, the solution to the time optimal problem was not unique; in fact, the requirements of an optimal solution could be appreciated by a simple analytical analysis of the behaviour of the state variables. In a sense, the application of the Pontryagin's Maximum Principle in this case was trivial; this is not true though, for the fixed-time problem.

REFERENCES

(1) Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V. and Mischenko, E.F., "The Mathematical Theory of Optimal Process", John Wiley & Sons, 1962.

(2) Ahmed, N.U., Jutras, G.J. and Yeo, S.H., "Optimal Control of an Economic System Modelled on the Basis of Cyclical Growth Theory", International Journal of Control, accepted for publication.

(3) Bergstrom, A.R., "Selected Economic Models and Their Analysis", American Elsevier Publ., 1967.

(4) Hermes, H. and Lasalle, J.P., "Functional Analysis and Time Optimal Control", pp. 113-132, Academic Press, 1969.

(5) Buchanan, L.F. and Norton, F.E., "Optimal Control Applications in Economic Systems",

(6) Friend, I. and Taubman, P., "A Short Term Forecasting Model", Review of Economics and Statistics, August, 1967.

(7) Mantell, L.H. and Sing, F.P., "Economics for Business Decisions", McGraw Hill, 1972.

(8) Naylor, T.H., "Computer Simulation Experiments with Models of Economic Systems", J. Wiley & sons, 1971.

(9) Andersen, L.C. and Carlson, K.M., "A Monetarist Model for Economic Stabilization", Federal Reserve Bank of St. Louis Review, April, 1970.

(10) Fletcher, R. and Powell, M.J.D., "A Rapidly Convergent Descent Method for Minimization", The Computer Journal, pp. 163-168, July, 1963.

(11) Rosenbrock, H.H., "An Automatic Method for Finding the Greatest or Least Value of a Function", The Computer Journal, pp.175, vol.3, 1960.

(12) Kalman, R.E. and Falb, P.L., "Topics in Mathematical System Theory", McGraw Hill, 1969.

VITA

Guy J. Jutras was born in Ottawa, Ontario on June 5, 1953. He studied Electrical Engineering at the University of Ottawa, obtaining a B.Sc.A. degree in 1974 and worked on his Masters thesis under the direction of Dr. N. U. Ahmed. He is currently employed at Bell-Northern Research, Systems Division in Montreal.