

BUCKLING ANALYSIS OF SANDWICH PIPES UNDER EXTERNAL PRESSURE

by

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Thesis submitted to the
Faculty of Graduate and Postdoctoral Studies
in partial fulfillment of the requirements
for Masters of Applied Science in Civil Engineering

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ABSTRACT

A general eigen-value buckling solution is developed for the buckling of long thick pipes subjected to internal and external hydrostatic pressure. The principle of stationary potential energy is used to formulate the conditions of equilibrium, neutral stability conditions, and associated boundary conditions using polar coordinates. The formulation accounts for shear deformation effects and is suited for composite pipe systems with thick cores. It involves destabilizing terms: one is due to the external hydrostatic pressure and incorporates the follower effects, and the other, is due to the pre-buckling stresses undergoing the nonlinear components of strains. The formulation adopts a work conjugate triplet consisting the Cauchy stress tensor, the Green Lagrange strain tensor, and constant constitutive relations. A Fourier series expansion of the displacement fields is adopted to transform the 2D problem into a series of independent 1D problems, thus keeping the computational effort to a minimum while preserving the accuracy of the solution. Two numerical solutions were developed and implemented under MATLAB; the first one is based on the finite difference technique and the second one is based on the finite element solution. Both solutions were shown to converge to the same solution, the finite difference from below, while the finite element converges from above.

The finite element solution is then applied to predict the buckling capacity of sandwich pipes consisting of two steel pipes with a soft core. A comprehensive verification study is conducted and the validity of the formulation was established through comparison with other solutions. A parametric study is then conducted to investigate the effect of hydrostatic internal pressure, core material, core thickness, and internal and external pipe thicknesses, on the external buckling pressure of sandwich pipes.

The core material properties were observed to be particularly influential on the buckling capacity of the system, increasing the internal and external pipe thickness were observed to improve the buckling capacity of the system. In most cases, increasing the thickness of the external pipe is found to be an effective measure to increase the buckling capacity of the system. It was observed when the internal and external pipes are thin and the core thickness increases, the system tends to buckle in higher modes. Also, the hydrostatic internal pressure acting on the inside surface of the sandwich pipe is found to increase the external buckling capacity of the system in a nearly linearly fashion.

ACKNOWLEDGMENTS

First all, I would like to express my sincere appreciation to my wonderful and knowledgeable supervisor Professor Magdi Mohareb, not only for his technical and financial support, helpful advice, and great patience, but also for being a great human and for his great attitude toward me and my wife. Beyond any doubt this thesis would have been impossible without him.

I am also delighted to express my words to my best friend and gorgeous wife Sepideh for her deepest love and patience. Being with her makes every moment of my life meaningful.

My deepest thanks go to my parents, Mojgan and Majid, and my brother, Touraj, for their invaluable sacrifices, love and support. Although they are far away, without them I couldn't have made it up to this point. I wish them all the best and long life.

At the end I want to express my gratitude to my friends, office mates, and especially my family in-laws, Nahid, Ahmadreza, and Saye for all the help and moral support they provided.

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LIST OF SYMBOLS

α	Total number of terms included in Fourier series
α_1, α_2	Coefficients used in Eq. (6.1)
$\varepsilon_\theta, \varepsilon_\theta^*$	Strain and total strain in tangential direction
$\varepsilon_{\theta,pi}, \varepsilon_{\theta,pe}$	Strain in tangential direction due to the pre-buckling internal and external pressure
$\varepsilon_r, \varepsilon_r^*$	Strain and total strain in radial direction
$\varepsilon_{r,pi}, \varepsilon_{r,pe}$	Strain in radial direction due to the pre-buckling internal and external pressure
$\varepsilon_{L,b}, \varepsilon_{NL,b}$	Linear and non-linear strain due to buckling deformation
$\varepsilon_{L,p}, \varepsilon_{NL,p}$	Linear and non-linear strain due to pre-buckling deformation
$\gamma_{r\theta}, \gamma_{r\theta}^*$	Shear strain and total shear strain
κ	Rotation associated to the center-line deformation about the tangential coordinate
θ	Tangential component of the polar coordinate system
λ	Load factor (Eigen value)
σ_θ	Normal stress in tangential direction
σ_r	Normal stress in radial direction
σ_{PK}	Second Piola-Kirchhoff stress tensor
$\tau_{r\theta}$	Shear stress

ν	Poisson's ratio
ν_c	Poisson's ratio of the core layer
ζ	$(nt/R)^2 / \left[(nt/R)^2 + 5(1-\nu) \right]$
$\omega_r, \omega_\theta, \omega_x$	Rotation along the radial, tangential, and longitudinal coordinates
Φ	Rotation associated to the center-line deformation about the radial coordinate
Δ	Length of each subdivision in finite difference solution
Π	Total potential energy of the system
Π_p	Pre-buckling total potential energy
e	Young elastic modulus ratio E_s/E_c
$\bar{e}$	Infinitesimal perturbation magnitude
$\{f\}$	A vector contains the values of $\{\bar{f}_i\}$ for each point used in finite difference solution
f_{1n}, f_{2n} g_{1n}, g_{2n}	Functions of r only for mode n of the Fourier expansion of displacements in Eq. (3.27)
$\{\bar{f}_i\}$	Two by one vector contains the values of $f_{1n}(r_i)$ and $g_{2n}(r_i)$ in finite difference solution
h_c	Thickness of the core layer
k	Buckling coefficient
k_f	Elastic foundation modulus
l	Total number of elements used in finite element solution
n	Fourier mode under consideration

p	Number of subdivisions in finite difference solution
q_f	Reaction of the core to the external pipe
r	Radial component of the polar coordinate system
r_1	Internal radius of a thick pipe or internal radius of the internal steel pipe of a sandwich pipe
r_2	External radius of a thick pipe or external radius of the internal steel pipe and internal radius of the core of a sandwich pipe
r_3	External radius of the core and internal radius of the external steel pipe of the sandwich pipe
r_4	External radius of the external steel pipe of the sandwich pipe
t	Thickness of the plane pipe
t_{ext}	Thickness of the external pipe
t_{int}	Thickness of the internal pipe
t^B	Traction vector acting on the outer surface
$u(r, \theta)$	Radial displacement
$u^*(r, \theta)$	Total radial displacement
$u_b(r, \theta)$	Buckling radial displacement
$u_{p,i}$	Pre-buckling displacement of Element i finite element solution
u_{pE}, u_{pI}	Pre-buckling radial displacement due to external and internal pressure respectively
$\hat{u}$	Center-line radial displacement

$v(r, \theta)$	Tangential displacement
$v^*(r, \theta)$	Total tangential displacement
$v_b(r, \theta)$	Buckling tangential displacement
$\hat{v}$	Center-line tangential displacement
x	Longitudinal component of the polar coordinate system
w	Displacement in longitudinal direction
$[A], [B]$	Matrices used in Eq. (4.28) and defined in Eqs. (4.29) and (4.30)
$A_{i,j}$	Functions of r assumed in Eqs. (3.19) and (3.20)
$B_{i,j}$	Functions of r and pre-buckling displacement assumed in Eqs. (3.19) and (3.20)
$C_{i,j}$	Functions of r assumed in Eqs. (3.21) to (3.26)
D	Diameter of the middle surface of the pipe cross section
$D_{i,j}$	Functions of r and pre-buckling displacement assumed in Eqs. (3.21) to (3.26)
E	Young's Modulus
E_c	Young's Modulus of the core layer
E_s	Young's Modulus of the steel pipes
F	Deformation gradient
$\bar{F}_{0,1}, \bar{F}_{0,2}$	Integration constants used in Eq. (4.3) for pre-buckling displacement function in finite difference solution

$\bar{F}_{i,1}, \bar{F}_{i,2}$	Coefficients used in Eq. (5.8) for pre-buckling displacement function in finite element solution
I	Moment of inertial ($t^3/12$)
ID	Inside diameter of the pipe cross section
J	Determinant Jacobian of the deformation gradient tensor
$[K_{j,E,i}]$	Matrices used in Eq. (4.19) where represent the elastic parts of the equation
$[K_{j,g,i}]$	Matrices used in Eq. (4.19) where represent the destabilizing parts of the equation
$[K_{j,EBC}]$	Matrices used in Eq. (4.24) where represent the elastic parts of the equation
$[K_{j,g,EBC}]$	Matrices used in Eq. (4.24) where represent the destabilizing parts of the equation
$[K_{j,IBC}]$	Matrices used in Eq. (4.22) where represent the elastic parts of the equation
$[K_{j,g,IBC}]$	Matrices used in Eq. (4.22) where represent the destabilizing parts of the equation
L	$Ev/(1+\nu)(1-2\nu)$
OD	Outside diameter of the pipe cross section
P_{cr}	Critical pressure; buckling is happening at this value
$P_{cr,n}$	Buckling pressure of Mode n
$P_{cr,R01}, P_{cr,R02}$	Critical pressure of the reference cases R01 and R02

$P_{cr0,R01}, P_{cr0,R02}$	Critical pressure of the reference cases R01 and R02 when internal pressure is zero
P_{ext}	Pressure acting on the outside surface of the pipe
P_{int}	Pressure acting on the inside surface of the pipe
R	Radius of the middle surface of the pipe cross section
R01, R02	Reference cases one and two
R_1, R_2	Correlation coefficients of Eqs. (6.6)
R_{ext}	Radius of the middle surface of the external pipe in a sandwich pipe system
R_{int}	Radius of the middle surface of the internal pipe in a sandwich pipe system
U	Internal strain energy of a system
U_p	Internal strain energy of a system due to pre-buckling deformation
V	Total volume of the system
$\vec{V}$	Vector of displacement fields
W	Load potential energy
W_I	The work done by the internal pressure
W_E	The work done by the external pressure
W_p	The work done by the pre-buckling internal and external pressure
W_{PF}	The work done by hydrostatic pressure

CHAPTER 1. INTRODUCTION

The growth in energy demands brings humans to the deep waters. Extracting oil and gas from the depth of more than 2000 m requires high resistance subsea pipelines system. In recent years sandwich pipe systems have become a viable substitution for expensive thick steel pipes in such application. This chapter provides a brief introduction to sandwich pipe systems, typical dimensions, methods of installation, technical challenges during the installation and throughout operation.

1.1 Sandwich Pipe Systems

The sandwich pipe system is made from an internal steel pipe, a core layer made from a relatively softer material, and external high strength steel pipe. The internal pipe, known as product pipe, is mostly designed to carry the internal pressure and transfer the products safely. The core layer is thick comparatively and is made from a softer material. The core layer is designed to provide thermal insulation, to improve the structural performance of the system by conveying the external pressure from the external pipe to the product pipe, or serve as a dual insulation and structural purpose. The outer pipe, known as sleeve pipe, is the structural part of the system and separates the product, product pipe, and core from the surrounding environment (Figure 1-1).

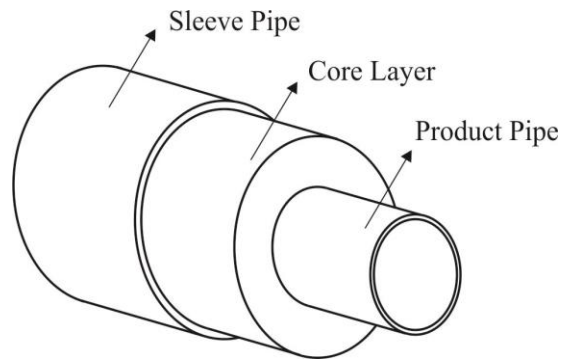


Figure 1-1. Typical Sandwich Pipe

1.1.1 Mechanical Properties of the Core

Depending on the function of the core layer, several materials have been used. Sato et al. (2008) considered three elastic materials a) Polymers b) Ceramics c) Advanced composites with Young's modulus ranging from 0.025 to 800 GPa. Polymers are the most common core materials in sandwich pipe system. Although the Young modulus values are low, the materials are low in density and act as a good insulator. Ceramics have been widely used. Although they are heavy weighted, they can improve the structural capacity of sandwich pipes. They serve as moderate insulators. Advanced composites are other types of materials used for the core layer. They have a Young's modulus as high as ceramics, but differ in weight and insulation. Table 1-1 provides the ranges of Young's modulus for the all three types of material.

Table 1-1 Range of Young's Modulus of the cores for different material types (from Sato et al. (2008))

Core Material	Young's modulus range (GPa)
Ceramics	2.5 to 800
Polymers	0.025 to 2.5
Advanced composites	30 to 190

In an experimental study, Estefen et al. (2005), adopted cement and polypropylene for the core layer. It was reported that both of these low cost materials are feasible options for ultra-deep-water applications.

Table 1-2 provides the physical properties of the sandwich pipes that have been used in the literature are presented. In Table 1-2 r_2 represents the internal radius of the core layer, r_3 is the external radius of the core layer, t_{int} is the thickness of the internal pipe, r_1 is the mean radius of the internal pipe, t_{ext} is the thickness of the external pipe, and r_4 mean radius of the external pipe.

Table 1-2 Physical properties of the sandwich pipes have been used in the literature

Source	r_2/r_3	t_{int}/r_1	t_{ext}/r_4	E_c/E_p	ν_p	ν_c
Estefen (2005)	0.69-0.86	0.06-0.07	0.04-0.05	0.18-0.21	-	-
Sato, Patel (2007)	0.6-0.85	0.03-0.09	0.03-0.05	0.0001-0.1	0.3	0.4
Sato et al. (2008)	0.2-0.9	0.01	0.01	0.0001-0.1	0.3	0.4
Castello et al. (2009)	0.72-0.92	0.04-0.08	0.036-0.052	-	-	-
Arjomandi (2011)	0.55-0.9	0.05	0.04-0.14	0.001-0.1	0.3	0.5

1.1.2 Installation of Sandwich Pipes

In general, there are two types of installation methods, the S-lay and J-lay. In the S-lay method (Figure 1-2) is suited for relatively shallow depths, the connected pipes take the form of S during the installation. The pipe leaves the vessel through a stinger, which is a structure that supports the pipe through the transition from horizontal position to the inclined suspended shape. Then it is bent twice on its way to the seabed. During this process the pipe is under tension due to the weight of suspended parts. The upper curved part is called the overbend. In this position, the pipe is also

subject to bending moments due to the shape of deformation. After that, the pipe continues in a straight line under tension and hydrostatic pressure, until it bends gradually in an opposite direction to lay on the seabed. The second curved part is called sagbend. When the depth of seabed increases, the suspended length of the pipe will also increase; this induces additional tension in the pipe.

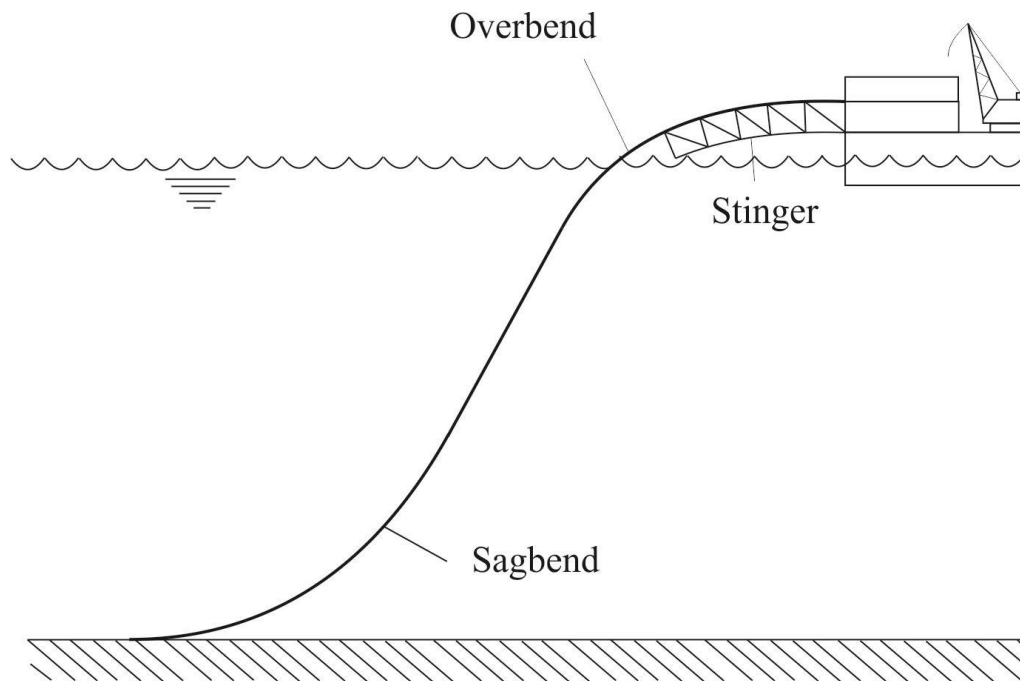


Figure 1-2 Schematic presentation of the applied loads during S-Lay pipeline (Kyriakides and Corona (2007))

Another installation method has been used in order to reduce the amount of tension; namely, the J-lay method (Figure 1-3). In J-lay, the pipes leave the vessel in a vertical position, then gradually bend to lay down on the seabed. This resembles the shape of J.

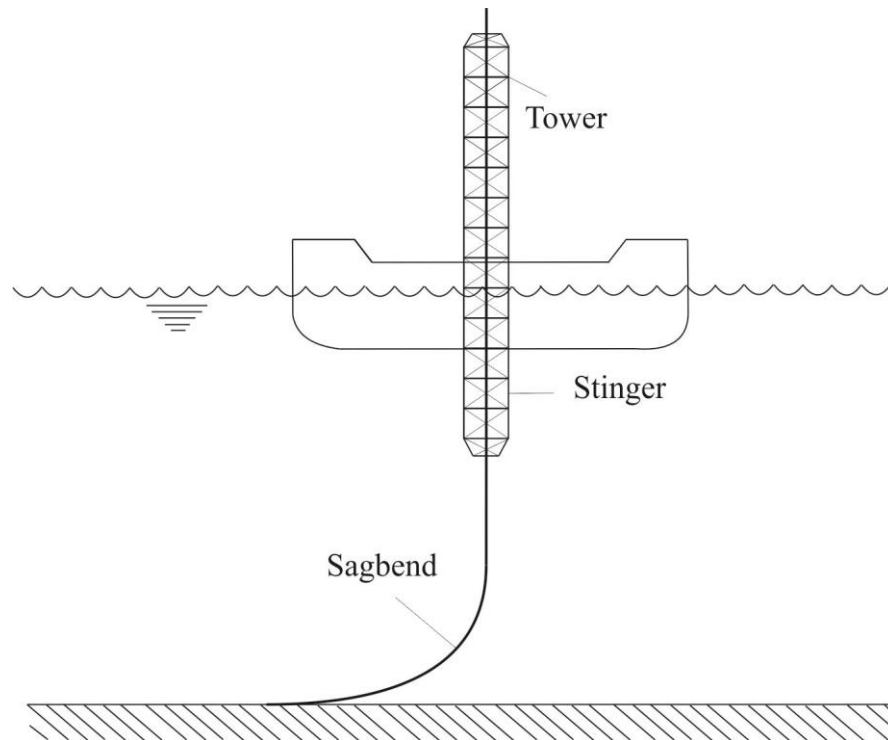


Figure 1-3 Schematic representation of J-lay pipeline installation and associated pipeline loading. (Kyriakides and Corona (2007))

During the installation process, in order to reduce the amount of tension, the pipes are sent down empty. When the seabed is flat, the empty pipe is only under external hydrostatic pressure.

According to Kyriakides (2007), the design of offshore pipelines depends on several factors such as steel grade, pipe manufacturing methods and the method of installation. Thermal properties and weight-strength ratio could also affect the design process. Frequently, the design is influenced by the external hydrostatic pressure. Thus, the primary focus of this study is on the buckling of a long pipe under external and internal hydrostatic pressure.

CHAPTER 2. Literature review

This chapter reviews basic concepts related to the objective of the study and summarizes relevant previous research. Section 2.1, presents basic concepts and mathematical preliminaries relevant to the formulations to be developed. Section 2.2 provides an overview of buckling studies of thin rings. While thin rings are not the primary focus of the present study, it is important to understand the behaviour of thin ring formulations, their bases, and limitations prior to developing solutions for thick rings or sandwich pipes. Sections 2.3 summarizes research about the buckling of thick rings under external pressure, while Section 2.4 surveys buckling studies for sandwich pipes.

2.1 Basic Concepts

In this section, a brief introduction into the Variational Calculus and non-linear elasticity is presented in Sub-sections 2.1.1 and 2.1.2, respectively. Both ingredients will be used in the formulation. Sub-section 2.1.3 provides a discussion on the effect of external pressure with emphasis on its follower effect which is key in the developments in the next chapters when investigating the buckling of a ring.

2.1.1 Variational Calculus

Variational principles are powerful tools for formulating governing equations and developing approximate solutions. They are based on minimizing or maximizing a functional quantity which itself is a function of several functions and their derivatives. Assume a functional Q as an integral expressed in the interval (a,b) where y is a function of x and the limits a and b are independent of y (Qian (1980))

$$Q = \int_a^b F(x, y_1, y_1', y_1'', \dots, y_1^{(n)}, y_2, y_2', y_2'', \dots, y_2^{(n)}, \dots, y_n, \dots, y_n^{(n)}) dx \quad (i=1, 2, 3, \dots, k) \quad (2.1)$$

where x is an independent variable, $y_i(x)$ is an unknown function of x , $y_i' = dy_i/dx$, $y_i'' = d^2 y_i/dx^2$, ..., and $y_i^{(n)} = d^n y_i/dx^n$. In order to satisfy the stationary condition, the first variation of functional Q should vanish, i.e.,

$$\bar{Q} = \int_a^b \bar{F} dx = \int_a^b \sum_{i=1}^k \left(\frac{\partial F}{\partial y_i} \bar{y}_i + \frac{\partial F}{\partial y_i'} \bar{y}_i' + \dots + \frac{\partial F}{\partial y_i^{(n)}} \bar{y}_i^{(n)} \right) dx = 0 \quad (2.2)$$

in which $\bar{y}_i$, $\bar{y}_i'$, $\bar{y}_i''$, ..., and $\bar{y}_i^{(n)}$ are arbitrary functions, which satisfy the boundary conditions

$$\begin{aligned} y_i(a) &= y_{i,a}, y_i'(a) = y_{i,a}', \dots, y_i^{(n)}(a) = y_{i,a}^{(n)} \\ y_i(b) &= y_{i,b}, y_i'(b) = y_{i,b}', \dots, y_i^{(n)}(b) = y_{i,b}^{(n)} \end{aligned} \quad (2.3)$$

Performing integration by parts and combining terms, Eq. (2.2) can be re-written as

$$\bar{Q} = \sum_{i=1}^k \int_a^b \left[\frac{\partial F}{\partial y_i} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y_i'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial y_i''} \right) \dots + (-1)^n \frac{\partial^n}{\partial x^n} \left(\frac{\partial F}{\partial y_i^{(n)}} \right) \right] \bar{y}_i dx + B.C. = 0 \quad (2.4)$$

where $B.C.$ denotes the boundary terms arising from the integration by parts. Equation (2.4) yields a set of governing equations and boundary conditions which represent the stationary condition of the system and take the form

$$\begin{aligned} \frac{\partial F}{\partial y_1} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y_1'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial y_1''} \right) \dots + (-1)^n \frac{\partial^n}{\partial x^n} \left(\frac{\partial F}{\partial y_1^{(n)}} \right) &= 0 \\ \vdots & \\ \frac{\partial F}{\partial y_i} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y_i'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial y_i''} \right) \dots + (-1)^n \frac{\partial^n}{\partial x^n} \left(\frac{\partial F}{\partial y_i^{(n)}} \right) &= 0 \end{aligned} \quad (2.5)$$

In elasticity, Q could represent the total potential energy, in which case Eqs. (2.5) would provide the equilibrium conditions. In a buckling problem, there is no guarantee that the equilibrium position will be stable. An equilibrium position is stable when its second variation is positive value i.e. $\overline{\overline{Q}} > 0$. Conversely, an unstable condition is attained when the second variation is negative, i.e. $\overline{\overline{Q}} < 0$, where here the second variation of Q , denoted as $\overline{\overline{Q}}$ is given by

$$\overline{\overline{Q}} = \int_a^b \overline{\overline{F}} dx = \int_a^b \sum_{j=1}^k \sum_{i=1}^k \sum_{m=0}^n \sum_{l=0}^n \left(\frac{\partial^2 F}{\partial y_i^m \partial y_j^l} \overline{y_i^m} \overline{y_j^l} \right) dx \quad (2.6)$$

Thus, in order to determine the border delineating the regions of stable and unstable equilibrium, the variation of the second variation of the total potential energy is set to zero i.e. $\delta(1/2 \overline{\overline{Q}}) = 0$, thus providing the condition of neutral stability.

2.1.2 Nonlinear Elasticity

For a system under a specific load and constraints, the linearized theory of elasticity gives a unique set of differential equations. However, a system under the same loading and constraint conditions could have several possible equilibrium positions. The difference between the classical solution, based on linear elasticity, and real response is attributed to neglecting the non-linear displacement terms when defining the strains. Neglecting the non-linear terms implies ignoring the effect of rotations throughout deformation, a key issue when capturing buckling phenomena. For a system under given loading and constraints, when multiple equilibrium positions are possible, such is the case in buckling, it is essential to capture the effects of rotation in the equations (since buckling is directly linked to the amount of rotation). At the point of buckling (i.e., when there are several positions of equilibrium possible), the equilibrium position found based on linear analysis is

normally unstable (Novozhilov (1953)). Consider a general body subjected to a specified load. As long as the applied load is less than a critical value, the equilibrium position is unique. When the applied load is increased and attains the critical value, other positions of equilibrium become possible. These additional equilibrium positions cannot be determined using a geometrically linear elastic analysis. According to Pearson (1956) an equilibrium position is stable if a) it does not have any other adjacent equilibrium in its neighborhood, and b) the total potential energy is a relative minimum.

2.1.2.1 Condition of Neutral Stability in Cartesian Coordinates

Novozhilov (1953) has developed the governing differential equations of neutral stability based on the non-linear elasticity in Cartesian coordinates. While the work in this thesis will be based on cylindrical coordinates, it is particularly instructive to review the work of Novozhilov under Cartesian Coordinates. Consider a body subjected to a critical load to have two infinitely close positions of equilibrium. Let $w_{1,0}, w_{2,0}, w_{3,0}$, be the displacements corresponding to the first position of equilibrium, respectively in x_1, x_2 , and x_3 directions. The displacement configuration provided by $w_{1,0}, w_{2,0}, w_{3,0}$, becomes unstable when the critical load is reached. Since the second equilibrium position is very close to the first one, the displacements corresponding to the second position could be written in terms of an infinitely small quantity α as

$$w_i = w_{i,0} + \alpha w_{i,1} \quad i = 1, 2, 3 \quad (2.7)$$

where $\alpha w_{1,1}, \alpha w_{2,1}, \alpha w_{3,1}$ are the displacements in going from first position of equilibrium to the second one. Like the displacements, the strains in the second position can be expressed in terms of the strains in the first position of equilibrium (Novozhilov (1953))

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \alpha \varepsilon'_{ij} + \alpha^2 \varepsilon''_{ij} \quad (2.8)$$

In Eq. 2.7, the strain component ε_{ij}^0 in the first equilibrium position is given by

$$\varepsilon_{ij}^0 = \frac{1}{2} \left(\frac{\partial w_{i,0}}{\partial x_j} + \frac{\partial w_{j,0}}{\partial x_i} + \frac{\partial w_{k,0}}{\partial x_i} \frac{\partial w_{k,0}}{\partial x_j} \right) \quad (2.9)$$

where the Einstein summation convention is assumed in this study. The strain component that depends both on the derivatives of $w_{i,0}$ and $w_{i,1}$ is given by

$$\varepsilon'_{ij} = \frac{1}{2} \left(\frac{\partial w_{i,1}}{\partial x_j} + \frac{\partial w_{j,1}}{\partial x_i} + \frac{\partial w_{k,0}}{\partial x_i} \frac{\partial w_{k,1}}{\partial x_j} + \frac{\partial w_{k,0}}{\partial x_j} \frac{\partial w_{k,1}}{\partial x_i} \right) \quad (2.10)$$

and the component of strains depending only on the derivatives of $w_{i,1}$ is

$$\varepsilon''_{ij} = \frac{1}{2} \left(\frac{\partial w_{k,1}}{\partial x_i} \frac{\partial w_{k,1}}{\partial x_j} \right) \quad (2.11)$$

According to the generalized Hooke's law, the stress-strain relationships are given by

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\varepsilon_{ij} + \delta_{ij} \frac{\nu}{1-2\nu} \varepsilon_{ii} \right) \quad (2.12)$$

where E is the modulus of elasticity and ν is Poisson's ratio and δ_{ij} is the Kronicker delta. It takes the value $\delta_{ij} = 1$ when $i = j$ and vanishes otherwise. From Eq.(2.12), by substituting into Eq.(2.8), one can write

$$\sigma_{ij} = \sigma_{ij}^0 + \alpha \sigma'_{ij} + \alpha^2 \sigma''_{ij} \quad (2.13)$$

where

$$\begin{aligned}
\sigma_{ij}^0 &= \frac{E}{1+\nu} \left(\varepsilon_{ij}^0 + \delta_{ij} \frac{\nu}{1-2\nu} \varepsilon_{ii}^0 \right) \\
\sigma'_{ij} &= \frac{E}{1+\nu} \left(\varepsilon'_{ij} + \delta_{ij} \frac{\nu}{1-2\nu} \varepsilon'_{ii} \right) \\
\sigma''_{ij} &= \frac{E}{1+\nu} \left(\varepsilon''_{ij} + \delta_{ij} \frac{\nu}{1-2\nu} \varepsilon''_{ii} \right)
\end{aligned} \tag{2.14}$$

Knowing that α in Eq. (2.8) is an infinitesimal value, the terms containing α^2 are infinitesimals of second order and can be neglected in the equations. Since the body is in equilibrium right before and after buckling, one can write the equilibrium equations for both configurations. Subtracting the two groups of equilibrium equations and canceling α provides a new set of equations. This set of equations contains the strains and stresses related to the first and second positions of equilibrium. Assuming an arbitrary magnitude of the pre-buckling loads (i.e., the reference loads), the pre-buckling strains and stresses related to the first equilibrium position could be solved in terms of coordinates and the applied load. Substituting them into the new set of equations, the following equations are recovered

$$\begin{aligned}
\frac{\partial}{\partial x_1} [\sigma'_{11} - \omega'_3 \sigma_{12}^0 + \omega'_2 \sigma_{13}^0] + \frac{\partial}{\partial x_2} [\sigma'_{12} - \omega'_3 \sigma_{22}^0 + \omega'_2 \sigma_{23}^0] + \frac{\partial}{\partial x_3} [\sigma'_{13} - \omega'_3 \sigma_{23}^0 + \omega'_2 \sigma_{33}^0] &= 0 \\
\frac{\partial}{\partial x_1} [\sigma'_{12} - \omega'_1 \sigma_{13}^0 + \omega'_3 \sigma_{11}^0] + \frac{\partial}{\partial x_2} [\sigma'_{22} - \omega'_1 \sigma_{23}^0 + \omega'_3 \sigma_{21}^0] + \frac{\partial}{\partial x_3} [\sigma'_{32} - \omega'_1 \sigma_{33}^0 + \omega'_3 \sigma_{31}^0] &= 0 \\
\frac{\partial}{\partial x_1} [\sigma'_{13} - \omega'_2 \sigma_{11}^0 + \omega'_1 \sigma_{12}^0] + \frac{\partial}{\partial x_2} [\sigma'_{23} - \omega'_2 \sigma_{21}^0 + \omega'_1 \sigma_{22}^0] + \frac{\partial}{\partial x_3} [\sigma'_{33} - \omega'_2 \sigma_{31}^0 + \omega'_1 \sigma_{32}^0] &= 0
\end{aligned} \tag{2.15}$$

where ω_i ($i=1,2,3$) denotes the rotation about i axis

$$\begin{aligned}
2\omega_1 &= \frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3} \\
2\omega_2 &= \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1} \\
2\omega_3 &= \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}
\end{aligned} \tag{2.16}$$

All the expressions presented in this section are restricted to Cartesian coordinates. Analogous expressions under cylindrical coordinates will be developed in Chapter 3.

2.1.3 Work done by the external pressure

Follower forces are in general non-conservative. While external or internal pressure are follower forces, (i.e., their direction changes as the body they act upon deforms), the buckling of a ring under external pressure happens to be a conservative system.

When pressure is applied to a ring, the force vector remains perpendicular to the surface throughout deformation (Figure 2-1(a)). Pearson (1956) formulated an expression for the work done by hydrostatic pressure W_{PF} acting on the external surface of a pipe. He assumed the displacements to grow at a constant rate from the onset of buckling configuration to the buckled configuration. Later on, Brush and Almroth (1975) reached the same expression by multiplying the radial traction induced by the external pressure P_{ext} acting on the outer surface of the pipe by the change in the area enclosed by the outer surface, before and after deformation

$$W_{PF} = -P_{ext} (\pi r_2^2 - A^*) \tag{2.17}$$

where P_{ext} is the radial traction on the outer surface taken as positive when acting inward, r_2 is the external radius, and A^* is the area enclosed by the external surface of the pipe after

deformation. An arbitrary point on the surface of a ring with initial Cartesian coordinates of (x_1, x_2) and cylindrical coordinates (r_2, θ) is assumed to undergo radial and tangential displacements u and v , respectively. The deformed coordinates (x_1^*, x_2^*) of the point are

$$\begin{aligned} x_1^* &= (r_2 + u) \cos \theta - v \sin \theta \\ x_2^* &= (r_2 + u) \sin \theta + v \cos \theta \end{aligned} \quad (2.18)$$

Based on the deformed coordinates, the area enclosed by the external surface of the ring is obtained by the integral expression

$$A^* = \frac{1}{2} \oint_C (-x_2^* dx_1^* + x_1^* dx_2^*) \quad (2.19)$$

which the integral is performed around the deformed configuration (C) . Equation (2.19) can be re-written as

$$A^* = \frac{1}{2} \int_0^{2\pi} \left(-x_2^* \frac{dx_1^*}{d\theta} + x_1^* \frac{dx_2^*}{d\theta} \right) d\theta \quad (2.20)$$

Taking the derivative of Eqs. (2.18) with respect to θ and substituting them into Eq. (2.20), one obtains

$$A^* = \frac{1}{2} \int_0^{2\pi} (r_2^2 + 2r_2 u + r_2 \dot{v} + u^2 + u \dot{v} + v^2 - \dot{u} v) d\theta \quad (2.21)$$

where all dots denote the differentiation of the argument functions with respect to θ . From Eq. (2.21), by substituting into Eq. (2.17), the work done by hydrostatic pressure acting on the external surface of ring W_{PF} is found to take the form

$$W_{PF} = P_{ext} \int_0^{2\pi} \left[r_2 u + \frac{1}{2} (u^2 + u\dot{v} + v^2 - \dot{u}v) \right] d\theta \quad (2.22)$$

In which r_2 is the outside radius of the pipe. It is noted that Eq. (2.22) is equally valid for both thin and thick pipe, under external hydrostatic pressure P_{ext} acting on the outside radius r_2 .

2.2 Buckling Formulations for Thin Rings under External Pressure

Several studies have been conducted on the buckling of thin rings. Unlike the treatment by Novozilov (1953), all of these studies are based on cylindrical coordinates. Since the present study primarily aims at investigating sandwich pipes involving thick cores, only a short review of thin ring solution is presented here.

2.2.1 The study of Bodner

Bodner (1958)) investigated three different pressure types. These are

- a) Hydrostatic pressure, where the pressure always remains perpendicular to the surface (Figure 2-1 (a)),
- b) Constant directional pressure, where the direction of the pressure is assumed to remain constant during deformations (Figure 2-1 (b)), and
- c) Centrally directed pressure, where the pressure is assumed to change directions during buckling but it always points at the initial center of the ring (Figure 2-1 (c)).

In the case of constant directional pressure, since the pressure is constant in direction throughout buckling, there is no rotation for the force vectors (Figure 2-1 (b)). This implies that only the linear terms of displacements have roles in the work of external pressure (Bodner (1958)). In case of centrally directed pressure, the direction of the force vectors change during buckling, but are

always pointing at the center of the ring in the un-deformed configuration (Figure 2-1 (c)). The work done by the radial tractions is the product of the external pressure by the change in the distance of the external surface from the old center of the ring (Bodner (1958)).

Bodner concluded that when the magnitude of the pressure remains constant during the buckling, the critical load will be conservative in each of the above cases.

All three scenarios above were treated (Bodner 1958) while assuming the magnitude of the pressure to remain constant throughout deformation and the ring to remain in-extensional. Thus, the difference in critical external pressure magnitudes reported between the three scenarios is solely due to the different directions of pressure throughout buckling.

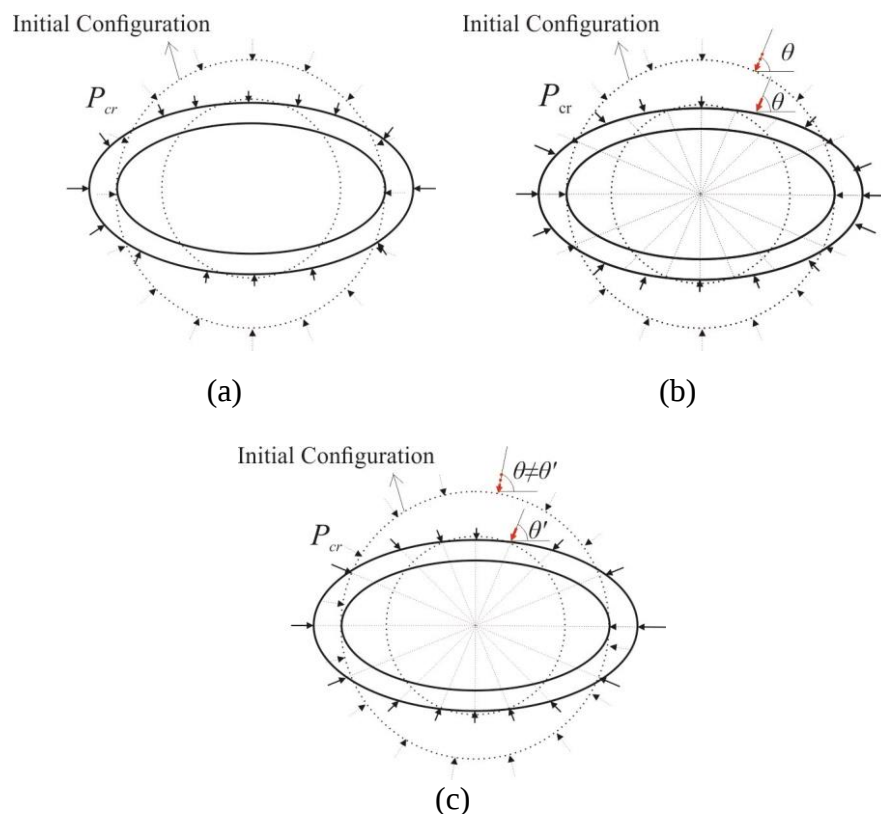


Figure 2-1 Different types of force systems acting on the external surface of a ring. (a) Hydrostatic pressure, (b) Constant directional pressure, (c) Centrally directed pressure.

Bodner (1958) has shown that the buckling pressure per unit length of a ring is given by $P_{cr} = k(EI/R^3)$, where I is the moment of inertia, R is the mean radius of the ring, and k is the buckling coefficients given as

- a) Hydrostatic pressure, $k = 3.0$
- b) Constant directional pressure, $k = 4.0$
- c) Centrally directed pressure, $k = 4.5$

2.2.2 Work by other researchers

Koiter (1963) developed a solution for the buckling and post-buckling analysis of thin rings. Wah (1967) studied the twisting-buckling of thin rings and developed a simple formula for in-plane buckling for a ring. Hutchinson (1968) investigated the buckling and post buckling behaviour of thin cylindrical shells with elliptical cross sections. Singer and Babcock (1970) investigated the buckling of the rings under constant directional and centrally directed pressure. Results were not in agreement with those of Bodner (1958). Amazigo and Fraser (1971) investigated the buckling and load-deflection behaviour of cylindrical shells with dimple shaped initial imperfections. Brush and Almroth (1975) studied the buckling of rings and cylindrical shells based on the nonlinear equilibrium equations of a ring under external hydrostatic pressure and proposed a solution based on a ring on an elastic foundations analogy which was later on applied to sandwich pipes.

Kyriakides and Babcock (1981), and Ju and Kyriakides (1991) have found that the critical external pressure P_{cr} for a thin ring under hydrostatic pressure is given by $P_{cr} = Et^3/4R^3$, where t is the thickness of the ring.

Kyriakides and Corona (2007) developed a solution for a long circular cylinder subjected to external pressure. The study accounts for geometric nonlinear effects, small strains, moderate rotations and is based on Sanders' shell theory (Sanders (1963)) i.e., the mid-surface strains are small and rotations are small but finite. The study also considered the effect of pipe imperfections and incorporated the plasticity behaviour in thin cylinders.

2.3 Studies on Buckling of Thick Rings

The studies in the previous section assumed that shear deformation effect through the thickness of the ring is negligible. However, when the thickness increases, the shear deformation and rotation will have a considerable effect on the behavior of the ring. Thus, the critical pressure predicted by the thin shell theory is expected to deviate significantly from that based on thick shell formulation.

In the following studies, a polar coordinate system has been adopted. Coordinates r , θ , and x are oriented along the radial, tangential, and longitudinal directions, and u , v , and w are the displacement components along those directions, respectively.

2.3.1 Kardomateas (1993)

Kardomateas (1993) presented an elasticity solution to the problem of buckling of orthotropic cylindrical thick shells subjected to external pressure. Using the second Piola-Kirchhoff stress tensor, he developed the equations of equilibrium by setting $\text{div}(\sigma_{PK} \cdot F^T) = 0$, where F is the deformation gradient

$$F = \begin{bmatrix} 1+e_{rr} & \frac{1}{2}e_{r\theta} - \omega_x & \frac{1}{2}e_{rx} + \omega_\theta \\ \frac{1}{2}e_{r\theta} + \omega_x & 1+e_{\theta\theta} & \frac{1}{2}e_{\theta x} - \omega_r \\ \frac{1}{2}e_{rx} - \omega_\theta & \frac{1}{2}e_{\theta x} + \omega_r & 1+e_{xx} \end{bmatrix} \quad (2.23)$$

and σ_{PK} is the second Piola-Kirchhoff stress tensor given by

$$\sigma_{PK} = JF^{-1}[\sigma_{ij}]F^{-T} \quad (2.24)$$

where $J = \det F$ is the determinant Jacobian of the deformation gradient tensor. The boundary conditions are also expressed as $(F\sigma_{PK}^T)\hat{n} = t_r(\vec{V})$, where $\hat{n}$ is a unit normal vector in polar coordinates at the outer surface of the cylinder before deformation, and t_r is the traction vector acting on the outer surface and $\vec{V} = (u, v, w)$ is the vector of displacement fields.

2.3.2 Fu and Waas (1995)

Fu and Waas (1995) developed buckling and initial post-buckling solutions for of a thick ring under external hydrostatic pressure. The study involved two sets of analyses; the first analysis was based on shear deformable model where a linear displacement pattern was assumed throughout the thickness of the ring i.e., the tangential displacement v and the radial displacement u were assumed to take the form

$$\begin{aligned} u &= \hat{u} + z\kappa \\ v &= \hat{v} + z\Phi \end{aligned} \quad (2.25)$$

where $\hat{u}$ and $\hat{v}$ are the center-line radial and tangential displacements, respectively, $\Phi = \partial v / \partial r$ and $\kappa = \partial u / (R \partial \theta)$ are rotations associated to the center-line deformations, and z is the distance

from the mid-surface of the ring. The deformations $\hat{u}$, $\hat{v}$, κ , and Φ and pressure P_{ext} were considered as a summation of four terms and each term is a coefficient of different power in an infinitesimal perturbation magnitude $\bar{e}$

$$\begin{aligned}\hat{u} &= u_0 + \bar{e}u_1 + \bar{e}^2u_2 + \bar{e}^3u_3 \\ \hat{v} &= \bar{e}v_1 + \bar{e}^2v_2 + \bar{e}^3v_3 \\ \kappa &= \bar{e}\kappa_1 + \bar{e}^2\kappa_2 + \bar{e}^3\kappa_3 \\ \Phi &= \bar{e}\Phi_1 + \bar{e}^2\Phi_2 + \bar{e}^3\Phi_3 \\ P_{ext} &= P_0 + \bar{e}P_1 + \bar{e}^2P_2 + \bar{e}^3P_3\end{aligned}\tag{2.26}$$

In Eq. (2.26), the terms independent of $\bar{e}$ denote the pre-buckling displacement or the pre-buckling external pressure. Terms v_0 , κ_0 , and Φ_0 have been assumed to vanish since the pre-buckling displacements are independent of θ . Coefficients of $\bar{e}^1$ are related to the buckling configuration and coefficients of $\bar{e}^2$ and $\bar{e}^3$ are second order and third order terms, which characterize post-buckling deformation. The equations were substituted into the Euler's equilibrium equation to enforce the stationarity condition of the total potential energy, leading to a series of differential equations in terms of displacement fields of the form

$$\begin{aligned}\text{Pre-buckling} + \bar{e}(\text{Buckling}_1) + \bar{e}^2(\text{Post-Buckling1}_1) + \bar{e}^3(\text{Post-Buckling2}_1) + \dots &\approx 0 \\ \bar{e}(\text{Buckling}_2) + \bar{e}^2(\text{Post-Buckling1}_2) + \bar{e}^3(\text{Post-Buckling2}_2) + \dots &\approx 0 \\ \bar{e}(\text{Buckling}_3) + \bar{e}^2(\text{Post-Buckling1}_3) + \bar{e}^3(\text{Post-Buckling2}_3) + \dots &\approx 0 \\ \bar{e}(\text{Buckling}_4) + \bar{e}^2(\text{Post-Buckling1}_4) + \bar{e}^3(\text{Post-Buckling2}_4) + \dots &\approx 0\end{aligned}\tag{2.27}$$

A numerical method was adopted to solve Eqs. (2.27) to obtain the critical buckling pressure. In second analysis, a 2D formulation was developed through a variational approach based on the procedure developed in Novozhilov (1953). The following equilibrium equations were recovered

$$\begin{aligned}\frac{A_6}{r} - \frac{\partial A_4}{\partial r} - \frac{1}{r} \frac{\partial A_5}{\partial \theta} &= 0 \\ \frac{A_3}{r} - \frac{\partial A_1}{\partial r} - \frac{1}{r} \frac{\partial A_2}{\partial \theta} &= 0\end{aligned}\quad (2.28)$$

and the associated boundary conditions were

$$\begin{aligned}\left[A_1 + P_{ext} r \left(1 + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \right]_{r=r_2} &= 0 \\ \left[A_4 + P_{ext} r \left(\frac{v}{r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \right) \right]_{r=r_2} &= 0 \\ A_1|_{r=r_1} &= 0 \\ A_4|_{r=r_1} &= 0\end{aligned}\quad (2.29)$$

where q is the critical load, u_r and u_θ are radial and tangential displacements, respectively, and

$$\begin{aligned}A_1 &= r \left[\sigma_{rr} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{r\theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) \right] \\ A_2 &= r \left[\sigma_{\theta\theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) + \sigma_{r\theta} \left(1 + \frac{\partial u_r}{\partial r} \right) \right] \\ A_3 &= r \left[\sigma_{\theta\theta} \left(1 + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \sigma_{r\theta} \left(\frac{\partial u_\theta}{\partial r} \right) \right] \\ A_4 &= r \left[\sigma_{rr} \left(\frac{\partial u_\theta}{\partial r} \right) + \sigma_{r\theta} \left(1 + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \right] \\ A_5 &= r \left[\sigma_{\theta\theta} \left(1 + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \sigma_{r\theta} \left(\frac{\partial u_\theta}{\partial r} \right) \right] \\ A_6 &= (-r) \left[\sigma_{\theta\theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) + \sigma_{r\theta} \left(1 + \frac{\partial u_r}{\partial r} \right) \right]\end{aligned}\quad (2.30)$$

The authors used the Galerkin solution technique to obtain a numerical solution for the eigenvalue problem. The results based on 1D and 2D analyses were compared. The authors observed that for

D/t less than 15, the 1D analysis does not properly describe the buckling behaviour and deformations of the system.

2.3.3 Papadakis (2008)

Papadakis (2008) developed a set of neutral stability conditions by considering the transverse shear due to the destabilizing effect caused by stresses undergoing non-linear strains. The solution is based on the higher order shell theory of Voyiadjis and Shi (1991). Papadakis (2008) used the 2D version of the 3D equilibrium equations of Kardomateas (1993) which include the non-linear terms of strains and rotation ω_x . He adopted the second Piola-Kirchhoff stress tensor with its conjugate strain, the Green-Lagrange strain tensor. The solution was based on integrating the differential equations of neutral stability through the thickness of a cylinder and solving them numerically. Three elastic solutions were proposed. The first solution, and the most general one, considered the buckling shear strains $e_{r\theta}$ and rotation ω_x in the buckling equations. The finite volume method was used to solve the equilibrium equations numerically. In the second solution, simplified stability equations were obtained by neglecting the buckling shear strains $e_{r\theta}$ and solved them numerically. In the third solution the simplified stability equations were analytically solved based on a higher order shell theory. Results based on the third solution are a simplified buckling equation for thick rings which account for the rotation effect and takes the form

$$P_{cr} = \frac{1}{4} \frac{E}{1-\nu^2} \left(\frac{t}{R} \right)^3 \frac{(1 - \zeta_{(n=2)})}{1 + \frac{t}{2R} - \zeta_{(n=2)} \left(\frac{5}{16} - \frac{25}{96} \frac{t}{R} \right)} \quad (2.31)$$

where $\zeta(n) = (nt/R)^2 / \left[(nt/R)^2 + 5(1-\nu) \right]$ and n represent the buckling mode number.

Table 2-1 represents the summary of the solutions of Papadakis (2008).

Table 2-1 Solutions presented in Papadakis (2008) and the parameters included or neglected in them

Number of Solution	Parameters		Method of solution
	Shear strain $e_{r\theta}$	Rotation ω_x	
First solution	included	included	Numerically
Second solution	neglected	included	Numerically
Third solution	neglected	included	Analytically

Papadakis compared the buckling pressure calculated by the three solutions. The first solution was found to agree with that by Kardomateas (2000) and gives the smallest critical pressure among the three solutions. The second solution agrees with the solution of Kardomateas (1993), and the third solution, based on (Eq. (2.31)), gives buckling pressures in close agreement but smaller than those based on the second method. For thicker pipes, considerable differences in buckling pressure were observed when the shear strains are neglected. It was concluded that as the thickness increases, the ratio of the shear strain $e_{r\theta}$ to the rotation ω_x increases and reaches its maximum at the mid surface of the ring. This suggests that the shear strain $e_{r\theta}$ cannot be neglected for thick rings or for sandwich pipes with thick cores. Thus, in the present study, the contribution of the shear strain $e_{r\theta}$ will be retained in the formulation.

2.3.4 Ji and Waas (2014)

Ji and Waas (2014) formulated three 2D plane strain elasticity solutions for thick orthotropic rings under external pressure. The first solution considered a conservative work potential approach for configuration dependent loading with Cauchy stress tensor, Green Lagrange strain tensor, and a

constant constitutive relations, depending on the buckled configuration in a way that makes the chosen strain and stress tensors work-conjugate. The results are found consistent with those based on the finite element software such as NASTRAN.

The second solution was aimed at demonstrating the effect of the follower force throughout buckling. The virtual work expression was written based on the Cauchy stress tensor (in the deformed configuration) and later the stresses were transformed to the second Piola-Kirchhoff stress tensor (in the un-deformed configuration). However, the terms related to the traction at the boundaries were not transformed to the deformed configuration, but kept in the un-deformed area.

The third solution was aimed at demonstrating the limitations of FEA software such as ABAQUS and ANSYS. In this treatment, they formulated a solution based upon the Jaumann rate of Kirchhoff stress tensor, the Green Lagrange Strain tensor, and a constant constitutive tensor. The buckling results calculated based on this solution were shown to perfectly match those of ABAQUS, which suggests that ABAQUS is adopting non-work-conjugate stress-strain tensors. They observed that Solutions 1 and 3 agree well for thin pipes. However, the approximation in Solution 3 was observed to induce discrepancies in the prediction of buckling pressures for thick pipes. In the present study, it is reasoned that such a discrepancy would appear for sandwich pipes with thick cores. As such, a formulation analogous to Solution 1 will be adopted in the present study.

2.4 Sandwich Pipe System

In a sandwich pipe system, the core layer has two main functions: a) thermal insulation and b) increasing structural resistance. The two most common materials that have been studied in recent years are polypropylene and cement. In the following, an overview of studies conducted to

investigate the effect of core materials on the performance of sandwich pipe system is first conducted (Section 2.4.1). An in-depth review of the solution of various formulations for critical pressure formulations for sandwich pipe is then provided (Sections 2.4.2 through 2.4.8) given their relevance to the subject of the thesis.

2.4.1 Studies aimed investigating effect of core material

Studies on composite pipes include the work of Netto et al. (2002), who investigated the effect of core and pipe thickness on mechanical properties of sandwich pipes on the external pressure capacity through small scale experiments and nonlinear finite element analysis. Pasqualino et al. (2002) conducted 3D nonlinear FEA analysis to develop interaction diagrams for sandwich pipes under bending and external pressure. Castelo and Estefen (2006) studied the effect of core layer adhesion on the ultimate strength of sandwich pipes under external pressure and bending. De Souza et al. (2007) numerically and experimentally investigated the effect of core material properties on the collapse pressure of sandwich pipes subject to bending, pressure, and temperature. Lourenço et al. (2008) experimentally and numerically investigated the contribution of the core layer properties and geometry on the propagation pressure of sandwich pipes under combinations of internal and external pressure. Castello et al. (2009) conducted a numerical investigation to evaluate adhesion strength between steel pipes and polyurethane core material with varying material properties and initial out-of roundness. An et al. (2013) provided a reviewed of recent research (2002-2012) on the collapse and buckling of sandwich pipes.

2.4.2 Closed form Solution by Brush and Almroth (1975)

Brush and Almroth (1975) formulated an approximate buckling pressure solution for sandwich pipe systems based on a ring on elastic foundation analogy. The study considered reaction q_f of

the core to the external pipe with a mean radius R and thickness t to be proportional to the displacement of the pipe in the radial direction u , i.e.,

$$q_f = -k_f u \quad (2.32)$$

where k_f is an elastic foundation modulus which depends solely on the sandwich core material properties. According to Sato and Patel (2007), k_f was given by

$$k_f = E_c \frac{2n(\nu_c - 1) - 2\nu_c + 1}{4\nu_c^2 + \nu_c - 3} \quad (2.33)$$

The solution neglects the contribution of the inside pipe and the thickness of the core material. When expressing the internal strain energy, the effect of the foundation was considered in the total potential energy of the system, and the stationary condition of the governing equilibrium equations were evoked. By solving these equations, the critical pressure P_{cr} was obtained as

$$P_{cr} = \frac{1}{12} \frac{n^2 - 1}{1 + \frac{1}{12} \left(\frac{t}{R} \right)^2} \frac{E_s}{1 - \nu_s} \left(\frac{t}{R} \right)^3 + \frac{1}{n^2 - 1} k_f \quad n = 2, 3, 4, \dots \quad (2.34)$$

2.4.3 Kardomateas (2001)

Kardomateas (2001) developed a closed form two-dimensional elasticity solutions for a cylindrical sandwich shell subjected to external pressure, internal pressure, and axial force. His work is an extension of the buckling solution of Lekhnitskii (1963) for monolithic homogeneous, orthotropic cylindrical shells to sandwich shells.

2.4.4 Kardomateas and Simites (2004)

Kardomateas and Simites (2004) developed a 3D solution buckling solution for three-layer sandwich cylindrical shells subjected to external pressure. The study is an extension of the 2D elasticity solution of Kardomateas (1993). The assumed displacement functions in all three layers were distinct and the slopes at the boundaries were assumed to be non-continuous. The results were compared to those based on shell theory with and without considering the transverse shear effect. It was observed that the omission of transverse shear effects leads to highly non-conservative critical pressures, while the retention of transverse effect was shown to lead to reasonable critical pressures.

2.4.5 Sato and Patel (2007)

Sato and Patel (2007) studied the elastic buckling pressure of sandwich pipes involving two thin pipes with an intermediate thick core layer. The study adopted the internal strain energy expression in Brush and Almroth (1975) for the outer and inner pipes. The stress fields of the core were expressed in terms of the displacements of the inside and outside pipes. Using the Euler-Lagrange equations, the governing field equations were formulated for elastic buckling of the system. As a simplification, the non-linear components of the strain-displacement relations were neglected in the core. This approximation lent itself to the use of the classical Airy stress function solution to characterize the behaviour of the core. A key difference between the study to be developed under this thesis and that by Sato and Patel is the retention of the non-linear strain terms. The critical pressure predicted in Sato and Patel (2007) were shown to correspond to higher buckling modes (i.e., not corresponding to ovalization) in some cases, depending on the ratio of elastic modulus of the core to that of the inner and outer pipes. For instance, when the ratio of the

mean radii of the two pipes is less than 0.78 and for radius to thickness ratio of 100 for steel pipes with $E_c/E_p = 10^{-3}$, the governing buckling pressure corresponds to Mode 15. The model suggests there are two possible types of buckling in sandwich pipes. One is the overall buckling in which the whole system buckles, and the other one, termed as the local buckling, when only the outer pipe undergoes buckling. This phenomenon takes place for buckling modes greater than 2 and when the deformation of the inner pipe is negligible compared to the outer pipe.

2.4.6 Arjomandi and Taheri (2010)

Based on a stress function approach, Arjomandi and Taheri (2010) developed an analytical solution for the critical buckling of sandwich pipes. Like Sato and Patel (2007), the solution omits the non-linear strain components in the core. Four cases were considered in the investigations: a) core is fully bonded to both inner and outer pipes, b) core is un-bonded to the outer pipe in the tangential direction, but is fully bonded to the inner pipe, c) core is un-bonded to the inner pipe in tangential direction, but is fully bonded to the outer pipe, and d) core can slide freely against both inner and outer pipe. According to their investigation, the contact mechanism between the core and the steel pipes has considerable effects on the buckling capacity of the system.

2.4.7 Arjomandi and Taheri (2011)

Based on a 3D finite element model within the software package ABAQUS, Arjomandi and Taheri (2011) investigated the elastic stability and post-buckling responses of sandwich pipes under hydrostatic external pressure for four conditions:

- (a) Core is fully bonded to the inner and outer pipe (Fully bonded)
- (b) Core is only bonded to the inner pipe (Outer surface unbonded)
- (c) Core is only bonded to the outer pipe (Inner surface unbonded)

(d) Core is bonded neither to the outer pipe nor to the inner pipe (Both surfaces unbonded)

The solution incorporated the effect of initial imperfection. The effects of core thickness and mechanical properties were investigated. A simplified equation was proposed for estimating the critical pressure of a sandwich pipe based on the FEA results

$$P_{cr} = \kappa P_{crs} + E \left(1 + \alpha_1 \nu_c^2 \right) \left(\frac{t_{ext}}{R_{ext}} \right)^{\alpha_2} (\psi_1 + \psi_2) \quad (2.35)$$

where

$$\begin{aligned} P_{crs} &= \frac{E}{4(1-\nu^2)} \left(\frac{t_{ext}}{R_{ext}} \right)^3 \\ \psi_1 &= \gamma_1 \left(\frac{E_c}{E} \right)^{\gamma_2} \left(1 - \frac{R_{int}}{R_{ext}} \right)^{\gamma_3} \\ \psi_2 &= \xi_1 \left(\frac{E_c}{E} \right)^{\xi_2} \left(1 - \frac{R_{int}}{R_{ext}} \right)^{\xi_3} \left(\frac{t_{int}}{R_{int}} \right)^{\xi_4} \end{aligned} \quad (2.36)$$

Tables 2-2 and 2-3 provide the magnitudes of the various constants appearing in Eqs. (2.35) and (2.36)

Table 2-2 Coefficients to be used in Eq. (2.35) and (2.36) for calculating the critical hydrostatic external pressure of the first category

	κ	α_1	α_2	γ_1	γ_2	γ_3	ξ_1	ξ_2	ξ_3	ξ_4
Fully bonded	0.9844	-0.5444	0.1	0.474	0.98	1.062	0.43	0.079	-0.1031	2.8
Outer unbonded	1.019	0.2461	-0.0904	0.816	0.982	3.146	0.1792	0.0329	-0.1062	2.929
Inner unbonded	0.9814	1.3922	0.083	0.712	0.962	2.827	0.202	0.041	-0.188	2.913
Core unbonded	0.9833	1.106	-0.0945	0.336	0.966	3.631	0.1589	0.0184	-0.0837	3.01

Table 2-3 Coefficients to be used in Eq. (2.35) and (2.36) for calculating the critical hydrostatic external pressure of the second category

	κ	α_1	α_2	γ_1	γ_2	γ_3	ξ_1	ξ_2	ξ_3	ξ_4
Fully bonded	1.0447	0.3259	0.1867	1.3043	1.153	2.289	0.1279	0.47	0.7295	0.5027
Outer unbonded	0.836	0.2461	-0.0904	0.816	0.982	3.146	0.0788	0.984	-1.1379	1.1799
Inner unbonded	-	-	-	-	-	-	-	-	-	-
Core unbonded	-	-	-	-	-	-	-	-	-	-

According to Arjomandi and Taheri (2011), the behaviour of sandwich pipes is divided into two categories depending on the external pressure-ovalization relationship. The first category involves systems where no maximum external pressure capacity could be obtained, i.e., where the external pressure increased monotonically as ovalization was increased. The second category consists of systems where external pressure peaks at a certain ovalization, after which further ovalization induces a reduction in external pressure magnitude.

2.4.8 Modelling Eigenvalue Buckling Problems in ABAQUS

ABAQUS/Standard is a powerful finite element software, widely used in academic research and industry. In the present study, ABAQUS is used for modeling 2D plane strain ring under internal and external pressure and the results will be compared.

2.4.8.1 Background about buckling solutions

ABAQUS is capable for estimating elastic buckling capacity through eigen-value extraction. This estimation is typically useful for stiff structures, where the pre-buckling response is linear (Simulia

2011). The analysis considers a base model with stresses σ^B in equilibrium with the surface tractions t^B . Assuming elastic deformation with small displacement gradients, the introduction of an extra surface tractions Δt will induce additional stresses $\Delta\sigma$ and boundary displacements Δu . Since linear behaviour is assumed, applying the tractions $\lambda\Delta t^B$ will induce stresses $\lambda\Delta\sigma$ and boundary displacements $\lambda\Delta u$. So the system, under the surface tractions $t^B + \lambda\Delta t$, experiences stresses equal to $\sigma^B + \lambda\Delta\sigma$. The interest of this buckling problem is to find λ such that the model buckles under $t^B + \lambda\Delta t$.

2.4.8.2 Magnitude to tractions throughout buckling

During deformation, the base model will go under some deformations and rotations. If the tractions remain constant before and after deformation, the model will experience a different amount of loading, since there would be a change in the area of the traction. In order to capture this effect, ABAQUS changes the amount of traction based on the gradient of deformation, in a way that the product of the traction to area of the surface remain constant during the analysis.

2.4.8.3 Direction tractions throughout buckling

As discussed under Section 2.1.3, hydrostatic pressure is a follower pressure. While, in general, ABAQUS is able to model follower forces in non-linear incremental analyses, it is presently not equipped to conduct an eigen-value buckling analysis for cases involving follower pressure (Simulia 2011). Thus, the present study treats the external and internal pressures as follower pressures, in an effort to fill this void.

2.5 Comparative Summary on Studies on Buckling of Sandwich Pipes

A summary of the features of the most relevant studies on the buckling analysis of sandwich pipes is provided in Table 2-4. These are the studies of Brush and Almroth (1975), Sato and Patel (2007), Arjomandi and Taheri (2011). Also shown are the features and limitations of ABAQUS FEA solution based on an eigen value analysis model. As shown in the table, the present study aims at capturing shear deformation effects in the steel pipe and the core, the destabilizing terms due to follower effects, and pre-buckling stresses undergoing non-linear strains, and contribution to the internal pipe. A distinctive feature not captured in previous model is the adoption of work conjugate stress-strain-constitutive model triplets.

2.6 Objective and Outline

The objective of this study is to develop an efficient solution for the buckling analysis of multi-layer elastic pipes under hydrostatic internal and external pressure, and use the developed model to conduct parametric investigations of design relevance. The variational principle, associated field equations and boundary conditions are developed in Chapter 3. Chapter 4 then provides a finite difference solution for the governing equations developed in Chapter 3. Chapter 5 develops a finite element solution for the buckling of sandwich pipe systems. The validity of the solutions developed in Chapters 3-5 are assessed in Chapter 6. Also given in Chapter 6 is a comprehensive parametric study for sandwich pipe systems. Chapter 7 then provides a summary and the conclusions of the study.

Table 2-4 Summary and comparison of various studies on the buckling analysis of sandwich pipes

Study	Shear Deformation		Destabilizing terms due to		Contribution of Internal Pipe Captured	Work Conjugate Triplets	Multiple Conditions at Pipe-core Interfaces
	Steel Pipe	Core	Follower Force Effect	Pre-buckling Stresses Undergoing Non-linear Strains			
Brush and Almroth (1975)	×	×	✓	×	×	×	×
Sato and Patel (2007)	×	✓	✓	×	✓	×	×
Arjomandi and Taheri (2011)	×	✓	✓	×	✓	×	✓
ABAQUS (CPE8R)	✓	✓	×	✓	✓	×	×
Present Study	✓	✓	✓	✓	✓	✓	×

CHAPTER 3. Buckling Formulation for a Thick Cylindrical Pipe under External Pressure

3.1 General

A sandwich pipe system consists of three concentric pipes; an outside steel casing, and an inside pipe which conveys the fluid, separated by a typically weaker core material. When the diameter to thickness for the outside and inside pipes is large, their structural behaviour can be reliably approximated through a thin pipe idealization. In sandwich pipes of common geometries, the inside core material is typically too thick to be considered as thin pipe. A thick pipe treatment needs to be implemented for a reliable prediction for the behaviour of the core layer. Within this context, this chapter develops a buckling formulation for a thick cylinder under polar coordinates. While the primary motivation is to examine the buckling solution for the thick core layer, the formulation remains equally valid for the outside and inside steel pipes, irrespective of their diameter to thickness ratio, and with minor modifications, they become applicable for the whole composite system.

Considering large deformations and non-linear strains, the governing equations of neutral stability are developed for a thick pipe subjected to internal and external pressure. The internal strain energy is developed based on the plane strain assumption in terms of the unknown displacement fields. Owing to the axi-symmetric nature of the pipe geometry, the formulation adopts polar coordinates. As will be shown, this treatment allows the transformation of the 2D plane strain problem, in which displacement fields are expressed in terms of two independent coordinates r, θ into a series of one-dimensional problems in the radial coordinate r . The displacements are expressed as Fourier

series. The variation of the second variation of the total potential energy is set to vanish to recover the field equations and boundary conditions at the buckling state.

3.2 Statement of the Problem

A long thick pipe with internal radius r_1 and external radius r_2 is considered in Figure 3-1. The pipe is assumed to be subjected to a constant internal hydrostatic pressure P_{int} acting on the inside radius r_1 and a gradually increasing external hydrostatic pressure P_{ext} acting on the outside radius r_2 . Sign conventions of both pressures are shown in Figure 3-1. It is required to determine the external pressure P_{ext} at which the thick pipe will buckle.

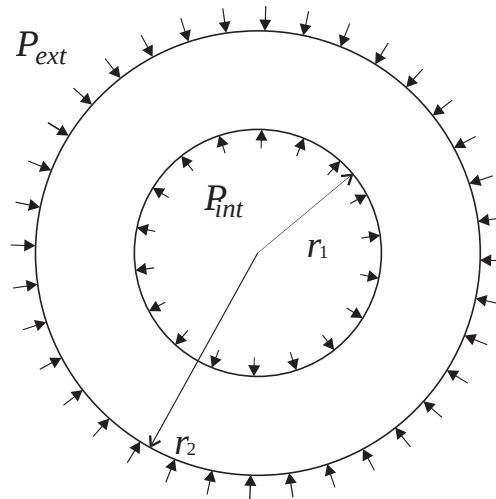


Figure 3-1 Internal and external pressure acting in a thick pipe

3.3 Assumptions

The following assumptions are made

- Since the pipe length is several orders of magnitude larger than the pipe radii, the problem can be idealized as a plane strain problem. This idealization is valid when no spanwise localized are permitted

- Pipe material is assumed linearly elastic and isotropic, and
- The pipe is subject only to hydrostatic pressures (both internal and external) and the hydrostatic pressure remains always perpendicular to the deformed pipe surface, i.e., the follower effect is to be captured in the model

3.4 Coordinates and Sign convention

Figure 3-2 depicts the adopted polar coordinates system (r, θ) and the corresponding radial and tangential displacements (u, v) . The r direction and its corresponding displacement u are oriented along the radial direction and coordinate θ and corresponding displacement v are oriented in the tangential direction.

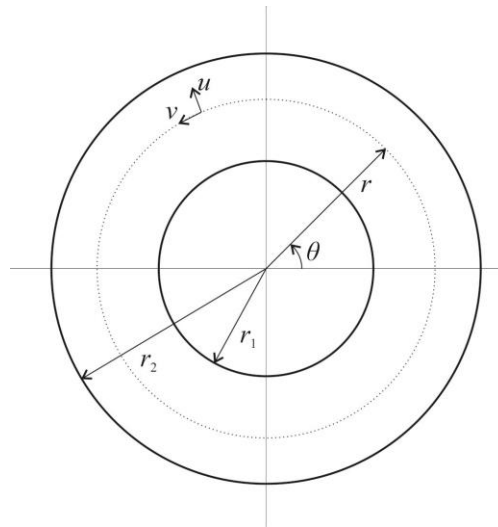


Figure 3-2 the coordinate system and displacements

3.5 General

3.5.1 Configurations

Consider a thick pipe at the initial configuration (Figure 3-3 (a)) is under no external loads. By applying the internal pressure P_{int} at the inside surface of the pipe (Figure 3-3 (b)), the pipe will expand with radial pre-buckling displacements u_{pl} as shown in Configuration (b). An external

pressure P_{ext} is then introduced to the outer surface of the pipe, causing it to deform from configuration (b) to Configuration (c). This deformation is associated with a new pre-buckling displacement $-u_{pE}$ to be superimposed on the pre-buckling displacement field u_{pI} , i.e., the displacement field at configuration (c) is $u_{pI} - u_{pE}$. Next, the internal pressure P_{int} is assumed to remain constant while the external pressure is assumed to increase until the pipe attains the state of onset of buckling at an external pressure λP_{ext} (Configuration (d)). The corresponding displacement associated with configuration (d) becomes $u_{pI} + \lambda u_{pE}$, in which λ is the eigenvalue sought. It is noted here that, u_{pE} is positive in the outward direction, consistently with the sign convention of the radial coordinate r . Thus, under the external pressure as defined in Figure 3-1, its magnitude will be negative. As the pipe buckles (i.e., goes from Configuration (d) to Configuration (e)), the applied pressures P_{int} and λP_{ext} are assumed to remain constant. However, throughout this stage the pipe undergoes buckling manifested through radial displacements u_b and tangential displacements v_b . At Configuration (e) the total displacement fields are obtained by summing the displacement field at the onset of buckling to those during buckling, yielding

$$\begin{aligned} u^* &= u_{pI} + \lambda u_{pE} + u_b \\ v^* &= 0 + v_b \end{aligned} \tag{3.1}$$

In the above, as a convention all displacement fields induced in the pre-buckling stage are denoted with a subscript p and all the displacement fields occurring during the buckling stage are denoted with subscript b . A superscript $*$ denotes displacements, in the final buckling state (i.e., in going from configuration (a) to configuration (e)). The above notation convention is also extended to other fields such as strains and stresses. As shown in Figure 3-2, the radial displacement is positive

outwards and the tangential displacement is positive in the counter-clockwise direction. Thus, when the ring moves towards the center point or in the counter clockwise direction, the associated displacements should have negative values.

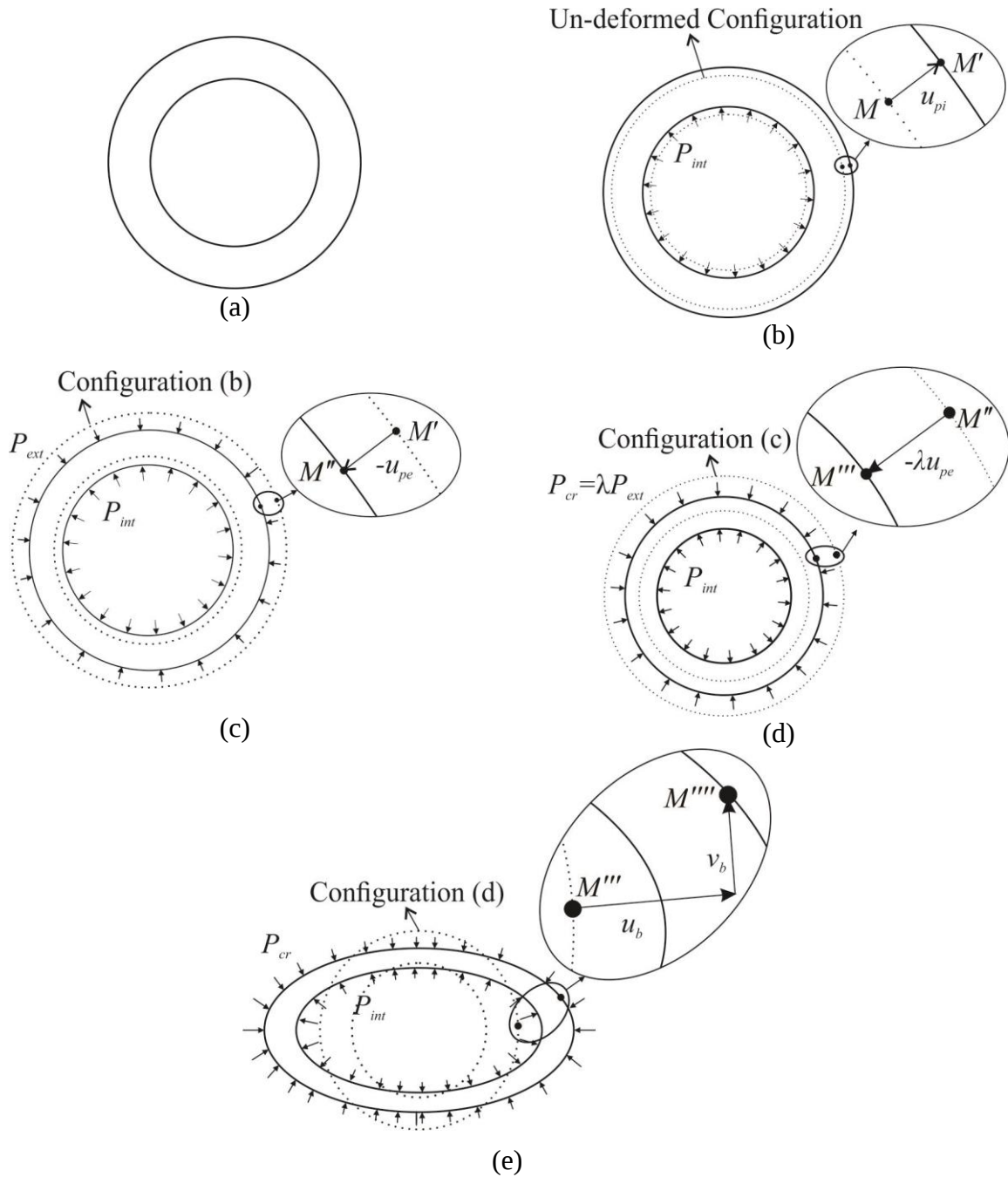


Figure 3-3 Kinematics of thick pipe (a) un-deformed configuration, (b) pre-buckling configuration under internal pressure, (c) pre-buckling configuration under internal and reference external pressure, (d) onset of buckling, and (e) buckled configuration

3.5.2 Strain in Terms of Displacements

The finite strain displacement relations in polar coordinates (e.g., Kardomateas (1993)) are

$$\begin{aligned}\varepsilon_r &= u' + \frac{1}{2}(u'^2 + v'^2) \\ \varepsilon_\theta &= \frac{1}{r}(u + \dot{v}) + \frac{1}{2r^2}[\dot{u}^2 + v^2 - 2\dot{u}v + u^2 + \dot{v}^2 + 2u\dot{v}] \\ \gamma_{r\theta} &= \frac{1}{r}(\dot{u} - v) + v' + \frac{1}{r}(\dot{u}u' - u'v + uv' + v'\dot{v})\end{aligned}\quad (3.2)$$

where all primes denote the differentiation of the argument function with respect to r and all dots denote the differentiation of with respect to θ .

3.5.3 Strain decomposition

From Eqs. (3.1), by substituting into Eq. (3.2), one obtains the total strains in terms of the pre-buckling and buckling displacements. Also, the strains can be divided into linear and nonlinear strains. As a convention, all the linear strain components are indicated with subscript L while nonlinear strain components are denoted with subscript NL . Using this notation, each of the total strains (i.e., in going from configuration (a) to configuration (d)) consists of four components

$$\begin{aligned}\varepsilon_r^* &= \varepsilon_{r,L,p} + \varepsilon_{r,NL,p} + \varepsilon_{r,L,b} + \varepsilon_{r,NL,b} \\ \varepsilon_\theta^* &= \varepsilon_{\theta,L,p} + \varepsilon_{\theta,NL,p} + \varepsilon_{\theta,L,b} + \varepsilon_{\theta,NL,b} \\ \gamma_{r\theta}^* &= \varepsilon_{r\theta,L,p} + \varepsilon_{r\theta,NL,p} + \varepsilon_{r\theta,L,b} + \varepsilon_{r\theta,NL,b}\end{aligned}\quad (3.3)$$

where ε_r^* is the total radial strain ε_θ^* is the total tangential strain and $\gamma_{r\theta}^*$ is the total shear strain.

Expressions for the components in Eq. (3.3) are given in Table 3-1

Table 3-1 Linear and non-linear strains are divided into pre-buckling and buckling stages. u_p is the pre-buckling displacement and subscript b indicates the buckling displacements

	Total Strains in Configuration (e)		Total Strains in Configuration (d)		Strains from Configuration (d) to Configuration (e)	
	Linear	Non-linear	Linear	Non-linear	Linear	Non-linear
ε_r^*	u'^*	$1/2(u'^{*2} + v'^{*2})$	$u'_{pI} + \lambda u'_{pE}$	$1/2(u'_{pI} + \lambda u'_{pE})^2$	u'_b	$1/2[u_b'^2 + v_b'^2 + 2(u'_{pI} + \lambda u'_{pE})u'_b]$
ε_θ^*	$\frac{1}{r}(u^* + \dot{v}^*)$	$(1/2r^2)(\dot{u}^{*2} + \dot{v}^{*2} - 2\dot{u}^*\dot{v}^* + u^{*2} + \dot{v}^{*2} + 2u^*\dot{v}^*)$	$\frac{1}{r}(u'_{pI} + \lambda u'_{pE})$	$\frac{1}{2r^2}(u_{pI} + \lambda u_{pE})^2$	$\frac{1}{r}(u_b + \dot{v}_b)$	$(1/2r^2)[\dot{u}_b^2 + \dot{v}_b^2 - 2\dot{u}_b\dot{v}_b + u_b^2 + 2(u_{pI} + \lambda u_{pE})u_b + \dot{v}_b^2 + 2(u_{pI} + \lambda u_{pE} + u_b)\dot{v}_b]$
$\gamma_{r\theta}^*$	$\frac{1}{r}(\dot{u}^* - \dot{v}^*) + v'^*$	$\frac{1}{r}(\dot{u}^*u'^* - u'^*\dot{v}^* + u^*v'^* + v'^*\dot{v}^*)$	-	-	$\frac{1}{r}(\dot{u}_b - \dot{v}_b) + v'_b$	$(1/r)[\dot{u}_b(u'_{pI} + \lambda u'_{pE} + u'_b) - (u'_{pI} + \lambda u'_{pE} + u'_b)v_b + (u_{pI} + \lambda u_{pE} + u_b)v'_b + v'_b\dot{v}_b]$

3.5.4 Strain-Stress Relationships

The generalized Hooke Constitutive law for the case of plane strain gives the relationships between stresses (σ_r, σ_θ) and strains ($\varepsilon_r, \varepsilon_\theta$) in radial and tangential directions, respectively, as well as the shear stress ($\tau_{r\theta}$) in terms of the shear strain ($\gamma_{r\theta}$). In a matrix form, it takes the form (e.g. Timoshenko (1970))

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} \quad (3.4)$$

in which E is Young's Modulus and ν is Poisson's Ratio.

3.6 Pre-Buckling Analysis

Before performing a buckling analysis, the configuration of the pipe under internal and external pressure (configuration (c) in Figure 3-3) needs to be determined from equilibrium considerations, which is a pre-buckling analysis. The pre-buckling total potential energy Π_p is the sum of internal strain energy U_p and the load potential gained by external loads W_p

$$\Pi_p = U_p + W_p \quad (3.5)$$

Assuming plane strain conditions, the internal strain energy is expressed as

$$U_p = \frac{1}{2} \int_V (\sigma_{r,p} \varepsilon_{r,p} + \sigma_{\theta,p} \varepsilon_{\theta,p} + \tau_{r\theta,p} \gamma_{r\theta,p}) dV \quad (3.6)$$

At configuration (d), the load potential gained by the external and internal pressure is given by

$$W_p = \lambda P_{ext} \int_0^{2\pi} u_{p,E}(r_2) r_2 d\theta - P_{int} \int_0^{2\pi} u_{p,I}(r_1) r_1 d\theta \quad (3.7)$$

The strain-stress relation is a special case of Eq. (3.4). The difference is in the shear strain, which is zero throughout pre-buckling deformations. Using these relations and substituting the stresses in terms of strains, Eq. (3.6) can be written as

$$U_p = \frac{L}{2\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ (1-\nu) \left[(\varepsilon_{r,pi} + \varepsilon_{r,pe})^2 + (\varepsilon_{\theta,pi} + \varepsilon_{\theta,pe})^2 \right] + 2\nu (\varepsilon_{r,pi} + \varepsilon_{r,pe})(\varepsilon_{\theta,pi} + \varepsilon_{\theta,pe}) r \right\} d\theta dr \quad (3.8)$$

In Eq. (3.8), the Lamé's constant $L = E\nu/(1+\nu)(1-2\nu)$ has been introduced for conciseness.

In Eqs. (3.7) and (3.8), by setting $\lambda = 0$, one obtains the energy expressions relevant to configuration b. Also, by setting $\lambda = 1$, one obtains the energy expression relevant to configuration (c).

By summing Eqs. (3.8) and (3.7), using the strain-displacement relations in Column 2 of Table 3.1 and taking the first variation of the total potential energy with respect to the pre-buckling displacement field, one obtains

$$\begin{aligned} \bar{\Pi}_p = & \frac{L}{2\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[(1-\nu) (2ru'_p \bar{u}'_p + 2\frac{1}{r} u_p \bar{u}_p) + 2\nu u_p \bar{u}'_p + 2\nu u'_p \bar{u}_p \right] d\theta dr \\ & + P_{ext} \int_0^{2\pi} (ru_p)_{r=r_2} d\theta - P_{int} \int_0^{2\pi} (ru_p)_{r=r_1} d\theta = 0 \end{aligned} \quad (3.9)$$

where $(\bar{\quad})$ denotes the first variation with respect the pre-buckling displacement. Note that the nonlinear strains are neglected in the pre-buckling analysis. This assumption is justified as long as the system is relatively rigid, as is the case herein the presence of steel pipes. By performing integration by parts one obtains the equilibrium equation

$$\frac{1}{r} u_p - u'_p - ru''_p = 0 \quad (3.10)$$

and the boundary conditions

$$\begin{aligned} \left\langle \left\{ \frac{2L\pi}{\nu} [(1-\nu)ru'_p + \nu u_p] + 2\pi P_{ext}r \right\} \overline{u_p} \right\rangle_{r=r_2} &= 0 \\ \left\langle \left\{ \frac{2L\pi}{\nu} [(1-\nu)ru'_p + \nu u_p] + 2\pi P_{int}r \right\} \overline{u_p} \right\rangle_{r=r_1} &= 0 \end{aligned} \quad (3.11)$$

The solution of Eq. (3.10) subject to the boundary conditions (3.11) provides the pre-buckling displacements in terms of internal and external pressure. The solution will be provided in Chapter 4 and 5.

3.7 Buckling Analysis

3.7.1 Total Potential Energy

Throughout buckling, the total potential energy equation is the summation of internal strain energy and work of pressure acting on the outer surface of the pipe. The generalized destabilizing term due to the internal and external pressure is (Kyriakides 2007)

$$\begin{aligned} W = \lambda P_{ext} \int_0^{2\pi} \left[u^* r + \frac{1}{2} (v^{*2} + u^{*2} - v^* \dot{u}^* + \dot{v}^* u^*) \right]_{r=r_2} d\theta \\ - P_{int} \int_0^{2\pi} \left[u^* r + \frac{1}{2} (v^{*2} + u^{*2} - v^* \dot{u}^* + \dot{v}^* u^*) \right]_{r=r_1} d\theta \end{aligned} \quad (3.12)$$

Thus, the total potential energy will be

$$\begin{aligned} \Pi = \frac{L}{2\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[(1-\nu)(\varepsilon_r^{*2} + \varepsilon_\theta^{*2}) + 2\nu \varepsilon_r^* \varepsilon_\theta^* + \frac{1-2\nu}{2} \gamma_{r\theta}^{*2} \right] r d\theta dr \\ + \lambda P_{ext} \int_0^{2\pi} \left[u^* r + \frac{1}{2} (v^{*2} + u^{*2} - v^* \dot{u}^* + \dot{v}^* u^*) \right]_{r=r_2} d\theta \\ - P_{int} \int_0^{2\pi} \left[u^* r + \frac{1}{2} (v^{*2} + u^{*2} - v^* \dot{u}^* + \dot{v}^* u^*) \right]_{r=r_1} d\theta \end{aligned} \quad (3.13)$$

3.7.2 Conditions of neutral stability

The condition of neutral stability is obtained by setting to zero the variation of the second variation of total potential energy for an arbitrary set of buckling deformations, i.e.,

$$\delta \left(\frac{1}{2} \overline{\overline{\Pi}} \right) = \delta \left[\frac{1}{2} \left(\overline{\overline{U}} + \overline{\overline{W}} \right) \right] = 0 \quad (3.14)$$

where $\overline{\overline{(\)}}$ indicates the second variation of the argument function (or functional), and δ denotes the variation of the argument functions with respect to buckling displacement. It is observed that the second variation of the total potential energy $\overline{\overline{\Pi}}$ consists of multiple terms of the form

$\frac{1}{2} \int_V E \varepsilon_j \varepsilon_k dV$, where the arguments $\varepsilon_j, \varepsilon_k$ ($j = r, \theta, r\theta, k = r, \theta, r\theta$) consist of three

components, e.g., $\varepsilon_j = \varepsilon_{j,L,p} + \varepsilon_{j,L,b} + \varepsilon_{j,NL,b}$ and $\varepsilon_{j,L,p}$ is the pre-buckling linear strain, $\varepsilon_{j,L,b}$ is the linear component of the buckling strain, and $\varepsilon_{j,NL,b}$ is the non-linear component of the buckling

strain. When $\varepsilon_j = \varepsilon_r$ and $\varepsilon_k = \varepsilon_\theta$, the term $\frac{1}{2} \int_V E \varepsilon_r \varepsilon_\theta dV$ can be shown (Appendix A.1) to take the form

$$\frac{1}{2} \int_V E \varepsilon_r \varepsilon_\theta dV = \int_V \frac{E}{2} \left(2 \overline{\varepsilon_{\theta,L}} \overline{\varepsilon_{r,L}} + \overline{\varepsilon_\theta} \overline{\varepsilon_{r,NL}} + \overline{\varepsilon_r} \overline{\varepsilon_{\theta,NL}} \right) dV \quad (3.15)$$

As a special case, when $\varepsilon_j = \varepsilon_k = \varepsilon_r$ or $\varepsilon_j = \varepsilon_k = \varepsilon_\theta$, Eq. (3.15) yields

$$\frac{1}{2} \left(\frac{1}{2} \int_V E \varepsilon_r \varepsilon_r dV \right) = \frac{1}{2} \int_V \left[E \left(\overline{\varepsilon_{r,L,b}} \right)^2 + E \varepsilon_{L,p} \overline{\varepsilon_{NL,b}} \right] dV \quad (3.16)$$

Thus, the second variation of the total potential energy is

$$\begin{aligned}
\frac{1}{2} \bar{\bar{\Pi}} = & \frac{L(1-\nu)}{2\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[\left(\bar{\varepsilon}_{r,L,b} \right)^2 + \left(\varepsilon_{r,L,p,i} + \varepsilon_{r,L,p,e} \right) \bar{\varepsilon}_{r,NL,b} \right] r d\theta dr \\
& + \frac{L(1-\nu)}{2\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[\left(\bar{\varepsilon}_{\theta,L,b} \right)^2 + \left(\varepsilon_{\theta,L,p,i} + \varepsilon_{\theta,L,p,e} \right) \bar{\varepsilon}_{\theta,NL,b} \right] r d\theta dr \\
& + \frac{1}{2} L \int_{r_1}^{r_2} \int_0^{2\pi} \left[2 \bar{\varepsilon}_{r,L,b} \bar{\varepsilon}_{\theta,L,b} + \left(\varepsilon_{r,L,p,i} + \varepsilon_{r,L,p,e} \right) \bar{\varepsilon}_{r,NL,b} \right. \\
& \quad \left. + \left(\varepsilon_{\theta,L,p,i} + \varepsilon_{\theta,L,p,e} \right) \bar{\varepsilon}_{\theta,NL,b} \right] r d\theta dr \\
& + \frac{L(1-2\nu)}{4\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[\left(\bar{\gamma}_{r\theta,L,b} \right)^2 + \gamma_{r\theta,L,p} \bar{\gamma}_{r\theta,NL} \right] r d\theta dr \\
& + \frac{1}{2} \lambda P_{ext} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{v}_b \bar{u}_b \right]_{r=r_2} d\theta \\
& - \frac{1}{2} P_{int} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{v}_b \bar{u}_b \right]_{r=r_1} d\theta
\end{aligned} \tag{3.17}$$

In Eq. (3.18), the pre-buckling radial linear strain $\varepsilon_{r,L,p}$ has been subdivided into two terms, i.e.,

$\varepsilon_{r,L,p} = \varepsilon_{r,L,p,i} + \varepsilon_{r,L,p,e}$ in which $\varepsilon_{r,L,p,i}$ is the contribution due to internal pressure and $\varepsilon_{r,L,p,e}$ is

the contribution due to external pressure. In a similar manner, the definitions

$\varepsilon_{\theta,L,p} = \varepsilon_{\theta,L,p,i} + \varepsilon_{\theta,L,p,e}$ have been introduced.

Note that the pre-buckling strains multiplied by Lamé's constant L i.e., $L\varepsilon_{r,L,p}$ and $L\varepsilon_{\theta,L,p}$

represent the pre-buckling stresses. Since the pressures are assumed to remain constant in going

from the onset of buckling (Configuration (d)) to the buckled configuration (Configuration (e)),

the pre-buckling strains will also be assumed to remain constant throughout buckling

3.7.3 Variation of the Second Variation of Total Potential Energy in Terms of Displacements

In Eq. (3.17), the second variation of the total potential energy was expressed in terms of strains. From the strain displacement expressions provided in Table 3-1, by substituting into Eq. (3.17), one can express the functional in terms of the displacement fields. Since the pre-buckling deformations are assumed to remain constant throughout buckling, their variations with respect to the buckling displacements will vanish (Appendix-A.2). The condition of neutral stability is given by $\delta(1/2\bar{\Pi}) = 0$ thus yielding

$$\begin{aligned}
\delta \frac{1}{2} \bar{\Pi} = & \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \left[\frac{1}{r} (1-\nu) \bar{u}_b + \nu \bar{u}_b' + \frac{1}{r} (1-\nu) \bar{v}_b \right] \delta \bar{u}_b + \left[\nu \bar{u}_b + (1-\nu) r \bar{u}_b' + \nu \bar{v}_b \right] \delta \bar{u}_b' \right. \\
& + \left[\frac{1}{r} \frac{(1-2\nu)}{2} \bar{\dot{u}}_b - \frac{1}{r} \frac{(1-2\nu)}{2} \bar{v}_b + \frac{(1-2\nu)}{2} \bar{v}_b' \right] \delta \bar{\dot{u}}_b \\
& + \left[-\frac{1}{r} \frac{(1-2\nu)}{2} \bar{\dot{u}}_b + \frac{1}{r} \frac{(1-2\nu)}{2} \bar{v}_b - \frac{(1-2\nu)}{2} \bar{v}_b' \right] \delta \bar{\dot{v}}_b \\
& + \left[\frac{(1-2\nu)}{2} \bar{\dot{u}}_b - \frac{(1-2\nu)}{2} \bar{v}_b + r \frac{(1-2\nu)}{2} \bar{v}_b' \right] \delta \bar{\dot{v}}_b' \\
& \left. + \left[\frac{1}{r} (1-\nu) \bar{u}_b + \nu \bar{u}_b' + \frac{1}{r} (1-\nu) \bar{v}_b \right] \delta \bar{\dot{v}}_b \right\} d\theta dr \\
& + \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \left[\frac{1}{r^2} (1-\nu) (u_{pI} + \lambda u_{pE}) \bar{u}_b + \frac{1}{r} \nu (u_{pI}' + \lambda u_{pE}') \bar{u}_b \right. \right. \\
& + \frac{1}{r^2} (1-\nu) (u_{pI} + \lambda u_{pE}) \bar{v}_b + \frac{1}{r} \nu (u_{pI}' + \lambda u_{pE}') \bar{v}_b \left. \right] \delta \bar{u}_b \\
& + \left[(1-\nu) r (u_{pI}' + \lambda u_{pE}') \bar{u}_b' + \nu (u_{pI} + \lambda u_{pE}) \bar{u}_b' \right] \delta \bar{u}_b' \\
& + \left[\frac{1}{r^2} (1-\nu) (u_{pI} + \lambda u_{pE}) \bar{\dot{u}}_b + \frac{1}{r} \nu (u_{pI}' + \lambda u_{pE}') \bar{\dot{u}}_b \right. \\
& \left. \left. - \frac{1}{r^2} (1-\nu) (u_{pI} + \lambda u_{pE}) \bar{v}_b - \frac{1}{r} \nu (u_{pI}' + \lambda u_{pE}') \bar{v}_b \right] \delta \bar{\dot{u}}_b \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{1}{r^2}(1-\nu)(u_{pI} + \lambda u_{pE})\bar{u}_b - \frac{1}{r}\nu(u'_{pI} + \lambda u'_{pE})\bar{u}_b \right. \\
& \quad \left. + \frac{1}{r^2}(1-\nu)(u_{pI} + \lambda u_{pE})\bar{v}_b + \frac{1}{r}\nu(u'_{pI} + \lambda u'_{pE})\bar{v}_b \right] \delta \bar{v}_b \\
& + \left[r(1-\nu)(u'_{pI} + \lambda u'_{pE})\bar{v}'_b + \nu(u_{pI} + \lambda u_{pE})\bar{v}'_b \right] \delta \bar{v}'_b \\
& + \left[\frac{1}{r^2}(1-\nu)(u_{pI} + \lambda u_{pE})\bar{u}_b + \frac{1}{r}\nu(u'_{pI} + \lambda u'_{pE})\bar{u}_b \right. \\
& \quad \left. + \frac{1}{r^2}(1-\nu)(u_{pI} + \lambda u_{pE})\bar{v}_b + \frac{1}{r}\nu(u'_{pI} + \lambda u'_{pE})\bar{v}_b \right] \delta \bar{v}_b \Big\} d\theta dr \\
& + \lambda P_{ext} \int_0^{2\pi} \left[\left(\bar{u}_b + \frac{1}{2}\bar{v}_b \right) \delta \bar{u}_b - \frac{1}{2}\bar{v}_b \delta \bar{u}_b + \left(\bar{v}_b - \frac{1}{2}\bar{u}_b \right) \delta \bar{v}_b + \frac{1}{2}\bar{u}_b \delta \bar{v}_b \right]_{r=r_2} d\theta \\
& - P_{int} \int_0^{2\pi} \left[\left(\bar{u}_b + \frac{1}{2}\bar{v}_b \right) \delta \bar{u}_b - \frac{1}{2}\bar{v}_b \delta \bar{u}_b + \left(\bar{v}_b - \frac{1}{2}\bar{u}_b \right) \delta \bar{v}_b + \frac{1}{2}\bar{u}_b \delta \bar{v}_b \right]_{r=r_1} d\theta
\end{aligned} \tag{3.18}$$

3.7.4 Conditions of Neutral Stability and Associated Boundary Conditions

By performing integration by parts on Eq. (3.18), and grouping terms with similar arbitrary functions, (Appendix-A.2). One recovers the conditions of neutral stability

$$\begin{aligned}
& \int_0^{2\pi} \left[\left(A_{1,1}\bar{u}_b + A_{1,2}\bar{u}'_b + A_{1,3}\bar{u}''_b + A_{1,4}\bar{u}_b + A_{1,5}\bar{v}_b + A_{1,6}\bar{v}'_b \right) \right. \\
& \quad \left. + \lambda \left(B_{1,1}\bar{u}_b + B_{1,2}\bar{u}'_b + B_{1,3}\bar{u}''_b + B_{1,4}\bar{u}_b + B_{1,5}\bar{v}_b \right) \right] d\theta = 0
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& \int_0^{2\pi} \left[\left(A_{2,1}\bar{u}_b + A_{2,2}\bar{u}'_b + A_{2,3}\bar{v}_b + A_{2,4}\bar{v}'_b + A_{2,5}\bar{v}''_b + A_{2,6}\bar{v}_b \right) \right. \\
& \quad \left. + \lambda \left(B_{2,1}\bar{u}_b + B_{2,2}\bar{v}_b + B_{2,3}\bar{v}'_b + B_{2,4}\bar{v}''_b + B_{2,5}\bar{v}_b \right) \right] d\theta = 0
\end{aligned} \tag{3.20}$$

and the boundary conditions

$$\left\{ \int_0^{2\pi} \left[\left(C_{1,1}\bar{u}_b + C_{1,2}\bar{u}'_b + C_{1,3}\bar{v}_b \right) + \lambda \left(D_{1,1}\bar{u}'_b \right) \right] d\theta \right\}_{r=r_1} = 0 \tag{3.21}$$

$$\left\{ \int_0^{2\pi} \left[(C_{2,1} \bar{\dot{u}}_b + C_{2,2} \bar{v}_b + C_{2,3} \bar{v}'_b) + \lambda (D_{2,1} \bar{v}'_b) \right] d\theta \right\}_{r=r_1} = 0 \quad (3.22)$$

$$\left\{ \int_0^{2\pi} \left[(C_{3,1} \bar{u}_b + C_{3,2} \bar{u}'_b + C_{3,3} \bar{\dot{v}}_b) + \lambda (D_{3,1} \bar{u}_b + D_{3,2} \bar{u}'_b + D_{3,3} \bar{\dot{v}}_b) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.23)$$

$$\left\{ \int_0^{2\pi} \left[(C_{4,1} \bar{\dot{u}}_b + C_{4,2} \bar{v}_b + C_{4,3} \bar{v}'_b) + \lambda (D_{4,1} \bar{\dot{u}}_b + D_{4,2} \bar{v}_b + D_{4,3} \bar{v}'_b) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.24)$$

$$\left[\left(C_{5,1} \bar{\dot{u}}_b + C_{5,2} \frac{1}{r} \bar{v}_b + C_{5,3} \bar{v}'_b \right) + \lambda (D_{5,1} \bar{\dot{u}}_b + D_{5,2} \bar{v}_b) \right]_{\theta=0}^{\theta=2\pi} = 0 \quad (3.25)$$

$$\left[(C_{6,1} \bar{u}_b + C_{6,2} \bar{u}'_b + C_{6,3} \bar{\dot{v}}_b) + \lambda (D_{6,1} \bar{u}_b + D_{6,2} \bar{\dot{v}}_b) \right]_{\theta=0}^{\theta=2\pi} = 0 \quad (3.26)$$

In Eqs. (3.19) to (3.26), the coefficients $A_{i,j} = A_{i,j}(r)$, $B_{i,j} = B_{i,j}(r)$, $C_{i,j} = C_{i,j}(r)$, and $D_{i,j} = D_{i,j}(r)$ are dependent upon the radius r and take the form

$$A_{1,1} = (1-\nu)/r$$

$$A_{2,1} = -(3-4\nu)/2r$$

$$A_{1,2} = -(1-\nu)$$

$$A_{2,2} = -1/2$$

$$A_{1,3} = -r(1-\nu)$$

$$A_{2,3} = (1-2\nu)/2r$$

$$A_{1,4} = -(1-2\nu)/2r$$

$$A_{2,4} = -(1-2\nu)/2$$

$$A_{1,5} = (3-4\nu)/2r$$

$$A_{2,5} = -(1-2\nu)r/2$$

$$A_{1,6} = -1/2$$

$$A_{2,6} = -(1-\nu)/r$$

$$B_{1,1} = (1-\nu)u_p/r^2 + \nu u'_p/r$$

$$B_{2,1} = -2(1-\nu)u_p/r^2 - 2\nu u'_p/r$$

$$B_{1,2} = -u'_p - (1-\nu)ru''_p \quad B_{2,2} = (1-\nu)u_p/r^2 + \nu u'_p/r$$

$$B_{1,3} = -\nu u_p - (1-\nu)ru'_p \quad B_{2,3} = -u'_p - r(1-\nu)u''_p$$

$$B_{1,4} = -(1-\nu)u_p/r^2 - \nu u'_p/r \quad B_{2,4} = -\nu u_p - r(1-\nu)u'_p$$

$$B_{1,5} = 2(1-\nu)u_p/r^2 + 2\nu u'_p/r \quad B_{2,5} = -(1-\nu)u_p/r^2 - \nu u'_p/r$$

$$C_{1,1} = \nu + P_{int}\nu/L \quad C_{2,1} = (1-2\nu)/2 - P_{int}\nu/L$$

$$C_{1,2} = (1-\nu)r_1 \quad C_{2,2} = -(1-2\nu)/2 + P_{int}\nu/L$$

$$C_{1,3} = \nu + P_{int}\nu/L \quad C_{2,3} = (1-2\nu)/2r_1$$

$$D_{1,1} = \nu u_p + (1-\nu)r_1u'_p \quad D_{2,1} = \nu u_p + (1-\nu)r_1u'_p$$

$$C_{3,1} = \nu \quad C_{4,1} = (1-2\nu)/2$$

$$C_{3,2} = (1-\nu)r_2 \quad C_{4,2} = -(1-2\nu)/2$$

$$C_{3,3} = \nu \quad C_{4,3} = (1-2\nu)r_2/2$$

$$D_{3,1} = \nu P_{ext}/L \quad D_{4,1} = -\nu P_{ext}/L$$

$$D_{3,2} = \nu u_p + (1-\nu)r_2u'_p \quad D_{4,2} = \nu P_{ext}/L$$

$$D_{3,3} = \nu P_{ext}/L \quad D_{4,3} = \nu u_p + (1-\nu)r_2u'_p$$

$$C_{5,1} = (1-2\nu)/2r \quad C_{6,1} = (1-\nu)/r$$

$$C_{5,2} = -(1-2\nu)/2r$$

$$C_{6,2} = \nu$$

$$C_{5,3} = (1-2\nu)/2$$

$$C_{6,3} = (1-\nu)/r$$

$$D_{5,1} = (1-\nu)u_p/r^2 + \nu u'_p/r$$

$$D_{6,1} = (1-\nu)u_p/r^2 + \nu u'_p/r$$

$$D_{5,2} = -(1-\nu)u_p/r^2 - \nu u'_p/r$$

$$D_{6,2} = (1-\nu)u_p/r^2 + \nu u'_p/r$$

3.7.5 Assumed displacement functions

Displacements u_b and v_b are functions of r and θ . Given their harmonic nature in coordinate θ , they are expressed in terms of Fourier series as

$$\begin{aligned} u_b(\theta, r) &= \sum_{n=1}^{\alpha} f_{1n}(r) \cos n\theta + \sum_{n=1}^{\alpha} f_{2n}(r) \sin n\theta \\ v_b(\theta, r) &= \sum_{n=1}^{\alpha} g_{1n}(r) \cos n\theta + \sum_{n=1}^{\alpha} g_{2n}(r) \sin n\theta \end{aligned} \quad (3.27)$$

where Eq. (3.27) becomes exact as $\alpha \rightarrow \infty$. For practical purposes the series will be truncated and the number of terms α becomes finite. Substituting the assumed displacement functions into Eqs. (3.19) and (3.20), multiplying each equation once by $\cos m\theta$ and once by $\sin m\theta$ ($m = 2, 3, 4 \dots$) and integrating both sides with respect to θ from $\theta = 0$ to $\theta = 2\pi$, will generate two sets of coupled equilibrium equations (Appendix-A.3). The first set of the equilibrium equations is observed to depend on functions f_{1n} and g_{2n} and takes the form

$$\begin{aligned} & \left(A_{1,1} f_{1n} + A_{1,2} f'_{1n} + A_{1,3} f''_{1n} - A_{1,4} n^2 f_{1n} + A_{1,5} n g_{2n} + A_{1,6} n g'_{2n} \right) \\ & + \lambda \left(B_{1,1} f_{1n} + B_{1,2} f'_{1n} + B_{1,3} f''_{1n} - B_{1,4} n^2 f_{1n} + B_{1,5} n g_{2n} \right) = 0 \end{aligned} \quad (3.28)$$

$$\begin{aligned} & \left(-A_{2,1}nf_{1n} - A_{2,2}nf'_{1n} + A_{2,3}g_{2n} + A_{2,4}g'_{2n} + A_{2,5}g''_{2n} - A_{2,6}n^2g_{2n} \right) \\ & + \lambda \left(-B_{2,1}nf_{1n} + B_{2,2}g_{2n} + B_{2,3}g'_{2n} + B_{2,4}g''_{2n} - B_{2,5}n^2g_{2n} \right) = 0 \end{aligned} \quad (3.29)$$

and the other set is observed to depend on f_{2n} and g_{1n} , i.e.,

$$\begin{aligned} & \left(-A_{1,1}f_{2n} - A_{1,2}f'_{2n} - A_{1,3}f''_{2n} + A_{1,4}n^2f_{2n} + A_{1,5}ng_{1n} + A_{1,6}ng'_{1n} \right) \\ & + \lambda \left(-B_{1,1}f_{2n} - B_{1,2}f'_{2n} - B_{1,3}f''_{2n} + B_{1,4}n^2f_{2n} + B_{1,5}ng_{1n} \right) = 0 \end{aligned} \quad (3.30)$$

$$\begin{aligned} & \left(A_{2,1}nf_{2n} + A_{2,2}nf'_{2n} + A_{2,3}g_{1n} + A_{2,4}g'_{1n} + A_{2,5}g''_{1n} - A_{2,6}n^2g_{1n} \right) \\ & + \lambda \left(B_{2,1}nf_{2n} + B_{2,2}g_{1n} + B_{2,3}g'_{1n} + B_{2,4}g''_{1n} - B_{2,5}n^2g_{1n} \right) = 0 \end{aligned} \quad (3.31)$$

where $n = 2, 3, 4, \dots, \alpha$. It is noted that the coefficients of both sets of equations are similar, with opposite signs in some cases. However, both systems predict the same buckling pressure and buckled configuration, albeit the buckling configuration from the system is rotated by 90 degrees from that obtained from the second system.

A similar procedure is applied to the boundary equations in Eqs. (3.21) to (3.26), to express them in terms of displacement fields f_{1n} , f_{2n} , g_{1n} and g_{2n} . Since the assumed displacements are harmonic, Eqs. (3.26) and (3.27) are automatically satisfied irrespective of the values of function f_{1n} , f_{2n} , g_{1n} and g_{2n} . By multiplying Eq. (3.21) and Eq. (3.23) by $\cos m\theta$ and Eq. (3.22) and Eq. (3.24) by $\sin m\theta$, one obtains the following boundary conditions related to f_{1n} and g_{2n}

$$\left\{ \int_0^{2\pi} \left[C_{1,1}f_{1n} + C_{1,2}f'_{1n} + C_{1,3}ng_{2n} + \lambda(D_{1,1}f'_{1n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (3.32)$$

$$\left\{ \int_0^{2\pi} \left[(-C_{2,1}nf_{1n} + C_{2,2}g_{2n} + C_{2,3}g'_{2n}) + \lambda(D_{2,1}g'_{2n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (3.33)$$

$$\int_0^{2\pi} \left\{ \left[(C_{3,1} f_{1n} + C_{3,2} f'_{1n} + C_{3,3} n g_{2n}) + \lambda (D_{3,1} f_{1n} + D_{3,2} f'_{1n} + D_{3,3} n g_{2n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.34)$$

$$\left\{ \int_0^{2\pi} \left[(-C_{4,1} n f_{1n} + C_{4,2} g_{2n} + C_{4,3} g'_{2n}) + \lambda (-D_{4,1} n f_{1n} + D_{4,2} g_{2n} + D_{4,3} g'_{2n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.35)$$

where $n = 2, 3, 4, \dots, \alpha$. Another set of boundary conditions is generated by multiplying Eq. (3.21) and Eq. (3.23) by $\sin m\theta$ and Eq. (3.22) and Eq. (3.24) by $\cos m\theta$, yielding the following boundary conditions related to f_{2n} and g_{1n}

$$\left\{ \int_0^{2\pi} \left[C_{1,1} f_{2n} + C_{1,2} f'_{2n} - C_{1,3} n g_{1n} + \lambda (D_{1,1} f'_{2n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (3.36)$$

$$\left\{ \int_0^{2\pi} \left[(C_{2,1} n f_{2n} + C_{2,2} g_{1n} + C_{2,3} g'_{1n}) + \lambda (D_{2,1} g'_{1n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (3.37)$$

$$\int_0^{2\pi} \left\{ \left[(C_{3,1} f_{2n} + C_{3,2} f'_{2n} - C_{3,3} n g_{1n}) + \lambda (D_{3,1} f_{2n} + D_{3,2} f'_{2n} - D_{3,3} n g_{1n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.38)$$

$$\left\{ \int_0^{2\pi} \left[(C_{4,1} n f_{2n} + C_{4,2} g_{1n} + C_{4,3} g'_{1n}) + \lambda (D_{4,1} n f_{2n} + D_{4,2} g_{1n} + D_{4,3} g'_{1n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (3.39)$$

3.8 Summary

This chapter developed the governing equations (Eq. (3.28) through Eq. (3.31)) and associated boundary equations (Eq. (3.32) through Eq. (3.39)) governing the buckling of a thick pipe. The resulting equations consist of 2α sets of coupled equations, each consisting of two equilibrium equations and four boundary equations. Comparing Eqs. (3.28) and (3.29) with Eqs. (3.30) and (3.31) and also the Eqs. (3.32) to (3.35) with Eqs. (3.36) to (3.39) one can observe that the

difference is only in some of the negative sign for the f_{1n}, f_{2n} functions. This means that the radial displacements in each set will be the negative of the other set. Considering the similarity of both sets of equations, one needs to consider only one of these sets, since the buckling load predicted by each case will be identical and one can obtain the corresponding buckled configuration by rotating that based on the other system by 90 degrees. Thus, only the system of Eqs. (3.28) and (3.29) and associated boundary conditions (Eqs. (3.32), (3.33), (3.34), and (3.35)) will be solved in the finite difference solution (Chapter 4) and the Finite Element formulation (Chapter 5) without any loss in the generality of the solution.

CHAPTER 4. Solution of Equilibrium Equations and Boundary Conditions

Prior to determining the buckling pressure, it is required first to determine the pre-buckling equilibrium configuration. Thus, in this chapter a closed form solution for the pre-buckling displacement will be presented. Subsequently, the results will be used to determine the buckling pressure. For a given reference external and internal pressures P_{ext} and P_{int} , the unknowns sought in Eqs. (3.30) to (3.35) are the displacements field functions f_{1n} and g_{2n} ($n=2, 3, 4 \dots$) and the buckling external pressure λP_{ext} . The buckling external pressure is determined by discretizing the neutral stability conditions using the finite difference technique (in the present chapter) and the finite element technique (in chapter 5) resulting in an Eigen value problem to be solved numerically.

4.1 Pre-buckling Solution

The governing differential equation of equilibrium found in Chapter 3 (i.e. Eq. (3.11)) is

$$\frac{1}{r}u_p - u_p' - ru_p'' = 0 \quad (4.1)$$

In order to solve this equation, the radial displacement is assumed to be $u_p(r) = f_0(r) = \bar{F}_0 r^{m_0}$. Substituting this function into (3.28) leads to the characteristic equation

$$1 - m_0^2 = 0 \quad (4.2)$$

which gives $m_{0,1} = 1$ and $m_{0,2} = -1$, and the displacement will be

$$u_p = \bar{F}_{0,1} r + \bar{F}_{0,2} r^{-1} \quad (4.3)$$

In the above equation, $\bar{F}_{0,1}$ and $\bar{F}_{0,2}$ are unknown integration constants to be obtained from the boundary conditions of the problem. By substituting Eq. (4.3) into the boundary equations (i.e. Eq. (3.11)) one can write

$$\begin{bmatrix} r_2 & -(1-2\nu)r_2^{-1} \\ r_1 & -(1-2\nu)r_1^{-1} \end{bmatrix} \begin{bmatrix} \bar{F}_{0,1} \\ \bar{F}_{0,2} \end{bmatrix} = \frac{\nu}{L} \begin{bmatrix} -r_2 P_{ext} \\ r_1 P_{int} \end{bmatrix} \quad (4.4)$$

and solving Eqs. (4.4) one obtains the integration constants in terms of the internal and external pressure as

$$\begin{aligned} \bar{F}_{0,1} &= \frac{\nu}{L} \frac{(r_2^2 P_{ext} - r_1^2 P_{int})}{(r_1^2 - r_2^2)} \\ \bar{F}_{0,2} &= \frac{\nu}{L(1-2\nu)} \frac{r_1^2 r_2^2 (P_{int} - P_{ext})}{(r_1^2 - r_2^2)} \end{aligned} \quad (4.5)$$

The pre-buckling displacement field is then expressed in terms of internal and external pressure as

$$u_p = u_{pI} + u_{pE} \quad (4.6)$$

where

$$\begin{aligned} u_{pI} &= -\frac{\nu}{L} \frac{r_1^2 P_{int}}{(r_1^2 - r_2^2)} r + \frac{\nu}{L(1-2\nu)} \frac{P_{int}}{(r_1^2 - r_2^2)} r^{-1} \\ u_{pE} &= \frac{\nu}{L} \frac{r_2^2 P_{ext}}{(r_1^2 - r_2^2)} r - \frac{\nu}{L(1-2\nu)} \frac{r_1^2 r_2^2 P_{ext}}{(r_1^2 - r_2^2)} r^{-1} \end{aligned} \quad (4.7)$$

4.2 Buckling Solution

The buckling displacements in terms of the unknown displacement functions $f_{1n}, f_{2n}, g_{1n}, g_{2n}$ are obtained by solving the governing differential equations of neutral stability (i.e. Eqs (3.28))

and (3.29))

$$\begin{aligned}
& \left(A_{1,1} f_{1n} + A_{1,2} f'_{1n} + A_{1,3} f''_{1n} - A_{1,4} n^2 f_{1n} + A_{1,5} n g_{2n} + A_{1,6} n g'_{2n} \right) \\
& + \lambda \left(B_{1,1} f_{1n} + B_{1,2} f'_{1n} + B_{1,3} f''_{1n} - B_{1,4} n^2 f_{1n} + B_{1,5} n g_{2n} \right) = 0 \\
& \left(-A_{2,1} n f_{1n} - A_{2,2} n f'_{1n} + A_{2,3} g_{2n} + A_{2,4} g'_{2n} + A_{2,5} g''_{2n} - A_{2,6} n^2 g_{2n} \right) \\
& + \lambda \left(-B_{2,1} n f_{1n} + B_{2,2} g_{2n} + B_{2,3} g'_{2n} + B_{2,4} g''_{2n} - B_{2,5} n^2 g_{2n} \right) = 0 \\
& n = 2, 3, \dots, \alpha
\end{aligned} \tag{4.8}$$

Equations (4.8) are a set of $\alpha - 1$ systems of ordinary coupled differential equations, in the field variables f_{1n}, g_{2n} . The solutions are mode specific given the dependence of the coefficients in n . It is noted that coefficients $A_{i,j}$ and $B_{i,j}$ are functions of the radial coordinate r and thus a closed form solution for the system of equations is not possible. Thus, the finite difference method is adopted in this chapter to solve the problem. The boundary conditions as given in Eqs. (3.32) to (3.35) are recalled here:

$$\begin{aligned}
& \left\{ C_{1,1} f_{1n} + C_{1,2} f'_{1n} + C_{1,3} n g_{2n} + \lambda \left(D_{1,1} f'_{1n} \right) \right\}_{r=r_1} = 0 \\
& \left\{ \left(-C_{2,1} n f_{1n} + C_{2,2} g_{2n} + C_{2,3} g'_{2n} \right) + \lambda \left(D_{2,1} g'_{2n} \right) \right\}_{r=r_1} = 0 \\
& \left\{ \left(C_{3,1} f_{1n} + C_{3,2} f'_{1n} + C_{3,3} n g_{2n} \right) + \lambda \left(\left[D_{3,1} f_{1n} + D_{3,2} f'_{1n} + D_{3,3} n g_{2n} \right] \right) \right\}_{r=r_2} = 0 \\
& \left\{ \left(-C_{4,1} n f_{1n} + C_{4,2} g_{2n} + C_{4,3} g'_{2n} \right) + \lambda \left(-D_{4,1} n f_{1n} + D_{4,2} g_{2n} + D_{4,3} g'_{2n} \right) \right\}_{r=r_2} = 0
\end{aligned} \tag{4.9}$$

4.3 Finite Difference Solution

The finite difference solution technique transforms the differential equations of equilibrium into algebraic equations of the unknown displacement functions at a number of discrete nodes $i = 1, \dots, p+1$ within the domain (Figure 4-1)

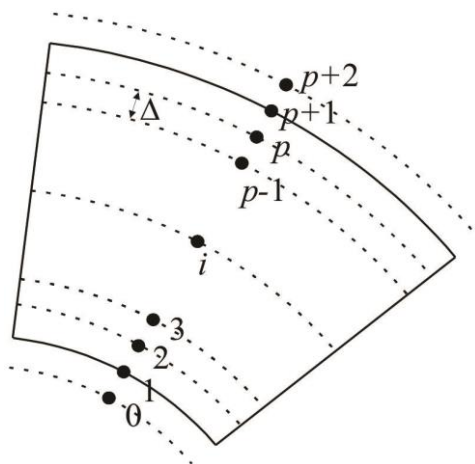


Figure 4-1 A portion of a thick pipe divided in to p parts with defining two imaginary nodes

In order to discretize the differential equations into algebraic equations, the following approximate formulas are developed (e.g. LeVeque (2007)). For the i th node, the derivatives of the field functions $f_{1n,i}$ and $g_{2n,i}$ can be approximated as

$$\begin{aligned} f'_{1n,i} &\approx \frac{f_{1n,i+1} - f_{1n,i-1}}{2\Delta}, & g'_{2n,i} &\approx \frac{g_{2n,i+1} - g_{2n,i-1}}{2\Delta} \\ f''_{1n,i} &\approx \frac{f_{1n,i+1} - 2f_{1n,i} + f_{1n,i-1}}{\Delta^2}, & g''_{2n,i} &\approx \frac{g_{2n,i+1} - 2g_{2n,i} + g_{2n,i-1}}{\Delta^2} \end{aligned} \quad (4.10)$$

where Δ is the length of each subdivision (Figure 4-1). From Eqs. (4.10), by substituting into Eqs. (4.8) and (4.9), one obtains,

$$\left[\left(-\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} \right) f_{i-1} - \frac{A_{1,6}n}{2\Delta} g_{i-1} + \left(A_{1,1} - \frac{2A_{1,3}}{\Delta^2} - A_{1,4}n^2 \right) f_i + A_{1,5}ng_i + \left(\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} \right) f_{i+1} + \frac{A_{1,6}n}{2\Delta} g_{i+1} \right] \quad (4.11)$$

$$+ \lambda \left[\left(-\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} \right) f_{i-1} + \left(B_{1,1} - \frac{2B_{1,3}}{\Delta^2} - B_{1,4}n^2 \right) f_i + B_{1,5}ng_i + \left(\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} \right) f_{i+1} \right] = 0$$

$$\left[\frac{A_{2,2}n}{2\Delta} f_{i-1} + \left(-\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} \right) g_{i-1} - A_{2,1}nf_i + \left(A_{2,3} - \frac{2A_{2,5}}{\Delta^2} - A_{2,6}n^2 \right) g_i - \frac{A_{2,2}n}{2\Delta} f_{i+1} + \left(\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} \right) g_{i+1} \right] \quad (4.12)$$

$$+ \lambda \left[\left(-\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} \right) g_{i-1} - B_{2,1}nf_i + \left(B_{2,2} - \frac{2B_{2,4}}{\Delta^2} - B_{2,5}n^2 \right) g_i + \left(\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} \right) g_{i+1} \right] = 0$$

The coefficients of the above equations can be arranged in a tabular form as given in Table 4-1 and Table 4-2

Table 4-1 coefficients of first discretized equilibrium equation (Eq. (4.11))

	f_{i-1}	g_{i-1}	f_i	g_i	f_{i+1}	g_{i+1}
	$-\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2}$	$-\frac{A_{1,6}n}{2\Delta}$	$A_{1,1} - \frac{2A_{1,3}}{\Delta^2} - A_{1,4}n^2$	$A_{1,5}n$	$\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2}$	$\frac{A_{1,6}n}{2\Delta}$
Terms multiplied by λ	$-\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2}$	0	$B_{1,1} - \frac{2B_{1,3}}{\Delta^2} - B_{1,4}n^2$	$B_{1,5}n$	$\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2}$	0

Table 4-2 coefficients of second discretized equilibrium equation (Eq. (4.12))

	f_{i-1}	g_{i-1}	f_i	g_i	f_{i+1}	g_{i+1}
	$\frac{A_{2,2}n}{2\Delta}$	$-\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2}$	$-A_{2,1}n$	$A_{2,3} - \frac{2A_{2,5}}{\Delta^2} - A_{2,6}n^2$	$-\frac{A_{2,2}n}{2\Delta}$	$\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2}$
Terms multiplied by λ	0	$-\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2}$	$-B_{2,1}n$	$B_{2,2} - \frac{2B_{2,4}}{\Delta^2} - B_{2,5}n^2$	0	$\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2}$

For the n th system of equations, (Eqs. (4.11) and (4.12)) are applied at each node $i = 1, \dots, p+1$, resulting into $2(p+1)$ equations into the unknown $f_1, g_1, \dots, f_{p+1}$, and g_{p+1} . When writing the discretized form of the equilibrium equations at nodes $i = 1$ and $i = p+1$, both lying on the boundaries, additional nodal variables $f_0, g_0, f_{p+2}, g_{p+2}$ arise. Thus, the $2(p+1)$ equations have $2(p+3)$ unknowns. The additional equations required are obtained by writing the boundary conditions (Eqs. (4.9)) in a discretized form, i.e.,

$$-\frac{C_{1,2}}{2\Delta} f_0 + C_{1,1} f_1 + C_{1,3} n g_1 + \frac{C_{1,2}}{2\Delta} f_2 + \lambda \left(-\frac{D_{1,1}}{2\Delta} f_0 + \frac{D_{1,1}}{2\Delta} f_2 \right) = 0 \quad (4.13)$$

$$\left(-\frac{C_{2,3}}{2\Delta} g_0 - C_{2,1} n f_1 + C_{2,2} g_1 + \frac{C_{2,3}}{2\Delta} g_2 \right) + \lambda \left(-\frac{D_{2,1}}{2\Delta} g_0 + \frac{D_{2,1}}{2\Delta} g_2 \right) = 0 \quad (4.14)$$

$$\left(-\frac{C_{3,2}}{2\Delta} f_p + C_{3,1} f_{p+1} + C_{3,3} n g_{p+1} + \frac{C_{3,2}}{2\Delta} f_{p+2} \right) + \lambda \left(-\frac{D_{3,2}}{2\Delta} f_p + D_{3,1} f_{p+1} + D_{3,3} n g_{p+1} + \frac{D_{3,2}}{2\Delta} f_{p+2} \right) = 0 \quad (4.15)$$

$$\left(-\frac{C_{4,3}}{2\Delta} g_p - C_{4,1} n f_{p+1} + C_{4,2} g_{p+1} + \frac{C_{4,3}}{2\Delta} g_{p+2} \right) + \lambda \left(-\frac{D_{4,3}}{2\Delta} g_p - D_{4,1} n f_{p+1} + D_{4,2} g_{p+1} + \frac{D_{4,3}}{2\Delta} g_{p+2} \right) = 0 \quad (4.16)$$

The coefficients of the above equations can be arranged in a tabular form as given in Table 4-3 to Table 4-6.

Table 4-3 coefficients of first discretized boundary equations (Eq. (4.13))

	f_0	g_0	f_1	g_1	f_2	g_2
	$-\frac{C_{1,2}}{2\Delta}$	0	$C_{1,1}$	$C_{1,3}n$	$\frac{C_{1,2}}{2\Delta}$	0
Terms multiplied by λ	$-\frac{D_{1,1}}{2\Delta}$	0	0	0	$\frac{D_{1,1}}{2\Delta}$	0

Table 4-4 coefficients of second discretized boundary equation (Eq. (4.14))

	f_0	g_0	f_1	g_1	f_2	g_2
	0	$-\frac{C_{2,3}}{2\Delta}$	$-C_{2,1}n$	$C_{2,2}$	0	$\frac{C_{2,3}}{2\Delta}$
Terms multiplied by λ	0	$-\frac{D_{2,1}}{2\Delta}$	0	0	0	$\frac{D_{2,1}}{2\Delta}$

Table 4-5 coefficients of third discretized boundary equation (Eq. (4.15))

	f_p	g_p	f_{p+1}	g_{p+1}	f_{p+2}	g_{p+2}
	$-\frac{C_{3,2}}{2\Delta}$	0	$C_{3,1}$	$C_{3,3}n$	$\frac{C_{3,2}}{2\Delta}$	0
Terms multiplied by λ	$-\frac{D_{3,2}}{2\Delta}$	0	$D_{3,1}$	$D_{3,3}n$	$\frac{D_{3,2}}{2\Delta}$	0

Table 4-6 coefficients of fourth discretized boundary equation (Eq. (4.16))

	f_p	g_p	f_{p+1}	g_{p+1}	f_{p+2}	g_{p+2}
	0	$-\frac{C_{4,3}}{2\Delta}$	$-C_{4,1}n$	$C_{4,2}$	0	$\frac{C_{4,3}}{2\Delta}$
Terms multiplied by λ	0	$-\frac{D_{4,3}}{2\Delta}$	$-D_{4,1}n$	$D_{4,2}$	0	$\frac{D_{4,3}}{2\Delta}$

The resulting system of discretized equilibrium equations leads at node i , can be expressed as

$$\left\{ \left[\begin{matrix} K_{1,E,i} \\ K_{2,E,i} \\ K_{3,E,i} \end{matrix} \right] \right\}_{2 \times 6} + \lambda \left[\begin{matrix} K_{1,g,i} \\ K_{2,g,i} \\ K_{2,g,i} \end{matrix} \right]_{2 \times 6} \left\{ \begin{matrix} \bar{f}_{i-1} \\ \bar{f}_i \\ \bar{f}_{i+1} \end{matrix} \right\}_{6 \times 1} = \{0\}_{6 \times 1} \quad (4.17)$$

where $i = 1, \dots, p+1$ in Eq. (4.17) in which $\{\bar{f}_i\} = \begin{Bmatrix} f_i \\ g_i \end{Bmatrix}$. Also, the sub-matrices

$$\begin{aligned} [K_{1,E,i}] &= \begin{bmatrix} -\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} & -\frac{A_{1,6}n}{2\Delta} \\ \frac{A_{2,2}n}{2\Delta} & -\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} \end{bmatrix} \\ [K_{2,E,i}] &= \begin{bmatrix} A_{1,1} - \frac{2A_{1,3}}{\Delta^2} - A_{1,4}n^2 & A_{1,5}n \\ -A_{2,1}n & A_{2,3} - \frac{2A_{2,5}}{\Delta^2} - A_{2,6}n^2 \end{bmatrix} \\ [K_{3,E,i}] &= \begin{bmatrix} \frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} & \frac{A_{1,6}n}{2\Delta} \\ -\frac{A_{2,2}n}{2\Delta} & \frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} \end{bmatrix} \end{aligned} \quad (4.18)$$

have been defined and represent the elastic part of the system, and the matrices

$$\begin{aligned} [K_{1,g,i}] &= \begin{bmatrix} -\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} & 0 \\ 0 & -\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} \end{bmatrix} \\ [K_{2,g,i}] &= \begin{bmatrix} B_{1,1} - \frac{2B_{1,3}}{\Delta^2} - B_{1,4}n^2 & B_{1,5}n \\ -B_{2,1}n & B_{2,2} - \frac{2B_{2,4}}{\Delta^2} - B_{2,5}n^2 \end{bmatrix} \\ [K_{3,g,i}] &= \begin{bmatrix} \frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} & 0 \\ 0 & \frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} \end{bmatrix} \end{aligned} \quad (4.19)$$

include the destabilizing terms due to external pressure.

Also, the boundary equations at the internal boundary are expressed in a matrix form as

$$\begin{aligned}
& \left\{ \left[\begin{matrix} K_{1,IBC} \\ K_{2,IBC} \\ K_{3,IBC} \end{matrix} \right] \right\}_{2 \times 6} \\
& + \lambda \left\{ \left[\begin{matrix} K_{1,g,IBC} \\ K_{2,g,IBC} \\ K_{2,g,IBC} \end{matrix} \right] \right\}_{2 \times 6} \left\{ \begin{matrix} \{\bar{f}_{-1}\} \\ \{\bar{f}_0\} \\ \{\bar{f}_1\} \end{matrix} \right\}_{6 \times 1} = \{0\}_{6 \times 1}
\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
\left[K_{1,IBC} \right] &= \begin{bmatrix} -C_{1,2}/2\Delta & 0 \\ 0 & -C_{2,3}/2\Delta \end{bmatrix} \\
\left[K_{2,IBC} \right] &= \begin{bmatrix} C_{1,1} & C_{1,3}n \\ -C_{2,1}n & C_{2,2} \end{bmatrix} \\
\left[K_{3,IBC} \right] &= \begin{bmatrix} C_{1,2}/2\Delta & 0 \\ 0 & C_{2,3}/2\Delta \end{bmatrix}
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
\left[K_{1,g,IBC} \right] &= \begin{bmatrix} -D_{1,1}/2\Delta & 0 \\ 0 & -D_{2,1}/2\Delta \end{bmatrix} \\
\left[K_{2,g,IBC} \right] &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\left[K_{3,g,IBC} \right] &= \begin{bmatrix} D_{1,1}/2\Delta & 0 \\ 0 & D_{2,1}/2\Delta \end{bmatrix}
\end{aligned} \tag{4.22}$$

and those at the external boundary, are explained in matrix form as

$$\begin{aligned}
& \left\{ \left[\begin{matrix} K_{1,EBC} \\ K_{2,EBC} \\ K_{3,EBC} \end{matrix} \right] \right\}_{2 \times 6} \\
& + \lambda \left\{ \left[\begin{matrix} K_{1,g,EBC} \\ K_{2,g,EBC} \\ K_{2,g,EBC} \end{matrix} \right] \right\}_{2 \times 6} \left\{ \begin{matrix} \{\bar{f}_p\} \\ \{\bar{f}_{p+1}\} \\ \{\bar{f}_{p+2}\} \end{matrix} \right\}_{6 \times 1} = \{0\}_{6 \times 1}
\end{aligned} \tag{4.23}$$

Where

$$\begin{aligned}
 [K_{1,EBC}] &= \begin{bmatrix} -C_{3,2}/2\Delta & 0 \\ 0 & -C_{4,3}/2\Delta \end{bmatrix} \\
 [K_{2,EBC}] &= \begin{bmatrix} C_{3,1} & C_{3,3}n \\ -C_{4,1}n & C_{4,2} \end{bmatrix} \\
 [K_{3,EBC}] &= \begin{bmatrix} C_{3,2}/2\Delta & 0 \\ 0 & C_{4,3}/2\Delta \end{bmatrix}
 \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
 [K_{1,g,EBC}] &= \begin{bmatrix} -D_{3,2}/2\Delta & 0 \\ 0 & -D_{4,3}/2\Delta \end{bmatrix} \\
 [K_{2,g,EBC}] &= \begin{bmatrix} D_{3,1} & D_{3,3}n \\ -D_{4,1}n & D_{4,2} \end{bmatrix} \\
 [K_{3,g,EBC}] &= \begin{bmatrix} D_{3,2}/2\Delta & 0 \\ 0 & D_{4,3}/2\Delta \end{bmatrix}
 \end{aligned} \tag{4.25}$$

Consolidating Eqs. (4.17), (4.20), and (4.23) in a single matrix notation, one obtains the Eigen-value problem

$$([A] + \lambda[B])\{f\} = \{0\} \tag{4.26}$$

in which $\langle f \rangle^T = \langle f_0 \ g_0 \ f_1 \ g_1 \ \cdots \ f_i \ g_i \ \cdots \ f_{p+2} \ g_{p+2} \rangle$ and the following matrices have been defined

$$[A] = \begin{bmatrix} [K_{1,IBC}] & [K_{2,IBC}] & [K_{3,IBC}] & & & & \\ [K_{1,E,1}] & [K_{2,E,1}] & [K_{3,E,1}] & & & & \\ & [K_{1,E,2}] & [K_{2,E,2}] & [K_{3,E,2}] & & & \\ & & \ddots & & & & \\ & & & [K_{1,E,i}] & [K_{2,E,i}] & [K_{3,E,i}] & \\ & & & & \ddots & & \\ & & & & & [K_{1,E,p}] & [K_{2,E,p+1}] & [K_{3,E,p+2}] \\ & & & & & [K_{1,EBC}] & [K_{2,EBC}] & [K_{3,EBC}] \end{bmatrix} \quad (4.27)$$

and

$$[B] = \begin{bmatrix} [K_{1,g,IBC}] & [K_{2,g,IBC}] & [K_{3,g,IBC}] & & & & \\ [K_{1,g,E,1}] & [K_{2,g,E,1}] & [K_{3,g,E,1}] & & & & \\ & [K_{1,g,E,2}] & [K_{2,g,E,2}] & [K_{3,g,E,2}] & & & \\ & & \ddots & & & & \\ & & & [K_{1,g,E,i}] & [K_{2,g,E,i}] & [K_{3,g,E,i}] & \\ & & & & \ddots & & \\ & & & & & [K_{1,g,E,p}] & [K_{2,g,E,p+1}] & [K_{3,g,E,p+2}] \\ & & & & & [K_{1,g,EBC}] & [K_{2,g,EBC}] & [K_{3,g,EBC}] \end{bmatrix} \quad (4.28)$$

For a given mode n , the above numerical eigenvalue problem is solved using MATLAB for the unknown Eigen-pair $\lambda_n, \{f\}_n$ corresponding to the smallest positive Eigen-value. The critical buckling pressure based on mode n is $P_{cr,n} = \lambda P_{ext}$ where we recall that P_{ext} is the reference external pressure adopted in the pre-buckling analysis. The process is repeated for modes $n = 2, 3, 4, \dots, \alpha$ and the critical pressure is given by $P_{cr} = \min(P_{cr,2}, P_{cr,3}, P_{cr,4}, \dots)$.

CHAPTER 5. Finite Element Formulation

5.1 General

The finite difference solution developed in Chapter 4 is applicable only for plain thick pipes. The solution can be extended to sandwich pipes, but requires additional manipulation. Within this context, the finite element formulation is more amenable to such modifications. Thus, the objective of this chapter is to develop a general solution based on a finite element formulation applicable to plain pipes as well as sandwich pipe systems. A finite element solution is developed by expressing the second variation of the total potential energy expression developed in Chapter 3 in terms of nodal displacements. In Section 5.2, a pre-buckling finite element analysis is first provided. The pre-buckling displacements obtained will then be used in Section 5.3 in order to conduct a buckling analysis.

5.2 Pre-Buckling Analysis

In Section 3.6, a pre-buckling analysis was conducted on a thick plain pipe. The solution is not applicable to sandwich pipes with multiple layers having distinct material properties. Thus, in the following section, a pre-buckling analysis will be conducted for a sandwich pipe system consisting multiple layers.

5.2.1 Variational Principle and Governing Equations

The variation of the total potential energy as provided in Eq. (3.9) needs to be generalized for a sandwich pipe system by writing the internal strain energy for each layer. Assuming a sandwich pipe system with $i = 1, 2, \dots, l$ layers, one can write

$$\begin{aligned} \bar{\Pi} = \sum_{i=1}^l \left\{ \int_{r_i(0)}^{r_i(d_i)} \int_0^{2\pi} \frac{L}{2\nu} \left[(1-\nu)(2ru'_p \bar{u}'_p + 2\frac{1}{r}u_p \bar{u}_p) + 2\nu u_p \bar{u}'_p + 2\nu u'_p \bar{u}_p \right] d\theta dr \right\} \\ + P_{ext} \int_0^{2\pi} (r\bar{u}_p)_{r_i(d_i)} d\theta - P_{int} \int_0^{2\pi} (r\bar{u}_p)_{r_i(0)} d\theta = 0 \end{aligned} \quad (5.1)$$

where d_i , $r_i(0)$, and $r_i(d_i)$ respectively denote the thickness, internal radius, and external radius of Layer i . Performing integration by parts on Eq. (5.1) yields l equilibrium equations of the form

$$\frac{1}{r}u_p - u'_p - ru''_p = 0 \quad r_i(0) \leq r \leq r_i(d_i), \quad i = 1, 2, \dots, l \quad (5.2)$$

Also, knowing that the pre-buckling radial displacement u_p at the outside radius $r_i(d_i)$ of layer i is equal to the radial displacement at the inside radius $r_{i+1}(0)$, i.e.,

$$(u_p)_{r_i(d_i)} - (u_p)_{r_{i+1}(0)} = 0 \quad i = 1, 2, \dots, l \quad (5.3)$$

and taking the variations of Eqs. (5.3), i.e., $(\bar{u}_p)_{r_i(d_i)} - (\bar{u}_p)_{r_{i+1}(0)} = 0$, one can express the boundary conditions and interface continuity conditions as

$$\begin{aligned} \left\{ -rP_{int} - \left[\frac{L(1-\nu)}{\nu} ru'_p + Lu_p \right]_{r_i(0)} \right\} (\bar{u}_p)_{r_i(0)} &= 0 \\ \left\{ \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_1(d_1)} - \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_2(0)} \right\} (\bar{u}_p)_{r_1(d_1)} &= 0 \\ \vdots & \\ \left\{ \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_i(d_i)} - \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_{i+1}(0)} \right\} (\bar{u}_p)_{r_i(d_i)} &= 0 \\ \vdots & \\ \left\{ \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_{l-1}(d_{l-1})} - \left[\frac{(1-\nu)}{\nu} Lru'_p + Lu_p \right]_{r_l(0)} \right\} (\bar{u}_p)_{r_{l-1}(d_{l-1})} &= 0 \end{aligned} \quad (5.4)$$

where $i = 1, 2, \dots, l$.

5.2.2 Pre-buckling Closed Form Solution

In order to solve Eqs. (5.3) and (5.4), the pre-buckling displacement for a layer i , $i = 1, 2, \dots, l$ is assumed take the form $u_{p,i}(r) = \sum_j \bar{F}_{i,j} r^{m_{i,j}}$. By substituting into Eq. (5.2), one obtains the characteristic equations

$$1 - m_{i,j}^2 = 0 \quad (5.5)$$

which gives $m_{i,j} = \pm 1$, and the pre-buckling radial displacement will take the form

$$u_{p,i} = \bar{F}_{i,1} r + \bar{F}_{i,2} r^{-1} \quad (5.6)$$

where $\bar{F}_{i,1}, \bar{F}_{i,2}$ are unknown integration constants to be determined from the boundary conditions and interlayer continuity conditions. From Eq. (5.6), by substituting into Eqs. (5.3) and (5.4) one obtains $l - 1$ displacement continuity equations

$$\begin{aligned} r_1(d_1)\bar{F}_{1,1} + r_1(d_1)^{-1}\bar{F}_{1,2} - r_2(0)\bar{F}_{2,1} - r_2(0)^{-1}\bar{F}_{2,2} &= 0 \\ \vdots \\ r_i(d_i)\bar{F}_{i,1} + r_i(d_i)^{-1}\bar{F}_{i,2} - r_{i+1}(0)\bar{F}_{i+1,1} - r_{i+1}(0)^{-1}\bar{F}_{i+1,2} &= 0 \\ \vdots \\ r_{l-1}(d_{l-1})\bar{F}_{l-1,1} + r_{l-1}(d_{l-1})^{-1}\bar{F}_{l-1,2} - r_l(0)\bar{F}_{l,1} - r_l(0)^{-1}\bar{F}_{l,2} &= 0 \end{aligned} \quad (5.7)$$

the $l + 1$ boundary conditions and interlayer natural continuity equations.

$$\begin{aligned}
& \left[-rP_{int} - \frac{L}{\nu} r\bar{F}_{1,1} + \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{1,2} \right]_{r_1(0)} = 0 \\
& \left[\frac{L}{\nu} r\bar{F}_{1,1} - \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{1,2} \right]_{r_1(d_1)} + \left[-\frac{L}{\nu} r\bar{F}_{2,1} + \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{2,2} \right]_{r_2(0)} = 0 \\
& \vdots \\
& \left[\frac{L}{\nu} r\bar{F}_{i,1} - \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{i,2} \right]_{r_i(d_i)} + \left[-\frac{L}{\nu} r\bar{F}_{i+1,1} + \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{i+1,2} \right]_{r_{i+1}(0)} = 0 \\
& \vdots \\
& \left[\frac{L}{\nu} r\bar{F}_{l-1,1} - \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{l-1,2} \right]_{r_{l-1}(d_{l-1})} + \left[-\frac{L}{\nu} r\bar{F}_{l,1} + \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{l,2} \right]_{r_l(0)} = 0 \\
& \left[\frac{L}{\nu} r\bar{F}_{l,1} - \frac{L}{\nu} (1-2\nu) r^{-1} \bar{F}_{l,2} + rP_{ext} \right]_{r_l(d_l)} = 0
\end{aligned} \tag{5.8}$$

The above $2l$ equations are solved for the unknown integration $2l$ constants $\bar{F}_{i,1}, \bar{F}_{i,2}$ and the pre-buckling radial displacements $u_{p,i}$ are thus determined for each layer.

5.3 Buckling Analysis

The second variation of total potential energy expression developed in Section 3.7.2 is to be expressed as a function of nodal displacements, thus expressing the neutral stability condition in a discretized form.

5.3.1 Second Variation of Total Potential Energy

The second variation of total potential energy as given by Eq. (3.18) takes the form.

$$\bar{\bar{\Pi}} = \bar{\bar{U}}_L + \bar{\bar{U}}_{NL} + \bar{\bar{W}}_I + \bar{\bar{W}}_E \tag{5.9}$$

where $\bar{\bar{U}}_L$, the second variation of the internal strain energy induced by the linear strains, is given by

$$\begin{aligned} \frac{1}{2} \bar{\bar{U}}_L = & \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \frac{L(1-\nu)}{2\nu} \left[\left(\bar{u}'_b \right)^2 + \frac{1}{r^2} \left(\bar{u}_b + \bar{v}_b \right)^2 \right] \right. \\ & \left. + L \bar{u}'_b \frac{1}{r} \left(\bar{u}_b + \bar{v}_b \right) + \frac{L(1-2\nu)}{4\nu} \left(\frac{1}{r} \bar{\dot{u}}_b - \frac{1}{r} \bar{\dot{v}}_b + \bar{v}'_b \right)^2 \right\} r d\theta dr \end{aligned} \quad (5.10)$$

$\bar{\bar{U}}_{NL}$, the second variation of the internal strain energy induced by the non-linear components of the strain fields, and is given by

$$\begin{aligned} \frac{1}{2} \bar{\bar{U}}_{NL} = & \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \frac{L(1-\nu)}{2\nu} \left[\frac{1}{r} (u_{p,I} + \lambda u_{p,E}) \frac{1}{2} \left(2\bar{u}'_b{}^2 + 2\bar{v}'_b{}^2 \right) + \frac{1}{2} (u'_{p,I} + \lambda u'_{p,E}) \left(2\bar{u}'_b{}^2 + 2\bar{v}'_b{}^2 \right) \right] \right. \\ & + \frac{1}{2} L \left[\frac{1}{2r} (u_{p,I} + \lambda u_{p,E}) \left(2\bar{u}'_b{}^2 + 2\bar{v}'_b{}^2 \right) \right. \\ & \left. \left. + \frac{1}{2r^2} (u'_{p,I} + \lambda u'_{p,E}) \left(2\bar{\dot{u}}_b{}^2 + 2\bar{\dot{v}}_b{}^2 - 4\bar{u}_b \bar{v}_b + 2\bar{u}_b{}^2 + 2\bar{v}_b{}^2 + 4\bar{u}_b \bar{v}_b \right) \right] \right\} r d\theta dr \end{aligned} \quad (5.11)$$

and $\bar{\bar{W}}_I$ and $\bar{\bar{W}}_E$ are the second variation of the load potential term caused by internal pressure and external pressure respectively

$$\begin{aligned} \frac{1}{2} \bar{\bar{W}}_I = & -\frac{1}{2} P_{int} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{\dot{v}}_b \bar{u}_b \right]_{r=r_1} d\theta \\ \frac{1}{2} \bar{\bar{W}}_E = & \frac{1}{2} \lambda P_{ext} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{\dot{v}}_b \bar{u}_b \right]_{r=r_2} d\theta \end{aligned} \quad (5.12)$$

The variation of the nodal displacements $\bar{u}_b(r, \theta), \bar{v}_b(r, \theta)$ can be expressed as the summation of an infinite number of functions $\bar{u}_{bn}(r), \bar{v}_{bn}(r)$ multiplied by the harmonic functions $\cos n\theta$ and

$\sin n\theta$, i.e.,

$$\begin{aligned}\bar{u}_b(r, \theta) &= \sum_{n=2}^{\infty} \bar{u}_{bn}(r) \cos n\theta \\ \bar{v}_b(r, \theta) &= \sum_{n=2}^{\infty} \bar{v}_{bn}(r) \sin n\theta\end{aligned}\tag{5.13}$$

5.3.2 Displacement Fields in Terms of Nodal Displacements

The buckling displacement fields $\bar{u}_{bn}(r)$ and $\bar{v}_{bn}(r)$ are written in terms of nodal displacements $\{U_n\}$ and $\{V_n\}$ as

$$\begin{aligned}\bar{u}_{bn} &= \langle N \rangle^T \{U_n\} = \langle N_1(r) \quad N_2(r) \rangle \begin{Bmatrix} u_n(r_1) \\ u_n(r_2) \end{Bmatrix} \\ \bar{v}_{bn} &= \langle N \rangle^T \{V_n\} = \langle N_1(r) \quad N_2(r) \rangle \begin{Bmatrix} v_n(r_1) \\ v_n(r_2) \end{Bmatrix}\end{aligned}\tag{5.14}$$

where N_1 and N_2 are the shape functions. A linear relationship is considered for the shape functions, yielding

$$\begin{aligned}N_1(r) &= \frac{r_2 - r}{d}, \quad N_2(r) = \frac{r - r_1}{d}, \\ N_1'(r) &= -\frac{1}{d}, \quad N_2'(r) = \frac{1}{d},\end{aligned}\tag{5.15}$$

Substituting equations (5.15) into Eq.(5.14), one can write the buckling displacements in terms of nodal displacements as

$$\begin{aligned}\bar{u}_b(r, \theta) &= \sum_{n=2}^{\infty} \langle N(r) \rangle^T \{U_n\} \cos n\theta \\ \bar{v}_b(r, \theta) &= \sum_{n=2}^{\infty} \langle N(r) \rangle^T \{V_n\} \sin n\theta\end{aligned}\tag{5.16}$$

During the next section the buckling displacements will be replaced by these functions in Eqs. (5.10) through (5.12).

5.3.3 Second Variation of Total Potential Energy in Terms of Nodal Displacements

From Eq. (5.16) by substituting into Eqs.(5.10), (5.11), and (5.12) one obtains the internal strain energy due to the linear components of strains, the internal strain energy due to the non-linear components of strains, and the load potential terms in terms of the nodal displacements as

$$\begin{aligned}
\frac{1}{2}\bar{U}_L = \sum_{n=2}^{\infty} & \left\{ \frac{L(1-\nu)}{2\nu} \left(r \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + \frac{1}{r} \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right. \right. \\
& + \frac{1}{r} n^2 \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + 2 \frac{1}{r} n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \Bigg) \\
& + L \left(\langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{C}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{D}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right) \\
& + \frac{L(1-2\nu)}{4\nu} \left(\frac{1}{r} n^2 \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \frac{1}{r} \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right. \\
& + r \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + 2 \frac{1}{r} n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \\
& \left. \left. - 2 \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{C}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta - 2n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{C}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right) \right\} r dr
\end{aligned}
\tag{5.17}$$

Also, one has

$$\begin{aligned}
\frac{1}{2} \overline{\overline{U}}_{NL} = & \sum_{n=2}^{\infty} \left[r(u'_{p,I} + \lambda u'_{p,E}) \frac{L(1-\nu)}{2\nu} \left(\langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right) \right. \\
& + \frac{1}{r^2} (u_{p,I} + \lambda u_{p,E}) \frac{L(1-\nu)}{2\nu} \left(n^2 \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right. \\
& + 2n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \\
& \left. + n^2 \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + 2n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right) \\
& + \frac{1}{r} (u'_{p,I} + \lambda u'_{p,E}) \frac{1}{2} L \left(n^2 \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right. \\
& + 2n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \\
& \left. + n^2 \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + 2n \langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{A}] dr \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right) \\
& \left. + (u_{p,I} + \lambda u_{p,E}) \frac{1}{2} L \left(\langle U_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta + \langle V_n \rangle^T \int_{r_1}^{r_2} [\bar{B}] dr \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta \right) \right] dr
\end{aligned}
\tag{5.18}$$

and the load potential terms take the form

$$\begin{aligned}
\overline{\overline{W}}_I = & - \sum_{n=2}^{\infty} \frac{1}{2} P_{int} \left(\langle V_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle U_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right. \\
& \left. + n \langle U_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + n \langle U_n \rangle^T [\bar{A}] \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right)_{r=r_1} \\
\overline{\overline{W}}_E = & \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{1}{2} \lambda P_{ext} \left(\langle V_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + \langle U_n \rangle^T [\bar{A}] \{U_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right. \\
& \left. + n \langle U_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \sin^2 n\theta d\theta + n \langle U_n \rangle^T [\bar{A}] \{V_n\} \int_0^{2\pi} \cos^2 n\theta d\theta \right)_{r=r_2}
\end{aligned}
\tag{5.19}$$

where the following matrices have been defined.

$$\begin{aligned} [\bar{A}] &= \{N\} \langle N \rangle^T, & [\bar{B}] &= \{N\}' \langle N \rangle'^T \\ [\bar{C}] &= \{N\} \langle N \rangle'^T, & [\bar{D}] &= \{N\}' \langle N \rangle^T \end{aligned} \quad (5.20)$$

Using the orthogonality condition $\int_0^{2\pi} \cos n\theta \cos m\theta d\theta = \int_0^{2\pi} \sin n\theta \sin m\theta d\theta = 0, n \neq m$ and the

properties $\int_0^{2\pi} \cos^2 n\theta d\theta = \int_0^{2\pi} \sin^2 n\theta d\theta = \pi$, one can eliminate the dependence of Eqs. (5.17)

through (5.19) on θ yielding

$$\begin{aligned} \frac{1}{2} \bar{\pi} &= L\pi \sum_{n=2}^{\infty} \left\langle \{U_n\}^T [C_{e,u,n}] \{U_n\} + \{V_n\}^T [C_{e,v,n}] \{V_n\} + \{U_n\}^T [C_{e,uv,n}] \{V_n\} \right. \\ &\quad + \{U_n\}^T [C_{g,u,n,I}] \{U_n\} + \{V_n\}^T [C_{g,v,n,I}] \{V_n\} + \{U_n\}^T [C_{g,uv,n,I}] \{V_n\} \\ &\quad \left. + \lambda \left(\{U_n\}^T [C_{g,u,n,E}] \{U_n\} + \{V_n\}^T [C_{g,v,n,E}] \{V_n\} + \{U_n\}^T [C_{g,uv,n,E}] \{V_n\} \right) \right\rangle \end{aligned} \quad (5.21)$$

where

$$\begin{aligned}
[C_{e,u,n}] &= \left[\frac{(1-\nu)}{2\nu} + \frac{(1-2\nu)}{4\nu} n^2 \right] [\bar{\bar{A}}] + \frac{(1-\nu)}{2\nu} [\bar{\bar{B}}] + [\bar{\bar{C}}] \\
[C_{e,v,n}] &= \left[\frac{(1-\nu)}{2\nu} n^2 + \frac{(1-2\nu)}{4\nu} \right] [\bar{\bar{A}}] + \frac{(1-2\nu)}{4\nu} [\bar{\bar{B}}] - \frac{(1-2\nu)}{2\nu} [\bar{\bar{C}}] \\
[C_{e,uv,n}] &= \left[\frac{(1-\nu)}{\nu} n + \frac{(1-2\nu)}{2\nu} n \right] [\bar{\bar{A}}] + n [\bar{\bar{C}}]^T - \frac{(1-2\nu)}{2\nu} n [\bar{\bar{C}}] \\
[C_{g,u,n,I}] &= \frac{(1-\nu)}{2\nu} (1+n^2) [\bar{\bar{A}}_{g1,I}] + \frac{1}{2} (1+n^2) [\bar{\bar{A}}_{g2,I}] + \frac{(1-\nu)}{2\nu} [\bar{\bar{B}}_{g1,I}] \\
&\quad + \frac{1}{2} [\bar{\bar{B}}_{g2,I}] - \frac{1}{2} \frac{P_{int}}{L} [\bar{A}]_{r=r_1} \\
[C_{g,v,n,I}] &= \frac{(1-\nu)}{2\nu} (1+n^2) [\bar{\bar{A}}_{g1,I}] + \frac{1}{2} (1+n^2) [\bar{\bar{A}}_{g2,I}] + \frac{(1-\nu)}{2\nu} [\bar{\bar{B}}_{g1,I}] \\
&\quad + \frac{1}{2} L [\bar{\bar{B}}_{g2,I}] - \frac{1}{2} \frac{P_{int}}{L} [\bar{A}]_{r=r_1} \\
[C_{g,uv,n,I}] &= 2 \frac{(1-\nu)}{\nu} n [\bar{\bar{A}}_{g1,I}] + 2n [\bar{\bar{A}}_{g2,I}] - \frac{P_{int}}{L} n [\bar{A}]_{r=r_1} \\
\\
[C_{g,u,n,E}] &= \frac{(1-\nu)}{2\nu} (1+n^2) [\bar{\bar{A}}_{g1,E}] + \frac{1}{2} (1+n^2) [\bar{\bar{A}}_{g2,E}] + \frac{(1-\nu)}{2\nu} [\bar{\bar{B}}_{g1,I}] \\
&\quad + \frac{1}{2} [\bar{\bar{B}}_{g2,E}] + \frac{1}{2} \frac{P_{ext}}{L} [\bar{A}]_{r=r_2} \\
[C_{g,v,n,E}] &= \frac{(1-\nu)}{2\nu} (1+n^2) [\bar{\bar{A}}_{g1,E}] + \frac{1}{2} (1+n^2) [\bar{\bar{A}}_{g2,E}] + \frac{(1-\nu)}{2\nu} [\bar{\bar{B}}_{g1,I}] \\
&\quad + \frac{1}{2} L [\bar{\bar{B}}_{g2,E}] + \frac{1}{2} \frac{P_{ext}}{L} [\bar{A}]_{r=r_2} \\
[C_{g,uv,n,E}] &= \left\{ 2 \frac{(1-\nu)}{\nu} n [\bar{\bar{A}}_{g1,E}] + 2n [\bar{\bar{A}}_{g2,E}] + \frac{P_{ext}}{L} n [\bar{A}]_{r=r_2} \right\}
\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}
\left[\bar{\bar{A}} \right] &= \left\{ \int_{r_1}^{r_2} \frac{1}{r} [\bar{A}] dr \right\}, \quad \left[\bar{\bar{B}} \right] = \left\{ \int_{r_1}^{r_2} r [\bar{B}] dr \right\}, \quad \left[\bar{\bar{C}} \right] = \left\{ \int_{r_1}^{r_2} [\bar{C}] dr \right\} \\
\left[\bar{\bar{A}}_{g1,I} \right] &= \left\{ \int_{r_1}^{r_2} u_{p,I} \frac{1}{r^2} [\bar{A}] dr \right\}, \quad \left[\bar{\bar{A}}_{g2,I} \right] = \left\{ \int_{r_1}^{r_2} \frac{u'_{p,I}}{r} [\bar{A}] dr \right\} \\
\left[\bar{\bar{B}}_{g1,I} \right] &= \left\{ \int_{r_1}^{r_2} u'_{p,I} r [\bar{B}] dr \right\}, \quad \left[\bar{\bar{B}}_{g2,I} \right] = \left\{ \int_{r_1}^{r_2} u_{p,I} [\bar{B}] dr \right\} \\
\left[\bar{\bar{A}}_{g1,E} \right] &= \left\{ \int_{r_1}^{r_2} u_{p,E} \frac{1}{r^2} [\bar{A}] dr \right\}, \quad \left[\bar{\bar{A}}_{g2,E} \right] = \left\{ \int_{r_1}^{r_2} \frac{u'_{p,E}}{r} [\bar{A}] dr \right\} \\
\left[\bar{\bar{B}}_{g1,E} \right] &= \left\{ \int_{r_1}^{r_2} u'_{p,E} r [\bar{B}] dr \right\}, \quad \left[\bar{\bar{B}}_{g2,E} \right] = \left\{ \int_{r_1}^{r_2} u_{p,E} [\bar{B}] dr \right\}
\end{aligned} \tag{5.23}$$

5.3.4 Condition of neutral stability

By setting the variation of Eq. (5.21) to zero, one obtains a series of independent eigenvalue problems for $n = 2, 3, \dots, \infty$, given by

$$\begin{aligned}
&\left\{ \left[\begin{array}{c|c} [C_{e,u}] + [C_{e,u}]^T + [C_{g,u,I}] + [C_{g,u,I}]^T & [0] \\ \hline [0] & [C_{e,uv}] + [C_{g,uv,I}] \end{array} \right] + \right. \\
&\quad \left. \lambda \left[\begin{array}{c|c} [C_{g,u,E}] + [C_{g,u,E}]^T & [0] \\ \hline [0] & [C_{g,uv,E}] \end{array} \right] \right\} \begin{Bmatrix} \{U_n\} \\ \{V_n\} \end{Bmatrix} = 0
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
&\left\{ \left[\begin{array}{c|c} [C_{e,uv}]^T + [C_{g,uv,I}]^T & [0] \\ \hline [0] & [C_{e,v}] + [C_{e,v}]^T + [C_{g,v,I}] + [C_{g,v,I}]^T \end{array} \right] + \right. \\
&\quad \left. \lambda \left[\begin{array}{c|c} [C_{g,uv,E}]^T & [0] \\ \hline [0] & [C_{g,v,E}] + [C_{g,v,E}]^T \end{array} \right] \right\} \begin{Bmatrix} \{U_n\} \\ \{V_n\} \end{Bmatrix} = 0
\end{aligned} \tag{5.25}$$

Unlike Eq. (5.21), which involves a summation on all modes n , Eqs. (5.24) and (5.25) do not involve a summation. Thus, for a given mode n , the corresponding eigenvalue problem involves

only the nodal displacement vectors $\{U_n\}$ and $\{V_n\}$, but not $\{U_m\}$, $\{V_m\}$ $m \neq n$. The size of the resulting eigenvalue problems arising is particularly small resulting in significant computational efficiency.

5.4 Summary

This chapter provided a closed form solution for the pre-buckling equilibrium conditions for a general multi-layer sandwich pipe system. Using linear interpolation functions, a discretized form of the conditions of neutral stability was obtained. The resulting Eigen value problem is observed to consist of series of small size independent Eigen value problems for each mode separately, i.e., no coupling is observed between any two distinct modes $n \neq m$, leading to a particularly efficient solution. The above procedure was implemented under MATLAB (R2011b) and the Eigen value problem was solved using the generalized Eigen solver within the MATLAB library. A listing of the program is provided in Appendix A.6.

CHAPTER 6. Verification and Parametric Runs

6.1 General

The solutions developed in previous chapters are verified in the present chapter. Towards this goal, results based on the finite difference solutions are compared to those based on the finite element solution for homogeneous pipe (Section 6.2). The finite element model is then used to predict the buckling pressure of thin rings, thick rings and sandwich pipes. Comparisons are provided with various solutions reported in the literature and ABAQUS (Section 6.3). A sample of input files for the runs conducted in this chapter is provided in Appendix A.7. The verified model is then adopted to conduct a series of parametric studies (Section 6.4).

6.2 Comparison of Finite Difference and Finite Element Solutions

This section provides a numeric comparison of the finite difference solution developed in Chapter 4 with the finite element solution developed in Chapter 5. Two cases are considered for the comparison. The first case consists of a pipe with a 200 mm diameter and a 5 mm thickness, and the second case consists of a pipe with a 200 mm diameter and a 20 mm thickness. Figure 6-1 and Figure 6-2 provide the normalized buckling pressure P_{cr}/E of the two pipes versus the number of elements in the finite element analysis (or intervals in the finite difference solution) for both cases.

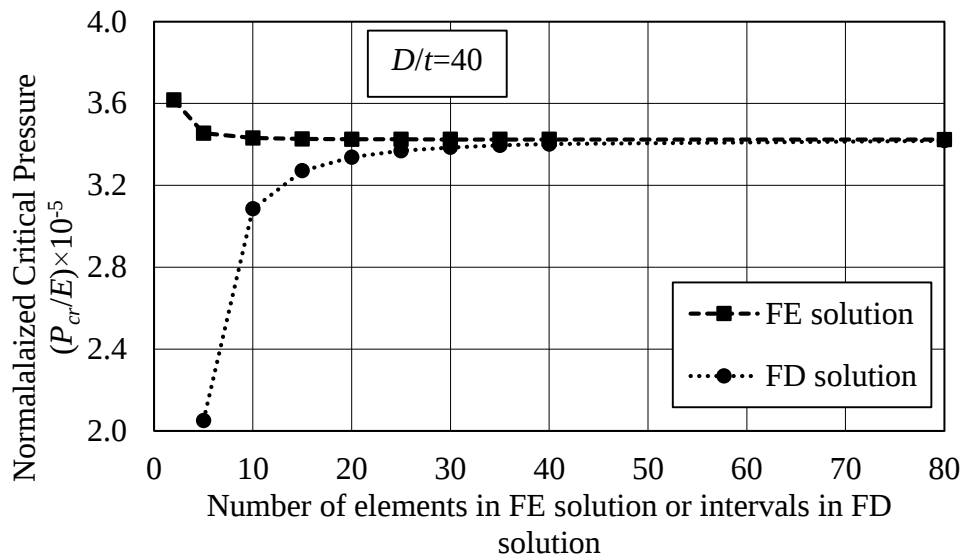


Figure 6-1 Convergence of FE and FD solutions for a pipe with diameter over thickness ratio (D/t) of 40

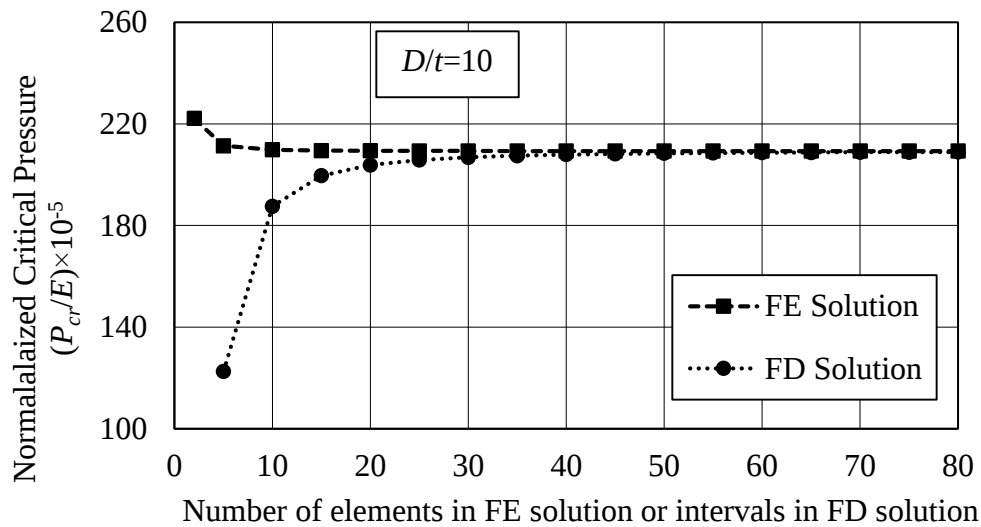


Figure 6-2 Convergence of FE and FD solutions for a pipe with diameter over thickness ratio (D/t) of 10

It is observed that both solutions are converging to the same buckling pressure. The finite element solution achieves convergence with 10 elements while in the finite difference solution, at least 40 intervals are required to achieve the same degree of accuracy. Also, while the finite element solution converges from above, the finite difference is observed to converge from below.

6.3 Verification

The validity of the finite element solution is assessed for two types of pipe configurations; homogeneous pipes (Section 6.3.1) and Sandwich Pipes (Section 6.3.2).

6.3.1 Homogeneous Pipes

In this section the buckling pressures of single layer rings predicted by the present study are compared to those based on published for thin shell theory, 2D ABAQUS solutions, and a refined simplified solution presented by Papadakis (2008). Based on thin shell theory (Pearson (1956)), the buckling pressure P_{cr} is given by

$$P_{cr} = \frac{3EI}{(1-\nu^2)R^3} \quad (6.1)$$

where $I = t^3/12$. By writing I in terms of the ring thickness and substituting R with $D/2$ one can rewrite the Eq.(6.1) in terms of D/t as follows

$$P_{cr} = \frac{2E}{(1-\nu^2)(D/t)^3} \quad (6.2)$$

The ABAQUS 2D solution is based on the CPE8R element which is used to idealize the problem as a plane strain ring under hydrostatic external pressure. As indicated in Figure 6-3, the ABAQUS mesh consists of 40 elements circumferentially by 12 elements radially.

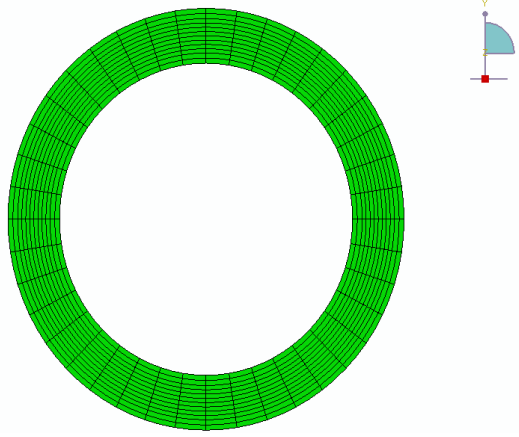


Figure 6-3 Un-deformed configuration of ABAQUS model with a mesh consisting of 40 elements circumferentially by 12 elements radially

In his investigation, Papadakis developed a 2D solution in which he considered the shear deformations and rotations induced by large deformations in developing an expression for the buckling pressure. However, his formulation neglected shear strains. According to Papadakis (Section 2.3.3), the buckling pressure of thick rings under hydrostatic pressure is given by

$$P_{cr} = \frac{1}{4} \frac{E}{1-\nu^2} \left(\frac{t}{R} \right)^3 \frac{(1-\zeta_{(n=2)})}{1 + \frac{t}{2R} - \zeta_{(n=2)} \left(\frac{5}{16} - \frac{25}{96} \frac{t}{R} \right)} \quad (6.3)$$

where $\zeta = (nt/R)^2 / \left[(nt/R)^2 + 5(1-\nu) \right]$.

A series of parametric analyses is conducted for various mean diameter D to thickness t ratios (D/t). Table 6-1 and Figure 6-4 provide a comparison based on all four solutions. The thin shell theory buckling solution takes into account the geometric nonlinearity and the coefficient 3 in Eq. (6.1) corresponds to the case where the pressure is treated as a follower force. However, the solution neglects shear deformation effects. By comparing the results based on the present study with those based on the thin-shell theory solution (i.e, Columns (2) and (3) in Table 6-1), it is observed that the thin-shell solution tends to be reliable to for relatively thin rings where $D/t > 40$.

In this range, very good agreement is observed between both solutions and with that based on the shell model (in Column (6)). All four solutions agree within 5% when $D/t > 40$.

Table 6-1 Buckling pressure of thin ring under hydrostatic pressure

(1) D/t	$(P_{cr}/E) \times 10^{-6}$			(5) Papadakis	(6) [3]/[2]	(7) [4]/[2]	(8) [5]/[2]
	(2) Present Study	(3) Thin shell Theory	(4) ABAQUS				
200	0.275	0.275	0.275	0.273	1.00	1.00	0.99
190	0.314	0.320	0.321	0.319	1.02	1.02	1.02
180	0.373	0.377	0.377	0.375	1.01	1.01	1.00
170	0.439	0.447	0.448	0.445	1.02	1.02	1.01
160	0.530	0.537	0.537	0.533	1.01	1.01	1.01
150	0.644	0.651	0.652	0.647	1.01	1.01	1.00
140	0.788	0.801	0.802	0.795	1.02	1.02	1.01
130	0.982	1.00	1.001	0.993	1.02	1.02	1.01
120	1.25	1.27	1.27	1.26	1.02	1.02	1.01
110	1.62	1.65	1.65	1.64	1.02	1.02	1.01
100	2.15	2.20	2.2	2.18	1.02	1.02	1.01
90	2.95	3.01	3.02	2.98	1.02	1.02	1.01
80	4.19	4.29	4.3	4.24	1.02	1.03	1.01
70	6.22	6.41	6.41	6.31	1.03	1.03	1.02
60	9.83	10.2	10.2	10.00	1.04	1.04	1.02
50	16.87	17.6	17.6	17.22	1.04	1.04	1.02
40	32.59	34.3	34.4	33.44	1.05	1.05	1.03
30	75.83	81.4	81.3	78.50	1.07	1.07	1.04
20	246	274	273	259.6	1.12	1.11	1.06
19	285	320	319	301.8	1.12	1.12	1.06
18	333	376	374	353.6	1.13	1.12	1.06
17	392	447	444	418.0	1.14	1.13	1.07
16	467	536	532	499.0	1.15	1.14	1.07
15	560	651	644	602.2	1.16	1.15	1.07

14	681	801	791	736.0	1.18	1.16	1.08
13	839	1000	986	912.3	1.19	1.18	1.09
12	1049	1272	1250	1150	1.21	1.19	1.10
11	1335	1651	1617	1476	1.24	1.21	1.11
10	1735	2198	2142	1939	1.27	1.23	1.12
9	2310	3015	2919	2615	1.31	1.26	1.13
8	3165	4292	4121	3643	1.36	1.30	1.15
7	4492	6408	6075	5281	1.43	1.35	1.18
6	6658	10175	9470	8049	1.53	1.42	1.21
5	10422	17582	15890	13090	1.69	1.52	1.26
4	17487	34341	29495	23193	1.96	1.69	1.33
3	32088	81400	63294	46154	2.54	1.97	1.44

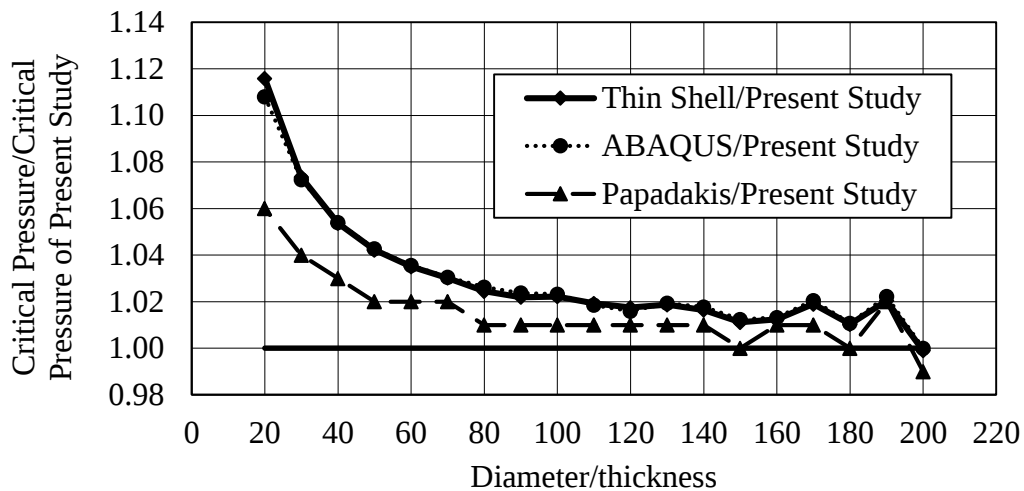


Figure 6-4 the ratio of the buckling pressure of different methods with respect to present study versus the ratio of the diameter to the thickness of the pipe for relatively thin rings

However, when the thickness increases (i.e. D/t is smaller than 40), the present solution yields lower buckling pressure predictions than that based on the thin shell theory. The difference between the two solutions grows as D/t decreases. The difference can be attributed to the neglect of shear deformation effect in thin shell theory. A similar conclusion can be made when comparing the ABAQUS solution with the shell theory solution. Again the difference is attributed to the

neglect of shear deformation effect. However, it is observed that the present solution does not agree with the ABAQUS predictions for thick pipes. This is the case because of two reasons:

- (a) While the present solution accounts for the follower effect, the ABAQUS solution is unable to capture the follower effect in an Eigen-value analysis (Simulia 2011).
- (b) As reported in Ji and Waas (2009, 2010, and 2014)) in Ji et al. (2013 and 2014), the stress-strain-constitutive measures adopted in ABAQUS (based on The Jaumann rate of the Kirchhoff stress tensor), the strain tensor (the Green Lagrange Strain) with a constant constitutive moduli are non-work-conjugate. In contrast, the present study adopts the Cauchy stress tensor, the Green Lagrange Strain tensor, and constant constitutive moduli are energy conjugate. Ji and Waas (2014) have numerically shown that the error involved by adopting in non-conjugate stress-strain-constitutive triplets in ABAQUS lead to an overestimation of the critical pressure. Using the critical pressure of the present study as a reference, the buckling pressure based on thin shell solution, and ABAQUS FEA analysis are shown in Figure 6-5. In a manner consistent with Ji and Waas (2014), the ABAQUS solution is observed to be consistently higher than that based on the present study.

According to Figure 6-5 the Papadakis solution predicts better results than thin shell theory and ABAQUS, and it was expected since the rotation or, in another word, some parts of the non-linear strains were assumed in its equations. Considerable difference is observed between the current study and Papadakis solution when thickness increases. The reasons for this difference are

- (a) Neglecting the shear strains as was assumed in Papadakis (2008).
- (b) Neglecting the radial pre-buckling stresses in the formulations.
- (c) Disregarding the follower force effect by adopting inaccurate boundary conditions.

(d) Neglecting the non-linear strain components makes the stress-strain constitutive measures non-conjugate

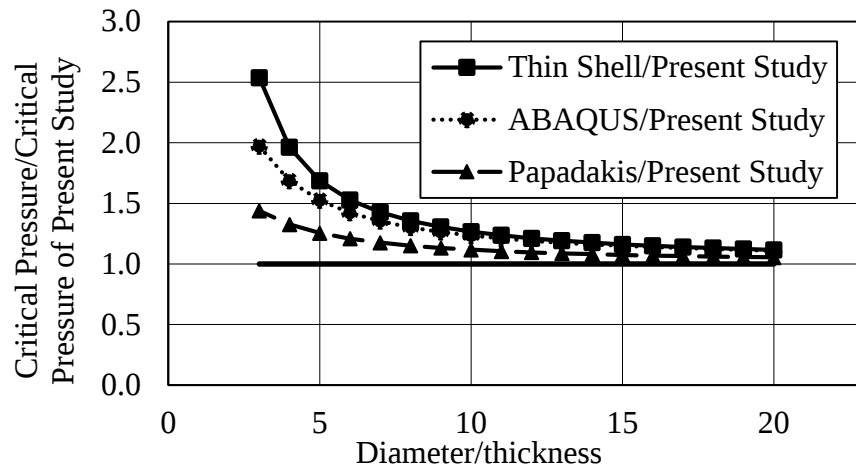


Figure 6-5 the ratio of the buckling pressure of different methods with respect to present study versus the ratio of the diameter to the thickness of the pipe for thick rings

6.3.2 Sandwich Pipes

In this section, the buckling pressures of sandwich pipe systems are provided based on (a) the formulations of the current study, (b) Sato and Patel (2007) solution, (c) ABAQUS CPE8R, (d) the simplified solution presented by Brush and Almroth (1975), and (e) Arjomandi and Taheri (2011) solution. In their investigations, Sato and Patel (2007) considered two thin pipes connected with a thick core layer as indicated in Section 2.4.4. They used the Euler-Lagrange equations to write the governing equations of elastic buckling of internal and external pipes in terms of the shear stress components due to the restraint of the core. Brush and Almroth (1975) considered the core as an elastic foundation for the external pipe. Thus, the buckling pressure of the system in their method only depended on the external pipe and core material properties (Section 2.4.2). Arjomandi and Taheri presented a simplified equation based on their FE analysis results (Section 2.4.7). Table 6-2 shows the buckling pressures and corresponding buckling mode numbers for a sandwich system with an external radius of 304.5 mm and a thickness of 9.0 mm,

and an internal pipe with a 9.0 mm thickness. The internal pipe radius was changed to vary the core thickness from 50 mm to 85 mm, to 115 mm. Also, the material modular ratio E_s/E_c was varied to take the values $10^4, 10^3$, and 20, resulting in nine parametric run combinations. As shown in Table 6-2, buckling pressure predictions based on the current study agree with that based on other methods when the core stiffness is relatively low relative to that of the internal and external pipes (i.e. $E_s/E_c = 1000$ and 10000). When the core stiffness increases (i.e. $E_s/E_c = 20$) the difference becomes significant especially when the core is thick (i.e., 115 mm). In this case, the Sato and Patel solution predicted that Mode 18 is the governing buckling mode, while the current study and ABAQUS both predicted that Mode 2 is the governing one.

Table 6-2 Comparison of Critical Pressure P_{cr} (MN/mm^2) ($r_4 = 304.5 \text{ mm}$, $t_{ext} = t_{int} = 9.0 \text{ mm}$)

$\frac{E_s}{E_c}$	h_c^* (mm)	(1) Current Study		(2) Sato and Patel		(3) ABAQUS		(4) Brush & Almroth		(5) Arjomandi and Taheri		(2) (1)	(3) (1)	(4) (1)	(5) (1)
		P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n				
10^4	115	10.0	2	10.6	2	10.1	2	8.29	3	11.4	2	1.06	1.01	0.83	1.14
	85	7.86	2	8.42	2	7.9	2			8.29	2	1.07	1.00	1.06	1.05
	50	5.66	2	6.07	2	5.69	2			5.94	2	1.07	1.00	1.46	1.05
10^3	115	36.4	2	40.5	2	38.5	2	36.5	5	37.3	2	1.11	1.06	1.00	1.03
	85	26.2	2	29.2	2	27.6	2			27.5	2	1.11	1.05	1.39	1.05
	50	16.8	2	18.5	2	17.4	2			17.7	2	1.10	1.04	2.17	1.05
20	115	910	2	570	18	1115	2	497	18	1279	2	0.62	1.22	0.55	1.40
	85	613	2	570	18	695	2			952	2	0.93	1.13	0.81	1.55
	50	293	2	324	2	309	2			581	2	1.10	1.05	1.7	1.98
Average												1.02	1.06	1.23	1.26
Standard Deviation												0.16	0.07	0.51	0.33

* h_c is the core thickness

A second set of runs was also considered for an external pipe radius of 304.5 mm and thickness of 9.0 mm. The internal pipe thickness was kept at 15 mm. As in the first set of parametric runs, the

modular ratio and the core thickness were varied. The corresponding results are provided in Table 6-3

Table 6-3 Comparison of Critical Pressure P_{cr} (MN/mm^2) ($r_4 = 304.5 \text{ mm}$, $t_{ext} = 9.0 \text{ mm}$, $t_{int} = 15.0 \text{ mm}$)

$\frac{E_s}{E_c}$	h_c^* (mm)	(1) Current Study		(2) Sato and Patel		(3) Abaqus		(4) Brush & Almroth		(5) Arjomandi and Taheri		$\frac{(2)}{(1)}$	$\frac{(3)}{(1)}$	$\frac{(4)}{(1)}$	$\frac{(5)}{(1)}$
		P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n				
10^4	115	12.1	4	12.3	4	12.2	4	8.29	3	35.4	2	1.02	1.01	0.69	2.93
	85	14.1	4	14.3	4	14.3	4			24.0	2	1.01	1.01	0.59	1.71
	50	15.0	2	16.3	2	15.2	2			16.5	2	1.08	1.01	0.55	1.10
10^3	115	45.7	6	41.7	6	48.3	6	36.5	5	66.7	2	0.91	1.05	0.80	1.46
	85	44.4	2	46.3	6	47.0	2			47.0	2	1.04	1.06	0.82	1.06
	50	28.8	2	32.6	2	30.2	2			30.9	2	1.13	1.05	1.27	1.07
20	115	993	2	570	18	1242	2	498	18	1351	2	0.57	1.25	0.50	1.36
	85	703	2	570	18	811	2			1010	2	0.81	1.15	0.71	1.43
	50	371	2	423	2	396	2			630	2	1.14	1.07	1.34	1.69
Average												0.97	1.07	0.81	1.54
Standard Deviation												0.18	0.08	0.30	0.58

When $E_s/E_c = 10000$ and the core thickness is 115 and 85 mm, all methods predict that Mode 4 is governing buckling mode (Graphical representations of typical mode shapes will be provided later on under Section 6.4.5.1). When $E_s/E_c = 1000$ and the core thickness is 115 mm, the governing buckling configuration corresponds to Mode 6. However, when the core stiffness increases, the differences become significant where with the core thickness of 115 mm and $E_s/E_c = 20$, the current study predicts a buckling pressure twice as larger than that predicted by Sato and Patel. Table 6-4 presents the buckling pressures of sandwich pipe systems with the internal pipe thickness of 9.0 mm and external pipe thickness of 15 mm and Table 6-5 shows the buckling pressure of sandwich pipes with internal and external pipe thickness of 15 mm. Similar conclusions can be drawn from both tables.

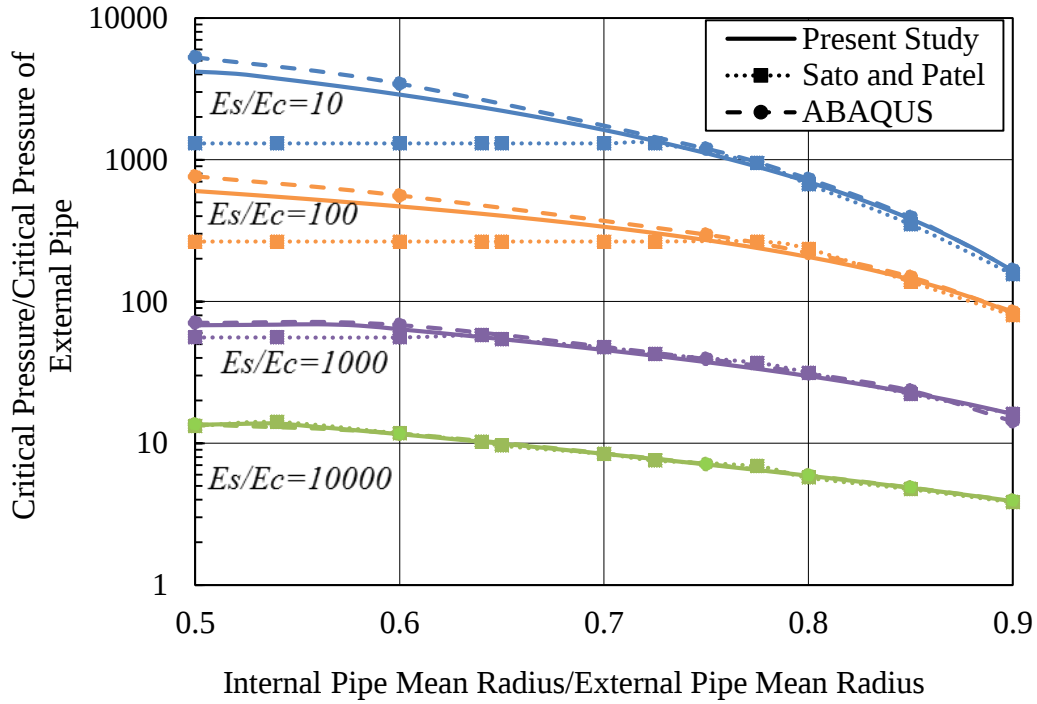
Table 6-4 Comparison of Critical Pressure P_{cr} (MN/mm^2) ($r_4 = 307.5 \text{ mm}$, $t_{ext} = 15.0 \text{ mm}$, $t_{int} = 9.0 \text{ mm}$)

$\frac{E_s}{E_c}$	h_c^* (mm)	(1) Current Study		(2) Sato and Patel		(3) Abaqus		(4) Brush & Almroth		(5) Arjomandi and Taheri		$\frac{(2)}{(1)}$	$\frac{(3)}{(1)}$	$\frac{(4)}{(1)}$	$\frac{(5)}{(1)}$
		P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n				
10^4	115	15.6	2	17.1	2	15.7	2	14.3	2	17.7	2	1.09	1.01	0.53	1.13
	85	13.5	2	14.8	2	13.5	2			14.2	2	1.10	1.00	0.61	1.05
	50	11.2	2	12.4	2	11.3	2			11.6	2	1.10	1.00	0.74	1.03
10^3	115	43.8	2	49.3	2	45.6	2	61.7	2	45.7	2	1.12	1.04	0.83	1.04
	85	33.4	2	37.5	2	34.5	2			35.1	2	1.12	1.03	1.09	1.05
	50	23.6	2	26.5	2	24.3	2			24.7	2	1.12	1.03	1.55	1.05
20	115	1022	2	970	11	1255	2	827	11	1386	2	0.95	1.23	0.49	1.36
	85	714	2	863	2	812	2			1041	2	1.21	1.14	0.70	1.46
	50	370	2	422	2	392	2			650	2	1.14	1.06	1.34	1.76
Average												1.11	1.06	0.87	1.21
Standard Deviation												0.07	0.07	0.37	0.26

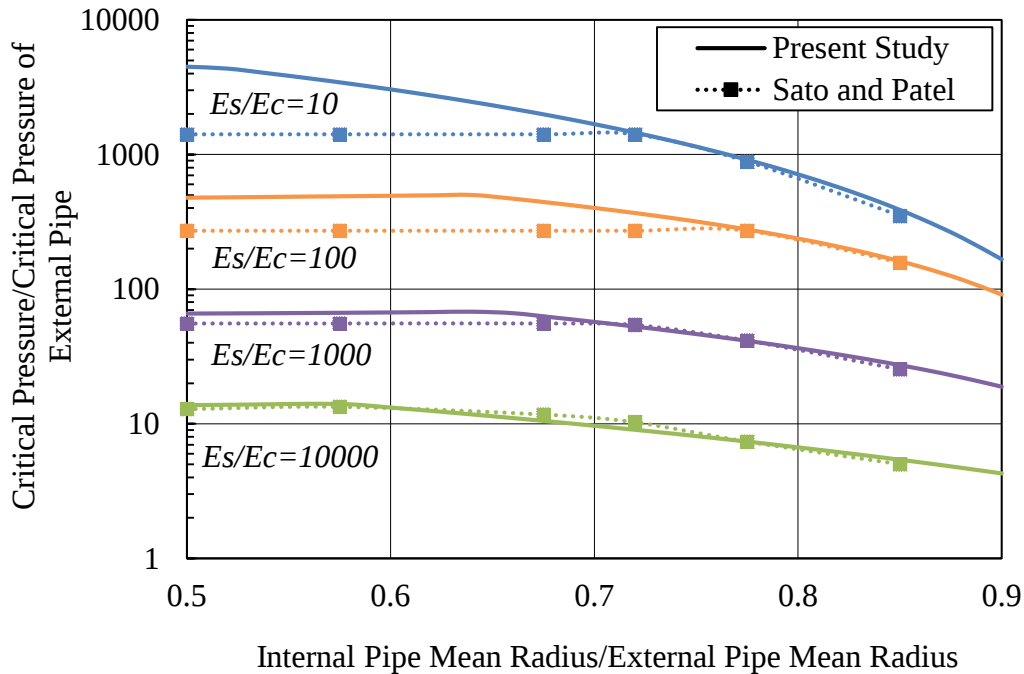
Table 6-5 Comparison of Critical Pressure P_{cr} (MN/mm^2) ($r_4 = 307.5 \text{ mm}$, $t_{ext} = t_{int} = 15.0 \text{ mm}$)

$\frac{E_s}{E_c}$	h_c^* (mm)	(1) Current Study		(2) Sato and Patel		(3) Abaqus		(4) Brush & Almroth		(5) Arjomandi and Taheri		$\frac{(2)}{(1)}$	$\frac{(3)}{(1)}$	$\frac{(4)}{(1)}$	$\frac{(5)}{(1)}$
		P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n	P_{cr}	n				
10^4	115	23.9	2	25.4	2	24.2	2	14.3	2	44.1	2	1.06	1.01	0.60	1.84
	85	23.9	2	26.1	2	24.1	2			31.4	2	1.09	1.01	0.60	1.32
	50	20.9	2	23.3	2	20.1	2			23.1	2	1.11	0.96	0.69	1.11
10^3	115	70.4	2	80.5	2	73.8	2	61.7	2	78.1	2	1.14	1.05	0.88	1.11
	85	52.4	2	60.5	2	54.6	2			56.4	2	1.15	1.04	1.18	1.08
	50	36.3	2	41.8	2	37.6	2			39.1	2	1.15	1.04	1.70	1.08
20	115	1107	2	970	11	944	2	827	11	1463	2	0.88	0.85	0.75	1.32
	85	811	2	976	11	943	2			1103	2	1.20	1.16	1.02	1.36
	50	464	2	545	2	500	2			702	2	1.17	1.08	1.78	1.51
Average												1.02	1.06	1.22	1.30
Standard Deviation												0.16	0.07	0.51	0.25

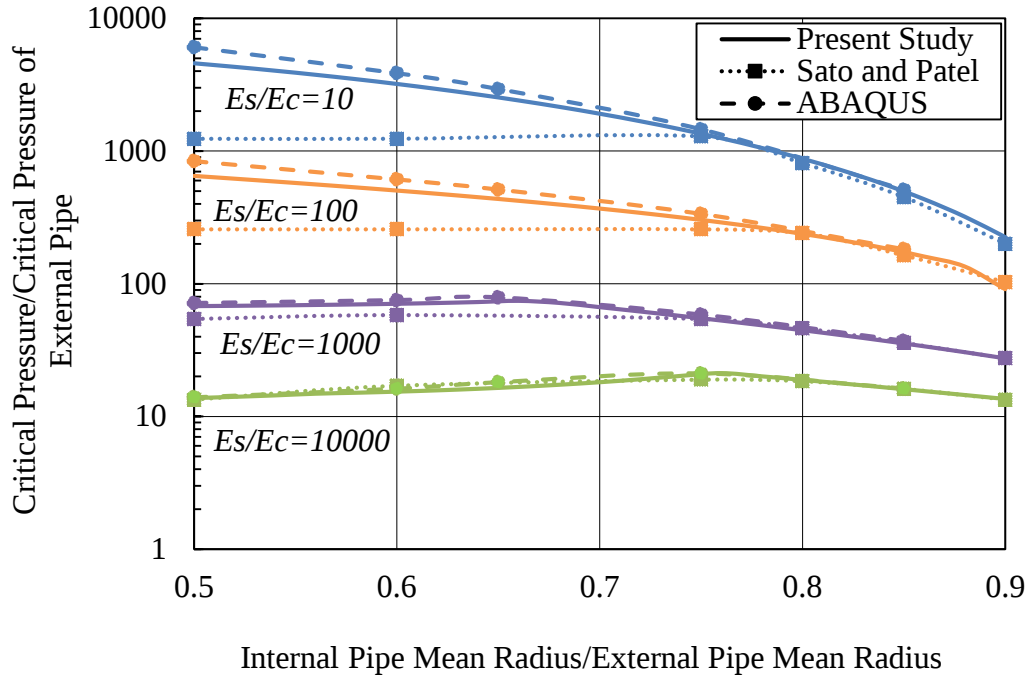
Another set of comparisons was provided against the results of Sato and Patel (Figure 6-6 a-e) where the ratio of the buckling pressure for the sandwich system normalized with respect to the buckling pressure of the external pipe is plotted against the ratio R_{int}/R_{ext} of the mean radius of internal pipe to the mean radius of external pipe. The continuous lines denote results based on the present formulation while the discrete points were read from the graphs published in Sato and Patel (2007). As a general observation, when the core stiffness is weak, the results are in good agreement. Also, when the core is thicker, the results are in good agreement except when $t_{ext}/R_{ext} = t_{int}/R_{ext} = 0.04$ and $\nu_c = 0.4$ (Figure 6-6 a), which show a larger discrepancy between both solutions. As identified in Section 2.4.5, these discrepancies are attributed to the fact that Sato and Patel neglected in their formulation the pre-buckling stresses undergoing non-linear strains. Where discrepancies between the present solution and that based on Sato and Patel results were observed, additional ABAQUS runs were conducted. In all cases, the present solution was found to be in closer agreement with ABAQUS results. Sato and Patel results were reported to agree well with those of Brush and Almroth (1975) which do not account for the stiffness of the inside pipe. In contrast, in the present work (as is the case in the ABAQUS solution), the interaction between the whole composite system is thoroughly incorporated. The difference in both approaches is a possible reason for the discrepancy observed.



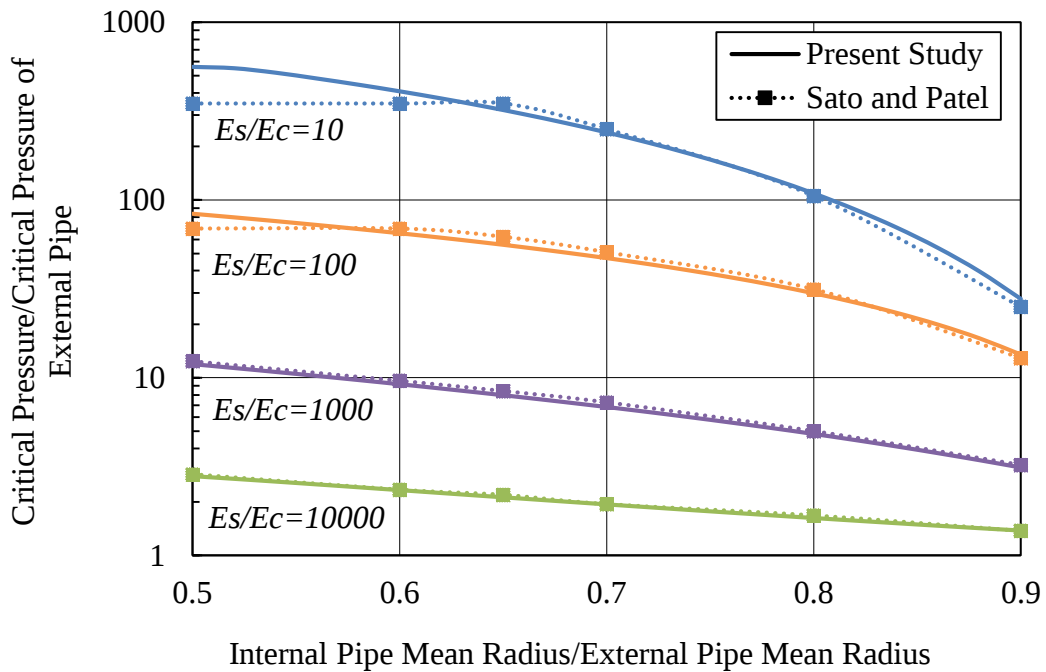
(a) External pipe radius/ External pipe thickness (R_{ext}/t_{ext})=50
External pipe radius/ Internal pipe thickness (R_{ext}/t_{int})=50
with Poisson ratio $\nu_c=0.4$



(b) External pipe radius/ External pipe thickness (R_{ext}/t_{ext})=50
External pipe radius/ Internal pipe thickness (R_{ext}/t_{int})=50
with Poisson ratio $\nu_c=0.1$



(c) External pipe radius/ External pipe thickness (R_{ext}/t_{ext})=50
 External pipe radius/ Internal pipe thickness (R_{ext}/t_{int})=25
 with Poisson ratio $\nu_c=0.4$



(d) External pipe radius/ External pipe thickness (R_{ext}/t_{ext})=25
 External pipe radius/ Internal pipe thickness (R_{ext}/t_{int})=50
 with Poisson ratio $\nu_c=0.4$

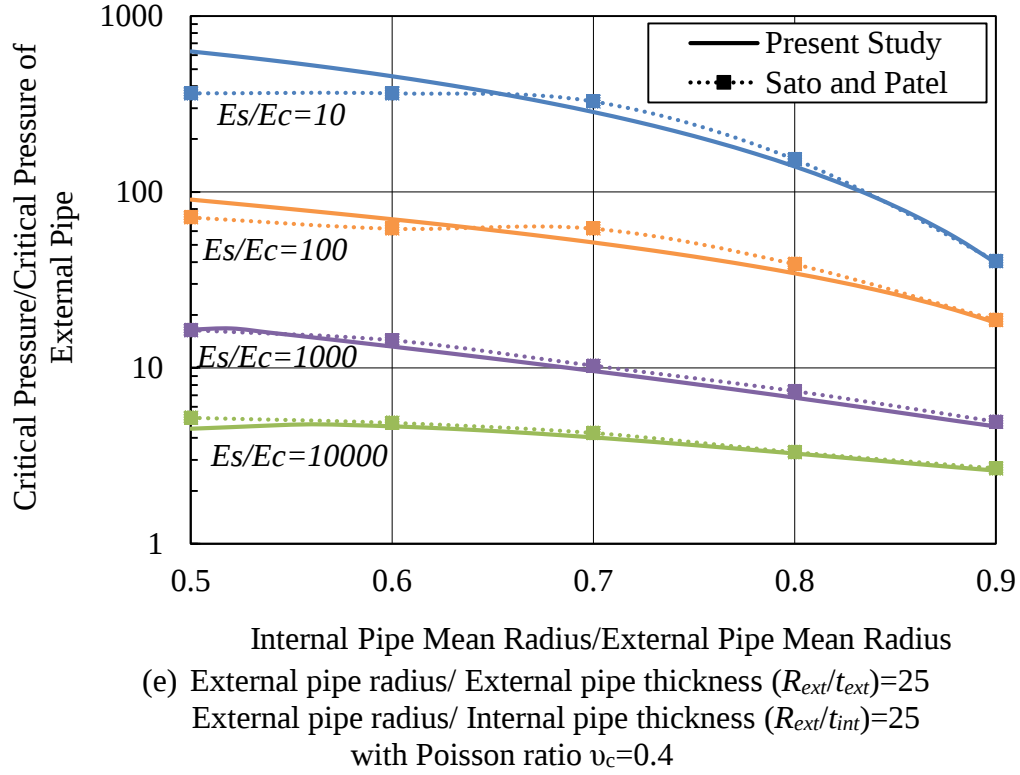


Figure 6-6 a-d Non-dimensional buckling pressure versus R_{int}/R_{ext} for (a) (R_{ext}/t_{ext}) = (R_{ext}/t_{int}) = 50 and $\nu_c = 0.40$, (b) (R_{ext}/t_{ext}) = (R_{ext}/t_{int}) = 50, and $\nu_c = 0.10$, (c) (R_{ext}/t_{ext}) = 50, (R_{ext}/t_{int}) = 25 and $\nu_c = 0.40$, (d) (R_{ext}/t_{ext}) = 25, (R_{ext}/t_{int}) = 50 and $\nu_c = 0.40$, and (e) (R_{ext}/t_{ext}) = (R_{ext}/t_{int}) = 25 and $\nu_c = 0.40$

6.4 Parametric Runs

6.4.1 Motivation

While the previous section has focused on establishing the validity of the model compared to other solutions, the present section adopts the model to generate a series of parametric runs.

When designing an offshore pipeline, the inside diameter (ID) of the inside pipe is known a-priori based on discharge requirements and is determined from hydraulic considerations. A structural designer may then be required in assessing several design scenarios involving core thicknesses, core material properties, outside pipe thickness, etc. on the critical external pressure of the system.

Thus, in the following sections, two reference cases are defined and various scenarios for the outside diameters are investigated in each case. Systematic deviations from the reference cases are

then considered while changing one parameter at a time, and the critical external pressure of the system is quantified.

6.4.2 Reference Cases

Two reference sandwich pipes with zero internal pressure are considered (R01 and R02 (Figure 6-7)). The composite systems are 1626 mm OD external steel pipe with 25.4 and 9.5 mm thicknesses (t_{ext}) and 1118 mm OD internal steel pipe with 23.8 and 8.7 mm thicknesses (t_{int}). The corresponding core layer thicknesses are 228.6 and 244.5 mm, respectively. Material properties are $E = 200,000MPa$, $\nu = 0.30$ and the core elastic material is $E_c = 200MPa$, $\nu = 0.40$ (Table 6-6). The differences between the two models are the thicknesses of the steel pipes. Both models are considered in order to showcase distinct behavioural characteristics. The systems are subject to an external pressure P_{ext} . The critical pressures $P_{cr,R01}$ and $P_{cr,R02}$ for two composite systems were sought. Note that indices R01 and R02 indicate that the parameter belongs to the reference case R01 or R02, respectively.

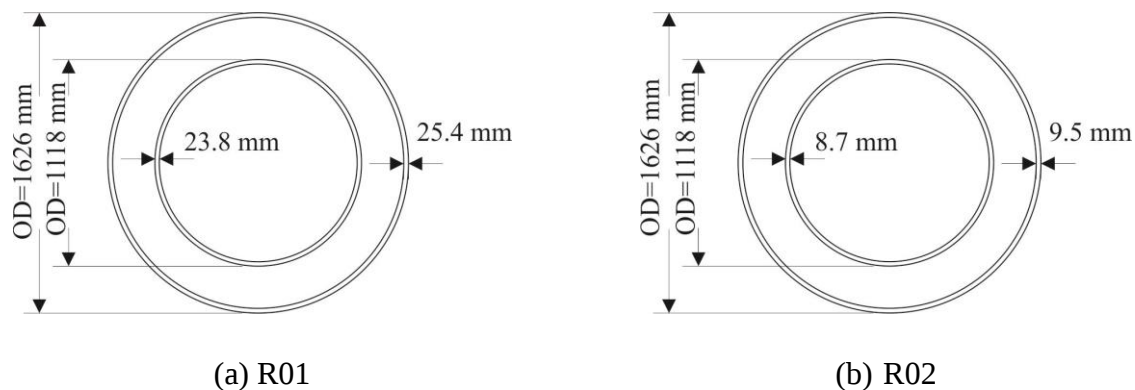
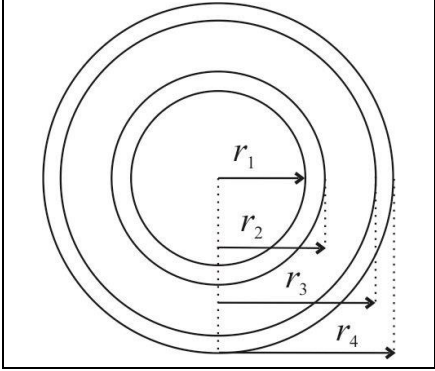


Figure 6-7 (a) Reference Case 1-(Thick Pipes) and (b) Reference Case 2 (Thin-Pipes)

Table 6-6 Reference Case 1 (R01) and Reference Case 2 (R02)

Reference Case	R01	R02
External Pipe OD (mm)	1626	1626
Internal Pipe OD (mm)	1118	1118
t_{ext} (mm)	25.4	9.5
t_{int} (mm)	23.8	8.7
E_s/E_c	1000	1000
ν_s	0.3	0.3
ν_c	0.4	0.4

Reference Case	R01	R02
r_1 (mm)	535.2	550.3
r_2 (mm)	559	559
r_3 (mm)	787.6	803.5
r_4 (mm)	813	813



6.4.3 Parameters investigated

According to Arjomandi and Taheri, adhesion between the core and both pipes is an important factor that affect buckling collapse capacity of composite pipes. However, this effect was not considered in the present study and thus the following parametric investigation exclude adhesion.

Thus, the external pressure capacity $P_{cr} = \lambda P_{ext}$ of a composite pipe system depends upon several parameters, i.e.,

$$\frac{P_{cr}}{E_s} = f_1 \left(\frac{P_{int}}{P_{cr}}, e, \nu_s, \nu_c, \frac{R_{ext}}{t_{ext}}, \frac{R_{int}}{t_{int}}, \frac{t_{int}}{t_{ext}} \right) \quad (6.4)$$

where $e = E_s/E_c$. Since the internal and external pipes are made of steel and elastic modulus E_s and Poisson's ratio ν_s exhibit little variability, one can rewrite Eq. (6.4) as

$$\frac{P_{cr}}{E_s} \approx f_2 \left(\frac{P_{int}}{P_{cr}}, e, \nu_c, \frac{R_{ext}}{t_{ext}}, \frac{R_{int}}{t_{int}}, \frac{t_{int}}{t_{ext}} \right) \quad (6.5)$$

In the following section, the effect of the parameters which influence the buckling capacity of the systems will be investigated. Firstly, the influence of the internal pressure P_{int} on the external buckling pressure of the system is investigated in Section 6.4.5.2. The effect of core material properties is then studied in Section 6.4.5.3. In Section 6.4.5.4, the thickness of the external pipe t_{ext} and thickness of internal pipe t_{int} will be varied and their effects on the buckling capacity will be investigated, while the core thickness effect is studied in Section 6.4.5.5.

6.4.4 Finite Element Mesh Study

Referring to the buckling formulation in Chapter 3, the 2D plane strain sandwich pipe system was modeled by 1D sets of nodes in the radial direction. Based on a mesh study conducted on the FE solution of the present study for the reference case 1 R01, using 2 elements in each layer of the sandwich pipe system, external pipe, internal pipe, and core layer gives a critical pressure with error of 1.5 percent which is considered acceptable. Figure 6-8 shows the results of the mesh study on the reference case R01

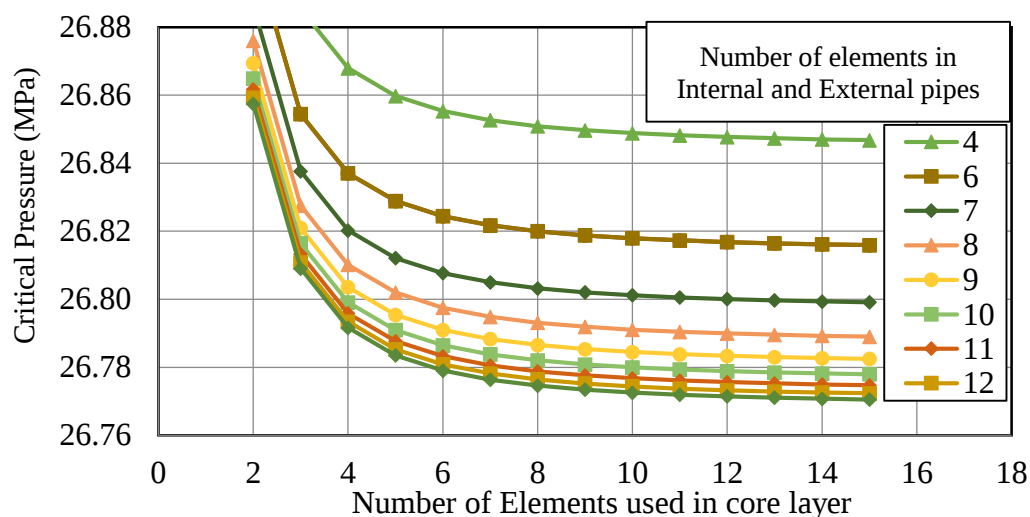


Figure 6-8 Mesh study for FE solution of the present study on R01

According to Figure 6-8, the buckling pressure is observed to be much more sensitive to the number of elements taken to model the internal and external pipe compared to that of the core layer. The figure suggests by taking 10 elements for each layer, reliable results are obtained for all

cases. Thus, in the following runs, 10 elements were taken per layer. Figure 6-9 represents the correspond schematic of the 1D model.

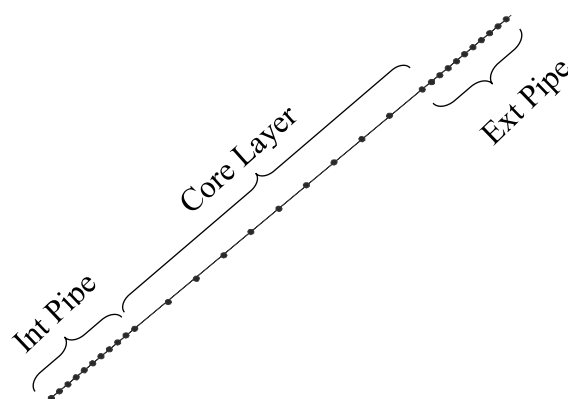


Figure 6-9 Schematic of 1D FE model in present study

6.4.5 Results

6.4.5.1 Buckling Pressures and Corresponding Modes

The FE buckling solution developed in Chapter 5 is used to determine the buckling pressure. Figure 6-10 represents the buckling pressure of the reference cases for various buckling modes $n = 2, 3, \dots$. To each mode n , corresponds a distinct buckling pressure and the critical pressure sought is that corresponding to the smallest value. Thus, for a given problem, it would be required to calculate the buckling pressure of different modes and select the mode corresponding to the minimum critical pressure. In these examples, the critical pressures are observed to take place at Mode 2 and take the values 26.8 and 18.34 MPa for cases 1 and 2, respectively.

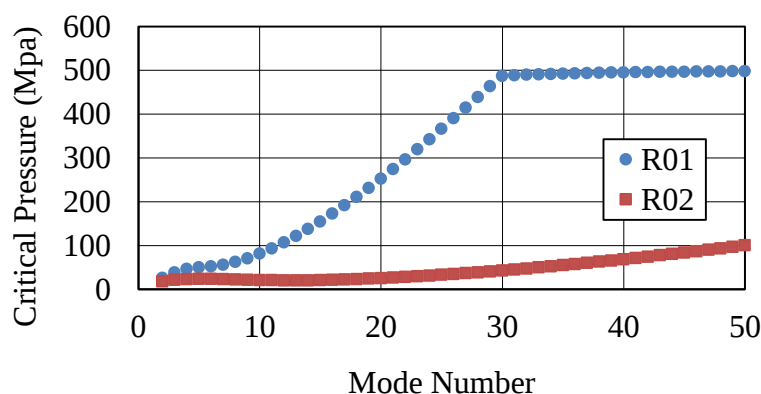
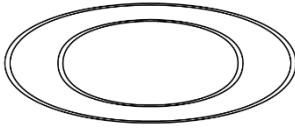
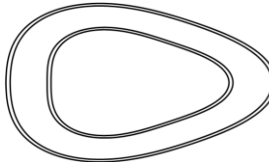
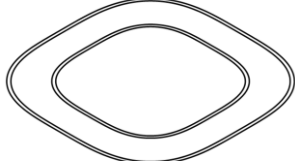
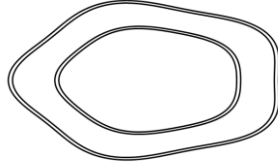
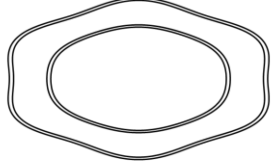


Figure 6-10 Buckling pressure of the reference cases R01 and R02 for different modes

Each mode represents a specific shape of deformation. In Mode 2, the system ovalizes while in the other modes have more complex buckling configurations (Table 6-7).

Table 6-7 Buckling Modes Configurations for reference cases R01 and R02 and associated buckling pressures.

Mode Number	Critical Pressure for Case R01(MPa)	Critical Pressure for Case R02 (MPa)	Buckled configuration
2	26.77	18.34	
3	39.08	22.25	
4	47.19	23.65	
5	50.61	24.19	
6	52.56	24.22	

6.4.5.2 Effect of internal pressure

In the previous section, the internal pressure was assumed to vanish. In this section, the effect of the internal pressure on the buckling pressure of the systems is investigated (Figure 6-11 and Figure 6-12).

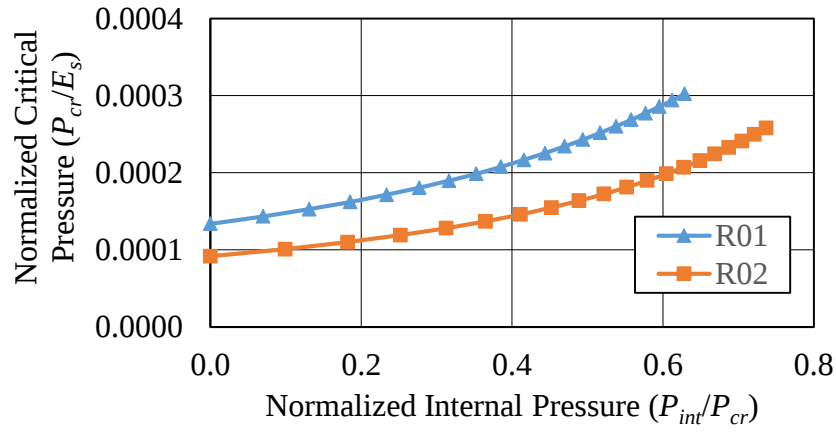


Figure 6-11 Effect of internal pressure on the buckling capacity of sandwich pipes

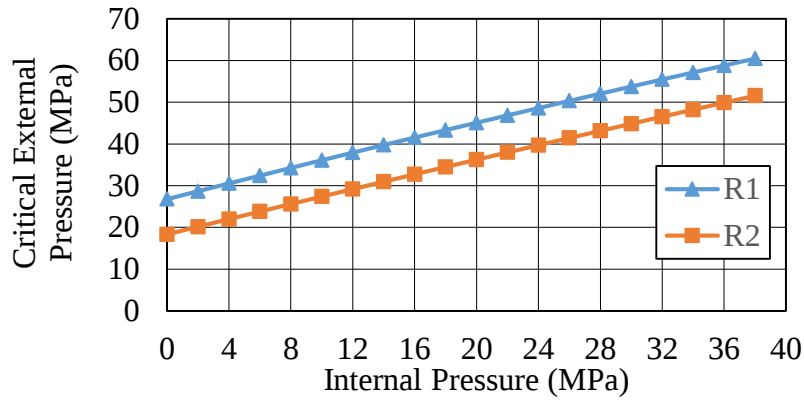


Figure 6-12 Effect of internal pressure on the buckling capacity of R01 and R02

As observed in Figure 6-12, adding the internal pressure to the system is observed to lead to higher buckling pressure. For the given problems, the critical external pressure-internal pressure relations are nearly linear. For the given reference cases, the critical pressures $P_{cr,R01}$ and $P_{cr,R02}$ can be approximately related to the critical pressure $P_{cr0,R01}$ and $P_{cr0,R02}$ at no internal pressure through

$$\begin{aligned} P_{cr,R01} &\approx P_{cr0,R01} + \alpha_1 P_{int} \\ P_{cr,R02} &\approx P_{cr0,R02} + \alpha_2 P_{int} \end{aligned} \quad (6.6)$$

where $\alpha_1 = 0.8863$, $\alpha_2 = 0.875$, and the correlation coefficients of Eqs. (6.6) are $R_1^2 = R_2^2 = 0.999$

6.4.5.3 Effect of the core material stiffness

This section investigates the effect of the core material properties on the buckling pressure of the sandwich pipe systems. Different combinations of elastic moduli E_c and Poisson ratio ν_c are considered for the core layer. In the first scenario (subsequently referred to as S1) a sandwich pipe system is considered with dimensions identical to those of R01 in which the core material parameters (E_c and ν_c) are varied from those of reference case R01 and the critical pressure is obtained for each case. In all scenarios, Mode 2 was observed to always correspond to the smallest buckling pressure (Figure 6-13 and Table 6-8)

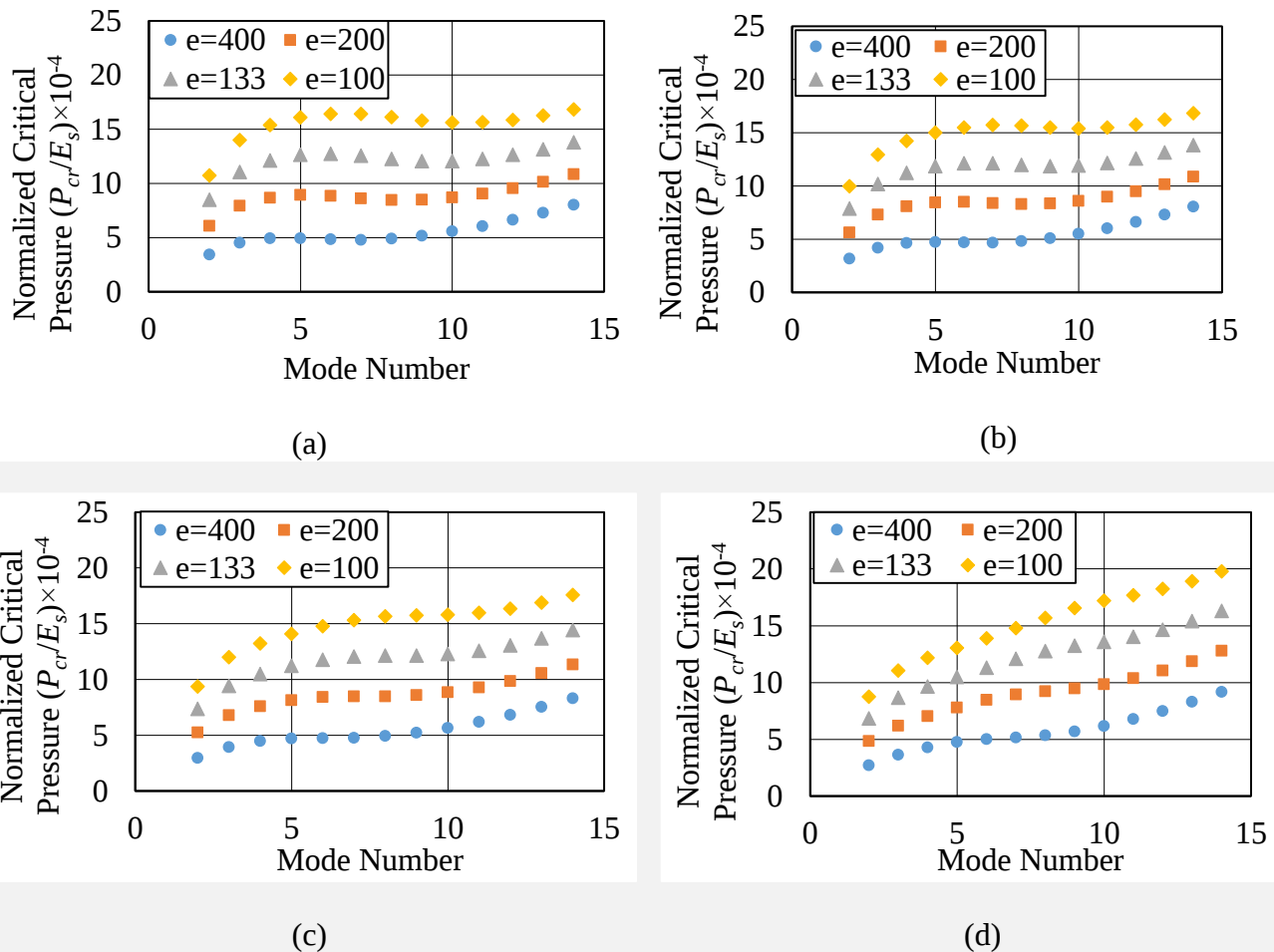
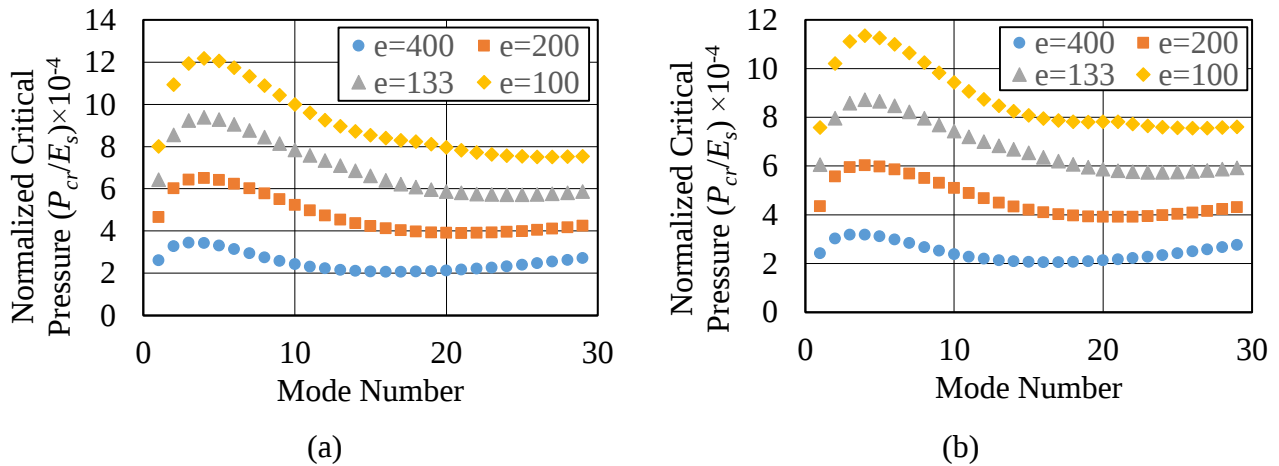


Figure 6-13 Effect of core material on first scenario S1 where the Poisson ratios are (a) $\nu_c = 0.1$, (b) $\nu_c = 0.2$, (c) $\nu_c = 0.3$, and (d) $\nu_c = 0.4$

Table 6-8 Buckling pressure and mode number for Scenario S1

e	$\nu_c = 0.1$		$\nu_c = 0.2$		$\nu_c = 0.3$		$\nu_c = 0.4$	
	P_{cr}/E_s	n	P_{cr}/E_s	n	P_{cr}/E_s	n	P_{cr}/E_s	n
400	0.00034	2	0.00032	2	0.00029	2	0.00027	2
200	0.00061	2	0.00056	2	0.00052	2	0.00048	2
133	0.00085	2	0.00079	2	0.00073	2	0.00068	2
100	0.00107	2	0.00100	2	0.00093	2	0.00088	2
80	0.00128	2	0.001197	2	0.001124	2	0.001058	2
50	0.001848	2	0.00174	2	0.001647	2	0.001566	2
20	0.003508	2	0.003364	2	0.003242	2	0.003144	2
10	0.005173	2	0.005037	2	0.004927	2	0.004847	2

In Figure 6-13, it is observed that a system with a stiffer core material, i.e., lower e (or higher E_c) gives a higher critical pressure. Also, a smaller Poisson ratio makes the core stiffer and leads to higher buckling pressures. In the second scenario, (referred to as S2) a system with the same geometry as the reference case R02 but with different core material properties $e = E_s/E_c$ ranging from 100-400 and ν_c ranging from 0.1 to 0.4 is considered. For the values of $e = E_s/E_c$ considered, the critical pressure is observed to correspond to other modes than Mode 2 (Figure 6-14). Also, the Poisson ratio effect is observed to affect the critical pressure only mildly.



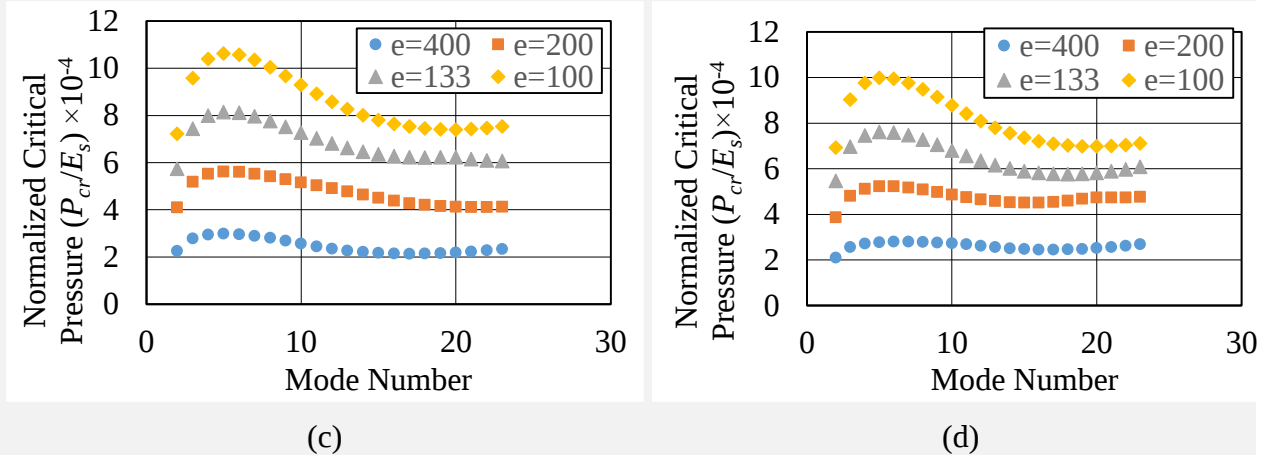


Figure 6-14 Effect of core material on second scenario S2 where the Poisson ratios are (a) $\nu_c = 0.1$, (b) $\nu_c = 0.2$, (c) $\nu_c = 0.3$, and (d) $\nu_c = 0.4$

The buckling pressures and their corresponding mode numbers are provided in Table 6-9 and illustrated in Figure 6-14. Additional runs were provided for $e = E_s/E_c = 10, 20, 50, 80$. Generally, when the core material is weaker (i.e., high e) and the Poisson ratio is smaller, the governing buckling pressure is observed to occur at modes higher than 2. Conversely, when the core material is stiffer (i.e., e is low), the critical pressure is observed to correspond to mode $n = 2$.

Table 6-9 Buckling pressure and mode number for Scenario S2

e	$\nu_c = 0.1$		$\nu_c = 0.2$		$\nu_c = 0.3$		$\nu_c = 0.4$	
	P_{cr}/E_s	n	P_{cr}/E_s	n	P_{cr}/E_s	n	P_{cr}/E_s	n
400	0.00021	18	0.0002	17	0.00021	17	0.00021	2
200	0.00039	22	0.00039	22	0.00041	2	0.00039	2
133	0.00058	23	0.00058	23	0.00057	2	0.00055	2
100	0.00077	23	0.00076	2	0.00072	2	0.00069	2
80	0.00094	2	0.00089	2	0.00085	2	0.00088	21
50	0.001291	2	0.00124	2	0.00120	2	0.00116	2
20	0.00215	2	0.00211	2	0.00208	2	0.00206	2
10	0.00299	2	0.00296	2	0.00296	2	0.00298	2

Figures 6-15 and 6-16 provide the normalized critical pressure as a function of elasticity modulus ratio $e = E_s/E_c$ varying from 10 to 400 (i.e., E_c varies from 500 MPa to 2000 MPa). A stiffer core material is shown to lead to a higher normalized critical pressure.

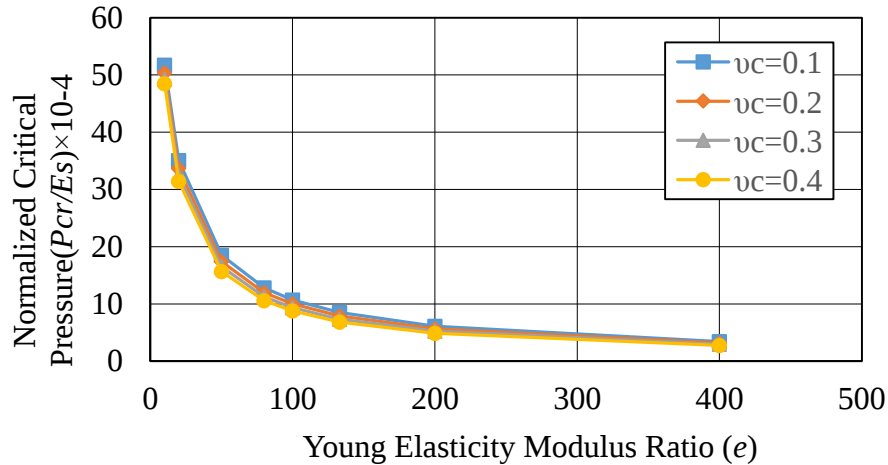


Figure 6-15 Normalized Critical Pressure (P_{cr}/E_s) versus Young Elasticity Modulus Ratio (e) in Scenario S1 for Poisson Ratios $\nu_c=0.1, 0.2, 0.3$, and 0.4

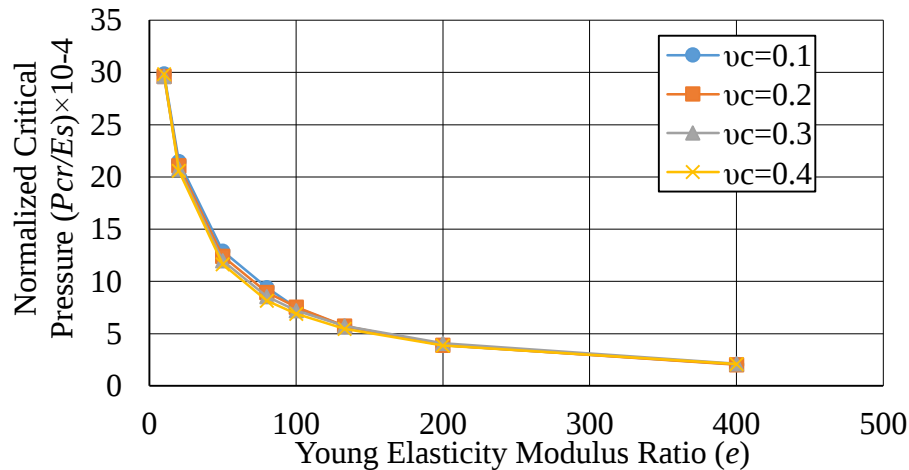


Figure 6-16 Normalized Critical Pressure (P_{cr}/E_s) versus Young Elasticity Modulus Ratio (e) in Scenario S2 for Poisson Ratios $\nu_c=0.1, 0.2, 0.3$, and 0.4

In Figures 6-15 and 6-16 one can observe that, in both scenarios S1 and S2, a stiffer Young modulus for the core layer significantly improves the buckling capacity. A similar conclusion was made by Arjomandi and Taheri (2011). Also, under both scenarios, the critical pressure is observed to be only mildly influenced by Poisson's ratio. In Scenario S1, when the Poisson ratio decreases,

the buckling pressure is found to increase. Irrespective of the Poisson's ratio, the governing buckling pressure is observed to always correspond to Mode 2. In contrast, for Scenario S2, the governing buckling pressure is observed to correspond to higher modes.

6.4.5.4 Effect of steel pipe thicknesses

The present section investigates the effect of internal and external pipe thickness on the buckling capacity of the sandwich pipe system. A range of thicknesses (6 to 30 mm) is considered for both the internal and external pipes. For each case, a buckling analysis was conducted for various buckling modes. The results are shown in Figures 6-17 to 6-23. In each figure, the outside diameters for internal and external pipes are identical to those of reference cases R01 or R02 and the external pipe thickness is kept constant for each figure (i.e., $t_{ext} = 6, 10, 14, 22, 26, 30 \text{ mm}$ respectively in Figures 6-17 to 6-23), while the internal pipe thickness is varied (from $t_{int} = 6$ to 30 mm). In Figure 6-17 where the external pipe is relatively thin, the governing buckling pressure occurs at Mode 20, which is about 20% less than the critical pressure at Mode 2.

When the external pipe thickness increases to 10 mm, and the internal pipe thickness is smaller than 24 mm the buckling is observed to occur at Mode 2. For a thickness greater than 24 mm, the governing buckling pressure takes place at Mode 12 (Figure 6-18). For systems with an external pipe thickness greater than 14 mm, the governing buckling pressure corresponds to Mode 2 (Figures 6-19 to 6-23). As a general observation, buckling in higher modes is observed to correspond to a significant reduction in the buckling capacity of the system, particularly when the thickness of the internal pipe is small.

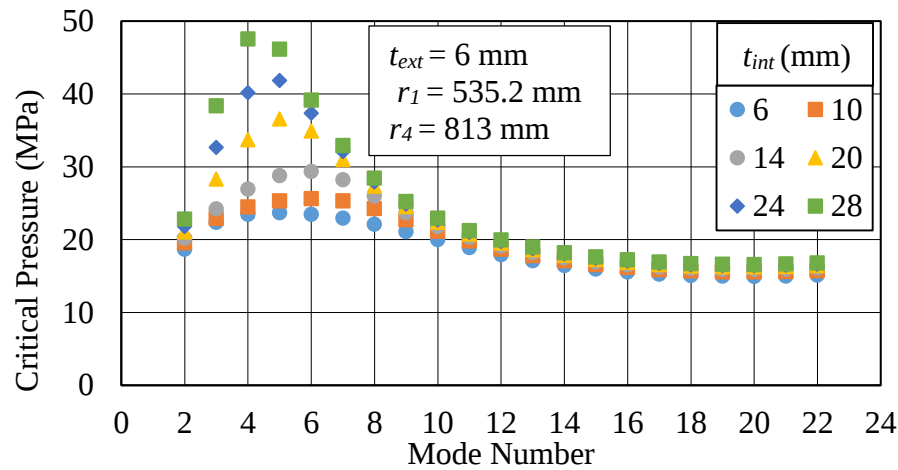


Figure 6-17 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 6 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure.

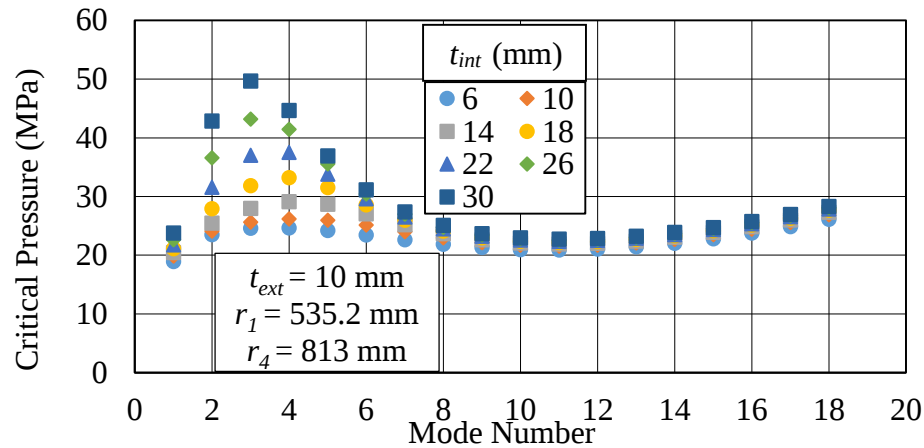


Figure 6-18 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 10 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure.

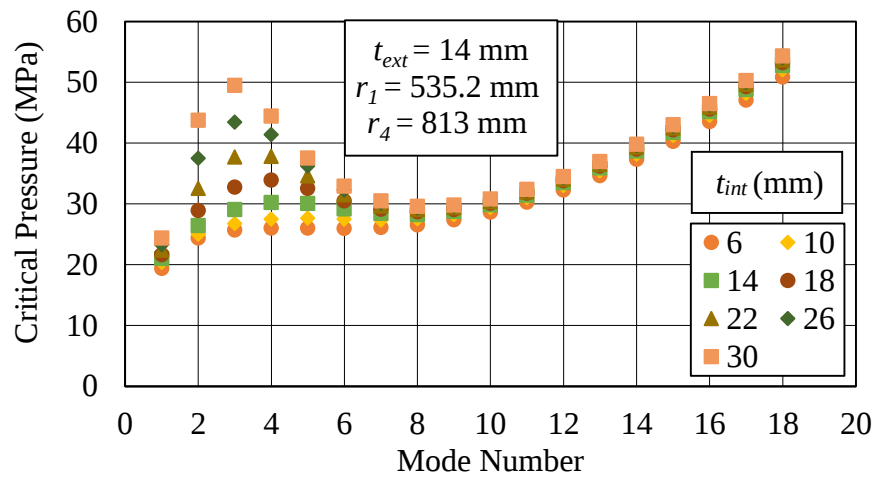


Figure 6-19 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 14 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure.

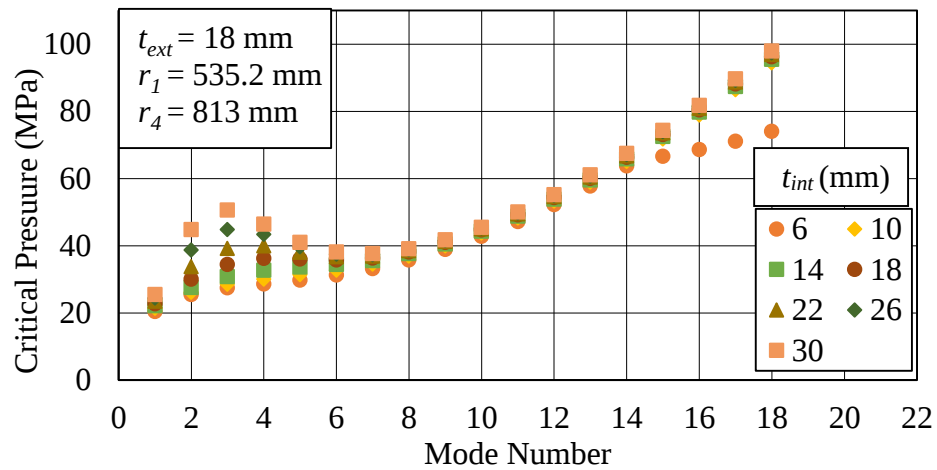


Figure 6-20 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 18 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure

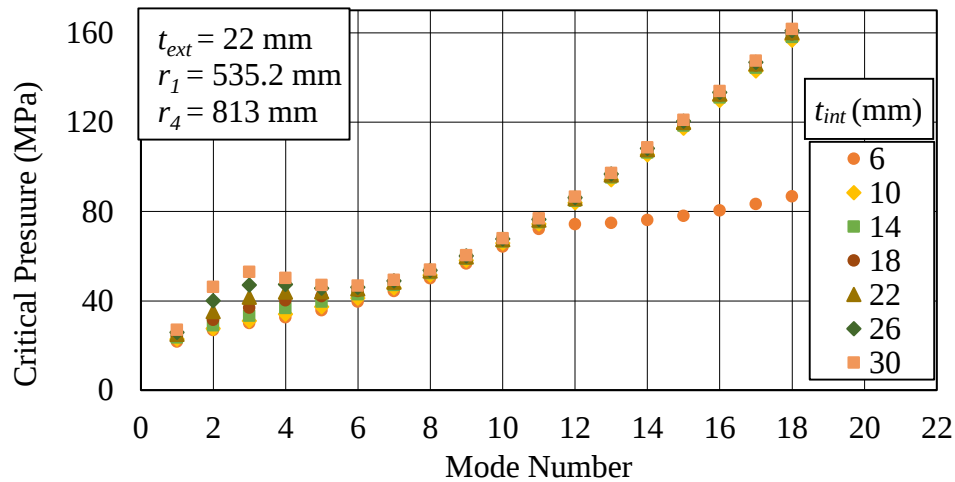


Figure 6-21 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 22 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure

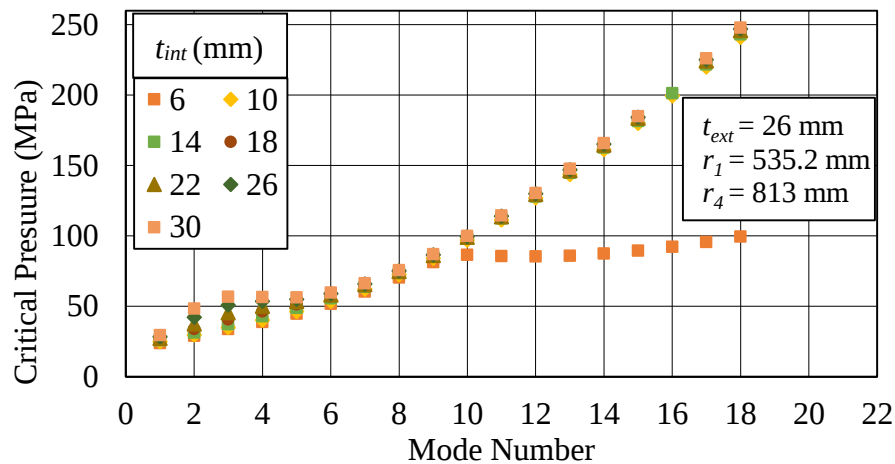


Figure 6-22 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 26 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure

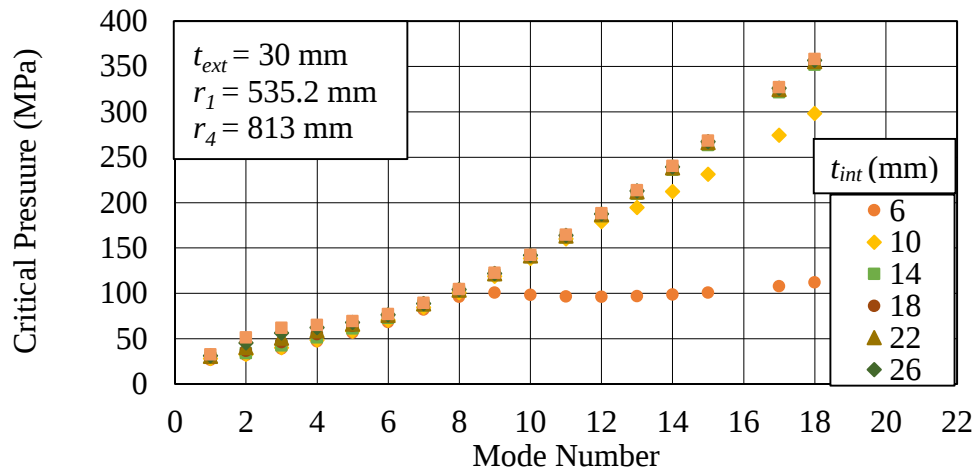
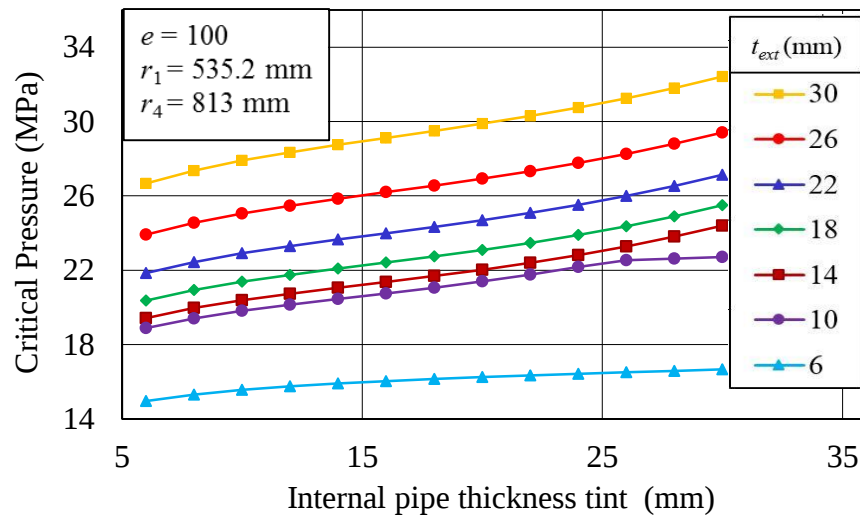


Figure 6-23 Critical pressure of sandwich pipes with external pipe thickness (t_{ext}) of 30 mm, internal radius (r_1) of 535.2 mm, and external radius (r_4) of 813 mm, and different internal pipe thickness (t_{int}) as indicated in the figure

From Figures 6-17 to 6-23, the governing pressure P_{cr} is extracted in each case and plotted against the internal and external pipe thicknesses and the results are presented in Figure 6-24. Figure 6-24 (b) shows that when the external pipe thickness is in the 5-10mm or the 14-30mm ranges the critical pressure increases significantly. In contrast, the gain in pressure capacity achieved in the range of 10mm-14mm is observed to be less significant. In this range, it would thus be more beneficial to increase the thickness of the internal pipe as evidenced by the steeper slopes observed in Figure 6-24 (a).



(a)

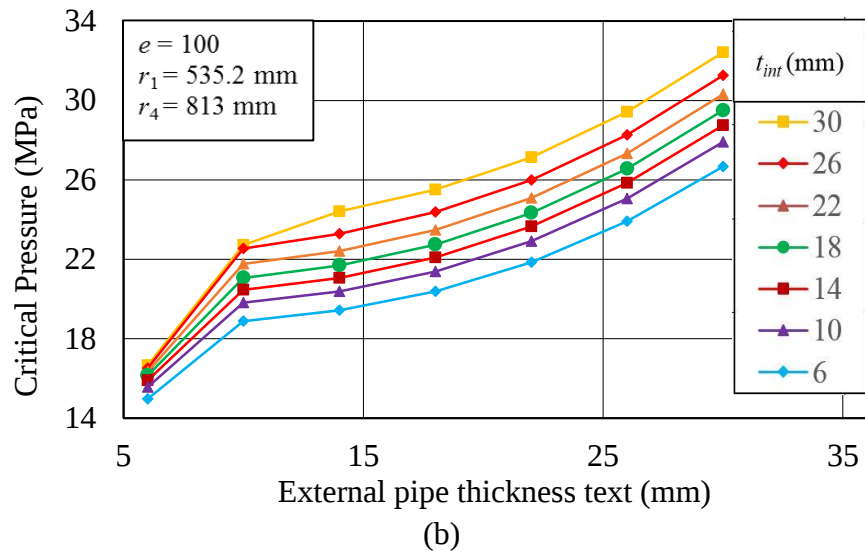


Figure 6-24 Effect of steel pipe thicknesses where internal radius r_1 is 535.2 mm, external radius r_4 is 813 mm, and Young elastic modulus ratio e is 1000, a) critical pressure versus internal pipe thickness, b) critical pressure versus external pipe thickness.

In another set of runs, the effect of thickness to radius ratio of the internal and external pipe t_{int}/R_{int} and t_{ext}/R_{ext} is investigated. Figure 6-25 provides the normalized buckling pressure (P_{cr}/E_s) of a system with Young modulus ratio (e) of 1000 and a thickness ratio t_{ext}/t_{int} of 1.1 where the other parameters (t_{int}/R_{int} and t_{ext}/R_{ext}) are varied such as indicated in the figure. It is observed that the normalized critical pressure decrease as the radius to thickness ratio of both pipes increase.

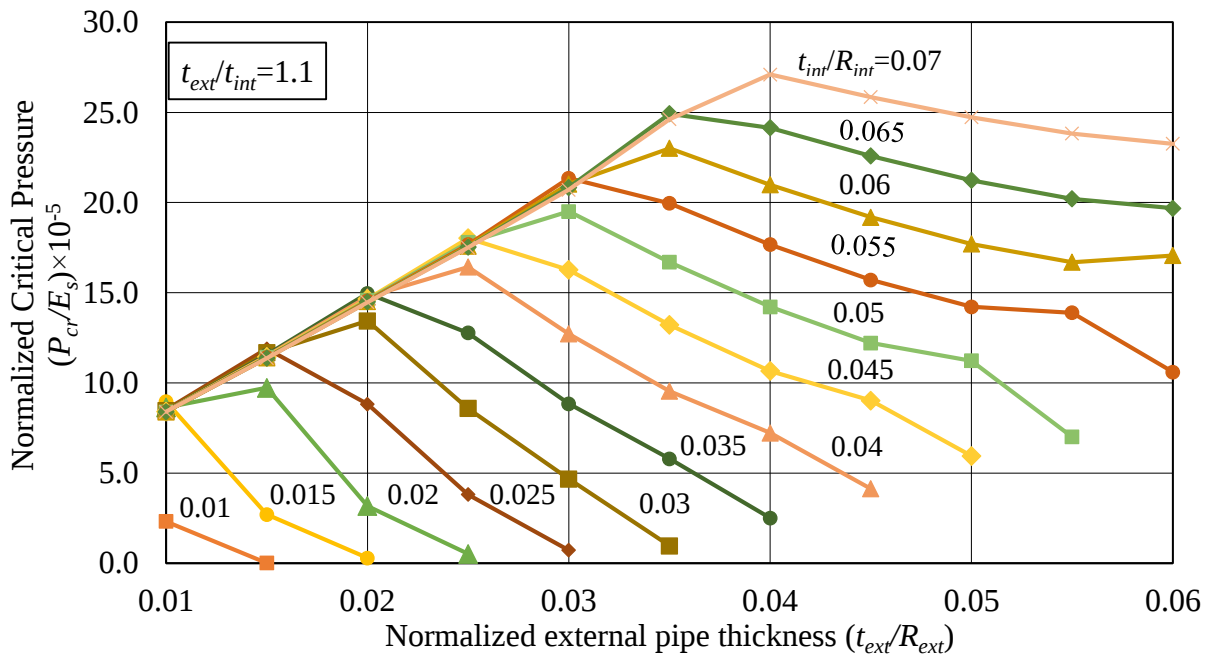


Figure 6-25 Normalized external pressure P_{cr}/E_s versus normalized external pipe thickness t_{ext}/R_{ext} for various internal thickness (t_{int}/R_{int} ranging from 0.015 to 0.07 as marked on figure and $t_{ext}/t_{int}=1.1$)

6.4.5.5 Effect of the core thickness

In this section, the effect of core thickness is assessed on the buckling pressure of the SP system. Two sets of parametric runs are conducted, in which all parameters were taken identical to those of reference cases R01 and R02 (Section 6.4.2) with the exception that the outside radius (R_{ext}) of the external pipe, which was varied in order to increase the core size. Figure 6-26 provides the critical non-dimensional pressures corresponding to the first 15 modes for the first set of runs (based on R01 geometries). The buckling pressures were observed to correspond to Mode 2. Within the range of R_{ext}/t_{ext} considered, the buckling capacity is observed to increase with the radius to thickness ratio R_{ext}/t_{ext} of the outside pipe. The trend of critical pressures suggest that for R_{ext}/t_{ext} higher than 40 this may not be the case. For the governing mode (Mode 2), the critical pressure magnitude shows a nearly linear trend with the R_{ext}/t_{ext} . Table 6-10 represents the numeric values provided in Figure 6-26.

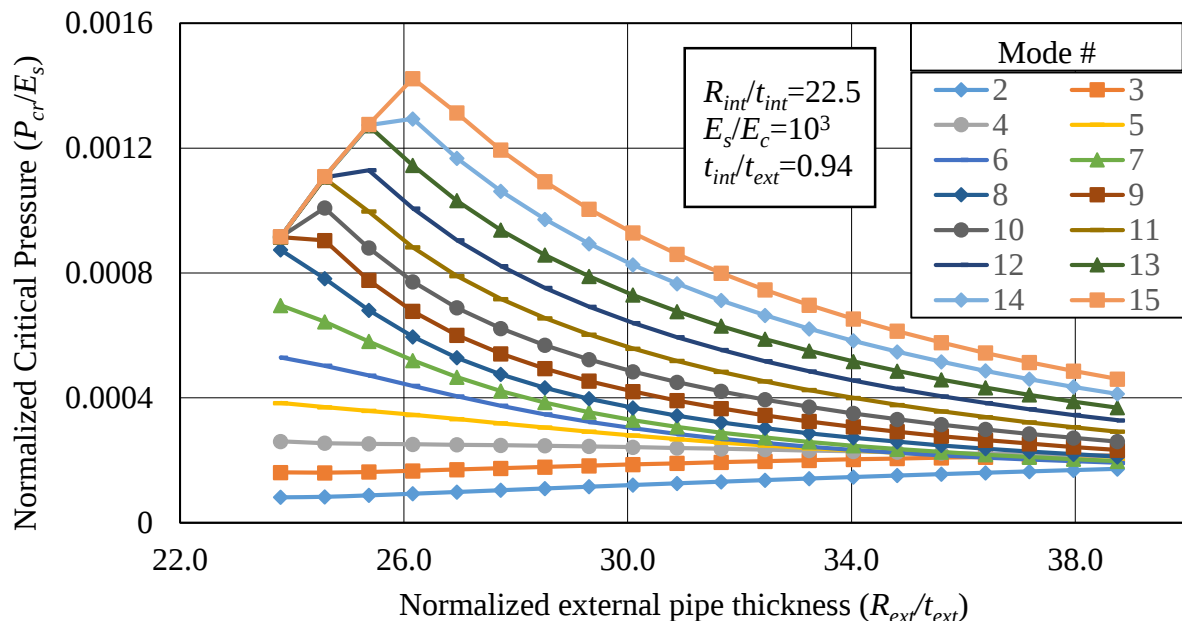


Figure 6-26 Effect of core thickness on critical pressure when the core thickness changes by varying the external pipe mean radius to its thickness ratio (R_{ext}/t_{ext}) while all other parameters (Young Elasticity Modulus ratio (e) is 1000, the internal pipe mean radius to its thickness ratio (R_{int}/t_{int}) is 22.5, and internal pipe thickness to external pipe thickness ratio (t_{int}/t_{ext}) is 0.94) are constant as indicated in the figure

Table 6-10 Numerical values of normalized critical pressure (P_{cr}/E_s) presented in Figure 6-26 where Young Elasticity Modulus ratio (e) is 1000, the internal pipe mean radius to its thickness ratio (R_{int}/t_{int}) is 22.5, and internal pipe thickness to external pipe thickness ratio (t_{int}/t_{ext}) is 0.94

R_{ext}/t_{ext}	Mode #													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
23.8	8.19	16.1	26.1	38.3	52.9	69.6	87.4	91.5	91.5	91.6	91.6	91.6	91.6	91.6
24.6	8.35	16.0	25.5	37.0	50.3	64.4	78.2	90.4	101	110	111	111	111	111
25.4	8.81	16.3	25.3	35.8	47.2	58.1	68.1	77.7	88.0	99.7	113	127	127	128
26.2	9.34	16.6	25.1	34.6	43.8	52.0	59.6	67.7	77.2	88.2	101	114	129	142
26.9	9.89	17.0	25.0	33.2	40.5	46.7	52.8	60.1	68.9	79.1	90.6	103	117	131
27.7	10.5	17.5	24.8	31.9	37.5	42.2	47.5	54.2	62.3	71.7	82.2	93.8	106	119
28.5	11.0	17.9	24.7	30.5	34.7	38.5	43.3	49.4	56.8	65.5	75.2	85.8	97.2	109
29.3	11.6	18.3	24.5	29.2	32.3	35.5	39.7	45.4	52.3	60.3	69.2	78.9	89.4	100
30.1	12.1	18.7	24.2	27.9	30.2	32.9	36.8	42.0	48.4	55.8	64.0	72.9	82.6	92.8
30.9	12.6	19.0	24.0	26.8	28.4	30.7	34.3	39.1	45.0	51.8	59.4	67.7	76.6	86.0
32.5	13.7	19.7	23.4	24.8	25.5	27.3	30.3	34.4	39.5	45.3	51.7	58.8	66.5	74.6
33.2	14.2	20.0	23.2	23.9	24.4	25.9	28.7	32.5	37.1	42.5	48.5	55.1	62.2	69.7
34.0	14.7	20.3	22.9	23.1	23.3	24.7	27.2	30.7	35.0	40.0	45.6	51.7	58.3	65.3
34.8	15.1	20.6	22.6	22.4	22.4	23.6	25.9	29.2	33.2	37.8	42.9	48.6	54.8	61.3
35.6	15.6	20.8	22.4	21.8	21.6	22.7	24.8	27.8	31.4	35.7	40.5	45.8	51.6	57.7
36.4	16.0	21.1	22.1	21.2	20.9	21.8	23.7	26.5	29.9	33.9	38.3	43.3	48.7	54.4
37.2	16.4	21.3	21.9	20.7	20.3	21.0	22.8	25.3	28.5	32.2	36.4	41.0	46.0	51.4
38.0	16.9	21.4	21.6	20.3	19.7	20.4	21.9	24.3	27.2	30.6	34.5	38.8	43.5	48.6
38.8	17.3	21.6	21.4	19.9	19.2	19.7	21.2	23.3	26.0	29.2	32.9	36.9	41.3	46.1

In another set of similar analyses, the effect of core thickness is investigated when $E_c=200$ MPa (Figure 6-27). It is observed that the buckling capacity of the system increases significantly when the core layer becomes stiffer. Also, the buckling pressure is observed to linearly vary with the radius to thickness ratio R_{ext}/t_{ext} of the external pipe (and hence the core thickness) in a manner similar to the first example shown in Figure 6-26.

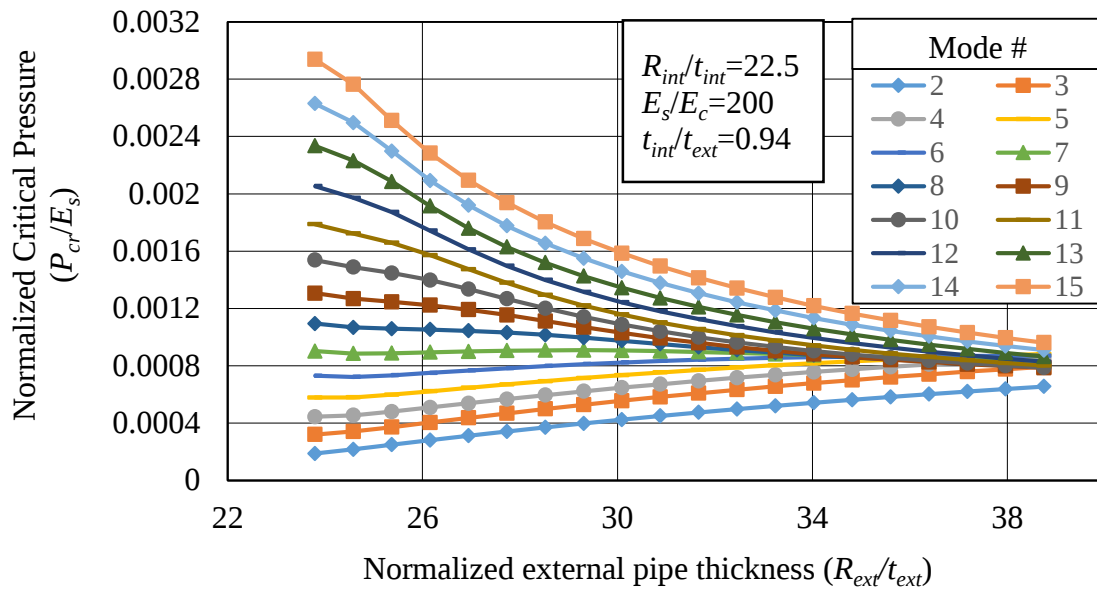


Figure 6-27 Effect of core thickness on critical pressure when the core thickness changes by varying the external pipe mean radius over its thickness ratio (R_{ext}/t_{ext}) while all other parameters (Young Elasticity Modulus ratio (e) is 200, the internal pipe mean radius over its thickness ratio (R_{int}/t_{int}) is 22.5, and internal pipe thickness over external pipe thickness ratio (t_{int}/t_{ext}) is 0.94) are constant as indicated in the figure

In Figure 6-28, the effect of core thickness on the buckling capacity is investigated through another R_{ext}/t_{ext} range of 60-100. Figure 6-28 represents the buckling capacity (P_{cr}/E_s) of the second set of parametric runs (based on the R02 geometries). The results based on the first 15 modes are provided. In the present runs, where the steel pipes are thinner than the steel pipes in Figure 6-26, the governing buckling mode does not always occur at Mode 2 (Figure 6-28). When the R_{ext}/t_{ext} is greater than 89, Modes 13-15 become the governing buckling modes, depending on the geometry of the external pipe.

Table 6-11 provides the numeric values of the results shown on Figure 6-28.

Table 6-11 Numerical values of normalized critical pressure (P_{cr}/E_s) $\times 10^{-5}$ presented in Figure 6-28 where Young Elasticity Modulus ratio (e) is 1000, the internal pipe mean radius to its thickness ratio (R_{int}/t_{int}) is 61.5, and internal pipe thickness to external pipe thickness ratio (t_{int}/t_{ext}) is 0.92

R_{ext}/t_{ext}	Mode #													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
23.8	1.8	2.7	3.5	4.3	5.2	6.3	7.5	8.8	10.3	11.9	13.7	15.6	17.6	19.8
24.6	2.6	3.6	4.4	5.2	6.1	7.0	8.2	9.4	10.8	12.3	13.9	15.6	17.5	19.4
25.4	3.4	4.5	5.3	6.1	6.9	7.8	8.9	10.0	11.3	12.6	14.0	15.5	17.0	18.5
26.2	4.1	5.4	6.2	7.0	7.7	8.6	9.5	10.5	11.6	12.7	13.9	15.0	16.1	17.3
26.9	4.9	6.3	7.1	7.8	8.5	9.3	10.1	11.0	11.8	12.7	13.5	14.3	15.1	16.0
27.7	5.6	7.1	7.9	8.6	9.2	9.9	10.6	11.3	11.9	12.5	13.0	13.6	14.2	14.9
28.5	6.3	7.9	8.7	9.3	9.9	10.5	11.0	11.4	11.8	12.2	12.5	12.9	13.3	13.9
29.3	6.9	8.6	9.4	10.0	10.5	10.9	11.3	11.5	11.7	11.8	12.0	12.3	12.7	13.2
30.1	7.5	9.3	10.1	10.6	11.0	11.3	11.5	11.5	11.5	11.5	11.6	11.8	12.1	12.5
30.9	8.1	10.0	10.8	11.2	11.5	11.6	11.6	11.4	11.3	11.2	11.2	11.3	11.6	11.9
31.7	8.7	10.6	11.4	11.8	11.9	11.9	11.6	11.4	11.1	10.9	10.9	10.9	11.1	11.4
32.5	9.3	11.3	12.0	12.2	12.3	12.0	11.7	11.3	10.9	10.7	10.6	10.6	10.7	11.0
33.2	9.8	11.8	12.5	12.7	12.6	12.2	11.7	11.2	10.7	10.5	10.3	10.3	10.4	10.6
34.0	10.3	12.4	13.0	13.1	12.8	12.3	11.7	11.1	10.6	10.3	10.1	10.0	10.1	10.3
34.8	10.8	12.9	13.5	13.5	13.0	12.4	11.6	11.0	10.4	10.1	9.9	9.8	9.9	10.0
35.6	11.3	13.4	14.0	13.8	13.2	12.4	11.6	10.9	10.3	10.0	9.7	9.6	9.6	9.7
36.4	11.8	13.9	14.4	14.1	13.4	12.5	11.6	10.8	10.2	9.8	9.6	9.4	9.4	9.5
37.2	12.2	14.4	14.8	14.4	13.5	12.5	11.5	10.7	10.1	9.7	9.4	9.3	9.2	9.3
38.0	12.6	14.8	15.1	14.6	13.6	12.5	11.5	10.7	10.0	9.6	9.3	9.1	9.0	9.1
38.8	13.1	15.3	15.5	14.8	13.7	12.5	11.5	10.6	10.0	9.5	9.2	9.0	8.9	8.9

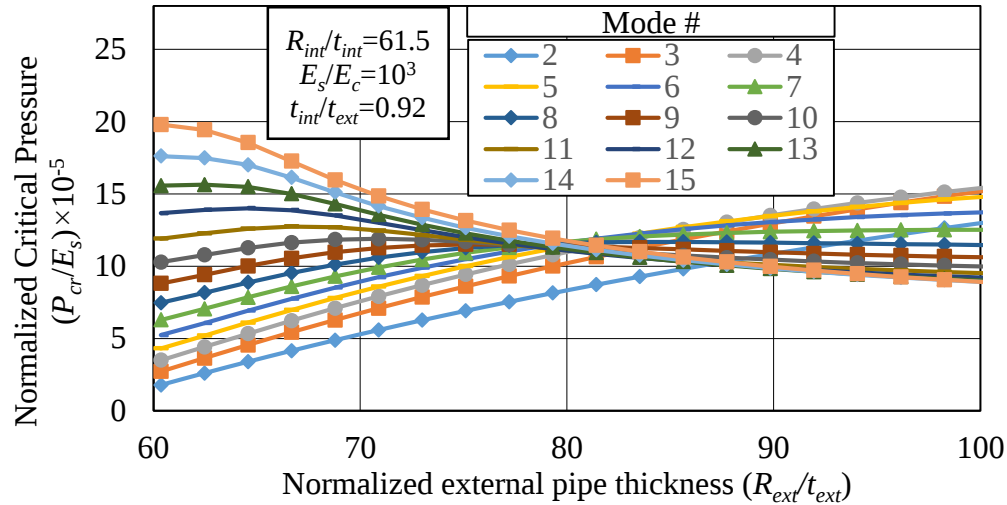


Figure 6-28 Effect of core thickness on critical pressure when the core thickness changes by varying the external pipe mean radius to its thickness ratio (R_{ext}/t_{ext}) while all other parameters (Elastic modulus ratio (e) is 1000, the internal pipe mean radius to its thickness ratio (R_{int}/t_{int}) is 61.5, and internal pipe thickness to external pipe thickness ratio (t_{int}/t_{ext}) is 0.92) are constant as indicated in the figure

Another set of parametric runs was conducted for the geometries similar to the R02 reference case, again while varying R_{ext}/t_{ext} but with a stronger core material ($E_s/E_c = 200$). Figure 6-29 suggests that increasing the core thickness considerably increases the buckling capacity up to a $R_{ext}/t_{ext} = 89$, after which a thicker core is observed to result in a slightly smaller buckling pressures. The findings are similar to those of Arjomandi (2010).

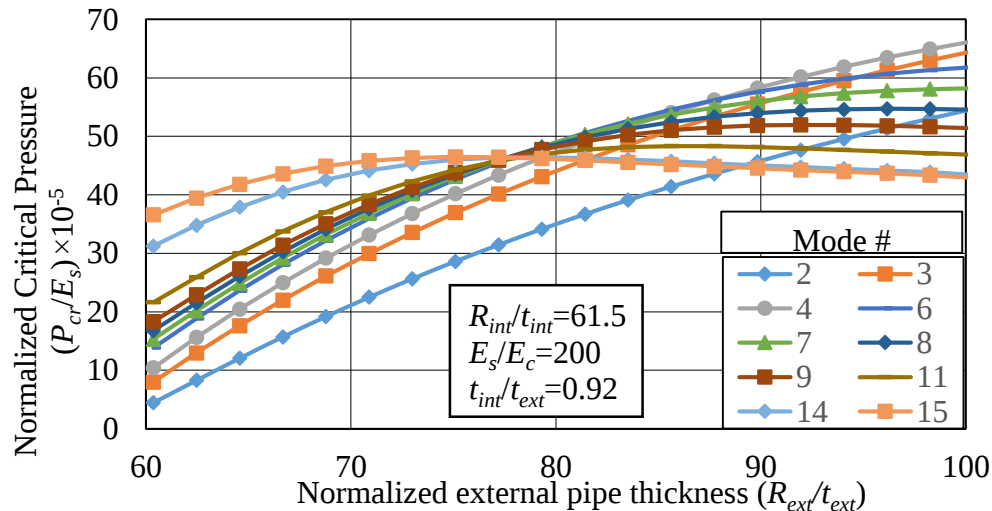


Figure 6-29 Effect of core thickness on critical pressure when the core thickness changes by varying the external pipe mean radius to its thickness ratio (R_{ext}/t_{ext}) while all other parameters (Young Elasticity Modulus ratio (e) is 200, the internal pipe mean radius to its thickness ratio (R_{int}/t_{int}) is 61.5, and internal pipe thickness to external pipe thickness ratio (t_{int}/t_{ext}) is 0.92) are constant as indicated in the figure

6.5 Conclusions

Based on the results presented in Section 6.4, the following conclusions are made

- a) The presence of hydrostatic internal pressure increases the external pressure buckling capacity of the system in a nearly linear manner.
- b) The modulus of elasticity of the core material significantly affects the capacity of sandwich pipes while its Poisson ratio has a minor influence on the buckling capacity.
- c) Generally, the increase of the internal and external pipe thickness improves the buckling capacity of the system. In most cases, the external pipe thickness has more influence.
- d) The core thickness effect on the buckling capacity depends on the thickness of internal and external pipes. When the internal and external pipes are thick, the buckling capacity is observed to improve with the core thickness. However, when the internal and external pipes are thin, increasing the core thickness may result in a decrease of the buckling capacity since higher mode configurations will tend to govern.

CHAPTER 7. Summary, Conclusion, and Recommendations

7.1 Summary

The principle of stationary potential energy was used to obtain the conditions of equilibrium, neutral stability conditions, and associated boundary conditions for thick pipes using polar coordinates. A Fourier series expansion of the displacement fields was adopted and found to be particularly useful in transforming the 2D problem into a series of independent 1D problems, thus preserving the accuracy of the solution while keeping the degrees of freedom involved to a minimum. Two numerical solutions were developed and implemented under MATLAB. These are the finite difference solution and finite element solution. Both solutions were shown to converge to the same results, the finite difference solution from below, and the finite element solution from above. The finite element solution was used to analyze sandwich pipe systems consisting of two steel pipes with a soft core. A comprehensive verification study was conducted through comparison with other solution and the validity of the formulation was established. Parametric runs were conducted to investigate the effect of hydrostatic internal pressure, core material properties, core thickness, and internal and external pipe thicknesses.

7.2 Features of the Formulation

The formulation has the following features

1. It accounts for shear deformation effects and is suited for composite pipe systems with thick cores.
2. It involves two destabilizing terms: one is due to the external hydrostatic pressure and incorporates the follower effects, and the other is due to the pre-buckling stresses

undergoing the nonlinear components of the strains. The second contribution has been neglected in most published studies.

3. It adopts Cauchy (True) stress tensor and Green Lagrange strain tensor with constant constitutive relation as a work conjugate stress-strain-constitutive relation, which are energy-conjugate triplets, and thus is judged to provide a superior solution in comparison with the FE software such as those in ABAQUS, in which energy conjugacy is achieved only in an approximate sense.
4. It is applicable to the buckling analysis of any multilayer elastic sandwich pipe system subjected to internal and external pressures.

7.3 Conclusions

Based on the results investigated in Section 6.4 the following conclusions can be drawn:

1. The hydrostatic internal pressure acting on the inside surface of the sandwich pipe increases the external buckling capacity of the system in a nearly linearly fashion.
2. The core material has the most influence on the buckling capacity of the system. An increase of the Young Elasticity Modulus of the core improves the buckling capacity of the composite system significantly. The Poisson's ratio was observed not to have much influence on the buckling capacity of the system.
3. Generally, increasing the internal and external pipe thickness improves the buckling capacity of the system. In most cases, external pipe thickness has more influence on the critical pressure than the internal pipe thickness. However, in a small range of thicknesses, increasing the internal pipe thickness is observed more effective in improving the capacity of the composite system.

4. The core thickness effect on the buckling capacity depends on the thickness of internal and external pipes. When the internal and external pipes are thick enough, the buckling capacity of the system improves by increasing the core thickness. However, when the internal and external pipe are thin, increasing the core thickness may result in decreasing the buckling capacity since higher mode configurations will govern the buckling capacity of the systems. When this is the case, the capacity of the system is not fully utilized.

A summary of the the results investigated under Section 6.4 (Table 7-1) indicates that the buckling capacity of the system is highly dependent on the the Young Modulus and thickness of the core, as well as the external pipe radius to thickness ratio, and moderately on the internal pipe radius to thickness ratio and internal pressure and mildly on the Poisson ratio of the core.

Table 7-1 Effect of different parameters on the buckling capacity of sandwich pipe

Core	Young modulus ratio	E_c / E_s	High
	Poisson Ratio	ν_c	Low
	Thickness	R_{int} / R_{ext}	High
Steel Pipes	Internal pipe radius to thickness ratio	R_{int} / t_{int}	Moderate
	External pipe radius to thickness ratio	R_{ext} / t_{ext}	High
Internal Pressure		P_{int}	Moderate

7.4 Future Work

1. The present study focused solely on the effect of internal and external pressure. It is of interest to expand the work to incorporate the effect of bending moments, and axial forces.
2. The present study focused on quantifying the peak pressure a system can withstand. In an offshore pipeline, it is of interest to quantify the propagation pressure, which is typically a

small fraction of the peak pressure. Such a propagation pressure needs to be quantified for sandwich pipe systems. DNV (2007) estimates the propagation pressure for plain pipes based on its elastic buckling capacity, and its plastic capacity. It is of interest to modify the DNV rules for sandwich pipes.

3. The present study dealt with an ideal sandwich pipe system with no initial geometric imperfections, resulting in an eigen-value buckling analysis. It is of practical interest to incorporate the effect of initial imperfections into the solution in an incremental geometrically nonlinear analysis of sandwich system to trace their post-peak response.

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Appendices

A.1 Second Variation of a Quadratic Functional

In Chapter 3, the total potential energy was shown to consist of the summation of several Π_{jk} terms of the form

$$\Pi_{jk} = \int_V \frac{1}{2} E \varepsilon_j \varepsilon_k dV \quad (\text{A.1.1})$$

in which $\varepsilon_j, \varepsilon_k$ are strain expressions expressed in terms of the displacement fields and their derivatives. It is required to find an expression of the second variation of Π_{jk} . By definition, the variation of the second variation of Π_{jk} is

$$\delta^2 \Pi_{jk} = \int_V \sum_l \sum_m \left(\frac{\partial^2 \Pi}{\partial u_l \partial u_m} \right) \delta u_l \delta u_m dV \quad (\text{A.1.2})$$

Substituting Eq. (A.1.1) into Eq. (A.1.2) the second variation of the potential energy can be shown to take the form

$$\delta^2 \left(\int_V \frac{1}{2} E \varepsilon_j \varepsilon_k dV \right) = \frac{1}{2} E \int_V \sum_l \sum_m \frac{\partial}{\partial u_l} \left[\frac{\partial (\varepsilon_j \varepsilon_k)}{\partial u_m} \right] \delta u_l \delta u_m dV \quad (\text{A.1.3})$$

Taking derivatives of the term in the round brackets, one obtains

$$\delta^2 \left(\int_V \frac{1}{2} E \varepsilon_j \varepsilon_k dV \right) = \frac{1}{2} E \int_V \sum_l \sum_m \frac{\partial}{\partial u_l} \left(\varepsilon_j \frac{\partial \varepsilon_k}{\partial u_m} + \varepsilon_k \frac{\partial \varepsilon_j}{\partial u_m} \right) \delta u_l \delta u_m dV \quad (\text{A.1.4})$$

Also, taking derivatives of the bracketed term in Eq. (A.1.4), one obtains

$$\delta^2 \left(\int_V \frac{1}{2} E \varepsilon_j \varepsilon_k dV \right) = \frac{1}{2} E \int_V \sum_l \sum_m \left(2 \frac{\partial \varepsilon_j}{\partial u_l} \frac{\partial \varepsilon_k}{\partial u_m} + \varepsilon_k \frac{\partial^2 \varepsilon_j}{\partial u_l \partial u_m} + \varepsilon_j \frac{\partial^2 \varepsilon_k}{\partial u_l \partial u_m} \right) \delta u_l \delta u_m dV \quad (\text{A.1.5})$$

Taking the term $\delta u_l \delta u_m$ inside of the bracket, one obtains

$$\begin{aligned} \delta^2 \left(\int_V \frac{1}{2} E \varepsilon_r \varepsilon_\theta dV \right) = \int_V \frac{E}{2} \sum_i \sum_j \left[2 \left(\frac{\partial \varepsilon_\theta}{\partial u_i} \delta u_i \right) \left(\frac{\partial \varepsilon_r}{\partial u_j} \delta u_j \right) \right. \\ \left. + \varepsilon_\theta \left(\frac{\partial^2 \varepsilon_r}{\partial u_i \partial u_j} \delta u_i \delta u_j \right) + \varepsilon_r \left(\frac{\partial^2 \varepsilon_\theta}{\partial u_i \partial u_j} \delta u_i \delta u_j \right) \right] dV \end{aligned} \quad (\text{A.1.6})$$

One can write $(\partial \varepsilon_\theta / \partial u_i) \delta u_i = \delta \varepsilon_\theta$ and $(\partial^2 \varepsilon_r / \partial u_i \partial u_j) \delta u_i \delta u_j = \delta^2 \varepsilon_r$, so

$$\delta^2 \left(\int_V \frac{1}{2} E \varepsilon_r \varepsilon_\theta dV \right) = \int_V \frac{E}{2} \left[2 (\delta \varepsilon_\theta) (\delta \varepsilon_r) + \varepsilon_\theta \delta^2 \varepsilon_r + \varepsilon_r \delta^2 \varepsilon_\theta \right] dV \quad (\text{A.1.7})$$

considering the fact that the strains have two components; a linear component and a non-linear component i.e.,

$$\begin{aligned} \varepsilon_r &= \varepsilon_{r,L} + \varepsilon_{r,NL} \\ \varepsilon_\theta &= \varepsilon_{\theta,L} + \varepsilon_{\theta,NL} \end{aligned} \quad (\text{A.1.8})$$

and assuming the non-linear component is much smaller than the linear part, one can write

$\delta \varepsilon_r \approx \delta \varepsilon_{r,L}$. Knowing that the linear strains have zero second variation (

$\delta^2 \varepsilon_r = \delta^2 \varepsilon_{r,L} + \delta^2 \varepsilon_{r,NL} = \delta^2 \varepsilon_{r,NL}$), Eq.(A.1.7) can be written as

$$\delta^2 \left(\int_V \frac{1}{2} E \varepsilon_r \varepsilon_\theta dV \right) = \int_V \frac{E}{2} \left[2 (\delta \varepsilon_{\theta,L}) (\delta \varepsilon_{r,L}) + \varepsilon_\theta \delta^2 \varepsilon_{r,NL} + \varepsilon_r \delta^2 \varepsilon_{\theta,NL} \right] dV \quad (\text{A.1.9})$$

In a special case if $\varepsilon_r = \varepsilon_\theta$ the above equation turns into

$$\frac{1}{2} \delta^2 \left(\int_V \frac{1}{2} E \varepsilon_r^2 dV \right) = \int_V \frac{E}{2} \left[\left(\delta \varepsilon_{r,L} \right)^2 + \varepsilon_r \delta^2 \varepsilon_{r,NL} \right] dV \quad (\text{A.1.10})$$

Equation (A.1.9) and (A.1.10) are used to determine the second variation of Eq. (3.14)

A.2 Deriving the 2D Neutral Stability Conditions and Boundary Conditions

This appendix substitutes the strain-displacement relations defined in Table 3-1, into Eq. (3.18), then sets the variation of the second variation of the resulting functional to zero and performs integration by parts to derive the equilibrium equations and boundary conditions of the system.

The second variation $\bar{\Pi}$ of the total potential energy in terms of displacements is given by

$$\begin{aligned}
 \frac{1}{2}\bar{\Pi} = & \frac{L(1-\nu)}{2\nu} \int_{r_1}^{r_4} \int_0^{2\pi} \left[\left(\delta \bar{u}'_b \right)^2 + \frac{1}{2} \lambda u'_p \left(2\bar{u}'_b{}^2 + 2\bar{v}'_b{}^2 \right) \right] r d\theta dr \\
 & + \frac{L(1-\nu)}{2\nu} \int_{r_1}^{r_4} \int_0^{2\pi} \left[\frac{1}{r^2} \left(\bar{u}_b + \bar{v}_b \right)^2 \right. \\
 & \quad \left. + \frac{1}{2r^3} \lambda u_p \left(2\bar{u}_b{}^2 + 2\bar{v}_b{}^2 - 4\bar{u}_b \bar{v}_b + 2\bar{u}_b{}^2 + 2\bar{v}_b{}^2 + 4\bar{u}_b \bar{v}_b \right) \right] r d\theta dr \\
 & + \frac{1}{2} L \int_{r_1}^{r_4} \int_0^{2\pi} 2\bar{u}'_b \times \frac{1}{r} \left(\bar{u}_b + \bar{v}_b \right) r d\theta dr \\
 & + \frac{1}{2} L \int_{r_1}^{r_4} \int_0^{2\pi} \lambda u'_p \frac{1}{2r^2} \left(2\bar{u}_b{}^2 + 2\bar{v}_b{}^2 - 4\bar{u}_b \bar{v}_b + 2\bar{u}_b{}^2 + 2\bar{v}_b{}^2 + 4\bar{u}_b \bar{v}_b \right) r d\theta dr \\
 & + \frac{1}{2} L \int_{r_1}^{r_4} \int_0^{2\pi} \frac{1}{r} \lambda u_p \frac{1}{2} \left(2\bar{u}'_b{}^2 + 2\bar{v}'_b{}^2 \right) r d\theta dr \\
 & + \frac{L(1-2\nu)}{4\nu} \int_{r_1}^{r_4} \int_0^{2\pi} \left(\frac{1}{r} \bar{u}_b - \frac{1}{r} \bar{v}_b + \bar{v}'_b \right)^2 r d\theta dr \\
 & + \frac{1}{2} \lambda P_{ext} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{v}_b \bar{u}_b \right]_{r=r_2} d\theta \\
 & - \frac{1}{2} P_{int} \int_0^{2\pi} \left[\left(\bar{v}_b \right)^2 + \left(\bar{u}_b \right)^2 - \bar{v}_b \bar{u}_b + \bar{v}_b \bar{u}_b \right]_{r=r_1} d\theta
 \end{aligned}
 \tag{A.2.1}$$

Eq. (A.2.2) is Evoking the condition of neutral stability by vanishing the variation of the Eq. (A.2.1)

$$\begin{aligned}
\delta \frac{1}{2} \overline{\overline{\Pi}} = & \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \left[\frac{1}{r} (1-\nu) \overline{u}_b + \nu \overline{u}'_b + \frac{1}{r} (1-\nu) \overline{\dot{v}}_b \right] \delta \overline{u}_b + \left[\nu \overline{u}_b + (1-\nu) r \overline{u}'_b + \nu \overline{\dot{v}}_b \right] \delta \overline{u}'_b \right. \\
& + \left[\frac{1}{r} \frac{(1-2\nu)}{2} \overline{\ddot{u}}_b - \frac{1}{r} \frac{(1-2\nu)}{2} \overline{\dot{v}}_b + \frac{(1-2\nu)}{2} \overline{\dot{v}}'_b \right] \delta \overline{\ddot{u}}_b \\
& + \left[-\frac{1}{r} \frac{(1-2\nu)}{2} \overline{\ddot{u}}_b + \frac{1}{r} \frac{(1-2\nu)}{2} \overline{\dot{v}}_b - \frac{(1-2\nu)}{2} \overline{\dot{v}}'_b \right] \delta \overline{\dot{v}}_b \\
& + \left[\frac{(1-2\nu)}{2} \overline{\ddot{u}}_b - \frac{(1-2\nu)}{2} \overline{\dot{v}}_b + r \frac{(1-2\nu)}{2} \overline{\dot{v}}'_b \right] \delta \overline{\dot{v}}'_b \\
& \left. + \left[\frac{1}{r} (1-\nu) \overline{u}_b + \nu \overline{u}'_b + \frac{1}{r} (1-\nu) \overline{\dot{v}}_b \right] \delta \overline{\dot{v}}_b \right\} d\theta dr \\
& + \lambda \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ \left[\frac{1}{r^2} (1-\nu) u_p \overline{u}_b + \frac{1}{r} \nu u'_p \overline{u}_b + \frac{1}{r^2} (1-\nu) u_p \overline{\dot{v}}_b + \frac{1}{r} \nu u'_p \overline{\dot{v}}_b \right] \delta \overline{u}_b \right. \\
& + \left[(1-\nu) r u'_p \overline{u}'_b + \nu u_p \overline{u}'_b \right] \delta \overline{u}'_b \\
& + \left[\frac{1}{r^2} (1-\nu) u_p \overline{\ddot{u}}_b + \frac{1}{r} \nu u'_p \overline{\ddot{u}}_b - \frac{1}{r^2} (1-\nu) u_p \overline{\dot{v}}_b - \frac{1}{r} \nu u'_p \overline{\dot{v}}_b \right] \delta \overline{\ddot{u}}_b \\
& + \left[-\frac{1}{r^2} (1-\nu) u_p \overline{\ddot{u}}_b - \frac{1}{r} \nu u'_p \overline{\ddot{u}}_b + \frac{1}{r^2} (1-\nu) u_p \overline{\dot{v}}_b + \frac{1}{r} \nu u'_p \overline{\dot{v}}_b \right] \delta \overline{\dot{v}}_b \\
& + \left[r (1-\nu) u'_p \overline{\dot{v}}'_b + \nu u_p \overline{\dot{v}}'_b \right] \delta \overline{\dot{v}}'_b \\
& \left. + \left[\frac{1}{r^2} (1-\nu) u_p \overline{u}_b + \frac{1}{r} \nu u'_p \overline{u}_b + \frac{1}{r^2} (1-\nu) u_p \overline{\dot{v}}_b + \frac{1}{r} \nu u'_p \overline{\dot{v}}_b \right] \delta \overline{\dot{v}}_b \right\} d\theta dr \quad (A.2.2) \\
& + \lambda P_{ext} \int_0^{2\pi} \left[\left(\overline{u}_b + \frac{1}{2} \overline{\dot{v}}_b \right) \delta \overline{u}_b - \frac{1}{2} \overline{\dot{v}}_b \delta \overline{\ddot{u}}_b + \left(\overline{\dot{v}}_b - \frac{1}{2} \overline{\ddot{u}}_b \right) \delta \overline{\dot{v}}_b + \frac{1}{2} \overline{u}_b \delta \overline{\dot{v}}_b \right]_{r=r_2} d\theta \\
& - P_{int} \int_0^{2\pi} \left[\left(\overline{u}_b + \frac{1}{2} \overline{\dot{v}}_b \right) \delta \overline{u}_b - \frac{1}{2} \overline{\dot{v}}_b \delta \overline{\ddot{u}}_b + \left(\overline{\dot{v}}_b - \frac{1}{2} \overline{\ddot{u}}_b \right) \delta \overline{\dot{v}}_b + \frac{1}{2} \overline{u}_b \delta \overline{\dot{v}}_b \right]_{r=r_1} d\theta = 0
\end{aligned}$$

Performing integration by parts on Eq. (A.2.2) results in Eq. (A.2.3)

$$\begin{aligned}
\delta \frac{1}{2} \overline{\overline{\Pi}} = & \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[\left(A_{1,1} \overline{u}_b + A_{1,2} \overline{u}'_b + A_{1,3} \overline{u}''_b + A_{1,4} \overline{\ddot{u}}_b + A_{1,5} \overline{\dot{v}}_b + A_{1,6} \overline{\dot{v}}'_b \right) \delta \overline{u}_b \right. \\
& + \left(A_{2,1} \overline{\ddot{u}}_b + A_{2,2} \overline{\ddot{u}}'_b + A_{2,3} \overline{\dot{v}}_b + A_{2,4} \overline{\dot{v}}'_b + A_{2,5} \overline{\dot{v}}''_b + A_{2,6} \overline{\ddot{v}}_b \right) \delta \overline{\dot{v}}_b \left. \right] d\theta dr \\
& + \lambda \frac{L}{\nu} \int_{r_1}^{r_2} \int_0^{2\pi} \left[\left(B_{1,1} \overline{u}_b + B_{1,2} \overline{u}'_b + B_{1,3} \overline{u}''_b + B_{1,4} \overline{\ddot{u}}_b + B_{1,5} \overline{\dot{v}}_b \right) \delta \overline{u}_b \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(B_{2,1} \bar{\dot{u}}_b + B_{2,2} \bar{v}_b + B_{2,3} \bar{v}'_b + B_{2,4} \bar{v}''_b + B_{2,5} \bar{\ddot{v}}_b \right) \delta \bar{v}_b \Big] d\theta dr \\
& + \left\{ \frac{L}{\nu} \int_0^{2\pi} \left[\nu \bar{u}_b + (1-\nu) r \bar{u}'_b + \nu \bar{\dot{v}}_b \right] \delta \bar{u}_b d\theta \right\}_{r=r_1}^{r=r_2} \\
& + \left\{ \frac{L(1-2\nu)}{\nu} \int_{r_1}^{r_2} \left[\frac{1}{r} \bar{\dot{u}}_b - \frac{1}{r} \bar{v}_b + \bar{v}'_b \right] \delta \bar{u}_b dr \right\}_{\theta=0}^{\theta=2\pi} \\
& + \left\{ \lambda \frac{L}{\nu} \int_0^{2\pi} \left[(1-\nu) r u'_p \bar{u}'_b + \nu u_p \bar{u}'_b \right] \delta \bar{u}_b d\theta \right\}_{r=r_1}^{r=r_2} \\
& + \left\{ \lambda \frac{L}{\nu} \int_{r_1}^{r_2} \left[\frac{1}{r^2} (1-\nu) u_p \bar{\dot{u}}_b + \frac{1}{r} \nu u'_p \bar{\dot{u}}_b - \frac{1}{r^2} (1-\nu) u_p \bar{v}_b - \frac{1}{r} \nu u'_p \bar{v}_b \right] \delta \bar{u}_b dr \right\}_{\theta=0}^{\theta=2\pi} \\
& + \left\{ \frac{L}{\nu} \int_0^{2\pi} \left[\frac{(1-2\nu)}{2} \bar{\dot{u}}_b - \frac{(1-2\nu)}{2} \bar{v}_b + r \frac{(1-2\nu)}{2} \bar{v}'_b \right] \delta \bar{v}_b d\theta \right\}_{r=r_1}^{r=r_2} \\
& + \left\{ \frac{L}{\nu} \int_{r_1}^{r_2} \left[\frac{1}{r} (1-\nu) \bar{u}_b + \nu \bar{u}'_b + \frac{1}{r} (1-\nu) \bar{\dot{v}}_b \right] \delta \bar{v}_b dr \right\}_{\theta=0}^{\theta=2\pi} \\
& + \left\{ \lambda \frac{L}{\nu} \int_0^{2\pi} \left[r (1-\nu) u'_p \bar{v}'_b + \nu u_p \bar{v}'_b \right] \delta \bar{v}_b d\theta \right\}_{r=r_1}^{r=r_2} \\
& + \left\{ \lambda \frac{L}{\nu} \int_{r_1}^{r_2} \left[\frac{1}{r^2} (1-\nu) u_p \bar{u}_b + \frac{1}{r} \nu u'_p \bar{u}_b + \frac{1}{r^2} (1-\nu) u_p \bar{\dot{v}}_b + \frac{1}{r} \nu u'_p \bar{\dot{v}}_b \right] \delta \bar{v}_b dr \right\}_{\theta=0}^{\theta=2\pi} \quad (A.2.3) \\
& + \lambda P_{ext} \int_0^{2\pi} \left[+ (\bar{u}_b + \bar{\dot{v}}_b) \delta \bar{u}_b + (\bar{v}_b - \bar{\dot{u}}_b) \delta \bar{v}_b \right]_{r=r_2} d\theta \\
& - P_{int} \int_0^{2\pi} \left[+ (\bar{u}_b + \bar{\dot{v}}_b) \delta \bar{u}_b + (\bar{v}_b - \bar{\dot{u}}_b) \delta \bar{v}_b \right]_{r=r_1} d\theta = 0
\end{aligned}$$

where the following constants have been defined

$$\begin{aligned}
A_{1,1} &= \frac{1}{r} (1-\nu) & A_{2,1} &= -\frac{1}{2r} (3-4\nu) \\
A_{1,2} &= -(1-\nu) & A_{2,2} &= -\frac{1}{2}
\end{aligned}$$

$$A_{1,3} = -r(1-\nu)$$

$$A_{2,3} = \frac{1}{2r}(1-2\nu)$$

$$A_{1,4} = -\frac{1}{2r}(1-2\nu)$$

$$A_{2,4} = -\frac{1}{2}(1-2\nu)$$

$$A_{1,5} = \frac{1}{2r}(3-4\nu)$$

$$A_{2,5} = -\frac{r}{2}(1-2\nu)$$

$$A_{1,6} = -\frac{1}{2}$$

$$A_{2,6} = -\frac{1}{r}(1-\nu)$$

$$B_{1,1} = \frac{1}{r^2}(1-\nu)u_p + \frac{1}{r}\nu u'_p$$

$$B_{2,1} = -\frac{2}{r^2}(1-\nu)u_p - \frac{2}{r}\nu u'_p$$

$$B_{1,2} = -u'_p - (1-\nu)ru''_p$$

$$B_{2,2} = \frac{1}{r^2}(1-\nu)u_p + \frac{1}{r}\nu u'_p$$

$$B_{1,3} = -\nu u_p - (1-\nu)ru'_p$$

$$B_{2,3} = -u'_p - r(1-\nu)u''_p$$

$$B_{1,4} = -\frac{1}{r^2}(1-\nu)u_p - \frac{1}{r}\nu u'_p$$

$$B_{2,4} = -\nu u_p - r(1-\nu)u'_p$$

$$B_{1,5} = \frac{2}{r^2}(1-\nu)u_p + \frac{2}{r}\nu u'_p$$

$$B_{2,5} = -\frac{1}{r^2}(1-\nu)u_p - \frac{1}{r}\nu u'_p$$

Since $\delta \bar{u}_b$ and $\delta \bar{v}_b$ are arbitrary functions, their coefficients must vanish leading to the following two equilibrium Equations

$$\begin{aligned} \int_0^{2\pi} \left[\left(A_{1,1} \bar{u}_b + A_{1,2} \bar{u}'_b + A_{1,3} \bar{u}''_b + A_{1,4} \bar{\ddot{u}}_b + A_{1,5} \bar{\dot{v}}_b + A_{1,6} \bar{\dot{v}}'_b \right) \right. \\ \left. + \lambda \left(B_{1,1} \bar{u}_b + B_{1,2} \bar{u}'_b + B_{1,3} \bar{u}''_b + B_{1,4} \bar{\ddot{u}}_b + B_{1,5} \bar{\dot{v}}_b \right) \right] d\theta = 0 \end{aligned} \quad (\text{A.2.4})$$

$$\int_0^{2\pi} \left[\left(A_{2,1} \bar{\dot{u}}_b + A_{2,2} \bar{\dot{u}}'_b + A_{2,3} \bar{v}_b + A_{2,4} \bar{v}'_b + A_{2,5} \bar{v}''_b + A_{2,6} \bar{\ddot{v}}_b \right) + \lambda \left(B_{2,1} \bar{\dot{u}}_b + B_{2,2} \bar{v}_b + B_{2,3} \bar{v}'_b + B_{2,4} \bar{v}''_b + B_{2,5} \bar{\ddot{v}}_b \right) \right] d\theta = 0 \quad (\text{A.2.5})$$

and the boundary conditions

$$\left\{ \int_0^{2\pi} \left[\left(C_{1,1} \bar{u}_b + C_{1,2} \bar{u}'_b + C_{1,3} \bar{\dot{v}}_b \right) + \lambda \left(D_{1,1} \bar{u}'_b \right) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.2.6})$$

$$\left\{ \int_0^{2\pi} \left[\left(C_{2,1} \bar{\dot{u}}_b + C_{2,2} \bar{v}_b + C_{2,3} \bar{v}'_b \right) + \lambda \left(D_{2,1} \bar{v}'_b \right) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.2.7})$$

$$\left\{ \int_0^{2\pi} \left[\left(C_{3,1} \bar{u}_b + C_{3,2} \bar{u}'_b + C_{3,3} \bar{\dot{v}}_b \right) + \lambda \left(D_{3,1} \bar{u}_b + D_{3,2} \bar{u}'_b + D_{3,3} \bar{\dot{v}}_b \right) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.2.8})$$

$$\left\{ \int_0^{2\pi} \left[\left(C_{4,1} \bar{\dot{u}}_b + C_{4,2} \bar{v}_b + C_{4,3} \bar{v}'_b \right) + \lambda \left(D_{4,1} \bar{\dot{u}}_b + D_{4,2} \bar{v}_b + D_{4,3} \bar{v}'_b \right) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.2.9})$$

$$\left[\left(C_{5,1} \bar{\dot{u}}_b + C_{5,2} \frac{1}{r} \bar{v}_b + C_{5,3} \bar{v}'_b \right) + \lambda \left(D_{5,1} \bar{\dot{u}}_b + D_{5,2} \bar{v}_b \right) \right]_{\theta=0}^{\theta=2\pi} = 0 \quad (\text{A.2.10})$$

$$\left[\left(C_{6,1} \bar{u}_b + C_{6,2} \bar{u}'_b + C_{6,3} \bar{\dot{v}}_b \right) + \lambda \left(D_{6,1} \bar{u}_b + D_{6,2} \bar{\dot{v}}_b \right) \right]_{\theta=0}^{\theta=2\pi} = 0 \quad (\text{A.2.11})$$

Where constants $C_{i,j}$ and $D_{i,j}$ have been defined as

$$C_{1,1} = \nu + \frac{\nu}{L} P_{int}$$

$$C_{2,1} = \frac{1}{2}(1-2\nu) - \frac{\nu}{L} P_{int}$$

$$C_{1,2} = (1-\nu)r_1$$

$$C_{2,2} = -\frac{1}{2}(1-2\nu) + \frac{\nu}{L} P_{int}$$

$$C_{1,3} = \nu + \frac{\nu}{L} P_{int}$$

$$C_{2,3} = \frac{1}{2}(1-2\nu)r_1$$

$$D_{1,1} = \nu u_p + (1-\nu)r_1 u'_p$$

$$D_{2,1} = \nu u_p + (1-\nu)r_1 u'_p$$

$$C_{3,1} = \nu$$

$$C_{4,1} = \frac{1}{2}(1-2\nu)$$

$$C_{3,2} = (1-\nu)r_2$$

$$C_{4,2} = -\frac{1}{2}(1-2\nu)$$

$$C_{3,3} = \nu$$

$$C_{4,3} = \frac{1}{2}(1-2\nu)r_2$$

$$D_{3,1} = \frac{\nu}{L} P_{ext}$$

$$D_{4,1} = -\frac{\nu}{L} P_{ext}$$

$$D_{3,2} = \nu u_p + (1-\nu)r_2 u'_p$$

$$D_{4,2} = \frac{\nu}{L} P_{ext}$$

$$D_{3,3} = \frac{\nu}{L} P_{ext}$$

$$D_{4,3} = \nu u_p + (1-\nu)r_2 u'_p$$

$$C_{5,1} = \frac{1}{2r}(1-2\nu)$$

$$C_{6,1} = \frac{1}{r}(1-\nu)$$

$$C_{5,2} = -\frac{1}{2r}(1-2\nu)$$

$$C_{6,2} = \nu$$

$$C_{5,3} = \frac{1}{2}(1-2\nu)$$

$$C_{6,3} = \frac{1}{r}(1-\nu)$$

$$D_{5,1} = \frac{1}{r^2}(1-\nu)u_p + \frac{1}{r}\nu u'_p$$

$$D_{6,1} = \frac{1}{r^2}(1-\nu)u_p + \frac{1}{r}\nu u'_p$$

$$D_{5,2} = -\frac{1}{r^2}(1-\nu)u_p - \frac{1}{r}\nu u_p'$$

$$D_{6,2} = \frac{1}{r^2}(1-\nu)u_p + \frac{1}{r}\nu u_p'$$

Equations (A.2.4) to (A.2.11) are the same as Eqs. (3.20) to (3.27).

A.3 Neutral Stability Equations and Boundary Conditions in Terms of the Radial Coordinate

The goal of this appendix is to substitute the displacement in the equilibrium equations and boundary conditions have been found in Section (3.7.4) with the displacement functions assumed in Eq. (3.28). Starting with the equilibrium equations, Eqs. (3.20) and (3.21) could re-written as

$$\begin{aligned}
 \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ A_{1,1} \left(\sum_{n=1}^{\alpha} f_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f_{2n} \sin n\theta \right) + A_{1,2} \left(\sum_{n=1}^{\alpha} f'_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f'_{2n} \sin n\theta \right) \right. \\
 + A_{1,3} \left(\sum_{n=1}^{\alpha} f''_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f''_{2n} \sin n\theta \right) + A_{1,4} \left(-\sum_{n=1}^{\alpha} n^2 f_{1n} \cos n\theta - \sum_{n=1}^{\alpha} n^2 f_{2n} \sin n\theta \right) \\
 + A_{1,5} \left(-\sum_{n=1}^{\alpha} n g_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n g_{2n} \cos n\theta \right) + A_{1,6} \left(-\sum_{n=1}^{\alpha} n g'_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n g'_{2n} \cos n\theta \right) \\
 + \lambda \left[B_{1,1} \left(\sum_{n=1}^{\alpha} f_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f_{2n} \sin n\theta \right) + B_{1,2} \left(\sum_{n=1}^{\alpha} f'_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f'_{2n} \sin n\theta \right) \right. \\
 + B_{1,3} \left(\sum_{n=1}^{\alpha} f''_{1n} \cos n\theta + \sum_{n=1}^{\alpha} f''_{2n} \sin n\theta \right) + B_{1,4} \left(-\sum_{n=1}^{\alpha} n^2 f_{1n} \cos n\theta - \sum_{n=1}^{\alpha} n^2 f_{2n} \sin n\theta \right) \\
 \left. \left. + B_{1,5} \left(-\sum_{n=1}^{\alpha} n g_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n g_{2n} \cos n\theta \right) \right] \right\} d\theta dr = 0
 \end{aligned} \tag{A.3.1}$$

$$\begin{aligned}
 \int_{r_1}^{r_2} \int_0^{2\pi} \left\{ A_{2,1} \left(-\sum_{n=1}^{\alpha} n f_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n f_{2n} \cos n\theta \right) + A_{2,2} \left(-\sum_{n=1}^{\alpha} n f'_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n f'_{2n} \cos n\theta \right) \right. \\
 + A_{2,3} \left(\sum_{n=1}^{\alpha} g_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g_{2n} \sin n\theta \right) + A_{2,4} \left(\sum_{n=1}^{\alpha} g'_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g'_{2n} \sin n\theta \right) \\
 + A_{2,5} \left(\sum_{n=1}^{\alpha} g''_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g''_{2n} \sin n\theta \right) + A_{2,6} \left(-\sum_{n=1}^{\alpha} n^2 g_{1n} \cos n\theta - \sum_{n=1}^{\alpha} n^2 g_{2n} \sin n\theta \right) \\
 + \lambda \left[B_{2,1} \left(-\sum_{n=1}^{\alpha} n f_{1n} \sin n\theta + \sum_{n=1}^{\alpha} n f_{2n} \cos n\theta \right) + B_{2,2} \left(\sum_{n=1}^{\alpha} g_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g_{2n} \sin n\theta \right) \right. \\
 + B_{2,3} \left(\sum_{n=1}^{\alpha} g'_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g'_{2n} \sin n\theta \right) + B_{2,4} \left(\sum_{n=1}^{\alpha} g''_{1n} \cos n\theta + \sum_{n=1}^{\alpha} g''_{2n} \sin n\theta \right) \\
 \left. \left. + B_{2,5} \left(-\sum_{n=1}^{\alpha} n^2 g_{1n} \cos n\theta - \sum_{n=1}^{\alpha} n^2 g_{2n} \sin n\theta \right) \right] \right\} d\theta dr = 0
 \end{aligned} \tag{A.3.2}$$

Now, in order to remove the summation sign in the above equations, of the following orthogonality relations will be used

$$\int_0^{2\pi} \cos m\theta \sin n\theta d\theta = 0$$

$$\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = \begin{cases} 0 & m \neq n \\ \cos^2 n\theta & m = n \end{cases} \quad (\text{A.3.3})$$

$$\int_0^{2\pi} \sin m\theta \sin n\theta d\theta = \begin{cases} 0 & m \neq n \\ \sin^2 n\theta & m = n \end{cases}$$

By multiplying Eq. (A.3.1) by $\cos m\theta$ and Eq. (A.3.2) by $\sin m\theta$, one obtains

$$\int_{r_1}^{r_2} \left[(A_{1,1} f_{1n} + A_{1,2} f'_{1n} + A_{1,3} f''_{1n} - A_{1,4} n^2 f_{1n} + A_{1,5} n g_{2n} + A_{1,6} n g'_{2n}) \right. \\ \left. + \lambda (B_{1,1} f_{1n} + B_{1,2} f'_{1n} + B_{1,3} f''_{1n} - B_{1,4} n^2 f_{1n} + B_{1,5} n g_{2n}) \right] dr = 0 \quad (\text{A.3.4})$$

$$\int_{r_1}^{r_2} \left[(-A_{2,1} n f_{1n} - A_{2,2} n f'_{1n} + A_{2,3} g_{2n} + A_{2,4} g'_{2n} + A_{2,5} g''_{2n} - A_{2,6} n^2 g_{2n}) \right. \\ \left. + \lambda (-B_{2,1} n f_{1n} + B_{2,2} g_{2n} + B_{2,3} g'_{2n} + B_{2,4} g''_{2n} - B_{2,5} n^2 g_{2n}) \right] dr = 0 \quad (\text{A.3.5})$$

Also if Eq. (A.3.1) is multiplied by $\sin m\theta$ and Eq. (A.3.2) is multiplied by $\cos m\theta$ another set of equilibrium equations are recovered as

$$(A_{1,1} f_{2n} + A_{1,2} f'_{2n} + A_{1,3} f''_{2n} - A_{1,4} n^2 f_{2n} - A_{1,5} n g_{1n} - A_{1,6} n g'_{1n}) \\ + \lambda (B_{1,1} f_{2n} + B_{1,2} f'_{2n} + B_{1,3} f''_{2n} - B_{1,4} n^2 f_{2n} - B_{1,5} n g_{1n}) = 0 \quad (\text{A.3.6})$$

$$\begin{aligned} & \left(A_{2,1}nf_{2n} + A_{2,2}nf'_{2n} + A_{2,3}g_{1n} + A_{2,4}g'_{1n} + A_{2,5}g''_{1n} - A_{2,6}n^2g_{1n} \right) \\ & + \lambda \left(B_{2,1}nf_{2n} + B_{2,2}g_{1n} + B_{2,3}g'_{1n} + B_{2,4}g''_{1n} - B_{2,5}n^2g_{1n} \right) = 0 \end{aligned} \quad (\text{A.3.7})$$

The same procedure is applicable to the boundary equations, they are re-written in terms of displacement fields. Each equation is multiplied once by $\cos m\theta$ and once by $\sin m\theta$. Note that since the assumed displacement functions are periodic, Eq. (A.2.10) and Eq. (A.2.11) vanish irrespective of the magnitude for the displacement fields, and they can thus be omitted. By multiplying Eq. (A.2.6) and Eq. (A.2.8) by $\cos m\theta$ and Eq. (A.2.7) and Eq. (A.2.9) by $\sin m\theta$, one obtains

$$\left\{ \int_0^{2\pi} \left[C_{1,1}f_{1n} + C_{1,2}f'_{1n} + C_{1,3}ng_{2n} + \lambda(D_{1,1}f'_{1n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.3.8})$$

$$\left\{ \int_0^{2\pi} \left[(-C_{2,1}nf_{1n} + C_{2,2}g_{2n} + C_{2,3}g'_{2n}) + \lambda(D_{2,1}g'_{2n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.3.9})$$

$$\int_0^{2\pi} \left\{ \left[(C_{3,1}f_{1n} + C_{3,2}f'_{1n} + C_{3,3}ng_{2n}) + \lambda(D_{3,1}f_{1n} + D_{3,2}f'_{1n} + D_{3,3}ng_{2n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.3.10})$$

$$\left\{ \int_0^{2\pi} \left[(-C_{4,1}nf_{1n} + C_{4,2}g_{2n} + C_{4,3}g'_{2n}) + \lambda(-D_{4,1}nf_{1n} + D_{4,2}g_{2n} + D_{4,3}g'_{2n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.3.11})$$

Again another set of boundary conditions can be produced.

$$\left\{ \int_0^{2\pi} \left[C_{1,1}f_{2n} + C_{1,2}f'_{2n} - C_{1,3}ng_{1n} + \lambda(D_{1,1}f'_{2n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.3.12})$$

$$\left\{ \int_0^{2\pi} \left[(C_{2,1}nf_{2n} + C_{2,2}g_{1n} + C_{2,3}g'_{1n}) + \lambda(D_{2,1}g'_{1n}) \right] d\theta \right\}_{r=r_1} = 0 \quad (\text{A.3.13})$$

$$\int_0^{2\pi} \left\{ \left[(C_{3,1} f_{2n} + C_{3,2} f'_{2n} - C_{3,3} n g_{1n}) + \lambda (D_{3,1} f_{2n} + D_{3,2} f'_{2n} - D_{3,3} n g_{1n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.3.14})$$

$$\left\{ \int_0^{2\pi} \left[(C_{4,1} n f_{2n} + C_{4,2} g_{1n} + C_{4,3} g'_{1n}) + \lambda (D_{4,1} n f_{2n} + D_{4,2} g_{1n} + D_{4,3} g'_{1n}) \right] d\theta \right\}_{r=r_2} = 0 \quad (\text{A.3.15})$$

Given similarity of the resulting identity, only equations (A.3.8) to (A.3.11) will be considered.

A.4 Matrices Used in Eq. (4.19)

In this appendix all the matrices have been used in Eq. (4.19) in chapter 4 are listed

$$[K_{E,i}] = \begin{bmatrix} -\frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} & -\frac{A_{1,6}n}{2\Delta} & A_{1,1} - \frac{2A_{1,3}}{\Delta^2} - A_{1,4}n^2 & A_{1,5}n & \frac{A_{1,2}}{2\Delta} + \frac{A_{1,3}}{\Delta^2} & \frac{A_{1,6}n}{2\Delta} \\ \frac{A_{2,2}n}{2\Delta} & -\frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} & -A_{2,1}n & A_{2,3} - \frac{2A_{2,5}}{\Delta^2} - A_{2,6}n^2 & -\frac{A_{2,2}n}{2\Delta} & \frac{A_{2,4}}{2\Delta} + \frac{A_{2,5}}{\Delta^2} \end{bmatrix} \quad (A.4.1)$$

$$[K_{g,E,i}] = \begin{bmatrix} -\frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} & 0 & B_{1,1} - \frac{2B_{1,3}}{\Delta^2} - B_{1,4}n^2 & B_{1,5}n & \frac{B_{1,2}}{2\Delta} + \frac{B_{1,3}}{\Delta^2} & 0 \\ 0 & -\frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} & -B_{2,1}n & B_{2,2} - \frac{2B_{2,4}}{\Delta^2} - B_{2,5}n^2 & 0 & \frac{B_{2,3}}{2\Delta} + \frac{B_{2,4}}{\Delta^2} \end{bmatrix} \quad (A.4.2)$$

$$\{\bar{f}_{E,i}\} = \{f_{i-1} \quad g_{i-1} \quad f_i \quad g_i \quad f_{i+1} \quad g_{i+1}\}^T \quad (A.4.3)$$

Note that the above equations have different values for each node, since they depend on the radius of the node and each node has different radius.

$$[K_{IBC}] = \begin{bmatrix} -\frac{C_{1,2}}{2\Delta} & 0 & C_{1,1} & C_{1,3}n & \frac{C_{1,2}}{2\Delta} & 0 \\ 0 & -\frac{C_{2,3}}{2\Delta} & -C_{2,1}n & C_{2,2} & 0 & \frac{C_{2,3}}{2\Delta} \end{bmatrix} \quad (A.4.4)$$

$$[K_{g,IBC}] = \begin{bmatrix} -\frac{D_{1,1}}{2\Delta} & 0 & 0 & 0 & \frac{D_{1,1}}{2\Delta} & 0 \\ 0 & -\frac{D_{2,1}}{2\Delta} & 0 & 0 & 0 & \frac{D_{2,1}}{2\Delta} \end{bmatrix} \quad (A.4.5)$$

$$[K_{EBC}] = \begin{bmatrix} -\frac{C_{3,2}}{2\Delta} & 0 & C_{3,1} & C_{3,3}n & \frac{C_{3,2}}{2\Delta} & 0 \\ 0 & -\frac{C_{4,3}}{2\Delta} & -C_{4,1}n & C_{4,2} & 0 & \frac{C_{4,3}}{2\Delta} \end{bmatrix} \quad (A.4.6)$$

$$\left[K_{g,EBC} \right] = \begin{bmatrix} -\frac{D_{3,2}}{2\Delta} & 0 & D_{3,1} & D_{3,3}n & \frac{D_{3,2}}{2\Delta} & 0 \\ 0 & -\frac{D_{4,3}}{2\Delta} & -D_{4,1}n & D_{4,2} & 0 & \frac{D_{4,3}}{2\Delta} \end{bmatrix} \quad (\text{A.4.7})$$

$$\left\{ \bar{f}_{IBC} \right\} = \left\{ f_0 \quad g_0 \quad f_1 \quad g_1 \quad f_2 \quad g_2 \right\}^T \quad (\text{A.4.8})$$

$$\left\{ \bar{f}_{EBC} \right\} = \left\{ f_p \quad g_p \quad f_{p+1} \quad g_{p+1} \quad f_{p+2} \quad g_{p+2} \right\}^T \quad (\text{A.4.9})$$

A.5 Finite Difference MATLAB Code

This appendix contains the codes have been used in MATLAB to solve the finite difference equations found in chapter 4. This program is able to find the buckling load of a plane thick pipe.

The following is the main code

```
clc
clear all

% Introducing the material properties of the pipe
E=200000;           % Modulus of elasticity
v=0.3;             % Poisson's ratio
L=E*v/( (1+v)*(1-2*v) );

% Defining the internal (r1) and external (r2) radius of the pipe
r1=200;
r2=400;

% Applying the internal (Pint) and external (Pext) pressure
Pint=0;
Pext=1;

% Defining the mode number of buckling (n)
n=2;

% Introducing the number of subdivisions (p) and calculating the length
% of each part (d)
p=50;
d=(r2-r1)/p;

% Boundary conditions at r1
% Using a pre-defined function in order to calculate the pre-buckling
% displacement (up) and its derivatives (dup and ddup) at r1
[up dup ddup]=prebucklingdisplacement(L,v,r1,r1,r2,Pext,Pint);
```

```

% Defining some coefficients for the boundary conditions at r1
C(1,1) = v+v/L*Pint;          C(2,1) = (1-2*v)/2-Pint*v/L;
C(1,2) = (1-v)*r1;           C(2,2) = -(1-2*v)/2+Pint*v/L;
C(1,3) = v+Pint*v/L;         C(2,3) = (1-2*v)*r1/2;
D(1,1) = v*up+(1-v)*dup*r1;   D(2,1) = v*up+(1-v)*dup*r1;

% Filling out the stiffness matrix [K] and geometric stiffness matrix [Kg]
K(1,1) = -C(1,2)/(2*d);       Kg(1,1) = -D(1,1)/(2*d);
K(1,3) = C(1,1);
K(1,4) = C(1,3)*n;
K(1,5) = C(1,2)/(2*d);       Kg(1,5) = D(1,1)/(2*d);

K(2,2) = -C(2,3)/(2*d);       Kg(2,2) = -D(2,1)/(2*d);
K(2,3) = -C(2,1)*n;
K(2,4) = C(2,2);
K(2,6) = C(2,3)/(2*d);       Kg(2,6) = D(2,1)/(2*d);

% Equilibrium Equations
% Defining a node counter (i)
i=1;

% Generating the equilibrium equations for each node trough a loop
for r=r1:d:r2

% Using a pre-defined function to calculate some coefficients for the
% stiffness matrix at r
[A B] = coefficients(r,r1,r2,L,v,Pext,Pint);

% Filling out the stiffness matrix and geometrical stiffness matrix
K(2*i+1,2*i-1) = -A(1,2)/2/d+A(1,3)/d^2;
K(2*i+1,2*i) = -A(1,6)*n/2/d;
K(2*i+1,2*i+1) = A(1,1)-A(1,4)*n^2-2*A(1,3)/d^2;
K(2*i+1,2*i+2) = A(1,5)*n;
K(2*i+1,2*i+3) = A(1,2)/2/d+A(1,3)/d^2;
K(2*i+1,2*i+4) = A(1,6)*n/2/d;
Kg(2*i+1,2*i-1) = -B(1,2)/2/d+B(1,3)/d^2;

```

```

Kg(2*i+1,2*i+1) = B(1,1)-2*B(1,3)/d^2-B(1,4)*n^2;
Kg(2*i+1,2*i+2) = B(1,5)*n;
Kg(2*i+1,2*i+3) = B(1,2)/2/d+B(1,3)/d^2;

K(2*i+2,2*i-1) = A(2,2)*n/2/d;
K(2*i+2,2*i) = -A(2,4)/2/d+A(2,5)/d^2;
K(2*i+2,2*i+1) = -A(2,1)*n;
K(2*i+2,2*i+2) = A(2,3)-2*A(2,5)/d^2-A(2,6)*n^2;
K(2*i+2,2*i+3) = -A(2,2)*n/2/d;
K(2*i+2,2*i+4) = A(2,4)/2/d+A(2,5)/d^2;
Kg(2*i+2,2*i) = -B(2,3)/2/d+B(2,4)/d^2;
Kg(2*i+2,2*i+1) = -B(2,1)*n;
Kg(2*i+2,2*i+2) = B(2,2)-2*B(2,4)/d^2-B(2,5)*n^2;
Kg(2*i+2,2*i+4) = B(2,3)/2/d+B(2,4)/d^2;

% Increasing the node number to find the equilibrium equation of the next
% node during the next loop
i=i+1;
end

% Boundary conditions at r2
% Using a pre-defined function in order to calculate the pre-buckling
% displacement (up) and its derivatives (dup and ddup) at r2
[up dup ddup]=prebucklingdisplacement(L,v,r,r1,r2,Pext,Pint);

% Defining some coefficients for the boundary conditions at r2
C(3,1) = v; C(4,1) = (1-2*v)/2;
C(3,2) = (1-v)*r2; C(4,2) = -(1-2*v)/2;
C(3,3) = v; C(4,3) = (1-2*v)*r2/2;
D(3,1) = Pext*v/L; D(4,1) = -Pext*v/L;
D(3,2) = v*up+(1-v)*dup*r2; D(4,2) = Pext*v/L;
D(3,3) = Pext*v/L; D(4,3) = v*up+(1-v)*dup*r2;

% completing the stiffness matrix [K] and geometric stiffness matrix [Kg]
K(2*i+1,2*i-3) = -C(3,2)/(2*d); Kg(2*i+1,2*i-3) = -D(3,2)/(2*d);
K(2*i+1,2*i-1) = C(3,1); Kg(2*i+1,2*i-1) = D(3,1);

```

```

K(2*i+1,2*i)    =  C(3,3)*n;          Kg(2*i+1,2*i)    =  D(3,3)*n;
K(2*i+1,2*i+1)  =  C(3,2)/(2*d);      Kg(2*i+1,2*i+1)  =  D(3,2)/(2*d);

K(2*i+2,2*i-2)  =  -C(4,3)/(2*d);     Kg(2*i+2,2*i-2)  =  -D(4,3)/(2*d);
K(2*i+2,2*i-1)  =  -C(4,1)*n;         Kg(2*i+2,2*i-1)  =  -D(4,1)*n;
K(2*i+2,2*i)    =  C(4,2);            Kg(2*i+2,2*i)    =  D(4,2);
K(2*i+2,2*i+2)  =  C(4,3)/(2*d);      Kg(2*i+2,2*i+2)  =  D(4,3)/(2*d);

% Solving the Eigen value buckling problem. D is the Eigen values and V is
% the Eigen vectors
[V,D] = eig(K,-Kg);
% Sorting the Eigen values to pick the smallest one which is the buckling
% load
D1=sort(diag(D));

% Printing the buckling load
if D1(1,1)>0
    disp(['The critical load is: ', num2str(Pext*D1(1,1))]);
else
    disp(['The critical load is: ', num2str(Pext*D1(2,1))]);
end

% Calculating the buckling load base on the thin shell theory
Pcrt=E*(r2-r1)^3/(4*(1-v^2)*((r1+r2)/2)^3)

```

The following is a function has been used in the main code to calculate the pre-buckling displacements

```

function [up,dup,ddup]=prebucklingdisplacement(L,v,r,r1,r2,Pext,Pint)

up=v/L*(r2^2*Pext+r1^2*Pint)/(r1^2-r2^2)*r...
    +v/L/(1-2*v)*r1^2*r2^2*(Pint+Pext)/(r1^2-r2^2)/r;
dup=v/L*(r2^2*Pext+r1^2*Pint)/(r1^2-r2^2)...

```

```

        -v/L/(1-2*v)*r1^2*r2^2*(Pint+Pext)/(r1^2-r2^2)/r^2;
ddup=2*v/L/(1-2*v)*r1^2*r2^2*(Pint+Pext)/(r1^2-r2^2)/r^3;
end

function [A,B]=coefficients(r,r1,r2,L,v,Pext,Pint)
A=zeros(2,6);
B=zeros(2,5);

[up dup ddup]=prebucklingdisplacement(L,v,r,r1,r2,Pext,Pint);

A(1,1)=(1-v)/r;                A(2,1)=-(3-4*v)/r/2;
A(1,2)=-(1-v);                A(2,2)=-1/2;
A(1,3)=-r*(1-v);              A(2,3)=(1-2*v)/r/2;
A(1,4)=-(1-2*v)/r/2;          A(2,4)=-(1-2*v)/2;
A(1,5)=(3-4*v)/r/2;            A(2,5)=-r*(1-2*v)/2;
A(1,6)=-1/2;                  A(2,6)=-(1-v)/r;
B(1,1)=(1-v)*up/r^2+v*dup/r;   B(2,1)=-2*(1-v)*up/r^2-2*v*dup/r;
B(1,2)=-dup-(1-v)*r*ddup;      B(2,2)=(1-v)*up/r^2+v*dup/r;
B(1,3)=-v*up-(1-v)*r*dup;      B(2,3)=-dup-r*(1-v)*ddup;
B(1,4)=-(1-v)*up/r^2-v*dup/r;  B(2,4)=-v*up-r*(1-v)*dup;
B(1,5)=2*(1-v)*up/r^2+2*v*dup/r; B(2,5)=-(1-v)*up/r^2-v*dup/r;
end

```

also another function is defined in order to calculate some coefficients used in the main part of the code.

```

function [A,B]=coefficients(r,r1,r2,L,v,Pext,Pint)
A=zeros(2,6);
B=zeros(2,5);

[up dup ddup]=prebucklingdisplacement(L,v,r,r1,r2,Pext,Pint);

A(1,1)=(1-v)/r;                A(2,1)=-(3-4*v)/r/2;
A(1,2)=-(1-v);                A(2,2)=-1/2;

```

```

A(1,3)=-r*(1-v);
A(1,4)=-(1-2*v)/r/2;
A(1,5)=(3-4*v)/r/2;
A(1,6)=-1/2;
B(1,1)=(1-v)*up/r^2+v*dup/r;
B(1,2)=-dup-(1-v)*r*ddup;
B(1,3)=-v*up-(1-v)*r*dup;
B(1,4)=-(1-v)*up/r^2-v*dup/r;
B(1,5)=2*(1-v)*up/r^2+2*v*dup/r;
A(2,3)=(1-2*v)/r/2;
A(2,4)=-(1-2*v)/2;
A(2,5)=-r*(1-2*v)/2;
A(2,6)=-(1-v)/r;
B(2,1)=-2*(1-v)*up/r^2-2*v*dup/r;
B(2,2)=(1-v)*up/r^2+v*dup/r;
B(2,3)=-dup-r*(1-v)*ddup;
B(2,4)=-v*up-r*(1-v)*dup;
B(2,5)=- (1-v)*up/r^2-v*dup/r;
end

```

A.6 Finite Element MATLAB Code

This appendix contains the codes have been used in MATLAB to solve the Finite Element equations developed in Chapter 5. The program consists of a main modul which calls several functions to solve the problem. The program reads all of the input data from an excel file and all the out puts, which includes the buckling pressure and its associated normalized displacements will be written in another excel file. Also, the deformed configuration of the system is saved on the same directory of the program. Below is the listing of the main program and all of its functions. An example of the input Excel data is provided in the end of this appendix.

```
clc
clear all
disp('Buckling of Sandwich Pipes FEM')

% Read file names
file_name1 = 'COREMATERIAL.xlsx';
Run=xlsread(file_name1, 2, 'B1');
Results=zeros(Run,1);
%
h=zeros(Run,1);

x=findcolumn(Run);

n=xlsread(file_name1, 2, ['C2:' char(x) '2']);
NL=xlsread(file_name1, 2, ['C3:' char(x) '3']);
Ri=xlsread(file_name1, 2, ['C4:' char(x) '4']);
Re=xlsread(file_name1, 2, ['C5:' char(x) '5']);
Pint=xlsread(file_name1, 2, ['C6:' char(x) '6']);
Pext=xlsread(file_name1, 2, ['C7:' char(x) '7']);

NE=zeros(NL(1,Run),Run);
E=zeros(NL(1,Run),Run);
v=zeros(NL(1,Run),Run);
t=zeros(NL(1,Run),Run);

for i=1:NL(1,1)
    i1=8+(i-1)*4;
    NE(i,:)=xlsread(file_name1, 2, ['C' num2str(i1) ':' char(x) num2str(i1)]);
    E(i,:)=xlsread(file_name1, 2, ['C' num2str(i1+1) ':' char(x) num2str(i1+1)]);
    v(i,:)=xlsread(file_name1, 2, ['C' num2str(i1+2) ':' char(x) num2str(i1+2)]);
    t(i,:)=xlsread(file_name1, 2, ['C' num2str(i1+3) ':' char(x) num2str(i1+3)]);
end
```

```

for RC=1:Run

DOF=2;
NS=NL(1,RC)+1;

TNE=0;
L=zeros(NL(1,RC),1);
d=zeros(NL(1,RC),1);
for i=1:NL(1,RC)
    TNE=TNE+NE(i,RC);
    L(i,1)=E(i,RC)*v(i,RC)/(1+v(i,RC))/(1-2*v(i,RC));
    d(i,1)=t(i,RC)/NE(i,RC);
end
TNS=TNE+1;

R=zeros(NS,1);
R(1,1)=Ri(1,RC);
for i=2:NS
    R(i,1)=R(i-1,1)+t(i-1,RC);
end

r=zeros(TNS,1);
r(1,1)=Ri(1,RC);
for i=2:TNS
    if i<=NE(1,RC)+1
        j=1;
    elseif i<=NE(1,RC)+NE(2,RC)+1
        j=2;
    else
        j=3;
    end
    r(i,1)=r(i-1,1)+d(j,1);
end

NDOF=2*TNS;

Ket = zeros(NDOF);
Kgit = zeros(NDOF);
Kget = zeros(NDOF);
[NSC] = Build_NSC(TNS);
[Fi]=prebucklingdisplacement(NL(1,RC),L,v(:,RC),R,0,Pint(1,RC));
[Fe]=prebucklingdisplacement(NL(1,RC),L,v(:,RC),R,Pext(1,RC),0);
for i = 1:TNE
    if i<=NE(1,1)
        j=1;
    elseif i<=NE(1,1)+NE(2,1)
        j=2;
    else
        j=3;
    end
    F1i = Fi(2*(j-1)+1,1);
    F2i = Fi(2*(j-1)+2,1);
    F1e = Fe(2*(j-1)+1,1);
    F2e = Fe(2*(j-1)+2,1);
    Li = L(j,1);

```

```

        vi = v(j,RC);
        [Ke, Kgi,
Kge]=Calculate_Local_Stiffness_pipe(n(1,RC),L(j,1),v(j,RC),r(i,1),...

d(j,1),Pint(1,RC),Pext(1,RC),i,TNE,F1i,F2i,F1e,F2e);
        Ket = Assemble_Kt(DOF, NDOF, i, NSC, Ke, Ket);
        Kgit = Assemble_Kt(DOF, NDOF, i, NSC, Kgi, Kgit);
        Kget = Assemble_Kt(DOF, NDOF, i, NSC, Kge, Kget);
end

[V,D]=eig(Ket+Kgit,-Kget);
[D1, IX]=sort(diag(D));
u1=zeros(TNS,1);
v1=zeros(TNS,1);
% Printing the buckling load
for j=1:NE*2
    if D1(j,1)>0
        V1=V(:, IX(j,1));
        for i=0:TNS-1
            u1(i+1,1)=V1(1+2*i,1);
            v1(i+1,1)=V1(2+2*i,1);
        end
        disp(['The critical load is: ', num2str(D1(j,1))]);
        Results(RC,1)=D1(j,1);

        % xlswrite(file_name1, u1(1,1), 1, ['C' num2str(RC)]);
        % xlswrite(file_name1, Fe(1,1)*Ri(1,RC)+Fe(2,1)/Ri(1,RC), 1, ['D'
num2str(RC)]);
        % xlswrite(file_name1, u1(TNS,1), 1, ['F' num2str(RC)]);
        % xlswrite(file_name1, Fe(5,1)*Re(1,RC)+Fe(6,1)/Re(1,RC), 1, ['G'
num2str(RC)]);
        break
    end
end

for j=0:TNS-1
    if j+1<=NE(1,1)
        i=1;
    elseif j+1<=NE(1,1)+NE(2,1)
        i=2;
    else
        i=3;
    end
    F1i = Fi(2*(i-1)+1,1);
    F2i = Fi(2*(i-1)+2,1);
    F1e = Fe(2*(i-1)+1,1);
    F2e = Fe(2*(i-1)+2,1);
    for i=0:360
        C(i+1,j+1,1,RC)=r(j+1)*cos(i*pi/180);

U(i+1,j+1,1,RC)=(u1(j+1,1)+(F1e*r(j+1)+F2e/r(j+1))+(F1i*r(j+1)+F2i/r(j+1)))*c
os(n(1,RC)*i*pi/180)*cos(i*pi/180)-
v1(j+1,1)*sin(n(1,RC)*i*pi/180)*sin(i*pi/180);
        C(i+1,j+1,2,RC)=r(j+1)*sin(i*pi/180);

U(i+1,j+1,2,RC)=(u1(j+1,1)+(F1e*r(j+1)+F2e/r(j+1))+(F1i*r(j+1)+F2i/r(j+1)))*c

```

```

os(n(1,RC)*i*pi/180)*sin(i*pi/180)+v1(j+1,1)*sin(n(1,RC)*i*pi/180)*cos(i*pi/180);
    end
end

X=200;
FinalC=C+X*U;
    xlswrite('FIGURES', FinalC(:, :, 1, RC), ['Model #' num2str(RC)], 'B2')
    xlswrite('FIGURES', FinalC(:, :, 2, RC), ['Model #' num2str(RC)], 'B370')
h(RC,1)=figure;
plot(FinalC(:, 1:1:TNS, 1, RC), FinalC(:, 1:1:TNS, 2, RC));
saveas(h(RC,1), ['model' num2str(RC)], 'jpg');
close
end
xlswrite(file_name1, Results, 1, 'B2');

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                                End of the main program                                %%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function [Kt] = Assemble_Kt(NCS, NDOF, EB, NSC, K, Kt)
%   Function that stores the pertinent elements of the member global
%   stiffness matrix [Kt] in the structure stiffness matrix [K]
EE=EB+1;
    for i = 1:(2*NCS)
        if i <= NCS
            i1 = (EB-1)*NCS + i;
        else
            i1 = (EE-1)*NCS + (i - NCS);
        end
        n1 = NSC(i1);
        if n1 <= NDOF
            for j = 1:(2*NCS)
                if j <= NCS
                    i1 = (EB-1)*NCS + j;
                else
                    i1 = (EE-1)*NCS + (j - NCS);
                end
                n2 = NSC(i1);
                if n2 <= NDOF
                    Kt(n1,n2) = Kt(n1,n2) + K(i,j);
                end
            end
        end
    end

```

```

end
end
end
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
End of the function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function [NSC] = Build_NSC(NS)
% Function that forms the structure coordinate number vector {NSC}
NSC=zeros(2*NS,1);
j = 0;
for i = 1:NS;
    for i1 = 1:2;
        i2 = (i - 1)*2 + i1;
        j = j + 1;
        NSC(i2) = j;
    end
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
End of the function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function [Ke, Kgi,Kge] = Calculate_Local_Stiffness_pipe(n,L,v,r1,d,Pint,...
Pext,i,TNE,F1i,F2i,F1e,F2e)
% Function that calculates the member local stiffness matrix for a plane
r2=r1+d;
A=1/d^2*[ (log(r2/r1)-1.5)*r2^2-0.5*r1^2+2*r2*r1 ....
0.5*(r2^2-r1^2)-r2*r1*log(r2/r1)
0.5*(r2^2-r1^2)-r2*r1*log(r2/r1) ...
0.5*r2^2+(log(r2/r1)+1.5)*r1^2-2*r2*r1];
B=1/2/d^2*(r2^2-r1^2)*[1 -1;-1 1];
C=1/d^2*[-0.5*(r2^2+r1^2)+r2*r1 0.5*(r2^2+r1^2)-r2*r1

```

```

-0.5*(r2^2+r1^2)+r2*r1 0.5*(r2^2+r1^2)-r2*r1];
Agil=1/d^2*Fli*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...
(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]+...
1/d^2*F2i*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];
Agiz=1/d^2*Fli*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...
(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]-...
1/d^2*F2i*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];
Bgi1=1/d^2*(Fli/2*(r2^2-r1^2)-F2i*log(r2/r1))*[1 -1; -1 1];
Bgi2=1/d^2*(Fli/2*(r2^2-r1^2)+F2i*log(r2/r1))*[1 -1; -1 1];

Age1=1/d^2*Fle*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...
(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]+...
1/d^2*F2e*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];
Age2=1/d^2*Fle*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...
(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]-...
1/d^2*F2e*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];
Bge1=1/d^2*(Fle/2*(r2^2-r1^2)-F2e*log(r2/r1))*[1 -1; -1 1];

```

```

Bge2=1/d^2*(F1e/2*(r2^2-r1^2)+F2e*log(r2/r1))*[1 -1; -1 1];

A1=[1 0; 0 0];
A2=[0 0; 0 1];

if i==1
    Pintf=Pint;
else
    Pintf=0;
end
if i==TNE
    Pextf=Pext;
else
    Pextf=0;
end

Ceuv=L*( (1-v)/v+(1-2*v)/4/v*n^2)*A+L*(1-v)/2/v*B+L*C-0.5*Pintf*A1;
Cev=L*( (1-v)/2/v*n^2+(1-2*v)/4/v)*A+L*(1-2*v)/4/v*B-L*(1-2*v)/2/v*C-0.5*Pintf*A1;
Ceuv=L*( (1-v)/v*n+(1-2*v)/2/v*n)*A+L*n*C'-L*(1-2*v)/2/v*n*C-Pintf*n*A1;
Cgui=L*(1-v)/2/v*(1+n^2)*Agil+L*0.5*(1+n^2)*Agi2+L*(1-v)/2/v*Bgil+L*0.5*Bgi2;
Cgvi=L*(1-v)/2/v*(1+n^2)*Agil+L*0.5*(1+n^2)*Agi2+L*(1-v)/2/v*Bgil+L*0.5*Bgi2;
Cguvi=L*2*(1-v)/v*n*Agil+L*2*n*Agi2;
Cgue=L*(1-v)/2/v*(1+n^2)*Age1+L*0.5*(1+n^2)*Age2+L*(1-v)/2/v*Bge1+L*0.5*Bge2+0.5*Pextf*A2;
Cgve=L*(1-v)/2/v*(1+n^2)*Age1+L*0.5*(1+n^2)*Age2+L*(1-v)/2/v*Bge1+L*0.5*Bge2+0.5*Pextf*A2;
Cguve=L*2*(1-v)/v*n*Age1+L*2*n*Age2+Pextf*n*A2;

Ke=[2*Ceu(1,1)          Ceuv(1,1)          Ceu(1,2)+Ceuv(2,1)  Ceuv(1,2);...
    Ceuv(1,1)          2*Cev(1,1)          Ceuv(2,1)
Cev(1,2)+Cev(2,1);...
    Ceu(2,1)+Ceuv(1,2)  Ceuv(2,1)          2*Ceu(2,2)          Ceuv(2,2);...
    Ceuv(1,2)          Cev(2,1)+Cev(1,2)  Ceuv(2,2)          2*Cev(2,2)];

Kgi=[2*Cgui(1,1)          Cguvi(1,1)          Cgui(1,2)+Cguvi(2,1)  Cguvi(1,2);...
    Cguvi(1,1)          2*Cgvi(1,1)          Cguvi(2,1)
Cgvi(1,2)+Cgvi(2,1);...

```

```

Cgui(2,1)+Cgui(1,2) Cguvi(2,1)          2*Cgui(2,2)          Cguvi(2,2);...
Cguvi(1,2)          Cgvi(2,1)+Cgvi(1,2) Cguvi(2,2)          2*Cgvi(2,2)];
Kge=[2*Cgue(1,1)          Cguve(1,1)          Cgue(1,2)+Cgue(2,1) Cguve(1,2);...
Cguve(1,1)          2*Cgve(1,1)          Cguve(2,1)
Cgve(1,2)+Cgve(2,1);...
Cgue(2,1)+Cgue(1,2) Cguve(2,1)          2*Cgue(2,2)          Cguve(2,2);...
Cguve(1,2)          Cgve(2,1)+Cgve(1,2) Cguve(2,2)          2*Cgve(2,2)];
end

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% End of the function %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function [Ke, Kg] = Calculate_Local_Stiffness_pipetest(n, L, v, r10, r20, r1,
r2, Pint, Pext, i, NL)

```

```

% Function that calculates the member local stiffness matrix for a plane
d=r2-r1;

```

```

s=v/L*(r20^2*Pext+r10^2*Pint)/(r10^2-r20^2);
h=v/L/(1-2*v)*(r10^2)*(r20^2)*(Pint+Pext)/(r10^2-r20^2);

```

```

A=1/d^2*[(log(r2/r1)-1.5)*r2^2-0.5*r1^2+2*r2*r1 ....
0.5*(r2^2-r1^2)-r2*r1*log(r2/r1)
0.5*(r2^2-r1^2)-r2*r1*log(r2/r1) ...
0.5*r2^2+(log(r2/r1)+1.5)*r1^2-2*r2*r1];

```

```

B=1/2/d^2*(r2^2-r1^2)*[1 -1;-1 1];

```

```

C=1/d^2*[-0.5*(r2^2+r1^2)+r2*r1 0.5*(r2^2+r1^2)-r2*r1
-0.5*(r2^2+r1^2)+r2*r1 0.5*(r2^2+r1^2)-r2*r1];

```

```

Ag1=1/d^2*s*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...
(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]+...
1/d^2*h*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];

```

```

Ag2=1/d^2*s*[(r2^2-r1^2)/2-2*r2^2+2*r1*r2+r2^2*log(r2/r1)...

```

```

(r2^2-r1^2)/2-r2*r1*log(r2/r1)
(r2^2-r1^2)/2-r2*r1*log(r2/r1)...
(r2^2-r1^2)/2+2*r1^2-2*r1*r2+r1^2*log(r2/r1)]-...
1/d^2*h*[log(r2/r1)+3/2-2*r2/r1+r2^2/2/r1^2 ...
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1
-log(r2/r1)-(r1+r2)/r2+(r1+r2)/r1+r1/2/r2-r2/2/r1 ...
log(r2/r1)-3/2+2*r1/r2-r1^2/2/r2^2];

Bg1=1/d^2*(s/2*(r2^2-r1^2)-h*log(r2/r1))*[1 -1; -1 1];
Bg2=1/d^2*(s/2*(r2^2-r1^2)+h*log(r2/r1))*[1 -1; -1 1];

A1=[1 0; 0 0];
A2=[0 0; 0 1];

if i==1
    Pintf=Pint;
else
    Pintf=0;
end
if i==NL
    Pextf=Pext;
else
    Pextf=0;
end

Ceuv=(1-v)/2/v+(1-2*v)/4/v*n^2)*A+(1-v)/2/v*B+C-0.5*Pintf/L*A1;
Cev=(1-v)/2/v*n^2+(1-2*v)/4/v)*A+(1-2*v)/4/v*B-(1-2*v)/2/v*C-0.5*Pintf/L*A1;
Ceuv=(1-v)/v*n+(1-2*v)/2/v*n)*A+n*C'-(1-2*v)/2/v*n*C-Pintf*n/L*A1;
Cgu=0.5*Pextf/L*A2;
Cgv=0.5*Pextf/L*A2;
Cguv=Pextf*n*A2/L;

Ke=[2*Ceuv(1,1)      Ceuv(1,1)      Ceuv(1,2)+Ceuv(2,1) Ceuv(1,2);...
    Ceuv(1,1)      2*Cev(1,1)      Ceuv(2,1)
    Ceuv(1,2)+Cev(2,1);...
    Ceuv(2,1)+Ceuv(1,2) Ceuv(2,1)      2*Ceuv(2,2)      Ceuv(2,2);...
    Ceuv(1,2)      Cev(2,1)+Cev(1,2) Ceuv(2,2)      2*Cev(2,2)];

```

```

Kg=[2*Cgu(1,1)          Cguv(1,1)          Cgu(1,2)+Cgu(2,1) Cguv(1,2);...
    Cguv(1,1)          2*Cgv(1,1)          Cguv(2,1)
Cgv(1,2)+Cgv(2,1);...
    Cgu(2,1)+Cgu(1,2) Cguv(2,1)          2*Cgu(2,2)          Cguv(2,2);...
    Cguv(1,2)          Cgv(2,1)+Cgv(1,2) Cguv(2,2)          2*Cgv(2,2)];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
End of the function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

function [x]=findcolumn(RC)
% This function helps to read the proper input data from the excel file
a=24;
b=26;
if RC<=a
    x=RC+66; end
if RC>a && RC<=a+b
    RCC=RC-a; x=[65 64+RCC]; end
if RC>a+b && RC<=a+2*b
    RCC=RC-a-b; x=[66 64+RCC]; end
if RC>a+2*b && RC<=a+3*b
    RCC=RC-a-2*b; x=[67 64+RCC]; end
if RC>a+3*b && RC<=a+4*b
    RCC=RC-a-3*b; x=[68 64+RCC]; end
if RC>a+4*b && RC<=a+5*b
    RCC=RC-a-4*b; x=[69 64+RCC]; end
if RC>a+5*b && RC<=a+6*b
    RCC=RC-a-5*b; x=[70 64+RCC]; end
if RC>a+6*b && RC<=a+7*b
    RCC=RC-a-6*b; x=[71 64+RCC]; end
if RC>a+7*b && RC<=a+8*b
    RCC=RC-a-7*b; x=[72 64+RCC]; end
if RC>a+8*b && RC<=a+9*b
    RCC=RC-a-8*b; x=[73 64+RCC]; end
if RC>a+9*b && RC<=a+10*b

```

```

        RCC=RC-a-9*b; x=[74 64+RCC]; end
if RC>a+10*b && RC<=a+11*b
        RCC=RC-a-10*b; x=[75 64+RCC]; end
if RC>a+11*b && RC<=a+12*b
        RCC=RC-a-11*b; x=[76 64+RCC]; end
if RC>a+12*b && RC<=a+13*b
        RCC=RC-a-12*b; x=[77 64+RCC]; end
if RC>a+13*b && RC<=a+14*b
        RCC=RC-a-13*b; x=[78 64+RCC]; end
if RC>a+14*b && RC<=a+15*b
        RCC=RC-a-14*b; x=[79 64+RCC]; end
if RC>a+15*b && RC<=a+16*b
        RCC=RC-a-15*b; x=[80 64+RCC]; end
if RC>a+16*b && RC<=a+17*b
        RCC=RC-a-16*b; x=[81 64+RCC]; end
if RC>a+17*b && RC<=a+18*b
        RCC=RC-a-17*b; x=[82 64+RCC]; end
if RC>a+18*b && RC<=a+19*b
        RCC=RC-a-18*b; x=[83 64+RCC]; end
if RC>a+19*b && RC<=a+20*b
        RCC=RC-a-19*b; x=[84 64+RCC]; end
if RC>a+20*b && RC<=a+21*b
        RCC=RC-a-20*b; x=[85 64+RCC]; end
if RC>a+21*b && RC<=a+22*b
        RCC=RC-a-21*b; x=[86 64+RCC]; end
if RC>a+22*b && RC<=a+23*b
        RCC=RC-a-22*b; x=[87 64+RCC]; end
if RC>a+23*b && RC<=a+24*b
        RCC=RC-a-23*b; x=[88 64+RCC]; end
if RC>a+24*b && RC<=a+25*b
        RCC=RC-a-24*b; x=[89 64+RCC]; end
if RC>a+25*b && RC<=a+26*b
        RCC=RC-a-25*b; x=[90 64+RCC]; end
if RC>a+26*b && RC<=a+27*b
        RCC=RC-a-26*b; x=[91 64+RCC]; end
if RC>a+27*b && RC<=a+28*b
        RCC=RC-a-27*b; x=[92 64+RCC]; end

```

```

if RC>a+28*b && RC<=a+29*b
    RCC=RC-a-28*b; x=[93 64+RCC]; end
if RC>a+29*b && RC<=a+30*b
    RCC=RC-a-29*b; x=[94 64+RCC]; end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                                End of the function                                %%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [F]=prebucklingdisplacement(NL,L,v,R,Pext,Pint)
NS=NL+1;
K=zeros(2*NL);
P=zeros(2*NL,1);

K(1,1)=L(1,1)*R(1,1)/v(1,1);
K(1,2)=-L(1,1)*(1-2*v(1,1))/v(1,1)/R(1,1);

for i=0:NS-3
    K(2*i+2,2*i+1)=L(i+1,1)*R(i+2,1)/v(i+1,1);
    K(2*i+2,2*i+2)=-L(i+1,1)*(1-2*v(i+1,1))/v(i+1,1)/R(i+2,1);
    K(2*i+2,2*i+3)=-L(i+2,1)*R(i+2,1)/v(i+2,1);
    K(2*i+2,2*i+4)=L(i+2,1)*(1-2*v(i+2,1))/v(i+2,1)/R(i+2,1);
    K(2*i+3,2*i+1)=R(i+2,1)^2;
    K(2*i+3,2*i+2)=1;
    K(2*i+3,2*i+3)=-R(i+2,1)^2;
    K(2*i+3,2*i+4)=-1;
end
K(2*NL,2*NL-1) = L(NL,1)*R(NS,1)/v(NL,1);
K(2*NL,2*NL) = -L(NL,1)*(1-2*v(NL,1))/v(NL,1)/R(NS,1);

P(1,1)=-R(1,1)*Pint;
P(2*NL,1)=-R(NS,1)*Pext;
F=K^-1*P;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

%%%%%%%%%

End of the function

%%%%%%%%%

%%%

Sample of the Excel input data

Number of Runs=	5	Model #1	Model #2	Model #3	Model #4	Model #5
Mode # (n)		2	3	4	5	6
Number of Layers		3	3	3	3	3
Internal Radius of Sandwich Pipe (mm)		535.2	535.2	535.2	535.2	535.2
External Radius of Sandwich Pipe (mm)		813	813	813	813	813
Internal Pressure (Mpa)		0	0	0	0	0
External Pressure (Mpa)		1	1	1	1	1
Layer #1	Number of EL	10	10	10	10	10
	Young's Modulus (MPa)	200000	200000	200000	200000	200000
	Poisson's Ration	0.3	0.3	0.3	0.3	0.3
	Thickness (mm)	16	16	16	16	16
Layer # 2	Number of EL	10	10	10	10	10
	Young's Modulus (MPa)	200	200	200	200	200
	Poisson's Ration	0.4	0.4	0.4	0.4	0.4
	Thickness (mm)	245.8	245.8	245.8	245.8	245.8
Layer # 3	Number of EL	10	10	10	10	10
	Young's Modulus (MPa)	200000	200000	200000	200000	200000
	Poisson's Ration	0.3	0.3	0.3	0.3	0.3
	Thickness (mm)	16	16	16	16	16

A.7 ABAQUS Parameterized Input File

The following is the input listing used to run the ABAQUS models

```
*HEADING
  BUCKLING OF SNADWICH PIPELINE UNDER HYDROSTATIC PRESSURE
*PARAMETER
#####
r4=300      #External Radius of Sandwich Pipe
r1=150      #Internal Radius of Sandwich Pipe
t1=25       #Thickness of Internal Pipe
t3=25       #Thickness of External Pipe
r2=r1+t1    #External Radius of Internal Pipe
r3=r4-t3    #Internal Radius of External Pipe
t2=r3-r2    #Thickness of the Core Layer
#####
E1=200000   #Modulus of elasticity of internal Pipe
E2=200000   #Modulus of elasticity of Core Layer
E3=200000   #Modulus of elasticity of External Pipe
v1=0.3      #Poisson's Ratio of internal Pipe
v2=0.3      #Poisson's Ratio of Core Layer
v3=0.3      #Poisson's Ratio of External Pipe
#####
n1=1        #Number of Elements in Internal Pipe (Radial Direction)
n2=2        #Number of Elements in Core Layer (Radial Direction)
n3=n1       #Number of Elements in External Pipe (Radial Direction)
mn1=2*n1    #Number of Nodes in Internal Pipe (Radial Direction)
mn2=2*n2    #Number of Nodes in Core Layer (Radial Direction)
mn3=2*n3    #Number of Nodes in External Pipe (Radial Direction)
m=20        #Number of Elements in Tangential Direction
nm=2*m      #Number of Nodes in Tangential Direction
#####
MN1B=1      #First Master Node at Internal Surface
```

```

MN1E=nm                #Last Master Node at Internal Surface
MN2B=MN1B+2*n1*nm      #First Master Node at Core Internal Surface
MN2E=MN2B+2*m-1        #Last Master Node at Core Internal
Surface
MN3B=MN2B+2*n2*nm      #First Master Node at Core External Surface
MN3E=MN3B+2*m-1        #Last Master Node at Core External
Surface
MN4B=MN3B+2*n3*nm      #First Master Node at External Surface
MN4E=MN4B+2*m-1        #Last Master Node at External Surface
MNEA=-360.00/nm        #Last Master Node End Angel
#####
MN1A90=MN1B+nm/4
MN1A180=MN1B+nm/2
MN1A270=MN1B+3*nm/4
MN4A90=MN4B+nm/4
MN4A180=MN4B+nm/2
MN4A270=MN4B+3*nm/4
#####
ME1B=1                #First Master Element at Internal Surface
ME1E=m                #Last Master Element at Internal Surface
ME2B=n1*m+1           #First Master Element at Second Surface
ME2E=ME2B+m-1         #Last Master Element at Second Surface
ME3B=(n1+n2)*m+1      #First Master Element at Third Surface
ME3E=ME3B+m-1         #Last Master Element at Third Surface
ME4B=(n1+n2+n3-1)*m+1 #First Master Element at External
Surface
ME4E=(n1+n2+n3)*m      #Last Master Element at External Surface
#####
###Defining the Nodes of Master Elements
ME1BN1=MN1B
ME1BN2=ME1BN1+2*nm
ME1BN3=ME1BN2+2
ME1BN4=ME1BN1+2
ME1BN5=ME1BN1+nm

```

```

ME1BN6=ME1BN2+1
ME1BN7=ME1BN5+2
ME1BN8=ME1BN1+1
#
ME1EN1=ME1BN1+nm-2
ME1EN2=ME1EN1+2*nm
ME1EN3=ME1BN2
ME1EN4=ME1BN1
ME1EN5=ME1EN1+nm
ME1EN6=ME1EN2+1
ME1EN7=ME1BN5
ME1EN8=ME1EN1+1
##
ME2BN1=ME1BN1+nm*mn1
ME2BN2=ME2BN1+2*nm
ME2BN3=ME2BN2+2
ME2BN4=ME2BN1+2
ME2BN5=ME2BN1+nm
ME2BN6=ME2BN2+1
ME2BN7=ME2BN5+2
ME2BN8=ME2BN1+1
#
ME2EN1=ME2BN1+nm-2
ME2EN2=ME2EN1+2*nm
ME2EN3=ME2BN2
ME2EN4=ME2BN1
ME2EN5=ME2EN1+nm
ME2EN6=ME2EN2+1
ME2EN7=ME2BN5
ME2EN8=ME2EN1+1
##
ME3BN1=MN3B
ME3BN2=ME3BN1+2*nm

```

```

ME3BN3=ME3BN2+2
ME3BN4=ME3BN1+2
ME3BN5=ME3BN1+nm
ME3BN6=ME3BN2+1
ME3BN7=ME3BN5+2
ME3BN8=ME3BN1+1
#
ME3EN1=ME3BN1+nm-2
ME3EN2=ME3EN1+2*nm
ME3EN3=ME3BN2
ME3EN4=ME3BN1
ME3EN5=ME3EN1+nm
ME3EN6=ME3EN2+1
ME3EN7=ME3BN5
ME3EN8=ME3EN1+1
#####
# (FOR ELGEN)
NEGEN=m-1  #NUMBER OF ELEMENTS IN EACH LAYER
NODEINC1=2
ELINC1=1
NODEINC2=2*nm
ELINC2=m
#####
*NODE, system=c
  <MN1B>, <r1>,    0.0
  <MN1E>, <r1>,    <MNEA>
  <MN2B>, <r2>,    0.0
  <MN2E>, <r2>,    <MNEA>
  <MN3B>, <r3>,    0.0
  <MN3E>, <r3>,    <MNEA>
  <MN4B>, <r4>,    0.0
  <MN4E>, <r4>,    <MNEA>
*****

```

```

*NGEN, LINE=C, NSET=INT
  <MN1B>, <MN1E>, 1,, 0, 0, 0, 0., 0., 1
*NGEN, LINE=C, NSET=B1
  <MN2B>, <MN2E>, 1,, 0, 0, 0, 0., 0., 1
*NGEN, LINE=C, NSET=B2
  <MN3B>, <MN3E>, 1,, 0, 0, 0, 0., 0., 1
*NGEN, LINE=C, NSET=EXT
  <MN4B>, <MN4E>, 1,, 0, 0, 0, 0., 0., 1
*****

*NFILL
  INT, B1 , <mn1>, <nm>
  B1 , B2 , <mn2>, <nm>
  B2 , EXT, <mn3>, <nm>
*****

*NSET, NSET=Eq0, GENERATE
  <MN1B>, <MN4B>, <nm>
*NSET, NSET=Eq90, GENERATE
  <MN1A90>, <MN4A90>, <nm>
*NSET, NSET=Eq180, GENERATE
  <MN1A180>, <MN4A180>, <nm>
*NSET, NSET=Eq270, GENERATE
  <MN1A270>, <MN4A270>, <nm>
*****

*ELEMENT, TYPE=CPE8R, ELSET=PIPE_INT

<ME1B>, <ME1BN1>, <ME1BN2>, <ME1BN3>, <ME1BN4>, <ME1BN5>, <ME1BN6>, <ME1BN7>, <ME1BN8>

<ME1E>, <ME1EN1>, <ME1EN2>, <ME1EN3>, <ME1EN4>, <ME1EN5>, <ME1EN6>, <ME1EN7>, <ME1EN8>
****

*ELGEN, ELSET=PIPE_INT
  <ME1B>, <NEGEN>, <NODEINC1>, <ELINC1>, <n1>, <NODEINC2>, <ELINC2>
  <ME1E>, <n1>, <NODEINC2>, <ELINC2>, 1

```

```

****

*SOLID SECTION,ELSET=PIPE_INT,MATERIAL=STEELINT
*MATERIAL,NAME=STEELINT
*ELASTIC
    <E1>, <v1>
*****

*ELEMENT,TYPE=CPE8R,ELSET=PIPE_core

<ME2B>,<ME2BN1>,<ME2BN2>,<ME2BN3>,<ME2BN4>,<ME2BN5>,<ME2BN6>,<ME
2BN7>,<ME2BN8>

<ME2E>,<ME2EN1>,<ME2EN2>,<ME2EN3>,<ME2EN4>,<ME2EN5>,<ME2EN6>,<ME
2EN7>,<ME2EN8>
*****

*ELGEN,ELSET=PIPE_core
    <ME2B>,<NEGEN>,<NODEINC1>,<ELINC1>,<n2>,<NODEINC2>,<ELINC2>
    <ME2E>,<n2>,<NODEINC2>,<ELINC2>,1
**

*SOLID SECTION,ELSET=PIPE_core,MATERIAL=core
*MATERIAL,NAME=core
*ELASTIC
    <E2>,<v2>
*****

*ELEMENT,TYPE=CPE8R,ELSET=PIPE_EXT

<ME3B>,<ME3BN1>,<ME3BN2>,<ME3BN3>,<ME3BN4>,<ME3BN5>,<ME3BN6>,<ME
3BN7>,<ME3BN8>

<ME3E>,<ME3EN1>,<ME3EN2>,<ME3EN3>,<ME3EN4>,<ME3EN5>,<ME3EN6>,<ME
3EN7>,<ME3EN8>
*****

*ELGEN,ELSET=PIPE_EXT
    <ME3B>,<NEGEN>,<NODEINC1>,<ELINC1>,<n3>,<NODEINC2>,<ELINC2>
    <ME3E>,<n3>,<NODEINC2>,<ELINC2>,1
****

```

```

*SOLID SECTION,ELSET=PIPE_EXT,MATERIAL=STEELEXT
*MATERIAL,NAME=STEELEXT
*ELASTIC
  <E3>, <v3>
*****
*ELSET, ELSET=el_int, GEN
  <ME1B>, <ME1E>, 1
*ELSET, ELSET=el_ext, GEN
  <ME4B>, <ME4E>, 1
*****
*EQUATION
  2
  Eq0,1,1.0,Eq180,1,1.0
*EQUATION
  2
  Eq90,2,1.0,Eq270,2,1.0
*EQUATION
  2
  Eq90,1,1.0,Eq270,1,-1.0
*****
*step
static preload for internal pressure
*static
1.0,1.0
*dload
  el_int, P4,0
*end step
*****
*step
*buckle
  5,,
*dload
  el_ext, P2, 1

```

```
*output,field
*node output
u,
*node file,global=yes
u,
*end step
*****
```