SOME FAMILIES OF STAR-FREE EVENTS

by

Zaw-Sing Su, B.A.Sc.

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Department of Electrical Engineering

Faculty of Pure and Applied Sciences

University of Ottawa

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ABSTRACT

The concept of regular expression as a mathematical model has been a convenient tool used for analysis and synthesis of sequential circuits. Regular expressions can be defined by using five operations (union, intersection, concatenation, iterate and complementation). This thesis is concerned with a subset of regular events, called the set of star-free events, which can be expressed by regular expressions without using the iterate operation. The importance of star-free events relies on the fact that the events can be realized by 'almost loop-free' sequential circuits.

Closure properties of certain families of star-free events have been studied in this thesis. The study was started with the set of finite events. New classes of events were introduced by embedding a given class of events in the smallest class closed under a given operation. Some finite products of finite and cofinite events have been further studied and it is shown that unique factorization is possible in some cases.

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CHAPTER I

INTRODUCTION

In 1956, Kleene[9] proved that every regular event can be realized by a finite, deterministic, synchronous automaton, and that any such automaton can be characterized by a regular expression. This result proved that the concept of regular expressions plays an important role and is a useful tool in the study of finite automata. Since then the concept of regular expressions has been studied and developed extensively.

An event is defined as a subset of the set of all possible input sequences to an automaton. An event is regular if, and only if, the set of sequences which constitutes the event can be described by a regular expression. In a recent paper by Papert and McNaughton[12], a subclass of the class of regular events has been studied. This subclass is called the class of non-counting events, that is,the class of events represented by automata which are not capable of counting the occurances of some event at the input modulo an integer. It was further proved that the class of non-counting events coincides with the class of star-free events, that is those events which can be denoted by regular expressions without using the star operation. The latter is of interests of study because it may be realized by 'almost loop-free' sequential circuits.

In the next chapter of this thesis, the fundamentals used in later developments are summerized. The third chapter studies algebraic closure properties for some families of star-free events under various regular operations. Finally, classification of some families of star-free events are studied in Chapter IV.

CHAPTER II

PRELIMINARIES

In this chapter, we give a brief review of definitions and known results relevant to this work.

§2.1 ALGEBRAIC SYSTEMS

We shall first define and give some fundamental properties of certain algebraic systems which are the mathematical models of the families of star-free events discussed in this thesis.

A semigroup is a non-empty set S on which a binary associative operation is defined. This operation is often called multiplication, or product, and is denoted by (.) or juxtaposition. For subsets P and Q of S, multiplication is defined by:

$$PQ = \{k \mid k = pq,\ p \in P \text{ and } q \in Q\}.$$

It is obvious that $PQ \subset S$. It is also clear from set theory that $P \cup Q$, $P \cap Q$ and $P - Q$ (union, intersection and relative complement respectively) are subsets of S. Furthermore, for subsets P, Q and R of semigroup S, distributive laws

$$P(Q \cup R) = PQ \cup PR \text{ and}$$

$$(P \cup Q)R = PR \cup QR$$

are true[10].

A monoid M is a semigroup in which there exists a unit element (or identity) e satisfying me = em = m for all m ε M. It can be proved that a semigroup can have at most one unit element[10]. Given a subset G of a monoid M, if $M = \bigcup_{n=0}^{\infty} G^n$ with $G^0 = \{e\}$, the single-element set containing the identity of M, and $G^2 = GG$, etc., we say that M is generated by G and G is a generating set of M. A generating set G of M is irreducible if, and only if, no proper subset of G generates M. A monoid with length is a monoid M in which to every m ε M, m ≠ e is assigned a positive integral length $\ell(m)$, $\ell(e) = 0$ and $\ell(mn) = \ell(m) + \ell(n)$ for all m, n ε M.

Another algebraic system of interest is Boolean algebra of subsets of any universal set I under intersection, union and complementation. The following properties hold:

1. Both intersection and union are commutative and associative;
2. Union and intersection are mutually distributive;
3. For any subset R of I

$\phi \cup R = R$,

$\phi \cap R = \phi$,

$I \cap R = R$,

$I \cup R = I$,

$R \cap \overline{R} = \phi$ and

$R \cup \overline{R} = I$,

where ϕ is the empty subset and $\overline{R}$ the complement of subset R.

§2.2 REGULAR EVENTS

We begin by considering sets of sequences of symbols from a finite alphabet. These sets will be called events or languages. Each sequence of symbols is called a word.

DEFINITION 2.2.1: The universal event I over a finite alphabet $A = \{a_1, a_2, \cdots, a_n\}$ is the free monoid generated by alphabet A. An event is a subset of the universal event I. The empty set of words is called the empty event Φ.

DEFINITION 2.2.2: The length of a word is defined to be the number of symbols in the sequence. λ, the empty word, is of zero length.

DEFINITION 2.2.3: Any finite event is an event which contains a finite number of words.

NOTATION 2.2.4: The length of the longest word in finite event F is denoted $\ell_M(F)$, and F_n will be used to denote a finite event F with $\ell_M(F) = n$.

DEFINITION 2.2.5: A cofinite event is an event whose complement is finite.

For each cofinite event C there exists an integer n such that $C \supseteq IA^{n+1}$, i.e., all words of length n+1 or more.

NOTATION 2.2.6: C_n denotes a cofinite event with the smallest possible integer n such that $C_n = F \cup IA^{n+1}$, where if the finite event F is not empty then $\ell_M(F) \leq n$. Note that C_n = complement of F_n for some F_n.

EXAMPLE 2.2.7: Let $A = \{0,1\}$. Then the cofinite event $\overline{\{01 \cup 10\}} = \{\lambda \cup 0 \cup 1 \cup 00 \cup 11\} \cup A^3I = F_2 \cup A^3I$ is denoted as C_2 according to our convention. Different events with the same length property are to be distinguished by primes.

DEFINITION 2.2.8: If $R = P_1P_2...P_i...P_m$, we say that each P_i is a factor of R.

DEFINITION 2.2.9: If $w = a_1a_2...a_n \in I$ and $1 \leq k \leq m \leq n$, $w' = a_ka_{k+1}...a_m$ is a subword of the word w, $p = a_1a_2...a_m$ is a prefix of w, and $s = a_ka_{k+1}...a_n$ is a suffix of w.

DEFINITION 2.2.10: Let P and Q be any two events, then we define operations:

Union or Inclusive or ($\cup$): a binary operation such that

$$P \cup Q = \{w \mid w \in P \text{ or } w \in Q \text{ or both}\};$$

Intersection or And ($\cap$): a binary operation such that

$$P \cap Q = \{w \mid w \in P \text{ and } w \in Q\};$$

Product or Concatentation ($\cdot$): a binary operation such that

$$P \cdot Q = \{w \mid w = uv,\ u \in P,\ v \in Q\},$$

the dot is often omitted for convenience;

(Absolute) complement or Negation ($\bar{\ }$): a unary operation such that

$$\bar{P} = \{w \mid w \notin P\} \text{ and}$$

Iterate or Star operation (*): a unary operation such that

$$P^* = \bigcup_{n=0}^{\infty} P^n, \text{ where } P^0 = \{\lambda\} \text{ and } P^2 = PP, \text{ etc.}$$

It is easy to show that all three binary operations defined above are associative and that product distributes over union.

DEFINITION 2.2.11: The reverse of a word $w = a_1a_2...a_n$ is denoted by $\overleftarrow{w}$ and defined as $\overleftarrow{w} = a_na_{n-1}...a_1$ and $\overleftarrow{\lambda} = \lambda$. The reverse of an event P is $\overleftarrow{P} = \{\overleftarrow{w} \mid w \in P\}$.

DEFINITION 2.2.12: A regular expression over alphabet $A = \{a_1,a_2,...,a_n\}$

is defined inductively. Let λ and ϕ be two distinct symbols not in A. Let $\cup$, $\cap$ and $\cdot$ be binary operations and let $*$ and $\bar{}$ be unary operations. The inductive definition of regular expressions is:

BASIS: $a_1, a_2, \ldots, a_n$, λ and ϕ are regular expressions.

INDUCTION STEPS: If P and Q are regular expressions then so are $(P \cup Q)$, $(P \cap Q)$, (PQ), P^* and $\bar{P}$.

Nothing else is a regular expression.

Regular expressions denote events according to the mapping $|\ |$ from regular expressions to events: $|a_i| = \{a_i\}$ for $i = 1,2,\ldots,n$; $|\lambda| = \{\lambda\}$; $|\phi| = \phi$; $|P \cup Q| = |P| \cup |Q|$; $|P \cap Q| = |P| \cap |Q|$; $|PQ| = |P||Q|$; $|P^*| = |P|^*$ and $|\bar{P}| = \overline{|P|}$.

An event P is <u>regular</u> if, and only if, there exists a regular expression R such that $|R| = P$.

DEFINITION 2.2.13: <u>Star-free events</u> are those regular events which can be denoted by a regular expression without using the star operation.

Obviously, the set of star-free events is contained in the set of regular events.

DEFINITION 2.2.14: Let $w \in I$ and $P \subseteq I$. The <u>left quotient</u> of P by w is defined as $w \backslash P = \{x \mid wx \in P\}$.

DEFINITION 2.2.15: Let P and Q be regular expressions. Then we define the <u>derivative</u> of a regular expression with respect to a word $u \in I$ recursively as follows:

BASIS: For $a \in A$, $D_a a = \lambda$, $D_a b = \phi$ for $b = \phi$, λ, or $b \in A$ and $b \neq a$.

INDUCTION STEPS: $D_a(P \cup Q) = D_a P \cup D_a Q$,

$D_a(P \cap Q) = D_a P \cap D_a Q$,

$D_a(PQ) = (D_a P)Q \cup \delta(P) D_a Q$, where $\delta(P) = \lambda$ if $\lambda \in P$ and $\delta(P) = \phi$ if $\lambda \notin P$,

$D_a(P^*) = (D_a P)P^*$ and

$D_a(\overline{P}) = \overline{D_a P}$.

For $\ell(w) \neq 1$ we have:

$D_\lambda P = P$ and

$D_{ua} P = D_a(D_u P)$.

EXAMPLE 2.2.16: Let $R = \{((0 \cup 1)10) \cap (0 \cup 010)\}$ and $v = 01$.

$$
\begin{aligned}
D_v R &= D_{01}\{((0 \cup 1)10) \cap (0 \cup 010)\} \\
&= D_1\{D_0((0 \cup 1)10) \cap D_0(0 \cup 010)\} \\
&= D_1\{(D_0(0 \cup 1)10 \cap \delta(0 \cup 1)D_0(10)) \cap (D_0(0) \cup D_0(010))\} \\
&= D_1\{((D_0(0) \cup D_0(1))10 \cup \phi) \cap (\lambda \cup 10)\} \\
&= D_1\{(\lambda \cup \phi)10\} \cap D_1\{\lambda \cup 10\} \\
&= 0 \cap (\phi \cup 0) = 0 \cap 0 = 0.
\end{aligned}
$$

The following results have been proved by Brzozowski[4].

THEOREM 2.2.17: Any derivative of a regular expression is a regular expression.

THEOREM 2.2.18: $|D_s R| = s\backslash|R|$ for any sequence $s \in I$.

THEOREM 2.2.19: A sequence s is contained in a regular expression R if, and only if, λ is contained in $D_s R$.

THEOREM 2.2.20: Every regular expression R can be written in the form: $R = \bigcup_{a \in A} a D_a R \cup \delta(R)$

where A is the alphabet and the terms in the union are disjoint.

THEOREM 2.2.21: The relationship between derivatives of R can be represented by a unique set of equations of the form:

$$D_s R = \bigcup_{a \in A} a D_{sa} R \cup \delta(D_s R)$$

where s ε A*. Such equations will be called the characteristic equations of R.

§2.3 AUTOMATA

DEFINITION 2.3.1: A *finite automaton* S is a quintuple $S = \langle Q,M,q_1,F,A\rangle$ where A is the alphabet,

$Q = \langle q_1,q_2,\cdots,q_n\rangle$ is a finite, non-empty set of *internal states* of S,

M is the *transition function* $M : Q \times A \to Q$,

$q_1 \varepsilon Q$ is the *starting* (or *initial*) *state* and

$F \subseteq Q$ is the set of *final* (or *accepting*) *states*.

A finite automaton is an abstract model of a sequential circuit. We shall assume that the behavior of the automaton is of interest only at discrete time moments $t = 1,2,\cdots$. At $t = 0$, before any input is applied, the automaton is in its starting state. For convenience we also assume that the automaton has all of its states accessible from its initial state.

DEFINITION 2.3.2: An input sequence s is *accepted by an automaton* S with starting state q_1 if, and only if, when s is applied to S in q_1, the resulting state is an *accepting state*. Otherwise, s is *rejected* by S. A sequence s is *accepted by a state* q_i of S if, and only if, when S is started in q_i and the state at the end of applying s is an accepting state.

DEFINITION 2.3.3: A finite automaton S *realizes* an event R if, and only if, the set of sequences accepted by S is exactly R.

If a finite automaton S realizes an event denoted by regular expression R, then each distinct state of S corresponds to one distinct derivative of R. Furthermore, each state accepts the derivative to which it corresponds[4]. A state corresponding to the empty event is called an <u>empty state</u>, otherwise, it is a <u>non-empty state</u>.

§2.4 DEFINITE AND REVERSE DEFINITE EVENTS[3]

DEFINITION 2.4.1: An event is <u>definite</u> if, and only if, it can be described by a regular expression of the form $R = E \cup IF$ which E and F are finite and I is the universal event. The family of definite event is denoted by $\mathcal{D}$.

For every definite event there exists an integer $k \geq \ell_M(F)$ such that the output of the finite automaton realizing this event depends only upon the last k symbols of any input sequence. The importance of definite events is that they are precisely the set of events having 'feed-back free' realizations.

DEFINITION 2.4.2: Let R be a definite event and r the smallest possible integer such that $R = E_{<r} \cup IF_r$. This form is unique and is called the <u>long canonical form</u> of R.

DEFINITION 2.4.3: The reverse of a definite event R is called a <u>reverse definite</u> event and has the form $\overleftarrow{R} = E \cup FI$ with E and F finite. The family of reverse definite events is denoted $\overleftarrow{\mathcal{D}}$.

For tests of definiteness of an event, the reader is referred to section 9 of reference [3].

§2.5 STAR[5] AND COMET[6] EVENTS

DEFINITION 2.5.1: An event S is a <u>star event</u> if, and only if, there exists an event $P \subseteq I$ such that $S = P^*$. P is called a <u>root</u> of S.

For an event $S \subseteq I$ to be a star event, each of the following equivalent conditions is necessary and sufficient:

(1) $S = S^*$

(2) $S = S^2$

(3) $\lambda \varepsilon S$, and for each $w \varepsilon I$, $\lambda \varepsilon D_w S$ if, and only if $D_w S \subseteq S$.

For any star event S there exists a unique minimum root $S_I = (S - \{\lambda\}) - (S - \{\lambda\})^2$[10], which is contained in every other root of S. It has also been proved that a star event S is regular if, and only if, its minimum root S_I is regular.

It is noticed that every star event is a monoid with length and its minimum root corresponds to the minimum generating set of the monoid.

DEFINITION 2.5.2: An event C is a <u>comet event</u> (or simply <u>comet</u>) if, and only if, there exists an event $T \subseteq I$ and a star event $S \neq \{\lambda\}$ such that $C = ST$, S is called the <u>front star</u> and T the <u>tail of the comet event</u> C. We shall also call a comet event $C = ST$ an <u>S-comet</u> when the reference to the star S is necessary.

The following results about comet events have been proved:

THEOREM 2.5.3: C is an S-comet if, and only if, $C = SC$.

THEOREM 2.5.4: An event P is a comet event if, and only if, there exists at least one derivative, D_aP, for $a \neq \lambda$, of P containing P.

THEOREM 2.5.5: For any comet event C there exists a unique largest star event S_M such that $C = S_MC$ and $S_M = \bigcup_i S_i$ for all star events S_i such that $C = S_iC$.

THEOREM 2.5.6: For every S-comet C there exists a unique smallest tail T_m (with respect to S) such that $C = ST_m$. Furthermore, $T_m = T_i - (S - \{\lambda\})T_i = \bigcap_i T_i$, for any tail T_i such that $C = ST_i$.

THEOREM 2.5.7: Every non-empty comet event C has a unique canonical form $C = S_MT_m$, where T_m is the smallest tail of C with respect to S_M, the largest left star of C.

The following theorem proved by Paz and Peleg[13] gives an algorithm for finding S_{LM}, the largest star on the left, for a comet event. The proof is not to be repeated here. An example is given to describe the algorithm. For the algorithm for finding the smallest tail of C with respect to S_{LM}, the reader is referred to reference [6].

THEOREM 2.5.8: Let C be a regular event and let $A(C) = \langle Q,M,q_1,F \rangle$ be the automaton recognizing C. If F_S be the set of all states $q_i \in Q$, whose corresponding derivatives D_{q_i} contain C, then S_M is the event recognized by automaton $A' = \langle Q,M,q_1,F_S \rangle$.

EXAMPLE 2.5.9: Let $A = \{0,1\}$ and $R = (0^*1)^*10(0 \cup 1)^*$. We have the characteristic equations for R:

$$R = 0D_0R \cup 1D_1R$$

$$D_0R = 0D_0R \cup 1R$$

$$D_1R = 0D_{10}R \cup 1D_1R$$

$$D_{10}R = 0D_{10}R \cup 1D_{10}R \cup \lambda.$$

One can find that $\{R, D_1R, D_{10}R\}$ is the set of derivatives which contain R. Then S_M of R is described by characteristic equations:

$$S = 0D_0S \cup 1D_1S \cup \lambda,$$

$$D_0S = 0D_0S \cup 1S,$$

$$D_1S = 0D_{10}S \cup 1D_1S \cup \lambda \text{ and}$$

$$D_{10}S = 0D_{10}S \cup 1D_{10}S \cup \lambda.$$

By the algorithm, discussed by Brzozowski[5], of finding the minimum root of S_M, we obtain:

$$S_M = (00^*1 \cup 1 \cup 10 \cup 100^*0)^* = (0^*1 \cup 100^*)^*.$$

The *minimum tail* T_m of R with respect to S_M can also be shown[6] to be 100^*1^*. Thus $R = (0^*1 \cup 100^*)^*100^*1^*$.

The reverse of any comet is called a *reverse comet*. Let $C = ST$ be a comet event. Then $\overleftarrow{C} = \overleftarrow{T}\overleftarrow{S} = FS'$ is a reverse comet with the minimum root of S' the reverse of the minimum root of S. F is called a *front* of $\overleftarrow{C}$. $F_m = \overleftarrow{T_m}$ is called the *minimum front* of $\overleftarrow{C}$ with respect to S'.

DEFINITION 2.5.10: An event W is a *two-sided comet* if, and only if, there exist two non-$\{\lambda\}$ stars S_L and S_R and an arbitrary event M such that $W = S_LMS_R$. S_L is called a *left star*, S_R a *right star* and M a *middle* of W.

Every two-sided comet also has a unique canonical form $S_{LM}M_mS_{RM}$ such that S_{LM} and S_{RM} are maximum stars on each side and M_m is the minimum middle with respect to S_{LM} and S_{RM}.

CHAPTER III

CLOSURE PROPERTIES

Closure properties of some families of star-free events under union, intersection, product, complementation and star operation are investigated in this chapter. The events considered here are over an arbitrary finite alphabet $A = \{a_1, a_2, \cdots, a_n\}$. The free monoid with length over A gives the universal event $I = A^*$. It is easy to observe that the family $M_1 = \{R \mid R = w, w \in I\}$ of events consisting of single words is closed only under multiplication.

In each of the following sections closure properties of a family of star-free events are investigated.

The family $F(M_1)$ of finite unions of events in M_1 will be considered first. Clearly $F(M_1)$ is the family of all finite events.

§3.1 FINITE AND COFINITE EVENTS

By definitions it is self-evident that the family $F(M_1)$ of finite events is closed under union, intersection and product. It is also obvious that a star event generated by any finite event containing at least one word of length greater than 0 is infinite. Since the universal event I is infinite the complement of a finite event can not be finite. Thus $F(M_1)$ is not closed under the star operation nor under complementation. Noting that union, intersection and product are associative we see that $F(M_1)$ is a semigroup under each of these operations. Furthermore, it is also a monoid under union, intersection and product with identities Φ, I and $\{\lambda\}$ respectively.

Clearly, to close the family $F(M_1)$ under complementation it is necessary and sufficient to include the family $C(M_1)$ of cofinite events. Below we consider the closure properties of the family $C(M_1)$.

In this chapter, unless otherwise stated, we shall denote a finite event by a capital letter E, F, G or H with a subscript, if necessary, to indicate its length property. Capital letter C is used to denote a cofinite event according to Notation 2.2.6. By definition, any cofinite event $C_n = (\bigcup_{i=0}^{n} A^i - F_n) \cup A^{n+1}I = E_{\le n} \cup IA^{n+1}$, where $E_{\le n}$ has words of length less than or equal to n.

Let m and n be integers such that $0 \le m \le n$, then we have the following.

1. Union:

$$C_m \cup C'_n = (E_{\le m} \cup IA^{m+1}) \cup (F_{\le n} \cup IA^{n+1})$$
$$= E_{\le m} \cup F_{\le n} \cup IA^{m+1} \cup IA^{n+1}$$
$$= G_{\le k} \cup IA^{k+1} = C''_k$$

where $k \le m$.

2. Intersection:

$$C_m \cap C'_n = (E_{\le m} \cup IA^{m+1}) \cap (F_{\le n} \cup IA^{n+1})$$
$$= (E_{\le m} \cap F_{\le n}) \cup (IA^{m+1} \cap F_{\le n}) \cup (E_{\le m} \cap IA^{n+1}) \cup (IA^{m+1} \cap IA^{n+1})$$
$$= G_{\le m} \cup G'_{\le n} \cup \Phi \cup IA^{n+1}$$
$$= H_{\le n} \cup IA^{n+1} = C''_n.$$

Note that if either or both of the finite events E and F in the above argument were empty, the result would not be affected.

3. Product:

$$C_m \cdot C'_n = (E \cup IA^{m+1})(F \cup IA^{n+1})$$
$$= EF \cup IA^{m+1}F \cup EIA^{n+1} \cup IA^{m+1}IA^{n+1}$$
$$= G \cup IA^{m+1}F \cup EIA^{n+1} \cup IA^{m+n+2}$$
$$= C''_{m+n+1}.$$

4. Complement: By definition of cofinite event, $\overline{C_n} = F_n$ is not cofinite.

5. Star operation: Let $C_n = (E_n \cup IA^{n+1})$. Then $(C_n)^* = (E_n \cup IA^{n+1})^*$. Since $IA^{n+1} \subset (C_n)^*$, $(C_n)^*$ must be cofinite. Therefore $C(M_1)$ is closed under the star operation.

§3.2 BOOLEAN ALGEBRA $\mathcal{B}_1$

In this section we consider the family $\mathcal{B}_1 = F(M_1) \cup C(M_1)$ $= \{R \mid R$ is a finite or cofinite event$\}$. Since this family includes $F(M_1)$ and $C(M_1)$, cases previously being proved closed must still be closed. For brevity these cases will not be considered again.

1. Union:

$$F \cup C_m = F \cup E \cup IA^{m+1}$$
$$= G \cup IA^{m+1} = C'_q$$

where $q \leq m$.

2. Intersection: Intersection of a finite event with any other event is finite.

3. Complement: By construction it is also closed under complementation.

Hence $\mathcal{B}_1$ is a Boolean algebra.

4. Product: Let C_m and F be any cofinite and finite event respectively.

i) The cofinite-finite event

$$C_m . F = (E \cup IA^{m+1})F$$
$$= EF \cup IA^{m+1}F$$
$$= G \cup IH.$$

As the following example shows this product needs not be either finite or cofinite.

EXAMPLE 3.2.1: Let $C_0 = I$, $F_1 = \{a_i\}$ with $a_i \in A = \{a_1, a_2, \cdots, a_n\}$, $n > 1$. Then,

$$C_0 F_1 = I\{a_i\}.$$

$I\{a_i\}$ can not be finite. On the other hand its complement $I(A - \{a_i\})$

$\cup$ λ is also infinite. Hence C_0F_1 is not cofinite.

LEMMA 3.2.2: Let $C_m = E \cup IA^{m+1}$ be an arbitrary cofinite event and F a finite event such that $\lambda \in F$. Then $C_mF = C'_n$ for $n \leq m$.

PROOF: $C_m.F = EF \cup IA^{m+1}F$

$= EF \cup IA^{m+1}(F \cup \lambda)$

$= EF \cup IA^{m+1}F \cup IA^{m+1}$

$= EF \cup IA^{m+1} = C'_n$

where $n = m$ if $A^m \not\subset EF$ or $n < m$ if $A^m \subset EF$.

ii) Similarly, in general, the finite-cofinite event $F.C_m = G \cup HI$ is neither finite nor cofinite. However, if $\lambda \in F$, then $F.C_m = C'_n$ for $n \leq m$.

5. The failure of closure of B_1 under the star operation is shown by the counter example $R = 1^*$. 1^* is obviously not finite. Its complement I0I is also not finite.

Incidentally the families of cofinite-finite and finite-cofinite events are sub-families of the families of definite ($E \cup IF$) and reverse definite ($E \cup FI$) events respectively. Also we remark that $B_1 = D \cap \overleftarrow{D} = \{R \mid R \text{ is definite and reverse definite}\}$. The argument is like this: The containment $B_1 \subseteq (D \cap \overleftarrow{D})$ is obvious, for any finite event is in $(D \cap \overleftarrow{D})$. Also since $A^nI = IA^n$ for $n \geq 0$, any cofinite event is in $(D \cap \overleftarrow{D})$. For proving the reverse containment, let definite event $D = E \cup IF$ be in its long canonical form. If $F = \phi$, then $E = D \in B_1$. If $F \neq \phi$, since $D \in \overleftarrow{D}$, we have $D = E \cup IF = E' \cup F'I$, where the last expression is assumed to be in the long canonical form for reverse definite expressions. Since they are equal we have $A^{\ell_M(F')}F = F'A^{\ell_M(F)}$ for the set of words of length $\ell_M(F')$ + $\ell_M(F)$. If $F \neq A^{\ell_M(F)}$ there must be at least one word of length $\ell_M(F)$ such that $f \notin F$. But $f'f \in FA^{\ell_M(F)}$ for some $f' \in F'$. Which is a contradiction. Therefore $F = A^{\ell_M(F)}$. Then $E \cup IA^{\ell_M(F)}$ is cofinite and $E \cup IA^{\ell_M(F)} = D \in \mathcal{B}_1$.

§3.3 MONOID M_2

From the results of the last section we see that to close B_1 under multiplication it is necessary to include the families of finite-cofinite (FC-) and cofinite-finite (CF-) events. It is easy to see that this is still not sufficient because an arbitrary finite product of finite and cofinite events need not be reducible to a finite, cofinite, finite-cofinite (FC-) or cofinite-finite (CF-) event. The example below shows this fact.

EXAMPLE 3.3.1: Let $A = \{0,1\}$ and $R = I0I$. Obviously R is a finite product of finite event $\{0\}$ and cofinite event I. It is obviously not finite and it is not cofinite for $\overline{I0I} = 1^*$. By the test for definite events used by Brzozowski[3], it can be shown R is neither definite nor reverse definite. Hence R is neither cofinite-finite nor finite-cofinite. Therefore a necessary and sufficient condition for closure under product is to include all <u>(F,C)-products</u> (finite products of finite and cofinite events). The resulting family will be called $M_2 = \{R \mid R \text{ is an } (F,C)\text{-product}\}$ and is a monoid with generating set B_1. Its closure properties are investigated below:

1. The closure under union can be disproved by the following counter example.

EXAMPLE 3.3.2: Let $L = 0 \cup I1$ and $A = \{0,1\}$. Suppose $L = 0 \cup I1 = P_1P_2\ldots P_n$ with each of its factors P_i cofinite or finite (subscripts here indicate consecutive ordering instead of the word length property). Since $0 \in L$, there must exist P_i such that $0 \in P_i$ and $\lambda \in P_j$, for $j \neq i$, because $\lambda \notin L$. If $P_i \neq P_1$ and $w \in P_k$, $\ell(w) > 0$ and $k < i$ then L contains w0. This is impossible and the only possibility is $i = 1$, i.e. $0 \in P_1$. Now, if some P_k, $k > 1$, is cofinite then $P_k = E \cup IA^n$. All the P_j, $j \neq 1$, contain λ and therefore L must contain $0A^n \supseteq 0A^{n-1}0$. Clearly, this is impossible. Thus no P_k can be cofinite for $k > 1$. Also P_1 cannot be cofinite for then $L \supseteq A^{n-1}0$. Thus all the P_j and hence L must be finite. However, this is a contradiction.

2. For closure under intersection the following counter example exists.

EXAMPLE 3.3.3: Let $A = \{0,1,2\}$ and $R = I(01 \cup 10)I \cap I2I$. It is found that the shortest words in R are 3-symbol words 012, 102, 201 and 210. Assume R is a product $P_1P_2\ldots P_n$ where P_i are finite or cofinite. For $012 \in R$, a possibility of symbol distribution is $0 \in P_i$, $1 \in P_j$, $2 \in P_k$ for $i < j < k$ and $\lambda \in P_q$ for $q \neq i$, j or k (refer to table 3.1, conjecture (1)). We also have $210 \in R$. No subword of 210 can appear in P_q for $Q \neq i$, j or k because, if this happens, one of P_i, P_j or P_k must contain λ. This will give a 2-symbol word which is not in R. Thus we have $2 \varepsilon P_i$, $1 \varepsilon P_j$ and $0 \varepsilon P_k$ as the only possibility left. But then $212 \varepsilon R$, which is a contradiction and this distribution is impossible. For $012 \varepsilon R$ another possibility of distribution (conjecture (2) in Table 3.1) is to have $0 \varepsilon P_i$, $12 \varepsilon P_j$, for $i < j$, and $\lambda \varepsilon P_q$ for $q \neq i$ or j. Again, $210 \varepsilon R$. If any subword of 210 is in P_q for $q \neq i, j$ it would give a 1- or 2-symbol word which is not in R. If $21 \varepsilon P_i$, $0 \varepsilon P_j$ then $00 \varepsilon R$ gives a contradiction. If $2 \varepsilon P_i$ and $10 \varepsilon P_j$, then $212 \varepsilon R$ gives a contradiction. A similar approach can disprove the possibility that $01 \varepsilon P_i$ and $2 \varepsilon P_j$ for $012 \varepsilon R$. Hence the only possibility left is the third conjecture in Table 3.1: one product P_i contains all the words 012, 102, 201 and 210, and $\lambda \varepsilon P_q$ for $q \neq i$. This is possible for 3-symbol words. But consider a word 1002 which is obviously in R. Some subword of this word must appear in P_i or P_i would contain λ and so would R. Now, if 002 or 100 appears in P_i then these words are also in R which is a contradiction. Thus we can only have $1002 \varepsilon P_i$. This is true also for 100^n2 for all n. Hence P_i cannot be finite. If P_i is cofinite then $P_i \supseteq IA^n$ for some n. Thus R would have to contain 0^n, for example, and this is a contradiction. Hence R is not a member of $\mathcal{M}_2$.

3. By construction $\mathcal{M}_2$ is clearly closed under product.

	P_1	P_2	···	P_i	···	P_j	···	P_k	···
Conjecture (1)	{λ,···}	{λ,···}	···	{0,·············}	···	{1,·····}	···	{2,···}	···
Conjecture (2)	{λ,···}	{λ,···}	···	{0,·············}	···	{1,2,···}	···	{λ,···}	···
Conjecture (3)	{λ,···}	{λ,···}	···	{012,102,201,210}	···	{λ,·····}	···	{λ,···}	···

Table 3.1 Conjectures of possible symbol distribution for Example 3.3.3

4. The failure of closure under complementation and under the star operation can be proved by the following example.

EXAMPLE 3.3.4: Clearly, both $\overline{I0I}$ and 1 ε M_2 = {all (F,C)-products}. Now, let R = 1* = $\overline{I0I}$. Since any cofinite event C $\supset$ $A^m I$ for some integer m and there can be no 0 in any word in 1*, there is no cofinite event as a factor in 1*. On the other hand 1* is not finite, therefore, 1* $\notin$ M_2. Thus the failure of closure under complementation and star operation are proved.

§3.4 $F(M_2)$

To parallel the study of $F(M_1)$ we investigate finite unions of elements of M_2. This family is denoted by $F(M_2)$. As proved by Example 3.3.2 M_2 is not closed under union, $M_2 \neq F(M_2)$ = {finite unions of (F,C)-products}.

$F(M_2)$ is closed under union and product by definition. But it is not closed under complementation as can be proved by Example 3.3.1 with the following explanation.

Let $R = \overline{I0I} = 1^*$. Since any cofinite event contains A^n for some positive integer n, no cofinite event can appear as a factor of any event contained in R. That means R does not contain any non-trivial (F,C)-product. Since R is also not finite this proves $F(M_2)$ is not closed under complementation.

By noticing that $1 \in F(M_2)$ and $1^* \notin F(M_2)$ which we have shown, it is clear that $F(M_2)$ is not closed under star operation.

The proof that $F(M_2)$ is closed under intersection is attempted below.

LEMMA 3.4.1: $F(M_2)$ = {R | R is a finite union of (w,I)-products} where a <u>(w,I)-product</u> is a finite product of words and universal event.

PROOF: By definition $F(M_2)$ = {R | R is a finite union of (F,C)-products}. Let P = FCF···CF be an arbitrary (F,C)-product. As noticed, any cofinite event is of the form $C = E \cup A^nI$ with E and A^n finite. This enables us to write P as a union of <u>(F,I)-products</u> (finite products of finite events and the universal event). Furthermore, since any finite event

F is a finite set of words and since product distributes over union any (F,I)-product can be expanded into a union of (w,I)-products. Hence, $F(M_2) \in \{R \mid R$ is a finite union of (w,I)-products$\}$. The reverse containment is trivial, for every word w is a finite event and the universal event I is cofinite.

We know the empty event $\Phi \in F(M_2)$. Now, let P, Q $\in F(M_2)$ and $P \cap Q = (\bigcup_{i=1}^{m} p_i) \cap (\bigcup_{j=1}^{n} q_j) \neq \Phi$ with p_i, q_j being (w,I)-products. Since intersection distributes over union we have P $\cap$ Q = $\bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (p_i \cap q_j)$. From this and the fact that $F(M_2)$ is closed under union, we know that to prove the closure of $F(M_2)$ under intersection it is sufficient to prove that the intersectinn of any two (w,I)-products if not empty is a finite union of (w,I)-products.

THEOREM 3.4.2: The intersection of any two (w,I)-products, if not empty, can be expressed as a finite union of (w,I)-products.

PROOF: Let $U = u_1Iu_2I\ldots Iu_m$ and $V = v_1Iv_2I\ldots Iv_n$ be any two (w,I)-products. For intersection $A = U \cap V$ not empty, either u_1 of v_1 must be a prefix of the other and u_m or v_n must be a suffix of the other. Now, we assume $A \neq \Phi$ and let $W_k = w_1Iw_2I\ldots Iw_\ell$. If each u_i and each v_j is a subword of some word w_r, $1 \leq r \leq \ell$ of W_k and the u_i and the v_j appear in the same order as in U and V respectively, then W_k must be contained in A. Let an event $A' = \bigcup_{k=1}^{p} W_k$ where W_k as described above. Then $A \supseteq A'$. Assume $A \not\subseteq A'$, there must be a word $w \in A$ but $w \notin A'$. Since $w \in A$, $w = w_1x_1w_2\ldots x_{k-1}w_kx_k\ldots x_{h-1}w_h$ such that each u_i and v_j is a subword of some w_k and each letter of every W_k is a letter of some u_i or v_j. Then $w \in W_q = w_1Iw_2I\ldots Iw_kI\ldots Iw_h$. If we expand A' to

$A'' = W_q \cup A' = \bigcup_{k=1}^{q} W_k$, $w \varepsilon A''$. Since there are only a finite number of u_i's and v_j's, and each u_i or v_j is a finite word, only a finite number of W_q's as the one constructed above are necessary to be added to A' to make $A' \supseteq A$. Since A also contains all these W_q's, $A' = A = \bigcup_{k=1}^{r} W_k$ for some finite r, where each W_k is a (w,I)-product.

EXAMPLE 3.4.3: $I10I \cap I0I1I = I10I1I \cup I0I10I$.

§3.5 $C(M_2)$

DEFINITION 3.5.1: $C(M_2) = \{P \mid \overline{P} \in F(M_2)\}$.

EXAMPLE 3.5.2: $\overline{\text{IOI}} = 1^* \in C(M_2)$.

By investigating the closure of $C(M_2)$ under union and intersection, we observe the following: Let F_1 and $F_2 \in F(M_2)$. By definition, $\overline{F_1}$ and $\overline{F_2} \in C(M_2)$. We know $\overline{F_1} \cup \overline{F_2} = \overline{F_1 \cap F_2}$. Since $F(M_2)$ is closed under intersection, $\overline{F_1 \cap F_2} = \overline{F_3}$ for some $F_3 \in F(M_2)$ and $\overline{F_3} \in C(M_2)$. Therefore $C(M_2)$ is closed under union. Similarly, since $\overline{F_1} \cap \overline{F_2} = \overline{F_1 \cup F_2}$ and $F(M_2)$ is closed under union, $\overline{F_1 \cup F_2} = \overline{F_4}$ for some $F_4 \in F(M_2)$. Therefore $C(M_2)$ is also closed under intersection.

By definition, $\overline{\text{IOI}} \in C(M_2)$. If IOI is in $C(M_2)$, then $\overline{\text{IOI}} \in F(M_2)$. But, as explained at the beginning of this section, $\overline{\text{IOI}} = 1^* \notin F(M_2)$. Hence a contradiction is established to prove the lack of closure of $C(M_2)$ under complementation.

By definition, any cofinite event C is in $F(M_2)$. Hence any finite event $F = \overline{C} \in C(M_2)$. By this we claim that each $a \in A$ is an element of $C(M_2)$. We have seen that $C(M_2)$ is closed under union. The assumption that $C(M_2)$ is also closed under both product and star operation would make $C(M_2)$ a family containing the family of regular events. This is a contradiction because $C(M_2)$ is merely a subset of the family of star-free events which in turn is a subset of the family of regular events. Hence $C(M_2)$ cannot be closed under both product and star operation.

§3.6 B_2

For parallel study, the family $B_2 = F(M_2) \cup C(M_2) = \{R \mid R \in F(M_2) \text{ or } R \in C(M_2)\}$ corresponds to Boolean algebra B_1. By the following counter example we prove the lack of closure of B_2 under union and hence, unfortunately, B_2 is not a Boolean algebra.

EXAMPLE 3.6.1: Let $R = I11I \cup \overline{I0101}$. From the expression we know that no $x \in 0^*0100^*$ is an element of R, and that $0^* \subseteq R$. Now, assume $R = \bigcup_{i=1}^{n} (FCFC\cdots)_i \in F(M_2)$ such that $\lambda \notin F$, any finite factor but probably factors of those products which are finite. This assumption is valid because of Corollary 3.2.2. Then 0^q, for some integer q, must be in some product P, in the above union. Therefore, every factor, finite or cofinite, of P must contain 0^k for some non-negative integer $k \leq q$. Since R is infinite, there is at least one (F,C)-product having a cofinite event as a factor. Since every cofinite event $C \supseteq A^n$ for $n \geq m$, m a non-negative integer, there must exist some $x \in 0^*0100^*$ being an element of R. This leads to a contradiction because $0^r10^s \notin R$, for any r and s. Therefore $0^q \notin P$ for any non-finite (F,C)-product P (a non-finite event). But 0^* is not finite. Therefore, $R \notin F(M_2)$. Next we assume that $R \in C(M_2)$, i.e., $\overline{I11I \cup \overline{I0101}} = \bigcup_{i=1}^{n} (FCFC\cdots)_i$. By DeMorgan's law we have $\overline{R} = \overline{I11I} \cap I0101 = \bigcup_{i=1}^{n} (FCFC\cdots)_i$.

Because of condition $\overline{I11I}$ (no consecutive 1's in any word), we can have no cofinite event as a factor of any (F,C)-product in $\overline{R}$. Since $\overline{R}$ is infinite, another contradiction is established. The lack of closure under union is thus proved.

Let $S = \overline{I11I} \cap I010I$. Comparing with Example 3.6.1, we see that $S = \overline{R}$. Since $\overline{R}$ and R are not in $F(M_2)$, S and $\overline{S}$ are not in $F(M_2)$ and a counter example to the closure of B_2 under intersection is established.

B_2 is obviously closed under complementation. But since it fails to be closed under union and intersection, it cannot be a Boolean algebra.

§3.7 SUMMARY

FAMILY OF EVENTS	CLOSED UNDER	NOT CLOSED UNDER
A = the set of alphabet		
$M_1 = \{R \mid R = \{w\},\ w \in A^*\} = \{\text{words}\}$	$\cdot$	$\cup,\ \bar{\ },\ \cap,\ *$
$F(M_1) = \{\text{finite events}\}$	$\cup,\ \cdot,\ \cap$	$\bar{\ },\ *$
$C(M_1) = \{\text{cofinite events}\}$	$\cup,\ \cdot,\ \cap,\ *$	$\bar{\ }$
$B_1 = F(M_1) \cup C(M_1)$ $= \{R \mid R \text{ is finite or cofinite}\}$	$\cup,\ \bar{\ },\ \cap$	$\cdot,\ *$
$M_2 = \{R \mid R$ is a finite product of finite and cofinite events$\}$	$\cdot$	$\cup,\ \bar{\ },\ \cap,\ *$
$F(M_2) = \{$finite unions of finite products of finite and cofinite events$\}$	$\cup,\ \cdot,\ \cap$	$\bar{\ },\ *$
$C(M_2) = \{R \mid \overline{R} \in F(M_2)\}$	$\cup,\ \cap$	$\bar{\ },\ \cdot$ and/or $*$
$B_2 = F(M_2) \cup C(M_2) = \{R \mid R \in F(M_2)$ or $R \in C(M_2)\}$	$\bar{\ }$	$\cup,\ \cap$

Table 3.2 Closure properties of some families of star free events

CHAPTER IV

STUDY OF SOME EVENTS IN $\mathcal{E}_2$

§4.1 FINITE EVENTS

LEMMA 4.1.1: Let F be a finite non-empty event over alphabet A and let $x \varepsilon I$. If $D_x F \neq \phi$, $D_{xy}F \not\leq D_x F$ for any y, $\ell(y) \geq 1$.

PROOF: Note that $\ell_M(D_{xy}F) < \ell_M(D_x F)$. Therefore $D_{xy}F \not\geq D_x F$.

DEFINITION 4.1.2: Let q be a state of the automaton A. A non-empty state q is <u>transient</u> if,and only if, once A leaves q it cannot re-enter q.

THEOREM 4.1.3: A finite automaton accepts a finite event if, and only if, all of its non-empty states are transient.

PROOF: Let automaton A accept a finite event R. Assume A has a non-transient state q accepting input x, and input y brings A from q back to q. Then A accepts event S = wy*x with w taking A from initial state to q. But S is not finite. Thus the assumption is proved false, and it is proved that a finite automaton accepts a finite event implies all of its non-empty states are transient. The reverse implication is obvious by the fact that transient states establish no loop. Since A is a finite automaton it accepts only a finite event.

Any finite event F is obviously a special case of both a definite event $D = F \cup IE$ and a reverse definite event $\overleftarrow{D} = F \cup EI$ with E empty.

§4.2 COFINITE EVENTS

The test whether an event is cofinite relies on the fact that its complement is finite. The finiteness of an arbitrary event can be tested by constructing an automaton[4] realizing this event. By Theorem 4.1.3 the automaton can have only transient non-empty states.

In the following we shall discuss some properties of cofinite events.

DEFINITION 4.2.1: Let event $B = \{b_1, b_2, \cdots, b_k\}$ be finite and $\ell_M(B) = M$. Also let $n_i = M - \ell(b_i)$ for $i = 1,2,\cdots,k$. We define the class $\mathcal{B}$ of finite events as the following: $B \varepsilon \mathcal{B}$ if, and only if, $b_1A^{n_1} \cup b_2A^{n_2} \cup \cdots \cup b_kA^{n_k} = A^M$.

In the following examples alphabet $A = \{0,1\}$ is assumed.

EXAMPLE 4.2.2: Let $B = \{10 \cup 11 \cup 0\}$.

$$10(A^0) \cup 11(A^0) \cup 0A = A^2.$$

Therefore $B \varepsilon \mathcal{B}$.

EXAMPLE 4.2.3: $B = \{01\} \notin \mathcal{B}$ since $01A^0 = 01 \neq A^2$.

LEMMA 4.2.4: Let A be a finite alphabet. $FI \supseteq A^n$ if, and only if, $FI \supseteq A^m$ for all $m \geq n$.

PROOF: Suppose $FI \supseteq A^n$. Let $x \varepsilon A^n$ and $x = x_1x_2$ with $x_1 \varepsilon F$, $x_2 \varepsilon I$. Since $x_2A \subset I$, $x_1x_2A = xA \subset FI$. This is true for all $x \varepsilon A^n$. Hence $FI \supseteq A^nA = A^{n+1}$. By induction $FI \supseteq A^m$ for all $m \geq n$. The reverse implication is cbvious.

LEMMA 4.2.5: $FI \supseteq A^n$, $n \geq 0$, implies $F \supseteq B$ for some $B \varepsilon \mathcal{B}$.

§4.2 COFINITE EVENTS

The test whether an event is cofinite relies on the fact that its complement is finite. The finiteness of an arbitrary event can be tested by constructing an automaton[4] realizing this event. By Theorem 4.1.3 the automaton can have only transient non-empty states.

In the following we shall discuss some properties of cofinite events.

DEFINITION 4.2.1: Let event $B = \{b_1, b_2, \cdots, b_k\}$ be finite and $\ell_M(B) = M$. Also let $n_i = M - \ell(b_i)$ for $i = 1,2,\cdots,k$. We define the class $\mathcal{B}$ of finite events as the following: $B \in \mathcal{B}$ if, and only if, $b_1A^{n_1} \cup b_2A^{n_2} \cup \cdots \cup b_kA^{n_k} = A^M$.

In the following examples alphabet $A = \{0,1\}$ is assumed.

EXAMPLE 4.2.2: Let $B = \{10 \cup 11 \cup 0\}$.

$$10(A^0) \cup 11(A^0) \cup 0A = A^2.$$

Therefore $B \in \mathcal{B}$.

EXAMPLE 4.2.3: $B = \{01\} \notin \mathcal{B}$ since $01A^0 = 01 \neq A^2$.

LEMMA 4.2.4: Let A be a finite alphabet. $FI \supset A^n$ if, and only if, $FI \supset A^m$ for all $m \geq n$.

PROOF: Suppose $FI \supset A^n$. Let $x \in A^n$ and $x = x_1x_2$ with $x_1 \in F$, $x_2 \in I$. Since $x_2A \subset I$, $x_1x_2A = xA \subset FI$. This is true for all $x \in A^n$. Hence $FI \supset A^nA = A^{n+1}$. By induction $FI \supset A^m$ for all $m \geq n$. The reverse implication is cbvious.

LEMMA 4.2.5: $FI \supset A^n$, $n \geq 0$, implies $F \supseteq B$ for some $B \in \mathcal{B}$.

PROOF: Suppose $FI \supseteq A^n$ for some $n \geq 0$. Let $x \in A^n$ and $x = x_1x_2$ with $x_1 \in F$, $x_2 \in I$. Then $x \in x_1A^{\ell(x_2)}$. Since this argument is similarly true for every $x \in A^n$, by definition $F \supseteq B$ for some $B = \{x_1 \mid x_1x_2 = x \in A^n\} \in \mathcal{B}$. Hence, $FI \supseteq A^n$ implies $F \supseteq B$.

LEMMA 4.2.6: $F \supseteq B$ for some $B \in \mathcal{B}$ implies $FI \supseteq A^{\ell_M(B)}I$.

PROOF: By definition 4.2.1, it is obvious that $BI \supseteq A^{\ell_M(B)}$. We know for any events P, Q and R, $P \supseteq Q$ implies $PR \supseteq QR$. Hence $F \supseteq B$ implies $FI \supseteq BI \supset A^{\ell_M(B)}$ and $FII \supseteq A^{\ell_M(B)}I$. But $FII = FI$; therefore $FI \supseteq A^{\ell_M(B)}I$.

THEOREM 4.2.7: Any cofinite event C can be written in the form $C = E \cup IA^n = (\lambda \cup IA^m)(E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1})$ for $m \geq n - \ell_m(E)$.

PROOF:
$$(\lambda \cup IA^m)(E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1})$$
$$= (E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1}) \cup IA^m(E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1})$$
$$= E \cup IA^mE \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1} \cup IA^m(A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1})$$
$$= E \cup IA^mE \cup IA^n = E \cup IA^n.$$

The last reduction step is justified by the fact that the assumption $m + \ell_m(E) \geq n$ implies $IA^mE \subseteq IA^n$.

COROLLARY 4.2.8: $\lambda \cup E \cup IA^n = (\lambda \cup IA^n)(\lambda \cup E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{2n-1})$.

LEMMA 4.2.9: $\lambda \cup IA^m = (A^m \cup A^{m+1} \cup \cdots \cup A^{2m-1})^*$.

PROOF: Let $n \geq m$ and $n = km + r$ for $0 \leq r < m$. L.H.S. $\subseteq$ R.H.S. because $A^n = A^{km+r} = A^{(k-1)m}A^{m+r}$, $m \leq m + r \leq 2m - 1$ and λ is contained in every star event. Since all sequences, except λ, in the R.H.S. are of lengths greater than or equal to m, the reverse containment is also true. The equality is thus proved.

THEOREM 4.2.10: Any cofinite event is a comet event with the maximum front star cofinite and the minimum tail finite.

PROOF: By Theorem 4.2.7 and Lemma 4.2.9, a cofinite event is also a comet event. Since the maximum front star S_L contains any other front star, $S_L \supseteq (A^n \cup \cdots \cup A^{2n-1})^* = \lambda \cup IA^n$. Thus, S_L is cofinite. Furthermore, since the minimum tail $T_m \subseteq (E \cup A^n \cup A^{n+1} \cup \cdots \cup A^{m+n-1})$, it is finite.

COROLLARY 4.2.11: Any event $R = CQ$, with a front event C cofinite and any tail Q, is a comet event with a cofinite maximum front star.

LEMMA 4.2.12: If E is finite, $E^* \cup IA^n = (E \cup A^n \cup \cdots \cup A^{2n-1})^*$.

PROOF: By Lemma 4.2.9, $E^* \cup IA^n = E^* \cup (\lambda \cup IA^n) = E^* \cup (A^n \cup \cdots \cup A^{2n-1})^*$. Since $EA^m \subset IA^n$ for any $m \geq n$, $E^* \cup (A^n \cup \cdots \cup A^{2n-1})^* = (E \cup A^n \cup \cdots \cup A^{2n-1})^*$.

THEOREM 4.2.13: Let $C = E \cup IA^n$ be a cofinite event. C is a star event if, and only if, $E^* \subset C$.

PROOF: If C is a star event, $C = (E \cup IA^n)^*$. Hence $E^* \subset C$. For the reverse implication, we assume $E^* \subset C$. Then

$$C = (E \cup IA^n) \cup E^* = E^* \cup IA^n.$$

By Lemma 4.2.12, $C = (E \cup A^n \cup \cdots \cup A^{2n-1})^*$.

Note that $E \cup A^n \cup \cdots \cup A^{2n-1}$ is not the minimum root if E is not empty.

Since the reverse of a cofinite event is also cofinite, any property of cofinite events also applies to the reverse of any cofinite event.

§4.3 FINITE-COFINITE AND COFINITE-FINITE EVENTS

From results in section 3.1 we know that the set of finite events and the set of cofinite events are both closed under product operation, i.e., a finite product of finite events is finite and a finite product of cofinite events is cofinite. This leaves us finite-cofinite (FC-) events and cofinite-finite (CF-) events among 2-factor products of finite and cofinite events to be studied in this section.

I. DEGENERATE CASES

It is found that some events exist in the families of finite-cofinite and cofinite-finite events, which degenerate into a simpler form — cofinite events. The necessary and sufficient condition for a finite-cofinite event being cofinite is studied below. The conditions for a cofinite-finite event being cofinite follows trivially by considering its reverse.

LEMMA 4.3.1: Let A be a finite alphabet for any events R and P. $PA \subseteq RA$ if, and only if, $P \subseteq R$.

PROOF: Let $p \in P$. Then $pa \in Pa$ for $a \in A$. Since $PA \subseteq RA$, we have $pa \in Ra$. Therefore $p \in R$ and $P \subseteq R$. The reverse implication is obvious.

THEOREM 4.3.2: A finite-cofinite event $R = FC$ is cofinite if, and only if, $B \subseteq F$ for some $B \in \mathcal{B}$.

PROOF:

1. Necessity: Let $C = E \cup A^m I$ with m being the least possible

such integer and E being finite such that $E \cap A^mI = \phi$. Then the finite-cofinite event $R = FC = FE \cup FA^mI$.

Assuming R is cofinite such that $R = FC = C' = E' \cup A^{m'}I$, we have $A^{m'}I \subseteq FC$. This implies $A^{m''} \subseteq FC$ for $m'' \geq m'$. For any specific case one can always choose the integer m'' large enough to be greater than $\ell_M(FE)$. By this assumption $A^{m'} \subset FC$ implies $A^{m''} \subset FA^mI$ and $A^{m''}A^m \subset FIA^m$. By Lemma 4.3.1 it follows that $A^{m''} \subset FI$. Finally, by Lemma 4.2.5 we conclude that $B \subseteq F$ for some $B \in \mathcal{B}$.

2. Sufficiency: Assume $B \subseteq F$ for some $B \in \mathcal{B}$. Let $\ell_M(F) = n$. Since $B \subseteq F$, $\ell_M(B) \leq \ell_M(F)$, by Lemma 4.2.6 $A^nI \subset FI$. Then, by $A^{n+m}I \subseteq FIA^m \subseteq FC$, we conclude that FC is cofinite.

Hereafter in this section we deal with only non-degenrate finite-cofinite and cofinite-finite events.

II. UNIQUE FACTORIZATION OF COFINITE-FINITE EVENTS

THEOREM 4.3.3: Any cofinite-finite event can be expressed as a comet event with a cofinite maximum front star and a finite minimum tail.

PROOF: Let R = CF be a cofinite-finite event. From the result of Theorem 4.2.10 we have $R = S_{MC}T_{mC}F$ where S_{MC} is the cofinite maximum front star of C and T_{mC}, the finite minimum tail of C with respect to S_{MC}. By Corollary 4.2.11 we have, for R, maximum cofinite front star S_{MR}. Furthermore, $S_{MR} \supseteq S_{MC}$ and $R = S_{MR}R = S_{MR}S_{MC}T_{mC}F$. By Lemma 2.5.2, $R = S_{MR}T_{mC}F$. Since the minimum tail T_{mR} with respect to S_{MR} is contained in $T_{mC}F$, T_{mR} is also finite.

Since the maximum front star is unique for any comet event, and the minimum tail with respect to the maximum front star is also unique, the factorization $R = S_{MR}T_{mR}$ is unique for a cofinite-finite event R.

III. <u>RELATIONS BETWEEN FINITE-COFINITE EVENTS AND COFINITE-FINITE EVENTS</u>

LEMMA 4.3.4: Let E, F be finite events and m', m be integers such that $m' > m + \ell_M(F)$. Then $IA^{m'}E \subseteq FA^mI$ implies $A^{m'}I \subseteq FA^mI$.

PROOF: Let $IA^{m'}E \subseteq FA^mI$ such that $m' > m + \ell_M(F)$. $A^{m'}E \subset IA^{m'}E \subseteq FA^mI$. Observing that a word $x \in X$, X an arbitrary event, implies $\overleftarrow{x} \in \overleftarrow{X}$, we have $\overleftarrow{E}A^{m'} \subset IA^m\overleftarrow{F}$. Let e be one of the longest words of $\overleftarrow{E}$. Then $D_e(IA^m\overleftarrow{F}) = IA^m\overleftarrow{F} \cup G$ with $\ell_M(G) < m + \ell_M(F)$ and $D_e(\overleftarrow{E}A^{m'}) = A^{m'} \cup H$ with $\ell_M(H) < m'$. Since $P \subset Q$ implies $D_aP \subseteq D_aQ$ for any events P and Q and word a, $A^{m'} \cup H \subset IA^m\overleftarrow{F} \cup G$. Hence $A^{m'} \subset IA^m\overleftarrow{F} \cup G$. By assumptions $m' > m + \ell_M(F)$ and $m + \ell_M(F) > \ell_M(G)$, $A^{m'} \subset IA^m\overleftarrow{F} \cup G$ implies $A^{m'} \subset IA^m\overleftarrow{F}$. Furthermore, $\overleftarrow{A^{m'}} \subset \overleftarrow{IA^m\overleftarrow{F}}$, hence $A^{m'} \subset FA^mI$ for $m' > m + \ell_M(F)$. By Lemma 4.2.4, therefore, $A^{m'}I \subseteq FA^mI$.

THEOREM 4.3.5: A finite-cofinite event is cofinite-finite if, and only if, it is cofinite.

PROOF:

1. Necessity: Assuming R is both finite-cofinite and cofinite-finite we have $R = FC = FE \cup FA^mI = C'F' = E'F' \cup IA^{m'}F'$ for some C, F, C' and F'. After deleting some short words, if necessary, from $IA^{m'}F'$ and obtaining $IA^{m''}F''$, we may have $IA^{m''}F'' \subseteq FA^mI$. By Lemma 4.3.4, $A^{m''}I \subseteq FA^mI$. Therefore $R \supseteq FA^mI \supseteq A^{m''}I$ and R is cofinite.

2. Sufficiency: When R is cofinite, it is trivially cofinite-finite and finite-cofinite.

IV. PROPERTIES OF FINITE-COFINITE EVENTS

LEMMA 4.3.6: Any finite-cofinite event must have a non-trivial front star ($\neq \{\lambda\}$).

PROOF: We know any finite-cofinite event R = FC can be expanded as $FC = FE \cup FA^m I$ with E finite and m the smallest such integer.

Let y be one of the longest words in FA^m. Then, the derivative $D_y(FC) = I$. Hence there exists at least one non-trivial derivative (i.e., $y \neq \lambda$) of R containing R. Therefore the front star cannot be trivial.

It is noticed that to uniquely factor any FC-event by Theorem 4.3.3 we can always find the largest star from the right and find the corresponding smallest front.

§4.4 COFINITE-FINITE-COFINITE (CFC-) EVENTS CHARACTERIZED AS TWO-SIDED COMETS

THEOREM 4.4.1: Any CFC-event R is a two-sided comet event $S_L M_m S_R$ with a finite minimum middle M_m and two cofinite maximum stars S_L and S_R on both sides.

PROOF: By Theorem 4.2.1 any cofinite event $C = (\lambda \cup A^nI)F$ where $(\lambda \cup A^nI) = (A^n \cup \cdots \cup A^{2n-1})*$ and F is finite. Therefore we can expand any CFC-event $R = (\lambda \cup A^nI)EFG(\lambda \cup A^mI)$ with E, F, G all finite. Since a maximum front star of a comet event contains all other stars which may preceed the comet event, we have $A^nI \subseteq (A^n \cup \cdots \cup A^{2n-1})* \subseteq S_L$ and S_L is cofinite. Furthermore, considering $(\lambda \cup A^nI) \subseteq S_L$ we let $S_L = \lambda \cup H \cup A^nI$, with H finite and $0 \leq \ell_M(H) < n$. Similarly, let $S_R = \lambda \cup H' \cup A^mI$ with H' finite and $0 \leq \ell_M(H') < m$. Then we have $R = (\lambda \cup H \cup A^nI)EFG(\lambda \cup H' \cup A^mI)$. Since EFG is finite and the minimum middle M_m is contained in any other middle with respect to S_L and S_R, M_m is also finite.

The factorization of CFC-events in Theorem 4.4.1 is unique because the cofinite maximum stars S_L and S_R on both sides are unique and the finite middle M_m is minimum with respect to S_L and S_R.

CHAPTER V

CONCLUSIONS AND PROBLEMS FOR FURTHER STUDY

In Chapter 3 closure properties for some families of star-free events were studied. The study was started with the family of finite events. By closing this family under Boolean operation and concatenation we generated some star-free events. The closure properties of these families are listed in Table 3.2. We found the closure properties of family M_2 = {finite products of finite and cofinite events} were similar to those of M_1.. This similarity led us to treat B_1 as an 'alphabet' of M_2. With the observation of further similarity between $F(M_2)$ and $F(M_1)$, a question was raised that whether this mathematical structure would recur infinitely. An attempt was made to answer this question. The fact that B_2 is not a Boolean algebra answered the question negatively.

In Chapter 4 attention was shifted to unique factorization of some families of star-free events. Methods for unique factorization of cofinite-finite, finite-cofinite and CFC-events were obtained. The problem of uniquely factoring FCF-events and the problem of uniquely factoring any finite product of finite, cofinite events are suggested for further study. The solution of these problems may provide a unique method of designing sequential circuits realizing certain families of star-free events.

This thesis also demonstrated how difficult it is to deal with Boolean operations and concatenation. The apparently simple families are in fact very difficult to characterize. It is felt that these problems deserve some further study.

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VITA

NAME: Zaw-Sing SU

BORN: 15 August, 1937. Chung-King, Szechueng, China

EDUCATED:

Primary: Shanghai, China

Secondary: Shanghai, China

University: Chiao-Tung University

Shanghai, China

University of Ottawa

Ottawa, Ontario, Canada

1966 Bachelor of Applied Sciences

(B.A.Sc)

Electrical Engineering