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POSTDOCTORAL STUDIES

Radu Bogdan Munteanu

AUTEUR DE LA THÈSE / AUTHOR OF THESIS

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Actions of Xerox Type on Araki-Woods Factors and Their Fixed Point Von Neumann Algebras

TITRE DE LA THÈSE / TITLE OF THESIS

Thierry Giordano

DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

EXAMINATEURS (EXAMINATRICES) DE LA THÈSE / THESIS EXAMINERS

Benoît Collins

James Mingo

David Hendelman

Vladimir Pestov

Gary W. Slater

Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

ACTIONS OF XEROX TYPE ON ARAKI-WOODS
FACTORS AND THEIR FIXED POINT VON NEUMANN
ALGEBRAS

Radu-Bogdan Munteanu

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In partial fulfillment of the requirements

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Abstract

The von Neumann algebras studied in this thesis arise as GNS-representation of fixed point algebras under a certain type of group actions, called xerox actions, on $A = \otimes M_k(\mathbb{C})$, the k^∞ -UHF algebra. The xerox actions are induced by unitary representations of compact groups on $M_k(\mathbb{C})$. The states used to perform the GNS representation are restrictions of diagonal product states on k^∞ -UHF algebras. First, we analyze xerox actions induced by diagonal representations of compact groups. In this case the von Neumann algebras studied here can be seen as fixed point algebras under xerox actions of compact groups on Araki-Woods factors. We obtain necessary and sufficient conditions for such von Neumann algebras to be factors and we determine their type

We also study the GNS representation of the fixed point algebra under the xerox action induced by non-diagonal representations of a compact groups G . We show that the von Neumann algebra obtained in this way can be identified up to isomorphism with the GNS representation of a fixed point algebra under a xerox action induced by a diagonal representation of a closed subgroup of G .

Any von Neumann algebra, N , studied here can be realized as $W^*(X, \mu, \mathcal{R})$, the von Neumann algebra associated to an equivalence relation \mathcal{R} on a measured space (X, μ) .

We give sufficient conditions for N to be isomorphic to an Araki-Woods factor. To prove this we show that the associated flow of \mathcal{R} is approximately transitive. We study also when the equivalence relation \mathcal{R} , that corresponds to N has Krieger's property A, and we prove that there exist equivalence relations which have property A but their associated flow is not approximately transitive.

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Dedication

To my parents

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Introduction

The infinite tensor product of factors was first introduced by J. von Neumann in the late 30s, to accommodate quantum field theory. The infinite tensor of factors has proven to be a rich source of enlightening examples. In 1967, R. Powers showed the existence of continuously many non-isomorphic factors of type III, by studying factors arising from infinite tensor products of 2×2 matrix algebras. His work was quickly extended to the classification of Araki-Woods factors (known also as infinite tensor products of finite type I factors, abbreviated ITPFI), by Araki and Woods in 1968, [AW].

The relationship between von Neumann algebras and ergodic theory has a long history, beginning with the pioneering work of Murray and von Neumann and their group measure space construction; this associates a von Neumann algebra to a group acting on a space (X, μ) . This construction was generalized by Krieger and later on, by Feldman and Moore who recognized the importance of the orbit structure rather than the group itself, and associated to an equivalence relation \mathcal{R} on a space (X, μ) a von Neumann algebra $W^*(X, \mu, \mathcal{R})$. A major step in the structure theory of approximately finite dimensional (abbreviated AFD) factors was taken by H. Dye in two articles, [Dye], by proving that the factor associated with the group measure space construction based on a finite measure preserving ergodic group of polynomial growth is indeed approximately finite dimensional and hence isomorphic to any other AFD factor of type II_1 .

Outstanding results of Connes, Krieger, Feldman and Moore established a complete correspondence between AFD von Neumann algebras (up to isomorphism), er-

godic equivalence relation (up to orbit equivalence) and ergodic flows (up to conjugacy). In 1980, Connes and Woods introduced a new property of ergodic actions called approximate transitivity (AT), to characterize among AFD von Neumann factors the Araki-Woods factors. More precisely, the Araki-Woods factors correspond to equivalence relations whose associated flow is AT.

In this thesis we study von Neumann algebras arising as GNS-representation of fixed point algebras under a certain type of group actions, called xerox actions, on k^∞ -UHF algebras. We obtain necessary and sufficient conditions for such a von Neumann algebra to be a factor and we determine the type of the factor obtained. The states used to perform the GNS representation will be restrictions of diagonal product states on k^∞ -UHF algebras. The von Neumann algebras obtained in this way are approximately finite dimensional.

We generalize previous results obtained by Powers and Baker in [BP1] and [BP2]. In the special case of the standard action of $SU(2)$ on the 2^∞ -UHF algebra, Baker and Powers, [BP1], give necessary and sufficient conditions for the factor obtained to be of type I, II, or type III. Further work of Baker and Giordano classified the type III factors in subtypes III_λ by defining a ratio set which generalizing the one defined by Araki-Woods for ITPFI factors [AW]. The groups considered in [BP2] are more general and they act by standard xerox actions on k^∞ -UHF algebras. In [BP2], Baker and Powers give necessary and sufficient conditions for the factors obtained to be of type I or of type II_1 . In this thesis, for product diagonal product states, we extend the results of [BP1] and [BP2] to more general xerox actions and give necessary and sufficient conditions for such a factor to be of type II_1 , II_∞ and III. Following [BG], we also define a ratio set to classify the factors of type III in subtypes.

Let us give a short overview of the content of this thesis. We consider $A = \otimes M_k(\mathbb{C})$ the k^∞ -UHF algebra, π a unitary representation of compact group G on $\mathcal{U}(M_k(\mathbb{C}))$ and a faithful product diagonal state $\varphi = \otimes \varphi_n$ on A . The representation π induces an action of G on A , called a xerox action, in the following way: $\alpha(g) =$

$\otimes \text{Ad } \pi(g)$. We denote by A^G the fixed point algebra under this action and by φ^G the restriction of φ to the fixed point algebra A^G . If $(\pi_{\varphi^G}, H_{\varphi^G})$ is the GNS representation of (A^G, φ^G) we denote $\pi_{\varphi^G}(A^G)''$ by N .

The fixed point algebra A^G is an AF-algebra that we can write either as $A(X, \Gamma)$ [SV1], or as $C^*(X, \mathcal{R})$ [Re] where $X = \prod\{0, 1, \dots, k-1\}$ and \mathcal{R} is an étale equivalence relation on X . We can identify $N = \pi_{\varphi^G}(A^G)''$ with $(\pi_{\varphi}(A)'')^G$, the fixed point subalgebra under the extension of the xerox action to $M = \pi_{\varphi}(A)''$.

An Araki-Woods factor $M = \pi_{\varphi}(A)''$ can be realized as $W^*(X, \mu, \mathcal{T})$ where $X = \prod\{0, 1, \dots, k-1\}$, \mathcal{T} , is the so-called tail equivalence relation on X , and μ the measure on X induced by the state φ . Then the subfactor N can be realized as $W^*(X, \mu, \mathcal{R})$ where \mathcal{R} is a subequivalence relation of \mathcal{T} .

We use Connes' invariant T to decide whether N is semifinite or not. When N is of type III, we define a ratio set, which extends the definition of the ratio set given by Araki-Woods for ITPFI factors and that defined by Baker-Giordano for the fixed point factors under the standard action of the torus on ITPFI₂ factors.

We study and give a partial answer to a very interesting question: When is a fixed point factor under a xerox action on an ITPFI factor isomorphic to an ITPFI? We prove that N is isomorphic to an ITPFI factor, we need by showing that the its flow of weights is approximately transitive (AT). We prove, under additional conditions, that if the sequence of density matrices of a product state on $A = \otimes M_2(\mathbb{C})$ does not have a limit point with non-zero entries, then the fixed point factor under the standard xerox action of the 1-dimensional torus is an ITPFI factor.

To determine the associated flow of an equivalence relation and to decide if it is AT, is in general, very difficult. In [K2], Krieger introduced the so-called property A to show the existence of ergodic transformations not of product type, or equivalently, in terms of von Neumann algebras, of Krieger factors not isomorphic to ITPFI factors. We state Krieger's property A for an equivalence relation and reformulate Krieger's result as follows: any tail equivalence of product type of type III has property A.

We show that the equivalence relation coming from the example of the fixed point subfactor of index 2, not isomorphic to an ITPFI factor, constructed by Giordano and Handelman, [GH], has property A, but by [GH], its flow is not AT. With this example we answer a question stated by Dooley and Hamachi in [DH].

Also, we construct a Bratteli diagram $B = (V, E)$ and an AF-measure μ on X , its tail equivalence relation \mathcal{R} does not have Krieger's property A. We show that the associated flow of (X, μ, \mathcal{R}) is measure preserving and not AT, by [CW]. Therefore, the factor associated to (X, μ, \mathcal{R}) , is not ITPFI. This example is easier to describe than the Krieger's example in [K2] and Dooley-Hamachi's example in [DH].

We now present the content of the thesis in details:

In Chapter I, we present the background material we will need.

In Chapter II, we consider groups acting by xerox actions induced by diagonal representations of compact groups and we give necessary and sufficient conditions for the GNS representation of their fixed point algebras to be factors. We determine the type of the factor obtained in terms of the entries of the density matrices of the product state.

In Chapter III, we pursue the classification in subtypes of the factors of type III discussed in Chapter II.

In Chapter IV we study the GNS representations of fixed point algebras under a xerox action induced by a non-diagonal representation of a compact group G . We show that the von Neumann algebra obtained by this construction can be identified, up to isomorphism, with the GNS representation of a fixed point algebra under a diagonal xerox action of a closed subgroup of G .

In Chapter V, we give sufficient conditions for a factor N arising as a fixed point algebra under a xerox action to be an ITPFI factor. To accomplish this, we realize N as the von Neumann algebra associated to an equivalence relation and we show that under suitable conditions, the associated flow of the equivalence relation is approximately transitive. By [CW], the fixed point factor is then isomorphic to an

ITPFI factor.

In Chapter VI, we focus on Krieger's Property A. We prove that there exist equivalence relations not of product type which have property A. At the end of the chapter we construct an example of an equivalence relation which is the tail equivalence on a Bratteli diagram and does not have property A; consequently the associated AFD factor is not ITPFI.

Chapter 1

Preliminaries

Most of the results in this chapter are known facts. We state them without proof but we indicate where details can be found.

1.1 AF Algebras and Dynamical Systems

1.1.1 Basic Facts

Definition 1.1.1. If a Banach algebra A admits a map $x \mapsto x^* \in A$ with the following properties:

- (i) $(x^*)^* = x$;
- (ii) $(x + y)^* = x^* + y^*$;
- (iii) $(\alpha x)^* = \bar{\alpha}x^*$;
- (iv) $(xy)^* = y^*x^*$;

for every $x, y \in A$ and $\alpha \in \mathbb{C}$, then A is called an *involution Banach algebra* and the map $x \mapsto x^*$ the *involution* of A . If the involution of A satisfies the following additional condition:

- (v) $\|x^*x\| = \|x\|^2, x \in A$,

then A is called a C^* -algebra.

If A is a C^* -algebra, then $\|x\| = \|x^*\|$, for all $a \in A$.

The first example is the complex numbers, with $*$ being complex conjugation and the usual norm. The second example is $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. The $*$ operation is conjugate transpose and the norm is

$$\|a\| = \sup\{\|a\xi\|_2, \xi \in \mathbb{C}^n, \|\xi\|_2 = 1\},$$

for all a in $M_n(\mathbb{C})$, where $\|\cdot\|_2$ denotes the l_2 -norm on \mathbb{C}^n . This example can be easily generalized to the algebra of bounded linear transformation on a complex Hilbert space, by replacing \mathbb{C}^n by the Hilbert space

Definition 1.1.2. ([Da], p.75) A C^* -algebra A is called *approximately finite dimensional* or *AF* if it is the norm-closure of an increasing sequence of finite dimensional C^* -subalgebras A_n i.e., $A = \overline{\bigcup_{n \geq 1} A_n}$.

Suppose that ϕ is a unital homomorphism of a finite dimensional C^* -algebra $B_1 \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ into another finite dimensional C^* -algebra $B_2 \simeq M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C})$. Then ϕ is determined up to an inner automorphism of B_2 by an $k \times l$ matrix $A = [a_{ij}]$ in $M_{lk}(\mathbb{N}_0)$ with nonnegative integer entries such that

$$\begin{bmatrix} n_1 & n_2 & \cdots & n_k \end{bmatrix} A = \begin{bmatrix} m_1 & m_2 & \cdots & m_l \end{bmatrix}$$

The integers a_{ij} is the multiplicity of imbedding the summand $M_{n_j}(\mathbb{C})$ into the summand $M_{m_i}(\mathbb{C})$. This allows us to describe the imbedding of B_1 into B_2 in a simple graphical way. Represent B_1 by the k -tuple $\{(1, 1), (1, 2), \dots, (1, k)\}$ and B_2 by the l -tuple $\{(2, 1), (2, 2), \dots, (2, l)\}$. Denote the imbedding B_1 into B_2 by drawing a_{ij} arrows from $(1, i)$ to $(2, j)$ to indicate the partial multiplicity of the imbedding.

The sequence of these pictures for a sequence of imbeddings of A_n into A_{n+1} is called the Bratteli diagram of the sequence. This is an infinite graph consisting of vertices and edges. This is also called the Bratteli diagram of the algebra, but this

is an abuse of terminology as the diagram is not unique. We will discuss again, in Subsection 1.1.4, about Bratteli diagrams, in the context of AF-equivalence relations, and we will see how to associate an AF-algebra to a Bratteli diagram.

Definition 1.1.3. An AF-algebra is called *uniformly hyperfinite* or *UHF* if it is the closure of an increasing union of unital algebras isomorphic to full matrix algebras $M_{k_n}(\mathbb{C})$.

Since a unital imbedding of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ requires $m|n$, we have an increasing sequence $k_1|k_2|k_3 \dots$. If $k_n = k^n$ for all k we call A the k^∞ -UHF algebra and in this case, as each $M_n \simeq \otimes_{i=1}^n M_k(\mathbb{C})$, we can write it as the C^* -algebra $\otimes M_k(\mathbb{C})$.

1.1.2 Group Actions on C^* algebras

Definition 1.1.4. An action $\alpha : G \rightarrow \text{Aut}(A)$ of a locally compact group G on a C^* algebra A is called *continuous* if $\lim_{g \rightarrow e} \|\alpha_g(x) - x\| = 0$ for any $x \in A$.

Definition 1.1.5. Let G be a compact group and $\pi : G \rightarrow \mathcal{U}(M_k(\mathbb{C}))$ a unitary representation of G (we denote by $\mathcal{U}(M_k(\mathbb{C}))$ the group of unitaries of $M_k(\mathbb{C})$) and $A = \otimes M_k(\mathbb{C})$ the k^∞ -UHF algebra. The action $\alpha : G \rightarrow \text{Aut}(A)$ given by $\alpha_g = \otimes \text{Ad } \pi_g$ will be called the *xerox action* of π . If π is a diagonal (resp. non-diagonal) representation of G , we call α a *diagonal (resp. non-diagonal) xerox action*. We will denote by A^G or A^α the fixed point algebra under the action α , $A^G = \{x \in A; \alpha_g(x) = x \text{ for all } x \in G\}$.

Proposition 1.1.1. *Let α be a xerox action of a compact group G on $A = \otimes M_k(\mathbb{C})$. Then the fixed point algebra A^G is an AF algebra.*

Proof. If $A_n = \otimes_{i=1}^n M_k(\mathbb{C})$, then $A_n^G = \{x \in A_n : \otimes_{i=1}^n \text{Ad } \pi(g)(x) = x\}$ is a finite dimensional algebra and it is clear that $A^G \supseteq \overline{\cup A_n^G}$. On the other hand, let $x \in A^G$. Then there exist $(x_n)_{n \geq 1}$ in $\cup A_n$ such that $x_n \rightarrow x$. Let $y_n := \int_G \alpha_g(x_n) dg$. Then $y_n \in \cup A_n^G$ and $\int_G \alpha_g(x) dg = x$, as $x \in A^G$, where dg is the Haar measure on G . Then:

$$\|y_n - x\| = \left\| \int_G \alpha_g(x_n) dg - x \right\| = \left\| \int_G \alpha_g(x_n) dg - \int_G \alpha_g(x) dg \right\| \leq \|x_n - x\| \rightarrow 0$$

Therefore, we have

$$A^G = \overline{\cup A_n^G}$$

and hence A^G is an AF -algebra. \square

1.1.3 Stratila-Voiculescu's Diagonalization for AF-algebras

In [SV1], Stratila and Voiculescu proved the following result.

Theorem 1.1.2. ([SV1], p. 17) *Given an AF-algebra A there exists:*

- (i) *a masa C (i.e., a maximal abelian subalgebra) in A ,*
- (ii) *a conditional expectation P of A with respect to C ,*
- (iii) *a subgroup U of the group $U(A)$ of unitaries of A ,*

such that

- (1) $u^*Cu = C$ for all $u \in U$,
- (2) $P(u^*xu) = u^*P(x)u$ for all $u \in U$, $x \in A$,
- (3) A is the norm closure of $\text{Span}_{\mathbb{C}}\{UC\}$.

There is thus an isomorphism of the group U onto a group Γ of $*$ -automorphisms of C , namely, for $u \in U$, the corresponding $*$ -automorphism $\gamma_u \in \Gamma$ is

$$\gamma_u : C \ni c \mapsto u^*cu \in C$$

We denote by X the Gelfand spectrum of the commutative C^* -algebra C . Then X is a compact topological space, $C \simeq C(X)$ and we may view Γ as a group of homeomorphisms on X . We thereby obtain a topological dynamical system (X, Γ) associated to the algebra A . We consider the Hilbert space $l^2(X)$ with orthonormal basis $\{e_x; x \in X\}$ and we denote by $(\cdot|\cdot)$ the standard inner product. Each function $f \in C(X)$ defines a “multiplication operator” T_f on $l^2(X)$ by

$$T_f(h) = fh \quad h \in l^2(X)$$

On the other hand, each element $\gamma \in \Gamma$ defines a “permutation operator” V_γ on $l^2(X)$ by

$$V_\gamma(h)(x) = h(\gamma^{-1}(x)), \quad x \in X; \quad h \in l^2(X)$$

Let us denote by $A(X, \Gamma)$ the C*-algebra generated in $B(l^2(X))$ by the operators T_f , $f \in C(X)$ and V_γ , $\gamma \in \Gamma$.

Theorem 1.1.3. ([SV1], p.17) *There is a *-isomorphism*

$$A \simeq A(X, \Gamma)$$

and moreover

$$P(a)(x) = (ae_x|e_x), \quad x \in X; a \in A$$

1.1.4 C*-algebra of an Étale Equivalence Relation

We begin with some notation and basic ideas and for this we follow [P]. On a compact metric space X , we consider an equivalence relation $\mathcal{R} \subseteq X \times X$. We will soon restrict to the case that \mathcal{R} has only countable equivalence classes. We let r and s (for range and source) denote the two canonical projections from \mathcal{R} to X ; $s(x, y) = x$, $r(x, y) = y$.

Definition 1.1.6. Let X be a compact metrizable space, \mathcal{R} an equivalence relation on X , and \mathcal{T} a topology on \mathcal{R} . We say that $(\mathcal{R}, \mathcal{T})$ is *étale* if

- (1) \mathcal{T} is Hausdorff, second countable and σ -compact,
- (2) the diagonal $\Delta = \{(x, x) | x \in X\}$ is open in \mathcal{R} ,
- (3) the maps $r, s : \mathcal{R} \rightarrow X$ are local homeomorphisms; that is, for every (x, y) in \mathcal{R} , we may find an open set U in \mathcal{T} such that $r(U)$ and $s(U)$ are open in X and $r : U \rightarrow r(U)$ and $s : U \rightarrow s(U)$ are homeomorphisms.
- (4) if U and V are open sets as above, then the set $UV = \{(x, z) | (x, y) \in U, (y, z) \in V, \text{ for some } y\}$ is also open and

(5) if U as above is open, then so is $U^{-1} = \{(x, y) | (y, x) \in U\}$.

It follows that the diagonal $\Delta = \{(x, x) | x \in X\}$ is a clopen subset of \mathcal{R} . If X is zero-dimensional, we may choose U in (3) above to be a compact open set. Also, Δ is homeomorphic to X , and so we are justified in identifying Δ with X . It is easily deduced that $r^{-1}(x) = \{(x, y) \in \mathcal{R}\}$, and that $s^{-1}(x) = \{(y, x) \in \mathcal{R}\}$ are (countable) discrete topological spaces in the relative topology for each $x \in X$. Clearly \mathcal{R} can be written as a union of graphs of local homeomorphisms of the form $s \circ r^{-1}$. When \mathcal{T} is understood, we simply say that \mathcal{R} is étale. An equivalence relation is also a principal groupoid. The term ‘étale’ is relatively recent; in the past these have also been known as r -discrete groupoids with counting measure as Haar system. (See [Re], [Pa], [GPS2]).

Definition 1.1.7. ([GPS2], p. 5) Let (X, \mathcal{R}) and (X', \mathcal{R}') be two étale equivalence relations.

- (1) We say that (X, \mathcal{R}) and (X', \mathcal{R}') are *orbit equivalent* and write $(X, \mathcal{R}) \sim (X', \mathcal{R}')$ if there is a homeomorphism $h : X \rightarrow X'$ such that $h \times h(\mathcal{R}) = \mathcal{R}'$. That is, the map h carries \mathcal{R} -equivalence classes exactly to \mathcal{R}' -equivalence classes.
- (2) We say that (X, \mathcal{R}) and (X', \mathcal{R}') are *isomorphic* and write $(X, \mathcal{R}) \simeq (X', \mathcal{R}')$ if there is a homeomorphism $h : X \rightarrow X'$ such that $h \times h(\mathcal{R}) = \mathcal{R}'$ and such that $h \times h : \mathcal{R} \rightarrow \mathcal{R}'$ is a homeomorphism.

If \mathcal{R} is an étale equivalence relation on a space X (not necessarily a Cantor set), we may construct a C^* -algebra as follows. Let $C_c(\mathcal{R})$ denote the set of continuous, compactly supported complex-valued functions on \mathcal{R} . It is a linear space in an obvious way. The product and involution are defined by

$$(1) f * g(x, y) = \sum_{z \in \mathcal{R}x} f(x, z)g(z, y)$$

$$(2) f^*(x, y) = \overline{f(y, x)}$$

for all f, g in $C_c(\mathcal{R})$ and (x, y) in \mathcal{R} . It is a subtle point here that the product $f * g$ is again in $C_c(\mathcal{R})$. The proof uses the étale property of \mathcal{R} .

The issue of a norm is more subtle. Let μ be a σ -finite measure on X and let $\nu(C) = \int |r^{-1}(x) \cap C| d\mu(x)$. For f in $C_c(\mathcal{R})$ and ξ in $L^2(\mathcal{R}, \nu)$, we set

$$\text{Ind } \mu(f)\xi(x, y) = \sum_{z \in \mathcal{R}x} f(x, z)\xi(z, y)$$

It is easy to check that $\text{Ind } \mu$ is a bounded representation and the collection of such representations is faithful on $C_c(\mathcal{R})$. The completion of $C_c(\mathcal{R})$ in the norm defined by the equation

$$\|f\|_r = \sup\{\|\text{Ind } \mu(f)\|, \mu \text{ is a measure on } X\}$$

will be denoted by $C_r^*(X, \mathcal{R})$ and will be called the *reduced algebra of the equivalence relation* \mathcal{R} . This is not quite the definition of $C_r^*(X, \mathcal{R})$ given in [Re], but it is equivalent to it ([MS], p. 45). We remark that if μ is a measure on X with $\text{supp}(\mu) = X$, then $\|\text{Ind}\mu(f)\| = \|f\|_r$, for all $f \in C_c(\mathcal{R})$. The reason for the subscript r and the term “reduced” is that there are other choices for the norm. For amenable equivalence relations, all (reasonable) norms are the same.

Remark 1.1.1. The elements of $C_r^*(X, \mathcal{R})$ can be seen as functions in $C_0(\mathcal{R})$ ([Re], p.99). We have a conditional expectation $P : C_r^*(X, \mathcal{R}) \rightarrow C(X)$ that associates to any function f in $C_r^*(\mathcal{R})$ its restriction to the diagonal $\Delta = \{(x, x) | x \in X\}$ ([Re], p. 104), where the diagonal Δ is identified with X as we remarked before.

We end this subsection by giving a characterization of the C^* -subalgebras of $C_r^*(X, \mathcal{R})$ which contain $C(X)$. We assume that \mathcal{R} is an amenable, (in the sense of [Re], II.3) étale equivalence relation on a compact space X . For an open subset D of \mathcal{R} we define:

$$A(D) = \{f \in C_c(\mathcal{R}), f = 0 \text{ on } \mathcal{R} \setminus D\}$$

$$C_c(D) = \{f \in C_c(\mathcal{R}), \text{supp}(f) \subseteq D\}$$

Remark 1.1.2. ([MS], p.67) With the above notation $C_c(D)$ is dense in $A(D)$ with respect to the norm on $C_r^*(X, \mathcal{R})$.

Theorem 1.1.4. ([MS], Prop 3.10) Every closed $C(X)$ -bimodule $B \subseteq C_r^*(X, \mathcal{R})$ is of the form $A(D)$ for a unique open subset D of \mathcal{R} .

Definition 1.1.8. If \mathcal{R} is an étale equivalence relation on X , an open subset \mathcal{S} of \mathcal{R} is a *subequivalence relation* of \mathcal{R} if \mathcal{S} is an equivalence relation and $\mathcal{S}(x) \subseteq \mathcal{R}(x)$ for every $x \in X$.

Theorem 1.1.5. ([MS], Theorem 4.1) For each subequivalence relation \mathcal{S} in \mathcal{R} , $A(\mathcal{S})$ is a C^* -subalgebra of $C_r^*(X, \mathcal{R})$ containing $C(X)$. Conversely, each C^* -subalgebra of $C_r^*(X, \mathcal{R})$ containing $C(X)$ is of the form $A(\mathcal{S})$ for a unique subequivalence \mathcal{S} . The correspondence $\mathcal{S} \mapsto A(\mathcal{S})$ is an inclusion preserving bijection between the collection of subequivalence of \mathcal{R} and C^* -subalgebras of $C_r^*(X, \mathcal{R})$ containing $C(X)$.

Proof. It is enough to show that $A(\mathcal{S})$ is a C^* -subalgebra containing $C(X)$ if and only if \mathcal{S} is a subequivalence relation, since everything else is clear. We need to check that $\mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$ and $\mathcal{S} = \mathcal{S}^{-1}$ if and only if $A(\mathcal{S}) \cdot A(\mathcal{S}) \subseteq A(\mathcal{S})$ and $A(\mathcal{S})^* = A(\mathcal{S})$. Suppose that $A(\mathcal{S}) \cdot A(\mathcal{S}) \subseteq A(\mathcal{S})$ and let $(x, y), (y, z) \in \mathcal{S}$. Choose f, g in $A(\mathcal{S})$ and U and V compact open neighborhoods of (x, y) and (y, z) such that $f|_U(x, y) \neq 0$ and $g|_V(y, z) \neq 0$ and the restriction of r and s to U are homeomorphisms. Then $f|_U$ and $g|_V$ belong to $A(\mathcal{S})$ by [MS], Proposition 3.6. As $s(x, u) = s(x, y) = x$ and $s|_U$ is one to one, $(x, u) \notin \mathcal{S}$ whenever $u \neq y$. Hence $f|_U(x, u) = 0$ if $u \neq y$. Similarly, $g|_V(u, z) = 0$ whenever $u \neq y$. Therefore, we have:

$$f|_U * g|_V(x, z) = \sum_{u \sim x} f|_U(x, u)g|_V(u, z) = f|_U(x, y) \cdot g|_V(y, z) \neq 0$$

Since $f|_U * g|_V$ lies in $A(\mathcal{S})$, we conclude that (x, z) lies in \mathcal{S} , i.e., $\mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$. On the other hand as $A(\mathcal{S})$ is $*$ -closed, it follows that $\mathcal{S} = \mathcal{S}^{-1}$. Clearly, $\Delta \subseteq \mathcal{S}$, because $A(\mathcal{S})$ contains $C(X)$. Therefore \mathcal{S} is a subequivalence relation of \mathcal{R} . For the converse, we see first that if $f, g \in C_c(\mathcal{S})$ then $f * g, f^* \in C_c(\mathcal{S})$. As $C_c(\mathcal{S})$ is dense in $A(\mathcal{S})$ (Remark 1.1.2), we conclude that $A(\mathcal{S})$ is a C^* -subalgebra of $C_r^*(X, \mathcal{R})$. \square

We will come back to this problem later on, in Section 2.5 and we will show that in fact $A(\mathcal{S}) \simeq C_r^*(X, \mathcal{S})$.

1.1.5 AF Equivalence Relations

Definition 1.1.9. An étale equivalence relation \mathcal{R} on X is an *AF-relation* if X is compact, metrizable and totally disconnected and if there is an increasing sequence \mathcal{R}_n of compact, open, subequivalence relations of \mathcal{R} such that $\bigcup \mathcal{R}_n = \mathcal{R}$.

We present now a general method for producing an AF-relation. For this we follow [GPS2], p.7. We begin with a special infinite directed graph $B = (V, E)$, called a Bratteli diagram, which consists of a vertex set V and an edge set E , where V and E can be written as a countable disjoint union of non-empty finite sets: $V = V_0 \cup V_1 \cup V_2 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$ with the following property: an edge e in E_n goes from a vertex in V_{n-1} to one in V_n , which we denote by $i(e)$ and $f(e)$, respectively. We call i the source map and f the range map. We require that $V_0 = \{v_0\}$, there is only one source, v_0 , i.e., $f^{-1}(v) \neq \emptyset$ for all $v \in V_1 \cup V_2 \cup \dots$ and there are no sinks, i.e., $i^{-1}(v) \neq \emptyset$ for all $v \in V$. It is convenient to give a diagrammatic presentation of a Bratteli diagram with V_n the vertices at level n and E_n the edges (downward directed) between V_{n-1} and V_n . We let $X = X_{(V,E)}$ denote the space of infinite paths in the diagram beginning at v_0 , i.e.,

$$X = \{(e_1, e_2, \dots) \mid i(e_{n+1}) = f(e_n); n \geq 1\}$$

which is given the relative topology of the product space $\prod_{n \geq 1} E_n$, and is therefore compact, metrizable and zero-dimensional. X has a basis consisting of clopen cylinder sets, i.e., sets of the form

$$U(e_1, \dots, e_m) = \{(f_1, f_2, \dots) \in X \mid f_1 = e_1, \dots, f_m = e_m\}$$

The equivalence \mathcal{R} on X shall be cofinal or tail equivalence: two paths are equivalent if they agree from some level on. For $N = 0, 1, 2, \dots$, let

$$\mathcal{R}_N = \{(e_1, e_2, \dots), (e'_1, e'_2, \dots) \in X \times X \mid e_k = e'_k \text{ for all } k > N\}.$$

Give \mathcal{R}_N the relative topology \mathfrak{T}_N of $X \times X$. Then \mathcal{R}_N is compact and is an open subset of \mathcal{R}_{N+1} for all N . Let $\mathcal{R} = \bigcup_{N \geq 1} \mathcal{R}_N$, and give \mathcal{R} the inductive limit topology \mathfrak{T} , so that a set U is in \mathfrak{T} if and only if $U \cap \mathcal{R}_N$ is in \mathfrak{T}_N for each N . This means that a sequence $\{(x_n, y_n)\}$ in \mathcal{R} converges to (x, y) in \mathcal{R} if and only if $\{x_n\}$ converges to x , $\{y_n\}$ converges to y (in X) and, for some N , $\{(x_n, y_n)\}$ is in \mathcal{R}_N for all but finitely many n . It is now a simple task to verify that $(\mathcal{R}, \mathfrak{T})$ is an étale equivalence relation.

Theorem 1.1.6. *[GPS2] If $B=(V,E)$ is a Bratteli diagram and $\mathcal{R} \subset X \times X$ is the tail equivalence described above then $C_r^*(X, \mathcal{R})$ is an AF-algebra which has $B=(V,E)$ as a Bratteli diagram.*

Theorem 1.1.7. *If (X, \mathcal{R}) and (X', \mathcal{R}') are two AF-equivalence relations then $(X, \mathcal{R}) \simeq (X', \mathcal{R}')$ (see Definition 1.1.7) if and only if $C_r^*(X, \mathcal{R}) \simeq C_r^*(X', \mathcal{R}')$*

Remark 1.1.3. Let $A = \overline{\cup A_n}$ an AF-algebra. By [SV1], in each A_n there exists a masa C_n and a group of unitaries U_n , with $U_n \subseteq U_{n+1}$ that normalizes C_n . We have that $C = \overline{\cup C_n}$ is a masa in A and the group $U = \cup U_n$ normalizes C . Let X be the Gelfand spectrum of C , i.e., $C \simeq C(X)$. Each U_n acts on C by $\text{Ad } u$, $u \in U_n$ and induces a group of homeomorphisms on X , denoted by Γ_n . Each Γ_n induces an equivalence relation on X , denoted \mathcal{R}_n , and we have an increasing sequence of equivalence relations on X . In fact, ([Re], III.1.15), $A_n \simeq C_r^*(X, \mathcal{R}_n)$ and therefore, $A \simeq \overline{\cup C_r^*(X, \mathcal{R}_n)} = C_r^*(X, \mathcal{R})$ where $\mathcal{R} = \cup \mathcal{R}_n$, and the conditional expectation $P : A \rightarrow C$ corresponds to the conditional expectation from $C_r^*(X, \mathcal{R})$ onto $C(X)$ (see Remark 1.1.1).

We will denote by $\{e_{ij}; 1 \leq i, j \leq k\}$ the standard system of matrix units (s.m.u) of $M_k(\mathbb{C})$

Definition 1.1.10. Let $(k_n)_{n \geq 1}$ be a sequence of integers greater or equal the 2 and let $A = \otimes M_{k_n}(\mathbb{C})$ the corresponding UHF-algebra. The masa C generated for $n \geq 1$ by $\{\otimes_{m=1}^n e_{ii}^m \otimes \bigotimes_{m \geq n+1} 1\}$ will be the *standard masa* of A .

Remark 1.1.4. The standard masa C of $A = \otimes M_{k_n}(\mathbb{C})$ coincides with the masa considered in [SV1].

Example 1.1.1. On $X = \prod\{0, 1, \dots, k_n - 1\}$ we consider the following equivalence relation denoted by \mathcal{T} :

$$x\mathcal{T}y \text{ if and only if there exists } n \in \mathbb{N} \text{ such that } x_m = y_m \text{ for all } m > n$$

We call this equivalence relation the *tail equivalence* on X .

Remark 1.1.5. With the same notations as in the previous definition we observe that we can write $\mathcal{T} = \cup \mathcal{T}_n$ where for $n \geq 1$, \mathcal{T}_n is the equivalence relation:

$$x\mathcal{T}_n y \text{ if and only } x_m = y_m \text{ for all } m > n$$

If $A = \otimes M_{k_n}(\mathbb{C})$ an UHF algebra, (see Remark 1.1.3) it is not difficult to see that $A \simeq C_r^*(X, \mathcal{T})$. In fact, we can regard \mathcal{T} as the tail equivalence on the Bratteli diagram of A .

More details about étale and AF-equivalence relations can be found in [Re], [M], [GPS2], [P] and [Pa].

1.2 Von Neumann Algebras

1.2.1 Definitions

Definition 1.2.1. A *von Neumann algebra* on a Hilbert space H is a $*$ -subalgebra M of $B(H)$, equal to its bicommutant (i.e., $M = M''$). A von Neumann algebra whose center, $Z(M) = M \cap M' = \mathbb{C}1$ is called a *factor*. A von Neumann algebra with separable predual is called *approximately finite dimensional (AFD)* if it is generated by an increasing sequence of finite-dimensional $*$ -subalgebras.

We mention that all von Neumann algebras considered here have separable predual.

If H is a Hilbert space then $B(H)$ is a von Neumann algebra. Here are some topologies on $B(H)$ we will use:

- (1) the weak operator topology defined by the following family of seminorms

$$x \rightarrow |(x\xi, \eta)|, \quad \xi, \eta \in H;$$

- (2) the strong operator topology defined by the following family of seminorms

$$x \rightarrow \|x\xi\|, \quad \xi \in H;$$

- (3) the σ -weak operator topology defined by following family of seminorms

$$x \rightarrow \left| \sum (x\xi_n, \eta_n) \right|, \quad \xi_n, \eta_n \in H, \quad \sum \|\xi_n\|^2 + \|\eta_n\|^2 < \infty.$$

When $M \subseteq B(H)$ is a von Neumann algebra, we can consider the restrictions of these topologies to M . The σ -weak topology on M does not depend on the Hilbert space on which M is acting and can be defined as the $\sigma(M, M_*)$ -topology, where M_* is the predual of M (see [Ta], vol. I for more details). The homomorphisms between von Neumann algebras will always be normal, or equivalently, $\sigma(M, M_*)$ -continuous.

Theorem 1.2.1. (*Von Neumann's Double Commutant Theorem*) *For any unital $*$ -algebra $A \subset B(H)$, A'' coincides with each of the strong and weak closures of A .*

More details can be found in [J1], [Ta], [S], or [Blk].

1.2.2 Projections in a von Neumann algebra

Definition 1.2.2. If p and q are projections in a von Neumann algebra M we say that p and q are *equivalent*, written $p \sim q$, if there is a partial isometry $u \in M$ with $uu^* = p$ and $u^*u = q$ and we say that p is *subordinate* to q , written $p \preceq q$ if $p \sim q' \leq q$.

Observe that \sim is an equivalence relation. The relation \preceq is a partial order on the equivalence classes of projections in a von Neumann algebra.

Theorem 1.2.2. *If M is a factor and p, q are projections in M , either $p \preceq q$ or $q \preceq p$.*

Definition 1.2.3. A projection p in a von Neumann algebra M is called *infinite* if $p \preceq q$ for some $q < p, p \neq q$. Otherwise p is called *finite*. A projection p is said to be *purely infinite* if there is no nonzero finite projection $q \leq p$ in M . If zp is infinite for every central projection $z \in M$ with $zp \neq 0$, then p is called *properly infinite*.

Definition 1.2.4. A von Neumann algebra M is said to be *finite* (respectively *infinite, properly infinite or purely infinite*) if 1 is a finite projection (respectively infinite, properly infinite or purely infinite).

Definition 1.2.5. (a) A von Neumann factor M is said to be of

- (i) *type I*, if M contains a nonzero minimal projections;
 - (ii) *type II*, if M contains nonzero finite projections and if M is not of type I;
 - (iii) *type III*, if M contains no nonzero finite projections;
- (b) If M is a factor of type II, then M is of *type II_1* (respectively *II_∞*), if M is finite (respectively infinite);
- (c) A factor M is *semifinite* if contains no nonzero finite projections.

We remark that factors of type I_∞ , type II_∞ , or type III are properly infinite; a factor is semifinite if and only if it is of type I or of type II.

1.2.3 The GNS Representation

Definition 1.2.6. A *state* on a C^* algebra or on a von Neumann algebra is a positive linear functional of norm one. If φ is a state on a von Neumann algebra M , we say that φ is

- (1) *normal*, if $\varphi(x) = \sup \varphi(x_i)$, whenever x is the supremum of a monotone increasing net (x_i) in M_+ ,

(2) *faithful* if $\varphi(x) \neq 0$ for all $x \in M_+$, $x \neq 0$.

Definition 1.2.7. A *weight* on a von Neumann algebra M is a map $\varphi : M_+ \rightarrow [0, \infty]$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in M_+$ and $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda > 0$ and $x \in M_+$.

We recall that any state on $M_k(\mathbb{C})$ is of the form $\text{tr}(h \cdot)$ with h positive matrix with trace one; h is the corresponding density matrix for the state.

To simplify notation, we introduce the following definition:

Definition 1.2.8. 1) Let $(k_n)_{n \geq 2}$ be a sequence of integers greater than or equal 2 and let $A = \otimes M_{k_n}(\mathbb{C})$ be the corresponding UHF-algebra. Let $(\varphi_n)_{n \geq 1}$ be a sequence of faithful states on $M_{k_n}(\mathbb{C})$ and let $(h_n)_{n \geq 1}$ be the corresponding sequence of density matrices.

The product state $\varphi = \otimes \varphi_n$ on A will be called a *faithful diagonal product state* if for each $n \geq 1$, $h_n = \text{diag}(h_1^n, \dots, h_{k_n}^n)$ where $h_j^n > 0$ for $1 \leq j \leq k_n$.

2) If $\varphi = \otimes \varphi_n$ is a product state on $A = \otimes M_k(\mathbb{C})$ and φ_n are all identical, φ is called a *symmetric state*.

Definition 1.2.9. Let $(h_n)_{n \geq 1}$, $h_n = \text{diag}(h_1^n, h_2^n, \dots, h_k^n)$ be a sequence of diagonal matrices, in $M_k(\mathbb{C})$. We say that $(h_n)_{n \geq 1}$ *converges* to $h \in M_k(\mathbb{C})$ if h is a diagonal matrix, $h = \text{diag}(h^1, h^2, \dots, h^k)$ and if

$$\lim_{n \rightarrow \infty} h_n^i = h^i, \text{ for } 1 \leq i \leq k. \quad (1.2.1)$$

As any diagonal matrix in $M_k(\mathbb{C})$ can be identified with a vector in \mathbb{C}^n , if $(h_n)_{n \geq 1}$ converges to h and h is a diagonal matrix with distinct (no zero) entries on the diagonal, we simply say that $(h_n)_{n \geq 1}$ converges to h and h has distinct (no zero) entries.

We note that for any norm, $\|\cdot\|$, on $M_k(\mathbb{C})$, $(h_n)_{n \geq 1}$ converges to h if and only if h is diagonal and (1.2.1) holds. Regarding each h_n , as a vector in \mathbb{C}^n , $h_n \rightarrow h$ in $l^2(\{1, 2, \dots, n\}) \simeq \mathbb{C}^n$.

Let A be a unital C^* -algebra φ be a state on A . The Gelfand-Naimark-Segal (GNS) representation of (A, φ) is defined as follows. To construct a representation of A , we need to construct a suitable Hilbert space. The space

$$N_\varphi = \{x \in A; \varphi(x^*x) = 0\}$$

is a closed left ideal of A and the formula

$$(x + N_\varphi, y + N_\varphi) = \varphi(y^*x)$$

defines an inner product on the vector space A/N_φ . The completion of A/N_φ is a Hilbert space, we denote by H_φ . The equation $\pi_\varphi(a)(x + N_\varphi) = ax + N_\varphi$, for $x \in A$, defines a bounded linear operator $\pi_\varphi(a)$ on A/H_φ (we have $\|\pi_\varphi(a)\| \leq \|a\|$, $a \in A$) which extends to H_φ . Then π_φ defines a representation of A on H_φ . Moreover, if $\xi_\varphi = 1 + N_\varphi \in H_\varphi$, then ξ_φ is cyclic vector for π_φ (i.e., $\pi_\varphi(A)\xi_\varphi$ is dense in H_φ) and $\varphi(a) = (\pi_\varphi(a)\xi_\varphi | \xi_\varphi)$, for $a \in A$.

We summarize the above construction in the following theorem:

Theorem 1.2.3. (*[S], Theorem 2.2.1*) *Let φ be a state on a C^* -algebra A . Then there exists a triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$, where*

- (i) H_φ is a Hilbert space,
- (ii) π_φ is a $*$ -algebra homomorphism of M into $B(H_\varphi)$,
- (iii) $\xi_\varphi \in H_\varphi$ and $H_\varphi = \overline{\pi_\varphi(A)\xi_\varphi}$, and
- (iv) $\varphi(x) = (\pi_\varphi(x)\xi_\varphi, \xi_\varphi)$ for all $x \in M$.

Such a triple is unique in the sense that if (H', π', ξ') is another such triple, there is a unique unitary operator $w : H_\varphi \rightarrow H'$ such that $w\xi_\varphi = \xi'$ and $\pi'(x) = w\pi_\varphi(x)w^$.*

1.2.4 Modular Group of Automorphisms, T-set and S-set

Let M a von Neumann algebra and φ a faithful normal state on M . We denote by η_φ the canonical injection from the pre-Hilbert space M endowed with the inner product given by $(x, y) \mapsto \varphi(y^*x)$ onto H_φ , the completion of M with respect to this inner product. Let $\eta(1) = \xi_\varphi$. For each $x \in M$, the function $\eta_\varphi(y) \mapsto \eta_\varphi(xy)$ can be extended to a bounded operator on H_φ denoted $\pi_\varphi(x)$. The function π_φ is an isomorphism of M onto a subalgebra $\pi_\varphi(M)$ of the set of bounded operators on H_φ . The vector ξ_φ is cyclic and separating and the triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ is the GNS representation associated to the state φ on M (see Theorem 1.2.3).

We remark that whenever φ is faithful normal state on a von Neumann algebra M , $\pi_\varphi(M)$ is a von Neumann algebra of operators on H_φ and π_φ is a σ -weakly continuous homomorphism of M onto $\pi_\varphi(M)$. The closure S_φ of the antilinear map $\pi_\varphi(x)\xi_\varphi \mapsto \pi_\varphi(x^*)\xi_\varphi$ has a polar decomposition $S_\varphi = J_\varphi\Delta_\varphi^{\frac{1}{2}} = \Delta_\varphi^{-\frac{1}{2}}J_\varphi$ where J_φ is an isometric involution (J_φ is antilinear and $J_\varphi^2 = 1$) and Δ_φ is an invertible and self-adjoint operator, called the *modular operator* of φ . We have $J_\varphi\pi_\varphi(M)J_\varphi = \pi_\varphi(M)'$ and for $t \in \mathbb{R}$,

$$\Delta_\varphi^{it}\pi_\varphi(M)\Delta_\varphi^{-it} = \pi_\varphi(M)$$

The one-parameter group of automorphisms of M given by

$$\pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it}\pi_\varphi(x)\Delta_\varphi^{-it}$$

for $x \in M$ and $t \in \mathbb{R}$ is called the *modular group of automorphisms* of φ . A similar construction can be done starting with a weight instead of a state.

Definition 1.2.10. Let $(k_n)_{n \geq 1}$, be a sequence of integers, $k_n \geq 2$ for all n and let φ_n be faithful states on $M_{k_n}(\mathbb{C})$.

If $A = \otimes M_{k_n}(\mathbb{C})$ and φ is the faithful product state $\otimes \varphi_n$, then let $\pi_\varphi(A)'' \subset B(H_\varphi)$ denote the GNS representation of A associated to φ .

The *ITPFI* or *Araki-Woods factor* $\otimes(M_{k_n}(\mathbb{C}), \varphi_n)$ will denote the factor $\pi_\varphi(A)'' \subset B(H_\varphi)$.

Example 1.2.1. 1) If $k_n = k$, for all $n \geq 1$, then $\otimes(M_k(\mathbb{C}), \varphi_n)$ is an ITPFI $_k$.

2) If the sequence $(k_n)_{n \geq 1}$ is uniformly bounded, then the corresponding ITPFI is called of *bounded type*. By Theorem 2.1, [GS1] any bounded type ITPFI is isomorphic to an ITPFI $_2$.

3) If $k_n = 2$, for all $n \geq 1$, and all the states ϕ_n are given by

$$\phi_n(x) = \text{tr}(hx); \quad h = \begin{bmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{bmatrix}$$

then $(\otimes M_2(\mathbb{C}), \varphi_n)$ is called the *Powers factor* R_λ . Here $0 \leq \lambda \leq 1$. For $\lambda = 1$ we obtain the hyperfinite type II_1 factor denoted by R .

Remark 1.2.1. (i) Any state on $M_k(\mathbb{C})$ is of the form $\text{tr}(h \cdot)$ where h is a positive matrix with trace one. In this case we have that $\sigma_t^\varphi(x) = h^{it} x h^{-it}$.

(ii) If M denotes the ITPFI factor $\otimes(M_{k_n}(\mathbb{C}), \varphi_n)$ and $\varphi = \otimes \varphi_n$ the corresponding product state, then the modular group of automorphisms σ_t^φ is given for $t \in \mathbb{R}$ by

$$\sigma_t^\varphi = \otimes \text{Ad } h_n^{it}, \quad \text{for } t \in \mathbb{R},$$

where h_n is the density matrix of φ_n (i.e., $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$).

Definition 1.2.11. ([J1], p.97) Let M be a von Neumann algebra. The subgroup of all $t \in \mathbb{R}$ for which σ_t^φ is inner (i.e., there exists a unitary u in M such that $\sigma_t^\varphi(x) = u x u^*$ for all x in M) is independent of the faithful normal state φ . This subgroup is an invariant of M and is called $T(M)$.

Theorem 1.2.4. *A factor M with separable predual is semifinite (type II_1 or type II_∞) if and only if $T(M) = \mathbb{R}$.*

The following result and definitions come from [C] (see for example [Ta], vol. III).

Proposition 1.2.5. ([Ta], vol. III p. 89) *If $M = \otimes(M_{k_n}(\mathbb{C}), \varphi_n)$ where $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$, $\text{tr}(h_n) = 1$, and u_n are unitaries in M_{k_n} that commute with h_n , then the automorphism $\otimes \text{Ad } u_n$ is inner if and only if $\sum_{n=1}^{\infty} (1 - |\text{tr}(h_n u_n)|) < \infty$.*

Definition 1.2.12. If M is a von Neumann algebra, the invariant $S(M)$ is the intersection over all faithful normal states φ of the spectra of their corresponding modular operators Δ_φ . In other words,

$$S(M) = \cap \{Sp(\Delta_\varphi); \varphi \text{ normal faithful state of } M\}$$

Theorem 1.2.6. *A factor is of type III if and only if $0 \in S(M)$.*

Theorem 1.2.7. *(Connes-van Daele) $S(M) \cap \mathbb{R}_+^*$ is a closed subgroup of \mathbb{R}_+^* .*

Definition 1.2.13. A factor M is of type

- (1) III_0 , if $S(M) = \{0, 1\}$,
- (2) III_λ , if $S(M) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ for $\lambda \in (0, 1)$,
- (3) III_1 , if $S(M) = [0, \infty)$.

Recall that if $S \subseteq \mathbb{R}_+^*$ is a closed subgroup, then its annihilator in \mathbb{R} is the set $T = \{t \in \mathbb{R}; \lambda^{it} = 1 \text{ for all } \lambda \in S\}$; T is a closed subgroup of \mathbb{R} and $S = \{\lambda \in \mathbb{R}_+^*; \lambda^{it} = 1 \text{ for all } t \in T\}$. The invariants $T(M)$ and $S(M)$ are connected by the following partial duality result:

Theorem 1.2.8. *If M is a factor and $S(M) \neq \{0, 1\}$ (i.e., M is not of type III_0), then $T(M)$ is the annihilator of $S(M) \cap \mathbb{R}_+^*$ in \mathbb{R} .*

There are only three kinds of closed subgroups of \mathbb{R}_+^* ,

Remark 1.2.2. (a) By Theorem 1.2.8, if M is a factor of

- (i) type III_λ , $0 < \lambda < 1$, then $T(M) = \{\frac{2k\pi}{\log \lambda}; k \in \mathbb{Z}\}$
- (ii) type III_1 , then $T(M) = \{0\}$.

(b) If $T(M)$ is not a closed subgroup of \mathbb{R} , then M is of type III_0 .

Theorem 1.2.9. *If $0 < \lambda < 1$, then the Powers factor R_λ is of type III_λ and any AFD factor of type III_λ factor, $0 < \lambda < 1$, is isomorphic to R_λ .*

There exists also a unique up to isomorphism AFD factor of type III_1 and it is isomorphic to $R_\lambda \otimes R_\mu$ with $\log \lambda$ and $\log \mu$ rationally independent.

More details and the proofs for this subsection can be found in [J1], [Blk], [Ta].

1.2.5 Group Actions on Von Neumann Algebras

Theorem 1.2.10. *([Blk], Theorem 3.2.2) Let G be a locally compact group, N a von Neumann algebra on a Hilbert space H , and $\alpha : G \rightarrow \text{Aut}(N)$ a homomorphism. The following are equivalent:*

- (i) α is continuous with respect to the point-weak (point-strong, etc) topology.*
- (ii) α is continuous with respect to the topology of pointwise norm-convergence on N_* .*
- (iii) The map $(g, \varphi) \mapsto \varphi \circ \alpha_g$ from $G \times N_*$ to N_* is norm-continuous.*

Definition 1.2.14. A homomorphism α satisfying these conditions is called a *continuous action* of G on N , and (N, G, α) is called a dynamical system.

Proposition 1.2.11. *Let A be a C^* -algebra, φ a state on A , and $\alpha : G \rightarrow \text{Aut}(A)$ a continuous action of a locally compact group G . If φ is α -invariant (i.e., $\varphi \circ \alpha_g = \varphi$, for all $g \in G$), then the action α on A induces an action on $\pi_\varphi(A)''$ denoted also by α which is a continuous action of G on $\pi_\varphi(A)''$.*

Proof. By [Ta], vol. I, p. 47, there is a continuous unitary representation U_φ of G on the Hilbert space H_φ , defined by:

$$U_\varphi(g)\pi_\varphi(x)\xi_\varphi = \pi_\varphi(\alpha_g(x))\xi_\varphi$$

and

$$U_\varphi(g)\pi_\varphi(x)U_\varphi(g)^* = \pi_\varphi(\alpha_g(x))$$

If we set now

$$\alpha_g(x) = U_\varphi(g)xU_\varphi(g)^*$$

this is a continuous action, ([Ta], vol. II, p. 238). \square

Definition 1.2.15. An action α of a group G on a von Neumann algebra M is said to be *ergodic* if $M^\alpha = \mathbb{C} \cdot 1$.

Theorem 1.2.12. [AHKT] *Let M be a von Neumann algebra equipped with a continuous action γ of a compact group G such that the fixed point algebra $M^\gamma = N$ is a properly infinite factor. Assume further that the group*

$$\text{Aut}_\gamma M = \{\varrho \in \text{Aut}M : \varrho\gamma_g = \gamma_g\varrho, g \in G\}$$

has a subgroup S acting ergodic on M in the sense

$$\{x \in M : \beta(x) = x \text{ for all } \beta \in S\} = \mathbb{C}$$

Then, if α is an automorphism of M commuting with S and such that $\alpha(x) = x$ for every $x \in M^\gamma$ there is an element g_α in G such that $\alpha = \gamma_{g_\alpha}$.

Proof. The same proof as in [AHKT] with N a properly infinite factor instead of a type III factor, because all the things we need are true for properly infinite factors, not just for type III factors. \square

1.2.6 Standard Equivalence Relations

In this section, we present the definitions and main results on measurable countable equivalence relations. Details can be found in [FM1], [FM2] and [Sch].

In this subsection (X, \mathfrak{B}) will always denote a standard Borel space and (X, \mathfrak{B}, μ) a measure space with (X, \mathfrak{B}) a standard space and μ a σ -finite measure on \mathfrak{B} .

Definition 1.2.16. (i) A Borel set $\mathcal{R} \subseteq X \times X$ is a *standard equivalence relation* on X if \mathcal{R} is an equivalence relation. We write \mathcal{C} for the restriction of $\mathfrak{B} \times \mathfrak{B}$ to \mathcal{R} .

(ii) We write $x\mathcal{R}y$ for $(x, y) \in \mathcal{R}$, and define $s(x, y) = x$, the left projection, and $r(x, y) = y$, the right projection of \mathcal{R} .

(iii) For any $x \in X$, $\mathcal{R}(x) = \{y \in X; (x, y) \in \mathcal{R}\}$ will denote the equivalence class of x . For a subset $A \in \mathcal{B}$, $\mathcal{R}(A) = \bigcup_{x \in A} \mathcal{R}(x)$ is called the *saturation* of A .

(iv) If \mathcal{S} is another equivalence relation on (X, \mathfrak{B}) then \mathcal{S} is a *subequivalence relation* of \mathcal{R} if $\mathcal{S}(x) \subseteq \mathcal{R}(x)$ for every $x \in X$.

Definition 1.2.17. Let \mathcal{R} be a standard equivalence relation on X and let μ be a measure on \mathfrak{B} . Then:

- (i) \mathcal{R} is μ -nonsingular if $\mu(\mathcal{R}(A)) = 0$ for every $A \in \mathfrak{B}$ with $\mu(A) = 0$.
- (ii) \mathcal{R} is μ -ergodic if $\mu(\mathcal{R}(A)^c) = 0$ whenever $A \in \mathfrak{B}$ and $\mu(A) > 0$.

Recall that an *isomorphism* T between two measured spaces (X, \mathcal{B}, μ) and (X', \mathcal{B}', μ') is a bimeasurable bijection between two conull subsets $Y \subseteq X$ and $Y' \subseteq X'$, preserving null sets.

Definition 1.2.18. Two nonsingular equivalence relations \mathcal{R}_i , on $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, 2$ are called *orbit equivalent* if there is an isomorphism T from $(X_1, \mathfrak{B}_1, \mu_1)$ onto $(X_2, \mathfrak{B}_2, \mu_2)$ such that $T(\mathcal{R}_1(x)) = \mathcal{R}_2(Tx)$ for almost all $x \in X_1$.

In the rest of this subsection, an equivalence relation on (X, \mathcal{B}, μ) or simply (X, μ) , will mean a nonsingular countable standard equivalence relation on (X, \mathfrak{B}, μ) .

Example 1.2.2. Let G be a countable group of automorphisms on (X, \mathfrak{B}) and let \mathcal{R}_G denote the countable equivalence relation induced by G , i.e.,

$$\mathcal{R}_G = \{(x, gx) : x \in X, g \in G\}.$$

It is clear that \mathcal{R}_G is a countable standard equivalence relation. If μ is a σ -finite measure on (X, \mathfrak{B}) , then:

- (i) the action of G on (X, \mathfrak{B}, μ) is nonsingular if and only if \mathcal{R}_G is μ -nonsingular
- (ii) the action of G on (X, \mathfrak{B}, μ) is ergodic if and only if \mathcal{R}_G is μ -ergodic.

Example 1.2.3. Let $X = \prod\{0, 1, \dots, k_n - 1\}$ endowed with the usual product σ -algebra and $\mu = \otimes\mu_n$ a product measure, with μ_n probability measures on $\{0, 1, \dots, k_n - 1\}$ and $\mu_n(i) > 0$ for $0 \leq i \leq k_n - 1$ and for all $n \geq 1$. On (X, μ) let \mathcal{T} denote the so-called tail equivalence defined for $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ by

$$x\mathcal{T}y \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for all } i > n$$

Tail equivalence \mathcal{T} is induced either by an action of $\oplus\mathbb{Z}_{k_n}$ or by an action of \mathbb{Z} , via the odometer or adding machine T . The odometer $T : X \rightarrow X$ is the automorphism defined by $T(1, 1, 1, \dots) = (0, 0, 0, \dots)$ and if $x \neq (1, 1, 1, \dots)$ and $N(x) := \min\{n \geq 1 : x_n < k_n - 1\}$ then

$$T(x)_n = \begin{cases} 0 & \text{if } n < N(x) \\ x_n + 1 & \text{if } n = N(x) \\ x_n & \text{if } n > N(x). \end{cases}$$

Notice that T is a homeomorphism of the compact space X (endowed with the product topology). The action of $G = \oplus\mathbb{Z}_{k_n}$ is given for $g = (g_n)_{n \geq 1} \in G$ and $x = (x_n)_{n \geq 1} \in X$ by

$$(gx)_n = x_n + g_n \pmod{k_n}, \quad g = (g_n)_{n \geq 1} \in G, \quad x \in X.$$

Example 1.2.4. Let $X = \prod\{0, 1\}$, $\mu = \otimes\mu_n$ with $\mu_n(0) = a_n$, $\mu_n(1) = 1 - a_n$, and $0 < a_n < 1$ for all $n \geq 1$ and let \mathcal{R}_∞ denote the subequivalence relation of tail equivalence given for $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ by:

$$x\mathcal{R}_\infty y \text{ if and only if } x\mathcal{T}y \text{ and } \sum_{i \geq 1} x_i = \sum_{i \geq 1} y_i$$

Let $S = \cup S_n$ be the group of finite permutations of $\mathbb{N}^* = \{k; k \geq 1\}$. Then S_∞ acts by homeomorphisms on X by:

$$\sigma(x_1, x_2, \dots) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots), \quad \sigma \in S_\infty.$$

The equivalence relation \mathcal{R}_∞ is induced by S_∞ .

Definition 1.2.19. The *full group* $[\mathcal{R}]$ of \mathcal{R} is the group of all nonsingular automorphisms V of (X, \mathfrak{B}, μ) with $(x, Vx) \in \mathcal{R}$ for μ -a.e. $x \in X$.

Theorem 1.2.13. [FM1] Let \mathcal{R} be an equivalence relation on (X, \mathfrak{B}, μ) .

(a) For any $C \in \mathcal{C}$, the function $x \mapsto |s^{-1}(x) \cap C|$ is Borel and the measure ν_l defined by

$$\nu_l(C) = \int_X |s^{-1}(x) \cap C| d\mu(x),$$

is σ -finite; it will be called the *left counting measure* of μ .

(b) The null sets of ν_l are exactly those $C \in \mathcal{C}$ such that $\mu(s(C)) = 0$.

(c) The *right counting measure* of μ defined by

$$\nu_r(C) = \int_X |r^{-1}(x) \cap C| d\mu(x)$$

satisfies $\nu_r = \nu_l \circ \theta$ with $\theta(x, y) = \theta(y, x)$ and we have $\nu_r \sim \nu_l$.

Definition 1.2.20. (i) The *Radon-Nikodym cocycle* of μ with respect to \mathcal{R} is the Borel function $\delta(x, y) = \frac{d\nu_l}{d\nu_r}(x, y)$ on \mathcal{R} . It is unique up to null sets of $\nu_l \sim \nu_r$.

(ii) We say that μ is \mathcal{R} -invariant if $\delta = 1$ ν -a.e.

Definition 1.2.21. A *partial Borel isomorphism* on X will be a Borel isomorphism defined on some $A \in \mathfrak{B}$ with range some $B \in \mathfrak{B}$.

Example 1.2.5. Let \mathcal{T} be the tail equivalence on (X, μ) the space considered in Example 1.2.3 (i). Then,

$$\delta(y, x) = \prod_{i \geq 1} \frac{\mu_i(y_i)}{\mu_i(x_i)}, \quad (y, x) \in \mathcal{T}.$$

Definition 1.2.22. An ergodic equivalence relation \mathcal{R} on (X, \mathfrak{B}, μ) is said to be of

- (i) *type I*, if μ is concentrated on one orbit, i.e., there exists a point $x \in X$ such that $\mu(\mathcal{R}(x)^c) = 0$;
- (ii) *type II₁*, if \mathcal{R} is not of type I and if there exists a finite \mathcal{R} -invariant measure on X equivalent to μ ;

- (iii) *type II_∞* , if \mathcal{R} is not of type I and if there exists an infinite σ -finite \mathcal{R} -invariant measure on X equivalent to μ ;
- (iv) *type III*, if there exists no invariant σ -finite measure on X equivalent to μ .

Recall that the *essential range* of a measurable function f on a measured space (X, \mathfrak{B}, μ) , denoted by $\sigma(f)$ is the set of all $\lambda \in \mathbb{C}$ such that for any $\varepsilon > 0$, $\mu(\{x \in X, |f(x) - \lambda| < \varepsilon\}) > 0$.

Definition 1.2.23. For an ergodic equivalence relation \mathcal{R} on (X, \mathfrak{B}, μ) ,

$$r_\infty(\mathcal{R}) = \bigcap_{E \in \mathfrak{B}, \mu(E) > 0} \sigma(\delta_E)$$

is called the *asymptotic ratio set* of \mathcal{R} , where δ_E is the restriction of the modulus δ to $\mathcal{R} \cap (E \times E)$.

Remark 1.2.3. (i) Let G be a countable group acting on (X, μ) . In [K4], Krieger associated to it its ratio set $r(G, X, \mu)$. \mathcal{R}_G denote the equivalence relation induced by G on (X, μ) then

$$r_\infty(\mathcal{R}) = r(G, X, \mu) \subseteq [0, \infty)$$

(ii) If \mathcal{R} is an ergodic equivalence relation then $r_\infty(\mathcal{R}) \cap \mathbb{R}_+^*$ is a closed subgroup of \mathbb{R}_+^* (see [Mo]).

Definition 1.2.24. [Mo] A nonsingular, ergodic, equivalence relation \mathcal{R} on (X, \mathfrak{B}, μ) is said to be of:

- (1) III_0 , if $r_\infty(\mathcal{R}) = \{0, 1\}$;
- (2) III_λ , $0 < \lambda < 1$, if $r_\infty(\mathcal{R}) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$;
- (3) III_1 , if $r_\infty(\mathcal{R}) = [0, \infty)$.

Properties of an equivalence relation can be translated into another construction associated to it, namely the associated flow.

We recall that a one parameter group of automorphisms $\{F_t\}_{t \in \mathbb{R}}$ of the standard space (X, \mathfrak{B}, μ) , (i.e., $F_{t+s}x = F_t(F_sx)$, $\forall t, s \in \mathbb{R}$ and $\forall x \in X$) is called a *flow* if the function $\psi : X \times \mathbb{R} \rightarrow \mathbb{R}$ given by the formula $\psi(x, t) = F_t x$ is measurable. Two flows $\{F_t^1\}_{t \in \mathbb{R}}$ and $\{F_t^2\}_{t \in \mathbb{R}}$ of automorphisms on $(X_1, \mathfrak{B}_1, \mu_1)$ resp. $(X_2, \mathfrak{B}_2, \mu_2)$, are conjugate if it exists an isomorphism $T : (X_1, \mathfrak{B}_1, \mu_1) \rightarrow (X_2, \mathfrak{B}_2, \mu_2)$ such that for all $t \in \mathbb{R}$ and for μ_1 -almost all $x \in X_1$, $F_t^2(T(x)) = T(F_t^1(x))$.

Definition 1.2.25. (*Associated flow*) Let \mathcal{R} be an equivalence relation on (X, \mathfrak{B}, μ) . Let $\tilde{\mathcal{R}}$ be the the equivalence relation on $(X \times \mathbb{R}, \mu \times e^u du)$ defined by $((x, s), (y, t)) \in \tilde{\mathcal{R}}$ if $(x, y) \in \mathcal{R}$ and $t = s - \log \delta(y, x)$, where $(x, s), (y, t) \in X \times \mathbb{R}$. By Y we denote the quotient space of $X \times \mathbb{R}$ by the measurable partition consisting of all ergodic components of $\tilde{\mathcal{R}}$. We let π be the natural surjection from $X \times \mathbb{R}$ to Y . By $\{T_t, t \in \mathbb{R}\}$, we denote the flow $T_t(x, s) = (x, s + t)$ for $(x, s) \in X \times \mathbb{R}$. By $\{F_t, t \in \mathbb{R}\}$, we denote the factor flow of $\{T_t, t \in \mathbb{R}\}$ to the quotient space Y through the factor map π , that is, $\pi T_t = F_t \pi$, for all $t \in \mathbb{R}$. The flow F_t is called the *associated flow* (or the *Poincaré flow*) of \mathcal{R} .

Remark 1.2.4. If \mathcal{R} is an equivalence relation of type III_λ , $\lambda \neq 0$, or of type II_∞ then the associated flow is transitive.

Proposition 1.2.14. *Let \mathcal{R}_1 and \mathcal{R}_2 be two ergodic equivalence relations on $(X_1, \mathfrak{B}_1, \mu_1)$ respectively $(X_2, \mathfrak{B}_2, \mu_2)$. If \mathcal{R}_1 and \mathcal{R}_2 are orbit equivalent then their corresponding flows are conjugated.*

1.2.7 The von Neumann Algebra Associated to an Equivalence Relation

Given a nonsingular standard equivalence relation \mathcal{R} on (X, \mathfrak{B}, μ) , Feldman and Moore associate to it a von Neumann algebra. We follow their description, given in [FM2].

Consider the space A of bounded Borel functions $f : \mathcal{R} \rightarrow C$ for which there exists $n \in \mathbb{N}$ such that for all $x \in X$ $\text{card}\{y, f(x, y) \neq 0\} \leq n$ and $\text{card}\{y, f(y, x) \neq 0\} \leq n$. Endowed with the operations,

$$(1) f * g(x, y) = \sum_{z \mathcal{R} x} f(x, z)g(z, y)$$

$$(2) f^*(x, y) = \overline{f(y, x)}$$

A becomes a $*$ -algebra. Let H denote the Hilbert space $L^2(\mathcal{R}, \nu)$ where ν is the right counting measure appearing in Theorem 1.2.13 and let L denote the left regular representation of A on $B(H)$, given for $f \in A$ and $\xi \in H$, by

$$L_f \xi(x, y) = \sum_{z \mathcal{R} x} f(x, z)\xi(z, y)$$

Definition 1.2.26. The operators $L_f, f \in A$ form a $*$ -algebra of operators; we denote its weak closure by $W^*(X, \mu, \mathcal{R})$.

Any function $f \in L^\infty(X, \mu)$, can be viewed as a function on \mathcal{R} , also denoted f , by defining for $(x, y) \in \mathcal{R}$,

$$f(x, y) = \begin{cases} f(x, x) & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Thus f is supported on the diagonal $\Delta \subseteq \mathcal{R}$ and belongs to A . Recall that Δ is a set of positive measure and $\nu|_\Delta$ is equivalent to μ . Let $\mathcal{A} = \{L_f, f \in L^\infty(X)\}$ denote the masa of $W^*(X, \mu, \mathcal{R})$; it is isomorphic to $L^\infty(X, \mu)$.

Remark 1.2.5. Let G be a countable group acting on (X, μ) . In [K5], Krieger associated to this dynamical system a von Neumann algebra $W^*(X, \mu, G)$ (see [Ta], vol. III, Chapter XIII). Then $W^*(X, \mu, G) \simeq W^*(X, \mu, \mathcal{R}_G)$. Moreover, if the G -action on (X, μ) is free, then $W^*(X, \mu, \mathcal{R})$ is isomorphic to Murray-von Neumann group measure space construction $L^\infty(X, \mu) \rtimes G$.

The relation between subequivalence relations and subalgebras of the von Neumann algebra associated to a standard equivalence relation is:

Theorem 1.2.15. (*[Su] or [MSS]*) *If \mathcal{R} is a standard equivalence relation on (X, μ) and N is a von Neumann subalgebra of $W^*(X, \mu, \mathcal{R})$ that contains $L^\infty(X, \mu)$, then N is of the form $W^*(X, \mu, \mathcal{S})$ for some measurable subequivalence relation \mathcal{S} of \mathcal{R} .*

We focused in the first section of this chapter on equivalence relations from topological point of view. But it is clear that a topological space carries with it also a Borel structure. Let us see how the two settings, topological and measurable, can be related.

Consider an étale equivalence relation \mathcal{R} on X and μ a σ -finite measure on X with $\text{supp}(\mu) = X$. Then $\text{Ind } \mu$ is faithful and, we can identify $C_r^*(X, \mathcal{R})$ with its image $\text{Ind } \mu(C_r^*(X, \mathcal{R}))$, ([MS],p 53). In this way, the weak closure of $C_r^*(X, \mathcal{R})$ is $W^*(X, \mu, \mathcal{R})$ the von Neumann algebra constructed by Feldman and Moore (therefore $W^*(X, \mu, \mathcal{R})$ is the weak closure of $C_c(X, \mathcal{R})$). If P is the conditional expectation from $C_r^*(X, \mathcal{R})$ onto $C(X)$, (see Remark 1.1.3) we can regard $\mu \circ P$ as a state φ_μ on $C_r^*(X, \mathcal{R})$ and $\text{Ind } \mu(C_r^*(X, \mathcal{R})) = \pi_{\varphi_\mu}(C_r^*(X, \mathcal{R}))$. We have:

Theorem 1.2.16. *With the above notation*

$$\pi_{\varphi_\mu}(C_r^*(X, \mathcal{R}))'' = W^*(X, \mu, \mathcal{R}).$$

Remark 1.2.6. If A is an AF-algebra, then we can write it as $A(X, \Gamma)$, (see Theorem 1.1.3) or $C_r^*(X, \mathcal{R})$ ([Re], III.1.15), with \mathcal{R} an AF-equivalence relation on X . If μ is a measure on X it corresponds to a state φ on $C \simeq C(X)$. Let ϕ be the state on A given by $\phi = \varphi \circ P$ where $P : A \rightarrow C$ is the conditional expectation from Theorem 1.1.3. As this conditional expectation corresponds to the conditional expectation from $C_r^*(X, \mathcal{R})$ onto $C(X)$, (see Remark 1.1.3), by the previous theorem $\pi_\phi(A)'' \simeq \pi_{\varphi_\mu}(C_r^*(X, \mathcal{R}))'' \simeq W^*(X, \mu, \mathcal{R})$.

Remark 1.2.7. If $A = \otimes M_{k_n}(\mathbb{C})$ then (see Remark 1.1.5), we can write it as $C^*(X, \mathcal{T})$ with \mathcal{T} the tail equivalence relation (see Definition 1.1.16) on $X = \prod\{0, 1, \dots, k_n - 1\}$. Let $\varphi = \text{tr}(a_n \cdot)$ be a faithful diagonal product state on A . Then $\pi_\varphi(A)''$ is a factor and we called such a factor ITPFI (see Definition 1.2.10).

Also, $\pi_\varphi(A)'' \simeq W^*(X, \mu, \mathcal{T})$ with $\mu = \otimes \mu_n$ and $\mu_n(i-1) = \varphi_n(e_{ii}^n)$ for $1 \leq i \leq k_n$ and $n \geq 1$.

We come back to measurable equivalence relation.

Theorem 1.2.17. *Let \mathcal{R} be an equivalence relation on (X, \mathfrak{B}, μ) . Then*

$W^(X, \mu, \mathcal{R})$ is factor if and only if \mathcal{R} is μ -ergodic*

In this case, $W^(X, \mu, \mathcal{R})$ is of type I (respectively of type $II_1, II_\infty, III_0, III_\lambda$, with $0 < \lambda < 1$, or type III_1) if and only if \mathcal{R} is of type I (respectively of type $II_1, II_\infty, III_0, III_\lambda$, with $0 < \lambda < 1$, or type III_1)*

Zimmer ([Z2], [Z3], [Z4]) initiated the study of amenability in the case of discrete group actions and countable equivalence relations. He introduced amenability through an adaptation of the classical fixed-point property and proved that $W^*(X, \mu, \mathcal{R})$ is an injective von Neumann algebra if and only if \mathcal{R} is amenable. In the discrete case, Zimmer proved the following equivalent characterization:

Definition 1.2.27. Let \mathcal{R} be a nonsingular equivalence relation on (X, \mathfrak{B}, μ) . A *left invariant mean* on \mathcal{R} is a map $P : L^\infty(\mathcal{R}, \nu_r) \rightarrow L^\infty(X, \mu)$ with the following properties: $P(1) = 1$, $P(f) \geq 0$ whenever $f \geq 0$, and $P(L_V f) = (Pf)V^{-1}$ for every $f \in L^\infty(\mathcal{R}, \nu_r)$, $V \in [\mathcal{R}]$, where $L_V f(x, y) = f(V^{-1}x, y)$. The equivalence relation \mathcal{R} is *amenable* if it admits a left invariant mean.

There are several other equivalent definitions of measurable equivalence relations (see for example [AR] or [AEG]), the deepest result in this sense being obtained by Connes-Feldman-Weiss:

Theorem 1.2.18. [Dye], [CFW] *Let \mathcal{R} be a nonsingular equivalence relation on (X, \mathfrak{B}, μ) . The following are equivalent:*

- (1) \mathcal{R} is amenable;

- (2) \mathcal{R} is hyperfinite, i.e., there exists an increasing sequence $\{\mathcal{R}_n, n \geq 1\}$ of finite subrelations of \mathcal{R} such that $\mathcal{R}(x) = \cup \mathcal{R}_n(x)$ for μ -a.e. $x \in X$;
- (3) $\mathcal{R} = \mathcal{R}_T = \{(x, T^n x), x \in X, n \in \mathbb{Z}\}$ for a nonsingular automorphism T of (X, \mathfrak{B}, μ) .

Before stating the next theorem let us recall (see [CT]) that if \mathcal{R} is an ergodic amenable equivalence relation of type II_∞ or III on (X, μ) , then its associated flow corresponds to the flow of weights of the factor $W^*(X, \mu, \mathcal{R})$.

Moreover if M is a von Neumann factor, with separable predual, then by results of Connes' for the type II and type III_λ , with $\lambda \neq 1$ and of Haagerup for type III_1 , M is injective if and only if it is AFD.

Krieger's result can be stated as follows.

Theorem 1.2.19. [K4] For $i = 1, 2$, let \mathcal{R}_i be an ergodic, amenable equivalence relation of type II_∞ or III. The following are equivalent:

- (1) \mathcal{R}_1 and \mathcal{R}_2 are orbit equivalent;
- (2) $W^*(X_1, \mu_1, \mathcal{R}_1)$ and $W^*(X_2, \mu_2, \mathcal{R}_2)$ are isomorphic;
- (3) the associated flows of \mathcal{R}_1 and \mathcal{R}_2 are conjugate;
- (4) the flows of weights of $W^*(X_1, \mu_1, \mathcal{R}_1)$ and $W^*(X_2, \mu_2, \mathcal{R}_2)$ are conjugate.

Remark 1.2.8. Any amenable equivalence relation \mathcal{R} on (X, μ) is orbit equivalent to tail equivalence \mathcal{T} on $\prod\{0, 1\}$ with a \mathcal{T} -ergodic measure μ . However, μ is not necessary a measure of product type (see [KW], for example).

In 1980, Connes and Woods introduced a new property of ergodic actions called approximate transitivity to characterize among AFD von Neumann factors the Araki-Woods or ITPFI factors.

Definition 1.2.28. An action of a group G on a measure space (X, μ) is *approximately transitive (AT)* if for all $\epsilon > 0$ and any sequence f_1, f_2, \dots, f_n of functions in $L^1(X, \mu)_+$, the space of positive integrable functions, there exists a function f in $L^1(X, \mu)_+$, finitely many elements $g_{i,j}$ of G and constants $\lambda_{i,j} \geq 0$ such that

$$\left\| f_i - \sum_j \lambda_{i,j} f \circ g_{i,j} \frac{d\mu \circ g_{i,j}}{d\mu} \right\|_1 < \epsilon$$

for each i . We will write $L_g(f)$ for the function defined by

$$L_g(f)(x) = f(g(x)) \frac{d\mu \circ g}{d\mu}(x)$$

Note that for positive functions f , $L_g(f)$ is an L^1 norm-preserving operator.

Example 1.2.6. 1) A rank one transformation is AT. In particular an odometer is AT.

2) A pure point spectrum \mathbb{R} -flow is AT.

Definition 1.2.29. An amenable ergodic equivalence relation \mathcal{R} on a measured space (X, \mathfrak{B}, μ) is of *product type* if \mathcal{R} is orbit equivalent to the tail equivalence on an infinite product of finite sets with an infinite product measure.

Theorem 1.2.20. [CW] *Let \mathcal{R} an ergodic amenable equivalence relation on a measured space (X, \mathfrak{B}, μ) . The following are equivalent:*

- (1) *the associated flow of \mathcal{R} is AT;*
- (2) *the factor $W^*(X, \mu, \mathcal{R})$ is isomorphic to an ITPFI factor;*
- (3) *\mathcal{R} is of product type.*

Remark 1.2.9. An amenable ergodic equivalence relation of product type cannot always be realized as tail equivalence on $X = \prod\{0, 1\}$ (see also Example 1.2.1).

Chapter 2

Actions of Xerox Type and Fixed Point Factors

Let G be a compact group and $\pi : G \rightarrow \mathcal{U}(M_k(\mathbb{C}))$ be a fixed unitary diagonal representation of G . Let $A = \otimes M_k(\mathbb{C})$ the k^∞ -UHF algebra, $\alpha(g) = \otimes \text{Ad } \pi(g)$ the xerox action of π and A^G the fixed point algebra under this action. In this chapter we study restrictions of faithful product diagonal states $\varphi = \otimes \varphi_n$ of the k^∞ -UHF algebra A to A^G . We give necessary and sufficient conditions for $\pi_{\varphi^G}(A^G)'' = GNS(A^G, \varphi|_{A^G})$ to be factor and we also give necessary and sufficient conditions for $\pi_{\varphi^G}(A^G)''$ to be type II_1, II_∞ or type III . In order to prove this, we approach the problem in a few ways.

We can identify A^G (Theorem 1.1.8) with $A(X, \Gamma)$ where $X = \prod\{0, 1, 2, \dots, k-1\}$ and Γ is a group of homeomorphisms on X ; by [Re], III.1.1.15, we also can see A^G as $C_r^*(X, \mathcal{R})$ where \mathcal{R} is the equivalence relation induced by Γ . This helps us to decide when $\pi_{\varphi^G}(A^G)''$ is factor. Also we can identify $\pi_{\varphi^G}(A^G)''$ with the fixed point algebra $N = M^G$ under the action induced by G on the ITPFI factor $M = GNS(A, \varphi)$. By using Connes' invariant T we can decide whether N is semifinite. We prove that any ITPFI₃ of type III can be obtained as the fixed point algebra under the standard action of T^2 . To obtain this result, we use the the associated flow of an equivalence relation.

In Section 2.3, we study xerox actions of finite groups, we analyze the standard action of T^2 in Section 2.4 and in the last section we show that the problem can be solved for arbitrary xerox actions induced by diagonal representations of compact groups.

2.1 The T-set for Fixed Point Factors under Actions of Xerox Type

In this section, A will denote the k^∞ -UHF algebra.

Let T^k be the k -dimensional torus. We denote by \mathbf{t} an element $(t_1, t_2, \dots, t_k) \in T^k$ where $t_j \in \mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$. We denote the identity of T^k by $\mathbf{1}$. Let H be $\prod_{n \geq 1} T^k$. An element $(\mathbf{t}^1, \mathbf{t}^2, \dots, \mathbf{t}^n, \dots)$ in $H = \prod_{n \geq 1} T^k$ will be denoted \mathbf{h} and the identity of H by \mathbf{e} . We further consider π to be the diagonal representation of T^k on $\mathcal{U}(M_k(\mathbb{C}))$, given by:

$$\pi(\mathbf{t}) = u_{\mathbf{t}} = \text{diag}(t_1, t_2, \dots, t_k)$$

and the action $\varrho : H \rightarrow \text{Aut}(A)$ of H on A given by:

$$\varrho_{\mathbf{h}} = \bigotimes_{n \geq 1} \text{Ad } \pi(\mathbf{t}^n)$$

We remark that $H_0 := \bigoplus_{n \geq 1} T^k$ is a dense subgroup of H . If $\mathbf{h} = (\mathbf{t}^1, \mathbf{t}^2, \dots, \mathbf{t}^n, \mathbf{1}, \dots) \in H_0$, we denote the unitary $\bigotimes_{i=1}^n u_{\mathbf{t}^i}$ by $u_{\mathbf{h}}$. With these notations we have:

Proposition 2.1.1. *ϱ is a continuous action of H on A .*

Proof. It is enough to prove the continuity at \mathbf{e} . If $x \in A_n = \bigotimes M_k(\mathbb{C}) \subset A$, there exists a neighborhood $V = V_1 \times V_2 \times \dots \times V_n \times T^k \times \dots$ of \mathbf{e} (V_i are neighborhoods of $\mathbf{1} \in T^k$) such that

$$\|\varrho_{\mathbf{h}}(x) - x\| < \epsilon \text{ for all } \mathbf{h} \in V \quad (2.1.1)$$

Indeed, this is clear when $x \in A_n$ is of the form $x_1 \otimes x_2 \otimes \dots \otimes x_n \otimes \mathbf{1} \otimes \dots$. As any element in A_n is a linear combination of elements of this form, we obtain that (2.1.1) is true for any $x \in A_n$. For an arbitrary $x \in A$ there exists $N \geq 1$ and $x_N \in A_N$ such

that $\|x - x_N\| < \epsilon$. As we saw before, there exists a neighborhood U of \mathbf{e} such that $\|\varrho_{\mathbf{h}}(x_N) - x_N\| < \epsilon$ for all $\mathbf{h} \in U$. Then

$$\|\varrho_{\mathbf{h}}(x) - x\| \leq \|\varrho_{\mathbf{h}}(x - x_N)\| + \|\varrho_{\mathbf{h}}(x_N) - x_N\| + \|x - x_N\| < 3\epsilon.$$

Hence, ϱ is continuous. \square

On A we consider the faithful diagonal product state $\varphi = \otimes \varphi_n$ with $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$.

In all this section, π_φ is the GNS representation corresponding to A and the product state φ . The actions induced by α and ϱ on $\pi_\varphi(A)''$ are denoted also by α and ϱ (see Theorem 1.2.11).

Theorem 2.1.2. *If α is a continuous action of a compact group G on A that leaves φ invariant, then $(\pi_\varphi(A)'')^\alpha = \pi_\varphi(A^\alpha)''$.*

Proof. Let $E : \pi_\varphi(A)'' \rightarrow (\pi_\varphi(A)'')^\alpha$ the conditional expectation given by $(E(x)_\varsigma | \eta) = \int_G (\alpha_g(x)_\varsigma | \eta) dg$ for $\varsigma, \eta \in H_\varphi$. Let $x \in (\pi_\varphi(A)'')^\alpha$. There exists $(x_n)_{n \geq 1}$ with $x_n \in A_n = \bigotimes_{i=1}^n M_k(\mathbb{C})$ such that $\pi_\varphi(x_n) \rightarrow x$ strongly. Then $E(\pi_\varphi(x_n)) = \int_G \alpha_g(\pi_\varphi(x_n)) dg \in \pi_\varphi(A^n)^\alpha \subset \pi_\varphi(A)^\alpha$ and $(E(\pi_\varphi(x_n))_\varsigma | \eta) \rightarrow (E(x)_\varsigma | \eta)$ for all $\varsigma, \eta \in H_\varphi$. Consequently, $E(x) \in \pi_\varphi(A^\alpha)''$. This shows that $(\pi_\varphi(A)'')^\alpha \subset \pi_\varphi(A^\alpha)''$. As the other inclusion is obvious we conclude that $(\pi_\varphi(A)'')^\alpha = \pi_\varphi(A^\alpha)''$. \square

Proposition 2.1.3. *If u is a unitary of A^G and β is an automorphism of $\pi_\varphi(A)''$ such that β is the identity on $\pi_\varphi(A^G)''$, then $\beta \circ \text{Ad } \pi_\varphi(u)(x) = \text{Ad } \pi_\varphi(u) \circ \beta(x)$ for all $x \in \pi_\varphi(A)''$.*

Proof. If $x \in \pi_\varphi(A)''$ then

$$\beta \circ \text{Ad } \pi_\varphi(u)x = \beta(\pi_\varphi(u))\beta(x)\beta(\pi_\varphi(u^*)) = \pi_\varphi(u)\beta(x)\pi_\varphi(u^*) = \text{Ad } \pi_\varphi(u) \circ \beta(x)$$

\square

From now on, G will be a compact group having a diagonal representation on $M_k(\mathbb{C})$ and α will be the xerox action induced by a such representation. We denote

by φ^G the restriction of the state φ on A to the fixed point algebra A^G and by π_{φ^G} the GNS representation associated to φ^G and A^G . We also denote $\pi_{\varphi}(A^G)'' = (\pi_{\varphi}(A)'')^{\alpha}$ by N .

Example 2.1.1. On the k^{∞} -UHF algebra $A = \otimes M_k(\mathbb{C})$, we consider the standard xerox action of the $(k-1)$ dimensional torus T^{k-1} :

$$\begin{aligned} \alpha : T^{k-1} &\rightarrow \text{Aut}(A); \quad \alpha(t_1, \dots, t_{k-1}) := \otimes \text{Ad } \pi(t_1, t_2, \dots, t_{k-1}), \text{ where} \\ \pi(t_1, \dots, t_{k-1}) &:= \text{diag}(1, e^{it_1}, e^{it_2}, \dots, e^{it_{k-1}}); \quad t_i \in [0, 2\pi) \end{aligned}$$

Proposition 2.1.4. *With the above notation ϱ commutes with α and $(\pi_{\varphi}(A)'')^{\varrho} = \pi_{\varphi}(C)''$ where C is the standard m.a.s.a of A .*

Proof. If $\mathbf{h} \in H_0$ then $u_{\mathbf{h}} \in A^G$ and $\varrho_{\mathbf{h}}$ is an inner automorphism $\text{Ad}\pi_{\varphi}(u_{\mathbf{h}})$ on $\pi_{\varphi}(A)''$. But then, by Proposition 2.1.3, $\varrho_{\mathbf{h}}\alpha_g = \alpha_g\varrho_{\mathbf{h}}$ for all $\mathbf{h} \in H_0$ and $g \in G$. Since H_0 is dense in H we have that $\varrho_{\mathbf{h}}\alpha_g = \alpha_g\varrho_{\mathbf{h}}$ for all $\mathbf{h} \in H$ and $g \in G$, which tells us that $\varrho_{\mathbf{h}}$ commutes with the action α for all \mathbf{h} in H . As H is a compact group and the state φ is invariant under the action of H , by Theorem 2.1.2 we deduce that $(\pi_{\varphi}(A)'')^{\varrho} = \pi_{\varphi}(A^{\varrho})''$. But $A^{\varrho} = C$ where C is the standard masa in A (and also a masa in A^G). Hence $(\pi_{\varphi}(A)'')^{\varrho} = \pi_{\varphi}(C)''$. \square

Assume that φ^G , the restriction of the product state φ to A^G , is a factor state. By Theorem 1.1.3, $A^G \simeq A(X, \Gamma)$ where $X = \prod_{n=1}^{\infty} \{0, 1, 2, \dots, k-1\}$ and Γ is a group of homeomorphisms on X which corresponds under the isomorphism above to a group of automorphisms of C (C is the standard masa of A and also masa of A^G) all of them of the form $\text{Ad } u$, with u in a group of unitaries of A^G denoted by U . However $\pi_{\varphi^G}(A^G)''$ is a factor and therefore Γ acts ergodically on X . Define

$$\mathcal{S} := \langle \varrho_t, t \in H; \text{Ad } \pi_{\varphi}(u), u \in U \rangle \subset \text{Aut}\pi_{\varphi}(A)''.$$

As the action α commutes with ϱ_t for all $t \in H$ and it commutes with every $\text{Ad } \pi_{\varphi}(u)$ with $u \in U$, the action α commutes with every element in \mathcal{S} .

We have an isomorphism from $\pi_\varphi(C)''$ onto $L^\infty(X, \mu)$, where $\mu = \otimes \mu_n$ is the product measure on X induced by the state φ , given by $\mu_n(i-1) = \varphi_n(e_{ii}^n)$, for $1 \leq i \leq n$ and $n \geq 1$. Hence $\pi_\varphi(C)'' \simeq L^\infty(X, \mu)$. If $x \in (\pi_\varphi(A)'')^S$, then x must be in $(\pi_\varphi(A)'')^\varrho$ i.e., x is in $\pi_\varphi(C)''$ and by the above isomorphism, it corresponds to a function f in $L^\infty(X, \mu)$. For any u in U we must also have $\text{Ad } \pi_\varphi(u)x = x$. As x was identified with a function in $L^\infty(X, \mu)$ this means $\gamma(f) = f$ for all γ in Γ . As Γ acts ergodically on X , we obtain that f is a scalar and consequently x is a scalar. This shows that $(\pi_\varphi(A)'')^S$ reduces to scalars. We just proved:

Proposition 2.1.5. *With the above notation, if G a compact group acting on A by a xerox action induced by a diagonal representation π of G on $\mathcal{U}(M_k(\mathbb{C}))$ and φ^G is a factor state, then \mathcal{S} acts ergodically on $\pi_\varphi(A)''$ and commutes with the action α .*

Proposition 2.1.6. *With the above notation, $N = (\pi_\varphi(A)'')^\alpha$ is a factor and*

$$\pi_{\varphi^G}(A^G)'' \simeq (\pi_\varphi(A)'')^\alpha.$$

Proof. If x is in $\pi_\varphi(A)''$ and commutes with $(\pi_\varphi(A)'')^\alpha$, then x commutes with all $\pi_\varphi(u_{\mathbf{h}})$ with $\mathbf{h} \in H_0$. As before, this means that x commutes with $\varrho_{\mathbf{h}}$ with \mathbf{h} in H_0 and then, as we had seen, x is fixed under the action ϱ and therefore x is in $\pi_\varphi(C)''$. But x must also commute with all $\pi_\varphi(u)$ with $u \in U$ and so, as before, x is a scalar multiple of the identity. We showed that any x in $\pi_\varphi(A)''$ which commutes with $(\pi_\varphi(A)'')^\alpha$ is a scalar. So the relative commutant of $(\pi_\varphi(A)'')^\alpha$ in $\pi_\varphi(A)''$ reduces to scalars. This implies that $(\pi_\varphi(A)'')^\alpha$ is a factor. Moreover, $\pi_{\varphi^G}(A^G)'' \simeq (\pi_\varphi(A)'')^\alpha$. Indeed, if $H_G = \pi(A^G)\xi$, let $e : H_\varphi \rightarrow H_G$ be the orthogonal projection on H_G . Then H_G is stable under e and therefore $e \in \pi(A^G)'$. As $\pi_\varphi(A^G)''$ is a factor, we have that $z(e) = 1$. By construction, $\pi_{\varphi^G}(A^G)'' \simeq \pi_\varphi(A^G)''_e$. Therefore $\pi_{\varphi^G}(A^G)'' \simeq \pi_\varphi(A^G)''_e \simeq (\pi_\varphi(A)'')^\alpha$. \square

Proposition 2.1.7. *Let $M = \otimes(M_k(\mathbb{C}), \varphi_n) = \pi_\varphi(A)''$, \mathcal{S} and α be as before such that N is properly infinite. Then for any $\beta \in \text{Aut}(M)$, such that $\beta(x) = x$ for all $x \in N$, there exists $g \in G$ such that $\beta = \alpha_g$.*

Proof. If $\beta \in \text{Aut}(M)$ and $\beta(x) = x, \forall x \in M^\alpha$, then β commutes with \mathcal{S} . Indeed, we already see in Proposition 2.1.3 that β commutes with all $\text{Ad}\pi_\varphi(u), u$ in A^G , and it commutes with all ϱ_t with $t \in H_0$. As H_0 is dense in H we have that β commutes with ϱ_t for all t in H . We conclude that β commutes with \mathcal{S} . By Theorem 1.2.12, we obtain the conclusion. \square

We recall that $T(N)$ denotes the T-set of N (Definition 1.2.11).

Proposition 2.1.8. *With the same notation as above, and N properly infinite*

$$t \in T(N) \text{ if and only if } \exists v \in U(N) \text{ and } g \in G \text{ such that } \sigma_t^\varphi = \text{Ad } v \circ \alpha_g$$

where σ_t^φ is the modular group of automorphisms of $M = \pi_\varphi(A)''$.

Proof. As the state is invariant for the action α , $\sigma_t^\varphi|_N$ is the modular group of automorphism of N . If $t \in T(N)$, then $\exists v \in U(N)$ such that $\sigma_t^\varphi|_N = \text{Ad } v$ or equivalently $\text{Ad } v^* \circ \sigma_t^\varphi|_N(x) = x$ for all $x \in N$. Let $\beta := \text{Ad } v^* \circ \sigma_t^\varphi$. We have that $\beta \in \text{Aut}(M)$ satisfies all conditions of Proposition 2.1.7. Hence, there exists $g \in G$ such that $\beta = \text{Ad } v^* \circ \sigma_t^\varphi = \alpha_g$. Therefore $\sigma_t^\varphi = \text{Ad } v \circ \alpha_g$. Conversely, if there exists $v \in U(N)$ such that $\sigma_t^\varphi = \text{Ad } v \circ \alpha_g$, then $\sigma_t^\varphi \circ \alpha_{g^{-1}} = \text{Ad } v$. For $x \in N$, $\alpha_{g^{-1}}(x) = x$ and then $\sigma_t^\varphi(x) = \text{Ad } v(x)$ i.e., $t \in T(N)$. \square

2.2 The Standard Action of the Torus

Let us first consider the standard action of the 1-dimensional torus on $A = \otimes M_2(\mathbb{C})$ given by:

$$\begin{aligned} \alpha & : T \rightarrow A; \alpha(t) := \otimes \text{Ad } \pi(t) \\ \pi(t) & : = \text{diag}(1, e^{it}); t \in [0, 2\pi). \end{aligned}$$

On A we consider the product state $\varphi(\cdot) = \otimes \text{tr}(h_n \cdot)$ with $h_n = \text{diag}(a_n, b_n), a_n \in (0, 1)$ and $b_n = 1 - a_n$. This was studied by Baker and Powers [BP1]. By Proposition 1.1.1,

A^T is an AF-algebra. We can describe A^T using Stratila-Voiculescu's procedure of diagonalization (see Theorem 1.1.7 1.1.3). For this let $A^T = \overline{\cup A_n^T}$ be the fixed point algebra under the action α . Let C_n be the m.a.s.a of A_n^T generated by the projections $e_{x_1x_1} \otimes e_{x_2x_2} \otimes \cdots \otimes e_{x_nx_n}$ with $x_i \in \{1, 2\}$ for all $1 \leq i \leq n$. Each A_n^T can be written as a direct sum, $A_n^T = \bigoplus_{k=0}^n A_n^k$ where A_n^k is isomorphic to $M_{n_k}(\mathbb{C})$ and $n_k = \binom{n}{k}$. For every $0 \leq k \leq n$, the set of minimal projections of $A_n^k \cap C_n$, is:

$$\{e_{x_1x_1} \otimes e_{x_2x_2} \otimes \cdots \otimes e_{x_nx_n}; x_1 + x_2 + \cdots + x_n = k + n\}.$$

Let $C = \overline{\cup C_n}$. We have that C is a masa in A^T and $C \simeq C(X)$ with $X = \prod_{n \geq 1} \{0, 1\}$. We have $C_n \subset C \simeq C(X)$ and we identify an element $e_{x_1x_1} \otimes e_{x_2x_2} \otimes \cdots \otimes e_{x_nx_n}$ of C_n with the function $\chi_{E(y_1, y_2, \dots, y_n)}$ in $C(X)$, where $E(y_1, y_2, \dots, y_n) = \{x \in X; y_i = x_i - 1 \text{ for all } 1 \leq i \leq n\}$. Following Stratila-Voiculescu's procedure of diagonalization we can find a group of unitaries, U_n , acting on $C_n \subset C$ by $\text{Ad } u$, $u \in U_n$. This group can be written explicitly and it corresponds to a group of homeomorphisms Γ_n acting on X . In fact Γ_n consists of all homeomorphisms γ on X , such that for any $x \in E(x_1, x_2, \dots, x_n)$, we have $\gamma(x_1x_2 \dots x_nx_{n+1} \dots) = (y_1y_2 \dots y_nx_{n+1} \dots)$ with $\sum_{i=1}^n (x_i - y_i) = 0$. We let $\Gamma = \cup \Gamma_n$ and by Theorem 1.1.3, we have $A^T = A(X, \Gamma)$.

On the other hand, by [Re] III.1.15, we can express A^T as $C_r^*(X, \mathcal{R}_\infty)$ where \mathcal{R}_∞ is an AF equivalence relation on X , which is the equivalence relation induced by the group Γ above. In fact each Γ_n induces an equivalence relation on X denoted by \mathcal{R}_n and $\mathcal{R}_\infty = \cup \mathcal{R}_n$. We have:

$$x\mathcal{R}_ny \text{ if and only if } x_i = y_i \text{ for } i > n \text{ and } \sum_{i=1}^n (x_i - y_i) = 0$$

and

$$x\mathcal{R}_\infty y \text{ if and only if } xTy \text{ and } \sum (x_i - y_i) = 0$$

We denote this equivalence relation by \mathcal{R}_∞ because it is also induced by the following action of the group S_∞ of all finite permutations of $\mathbb{N}^* = \mathbb{N} - \{0\}$ acting on X by:

$$\sigma(x_1, x_2, \dots) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots), \quad \sigma \in S_\infty.$$

The topology of \mathcal{R}_∞ is defined as follows: give each \mathcal{R}_n the product topology (as a subspace of $X \times X$) and say that a subset $U \subseteq \mathcal{R}_\infty$ is open if and only if $U \cap \mathcal{R}_n$ is open for every n . This defines a topology on \mathcal{R}_∞ called the inductive limit topology.

For A^T , let $B = (V, E)$ be the corresponding Bratteli diagram which is in this case the infinite Pascal triangle. For $n \geq 0$, let $V_n = \{(n, 0), (n, 1), \dots, (n, n)\}$ (we count the vertices of each level of the diagram from the left to the right) and for every $n \geq 1$ $E_n = \{(n, k, k), (n, k, k+1); k = 0, \dots, n-1\}$ (for $j = k$ or $j = k+1$, (n, k, j) denotes the edge from the vertex $(n-1, k) \in V_{n-1}$ to the vertex $(n, j) \in V_n$). The path space of the Bratteli diagram of A^T is:

$$X^T = \{((1, k_0, k_1), (2, k_1, k_2), \dots, (n, k_{n-1}, k_n) \dots); (n, k_{n-1}, k_n) \in E_n \text{ for } n \geq 1\} \subseteq \prod_{n \geq 1} E_n$$

On X^T we consider the tail equivalence relation and we denote it by \mathcal{R}^T . With the notations introduced in Subsection 1.1.5, $\mathcal{R}^T = \cup \mathcal{R}_n^T$. If $X = \prod_{n \geq 1} \{0, 1\}$, let

$$h : X^T \rightarrow X$$

$$h((1, k_0, k_1), (2, k_1, k_2), \dots, (n, k_{n-1}, k_n) \dots) = (k_1 - k_0, k_2 - k_1, \dots, k_{n+1} - k_n \dots)$$

Then h is a homeomorphism and moreover we have that $x, y \in X^T$ are tail equivalent if and only if $h(x)$ and $h(y)$ are \mathcal{R}_∞ equivalent. This means that $h \times h(\mathcal{R}^T) = \mathcal{R}_\infty$. Clearly, $h \times h : X^T \times X^T \rightarrow X \times X$ is a homeomorphism and $h \times h(\mathcal{R}_n^T) = \mathcal{R}_n$. We show that in fact $h \times h : \mathcal{R}^T \rightarrow \mathcal{R}_\infty$ is a homeomorphism.

Let $U \subseteq \mathcal{R}^T$ be an open set. Then $U \cap \mathcal{R}_n^T$ is an open subset of \mathcal{R}_n^T for all n . Hence it exists an open set $G_n \subseteq X^T \times X^T$ such that $U \cap \mathcal{R}_n^T = G_n \cap \mathcal{R}_n^T$. As $h \times h(\mathcal{R}_n^T \cap U) = h \times h(\mathcal{R}_n^T \cap G_n) = \mathcal{R}_n \cap h \times h(G_n) = \mathcal{R}_n \cap h \times h(U)$, and $h \times h$ is homeomorphism we get that $h(U) \cap \mathcal{R}_n$ is open in \mathcal{R}_n for all $n \geq 1$. Hence $h(U)$ is open in \mathcal{R}_∞ . Similarly, if $U \subseteq \mathcal{R}_\infty$ is open we can prove that $(h \times h)^{-1}(U)$ is open in \mathcal{R}^T . In conclusion $h \times h$ is a homeomorphism from \mathcal{R}^T onto \mathcal{R}_∞ . Hence, (X, \mathcal{R}_∞) and (X^T, \mathcal{R}^T) are isomorphic and therefore, we can identify the path space of the Bratteli diagram X^T , with X and the tail equivalence on X^T can be identified with the equivalence relation \mathcal{R}_∞ on X (see Figure 2.1)

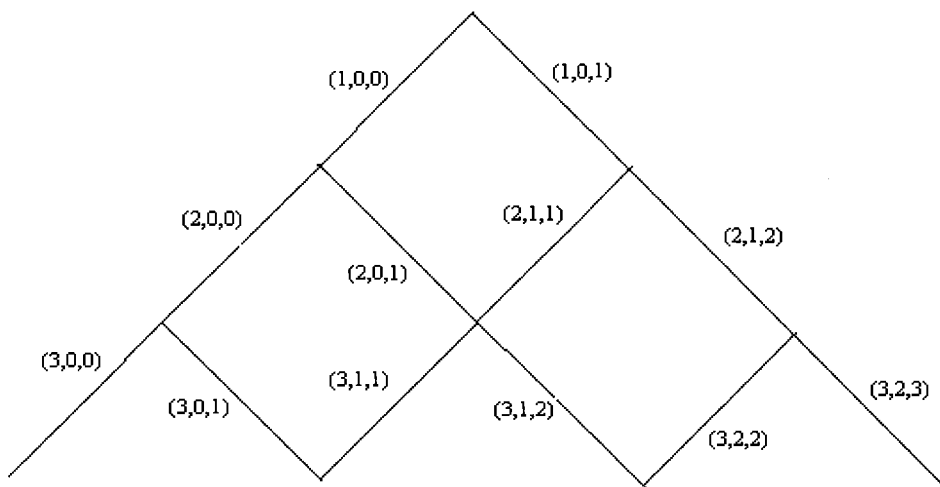


Figure 2.1: The Bratteli diagram of A^T and the path space X^T

We have in fact that $A^T \simeq A(X, \Gamma) \simeq C_r^*(X, \mathcal{R}_\infty)$ and $\pi_{\phi^T}(A^T)'' \simeq W^*(X, \mu, S_\infty) \simeq W^*(X, \mu, \mathcal{R}_\infty)$ (see Theorem 1.2.16). For other situations of this type it is easier to describe the equivalence relation than to effectively find the group and therefore it will be enough to prove that the equivalence relation is ergodic. The factoriality is equivalent to the fact that \mathcal{R}_∞ is μ -ergodic and this results either from [AP] or from [SV1].

Theorem 2.2.1. $\pi_{\phi^T}(A^T)''$ is a factor if and only if $\sum a_n(1 - a_n) = \infty$.

To determine the type of the factor in this case we can use either another result of Stratila-Voiculescu [SV2] or [BP1], namely,

Theorem 2.2.2. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers, $0 < a_n < 1$, for all $n \geq 1$. If $\mu = \otimes \mu_n$ is the product measure on $X = \prod \{0, 1\}$ with $\mu_n(0) = a_n$, $\mu_n(1) = b_n = 1 - a_n$ and $\sum_{n=1}^{\infty} a_n(1 - a_n) = \infty$, then

1) there is a finite invariant measure with respect to \mathcal{R}_∞ equivalent with μ if and only if $\sum (a_n - a)^2 < \infty$ for some $a \in (0, 1)$,

2) there is a σ -finite measure invariant with respect to \mathcal{R}_∞ equivalent with μ if and only if $\sum a_n(1 - a_n)(a - a_n)^2 < \infty$ for some $a \in (0, 1)$,

3) there is no σ -finite invariant measure with respect to \mathcal{R}_∞ and equivalent with μ if and only if $\sum a_n(1 - a_n)(a - a_n)^2 = \infty$ for all $a \in (0, 1)$.

We can obtain here an important information that helps us to classify factors that appear as fixed point algebras under xerox actions. As seen in the previous section, we can identify our factor with $N = (\pi_\phi(A))''^T$. By [BG] and Theorem 2.1.8, when N is properly infinite we have that $t \in T(N)$ if and only if there exists $s \in [0, 2\pi)$ such that

$$\sum_n a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) < \infty.$$

As a result,

Proposition 2.2.3. *With the above notation, if N is properly infinite, by Theorem 1.2.4, N is of type III if and only if $\exists t \in \mathbb{R}$ such that for any $s \in [0, 2\pi)$ we have*

$$\sum a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) = \infty.$$

2.3 Xerox Actions Induced by Diagonal Representations of Finite Groups

The following theorem will treat the case when $G = \mathbb{Z}_2$. In the second part of the section we explain how the problem can be solved for actions of other finite groups.

Theorem 2.3.1. *Let $A = \otimes M_2(\mathbb{C})$ be the 2^∞ -UHF algebra and let φ be the product state $\varphi(\cdot) = \otimes \text{tr}(h_n \cdot)$, where $h_n = \text{diag}(a_n, b_n)$, $a_n + b_n = 1$, $a_n, b_n > 0$. Let $A^{\mathbb{Z}_2}$ be the fixed point algebra under the xerox action of \mathbb{Z}_2 induced by the involutive automorphism*

$$\alpha = \otimes \text{Ad} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$ is factor if $\sum a_n b_n = \infty$.

We have that $C \simeq C(X)$ with $X = \prod\{0, 1\}$, is a masa in $A^{\mathbb{Z}_2}$ and following [SV1], we also have a group of homeomorphism Γ such that $A^{\mathbb{Z}_2} = A(X, \Gamma)$, where Γ is an increasing chain of finite subgroups Γ_n which we are not going to define here. It is easier to describe the equivalence relation \mathcal{R} induced by Γ on X than to describe Γ . As in Section 2.2, we have that $A^{\mathbb{Z}_2} \simeq C_r^*(X, \mathcal{R})$ and the equivalence relation \mathcal{R} is:

$$x\mathcal{R}y \text{ if and only if } x\mathcal{T}y \text{ and } \sum x_n - y_n = 0 \pmod{2}$$

where \mathcal{T} is the tail equivalence on X . Also, the equivalence \mathcal{R} is isomorphic to the tail equivalence on the Bratteli diagram, as shown in Figure 2.2. The state φ induces a probability measure $\mu = \otimes \mu_n$ on X with $\mu_n(0) = a_n$ and $\mu_n(1) = b_n$ for all $n \geq 1$. We have that $\pi_{\varphi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})'' \simeq W^*(X, \mu, \mathcal{R})$, (Theorem 1.2.16). The fact that $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$ is factor then follows from Proposition 2.3.4 and Theorem 1.2.17.

Definition 2.3.1. We say that x and y have *infinitely many encounters* if there are infinitely many n such that $\sum_{i=1}^n (x_i - y_i) = 0 \pmod{2}$.

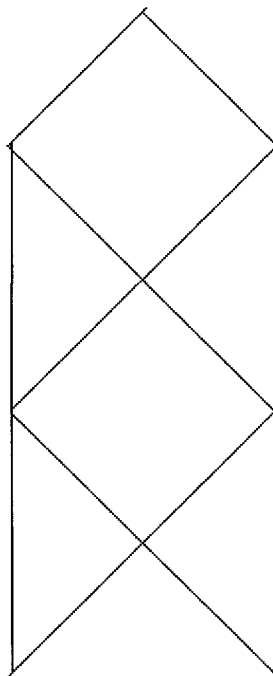
Lemma 2.3.2. *If $A \subseteq X$ with $\mu(A) > 0$ and $E_x^n = \{y \in X; x_i = y_i, 1 \leq i \leq n\}$ then*

$$\lim_{n \rightarrow \infty} \frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} = 1$$

for almost all $x \in A$

Proof. We denote by \mathfrak{B} the σ -algebra generated by all elementary cylinders of X . If we denote \mathfrak{B}_n the σ -algebra generated by $\{C_j; 1 \leq j \leq 2^n\}$, the set of elementary cylinders of length n , then the σ -algebra \mathfrak{B} on the product space $X = \prod\{0, 1\}$ is nothing else than $\sigma\{B_n; n \geq 1\}$. We have

$$E(\chi_A | B_n) = \sum_{j=1}^{2^n} \frac{\mu(A \cap C_j)}{\mu(C_j)} \chi_{C_j}$$

Figure 2.2: The Bratteli diagram of $A^{\mathbb{Z}_2}$

where $E(\chi_A|B_n)$ denotes the conditional expectation of χ_A with respect to the σ -algebra \mathfrak{B}_n . By [B], Theorem 35.6, we have that $E(\chi_A|B_n) \rightarrow E(\chi_A|B) = \chi_A$ almost everywhere when $n \rightarrow \infty$ and therefore

$$\lim_{n \rightarrow \infty} \frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} = \lim_{n \rightarrow \infty} E(\chi_A|B_n)(x) = 1$$

for almost all $x \in A$. □

Proposition 2.3.3. *With the above notation, the equivalence relation \mathcal{R} on X , is μ -ergodic.*

Proof. Consider the following sets in $X \times X$:

$$C_n = \{(x, y) \in X \times X; x_n = 0, y_n = 1\}.$$

Then C_n are independent events in $X \times X$ and $\sum \mu \times \mu(C_n) = \sum a_n b_n = \infty$. By Borel Cantelli lemma we get that for almost all $(x, y) \in X \times X$, $x_n = 0$ and $y_n = 1$ infinitely many often. But this means that x and y have infinitely many encounters. Let A be a \mathcal{R} -invariant set with $0 < \mu(A) < 1$. Assume that A does not have measure 1. Hence $A^c = X - A$ is also a \mathcal{R} -invariant set of nonnegative measure. Choose now $x \in A$ and $y \in A^c$ with infinitely many encounters and $N > 1$ such that for any $n > N$

$$\frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} > \frac{1}{2},$$

$$\frac{\mu(A^c \cap E_y^n)}{\mu(E_y^n)} > \frac{1}{2},$$

This is possible by Lemma 2.3.3. As x, y intersect infinitely many often, there exists $n > N$ such that $\sum_{i=1}^n (x_i - y_i) = 0 \pmod{2}$. There exists $\phi \in [\mathcal{R}]$ such that $\phi E_x^n = E_y^n$. We have:

$$\begin{aligned} \mu(A \cap E_y^n) &= \int_{A \cap E_y^n} d\mu = \int_{\phi(A \cap E_x^n)} d\mu = \int_{A \cap E_x^n} \frac{d\mu \circ \phi}{d\mu}(x) d\mu(x) \\ &= \frac{\mu(E_y^n)}{\mu(E_x^n)} \mu(A \cap E_x^n) > \frac{1}{2} \mu(E_y^n) \end{aligned}$$

Hence we have that

$$\frac{\mu(A \cap E_y^n)}{\mu(E_y^n)} > \frac{1}{2}.$$

But then,

$$1 = \frac{\mu(A \cap E_y^n)}{\mu(E_y^n)} + \frac{\mu(A^c \cap E_y^n)}{\mu(E_y^n)} > \frac{1}{2} + \frac{1}{2} = 1.$$

This is a contradiction. Therefore A has full measure and this shows that our equivalence relation is ergodic. \square

In the same way, we can analyze xerox actions on k^∞ -UHF algebras induced by diagonal representations of finite abelian groups. To prove that the fixed point algebra is a factor it is enough, as before, to show that an equivalence relation \mathcal{R} on $X = \prod \{0, 1, \dots, k-1\}$ is ergodic.

Let us consider for example the action of \mathbb{Z}_3 on $A = \otimes M_3(\mathbb{C})$ induced by the automorphism

$$\alpha = \otimes \text{Ad} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{bmatrix}$$

where $\epsilon = e^{\frac{2\pi i}{3}}$. On A we consider the product state $\varphi(\cdot) = \otimes \text{tr}(h_n \cdot)$ where $h_n = \text{diag}(a_n^0, a_n^1, a_n^2)$, $a_n^0 + a_n^1 + a_n^2 = 1$, $a_n^0, a_n^1, a_n^2 > 0$. Assume that

$$\sum_{n=1}^{\infty} (a_n^0 a_n^1 + a_n^0 a_n^2 + a_n^1 a_n^2) = \infty$$

By Theorem 1.1.3, the fixed point algebra $A^{\mathbb{Z}_3}$ can be identified with $A(X, \Gamma)$ where $X = \prod\{0, 1, 2\}$ and $\Gamma = \cup \Gamma_n$. As before it is easier to describe the equivalence relation \mathcal{R} induced by Γ . This is:

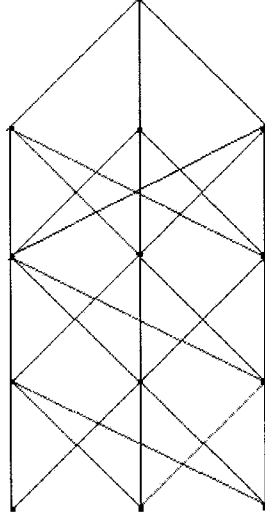
$$x\mathcal{R}y \text{ if and only if } x\mathcal{T}y \text{ and } \sum (x_n - y_n) = 0 \pmod{3}$$

where \mathcal{T} is the tail equivalence on X . We also have that $A^{\mathbb{Z}_3} \simeq C_r^*(X, \mathcal{R})$, and \mathcal{R} can be identified with the tail equivalence on the Bratteli diagram, as shown in Figure 2.3

On X , the state induces a product measure $\mu = \otimes \mu_n$ with $\mu_n(i) = a_n^i$, $0 \leq i \leq 2$ and $\pi_{\varphi^{\mathbb{Z}_3}}(A^{\mathbb{Z}_3})'' \simeq W^*(X, \mu, \mathcal{R})$. The proof follows the same idea as in the case of the \mathbb{Z}_2 -action but this time it is more difficult to show that almost all x and y have infinitely many encounters. First we prove the following lemma:

Lemma 2.3.4. *If $Z = \prod\{0, 1, 2\}$ and $\nu = \otimes \nu_n$ where $\alpha_n = \nu_n(0)$, $\beta_n = \nu_n(1)$, $\gamma_n = \nu_n(2)$ and $\alpha_n, \beta_n, \gamma_n \geq r > 0$ then for $\nu \times \nu$ almost all $(x, y) \in Z \times Z$ there are infinitely many n such that $\sum_{i=1}^n (x_i - z_i) = 0 \pmod{3}$ i.e., for $\nu \times \nu$ -almost all $(x, y) \in Z \times Z$, x and y have infinitely many encounters.*

Proof. Let $A_n = \{(x, y) \in Z \times Z : x_{2n-1} = 1, x_{2n} = 1, y_{2n} = 0, y_{2n+1} = 0\}$, $n \geq 1$. We have that $\sum_{n=1}^{\infty} \nu \times \nu(A_n) = \infty$. By Borel-Cantelli lemma we have that for

Figure 2.3: The Bratteli diagram of $A^{\mathbb{Z}_3}$

almost all $(x, y) \in Z \times Z$, $(x, y) \in A_n$ for infinitely many n . But if $x, y \in A_n$ we have that $\sum_{i=1}^{2n-2}(x_i - y_i) = 0 \pmod{3}$ or $\sum_{i=1}^{2n-1}(x_i - y_i) = 0 \pmod{3}$ or $\sum_{i=1}^{2n}(x_i - y_i) = 0 \pmod{3}$. Hence, whenever $x, y \in A_n$, x and y have an encounter. Therefore for $\nu \times \nu$ -almost all (x, y) , there are infinitely many $n \in \mathbb{N}$ such that $\sum_{i=1}^n(x_i - y_i) = 0 \pmod{3}$. \square

Consider now the general case.

For $k \in \{0, 1, 2\}$ we define $V_{i,j}^k := \{x = x_i x_{i+1} \dots x_j : \sum_{n=i}^j x_n = k \pmod{3}\}$ and $E_{i,j}^k := \bigcup \{E_x; x \in V_{i,j}^k\}$ where, $E_x = \{y \in X; y_l = x_l, \text{ for } i \leq l \leq j\}$, if $x \in V_{i,j}^k$. We will prove that

$$\lim_{j \rightarrow \infty} \mu(E_{i,j}^k) = \frac{1}{3} \text{ for all } k = 0, 1, 2.$$

We have that

$$\prod_{n=i}^j (a_n^0 + a_n^1 + a_n^2) = \mu(E_{i,j}^0) + \mu(E_{i,j}^1) + \mu(E_{i,j}^2), \quad (2.3.1)$$

$$\prod_{n=i}^j (a_n^0 + \epsilon a_n^1 + \epsilon^2 a_n^2) = \mu(E_{i,j}^0) + \epsilon \mu(E_{i,j}^1) + \epsilon^2 \mu(E_{i,j}^2), \quad (2.3.2)$$

$$\prod_{n=i}^j (a_n^0 + \epsilon^2 a_n^1 + \epsilon a_n^2) = \mu(E_{i,j}^0) + \epsilon^2 \mu(E_{i,j}^1) + \epsilon \mu(E_{i,j}^2). \quad (2.3.3)$$

Indeed, the first equality is clear. For the second one

$$\begin{aligned} \prod_{n=i}^j (a_n^0 + a_n^1 \epsilon + a_n^2 \epsilon^2) &= \sum_{x_i, x_{i+1}, \dots, x_j \in \{0,1,2\}} a_i^{x_1} a_{i+1}^{x_2} \cdots a_j^{x_j} \epsilon^{x_i + \cdots + x_j} \\ &= \sum_{k=0}^2 \sum_{x \in V_{i,j}^k} a_i^{x_1} a_{i+1}^{x_2} \cdots a_j^{x_j} \epsilon^k \\ &= \mu(E_{i,j}^0) + \mu(E_{i,j}^1) \epsilon + \mu(E_{i,j}^2) \epsilon^2 \end{aligned}$$

and the third one can be proved in the same way. On the other hand:

$$\begin{aligned} (a_n^0 + \epsilon a_n^1 + \epsilon^2 a_n^2)(a_n^0 + \bar{\epsilon} a_n^1 + \bar{\epsilon}^2 a_n^2) &= (a_n^0 + \epsilon a_n^1 + \epsilon^2 a_n^2)(a_n^0 + \epsilon^2 a_n^1 + \epsilon a_n^2) \\ &= (a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 + (a_n^0 a_n^1 + a_n^0 a_n^2 + a_n^1 a_n^2) \epsilon^2 + (a_n^0 a_n^1 + a_n^0 a_n^2 + a_n^1 a_n^2) \epsilon^2 \\ &= (a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 + a_n^0 a_n^1 (\epsilon + \bar{\epsilon}) + a_n^0 a_n^2 (\epsilon + \bar{\epsilon}^2) + a_n^1 a_n^2 (\epsilon + \bar{\epsilon}) \\ &= (a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 - a_n^0 a_n^1 - a_n^0 a_n^2 - a_n^1 a_n^2 \end{aligned}$$

Hence

$$\lim_{j \rightarrow \infty} \prod_{n=i}^j |a_n^0 + \epsilon a_n^1 + \epsilon^2 a_n^2| = 0 \quad (2.3.4)$$

because,

$$\begin{aligned} \lim_{j \rightarrow \infty} \sum_{n=i}^j (1 - |a_n^0 + \epsilon a_n^1 + \epsilon^2 a_n^2|) &= \lim_{j \rightarrow \infty} \sum_{n=i}^j \{1 - [(a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 - a_n^0 a_n^1 - a_n^0 a_n^2 - a_n^1 a_n^2]^{\frac{1}{2}}\} \\ &= \lim_{j \rightarrow \infty} \sum_{n=i}^j \{a_n^0 + a_n^1 + a_n^2 - [(a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 - a_n^0 a_n^1 - a_n^0 a_n^2 - a_n^1 a_n^2]^{\frac{1}{2}}\} \\ &= 3 \cdot \lim_{j \rightarrow \infty} \sum_{n=i}^j \frac{a_n^0 a_n^1 + a_n^0 a_n^2 + a_n^1 a_n^2}{1 + [(a_n^0)^2 + (a_n^1)^2 + (a_n^2)^2 - a_n^0 a_n^1 - a_n^0 a_n^2 - a_n^1 a_n^2]^{\frac{1}{2}}} \\ &\geq \lim_{j \rightarrow \infty} \sum_{n=i}^j [a_n^0 a_n^1 + a_n^0 a_n^2 + a_n^1 a_n^2] = \infty \end{aligned}$$

Similarly,

$$\lim_{j \rightarrow \infty} \prod_{n=i}^j |a_n^0 + \epsilon^2 a_n^1 + \epsilon a_n^2| = 0 \quad (2.3.5)$$

Therefore from (2.3.1), (2.3.2), (2.3.3), (2.3.4) and (2.3.5) we obtain:

$$(2.3.1) + (2.3.2) + (2.3.3) \Rightarrow \lim_{j \rightarrow \infty} \mu(E_{ij}^0) = \frac{1}{3}$$

$$(2.3.1) + \epsilon(2.3.2) + \epsilon^2(2.3.3) \Rightarrow \lim_{j \rightarrow \infty} \mu(E_{ij}^2) = \frac{1}{3}$$

$$(2.3.1) + \epsilon^2(2.3.2) + \epsilon(2.3.3) \Rightarrow \lim_{j \rightarrow \infty} \mu(E_{ij}^1) = \frac{1}{3}$$

Inductively we build a sequence $1 = i_1 < i_2 < i_3 < \dots$ with $\mu(E_{i_n, i_{n+1}-1}^k) \geq \frac{1}{4}$ and $\sum_{k=0}^2 \mu(E_{i_n, i_{n+1}-1}^k) = 1$ for $k = 0, 1, 2$. Let $Z := \prod\{0, 1, 2\}$. On Z we consider the product measure $\nu = \otimes \nu_n$ where $\nu_n(k) = \mu(E_{i_n, i_{n+1}-1}^k)$. Let $\pi : (X, \mu) \rightarrow (Z, \nu)$ defined by

$$\pi(y_1, y_2 \dots y_{i_1-1}, y_{i_1} \dots y_{i_2-1} \dots) = (k_1, k_2 \dots)$$

with $y_{i_n} + y_{i_{n+1}} + \dots y_{i_{n+1}-1} = k_n \pmod{3}$ for all $n \geq 1$. By the previous lemma, for $\nu \times \nu$ -almost all $(z, w) \in Z \times Z$, there are infinitely many $n \in \mathbb{N}$ such that $\sum_{i=1}^n (z_i - w_i) = 0 \pmod{3}$. If $x \in \pi^{-1}(z)$, $y \in \pi^{-1}(w)$, we have also the property that $\sum_{i=1}^n (x_i - y_i) = 0 \pmod{3}$. As $\nu = \mu \circ \pi^{-1}$, we have that for almost all $(x, y) \in X \times X$ $\sum_{i=1}^n (x_i - y_i) = 0 \pmod{3}$. We conclude that the equivalence relation is ergodic, in the same way we did in Proposition 2.3.4.

We now determine the type of the factor $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$.

Theorem 2.3.5. *With the above notations, if $\sum a_n(1 - a_n) = \infty$, $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$ is a factor of*

1) type II_1 if and only if $\sum (a_n - \frac{1}{2})^2 < \infty$

If $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$ is not type II_1 then it is a factor of

2) type II_∞ if and only if $\sum a_n(1 - a_n)(a_n - \frac{1}{2})^2 < \infty$

3) type III if and only if $\sum a_n(1 - a_n)(a_n - \frac{1}{2})^2 = \infty$

The proof follows mainly the same lines as the one in the next section when we consider the standard action of the 2-dimensional torus. As we saw before, we can also identify $\pi_{\phi^{\mathbb{Z}_2}}(A^{\mathbb{Z}_2})''$ with $(\pi_\phi(A'')^{\mathbb{Z}_2})''$. An easy computation shows that the action

of \mathbb{Z}_2 extended to the ITPFI $\pi_\varphi(A)''$ is outer and therefore, the group being finite, the fixed point algebra is a factor. This is known but the proof results using completely different arguments. (see for example [J2], p.6).

2.4 The Standard Action of the Torus T^2

We recall that the standard xerox action of the $(k-1)$ dimensional torus T^{k-1} on $A = \otimes M_k(\mathbb{C})$, the k^∞ -UHF algebra, is given by:

$$\begin{aligned} \alpha : T^{k-1} &\rightarrow \text{Aut}(A); \quad \alpha(t_1, \dots, t_{k-1}) := \otimes \text{Ad } \pi(t_1, t_2, \dots, t_{k-1}), \text{ where} \\ \pi(t_1, \dots, t_{k-1}) &:= \text{diag}(1, e^{it_1}, e^{it_2}, \dots, e^{it_{k-1}}); \quad t_i \in [0, 2\pi) \end{aligned}$$

To simplify the notation, for $k = 3$ we denote:

$$\begin{aligned} \alpha &: T^2 \rightarrow \text{Aut}(A); \quad \alpha(s, u) := \otimes \text{Ad } \pi(s, u) \\ \pi(s, u) &:= \text{diag}(1, e^{is}, e^{iu}); \quad s, u \in [0, 2\pi) \end{aligned}$$

On $A = \otimes M_3(\mathbb{C})$ we consider the faithful diagonal product state $\varphi = \otimes \varphi_n$ given by $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$, where

$$h_n = \text{diag}(a_n, b_n, c_n) \quad a_n, b_n, c_n \in (0, 1) \text{ and } a_n + b_n + c_n = 1 \quad (2.4.1)$$

As φ is a diagonal state, φ is invariant under α , i.e., $\varphi \circ \alpha(s, t) = \varphi$ for all $s, t \in [0, 2\pi)$. As in Section 2.2, we will denote by M the GNS representation $\pi_\varphi(A)''$. Then $\pi_\varphi(A)'' = \otimes (M_3(\mathbb{C}), \varphi_n)$. Let N be the fixed point subalgebra $M^{T^2} = (\pi_\varphi(A)'')^{T^2}$. In fact $N = \pi_\varphi(A^{T^2})'' \simeq \pi_{\varphi_{T^2}}(A^{T^2})''$.

Let $X = \prod_{n=1}^{\infty} \{0, 1, 2\}$ and let \mathcal{R}_∞ be the following equivalence relation:

$x \mathcal{R}_\infty y$ if and only if there exists $n \geq 1$ such that $x_i = y_i$ for $i > n$ and

$$\text{card}\{i; 1 \leq i \leq n, x_i = k\} = \text{card}\{i; 1 \leq i \leq n, y_i = k\} \text{ for } k \in \{0, 1, 2\}$$

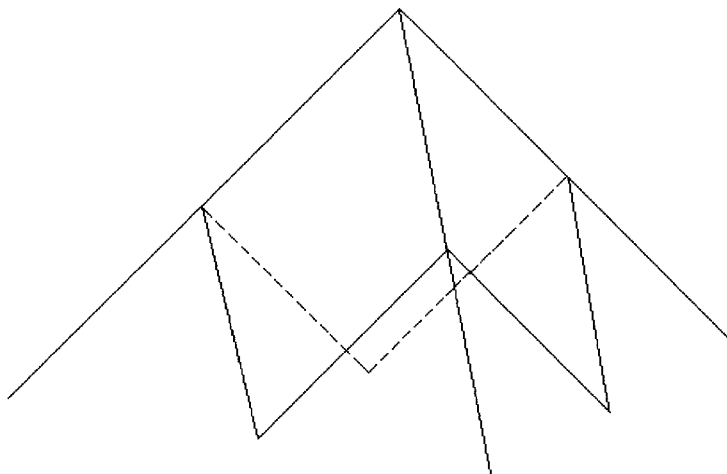


Figure 2.4: The Bratteli diagram of A^{T^2}

We can identify X with the path space of the Bratteli diagram of A^{T^2} , in the same way we did in Section 2.2, and again the equivalence relation \mathcal{R}_∞ corresponds to tail equivalence on the diagram, as shown in Figure 2.4

We also can identify A^{T^2} with $C_r^*(X, \mathcal{R}_\infty)$ and \mathcal{R}_∞ is induced by the following action of S_∞ on X :

$$\sigma(x_1, x_2, \dots) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots), \quad \sigma \in S_\infty$$

The state φ induces a product measure $\mu = \otimes \mu_n$ on X with μ_n the probability measure on $\{0, 1, 2\}$ given by $\mu_n(0) = a_n$, $\mu_n(1) = b_n$, $\mu_n(2) = c_n$. We have that $\pi_{\varphi^X}(A^T)'' \simeq W^*(X, \mu, \mathcal{R}_\infty)$ (see Theorem 1.2.16). The factoriality is equivalent to the ergodicity of \mathcal{R}_∞ with respect to μ and by [AP] we have that \mathcal{R}_∞ is μ -ergodic if and only if $\sum a_n(1 - a_n) = \infty$, $\sum b_n(1 - b_n) = \infty$, $\sum c_n(1 - c_n) = \infty$. Because the measure μ is non atomic the factor N is either of type II or of type III.

First we will give a necessary and sufficient condition for N to be of type II_1 .

Proposition 2.4.1. *With the above notation, N is of type II_1 if and only if there*

exist $a, b, c \in (0, 1)$, $a + b + c = 1$ such that

$$\sum (a_n - a)^2 + (b_n - b)^2 + (c_n - c)^2 < \infty \quad (2.4.2)$$

Proof. N is a factor of type II_1 if and only if there is an \mathcal{R}_∞ -invariant ergodic probability measure $\nu \sim \mu$ on X , equivalently, if and only if there exists an \mathcal{R}_∞ extremal invariant measure ν equivalent to μ , on X . By [W], the ergodic invariant measures are in bijection with the extremal traces on A^{T^2} , and by [Pr], these are restrictions of symmetric states on A (see Definition 1.2.10). Hence, if N is a factor of type II_1 , there exists a measure ν , on $\prod\{0, 1, 2\}$, $\nu \sim \mu$ such that $\nu = \otimes \nu_0$, with $\nu_0(0) = a$, $\nu_0(1) = b$, $\nu_0(2) = c$ and $abc \neq 0$. By Kakutani's Theorem, ([HS], Theorem 22.36), we have

$$\sum (a_n - a)^2 + (b_n - b)^2 + (c_n - c)^2 < \infty.$$

Conversely, if (2.4.2) holds, the conclusion follows again by Kakutani's Theorem. \square

Notice that in this situation, [BP2], gives only necessary and sufficient conditions for N to be of type II_1 .

When N is properly infinite, we will use the computation of Connes' invariant T to determine if N is of type II_∞ or III .

Proposition 2.4.2. *If $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ where h_n are as in (2.4.1), and N is properly infinite, then $t \in T(N)$ if and only if there exist $s, u \in [0, 2\pi)$ such that*

$$\sum a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) + a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} - u)) + b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - s + u)) < \infty \quad (2.4.3)$$

Proof. We recall that $\sigma_t^\varphi = \otimes \text{Ad } h_n^{it}$. For s, t, u fixed, we have $\sigma_t^\varphi \circ \alpha(-s, -u) = \otimes \text{Ad } v_n$, where

$$v_n = \begin{bmatrix} a_n^{it} & & \\ & b_n^{it} e^{-is} & \\ & & c_n^{it} e^{-iu} \end{bmatrix}.$$

By Theorem 1.2.5, we have that $\otimes \text{Ad } v_n$ defines an inner automorphism if and only if $\sum (1 - |\varphi_n(v_n)|) < \infty$. We have,

$$\begin{aligned}
1 - |\varphi_n(v_n)| &= 1 - |\varphi_n(v_n)| = 1 - |a_n^{it+1} + b_n^{it+1}e^{-is} + c_n^{it+1}e^{-iu}| \\
&= 1 - |a_n \cos(t \log a_n) + ia_n \sin(t \log a_n) + b_n \cos(t \log b_n - s) + \\
&\quad + ib_n \sin(t \log b_n - s) + c_n \cos(t \log c_n - u) + ic_n \sin(t \log c_n - u)| \\
&= (a_n + b_n + c_n) - |a_n \cos(t \log a_n) + ia_n \sin(t \log a_n) + b_n \cos(t \log b_n - s) + \\
&\quad + ib_n \sin(t \log b_n - s) + c_n \cos(t \log c_n - u) + ic_n \sin(t \log c_n - u)| \\
&= (a_n + b_n + c_n) - \{[a_n \cos(t \log a_n) + b_n \cos(t \log b_n - s) + c_n \cos(t \log c_n - u)]^2 + \\
&\quad [a_n \sin(t \log a_n) + b_n \sin(t \log b_n - s) + c_n \sin(t \log c_n - u)]^2\}^{\frac{1}{2}} < \infty
\end{aligned}$$

Then the series

$$\begin{aligned}
&\sum (a_n + b_n + c_n) - [a_n^2 + b_n^2 + c_n^2 + 2a_nb_n \cos(t \log \frac{b_n}{a_n} - s) + \\
&\quad + 2c_na_n(t \log \frac{c_n}{a_n} - u) + 2b_nc_n \cos(t \log \frac{b_n}{c_n} - s + u)]^{\frac{1}{2}}
\end{aligned}$$

is convergent if and only if the following one is convergent

$$\sum a_nb_n(1 - \cos(t \log \frac{b_n}{a_n} - s)) + a_nc_n(1 - \cos(t \log \frac{c_n}{a_n} - u)) + b_nc_n(1 - \cos(t \log \frac{b_n}{c_n} - s + u))$$

Therefore, we have that $\otimes \text{Ad } v_n$ is inner if and only if

$$\sum a_nb_n(1 - \cos(t \log \frac{b_n}{a_n} - s)) + a_nc_n(1 - \cos(t \log \frac{c_n}{a_n} - u)) + b_nc_n(1 - \cos(t \log \frac{b_n}{c_n} - s + u)) < \infty$$

Now, if the sum (2.4.3) is finite then

$$\bigotimes_{n=1}^{\infty} \begin{pmatrix} a_n^{it} & & \\ & b_n^{it}e^{-is} & \\ & & c_n^{it}e^{-iu} \end{pmatrix} = v \in M$$

As by definition $v \in N$, we have that $\sigma_t^\varphi = \text{Ad } v \circ \alpha(s, u)$. Hence, by Proposition 2.1.8, $t \in T(N)$.

Conversely, if $t \in T(N)$ then again by Proposition 2.1.8, there exist $s, u \in [0, 2\pi)$ and $v \in U(N)$ such that $\sigma_t^\varphi(x) = \text{Ad } v \circ \alpha(s, u)(x)$ for all $x \in M$. Hence $\sigma_t^\varphi \circ \alpha(-s, -u)$ is an inner automorphism and therefore the sum (2.4.3) is finite. \square

Proposition 2.4.3. *Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ sequences of nonnegative real numbers. Suppose that $\lim_{n \rightarrow \infty} a_n = 0$, $\sum a_n = \infty$ and $\lim_{n \rightarrow \infty} b_n = b \neq 0$. Then $\exists t \in \mathbb{R}$ such that for any $s \in [0, 2\pi)$, $\sum a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) = \infty$.*

Proof. Let $\psi(\cdot) = \otimes \text{tr}(k_n \cdot)$ be the faithful diagonal state on $B = \otimes M_2(\mathbb{C})$ given by $k_n = \text{diag}(\frac{a_n}{a_n + b_n}, \frac{b_n}{a_n + b_n})$. Let $P := \pi_\psi(B)''$ and $P^\alpha = Q$ (here α is the action of the 1-dimensional torus on B). As, $\sum \frac{a_n}{a_n + b_n} \cdot \frac{b_n}{a_n + b_n} = \infty$, Q is a factor by Theorem 2.2.1. By Theorem 2.2.2, Q is a factor of type III because

$$\sum \frac{a_n}{a_n + b_n} \left(1 - \frac{a_n}{a_n + b_n}\right) \left(\frac{a_n}{a_n + b_n} - c\right)^2 = \infty \text{ for all } c \in (0, 1).$$

Therefore, by Proposition 2.2.3, there exists $t \notin T(N)$. This means that there exists $t \in \mathbb{R}$ such that for every $s \in [0, 2\pi)$ we have $\sum \frac{a_n b_n}{(a_n + b_n)^2} (1 - \cos(t \log \frac{b_n}{a_n} - s)) = \infty$, or, equivalently, $\sum_n a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) = \infty$. \square

Corollary 2.4.4. *Let $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ where h_n as in (2.4.1). If $(a_n, b_n, c_n)_{n \geq 1}$ has a subsequence $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ such that $a_{n_k} \rightarrow 0$, $\sum_{k=1}^{\infty} a_{n_k} = \infty$ and either b_{n_k} (or c_{n_k}) converges to a nonzero number, then N is of type III. If N is of type II_∞ and $(a_n, b_n, c_n)_{n \geq 1}$ has a subsequence $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ with $a_{n_k} \rightarrow 0$ and either b_{n_k} (or c_{n_k}) converges to a nonzero number, then $\sum_{k=1}^{\infty} a_{n_k} < \infty$.*

Proof. By Proposition 2.4.3 there exists $t \in \mathbb{R}$ such that for every $s \in [0, 2\pi)$, $\sum a_{n_k} b_{n_k} (1 - \cos(t \log \frac{b_{n_k}}{a_{n_k}} - s)) = \infty$ and then there exists $t \in \mathbb{R}$ such the sum (2.4.3) is infinite for every s . Hence $T(N) \neq \mathbb{R}$. By Theorem 1.2.4, N is of type III. \square

Remark: the role of a_n , b_n and c_n can clearly be interchanged.

Proposition 2.4.5. *If $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ is as in (2.4.1) and $(a_n, b_n, c_n)_{n \geq 1}$ has two distinct limit points (a^1, b^1, c^1) and (a^2, b^2, c^2) with $a^i b^i \neq 0$ for $i = 1, 2$ and $\frac{b^1}{a^1} \neq \frac{b^2}{a^2}$ then N is of type III.*

Proof. By assumption, let $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ and $(a_{n_j}, b_{n_j}, c_{n_j})_{j \geq 1}$ be two subsequences of $(a_n, b_n, c_n)_{n \geq 1}$ converging to (a^1, b^1, c^1) and (a^2, b^2, c^2) with $a_i b_i \neq 0$, $i = 1, 2$.

If $N = M^{T^2}$ is semifinite, then $T(N) = \mathbb{R}$ and therefore for every $t \in \mathbb{R}$, there exists $s \in [0, 2\pi)$ such that $\sum a_n b_n (1 - \cos(t \log \frac{a_n}{b_n} - s)) < \infty$. Therefore, $\lim_{j \rightarrow \infty} \cos(t \log \frac{b_{n_j}}{a_{n_j}} - s) = \cos(t \log \frac{b^1}{a^1} - s) = 0$ and $\lim_{k \rightarrow \infty} \cos(t \log \frac{b_{n_k}}{a_{n_k}} - s) = \cos(t \log \frac{b^2}{a^2} - s) = 0$. Then $s - t \log \frac{b^1}{a^1} \in 2\pi\mathbb{Z}$ and $s - t \log \frac{b^2}{a^2} \in 2\pi\mathbb{Z}$. Let $t \in \mathbb{R} = T(N)$ such that $t \notin \frac{2\pi\mathbb{Z}}{\log \frac{a^2 b^1}{a^1 b^2}}$. As we saw before, there exists $s \in [0, 2\pi)$ such that $s - t \log \frac{a^1}{b^1} \in 2\pi\mathbb{Z}$ and $s - t \log \frac{a^2}{b^2} \in 2\pi\mathbb{Z}$. It results that $t \in \frac{2\pi\mathbb{Z}}{\log \frac{a^2 b^1}{a^1 b^2}}$ which is a contradiction. Hence, N is of type III.

Remark: the role of a_n , b_n and c_n can clearly be interchanged. □

Lemma 2.4.6. *Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences of nonnegative real numbers such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$ with $ab \neq 0$. If $t \neq 0$, then the series $\sum a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - t \log \frac{b}{a}))$ and $\sum a_n b_n (a_n b - b_n a)^2$ are either both convergent or both divergent.*

Proof. The lemma results from:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - t \log \frac{b}{a}))}{a_n b_n (b_n a - a_n b)^2} &= \lim_{n \rightarrow \infty} \frac{2 \sin^2(\frac{t}{2} \log \frac{b_n a}{a_n b})}{a_n^2 b^2 (\frac{b_n a}{a_n b} - 1)^2} \\ &= \lim_{n \rightarrow \infty} \frac{t^2}{2a_n^2 b^2} \frac{\sin^2(\frac{t}{2} \log \frac{b_n a}{a_n b})}{(\frac{t}{2} \log \frac{b_n a}{a_n b})^2} \left(\frac{\log \frac{b_n a}{a_n b}}{\frac{b_n a}{a_n b} - 1} \right)^2 = \frac{t^2}{2a^2 b^2} \end{aligned}$$

□

We recall that the conditions for N to be a factor are:

$$\sum a_n (1 - a_n) = \infty, \quad \sum b_n (1 - b_n) = \infty, \quad \sum c_n (1 - c_n) = \infty$$

Proposition 2.4.7. *With the above notation, $T(N) = \mathbb{R}$, i.e., N is semifinite if and only if there is a unique $(a, b, c) \in (0, 1)^3$ with $a + b + c = 1$ and*

$$\sum a_n b_n (a_n b - b_n a)^2 + \sum a_n c_n (a_n c - c_n a)^2 + \sum b_n c_n (c_n b - b_n c)^2 < \infty \quad (2.4.4)$$

Proof. Suppose first that N is a factor of type II_∞ . To have a geometric picture we can see the sequence $(a_n, b_n, c_n)_{n \geq 1}$ inside the triangle determined by the vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$. We denote by (AB) the interior of the line segment determined by A and B and similarly, for (AC) and (BC) . We call (AB) , (AC) and (BC) edges. By Proposition 2.4.5, the sequence $(a_n, b_n, c_n)_{n \geq 1}$ has at most a limit point in the interior of the triangle ABC and it has at most a limit point on each edge.

Case I. Assume first that the sequence $(a_n, b_n, c_n)_{n \geq 1}$ has a limit point (a, b, c) in the interior of the triangle (ABC) , (i.e., $abc \neq 0$). By Proposition 2.4.5, if $(m, n, 0)$ is a limit point on (AB) , then $m = \frac{a}{a+b}$ and $n = \frac{b}{a+b}$. Similarly, if $(0, f, g)$ is a limit point on (AC) , we obtain that $f = \frac{a}{a+c}$ and $g = \frac{c}{a+c}$, and if $(0, q, r)$ is a limit point on (BC) , $q = \frac{b}{b+c}$ and $r = \frac{c}{b+c}$. Therefore if (a, b, c) is a limit point inside (ABC) , then, the possible limit points are: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{a}{a+b}, \frac{b}{a+b}, 0)$, $(\frac{a}{a+c}, 0, \frac{c}{a+c})$, $(0, \frac{b}{b+c}, \frac{c}{b+c})$, (a, b, c) . We divide the set of non negative integers in mutually disjoint sets E_1, E_2, \dots, E_7 which are either infinite or empty, and such that

$$\begin{aligned} (a_n, b_n, c_n) &\xrightarrow[n \in E_1]{} (1, 0, 0); & (a_n, b_n, c_n) &\xrightarrow[n \in E_2]{} (0, 1, 0); \\ (a_n, b_n, c_n) &\xrightarrow[n \in E_3]{} (0, 0, 1); & (a_n, b_n, c_n) &\xrightarrow[n \in E_4]{} \left(\frac{a}{a+b}, \frac{b}{a+b}, 0 \right); \\ (a_n, b_n, c_n) &\xrightarrow[n \in E_5]{} \left(\frac{a}{a+c}, 0, \frac{c}{a+c} \right); & (a_n, b_n, c_n) &\xrightarrow[n \in E_6]{} \left(0, \frac{b}{b+c}, \frac{c}{b+c} \right); \\ & & (a_n, b_n, c_n) &\xrightarrow[n \in E_7]{} (a, b, c) \end{aligned}$$

For $i \in \{1, 2, \dots, 7\}$ and $E_i \neq \emptyset$ let

$$R_i := \sum_{E_i} a_n b_n (a_n b - b_n a)^2 + \sum_{E_i} a_n c_n (a_n c - c_n a)^2 + \sum_{E_i} b_n c_n (c_n b - b_n c)^2$$

If $E_1 \neq \emptyset$, as N is of type II_∞ , by Corollary 2.4.4, we must have $\sum_{n \in E_1} b_n < \infty$ and $\sum_{n \in E_1} c_n < \infty$. This is enough to ensure that $R_1 < \infty$. Similarly, if E_2 or E_3 are nonempty we obtain that $R_2 < \infty$ and $R_3 < \infty$.

According to our assumption, $E_7 \neq \emptyset$. Let $t \in \mathbb{R}$. As $T(N) = \mathbb{R}$, there exists $s \in [0, 2\pi)$ and $u \in [0, 2\pi)$ such that (2.4.3) holds. We show now that R_7 is finite. Based on $\sum_{E_7} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) < \infty$, we have $\cos(t \log \frac{b_n}{a_n} - s) \xrightarrow{n \in E_7} 0$. Hence¹, $s = t \log \frac{b}{a} \pmod{2\pi}$ and the sum becomes $\sum_{E_7} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - t \log \frac{b}{a})) < \infty$. By Lemma 2.4.6, $\sum_{E_7} a_n b_n (a_n b - b_n a)^2 < \infty$. As (2.4.3) holds, we also must have $\sum_{E_7} a_n c_n (1 - \cos(\log \frac{c_n}{a_n} - u)) < \infty$. This implies that $u = t \log \frac{c}{a} \pmod{2\pi}$ and again by Lemma 2.4.6, $\sum_{E_7} a_n c_n (a_n c - c_n a)^2 < \infty$. Also, $\sum_{E_7} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - s + u)) = \sum_{E_7} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - t \log \frac{b}{a} + t \log \frac{c}{a})) = \sum_{E_7} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - t \log \frac{b}{c})) < \infty$. By Lemma 2.4.6 we obtain $\sum_{E_7} b_n c_n (b_n c - c_n b)^2 < \infty$. Consequently, R_7 is finite.

For s and u determined above we have to show now that for $i \in \{4, 5, 6\}$, R_i are finite whenever E_i is nonempty.

If $E_4 \neq \emptyset$, then we must have $\sum_{E_4} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - t \log \frac{b}{a}))$, and by Lemma 2.4.6, $\sum_{E_4} a_n b_n (a_n \frac{b}{a+b} - b_n \frac{a}{a+b})^2 < \infty$ and therefore $\sum_{E_4} a_n b_n (a_n b - b_n a)^2 < \infty$. Also, as $T(N) = \mathbb{R}$, by Corollary 2.4.4, $\sum_{E_4} c_n < \infty$ and therefore $\sum_{E_4} b_n c_n (b_n c - c_n b)^2 < \infty$ and $\sum_{E_4} a_n c_n (a_n c - c_n a)^2 < \infty$. Consequently, R_4 is finite. Similarly if E_5 or E_6 are non void we have $R_5 < \infty$ and $R_6 < \infty$.

Therefore, as all R_i are finite we have that the sum (2.4.4) is finite.

For uniqueness, suppose that it exists $(p, q, r) \in (0, 1)^3$, $p + q + r = 1$ such that

$$\sum a_n b_n (a_n q - b_n p)^2 + \sum a_n c_n (a_n r - c_n p)^2 + \sum b_n c_n (c_n q - b_n r)^2 < \infty$$

Let $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ be a subsequence of $(a_n, b_n, c_n)_{n \geq 1}$ that converges to (a, b, c) . We must have $\lim_{n \rightarrow \infty} (a_{n_k} q - b_{n_k} p)^2 = aq - bp = 0$ and therefore $\frac{a}{b} = \frac{p}{q}$. Similarly $\frac{a}{c} = \frac{m}{p}$. This is enough to conclude that $(a, b, c) = (m, n, p)$.

¹if $t \in \mathbb{R}$ by $t \pmod{2\pi}$ we denote the unique real number $s \in [0, 2\pi)$ such that $s - t \in 2\pi\mathbb{Z}$

Case 2. Let us assume now that there is no limit point in the interior of the triangle ABC. Then:

Claim I. On at least two edges there exist a limit point of the sequence $(a_n, b_n, c_n)_{n \geq 1}$

Proof. Assume the opposite, i.e., there is at most one edge on which it exists a limit point of the sequence $(a_n, b_n, c_n)_{n \geq 1}$. Let us assume that this point is on (AB) . Then, as we already assumed that there is no limit point in the interior of the triangle, the only possible limit points are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(p, q, 0)$. We divide $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ in mutually disjoint sets E_1, E_2, E_3, E_4 which are either infinite or empty, and such that:

$$\begin{aligned} (a_n, b_n, c_n) &\xrightarrow[n \in E_1]{} (1, 0, 0); & (a_n, b_n, c_n) &\xrightarrow[n \in E_2]{} (0, 1, 0); \\ (a_n, b_n, c_n) &\xrightarrow[n \in E_3]{} (0, 0, 1); & (a_n, b_n, c_n) &\xrightarrow[n \in E_4]{} (p, q, 0); \end{aligned}$$

Case a. If $E_3 = \emptyset$ then by Corollary 2.4.4 we get that $\sum_{n \in E_1 \cup E_2 \cup E_4} c_n < \infty$ and therefore $\sum_{n \geq 1} c_n(1 - c_n) < \infty$, which contradicts the conditions for factoriality.

Case b. If $E_3 \neq \emptyset$ then again by Corollary 2.4.4 we have that $\sum_{n \in E_3} a_n + b_n < \infty$ and $\sum_{n \in E_1 \cup E_2 \cup E_4} c_n < \infty$ and therefore $\sum_{n \in \mathbb{N}} c_n(a_n + b_n) = \sum_{n \in \mathbb{N}} c_n(1 - c_n) < \infty$ which is a contradiction. \square

So, if we don't have a limit point in the interior of (ABC) , without loss of generality, we can assume that there is one limit point on (AB) and another one on (AC) . We denote these points by $(m, n, 0)$ and $(q, 0, r)$ and take (a, b, c) the point situated at the intersection of the lines determined by these points and their opposite vertices in our triangle. It is also possible to have a limit point on (BC) . If such a point exists, we denote it by $(0, f, g)$ and we have:

Claim II. With the above notations $\frac{f}{g} = \frac{b}{c}$.

Proof. From the way we defined (a, b, c) we must have $\frac{b}{a} = \frac{n}{m}$ and $\frac{c}{a} = \frac{r}{q}$. Let $(a_{n_k}, b_{n_k}, c_{n_k})_{n \geq 1}$ be a subsequence of $(a_n, b_n, c_n)_{n \geq 1}$ converging to $(m, n, 0)$. We have

that $\sum a_{n_k} b_{n_k} (1 - \cos(t \log \frac{b_{n_k}}{a_{n_k}} - s)) < \infty$ and necessary $s = t \log \frac{b}{a} \pmod{2\pi}$. Similarly $u = t \log \frac{c}{a} \pmod{2\pi}$. Let $(a_{n_j}, b_{n_j}, c_{n_j})_{n \geq 1}$ be a subsequence of $(a_n, b_n, c_n)_{n \geq 1}$ converging to $(0, f, g)$. But then $\sum b_{n_j} c_{n_j} (1 - \cos(t \log \frac{b_{n_j}}{c_{n_j}} - s + u)) = \sum b_{n_j} c_{n_j} (1 - \cos(t \log \frac{b_{n_j}}{c_{n_j}} - t \log \frac{b}{a} + t \log \frac{c}{a})) < \infty$. Hence $\lim_{j \rightarrow \infty} \cos(t \log \frac{b_{n_j} c}{c_{n_j} b}) = \cos(t \log \frac{f c}{g b}) = 1$ for all t in \mathbb{R} . Therefore we must have $\frac{b}{c} = \frac{f}{g}$. \square

In other words, $(0, f, g) = (0, \frac{b}{c+b}, \frac{c}{b+c})$. From the way (a, b, c) is defined, in terms of a and b , the other two limit points on the edges can be written as $(\frac{a}{a+b}, \frac{b}{a+b}, 0)$ and $(\frac{a}{a+c}, 0, \frac{c}{a+c})$.

Therefore the possible limit points are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{a}{a+b}, \frac{b}{a+b}, 0)$, $(\frac{a}{a+c}, 0, \frac{c}{a+c})$, $(0, \frac{b}{b+c}, \frac{c}{b+c})$. We divide again the set of non negative integers in mutually disjoint sets, E_1, E_2, \dots, E_6 which are infinite or empty, and such that:

$$\begin{aligned} (a_n, b_n, c_n) &\xrightarrow[n \in E_1]{} (1, 0, 0); & (a_n, b_n, c_n) &\xrightarrow[n \in E_2]{} (0, 1, 0); \\ (a_n, b_n, c_n) &\xrightarrow[n \in E_3]{} (0, 0, 1); & (a_n, b_n, c_n) &\xrightarrow[n \in E_4]{} \left(\frac{a}{a+b}, \frac{b}{a+b}, 0 \right); \\ (a_n, b_n, c_n) &\xrightarrow[n \in E_5]{} \left(\frac{a}{a+c}, 0, \frac{c}{a+c} \right); & (a_n, b_n, c_n) &\xrightarrow[n \in E_6]{} \left(0, \frac{b}{b+c}, \frac{c}{b+c} \right) \end{aligned}$$

For $i \in \{1, 2, \dots, 6\}$, R_i will denote the same sums as in Case 1. If $E_i \neq \emptyset$, $i = 1, 2, 3$ then, as in Case 1, we obtain that $R_i < \infty$, $i = 1, 2, 3$.

According to our assumption, E_4 and E_5 are non void.

Let $t \in \mathbb{R}$. As $T(N) = \mathbb{R}$, there exists $s \in [0, 2\pi)$ and $u \in [0, 2\pi)$ such that (2.4.3) is finite. We must have $\sum_{E_4} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) < \infty$. This implies that $s = t \log \frac{b}{a} \pmod{2\pi}$ and by Lemma 2.4.6, $\sum_{E_4} a_n b_n (a_n b - b_n a)^2 < \infty$. As $T(N) = \mathbb{R}$, by Corollary 2.4.4, $\sum_{E_4} c_n < \infty$ and therefore R_4 is finite.

We must also have $\sum_{E_5} a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} - u)) < \infty$. This implies that $u = t \log \frac{c}{a} \pmod{2\pi}$ and by Lemma 2.4.6, $\sum_{E_5} a_n c_n (a_n c - c_n a)^2 < \infty$. By Corollary 2.4.4, $\sum_{E_5} b_n < \infty$. Therefore R_5 is finite.

If $E_6 \neq \emptyset$ we must have $\sum_{E_6} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - s + u)) = \sum_{E_6} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - t \log \frac{b}{a} + t \log \frac{c}{a})) = \sum_{E_6} b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - t \log \frac{b}{c}))^2 < \infty$ and, by Lemma 2.4.6, we

have $\sum_{E_6} b_n c_n (b_n c - c_n b)^2 < \infty$. On the other hand, we must have $\sum_{E_6} a_n < \infty$ and we conclude that R_6 is finite.

For the uniqueness we proceed as in Case I.

We prove now the converse. Hence, we assume that the sum (2.4.4) is finite. Suppose that we have a limit point $(p, q, 0)$ with $pq \neq 0$. Then it exists a subsequence $(a_{n_k}, b_{n_k}, c_{n_k}) \rightarrow (p, q, r)$. As $\sum a_{n_k} b_{n_k} (a_{n_k} b - b_{n_k} a) < \infty$, we must have $\lim(a_{n_k} b - b_{n_k} a) = pb - qa = 0$ and consequently $(p, q, 0) = (\frac{a}{a+b}, \frac{b}{a+b}, 0)$. Similarly, if $(0, q, r)$ is a limit point, $(0, q, r) = (0, \frac{b}{b+c}, \frac{c}{b+c})$, if $(p, 0, r)$ is a limit point then $(p, 0, r) = (\frac{a}{a+c}, 0, \frac{c}{a+c})$ and if (p, q, r) with $pqr \neq 0$ is a limit point then $(p, q, r) = (a, b, c)$. Therefore the possible limit points are: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{a}{a+b}, \frac{b}{a+b}, 0)$, $(\frac{a}{a+c}, 0, \frac{c}{a+c})$, $(0, \frac{b}{b+c}, \frac{c}{b+c})$, (a, b, c) . Again we divide the set of non negative integers in mutually disjoint sets, E_1, E_2, \dots, E_7 which are either infinite or empty, and such that:

$$\begin{aligned} (a_n, b_n, c_n) &\xrightarrow{n \in E_1} (1, 0, 0); & (a_n, b_n, c_n) &\xrightarrow{n \in E_2} (0, 1, 0); \\ (a_n, b_n, c_n) &\xrightarrow{n \in E_3} (0, 0, 1); & (a_n, b_n, c_n) &\xrightarrow{n \in E_4} \left(\frac{a}{a+b}, \frac{b}{a+b}, 0\right); \\ (a_n, b_n, c_n) &\xrightarrow{n \in E_5} \left(\frac{a}{a+c}, 0, \frac{c}{a+c}\right); & (a_n, b_n, c_n) &\xrightarrow{n \in E_6} \left(0, \frac{b}{b+c}, \frac{c}{b+c}\right); \\ (a_n, b_n, c_n) &\xrightarrow{n \in E_7} (a, b, c) \end{aligned}$$

Let $t \in \mathbb{R}$. Let $s := t \log \frac{b}{a} \pmod{2\pi}$ and $u := t \log \frac{c}{a} \pmod{2\pi}$. For $i \in \{1, 2, \dots, 7\}$, if $E_i \neq \emptyset$, let

$$\begin{aligned} S_i := \sum_{E_i} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) &+ a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} - u)) \\ &+ b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - s + u)) \end{aligned}$$

We will show that for t, s, u as above all S_i are finite.

If $E_1 \neq \emptyset$ we must have $\sum_{E_1} a_n b_n (a_n b - b_n a)^2 < \infty$. Because $a_n (a_n b - b_n a)^2 \xrightarrow{n \in E_1} b^2$ we have $\sum_{E_1} b_n < \infty$. Similarly, $\sum_{E_1} c_n < \infty$. Therefore $\sum_{n \in E_1} a_n b_n < \infty$, $\sum_{n \in E_1} a_n c_n < \infty$

and $\sum_{n \in E_1} b_n c_n < \infty$ and consequently S_1 is finite. Similarly, S_2 and S_3 are finite if $E_2 \neq \emptyset$, respectively $E_3 \neq \emptyset$.

If $E_4 \neq \emptyset$, then $\sum_{E_4} a_n b_n (a_n b - b_n a)^2 < \infty$. Then, $\sum_{E_4} a_n b_n (a_n \frac{b}{a+b} - b_n \frac{a}{a+b})^2 < \infty$ and by Lemma 2.4.6, we have $\sum_{E_4} a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - t \log \frac{b}{a})) < \infty$. By (2.4.4), $\sum_{E_4} a_n c_n (a_n c - c_n a)^2 < \infty$ and because $a_n (a_n c - c_n a)^2 \xrightarrow{n \in E_4} \frac{a^3}{(a+b)^3} c > 0$ we have $\sum_{E_4} c_n < \infty$. Consequently, $\sum_{E_4} a_n c_n$ and $\sum_{E_4} b_n c_n$ are finite. Therefore S_4 is finite.

If $E_5 \neq \emptyset$, then $\sum_{E_5} a_n c_n (a_n c - c_n a)^2 < \infty$. Then $\sum_{E_5} a_n c_n (a_n \frac{c}{a+c} - c_n \frac{a}{a+c})^2 < \infty$ and by Lemma 2.4.6 we have $\sum_{E_5} a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} - t \log \frac{c}{a})) < \infty$. We have also that $\sum_{E_5} b_n < \infty$, and we obtain that $S_5 < \infty$.

Similarly, S_6 and S_7 are finite if E_6 respectively E_7 are nonempty. Therefore the sum (2.4.3) is finite. This shows that $t \in T(N)$. As this is true for all $t \in \mathbb{R}$, we obtain that $T(N) = \mathbb{R}$ and therefore N is of type II_∞ . □

We summarize the results of this section in the following:

Theorem 2.4.8. *Let $M = \pi_\varphi(A)'' = \otimes(M_3(\mathbb{C}), \varphi_n)$, where $\varphi = \otimes \varphi_n$, $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ with h_n as in (2.4.1), $N = M^\alpha$.*

Then N is a factor if and only if

$$\sum a_n(1 - a_n) = \infty, \sum b_n(1 - b_n) = \infty \text{ and } \sum c_n(1 - c_n) = \infty.$$

If N is a factor then:

(i) *N is type II_1 if and only if there exist $a, b, c \in (0, 1)$, $a + b + c = 1$ such that*

$$\sum (a_n - a)^2 + (c_n - c)^2 + (b_n - b)^2 < \infty$$

If N is not of type II_1 then

(ii) *N is type II_∞ if and only if there exist $a, b, c \in (0, 1)$, $a + b + c = 1$ such that*

$$\sum a_n b_n (a_n b - b_n a)^2 + \sum a_n c_n (a_n c - c_n a)^2 + \sum b_n c_n (c_n b - b_n c)^2 < \infty$$

(iii) *N is a factor of type III otherwise.*

For the standard action of the $(k-1)$ -dimensional torus we have the following result:

Theorem 2.4.9. *Let $A = \otimes M_k(\mathbb{C})$ be the k^∞ -UHF algebra. On A we consider a faithful diagonal product state $\varphi = \otimes \varphi_n$, $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ where $h_n = \text{diag}(a_n^0, a_n^1, \dots, a_n^{k-1})$. Let $A^{T^{k-1}}$ the fixed point algebra under the standard action of the $(k-1)$ -dimensional torus, given in the beginning of this subsection and $N = \pi_{\varphi^{T^{k-1}}}(A^{T^{k-1}})$.*

Then, by [AP], N is factor if and only if $\sum_{n=1}^{\infty} \left\{ \sum_{i \in K} a_n^i (1 - \sum_{i \in K} a_n^i) \right\} = \infty$ for every $K \subset \{0, 1, \dots, k-1\}$. In this case:

(i) N is of type II_1 if and only if there exist a^0, a^1, \dots, a^{k-1} in $(0, 1)$ with $\sum_{i=0}^{k-1} a_i = 1$ such that

$$\sum (a_n^i - a^i)^2 < \infty$$

If N is not of type II_1 then

(ii) N is of type II_∞ if and only if there exist a^0, a^1, \dots, a^{k-1} in $(0, 1)$ with $\sum_{i=0}^{k-1} a_i = 1$ such that

$$\sum_{n=1}^{\infty} \sum_{i,j} a_n^i a_n^j (a_n^i a_n^j - a_n^j a_n^i)^2 < \infty$$

(iii) N is of type III otherwise.

We will see now that any $ITPFI_3$ factor can be obtained as the fixed point factor under the standard action of T^2 .

Theorem 2.4.10. *Let $A = \otimes M_3(\mathbb{C})$ and let $\varphi = \otimes \varphi_n$ where $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ be a faithful product state where $h_{2n} = \text{diag}(a_0^n, a_1^n, a_2^n)$, $\text{tr}(h_n) = 1$, and $h_{2n+1} = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then $\pi_\varphi(A^{T^2})''$ is isomorphic with an $ITPFI_3$ factor and therefore any $ITPFI_3$ can be obtained in this way.*

Proof. Let $X = \prod \{0, 1, 2\}$. On X we consider the product measure $\eta = \otimes \eta_n$ where $\eta_{2n}(i) = \frac{1}{3}$, $\eta_{2n+1}(i) = a_i^n$ for all $i \in \{0, 1, 2\}$ and for all $n \geq 1$. We can identify (X, η) with $(X_1 \times X_2, \mu \times \nu)$ where $X_1 = X_2 = \prod \{0, 1, 2\}$ and the measures $\mu = \otimes \mu_n$

and $\nu = \otimes \nu_n$ are given by $\mu_n(i) = a_i^n$, $\nu_n(i) = \frac{1}{3}$ for $n \geq 1$ and $i \in \{0, 1, 2\}$. The equivalence relation \mathcal{R}_∞ on X is given by:

$$x\mathcal{R}_\infty y \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for } i > n \text{ and} \\ \text{card}\{i; 1 \leq i \leq n, x_i = k\} = \text{card}\{i; 1 \leq i \leq n, y_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

Also, \mathcal{R}_∞ can be seen as an equivalence relation on $X_1 \times X_2$, denoted also by \mathcal{R}_∞ .

For $(x, y) \in X_1 \times X_2$, with $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$,

$$(x, y)\mathcal{R}_\infty(x', y') \text{ if and only if there exists } n \in \mathbb{N} \text{ such that } x_i = x'_i, y_i = y'_i \text{ for } i > n \\ \text{and } \text{card}\{i; 1 \leq i \leq n, x_i = k\} + \text{card}\{i; 1 \leq i \leq n, y_i = k\} \\ = \text{card}\{i; 1 \leq i \leq n, x'_i = k\} + \text{card}\{i; 1 \leq i \leq n, y'_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

Then, (see Section 2.4),

$$\pi_\varphi(A^{T^2})'' \simeq W^*(X_1 \times X_2, \mu \times \nu, \mathcal{R}_\infty).$$

The Lebesgue measure on \mathbb{R} is denoted by λ . With these notations, we consider also the equivalence relation $\tilde{\mathcal{R}}_\infty$ on $X_1 \times X_2 \times \mathbb{R}$, given by:

$$(x, y, s)\tilde{\mathcal{R}}_\infty(x', y', t) \text{ if and only if} \\ (x, y)\mathcal{R}_\infty(x', y') \text{ and } t = s - \log \delta((x', y'), (x, y)).$$

We also consider the following equivalence relation on X_2 , denoted by \mathcal{S}_∞ :

$$y\mathcal{S}_\infty y' \text{ if and only if there exists } n \in \mathbb{N} \text{ such that } y_i = y'_i \text{ for } i > n \text{ and} \\ \text{card}\{i; 1 \leq i \leq n, y_i = k\} = \text{card}\{i; 1 \leq i \leq n, y'_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

As ν is \mathcal{S}_∞ -invariant, we have that

$$(x, y, t)\tilde{\mathcal{R}}_\infty(x, y', t) \text{ for all } x, t \text{ and all } (y, y') \in \mathcal{S}_\infty$$

On X_1 we consider the following equivalence relations

$$x\mathcal{T}_n y \text{ if and only if } x_i = y_i, \text{ for all } i > n$$

and

$x\mathcal{T}y$ if and only if there exists n such that $x\mathcal{T}_n y$

Let now $\tilde{\mathcal{T}}$ be the following equivalence relation $(X_1 \times \mathbb{R}, \mu \times \lambda)$

$(x, s)\tilde{\mathcal{T}}(x', t)$ if there exists n such that $x\mathcal{T}_n x'$ and $t = s - \log \prod_{i=1}^n \frac{a_{x'_i}^i}{a_{x_i}^i} = s - \log \delta(x', x)$.

Let $f \in L^\infty(X_1 \times X_2 \times \mathbb{R}, \nu \times \mu \times \lambda)^{\tilde{R}_\infty}$, the space of all essentially bounded functions defined on $(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)$, invariant under \tilde{R}_∞ . We can assume that f is bounded and invariant, ([Z1], p.21), as any invariant function is equal almost everywhere with an invariant bounded function. Because ν is \mathcal{S}_∞ -invariant, we have

$$f(x, y, t) = f(x, y', t) \text{ for all } (x, y, t) \in X_1 \times X_2 \times \mathbb{R}, y\mathcal{S}_\infty y'.$$

As by [AP], \mathcal{S}_∞ is ν -ergodic we have that for all (x, t) , $y \mapsto f(x, y, t)$, is constant ν -a.e. on X_2 . We denote this value by $\mathbf{f}(x, t)$. Hence, we have a function $\mathbf{f} : X_1 \times \mathbb{R} \rightarrow \mathbb{R}$. Fix $(x, t) \in X_1 \times \mathbb{R}$ with $x = (x_1, x_2, \dots) \in X_1$. Let $(y, t') \in X_1 \times \mathbb{R}$ where $y = (y_1, y_2, \dots, y_m, x_{m+1}, \dots) \in X_1$ and $t' := t - \log \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i}$. In other words $(x, t)\tilde{\mathcal{T}}(y, t')$.

Let us denote by C_1 the cylinder set $C(y_1, \dots, y_m) \subset X_2$. As $z \mapsto f(x, z, t)$ is constant ν -a.e z , there exists $A_1 \subset X_2$, $\nu(A_1) = 0$ such that $f(x, z, t) = \mathbf{f}(x, t)$ for all $z \in C_1 - A_1$. Let C_2 be the cylinder set $C(x_1, \dots, x_m) \subset X_2$. Similarly, as $f(y, z, t') = \mathbf{f}(z, t')$ for ν -a.e. $z \in X_2$, there exists also $A_2 \subset X_2$ $\nu(A_2) = 0$ such that $f(y, z, t') = \mathbf{f}(y, t')$ for all $z \in C_2 - A_2$. Let ϕ be a Borel isomorphism on X_2 that affects only the first m coordinates and such that $\phi(C_1) = C_2$ i.e., $\phi(y_1, y_2, \dots, y_m, y_{m+1}, \dots) = (x_1, x_2, \dots, x_m, y_{m+1}, \dots)$ when $y \in C_1$. As $\nu(A_2) = 0$, we also have that $\nu(\phi^{-1}(A_2)) = 0$. Therefore $\nu(C_1 - A_1 - \phi^{-1}(A_2)) > 0$. Let $z \in C_1 - A_1 - \phi^{-1}(A_2)$. Then $\phi(z) \in C_2 - A_2$. We denote $\phi(z)$ by \hat{z} . It also exists σ a partial Borel automorphism, σ on $X_1 \times X_2$ with $\text{Graph}(\sigma) \in \mathcal{R}_\infty$ such that $\sigma(x, z) = (y, \hat{z})$. We have $f(x, z, t) = \mathbf{f}(x, t)$, $f(y, \hat{z}, t') = \mathbf{f}(y, t')$ and

$$\delta((y, \hat{z})(x, z)) = \frac{d\mu \times \nu \circ \sigma}{d\mu \times \nu}(x, z) = \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} = t'.$$

We have:

$$\mathbf{f}(x, t) = f(x, z, t) = f(y, \hat{z}, t') = \mathbf{f}(y, t')$$

which proves that $(x, t) \mapsto \mathbf{f}(x, t)$ is $\tilde{\mathcal{T}}$ invariant. The function \mathbf{f} is bounded and measurable, as $\mathbf{f}(x, y) = \int f(x, y, t) d\nu(y)$.

Therefore $L^\infty(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)^{\tilde{\mathcal{R}}_\infty}$ can be identified with $L^\infty(X_1 \times \mathbb{R}, \mu \times \lambda)^{\tilde{\mathcal{T}}}$. Moreover the action of \mathbb{R} by translation on $L^\infty(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)^{\tilde{\mathcal{R}}_\infty}$ corresponds to the action of \mathbb{R} on $L^\infty(X_1 \times \mathbb{R}, \mu \times \lambda)^{\tilde{\mathcal{T}}}$. In other words, the space $X_1 \times X_2 \times \mathbb{R} / \tilde{\mathcal{R}}_\infty$ on which \mathbb{R} acts by translation is identified with the space $X_1 \times \mathbb{R} / \tilde{\mathcal{T}}$ on which \mathbb{R} acts by translation. This means that the associated flow of $(\mathcal{R}_\infty, X_1 \times X_2, \mu \times \nu)$ is identified with the space $X_2 \times \mathbb{R} / \tilde{\mathcal{T}}$ on which \mathbb{R} acts by translation and therefore, by Theorem 1.2.19,

$$W^*(X_1 \times X_2, \mu \times \nu, \mathcal{R}_\infty) \simeq W^*(X_1, \mu, \mathcal{T}).$$

The last one is an ITPFI factor. Therefore, we conclude that $\pi_\varphi(A^{T^2})''$ is isomorphic to an ITPFI factor.

□

2.5 Other Examples

Let G be a compact group and $\pi : G \rightarrow \mathcal{U}(M_k(\mathbb{C}))$ a diagonal unitary representation of G . Let $\alpha = \text{Ad } \pi(g)$ be the xerox action of π on $A = \otimes M_k(\mathbb{C})$ and $\varphi = \otimes \varphi_n$ a faithful diagonal product state on A . Let us denote $N = \pi_{\varphi^G}(A^G)''$. Then α induces an action denoted also by α on $M = \pi_{\varphi}(A)''$ and $N \simeq (\pi_{\varphi}(A)'')^{\alpha}$. On the other hand, $A^G \simeq C_r^*(X, \mathcal{R})$ where $X = \prod\{0, 1, \dots, k-1\}$ and \mathcal{R} is an AF equivalence relation on X . Consequently $N \simeq W^*(X, \mu, \mathcal{R})$ with $\mu = \otimes \mu_n$ the product measure on X induced by φ given by $\mu_n(i-1) = \varphi_n(e_{ii}^k)$ for $1 \leq i \leq k$ and for all n . In order to find conditions for N to be a factor and to determine its type we proceed as follows:

- (1) We find conditions for \mathcal{R} to be μ -ergodic,
- (2) We determine the \mathcal{R} -invariant ergodic measures on X to find necessary and sufficient conditions for N to be of type II_1 ,
- (3) If N is not a factor of type II_1 , we use Connes' invariant T to determine when N is of type II_{∞} or type III. We have that N is of type II_{∞} if and only if $T(N) = \mathbb{R}$ and N is of type III if and only if $T(N) \neq \mathbb{R}$.

In Section 2.4 we analyzed the fixed point algebra under the standard action of the 2-torus. Before considering another example we give a sufficient condition for factoriality.

We give a better characterization of a C^* -subalgebra of a C^* -algebra $C_r^*(X, \mathcal{R})$, associated to an étale equivalence relation \mathcal{R} that contains $C(X)$, than the one presented in Section 1.1.4. In Section 1.1.4, we see that such a C^* -subalgebra is of the form $A(\mathcal{S})$, for a unique (open) subequivalence relation \mathcal{S} of \mathcal{R} . We show that any C^* -subalgebra of the form $A(\mathcal{S})$ with \mathcal{S} open subequivalence relation of \mathcal{R} is isomorphic with $C_r^*(X, \mathcal{S})$, and therefore we have the following result:

Theorem 2.5.1. *For each subequivalence relation \mathcal{S} of \mathcal{R} , $A(\mathcal{S})$ is a C^* -subalgebra of $C_r^*(X, \mathcal{R})$ containing $C(X)$. Conversely, each C^* -subalgebra of $C_r^*(X, \mathcal{R})$ contain-*

ing $C(X)$ is of the form $A(\mathcal{S})$ for a unique subequivalence \mathcal{S} . The correspondence $\mathcal{S} \mapsto A(\mathcal{S})$ is an inclusion preserving bijection between the collection of subequivalence relations of \mathcal{R} and C^* -subalgebras of $C_r^*(X, \mathcal{R})$ containing $C(X)$. Moreover, $C_r^*(X, \mathcal{S}) \simeq A(\mathcal{S})$, i.e., any C^* -subalgebra of $C_r^*(X, \mathcal{R})$ containing $C(X)$ is isomorphic to $C_r^*(X, \mathcal{S})$, for a unique subequivalence relation \mathcal{S} of \mathcal{R} .

Proof. Let μ be a σ -finite measure on X with $\text{supp}(\mu) = X$. We denote by $\text{Ind}^{\mathcal{R}}\mu$ respectively $\text{Ind}^{\mathcal{S}}\mu$ the representations induced by μ for the equivalence relations \mathcal{R} resp. \mathcal{S} (see Section 1.1.4). The corresponding measures on \mathcal{R} resp. \mathcal{S} are denoted by $\nu_{\mathcal{R}}$ resp. $\nu_{\mathcal{S}}$ and it is clear that $\nu_{\mathcal{S}}$ is $\nu_{\mathcal{R}}$ restricted to \mathcal{S} . We can identify $L^2(\mathcal{S}, \nu_{\mathcal{S}})$ with a subspace of $H = L^2(\mathcal{R}, \nu_{\mathcal{R}})$, denoted by H_0 (by identifying a function defined on \mathcal{S} with a function defined on \mathcal{R} which is zero on $\mathcal{R} \setminus \mathcal{S}$). We denote by π the restriction to H_0 , of the representation $\text{Ind}^{\mathcal{R}}\mu$ of $C_c(\mathcal{S})$. But then, π can be regarded as $\text{Ind}^{\mathcal{S}}\mu$. We can identify $C_r^*(X, \mathcal{S})$ with the norm closure of $\pi(C_c(\mathcal{S}))$ in $B(H_0)$. On the other hand, by Remark 1.1.2, $C_c(\mathcal{S})$ is dense in $A(\mathcal{S})$, we can identify $A(\mathcal{S})$ with the norm-closure of $\text{Ind}^{\mathcal{R}}\mu(C_c(\mathcal{S}))$ in $B(H)$. Let N be the von Neumann algebra, acting on H , obtained as the weak closure of $A(\mathcal{S})$ and let e be the orthogonal projection from H to H_0 which is $e(f) = f|_{\mathcal{S}}$, for all $f \in H$. We show that $e \in N'$ and $z(e) = 1$ and therefore we have $N_e \simeq N$ with N_e acting on H_0 . The isomorphism from N to N_e is given by $x \mapsto exe$. As this correspondence is norm preserving we must have $\|\text{Ind}^{\mathcal{R}}\mu(f)\| = \|\pi(f)\|$ for all $f \in C_c(\mathcal{S})$. Therefore, we obtain $A(\mathcal{S}) \simeq C_r^*(X, \mathcal{S})$.

We prove now that $e \in N'$ and $z(e) = 1$. The fact that $e \in N'$ is clear because N leaves H_0 invariant. The weak closure of $\text{Ind}^{\mathcal{R}}\mu(C_c(\mathcal{R}))$ in $B(H)$ is $W^*(X, \mu, \mathcal{R})$. We denote it by M . We have $L^\infty(X, \mu) \subset N \subset M$ and $L^\infty(X, \mu)$ is maximal abelian in M . Consequently, $L^\infty(X, \mu)$ is maximal abelian in N and therefore $Z(N)$ is contained in $L^\infty(X, \mu)$. Hence, we can identify $z(e)$ with a function $\chi_A \in L^\infty(X, \mu)$. Let $B \subseteq X$, $\mu(B) < \infty$. Then $\chi_{A \cap B} = z(e)e(\chi_B) = e(\chi_B) = \chi_B$. As this is true for all B with $\mu(B) < \infty$, we have $\mu(A^c) = 0$ and therefore $z(e) = 1$. \square

Proposition 2.5.2. *If A is the k^∞ -UHF algebra and G is a compact group acting on*

A by a diagonal zero action induced by a representation π of G on $\mathcal{U}(M_k(\mathbb{C}))$, then A^G is a C^* -subalgebra of $A^{T^{k-1}}$.

Proof. First we see that the fixed point algebra $A^{T^{k-1}}$ can be obtained also by acting on A with the following action of the k -dimensional torus T^k :

$$\alpha : T^k \rightarrow \text{Aut}(A); \quad \alpha(t_1, \dots, t_k) := \otimes \text{Ad } \tilde{\pi}_{t_1, \dots, t_k}, \text{ where}$$

$$\tilde{\pi}_{t_1, \dots, t_k} := \text{diag}(e^{it_1}, e^{it_2}, e^{it_3}, \dots, e^{it_k}); \quad t_i \in [0, 2\pi).$$

Clearly, $\pi(G)$ is a subgroup of the k -dimensional torus and therefore, $A^{T^{k-1}} \subseteq A^G$. Moreover, both of these two C^* -algebras contain $C(X)$. \square

With the above notations, we recall that φ^G (resp $\varphi^{T^{k-1}}$) denote the restriction of a product diagonal state $\varphi = \otimes \varphi_n$ on A to A^G (resp. $A^{T^{k-1}}$).

Proposition 2.5.3. *If $\varphi^{T^{k-1}}$ is a factor state then φ^G is also a factor state.*

Proof. We have $A^{T^{k-1}} \simeq C_r^*(X, \mathcal{R}_\infty)$ and $A^G \simeq C_r^*(X, \mathcal{R})$. Here $X = \prod \{0, 1, \dots, k-1\}$ and \mathcal{R}_∞ is the equivalence relation on X given by:

$$x \mathcal{R}_\infty y \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for } i > n \text{ and}$$

$$\text{card}\{i; 1 \leq i \leq n; x_i = j\} = \text{card}\{i; 1 \leq i \leq n; y_i = j\} \text{ for } 0 \leq j \leq k-1.$$

The equivalence relation is induced by the action of the group S_∞ of finite permutations acting on X . By Theorem 2.5.1, $\mathcal{R} \supseteq \mathcal{R}_\infty$. Let $\mu = \otimes \mu_n$ the product measure on X induced by φ given by $\mu_n(i-1) = \varphi_n(e_{ii}^k)$ for $i \in \{1, \dots, k\}$ and $n \geq 1$. On the other hand, $\pi_{\varphi^{T^{k-1}}}(A^{T^{k-1}}) \simeq W^*(X, \mu, \mathcal{R})$ and $\pi_{\varphi^G}(A^G) \simeq W^*(X, \mu, \mathcal{R})$. But $W^*(X, \mu, \mathcal{R})$ is a factor if and only if \mathcal{R}_∞ is μ -ergodic. Consequently, as $\mathcal{R}_\infty \subseteq \mathcal{R}$, \mathcal{R} is also μ -ergodic, $W^*(X, \mathcal{R}, \mu)$ is a factor or equivalently, φ^G is a factor state. \square

If $\mu = \otimes \mu_n$ is the product measure on X induced by the state φ , then, by [AP], \mathcal{R}_∞ is μ -ergodic if and only if

$$\sum_{n=1}^{\infty} \left(\sum_{i \in J} \mu_n(i) (1 - \sum_{i \in J} \mu_n(i)) \right) = \infty.$$

for all $J \subset \{0, 1, \dots, k-1\}$. Therefore with the above notations, we have a sufficient condition for factoriality.

For any given xerox action induced by a diagonal representation π of a compact group G we can classify in types the GNS representation of the fixed point algebra. The technique can be used in general, as explained in the beginning of this section. We already used these techniques in Section 2.4, where we studied the standard action of the 2-dimensional torus. We will consider here a different example. The xerox action considered is also different from the type of actions studied in [BP2].

Let us consider now $A = \otimes M_3(\mathbb{C})$ and $\alpha : T \rightarrow \text{Aut}(A)$ the xerox action given by:

$$\alpha(t) = \otimes \text{Ad} \begin{bmatrix} 1 & & \\ & e^{it} & \\ & & e^{-it} \end{bmatrix}$$

Let A^α be the fixed point algebra under this action.

Let $\pi : SU(2) \rightarrow M_3(\mathbb{C})$ given by

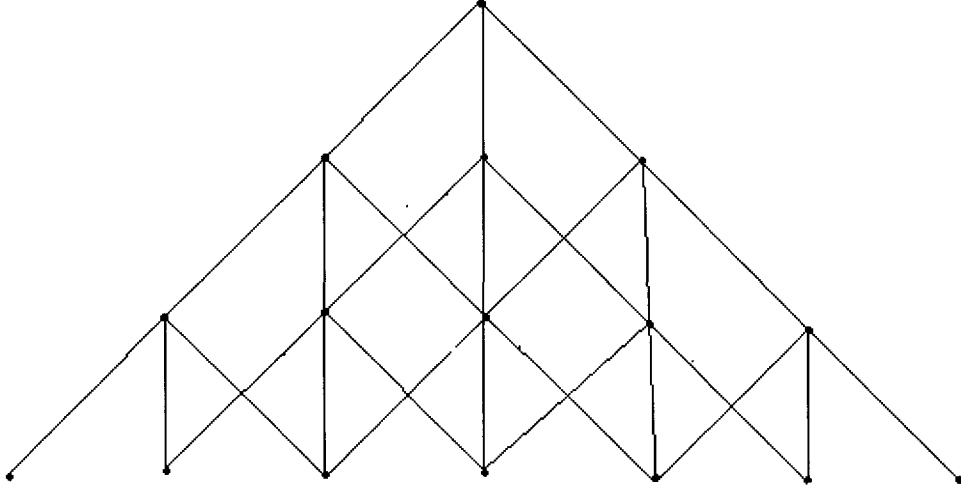
$$\pi \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{bmatrix}$$

In fact, the xerox action α considered above is the restriction to the maximal torus T of $SU(2)$ of the xerox action on A induced by the above representation π of $SU(2)$.

As usual, we have $A^\alpha \simeq A(X, \Gamma) \simeq C_r^*(X, \mathcal{R})$, where Γ is a countable group of homeomorphisms on $X = \prod\{0, 1, 2\}$ which generates the equivalence relation \mathcal{R} given by:

$$x\mathcal{R}y \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i, \text{ for } i \geq n+1 \text{ and} \quad (2.5.1)$$

$$\begin{aligned} \frac{1}{2} (\text{card} \{x_i = 0; i = 1 \leq n\} - \text{card} \{y_i = 0; 1 \leq i \leq n\}) &= \text{card} \{y_i = 1; 1 \leq i \leq n\} - \\ - \text{card} \{x_i = 1; 1 \leq i \leq n\} &= \text{card} \{y_i = 2; 1 \leq i \leq n\} - \text{card} \{x_i = 2; 1 \leq i \leq n\}. \end{aligned}$$

Figure 2.5: The Bratteli diagram of A^α

As in Section 2.2, we can identify \mathcal{R} with the tail equivalence on the Bratteli diagram of A^α , (i.e., \mathcal{R} is isomorphic to the tail equivalence on the Bratteli diagram) as in Figure 2.5 (the first 3 levels of the diagram).

On A we consider the product state:

$$\varphi = \otimes \text{tr}(h_n \cdot) \text{ where } h_n = \text{diag}(a_n, b_n, c_n), \text{ where } a_n, b_n, c_n > 0 \text{ and } \text{tr}(h_n) = 1. \quad (2.5.2)$$

The state φ induces a product measure $\mu = \otimes \mu_n$ on X , where μ_n are probability measures on $\{0, 1, 2\}$ with $\mu_n(0) = a_n$, $\mu_n(1) = b_n$, $\mu_n(2) = c_n$ for all $n > 0$. We have that $\pi_{\varphi^\alpha}(A^\alpha)'' \simeq W^*(X, \mu, \mathcal{R})$. With the usual notation (see, Section 2.1), $M = \otimes (M_k(C), \varphi_n) = \pi_\varphi(A)''$ and $N = (\pi_\varphi(A)'')^\alpha = M^\alpha \simeq \pi_{\varphi^\alpha}(A^\alpha)''$. We prove first the following lemma:

Lemma 2.5.4. *Let $X = \prod \{0, 1, 2\}$, $\mu = \otimes \mu_n$ with $\mu_n(0) = a_n$, $\mu_n(1) = b_n$, $\mu_n(2) = c_n$. Then \mathcal{R} defined in (2.5.1) is μ -ergodic if and only if $\sum_{n=1}^{\infty} a_n(1 - a_n) = \infty$.*

Proof. Assume that $\sum_{n=1}^{\infty} a_n(1 - a_n) = \infty$.

Case 1. First, let us also assume, that $\sum c_n < \infty$. We consider the set $A = \{x \in X; x_n = 0 \text{ or } 1\}$. As $\sum c_n < \infty$, we have $\prod_{n=1}^{\infty} (1 - c_n) > 0$ and therefore $\mu(A) > 0$. We denote by $\mathcal{R}(A)$ and $\mathcal{T}(A)$ the saturation of A with respect to \mathcal{R} and \mathcal{T} , where \mathcal{T} is the tail equivalence on X .

First, we show that any x in $\mathcal{T}(A)$ which has infinitely many 0's and 1's is in $\mathcal{R}(A)$. Indeed, let us take $x \in \mathcal{T}(A)$; then x has finitely many 2's. Assume that the number of 2's is equal to k . Since x has infinitely many 1's, we replace the k 2's and k 1's with $2k$ zeros; we obtain a $y \in A$, which is \mathcal{R} -equivalent to x . Hence, x is in $\mathcal{R}(A)$.

Since the set of paths which have only finitely many 0's or finitely many 1's has measure zero, $\mathcal{R}(A) = \mathcal{T}(A)$ up to a set of measure zero. On the other hand, by Kolmogorov's zero-one law $\mu(\mathcal{T}(A)) = 1$ and so $\mu(\mathcal{R}(A)) = 1$.

On A we consider the following equivalence relation, denoted by \mathcal{S}_{∞} :

$x\mathcal{S}_{\infty}y$ if and only if there exists $n \geq 1$ such that :

$$x_i = y_i, \text{ for } i > n \text{ and } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

Let us consider the product probability measure $\nu = \otimes \nu_n$ on A , with $\nu_n(i) = \frac{1}{1-c_n} \mu_n(i)$, $i \in \{0, 1\}$. As $\sum c_n < \infty$ we have $\sum \nu_n(0)(1 - \nu_n(0)) = \sum \frac{a_n b_n}{(1-c_n)^2} = \infty$ and therefore, by [AP], \mathcal{S}_{∞} is ν -ergodic. On the other hand, as $\prod (1 - c_n) < \infty$, $\nu \sim \mu|_A$, and therefore $\mu|_A$ is \mathcal{S}_{∞} -ergodic.

Let f be a \mathcal{R} -invariant measurable function on X . We want to show that f is constant on X , μ -a.e. Clearly, f restricted to A is \mathcal{S}_{∞} -invariant. Hence f restricted to A is constant μ -a.e and therefore the restriction of f to $A \subset X$ is constant μ -a.e. But then f is μ -a.e. constant on $\mathcal{R}(A)$. As seen, $\mu(\mathcal{R}(A)) = 1$. Hence, f is constant μ -a.e. on X . We conclude that \mathcal{R} is μ -ergodic.

Case 2. Assume $\sum (1 - c_n) < \infty$. Then, $\sum b_n < \infty$ and we prove the ergodicity as in the previous case, by replacing c_n with b_n .

Case 3. Assume $\sum a_n(1 - a_n) = \infty$, $\sum c_n(1 - c_n) < \infty$ and both sets $I :=$

$\{n; c_n < 1/2\}$ and I^c are infinite. This situation corresponds to the case when c_n has both 0 and 1 as limits points. As $\sum c_n(1 - c_n) < \infty$, we have $\sum_{n \in I} c_n < \infty$. But $\sum_{n \in I^c} (1 - c_n) = \sum_{n \in I^c} (a_n + b_n) < \infty$. Hence, $\sum_{n \in I^c} a_n b_n < \infty$ and consequently $\sum_{n \in I} a_n b_n = \infty$ (otherwise $\sum a_n(1 - a_n)$ would be finite).

Let $X_1 = \prod_{n \in I} \{0, 1, 2\}$, $X_2 = \prod_{n \in I^c} \{0, 1, 2\}$, $\mu_1 = \bigotimes_{n \in I} \mu_n$ and $\mu_2 = \bigotimes_{n \in I^c} \mu_n$. We can identify X with $X_1 \times X_2$, μ with $\mu_1 \times \mu_2$ and via this identification we can see \mathcal{R} as an equivalence relation on $X_1 \times X_2$. Let \mathcal{T}_2 be the tail equivalence on (X_2, μ_2) and let \mathcal{R}_1 the equivalence relation given by (2.5.1) but on the space (X_1, μ_1) . First, from Case 1, $L^\infty(X_1, \mu_1)^{\mathcal{R}_1}$, the space of all essentially bounded functions on (X_1, μ_1) invariant under \mathcal{R}_1 reduces to scalars. Using similar arguments to those in the proof of the Theorem 2.4.10 we have

$$L^\infty(X_1 \times X_2, \mu_1 \times \mu_2)^{\mathcal{R}} \simeq L^\infty(X_2, \mu_2)^{\mathcal{T}}$$

By the Kolmogorov's zero-one law, $L^\infty(X_2, \mu_2)^{\mathcal{T}}$ reduces to scalars and hence we conclude that $L^\infty(X_1 \times X_2, \mu_1 \times \mu_2)^{\mathcal{R}}$ reduces to scalars.

From cases 1, 2 and 3, we conclude that if $\sum a_n(1 - a_n) = \infty$ and $\sum c_n(1 - c_n) < \infty$ then \mathcal{R} is μ -ergodic. Similarly, we can prove ergodicity if $\sum a_n(1 - a_n) = \infty$ and $\sum b_n(1 - b_n) < \infty$.

Assume now that $\sum a_n(1 - a_n) = \infty$, $\sum b_n(1 - b_n) = \infty$ and $\sum c_n(1 - c_n) = \infty$. In this case we know that \mathcal{R}_∞ is μ -ergodic, (\mathcal{R}_∞ is the equivalence relation considered in Section 2.4). As $\mathcal{R}_\infty \subset \mathcal{R}$ we obtain that \mathcal{R} is μ -ergodic.

For the converse, let us assume that $\sum a_n(1 - a_n) < \infty$. Let $B = \{x \in X; x_n = 1 \text{ or } x_n = 2 \text{ for all } n \geq 1\}$ and $C = \{x \in X; x_1 = 0, x_n = 1 \text{ or } x_n = 2 \text{ for all } n \geq 2\}$. Then $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are disjoint invariant sets of positive measure. Therefore \mathcal{R} is not μ -ergodic. \square

Proposition 2.5.5. $N \simeq W^*(X, \mu, \mathcal{R})$ is a factor if and only if $\sum a_n(1 - a_n) = \infty$.

Proof. The proof follows from Lemma 2.5.4 and Theorem 1.2.17. \square

In the same way we did for the standard xerox action of the 2-torus, we can determine the type of the factor obtained. Since the measure μ is non atomic, the factor N is either of type II or of type III.

Proposition 2.5.6. *With the above notation, N is type II_1 if and only if it exists $x \in (0, 1)$ such that*

$$\sum (a_n - a)^2 + (b_n - b)^2 + (c_n - c)^2 < \infty \quad (2.5.3)$$

where $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$, and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$.

Proof. N is a factor of type II_1 if and only if there is a \mathcal{R} -invariant ergodic probability measure $\nu \sim \mu$ on X , equivalently, if and only if there exists a \mathcal{R} -extremal invariant measure ν equivalent to μ , on X . By [W], extremal invariant measures are in bijection with the extremal traces on A^α . By [Pr], diagonal extremal traces are restrictions of symmetric states (see Definition 1.2.10) and therefore any invariant ergodic measure should be of the form $\nu = \otimes \nu_0$, with $\nu_0(0) = a$, $\nu_0(1) = b$, $\nu_0(2) = c$ and $abc \neq 0$. Such a measure is \mathcal{R} -invariant if and only if $a^2 = bc$. As $a + b + c = 1$, if we denote $\frac{b}{a}$ by x , we have $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$ and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$. By Kakutani's Theorem such a measure is equivalent to μ if and only if (2.5.3) holds.

As a remark, this proposition can also be proved using [H2]. □

When N is properly infinite, we use the computation of Connes' invariant T to determine whether N is of type II_∞ or III . First, we assume that $\sum a_n(1 - a_n) = \infty$, $\sum b_n(1 - b_n) = \infty$ and $\sum c_n(1 - c_n) = \infty$.

Proposition 2.5.7. *If $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ with h_n as in (2.5.2) and N properly infinite, then $t \in T(N)$ if and only if there exists $s \in [0, 2\pi)$ such that*

$$\sum a_n b_n (1 - \cos(t \log \frac{b_n}{a_n} - s)) + a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} + s)) + b_n c_n (1 - \cos(t \log \frac{b_n}{c_n} - 2s)) < \infty \quad (2.5.4)$$

Proof. The proof is similar to the proof of Proposition 2.4.2. □

Proposition 2.5.8. *If $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ is as in (2.5.2) and $(a_n, b_n, c_n)_{n \geq 1}$ has two distinct limit points (a^1, b^1, c^1) and (a^2, b^2, c^2) with $a^i b^i \neq 0$ for $i = 1, 2$ and $\frac{b^1}{a^1} \neq \frac{b^2}{a^2}$, then N is of type III.*

Proof. Similar to the proof of Proposition 2.4.5. □

Proposition 2.5.9. *Let $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ with h_n as in (2.5.2). If $(a_n, b_n, c_n)_{n \geq 1}$ has a subsequence $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ such that $a_{n_k} \rightarrow 0$, $\sum_{k=1}^{\infty} a_{n_k} = \infty$ and either b_{n_k} (or c_{n_k}) converges to a nonzero number, then N is of type III. If N is of type II_{∞} and $(a_n, b_n, c_n)_{n \geq 1}$ has a subsequence $(a_{n_k}, b_{n_k}, c_{n_k})_{k \geq 1}$ with $a_{n_k} \rightarrow 0$ and either b_{n_k} (or c_{n_k}) converges to a nonzero number, then $\sum_{k=1}^{\infty} a_{n_k} < \infty$.*

Proof. The proof is similar to the proof for Corollary 2.4.4. □

Proposition 2.5.10. *With the above notation, $T(N) = \mathbb{R}$, i.e., N is semifinite, if and only if there exists a unique $x > 0$ such that:*

$$\sum a_n b_n (a_n b - b_n a)^2 + \sum a_n c_n (a_n c - c_n a)^2 + \sum b_n c_n (c_n b - b_n c)^2 < \infty \quad (2.5.5)$$

where $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$, and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$

Proof. Assume that N is of type II_{∞} , i.e., (2.5.5) holds. To have a geometric picture, as in the proof of Proposition 2.4.7, we can see the sequence $(a_n, b_n, c_n)_{n \geq 1}$ inside the triangle determined by the vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$. We denote by (AB) the interior of the line segment determined by A and B and similarly for (AC) and (BC) . We call (AB) , (AC) and (BC) edges. By Proposition 2.5.8 the sequence $(a_n, b_n, c_n)_{n \geq 1}$ has at most a limit point in the interior of the triangle ABC and it has at most a limit point on each edge.

First, let us assume that the sequence $(a_n, b_n, c_n)_{n \geq 1}$ does not have a limit point in the interior of the triangle ABC . Then:

Claim I. On at least two edges there exists a limit point of the sequence $(a_n, b_n, c_n)_{n \geq 1}$

Proof. The same proof as for Claim I, Proposition 2.4.7. \square

Hence, if the sequence $(a_n, b_n, c_n)_{n \geq 1}$ does not have a limit point in the interior of (ABC), it has one limit point on at least two edges. We will assume that there is one limit point on (AB) and another one on (AC). The other possible cases follow in the same way. We denote these points by $(m, n, 0)$ and $(q, 0, r)$ and take (a, b, c) the point situated at the intersection of the lines determined by these points and their opposite vertices in the triangle. By definition, let $x = \frac{b}{a}$. It is also possible to have a limit point on (BC). If such a point exists we denote it by $(0, f, g)$ and we have:

Claim II. With the above notations $\frac{f}{g} = x^2$.

Proof. Let $(a_{n_k}, b_{n_k}, c_{n_k})_{n \geq 1}$ be a subsequence of $(a_n, b_n, c_n)_{n \geq 1}$ converging to $(m, n, 0)$. We have that $\sum a_{n_k} b_{n_k} (1 - \cos(t \log \frac{b_{n_k}}{a_{n_k}} - s)) < \infty$ and necessary s must be equal to $t \log \frac{b}{a} \pmod{\pi}$. If $(a_{n_i}, b_{n_i}, c_{n_i})_{n \geq 1}$ is a subsequence converging to $(q, 0, r)$ then we must have $\sum a_{n_i} c_{n_i} (1 - \cos(t \log \frac{c_{n_i}}{a_{n_i}} + s)) < \infty$ with s defined above. It follows that $\cos(t \log \frac{rx}{q}) = 1$ for all $t \in \mathbb{R}$ and therefore $\frac{r}{q} = \frac{1}{x}$.

Let $(a_{n_j}, b_{n_j}, c_{n_j})_{n \geq 1}$ be a subsequence of $(a_n, b_n, c_n)_{n \geq 1}$ converging to $(0, f, g)$. But then $\sum b_{n_j} c_{n_j} (1 - \cos(t \log \frac{b_{n_j}}{c_{n_j}} - 2s)) = \sum b_{n_j} c_{n_j} (1 - \cos(t \log \frac{b_{n_j}}{c_{n_j}} - t \log x^2)) < \infty$. Hence $\lim_{j \rightarrow \infty} \cos(t \log \frac{b_{n_j}}{c_{n_j} x^2}) = \cos(t \log \frac{f}{g x^2}) = 1$ for all t in \mathbb{R} . Therefore, we must have $\frac{f}{g} = x^2$. \square

In other words, $(0, f, g) = (0, \frac{x^2}{1+x^2}, \frac{1}{1+x^2})$ and in terms of x , the other two limit points on the edges can be written as $(\frac{1}{1+x}, \frac{x}{1+x}, 0)$ and $(\frac{1}{1+\frac{1}{x}}, 0, \frac{\frac{1}{x}}{1+\frac{1}{x}})$. Therefore, if the sequence $(a_n, b_n, c_n)_{n \geq 1}$ does not have a limit point in the interior of the triangle ABC, there exists $x > 0$ such that the possible limit points are: $(1, 0, 0), (0, 1, 0), (0, 0, 1), (\frac{1}{1+x}, \frac{x}{1+x}, 0), (\frac{1}{1+\frac{1}{x}}, 0, \frac{\frac{1}{x}}{1+\frac{1}{x}}), (0, \frac{x^2}{1+x^2}, \frac{1}{1+x^2})$. Also, from the way we defined (a, b, c) , (the point situated at the intersection of the line determined by $(0,0,1)$ and $(m,n,0)$ with the line determined by $(0,1,0)$ and $(q,0,r)$) in terms of x , $(a, b, c) = (\frac{1}{1+x+\frac{1}{x}}, \frac{x}{1+x+\frac{1}{x}}, \frac{\frac{1}{x}}{1+x+\frac{1}{x}})$.

If there exists a limit point (a, b, c) in the interior of the triangle ABC, by Proposition 2.5.8 we can not have another limit point in the interior of the triangle and

we can have at most one limit point on (AB), (AC) or (BC). On the other hand we can conclude, using similar arguments as in the proof of Claim II above, that if we define $x = \frac{b}{a}$, then $\frac{c}{a} = \frac{1}{x}$ and $\frac{b}{c} = x^2$. As $a + b + c = 1$, we have that $a(1 + x + \frac{1}{x}) = 1$ and we obtain that $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$, and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$. By Proposition 2.5.8, we can conclude that the only possible limit points of the sequence $(a_n, b_n, c_n)_{n \geq 1}$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{1}{1+x}, \frac{x}{1+x}, 0)$, $(\frac{1}{1+\frac{1}{x}}, 0, \frac{\frac{1}{x}}{1+\frac{1}{x}})$, $(0, \frac{x^2}{1+x^2}, \frac{1}{1+x^2})$ and $(\frac{1}{1+x+\frac{1}{x}}, \frac{x}{1+x+\frac{1}{x}}, \frac{\frac{1}{x}}{1+x+\frac{1}{x}})$. The rest of the proof follows in the same way as the proof of Proposition 2.4.7 \square

Theorem 2.5.11. *Let $M = \pi_\varphi(A)'' = \otimes(M_3(\mathbb{C}), \varphi_n)$, where $\varphi = \otimes \varphi_n$, $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ with h_n as in (2.5.2), and $N = M^\alpha$. If $\sum a_n(1 - a_n) = \infty$, $\sum b_n(1 - b_n) = \infty$ and $\sum c_n(1 - c_n) = \infty$ then N is a factor. In this case we have:*

(i) *N is type II_1 if and only if there exists $x > 0$ such that:*

$$\sum (a_n - a)^2 + (c_n - c)^2 + (b_n - b)^2 < \infty,$$

where $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$, and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$.

If N is not of type II_1 then:

(ii) *N is type II_∞ if and only if there exists $x > 0$ such that:*

$$\sum a_n b_n (a_n b - b_n a)^2 + \sum a_n c_n (a_n c - c_n a)^2 + \sum b_n c_n (c_n b - b_n c)^2 < \infty$$

where $a = \frac{1}{1+x+\frac{1}{x}}$, $b = \frac{x}{1+x+\frac{1}{x}}$, and $c = \frac{\frac{1}{x}}{1+x+\frac{1}{x}}$.

(iii) *N is a factor of type III otherwise.*

As previously seen, for N to be factor we need $\sum a_n(1 - a_n) = \infty$ but we can have $\sum b_n(1 - b_n) < \infty$ or $\sum c_n(1 - c_n) < \infty$.

If $\sum b_n(1 - b_n) = \sum b_n a_n + b_n c_n < \infty$, then we must have $\sum a_n c_n = \infty$, because $\sum a_n(1 - a_n) = \infty$. In this case, from Proposition 2.5.5 we obtain that $t \in T(N)$ if and only if

$$\sum a_n c_n (1 - \cos(t \log \frac{c_n}{a_n} - 2s)) < \infty$$

for some $s \in [0, 2\pi)$. Therefore, in these conditions, we have:

Proposition 2.5.12. *N is of type II_∞ if and only if*

$$\sum a_n c_n (a_n c - c_n a)^2 < \infty$$

for a unique pair (a, c) with $a + c = 1$. Otherwise, N is of type III .

Proof. The proof is an immediate consequence of Proposition 2.2.3. \square

Similarly, if $\sum c_n(1 - c_n) < \infty$, then we must have $\sum a_n b_n = \infty$ and the following holds:

Proposition 2.5.13. *N is of type II_∞ if and only if*

$$\sum a_n b_n (a_n b - b_n a)^2 < \infty,$$

for a unique pair (a, b) with $a + b = 1$. Otherwise, N is of type III .

We end this section with another example. As the proof is similar to the case studied above we omit details.

Example 2.5.1. Let us consider $\beta : T \rightarrow \text{Aut}(A)$, where $A = \otimes M_3(\mathbb{C})$ given by:

$$\beta(t) = \otimes \text{Ad} \begin{bmatrix} 1 & & \\ & e^{it} & \\ & & e^{it} \end{bmatrix}$$

The restriction of φ to the fixed point algebra under this action denoted by φ^β , is a factor state if and only if $\sum a_n(1 - a_n) = \infty$.

First, $\pi_{\varphi^\beta}(A^\beta)''$ is a factor of type II_1 if and only if there exists (a, b, c) with $abc \neq 0$, $b = c$, $a + b + c = 1$, such that

$$\sum (a_n - a)^2 + (b_n - b)^2 + (c_n - c)^2 < \infty.$$

Assume that the factor obtained is not of type II_1 . To distinguish between type II_∞ and type III we have again two possibilities. If at least two of the following sums

$\sum a_n b_n$, $\sum a_n c_n$ and $\sum b_n c_n$ are infinite, (note that $\sum a_n(1 - a_n) = \infty$ and so, the restriction of the state to the fixed point algebra is a factor state), then $\pi_{\varphi^\beta}(A^\beta)''$ is of type II_∞ if and only if there exists a unique $(a, b, c) \in (0, 1)^3$, with $b = c$, $abc \neq 0$, $a + b + c = 1$ such that:

$$\sum a_n b_n (a_n b - b_n a)^2 + c_n b_n (c_n b - b_n c)^2 + a_n c_n (a_n c - c_n a)^2 < \infty,$$

and of type III otherwise.

If only one of the above sums is finite, then, since $\sum a_n(1 - a_n) = \infty$, we can have $\sum a_n b_n = \infty$ or $\sum a_n c_n = \infty$. If only $\sum a_n b_n = \infty$ then we obtain factor of type II_∞ if and only if

$$\sum a_n b_n (a_n b - b_n a)^2 < \infty$$

for a unique pair (a, b) with $a + b = 1$ and a factor of type III otherwise.

If only $\sum a_n c_n = \infty$ then we obtain a factor of type II_∞ if and only if

$$\sum a_n c_n (a_n c - c_n a)^2 < \infty$$

for a unique pair (a, c) with $a + c = 1$ and a factor of type III otherwise.

Chapter 3

Fixed Point Factors of Type III

In this chapter we classify the factors of type III obtained as fixed point algebras under xerox actions induced by diagonal representations of compact groups G , in subtypes III_λ with $0 \leq \lambda \leq 1$ (in Chapter II we were capable only to make distinction between type II_1 , II_∞ and type III). For this, following Araki-Woods and Baker-Giordano we construct a ratio set in terms of the entries of the density matrices of the product state.

3.1 Ratio Set

In [BG], Baker-Giordano extended Araki-Woods definition of the ratio set, [AW], to the factors obtained as fixed point algebras under the standard xerox action of the 1-dimensional torus on $ITPFI_2$ factors. In definition 3.1.1, we extend this definition to factors obtained as fixed point algebras under xerox actions induced by diagonal representations of compact groups.

In this chapter, G is a compact group acting on $A = \otimes M_k(\mathbb{C})$, the k^∞ -UHF algebra, by a xerox action α , induced by a diagonal unitary representation π of G on $M_k(\mathbb{C})$, $\varphi = \otimes tr(h_n \cdot)$ is a faithful diagonal product state on A (see Definition 2.1.10), $M = \pi_\varphi(A)'' = \otimes (M_k(\mathbb{C}), \varphi_n)$ and $N = \pi_\varphi(A)'' \simeq \pi_{\varphi^G}(A^G)''$ (see Section 2.1).

We denote the matrix units (in the canonical base) of the k^{th} term in the tensor product by ϵ_{ij}^k , $i, j \in \{1, 2, \dots, k\}$.

Let $I = \{i_1, i_2, \dots, i_n\} \subseteq \mathbb{N}$ with $\text{card}(I) = n$, $M(I) = \otimes_1^n M_k(\mathbb{C}) \simeq M_{k^n}(\mathbb{C})$, and $\alpha^n = \bigotimes_{i=1}^n \text{Ad } \pi(g)$ the product action on $M(I)$. Then $M(I)^{\alpha^n}$ is a unital subalgebra of $M(I)$, that we can write $M_{m_1^n}(\mathbb{C}) \oplus M_{m_2^n}(\mathbb{C}) \oplus \dots \oplus M_{m_{k(n)}^n}(\mathbb{C})$, where $k(n) \geq 1$. Notice that the decomposition of $M(I)^{\alpha^n}$ depends only on the length n of I and $m_1^n + m_2^n + \dots + m_{k(n)}^n = k^n$. Let $\varphi(I)$ denote the state $\otimes_{i \in I} \varphi_i$ on $M(I)$ and $Sp(\varphi(I))$ denote the set of eigenvalues of the density matrix of $\varphi(I)$. Let $E(I)$ be the set of (minimal) projections of $M(I)$ of the form $e_1 \otimes \dots \otimes e_n$ with $e_j \in \{\epsilon_{11}^{i_j}, \epsilon_{22}^{i_j}, \dots, \epsilon_{k k}^{i_j}\}$. According to the decomposition of $M(I)^{\alpha^n}$, we have that $E(I) = E(I_{m_1^n}) \cup E(I_{m_2^n}) \cup \dots \cup E(I_{m_{k(n)}^n})$ and $Sp(\varphi(I)) = I_{m_1^n} \cup I_{m_2^n} \cup \dots \cup I_{m_{k(n)}^n}$. Let λ be the function that associates to each minimal projection $e \in E(I)$ the corresponding eigenvalue $\lambda(e)$ in $Sp(\varphi(I))$ (in fact $\lambda(e) = \varphi(e)$).

Keeping the above notation we now give the definition of the ratio set that extends Araki-Woods' definition of the ratio set for ITPFI factors, ([AW], Definition 3.2) and Baker-Giordano's definition, [BG], of the ratio set for fixed point factors under the standard action of the torus.

Definition 3.1.1. The *asymptotic ratio set* of \mathbb{N} , denoted by $r_\infty(N, \varphi)$, is the set of all $x \in [0, \infty)$ for which there exists a sequence of disjoint subsets I_n of \mathbb{N} , each I_n having length c_n , mutually disjoint subsets $K_n^1, K_n^2 \subseteq E(I_n)$ and bijections $\psi_n : K_n^1 \rightarrow K_n^2$ satisfying:

- (1) $\sum_{n=1}^{\infty} \lambda(K_n^1) = \infty$, where $\lambda(K_n^1) = \sum_{e \in K_n^1} \lambda(e)$,
- (2) $\psi_n(K_n^1 \cap E(m_i^{c_n})) \subset K_n^2 \cap E(m_i^{c_n})$ for all $1 \leq i \leq k(c_n)$,
- (3) $\lim_{n \rightarrow \infty} \sup_{e \in K_n^1} |x - \frac{\lambda(\psi_n(e))}{\lambda(e)}| = 0$.

Such a sequence, $(I_n, K_n^1, K_n^2, \psi_n)$ is called an x -sequence.

We can also generalize the notion of maximal partial bijection defined by Connes ([C], Theorem 3.6.1), as follows:

Definition 3.1.2. Let V be a compact of $(0,1)$. A *partial bijection* (I, K^1, K^2, ψ) is said to be V maximal if $\frac{\lambda(\psi(e))}{\lambda(e)} \in V$ for all $e \in K^1$ and if there exists no partial bijection $(I, K^{1'}, K^{2'}, \psi')$ such that K^1 is strictly included in $K^{1'}$, $\psi'(e) = \psi(e)$ for all $e \in K^1$ and $\frac{\lambda(\psi'(e))}{\lambda(e)} \in V$ for all $e \in K^{1'}$.

Lemma 3.1.1. *Let $0 < x < 1$. If $x \in S(N)$, then $x \in r_\infty(N, \varphi)$.*

Proof. To prove the lemma it is enough to show that for any compact neighborhood $V \subset (0, 1)$ of x , for any finite set $I_0 \subseteq \mathbb{N}$ and any $\delta > 0$ there is a partial bijection (I, K^1, K^2, ψ) such that $I \cap I_0 = \emptyset$, $\frac{\lambda(\psi(e))}{\lambda(e)} \in V$, $e \in K^1$ and $\lambda(K^1) + \lambda(K^2) > 1 - \delta$. We have that $N \simeq W^*(X, \mu, \Gamma)$ where $X = \prod_{n=1}^{\infty} \{0, 1, \dots, k-1\}$ and $\Gamma = \cup \Gamma_n$ is a countable group (Γ can be obtained by applying Stratila-Voiculescu procedure of diagonalization, Theorem 1.1.8). We can identify a projection $\epsilon_{i_1 i_1}^1 \otimes \dots \otimes \epsilon_{i_n i_n}^n$ in N with the characteristic function $\chi_{E(k_1 k_2 \dots k_n)}$ of the cylinder $E(k_1 k_2 \dots k_n) = \{x \in X | x_1 = k_1, \dots, x_n = k_n\}$, where $k_j = i_j - 1$ for $1 \leq j \leq n$. By [C], Corollary 3.3.4, we have that $S(N) = r(\Gamma, X, \mu)$ where $r(\Gamma, X, \mu)$ is Krieger ratio set. If $x \in r(\Gamma, X, \mu)$, then it is easy to see that $x \in r(\Gamma, X_1, \mu_1)$, where $X_1 = \prod_{n \in \mathbb{N} - I_0} \{0, 1, \dots, k-1\}$, $\Gamma = \cup \Gamma_n$, Γ is acting on X , resp. on X_1 and each Γ_n affects only the first n coordinates. Hence, it's enough to proof the assertion when $I_0 = \emptyset$. In fact it is enough to prove that if V is a compact neighborhood of x and $(I_n, K_n^1, K_n^2, \psi_n)$ is a sequence of V -maximal partial bijections such that :

- (1) $\bigcup_{n \in \mathbb{N}} I_n = \mathbb{N}$,
- (2) $I_n \subseteq I_{n+1}$, for all $n \in \mathbb{N}$,
- (3) $(I_{n+1}, K_{n+1}^1, K_{n+1}^2, \psi_{n+1})$, extends the partial bijection $(I_n, K_n^1, K_n^2, \psi_n)$ and the identical bijection of $E(I_{n+1} - I_n)$,

then $C_n = \lambda(K_n^1) + \lambda(K_n^2) \rightarrow 1$ when $n \rightarrow \infty$. We should remark first that such a family exists; as x is in the Krieger ratio set, we can choose $(I_1, K_1^1, K_2^1, \psi_1)$, and we continue by induction. One has $K_n^1 \times E(I_{n+1} - I_n) \subseteq K_{n+1}^1$. Therefore $C_n \leq C_{n+1}$ for

all n . Assume that $C_n \leq C < 1$ for all n . We will prove that we get a contradiction.

Let $p_n = \sum_{e \in K_n^1 \cup K_n^2} e$ and $q_n = 1 - p_n$. Of course p_n and q_n are projections of $M(I_n)$. The sequence $p'_n = p_n \otimes 1_{M(N-I_n)}$ is an increasing sequence of projections with $\varphi(p'_n) = \lambda(K_n^1) + \lambda(K_n^2) = C_n \leq C$. Therefore, $q' = \bigwedge (1 - p'_n) \neq 0$.

Let $A \subset X$ be the measurable set corresponding to q' . One has $\mu(A) = \varphi(q') > 0$. Let $\varepsilon > 0$ be such that $(x - \varepsilon, x + \varepsilon) \subset V$. As $x \in S(N) = r(\Gamma, X, \mu)$, there exists $B \subseteq A, \mu(B) > 0, g \in \Gamma$ such that $gB \subseteq A$ and

$$\left| \frac{d\mu \circ g}{d\mu}(y) - x \right| < \varepsilon$$

for all $y \in B$. Let A_n be the measurable set corresponding to the projection $q_n \otimes 1_{M(N-I_n)}$. Each A_n is a union of elementary cylinders of length $|I_n|$ and we have $A = \bigcap A_n$. We can find n such that $g \in \Gamma_{|I_n|}$. As $B \subseteq A \subset A_n$ and A_n is a union of elementary cylinders of length I_n , there exists an elementary cylinder E_1 contained in A_n such that $\mu(B \cap E_1) > 0$. But then $\mu(gB \cap E_2) > 0$ with $E_2 = gE_1$. Hence $0 < \mu(gB \cap E_2) \leq \mu(A_n \cap E_2)$ and consequently, as an elementary cylinder is contained in A_n or not, it follows that $E_2 \subseteq A_n$. Therefore the projections e_1 and e_2 corresponding to E_1 and E_2 are not in $K_n^1 \cup K_n^2$. As for any $y \in B$ we have $|\frac{d\mu \circ g}{d\mu}(y) - x| < \varepsilon$ and the derivative $\frac{d\mu \circ g}{d\mu}$ is constant on cylinders of length $|I_n|$ (because $g \in \Gamma_{|I_n|}$) we have that $|\frac{d\mu \circ g}{d\mu}(y) - x| < \varepsilon$ for all $y \in E_1$. Hence, $x - \varepsilon < \frac{d\mu \circ g}{d\mu}(y) < x + \varepsilon$ on E_1 and therefore, by integration, we get

$$(x - \varepsilon)\mu(E_1) \leq \mu(E_2) = \int_{E_1} d\mu \circ g(y) \leq (x + \varepsilon)\mu(E_1)$$

Consequently, $|\frac{\lambda(e_2)}{\lambda(e_1)} - x| < \varepsilon$, which contradicts the V-maximality of $(I_n, K_n^1, K_n^2, \psi_n)$. \square

We recall now the Araki-Woods' definition ([AW], Definition 6.1) of the ratio set for arbitrary factors.

Definition 3.1.3. For an arbitrary factor P , let $r_\infty(P)$ denote the set of all $x \geq 0$ such that $P \simeq P \otimes R_x$ if $x \in [0, 1]$ and $P \simeq P \otimes R_{1/x}$ if $x \in [1, \infty]$ (see also Example 1.2.1).

For the sake of completeness, we include the proof of the following lemma:

Lemma 3.1.2. [BG] *Let P and Q be two von Neumann algebras and let $\alpha : G \rightarrow \text{Aut}(Q)$ be a continuous action of a compact group. Then $(P \otimes Q)^{1 \otimes \alpha} \simeq P \otimes Q^\alpha$*

Proof. We can see P acting on a Hilbert space H and Q acting on a Hilbert space K . Then both $(P \otimes Q)^{1 \otimes \alpha}$ and $P \otimes Q^\alpha$ are von Neumann algebras acting on the same Hilbert space $H \otimes K$ and we have to prove that $(P \otimes Q)^{1 \otimes \alpha} = P \otimes Q^\alpha$. Clearly, $P \otimes Q^\alpha \subseteq (P \otimes Q)^{1 \otimes \alpha}$. We only need to show that $P \otimes Q^\alpha$ is σ -weakly dense in $(P \otimes Q)^{1 \otimes \alpha}$. Let $x \in (P \otimes Q)^{1 \otimes \alpha}$. It exists $y_n = \sum_m p_{n,m} \otimes q_{n,m}$ (for all n , the sums have finitely many terms), such that $y_n \rightarrow x$ in the σ -weak topology. For all $g \in G$ we also have that

$$(1 \otimes \alpha_g)(y_n) = \sum_m p_{n,m} \otimes \alpha_g(q_{n,m}) \rightarrow (1 \otimes \alpha_g)(x) = x.$$

in the σ -weak topology. Hence,

$$\int_G (1 \otimes \alpha_g)(y_n) dg \rightarrow x$$

in the σ -weak topology, where dg is the Haar measure on G . We have

$$\int_G (1 \otimes \alpha_g)(y_n) dg = \sum_m p_{n,m} \otimes \int_G \alpha_g(q_{n,m}) dg$$

Let $r_{n,m} := \int_G \alpha_g(q_{n,m}) dg$. Then $r_{n,m} \in Q^\alpha$. Hence, $P \otimes Q^\alpha$ is σ -weakly dense in $(P \otimes Q)^{1 \otimes \alpha}$. We conclude that $(P \otimes Q)^{1 \otimes \alpha} = P \otimes Q^\alpha$. \square

Lemma 3.1.3. *Let $0 < x < 1$. If $x \in r_\infty(N, \varphi)$, then $N \simeq N \otimes R_x$.*

Proof. Let $(\varepsilon_m)_{m \geq 1}$ be a sequence with $\sum_{m \geq 1} \varepsilon_m < \infty$. As in [AW], Lemma 3.3, there exists an x -sequence $(I_m, K_m^1, K_m^2, \psi_m)_{m \geq 1}$ with $|1 - \lambda(K_m^1) - \lambda(K_m^2)| < \varepsilon_m$, for all $m \geq 1$. Let $M = \bigotimes_{n \geq 1} (M_k(\mathbb{C}), \varphi_n)$ and $\alpha : G \rightarrow \text{Aut}(M)$ be the xerox action. One has:

$$M = \left(\bigotimes_{n \in \mathbb{N}} (M(I_n), \varphi(I_n)) \right) \otimes \left(\bigotimes_{m \notin \cup_{n \in \mathbb{N}} I_n} (M_k(\mathbb{C}), \varphi_m) \right).$$

Let $c_n = |I_n|$. Then, we have that $M(I_n) \simeq M_{k^{c_n}}(\mathbb{C})$ and $\bigotimes_{n=1}^{c_n} \pi(g) \simeq \pi_n(g)$.

Let M_1 denote $\bigotimes_{n \geq 1} (M(I_n), \varphi(I_n))$ with the corresponding action $\alpha_1 = \bigotimes_{n \geq 1} \text{Ad } \pi_n(g)$ and $M_2 = \bigotimes_{m \notin \bigcup_{n=1}^{\infty} I_n} (M_k(\mathbb{C}), \varphi_m)$ with the corresponding action $\alpha_2 = \bigotimes_{n \geq 1} \text{Ad } \pi(g)$.

We have:

$$M^\alpha = (M_1 \otimes M_2)^{\alpha_1 \otimes \alpha_2}$$

Let $p_n = \sum_{e \in K_n^1 \cup K_n^2} e$. We have that $p_n \in M(I_n)$ and if $p := \bigotimes_{n \geq 1} p_n \in M_1$ then $p \neq 0$ because $\sum \varepsilon_n < \infty$. By construction $\alpha_1(p) = p$. Let $\tilde{p} = p \otimes 1 \in M_1 \otimes M_2$. Clearly, $\tilde{p} \in M^\alpha = N$.

For $n \geq 1$, let φ_{p_n} be the state on $M(I_n)_{p_n}$ given by

$$\varphi_{p_n} = \frac{1}{\varphi(I_n)(p_n)} \varphi(I_n)|_{M(I_n)_{p_n}}$$

With $\alpha_1^n = \text{Ad } p_n \pi_n(g) p_n$, one has

$$(M_1)_p \simeq \bigotimes_{n \geq 1} (M(I_n)_{p_n}, \varphi_{p_n}) \text{ and } \alpha_1 \simeq \bigotimes_{n \geq 1} \alpha_1^n.$$

For each $n \geq 1$, let $f_1^n = \sum_{e \in K_n^1} e$ and $f_2^n = \sum_{e \in K_n^2} e$. They are orthogonal projections of $M(I_n)_{p_n}^{\alpha_1^n}$ with $f_1^n + f_2^n = p_n$. Let ω_n be the partial isometry of $M(I_n)_{p_n}^{\alpha_1^n}$ such that $\omega_n^* \omega_n = f_1^n$ and $\omega_n \omega_n^* = f_2^n$. Let now A_n be the type I_2 subfactor of $M(I_n)_{p_n}$ generated by ω_n . With this,

$$M(I_n)_{p_n} \simeq A_n \otimes \{A'_n \cap M_n\} \simeq M_2(C) \otimes \widetilde{M}_n$$

where $\widetilde{M}_n = M_{|K_n^1|}(\mathbb{C})$. As $A_n \subset M(I_n)_{p_n}^{\alpha_1^n}$, via this isomorphism, the action splits in $1 \otimes \tilde{\alpha}_n$. Hence we can write $\alpha_1^n \simeq 1 \otimes \tilde{\alpha}_n$

To construct ω_n , let $K_n^1(k) = K_n^1 \cap E(m_k^{c_n})$, $K_n^2(k) = K_n^2 \cap E(m_k^{c_n})$ for $1 \leq k \leq k(c_n)$. We can denote the matrix units of $M(I_n)_{p_n}$ by e_{ij}^n with $1 \leq i, j \leq 2|K_n^1|$ in such a way that $\{e; e \in K_n^1\}$ is in bijection with $\{e_{ii}^n, 1 \leq i \leq |K_n^1|\}$ and $\{e; e \in K_n^2\}$ is in bijection with $\{e_{ii}^n, |K_n^1| + 1 \leq i \leq 2|K_n^1|\}$. We can see then, the bijection ψ_n as a bijection from $\{1, 2, \dots, |K_n^1|\}$ onto $\{|K_n^1| + 1, 2, \dots, 2|K_n^1|\}$. Accordingly, each $K_n^1(k)$ corresponds to a subset $L_n^1(k)$ of $\{i, 1 \leq i \leq |K_n^1|\}$ and each $K_n^2(k)$ corresponds to a subset $L_n^2(k)$ of $\{|K_n^1| + 1, 2, \dots, 2|K_n^1|\}$. Then, we can write K_n^2 as the set

$\{e_{\psi_n(i)\psi_n(i)}^n; 1 \leq i \leq |K_n^1|\}$. With the above identifications, let $f_1^n(k) = \sum_{j \in L_n^1(k)} e_{jj}^n$ and $f_2^n(k) = \sum_{j \in L_n^2(k)} e_{\psi_n(j)\psi_n(j)}^n$. Then $f_1 = \bigoplus_{k=1}^{k(c_n)} f_1^n(k)$ and $f_2 = \bigoplus_{k=1}^{k(c_n)} f_2^n(k)$. If $\omega_n(k) = \sum_{j \in L_n^1(k)} e_{\psi_n(j)j}$ we put $\omega_n = \bigoplus_{k=1}^{k(c_n)} \omega_n(k)$.

As before, let $\{e_{ii}^n\}$ be the elements of K_n^1 and for $1 \leq i \leq |K_n^1|$ let $\lambda_i^n = \lambda(e_{ii}^n)$ be the elements of $\lambda(K_n^1)$ and $\psi_n(\lambda_i^n) := \lambda(\psi_n(e_{ii}^n))$. Let $\tilde{\varphi}_n$ be the state on \tilde{M}_n given by the density matrix $\frac{1}{\varphi(f_1^n)} \text{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_{|K_n^1|}^n)$. Let χ_x be the Power state on $M_2(\mathbb{C})$, i.e., χ_x has the density matrix $\text{diag}(\frac{1}{1+x}, \frac{x}{1+x})$. Identifying $M(I_n)_{p_n}$ and $M_2(\mathbb{C}) \otimes \tilde{M}_n$ (as we saw before) we get that the density matrices of φ_{p_n} and $\chi_x \otimes \tilde{\varphi}_n$ are given by

$$\begin{aligned} h_{p_n} &= \frac{1}{\varphi(p_n)} \text{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_{|K_n^1|}^n, \psi_n(\lambda_1^n), \dots, \psi_n(\lambda_{|K_n^1|}^n)) \\ k_n &= \frac{1}{(1+x)\varphi(f_1^n)} \text{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_{|K_n^1|}^n, x\lambda_1^n, x\lambda_2^n, \dots, x\lambda_{|K_n^1|}^n) \end{aligned}$$

We want to show now that $\otimes(M_n(I_n)_{p_n}, \varphi_{p_n})$ and $\otimes((M_k, \chi_k) \otimes (\tilde{M}_n, \tilde{\varphi}_n))$ are unitary equivalent.

Indeed,

$$\begin{aligned} \text{tr}(h_n^{\frac{1}{2}} k_n^{\frac{1}{2}}) &= \frac{1}{\varphi(p_n)^{\frac{1}{2}} (1+x)^{\frac{1}{2}} \varphi(f_1^n)^{\frac{1}{2}}} \left(\sum_{j=1}^{|K_n^1|} \lambda_j^n + \psi_n(\lambda_j^n)^{\frac{1}{2}} x^{\frac{1}{2}} (\lambda_j^n)^{\frac{1}{2}} \right) \\ &= \frac{1}{\varphi(p_n)^{\frac{1}{2}} (1+x)^{\frac{1}{2}} \varphi(f_1^n)^{\frac{1}{2}}} \left(\varphi(f_1^n) + \sum \lambda_j^n x^{\frac{1}{2}} \left(\frac{\psi_n(\lambda_j^n)}{\lambda_j^n} \right)^{\frac{1}{2}} \right) \\ &\geq \frac{1}{\varphi(p_n)^{\frac{1}{2}} (1+x)^{\frac{1}{2}} \varphi(f_1^n)^{\frac{1}{2}}} \left(\varphi(f_1^n) (1 + x^{\frac{1}{2}} (x - \varepsilon_n)^{\frac{1}{2}}) \right) \\ &= \frac{\varphi(f_1^n)^{\frac{1}{2}}}{\varphi(p_n)^{\frac{1}{2}} (1+x^{\frac{1}{2}})} (1 + x^{\frac{1}{2}} (x - \varepsilon_n)^{\frac{1}{2}}) \\ &\geq \frac{1}{(1+(x+\varepsilon_n))^{\frac{1}{2}} (1+x)^{\frac{1}{2}}} (1 + x^{\frac{1}{2}} (x - \varepsilon_n)^{\frac{1}{2}}) \\ &\geq \frac{1}{(1+(x+\varepsilon_n))^{\frac{1}{2}} (1+x)^{\frac{1}{2}}} (1+x-\varepsilon_n) \geq \frac{1}{(1+\frac{\varepsilon_n}{1+x})^{\frac{1}{2}}} \left(1 - \frac{\varepsilon_n}{1+x}\right) \\ &\geq \frac{1}{1+\frac{\varepsilon_n}{1+x}} \left(1 - \frac{\varepsilon_n}{1+x}\right) \end{aligned}$$

Therefore we have

$$\sum (1 - \text{tr}(h_{p_n}^{\frac{1}{2}} k_n^{\frac{1}{2}})) \leq \sum_n \frac{2\varepsilon_n}{1+x} < \infty.$$

By [AW], Lemma 2.13, we obtain that $\otimes(M_n(I_n)_{p_n}, \varphi_{p_n})$ and $\otimes((M_k, \chi_k) \otimes (\widetilde{M}_n, \widetilde{\varphi}_n))$ are unitary equivalent.

As we already saw before, $\alpha_1^n \simeq 1 \otimes \widetilde{\alpha}_n$. Consequently, $\alpha_1 := \otimes_{n \geq 1} \alpha_1^n \simeq \otimes_{n \geq 1} (1 \otimes \widetilde{\alpha}_n)$.

Therefore

$$\otimes(M_n(I_n)_{p_n}, \varphi_{p_n}) \simeq (\otimes(M_2(C), \chi_x)) \otimes (\otimes(\widetilde{M}_n, \widetilde{\varphi}_n)) \simeq R_x \otimes (\otimes(\widetilde{M}_n, \widetilde{\varphi}_n)),$$

$$\alpha_1 \simeq 1 \otimes (\otimes \widetilde{\alpha}_n) = 1 \otimes \widetilde{\alpha} \text{ where } \widetilde{\alpha} := \otimes \widetilde{\alpha}_n$$

Then, if $\overline{M}_2 := (\otimes(\widetilde{M}_n, \widetilde{\varphi}_n)) \otimes M_2$ and $\overline{\alpha}_2 := \widetilde{\alpha} \otimes \alpha_2$,

$$M_{\overline{p}} = (M_1)_p \otimes M_2 \simeq R_x \otimes (\otimes(\widetilde{M}_n, \widetilde{\varphi}_n)) \otimes M_2 = R_x \otimes \overline{M}_2,$$

$$\alpha = \alpha_1 \otimes \alpha_2 \simeq 1 \otimes (\widetilde{\alpha} \otimes \alpha_2) = 1 \otimes \overline{\alpha}_2$$

We have

$$N_{\overline{p}} = (M^\alpha)_{\overline{p}} = (M_{\overline{p}})^\alpha \simeq (R_x \otimes \overline{M}_2)^{1 \otimes \overline{\alpha}_2} \simeq R_x \otimes \overline{M}_2^{\overline{\alpha}_2}$$

and therefore

$$N \simeq N_{\overline{p}} \simeq R_x \otimes \overline{M}_2^{\overline{\alpha}_2} \simeq R_x \otimes R_x \otimes \overline{M}_2^{\overline{\alpha}_2} \simeq R_x \otimes N_{\overline{p}} \simeq R_x \otimes N$$

Hence the lemma is proved. □

Proposition 3.1.4. *Let N be as above. Then $r_\infty(N, \varphi) \cap (0, 1) = S(N) \cap (0, 1)$.*

Proof. By Lemma 3.1.6, we have $(0, 1) \cap r_\infty(N, \varphi) \subseteq (0, 1) \cap r_\infty(N)$. By [C], Theorem 3.6.1, we then have that $(0, 1) \cap r_\infty(N) = (0, 1) \cap S(N)$. By Lemma 3.1.3, we conclude that $(0, 1) \cap r_\infty(N, \varphi) = (0, 1) \cap S(N)$. □

Following Definition 1.2.13, we have:

Theorem 3.1.5. *If N is of type III then:*

(i) *N is of type III₀ if and only if $(0, 1) \cap r_\infty(N, \varphi) = \emptyset$;*

- (ii) N is of type III_λ for some $\lambda \in (0, 1)$ if and only if $(0, 1) \cap r_\infty(N, \varphi) = \{\lambda^n, n > 0\}$;
- (iii) N is of type III_1 if and only if $(0, 1) \cap r_\infty(N, \varphi) = (0, 1)$.

3.2 Examples

In these examples if $A = \otimes M_3(\mathbb{C})$ then A^{T^2} denotes the fixed point algebra under the xerox action induced by the standard representation of T^2 on $M_3(\mathbb{C})$ as defined in Section 2.4. If $A = \otimes M_2(\mathbb{C})$ then A^T denotes the fixed point algebra under the standard xerox action of 1-dimensional torus considered in Section 2.2.

Example 3.2.1. If $h_n = \text{diag}(\frac{1}{1+\lambda+\lambda^2}, \frac{\lambda}{1+\lambda+\lambda^2}, \frac{\lambda^2}{1+\lambda+\lambda^2})$, we consider on $A = \otimes M_3(\mathbb{C})$ the product $\varphi = \otimes \varphi_n$, with $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$. Then $\pi_{\varphi|_{A^{T^2}}}(A^{T^2})''$ is a factor of type II_1 and $\pi_{\phi|_{A^\beta}}(A^\beta)''$ is a factor of type III_λ where β is the action considered in Section 2.5.

Example 3.2.2. If $h_n = \text{diag}(\frac{1}{2+n}, \frac{1}{2+n}, \frac{n}{2+n})$, $\phi(\cdot) = \otimes \text{tr}(h_n \cdot)$ we consider $\varphi = \otimes \varphi_n$ the product state on $A = \otimes M_3(\mathbb{C})$ with $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$. Then $\pi_{\varphi|_{A^{T^2}}}(A^{T^2})''$ is a factor of type III_1 .

We give now some examples where we see also that in the case of inclusions of factors $M \supseteq N$ of infinite index we can have M of type III_1 and N of type III_λ with $0 < \lambda < 1$ or we can have M of type III_λ with $0 < \lambda < 1$ and N of type III_0 . This is different to the situation of inclusions of finite index, [PhL].

Example 3.2.3. Consider the sequence:

$$1, 1, \dots, 2^k, 2^k, \dots, 2^k, 2^{k+1}, \dots, 2^{k+1}, 2^{k+2}, \dots$$

where 2^k appears n_k times, and the n_k are chosen large enough to ensure that

$$\sum_{n=1}^{\infty} a_n = \infty$$

where $a_n = \frac{1}{e^{2k}}$ on the k -th block. On $A = \otimes M_3(\mathbb{C})$ we consider the product state $\varphi = \otimes \varphi_n$ with $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ and

$$\begin{aligned} h_{2n} &= \text{diag}\left(\frac{1}{1+a+ba_n}, \frac{a}{1+a+ba_n}, \frac{ba_n}{1+a+ba_n}\right) \\ h_{2n+1} &= \text{diag}\left(\frac{1}{a+b+1}, \frac{a}{a+b+1}, \frac{b}{a+b+1}\right) \end{aligned}$$

Then $M = \pi_\varphi(A)''$ is a factor of type III_1 because $\log a$ and $\log b$ are rationally independent. By Proposition 2.4.8, N is of type III and, by Remark 1.2.24 (b), $N = \pi_\varphi(A^{T^2})''$ is a factor of type III_0 because $T(N) \supseteq \{\frac{2k\pi}{2^n}, n \geq 0, k \in \mathbb{Z}\}$.

Example 3.2.4. In the same conditions and the same notations as in Example 3.2.3, with the difference that this time $b = a^2$ we have that $M = \pi_\varphi(A)''$ is type $III_{\frac{1}{e}}$ and $N = \pi_\varphi(A^{T^2})''$ is type III_0 . If we define $a_n = \frac{1}{\alpha^{2k}}$ on the k -th block, then M is of type III_α and N is of type III_0 .

Example 3.2.5. On $A = \otimes M_3(\mathbb{C})$ we consider the product state $\varphi = \otimes \varphi_n$ where $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ and

$$\begin{aligned} h_{2n} &= \text{diag}\left(\frac{1}{1+\lambda+\mu}, \frac{\lambda}{1+\lambda+\mu}, \frac{\mu}{1+\lambda+\mu}\right) \\ h_{2n+1} &= \text{diag}\left(\frac{1}{1+\alpha\lambda+\alpha^2\mu}, \frac{\alpha\lambda}{1+\alpha\lambda+\alpha^2\mu}, \frac{\alpha^2\mu}{1+\alpha\lambda+\alpha^2\mu}\right) \end{aligned}$$

We assume that $\log \lambda$ and $\log \mu$ rationally independent. Then $M = \pi_\varphi(A)''$ is a factor of type III_1 and $N = \pi_\varphi(A^{T^2})''$ is a factor of type III_α . Indeed we can see that α is in the ratio set, $T(N) = \{\frac{2k\pi}{\log \alpha}, k \in \mathbb{Z}\}$ and therefore N is of type III_α .

Example 3.2.6. On $A = \otimes M_2(\mathbb{C})$ we consider the product state $\varphi = \otimes \varphi_n$ with $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ and

$$\begin{aligned} h_{2n} &= \text{diag}\left(\frac{1}{1+\mu}, \frac{\mu}{1+\mu}\right) \\ h_{2n+1} &= \text{diag}\left(\frac{1}{1+\lambda\mu}, \frac{\lambda\mu}{1+\lambda\mu}\right) \end{aligned}$$

We assume also that $\log \lambda$ and $\log \mu$ rationally independent. Then $M = \pi_\varphi(A)''$ is of type III_1 and $N = \pi_\varphi(A^T)''$ is of type III_λ . This is true because λ is in the ratio set and also $T(N) = \{\frac{2k\pi}{\log \lambda}, k \in \mathbb{Z}\}$.

Chapter 4

Non-diagonal Actions of Xerox Type

Let G be a compact group and π a unitary representation of G on $M_k(\mathbb{C})$, $k \geq 2$, such that the matrices $\pi(g)$ are not necessarily diagonal. Let $\alpha : G \rightarrow \text{Aut}(A)$ be the xerox action induced by π on the k^∞ -UHF algebra $A = \otimes M_k(\mathbb{C})$ and A^G be the fixed point subalgebra of A . Let also φ be a faithful diagonal state on A and φ^G be the restriction of φ to A^G . We study the GNS representation of (A^G, φ^G) .

4.1 Induced Covariant Representations and Imprimitivity Systems

In this section we recall some definitions and some results from [AHKT]. We need them later on, in order to study xerox actions induced by representation of compact groups, not necessarily diagonal. In this section A is a C^* -algebra, $\alpha : G \rightarrow \text{Aut}(A)$ is continuous action of a compact group G and φ is a state on A .

Definition 4.1.1. If A is a C^* -algebra and φ is a state on A , then the subgroup

$$G_\varphi := \{g \in G; \varphi \circ \alpha_g = \varphi\}$$

of G , is called the *stabilizer* of φ . We note that G_φ is a closed subgroup of G .

The action of G_φ on A is canonically implemented in the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ of A by the unitary representation U_φ of G_φ on H_φ given by:

$$U_\varphi(g)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(\alpha_g(a))\xi_\varphi$$

To the pair (A, G_φ) we have associated the covariant representation $\{\pi_\varphi, U_\varphi, H_\varphi\}$. We also can associate a covariant representation $\{\pi^\sim, U^\sim, H^\sim\}$ to the pair (A, G) . The Hilbert space H^\sim consists of all H_φ -valued square integrable functions ξ on G such that

$$\xi(hg) = U_\varphi(h)\xi(g), \quad h \in G_\varphi, g \in G$$

and the inner product in H^\sim is given by:

$$(\xi, \eta) = \int_G (\xi(g), \eta(g)) dg$$

The representation (π^\sim, U^\sim) is given by

$$\{\pi^\sim(a)\xi\}(g) = \pi_\varphi(\alpha_g(a))\xi(g), \quad a \in A, \quad g \in G$$

$$\{U^\sim(g)\xi\}(h) = \xi(hg), \quad g, h \in G.$$

We also define a vector ξ^\sim in H^\sim by

$$\xi^\sim(g) = \xi_\varphi, \quad g \in G.$$

Definition 4.1.2. The covariant representation $\{\pi^\sim, U^\sim, H^\sim\}$ defined above is called the *induced covariant representation* associated to $\{\pi_\varphi, U_\varphi, H_\varphi\}$ and we write

$$\{\pi^\sim, U^\sim, H^\sim\} = \text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}.$$

We further introduce operators $\theta(f)$, $f \in L^\infty(G_\varphi \backslash G)$, as follows:

$$\theta(f)\xi(g) = f(\hat{g})\xi(g), \quad g \in G, \quad f \in L^\infty(G_\varphi \backslash G)$$

where $\hat{g} = G_\varphi g \in G_\varphi \backslash G$. It then follows that θ is a σ -weakly continuous faithful representation of $L^\infty(G_\varphi \backslash G)$ on H^\sim .

Definition 4.1.3. The range B of θ is called the *imprimitivity system* of the induction.

Let us consider also the state ω on A , given by:

$$\omega = \int_G \varphi \circ \alpha_g dg$$

where dg is the normalized Haar measure dg of G . The action of G on A is canonically implemented in the GNS representation $(\pi_\omega, H_\omega, \xi_\omega)$ of A by the unitary representation U_ω on H_ω given by:

$$U_\omega(g)\pi_\omega(a)\xi_\omega = \pi_\omega(\alpha_g(a))\xi_\omega. \quad (4.1.1)$$

We want to describe the relation between $\{\pi_\omega, H_\omega, \xi_\omega\}$ and $\{\pi_\varphi, U_\varphi, H_\varphi\}$.

The proofs for the following two results are as in Theorem III.2.1 and Corollary III.2.3, [AHKT]. For the sake of completeness, we include them here.

Theorem 4.1.1. *Let A be a C^* -algebra, α an action of a compact group G and A^G the fixed point algebra under this action. Let φ a state on A and ω be as in (4.1.1). Let*

$$\{\pi^\sim, U^\sim, H^\sim\} = \text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}$$

Assume further that the imprimitivity system of induction B , is contained in $\pi^\sim(A)' \cap \pi^\sim(A)''$. Then

$$\{\pi_\omega, U_\omega, H_\omega\} \simeq \{\pi^\sim, U^\sim, H^\sim\}$$

in the sense that there exists a unitary $W : H_\omega \rightarrow H^\sim$ which intertwines $\{\pi_\omega, U_\omega\}$ and $\{\pi^\sim, U^\sim\}$.

Proof. We notice that by construction, $B \subseteq \pi^\sim(A)'$.

Let

$$\{\pi^\sim, U^\sim, H^\sim\} = \text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}$$

It then follows that

$$(\pi^\sim(a)\xi^\sim, \xi^\sim) = \int_G (\pi_\varphi \circ \alpha_g(a)\xi_\varphi | \xi_\varphi) dg = \int_G \varphi \circ \alpha_g(a) dg = \omega(a), \quad a \in A$$

Hence the map:

$$\pi_\omega(a)\xi_\omega \rightarrow \pi^\sim(a)\xi^\sim$$

extends to an isometry of H_ω into H^\sim , which we shall denote by W . We want to show that W is onto. For each operator $A \in U_\varphi(G_\varphi)'$ we put

$$A^\sim \xi(g) = A\xi(g), \quad \xi \in H^\sim, \quad g \in G \quad (4.1.2)$$

It is then known, [Ma] (see also [Ta], vol. II) that the mapping $A \in U_\varphi(G_\varphi)' \mapsto A^\sim \in B(H^\sim)$ gives rise to an isomorphism of $U_\varphi(G_\varphi)'$ onto $U^\sim(G)' \cap B'$. Since the range of W is invariant under $U^\sim(G)$ and $\pi^\sim(A)$, and B , is contained in $\pi^\sim(A)' \cap \pi^\sim(A)''$, it is also invariant under B , so that the projection of H^\sim onto WH_ω must be of the form P^\sim for some projection $P \in U_\varphi(G_\varphi)'$. But $\{(W\xi)(g); \xi \in H_\omega\}$ contains $\pi_\varphi \circ \alpha_g(A)\xi_\varphi$ for all $g \in G$ and is thus dense in H_φ so that P must be the identity 1. Hence we conclude that W is an isometry of H_ω onto H^\sim which intertwines $\{\pi_\omega, U_\omega\}$ and $\{\pi^\sim, U^\sim\}$. This completes the proof. \square

We will now identify the covariant representations $(\pi_\omega, U_\omega, H_\omega, \xi_\omega)$ with the induced covariant representation $(\pi^\sim, U^\sim, H^\sim, \xi^\sim)$. We define

$$\begin{aligned} M^\sim &= \pi^\sim(A)'', & M &= \pi_\varphi(A)'' \\ \alpha_g^\sim(A) &= U^\sim(g)AU^\sim(g)^*, & A &\in M, \quad g \in G \\ \alpha_g^\varphi(A) &= U_\varphi(g)AU_\varphi(g)^*, & A &\in M, \quad g \in G_\varphi \end{aligned}$$

Theorem 4.1.2. ([AHKT], Corollary 3.2.3) *In the same conditions as in Theorem 4.1.4, we have*

- (i) $\{M^\sim, G, \alpha^\sim\} \simeq \text{Ind}_{G_\varphi \uparrow G} \{M, G_\varphi, \alpha^\varphi\}$ where the definition of $\text{Ind}_{G_\varphi \uparrow G} \{M, G_\varphi, \alpha^\varphi\}$ is recalled in the proof.
- (ii) $M^{\sim G} \simeq M^{G_\varphi}$ under the correspondence given by (4.1.2).
- (iii) $\pi_\varphi(A^{G_\varphi})'' = \pi_\varphi(A^G)''$.

Proof. Let $\{M^-, G, \alpha^-\} = \text{Ind}_{G^\varphi \uparrow G} \{M, G_\varphi, \alpha^\varphi\}$. By definition, M^- is the subalgebra of $M \otimes L^\infty(G) = L^\infty(G, dg, M)$ consisting of all elements such that

$$x(hg) = \alpha_h^\varphi(x(g)), \quad h \in G_\varphi, \quad g \in G$$

The action α^- of G on M^- is defined by

$$\alpha_g^-(x)(k) = x(kg), \quad k, g \in G$$

For each $A = \pi^\sim(a)$, $a \in A$, let

$$A(g) = \{\pi_\varphi \circ \alpha_g\}(a) \quad g \in G$$

It follows that $A(\cdot)$ belongs to M^- and the correspondence: $A \leftrightarrow A(\cdot)$ is a normal isomorphism. Identifying A and $A(\cdot)$, M^\sim is regarded as a von Neumann subalgebra of M^- , globally invariant under α_g^- and containing B , so that ([Tak], Proposition 10.4) yields that M^\sim must be obtained from a von Neumann subalgebra N of M as $M^\sim = \{x \in M^- : x(g) \in N, g \in G\}$. This means, however, that N contains $\pi_\varphi(A)$. Thus $N = M$ and so $M^- = M^\sim$. This completes the proof of (i). We now prove (ii). By definition

$$M^{\sim G} = M^\sim \cap U^\sim(G)' \quad M^{G_\varphi} = M \cap U_\varphi(G_\varphi)'$$

Since, M^\sim contains the system B , of imprimitivity in its center, $M^{\sim G}$ is contained in $B' \cap U^\sim(G)'$. Hence $M^{\sim G}$ corresponds to a von Neumann subalgebra of $U_\varphi(G_\varphi)'$ under the correspondence (4.1.2). Under the identification of M^\sim with M^- , the correspondence (4.1.2) means that

$$A \in U_\varphi(G_\varphi)' \leftrightarrow A^\sim = A \otimes 1$$

Thus the subalgebra of $U_\varphi(G_\varphi)'$ corresponding to $M^{\sim G}$ must be M^{G_φ} . In order to prove that $\pi_\varphi(A^{G_\varphi})'' = \pi_\varphi(A^G)''$ it suffices to check that the images of these von Neumann algebras under the isomorphism (4.1.2) coincide. We just showed that the image of $\pi_\varphi(A^{G_\varphi})'' = M^{G_\varphi}$ is $M^{\sim G}$. On the other hand, since the image of $\pi_\varphi(A^G)$ is $\pi^\sim(A^G)$, the image of $\pi_\varphi(A^G)''$ is $\pi^\sim(A^G)'' = M^{\sim G}$. \square

4.2 Mixing States and Pairs of States on C*-algebras

In this section we define the notion of mixing for a state and for a pair of states on a C*-algebra A and we prove some elementary facts.

Definition 4.2.1. Let A be a C*-algebra and $(\sigma_n)_{n \geq 1}$ a sequence of automorphisms of A .

- (1) a state ρ is (pointwise) mixing with respect to $(\sigma_n)_{n \geq 1}$ if for all $a, b, c \in A$:

$$\lim_{n \rightarrow \infty} \rho(b\sigma_n(a)c) = \rho(bc)\rho(a)$$

- (2) a pair (ψ, ρ) of states on A (pointwise) mixing with respect to $(\sigma_n)_{n \geq 1}$ if for all $a, b, c \in A$

$$\lim_{n \rightarrow \infty} \psi(b\sigma_n(a)c) = \psi(bc)\rho(a)$$

Lemma 4.2.1. Let A be a C*-algebra, $\alpha : G \rightarrow \text{Aut}(A)$ an action of a compact group G on A , $(\sigma_n)_{n \geq 1}$ a sequence of automorphisms of A commuting with the action α and ρ a state on A , mixing with respect to $(\sigma_n)_{n \geq 1}$. If $a, b \in A$, then, the family $\{f_n\}_{n \geq 1}$ of continuous functions $f_n : G \rightarrow \mathbb{C}$ given by $f_n(g) = \rho(\alpha_g(a\sigma_n(b)))$ is equicontinuous and uniformly bounded.

Proof. It is enough to prove the equicontinuity at e , the neutral element of G . Let $\epsilon > 0$. As α is a continuous action, there exists a neighborhood V of e such that $\|\alpha_g(a) - a\| < \frac{\epsilon}{2\|a\|}$ and $\|\alpha_g(b) - b\| < \frac{\epsilon}{2\|b\|}$ for all $g \in V$. For $g \in V$ we have:

$$\begin{aligned} |f_n(g) - f_n(e)| &= |\rho(\alpha_g(ag_n(b)) - ag_n(b))| \leq \|\alpha_g(ag_n(b)) - ag_n(b)\| \\ &\leq \|\alpha_g(a)\alpha_g(g_n(b)) - \alpha_g(a)g_n(b)\| + \|\alpha_g(a)g_n(b) - ag_n(b)\| \\ &\leq \|\alpha_g(a)(\alpha_g(g_n(b)) - g_n(b))\| + \|(\alpha_g(a) - a)g_n(b)\| \\ &\leq \|\alpha_g(a)\| \|g_n(\alpha_g(b) - b)\| + \|\alpha_g(a) - a\| \|g_n(b)\| \\ &\leq \|a\| \|\alpha_g(b) - b\| + \|b\| \|\alpha_g(a) - a\| \leq \epsilon \end{aligned}$$

Therefore $\{f_n\}_{n \geq 1}$ is equicontinuous at e and consequently on G . As $|f_n(g)| \leq \|a\| \|b\|$ for all $g \in G$ and $n \geq 1$, we also have that $\{f_n\}_{n \geq 1}$ is uniformly bounded. \square

Let us denote by λ (resp. ϱ) the left (resp. right) regular representation of G on $L^2(G)$ (and by restriction, on $C(G)$). Recall that for $f \in L^2(G)$,

$$[\lambda(h)f](g) = f(h^{-1}g), \quad [\varrho(h)f](g) = f(gh)$$

for $h, g \in G$. If K is a closed subgroup of G , let $K \backslash G$ denote the homogeneous space of right cosets Kg , $g \in G$. The canonical surjection $\pi : G \rightarrow K \backslash G$ induces an identification

$$\tilde{f} \in C(K \backslash G) \mapsto \tilde{f} \circ \pi \in A(K) = \{f \in C(G), \lambda(k)f = f \text{ for all } k \in K\}.$$

We now have the following lemma, proved in [AHKT].

Lemma 4.2.2. ([AHKT], Lemma A.1) *The following correspondences establish a bijection between the closed subgroups K of G and the C^* -subalgebras A of $C(G)$ globally invariant under all right translations (i.e., with $\varrho(g)f \in A$ for all $f \in A$ and $g \in G$) :*

$$\begin{aligned} K \mapsto A(K) &= C(K \backslash G) = \{f \in C(G), \lambda(k)f = f \text{ for all } k \in K\} \\ A \mapsto K &= \{k \in G; f(kg) = f(g) \text{ for all } f \in A \text{ and } g \in G\} \end{aligned}$$

This bijection associates to the normal subgroups of G those globally right translation invariant C^ -subalgebras of $C(G)$ which are also globally left translation invariant.*

Lemma 4.2.3. *Let A be a C^* -algebra, $\alpha : G \rightarrow \text{Aut}(A)$ an action of a compact group G on A and $(\sigma_n)_{n \geq 1}$ a sequence of automorphisms of A commuting with α . Let also ρ be a state on A , mixing with respect to $(\sigma_n)_{n \geq 1}$. If $C_\rho(G)$ denotes $\{g \in G \mapsto f_a^\rho(g) = \rho(\alpha_g(a)); a \in A\}$, then*

$$\overline{C_\rho(G)} = C(G_\rho \backslash G) = A(G_\rho),$$

where G_ρ is the stabilizer of ρ in G .

Proof. Let $a, b \in A$ and let c be a complex number. Since $f_a^\rho + f_b^\rho = f_{a+b}^\rho$ and $cf_a^\rho = f_{ca}^\rho$, $C_\rho(G)$ is a linear subspace of $C(G)$. Since $f_{a^*}^\rho = \overline{f_a^\rho}$, $C_\rho(G)$ is closed under complex conjugation.

For all $g \in G$, $\lim_{n \rightarrow \infty} f_{a\sigma_n(b)}^\rho(g) = f_a^\rho(g)f_b^\rho(g)$. As by Lemma 4.2.1, $\{f_{a\sigma_n(b)}^\rho; n \in \mathbb{N}\}$ is an equicontinuous and uniformly bounded sequence, $f_{a\sigma_n(b)}^\rho$ converges uniformly to $f_a^\rho \cdot f_b^\rho$ (by the Arzela-Ascoli Theorem). Hence $f_a^\rho \cdot f_b^\rho \in \overline{C_\rho(G)}$, and $\overline{C_\rho(G)}$ is a C*-subalgebra of $C(G)$. For $h, g \in G$, we have $\varrho(h)(f_a^\rho) = f_{\alpha_h(a)}^\rho$. Hence $C_\rho(G)$ is globally right invariant. As

$$\begin{aligned} & \{k \in G; f(kg) = f(g), \forall g \in G, f \in \overline{C_\rho(G)}\} \\ &= \{k \in G; f_a^\rho(kg) = f_a^\rho(g), \forall g \in G, \forall a \in A\} \\ &= \{k \in G; \rho(\alpha_h(\alpha_g(a))) = \rho(\alpha_g(a)), \forall g \in G, \forall a \in A\} \end{aligned}$$

is equal to the stabilizer of ρ , by Lemma 4.2.2, we have:

$$\overline{C_\rho(G)} = A(G_\rho) = C(G_\rho \backslash G).$$

□

Lemma 4.2.4. *Let A be a C*-algebra, $\alpha : G \rightarrow \text{Aut}(A)$ an action of a compact group G on A , $(\sigma_n)_{n \geq 1}$ a sequence of automorphisms of A commuting with the action α and (ψ, ρ) a pair of states on A mixing with respect to $(\sigma_n)_{n \geq 1}$. Then*

$$\pi_\psi(\sigma_n(a)) \rightarrow \pi_\psi(\rho(a)1) \text{ weakly}$$

where π_ψ is the GNS-representation of (A, ψ) .

Proof. For $a, b, c \in A$, we have :

$$\begin{aligned} \lim_{n \rightarrow \infty} (\pi_\psi(\sigma_n(a))\pi_\psi(b)\xi_\psi | \pi_\psi(c)\xi_\psi) &= \lim_{n \rightarrow \infty} \psi(c^*\sigma_n(a)b) = \psi(c^*b)\rho(a) \\ &= \rho(a) (\pi_\psi(b)\xi_\psi | \pi_\psi(c)\xi_\psi) \\ &= (\pi_\psi(\rho(a)1)\pi_\psi(b)\xi_\psi | \pi_\psi(c)\xi_\psi). \end{aligned}$$

Hence $\pi_\psi(\sigma_n(a)) \rightarrow \pi_\psi(\rho(a)1)$ weakly. □

4.3 Non-diagonal Actions of Xerox Type

On $A = \otimes M_k(\mathbb{C})$ we act by a xerox action $\alpha_g = \otimes \text{Ad } \pi(g)$, induced by a representation π of a compact group G . The representation π is not necessarily diagonal.

Definition 4.3.1. Let $S_\infty = \cup S_n$ be the group of finite permutations of $\mathbb{N}^* = \{1, 2, \dots\}$. Each $\sigma \in S_\infty$ induces an automorphism on $A = \otimes M_k(\mathbb{C})$, given by:

$$\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \cdots) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \otimes \cdots$$

If $\sigma_n \in S_\infty$ is the permutation given by

$$\sigma_n(i) = \begin{cases} n+i & \text{if } 1 \leq i \leq n \\ i-n & \text{if } n < i \leq 2n \\ i & \text{if } i > 2n \end{cases}$$

then the corresponding automorphism induced on A is also denoted σ_n . We obtain a sequence $(\sigma_n)_{n \geq 1}$ of automorphisms of A which commute with the action α .

In this section $(\sigma_n)_{n \geq 1}$ denotes the sequence of automorphisms defined above.

Lemma 4.3.1. *Let $\psi = \otimes \psi_n$ a faithful diagonal product state on $A = \otimes M_k(\mathbb{C})$ and let $(h_n)_{n \geq 1}$ be the corresponding sequence of density diagonal matrices. We assume that $(h_n)_{n \geq 1}$ converges to h . Let ρ be the homogeneous state on A given by $\rho = \otimes \text{tr}(h \cdot)$. Then:*

(i) ρ is pointwise mixing w.r.t $(\sigma_n)_{n \geq 1}$, i.e., $\lim_{n \rightarrow \infty} \rho(b\sigma_n(a)c) = \rho(bc)\rho(a)$, for all $a, b, c \in A$.

(ii) the pair of states (ψ, ρ) is mixing w.r.t $(\sigma_n)_{n \geq 1}$, i.e., $\lim_{n \rightarrow \infty} \psi(b\sigma_n(a)c) = \psi(bc)\rho(a)$, for all $a, b, c \in A$.

Proof. We prove only the last part of statement (ii) as all the other claims follow by similar arguments.

We can assume that $\|a\|, \|b\|, \|c\| < 1$. Let $\varepsilon > 0$ be given. Then there exist $m \in \mathbb{N}$ and $a', b', c' \in \bigotimes_{i=1}^m M_k(\mathbb{C}) \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots$ such that

$$\|a - a'\|, \|b - b'\|, \|c - c'\| < \varepsilon/7 \text{ and } \|a'\|, \|b'\|, \|c'\| < 1.$$

For $n \geq m$, we have :

$$\begin{aligned} \|\psi(bg_n(a)c - b'g_n(a')c')\| &\leq \|\psi(b(g_n(a) - g_n(a'))c)\| + \|\psi((b - b')g_n(a')c)\| + \\ &\quad + \|\psi(b'g_n(a')(c - c'))\| \\ &< \|b\|\|a - a'\|\|c\| + \|(b - b')\|\|a'\|\|c\| + \|b'\|\|a'\|\|c - c'\| \\ &< \varepsilon/7 + \varepsilon/7 + \varepsilon/7 = \frac{3}{7}\varepsilon \end{aligned}$$

Similarly,

$$\|\rho(a)\psi(bc) - \rho(a')\psi(b'c')\| < \frac{3}{7}\varepsilon$$

As $\lim_{n \rightarrow \infty} h_n = h$, we have

$$\lim_{n \rightarrow \infty} \psi(b'g_n(a')c') = \rho(a')\psi(b'c').$$

Hence there exists $M > m$ such that for $n \geq M$,

$$|\psi(b'g_n(a')c') - \rho(a')\psi(b'c')| < \frac{\varepsilon}{7}.$$

Consequently, for any $n > M$ we have

$$\begin{aligned} \|\psi(bg_n(a)c - \rho(a)\psi(bc))\| &\leq \|\psi(bg_n(a)c - b'g_n(a')c')\| + \\ &\|\psi(b'g_n(a')c') - \rho(a')\psi(b'c')\| + \|\rho(a)\psi(bc) - \rho(a')\psi(b'c')\| < \varepsilon \end{aligned}$$

and the claim is proved. \square

Let $A = \bigotimes M_k(\mathbb{C})$ be the k^∞ -UHF algebra and $\varphi = \bigotimes \varphi_n$ be a product state on A . If $J \subset \mathbb{N}$ let us define

$$\begin{aligned} B &= \bigotimes_{n \in J} M_k(\mathbb{C}), \quad C = \bigotimes_{n \in J^c} M_k(\mathbb{C}) \\ \psi &= \bigotimes_{n \in J} \varphi_n, \quad \eta = \bigotimes_{n \in J^c} \varphi_n. \end{aligned}$$

We then have the following result whose proof is straightforward:

Lemma 4.3.2. *With the above notation we have:*

- (i) *there exists a canonical isomorphism $\beta : A \rightarrow B \otimes C$ such that $(\psi \otimes \eta) \circ \beta = \varphi$*
- (2) *if $(\pi_\varphi, H_\varphi, \xi_\varphi)$ and $(\pi_{\psi \otimes \eta}, H_{\psi \otimes \eta}, \xi_{\psi \otimes \eta})$ denote the GNS representations of (A, φ) and $(B \otimes C, \psi \otimes \eta)$, then β induces a unitary $U : H_\varphi \rightarrow H_{\psi \otimes \eta}$, defined by*

$$U\pi_\varphi(a)\xi_\varphi = \pi_{\psi \otimes \eta}(\beta(a))\xi_{\psi \otimes \eta} \text{ for all } a \in A.$$

We consider also the following subgroup of G :

$$G_0 := \{g \in G; \pi(g) \text{ is a diagonal matrix}\}$$

Let $\varphi = \otimes \varphi_n$ a faithful product state on A and let $(h_n)_{n \geq 1}$ be the corresponding sequence of density matrices.

From now on, we suppose that the sequence $(h_n)_{n \geq 1}$ has a subsequence $(h_{n_k})_{k \geq 1}$ convergent to h , and h has distinct entries (notice that $(h_n)_{n \geq 1}$ has at least a limit point, but it is possible that the limit point does not have distinct entries).

Let $J = \{n_1 < n_2 < \dots < n_k < \dots\}$ and $J^c = \{m_1 < m_2 < \dots < m_k < \dots\}$. By Lemma 4.3.2, we have $A = B \otimes C$, $\varphi = \rho \otimes \psi$. We remark that the stabilizers G_φ , G_ρ and G_ψ are all equal to G_0 .

Lemma 4.3.3. *With the above notation, the imprimitivity system B of the induction $\{\pi^\sim, U^\sim, H^\sim\} = \text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}$ is contained in the centre of $\pi^\sim(A)''$.*

Proof. By Lemma 4.2.4 we have

$$\lim_{n \rightarrow \infty} \pi_{\psi \otimes \eta}(\{\sigma_n \otimes 1\}(a \otimes 1)) = \lim_{n \rightarrow \infty} \pi_{\psi \otimes \eta}(\sigma_n(a) \otimes 1) = \pi_{\psi \otimes \eta}(\rho(a)1)$$

weakly. Let $\tilde{\sigma}_n := \beta^{-1} \circ \{\sigma_n \otimes 1\} \circ \beta$. Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \pi_\varphi(\beta^{-1} \circ \{\sigma_n \otimes 1\} \circ \beta(\beta^{-1}(a \otimes 1))) &= \lim_{n \rightarrow \infty} \pi_\varphi(\beta^{-1}(\sigma_n(a) \otimes 1)) \\ &= \lim_{n \rightarrow \infty} U^* \pi_{\psi \otimes \eta}(\sigma_n(a) \otimes 1) U = U^* \pi_{\psi \otimes \eta}(\rho(a)1) U = \pi_\varphi(\rho(a)1) \end{aligned}$$

weakly. Now, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (\pi^\sim \circ \tilde{\sigma}_n(\beta^{-1}(a \otimes 1))\xi|\eta) &= \lim_{n \rightarrow \infty} \int_G (\pi_\varphi \circ \alpha_g \circ \tilde{\sigma}_n(\beta^{-1}(a \otimes 1))\xi(g)|\eta(g))dg \\ &= \lim_{n \rightarrow \infty} \int_G (\pi_\varphi \circ \tilde{\sigma}_n \circ \alpha_g(\beta^{-1}(a \otimes 1))\xi(g)|\eta(g))dg \\ &= \int_G \rho \circ \alpha_g(a)(\xi(g)|\eta(g))dg = \int_G f_a^\rho(g)(\xi(g)|\eta(g))dg = (\theta(f_a^\rho)\xi|\eta) \end{aligned}$$

where $f_a^\rho(g) = \rho \circ \alpha_g(a)$. Hence $\theta(f_a^\rho)$ belongs to $\pi^\sim(A)''$. Since $\{f_a^\rho, a \in A\} = C_\rho(G)$ is dense in $C(G_0 \setminus G)$ by Lemma 4.2.3, θ maps $C(G_0 \setminus G)$ into $\pi^\sim(A)''$. The σ -weak continuity of θ yields that $B \subseteq \pi^\sim(A)''$. By its construction, $B \subseteq \pi^\sim(A)'$. Hence $B \subseteq \pi^\sim(A)' \cap \pi^\sim(A)''$. \square

Theorem 4.3.4. *If G is acting on A by a non-diagonal xerox action, $\varphi = \otimes \varphi_n$ is a product state with corresponding sequence of density matrices $(h_n)_{n \geq 1}$, and $(h_n)_{n \geq 1}$ has a limit point with distinct entries, then $\pi_\varphi(A^G)'' = \pi_\varphi(A^{G_\varphi})''$ where π_φ is the GNS representation of (A, φ)*

Proof. Let $\{\pi_\varphi, U_\varphi, H_\varphi\}$ be the covariant representation of (A, G_φ) and $(\pi^\sim, U^\sim, H^\sim) = \text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}$. By Lemma 4.3.4 the imprimitivity system B of $\text{Ind}_{G_\varphi \uparrow G} \{\pi_\varphi, U_\varphi, H_\varphi\}$ is contained in the center of $\pi^\sim(A)''$. The proof follows now from Theorem 4.1.4 and Theorem 4.1.2. We notice that $G_\varphi = G_0$. \square

Theorem 4.3.5. *Suppose $\varphi = \otimes \text{tr}(h_n \cdot)$ is a faithful diagonal product state on $A = \otimes M_k(\mathbb{C})$, the sequence $(h_n)_{n \geq 1}$ of the density matrices has as limit point with distinct entries and G is a compact group acting on A by the xerox action $\alpha(g) = \text{Ad } \pi(g)$ with π a representation of G on $\mathcal{U}(M_k(\mathbb{C}))$ such that $\varphi^{G_\varphi} = \varphi|_{A^{G_\varphi}}$ is a factor state. Then $\varphi^G = \varphi|_{A^G}$ is a factor state and $\pi_{\varphi^G}(A^G)'' \simeq \pi_{\varphi^{G_\varphi}}(A^{G_\varphi})''$. In addition, $G_\varphi = G_0$.*

Proof. By Theorem 4.3.4, we have that $\pi_\varphi(A^G)'' = \pi_\varphi(A^{G_\varphi})''$. Let π_1 be the restriction of π_φ to A^{G_φ} acting on H_1 which is the closed span of $\{\pi_\varphi(A^{G_\varphi})''\xi_\varphi\} = \{\pi_\varphi(A^G)''\xi_\varphi\}$. Note that $\xi_\varphi \in H_1$ is a cyclic vector for π_1 and ξ_φ is cyclic in H_1 for A^G so the restriction of π_1 to A^G is unitary equivalent to π_{φ^G} . Hence $\pi_{\varphi^G}(A^G)''$ is $*$ -isomorphic to

$\pi_1(A^G)''$. Similarly $\pi_{\varphi_{G\varphi}}(A^{G\varphi})''$ is $*$ -isomorphic with $\pi_1(A^{G\varphi})''$. Since H_1 is an invariant subspace of H for $\pi_{\varphi}(A^{G\varphi})$ and $\pi_{\varphi}(A^G)'' = \pi_{\varphi}(A^{G\varphi})''$ we have $\pi_1(A^G)'' = \pi_1(A^{G\varphi})''$ and consequently $\pi_{\varphi_G}(A^G)'' \simeq \pi_{\varphi_{G\varphi}}(A^{G\varphi})''$. \square

Remarks. In [BP2], Baker and Powers considered:

- (1) standard representations on $\mathcal{U}(M_k(\mathbb{C}))$ of groups G , which are unitary groups of $*$ -subalgebras of $M_k(\mathbb{C})$,
- (2) product states $\varphi = \otimes \varphi_n$ on $A = \otimes M_k(\mathbb{C})$.

If $(\pi_{\varphi}, H_{\varphi})$ is the GNS representation for (A, φ) and H is the subgroup of G defined by $H = \{g \in G; \pi_{\varphi} \underset{q}{\sim} \pi_{\varphi} \circ \alpha_g\}$, (here $\pi \underset{q}{\sim} \pi \circ \alpha_g$ means quasiequivalence of representations) they showed that $\pi_{\varphi}(A^G)'' = \pi_{\varphi}(A^H)''$.

If φ is a diagonal product state whose sequence of density matrices has a limit point with distinct entries and we consider a xerox action as in [PB2], then H coincides with $G_0 = \{g \in G, \pi(g) \text{ diagonal}\}$. The tools used by Powers and Baker allow us to conclude that we can have equality even when we impose weaker conditions for the sequence of density matrices of the product state. On the other hand, we considered arbitrary xerox actions (not only of the type considered in [BP2]). We showed that if the xerox action is induced by a representation π of a compact group G , and $G_0 = \{g \in G; \pi(g) \text{ is a diagonal matrix}\}$ then $\pi_{\varphi}(A^G)'' = \pi_{\varphi}(A^{G_0})''$ whenever π_{φ} is the GNS representation for (A, φ) with φ faithful diagonal product state whose sequence of density matrices has a limit point with distinct entries. To find a smaller subgroup H of G such that $\pi_{\varphi}(A^G)'' = \pi_{\varphi}(A^H)''$ we don't need to consider a diagonal product state. We consider only diagonal states is motivated by the fact that in this way we can use the tools developed in Chapter II and Chapter III and by Theorem 4.3.6 we can also determine the type of the factor obtained.

In particular, if G is a compact connected Lie group with maximal torus T , $\varphi = \otimes \varphi_n$ is a product diagonal state and the sequence of density matrices $(h_n)_{n \geq 1}$ has a limit point with distinct entries, then $\pi_{\varphi}(A^T)'' = \pi_{\varphi_G}(A^G)''$.

4.4 Examples

We present now some concrete situations. On $A = \otimes M_3(\mathbb{C})$ we consider the product state $\varphi = \otimes \varphi_n$ $\varphi_n(\cdot) = \text{tr}(h_n \cdot)$ and $h_n = \text{diag}(a_n, b_n, c_n)$, $a_n, b_n, c_n > 0$ $a_n + b_n + c_n = 1$.

Example 4.4.1. Let $SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$.

Let $\pi : SU(2) \rightarrow M_3(\mathbb{C})$ given by

$$\pi \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{bmatrix}$$

We consider the fixed point algebra under the xerox action induced by this representation and we denote it by $A^{SU(2)}$. The maximal torus of $SU(2)$ is T the 1-dimensional torus and the fixed point algebra under the restriction of the action to T is denoted by A^T . In fact, the restriction of the action to the maximal torus is the xerox action induced by the representation $\pi|_T : T \rightarrow M_3(\mathbb{C})$

$$\pi|_T(t) = \begin{bmatrix} 1 & & \\ & e^{it} & \\ & & e^{-it} \end{bmatrix}$$

This was studied in Section 2.5. If (a_n, b_n, c_n) has a limit point (a, b, c) with a, b, c all distinct then $SU(2)_\varphi = T$ and consequently,

$$\pi_\varphi(A^T)'' = \pi_\varphi(A^{SU(2)})''.$$

In fact, in this case the same conclusion holds if we only assume that (a_n, b_n, c_n) has a limit point (a, b, c) with $b \neq c$.

Example 4.4.2. We consider now $SU(3)$ acting on A by the xerox action induced by its standard representation on $M_3(\mathbb{C})$. The maximal torus of $SU(3)$ is T^2 . If (a_n, b_n, c_n) has a limit point (a, b, c) with a, b and c mutually distinct, then $SU(3)_\varphi =$

$\{\text{diag}(-t - s, s, t)\}$. As $A^{SU(3)_\varphi} = A^{T^2}$, where A^{T^2} is the fixed point algebra under the standard xerox action of the 2-torus studied in Section 2.4, we have

$$\pi_\varphi(A^{T^2})'' = \pi_\varphi(A^{SU(3)})''.$$

Chapter 5

Approximate Transitivity

We recall that if $M = \otimes(M_k(\mathbb{C}), \varphi_n)$ is an ITPFI $_k$ factor and $\alpha : G \rightarrow \text{Aut}(M)$ is a xerox action, then the fixed point algebra $N = M^\alpha$ is an AFD von Neumann algebra. In Chapter II, we gave conditions for N to be a factor. In this chapter, we look for sufficient conditions for N to be (isomorphic to) an ITPFI. In [BG], Baker and Giordano give sufficient conditions for the fixed point factor N under the standard action of the 1-torus to be isomorphic to an ITPFI factor. They show that this is true if the sequence of density matrices of the product state $\varphi = \otimes \varphi_n$ has a limit point with no zero entries on the diagonal. They also prove that any bounded ITPFI can be obtained as fixed point factor under the standard action of the 1-dimensional torus.

We generalize these results for the standard action of the 2-torus. Let M be an ITPFI $_3$ factor and N be the subfactor of M obtained as the fixed point algebra under the standard action of T^2 . We show that N is isomorphic to an ITPFI factor if the sequence of density matrices of the product state has a limit point with nonzero entries on the diagonal. Using the same techniques as in [BG], we realize the inclusion $N \subseteq M$ as $W^*(X, \mu, \mathcal{R}_\infty) \subseteq W^*(X, \mu, \mathcal{T})$ where $X = \coprod\{0, 1, 2\}$ and \mathcal{R}_∞ is a subequivalence relation of tail equivalence \mathcal{T} on X . As in the case of the 1-dimensional torus, \mathcal{R}_∞ is induced by an action of the group S_∞ , of finite permutations acting on X (see Section 2.2 and 2.4). In order to show that N is isomorphic to an ITPFI factor, we prove

that the associated flow of \mathcal{R}_∞ is approximately transitive. We have already shown in Sections 2.4 that any ITPFI₃ can be obtained as such a fixed point factor.

For the standard action of the 1-dimensional torus, if the sequence of density matrices does not have a limit point with nonzero entries, by imposing additional conditions we show that the fixed point factor is isomorphic to an ITPFI factor. To obtain this result, we realize again our factor as $W^*(X, \mu, \mathcal{R}_\infty)$, we compute the associated flow of \mathcal{R}_∞ and we show that it is AT. If the sequence of density matrices does not have a limit point with no zero entries and without additional conditions, it is still not known if the fixed point factor it is an ITPFI factor or not.

In [GH], Giordano and Handelman show the existence of an action of \mathbb{Z}_2 on an unbounded ITPFI for which the fixed point algebra N is not an ITPFI factor. They prove it by showing that the flow of weights of N is not AT. We construct here examples of actions of \mathbb{Z}_n , $n \geq 1$ on unbounded ITPFI factors, whose fixed point algebra is isomorphic to an ITPFI factor. To obtain this result, we realize the fixed point factor as $W^*(X, \mu, \mathcal{R})$, with a subequivalence relation \mathcal{R} , of tail equivalence, we compute the associated flow of \mathcal{R} and we prove that it is AT. The difference between our examples and the example built in [GH] consists in the choice of the density matrices of the product state.

5.1 Fixed Point Factors Isomorphic to ITPFI Factors

The first result of this section is the following:

Theorem 5.1.1. *Let $A = \otimes M_3(\mathbb{C})$ and let $\varphi = \otimes \text{tr}(h_n \cdot)$ be a faithful diagonal product state on A .*

a) If the sequence $(h_n)_{n \geq 1}$ of density matrices is given by $h_{2n} = \text{diag}(a_0^n, a_1^n, a_2^n)$ and $h_{2n+1} = \text{diag}(b_0, b_1, b_2)$, then $\pi_\varphi(A^{T^2})''$ is isomorphic to an ITPFI factor.

b) If the sequence $(h_n)_{n \geq 1}$ of density matrices has a limit point $h > 0$ then the same result holds.

As discussed in Section 2.4, $\pi_\varphi(A^{T^2})$ is isomorphic to $W^*(X, \eta, \mathcal{R}_\infty)$, where $X = \prod\{0, 1, 2\}$, $\eta = \otimes \eta_n$ is the product measure on X with $\eta_{2n}(i) = b_i$, $\eta_{2n+1}(i) = a_i^n$ for all $i \in \{0, 1, 2\}$ and $n \geq 1$ and the equivalence relation \mathcal{R}_∞ is given by:

$$x \mathcal{R}_\infty y \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for } i > n \text{ and}$$

$$\text{card}\{i; 1 \leq i \leq n, x_i = k\} = \text{card}\{i; 1 \leq i \leq n, y_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

By Theorem 1.2.20, Theorem 5.1.1 will follow from:

Lemma 5.1.2. *With the above notation, the associated flow of $(X, \mu, \mathcal{R}_\infty)$ is AT.*

Proof. We identify (X, η) with $(X_1 \times X_2, \mu \times \nu)$ where $X_1 = X_2 = \prod\{0, 1, 2\}$ and the measures $\mu = \otimes \mu_n$ and $\nu = \otimes \nu_n$ are given by $\mu_n(i) = a_i^n$, $\nu_n(i) = b_i$ for $n \geq 1$ and $i \in \{0, 1, 2\}$. Then, the equivalence relation \mathcal{R}_∞ can be seen as an equivalence relation on $X_1 \times X_2$ denote also by \mathcal{R}_∞ and given by:

$$\text{for } (x, y) \in X_1 \times X_2, \text{ with } x = (x_1, x_2, \dots) \text{ and } y = (y_1, y_2, \dots),$$

$$(x, y) \mathcal{R}_\infty (x', y') \text{ if and only if there exists } n \in \mathbb{N} \text{ such that } x_i = x'_i, y_i = y'_i \text{ for } i > n$$

$$\text{and } \text{card}\{i; 1 \leq i \leq n, x_i = k\} + \text{card}\{i; 1 \leq i \leq n, y_i = k\}$$

$$= \text{card}\{i; 1 \leq i \leq n, x'_i = k\} + \text{card}\{i; 1 \leq i \leq n, y'_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

The Lebesgue measure on \mathbb{R} is denoted by λ . With these notations, we consider also the equivalence relation $\tilde{\mathcal{R}}_\infty$ on $X_1 \times X_2 \times \mathbb{R}$, given by:

$$(x, y, s) \tilde{\mathcal{R}}_\infty (x', y', t) \text{ if and only if } (x, y) \mathcal{R}_\infty (x', y') \text{ and } t = s - \log \delta((x', y'), (x, y)).$$

We also consider the following equivalence relation on X_2 , denoted by \mathcal{S}_∞ :

$$y \mathcal{S}_\infty y' \text{ if and only if there exists } n \in \mathbb{N} \text{ such that } y_i = y'_i \text{ for } i > n \text{ and}$$

$$\text{card}\{i; 1 \leq i \leq n, y_i = k\} = \text{card}\{i; 1 \leq i \leq n, y'_i = k\} \text{ for } k \in \{0, 1, 2\}.$$

As ν is \mathcal{S}_∞ -invariant we have that

$$(x, y, t) \tilde{\mathcal{R}}_\infty(x, y', t) \text{ for all } x, t \text{ and all } y \mathcal{S}_\infty y'$$

On X_1 , we consider the following equivalence relations

$$x \mathcal{T}_n y \text{ if and only if } x_i = y_i, \text{ for all } i > n$$

and

$$x \mathcal{T} y \text{ if and only if there exists } n \text{ such that } x \mathcal{T}_n y$$

Let now $\tilde{\mathcal{T}}$ be the equivalence relation on $(X_1 \times \mathbb{R}, \mu \times \lambda)$, given by:

$$(x, s) \tilde{\mathcal{T}}(y, t) \text{ if it exists } m \text{ such that } x \mathcal{T}_m y \text{ and } t = s - \log \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} \prod_{i=1}^m \frac{b_{x_i}}{b_{y_i}}.$$

Let $f \in L^\infty(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)^{\tilde{\mathcal{R}}_\infty}$, the space of all essentially bounded functions defined on $(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)$, invariant under $\tilde{\mathcal{R}}_\infty$. As any invariant function is equal almost everywhere to an invariant bounded function, ([Z1], p.21), we can assume that f is bounded and invariant. Then we have

$$f(x, y, t) = f(x, y', t) \text{ for all } (x, y, t) \in X_1 \times X_2 \times \mathbb{R}, \quad y \mathcal{S}_\infty y'.$$

As by [AP], \mathcal{S}_∞ is ν -ergodic, we have that for all (x, t) , $y \mapsto f(x, y, t)$, is constant ν -a.e. We denote this value by $\mathbf{f}(x, t)$. Hence, we have a function $\mathbf{f} : X_1 \times \mathbb{R} \rightarrow \mathbb{R}$. Fix $(x, t) \in X_1 \times \mathbb{R}$ with $x = (x_1, x_2, \dots) \in X_1$. Let $(y, t') \in X_1 \times \mathbb{R}$ where $y = (y_1, y_2, \dots, y_m, x_{m+1}, \dots) \in X_1$ and $t' := t - \log \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} \prod_{i=1}^m \frac{b_{x_i}}{b_{y_i}}$. In other words $(x, t) \tilde{\mathcal{T}}(y, t')$.

Let us denote by C_1 the cylinder set $C(y_1, \dots, y_m) \subset X_2$. As $z \mapsto f(x, z, t)$ is constant for ν -a.e $z \in X_2$, there exists $A_1 \subset X_2$, $\nu(A_1) = 0$ such that $f(x, z, t) = \mathbf{f}(x, t)$ for all $z \in C_1 - A_1$. Let C_2 be the cylinder set $C(x_1, \dots, x_m) \subset X_2$. As above, $f(y, z, t') = \mathbf{f}(z, t')$ for ν -a.e. $z \in X_2$ and there exists $A_2 \subset X_2$, $\nu(A_2) = 0$ such that $f(y, z, t') = \mathbf{f}(y, t')$ for all $z \in C_2 - A_2$. There exists ϕ , an automorphism of (X_2, ν) which affects only the first m coordinates such that $\phi(C_1) = C_2$, i.e.,

$\phi(y_1, y_2, \dots, y_m, y_{m+1}, \dots) = (x_1, x_2, \dots, x_m, y_{m+1}, \dots)$ when $y \in C_1$. As $\nu(A_2) = 0$, we also have that $\nu(\phi^{-1}(A_2)) = 0$. Therefore $\nu(C_1 - A_1 - \phi^{-1}(A_2)) > 0$. Let $z \in C_1 - A_1 - \phi^{-1}(A_2)$. Then $\phi(z) \in C_2 - A_2$. We denote $\phi(z)$ by \hat{z} . It also exists a partial Borel isomorphism σ with $\text{Graph } \sigma \subseteq \mathcal{R}_\infty$ and such that $\sigma(x, z) = (y, \hat{z})$. We have $f(x, z, t) = \mathbf{f}(z, t)$, $f(y, \hat{z}, t') = \mathbf{f}(y, t')$ and

$$\delta((y, \hat{z})(x, z)) = \frac{d\mu \times \nu \circ \sigma}{d\mu \times \nu}(x, z) = \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} \prod_{i=1}^m \frac{b_{x_i}}{b_{y_i}} = t'.$$

Now, we have:

$$\mathbf{f}(x, t) = f(x, z, t) = f(y, \hat{z}, t') = \mathbf{f}(y, t')$$

which proves that $(x, t) \mapsto \mathbf{f}(x, t)$ is $\tilde{\mathcal{T}}$ -invariant. The function \mathbf{f} is bounded and measurable, as $\mathbf{f}(x, y) = \int f(x, y, t) d\nu(y)$.

Therefore $L^\infty(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)^{\tilde{R}_\infty}$ can be identified with $L^\infty(X_1 \times \mathbb{R}, \mu \times \lambda)^{\tilde{\mathcal{T}}}$. Moreover the action of \mathbb{R} by translation on $L^\infty(X_1 \times X_2 \times \mathbb{R}, \mu \times \nu \times \lambda)^{\tilde{R}_\infty}$ corresponds to the action of \mathbb{R} on $L^\infty(X_1 \times \mathbb{R}, \mu \times \lambda)^{\tilde{\mathcal{T}}}$. In other words, the space $X_1 \times X_2 \times \mathbb{R} / \tilde{R}_\infty$ on which \mathbb{R} acts by translation is identified with the space $X_2 \times \mathbb{R} / \tilde{\mathcal{T}}$ on which \mathbb{R} acts by translation. This means that the associated flow of $(X_1 \times X_2, \mathcal{R}_\infty)$ is identified with the space $X_2 \times \mathbb{R} / \tilde{\mathcal{T}}$ on which \mathbb{R} acts by translation. On the other hand, $\tilde{\mathcal{T}}$ is generated by the following action of $G = \oplus \mathbb{Z}_3$ on $(X_1 \times \mathbb{R}, \mu \times \lambda)$:

$$\alpha_g(x, t) := \left(gx, t - \log \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} \prod_{i=1}^m \frac{b_{x_i}}{b_{y_i}} \right)$$

where $g = (g_1, g_2, \dots, g_m, 0, \dots) \in G$ and $y_i = x_i + g_i \pmod{3}$.

Therefore $X_1 \times \mathbb{R} / \tilde{\mathcal{T}} = X_1 \times \mathbb{R} / \alpha$. To show that the associated flow is AT, by [CW] p.206, it is enough, to prove that the action γ of $G \times \mathbb{R}$ on $X_1 \times \mathbb{R}$ given by:

$$\gamma_{g,s}(x, t) = (y, t + s - \log \prod_{i=1}^m \frac{a_{y_i}^i}{a_{x_i}^i} \prod_{i=1}^m \frac{b_{x_i}}{b_{y_i}}),$$

where $y = gx$ and g acts on the first m components, is AT. Let

$$L_{g,s}(f) = f \circ \gamma_{g,s} \frac{d(\mu \times \lambda) \circ \gamma_{g,s}}{d(\mu \times \lambda)}$$

We show first that if

$$f = \chi_{A \times [0, \epsilon]}, \quad h = \chi_{B \times [c, d]}$$

with $d - c = \epsilon$, $A = C(0, 0, \dots, 0) = \{x \in X; x_i = 0, \text{ for } 1 \leq i \leq m\}$ and $B = C(y_1, y_2, \dots, y_m)$, there exist $(g, s) \in G \times \mathbb{R}$ and $\beta > 0$ such that $\beta \cdot L_{g,s}(f) = h$. Indeed, first we can choose $g \in G$ such that $\chi_A \circ g = \chi_B$ and g affects only the first m coordinates, i.e., $gB = A$. As the Radon-Nikodym derivative is constant on cylinder sets, and the Lebesgue measure is measure preserving, for $x \in B$ and $t \in \mathbb{R}$, we have

$$\frac{d\mu \circ g}{d\mu}(x) = \prod_{i=1}^m \frac{a_0^i}{a_{y_i}^i} = \frac{d(\mu \times \lambda) \circ \gamma_{g,s}(x, t)}{d(\mu \times \lambda)}(x, t).$$

We denote $\log \prod_{i=1}^m \frac{a_0^i}{a_{y_i}^i} \prod_{i=1}^m \frac{b_{y_i}}{b_0}$ by k and $\prod_{i=1}^m \frac{a_0^i}{a_{y_i}^i}$ by β^{-1} . Let us take s such that $[0, \epsilon] = [c, d] + s - k$, i.e., $s = -c + k$. We have that

$$\chi_{A \times [0, \epsilon]} \circ \gamma_{g,s} = \chi_{B \times [c, d]}$$

Then we have

$$\beta \cdot f \circ \gamma_{g,s} \frac{d\mu \times \lambda \circ \gamma(g, s)}{d\mu \times \lambda} = \beta \cdot L_{g,s}(f) = h$$

Let f, g be two functions in $L^1(X \times \mathbb{R})_+$. Then there exist m sufficiently large and ϵ sufficiently small such that f and g can be approximated by a finite linear combination with positive coefficients by a sum of functions of the form $\chi_{C(x_1, x_2, \dots, x_m) \times [a, b]}$ with $[a, b]$ intervals of length ϵ . On the other hand, as seen before, any such a function is approximated by a function of the form $\beta \cdot L_{g,s}(\chi_{C(0,0,\dots,0) \times [0, \epsilon]})$ with $\beta > 0$ and $(g, s) \in G \times \mathbb{R}$. Hence, f and g are approximated in L^1 -norm by a finite sum of functions of the form $\beta' \cdot L_{g,s}(\chi_{C(0,0,\dots,0) \times [0, \epsilon]})$ with $\beta' > 0$ and therefore the action γ is approximately transitive. Consequently, the action of \mathbb{R} by translation on the space $X_1 \times X_2 \times \mathbb{R} / \tilde{R}_\infty$ is AT.

To prove b) we use a) and Kakutani's Theorem. □

In general we have the following result:

Proposition 5.1.3. *Let $A = \otimes M_k(\mathbb{C})$ and let $\varphi = \otimes \varphi_n$ be a faithful diagonal product state on A . If the sequence $(h_n)_{n \geq 1}$ has a limit point with no zero entries and α is a zero action induced by a diagonal representation of a compact group G , then $\pi_\varphi(A^\alpha)''$ is an ITPFI factor.*

We proceed now to prove the main result of this chapter. We consider the case of the standard action of the 1-dimensional torus \mathbb{T} , on $A = \otimes M_2(\mathbb{C})$. Let φ be a faithful diagonal product state on $A = \otimes M_2(\mathbb{C})$ that does not have a limit point with no zero entries. We will show that $\pi_\varphi(A^T)''$ can still be isomorphic to an ITPFI factor in this case. As we already mentioned, when the sequence of density matrices has a limit point with no zero entries Baker and Giordano [BG] showed that $\pi_\varphi(A^T)''$ is isomorphic to an ITPFI factor.

Let us recall the following definition:

Definition 5.1.1. If \mathcal{R} is an equivalence relation on (X, μ) , a measure ν equivalent to μ is *lacunary* if there exists $\epsilon > 0$ such that $\log \delta_\nu(x, y) = 0$ or $|\log \delta_\nu(x, y)| > \epsilon$ for $(x, y) \in \mathcal{R}$ (where δ_ν is the modulus with respect to ν).

Hamachi-Osikawa proved the following result that we reformulate using an equivalence relation:

Theorem 5.1.4. [HO] *Let \mathcal{R} be a countable equivalence relation of type III on a Lebesgue space (X, \mathfrak{B}, μ) and ν a finite lacunary measure equivalent to μ . Consider on X the equivalence relation \sim :*

$$x \sim y \text{ if and only if } x\mathcal{R}y \text{ and } \log \delta(x, y) = 0,$$

Then:

(1) *there exists a measurable positive function $\phi(x)$ defined on the quotient space X/\sim with $\phi([x]) = \min\{\log \delta(x', x); (x', x) \in \mathcal{R}, \log \delta(x', x) > 0\}$ where $[x]$ is an element of the partition X/\sim containing x ;*

(2) there exists an ergodic automorphism U of X/\sim such that $U([x]) = [x']$ if $(x, x') \in \mathcal{R}$ satisfies $\log \delta(x', x) = \phi([x])$;

(3) the associated flow of \mathcal{R} is the flow built under the ceiling function ϕ with the base automorphism U .

The following lemma will be used later:

Lemma 5.1.5. ([HO], Lemma 2) *Let p and q be positive numbers with $p + q = 1$, and $0 < \varepsilon < 1$. Then, if n and M are positive numbers with $M^2 < npq\varepsilon/8$,*

$$\sum_{k=0}^{n-M} \left| \binom{n}{k} p^k q^{n-k} - \binom{n}{k+M} p^{k+M} q^{n-k-M} \right| < \varepsilon + 128M^2/npq\varepsilon^2$$

We start now to construct an example of a fixed point factor under the standard action of the 1-dimensional torus, isomorphic to an ITPFI factor.

Let $0 < \lambda < 1$ and let $(l_n)_{n \geq 1}$ be an increasing sequence of positive integers and $(\lambda_n)_{n \geq 1}$ a rapidly decreasing sequence of positive real numbers, each of them being a positive integer power of λ such that

$$2 \sum_{i=1}^{n-1} l_i |\log \lambda_i| < |\log \lambda_n|$$

and

$$\lim_{n \rightarrow \infty} l_n \lambda_n = \infty.$$

We put $l_0 = 0$ and for $n \geq 0$, let $L_n := l_0 + l_1 + l_2 + \cdots + l_n$.

Let $M = \otimes (M_2(\mathbb{C}), \text{tr}(a_i \cdot))$ with

$$a_i = \text{diag}\left(\frac{1}{1 + \lambda_n}, \frac{\lambda_n}{1 + \lambda_n}\right)$$

for $L_{n-1} < i \leq L_n$. On M we act by the standard action of the 1-dimensional torus (see Section 2.2) and we denote by N the fixed point algebra under this action. Our goal is to prove:

Theorem 5.1.6. *With the above notations N is isomorphic to an ITPFI factor.*

As seen in Section 2.2, $N \simeq W^*(X, \mu, \mathcal{R}_\infty)$ where $(X, \mu) = \prod_{i \geq 1} (\{0, 1\}, \mu_i)$, with μ_i probability measure on $\{0, 1\}$ given by:

$$\mu_i(0) = \frac{1}{1 + \lambda_n} \quad \mu_i(1) = \frac{\lambda_n}{1 + \lambda_n},$$

for $L_{n-1} + 1 \leq i \leq L_n$. Moreover, \mathcal{R}_∞ is the equivalence relation induced by the action of the group S_∞ , of finite permutations acting on X . The measure μ is lacunary because each λ_n is a positive integer power of λ . By [AW], M is of type III₀ and as $N \simeq W^*(X, \mu, \mathcal{T})$, we have $r_\infty(X, \mu, \mathcal{T}) = \{0, 1\}$. By Proposition 2.2.3, N is of type III, and as $r_\infty(X, \mu, \mathcal{R}_\infty) \subseteq r_\infty(X, \mu, \mathcal{T})$ it results that $r_\infty(X, \mu, \mathcal{R}_\infty) = \{0, 1\}$ and therefore N is a factor of type III₀. To show that $W^*(X, \mu, \mathcal{R}_\infty)$ is isomorphic to an ITPFI factor, we compute the associated flow of \mathcal{R}_∞ and show that it is approximately transitive.

We will need the following technical results.

Lemma 5.1.7. *Let $m \geq 1$ and $|\alpha_i|, |\beta_i| \leq l_i$. If $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_m^{\alpha_m} = \lambda_1^{\beta_1} \lambda_2^{\beta_2} \dots \lambda_m^{\beta_m}$ then $\alpha_i = \beta_i$ for all $1 \leq i \leq m$.*

Proof. Let us assume that $\alpha_m \neq \beta_m$. If $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_m^{\alpha_m} = \lambda_1^{\beta_1} \lambda_2^{\beta_2} \dots \lambda_m^{\beta_m}$. Consequently, $\sum_{i=1}^m \alpha_i \log \lambda_i = \sum_{i=1}^m \beta_i \log \lambda_i$. Hence, $|\log \lambda_m| \leq |\alpha_m - \beta_m| \cdot |\log \lambda_m| \leq \sum_{i=1}^{m-1} |\alpha_i - \beta_i| \cdot |\log \lambda_i| \leq 2l_i |\log \lambda_i|$. Contradiction. The lemma follows by induction. \square

Let $Y = \prod_{n \geq 1} \{0, 1, \dots, l_n\}$ with the product measure $\nu = \otimes \nu_n$ where

$$\nu_n(i) = \binom{l_n}{i} \frac{\lambda_n^i}{(1 + \lambda_n)^{l_n}}, \quad 1 \leq i \leq l_n$$

We consider $\pi : X \rightarrow Y$ given by

$$\pi(x_1, x_2, \dots) = (y_1, y_2, \dots)$$

with $y_n = \sum_{i=L_{n-1}+1}^{L_n} x_i$. In other words y_n is the number of m , $L_{n-1} + 1 \leq m \leq L_n$ such that $x_m = 1$. By $[y_1 y_2 \dots y_n]$ we denote the elementary cylinder $\{z \in Y; z_i = y_i, 1 \leq i \leq n\}$.

Let us define on X the following subequivalence relation of \mathcal{R}_∞ , denoted by \mathcal{R}

$$x\mathcal{R}y \text{ if and only if } x\mathcal{R}_\infty y \text{ and } \delta(x, y) = 1.$$

Lemma 5.1.8. *With the above notations, $\pi : X \rightarrow Y$ is a \mathcal{R} -factor map and therefore X/\mathcal{R} can be identified with Y .*

Proof. Assume that $x\mathcal{R}y$. Then, there exists $n \geq 1$ such that $x_i = y_i$ for all $i > L_n$. Therefore $\log \delta(x, y) = \sum_{k=1}^n \sum_{i=L_{k-1}+1}^{L_k} (y_i - x_i)\lambda_n = 0$. Hence, by the previous lemma,

$$\sum_{i=L_{k-1}+1}^{L_k} (y_i - x_i) = 0 \text{ for all } 1 \leq k \leq n. \text{ In other words, if } x\mathcal{R}y \text{ then } \pi(x) = \pi(y).$$

Let $A \subset X$ be a \mathcal{R} -invariant set. We will show that A may be arbitrarily closely approximated by taking a union of sets of the form $\pi^{-1}[y_1 y_2 \dots y_n]$. We observe first that a set of the form $\pi^{-1}[y_1 y_2 \dots y_n]$ is a union of elementary cylinders in X of length L_n . Let $0 < \varepsilon < 1$. We can find $n > 0$ and a finite number of cylinders, C_1, C_2, \dots, C_k all of length L_n such that if $E = C_1 \cup C_2 \cup \dots \cup C_k$ then $\mu(A \Delta E) < \varepsilon^2$. Let F be the union of all cylinders C , of length L_n , contained in E such that $\frac{\mu(A \cap C)}{\mu(C)} > 1 - \varepsilon$ and let H the union of all cylinders C of length L_k , contained in E such that $\frac{\mu(A \cap C)}{\mu(C)} < 1 - \varepsilon$ or equivalently, $\frac{\mu(C \setminus A)}{\mu(C)} > \varepsilon$. Hence $\frac{\mu(H \setminus A)}{\mu(H)} > \varepsilon$, and therefore $\mu(H) < \frac{1}{\varepsilon} \mu(H \setminus A) < \frac{1}{\varepsilon} \mu(E \setminus A) < \varepsilon$. Hence, $\mu(A \Delta F) < \mu(A \Delta E) + \mu(H) < \varepsilon^2 + \varepsilon < 2\varepsilon$. Let C be a cylinder of length L_n included in $\pi^{-1}[y_1 y_2 \dots y_n]$ such that $\frac{\mu(A \cap C)}{\mu(C)} > 1 - \varepsilon$. Let C' a cylinder of length L_n also included in $\pi^{-1}[y_1 y_2 \dots y_n]$. Hence, $C = [x_1 x_2 \dots x_{L_n}]$ and $C' = [z_1 z_2 \dots z_{L_n}]$ with $\sum_{i>L_{k-1}}^{L_k} x_i = \sum_{i>L_{k-1}}^{L_k} z_i = y_k$ for $1 \leq k \leq n$. Consequently there exists $\phi \in [\mathcal{R}_\infty]$, which affects only the first n coordinates such that $\phi(C) = C'$ (ϕ is measure preserving on C). As A is \mathcal{R} -invariant, it follows that $\frac{\mu(A \cap C')}{\mu(C')} > 1 - \varepsilon$.

Therefore

$$\frac{\mu(\pi^{-1}[y_1 y_2 \dots y_n] \cap A)}{\mu(\pi^{-1}[y_1 y_2 \dots y_n])} > 1 - \varepsilon$$

whenever $\pi^{-1}[y_1 y_2 \dots y_n]$ contains a cylinder of length L_n of F . Let F' the union of all $\pi^{-1}[y_1 y_2 \dots y_n]$ that intersects F . We have that $\frac{\mu(A \cap F')}{\mu(F')} > 1 - \varepsilon$ and therefore $\mu(F' \setminus A) < \varepsilon \mu(F') < \varepsilon$. We also have that $\mu(A \setminus F') < \mu(A \setminus F) < 2\varepsilon$ and therefore

$\mu(A\Delta F') < 3\varepsilon$. Hence any invariant set can be arbitrarily closely approximated by a union of elements of the form $\pi^{-1}[y_1 y_2 \dots y_n]$. As $\mu \circ \pi^{-1} = \nu$, we also have that for $A \subset Y$, $\mu \circ \pi^{-1}(A) = 0$ if and only if $\nu(A) = 0$. Therefore π is a \mathcal{R} -factor map that allows us to identify X/\mathcal{R} with Y . \square

We compute now the associated flow for \mathcal{R}_∞ using Theorem 5.1.4. By Lemma 5.1.8, we can identify X/\mathcal{R} with Y . As μ is lacunary, we can define a function ϕ on X by:

$$\phi(x) = \min\{\log \delta(x', x); x\mathcal{R}_\infty y; \log \delta(x', x) > 0\}.$$

Then ϕ is bounded below by $-\log \lambda$ and can be seen as a function on the quotient Y . By Theorem 5.1.4 we define an automorphism U on Y by: $U(\pi(x)) = \pi(x')$ if $x\mathcal{R}_\infty x'$ satisfies $\log \delta(x', x) = \phi(x)$. On $(D, \nu \times \lambda)$, with $D = \{(y, t); Y \in Y, 0 < t, \phi(z)\}$ and λ the Lebesgue measure, let F_s be the flow built under the ceiling function ϕ with the base transformation $U : Y \rightarrow Y$, i.e., for $s \geq 0$:

$$F_s(z, t) = \begin{cases} (z, t + s) & 0 \leq s + t < \phi(z) \\ (U(z), t + s - \phi(z)) & \phi(z) \leq s + t < \phi(U(z)) + \phi(z) \\ \dots & \dots \end{cases} \quad (5.1.1)$$

By Theorem 5.1.4,(3), the associated flow of \mathcal{R}_∞ is the flow F_s , $s \in \mathbb{R}$. Then we have:

Lemma 5.1.9. *Let $x, x' \in X$, $x\mathcal{R}_\infty x'$, $\alpha := \log \delta(x', x)$, $z = \pi(x)$, $z' = \pi(x')$ and $0 < t < -\log \lambda$ (such that (z, t) and (z', t) are in D). Then $F_s(z, t) = (z', t)$.*

Proof. Let us assume first that $\alpha > 0$. Because the measure μ is lacunary, there are only finitely many values of $\log \delta(z, x)$ between 0 and α , and without loss of generality we can assume that there is only one which we denote by β . Hence there exists $y \in X$, $\log \delta(y, x) = \beta$. Let $v := \pi(y)$. But then $U(z) = v$ and $\phi(z) = \beta$. As $0 < t < -\log \lambda$, $(v, t) \in D$, by (5.1.1), we have $F_{\phi(z)}(z, t) = (v, t)$. Also,

$$\log \delta(x', x) = \log \delta(x', y) + \log \delta(y, x)$$

But $\log \delta(x', y)$ is in fact $\phi(v) = \phi(U(z))$ and $F_{\phi(v)}(v, t) = (z', t)$. Therefore:

$$\phi(z) + \phi(U(z)) = \alpha$$

and

$$F_\alpha(z, t) = F_{\phi(z)+\phi(U(z))}(z, t) = F_{\phi(U(z))}(U(z), t) = (U^2(z), t) = (z', t).$$

If $s < 0$, then $-s > 0$ and as above, $F_{-s}(z', t) = (z, t)$ and therefore $F_s(z, t) = F_s(F_{-s}(z, t)) = (z', t)$. \square

By definition of U , for any $y \in Y$, $U(y)$ and y are cofinal. Therefore, we have:

Remark 5.1.1. If

$$y = (y_1, y_2, \dots, y_m, y_{m+1}, \dots)$$

and for some $n \in \mathbb{Z}$, $U^n y = z$ with

$$z = (z_1, z_2, \dots, z_m, y_{m+1}, \dots),$$

then

$$\frac{d\nu \circ U^n}{d\nu}(y) = \frac{\nu[z_1 z_2 \dots, z_m]}{\nu[y_1 y_2 \dots y_m]} = \prod_{i=1}^m \frac{\nu_i(z_i)}{\nu_i(y_i)}$$

Lemma 5.1.10. Let $x, x' \in X$, $x \mathcal{R}_\infty x'$, $x_i = x'_i$ if $i > L_k$, $y = \pi(x)$, $z = \pi(x')$ $0 < t < -t \log \lambda$ and $s = \log \delta(x', x)$ (in fact $\log \delta(x', x) = \sum_{i=1}^k (z_i - y_i) \log \lambda_i$). Then $F_s(y, t) = (z, t)$ and

$$\frac{d(\nu \times \lambda) \circ F_s}{d(\nu \times \lambda)}(y, t) = \frac{\nu[z_1, z_2, \dots, z_k]}{\nu[y_1, y_2, \dots, y_k]}.$$

Proof. We remark first that $z_i = y_i$ if $i > k$. It exists $n \in \mathbb{Z}$ such that $z = U^n(y)$. As λ is measure preserving we get:

$$\frac{d(\nu \times \lambda) \circ F_s}{d(\nu \times \lambda)}(y, t) = \frac{d\nu \circ U^n}{d\nu}(y)$$

The lemma follows from Remark 5.1.1. \square

Proposition 5.1.11. *The flow F_s on D , defined in (5.1.1) is AT.*

Proof. We denote by $[0]^M$ the cylinder set $\{y \in Y; y_i = 0, 1 \leq i \leq M\}$. We show first that for small ζ and $M > 0$ $\chi_{[0]^M \times [0, \zeta]}$ can approximate in L^1 -norm any function of the form $\chi_{[a_1 a_2 \dots a_M] \times [0, \zeta]}$. Let us assume that $\zeta < |\log \lambda|$. Let $\varepsilon > 0$, let $[a_1 a_2 \dots a_M]$ be a cylinder of length M and $n > M$. Let also,

$$m := a_1 + a_2 + \dots + a_M, \quad s := \sum_{i=1}^M a_i \log \lambda_i - m \log \lambda_n,$$

$$\beta := \frac{\nu[a_1 a_2 \dots a_M]}{\nu[0]^M} = \frac{\nu_1(a_1) \nu_2(a_2) \dots \nu_M(a_M)}{\nu_1(0) \nu_2(0) \dots \nu_M(0)}$$

$$g = \chi_{[a_1 \dots a_M] \times [0, \zeta]}$$

and

$$f = \chi_{[0]^M \times [0, \zeta]}.$$

We have

$$\begin{aligned} \beta \cdot L_{-s} f(z, t) &= \beta \cdot \chi_{[0]^M \times [0, \zeta]}((F_{-s}(z, t)) \frac{d(\nu \times \lambda) \circ F_{-s}}{d(\nu \times \lambda)}(z, t)) \\ &= \beta \cdot \chi_{F_s([0]^M \times [0, \zeta])}(z, t) \frac{d(\nu \times \lambda) \circ F_{-s}}{d\nu \times \lambda}(z, t) \end{aligned}$$

Let $z \in [0]^M$ with $z_n \geq m$ and z' with $z'_i = a_i$ if $0 \leq i \leq M$, $z'_n = z_n - m$ and $z'_i = z_i$ otherwise. We can find x and x' such that $x \mathcal{R}_\infty x'$, $\pi(x) = z$, $\pi(x') = z'$ and $\log \delta(x', x) = \sum_{i=1}^M a_i \log \lambda_i - m \log \lambda_n$. Then, by Lemma 5.1.9, $F_s(z, t) = (z', t)$ and $F_{-s}(z', t) = (z, t)$. Also, if $z_n = k + m$, with $k \geq 0$, by Lemma 5.1.10,

$$\begin{aligned} \frac{d(\nu \times \lambda) \circ F_{-s}}{d(\mu \times \lambda)}(z', t) &= \frac{\nu_1(0) \nu_2(0) \dots \nu_M(0)}{\nu_1(a_1) \nu_2(a_2) \dots \nu_M(a_M)} \frac{\nu_n[k + m]}{\nu_n[k]} \\ &= \beta^{-1} \binom{l_n}{k}^{-1} \frac{(1 + \lambda_n)^{l_n}}{\lambda_n^k} \binom{l_n}{k + m} \frac{\lambda_n^{k+m}}{(1 + \lambda_n)^{l_n}} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d(\nu \times \lambda) \circ F_{-s}}{d(\mu \times \lambda)}(z, t) &= \frac{\nu_1(0) \nu_2(0) \dots \nu_M(0)}{\nu_1(a_1) \nu_2(a_2) \dots \nu_M(a_M)} \frac{\nu_n[k + m]}{\nu_n[k]} \\ &= \beta^{-1} \cdot \binom{l_n}{k}^{-1} \frac{(1 + \lambda_n)^{l_n}}{\lambda_n^k} \binom{l_n}{k + m} \frac{\lambda_n^{k+m}}{(1 + \lambda_n)^{l_n}} \end{aligned}$$

whenever $(z, t) \in [a_1 a_2 \cdots a_M] \cap [z_n = k] \times [0, \zeta]$ and $0 \leq k \leq l_n - m$. Also,

$$\chi_{F_s([0]^M \times [0, \zeta] \cap [z_n = k+m])} = \chi_{[a_1 a_2 \cdots a_M] \cap [z_n = k]}$$

Let $A := \{z \in [0]^k : z_n \geq m\} \times [0, \zeta]$, $A' := \{z \in [0]^M : z_n < m\} \times [0, \zeta]$, $B := \{z \in [a_1 a_2 \cdots a_M] : z_n \leq l_n - m\} \times [0, \zeta]$ and $B' := \{z \in [a_1 a_2 \cdots a_M] : z_n > l_n - m\} \times [0, \zeta]$.

Let $B_k = [a_1 \dots a_M] \cap [z_n = k] \times [0, \zeta]$ and $A_k = [0]^M \cap [z_n = k + m] \times [0, \zeta]$. For $0 \leq k \leq l_n - m$, we have:

$$\begin{aligned} \|\chi_{B_k} - L_{-s} \beta \chi_{A_k}\|_1 &= \left\| \chi_{B_k} - \beta \cdot \chi_{B_k} \beta^{-1} \binom{l_n}{k}^{-1} \frac{(\lambda_n + 1)^{l_n}}{\lambda_n^k} \binom{l_n}{k+m} \frac{\lambda_n^{k+m}}{(\lambda_n + 1)^{l_n}} \right\|_1 \\ &= \|\chi_{B_k}\|_1 \cdot \left[1 - \binom{l_n}{k}^{-1} \frac{(\lambda_n + 1)^{l_n}}{\lambda_n^k} \binom{l_n}{k+m} \frac{\lambda_n^{k+m}}{(\lambda_n + 1)^{l_n}} \right] \\ &= \|g\|_1 \cdot \left[1 - \binom{l_n}{k}^{-1} \frac{(\lambda_n + 1)^{l_n}}{\lambda_n^k} \binom{l_n}{k+m} \frac{\lambda_n^{k+m}}{(\lambda_n + 1)^{l_n}} \right] \cdot \binom{l_n}{k} \frac{\lambda_n^k}{(\lambda_n + 1)^{l_n}} \\ &= \|g\|_1 \cdot \left[\binom{l_n}{k} \frac{\lambda_n^k}{(\lambda_n + 1)^{l_n}} - \binom{l_n}{k+m} \frac{\lambda_n^{k+m}}{(\lambda_n + 1)^{l_n}} \right] \end{aligned}$$

Also,

$$\begin{aligned} \|\beta L_{-s} \chi_{A'}\|_1 &= \beta \|\chi_{[0]^M \cap [z_n \in \{0, 1, \dots, m-1\}] \times [0, \zeta]}\|_1 = \beta \|\chi_{[0]^M \times [0, \zeta]}\|_1 \sum_{i=0}^{m-1} \frac{(\lambda_n)^i}{(1 + \lambda_n)^{l_n}} \binom{l_n}{i} \\ &= \|g\|_1 \cdot \sum_{i=0}^{m-1} \frac{\lambda_n^i}{(1 + \lambda_n)^{l_n}} \binom{l_n}{i} \end{aligned}$$

and

$$\begin{aligned} \|\chi_{B'}\|_1 &= \|\chi_{[a_1 a_2 \cdots a_M] \cap [z_n \in \{l_n - m + 1, \dots, l_n\}] \times [0, \zeta]}\|_1 \\ &= \|g\|_1 \sum_{i=l_n - m + 1}^{l_n} \frac{\lambda_n^i}{(1 + \lambda_n)^{l_n}} \binom{l_n}{i} \end{aligned}$$

As $(1 + \lambda_n)^{l_n} \rightarrow \infty$ when $n \rightarrow \infty$, for n very large,

$$\sum_{i=0}^{m-1} \frac{\lambda_n^i}{(1 + \lambda_n)^{l_n}} \binom{l_n}{i} < \varepsilon$$

$$\sum_{i=l_n-m+1}^{l_n} \frac{\lambda_n^i}{(1+\lambda_n)^{l_n}} \binom{l_n}{i} < \varepsilon$$

We choose n large enough such that also

$$m^2 < l_n \lambda_n \varepsilon / 8(1 + \lambda_n)^2, \text{ and } 128m^2(1 + \lambda_n)^2 / l_n \lambda_n < \varepsilon^3.$$

But $A = \cup_{k=0}^{l_n-m} [0]^M \cap [z_n = k+m] \times [0, \zeta] = \cup_{k=0}^{l_n-m} A_k$ and $B = \cup_{k=0}^{l_n-m} [a_1 a_2 \dots a_M] \cap [z_n = k] \times [0, \zeta] = \cup_{k=0}^{l_n-m} B_k$. Applying Lemma 5.1.5 for l_n , m , $p = \frac{\lambda_n}{1+\lambda_n}$ and $q = \frac{1}{1+\lambda_n}$ we have

$$\begin{aligned} \|\chi_B - \beta L_{-s} \chi_A\| &\leq \sum_{k=0}^{l_n-m} \|\chi_{[a_1 a_2 \dots a_M] \cap [z_n = k] \times [0, \zeta]} - \alpha L_{-s} \chi_{[0]^M \cap [z_n = k+m] \times [0, \zeta]}\|_1 \\ &= \|g\|_1 \sum_{k=0}^{l_n-m} \left| \binom{l_n}{k+m} \frac{\lambda_n^{k+m}}{(1+\lambda_n)^{l_n}} - \binom{l_n}{k} \frac{\lambda_n^k}{(1+\lambda_n)^{l_n}} \right| \\ &< \|g\|_1 (\varepsilon + 128m^2(1+\lambda_n)^2 / l_n \lambda_n \varepsilon^2) < \|g\|_1 \cdot 2\varepsilon \end{aligned}$$

As L_{-s} is norm preserving, we conclude

$$\begin{aligned} \|\beta L_{-s} \chi_{[0]^M \times [0, \zeta]} - \chi_{[a_0 a_1 \dots a_M] \times [0, \zeta]}\|_1 &\leq \|\beta L_{-s} \chi_A - \chi_B\|_1 + \|\chi_B\|_1 + \beta \|\chi_A\|_1 \\ &< \|g\|_1 (2\varepsilon + \varepsilon + \varepsilon) = 4\|g\|_1 \varepsilon. \end{aligned}$$

On the other hand, if $(x, u) \in D$ and $(x, u+t) \in D$ then

$$\frac{d(\nu \times \lambda) \circ F_t}{d(\nu \times \lambda)}(x, u) = 1,$$

as the Radon-Nikodym derivative with respect to the Lebesgue measure is 1. Therefore, if there exists $s \in \mathbb{R}$ and $\beta > 0$ such that $\beta L_s(\chi_{[0]^M \times [0, \zeta]})$ can approximate $\chi_{[a_1 a_2 \dots a_M] \times [0, \zeta]}$ then $\beta L_{-s-t}(\chi_{[0]^M \times [0, \zeta]})$ approximate $\chi_{[a_1 a_2 \dots a_M]^M \times [t, t+\zeta]}$. Consequently there exists $s \in \mathbb{R}$ and $\beta > 0$ such that $\beta L_s(\chi_{[0]^M \times [0, \zeta]})$ approximate in L^1 norm $\chi_{[a_1 a_2 \dots a_M] \times [t, t+\zeta]}$ whenever $[a_1 a_2 \dots a_M] \times [t, t+\zeta] \subset D$. Let $f, g \in L^1(D)_+$. There exist $M > 1$ sufficiently large and $\zeta > 0$ sufficiently small such that f and g are approximated in L^1 norm by a finite sum of functions of the form $c \cdot \chi_{B \times [0, \zeta]}$ with B cylinders of length M , $[a, b]$ intervals of length ε and $c > 0$. As seen before,

any function of the form $\chi_{[a_1 a_2 \dots a_M] \times [t, t+\zeta]}$ can be approximated by $\beta L_s(\chi_{[0]^M \times [0, \zeta]})$ for some $s \in \mathbb{R}$ and $\beta > 0$. We conclude that f and g can be approximated by a sum of the form of $\beta' \cdot L_s(\chi_{[0]^M})$ with $s \in \mathbb{R}$ and $\beta' > 0$. Therefore, the associated flow is approximately transitive. \square

Proof. (Theorem 5.1.6) As the associated flow of \mathcal{R} is AT, by Theorem 1.2.20, $N \simeq W^*(X, \mu, \mathcal{R}_\infty)$ is isomorphic to an ITPFI factor. \square

5.2 Fixed Point Subfactors of Unbounded ITPFI Factors

As announced in the beginning of the chapter, we will show that there exist fixed point factors under actions of \mathbb{Z}_n on unbounded ITPFI that are isomorphic to ITPFI factors.

Theorem 5.2.1. *There exist subfactors of any integer index $k \geq 2$ of unbounded ITPFI that appear as fixed point algebras under action of some finite groups.*

Proof. Consider r_n and l_n nonnegative integers increasing to infinity such that $\log r_n - 2 \sum_{i=1}^{n-1} l_i \log r_i > 0$ and r_n are positive integer powers of $\lambda > 1$ with $\lambda \in \mathbb{Z}$ for $n \geq 1$. We also choose l_n such that $l_n = 0 \pmod{k}$ for $n \geq 1$. Let $l_0 = k$. Let $A = \otimes_{k \geq 0} M_{k_n}(\mathbb{C})$ where $k_0 = k$ and $k_n = \sum_{i=1}^{l_n} r_n^i$ if $n > 0$. On this C*-algebra we act by the group action generated by the automorphism $\alpha = \otimes_{n \geq 0} \text{Ad} u_n$, where

$$u_0 = \text{diag}(1, \alpha, \alpha^2, \dots, \alpha^{k-1})$$

and for $n \geq 1$,

$$u_n = \text{diag}(I_{r_n}, \alpha I_{r_n^2}, \dots, \alpha^{k-1} I_{r_n^k}, I_{r_n^{k+1}}, \alpha I_{r_n^{k+2}}, \dots, \alpha^{k-1} I_{r_n^{2k}}, \dots, I_{r_n^{l_n-k+1}}, \dots, \alpha^{k-1} I_{r_n^{l_n}})$$

and $\alpha = e^{\frac{2\pi i}{k}}$. This automorphism produces an action of \mathbb{Z}_k on A . We consider A^α the fixed point algebra under this action. On A we consider the faithful diagonal product state $\varphi = \otimes \varphi_n$ with $\varphi_n(\cdot) = \text{tr}(a_n \cdot)$. We put $a_0 = \frac{1}{k} I_k$ and for $n > 1$

$$a_n(i, i) = \frac{1}{r_n l_n}; 1 \leq i \leq r_n,$$

$$a_n(i, i) = \frac{1}{r_n^j l_n}; r_n^1 + \dots + r_n^{j-1} + 1 \leq i \leq r_n^1 + \dots + r_n^j, 2 \leq j \leq l_n$$

If M is the ITPFI obtained by performing $\text{GNS}(A, \varphi)$ and $N = \text{GNS}(A^\alpha, \varphi|_{A^\alpha})$ we have that N is a factor. This can be seen as in Section 2.3, or by showing that the action on M is outer and consequently, by [J2], N is a subfactor of M of index k . We will see that N is isomorphic to an ITPFI factor. Let $X = \prod_{n \geq 0} \{0, 1, 2, \dots, k_n - 1\}$ and $\mu = \otimes \mu_n$, the product measure on X defined in the following way:

$$\mu_0(i) = \frac{1}{k}; i = 0, 1, \dots, k - 1$$

and for $n > 0$

$$\mu_n(i) = \frac{1}{r_n l_n}; 0 \leq i \leq r_n - 1,$$

$$\mu_n(i) = \frac{1}{r_n^j l_n}; r_n + \dots + r_n^{j-1} \leq i \leq r_n + \dots + r_n^j - 1; 2 \leq j \leq l_n$$

Let now $Y = \prod \{0, 1, \dots, l_n - 1\}$ and $\pi : X \rightarrow Y$,

$$\pi(x_1, x_2, \dots) = (y_1, y_2, \dots)$$

such that $y_1 = x_1$ and for $n \geq 2$, $y_n = j - 1$ if $x_n \in \{i : r_n + \dots + r_n^{j-1} \leq i \leq r_n + \dots + r_n^j - 1\}$ and $1 \leq j \leq l_n - 1$. We denote by \mathcal{R} the following equivalence relation on X :

$$x \mathcal{R} y \text{ if and only if } x \mathcal{T} y \text{ and } \sum \pi(x)_i - \pi(y)_i = 0 \pmod{k}$$

where \mathcal{T} is the tail equivalence on X (Example 1.1.1). Then $N = W^*(X, \mu, R)$. To see that N is an ITPFI we show that the associated flow is AT. We will follow the same idea as in Theorem 5.1.7 and that why this time we omit some details. The

base space can be identified with $Y = \prod\{0, 1, 2, \dots, l_n - 1\}$ with the measure $\nu = \otimes \nu_n$ where $\nu_n(i) = \frac{1}{l_n}$ for $0 \leq i \leq l_n - 1$. The base transformation is defined as follows: $U(\pi(x)) = \pi(x')$ if $x \mathcal{R}_\infty x'$ satisfies $\log \delta(x', x) = \phi(x)$. In fact if $y = (y_0, y_1, y_2, \dots)$ let $N(y) = \min\{n \geq 1; y_n > 0\}$. Then, $U(y)_i = l_i$ for $1 \leq i < n$, $U(y)_n = y_n - 1$ and $U(y)_i = y_i$ for $i > n$; $U(y)_0 = z_0$ in such way that $z_0 + U(y)_1 + \dots + U(y)_n$ differs from $y_0 + y_1 + \dots + y_n$ by an integer multiple of k . We observe also that the automorphism U is measure preserving. The ceiling function, ϕ is givenm by

$$\phi(x) = \min\{\log \delta(x', x); x \mathcal{R} y; \log \delta(x', x) > 0\}.$$

and the associated flow is the flow F_s which is the flow built under the ceiling function ϕ with the base transformation $U : Y \mapsto Y$. This flow which is also measure preserving is indeed AT. The fact that the flow is measure preserving makes the computations a lot easier than in Proposition 5.1.13.

In order to prove that the flow is AT, as in the proof of Proposition 5.1.13, it is enough to show that $f = \chi_{[00\dots 0]^m \times [0, \zeta]}$ with ζ small, can approximate in L^1 -norm any function of the form $\chi_{[a_0 a_1 \dots a_m]^m \times [0, \zeta]}$, where $[a_0 a_1 \dots a_m] = \{y \in Y, y_i = a_i, 0 \leq i \leq m\}$. Let us assume first that $a_0 + a_1 + \dots + a_m = 0 \pmod{k}$. If $s = -\sum_{i=1}^m a_i \log r_i$ with $0 \leq a_i \leq l_i - 1$ we have

$$L_{-s} f(z, t) = \chi_{[00\dots 0]^m \times [0, \zeta]}(F_{-s}(z, t)) \frac{d(\nu \times \lambda) \circ F_{-s}}{d(\nu \times \lambda)}(z, t)$$

We have that $\chi_{[00\dots 0]^m \times [0, \zeta]}(F_{-s}(z, t)) = \chi_{[a_0 a_1 \dots a_m]^m \times [0, \zeta]}(z, t)$ and therefore we can approximate in L^1 any function $\chi_{[b_0 b_1 \dots b_m]^m \times [0, \zeta]}$ with $b_0 + b_1 + \dots + b_m = 0 \pmod{k}$.

We want to show now that we can approximate in L^1 any function of the form $\chi_{[d00\dots 0]^m \times [0, \zeta]}$ with $1 \leq d \leq k - 1$. Let $p = k - d$. We take a very large n and consider $L_{p \log r_n} f$. Then, we have $f(F_{p \log r_n}) = \chi_{F_{-p \log r_n}([00\dots 0]^k \times [0, \zeta])}$. If $(z, t) \in [00\dots 0]^m \times [0, \zeta]$ and $z_n \leq l_n - p - 1$, then $F_{-p \log r_n}(z, t) = (z', t)$ where $z'_i = z_i$ for $i \neq \{0, n\}$, $z'_n = z_n + p$ and $z'_0 = d$.

Let $A = \{(z, t) \in [00\dots 0]^m \times [0, \zeta]; z_n \leq l_n - p - 1\}$ and $A' = [00\dots 0]^m \times [0, \zeta] \setminus A$. We have that $f = \chi_A + \chi_{A'}$ and $\chi_A \circ F_{p \log r_n} = \chi_{-F_{p \log r_n}(A)}$. We see that $F_{-p \log r_n}(A)$

is equal to $B = \{(z, t) \in [d00 \dots 0]^k \times [0, \zeta]; z_n \geq p\}$. It follows that $L_{-p \log r_n} \chi_A = \chi_B$. Let $B' = [d00 \dots 0]^m \times [0, \zeta] \setminus B$. Since $\|\chi_{A'}\|_1 = \|\chi_{B'}\|_1 = \frac{p}{l_n} \|f\|_1$ and $L_{p \log r_n}$ is linear and norm preserving, we have

$$\|L_{p \log r_n} f - \chi_{[d00 \dots 0]^m \times [0, \zeta]}\|_1 \leq \|L_{p \log r_n} \chi_A - \chi_B\|_1 + \|\chi_{A'}\|_1 + \|\chi_{B'}\|_1 \leq \frac{2p}{l_n} \|f\|_1.$$

This shows that for all $d \in \{1, 2, \dots, k-1\}$ we can obtain an approximation of $\chi_{[d00 \dots 0]^m \times [0, \zeta]}$ as good as we want. In the same way we showed that $\chi_{[00 \dots 0]^m \times [0, \zeta]}$ can approximate in L^1 any function $\chi_{[a_0 a_1 \dots a_m] \times [0, \zeta]}$ with $a_0 + a_1 + \dots + a_m = 0 \pmod{k}$, we can show that $\chi_{[d00 \dots 0]^m \times [0, \zeta]}$ can approximate in L^1 any function of the form $\chi_{[a_0 a_1 \dots a_m] \times [0, \zeta]}$ with $a_0 + a_1 + \dots + a_m = d \pmod{k}$. In conclusion, the associated flow of \mathcal{R} is AT and therefore, the factor N is isomorphic with an ITPFI factor. \square

Chapter 6

Krieger's Property A

In order to show that there exist groups not of product type, Krieger, [K2], introduced the so-called property A. He showed that any odometer of type III has this property and that property A is an invariant of orbit equivalence. Moreover, he built an example of a nonsingular ergodic transformation which does not have property A and therefore is not orbit equivalent to any odometer.

Let us recall first Krieger's definition of property A. Let (X, \mathcal{B}, μ) be a standard space and T a nonsingular ergodic transformation of (X, \mathcal{B}, μ) . For $A \in \mathfrak{B}$, $\mu(A) > 0$, we denote by T_A the induced automorphism given by:

$$T_A(x) = T^{r_A}(x), \text{ for } \mu\text{-a.e } x \in A,$$

where

$$r_A(x) = \min\{i \in \mathbb{N} : T^i(x) \in A\}, \text{ for } \mu\text{-a.e } x \in A.$$

We set:

$$\Lambda_{\nu, A, T}(x) = \left\{ \log \frac{d\nu \circ T_A^i(x)}{d\nu} : i \in \mathbb{Z} \right\}, \text{ for } \mu\text{-a.e } x \in A.$$

For a σ -finite measure $\nu \sim \mu$, $A \in \mathfrak{B}$ of positive measure and $s, \zeta > 0$, we set

$$K_{\nu, T}(A, s, \zeta) = \{x \in A : (e^{s-\zeta}, e^{s+\zeta}) \cap \Lambda_{\nu, A, T}(x) \neq \emptyset\} \cup \\ \{x \in A : (-e^{s+\zeta}, -e^{s-\zeta}) \cap \Lambda_{\nu, A, T}(x) \neq \emptyset\}$$

We say that T has *property A* if there exists a measure $\nu \sim \mu$ and $\eta, \zeta > 0$ such that the following holds: every set $A \in \mathcal{B}$ of positive measure contains a set $B \in \mathcal{B}$ of positive measure such that

$$\limsup_{s \rightarrow \infty} K_{\nu, T}(B, s, \zeta) > \eta \cdot \nu(B)$$

Suppose that \mathcal{R} is a countable amenable equivalence relation on (X, \mathcal{B}, μ) . If ν is a measure on X , equivalent to μ , we denote by δ the modulus of (\mathcal{R}, X, ν) . For $x \in A$, we define

$$\begin{aligned} \Lambda_{\nu, A, \mathcal{R}}(x) &= \{\log \delta(y, x) : (x, y) \in \mathcal{R} \text{ and } y \in A\} \\ &= \left\{ \log \frac{d\nu \circ \phi}{d\nu}(x) : \phi \in [\mathcal{R}], (x, \phi(x)) \in \mathcal{R} \text{ and } \phi(x) \in A \right\} \end{aligned}$$

For a σ -finite measure $\nu \sim \mu$, $A \in \mathcal{B}$ of positive measure and $s, \zeta > 0$, we set

$$\begin{aligned} K_{\nu, \mathcal{R}}(A, s, \delta) &= \{x \in A : (e^{s-\zeta}, e^{s+\zeta}) \cap \Lambda_{\nu, A, \mathcal{R}}(x) \neq \emptyset\} \cup \\ &\quad \{x \in A : (-e^{s+\zeta}, -e^{s-\zeta}) \cap \Lambda_{\nu, A, \mathcal{R}}(x) \neq \emptyset\} \end{aligned}$$

Definition 6.0.1. Let (X, \mathfrak{B}, μ) and \mathcal{R} be as above. Then \mathcal{R} has *property A* if there exists a measure $\nu \sim \mu$ and $\eta, \zeta > 0$ such that: every set $A \in \mathfrak{B}$ of positive measure contains a set $B \in \mathfrak{B}$ of positive measure such that

$$\limsup_{s \rightarrow \infty} K_{\nu, \mathcal{R}}(B, s, \zeta) > \eta \cdot \nu(B)$$

As the equivalence relation \mathcal{R} is amenable, there exists an automorphism T , on (X, \mathcal{B}, μ) such that T generates the equivalence relation \mathcal{R} (see Theorem 1.2.18). With the above notations, we observe that in fact

$$\Lambda_{\nu, A, \mathcal{R}}(x) = \Lambda_{\nu, A, T}(x) \text{ and } K_{\nu, \mathcal{R}}(A, s, \zeta) = K_{\nu, T}(A, s, \zeta).$$

Therefore if we define property A for amenable equivalence relation, as above, it coincides with the definition given by Krieger.

Then, Krieger's result, [K3], can be reformulated as follows: any tail equivalence of product type of type III has property A. In other words, if \mathcal{R} is an amenable equivalence relation whose associated flow is AT, then \mathcal{R} has property A.

In this section we will consider fixed point subfactors of bounded and unbounded ITPFI, which are the fixed point algebras under diagonal xerox actions of finite groups. By realizing the fixed point subfactor as $W^*(X, \mu, \mathcal{R})$, we will prove that \mathcal{R} has property A. We remark that \mathcal{R} is a subequivalence relation of the tail equivalence on X . As a consequence we will see that there are equivalence relations with property A, but whose associated flow is not AT. The equivalence relation \mathcal{R} will be the one coming from the example of the fixed point subfactor of index 2, not isomorphic to an ITPFI factor, built by Giordano and Handelmann, [GH]. With this example we answer a question stated by Dooley and Hamachi in [DH].

In the second part of this chapter we construct a Bratteli diagram such that the corresponding tail equivalence relation on the path space does not have property A.

6.1 Fixed Point Factors and Property A

The first result of this section is:

Theorem 6.1.1. *Let $M = \otimes(M_2(\mathbb{C}), \varphi_n)$ be an ITPFI₂ of type III, where $\varphi = \otimes \varphi_n$ is a faithful diagonal product. Assume that $\varphi_n(\cdot) = \text{tr}(\text{diag}(q_{n,0}, q_{n,1}) \cdot)$, $q_{n,1} + q_{n,0} = 1$, $q_{n,0}, q_{n,1} > 0$ and $\sum_{n \geq 1} q_{n,0} \cdot q_{n,1} = \infty$. Let N be the fixed point factor under the xerox action of \mathbb{Z}_2 , induced by the involutive automorphism of M :*

$$\alpha = \otimes \text{Ad} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then N is a factor isomorphic to $W^(X, \mu, \mathcal{R})$, where \mathcal{R} is a standard amenable equivalence relation of type III on a measured space (X, μ) and \mathcal{R} has property A.*

Proof. On $X = \prod_{n \geq 1} X_n = \prod_{n \geq 1} \{0, 1\}$ let $\mu = \otimes_{n \geq 1} p_n$ with $p_n(0) = q_{n,0}$, $p_n(1) = q_{n,1}$. Let \mathcal{R} be the following equivalence relation on X :

$x \mathcal{R} y$ if there exists $n \geq 1$ such that $x_i = y_i$ for $i \geq n+1$ and $\sum_{i=1}^n x_i - y_i = 0 \pmod{2}$.

By Theorem 2.3.1, $N \simeq W^*(X, \mu, \mathcal{R})$ and N is a factor. On the other hand, for almost all $x \in X$ let $N(x) = \min\{i \geq 2 : x_i < 1\}$. We define, for a.e. $x \in X$, T , as follows:

$$T(x_1, x_2, \dots, x_{N(x)}, x_{N(x)+1}, \dots) = ((Tx)_1, 0, 0, \dots, 0, x_{N(x)} + 1, x_{N(x)+1}, \dots)$$

where

$$(Tx)_1 = x_1 + N - 1 \pmod{2}$$

Then we can see that T is an automorphism of X and that T generates the equivalence relation \mathcal{R} . We have $W^*(X, \mathcal{R}, \mu) \simeq W^*(X, T, \mu)$ and we will show that \mathcal{R} , or equivalently T , has property A. As M is of type III, N is also of type III (see for example [J2], p.6), and consequently, by Theorem 1.2.17, T and \mathcal{R} are of type III. If $\Gamma \subseteq \mathbb{N}^*$ is finite and $A \subseteq \prod_{i \in \Gamma} X_i$ we use the following notation:

$$Z_{hA} := \{x \in X; (x_i)_{i \in \Gamma} \in A\}$$

We need the following auxiliary result:

Lemma 6.1.2. *Let $1 < N < I$. Consider $E \subseteq \{v \in \prod_{i=I+1}^N X_i\}$, $u \in \prod_{i=1}^I X_i$ and $B \subseteq Z_u$ such that $\mu(B) > (1 - \frac{\xi}{128})\mu(Z_u)$ and $\mu(Z_E) > \frac{\xi}{16}$. Let $E^0 = \{v \in E : \mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v)\}$. Then $\mu(Z_{E^0}) > \frac{1}{2}\mu(Z_E)$.*

Proof. Suppose by contradiction that $\mu(Z_{E^0}) \leq \frac{1}{2}\mu(Z_E)$.

$$\mu(Z_{E-E^0}) \geq \frac{1}{2}\mu(Z_E)$$

We have:

$$\frac{1}{4}\mu(Z_u)\mu(Z_{E-E^0}) \leq \mu(Z_u - B).$$

Indeed, if $v \in E - E^0$ then $\mu(Z_v \cap B) \leq \frac{3}{4}\mu(Z_u)\mu(Z_v)$ and we have

$$\mu(B \cap Z_{E-E^0}) \leq \frac{3}{4}\mu(Z_u)\mu(Z_{E-E^0}).$$

and therefore

$$\mu(Z_{E-E^0} \cap (Z_u - B)) \geq \frac{1}{4}\mu(Z_u)\mu(Z_{E-E^0})$$

Hence,

$$\mu(Z_u - B) \geq \mu(Z_{E-E^0} \cap (Z_u - B)) \geq \frac{1}{4} \mu(Z_u) \mu(Z_{E-E^0}) \geq \frac{\xi}{128} \mu(Z_u).$$

Therefore,

$$\mu(Z_u) \left(1 - \frac{\xi}{128}\right) \geq \mu(B)$$

which is a contradiction. The lemma is proved. \square

We recall that Krieger, [K3] proved that any tail equivalence of product type (and type III) has property A. We generalize his proof and we show that \mathcal{R} has property A. We define

$$a_{p,j}(x) := -\log p_j(x_j), \quad j \geq 1.$$

By [K3], if \mathcal{T} , the tail equivalence on (X, μ) , is of type III there exist sequences $b(i) > 0$, $c(i) \in \mathbb{R}$ and $0 < \beta < \frac{1}{2}$ with the following properties

$$\sup_{i \geq 1} b(i) = \infty, \tag{6.1.1}$$

$$\mu\{x \in X : |c(i) + \sum_{j=1}^i a_{p,j}(x)| \leq b(i)\} \geq 1 - 2\beta, \tag{6.1.2}$$

$$\mu\{x \in X : c(i) + \sum_{j=1}^i a_{p,j}(x) \geq b(i)\} \geq \beta, \tag{6.1.3}$$

$$\mu\{x \in X : c(i) + \sum_{j=1}^i a_{p,j}(x) \leq -b(i)\} \geq \beta. \tag{6.1.4}$$

Hence there exist $c(i), b(i), \beta$ as above. We define

$$\xi := \min\{\beta, 1 - 2\beta\}$$

By (6.1.1) we can choose a sequence $i(k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} b(i(k)) = \infty \tag{6.1.5}$$

and such that, as k goes to infinity, the random variables

$$b(i(k))^{-1}(c(i(k))) + \sum_{j=1}^{i(k)} a_{p,j}$$

converge in distribution with a limit measure to be denoted by λ (by Helly's Selection Theorem, [B], Theorem 29.3). By (6.1.2),

$$\lambda([-1, 1]) \geq \xi. \quad (6.1.6)$$

We choose ρ , $|\rho| \leq 1$, such that:

$$\lambda\left(\rho - \frac{1}{2}, \rho + \frac{1}{2}\right) \geq \frac{\xi}{3} \quad (6.1.7)$$

Let $A \subseteq X$. There exists $I \in \mathbb{N}$ and $u \in \prod_{i=1}^I X_i$ such that for

$$B = A \cap Z_u, \text{ we have } \mu(B) > \left(1 - \frac{\xi}{128}\right)\mu(Z_u).$$

It follows that the random variables

$$b(i(k))^{-1}(c(i(k))) + \sum_{j=I+1}^{i(k)} a_{p,j}$$

also converges in distribution and the limit measure is again λ . Let

$$L_{I,n} := \left\{v \in \prod_{i=I+1}^n X_i : v = v_{I+1}v_{I+2}\dots v_n, \sum_{i=I+1}^n v_i \equiv 0 \pmod{2}\right\}$$

$$R_{I,n} := \left\{v \in \prod_{i=I+1}^n X_i : v = v_{I+1}v_{I+2}\dots v_n, \sum_{i=I+1}^n v_i \equiv 1 \pmod{2}\right\}$$

We have

$$\lim_{n \rightarrow \infty} \mu(Z_{L_{I,n}}) = \lim_{n \rightarrow \infty} \mu(Z_{R_{I,n}}) = \frac{1}{2} \quad (6.1.8)$$

Indeed, since

$$|\mu(Z_{L_{I,n}}) - \mu(Z_{R_{I,n}})| = \prod_{i=I+1}^n (q_{i,0} - q_{i,1})$$

and $\lim_{n \rightarrow \infty} \sum_{i=I+1}^n (q_{i,0} + q_{i,1}) - |q_{i,0} - q_{i,1}| = 2 \lim_{n \rightarrow \infty} \sum_{i=I+1}^n \max(q_{i,0}, q_{i,1}) = \infty$, (6.1.8) holds.

We pick N sufficiently large such that

$$\mu(Z_{L_{I+1,N}}) > \frac{1}{4}, \quad \mu(Z_{R_{I+1,N}}) > \frac{1}{4}. \quad (6.1.9)$$

We define

$$\log r := \max_{x, y \in \prod_{i=I+1}^N X_i} \left| \sum_{i=I+1}^N \log p_i(x_i) - \sum_{i=I+1}^N \log p_i(y_i) \right|$$

Let $M > 1$ such that $e^{M-2} > \log r$. We can choose $k \in \mathbb{N}$ such that

$$b(i(k)) > e^{M+1}, \quad i(k) > N \quad (6.1.10)$$

and that

$$b(i(k)) > -4 \sum_{j=1}^I \min_{x \in X_j} p_j(x) \quad (6.1.11)$$

and such that with

$$\Gamma = \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : \left(\rho - \frac{1}{2} \right) b(i(k)) \leq c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq \left(\rho + \frac{1}{2} \right) b(i(k)) \right\}$$

one has

$$\mu(Z_\Gamma) > \frac{\xi}{4}. \quad (6.1.12)$$

Let

$$\begin{aligned} \Gamma_- &= \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq -\frac{3}{4} b(i(k)) \right\}, \\ \Gamma_+ &= \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \geq \frac{3}{4} b(i(k)) \right\} \end{aligned}$$

From (6.1.3), (6.1.4) and (6.1.11) we have

$$\mu(Z_{\Gamma_-}) \geq \xi \quad \mu(Z_{\Gamma_+}) \geq \xi. \quad (6.1.13)$$

By Lemma 6.1.2, we can find

$$v^- \in \Gamma_- \text{ and } v^+ \in \Gamma_+, \quad (6.1.14)$$

such that

$$\begin{aligned}\mu(B \cap Z_{v^-}) &> \frac{3}{4}\mu(Z_u)\mu(Z_{v^-}) \\ \mu(B \cap Z_{v^+}) &> \frac{3}{4}\mu(Z_u)\mu(Z_{v^+})\end{aligned}\tag{6.1.15}$$

Assume that $\rho \geq 0$. We define:

$$\begin{aligned}V^0 &= \{v \in \prod_{i=I+1}^{i(k)} X_i : \sum_{j=I+1}^{i(k)} v_j - v_j^- = 0 \pmod{2}\}, \\ V^1 &= \{v \in \prod_{i=I+1}^{i(k)} X_i : \sum_{j=I+1}^{i(k)} v_j - v_j^- = 1 \pmod{2}\}, \\ \Psi &= \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : \left(\rho - \frac{1}{2} \right) b(i(k)) - \log r \leq \right. \\ &\quad \left. \leq c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq \left(\rho + \frac{1}{2} \right) b(i(k)) + \log r \right\}.\end{aligned}\tag{6.1.16}$$

We define

$$s := \log \left| c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j^-) \right|.\tag{6.1.17}$$

As $v^- \in \Gamma_-$ we have:

$$e^s \geq \frac{3}{4}b(i(k))\tag{6.1.18}$$

and by (6.1.10)

$$s > M\tag{6.1.19}$$

We want to show that there exists a set $E^0 \subseteq \Psi \cap V^0$ such that $\mu(Z_{E^0}) > \frac{\xi}{32}$ and

$$\mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v) \text{ for all } v \in E^0.\tag{6.1.20}$$

We have that $\mu(Z_\Gamma) > \frac{\xi}{4}$. There are two possibilities:

Case I. If $\mu(Z_{\Gamma \cap V^0}) > \frac{\xi}{16}$ we define $E := \Gamma \cap V^0$.

Case II. Let us assume that $\mu(Z_{\Gamma \cap V^0}) < \frac{\xi}{16}$. Then, by (6.1.12), $\mu(Z_{\Gamma \cap V^1}) \geq \frac{3\xi}{16}$.

Without loss of generality, we can assume that v^- is such that $\sum_{i=I+1}^{i(k)} v_i^- = 0 \pmod{2}$.

(If $\sum_{i=I+1}^{i(k)} v_i^- = 1 \pmod{2}$ we can proceed in a similar way).

Let

$$\Gamma_0 = \{v \in \Gamma, \sum_{i=I+1}^N v_i = 0 \pmod{2}\},$$

$$\Gamma_1 = \{v \in \Gamma, \sum_{i=I+1}^N v_i = 1 \pmod{2}\},$$

and the canonical projections:

$$\pi_1 : \prod_{i=I+1}^{i(k)} X_i \rightarrow \prod_{i=I+1}^N X_{I+1}$$

$$\pi_2 : \prod_{i=I+1}^{i(k)} X_i \rightarrow \prod_{i=N+1}^{i(k)} X_i.$$

We have:

$$\Gamma_0 \subseteq \pi_1(\Gamma_0) \times \pi_2(\Gamma_0) \subseteq L_{I,n} \times \pi_2(\Gamma_0),$$

$$\Gamma_1 \subseteq \pi_1(\Gamma_1) \times \pi_2(\Gamma_1) \subseteq R_{I,n} \times \pi_2(\Gamma_1).$$

Let

$$A_0 = \{v \in \pi_2(\Gamma_0); \sum_{i=N+1}^{i(k)} v_i = 0 \pmod{2}\}, \quad A_1 = \{v \in \pi_2(\Gamma_0); \sum_{i=N+1}^{i(k)} v_i = 1 \pmod{2}\},$$

$$B_0 = \{v \in \pi_2(\Gamma_1); \sum_{i=N+1}^{i(k)} v_i = 0 \pmod{2}\}, \quad B_1 = \{v \in \pi_2(\Gamma_1); \sum_{i=N+1}^{i(k)} v_i = 1 \pmod{2}\}.$$

It follows that

$$\pi_2(\Gamma_0) = A_0 \cup A_1, \quad \pi_2(\Gamma_1) = B_0 \cup B_1$$

and

$$\Gamma \cap V^1 \subseteq L_{I,N} \times A_1 \cup R_{I,N} \times B_0.$$

We will show that:

$$L_{I,N} \times B_0 \cup R_{I,N} \times A_1 \subseteq \Psi \cap V^0.$$

Let $y \in R_{I,n} \times A_1$, $y = y_{I+1}y_{I+2} \cdots y_N y_{N+1} \cdots y_{i(k)}$ with $y_{N+1}y_{N+2} \cdots y_{i(k)} \in A_1$.

Hence, there exists $z \in \Gamma_0$, $z = z_{I+1} \cdots z_N z_{N+1} \cdots z_{i(k)}$ and $z_i = y_i$ for $i = N +$

$1, \dots, i(k)$. Therefore, $|\sum_{i=I+1}^{i(k)} (\log p_i(z_i) - \log p_i(y_i))| = |\sum_{i=I+1}^N (\log p_i(z_i) - \log p_i(y_i))| <$

r . Hence, $y \in \Psi$. As $y \in R_{I,n} \times A_1$, it follows that $\sum_{i=I+1}^{i(k)} (y_i - v_i^-) = 0 \pmod{2}$. Hence $z \in \Psi \cap V^0$.

Similarly, if $y \in L_{I,n} \times B_0$ then $y \in \Psi$ and $\sum_{i=I+1}^{i(k)} (y_i - v_i^-) = 0 \pmod{2}$, i.e. $y \in \Psi \cap V^0$.

By (6.1.9),

$$\frac{3\xi}{16} < \mu(Z_{\Gamma \cap V^1}) < \mu(Z_{L_{I,N} \times A_0 \cup R_{I,N} \times B_1}) < \frac{3}{4}\mu(Z_{A_0}) + \frac{3}{4}\mu(Z_{B_1}).$$

Therefore:

$$\mu(Z_{A_1}) + \mu(Z_{B_0}) > \frac{\xi}{4}$$

Consequently,

$$\mu(Z_{L_{I,N} \times B_0 \cup R_{I,N} \times A_1}) > \frac{\xi}{16}.$$

In this case we define $E := L_{I,N} \times B_0 \cup R_{I,N} \times A_1 \subseteq \Psi$ and we have just showed that $\mu(Z_E) > \frac{\xi}{16}$.

Therefore, we can always find $E \subseteq \Psi \cap V^0$ with $\mu(Z_E) > \frac{\xi}{16}$. By Lemma 6.1.2, $\mu(Z_{E^0}) > \frac{\xi}{32}$, $E^0 \subseteq E \subseteq \Psi \cap V^0$ and $\mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v)$ for all $v \in E^0$.

For $v \in E^0$, we have $\sum_{i=I+1}^{i(k)} v_i^- - v_i = 0 \pmod{2}$ and therefore there exists $S_v \in [\mathcal{R}]$ such that:

$$\begin{aligned} S_v : Z_u \cap Z_{v^-} &\mapsto Z_u \cap Z_v \\ (S_v x)_j &= x_j, \quad j > i(k). \end{aligned}$$

By (6.1.15) and (6.1.20) we have

$$\mu(B \cap Z_v \cap S_v(B \cap Z_{v^-})) > \frac{1}{2}\mu(Z_u)\mu(Z_v), \quad (6.1.21)$$

for all $v \in E^0$. We want to prove that:

$$B \cap Z_v \cap S_v(B \cap Z_{v^-}) \subseteq K_{\mathcal{R}, \mu}(B, s, 3) \quad (6.1.22)$$

for all $v \in E^0$. First we show that:

$$\left| \log \left(\sum_{I < j \leq i(k)} (\log p_j(v_j) - \log p_j(v_j^-)) \right) - s \right| < 3. \quad (6.1.23)$$

To prove (6.1.23), we have, by (6.1.18):

$$\begin{aligned} \sum_{j=I+1}^{i(k)} \log p_j(v_j^-) - \log p_j(v_j) &= c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) + e^s \\ &\leq \left(\rho + \frac{1}{2}\right)b(i(k)) + e^s + \log r \leq e^{s+2} + \log r < e^{s+3} \end{aligned}$$

From (6.1.18) and the fact that $\rho \geq 0$

$$\begin{aligned} \sum_{j=I+1}^{i(k)} (\log p_j(v_j^-) - \log p_j(v_j)) &= c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) + e^s \\ &\geq \left(\rho - \frac{1}{2}\right)b(i(k)) + e^s - \log r \geq e^s - \frac{2}{3}e^s - \log r > \frac{1}{3}e^s - \log r > e^{s-3} \end{aligned}$$

So by (6.1.21), (6.1.22) we conclude that:

$$\mu(K_{\mathcal{R},\mu}(B, s, 3)) > \frac{1}{2}\mu(Z_u)\frac{\xi}{32} \geq \frac{\xi}{64}\mu(B)$$

If $\rho \leq 0$ we proceed in a similar way v^+ instead of v^- . Hence \mathcal{R} has property A. \square

Remark 6.1.1. In [GH], it is shown that the flow of M^α is AT. Consequently, \mathcal{R} in the above theorem is orbit equivalent to a tail equivalent of product type and by [K3], \mathcal{R} has property A. Theorem 6.1.1 is a direct proof of this fact.

Remark 6.1.2. Let $M = \otimes(M_3(\mathbb{C}), \varphi_n)$ be an $ITPFI_3$ of type III, where $\varphi = \otimes \varphi_n$ is a faithful diagonal product. Assume that $\varphi_n(\cdot) = \text{tr}(\text{diag}(a_n, b_n, c_n) \cdot)$, $a_n + b_n + c_n = 1$, $q_{n,0}, q_{n,1} > 0$ and $\sum_{n \geq 1} q_{n,0} \cdot q_{n,1} = \infty$. Let N be the fixed point factor under the xerox action of \mathbb{Z}_3 , induced by the automorphism of M :

$$\alpha = \otimes \text{Ad} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{bmatrix}$$

where $\epsilon = e^{\frac{2\pi i}{3}}$

Then N is a factor isomorphic to $W^*(X, \mu, \mathcal{R})$, where \mathcal{R} is a standard amenable equivalence relation of type III on a measured space (X, μ) and \mathcal{R} has property A.

The proof follows the same idea as in the previous theorem and we need some techniques used in Section 2.3. In this case it is not known whether N is an ITPFI factor, i.e., it is not known whether \mathcal{R} is of product type.

We consider now the subfactor of index 2 not isomorphic to an ITPFI factor considered by Giordano and Handelman, [GH]. For $n \geq 1$ let $k_n = 1 + 5^{n-1}$ and $u_n = \text{diag}(-1, 1, 1, \dots, 1) \in M_{k_n}(\mathbb{C})$. Let M denote the ITPFI factor $\otimes(M_{k_n}(\mathbb{C}), \varphi_n)$, where the density matrices are given by

$$\text{diag} \left(\frac{1}{2}, \frac{1}{2 \cdot 5^{n-1}}, \frac{1}{2 \cdot 5^{n-1}}, \dots, \frac{1}{2 \cdot 5^{n-1}} \right).$$

It is known that M is an unbounded ITPFI of type III (see [GSW] or [GH]). If α denotes the automorphism $\otimes \text{Ad } u_n$, then, by [GH], the fixed point subfactor $N = M^\alpha$ is not an ITPFI.

Theorem 6.1.3. *Let M and α as above. Then the fixed point subfactor $N = M^\alpha$ can be realized as $W^*(X, \mu, \mathcal{R})$, where \mathcal{R} is a standard amenable equivalence relation of type III on (X, μ) and \mathcal{R} has property A.*

We have the following result, whose proof is straightforward. It follows from Theorem 6.1.4.

Corollary 6.1.4. *There exist amenable equivalence relations that have property A and their associated flow is not AT, or equivalently they are not orbit equivalent to a tail equivalence of product type.*

Proof. (of Theorem 6.1.3) For $n > 0$ let $X_n = \{0, 1, \dots, 5^{n-1}\}$ and p_n the measure on X_n given by

$$p_n(0) = \frac{1}{2}, \quad p_n(i) = \frac{1}{2 \cdot 5^{n-1}} \quad 1 \leq i \leq 5^{n-1}$$

Let also $Y = \prod\{0, 1\}$ and

$$\pi_n : X_n \rightarrow Y_n, \quad \pi_n(0) = 0, \quad \pi_n(i) = 1 \text{ if } i \neq 0$$

On $X = \prod X_n$ we consider the product measure $\mu = \otimes \mu_n$ and the equivalence relation \mathcal{R} given by:

$$x \mathcal{R} y \text{ if there exists } n > 0 \text{ such that } x_i = y_i \text{ for } i > n \text{ and } \sum_{i=1}^n \pi_i(x_i) - \pi_i(y_i) = 0 \pmod{2}.$$

Let us define an automorphism T of (X, μ) such that $\mathcal{R} = \mathcal{R}_T$. For μ -a.e. $x \in X$, let $N(x) = \min\{i \geq 2 : x_i < 5^i + 1\} < \infty$. For a.e. $x \in X$ we define T by

$$T(x_1, x_2, \dots, x_{N(x)}, x_{N(x)+1}, \dots) = ((Tx)_1, 0, 0, \dots, 0, x_{N(x)} + 1, x_{N(x)+1}, \dots)$$

where and we define $T(x)_1$ such that Tx and x are \mathcal{R} equivalent, i.e.,

$$\sum_{i=1}^n \pi_i((Tx)_i) - \pi_i(x_i) = 0 \pmod{2}$$

Then it is easy to see that T is a nonsingular transformation on X and T generates the equivalence relation \mathcal{R} . We have $N \simeq W^*(X, \mu, \mathcal{R}) = W^*(X, \mu, T)$ and we will prove that \mathcal{R} has property A or equivalently, T has property A. As in the proof of Theorem 6.1.1, T and \mathcal{R} are of type III.

If $\Gamma \subseteq \mathbb{N}^*$ is finite and $A \subseteq \prod_{i \in \Gamma} X_i$ we use the following notation:

$$Z_A := \{x \in X; (x_i)_{i \in \Gamma} \in A\}$$

We need the following lemma, whose proof is similar to the proof of Lemma 6.1.2.

Lemma 6.1.5. *Let $1 < N < I$. Consider $E \subseteq \{v \in \prod_{i=I+1}^N X_i\}$, $u \in \prod_{i=1}^I X_i$ and $B \subseteq Z_u$ such that $\mu(B) > (1 - \frac{\xi}{128})\mu(Z_u)$ and $\mu(Z_E) > \frac{\xi}{16}$. Let $E^0 = \{v \in E : \mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v)\}$. Then $\mu(Z_{E^0}) > \frac{1}{2}\mu(Z_E)$.*

We define

$$a_{p,j}(x) := -\log p_j(x_j), \quad j \geq 1.$$

To prove the theorem we follow the same idea as in Theorem 6.1.1. Let \mathcal{T} be the tail equivalence on (X, μ) . As \mathcal{T} is of type III, by [K3], there exist sequences $b(i) > 0$, $c(i) \in \mathbb{R}$ and $0 < \beta < \frac{1}{2}$ with the following properties

$$\sup_{i \geq 1} b(i) = \infty, \quad (6.1.24)$$

$$\mu\{x \in X : |c(i) + \sum_{j=1}^i a_{p,j}(x)| \leq b(i)\} \geq 1 - 2\beta, \quad (6.1.25)$$

$$\mu\{x \in X : c(i) + \sum_{j=1}^i a_{p,j}(x) \geq b(i)\} \geq \beta, \quad (6.1.26)$$

$$\mu\{x \in X : c(i) + \sum_{j=1}^i a_{p,j}(x) \leq -b(i)\} \geq \beta. \quad (6.1.27)$$

We define

$$\xi := \min\{\beta, 1 - 2\beta\}$$

By (6.1.24) we can choose a sequence $i(k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} b(i(k)) = \infty \quad (6.1.28)$$

and such that, as k goes to infinity, the random variables

$$b(i(k))^{-1} \left(c(i(k)) + \sum_{j=1}^{i(k)} a_{p,j} \right)$$

converge in distribution with a limit measure to be denoted by λ (by Helly's Selection Theorem, [B], Theorem 29.3). By (6.1.25),

$$\lambda([-1, 1]) \geq \xi. \quad (6.1.29)$$

We choose ρ , $|\rho| \leq 1$, such that:

$$\lambda\left(\rho - \frac{1}{2}, \rho + \frac{1}{2}\right) \geq \frac{\xi}{3} \quad (6.1.30)$$

Let $A \subseteq X$. There exists $I \in \mathbb{N}$ and $u \in \prod_{i=1}^I X_i$ such that for

$$B = A \cap Z_u, \text{ we have } \mu(B) > \left(1 - \frac{\xi}{128}\right) \mu(Z_u).$$

It follows that the random variables

$$b(i(k))^{-1}(c(i(k)) + \sum_{j=I+1}^{i(k)} a_{p,j})$$

also converges in distribution and the limit measure is again λ .

Let $M > 1$ such that $e^{M-2} > I \log 5$. We can choose $k \in \mathbb{N}$ such that

$$b(i(k)) > e^{M+1}, \quad i(k) > N \quad (6.1.31)$$

and that

$$b(i(k)) > -4 \sum_{j=1}^I \min_{x \in X_j} p_j(x) \quad (6.1.32)$$

and such that with

$$\Gamma = \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : \left(\rho - \frac{1}{2}\right)b(i(k)) \leq c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq \left(\rho + \frac{1}{2}\right)b(i(k)) \right\}$$

one has

$$\mu(Z_\Gamma) > \frac{\xi}{4}. \quad (6.1.33)$$

Let

$$\Gamma_- = \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq -\frac{3}{4}b(i(k)) \right\},$$

$$\Gamma_+ = \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \geq \frac{3}{4}b(i(k)) \right\}$$

From (6.1.26), (6.1.27) and (6.1.32) we have

$$\mu(Z_{\Gamma_-}) \geq \xi \quad \mu(Z_{\Gamma_+}) \geq \xi. \quad (6.1.34)$$

By Lemma 6.1.7, we can find

$$v^- \in \Gamma_- \text{ and } v^+ \in \Gamma_+, \quad (6.1.35)$$

such that

$$\mu(B \cap Z_{v^-}) > \frac{3}{4}\mu(Z_u)\mu(Z_{v^-}) \quad (6.1.36)$$

$$\mu(B \cap Z_{v^+}) > \frac{3}{4}\mu(Z_u)\mu(Z_{v^+})$$

Assume that $\rho \geq 0$. We define:

$$\begin{aligned} V^0 &= \left\{ v \in \prod_{j=I+1}^{i(k)} X_j : \sum_{j=I+1}^{i(k)} \pi_j(v_j) - \pi_j(v_j^-) = 0 \pmod{2} \right\}, \\ V^1 &= \left\{ v \in \prod_{j=I+1}^{i(k)} X_j : \sum_{j=I+1}^{i(k)} \pi_j(v_j) - \pi_j(v_j^-) = 1 \pmod{2} \right\}, \\ \Psi &= \left\{ v \in \prod_{i=I+1}^{i(k)} X_i : \left(\rho - \frac{1}{2} \right) b(i(k)) - \log r \leq \right. \\ &\quad \left. \leq c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) \leq \left(\rho + \frac{1}{2} \right) b(i(k)) + I \log 5 \right\}. \end{aligned} \quad (6.1.37)$$

We define:

$$s := \log \left| c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j^-) \right|. \quad (6.1.38)$$

As $v^- \in \Gamma_-$ we have:

$$e^s \geq \frac{3}{4}b(i(k)) \quad (6.1.39)$$

and by (6.1.31)

$$s > M \quad (6.1.40)$$

We show that there exists a set $E^0 \subseteq \Psi \cap V^0$ such that $\mu(Z_{E^0}) > \frac{\xi}{32}$ and

$$\mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v) \text{ for all } v \in E^0. \quad (6.1.41)$$

We have that $\mu(Z_\Gamma) > \frac{\xi}{4}$. There are two possibilities:

Case I. If $\mu(Z_{\Gamma \cap V^0}) > \frac{\xi}{16}$ we define $E := \Gamma \cap V^0$.

Case II. Let us assume that $\mu(Z_{\Gamma \cap V^0}) < \frac{\xi}{16}$. Then, by (6.1.33), $\mu(Z_{\Gamma \cap V^1}) \geq \frac{3\xi}{16}$.

Without loss of generality, we can assume that v^- is such that $\sum_{i=I+1}^{i(k)} \pi_i(v_i^-) = 0 \pmod{2}$

2). If $\sum_{i=I+1}^{i(k)} \pi_i(v_i^-) = 1 \pmod{2}$ we proceed in a similar way.

We define the following sets:

$$A_0^0 = \{v \in \Gamma; x_{I+1} = 0, \sum_{i=I+2}^{i(k)} \pi_i(v_i) = 0 \pmod{2}\}$$

$$A_1^0 = \{v \in \Gamma; \pi_{I+1}(x_{I+1}) = 1, \sum_{i=I+2}^{i(k)} \pi_i(v_i) = 0 \pmod{2}\}$$

$$A_0^1 = \{v \in \Gamma; x_{I+1} = 0, \sum_{i=I+2}^{i(k)} \pi_i(v_i) = 1 \pmod{2}\}$$

$$A_1^1 = \{v \in \Gamma; \pi_{I+1}(x_{I+1}) = 1, \sum_{i=I+2}^{i(k)} \pi_i(v_i) = 1 \pmod{2}\}$$

and

$$B_1^1 = \{x \in \prod_{i=I+1}^{i(k)} X_i; \exists v \in A_0^1 \text{ such that } x_i = v_i, \text{ if } i \geq I+2, \text{ and } x_{I+1} \in \{1, 2, \dots, 5^I\}\}$$

$$B_0^0 = \{x \in \prod_{i=I+1}^{i(k)} X_i; \exists v \in A_1^0 \text{ such that } x_i = v_i, \text{ if } i \geq I+2, \text{ and } x_{I+1} = 0\}$$

Then,

$$\Gamma \cap V^0 = A_0^0 \cup A_1^1 \text{ and } \Gamma \cap V^1 = A_0^1 \cup A_1^0$$

As, $\mu(Z_{V^1 \cap \Gamma}) \geq 3 \frac{\xi}{16}$ and $B_0^0 \cup B_1^1 \supseteq A_0^1 \cup A_1^0$ we also have:

$$\mu(Z_{B_0^0 \cup B_1^1}) \geq 3 \frac{\xi}{16} > \frac{\xi}{16}.$$

Moreover,

$$B_0^0 \cup B_1^1 \subseteq \Psi$$

Indeed, let $x \in B_1^1$. Then, $x_{I+1} \in \{1, 2, \dots, 5^I\}$ and it exists $v \in A_0^1$ such that $x_i = v_i$, if $i \geq I+2$, and $v_{I+1} = 0$. Hence

$$\sum_{i=I+1}^{i(k)} \log p_i(x_i) - \log p_i(v_i) = \log p_{I+1}(x_{I+1}) - \log p_{I+1}(v_{I+1}) = -\log \frac{1}{2} + \log \frac{1}{2 \cdot 5^I} = -I \log 5$$

and therefore, $x \in \Psi$. We also have that $\sum_{j=I+1}^{i(k)} \pi_j(x_j) - \pi_j(v_j^-) = 0 \pmod{2}$ and therefore $x \in V^0$. Similarly, if $x \in B_0^0$ then $x \in \Psi \cap V^0$. Let $E = B_0^0 \cup B_1^1$. Therefore, $E \subseteq \Psi \cap V^0$ and $\mu(Z_E) \geq \frac{\xi}{16}$.

By Lemma 6.1.7, $\mu(Z_{E^0}) > \frac{\xi}{32}$, $E^0 \subseteq E \subseteq \Psi \cap V^0$ and $\mu(B \cap Z_v) > \frac{3}{4}\mu(Z_u)\mu(Z_v)$ for all $v \in E^0$.

For $v \in E^0$, as $\sum_{i=I+1}^{i(k)} \pi_i(v_i) - \pi_i(v_i^-) = 0 \pmod{2}$ there exists $S_v \in [\mathcal{R}]$ such that:

$$\begin{aligned} S_v : Z_u \cap Z_{v^-} &\mapsto Z_u \cap Z_v \\ (S_v x)_j &= x_j, \quad j > i(k). \end{aligned}$$

By (6.1.36) and (6.1.41) we have

$$\mu(B \cap Z_v \cap S_v(B \cap Z_{v^-})) > \frac{1}{2}\mu(Z_u)\mu(Z_v), \quad (6.1.42)$$

for all $v \in E^0$. We want to prove that:

$$B \cap Z_v \cap S_v(B \cap Z_{v^-}) \subseteq K_{\mathcal{R}, \mu}(B, s, 3) \quad (6.1.43)$$

for all $v \in E^0$. First we show that:

$$\left| \log \left(\sum_{I < j \leq i(k)} (\log p_j(v_j) - \log p_j(v_j^-)) \right) - s \right| < 3. \quad (6.1.44)$$

To prove (6.1.44), we have, by (6.1.39):

$$\begin{aligned} \sum_{j=I+1}^{i(k)} \log p_j(v_j^-) - \log p_j(v_j) &= c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) + e^s \\ &\leq \left(\rho + \frac{1}{2}\right)b(i(k)) + e^s + I \log 5 \leq e^{s+2} + I \log 5 < e^{s+3} \end{aligned}$$

From (6.1.16) and the fact that $\rho \geq 0$

$$\begin{aligned} \sum_{j=I+1}^{i(k)} (\log p_j(v_j^-) - \log p_j(v_j)) &= c(i(k)) - \sum_{j=I+1}^{i(k)} \log p_j(v_j) + e^s \\ &\geq \left(\rho - \frac{1}{2}\right)b(i(k)) + e^s - \log r \geq e^s - \frac{2}{3}e^s - I \log 5 > \frac{1}{3}e^s - I \log 5 > e^{s-3} \end{aligned}$$

So by (6.1.42), (6.1.43) we conclude that:

$$\mu(K_{\mathcal{R}, \mu}(B, s, 3)) > \frac{1}{2}\mu(Z_u)\frac{\xi}{32} \geq \frac{\xi}{64}\mu(B)$$

If $\rho \leq 0$ we proceed in a similar way v^+ instead of v^- . Hence \mathcal{R} has property A. \square

In general we strongly believe that the following result is true:

Theorem 6.1.6. *Let $(k_n)_{n \geq 1}$ be a sequence of integers $k_n \geq 2$ and $(u_n)_{n \geq 1}$ be a sequence of involutive diagonal unitaries in $M_{k_n}(\mathbb{C})$ not equal to $\pm 1_{k_n}$.*

Let $M = (M_{k_n}(\mathbb{C}), \varphi_n)$ be an ITPFI of type III, where $\varphi = \otimes \varphi_n$ is faithful diagonal product state and $\alpha = \otimes \text{Ad } u_n \in \text{Aut}(M)$. If $N = M^\alpha$ is the fixed point subalgebra under α , then N is a factor isomorphic to $W^(X, \mu, \mathcal{R})$, where \mathcal{R} is a standard amenable equivalence relation of type III on a measured space (X, μ) and \mathcal{R} has property A.*

6.2 An Example of a Factor not Isomorphic to an ITPFI Factor

Recall that any AFD factor M with separable predual can be realized as $W^*(X, \mu, \mathcal{R})$ where:

- (i) X is the path space of a Bratteli diagram B ;
- (ii) μ is a so-called AF-measure on X (see for example [GH], Section 6);
- (iii) \mathcal{R} is the equivalence relation given by tail equivalence on X .

In this section we construct a Bratteli diagram $B = (V, E)$ and an AF-measure μ on X , such that \mathcal{R} does not have Krieger's property A and therefore the factor $W^*(X, \mu, \mathcal{R})$ is not ITPFI. The proof follows the same idea as in [K2] and [DH]. Then we describe the associated flow of (X, μ, \mathcal{R}) which is measure preserving and by Theorem 1.2.20, is not AT.

Let us consider $(r_m)_{m \geq 1}$ an increasing sequence of positive integers with the following properties:

$$\log \left(\frac{r_{m-2}}{\prod_{j=1}^{m-3} r_j} \right) \log \left(\prod_{i=1}^{m-1} r_i \right) < \log \left(\frac{r_m}{\prod_{j=1}^{m-1} r_j^2} \right), m \geq 4$$

$$\prod_{j=1}^{m-1} r_j^2 < r_m, \quad m \geq 2$$

We then have the following three relations that we will need:

$$\log r_m > 2 \sum_{i=1}^{m-1} \log r_i, \quad m \geq 2 \quad (6.2.1)$$

$$\sum_{i=1}^{m-1} \log r_i (\log r_{m-2} - \sum_{j=1}^{m-3} \log r_j) < \log r_m - 2 \sum_{i=1}^{m-1} \log r_i, \quad m \geq 4 \quad (6.2.2)$$

$$\log r_{m-1} - \sum_{i=1}^{m-2} \log r_i < \log r_m - \sum_{i=1}^{m-1} \log r_i, \quad m \geq 3. \quad (6.2.3)$$

We need the following technical results:

Lemma 6.2.1. *Let $m \geq 1$ and $\alpha_i, \beta_i \in \{0, 1, -1\}$. If $r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_m^{\alpha_m} = r_1^{\beta_1} r_2^{\beta_2} \cdots r_m^{\beta_m}$ then $\alpha_1 = \beta_1, \dots, \alpha_m = \beta_m$.*

Proof. If $\alpha_m \neq \beta_m$ then

$$\sum_{i=1}^{m-1} |\alpha_i - \beta_i| \log r_i \geq |\alpha_m - \beta_m| \log r_m \geq \log r_m > 2 \sum_{i=1}^{m-1} \log r_i \geq \sum_{i=1}^{m-1} |\alpha_i - \beta_i| \log r_i$$

which is a contradiction. The lemma follows by induction. \square

We define $I_m = [\log r_m - \sum_{i=1}^{m-1} \log r_i, \sum_{i=1}^m \log r_i]$, for $m \geq 2$. By (6.2.1), these intervals are mutually disjoint.

Let $\delta > 0$ be any fixed real number and choose an integer $N > 1$ such that:

$$\log r_N - \sum_{i=1}^{N-1} \log r_i > e^{2\delta}. \quad (6.2.4)$$

We have:

Proposition 6.2.2. *Let $\delta > 0$ and $N > 1$ such that (6.2.4) holds. For $s > 0$ such that $e^{s-\delta} > \sum_{i=1}^{N+1} \log r_i$ and for $m \geq 1$ satisfying $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \emptyset$ we have $(e^{s-\delta}, e^{s+\delta}) \cap I_j = \emptyset$, for all $j \neq m$.*

Proof. Since $e^{s-\delta} > \sum_{i=1}^{N+1} \log r_i$, we have $I_m \cap (e^{s-\delta}, e^{s+\delta}) = \emptyset$ for $m \leq N+1$.

It is enough to prove that if $m \geq N+2$ and if $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \emptyset$ then $(e^{s-\delta}, e^{s+\delta}) \cap I_{m+1} = \emptyset$ and $(e^{s-\delta}, e^{s+\delta}) \cap I_{m-1} = \emptyset$.

As $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \emptyset$, we have $e^{s-\delta} < \sum_{i=1}^m \log r_i$. We first prove that $\log r_{m+1} - 2 \sum_{i=1}^m \log r_i > e^{s+\delta} - e^{s-\delta}$. By (6.2.1) and (6.2.2), we have:

$$\begin{aligned} e^{s+\delta} - e^{s-\delta} &= e^{s-\delta} (e^{2\delta} - 1) \leq \sum_{i=1}^m \log r_i \left\{ \log r_N - \sum_{i=1}^{N-1} \log r_i \right\} \\ &< \sum_{i=1}^m \log r_i \left\{ \log r_{m-1} - \sum_{i=1}^{m-2} \log r_i \right\} < \log r_{m+1} - 2 \sum_{i=1}^m \log r_i. \end{aligned}$$

Therefore:

$$\begin{aligned} e^{s+\delta} &= e^{s+\delta} - e^{s-\delta} + e^{s-\delta} < \{\log r_{m+1} - 2 \sum_{i=1}^m \log r_i\} + \sum_{i=1}^m \log r_i \\ &= \log r_{m+1} - \sum_{i=1}^m \log r_i \end{aligned}$$

this proves $I_{m+1} \cap (e^{s-\delta}, e^{s+\delta}) = \emptyset$.

We prove that $(e^{s-\delta}, e^{s+\delta}) \cap I_{m-1} = \emptyset$ by contradiction. Suppose that $(e^{s-\delta}, e^{s+\delta}) \cap I_{m-1} \neq \emptyset$. Then,

$$e^{s-\delta} < \sum_{i=1}^{m-1} \log r_i$$

As $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \emptyset$, we also have

$$e^{s+\delta} > \log r_m - \sum_{i=1}^{m-1} \log r_i.$$

By (6.2.2) and (6.2.3), it follows that

$$e^{2\delta} = \frac{e^{s+\delta}}{e^{s-\delta}} > \frac{\log r_m - \sum_{i=1}^{m-1} \log r_i}{\sum_{i=1}^{m-1} \log r_i} > \log r_{m-2} - \sum_{i=1}^{m-3} \log r_i \geq \log r_N - \sum_{i=1}^{N-1} \log r_i.$$

Contradiction with the choice of N . □

First we consider the set $V = \cup V_n$ with $V_n = \{(n, 0), (n, 1), \dots, (n, n)\}$ for all $n \geq 0$. Let $\psi : \mathbb{N}^* \rightarrow V \setminus \{(n, 0); n \geq 0\}$ the bijection defined by $\psi(m) = (n, k)$ where n and k are uniquely determined in such a way that $1 + 2 + \dots + n - 1 + k = m$ and $1 \leq k \leq n$. We have $\psi(1) = (1, 1)$, $\psi(2) = (2, 1)$, $\psi(3) = (2, 2)$, $\psi(4) = (3, 1)$, $\psi(5) = (3, 2)$ and so on.

For $n \geq 1$ and $1 \leq k \leq n$, we define $\lambda_{n,k} := r_{\psi^{-1}(n,k)}$.

We build a Bratteli diagram in the following way: the set of vertices is given by $V = \cup V_n$, defined above. The set of edges is $E = \cup_{n \geq 1} E_n$ with E_n given by:

$$E_n = \{(n, k, k+1, i_j); 1 \leq j \leq \lambda_{n,k+1}, 0 \leq k \leq n-1\} \cup \{(n, k, k, 0); 0 \leq k \leq n-1\}.$$

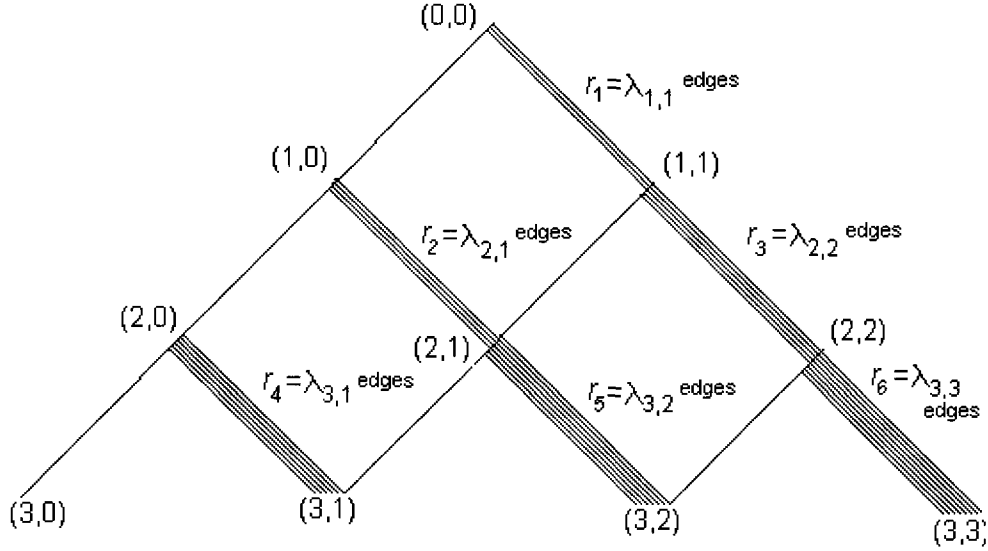


Figure 6.1: The Bratteli diagram

For $n \geq 1$ and $0 \leq k \leq n-1$, $i(n, k, k+1, i_j) = (n-1, k)$, if $1 \leq i_j \leq \lambda_{n, k+1}$, $i(n, k, k, 0) = (n-1, k)$, $f(n, k, k+1, i_j) = (n, k+1)$ if $1 \leq i_j \leq \lambda_{n, k+1}$ and $f(n, k, k, 0) = (n, k)$. The path space of the Bratteli diagram will be:

$$X = \{((1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n) \dots); (m, k_{m-1}, k_m, i_m) \in E_m\}$$

where $0 \leq k_n \leq n$, $k_n = k_{n+1}$ or $k_n + 1 = k_{n+1}$; $i_n = 0$ if $k_n = k_{n+1}$ and $1 \leq i_n \leq \lambda_{n, k_{n+1}}$ if $k_n + 1 = k_{n+1}$ (i and f are as in Subsection 1.1.5). When necessary, we denote a path $x \in X$ by $x = x_1, x_2, \dots, x_n \dots$, where $x_n \in E_n$ and $f(x_n) = i(x_{n+1})$. The transition matrix from V_{n-1} to V_n is $A_n = \{a_n^{i,j}, 0 \leq i \leq n-1, 0 \leq j \leq n\}$ where $a_n^{k-1, k-1} = 1$, $a_n^{k-1, k} = \lambda_{n, k}$ for $k = 1, \dots, n$ and all the other entries are equal to zero. This Bratteli diagram (the first 3 levels of the diagram are drawn in Figure 6.1) looks like the infinite Pascal triangle with the difference that from any vertex we have one edge going to the left and multiple edges going to the right.

We consider the von Neumann algebra $W^*(X, \mu, \mathcal{R})$, with \mathcal{R} the tail equivalence on the path space X . On X we define a measure μ in the following way: for an

arbitrary cylinder

$$C = [(1, 0, k_1, i_1)(2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n)],$$

we define

$$\mu(C) = \frac{1}{2^n} \frac{1}{a_1^{0k_1} a_2^{k_1 k_2} \dots a_n^{k_{n-1} k_n}}.$$

We note that if A is the C^* -algebra with the above Bratteli diagram, then $A \simeq C_r^*(X, \mathcal{R})$ and there exists a conditional expectation $P : A \rightarrow C(X)$. If we see μ as a state on $C(X)$, then $P \circ \mu$ is a state on A denoted by φ . Hence if $(\pi_\varphi, \xi_\varphi, H_\varphi)$ is the GNS representation corresponding to (A, φ) , then $W^*(X, \mu, \mathcal{R}) \simeq \pi_\varphi(A)''$.

Lemma 6.2.3. *If $(x, y) \in \mathcal{R}$ then $|\log \delta(y, x)| \in \cup_{m \geq 1} I_m$.*

Let $(y, x) \in \mathcal{R}$. Then, as x and y are tail equivalent, we can write

$$x = (1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots$$

$$y = (1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (n, j_{n-1}, k_n, l_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots$$

therefore,

$$\delta(y, x) = \frac{a_1^{0k_1} a_2^{k_1 k_2} \dots a_n^{k_{n-1} k_n}}{a_1^{0j_1} a_2^{j_1 j_2} \dots a_n^{j_{n-1} k_n}}.$$

As each a_n^{ij} which appears in the above relation is in fact equal to 1 or to a certain r_i^{-1} it results that $\delta(y, x) = r_1^{\alpha_1} r_2^{\alpha_2} \dots r_m^{\alpha_m}$ for some m , with $\alpha_i \in \{0, 1, -1\}$ for $1 \leq i \leq m$ and $\alpha_m \neq 0$ (i.e. $|\alpha_m| = 1$). Then, $|\log \delta(y, x)| = |\alpha_1 \log r_1 + \alpha_2 \log r_2 + \dots + \alpha_m \log r_m| \leq \sum_{i=1}^m \log r_m$. On the other hand, $|\log \delta(y, x)| \geq \log r_m - \sum_{i=1}^{m-1} \log r_i$. Therefore, there exists m such that $|\log \delta(x, y)| \in I_m$. Hence, the lemma is proved.

Lemma 6.2.4. *Let $F_{n,k}$ be the set of all $x \in X$ that cross the vertex (n, k) with $k \neq 0$.*

Then:

$$K_m = \{x \in X : \exists y \in X, (y, x) \in \mathcal{R} \text{ and } \alpha_1, \alpha_2, \dots, \alpha_m \in \{0, 1, -1\} \text{ with } \alpha_m \neq 0 \\ \text{such that } \delta(y, x) = r_1^{\alpha_1} r_2^{\alpha_2} \dots r_m^{\alpha_m}\} \subseteq F_{\psi(m)}$$

Proof. Let $x \in K_m$ and $(y, x) \in \mathcal{R}$ such that $\delta(y, x) = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_m^{\alpha_m}$ with $\alpha_m \neq 0$. As $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are tail equivalent, there exists $n \geq 1$ such that $x_i = y_i$ for $i > n$. On the other hand, before the edges of x and y coincide it is possible for the two paths to cross the same vertices. Let p such that $f(x_i) = f(y_i)$, if $i \geq p$ and $i(x_p) \neq i(y_p)$ (i.e., starting with the vertex $f(x_p) = f(y_p)$, x and y cross the same vertices). Clearly $p \leq n$ and $p > 1$ (otherwise $\delta(y, x)$ would be 1). We can write this in the following way:

$$\begin{aligned} x &= (1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (p, k_{p-1}, k_p, i_p), (p+1, k_p, k_{p+1}, i_{p+1}), \dots \\ &\quad \dots, (n, k_{n-1}, k_n, i_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots \\ y &= (1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (p, j_{p-1}, k_p, l_p), (p+1, k_p, k_{p+1}, l_{p+1}), \dots \\ &\quad \dots, (n, k_{n-1}, k_n, l_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots \end{aligned}$$

therefore,

$$\delta(y, x) = \frac{a_1^{0k_1}}{a_1^{0j_1}} \frac{a_2^{k_1k_2}}{a_2^{j_1j_2}} \cdots \frac{a_p^{k_{p-1}k_p}}{a_p^{j_{p-1}k_p}}.$$

Each a_m^{ij} which appears in the above relation is in fact equal to 1 or to a certain r_i for $1 \leq i \leq \psi^{-1}(p, k_p)$. Also, the last term should be equal to a nonzero power of $r_{\psi^{-1}(p, k_p)} = \lambda_{p, k_p}$ (because $a_p^{j_{p-1}k_p} = \lambda_{p, k_p}$ and $a_p^{k_{p-1}k_p} = 1$ or vice versa). Therefore, we can write

$$\delta(y, x) = \prod_{i=1}^{\psi^{-1}(p, k_p)} r_i^{\beta_i}$$

with $\beta_i \in \{0, 1, -1\}$. We assumed in the beginning that

$$\delta(y, x) = \prod_{i=1}^m r_i^{\alpha_i}$$

where $\alpha_m \neq 0$. By Lemma 6.2.1, $\psi^{-1}(p, k_p) = m$, i.e., $(p, k_p) = \psi(m)$. Hence, we proved that if $x \in K_m$ then x crosses the vertex $\psi(m) = (p, k_p)$, i.e. $x \in F_{\psi(m)}$. \square

We define a map $\pi : X \rightarrow Y$ where $Y = \prod_{n=1}^{\infty} \{0, 1\}$ in the following way:

$$\pi((1, k_0, k_1, i_1)(2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n) \dots) = (k_1 - k_0, k_2 - k_1, \dots, k_n - k_{n-1}, \dots)$$

Also, we define a measure $\hat{\mu} = \otimes \hat{\mu}_n$ on Y by $\hat{\mu}_n(0) = \hat{\mu}_n(1) = \frac{1}{2}$.

Lemma 6.2.5. *We have $\lim_{m \rightarrow \infty} \mu(F_{\psi(m)}) = 0$.*

Proof. Let $\psi(m) = (n, k)$ and $A_{n,k} = \{y \in Y, \sum_{i=1}^n y_i = k\}$. We have:

$$\hat{\mu}(A_{n,k}) = \frac{1}{2^n} \binom{n}{k}$$

A path x crosses the vertex (n, k) if and only if $\sum_{i=1}^n \pi(x)_i = k$ or, equivalently, if and only if $x \in \pi^{-1}(A_{n,k})$. Therefore,

$$\mu(F_{\psi(m)}) = \mu(F_{n,k}) = \mu \circ \pi^{-1}(A_{n,k}) = \hat{\mu}(A_{n,k}) = \frac{1}{2^n} \binom{n}{k}.$$

Let:

$$a_n = \max \left\{ \frac{1}{2^n} \binom{n}{k} ; 1 \leq k \leq n \right\}.$$

By Stirling's formula, we have $\lim_{n \rightarrow \infty} a_n = 0$. From the way the function ψ was defined, we conclude that $\lim_{m \rightarrow \infty} \mu(F_{\psi(m)}) = 0$. \square

Before showing that \mathcal{R} does not have property A we show that \mathcal{R} is ergodic and of type III.

Proof. We need the following lemma :

Lemma 6.2.6. ([Ke], Lemma 17.2) *If $Y = \prod \{0, 1\}$ and on Y we have the product measure $\hat{\mu} = \otimes \hat{\mu}$ with $\hat{\mu}_n(0) = \hat{\mu}_n(1) = \frac{1}{2}$ then for $\hat{\mu} \times \hat{\mu}$ almost all $(z, w) \in Y \times Y$, there exist infinitely many $n > 1$ such that $\sum_{i=1}^n w_i = \sum_{i=1}^n z_i$.*

Proposition 6.2.7. *The equivalence relation \mathcal{R} on X is ergodic.*

As seen before we have a map π from X onto $Y = \prod_{n=1}^{\infty} \{0, 1\}$ given by:

$$\pi((1, k_0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n) \dots) = (k_1 - k_0, k_2 - k_1, \dots, k_n - k_{n-1}, \dots)$$

The measure $\hat{\mu} = \otimes \hat{\mu}_n$ on Y is given by $\hat{\mu}_n(0) = \hat{\mu}_n(1) = \frac{1}{2}$. Two paths

$$x = (x_1, x_2, \dots, x_n, \dots) = ((1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n) \dots)$$

$$y = (y_1, y_2, \dots, y_n, \dots) = ((1, 0, h_1, l_1), (2, h_1, h_2, l_2), \dots, (n, h_{n-1}, h_n, l_n) \dots)$$

of X have an encounter (i.e., they intersect) if there exists $n \geq 1$ such that:

$$f(n, k_{n-1}, k_n, i_n) = f(n, l_{n-1}, h_n, l_n), \text{ i.e., } k_n = h_n.$$

This is equivalent to the fact that $\sum_{i=1}^n z_i = \sum_{i=1}^n w_i$, where $z = \pi(x)$ and $w = \pi(y)$. As $\mu \circ \pi^{-1} = \hat{\mu}$, by the previous lemma, for almost all $(z, w) \in Y \times Y$, there exist infinitely many $n > 1$ such that $\sum_{i=1}^n z_i = \sum_{i=1}^n w_i$. As $\hat{\mu} = \mu \circ \pi^{-1}$, we conclude that almost all $(x, y) \in X \times X$ have infinitely many encounters (there are infinitely many n such that $f(x_n) = f(y_n)$). Using similar arguments as in the proof of Proposition 2.3.4, we conclude that the equivalence relation \mathcal{R} is ergodic and consequently, $W^*(X, \mu, \mathcal{R})$ is a factor. \square

Proposition 6.2.8. \mathcal{R} is of type III.

Proof. Let $S_n = \pi^{-1}(z_n = 0)$. We have $\mu(S_n) = \mu(S_n^c) = \frac{1}{2}$. Suppose that we can find a σ -finite \mathcal{R} -invariant measure $\nu \sim \mu$. Let $f(x) = \frac{d\nu}{d\mu}(x)$. If $\phi \in [\mathcal{R}]$, then

$$\frac{d\mu \circ \phi}{d\mu}(x) = \frac{f(x)}{f(\phi x)}$$

a.e. $x \in X$ and that if

$$\log \frac{d\mu \circ \phi}{d\mu}(x) \in (-\log r_1, \log r_1),$$

then

$$\log \frac{d\mu \circ \phi}{d\mu}(x) = 0$$

Now X may be decomposed into a countable number of disjoint subsets $\{x \in X : \alpha_n r_1^{-\frac{1}{2}} < f(x) < \alpha_n r_1^{\frac{1}{2}}\}$. Hence, we can choose $\alpha > 0$ such that the subset $A = \{x \in X : \alpha r_1^{-\frac{1}{2}} < f(x) < \alpha r_1^{\frac{1}{2}}\}$ is of positive measure.

For $n > 0$, we have either $\mu(A \cap S_n) = 0$ or $\mu(A \cap S_n^c) = 0$. Indeed, suppose that $\mu(A \cap S_n) > 0$ and $\mu(A \cap S_n^c) > 0$ for some $n > 0$. We note that if $x \in A$ and if $\phi \in [\mathcal{R}]$ satisfy $\phi x \in A$ then:

$$\frac{d\mu \circ \phi}{d\mu}(x) = \frac{f(x)}{f(\phi x)} \in (r_1^{-1}, r_1),$$

and hence

$$\frac{d\mu \circ \phi}{d\mu}(x) = 1.$$

By Proposition 6.2.7, \mathcal{R} is ergodic and therefore, we obtain an integer $N \geq n$, such that there exist

$$C = [(1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (N-1, k_{N-2}, k_{N-1}, i_{N-1}), (N, k_{N-1}, k_N, i_N)],$$

$$D = [(1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (N-1, j_{N-2}, j_{N-1}, l_{N-1}), (N, j_{N-1}, k_N, l_N)],$$

and a partially defined invertible nonsingular map with $\text{Graph}(\xi) \subseteq \mathcal{R}$, $\text{Dom } \xi \subseteq A \cap S_n \cap C$ and $\text{Im } \xi \subseteq A \cap S_n^c \cap D$ satisfying $(\xi x)_i = x_i$, for $i \geq N+1$. As $\text{Dom } \xi \subseteq S_n$ and $\text{Im } \xi \subseteq S_n^c$ we have

$$a_n^{k_{n-1}k_n} = 1 \text{ and } a_n^{j_{n-1}j_n} = \lambda_{n,j_n} \quad (6.2.5)$$

As pointed out above, $\frac{d\mu \circ \xi}{d\mu}(x) = 1$, for all $x \in \text{Dom } \xi$. But if $x \in \text{Dom}(\xi)$,

$$\frac{d\mu \circ \xi}{d\mu}(x) = \frac{a_1^{0k_1} a_2^{k_1k_2}}{a_1^{0j_1} a_2^{j_1j_2}} \cdots \frac{a_n^{k_{n-1}k_n}}{a_n^{j_{n-1}j_n}} \cdots \frac{a_N^{k_{N-1}k_N}}{a_N^{j_{N-1}k_N}}.$$

This can be written as a product of $r_i^{\alpha_i}$ with $1 \leq i \leq r_{\psi^{-1}(N, k_N)}$. By (6.2.5), in this product, the exponent of $r_{\psi^{-1}(n, j_n)}$ is -1 . This is in contradiction with Lemma 6.2.1, which tells us that if such a product is 1, all the exponents are equal to 0.

Therefore $A \subseteq S_n$ or $A \subseteq S_n^c$ up to a set of measure zero. If $S_n^0 = S_n$ and $S_n^1 = S_n^c$ then we can choose a 0-1 sequence $(k_n)_{n \geq 1}$ such that $A \subseteq S_n^{k_n}$, for all $n \geq 1$. Then,

$$\mu(A) \leq \mu(\bigcap_{i=1}^n S_n) = \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Contradiction with the fact that $\mu(A) > 0$. □

Lemma 6.2.9. *Let $\delta > 0$ and $N > 1$ such that (6.2.4) holds. Let s be such that $e^{s-\delta} > \sum_{i=1}^{N+1} \log r_i$, and let $m \geq 1$ be such that $I_m \cap (e^{s-\delta}, e^{s+\delta}) \neq \emptyset$. If $x \in X$ and there exists $y \in X$, $(y, x) \in \mathcal{R}$ such that $|\log \delta(y, x)| \in (e^{s-\delta}, e^{s+\delta})$ then $\delta(x, y)$ is of the form $r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_m^{\alpha_m}$, with $\alpha_m \neq 0$.*

Proof. By Lemma 6.2.3, if $|\log \delta(y, x)| \in (e^{s-\delta}, e^{s+\delta})$ then $|\log \delta(y, x)|$ is in some I_n , with $n \geq 2$. Hence, $\delta(y, x) = r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n}$, with $\beta_n \neq 0$. Therefore, by Lemma 6.2.2, $n = m$. \square

We show now that \mathcal{R} does not have property A. To prove this we use the fact that if the Radon-Nikodym derivatives are in a certain interval $(e^{s-\delta}, e^{s+\delta})$, then they could be only in an interval I_m (Lemma 6.2.9).

Proposition 6.2.10. *\mathcal{R} does not have property A and $W^*(X, \mu, \mathcal{R})$ is not isomorphic to any ITPFI factor. Equivalently, the automorphism T that generates \mathcal{R} is not of product type.*

Proof. Let $\delta > 0$. We choose $N > 1$ such that (6.2.4) holds. If s is large enough and if we take x satisfying

$$\log \frac{d\mu \circ \phi}{d\mu}(x) \in (e^{s-\delta}, e^{s+\delta})$$

for some $\phi \in [\mathcal{R}]$, then, by Lemma 6.2.3 there exists an integer $m \geq 1$ such that

$$\log \frac{d\mu \circ \phi}{d\mu}(x) \in I_m,$$

and hence $I_m \cap (e^{s-\delta}, e^{s+\delta}) \neq \emptyset$. Moreover, by Lemma 6.2.9, m depends only on s and is uniquely determined by $I_m \cap (e^{s-\delta}, e^{s+\delta}) \neq \emptyset$. We can write $m = m(s)$ and then $m(s) \rightarrow \infty$ when $s \rightarrow \infty$. We have:

$$\begin{aligned} & \{x \in X : \exists \phi \in [\mathcal{R}] \text{ such that } |\log \frac{d\mu \circ \phi}{d\mu}(x)| \in (e^{s-\delta}, e^{s+\delta})\} \\ & \subseteq \{x \in X : \exists \phi \in [\mathcal{R}] \text{ such that } |\log \frac{d\mu \circ \phi}{d\mu}(x)| \in I_{m(s)}\} \\ & \subseteq \{x \in X : \exists \phi \in [\mathcal{R}] \text{ and } \alpha_1, \alpha_2, \dots, \alpha_{m(s)} \in \{-1, 0, 1\} \text{ with } \alpha_{m(s)} \neq 0 \text{ such that} \\ & \quad \frac{d\mu \circ \phi}{d\mu}(x) = r_1^{\alpha_1} \cdots r_{m(s)}^{\alpha_{m(s)}}\} \subseteq F_{\psi(m(s))}. \end{aligned}$$

Then, it follows:

$$\begin{aligned} & \lim_{s \rightarrow \infty} \mu \{x \in X : \exists \phi \in [\mathcal{R}] \text{ such that } |\log \frac{d\mu \circ \phi}{d\mu}(x)| \in (e^{s-\delta}, e^{s+\delta})\} \\ & \leq \lim_{s \rightarrow \infty} \mu \{x \in X : \exists \phi \in [\mathcal{R}] \text{ and } \alpha_1, \alpha_2, \dots, \alpha_m \in \{-1, 0, 1\} \text{ with } \alpha_m \neq 0 \text{ such that} \\ & \quad \frac{d\mu \circ \phi}{d\mu}(x) = r_1^{\alpha_1} \cdots r_{m(s)}^{\alpha_{m(s)}}\} \leq \lim_{s \rightarrow \infty} \mu(F_{\psi(m(s))}) = 0 \end{aligned}$$

Therefore, \mathcal{R} does not have property A , and consequently the factor $W^*(X, \mu, \mathcal{R})$ is not isomorphic with any ITPFI. \square

The Computation of the Associated Flow. Consider the associated flow of \mathcal{R} . This flow gives us a nice example of a measure preserving flow which does not satisfy AT. First the measure μ is lacunary as already observed in the proof of Lemma 6.2.8. As in Section 5.1, we will see the associated flow as a flow built under a ceiling function.

Let's identify first the space X/\sim where \sim is the equivalence relation on X defined by : $x \sim y$ if and only if $x\mathcal{R}y$ and $\delta(y, x) = 1$. Then the quotient space is $(Y, \hat{\mu})$ defined previously, i.e. $Y = \prod\{0, 1\}$ and $\hat{\mu} = \otimes \hat{\mu}_n$ with $\hat{\mu}_n(0) = \hat{\mu}_n(1) = \frac{1}{2}$ for all $n \geq 1$.

Let $\phi : X \rightarrow \mathbb{R}$, $\phi(x) := y$ where:

$$\phi(x) = \min\{\log \delta(x', x); \log \delta(x', x) > 0\}$$

This may be regarded as a function on the quotient Y . We also define an automorphism U on Y such that $U[x] = [x']$, if $(x', x) \in \mathcal{R}$ satisfies $\log \delta(x', x) = \phi(x)$, where $[x]$ is an element of the partition X/\sim that contains x . Then, the associated flow is the flow F_s which is the suspension flow of $U : Y \mapsto Y$ under the ceiling function $\phi(y)$. More precisely, the flow is defined on the set

$$D = \{[z, t] : z \in Y, 0 \leq t < \phi(z)\} \subseteq Y \times \mathbb{R}$$

with the measure $\nu \times \lambda$ (λ is the Lebesgue measure) restricted on D , and $F_s(z, t)$ is defined as in (5.1.1).

In fact U has a very nice description, namely:

$$U(1^m 0^n 01 \dots) = (0^n 1^m 10 \dots).$$

As N is not isomorphic to an ITPFI, the above flow with this nice base transformation, fails having AT-property.

The example constructed here appears naturally by starting with a Bratteli diagram. The two previous existing examples are due to Krieger [K2] and to Dooley-Hamachi [DH]. Both of them are hard to visualize. The associated flow for Krieger's example is also more complicated and it was computed by Giordano, [G].

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