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A SURVEY ON THE ORTHOGONAL EXPANSION OF THRESHOLD FUNCTIONS

by

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ABSTRACT

The classification and study of the structure of switching functions and their realizations have received considerable attention in the past ten years. They were studied under various names such as the coordinate representation (13), the discrete Fourier transform (3), the Rademacher-Walsh expansion (7, 16) and the orthogonal expansion (2) of the functions. A concise survey on this field is presented in the first part of this work. The last part of the work is to apply these theories for the testing and realizing method of threshold functions.

The (n+1) Chow parameters can uniquely characterize a threshold function of n variables. If we extend the number of Chow parameters to 2^n with each representing the number of vertices in certain subcube, which is represented by a Boolean product of uncomplemented variables only, and if we arrange them in groups with each group having subcubes of the same dimension, then an augmented vector can be constructed. Such a vector will uniquely characterize the corresponding function and this can be developed for testing and realization of threshold functions.

Without loss of generalities, we confine us to positive prime functions. A set of positive weights is first assigned to the variables. Their magnitudes will be in descending order. The components in the second group of the augmented vector corresponding to the number of vertices in the (n-2)-subcubes are listed in a table. In the table, each row and each column should be in descending order. Otherwise, the given function is not a threshold function. By comparing the entries in different rows and columns, a set of inequalities concerning the weights can be derived. This set of inequalities can be further reduced by considering the relating components of the augmented vector corresponding to those subcubes with dimensions less than (n-2). At length, the inequalities can be solved, resulting in a minimal integral realization.

CONTENTS

CHAPTER 1

INTRODUCTION

In the rapid development of the electronic digital computer and other modern digital devices, switching circuits are playing an increasingly important role. Traditionally, the basic building blocks used for physical realization of these circuits are the conventional AND, OR, NAND, NOR and other gates. There exists a well-developed boolean switching theory for this concern. However, in the late 1950's several variations of a new type of circuits called "threshold gates" were developed, and ever since considerable attention has been drawn on these gates.

A threshold gate allows it s inputs to have different weights, and an output is obtained from the gate only when the sum of the weighted input attains or exceeds a given value called the "threshold". Thus. the threshold gate performs a much more sophisticated decision process than the conventional gates. In other words, the threshold gate is logically more powerful, and fewer number of gates are needed to realize a given switching function. This leads to the reductions in the number of gates the number of interconnections, and therefore in the total cost of the overall circuits.

The present work is concerned, first with the orthogonal expansion of switching functions. Next, the augmented Chow-vector for a switching function is defined and its properties are studied. Finally, a testing and realizing method for threshold functions is derived by using such a vector.

In Chapter 2, the orthogonal expansion and threshold functions are defined. Special notations which will be used throughout this work are introduced.

In Chapter 3, a survey on the Rademacher Walsh functions is made and various properties relating to these functions are presented.

In Chapter 4, a short study on the characteristic parameters as well as the invariance operations is furnished. The augmented Chow-vector is defined and its relations with the spectrum by Dertouzos are worked out.

In Chapter5, a testing and realizing method for threshold functions is introduced. An algorithm for this method is provided. Several examples are worked out for illustration.

In Chapter 6, certain conclusions are drawn.

CHAPTER 2
ORTHOGONAL EXPANSION AND THRESHOLD FUNCTIONS

2.1. Introduction

This chapter is to introduce certain special notations as well as to summarize the principal definitions and concepts which form the background for this work. A basic understanding of switching theory and reasonable knowledge of threshold functions are required. We begin with the derivation of orthogonal expansion of switching functions and then give the definition of a threshold function. Furthermore, we also define compound-weight threshold functions and threshold functions of higher orders.

2.2. Orthogonal Expansion of Switching Functions

For the purpose of this work, a switching function may have its range and the domain of its arguments or variables from either the set (0, 1) or the set (-1, 1). In order to avoid ambiguity we denote them differently as follows:

F(X): A sw. f. whose range is (0, 1) and the domain of its argument is (0, 1)
F(Y): A sw. f. whose range is (0, 1) and the domain of its argument is (-1, 1)
G(X): A sw. f. whose range is (-1, 1) and the domain of its argument is (0, 1)
G(Y): A sw. f. whose range is (-1, 1) and the domain of its argument is (-1, 1)

In other words, x_i, $F \in (0, 1)$. but y_i, $G \in (-1, 1)$. It is not difficult to see that a linear transformation relating x_i to y_i and F to G exists, such that

$$y_i = 2x_i - 1 \quad ; \quad x_i = \frac{1}{2}(y_i + 1)$$

$$\text{and} \quad G = 2F - 1 \quad ; \quad F = \frac{1}{2}(G + 1) \qquad (2.2\text{-}1)$$

Moreover, we will find it convenient to introduce a specific notation $F_\alpha(X)$ for evaluating F at a particular vertex X, where α is the decimal number equivalent to the binary representation formed by the n variables in this order $x_n x_{n-1} \cdots x_1$. If Y is used instead of X, then α is the corresponding decimal number by simply replacing each occurrence of -1 with 0. Therefore, F(X) evaluated at $x_3 = 1$, $x_2 = 0$, $x_1 = 1$ is $F(x_3=1, x_2=0, x_i=1) = F_5(X)$, and F(Y) evaluated at $y_3 = 1$, $y_2 = -1$, $y_1 = -1$ is $F(y_3=1, y_2=-1, y_1=-1) = F_4(Y)$. It is quite obvious that $0 \leq \alpha \leq 2^n - 1$. We will also write $x_{i,\alpha}$ or $y_{i,\alpha}$ to indicate the value of the variable x_i or y_i when α is the subscript of F. For example, $x_{1,5} = 1$, $x_{2,5} = 0$; $y_{1,4} = -1$, $y_{2,4} = -1$.

We want to point out that the operations are the normal additions and multiplications when we deal with y_i and G; but they might be either normal or Boolean operations when x_i and F are operated.

Now G(Y) can be expressed into a series of orthogonal functions ϕ_i's as follows:

$$G(Y) = \sum_{i=0}^{2^n-1} a_i \phi_i \tag{2.2-2}$$

where a_i is an integer.

And we define a set of orthogonal functions (ϕ_i) for i = 0, 1, 2,.. .... 2^n-1, such that

$$\sum_{\alpha=0}^{2^n-1} \phi_{j,\alpha} \phi_{k,\alpha} = 2^n \delta_{jk} \tag{2.2-3}$$

and

$$\sum_{i=0}^{2^n-1} \phi_{i,j} \phi_{i,k} = 2^n \delta_{jk} \tag{2.2-4}$$

where δ_{jk} is the Kronecker delta function, that is

$$\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

It is well known that the following set of functions called Rademacher-Walsh functions is a set of orthogonal functions satisfying both Equations (2.2-3) and (2.2-4)

$\phi_0 = 1$ a constant

$\phi_1 = y_1$

$\phi_2 = y_2$

$$\begin{aligned}
\phi_3 &= y_2 y_1 \\
\phi_4 &= y_3 \\
\phi_5 &= y_3 y_1 \\
\phi_6 &= y_3 y_2 \\
&\vdots \\
\phi_{2^n-1} &= y_n y_{n-1} \cdots\cdots y_1
\end{aligned} \qquad (2.2\text{-}5)$$

For a particular vertex Y_α, there exists a corresponding decimal number α. Thus

$$G_\alpha(Y) = \sum_{i=0}^{2^n-1} a_i \phi_{i,\alpha} \qquad (2.2\text{-}6)$$

Multiply $\phi_{k,\alpha}$ by Equation (2.2-6) and sum over α.

$$\begin{aligned}
\sum_{\alpha=0}^{2^n-1} G_\alpha(Y)\, \phi_{k,\alpha} &= \sum_{\alpha=0}^{2^n-1} \sum_{i=0}^{2^n-1} a_i\, \phi_{i,\alpha}\, \phi_{k,\alpha} \\
&= \sum_{i=0}^{2^n-1} a_i \sum_{\alpha=0}^{2^n-1} \phi_{i,\alpha}\, \phi_{k,\alpha} \\
&= \sum_{i=0}^{2^n-1} a_i\, 2^n\, \delta_{ik} = 2^n a_k
\end{aligned}$$

$$a_k = 2^{-n} \sum_{\alpha=0}^{2^n-1} G_\alpha(Y)\, \phi_{k,\alpha} \qquad (2.2\text{-}7)$$

where $k = 0, 1, 2, \cdots\cdots 2^n-1$ and $\phi_{0,\alpha} = 1$.

Substituting Eq. (2.2-5) into Eq. (2.2-7), we have

$$\begin{aligned}
a_0 &= 2^{-n} \sum_{\alpha=0}^{2^n-1} G_\alpha(Y) \\
a_1 &= 2^{-n} \sum_{\alpha=0}^{2^n-1} G_\alpha(Y)\, y_{1,\alpha} \\
&\vdots \\
a_{2^n-1} &= 2^{-n} \sum_{\alpha=0}^{2^n-1} G_\alpha(Y)\, y_{n,\alpha} y_{n-1,\alpha} \cdots\cdots y_{1,\alpha}
\end{aligned} \qquad (2.2\text{-}8)$$

Thus, we conclude that any switching function can be expanded as a sum of orthogonal functions and their products as shown below:

$$G(Y) = \sum_{i=0}^{2^n-1} a_i \phi_i$$

$$= a_0 + a_1 y_1 + a_2 y_2 + a_3 y_2 y_1 + \cdots\cdots + a_{2^n-1} y_n y_{n-1} \cdots y_1 \quad (2.2\text{-}9)$$

where a_i has the value as in Eq. (2.2-8) for $0 \le i \le 2^n - 1$.

And we define the a-vector of a switching function as the ordered set of the coefficients for its orthogonal expansion:

$$\vec{a} = (a_0,\ a_1, \cdots\cdots a_{2^n-1}) \quad (2.2\text{-}10)$$

There is a relation between the coefficients and the corresponding product of the orthogonal functions. Express the subscript of a's as a binary number and use only those position numbers of the bits which are 1 for the subscript of y. The position number is in descending order from left to right beginning with n. For example, $(6)_{10} = (110)_2$ therefore a_6 is the coefficient for $y_3 y_2$.

Dertouzos(3) has matched the subscripts for the coefficients and the functions, also reordered the coefficients to form his augmented b-vector, or the spectrum.

$$\vec{\mathcal{B}} = (b_1, b_2, \cdots b_n; b_0 / b_{12}, b_{13}, \cdots / \cdots / b_{12\cdots n}) \quad (2.2\text{-}11)$$

and $G(Y) = 2^{-n}(b_0 + b_1 y_1 + \cdots + b_n y_n + b_{12} y_1 y_2 + \cdots + b_{12\cdots n} y_1 y_2 \cdots y_n)$ (2.2-12)

We can easily find that a_i's and b_i's are related as follows:

$$2^n a_0 = b_0 = \langle G_a(Y) \rangle$$

$$2^n a_1 = b_1 = \langle G_a(Y) y_{1,a} \rangle$$

$$2^n a_2 = b_2 = \langle G_a(Y) y_{2,a} \rangle$$

$$2^n a_3 = 2^n a_{(11)_2} = b_{21} = b_{12} = \langle G_\alpha(Y)\, y_{1,\alpha} y_{2,\alpha} \rangle$$

$$2^n a_4 = 2^n a_{(100)_2} = b_3 = \langle G_\alpha(Y)\, y_{3,\alpha} \rangle \qquad (2.2\text{-}13)$$

$$2^n a_5 = 2^n a_{(101)_2} = b_{31} = b_{13} = \langle G_\alpha(Y)\, y_{1,\alpha} y_{3,\alpha} \rangle$$

$$\vdots$$

$$2^n a_{2^n-1} = 2^n a_{(11\cdots1)_2} = b_{12\cdots n} = \langle G_\alpha(Y)\, y_{1,\alpha} y_{2,\alpha} \cdots y_{n,\alpha} \rangle$$

where the summing operation $\langle\ \rangle$ over all the vertices introduced by Dertouzos is defined as:

$$\langle G_\alpha(Y) \rangle = \sum_{\alpha=0}^{2^n-1} G_\alpha(Y) \qquad (2.2\text{-}14)$$

2. 3. Single Threshold Functions

A number of different but equivalent definitions for single threshold functions are used by different authors. Since they differ substantially in form, we give the most common one and then modify it slightly.

Definition 2. 3-1 A switching function of n variables $F_\alpha(X)$ is a single threshold function iff there exists a set of real numbers $w_1, w_2, \ldots, w_n$ called the weights forming the weight vector ($\vec{w}$) and a real number T called the threshold such that

$$\sum_{i=1}^{n} w_i x_{i,\alpha} \geq T \qquad \text{iff } F_\alpha(X) = 1$$

$$\qquad\qquad\qquad\qquad (2.3\text{-}1)$$

$$\sum_{i=1}^{n} w_i x_{i,\alpha} < T \qquad \text{iff } F_\alpha(X) = 0$$

If we now let $w_0 = -T$ and let $x_{0,\alpha} = 1$ for all α, then we can restate Definition 2. 3-1 as follows:

<u>Definition 2.3-2.</u> A switching function of n variables $F_\alpha(X)$ is a single threshold function iff there exists a set of real numbers $w_0, w_1, \ldots, w_n$ such that

$$\sum_{i=0}^{n} w_i x_{i,\alpha} \geq 0 \qquad \text{iff} \quad F_\alpha(X) = 1$$

$$\sum_{i=0}^{n} w_i x_{i,\alpha} < 0 \qquad \text{iff} \quad F_\alpha(X) = 0 \qquad (2.3\text{-}2)$$

2.4. Compound-Weight Threshold Functions and Threshold Functions of Higher Order

Among all switching functions, the percentage of single threshold functions diminishes and approaches zero as the number of variables increases. Thus an extension of the realization of switching functions other than that for single threshold functions is necessary. Spann(16) has allowed an extension of the type of summation involved and introduced the compound weights $w_{ij\ldots m}$ in addition to the linear weights as specified in Eqs. (2.3-1) and (2.3-2).

Consider a summation of the following form:

$$g_\alpha = 2^{-n}\left(\sum_{i=0}^{n} w_i y_{i,\alpha} + \sum_{j>i\geq 1}^{n} w_{ij} y_{i,\alpha} y_{j,\alpha} + \cdots + w_{12\ldots n} y_{1,\alpha} y_{2,\alpha} \cdots y_{n,\alpha}\right)$$

$$= 2^{-n}\left(w_0 + \sum_{j=1}^{n} \sum_{i_j > i_{j-1} > \cdots > i_1 \geq 1}^{n} w_{i_1 i_2 \cdots i_j} y_{i_1,\alpha} y_{i_2,\alpha} \cdots y_{i_j,\alpha}\right) \qquad (2.3\text{-}3)$$

Comparing the above summation (2.3-3) with the orthogonal expansion (2.2-12), we will find that if we let each weight w_k equal the corresponding element b_k of the spectrum vector, then Eqs. (2.3-3) and (2.2-12) will be equal to each other. Hence, it is clear that every switching function has a compound weight realization of the form

$$g_\alpha \geq 0 \qquad \text{iff} \quad G_\alpha = 1$$

$$g_\alpha < 0 \qquad \text{iff} \quad G_\alpha = -1 \qquad (2.3\text{-}4)$$

However, the realization by simply using b_k for w_k is not always optimal. In this respect, there remains the problem of finding realizations which are optimal according to some measures. One measure of optimality might be the number of non-zero weights needed to realize a function. Another measure, which was used by Krishnan (11) as the definition of threshold order, might be the largest number of subscripts on a non-zero weight.

Definition 2.4-1. A switching function which can be realized by a ϕ-function of degree r is called a rth order threshold function, where the ϕ - function of degree r is defined as

$$\phi_{\alpha}(X) = 2^{-n}\Big(\sum_{i_1=0}^{n} w_{i_1} x_{i_1,\alpha} + \sum_{i_2>i_1\geq 1}^{n} w_{i_1 i_2} x_{i_1,\alpha} x_{i_2,\alpha} + \cdots\cdots\cdots + \sum_{i_r>i_{r-1}>\ldots i_1\geq 1}^{n} w_{i_1 i_2 \ldots i_r} x_{i_1,\alpha} x_{i_2,\alpha} \ldots x_{i_r,\alpha}\Big)$$

$$= 2^{-n}\Big(w_0 + \sum_{j=1}^{r} \sum_{i_j>i_{j-1}>\ldots i_1\geq 1}^{n} w_{i_1 i_2 \ldots i_j} x_{i_1,\alpha} x_{i_2,\alpha} \ldots x_{i_j,\alpha}\Big)$$

It has been proved that any switching function of n variables can be realized as a threshold function of order r, where $r \leq n$. (11) In fact, when $r = n$, the ϕ -function coincides with the above mentioned summation (2.3-3).

CHAPTER 3
RADEMACHER WALSH FUNCTIONS

3.1. Introduction

Any switching function has a unique orthogonal expansion. The coefficients (or spectra) specify the switching function. The relation between the switching function and the spectra is an orthogonal one and can be expressed in a neat and ordered matrix form. This is the so called Walsh matrix. We shall first derive the Walsh matrix, then define the various forms. Lastly, we shall investigate its properties.

3.2. Boolean and Real Vectors

It is well known that a switching function can be expressed uniquely by a disjunctive canonical form, as a Boolean sum of disjoint minterms

$$F(x_1, x_2, \ldots\ldots x_n) = \sum_{i=0}^{2^n-1} a_i m_i(x_1, x_2, \ldots\ldots x_n) \qquad (3.2\text{-}1)$$

where Σ is the Boolean sum, $m_i(x_1, x_2, \ldots\ldots x_n)$ the minterm and $a_i \in (0, 1)$ the coefficients for $i=0, 1, 2, \ldots\ldots 2^n-1$.

Definition 3.2-1. A boolean vector of a switching function F denoted as B_F is an ordered set of the 2^n-coefficients when the switching function is expressed in the disjunctive canonical form. All its components belong to the set (0, 1).

Example 3.2-1.

$$F_1(x_1, x_2, x_3) = x_1 + x_2x_3 = \Sigma m(1,3,5,6,7)$$

$$B_{F_1} = (0,1,0,1,0,1,1,1)$$

For a switching function of n variables its Boolean vector has a dimension of 2^n. It is obvious that the Boolean vectors have the following properties:

A) $B_{f \vee g} = B_f \vee B_g = (a_0^f \vee a_0^g, \; a_1^f \vee a_1^g, \ldots\ldots\ldots a_{2^n-1}^f \vee a_{2^n-1}^g)$

B) $B_{f \wedge g} = B_f \wedge B_g = (a_0^f \wedge a_0^g, \; a_1^f \wedge a_1^g, \ldots\ldots\ldots a_{2^n-1}^f \wedge a_{2^n-1}^g)$

C) $B_0 = 0 = (0, 0, \ldots\ldots 0)$ a 2^n- vector
$B_1 = I = (1, 1, \ldots\ldots\ldots 1)$

D) $B_{\bar{f}} = \overline{B}_f$

E) $B_{f \oplus g} = B_f \oplus B_g = (a_0^f \oplus a_0^g, \; a_1^f \oplus a_1^g, \ldots\ldots a_{2^n-1}^f \oplus a_{2^n-1}^g)$

where $\vee$, $\wedge$ are used for Boolean addition and multiplication; $\oplus$ for exclusive-or operation.

Definition 3.2-2. A real vector of a switching function F denoted as R_F is defined as

$$R_F = 2B_F - I$$

The real vector has the same dimension as the boolean vector but its components belong to the set (-1, 1). Actually, the real vector is ? an isomorphism to its corresponding Boolean vector.

Example 3.2-2.

For the same function in EX. 3.2-1.

$R_F = (-1, 1, -1, 1, -1, 1, 1, 1)$

A set of properties for the real vectors similar to that of the Boolean vectors will be shown as follows. Once again, for the switching function as well as the Boolean vectors we deal with either the Boolean operations, such as $\vee$, $\wedge$, $^-$, $\oplus$, or normal additions or multiplications;

but for real vectors only normal additions and multiplications will be used. Since Boolean algebra is nothing but a subset of the algebra for the real numbers, the Boolean operations can be changed to normal operations.

A') $R_{f \vee g} = \frac{1}{2}(R_f + R_g - R_f R_g + I)$

Proof: $R_{f \vee g} = 2\, B_{f \vee g} - I = 2(B_f \vee B_g) - I$

$$= 2(B_f + B_g - B_f B_g) - I$$

$$= 2\left[\frac{1}{2}(R_f + I) + \frac{1}{2}(R_g + I) - \frac{1}{2}(R_f + I)\,\frac{1}{2}(R_g + I)\right] - I$$

$$= R_f + I + R_g + I - \frac{1}{2}\, R_f R_g - \frac{1}{2}R_f - \frac{1}{2}R_g - \frac{1}{2}I - I$$

$$= \frac{1}{2}(R_f + R_g - R_f R_g + I)$$

B') $R_{f \wedge g} = \frac{1}{2}(R_f + R_g + R_f R_g - I)$

Proof: $R_{f \wedge g} = 2B_{f \wedge g} - I = 2(B_f \wedge B_g) - I$

$$= 2\, B_f B_g - I$$

$$= 2\,\frac{1}{2}(R_f + I)\,\frac{1}{2}(R_g + I) - I$$

$$= \frac{1}{2}(R_f + R_g + R_f R_g - I)$$

C') $R_0 = 2B_0 - I = -I$ a 2^n - vector

$R_1 = 2B_1 - I = 2\,I - I = I$

D') $R_{\bar{f}} = -R_f$

Proof: $R_{\bar{f}} = 2B_{\bar{f}} - I = 2\bar{B}_f - I$

$$= 2\,(I - B_f) - I$$

$$= I - 2 \cdot \frac{1}{2}(R_f + I) = -R_f$$

E') $R_{f \oplus g} = -R_f R_g = -R_g R_f$

Proof : $R_{f \oplus g} = 2B_{f \oplus g} - I = 2(B_f \oplus B_g) - I$

$= 2[\bar{B}_f B_g + B_f \bar{B}_g - (\bar{B}_f B_g)(B_f \bar{B}_g)] - I$

$= 2(\bar{B}_f B_g + B_f \bar{B}_g) - I$

$= 2[(I - B_f) B_g + B_f (I - B_g)] - I$

$= 2(B_f + B_g - 2B_f B_g) - I$

$= (R_f + I) + (R_g + I) - (R_f + I)(R_g + I) - I$

$= -R_f R_g = -R_g R_f$

3.3. Rademacher and Walsh Functions

Definition 3.3-1. Denote the 2^n linear switching functions of n variables by l_0; $l_1, l_2, \ldots\ldots, l_n$; $l_{12}, l_{13}, \ldots\ldots$; $l_{12\cdots n}$ and define $l_{ij\cdots p} = x_i \oplus x_j \oplus \cdots\cdots \oplus x_p$, where $l_0 = 1$, a constant.

For all these l-functions it is not difficult to find their corresponding real vectors by first writing out the binary representations for n variables and then applying the property E') of the previous section. The total set of the 2^n vectors forms an orthogonal system which is called a Walsh system. Among these vectors only the first $(n+1)$ vectors are independent and the rest of them can be generated by direct multiplication. These $(n+1)$ independent vectors form another system called Rademacher system. Thus the Rademacher system is a subsystem of the Walsh system.

For $n = 3$

$l_0 = 1$ | $l_{12} = x_1 \oplus x_2$
$l_1 = x_1$ | $l_{13} = x_1 \oplus x_3$
$l_2 = x_2$ | $l_{23} = x_2 \oplus x_3$
$l_3 = x_3$ | $l_{123} = x_1 \oplus x_2 \oplus x_3$

Decimal Number	x_3	x_2	x_1
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

$$R_{1_0} = (1, 1, 1, 1, 1, 1, 1, 1)$$
$$R_{1_1} = (-1, 1, -1, 1, -1, 1, -1, 1)$$
$$R_{1_2} = (-1, -1, 1, 1, -1, -1, 1, 1)$$
$$R_{1_3} = (-1, -1, -1, -1, 1, 1, 1, 1)$$
$$R_{1_{12}} = R_{1_1 \oplus 1_2} = -R_{1_1} R_{1_2} = (-1, 1, 1, -1, -1, 1, 1, -1)$$
$$R_{1_{13}} = R_{1_1 \oplus 1_3} = -R_{1_1} R_{1_3} = (-1, 1, -1, 1, 1, -1, 1, -1)$$
$$R_{1_{23}} = R_{1_2 \oplus 1_3} = -R_{1_2} R_{1_3} = (-1, -1, 1, 1, 1, 1, -1, -1)$$
$$R_{1_{123}} = R_{1_1 \oplus 1_2 \oplus 1_3} = R_{1_1} R_{1_2} R_{1_3} = (-1, 1, 1, -1, 1, -1, -1, 1)$$

3.4. Walsh Matrix and Its Various Forms

The fundamental Walsh matrix or simply Walsh matrix is formed by the real vectors of the l-functions as the rows. Since the real vector R_1 is of dimension 2^n and there are exactly 2^n such vectors, the Walsh matrix [W] is thus a square matrix of the order 2^n. Furthermore, all the row (or column) vectors of the matrix are mutually orthogonal. Actually, the Walsh matrix is a subclass of Hadamard matrices (6), whose orders might be one, two or a multiple of four,

$$[W] = \begin{pmatrix} R_{1_0} \\ R_{1_1} \\ \vdots \\ R_{1_n} \\ R_{1_{12}} \\ \vdots \\ R_{1_{12..n}} \end{pmatrix} = \begin{pmatrix} R_{1_0}(0) & R_{1_0}(1) & \cdots\cdots & R_{1_0}(2^n-1) \\ R_{1_1}(0) & R_{1_1}(1) & \cdots\cdots & R_{1_1}(2^n-1) \\ \vdots & \vdots & \vdots & \vdots \\ R_{1_n}(0) & R_{1_n}(1) & \cdots\cdots & R_{1_n}(2^n-1) \\ R_{1_{12}}(0) & R_{1_{12}}(1) & \cdots\cdots & R_{1_{12}}(2^n-1) \\ \vdots & \vdots & \vdots & \vdots \\ R_{1_{12..n}}(0) & R_{1_{12..n}}(1) & \cdots\cdots & R_{1_{12..n}}(2^n-1) \end{pmatrix}$$

Somtimes for certain purposes, it is more convenient to order these real vectors in a different way. One of the most natural ways is the decimal order of binary representation. Write out a truth table for the n variables. Let the ith function be $h_i = 1_{i_1 i_2 \ldots\ldots i_n} = x_{j_1} \oplus x_{j_2} \oplus \ldots\ldots \oplus x_{j_n}$, where $i_1 i_2 \ldots\ldots i_n$ is the binary representation of i, or the ith row of the truth table. Thus

$$x_{j_k} = \begin{cases} x_k & \text{if } i_k = 1 \\ 0 & \text{if } i_k = 0 \end{cases} \qquad \text{for } k = 1, 2, \ldots\ldots\ldots n$$

Therefore, we reorder the 1-functions in the following way:

$$h_0 = h_{0\ldots\ldots 0} = 1_0$$

$$h_1 = h_{0\ldots\ldots 01} = 1_1$$

$$h_2 = h_{0\ldots\ldots 010} = 1_2$$

$$h_3 = h_{0\ldots\ldots 011} = 1_{12}$$

$$h_4 = h_{0\ldots\ldots 100} = 1_3$$

$$\vdots$$

$$h_{2^n-1} = h_{1\ldots\ldots 111} = 1_{12\ldots n}$$

The matrix formed by the real vectors of h-functions is called the modified Walsh matrix and denoted as [W'].

$$[W'] = \begin{pmatrix} R_{h_0} \\ R_{h_1} \\ \vdots \\ R_{h_{2^n-1}} \end{pmatrix} = \begin{pmatrix} R_{h_0}(0) & R_{h_0}(1) \cdots\cdots & R_{h_0}(2^n-1) \\ R_{h_1}(0) & R_{h_1}(1) \cdots\cdots & R_{h_1}(2^n-1) \\ \vdots & \vdots \quad \cdots\cdots & \vdots \\ R_{h_{2^n-1}}(0) & R_{h_{2^n-1}}(1) \cdots\cdots & R_{h_{2^n-1}}(2^n-1) \end{pmatrix} \quad (3.4\text{-}3)$$

If we introduce the factor $2^{-n/2}$, we have the normalized Walsh matrix. To distinguish the Walsh matrix from its normalized form, we attached a subscript N to the normalized matrix. Thus

$$[W_N] = 2^{-n/2}[W]$$

and

$$[W_N'] = 2^{-n/2}[W'] \quad (3.4\text{-}4)$$

Henderson(7) has shown that a Walsh matrix can be converted directly from the fundamental form to its modified form, and vice versa, by means of a permutation matrix, which has only a single 1 in each row and each column. Moreover, because of the orthogonality of the Walsh matrix, the two permutation matrices are also orthogonal to each other. Therefore, we have

$$\begin{aligned} [W'] &= [T_{FM}][W] = [T_{MF}]_t[W] \\ [W] &= [T_{MF}][W'] = [T_{FM}]_t[W'] \end{aligned} \quad (3.4\text{-}5)$$

where $[T_{FM}]$ and $[T_{MF}]$ both are permutation matrices, and the subscript t denotes the transpose of a matrix.

3.5. The Properties of Walsh Matrix

Theorem 3.5-1 The normalized fundamental and modified Walsh matrices are orthogonal.

<u>Proof</u>: Let $[A] = [W'][W']_t$, then it is well known that any element of $[A]$

$$a_{ij} = \sum_{k=0}^{2^n-1} R_{h_i}(k)\, R_{h_j}(k) \qquad \text{for } i, j = 0, 1, \cdots 2^n-1$$

$$= 2^n \delta_{ij}$$

where δ_{ij} is the Kronecker delta function

$$[W_N'][W_N']_t = 2^{-n}[W'][W']_t$$

$$= 2^{-n}[A]$$

$$= [I] \qquad \text{an identity matrix}$$

Thus, $[W_N']$ is orthogonal.

Let $[C] = [W_N][W_N]_t$, then

$$[C] = 2^{-n}[W][W]_t$$

$$= 2^{-n}[T_{MF}][W'][W']_t [T_{MF}]_t$$

$$= 2^{-n}[T_{MF}]\, 2^n\, [I][T_{MF}]_t$$

$$= [T_{MF}][T_{MF}]_t$$

where $[T_{MF}]$, a permutation matrix, has a single 1 in each row and each column. Assume that in $[T_{MF}]$, $t_{ij} = 1$, hence t'_{ji} in $[T_{MF}]_t$ must also be equal to 1 and the rest of ith row in $[T_{MF}]$ as well as the jth column in $[T_{MF}]_t$ are all zeroes. Thus, any element in $[C]$

$$c_{ik} = \sum_j t_{ij}\, t'_{jk} = \delta_{ik}$$

$$[W_N][W_N]_t = [C] = [I]$$

Thus, $[W_N]$ is orthogonal.

Q. E. D.

<u>Theorem 3.5-2</u> Let $[W_n']$ denote a modified Walsh matrix with n variables for n = 0, 1, 2,······. Then

$$[W_n'] = \left[\begin{array}{c|c} W_{n-1}' & W_{n-1}' \\ \hline W_{n-1}' & -W_{n-1}' \end{array}\right]$$

and we define $[W_0'] = [1]$, a matrix with a single element 1.

Proof : Let us denote the number of variables by using a superscript. For example, h_i^n is the ith h-function of n variables. The subscript of a h-function is only a binary representation with the leftmost bit having the highest weight. Therefore, for a positive integer $i<2^{n-1}$, the functions h_i^{n-1} and h_i^n will have the same form and the variable x_n will not appear in either function. For instances, $h_1^2 = x_1$, $h_1^3 = x_1$; $h_6^3 = x_3 \oplus x_2$, $h_6^4 = x_3 \oplus x_2$. However, they are in different spaces, one is in the n-space and the other in (n-1)-space. The dimensions of the real vectors $R_{h_i}^n$ and $R_{h_i}^{n-1}$ are respectively 2^n and 2^{n-1}. Because h_i^n is independent of x_n, therefore, the first half and the last half of the real vector $R_{h_i}^n$ are equal. Thus, for $0 \le i < 2^{n-1}$, $R_{h_i}^n$ is equal to the concatenation of $R_{h_i}^{n-1}$ with itself. This forms the following equation :

$$R_{h_i}^n = (R_{h_i}^{n-1}, R_{h_i}^{n-1}) \qquad \text{for } 0 \le i < 2^{n-1} \qquad (3.5\text{-}1)$$

As for $i \ge 2^{n-1}$, the variable x_n will definitely appear in h_i^n. Let $k = i - 2^{n-1}$, then we can express h_i^n as follows :

$$h_i^n = x_n \oplus h_k^n = 1_n^n \oplus h_k^n \qquad \text{for } 2^{n-1} \le i < 2^n, \text{ and } k = i - 2^{n-1} \quad (3.5\text{-}2)$$

By applying the property E') of Section 3.2., we have

$$R_{h_i}^n = - R_{1_n}^n \cdot R_{h_k}^n \qquad \text{for } 2^{n-1} \le i < 2^n \text{ and } k = i - 2^{n-1} \quad (3.5\text{-}3)$$

Furthermore, since $1_n^n = x_n$, it is quite obvious that

$$R_{1_n}^n = (-1, -1, \cdots -1, 1, 1, \ldots 1) = (-I^{n-1}, I^{n-1}) \qquad (3.5\text{-}4)$$

Next, the range of k can be found to be $0 \le k < 2^{n-1}$. Thus, Eq. (3.5-1) is also valid for $R_{h_k}^n$. Substituting Eq. (3.5-4) into Eq. (3.5-3) we have

$$R^n_{h_i} = -(-I^{n-1},\ I^{n-1})(R^{n-1}_{h_k}, R^{n-1}_{h_k})$$
$$= (R^{n-1}_{h_k}, -R^{n-1}_{h_k}) \qquad \text{for } 2^{n-1} \le i < 2^n \text{ and } k = i - 2^{n-1}$$

(3.5-5)

By Eqs. (3.5-1) and (3.5-5) we prove that

$$[W'_n] = \begin{pmatrix} R^n_{h_0} \\ R^n_{h_1} \\ \vdots \\ R^n_{h_{2^{n-1}-1}} \\ \hdashline R^n_{h_{2^{n-1}}} \\ R^n_{h_{2^{n-1}+1}} \\ \vdots \\ R^n_{h_{2^n-1}} \end{pmatrix} = \begin{pmatrix} R^{n-1}_{h_0} & R^{n-1}_{h_0} \\ R^{n-1}_{h_1} & R^{n-1}_{h_1} \\ \vdots & \vdots \\ R^{n-1}_{h_{2^{n-1}-1}} & R^{n-1}_{h_{2^{n-1}-1}} \\ \hdashline R^{n-1}_{h_0} & -R^{n-1}_{h_0} \\ R^{n-1}_{h_1} & -R^{n-1}_{h_1} \\ \vdots & \vdots \\ R^{n-1}_{h_{2^{n-1}-1}} & -R^{n-1}_{h_{2^{n-1}-1}} \end{pmatrix}$$

$$= \begin{pmatrix} W'_{n-1} & W'_{n-1} \\ \hdashline W'_{n-1} & -W'_{n-1} \end{pmatrix} \qquad (3.5\text{-}6)$$

Q. E. D.

CHAPTER 4

CHARACTERISTIC PARAMETERS

4.1. Introduction

We start this chapter with the introduction of Chow parameters and wind it up with the augmented Chow-vector. In between, the Dertouzos' vector as well as the spectrum of a switching function is introduced. There exists a one-to-one correspondence for a given switching function between the Chow parameters and the Dertouzos' vector, also the augmented Chow-vector and the spectrum. These relations are worked out entirely.

Without the invariance operations, which preserve the single threshold realizability of functions, the usefulness of characteristic parameters will be largely diminished. However, these operations have already been thoroughly discussed by many others. For the sake of completeness, we provide here, in the form of a table, a summary of the results regarding characteristic parameters.

4.2. Chow Parameters

A set of (n+1) parameters, which are named Chow parameters after its contributor, have the potential of characterizing uniquely the threshold functions of n variables. These parameters were first derived by Chow in 1961 (1), and studied by Coleman(2), Winder(17), Kaszerman(10), Dertouzos(3) and many others.

Before going to the definition of Chow parameters, we will first introduce some notations. Let F be a switching function of n variables. The measure of F, written m(F), is the number of true vertices in F. Obviously, it will never be negative. Since there are a total of 2^n vertices for n variables and $F + \overline{F}$ constitutes the universal set of the entire vertices, we have

$$m(F) + (\overline{F}) = 2^n \qquad (4.2\text{-}1)$$

By the reduced function of F along a variable x_i, written F_{x_i}, we mean the function of (n-1) arguments $F(x_1, x_2, \ldots x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. It is well known that F can be decomposed to reduced functions along any one of its variables as follows:

$$F = x_i F_{x_i} + \bar{x}_i F_{\bar{x}_i} \qquad (4.2\text{-}2)$$

Hence, it is trivial that

$$m(F) = m(F_{x_i}) + m(F_{\bar{x}_i}) \qquad (4.2\text{-}3)$$

and $$m(F_{x_i}) + m(\bar{F}_{x_i}) = 2^{n-1} \qquad (4.2\text{-}4)$$

Definition 4.2-1. The (n+1) Chow parameters are defined as:

$$c_0 = m(F) = \langle F_a(X) \rangle$$

$$c_i = m(F_{x_i}) = \langle x_{i,a} F_a(X) \rangle \text{ for } i = 1, 2, \cdots, n \qquad (4.2\text{-}5)$$

Definition 4.2-2. Two switching functions F and H are said to be equipollent, iff they have the same Chow parameters.

We conclude the results of Chow's work by the following theorem:

Theorem 4.2-1. Let two switching functions F and H be equipollent.

(1) If either F or H is a threshold function, then F = H

(2) If either F or H is not a threshold function, then the other is also not a threshold function.

(3) If F and H are distinct, then they are not threshold functions.

We call a variable x_i nonessential to a function F when the function F of n variables can be expressed as one of the following forms:

(1) $F = x_i H$

(2) $F = \bar{x}_i H$

(3) $F = x_i + H$

(4) $F = \bar{x}_i + H$

where H is a function of (n-1) variables and is independent of x_i.

It has been shown by Elgot (4) that F is realizable iff H is realizable. Considering the Chow parameters for the above four cases, it can be easily shown that if one of the parameters c_i is equal to 0, c_0, 2^{n-1} or $c_0 - 2^{n-1}$, then the corresponding variable x_i is nonessential. In what follows, we will be interested in ~~those~~ switching functions of essential variables only.

4. 3. Dertouzos' Characteristic Vector

Definition 4. 3-1 Dertouzos' characteristic vector $\vec{b}$ of a switching function G(Y) of n variables is defined as :

$$\vec{b} = (b_0 ; b_1 , b_2 , \cdots\cdots b_n)$$

$$\text{where } b_0 = \langle G_\alpha(Y) \rangle$$

$$b_i = \langle G_\alpha(Y)\, y_{i,\alpha} \rangle \quad \text{for } i = 1,\ 2, \cdots\cdots n. \qquad (4.3\text{-}1)$$

Comparing this definition with the one for Chow parameters as in Def. 4. 2-1, we will see that they are quite similar to each other. The only difference lies in that the Chow parameters are defined on a configuration of 0 and 1, whereas the Dertouzos' vector is on 1 and -1. Thus the relation between them can be obtained by applying Eq. (2. 2-1).

$$\begin{aligned} b_0 &= \langle G_\alpha(Y) \rangle \\ &= \langle (2F_\alpha(X) - 1) \rangle \\ &= 2\langle F_\alpha(X) \rangle - 2^n \\ &= 2\, c_0 - 2^n \end{aligned} \qquad (4.3\text{-}2)$$

$$\text{or} \quad c_0 = \frac{1}{2}(2^n + b_0) \qquad (4.3\text{-}2a)$$

$$\begin{aligned} \text{and } b_i &= \langle G_\alpha(Y)\, y_{i,\alpha} \rangle \\ &= \langle (2F_\alpha(X) - 1)(2x_{i,\alpha} - 1) \rangle \\ &= \langle 4F_\alpha(X)\, x_{i,\alpha} - 2F_\alpha(X) - 2x_{i,\alpha} + 1 \rangle \\ &= \langle 4F_\alpha(X)\, x_{i,\alpha} - 2x_{i,\alpha} - 2F_\alpha(X) \rangle + 2^n \\ &= 4\, c_i - 2 \cdot 2^{n-1} - 2\, c_0 + 2^n \\ &= 4\, c_i - 2\, c_0 \end{aligned} \qquad (4.3\text{-}3)$$

$$\text{or} \quad c_i = \frac{1}{4}(b_0 + b_i + 2^n) \; . \qquad (4.3\text{-}3a)$$

4.4. The Augmented C - Vector

As mentioned in Chapter 2, Dertouzos has augmented his b-vector from (n+1) elements to 2^n elements and called it the spectrum of the function. The elements of a spectrum were defined by Eq. (2.2-12).

Similarly, we can augment the Chow parameters to a total number of 2^n. And, if we order them in the same manner as the Dertouzos' spectrum, then a vector of 2^n elements is formed. We name such a vector Chow vector or simply c-vector. Therefore, we have

$$\vec{c} = (c_0;\, c_1,\, c_2, \ldots c_n / c_{12},\, c_{13}, \ldots c_{(n-1)n} / \cdots / c_{12\cdots n})$$

where the first (n+1) c_i's for i = 0 to n are Chow parameters defined as Eq. (4.2-5), and

$$c_{i_1 i_2} = m(F_{x_{i_1} x_{i_2}}) = \langle F_a(X)\, x_{i_1,a}\, x_{i_2,a} \rangle$$

$$c_{i_1 i_2 i_3} = m(F_{x_{i_1} x_{i_2} x_{i_3}}) = \langle F_a(X)\, x_{i_1,a}\, x_{i_2,a}\, x_{i_3,a} \rangle$$

$$\vdots$$

$$c_{i_1 i_2 \ldots i_k} = m(F_{x_{i_1} x_{i_2} \ldots x_{i_k}}) = \langle F_a(X)\, x_{i_1,a}\, x_{i_2,a} \cdots x_{i_k,a} \rangle$$

$$\vdots$$

$$c_{i_1 i_2 \ldots i_n} = m(F_{x_{i_1} x_{i_2} \ldots x_{i_n}}) = \langle F_a(X)\, x_{i_1,a}\, x_{i_2,a} \cdots x_{i_n,a} \rangle$$

(4.4-1)

Starting from Eq. (2.2-12) and applying Eq. (2.2-1), we can obtain the relation between $c_{i_1 i_2 \ldots i_k}$ and $b_{i_1 i_2 \ldots i_k}$, for k = 2, 3, $\cdots\cdots$ n.

$$b_{i_1 i_2} = \langle G_a(Y)\, y_{i_1,a}\, y_{i_2,a} \rangle$$

$$= \langle (2F_a(X) - 1)(2x_{i_1,a} - 1)(2x_{i_2,a} - 1) \rangle$$

$$= \langle 8F_a(X)\, x_{i_1,a} x_{i_2,a} - 4x_{i_1,a} x_{i_2,a} - 4F_a(X)\, x_{i_1,a} - 4F_a(X) x_{i_2,a}$$

$$+ 2x_{i_1,a} + 2x_{i_2,a} + 2F_a(X) - 1 \rangle$$

$$= 8c_{i_1 i_2} - 4\cdot 2^{n-2} - 4c_{i_1} - 4c_{i_2} + 2\cdot 2^{n-1} + 2\cdot 2^{n-1} + 2c_0 - 2^n$$

$$= 8c_{i_1 i_2} - 4(c_{i_1} + c_{i_2}) + 2c_0$$

or $$c_{i_1 i_2} = 2^{-3}(b_{i_1 i_2} + b_{i_1} + b_{i_2} + b_0 + 2^n)$$

$$b_{i_1 i_2 i_3} = \langle G_a(Y)\, y_{i_1,a}\, y_{i_2,a}\, y_{i_3,a} \rangle$$

$$= \langle (2F_a(X) - 1)(2x_{i_1,a} - 1)(2x_{i_2,a} - 1)(2x_{i_3,a} - 1) \rangle$$

$$= \langle 16F_a(X) x_{i_1,a} x_{i_2,a} x_{i_3,a} - 8x_{i_1,a} x_{i_2,a} x_{i_3,a} - 8F_a(X)$$

$$x_{i_1,a} x_{i_2,a} - 8F_a(X) x_{i_1,a} x_{i_3,a} - 8F_a(X) x_{i_2,a} x_{i_3,a} +$$

$$4x_{i_1,a} x_{i_2,a} + 4x_{i_1,a} x_{i_3,a} + 4x_{i_2,a} x_{i_3,a} + 4F_a(X) x_{i_1,a}$$

$$+ 4F_a(X) x_{i_2,a} + 4F_a(X) x_{i_3,a} - 2x_{i_1,a} - 2x_{i_2,a} - 2x_{i_3,a}$$

$$-2F_a(X) + 1 \rangle$$

$$= 16c_{i_1 i_2 i_3} - 8\cdot 2^{n-3} - 8c_{i_1 i_2} - 8c_{i_1 i_3} - 8c_{i_2 i_3} + 4\cdot 2^{n-2}$$

$$+ 4\cdot 2^{n-2} + 4\cdot 2^{n-2} + 4c_{i_1} + 4c_{i_2} + 4c_{i_3} - 2\cdot 2^{n-1} - 2\cdot 2^{n-1}$$

$$- 2\cdot 2^{n-1} - 2c_0 + 2^n$$

$$= 16c_{i_1 i_2 i_3} - 8(c_{i_1 i_2} + c_{i_1 i_3} + c_{i_2 i_3}) + 4(c_{i_1} + c_{i_2} + c_{i_3}) - 2c_0$$

or $$c_{i_1 i_2 i_3} = 2^{-4}(b_{i_1 i_2 i_3} + b_{i_1 i_2} + b_{i_1 i_3} + b_{i_2 i_3} + b_{i_1} + b_{i_2} + b_{i_3} + b_0 + 2^n)$$

And this relation can be generalized for all k's from 1 up to n, such as:

$$b_{i_1 i_2 \cdots i_k} = 2^{k+1} c_{i_1 i_2 \cdots i_k} - 2^k \sum_{j(k-1) > j(k-2) > \cdots > j1 \geq 1}^{k} c_{i_{j1} i_{j2} \cdots i_{j(k-1)}} - 2^{k-1} \sum_{j(k-2) > j(k-3) > \cdots > j1 \geq 1}^{k} c_{i_{j1} i_{j2} \cdots i_{j(k-2)}} + \cdots\cdots + (-1)^k 2c_0 \quad (4.4\text{-}2)$$

$$c_{i_1 i_2 \cdots i_k} = 2^{-(k+1)} \{ \sum_{m=1}^{k} \sum_{jm > j(m-1) > \cdots > j1 \geq 1}^{k} b_{i_{j1} i_{j2} \cdots i_{jm}} + b_0 + 2^n \} \quad (4.4\text{-}2a)$$

Thus we show that the augmented c-vector can be converted to the spectrum and vice versa.

Example 4.4-1.

Given a switching function of five variables, whose spectrum is

$\vec{b}$ = (2; 22, 10, 10, 6, 2; -2, -2, 2, -2, 2, -2, 2, -2, 2, -2,; -10, -6, -2, -6, -2, 2, 6, 2, -2, -2; 2, -2, 2, 2, -2; 2),

the corresponding c-vector calculated by applying Eq. (4.3-2a) and Eq. (4.4-2a) is

$\vec{c}$ = (17; 14, 11, 11, 10, 9; 8, 8, 8, 7, 7, 6, 6, 6, 6, 5; 4, 4, 4, 4, 4, 4, 4, 4, 3, 3; 2, 2, 2, 2, 2; 1).

Due to the fact that the c-vector is defined on the configuration of 0 and 1, it is much easier to compute the c_i's than the b's for a given switching function, especially for the case of fewer variables.

There are three systematic ways of computing the c-vector of a switching function. The first one is to obtain the total 2^n-1 reduced functions and calculate the number of true vertices in each of them plus the one in the function itself. This method is good and straightforward for those functions that have five variables or less. Besides, it is restricted to manual computation only.

The second method was first introduced by Golomb(5), and later modified by Dertouzos(3). It was called overlays method by Dertouzos. The details of this method can be found in the literature. The configuration of 1 and -1 were used by Dertouzos. Since the configuration of 0 and 1 is used in the c-vector, a simpler procedure will be expected. This method is also designed for manual evaluation only.

The third method will be called the matrix method. The matrix is a square matrix of order 2^n. The first column of the matrix contains only one's, the next n columns consist of the truth table of n variables, and the rest are formed by column multiplications of the n columns in the truth table. The multiplications shall be taken successively in ascending order, for instance, two at a time, three at a time up to n at a time. The column multiplication is defined as multiplications between the corresponding entries in different columns. Now, express a switching function of n variables as a Boolean vector of 2^n elements and let it be a row vector. Thus, by premultiplying the matrix by the Boolean vector of the function, we will get the c-vector for the corresponding function. Compared to the two previously mentioned methods, this one is more systematic and easier for programming, especially in the case of larger number of variables.

Assigning the job of finding the c-vector for a given function to a computer, the input data should contain n, the number of variables of the function, and the Boolean vector of the function. By specifying n, the computer can generate the matrix automatically. Besides, the number n can be used for computing 2^n and thus define the dimension of the Boolean vector for read-in purpose. Once the matrix is established and the given function is in the required format as a Boolean vector, the computer can execute matrix multiplication and give the result right away.

<u>Example 4.4-2.</u>

Given a switching function F of 4 variables, find the corresponding c-vector.

$$F = x_1(x_2 + x_3 + x_4) + x_3x_4$$

$$= \Sigma m\,(3,5,7,9,11,13,14,15)$$

$$B_F = (0,0,0,1,0,1,0,1,0,1,0,1,0,1,1,1)$$

$$\vec{c} = [0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1] \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0 \\ 1&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&1&1&1&0&0&0&1&1&1&0&0&1&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&1&0&0&1&0&0&0&1&0 \\ 1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&1&1&0&1&0&1&0&0&1&0&0&0 \\ 1&1&1&1&0&1&1&0&1&0&0&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

$$= (8;\ 5,5,5,7;3,3,4,3,4,4;2,2,2,2;1)$$

4.5. Invariance Operations and Positive Prime Functions

By invariance operations, we usually refer to those operations that transform a given switching function in a canonical form to all its variant forms of the same family, and vice versa. In this work, the family will be the one of single threshold realizable switching functions. Alternatively speaking, invariance operations are those operations that preserve single threshold realizability.

There are altogether five invariance operations, namely, the complementation of input variables, the permutation of variables, the complementation of the function, dualization of the function and the equidualization. Their properties as well as the effects on the characteristic vectors have been discussed thoroughly by Dertouzos(3). We will omit the details and provide only a summary by Table 4.5-1 shown in the next page.

In Table 4.5-1, we list only the corresponding change in the spectrum of the switching function. It can be seen that the changes are simple for any one of the operations. There does exist a corresponding change in the augmented c-vector of the switching function for each operation due to the fact that there is a one-to-one correspondence between the spectrum and the augmented c-vector. However, the relation will be more complicated than the one for the spectrum. Thus, so far as invariance operations are concerned, we have to use the spectrum of a function as a springboard.

Definition 4.5-1.

A Boolean function F of n variables is said to be prime if the following inequalities hold:

$$2^{n-2} \geq c_0/2 \geq c_1 \geq c_2 \geq \cdots\cdots \geq c_n \geq 1$$

This definition was first used by Chow in his famous work(1) for threshold functions. Since the aforementioned invariance operations preserve the realizability of a single threshold function, therefore, any non-prime function of no non-essential variables can be transformed to a prime function by appropriate invariance operations without losing its realizability.

However, the weights of all prime functions defined above will have negative values as far as single threshold realization is concerned. Because of the convenience in dealing with positive numbers rather than the negatives, we will modify the definition as follows:

Invariance Operations	Functional Change	Change in Spectrum	Change in Weight-Threshold
Original Function and Vectors	$F = F(x_1, x_2, \ldots, x_n)$	$\vec{b} = (b_0; b_1, \ldots b_n; b_{12}, \ldots; b_{1\ldots n})$	$w_1,\ w_2,\ \ldots,\ w_n;\ T$
Complementation of an input variable, x_i	$F' = F(x_1, \ldots \bar{x}_i, \ldots x_n)$	Change sign of all spectral terms having i in their subscript.	$w_1, \ldots -w_i, \ldots w_n;\ T - w_i$
Permutation of input variables, x_i and x_j	$F' = F(x_1, \ldots x_j, x_i, \ldots x_n)$	Interchange i with j and j with i in subscripts of spectral terms.	$w_1, \ldots w_j, w_i, \ldots w_n;\ T$
Complementation of the function	$F' = \overline{F(x_1, x_2, \ldots, x_n)}$	Change signs of all spectral terms.	$-w_1,\ -w_2, \ldots,\ -w_n;\ -T+1$
Dualization of the function	$F' = \overline{F(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)}$	Change signs of all spectral terms having an even number of subscripts.	$w_1, w_2, \ldots, w_n;\ \sum_{i=1}^{n} w_i - T$
Equidualization of the function	$F' = \overline{x_k \oplus F(\overline{x_1 \oplus x_k}, \ldots \overline{x_n \oplus x_k})}$	Append k to subscripts having even number of literals other than k	$w_1, \ldots w_{k-1},\ \sum_{i=1}^{n} w_i - 2T, w_{k+1} \ldots w_n; \sum_{i=1, i \neq k}^{n} w_i - T$

Table 4.5-1

Definition 4. 5-2.

A Boolean function of n variables is said to be positive prime if the following inequalities hold:

$$2^{n-1} > c_1 \geq c_2 \geq \cdots\cdots \geq c_{n-1} \geq c_n \geq c_0/2 \geq 2^{n-2}.$$

Actually, the prime function and the positive prime function are interchangeable by one of the invariance operations --- the complementation of functions. Thus, we can also conclude that any non-positive-prime functions of no non-essential variables can be transformed to a positive prime function by appropriate invariance operations and preserving the single threshold realizability.

CHAPTER 5

A TESTING AND REALIZING METHOD FOR THRESHOLD FUNCTIONS BY USING AUGMENTED C-VECTOR

5.1. Introduction

The problem of testing and realizing threshold functions is a practical requirement for logical design with threshold elements. For a switching function of 6 variables or less, the problem can be solved simply by first calculating the Chow parameters and then applying table-look-up techniques on the existing table(17) for solutions. There does exist such a table for 7 or 8 variables. However, due to the fact that the number of threshold functions increases rapidly, the table-look-up will be cumbersome and thus lose its original merit of simplicity.

For a function of larger number of variables, the normal procedure is to generate a system of linear inequalities by the definition of a threshold function, then reduce it, and finally solve it. Our method is based on the same principles. However, the augmented c-vector provides a powerful tool to help set up the reduced set of inequalities. The general methods for solving a set of inequalities have been summarized in detail by Hu(9).

In the first 5 sections of this chapter we will make a brief discussion on the relating properties of threshold functions. The testing and realizing method is developed in the subsequent sections. Examples are provided for illustration.

5.2. Properties of Threshold Functions

Unateness, monotonicity and asummability are the three important properties of a threshold function. They have already been studied extensively by many authors working in this field. It becomes a necessity to investigate them for testing and realizing of threshold functions. Thus, before going to develop a new testing and realizing method we will give a brief summary on these properties. Only those properties relevant to our testing and realizing method are covered.

Definition 5.2-1. A switching function F of n variables is said to be unate iff F has an irredundant normal disjointive form (abbreviated INDF) in which none of the variables appears both in complemented and uncomplemented form.

Example 5.2-1.

(A) $F(x_1, x_2, x_3) = x_1 + \bar{x}_2 x_3$ is a unate function.

(B) $F(x_1, x_2, x_3, x_4) = x_1 x_2 + \bar{x}_2 x_3 + \bar{x}_1 x_4$ is not a unate function.

Since all threshold functions are unate functions, the set of all threshold function forms a proper subset of unate functions. Alternatively speaking, unateness is a necessary condition for threshold functions. Due to the fact that the unateness of a given switching function is relatively easy to detect, it turns out to be the first step for the testing of a threshold function.

Definition 5.2-2. A switching function F of n variables is said to be k-monotonic, iff the reduced functions of F expanded along any i variables for $1 \leq i \leq k$, are comparable, and it is said to be completely monotonic iff k = n.

Example 5.2-2.

(A) $F(x_1, x_2, x_3) = x_1 + \bar{x}_2 x_3$

$F_{x_1} = 1,$ $\quad F_{\bar{x}_1} = \bar{x}_2 x_3$ $\quad$ implies $\quad F_{\bar{x}_1} \subset F_{x_1}$

$F_{x_2} = x_1$, $F_{\bar{x}_2} = x_1 + x_3$ implies $F_{x_2} \subset F_{\bar{x}_2}$

$F_{x_3} = x_1 + \bar{x}_2$, $F_{\bar{x}_3} = x_1$ implies $F_{\bar{x}_3} \subset F_{x_3}$

These conclude that F is 1-monotonic.

$F_{x_1x_2} = 1$, $F_{x_1\bar{x}_2} = 1$, $F_{\bar{x}_1x_2} = 0$, $F_{\bar{x}_1\bar{x}_2} = x_3$ implies

$$F_{\bar{x}_1x_2} \subset F_{\bar{x}_1\bar{x}_2} \subset F_{x_1\bar{x}_2} \subseteq F_{x_1x_2}$$

$F_{x_1x_3} = 1$, $F_{x_1\bar{x}_3} = 1$, $F_{\bar{x}_1x_3} = \bar{x}_2$, $F_{\bar{x}_1\bar{x}_3} = \bar{x}_2$ implies

$$F_{\bar{x}_1\bar{x}_3} \subseteq F_{\bar{x}_1x_3} \subset F_{x_1\bar{x}_3} \subseteq F_{x_1x_3}$$

$F_{x_2x_3} = x_1$, $F_{x_2\bar{x}_3} = x_1$, $F_{\bar{x}_2x_3} = 1$, $F_{\bar{x}_2\bar{x}_3} = x_1$ implies

$$F_{x_2\bar{x}_3} \subseteq F_{x_2x_3} \subseteq F_{\bar{x}_2\bar{x}_3} \subset F_{\bar{x}_2x_3}$$

These conclude that F is 2-monotonic.

$F_{x_1x_2x_3} = 1$, $F_{x_1x_2\bar{x}_3} = 1$, $F_{x_1\bar{x}_2x_3} = 1$, $F_{x_1\bar{x}_2\bar{x}_3} = 1$,

$F_{\bar{x}_1x_2x_3} = 0$, $F_{\bar{x}_1x_2\bar{x}_3} = 0$, $F_{\bar{x}_1\bar{x}_2x_3} = 1$, $F_{\bar{x}_1\bar{x}_2\bar{x}_3} = 0$ implies

$$F_{\bar{x}_1\bar{x}_2\bar{x}_3} \subseteq F_{\bar{x}_1x_2\bar{x}_3} \subseteq F_{\bar{x}_1x_2x_3} \subset F_{\bar{x}_1\bar{x}_2x_3} \subseteq F_{x_1\bar{x}_2\bar{x}_3} \subseteq F_{x_1\bar{x}_2x_3}$$
$$\subseteq F_{x_1x_2\bar{x}_3} \subseteq F_{x_1x_2x_3}$$

Therefore, F is complete monotonic.

(B) $F = x_1x_2 + x_3x_4$

$F_{x_1} = x_2 + x_3x_4$ $F_{\bar{x}_1} = x_3x_4$ implies $F_{\bar{x}_1} \subset F_{x_1}$

$F_{x_2} = x_1 + x_3x_4$ $F_{\bar{x}_2} = x_3x_4$ implies $F_{\bar{x}_2} \subset F_{x_2}$

$F_{x_3} = x_1x_2 + x_4$ $F_{\bar{x}_3} = x_1x_2$ implies $F_{\bar{x}_3} \subset F_{x_3}$

$F_{x_4} = x_1x_2 + x_3$ $F_{\bar{x}_4} = x_1x_2$ implies $F_{\bar{x}_4} \subset F_{x_4}$

These show that F is 1-monotonic.

But $F_{x_1x_3} = x_2 + x_4$, $F_{x_1\bar{x}_3} = x_2$, $F_{\bar{x}_1x_3} = x_4$, $F_{\bar{x}_1\bar{x}_3} = 0$

$\because$ $F_{x_1\bar{x}_3}$ and $F_{\bar{x}_1\bar{x}_3}$ are not comparable

$\therefore$ F is not 2-monotonic.

It can be seen that monotonicity is another necessary condition for threshold functions. And it is a more restrictive one than unateness. In fact, 1-monotonicity is equivalent to unateness. As shown in the examples, we check k-monotonicity in an ascending order of k. This also implies that the unateness is the first step for testing threshold functions. Besides, it is well known that an $[n/2]$ - monotonic switching function F of n variables is completely monotonic, where $[n/2]$ is the greatest integer $\leq n/2$. Thus, for complete monotonicity we have to check k-monotonicity starting from $k = 1$ up to $[n/2]$ only. Next, we wish to state two important theorems which will be used frequently in the following testing and realizing method. The proofs for them can be found in Sheng's book(15).

<u>Theorem 5.2-1.</u> Let F be a 2-monotonic switching function of n variables, $F_{x_j\bar{x}_k}$ and $F_{\bar{x}_jx_k}$ be two reduced functions of F expanded along two arbitrary variables x_j and x_k, and c_j, c_k be the coressponding Chow parameters for these two variables. Then

$$F_{x_j\bar{x}_k} \subset F_{\bar{x}_jx_k} \qquad \text{iff} \quad c_j > c_k$$

$$F_{x_j\bar{x}_k} = F_{\bar{x}_jx_k} \qquad \text{iff} \quad c_j = c_k .$$

<u>Theorem 5.2-2.</u> The reduced functions of a threshold function of n variables, expanded along any one subset of variables can be realized by the same weight vector.

Definition 5.2-3. A switching function F of n variables is said to be asummable, iff for some integers $j \geq 2$, it is impossible to have j vertices $\{X_1, \cdots\cdots X_j\}$ in F and j vertices $\{Y_1, \cdots\cdots Y_j\}$ in $\bar{F}$, not necessarily distinct, to fulfill the following condition:

$$\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} Y_i .$$

Asummability, which is derived from the concept of disjoint convex hulls, is a necessary and sufficient condition for threshold functions. However, it is hard to apply.

5.3. Some Aspects of Set Theory

A set is a collection of objects which have common and distinguishing properties. The objects of the set will be called its elements. If all elements of a set M are at the same time the elements of N, then M is called a subset of N denoted as $M \subseteq N$. Among all the sets there exists a greatest set S called space or universe, which contains all the sets. In other words, all the sets will be subsets of the space S. For example, a switching function F of n variables can be considered as a set of true vertices in the n-space S_n. This space has 2^n vertices formed by all kinds of combinations of the n variables. The reduced function F_{x_i} of F is a function of (n-1) variables and thus a set of true vertices in the (n-1)-space. However, we can consider F_{x_i} as a function of n variables independent of the variable x_i. Similarly, the function x_i can be considered as a function of n variables independent of the rest (n-1) variables. In order to have the reduced function being a set in the n-space as the function F, we shall take the intersection of x_i and F_{x_i}, where both of them are considered as functions of n variables, that is, sets in n-space. Then, all these products or intersections such as $x_i F_{x_i}$, $\bar{x}_i F_{\bar{x}_i}$, and $x_{i_1} \ldots x_{i_j} \bar{x}_{k_1} \ldots \bar{x}_{k_m} F_{x_{i_1} \ldots x_{i_j} \bar{x}_{k_1} \ldots \bar{x}_{k_m}}$ are subsets of the set F. The set that contains no elements will be called empty set or null set denoted by $\emptyset$. Naturally, it is a subset of any other sets.

Now let us consider any two arbitrary sets M and N, and define the different kinds of set operations. (1) If M is a subset of N, and N is a a subset of M, then they are said to be equal denoted as $M = N$. (2) The union $M \cup N$ is a set whose elements belong to at least one of the sets M and N. (3) The intersection $M \cap N$ is a set whose elements are common to M and N. (4) M and N are said to be mutually exclusive (or disjoint) if they have no common elements, i.e., if their intersection is empty. (5) The complement $\overline{M}$ of a set M is defined as the set consisting of all elements of the space S that are not in M. (6) The difference $M - N$ is a set consisting of the elements of M that are not in N. Thus,

$$M - N = M\overline{N} = M - MN$$

Theorem 5.3-1. If $M_1, M_2, \ldots M_m$ and $N_1, N_2, \ldots N_n$ all are the sets in the space S, then the following identity holds.

$$M_1M_2\ldots M_m\overline{N}_1\overline{N}_2\ldots\overline{N}_n = M_1M_2\ldots M_m - \sum_{i_1=1}^{n} M_1M_2\ldots M_mN_{i_1} + \sum_{i_2>i_1\geq 1}^{n} M_1\ldots M_mN_{i_1}N_{i_2} - \sum_{i_3>i_2>i_1\geq 1}^{n} M_1\ldots M_mN_{i_1}N_{i_2}N_{i_3} + \cdots + (-1)^k \sum_{i_k>i_{k-1}>\cdots>i_1\geq 1}^{n} M_1\ldots M_mN_{i_1}\ldots N_{i_k} + \cdots + (-1)^n M_1\ldots M_mN_1\ldots N_n \quad (5.3\text{-}1)$$

Proof: Let $M_1M_2\ldots M_m$ be replaced by M and we will prove this theorem by induction.

For $n = 1$ $\quad M\bar{N}_1 = M - MN_1$

For $n = 2$ $\quad M\bar{N}_1\bar{N}_2 = (M - MN_1)\bar{N}_2 = (M - MN_1) - (M - MN_1)N_2$

$$= M - \sum_{i_1=1}^{2} MN_{i_1} + MN_1N_2$$

$\vdots$

Assume that the theorem is true for n, we are going to prove that it is also true for $(n+1)$.

$$M\bar{N}_1\bar{N}_2\ldots\bar{N}_n\bar{N}_{n+1} = [M - \sum_{i_1=1}^{n} MN_{i_1} + \sum_{i_2>i_1\geq 1}^{n} MN_{i_1}N_{i_2} - \cdots\cdots$$

$$+(-1)^k \sum_{i_k>i_{k-1}>\cdots>i_1\geq 1}^{n} MN_{i_1}\ldots N_{i_k} + \cdots\cdots$$

$$+(-1)^n MN_1\ldots N_n]\bar{N}_{n+1}$$

$$= M - \sum_{i_1=1}^{n} MN_{i_1} + \sum_{i_2>i_1\geq 1}^{n} MN_{i_1}N_{i_2} + \cdots + (-1)^k$$

$$\sum_{i_k>i_{k-1}>\cdots>i_1\geq 1}^{n} MN_{i_1}\ldots N_{i_k} + \cdots + (-1)^n MN_{i_1}\ldots$$

$$N_n - [MN_{n+1} - (\sum_{i_1=1}^{n} MN_{i_1})N_{n+1} + (\sum_{i_2>i_1\geq 1}^{n} MN_{i_1}N_{i_2})$$

$$N_{n+1} - \cdots + (-1)^k(\sum_{i_k>\cdots>i_1\geq 1}^{n} MN_{i_1}\ldots N_{i_k})N_{n+1}$$

$$+ \cdots + (-1)^n MN_1N_2\ldots N_nN_{n+1}]$$

$$= M - \sum_{i_1=1}^{n+1} MN_{i_1} + \sum_{i_2>i_1\geq 1}^{n+1} MN_{i_1}N_{i_2} - \cdots + (-1)^k$$

$$\sum_{i_k>\cdots>i_1\geq 1}^{n+1} MN_{i_1}\ldots N_{i_k} + \cdots + (-1)^{n+1} MN_{i_1}\ldots N_{n+1}$$

Therefore, we conclude that Eq. (5. 3-1) is valid for all n.

Q. E. D.

5.4. Weight Vectors of Threshold Functions

As defined in Chapter 2, a weight vector for a threshold function of n variables is a set of n real numbers which meets the requirement of Eq. (2. 3-1). We will call a weight vector $\vec{w} = (w_1, w_2, \ldots w_n)$ non-negative iff $w_i \geq 0$ for $i = 1, 2, \cdots n$. It is said to be canonical iff $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Furthermore, it is said to be integral iff all its elements are integers. The aim of realizing a threshold function is to find such a weight vector together with a threshold under certain criterion. The most commonly used criterion is to minimize the summation over the absolute values of all the elements of a weight vector and the threshold. If the weight vector for a certain threshold function is integral and a minimum under the above-mentioned criteria, then such a vector plus the threshold will be called minimal-integral realization for the threshold function. In the following section we are going to develop a testing and realizing method for obtaining such a realization.

Chow(1) has shown that the weight vector of a threshold function is strongly related to the Chow parameters. We will restate his results in a theorem without proof.

Theorem 5.4-1.

Let F be a threshold function of n variables, $\vec{w} = (w_1, w_2, \ldots w_n)$ be its weight vector, and $c_0, c_1, c_2, \ldots c_n$ be the corresponding Chow parameters. Then, for $i, j, = 1, 2, \cdots n$ and $i \neq j$

(1) if $c_i < c_0/2$, then $w_i < 0$

(2) if $c_i > c_0/2$, then $w_i > 0$

(3) if $c_i = c_0/2$, then F is not dependent on x_i

(4) if $c_i < c_j$, then $w_i < w_j$ and

(5) if $c_i = c_j$, then it is admissable that $w_i = w_j$.

Since we are interested in minimal-integral realization of a threshold function, an extended relation between the weights will be found useful under this constraint. For the following theorem, we will establish equality relations instead of inequality ones between the weights. And by this theorem the number of inequalities as well as the number of unknowns will be reduced, thus the problem for realizing a threshold function will be simplified.

Theorem 5.4-2. Let F be a threshold function of n variables, $\vec{w} = (w_1, w_2, \ldots w_n)$ be its minimal-integral weight vector, and $c_1, c_2, \ldots c_n$ be the corresponding Chow parameters. If $c_i = c_j + 1$, for i, j = 1, 2, $\cdots$ n and $i \neq j$, then $w_i = w_j + 1$.

Proof: Since $c_i = c_j + 1$, that is $c_i > c_j$, therefore by Theorem 5.4-1. we have $w_i > w_j$.

According to the definition of Chow parameters, all the c_i's are nonnegative integers. Thus, if $c_i = c_j + 1$, there will not exist any c_k such that $c_i > c_k > c_j$. By Theorem 5.4-1., this implies that there will not have such inequalities as $w_i > w_k > w_j$.

Therefore, for a minimal-integral realization the difference between w_i and w_j should be a minimum and an integer, that is 1. This proves that $w_i = w_j + 1$.

Q. E. D.

5.5 The Incremental Weights

For a positive prime function of n variables its weight vector can be ordered as the following according to Theorem 5.4-1.

$$w_1 \geq w_2 \cdots \geq w_n \geq 0$$

If we introduce the incremental weights which are defined as the differences between sucessive weights as shown below.

$$\begin{aligned} \Delta w_1 &= w_1 - w_2 \\ \Delta w_2 &= w_2 - w_3 \\ &\vdots \\ \Delta w_{n-2} &= w_{n-2} - w_{n-1} \\ \Delta w_{n-1} &= w_{n-1} - w_n \end{aligned} \qquad (5.5\text{-}1)$$

then, we can express all the weights in term of the incremental weights and the minimum weight w_n as follows:

$$\begin{aligned} w_1 &= \Delta w_1 + \Delta w_2 + \Delta w_3 + \cdots\cdots \Delta w_{n-1} + w_n \\ w_2 &= \Delta w_2 + \Delta w_3 + \cdots\cdots \Delta w_{n-1} + w_n \\ &\vdots \\ w_{n-2} &= \Delta w_{n-2} + \Delta w_{n-1} + w_n \\ w_{n-1} &= \Delta w_{n-1} + w_n \\ w_n &= w_n \end{aligned} \qquad (5.5\text{-}2)$$

Now, if we are going to solve a set of inequalities about the weights, we can first replace them by the incremental and the minimum weights and then solve for these new variables instead. Normally, the change of variables will lead to a simpler set of inequalities and thus more easily solved. This will be seen in the following sections.

5.6. Establishing Inequalities by Using the Augmented C-Vector

Let F be a switching function of n variables. Expanding F along any one of the variables, say x_{k_1}, we have

$$F = x_{k_1} F_{x_{k_1}} + \bar{x}_{k_1} F_{\bar{x}_{k_1}}$$

or

$$\bar{x}_{k_1} F_{\bar{x}_{k_1}} = F - x_{k_1} F_{x_{k_1}} .$$

This relation can be applied successively for two or more variables. For two variables,

$$x_{i_1} F_{x_{i_1}} = x_{i_1} (x_{k_1} F_{x_{i_1} x_{k_1}} + \bar{x}_{k_1} F_{x_{i_1} \bar{x}_{k_1}})$$

$$x_{i_1} \bar{x}_{k_1} F_{x_{i_1} \bar{x}_{k_1}} = x_{i_1} (F_{x_{i_1}} - x_{k_1} F_{x_{i_1} x_{k_1}})$$

and

$$\bar{x}_{k_1} F_{\bar{x}_{k_1}} = F - x_{k_1} F_{x_{k_1}} = \bar{x}_{k_1} (x_{k_2} F_{\bar{x}_{k_1} x_{k_2}} + \bar{x}_{k_2} F_{\bar{x}_{k_1} \bar{x}_{k_2}})$$

$$\begin{aligned}
\bar{x}_{k_1} \bar{x}_{k_2} F_{\bar{x}_{k_1} \bar{x}_{k_2}} &= F - x_{k_1} F_{x_{k_1}} - \bar{x}_{k_1} x_{k_2} F_{\bar{x}_{k_1} x_{k_2}} \\
&= F - x_{k_1} F_{x_{k_1}} - x_{k_2} (F_{x_{k_2}} - x_{k_1} F_{x_{k_1} x_{k_2}}) \\
&= F - x_{k_1} F_{x_{k_1}} - x_{k_2} F_{x_{k_2}} - x_{k_1} x_{k_2} F_{x_{k_1} x_{1_2}} ,
\end{aligned}$$

Similarly, for three variables,

$$x_{i_1} x_{i_2} \bar{x}_{k_1} F_{x_{i_1} x_{i_2} \bar{x}_{k_1}} = x_{i_2} x_{i_2} (F_{x_{i_1} x_{i_2}} - x_{k_1} F_{x_{i_1} x_{i_2} x_{k_1}})$$

$$\begin{aligned}
x_{i_1} \bar{x}_{k_1} \bar{x}_{k_2} F_{x_{i_1} \bar{x}_{k_1} \bar{x}_{k_2}} = x_{i_1} (F_{x_{i_1}} &- x_{k_1} F_{x_{i_1} x_{k_1}} - x_{k_2} F_{x_{i_1} x_{k_2}} \\
&+ x_{k_1} x_{k_2} F_{x_{i_1} x_{k_1} x_{k_2}})
\end{aligned}$$

$$\bar{x}_{k_1}\bar{x}_{k_2}\bar{x}_{k_3}F_{\bar{x}_{k_1}\bar{x}_{k_2}\bar{x}_{k_3}} = F - x_{k_1}F_{x_{k_1}} - x_{k_2}F_{x_{k_2}} - x_{k_3}F_{x_{k_3}} + x_{k_1}x_{k_2}F_{x_{k_1}x_{k_2}} + x_{k_1}x_{k_3}F_{x_{k_1}x_{k_3}} + x_{k_2}x_{k_3}F_{x_{k_2}x_{k_3}} - x_{k_1}x_{k_2}x_{k_3}F_{x_{k_1}x_{k_2}x_{k_3}}.$$

To generalize this relation for any number of variables less than n, we will obtain ~~the similar~~ a result ~~as in Theorem 5.3-1.~~ similar to that of Th. 5.3-1.

$$x_{i_1}\cdots x_{i_j}\bar{x}_{k_1}\cdots\bar{x}_{k_s}F_{x_{i_1}\cdots x_{i_j}\bar{x}_{k_1}\cdots\bar{x}_{k_s}} = x_{i_1}\cdots x_{i_j}\Big(F_{x_{i_1}\cdots x_{i_j}} - \sum_{m1=1}^{s} x_{k_{m1}}F_{x_{i_1}\cdots x_{i_j}x_{k_{m1}}} + \sum_{m2>m1\geq 1}^{s} x_{k_{m1}}x_{k_{m2}}F_{x_{i_1}\cdots x_{i_j}x_{k_{m1}}x_{k_{m2}}} - \cdots + (-1)^s x_{k_1}\cdots x_{k_s}F_{x_{i_1}\cdots x_{i_j}x_{k_1}\cdots x_{k_s}}\Big) \quad (5.6\text{-}1)$$

It is quite obvious but very important that all the subscripts of the oolean variables shall be distinct.

Extending the relation in Eq. (4.4-1) to the complemented form of variables, we can denote $c_{i_1\cdots i_j\bar{k}_1\cdots\bar{k}_s} = m(F_{x_{i_1}\cdots x_{i_j}\bar{x}_{k_1}\cdots\bar{x}_{k_s}}) = \langle F_a(X)\, x_{i_1,a}\cdots x_{i_j,a}\bar{x}_{k_1,a}\cdots\bar{x}_{k_s,a}\rangle$, where a is the decimal index of a vertex as defined in Chapter 2 and $m(F_{x_{i_1}\cdots x_{i_j}\bar{x}_{k_1}\cdots\bar{x}_{k_s}})$ denotes the number of vertices in the (n - j - s)-variables reduced function $F_{x_{i_1}\cdots x_{i_j}\bar{x}_{k_1}\cdots\bar{x}_{k_s}}$. However, we can treat the reduced function as a function of n variables in which $x_{i_1}, x_{i_2}, \cdots x_{i_j}$ and $x_{k_1}, \cdots x_{k_s}$ are non-essential variables. By this reasoning, we have

$$c_{i_1 \dots i_j \bar{k}_1 \dots \bar{k}_s} = m(F_{x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s}})$$

$$= \langle x_{i_1,a} \dots x_{i_j,a} \bar{x}_{k_1,a} \dots \bar{x}_{k_s,a} F_a(X) \rangle$$

$$= \langle x_{i_1,a} \dots x_{i_j,a} \bar{x}_{k_1,a} \dots \bar{x}_{k_s,a} (x_{i_1,a} \dots x_{i_j,a} \bar{x}_{k_1,a} \dots \bar{x}_{k_s,a} F_{x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s},a} + \Sigma x^*_{i_1,a} \dots x^*_{i_j,a} \bar{x}^*_{k_1,a} \dots \bar{x}^*_{k_s,a} F_{x^*_{i_1} \dots x^*_{i_j} \bar{x}^*_{k_1} \dots \bar{x}^*_{k_s},a}) \rangle$$

$$= \langle x_{i_1,a} \dots x_{i_j,a} \bar{x}_{k_1,a} \dots \bar{x}_{k_s,a} F_{x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s},a} \rangle \tag{5.6-2}$$

where $\Sigma x^*_{i_1} \dots x^*_{i_j} \bar{x}^*_{k_1} \dots \bar{x}^*_{k_s} = \overline{x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s}}$, and $F_{x^*_{i_1} \dots x^*_{i_j} \bar{x}^*_{k_1} \dots \bar{x}^*_{k_s}}$'s are the corresponding reduced functions. It is not difficult to see that each term of the summation is a disjoint set with the set $x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s}$.

Now, if we sum up Eq. (5.6-1) over the total space of 2^n vertices and substitute Eq.(5.6-2) into it, then we will have,

$$c_{i_1 \dots i_j \bar{k}_1 \dots \bar{k}_s} = m(F_{x_{i_1} \dots x_{i_j} \bar{x}_{k_1} \dots \bar{x}_{k_s}})$$

$$= m(F_{x_{i_1} \dots x_{i_j}}) - \sum_{m1=1}^{s} m(F_{x_{i_1} \dots x_{i_j} x_{k_{m1}}})$$

$$+ \sum_{m2 > m1 \geq 1}^{s} m(F_{x_{i_1} \dots x_{i_j} x_{k_{m1}} x_{k_{m2}}}) + \cdots$$

$$+ (-1)^s m(F_{x_{i_1} \dots x_{i_j} x_{k_1} \dots x_{k_s}})$$

$$= c_{i_1 \ldots i_j} - \sum_{m1=1}^{s} c_{i_1 \ldots i_j k_{m1}} + \sum_{m2>m1\geq 1}^{s} c_{i_1 \ldots i_j k_{m1} k_{m2}}$$

$$\ldots + (-1)^s c_{i_1 \ldots i_j k_1 \ldots k_s} \qquad (5.6\text{-}3)$$

From the results we have derived so far, we can see that the c-vector of a given switching function contains all the information about the function and thus we might conclude that the c-vector completely characterizes the function. Before establishing the inequalities between the weights assigned to each variable of the function, we are going to prove a lemma and a theorem.

Lemma 5.6-1. If $F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}}$ is one of the reduced functions of a switching function F of n variables, where $s \leq n$, and if $(u_{s+1}, u_{s+2}, \ldots u_n)$ is a false (true) vertex of $F_{x_{i_1} \ldots x_{i_j} \bar{x}_{j+1} \ldots \bar{x}_s}$, then the vertex $V_u = (\underbrace{1, 1, \cdots 1}_{j}, \underbrace{0, 0, \cdots 0}_{(s-j)}, u_{s+1}, u_{s+2}, \cdots u_n)$ is a false (true) vertex of F, where u_i's $\in (0, 1)$.

Proof: The function F can be expanded as follows,

$$F = x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s} F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}} + \Sigma x^*_{i_1} \ldots x^*_{i_s} F_{x^*_{i_1} \ldots x^*_{i_s}}$$

where $\Sigma x^*_{i_1} \ldots x^*_{i_s} = \overline{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}}$ and $F_{x^*_{i_1} \ldots x^*_{i_s}}$ are the corresponding reduced functions.

If $(u_{s+1}, \ldots, u_n)$ is a true vertex of $F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}}$, then it is quite obvious to see that

$F(V_u) = 1$, i.e., V_u is a true vertex of F.

If $(u_{s+1}, \ldots, u_n)$ is a false vertex of $F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}}$, then for the vertex V_u $\Sigma x^*_{i_1} \ldots x^*_{i_s} = \overline{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}} = 0$. This implies that each term in the summation is equal to zero. Thus,

$F(V_u) = 0$, i.e., V_u is a false vertex of F.

Q. E. D.

Theorem 5.6-1. For a threshold function F of n variables, if

$$c_{i_1 \ldots i_j \bar{i}_{j+1} \ldots \bar{i}_s} > c_{\bar{i}_1 \ldots \bar{i}_j i_{j+1} \ldots i_s},$$

then $w_{i_1} + w_{i_2} + \cdots + w_{i_j} > w_{i_{j+1}} + w_{i_{j+2}} + \cdots + w_{i_s}$.

Proof: Since F is a threshold function, all its reduced functions expanded along the same variables are comparable.

$$\because \ c_{i_1 \ldots i_j \bar{i}_{j+1} \ldots \bar{i}_s} > c_{\bar{i}_1 \ldots \bar{i}_j i_{j+1} \ldots i_s}$$

$$\therefore \ F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}} \supset F_{\bar{x}_{i_1} \ldots \bar{x}_{i_j} x_{i_{j+1}} \ldots x_{i_s}}$$

Let $(u_{s+1}, \cdots u_n)$ be a true vertex of $F_{x_{i_1} \ldots x_{i_j} \bar{x}_{i_{j+1}} \ldots \bar{x}_{i_s}} \cap \bar{F}_{\bar{x}_{i_1} \ldots \bar{x}_{i_j} x_{i_{j+1}} \ldots x_{i_s}}$. Then, by lemma 5.6-1, $(\underbrace{1, 1, \cdots 1}_{j}, \underbrace{0, 0, \cdots 0}_{(s-j)}, u_{s+1}, \cdots u_n)$ is a true vertex of F while $(\underbrace{0, 0, \cdots 0}_{j}, \underbrace{1, 1, \cdots 1}_{(s-j)}, u_{s+1}, \cdots u_n)$ is a false vertex of F.

Thus, if $(w_{i_1}, w_{i_2}, \cdots w_{i_s}, w_{i_{s+1}}, \cdots w_n; T)$ is the weight threshold vector of the function F, then by the definition of a threshold function (Def. 2.3-1), we have

$$w_{i_1} + \cdots + w_{i_j} + \sum_{i=s+1}^{n} u_i w_i \geq T$$

$$w_{i_{j+1}} + \cdots + w_{i_s} + \sum_{i=s+1}^{n} u_i w_i < T$$

$$\therefore\ w_{i_1} + w_{i_2} + \cdots + w_{i_j} > w_{i_{j+1}} + w_{i_{j+2}} + \cdots + w_{i_s}$$

Q. E. D.

5.7. A Testing and Realizing Method for Threshold Functions

A testing and realizing method for threshold functions will be developed in this section. Without losing any generality due to the invariance operations, we will assume that all the switching functions to be tested are positive prime functions. By using any one of the methods we stated in Section 4.4, the augmented c-vector of the function can be calculated. According to the definition of a positive prime function, we will find the following ordering existentbetween the first (n+1) elements of the c-vector:

$$2^{n-1} - 1 \geq c_1 \geq c_2 \geq \cdots\cdots \geq c_{n-1} \geq c_n \geq c_0/2 \geq 2^{n-2} \qquad (5.7\text{-}1)$$

Now, let us consider the n reduced functions of F, namely, F_{x_1}, F_{x_2}, F_{x_3}, ... F_{x_n}. Take any one of these reduced functions, say F_{x_i}, and relate the Chow parameters for F_{x_i} to the elements of c-vector for F. In order to distinguish one from the other, we will use a superscript i plus the brackets for the Chow parameters of F_{x_i}. Thus by Definition 4.2-5 and Eq.(4.4-1), we have

$$\begin{aligned} c_0^{(i)} &= m(F_{x_i}) = c_i \\ c_j^{(i)} &= m[(F_{x_i})_{x_j}] = m(F_{x_i x_j}) = c_{ij} \quad \text{for } j \neq i;\ i,j = 1, 2 \cdots n \end{aligned} \qquad (5.7\text{-}2)$$

It is known that if F is a threshold function, then all it s reduced functions are threshold functions. Furthermore, by theorem 5.2-2 we can conclude that if F is a positive prime threshold function, then all its reduced functions are also positive prime threshold functions. Thus we have the following ordering for F_{x_i}.

$$c_1^{(i)} \geq c_2^{(i)} \geq \cdots\cdots \geq c_{i-1}^{(i)} \geq c_{i+1}^{(i)} \geq \cdots\cdots \geq c_n^{(i)} \geq c_0^{(i)}/2 \qquad (5.7\text{-}3)$$

By applying Eq. (5.7-2), we have

$$c_{i1} \geq c_{i2} \geq \cdots\cdots \geq c_{i(i-1)} \geq c_{i(i+1)} \geq \cdots\cdots \geq c_{in} \geq c_i/2 \qquad (5.7\text{-}4)$$

A testing table can now be established. The table will have n rows starting from one upto n and (n+1) columns starting from 1 upto (n+1). The (n+1)th column of the table will be the halves of the c_i's, whereas the entry in the ith row and jth column will be c_{ij} for $i \neq j$ as shown in Table 5.7-1.

column no. / row no.	1	2	3 ·····	i ·····	(n-1)	n	n+1
1	---	c_{12}	c_{13} ····	c_{1i} ····	$c_{1(n-1)}$	c_{1n}	$c_{1(n+1)} = \frac{c_1}{2}$
2	c_{21}	---	c_{23} ····	c_{2i} ····	$c_{2(n-1)}$	c_{2n}	$c_{2(n+1)} = \frac{c_2}{2}$
3	c_{31}	c_{32}	--- ····	c_{3i} ····	$c_{3(n-1)}$	c_{3n}	$c_{3(n+1)} = \frac{c_3}{2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	c_{i1}	c_{i2}	c_{i3} ····	--- ····	$c_{i(n-1)}$	c_{in}	$c_{i(n+1)} = \frac{c_i}{2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
(n - 1)	$c_{(n-1)1}$	$c_{(n-1)2}$	$c_{(n-1)3}$ ··	$c_{(n-1)i}$ ··	---	$c_{(n-1)n}$	$c_{(n-1)(n+1)} = \frac{c_{n-1}}{2}$
n	c_{n1}	c_{n2}	c_{n3} ····	c_{ni} ····	$c_{n(n-1)}$	---	$c_{n(n+1)} = \frac{c_n}{2}$

Table 5.7-1

Due to the fact that c_{ij} is equal to c_{ji}, and that we have c_{ij} for $j > 1$ only in the c-vector, therefore, we will eliminate the left lower half of the table. Thus, the table will have n rows starting from 1, and n columns starting from 2. Such a table is shown in table 5.7-2.

column no. / row no.	2	3 ·····	i ·····	(n-1)	n	(n+1)
1	c_{12}	c_{13} ···	c_{1i} ····	$c_{1(n-1)}$	c_{1n}	$c_{1(n+1)} = \frac{c_1}{2}$
2	—	c_{23} ···	c_{2i} ····	$c_{2(n-1)}$	c_{2n}	$c_{2(n+1)} = \frac{c_2}{2}$
3	—	— ···	c_{3i} ····	$c_{3(n-1)}$	c_{3n}	$c_{3(n+1)} = \frac{c_3}{2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	—	— ···	— ····	$c_{i(n-1)}$	c_{in}	$c_{i(n+1)} = \frac{c_i}{2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
(n-1)	—	— ···	— ····	—	$c_{(n-1)n}$	$c_{(n-1)(n+1)} = \frac{c_{(n-1)}}{2}$
n	—	— ···	— ····	—	—	$c_{n(n+1)} = \frac{c_n}{2}$

Table 5.7-2.

By Eq. (5.7-3) we can see that the entries of Table 5.7-2 are always non-increasing from left to right as well as from top to bottom.

$$c_{ij} \geq c_{km} \quad \text{if } i \leq k \text{ and } j \leq m \qquad (5.7\text{-}5)$$

$$\text{for } i, k = 1, 2, \cdots n \ ; \ j, m = 2, 3, \cdots (n+1)$$

For a given switching function F of n variables, we can assign a weight vector $\vec{w} = (w_1, w_2, \ldots, w_n)$ and a threshold T. According to the definition for a threshold function, there will be 2^n inequalities corresponding to the 2^n vertices. Within the 2^n inequalities, there exist many redundant ones. Thus, the natural way of finding the weights and the threshold is to reduce the number of inequalities as much as possible before going to solve them. Since we assume that the given function is positive prime, the ordering of (5. 7-1) holds. And by Theorem 5. 4-1 we have a similar ordering for the weights,

$$w_1 \geq w_2 \geq \cdots \cdots \geq w_n \geq 0$$

In order to set up a reduced set of inequalities from the testing table for finding the weights, we classify the inequalities into trivial and non-trivial ones, and further to critical non-trivial and non-critical non-trivial ones. We call an inequality trivial if the relation (5. 7-5) holds. Thus by non-trivial inequalities we mean those inequalities that satisfy the following relations :

$$c_{ij} > c_{km} \quad \text{for either} \quad i<k, \quad j>m \quad \text{or} \quad i>k, \quad j<m \qquad (5.7\text{-}7)$$

Geometrically speaking, the left hand side and the right hand side (abbreviated LHS and RHS) of the non-trivial inequality shall not be in the same row or the same column of the testing table. Furthermore, the RHS must be in the right upper side of the testing table relative to the LHS.

Consider the following two non-trivial inequalities

$$c_{i_1j_1} > c_{km} \qquad (5.7\text{-}8)$$

$$c_{i_2j_2} > c_{km} \qquad (5.7\text{-}8a)$$

If $i_1 \leq i_2$ and $j_1 \leq j_2$, then by (5. 7-5) we have $c_{i_1j_1} \geq c_{i_2j_2}$. Thus (5. 7-8a) implies (5. 7-8). In other words, the relation specified by (5. 7-8)is covered by the one specified by (5. 7-8a). We call such an inequality as in Eq. (5. 7-8a) a critical non-trivial one. Similarly, if we have two non-trivial inequalities as shown below :

$$c_{km} > c_{i_1 j_1} \quad (5.7\text{-}9)$$

$$c_{km} > c_{i_2 j_2} \quad (5.7\text{-}9a)$$

where $i_1 \leq i_2$ and $j_1 \leq j_2$ still holds. Then contrary to the previous results we have (5. 7-9) being the critical non-trivial inequality instead of (5. 7-9a).

In conclusion, if an inequality is critical and non-trivial, then it shall first of all satisfy (5. 7-7) and it shall not be covered or implied by any other inequalities.

5. 8. An Algorithm for Obtaining All the Critical Non-Trivial Inequaties in the Testing Table

The algorithm given below is quite straightforward and self-explanatory. One thing we want to point out is that all the entries except those of the (n+1)th column in Table 5. 7-2 are positive integers. Due to the fact that we are making comparison on the number of vertices in specific sets, the entries of the (n+1)th column have to be modified, either to drop the fractional part or to round it up. This algorithm is designed to establish critical non-trivial inequalities. For the case that an entry of the (n+1)th column is greater than some of the other columns, we shall drop the fractional part of this entry, while in the less than case we shall round it up. And, it is not difficult to realize that these fractional part to be modified are always equal to $\frac{1}{2}$. Therefore, by using the notation [ ] which means taking the integral part of, it will be very easy to execute such adjustments.

Step I Read in the given switching function of n variables. Find its augmented c-vector and store.

Step II Set $c(i, n+1) = [\frac{1}{2}c(i)]$ for $i = 1, 2, \cdots, n$. Combine this with those elements having two arguments in the augmented c-vector to form the testing table.

Step III Check the entries of each row and each column in the testing table. If all of them are in non-increasing order, then go to Step IV, otherwise stop the program for the reason that F is not a threshold function.

Step IV Set up two groups for appropriate LHS's : one for the case LHS > RHS called G-group, other for the case LHS < RHS called L-group. Scan each row in the testing table except the first and the last rows up to the nth column. The first entry of each row will automatically be in the L-group while the last entry of the scanning (i. e., the entry in the nth column) will be in the G-group. When scanning along each row, if there is a change in the values of two adjacent entries put the former in the G-group and the latter in the L-group.

Step V Divide the elements in the L-group into subgroups according to their values. If there is only one element in the subgroup, leave it as it is. If there are more than one elements in a certain subgroup, then compare the column numbers of the elements in this subgroup. If they are all different, leave them as they are. If some of them are equal, then eliminate those elements with greater row numbers.

Step VI Compare each element in the L-group with its upper right counterpart in the testing table (i. e., the entry in the next column but previous row).

A. If the element is less than its counterpart, then search for the rightmost entry in the same row as the counterpart that still preserves the relation. This will be the corresponding RHS of the inequality for that element as the LHS.

B. If the element is greater than or equal to its counterpart, then search upwards in the same column as the counterpart until an entry is found such that the entry is greater than the element and thus become the corresponding RHS; otherwise abandon this element from the group.

Step VII Do the same as Step V for the G-group except that instead of keeping the element in the subgroup with the least row number restore the one with the greatest row number for each column.

Step VIII Set $c(i, n+1) = [\frac{1}{2} c(i) + \frac{1}{2}]$ for $i = 1, 2, \ldots\ldots n$.

Step IX Compare each element in the G-group with its upper right counterpart in the testing table.

A. If the element is greater than its counterpart, then search for the upmost entry in the same column as the counterpart that still preserves the relation. This entry will be the corresponding RHS of the inequality for this element as the LHS.

B. If the element is less than or equal to its counterpart, then search rightwards in the same row as the counterpart until an entry is found such that this entry is greater than the element and thus become the corresponding RHS; otherwise abandon this element from the group.

An example is given below for illustrating this algorithm.

Example 5.8-1. Given a switching function of 7 variables

$$F = x_1 \{x_2(x_3+x_4+x_5+x_6+x_7) + x_3(x_4+x_5+x_6x_7) + x_4(x_5+x_6x_7) + x_5x_6x_7\}$$
$$+ x_2 \{x_3(x_4+x_5+x_6) + x_4(x_5+x_6x_7) + x_5x_6x_7\}$$
$$+ x_3 \{x_4[x_5(x_6+x_7) + x_6x_7] + x_5x_6x_7\} + x_4x_5x_6x_7$$

Step I The corresponding augmented c-vector is

$\vec{c}$ = (78; 52, 51, 50, 47, 47, 45, 44; 31, 31, 29, 29, 28, 28, 30, 29, 29, 28, 27, 28, 28, 28, 27, 28, 26, 26, 26, 26, 26; 16, 16, 16, 16, 15, 16, 16, 16, 15, 16, 16, 15, 15, 15, 15, 16, 16, 15, 15, 16, 16, 15, 16, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15; $\underbrace{8, 8, \cdots 8}_{35}$; $\underbrace{4, 4, \cdots 4}_{21}$; $\underbrace{2, 2, \cdots 2}_{7}$; 1)

Step II c(1, 8) = 26, c(2, 8) = 25, c(3, 8) = 25, c(4, 8) = 23, c(5, 8) =23, c(6, 8) = 22, c(7, 8) = 22

The testing table will be:

row no. \ column no.	2	3	4	5	6	7	8 for L-group	8 for G-group
1	31	31	29	29	28	28	26	26
2		30	29	29	28	27	25	26
3			28	28	28	27	25	25
4				28	26	26	23	24
5					26	26	23	24
6						26	22	23
7							22	22

Step III Each column and each row in the testing table are non-increasing. Go to Step IV.

Step IV G: c(2, 3), c(2, 5), c(2, 6), c(2, 7), c(3, 6), c(3, 7), c(4, 5), c(4, 7), c(5, 7), c(6, 7)

L: c(2, 3), c(2, 4), c(2, 6), c(2, 7), c(3, 4), c(3, 7), c(4, 5), c(4, 6), c(5, 6), c(6, 7)

Step V	Subgroups :	L_1	L_2	L_3	L_4	L_5
	Values :	30	29	28	27	26
	Entries :	c(2,3)	c(2,4)	c(2,6)	c(2,7)	c(4,6)
				c(3,4)	~~c(3,7)~~	~~c(5,6)~~
				c(4,5)		c(6,7)

Step VI (1) c(3,4) < c(2,5)

(2) c(4,6) < c(3,7)

Step VII	Subgroups :	G_1	G_2	G_3	G_4	G_5
	Values :	30	29	28	27	26
	Entries :	c(2,3)	c(2,5)	~~c(2,6)~~	~~c(2,7)~~	~~c(4,7)~~
				c(3,6)	c(3,7)	~~c(5,7)~~
				c(4,5)		c(6,7)

Step VIII We include this step in the testing table shown in Step II.

Step IX (1) c(2,3) > c(1,4)

(2) c(2,5) > c(1,6)

(3) c(3,6) > c(2,7)

(4) c(4,5) > c(3,7)

(5) c(3,7) > c(1,8)

(6) c(6,7) > c(3,8)

5.9. From the Critical Non-Trivial Inequalities to Inequalities of the Weights

In the last section we ~~have~~ developed an algorithm to set up a set of critical non-trivial inequalities from the testing table. Each of the inequalities has one of the following forms.

$$c_{i_1 i_2} > c_{k_1 k_2} \tag{5.9-1}$$

$$c_{i_1 i_2} > \frac{1}{2} c_k \tag{5.9-2}$$

$$\frac{1}{2} c_k > c_{i_1 i_2} \tag{5.9-3}$$

Since they are critical and non-trivial, from (5.9-1) we have either that $i_1 > k_1$ and $i_2 < k_2$ or that $i_1 < k_1$ and $i_2 > k_2$. However, without losing any generality we can assume that $i_1 > k_1$ and $i_2 < k_2$. And from (5.9-2) and (5.9-3) we have $i_1, i_2 > k$. Now let us consider the following :

$$\begin{aligned}
c_{i_1 i_2 \bar{k}_1 \bar{k}_2} - c_{\bar{i}_1 \bar{i}_2 k_1 k_2} &= c_{i_1 i_2} - c_{i_1 i_2 k_1} - c_{i_1 i_2 k_2} + c_{i_1 i_2 k_1 k_2} \\
&\quad - (c_{k_1 k_2} - c_{i_1 k_1 k_2} - c_{i_2 k_1 k_2} + c_{i_1 i_2 k_1 k_2}) \\
&= (c_{i_1 i_2} - c_{k_1 k_2}) + (c_{i_1 k_1 k_2} - c_{i_1 i_2 k_1}) + \\
&\quad (c_{i_2 k_1 k_2} - c_{i_1 i_2 k_2}) \\
&= D_1 + D_2 + D_3
\end{aligned} \tag{5.9-4}$$

Since $i_1 > k_1$ and $i_2 < k_2$, we have D_2 being negative and D_3 being positive. In other words, the differences D_2 and D_3 are compensating with each other. Furthermore, the magnitudes of those c_i's in D_2 and D_3 are about the half of the c_i's in D_1. Therefore, if (5.9-1) holds, i.e. D_1 is positive, then (5.9-4) is most likely to be positive. ? Thus we have,

$$c_{i_1 i_2 \bar{k}_1 \bar{k}_2} > c_{\bar{i}_1 \bar{i}_2 k_1 k_2}$$

and by Theorem 5.6-2 we get an inequality of the weights,

$$w_{i_1} + w_{i_2} > w_{k_1} + w_{k_2}$$

Next, we will consider,

$$c_{i_1 i_2 \bar{k}} - c_{\bar{i}_1 \bar{i}_2 k} = c_{i_1 i_2} - c_{i_1 i_2 k} - (c_k - c_{i_1 k} - c_{i_2 k} + c_{i_1 i_2 k})$$

$$= (c_{i_1 i_2} - \frac{1}{2} c_k) + (c_{i_1 k} - \frac{1}{2} c_k) + 2(\frac{1}{2} c_{i_2 k} - c_{i_1 i_2 k})$$

$$= D_1' + D_2' + D_3' \qquad (5.9\text{-}5)$$

It is quite obvious to see that D_2' is positive but D_3' is negative. They are compensating with each other. Thus, whether (5.9-5) is positive or not depends largely upon the sign of D_1'. Therefore, if (5.9-2) holds, it is most likely to have

$$c_{i_1 i_2 \bar{k}} > c_{\bar{i}_1 \bar{i}_2 k}$$

and $w_{i_1} + w_{i_2} > w_k$

On the other hand, if Eq. (5.9-3) holds, it is most likely to have

$$c_{i_1 i_2 \bar{k}} < c_{\bar{i}_1 \bar{i}_2 k}$$

and $w_{i_1} + w_{i_2} < w_k$

Therefore, after a set of critical non-trivial inequalities was established, we have to further check by (5.9-4) and (5.9-5). If some of them turn out to be equalities rather than inequalities as before, then the corresponding inequalities shall be eliminated from the set. Otherwise, these inequalities will lead to inequalities among the weights. In other words, the set of critical non-trivial inequalities can be transformed or further reduced to a set of inequalities among the weights. This set of inequalities combined with the set of inequalities obtained directly from the Chow parameters are sufficient to solve for minimal integral weights which will realize the given threshold function. Generally, the number of inequalities in this reduced set will be quite small. To solve such a set of inequalities will be very simple and straightforward. This will be seen in the following examples.

Once the minimal integral weights for a given threshold function are found, it is not difficult to find the threshold for this function. One of the simplest ways is to find a minimum weighted truth vertex of the given function and then the summation of the weights of this vertex will become the threshold.

<u>Example 5.9-1</u>. As given in Example 5.8-1 find the inequalities of the weights and transform them in terms of incremental and minimum weights.

Among the set of critical non-trivial inequalities obtained in Example 5.8-1, three of them become equalities when Eqs. (5.9-4) and (5.9-5) are applied, thus they shall be eliminated from the set. The remaining five are shown below.

$$c_{46\bar{3}\bar{7}} < c_{\bar{4}\bar{6}37} \quad \text{implies} \quad w_4 + w_6 < w_3 + w_7$$

$$c_{23\bar{1}\bar{4}} > c_{\bar{2}\bar{3}14} \quad \text{implies} \quad w_2 + w_3 > w_1 + w_4$$

$$c_{25\bar{1}\bar{6}} > c_{\bar{2}\bar{5}16} \quad \text{implies} \quad w_2 + w_5 > w_1 + w_6$$

$$c_{37\bar{1}} > c_{\bar{3}\bar{7}1} \quad \text{implies} \quad w_3 + w_7 > w_1$$

$$c_{67\bar{3}} > c_{\bar{6}\bar{7}3} \quad \text{implies} \quad w_6 + w_7 > w_3$$

If the incremental weights are introduced, we have,

$$\Delta w_3 > \Delta w_6 \tag{5.9-6}$$

$$\Delta w_3 > \Delta w_1 \tag{5.9-7}$$

$$\Delta w_5 > \Delta w_1 \tag{5.9-8}$$

$$w_7 > \Delta w_1 + \Delta w_2 \tag{5.9-9}$$

$$w_7 > \Delta w_3 + \Delta w_4 + \Delta w_5 \tag{5.9-10}$$

5.10. Procedure for the Testing and Realizing Method

Besides the set of inequalities obtained in the last sections there exists another set of inequalities which can be obtained by comparing the Chow parameters and applying Threoms 5.4-1 and 5.4-2. Combining these two sets of inequalities together, then we can seek for a solution under the constraint of minimal integral weights.

The solution of the set of inequalities is based on Fan's principle of bounding solutions. The detailed explanation of the application of Fan's principle is given by Hu (9), and will not be discussed here. This method introduced by Hu is quite general but not practicable unless the number of variables of the given function is very small.

Another general method of solving the set of inequalities is by successive elimination of the unknowns w_n, Δw_{n-1}, $\cdots$, Δw_2. In the process of elimination certain inequalities will turn out to be redundant and can be deleted. Finally, there will be a set of inequalities involving only one unknown Δw_1. A value can be assigned to Δw_1 to satisfy this set of inequalities. By going backwards, values can be assigned to $\Delta w_2, \cdots, \Delta w_{n-1}, w_n$ successively. The disadvantage of this method is the huge number of operations involved. The detailed procedure of this method is given in Hu (9) and will not be discussed here either.

Now, the procedure for the testing and realizing method can be stated as follows:

I. Find the augmented c-vector of the given function of n variables.

II. Check the function whether it is positive prime. By comparing the Chow parameters and by applying Theorems 5.4-1 and 5.4-2, obtain a set of inequalities or, equalities between the successive weights in either one of the following three forms. Then, transform these inequalities or equalities in terms of incremental weights.

$w_{i+1} < w_i$	implies	$\Delta w_i > 0$
$w_{i+1} = w_i$	implies	$\Delta w_i = 0$
$w_i = w_{i+1} + 1$	implies	$\Delta w_i = 1$

III. Set up the testing table as defined in Section 5.7. If there exist any contradictions, then the given function is not a threshold function.

IV. Apply the algorithm specified in Section 5.8. and obtain a set of critical non-trivial inequalities.

V. (A) If the inequality obtained by Step IV is in the form $c_{ij} > c_{km}$ check whether $c_{ij\bar{k}\bar{m}} > c_{\bar{i}\bar{j}km}$. If it is not true, eliminate this inequality from the set.

(B) If the inequality obtained by Step IV is is the form $c_{ij} > \frac{1}{2} c_k$ or $c_{ij} < \frac{1}{2} c_k$ check whether it is consistent for $c_{ij\bar{k}}$ and $c_{\bar{i}\bar{j}k}$. If it is false, eliminate this inequality from the set.

VI. Transform all the inequalities obtained from Step V. into inequalities relating the incremental and minimum weights.

VII. Combine the set of inequalities and equalities obtained from Step II with those from Step VI. If there exists contradiction between the inequalities, then the given function is not a threshold function. Solve these inequalities using the constraint of minimal integral weights.

VIII. Find a minimum weighted truth vertex of the given function and compute the threshold. Then, the given function can be realized by this threshold and the weights obtained in Step VII.

<u>Example 5.10-1.</u> Find the minimal integral realization for the function given in Ex. 5.8-1.

I. This step was completed in Ex. 5.8-1.

II. Comparing the Chow parameters of the function, we have

$$\Delta w_1 = 1 \quad (5.10\text{-}1)$$

$$\Delta w_2 = 1 \quad (5.10\text{-}2)$$

$$\Delta w_3 > 0 \quad (5.10\text{-}3)$$

$$\Delta w_4 = 0 \quad (5.10\text{-}4)$$

$$\Delta w_5 > 0 \quad (5.10\text{-}5)$$

$$\Delta w_6 = 1 \quad (5.10\text{-}6)$$

$$w_7 > 0 \quad (5.10\text{-}7)$$

III. IV. V. VI. these steps were completed in Ex. 5.9-1.

VII. Substituting (5.10-1), (5.10-2), (5.10-4), (5.10-6) into (5.9-6) up to (5.9-10) we have

$$\Delta w_3 > 1 \quad (5.10\text{-}8)$$

$$\Delta w_3 > 1$$

$$\Delta w_5 > 1 \quad (5.10\text{-}9)$$

$$w_7 > 2 \quad (5.10\text{-}10)$$

$$w_7 > \Delta w_3 + \Delta w_5 \quad (5.10\text{-}11)$$

These inequalities combined with (5.10-3), (5.10-5), (5.10-7) can be easily solved and the solutions are $\Delta w_3 = 2$, $\Delta w_5 = 2$, $w_7 = 5$. Thus, the weight vector that will realize the given function is $\vec{w} =$ (12, 11, 10, 8, 8, 6, 5).

VIII. The minimum weighted truth vertex is $\bar{x}_1 \bar{x}_2 \bar{x}_3 x_4 x_5 x_6 x_7$. Thus,

$$T = w_4 + w_5 + w_6 + w_7 = 8 + 8 + 6 + 5 = 27$$

Example 5.10-2. Given the following switching function of 8 variables, test whether it is a threshold function; if it is, then find the minimal integral realization for this function.

$$F = x_1(x_2+x_3+x_4+x_5+x_6+x_7)+x_2(x_3+x_4+x_5+x_6)+x_3(x_4+x_5$$
$$+x_6x_7+x_6x_8)+x_4(x_5x_6+x_5x_7+x_5x_8+x_6x_7+x_6x_8)+x_5x_6x_7$$

I. $\vec{c} = ($ 225; 126, 124, 123, 122, 121, 120, 116, 114; 64, 64, 64, 64, 64, 64, 63, 64, 64, 64, 64, 62, 62, 64, 64, 63, 62, 62, 63, 63, 62, 62, 62, 62, 61, 62, 61, 58; $\underbrace{32, 32, \cdots, 32}_{35}$, 31, 32, $\underbrace{32, 32, \cdots, 32}_{8}$, 31, $\underbrace{32, 32, \cdots, 32}_{5}$, 31, 32, 31, 31, 31; 16, $\underbrace{16, 16, \cdots, 16}_{69}$; $\underbrace{8, 8, \cdots, 8}_{56}$; $\underbrace{4, 4, \cdots, 4}_{28}$; $\underbrace{2, 2, \cdots, 2}_{8}$; 1)

II. Since $\frac{1}{2}c_0 = 112.5 < c_8 < c_7 < \cdots < c_1$, the given function is positive prime, and

$$\Delta w_1 > 0 \qquad (5.10\text{-}12)$$

$$\Delta w_2 = 1 \qquad (5.10\text{-}13)$$

$$\Delta w_3 = 1 \qquad (5.10\text{-}14)$$

$$\Delta w_4 = 1 \qquad (5.10\text{-}15)$$

$$\Delta w_5 = 1 \qquad (5.10\text{-}16)$$

$$\Delta w_6 > 0 \qquad (5.10\text{-}17)$$

$$\Delta w_7 > 0 \qquad (5.10\text{-}18)$$

$$w_8 > 0 \qquad (5.10\text{-}19)$$

III. The Testing table

column no. / row no.	2	3	4	5	6	7	8	9 for L-group	9 for G-group
1	64	64	64	64	64	64	63	63	63
2		64	64	64	64	62	62	62	62
3			64	64	63	62	62	61	62
4				63	63	62	62	61	61
5					62	62	61	60	61
6						62	61	60	60
7							58	58	58
8								57	57

IV. The set of critical non-trivial inequalities obtained by applying the algorithm in Section 5. 8. is shown below :

$$
\begin{aligned}
&c_{36} < c_{17} \\
&c_{45} < c_{26} \\
&c_{27} < \frac{1}{2} c_{1} \\
&c_{58} < \frac{1}{2} c_{2} \\
&c_{78} < \frac{1}{2} c_{6} \\
&c_{26} > c_{18} \\
&c_{46} > c_{27} \\
&c_{67} > c_{58}
\end{aligned}
\qquad (5.\ 10\text{-}20)
$$

V.

$$
\begin{aligned}
&c_{36\bar{1}\bar{7}} < c_{\bar{3}\bar{6}17} \\
&c_{45\bar{2}\bar{6}} < c_{\bar{4}\bar{5}26} \\
&c_{58\bar{2}} < c_{\bar{5}\bar{8}2} \\
&c_{78\bar{6}} < c_{\bar{7}\bar{8}6} \\
&c_{26\bar{1}\bar{8}} > c_{\bar{2}\bar{6}18} \\
&c_{46\bar{2}\bar{7}} > c_{\bar{4}\bar{6}27}
\end{aligned}
\qquad (5.\ 10\text{-}21)
$$

VI.

$$
\begin{aligned}
&\Delta w_6 < \Delta w_1 + \Delta w_2 \\
&\Delta w_5 < \Delta w_2 + \Delta w_3 \\
&w_8 < \Delta w_2 + \Delta w_3 + \Delta w_4 \\
&w_8 < \Delta w_6 \\
&\Delta w_6 + \Delta w_7 > \Delta w_1 \\
&\Delta w_6 > \Delta w_2 + \Delta w_3
\end{aligned}
\qquad (5.\ 10\text{-}22)
$$

VII. Substitute (5. 10-13) up to (5. 10-16) into the inequalities in Step VI. , and eliminate those inequalities which are repeated and trivial.

$\Delta w_6 < \Delta w_1 + 1$

$w_8 < 3$

$w_8 < \Delta w_6$

$\Delta w_6 + \Delta w_7 > \Delta w_1$

$\Delta w_6 > 2$

Combining these inequalities with (5. 10-12), (5. 10-17), (5. 10-18) and (5. 10-19) we can solve them easily and obtain the following solutions:

$\Delta w_1 = 3$

$\Delta w_6 = 3$

$\Delta w_7 = 1$

$w_8 = 2$

Therefore, the minimal-integral weight vector for the function is

$\vec{w} = (13, 10, 9, 8, 7, 6, 3, 2)$

VIII. One of the minimum weighted truth vertex is $x_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6x_7\bar{x}_8$.

Thus,

$T = w_1 + w_7 = 13 + 3 = 16$

CHAPTER 6
CONCLUSIONS

In this thesis, we have investigated a number of topics concerning orthogonal expansion and the realizability of threshold functions. All the work has been done by using the characteristic vectors such as Chow parameters, augmented Chow-vector, etc. rather than using the switching function itself. The author believes that these vectors provide a fertile area for further work on the basic properties of threshold functions. The followings are some remarks referring to certain chapters.

In Chapter 3 we have investigated the Rademacher Walsh functions and the Walsh matrix. All the properties of a Walsh matrix are mathematically sound and beautiful in form. Besides, the generation of such a matrix in modified form is quite easy and straighforward. However, its practical uses have not been fully developed up to the time being. Further work on the practical uses of the Walsh matrix might be worthy.

In Chapter 4 we have investigated the Chow parameters as well as Dertouzos' vector, and have extended the ideas to forming our augmented Chow-vector. Since the Chow parameters can uniquely characterize a threshold function, the augmented Chow-vector will characterize any given switching function. Thus, certain specific properties of a switching function will also be involved in it augmented Chow-vector. It might happen that some implicit properties in the function will turn out to be explicit in its augmented Chow-vector. We also suggest this topic for further study.

In Chapter 5 we have worked out a method for testing and realizing of threshold functions. Although the method is based on the same principles as many others by first establishing a reduced set of linear inequalities and eventually solving them for integral weights and threshold, it has the following advantages :

(1) It is simple and straightforward, thus it is easy to understand and to apply.

(2) It can be applied to switching functions of any number of variables.

(3) Although the procedure seems to be lengthy, it is programmable and can be manipulated by computers. Therefore, this method will save both labour and time.

REFERENCES

(1) C. K. Chow, " On The Characteristic Of Threshold Functions", Proc. on Switching Circuit Theory and Logical Design S-134, pp. 34 - 38, Sept. 1961

(2) R. P. Coleman, " Orthogonal Functions For The Logical Design Of Switching Circuits ", IRE Trans. on Elec. Computers vol. EC-10, pp. 379 -383, Sept. 1961

(3) M. L. Dertouzos, " Threshold Logic : A Synthesis Approach ", M. I. T. Press, Research Monograph No. 32, Cambridge, Mass. 1965

(4) C. C. Elgot, " Truth Functions Realizable by Single Threshold Organs ", AIEE Conference Paper No. 60 - 1311, Oct. 1960

(5) S. W. Golomb, " On The Classification Of Boolean Functions ", IRE Trans. on Circuit Theory, vol. CT-6 (Special Suppl.) pp. 176 - 186, May 1959

(6) S. W. Golomb et. al. , " Digital Communications ", Prentice-Hall 1964

(7) L. Hatfield, " Classification And Threshold Realizability Of Boolean Functions Using Spectral Analysis ", Master Thesis, Elec. Eng. Dept. , M. I. T. , Cambridge, Mass. May 1967

(8) K. W. Henderson, " Some Notes On The Walsh Functions ", IEEE Trans. on Elec. Computers vol. Ec-13, pp. 50 - 52, Feb. 1964

(9) S. T. Hu, " Threshold Logic ", Univ. of California Press, Berkeley, Calif. 1965

(10) P. Kaszerman, " A Region Concept And Its Application To Threshold Logic ", Information and Control, vol. 8, No. 5, pp. 531 - 551, Oct. 1965

(11) T. Krishnan, " On The Threshold Order Of A Boolean Function ", IEEE Trans. on Elec. Computers, vol. EC-15 pp. 369 - 372, June 1966

(12) S. Kaczmarz and H. Steinhaus, " Theorie der Orthogonalreihen ", Chelsea Publishing Co., New York, N. Y., 2nd Edition, pp. 125-134, 1951

(13) I. Ninomiya, " A Study Of The Structure Of Boolean Functions And Its Applications To The Synthesis Of Switching Circuits ", Memoirs of the Faculty of Engineering, Nagoya Univ., vol. 13, No. 2, pp. 149 - 363, Nov. 1961

(14) D. Sarma, " Studies On Some Aspects Of Threshold Logic ", Doctoral Dessertation, Institute of Radio Physics and Electronics, Univ. of Calcutta, Apr. 1969

(15) C. L. Sheng, " Threshold Logic ", The Ryerson Press, Toronto, Ontario, 1969

(16) R. N. Spann, " Generalized Threshold Functions ", Doctoral Dissertation, Elec. Eng. Dept., M. I. T. Cambridge, Mass., Feb. 1966

(17) R. O. Winder, " Threshold Logic ", Doctoral Dissertation, Mathe. Dept., Princeton Univ. Princeton, N. J., May 1962

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