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COMPLEMENTED BANACH*-ALGEBRAS

A thesis submitted

by

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to

the Faculty of Pure and Applied Science

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ABSTRACT

General complemented Banach algebras have been studied by B.J. Tomiuk [Canadian J. of Math. Vol. 14 (1962), 651 - 659], and recently he and F.E. Alexander have contributed to the study of complemented B*-algebras [Trans. AMS 1969]. In this thesis we extend their work to complemented Banach*-algebras and show their relation to annihilator and dual Banach*-algebras.

If A is a semi-simple complemented Banach*-algebra in which $x^*x = 0$ implies $x = 0$, then A is an A*-algebra which is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$; $\mathfrak{A}$ is uniquely determined up to *-isomorphism. We give several characterizations of duality for A*-algebras, some of which are expressed in terms of complementors. We show, in particular that if A is a dense 2-sided ideal of a B*-algebra, then A is dual if and only if it is complemented. Every complemented completely continuous A*-algebra is dual.

Let A be an A*-algebra contained in a B*-algebra $\mathfrak{A}$, and let p , q be complementors on $\mathfrak{A}$ and A respectively. Using the properties of continuous complementors on B*-algebras, we obtain conditions on $\mathfrak{A}$, A and the complementors p and q such that : (a) The mapping $I \longrightarrow cl(I)^p \cap A$ on the closed right ideals I of A is a complementor on A (called the complementor on A induced by p). (b) The mapping $R \longrightarrow cl(R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$ (called the complementor on $\mathfrak{A}$ induced by q). Finally we discuss an example of a complemented A*-algebra.

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Chapter I : Preliminaries

Chapter II : Complemented Banach algebras

Chapter III : Annihilator and dual Banach*-algebras

Chapter IV : Complemented A*-algebras

INTRODUCTION

In this thesis, we are concerned with the study of complemented Banach*-algebras, and their relation to annihilator and dual Banach*-algebras.

The contents of the thesis may briefly be discribed as follows. In Chapter I, we assemble together some basic definitions and results which are used throughout the work. In Chapter II, we study complemented Banach*-algebras in which $x^*x = 0$ implies $x = 0$. We show that if A is such an algebra then every closed 2-sided ideal of A is a *-ideal. Using this fact we obtain a structure theorem (Theorem(2.2.3)) for A which states that if A is semi-simple then A can be expressed as a topological direct sum of minimal closed 2-sided ideals each of which is a simple complemented Banach*-algebra. Since a simple Banach*-algebra in which $x^*x = 0$ implies $x = 0$ admits a faithful *-representation on a Hilbert space, it follows that, if A is semi-simple, then an auxiliary norm can be introduced in A making A into an A*-algebra, moreover A is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$, which is determined uniquely up to *-isomorphism (Theorem(2.2.5)).

Chapter III is devoted to the discussion of annihilator and dual Banach*-algebras. A Banach*-algebra A is said to have the weak (β_k) property if for every minimal left ideal I of A there exists a constant $k > 0$ such that

$||x||^2 \leq k||x^*x||$ for all $x \in I$. This concept is introduced in this chapter, where we also show its relation to annihilator properties in Banach*-algebras. It follows that if A is an A*-algebra which is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$, then A has the weak (β_k) property if and only if, for every minimal self-adjoint idempotent e in A, $Ae = \mathfrak{A}e$. The condition $Ae = \mathfrak{A}e$ holds for every dual A*-algebra which is a dense 2-sided ideal of a B*-algebra. Thus, if A is such an algebra, then A has the weak (β_k) property. A semi-simple Banach*-algebra is an annihilator algebra in which $x^*x = 0$ implies $x = 0$ if and only if it has a dense socle and the weak (β_k) property. Using this result we obtain the following lemma (Lemma(3.2.1)) which leads us to several characterizations of duality in A*-algebras : Let A be an annihilator A*-algebra, I a closed right ideal of A and $\mathfrak{A}$ the completion of A in an auxiliary norm $|\cdot|$. Then $\mathfrak{A}$ is dual, whose socle $\mathfrak{S}$ coincides with the socle of A , $cl(I)\mathfrak{S} \subset I$ and $cl(I) \cap A = r_A(\ell_A(I))$. With the help of this lemma we show, for example, that if A is an annihilator A*-algebra then A is dual if and only if $xx^* \in I$ implies $x \in I$ for all closed right ideals I of A and $x \in A$. Another result in this direction which we obtain by using this lemma is Theorem(4.1.1).

In Chapter IV, we investigate complemented A*-algebras. We show that a semi-simple complemented Banach*-algebra with weak (β_k) property is a dual A*-algebra (Theorem(4.1.1)). Several characterizations of duality for A*-algebras are given in terms of complementors. We show, in partcular, that if A is a dense 2-sided ideal of a B*-algebra then A is dual if and only if it is complemented (Theorem(4.1.4)). Every complemented completely continuous A*-algebra is dual. Let A be a complemented A*-algebra with a right complementor p. An idempotent e in A is said to be a p-projection if $(eA)^p = (1 - e)A$. If e is a minimal idempotent with this property, we say that e is a minimal p-projection (see Chapter II). It follows that a complemented A*-algebra is dual if and only if every non-zero right ideal contains a minimal p-projection.

Let A be a complemented A*-algebra which is a dense 2-sided ideal of the algebra of all completely continuous linear operators on a complex Hilbert space H. Then A is a 2-sided ideal of the algebra of all continuous linear operators on H (Theorem(4.1.7)). Using this fact and properties of continuous and uniformly continuous complementors (Definition(4.2.1)), we are thus led to the study of induced complementors. More precisely, let A be an A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$, and let p be a complementor on $\mathfrak{A}$ and q a complementor on A. We find conditions on $\mathfrak{A}$, A and the complementors p and q such that : (a) The mapping $I \rightarrow cl(I)^p \cap A$ on the closed the complementor on A induced by p). (b) The mapping $R \rightarrow cl((R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$ (called the complementor on $\mathfrak{A}$ induced by q). Finally we discuss an example of a complemented A*-algebra.

Chapter I

Preliminaries

§1. Notation and terminology

Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A. Following [14], we shall say that A is a right complemented Banach algebra if there exists a mapping $p : R \longrightarrow R^p$ of L_r into itself having the following properties:

(C_1) $\quad R + R^p = A \qquad (R \in L_r);$

(C_2) $\quad R \cap R^p = (0) \qquad (R \in L_r);$

(C_3) $\quad (R^p)^p = R \qquad (R \in L_r);$

(C_4) $\quad$ if $R_1 \supseteq R_2$, then $R_1^p \subseteq R_2^p \qquad (R_1 , R_2 \in L_r)$.

The mapping p is called a right complementor on A. Analogously we define a left complemented Banach algebra and a left complementor. Thus a complex Banach algebra is left (right) complemented if and only if it has a left (right) complementor defined on it. A left and right complemented Banach algebra is called bicomplemented. We shall restrict our attention to right complemented Banach algebras. Therefore, unless otherwise mentioned, a complementor on a Banach algebra will always mean a right complementor and a complemented Banach algebra will always mean a right complemented Banach algebra. All Banach algebras and Banach spaces under consideration are over the complex field C.

For any set S in a Banach algebra A, let $\ell(S) = \{x \in A : xS = (0)\}$ and $r(S) = \{x \in A : Sx = (0)\}$. Then $\ell(S)$ $(r(S))$ is called the left (right) annihilator of S in A. It is clear that $\ell(S)$ and $r(S)$ are closed left and right ideals respectively. If B is a subalgebra of A and S is a subset of B, the left(right) annihilator of S in B will be denoted by $\ell_B(S)$ $(r_B(S))$. A Banach algebra A is called an annihilator algebra if $\ell(A) = r(A) = (0)$, and if for every proper closed right ideal I and every proper closed left ideal J, $\ell(I) \neq (0)$ and $r(J) \neq (0)$. If, in addition, $r(\ell(I)) = I$ and $\ell(r(J)) = J$, then A is called a dual algebra.

A Banach algebra A is said to be semi-simple if the Jacobson radical is zero [11, P.55]. A is called simple if it is semi-simple and if (0) and A are the only closed 2-sided ideals of A. An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semi-simple, this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A.

A Banach algebra with an involution $x \longrightarrow x^*$ is called a Banach*-algebra. A Banach*-algebra A is called a B*-algebra if the norm and the involution satisfy the condition $||x^*x|| = ||x||^2$, $x \in A$. If A is a Banach*-algebra on which there is defined a second norm $|\cdot|$ which satisfies, in addition to the multiplicative condition $|xy| \leq |x|\,|y|$, the B*-algebra condition $|x|^2 = |x^*x|$, is called an A*-algebra. The second norm is called an auxiliary norm. Let A be an A*-algebra. Then A is semi-simple, the involution in A is continuous with respect the given norm $||\cdot||$ and the auxiliary norm $|\cdot|$, and $|\cdot| \leq \beta||\cdot||$ for a real constant β (cf. [11, P.187]).

Let $\{I_\lambda : \lambda\in\Lambda\}$ be a family of left (right) ideals of an algebra A . Then the smallest left (right) ideal of A which contains every I_λ is called the sum of the ideals I_λ and will be denoted by $\Sigma_\lambda I_\lambda$, $\Sigma_\lambda I_\lambda$ evidently consists of all finite sum of elements from the ideals I_λ. If A is a topological algebra, then the closure of $\Sigma_\lambda I_\lambda$ is called their topological sum. If each I_λ is closed and intersects the topological sum of the remaining ideals in the zero element then the topological sum is called a direct topological sum.

Let H be a Hilbert space with inner product (,). If x and y are elements of H then $x\otimes y$ will denote the operator on H defined by the relation $(x\otimes y)(h) = (h,y)x$ for all $h \in H$. Let L(H) be the algebra of all continuous linear operators on H into itself with the usual operator bound norm. LC(H) will denote the subalgebra of L(H) consisting of all compact operators on H.

Let $\{A_\lambda : \lambda\in\Lambda\}$ be a family of Banach algebras A_λ, and let $(\sum_\lambda A_\lambda)_0$ be the set of all functions f defined on Λ such that $f(\lambda) \in A_\lambda$ for each $\lambda\in\Lambda$ and such that, for arbitrary $\varepsilon > 0$, the set $\{\lambda : ||f(\lambda)|| \geq \varepsilon\}$ is finite. It is easy to see that $(\sum_\lambda A_\lambda)_0$ is closed under the usual operations of addition, multiplication, and multiplication by scalars for functions : $(f + g)(\lambda) = f(\lambda) + g(\lambda)$, $(fg)(\lambda) = f(\lambda)g(\lambda)$ and $(\alpha f)(\lambda) = \alpha f(\lambda)$ for all $f,g \in (\sum_\lambda A_\lambda)_0$,

$\lambda \in \Lambda$ and α scalars. $(\sum_\lambda A_\lambda)_0$ is a Banach algebra under the norm $\|f\| = \sup\{\|f(\lambda)\| : \lambda \in \Lambda\}$. If each A_λ is a B*-algebra, then $(\sum_\lambda A_\lambda)_0$ is also a B*-algebra under the above norm $\|f\|$ and the involution $f \longrightarrow f^*$ given by $(f^*)(\lambda) = f(\lambda)^{*_\lambda}$, where $*_\lambda$ is the involution on A_λ. $(\sum_\lambda A_\lambda)_0$ is called the B*-(∞)-sum of A_λ. If, in addition, A_λ are dual, then $(\sum_\lambda A_\lambda)_0$ is also dual [6, Lemma 2.4].

Let A be a dual B*-algebra and $\{I_\lambda\}$ the family of all minimal closed 2-sided ideals of A. Since I_λ is isometrically *-isomorphic to $LC(H_\lambda)$, for some Hilbert space H_λ, $(\lambda \in \Lambda)$, we see that A is isometrically *-isomorphic to $(\sum_\lambda LC(H_\lambda))_0$ [6, Lemma 2.3].

Let X be a topological space and S a subset of X . Then cl(S) will denote the closure of S in A. The norm in a B*-algebra will usually be denoted by $|\cdot|$.

§2. Some lemmas

In this section we shall state several lemmas which will be used throughout the thesis.

Lemma(1.2.1): Let A be a Banach algebra and p a mapping of L_r into itself having properties (C_3) and (C_4). If $\{R_\lambda : \lambda \in \Lambda\}$ is a family of closed right ideals of A , then $(\bigcap_\lambda R_\lambda^p)^p = cl(\sum_\lambda R_\lambda)$.

Proof: [1] Since, for each $\lambda \in \Lambda$, $R_\lambda^p \supset \bigcap_\lambda R_\lambda^p$, we have $(\bigcap_\lambda R_\lambda^p)^p \supset R_\lambda$ and hence $(\bigcap_\lambda R_\lambda^p)^p \supset \sum_\lambda R_\lambda$. Thus $(\bigcap_\lambda R_\lambda^p)^p \supset cl(\sum_\lambda R_\lambda)$. But $cl(\sum_\lambda R_\lambda) \supset R_\lambda$ for each $\lambda \in \Lambda$.

Hence $(\mathrm{cl}(\sum R_\lambda))^p \subset \bigcap_\lambda R_\lambda{}^p$ and consequently $\mathrm{cl}(\sum R_\lambda) \supset (\bigcap_\lambda R_\lambda{}^p)^p$. Therefore $(\bigcap_\lambda R_\lambda{}^p)^p = \mathrm{cl}(\sum R_\lambda)$.

Lemma(1.2.2) : Let A be a simple Banach algebra and let e_1, e_2 be minimal idempotents of A . Then the algebra e_1Ae_2 is isomorphic to the complex field C if $e_2e_1 \neq 0$ and is a one-dimensional zero algebra over C if $e_2e_1 = 0$.

Proof : Since A is simple, $e_1Ae_2 \neq (0)$ and the socle is dense in A (see [11, Lemma(2.1.12)]). Hence there exists an element of the form ex in A, where e is a minimal idempotent, such that $e_1exe_2 \neq 0$. Since eAe is isomorphic to C and, by [11, Lemma(2.1.8)], $e_1Ae_2 = e_1eAexe_2$, it follows that e_1Ae_2 is a one-dimensional algebra over C . Hence if $z = e_1ye_2 \in e_1Ae_2$ and $z \neq 0$, then every $w \in e_1Ae_2$ is of the form $w = \lambda z$ ($\lambda \in C$) . Now, if $e_2e_1 \neq 0$, then $z^2 \neq 0$. In fact, if $z^2 = 0$, then $(Ae_1ye_2)(e_1ye_2A) = (Ae_2)(e_1A) = (0)$ and in particular $e_2e_1 = 0$; a contradiction. Hence $z^2 = \lambda z$, $\lambda \neq 0$. Let $f = \frac{1}{\lambda}z$. Then f is an idempotent and every $w \in e_1Ae_2$ can be expressed uniquely as $w = \alpha f$, $\alpha \in C$. It is now easy to verify that e_1Ae_2 is isomorphic to C . If $e_2e_1 = 0$, then clearly e_1Ae_2 is a one-dimensional zero algebra over C.

Lemma(1.2.3) : Let $\mathfrak{A}$ be a B*-algebra under the norm $|\cdot|$, and let A be a dense 2-sided ideal of $\mathfrak{A}$ which is a Banach algebra under the norm $\|\cdot\|$. Then there exists a constant k such that $\|xy\| \leq k\|x\|\,|y|$ and $\|yx\| \leq k\|x\|\,|y|$ for all $x \in A$ and all $y \in \mathfrak{A}$.

Proof: [7] Let $x \in A$. We first show that the mapping $T_x : y \to xy$ from $\mathfrak{A}$ into A is continuous. Let $\{y_n\}$ be a sequence in $\mathfrak{A}$ such that $|y_n - y| \to 0$ and $\|xy_n - z\| \to 0$ for some $z \in A$. Then, by [11, Corollary (4.1.16)], $|xy_n - z| \to 0$. Since $|xy_n - xy| \to 0$, $xy = z$. Hence the mapping T_x is closed and so, by the Closed Graph Theorem, T_x is continuous. Therefore there exists a constant c_x (depending on x) such that $\|xy\| \leq c_x \|x\| |y|$ for all $y \in \mathfrak{A}$. Similarly, we can show that, for each $y \in \mathfrak{A}$, the linear mapping T'_y : $x \to xy$ on A into A is continuous and hence there exists a constant c'_y (depending on y) such that $\|xy\| \leq$ $\leq c'_y \|x\| |y|$ for all $x \in A$. Hence $\|T_x y\| \leq c'_y |y|$ for all $\|x\| = 1$. Therefore, by the Principle of Uniform Boundedness, $\sup\{\|T_x\| : \|x\| = 1\} \leq k_1$ for some constant k_1, where $\|T_x\| = \sup\{\|T_x y\| : |y| \leq 1\}$. Thus $\|xy\| \leq k_1 \|x\| |y|$ for all $x \in A$, $y \in \mathfrak{A}$. Similarly, we can show that there exists a constant k_2 such that $\|yx\| \leq k_2 \|x\| |y|$ for all $x \in A$, $y \in \mathfrak{A}$. Then $k = \max(k_1, k_2)$ clearly satisfies the inequalities in the lemma.

Lemma (1.2.4) : Let A be a semi-simple Banach algebra which is a dense 2-sided ideal of a semi-simple Banach algebra $\mathfrak{A}$. Then the minimal left(right) ideals of A and $\mathfrak{A}$ are identical.

Proof: Since A is a dense 2-sided ideal of $\mathfrak{A}$, it is easy to see that A and $\mathfrak{A}$ have the same minimal idempotents. Now every minimal left ideal of A (resp. of $\mathfrak{A}$)

is of the form Ae (resp. $\mathfrak{A}e$) [11, Lemma (2.1.5)] . Clearly $Ae \subset \mathfrak{A}e$. Since A is a 2-sided ideal of $\mathfrak{A}$, $\mathfrak{A}e \subset Ae$ and therefore $Ae = \mathfrak{A}e$.

Lemma (1.2.5): Let A be a dual B*-algebra and B a maximal commutative *-subalgebra of A . Then every self-adjoint idempotent $e \in B$ is minimal in B if and only if it is minimal in A .

Proof: By [8, Theorem 1] , B is a dual B*-algebra . Suppose that e is a minimal self-adjoint idempotent in B. It is easy to show that there exists a minimal self-adjoint idempotent f in B such that $ef = fe = 0$. Therefore, by Zorn's Lemma, there exists a maximal orthogonal family $\{e_\alpha\}$ of minimle self-adjoint idempotents in B containing e. It is clear that $\{e_\alpha\}$ is actually maximal in B . By [7, Theorem 16] , for all $x \in B$, we have $x = \sum_\alpha xe_\alpha = \sum_\alpha \lambda_\alpha e_\alpha$, where $\lambda_\alpha \in C$. Therefore $(eae)x = x(eae)$ for all $a \in A$, $x \in B$. Thus, by the maximality of B , $eae \in B$ and so $eae = \lambda e$ for some $\lambda \in C$. Hence $eAe = Ce$ and therefore e is minimal in A . The converse of the lemma is clear since if $e \in B$ is a minimal idempotent in A , then $eBe \subset eAe = Ce$.

Lemma (1.2.6) : Let A be a semi-simple Banach algebra with a dense socle. If I is a minimal right ideal of A , then the closed 2-sided ideal generated by I is a minimal closed 2-sided ideal of A .

Proof : The proof of this lemma is the same as that given in [2, Theorem 5] .

Chapter II

Complemented Banach algebras

§ 1 . Annihilator complemented Banach algebras

In this section, as well as in the rest of the thesis, a complemented Banach algebra will always mean a right complemented Banach algebra.

Lemma(2.1.1) : Let A be a complemented Banach algebra amd M a modular closed right ideal. Then there exists a unique idempotent $e \in A$ such that $M = (1 - e)A$ and $M^p = eA$.

Proof : [1] let u be a left idemtity modulo M and write $u = d + e$ with $d \in M$, $e \in M^p$. Let $x \in M^p$. Since $x - ex = (1 - u)x + dx \in M \cap M^p$, $x = ex$; in particular $e = e^2$. Since $ey = y + dy - (1 - u)y \in M \cap M^p$, for all $y \in M$, $eM = (0)$. Therefore $M = (1 - e)A$ and $M^p = eA$. We show that e is unique. Suppose there is an idempotent f with $fA = M^p$ and $(1 - f)A = M$. Then $f \in \ell(M) = Ae$, and so $f = fe$. Since $e, f \in R^p$, $fe - e \in R \cap R^p = (0)$ and so $fe = e$. Therefore $e = fe = f$.

Let A be a complemented Banach algebra with a complementor p . We shall call an idempotent e in A a p-projection if $(eA)^p = (1 - e)A$. If, moreover e is a minimal idempotent, we shall say that e is a minimal p-projection. (In [14] , a p-projection is called a left projection.) We see from Lemma(2.1.1) that if I is a modular closed right ideal then there exists a p-projection e such that $I = (1 - e)A$.

Lemma(2.1.2) : Let A be a semi-simple annihilator complemented Banach algebra with a complementor p . Then every non-zero right ideal I contains a minimal p-projection. Moreover, if I is a closed non-zero right ideal and $\{e_\alpha\}$ is the family of all minimal p-projections in I , then $I = cl(\sum_\alpha e_\alpha A)$.

<u>Proof</u> : By [14, Corollary Theorem 1] , I contains a minimal right ideal R . Since R^p is a maximal closed right ideal, by [2, Theorem 1] , R^p is modular, The existence of a minimal p-projection in I now follows from Lemma(2.1.1). To prove the second part of the lemma, suppose that $I \neq cl(\sum_\alpha e_\alpha A)$, let $J = cl(\sum_\alpha e_\alpha A)$. Then there exists $x \in I$ such that $x \notin J$. Write $x = x_1 + x_2$ with $x_1 \in J$ and $x_2 \in J^p$. Then $0 \neq x_2 = x - x_1 \in I$ and so $I \cap I^p \neq (0)$. Hence there exists a minimal p-projection e in $I \cap J^p \subset I$ which does not belong to J ; a contradiction. Therefore $I = J$.

Combining Lemmas (2.2.2) and (1.2.1) , we have the following result :

<u>Corollary(2.1.3) :</u> Let A be an annihilator semi-simple complemented Banach algebra. Then every closed right ideal of A is the intersection of maximal modular right ideals containing it.

<u>Theorem(2.1.4) :</u> Let A be a semi-simple complemented Banach algebra with a complementor p . Then the following statements are equivalent :

(i) A is an annihilator algebra .

(ii) Every non-zero right ideal contains a minimal p-projection.

(iii) Every maximal closed right ideal is modular.

(iv) Every maximal closed right ideal has a non-zero left annihilator.

Proof : (i) $\Longrightarrow$ (ii) . Follows from Lemma(2.1.2).

(ii) $\Longrightarrow$ (iii). Let M be a maximal closed right ideal of A. Since M^p is a minimal right ideal of A and contains a minimal p-projection e , $M^p = eA$. Thus $M = (1 - e)A$ and so M is modular.

(iii) $\Longrightarrow$ (iv) . Let M be a maximal closed right ideal of A . Then M is modular and so, by Lemma(2.1.1), $M = (1 - e)A$, for some idempotent e . Clearly $e \in \ell(M)$ and so $\ell(M) \neq (0)$.

(iv) $\Longrightarrow$ (i). Let I be a proper closed right ideal of A. Then, by [14, Corollary Theorem 1] , I^p contains a minimal right ideal R . Since R^p is a maximal closed right ideal, $\ell(R^p) \neq (0)$. Since $R^p \supset I$, $\ell(I) \neq (0)$. By [14, Theorem 8] , A is an annihilator algebra. This completes the proof of the theorem.

Theorem(2.1.5) : Let A be a semi-simple annihilator complemented Banach algebra. Then every closed 2-sided ideal of A is a semi-simple annihilator complemented Banach algebra.

Proof : Let M be a closed 2-sided ideal of A . By [14, Lemma 1] , $M^p = \ell(M) = r(M)$. Hence every closed left (right) ideal of M is a closed left (right) ideal of A and so M is semi-simple. Again, by [14, Lemma 1], the mapping $p_M : I \longrightarrow I^{p_M} = I^p \cap M$ on the closed right ideals of M is a (right) complementor on M . If I is a maximal closed 2-sided ideal of M, then I^{p_M} is a minimal right ideal of M and hence a minimal right ideal of A. Thus $(I^{p_M})^p$ is a maximal closed right ideal of A and therefore modular. But, by Lemma(2.1.1), $(I^{p_M})^p = \{x - ex : x \in A\}$, where e is an idempotent in I^{p_M}. Hence, since $I = (I^{p_M})^p \cap M$, it follows that $I = \{x - ex : x \in M\}$, i.e. I is modular. Therefore M is an annihilator algebra by Theorem (2.1.4).

§2. Semi-simple complemented Banach*-algebras

Lemma (2.2.1) : Let A be a semi-simple complemented Banach*-algebra. Then the involution in A is continuous and A is a bicomplemented algebra.

Proof: By [14, Lemma 5], the socle of A is dense in A and therefore, by [11, Corollary (2.5.8)], A has a unique norm topology. Hence $x \rightarrow x^*$ is continuous. Let p be the given right complementor on A. Then the mapping $q : J \rightarrow J^q = ((J^*)^p)^*$ on the closed left ideals J of A is a left complementor on A.

Throughout the rest of this section, unless otherwise mentioned, A will be a complemented Banach*-algebra with a complementor p, in which $x^*x = 0$ implies $x = 0$.

Lemma (2.2.2) : Every closed 2-sided ideal I of A is a complemented Banach*-algebra.

Proof: Since $x^*x = 0$ implies $x = 0$, $\ell(A) = 0$ and therefore, by [14, Lemma 1], I is a complemented algebra, and $\ell(I) = r(I) = I^p$. It remains to show that $I^* = I$.

Let $x \in I$ and $y \in I^p$. Since $I \cap I^p = (0)$, $(x^*y)^*(x^*y) = (y^*xx^*)y = 0$. Therefore $x^*y = 0$ and so $x^* \in \ell(I^p) = I^{pp} = I$ for all $x \in I$. Hence $I^* = I$.

Theorem (2.2.3) : (Structure Theorem) Let A be semi-simple. Then A is the direct topological sum of its minimal closed 2-sided ideals, each of which is a simple complemented Banach*-algebra.

Proof: Follows from [14, Theorem 4] and Lemma (2.2.2).

Lemma (2.2.4): Let A be simple. Then there exists a faithful *-representation of A on a Hilbert space H such that the image of A' of A in $L(H)$ is a dense subalgebra of $LC(H)$; A is an A*-algebra.

Proof : By [14, Theorem 1], A contains a minimal left ideal I, and, by [11, Lemma (4.10.1)], we can write $I = Ae$ with e a (unique) self-adjoint minimal idempotent. The proof of [11, Theorem (4.10.3)] shows that the scalar-valued function (x,y) defined by $(x,y)e = y^*x$ is an inner product on I with respect to which the left regular representation of A on I is a *-representation; moreover, every operator $T_a^o : a \longrightarrow ax$ $(x \in I)$ is bounded relative to the inner product norm $|x|_o = (x,x)^{\frac{1}{2}}$. Since A is simple, this representation is faithful. Let H be the Hilbert space completion of I under the norm $|x|_o$. Since T_a^o is bounded relative to $|x|_o$, T_a^o can be uniquely extended to an operator $T_a \in L(H)$. Hence $a \longrightarrow T_a$ is a faithful *-representation of A in $L(H)$ and, by [11, Theorem(4.10.5)], the image A' of A in $L(H)$ contains every operator of the form $g \otimes h$, where $g, h \in I$. For each $y \in A$, let $|y| = |T_y|$, where $|T_y|$ denotes the operator bound of T_y with respect to the norm $|x|_o$ on H. Then A is an A*-algebra with an auxiliary norm given by $|\cdot|$. Let $cl(A')$ be the closure of A' in $L(H)$. We show that $cl(A') = LC(H)$. By [14, Lemma 5], the socle $\mathfrak{S}$ of A is dense in A with respect to $\|\cdot\|$. If $\{e_\alpha\}$ is the family of all minimal idempotents in A, then every $x \in \mathfrak{S}$ is of the form $x = e_{\alpha_1}x_1 + \ldots + e_{\alpha_n}x_n$, where $x_i \in A$ $(i = 1,\ldots,n)$. Hence, by Lemma (1.2.2), T_x^o is an operator of finite rank on I for all $x \in \mathfrak{S}$. Thus the extension T_x of T_x^o to H is an operator of finite rank on H. Let $y \in A$. Then, for every $\varepsilon > 0$, there exists $x \in \mathfrak{S}$ such that $\|x - y\| < \varepsilon$. By [11, Corollary (4.1.16)], $|T_x - T_y| = |x - y| \leq \beta\|x - y\| < \beta\varepsilon$. Since $T_x \in LC(H)$, we have $T_y \in LC(H)$, and consequently $A' \subset LC(H)$. It remains to show that $LC(H) \subset cl(A')$. Let $x, y \in H$ and $\varepsilon > 0$. Then there exist $g, h \in I$ such that $|x - g|_o < \varepsilon$ and $|y - h|_o < \varepsilon$. Then $|x \otimes y - g \otimes h| \leq |x \otimes y - x \otimes h| + |x \otimes h - g \otimes h| \leq |x|_o|y - h|_o + |h|_o|x - g|_o \leq \varepsilon(|x|_o + |h|_o)$. Since $g \otimes h \in A'$, $x \otimes y \in cl(A')$. Thus $cl(A')$ contains all operators of finite rank on H and therefore $LC(H) \subset cl(A')$. This completes the proof.

Let A be a Banach*-algebra. Following [7], we shall say that A has (β_k) property if there exists a constant k such that $\|x\|^2 \leq k\|x^*x\|$ for all $x \in A$.

<u>Theorem (2.2.5):</u> Let A be semi-simple. Then A is an A*-

algebra which is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$; $\mathfrak{A}$ is uniquely determined up to *-isomorphism. A is a dense 2-sided ideal of $\mathfrak{A}$ if and only if the norm $||x||_1 =$ $= \sup\{||xy|| : ||y|| = 1\}$ on A satisfies the (β_k) property.

Proof : By Theorem (2.2.3) , A is the direct topological sum of $\{I_\lambda : \lambda \in \Lambda\}$, the family of all minimal closed 2-sided ideals I_λ of A . By Lemma (2.2.4), each I_λ may be considered as a dense subalgebra of $LC(H_\lambda)$ for some Hilbert space H_λ . Let $\mathfrak{A} = (\sum_\lambda LC(H_\lambda))_0$. Since $\mathfrak{A}$ can be considered as the topological direct sum of $LC(H_\lambda)$, A can be identified as a subalgebra of $\mathfrak{A}$. Hence A is an A*-algebra with the auxiliary norm given by the norm $|\cdot|$ on A . Considering A as a subalgebra of A, Lemma(2.2.4) implies that $LC(H_\lambda) \subset cl(A)$ for all λ . This shows that $\mathfrak{A} \subset cl(A)$ and therefore A is dense in $\mathfrak{A}$. Since the socle of A is dense in A , by [7, Theorem 3], A has a unique auxiliary norm topology. Hence $\mathfrak{A}$ is uniquely determined up to *-isomorphism. The last part of the theorem follows from [7, Theorem 18] .

Let A be a Banach*-algebra. The intersection of the kernals of all topologically irreducible *-representations of A on Hilbert space is called the *-radical of A and is denoted by $R^{(*)}$. If $R^{(*)} = (0)$, A is said to be *-semi-simple (see [11, P. 210]).

Theorem(2.2.6) : The (Jacobson) radical $\mathcal{R}$ and the *-radical $\mathcal{R}^{(*)}$ of A coincide.

Proof : By [11, Theorem(4.4.10)] , $\mathcal{R}^{(*)} \supset \mathcal{R}$. We may assume that $\mathcal{R} \neq A$; for if $\mathcal{R} = A$, then $\mathcal{R} = \mathcal{R}^{(*)} = A$. By [14, Theorem 2] , and Theorem(2.2.2), $\mathcal{R}^p$ is a semi-simple right complemented Banach*-algebra; clearly, $x^*x = 0$ implies $x = 0$ for all $x \in \mathcal{R}^p$. Hence, by Theorem(2.2.5), $\mathcal{R}^p$ is an A*-algebra. It is easy to show that the natural homomorphism $x \longrightarrow x'$ (where $x' = x + \mathcal{R}$) is a *-isomorphism of $\mathcal{R}^p$ onto $A/\mathcal{R}$. Therefore $A/\mathcal{R}$ is an A*-algebra and, by [11, Corollary(4.8.12)] , $A/\mathcal{R}$ is *-semi-simple. Hence $\mathcal{R}^{(*)}/\mathcal{R} = (0)$ and so $\mathcal{R}^{(*)} = \mathcal{R}$.

Chapter III

Annihilator and dual Banach*-algebras

§1. Annihilator and weak (β_k) properties in Banach*-algebras

If A is a Banach*-algebra in which $x^*x = 0$ implies $x = 0$, then, by [11, Lemma(4.10.1)], every minimal left ideal I of A is of the form $I = Ae$, where e is a minimal self-adjoint idempotent. A similar result holds for minimal right ideals. It follows from the proof of [11, Theorem (4.10.3)] that the scalar-valued function (x,y) defined by $(x,y)e = y^*x$ $(x,y \in I)$ is an inner product on I. Hence $|x|_0 = (x,x)^{\frac{1}{2}}$ is a norm on I. Since this inner product will be used on several occasions in the rest of the thesis, to avoid repeating ourselves in the future, we will adopt the following notation : The bracket (,) will always denote the inner product on the minimal left ideal I defined by $(x,y)e = y^*x$ $(x,y \in I)$ and $|\cdot|_0$ the inner product norm on I given by $|x|_0 = (x,x)^{\frac{1}{2}}$ for all $x \in I$.

It is easy to see that if A is a B*-algebra, then the norm $|\cdot|_0$ coincides with the given norm on every minimal left ideal of A.

Definition(3.1.1): A Banach*-algebra A is said to have weak (β_k) property if, for every minimal left ideal I of A, there exists a constant k (depending on I) such that $||x||^2 \leq k||x^*x||$, for all $x \in I$.

Theorem(3.1.2): Let A be an A*-algebra which is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$. Then A has weak (β_k)

property if and only if, for every minimal self-adjoint idempotent e in A, $Ae = \mathfrak{A}e$.

Proof: Suppose $Ae = \mathfrak{A}e$ for every minimal self-adjoint idempotent e in A, and let I be any minimal left ideal of A. Since $|\cdot| \leq \beta \|\cdot\|$, the identity map i of I with norm $\|\cdot\|$ onto I with norm $|\cdot|$ is continuous. As I is a Banach space under both norms, the Open Mapping Theorem shows that the inverse map of i is also continuous. Hence there exists a constant α such that $\|x\| \leq \alpha |x|$, for all $x \in I$. Therefore $\|\cdot\|$ and $|\cdot|$ are equivalent on I and hence, since $|x|^2 = |x^*x|$ $(x \in \mathfrak{A})$, there exists a constant k such that $\|x\|^2 \leq k\|x^*x\|$, for all $x \in I$. Thus A has weak (β_k) property.

Conversely, suppose that A has weak (β_k) property, and let e be a minimal self-adjoint idempotent in A. Let $I=Ae$, $J = \mathfrak{A}e$. Since A is dense in $\mathfrak{A}$ and eAe is one-dimensional, $e\mathfrak{A}e = eAe$, and hence e is a minimal idempotent of $\mathfrak{A}$. Consequently J is a minimal left ideal of $\mathfrak{A}$, and clearly $J = \mathrm{cl}(I)$, the closure of I in $\mathfrak{A}$. Since A has weak (β_k) property, [11, Theorem(4.10.6)] shows that the inner product norm $|x|_0$ is equivalent to the given norm on I. But the inner product norm $|\cdot|_0$ on J is equal to the norm $|\cdot|$ on J. Hence $|\cdot|$ and $\|\cdot\|$ are equivalent on I, so that I is closed in $|\cdot|$. Since I is dense in J, we have $I = J$, i.e., $Ae = \mathfrak{A}e$.

Corollary(3.1.3): Let A be an A*-algebra which is a dense 2-sided ideal of a dual B*-algebra $\mathfrak{A}$. Then A has weak

(β_k) property.

Proof: Follows from Lemma(1.2.4) and Theorem(3.1.2).

Lemma(3.1.4): Let A be a Banach*-algebra with socle σ such that $a\sigma = (0)$ implies $a = 0$. If A has weak (β_k) property, then $x^*x = 0$ implies $x = 0$.

Proof: By [11, Corollary(2.5.8)], A has a unique norm topology and hence the involution is continuous. Let $x \in A$ be such that $x^*x = 0$, and let I be any minimal left ideal of A. Then, for each $a \in I$, $(xa)^*(xa) = a^*x^*xa = 0$. Hence, by the weak (β_k) property of A, $\|xa\| = 0$ which gives $xa = 0$ and therefore $xI = (0)$. As I is an arbitrary minimal left ideal of A, it follows that $x\sigma = (0)$ and consequently $x = 0$.

Lemma(3.1.5): Let A be a semi-simple Banach algebra. If the socle of A is dense in A, then, for every proper closed 2-sided ideal I of A, $\ell(I) = r(I) \neq (0)$. Moreover, every closed left (right) ideal of the algebra I is also a closed left (right) ideal of A.

Proof: If A is simple, the lemma is clearly true. Now, suppose that A is not simple. By [11, Lemma(2.1.5)], each minimal right ideal of A is of the form eA for some minimal idempotent e in A. Let I be a proper closed 2-sided ideal of A. Since the socle is dense in A, there exists a minimal idempotent $e \in A$ such that $e \notin I$. Let $J = cl(AeA)$. By Lemma(1.2.6), J is a minimal closed 2-sided ideal of A. Since $e \notin I$, $I \cap J = (0)$ and so $J \subset \ell(I)$ which shows $\ell(I) \neq (0)$. By the proof of [11, Lemma(2.8.10)], we have $\ell(I) = r(I)$ and, if $R = cl(I + \ell(I))$, then $\ell(R) = (0)$. Since every proper closed 2-sided ideal of A has a non-zero annihilator, we must have R = A. The last part of the lemma follows now from the proof of [11, Lemma(2.8.11)].

Theorem(3.1.6): Let A be a semi-simple Banach*-algebra. Then the following statements are equivalent :

(i) A is an annihilator algebra in which $x^*x = 0$ implies $x = 0$.

(ii) A has weak (β_k) property and the socle $\mathfrak{S}$ of A is dense in A.

Proof: (i) ⇒ (ii). Suppose (i) holds. By [2, Theorem4], the socle $\mathfrak{S}$ of A is dense in A and therefore the involution is contiuous. Let I be a minimal left ideal of A. By [11, Lemma(4.10.1)], we may write I = Ae, where e is a minimal self-adjoint idempotent. Let J = cl(AeA). Then J is a minimal closed 2-sided ideal of A with $J^* = J$ and therefore a simple annihilator Banach*-algebra; moreover, I is a minimal left ideal of J. Hence, by the proof of [11, Theorem(4.10.16)], I is complete under the inner product norm $|\cdot|_0$ and so, by [11, Theorem(4.10.6)], there exists a constant k (depending on I) such that $||x||^2 \leq k||x^*x||$ for all $x \in I$. Thus A has weak (β_k) property.

(ii) ⇒ (i). Suppose (ii) holds. Since $A = cl(\mathfrak{S})$, the involution in A is continuous. By Lemma(3.1.4), $x^*x = 0$ implies $x = 0$. Assume first that A is simple, and let I be a minimal left ideal of A. Since A has weak (β_k)

property , the inner product norm $|\cdot|_0$ is equivalent to the given norm on I and hence I is a Hilbert space under the inner product (,). Now, by the proof of [11, Theorem (4.10.5)], the left regular representation $x \longrightarrow T_x$ of A on I is a faithful *-representation on the Hilbert space I and the image A' of A by this representation contains the set F of all operators of finite rank on I . Since, by Lemma(1.2.2), the elements of the socle give rise to operators of finite rank on I and, since $A = cl(\mathfrak{S})$, F is dense in A' relative to the norm $||\cdot||$. Hence,by [11, Theorem (2.8.23)], A' is an annihilator algebra and therefore A is an annihilator algebra; in fact A is an annihilator A*-algebra.

Now, suppose that A is not simple. Let I be a minimal left ideal of A and let J be the closed 2-sided ideal generated by I . By Lemma(1.2.6) , J is a minimal closed 2-sided ideal of A. Since $x^*x = 0$ implies $x = 0$, $I = Ae$, where e is a self-adjoint idempotent, and hence, since $J = cl(AeA)$, we have $J^* = J$ (by the continuity of the involution in A). Moreover, since $A = cl(\mathfrak{S})$, Lemma(3.1.5) shows that every closed left (right) ideal of J is a closed left (right) ideal of A . Since $e \in J$, it follows that J is a simple Banach*-algebra with a socle (by [11. Lemma(2.1.12)]) which must necessarily be dense in J . Therefore J is an annihilator algebra and consequently A is an annihilator algebra by [11, Theorem(2.8.29)] .

§2. Dual A*-algebras

In this section, we shall give several characterizations of duality in A*-algebras.

Lemma(3.2.1): Let A be an annihilator A*-algebra, I a closed right ideal of A and $\mathfrak{A}$ the completion of A in an auxiliary norm $|\cdot|$. Then the following statements are true:

(i) $\mathfrak{A}$ is a dual B*-algebra which is uniquely determined up to *-isomorphism.

(ii) A and $\mathfrak{A}$ have the same socle.

(iii) If $\mathfrak{S}$ is the socle of A, then $cl(I)\mathfrak{S} \subset I$.

(iv) $\ell(cl(I)) = cl(\ell_A(I))$.

(v) $cl(I) \cap A = r_A(\ell_A(I))$.

(where $cl(S)$ (resp. $cl_A(S)$) denotes the closure of the set S in $\mathfrak{A}$ (resp. in A), $\ell(S)$ (resp. $\ell_A(S)$) the left annihilator of S in $\mathfrak{A}$ (resp. in A).)

Proof: (i). By [2, Theorem 4], $\mathfrak{S}$ is dense in A and so, by [7, Theorem 3], A has a unique auxiliary norm. Therefore the B*-algebra $\mathfrak{A}$ is uniquely determined up to *-isomorphism. Let J be a minimal left ideal of A. Then $J = Ae$, where e is a minimal self-adjoint idempotent in A. Since, by Theorem(3.1.6), A has weak (β_k) property, by Theorem(3.1.2), J is also a minimal left ideal of $\mathfrak{A}$ and $J = \mathfrak{A}e$. It follows that the socle of $\mathfrak{A}$ exists and contains $\mathfrak{S}$ and, since $\mathfrak{S}$ is dense in A, we have $\mathfrak{A} = cl(\mathfrak{S})$. Thus the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$ and so $\mathfrak{A}$ is dual by [6, Theorem 2.1]. This proves (i).

(ii). By the proof of (i) and [11, Lemma(4.10.1)], it clearly suffices to show that every minimal self-adjoint idempotent of $\mathfrak{A}$ belongs to A. Let f be a minimal self-adjoint idempotent in $\mathfrak{A}$ and let $\{e_\alpha\}$ be the family of all minimal self-adjoint idempotents in A. Let $M = \mathrm{cl}(\mathfrak{A} f \mathfrak{A})$ and $M_\alpha = \mathrm{cl}(\mathfrak{A} e_\alpha \mathfrak{A})$ for all α. Since $\mathfrak{S}$ is dense in $\mathfrak{A}$, $M \cap M_\alpha \neq (0)$ for at least one α, say α_0. By the minimality of M and M_{α_0}, we have $M = M_{\alpha_0}$. Let $N = \mathrm{cl}_A(Ae_{\alpha_0}A)$ and let H be the Hilbert space Ae_{α_0} with the inner product $(\ ,\ )$. (By the weak (β_k) property, the norms $|\cdot|$ and $\|\cdot\|$ are equivalent on H). Since N is a simple annihilator A^*-algebra, its left regular representation on H contains all operators of finite rank, and, since $Ae_{\alpha_0} = \mathfrak{A} e_{\alpha_0}$, M is $*$-isomorphic to $LC(H)$. Hence $f \in N \subset A$, and the proof of (ii) is complete.

(iii). Since every $x \in \mathrm{cl}(I)\mathfrak{S}$ is of the form $x = \sum_{i=1}^{n} x_i y_i e_i$, where $x_i \in \mathrm{cl}(I)$, $y_i \in \mathfrak{S}$ and e_i a minimal idempotent $(i = 1, 2, \ldots, n)$, it suffices to show that, for $x \in \mathrm{cl}(I)$, $y \in \mathfrak{S}$, and a minimal idempotent e, we have $xye \in I$. Now $Ae = \mathfrak{A} e$ and the two norms $|\cdot|$ and $\|\cdot\|$ are equivalent on Ae (by the weak (β_k) property in A). Hence $\|xye\| \leq c\,|x|\,\|ye\|$, for some constant c. Let $\{x_n\}$ be a sequence in I such that $|x_n - x| \to 0$ as $n \to \infty$. Since $\|x_n ye - xye\| \leq c|x_n - x|\,\|ye\|$, $\|x_n ye - xye\| \to 0$ as $n \to \infty$, which shows that $xye \in I$. Hence $\mathrm{cl}(I)\mathfrak{S} \subset I$.

(iv). It is clear that $\mathrm{cl}(I)$ is a closed right ideal of $\mathfrak{A}$. Let $\{e_\beta\}$ be the set of all minimal self-adjoint idempotents in $\ell(\mathrm{cl}(I))$. Since $e_\beta \in \mathfrak{S} \subset A$, $e_\beta \in \ell(\mathrm{cl}(I)) \cap A = \ell_A(I)$ for all β. Now, by [7, Lemma 6], $\mathrm{cl}(\sum_\beta \mathfrak{A}e_\beta) = \ell(\mathrm{cl}(I))$ and so $\mathrm{cl}(\ell_A(I)) \supset \mathrm{cl}(\sum_\beta Ae_\beta) = \mathrm{cl}(\sum_\beta \mathfrak{A}e_\beta) = \ell(\mathrm{cl}(I))$. But $\ell_A(I) \subset \ell(\mathrm{cl}(I))$. Hence $\mathrm{cl}(\ell_A(I)) = \ell(\mathrm{cl}(I))$.

(v). By the duality of $\mathfrak{A}$ and (iv), we have

$$r_A(\ell_A(I)) = r(\ell_A(I)) \cap A = r(\ell(\mathrm{cl}(I))) \cap A = \mathrm{cl}(I) \cap A .$$

This completes the proof of the lemma.

From Theorem(2.2.5) and Lemma(3.2.1) we see that if A is either a complemented or an annihilator A*-algebra, then A can be imbeded as a dense subalgebra in a unique (up to *-isomorphism) B*-algebra $\mathfrak{A}$. From now on we shall refer to $\mathfrak{A}$ as the <u>completion</u> of A.

<u>Theorem(3.2.2)</u> : Let A be an annihilator A*-algebra. Then the following statements are equivalent:

(i) A is dual.

(ii) x belongs to the closure of xA for all $x \in A$.

(iii) For every closed right ideal I of A and $x \in A$, $xx^* \in I$ implies $x \in I$.

<u>Proof</u> : (i) $\Longrightarrow$ (ii). This is [11, Corollary(2.8.3)].

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds. Let $\mathfrak{A}$ be the completion of A and let $\mathrm{cl}(S)$ (resp. $\mathrm{cl}_A(S)$) denotes the closure of the set S in $\mathfrak{A}$ (resp. A). Since the socle $\mathfrak{S}$ of A is dense in A, $\mathrm{cl}_A(x\mathfrak{S}) = \mathrm{cl}_A(xA)$ for all $x \in A$. If $xx^* \in I$, then $xx^* \in \mathrm{cl}(I)$ and therefore, by [11, Corollary(4.9.3)], $x \in \mathrm{cl}(I) \cap A$. Applying Lemma(3.2.1) (iii), we obtain $x \quad \mathrm{cl}_A(x\quad) \quad \mathrm{cl}_A(\mathrm{cl}(I)\quad) \quad I$.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds, and let I be a closed right ideal of A. By Lemma(3.2.1) (v), $cl(I) \cap A = r(\ell(I))$. Let $x \in cl(I) \cap A$. Since $xx^* \in cl_A(xA)$, $x \in cl_A(xA) = cl_A(x\mathfrak{A}) \subset cl_A(cl(I)\mathfrak{A}) \subset I$ (by Lemma(3.2.1) (iii)), which shows that $x \in I$, and hence $cl(I) \cap A \subset I$. Therefore $I = cl(I) \cap A = r(\ell(I))$. Since the involution is continuous, it follows that A is dual.

Theorem(3.2.3): Let A be an A*-algebra. Then A is dual if and only if the following conditions are satisfied:

(i) every closed right ideal I of A is the intersection of maximal modular right ideals containing it ;

(ii) for every maximal modular right ideal M, we have $\ell(M) \neq (0)$.

Proof: Suppose A is dual and let $\mathfrak{A}$ be the completion of A. By Lemma(3.2.1), $\mathfrak{A}$ is dual and hence complemented ([1, Theorem 3.6]). Let I be a closed right ideal of A and let $R = cl(I)$, the closure of I in $\mathfrak{A}$. By Corollary (2.1.3), $R = \bigcap_\alpha N_\alpha$, where $\{N_\alpha\}$ is the family of all maximal modular right ideals of $\mathfrak{A}$ containing R. By Lemma (2.1.1), $N_\alpha = \{x - e_\alpha x : x \in \mathfrak{A}\}$, where e_α is a minimal idempotent in $\mathfrak{A}$ for all α. By Lemma(3.2.1) (ii), e_α is also a minimal idempotent of A. Hence $M_\alpha = N_\alpha \cap A$ is a maximal modular right ideal of A, and $I \subset M_\alpha$ for all α. From the proof of Theorem(3.2.2), $I = R \cap A$ and hence, since $R \cap A = \bigcap_\alpha (N_\alpha \cap A) = \bigcap_\alpha M_\alpha$, we obtain $I = \bigcap_\alpha M_\alpha$. This proves (i). (ii) is clearly true.

Conversely, suppose (i) and (ii) hold. Then clearly A

is an annihilator algebra. Therefore, by [2, Theorem 1], every maximal closed right ideal M of A is of the form $M = \{ x - ex : x \in A \}$, where e is a minimal idempotent. By Lemma(3.2.1) (ii), $N = \{ x - ex : x \in \mathfrak{A} \}$ is a maximal closed right ideal of $\mathfrak{A}$,and clearly $N = cl(M)$ and $M = N \cap A$. Now let I be a closed right ideal of A and $\{ M_\alpha \}$, the family of maximal closed right ideals containing I . Let $N_\alpha = cl(M_\alpha)$. It is easy to see that $cl(I) = \bigcap_\alpha N_\alpha$ and that $cl(I) \cap A = \bigcap_\alpha (N_\alpha \cap A) = \bigcap_\alpha M_\alpha = I$. It follows now from Lemma(3.2.1) (v) and the continunity of the involution that A is dual.

Theorem(3.2.4): Let A be an A*-algebra which is a dense subalgebra of a dual B*-algebra $\mathfrak{A}$. Then A is dual if and only if the following conditions are satisfied :

(i) A and $\mathfrak{A}$ have the same self-adjoint minimal idempotents;

(ii) $cl(I) \cap A \neq A$ for every proper closed right ideal I of$A$;

(iii) for every closed right ideal I of A and $x \in A$, $xx^* \in I$ implies $x \in I$.

Proof: (We use the notation of Lemma(3.2.1)) Suppose that A is dual. (i) follows from Lemma(3.2.1). Let I be a proper closed right ideal of A . By the proof of Theorem(3.2.2), $I = cl(I) \cap A \neq A$. This proves (ii). (iii) follows from Theorem(3.2.2).

Conversely suppose that (i),(ii) and (iii) hold. Let I be a proper closed right ideal of A . By (ii) $cl(I)$ is a proper closed right ideal of $\mathfrak{A}$. Let $\{e_\beta\}$ be the set of all minimal idempotents in $\ell(cl(I))$. Then $\{ e_\beta \} \subset$

$\subset \ell(\mathrm{cl}(I)) \cap A = \ell_A(I)$. By the proof of Lemma(3.2.1) (iv), we have $\mathrm{cl}(\ell_A(I)) = \ell(\mathrm{cl}(I))$ which shows that $\ell_A(I) \neq (0)$. By the continunity of the involution in A , A is an annihilator algebra. Thus (iii) and Theorem(3.2.2) shows that A is dual.

Lemma(3.2.5): Let A be an annihilator A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$. A commutative $*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is maximal in $\mathfrak{A}$ if and only if $\mathfrak{B}$ is of the form $\mathrm{cl}(B)$, where B is a maximal commutative $*$-subalgebra of A .

Proof: By Lemma(3.2.1), $\mathfrak{A}$ is dual. Let B be a maximal commutative $*$-subalgebra of A and $\mathfrak{B} = \mathrm{cl}(B)$. We show that $\mathfrak{B}$ is a maximal commutative $*$-subalgebra of $\mathfrak{A}$. In fact, let M be a maximal commutative $*$-subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$. By [8, Theorem 1] , M is dual. Let e be a self-adjoint minimal idempotent of M . By Lemma(1.2.5) , e is minimal in $\mathfrak{A}$ and so $e \in A$ (Lemma(3.2.1)). Since $B \subset M$, e commutes with B and therefore, by the maximality of B , $e \in B$. Let $x,z \in B$ and $y \in M$, then $(exy)z = z(exy)$. Now, by the proof of Lemma(3.2.1), $e\mathfrak{A} = eA$, hence $exy \in A$ and the maximality of B gives $exy \in B$. Hence eB is a closed right ideal of M and so $eB = eM$. Since the socle of M is dense in M , B is dense in M and hence $\mathfrak{B} = \mathrm{cl}(B) = M$.

Conversely, let $\mathfrak{B}$ be a maximal commutative $*$-subalgebra of $\mathfrak{A}$ and let $B = \mathfrak{B} \cap A$. By [8, Theorem 1] , $\mathfrak{B}$ is dual. We show that $\mathrm{cl}(B) = \mathfrak{B}$ and B is a maximal commutative $*$-subalgebra of A . In fact, let e be a self-adjoint minimal idempotent of $\mathcal{B}$. Lemma(1.2.5) shows that e is minimal in $\mathfrak{A}$ and hence in A and consequently $e \in \mathcal{B} \cap A = B$. But, by the proof of Lemma(3.2.1) , $e\mathfrak{A} = eA$. Hence $e\mathcal{B} \subset A \cap \mathcal{B} = B$ and since the socle of $\mathcal{B}$ is dense in $\mathcal{B}$, it follows that $cl(B) = \mathcal{B}$. Let N be any commutative *-subalgebra of A containing B . Then $cl(N) \supset cl(B) = \mathcal{B}$ and so, by the maximality of $\mathcal{B}$, $cl(N) = \mathcal{B}$. Thus $N \subset cl(N) \cap A = \mathcal{B} \cap A = B$ which shows that B is maximal in A and the proof is complete.

Theorem(3.2.6): Let A be an A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$. Then A is dual if and only if the following conditions are satisfied :

(i) every maximal commutative *-subalgebra $\mathcal{B}$ of $\mathfrak{A}$ is of the form $cl(B)$, where B is a maximal commutative *-subalgebra of A ;

(ii) for every maximal commutative *-subalgebra B of A , B and $\mathcal{B} = cl(B)$ have the same socle $\mathfrak{S}$ such that $cl(\mathfrak{S}) = \mathcal{B}$;

(iii) $cl(I) \cap A \neq A$ for every proper closed right ideal I of A ;

(iv) for every closed right ideal I of A and $x \in A$, $xx^* \in I$ implies $x \in I$.

Proof: Suppose that A is dual. Then $\mathfrak{A}$ is dual. (i) and (ii) follows from the proof of Lemma(3.2.5) and (iii) and (iv) are given by Theorem(3.2.4).

Conversely suppose (i), (ii), (iii) and (iv) hold. By (i) and (ii) , every maximal commutative *-subalgebra $\mathcal{B}$ of $\mathfrak{A}$ is dual and so $\mathfrak{A}$ is dual ([8, Theorem 1]). Let e

be a self-adjoint minimal idempotent of $\mathfrak{A}$ and let $\mathfrak{B}$ be a maximal commutative *-subalgebra of $\mathfrak{A}$ containing e . By Lemma(1.2.5), e is minimal in $\mathfrak{B}$ and therefore by (ii), $e \in A$. Thus A and $\mathfrak{A}$ have the same self-adjoint minimal idempotents and hence, by Theorem(3.2.4), A is dual.

Theorem(3.2.7): Let A be an A*-algebra which is a dense 2-sided ideal of a B*-algebra $\mathfrak{A}$. Then A is dual if and only if the following conditions are satisfied :

(i) every maximal commutative *-subalgebra B of A is dual,

(ii) every maximal commutative *-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is of the form $\mathfrak{B} = \mathrm{cl}(B)$, where B is a maximal commutative *-subalgebra of A ;

(iii) for every closed right ideal I of A and $x \in A$, $xx^* \in I$ implies $x \in I$.

Proof: Suppose that A is dual. Then every maximal commutative *-subalgebra of A is dual by [7, Theorem 19], whence (i). (ii) and (iii) follow from Lemma(3.2.5) and Theorem(3.2.2) respectively.

Conversely, suppose that conditions (i), (ii) and (iii) hold in A . By (i) and (ii) and Lemma(3.2.1), every maximal commutative *-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is dual and hence $\mathfrak{A}$ is dual by [8, Theorem 1]. Let $x \in A$. Since $xx^* \in \mathrm{cl}_A(xA)$, the closure of xA in A ,by (iii), $x \in \mathrm{cl}_A(xA)$ and therefore, by [7, Lemma 8], A is dual.

Theorem(3.2.8): Let A be a dual A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$. Then A is a 2-sided ideal of $\mathfrak{A}$ if and only if there exists a maximal orthogonal family of self-adjoint minimal idempotents $\{e_\alpha\}$ in A such that

$\sum_{\alpha} xye_{\alpha}$ exists in the norm $||\cdot||$ for all $x \in A$ and $y \in \mathfrak{A}$.

Proof: Since, by Lemma(3.2.1), $\mathfrak{A} e_{\alpha} = Ae_{\alpha}$ for all α, $xye_{\alpha} \in A$ $(x \in A, y \in \mathfrak{A})$. Suppose that $\sum_{\alpha} xye_{\alpha}$ exists in the norm $||\cdot||$ for all $x \in A$, $y \in \mathfrak{A}$. By [7, Theorem 16], $\sum_{\alpha} xye_{\alpha} = xy$, where the summation is taken in the norm $|\cdot|$ and so $xy \in A$. Since $A = A^*$, A is a 2-sided ideal of $\mathfrak{A}$. The converse of the theorem follows from [7, Theorem 16].

§3. w.c.c. A*-algebras

Definition(3.3.1): A Banach algebra A is said to be weakly completely continuous (w.c.c) if the left- and right- multiplication operators of every element in A are weakly completely continuous on A.

Lemma(3.3.2): The set B of all w.c.c. elements of a Banach algebra A is a closed 2-sided ideal of A.

Proof: [10] We may assume that $B \neq (0)$. It is well-known that the set of all weakly completely continuous operators on A is closed in the uniform topology (4 Corollary, P. 483, Linear operators, Part I, by N. Dunford and J. Schwartz) Thus B is closed. Let $x \in A$ and $y \in B$. We show that $xy \in B$. For each continuous linear functional f on A, let g be a linear functional on A given by $g(a) = f(xa)$ for all $a \in A$. Then g is continuous on A. Let $\{z_n\}$ be a bounded sequence in A. Since $y \in B$, there exists a subsequence $\{z_{n_k}\}$ such that yz_{n_k} converges weakly to an element $u \in A$. Thus $f(xyz_{n_k} - xu) =$

$= g(yz_{n_k} - u) \longrightarrow 0$ as $n_k \longrightarrow \infty$ and so $xy \in B$. Similarly we can show $yx \in B$.

Theorem(3.3.3) : An annihilator A*-algebra A is w.c.c.

Proof: Let $\mathfrak{A}$ be the completion of A in the norm $|\cdot|$. Then $\mathfrak{A}$ is dual and hence w.c.c. by [7, Theorem 8]. Let e be a minimal idempotent in A. By Lemma(3.2.1), we have $eA = e\mathfrak{A}$ and from its proof that $\|ex\| \leq c\|e\|\,|x|$ for all $x \in \mathfrak{A}$, where c is a constant (depending on e). Let $y \in A$ and let $\{y_n\}$ be any bounded sequence in A. Since $\mathfrak{A}$ is w.c.c. and $\{y_n\}$ is bounded in $|\cdot|$, there exists a subsequence $\{y_{n_k}\}$ such that $\{yy_{n_k}\}$ converges weakly to an element $z \in \mathfrak{A}$. For each continuous linear functional f on A, let g be a linear functional on $\mathfrak{A}$ given by $g(x) = f(ex)$ $(x \in \mathfrak{A})$. Since $|g(x)| = |f(ex)| \leq$ $\leq \|f\|\,\|ex\| \leq c\|f\|\,\|e\|\,|x|$, where $\|f\|$ denotes the norm of f with respect to $\|\cdot\|$, it follows that g is continuous on $\mathfrak{A}$. Now $ez \in A$ and $f(eyy_{n_k} - ez) =$ $= g(yy_{n_k} - z) \longrightarrow 0$ as $n_k \longrightarrow \infty$, and so ey is a w.c.c. element of A. This shows that every element of $\mathfrak{S}$ is a w.c.c. element of A. Since $\mathfrak{S}$ is dense in A and the set of all w.c.c. elements of A is a closed 2-sided ideal of A (Lemma(3.3.2)), A is w.c.c.

Theorem(3.3.4): Let A be an A*-algebra which is a dense 2-sided ideal of a B*-algebra $\mathfrak{A}$. Then A is an annihilator algebra if and only if A is w.c.c. and A^2 is dense in A.

Proof : If A is an annihilator algebra, Theorem(3.3.3) shows that A is w.c.c., and , since A^2 contains the socle of A , A^2 is dense in A . Conversely, suppose that A is w.c.c. and A^2 is dense in A . Then, by [7, Lemmas 9 and 10], $\mathfrak{A}$ is w.c.c. and hence , by Corollary(3.1.3), A has weak (β_k) property. Let $\mathfrak{S}$ be the socle of A and let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents in $\mathfrak{A}$. Then, for all $x, y \in A$, we have $xy = \sum_\alpha e_\alpha xy$, the summation being taken relative to the norm $\|\cdot\|$. (see [7, proof of Theorem 16]). Thus we have that, $xy \in cl_A(\mathfrak{S})$,(the closure of $\mathfrak{S}$ in A), which shows that $cl_A(\mathfrak{S}) = cl_A(A^2) = A$. This completes the proof of the theorem.

Chapter IV

Complemented A*-algebras

§1. Complementors and duality in A*-algebras

In this section we shall give several characterizations of duality in A*-algebra in terms of complementors. Throughout this section, we shall use the notation of Lemma(3.2.1).

Theorem(4.1.1): Let A be a semi-simple complemented Banach*-algebra with weak (β_k) property. Then A is a dual A*-algebra.

Proof: By [14, Lemma 5] and Theorem(2.2.5) and Theorem (3.1.6) , A is an annihilator A*-algebra. Let $\mathfrak{A}$ be the completion of A and let I be a closed right ideal of A. We claim that $cl(I) \cap A = I$. Let $J = cl(I) \cap A$. Then J is a closed right ideal of A, and clearly $I \subset J$. Let $\{ e_\alpha \}$ be the family of all minimal p-projections contained in I. By Lemma(2.1.2), $I = cl_A(\sum_\alpha e_\alpha A)$, and hence $cl(I) = cl(\sum e_\alpha A)$. If $I \neq J$, then $I^p \cap J \neq (0)$ and so, by Lemma(2.1.2), there exists a minimal p-projection $f \in I^p \cap J$. Since $e_\alpha \in I$ and $f \in I^p$, we have $e_\alpha f = f e_\alpha = 0$ for all α, which shows that $f cl(I) = (0)$. But this is a contradiction since $f \in cl(I)$ and f is an idempotent $\neq 0$. Hence $J = I$ and consequently, by Lemma(3.2.1), $I = \ell_A(r_A(I))$. Applying now the continuity of the involution, we obtain that A is dual.

Corollary(4.1.2) : An annihilator complemented A*-algebra is dual.

Proof : Follows from Theorems(3.1.6) and (4.1.1).

From Theorem(2.1.4) and Corollary(4.1.2), we have the following result :

Theorem(4.1.3) : Let A be a complemented A*-algebra. Then the following statements are equivalent :

(i) A is dual.

(ii) Every non-zero right ideal contains a minimal p-projection.

(iii) Every maximal closed right ideal is modular.

(iv) Every maximal closed right ideal has a non-zero left annihilator.

A B*-algebra is dual if and only if it is complemented ([1, Theorem 3.6]). In next theorem, we shall show that similar result holds for an A*-algebra which is a dense 2-sided ideal of a B*-algebra. We shall often use, without explicitly mentioning, the following fact about dual B*-algebra : If A is a dual B*-algebra, then the mapping $R \longrightarrow \ell(R)^*$ on the set of all closed right ideals R of A is a complementor on A (see [14, P. 652]).

Theorem(4.1.4) : Let A be an A*-algebra which is a dense 2-sided ideal of a B*-algebra $\mathfrak{A}$. Then A is dual if and only if A is complemented.

Proof : Suppose A is complemented. Then, by Theorem (2.2.5), $\mathfrak{A}$ is dual and therefore, by Corollary(3.1.3), A has weak (β_k) property. Theorem(4.1.1) now shows that A is dual.

Conversely, suppose A is dual. Let I be a closed right ideal of A and let $R = cl(I)$; R is a closed right ideal of $\mathfrak{A}$. Let $\{e_\alpha\}$ (resp. $\{e_\beta\}$) be a maximal orthogonal family of minimal self-adjoint idempotents contained in R (resp. in $\ell(R)^*$). From [7, Lemma 6] , $R = cl(\sum_\alpha e_\alpha \mathfrak{A})$ and $\ell(R)^* = cl(\sum_\beta e_\beta \mathfrak{A})$. Since $\{e_\beta\} \subset \subset \ell(R)$, $\{e_\beta\}$ is orthogonal to $\{e_\alpha\}$. Let $\{e_\gamma\} = \{e_\alpha\} \cup \{e_\beta\}$. If a is an element of $\mathfrak{A}$ such that $ae_\gamma = 0$ for all γ , then $a(R + \ell(R)^*) = a\mathfrak{A} = = (0)$, which gives $a = 0$. Thus $\{e_\gamma\}$ is a maximal orthogonal family of minimal self-adjoint idempotents in $\mathfrak{A}$. Since A is a dense 2-sided ideal of A , by Lemma(1.2.4) , each $e_\gamma \in A$ and therefore $\{e_\gamma\}$ is also a maximal orthogonal family of minimal self-adjoint idempoatents in A ; clearly $\{e_\alpha\} \subset R \cap A = I$ and $\{e_\beta\} \subset \ell(R)^* \cap A = = \ell_A(I)^*$ (see proof of Theorem(3.2.2)). Let $x \in A$. By [7, Theorem 16] , $\sum_\alpha e_\alpha x$, $\sum_\beta e_\beta x$ and $\sum_\gamma e_\gamma x$ are all summable relative to $||\cdot||$, and $x = \sum_\gamma e_\gamma x$. Since I and $\ell_A(I)$ are closed in A , $\sum_\alpha e_\alpha x \in I$ and $\sum_\beta e_\beta x \in \ell_A(I)^*$. Now $\mathfrak{A} = R \oplus \ell(R)^*$ so that

$x = x_1 + x_2$ with $x_1 \in R$ and $x_2 \in \ell(R)^*$. But, by [7, Lemma 6] , $x_1 = \sum_{\alpha} e_\alpha x_1$ and $x_2 = \sum_{\beta} e_\beta x_2$, where the summations are taken relative to $\|\cdot\|$. Hence $e_\alpha x = e_\alpha x_1$ and $e_\beta x = e_\beta x_2$ for all α , β . Therefore $\sum_{\alpha} e_\alpha x = \sum_{\alpha} e_\alpha x_1$ and so $x_1 = \sum_{\alpha} e_\alpha x \in I$; similarly $x_2 = \sum_{\beta} e_\beta x \in \ell_A(I)^*$. Hence $A = I + \ell_A(I)^*$. If $x \in I \cap \ell_A(I)^*$, then $x^*x = 0$ and so $x = 0$. Hence $I \cap \ell_A(I)^* = (0)$; moreover ,

$$\ell_A(\ell_A(I)^*)^* = \ell_A(r_A(I^*))^* = I^{**} = I .$$

Also, if $I_1 \supset I_2$, then $\ell_A(I_2)^* \supset \ell_A(I_1)^*$. Therefore, the mapping $I \longrightarrow \ell_A(I)^*$ is a right complementor on A and the proof is complete.

Theorem(4.1.5) : Let A be an annihilator A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$. Then the following statements are equvalent :

(i) A is dual.

(ii) There exists a complementor p on $\mathfrak{A}$ such that

$$(\#) \qquad I = cl(cl(I)^p \cap A)^p \cap A ,$$

holds for every closed right ideal I of A .

Proof : (i) $\Longrightarrow$ (ii) . Suppose (i) holds. By Lemma(3.2.1), $\mathfrak{A}$ is dual and hence the mapping $p : R \longrightarrow \ell(R)^*$ is a complementor on $\mathfrak{A}$. Therefore using the duality of A and the fact that $\ell(cl(I)) \cap A = \ell_A(I)$ and $r(cl(I)^*) \cap A = r_A(I^*)$, we can easily show that

$$I = cl(cl(I)^p \cap A)^p \cap A .$$

(In fact, $cl(I)^p \cap A = \ell(cl(I))^* \cap A = r(cl(I)^*) \cap A = r_A(I^*)$ and so

$$cl(cl(I)^p \cap A)^p \cap A = \ell(cl(r_A(I^*)))^* \cap A =$$

$$= \ell_A(r_A(I^*))^* \cap A = \ell_A(r_A(I^*))^* = I \quad).$$

(ii) $\Longrightarrow$ (i) . Now suppose (ii) holds. Then $cl(I) \subset \subset cl(cl(I)^p \cap A)^p$ and hence

$$I \subset cl(I) \cap A \subset cl(cl(I)^p \cap A)^p \cap A = I ,$$

which shows that $I = cl(I) \cap A$. But, by Lemma(3.2.1), $cl(I) \cap A = r_A(\ell_A(I))$. Therefore $I = r_A(\ell_A(I))$ and

so, by the continunity of the involution, A is dual. This completes the proof of the theorem.

<u>Theorem(4.1.6)</u> : Let A be an A*-algebra which is a dense 2-sided ideal of a B*-algebra $\mathfrak{A}$. Then A is dual if and only if there exists a complementor p on $\mathfrak{A}$ satisfying condition (#) of Theorem(4.1.5).

<u>Proof</u> : If A is dual, the existence of the complementor p with the required property follows from Theorem(4.1.5).

Conversely, suppose that p is such a complementor on $\mathfrak{A}$. Then $\mathfrak{A}$ is dual, and clearly A and $\mathfrak{A}$ have the same socle $\mathfrak{S}$. From condition (#) we have that $cl_A(\mathfrak{S}) = A$, and from Corollary(3.1.3) that A has weak (β_k) property. Hence, by Theorem(3.1.6), A is an annihilator algebra and therefore dual by Theorem(4.1.5).

Remark : Let A be as in Theorem(4.1.5) or as in Theorem (4.1.6). It is easy to show that if there exists a complementor p on $\mathfrak{A}$ such that (#) holds for p, then (#) holds for every complementor q on $\mathfrak{A}$. For if p is such a complementor on $\mathfrak{A}$, then A is dual. Let $R = (cl(I))^q$ and $\{e_\alpha\}$ the family of all minimal self-adjoint idempotents contained in R. By [7, Lemma 6], $R = cl(\sum_\alpha e_\alpha \mathfrak{A})$. Since, by Lemma(3.2.1), $e_\alpha \mathfrak{A} \subset R \cap A$ for all α, $R \subset cl(R \cap A)$ which gives $R = cl(R \cap A)$. Thus, by the duality of A, we have

$$cl(\,cl(I\,)^q \cap A\,)^q \cap A = cl(R \cap A\,)^q \cap A =$$

$$= R^q \cap A = cl(I) \cap A = I\,.$$

Theorem(4.1.7) : Every complemented A*-algebra A which is a dense 2-sided ideal of LC(H) is a 2-sided ideal of L(H).

Proof : By Theorem(4.1.4), A is dual. Let $x \in A$, $y \in L(H)$ and let $\{e_\alpha\}$ be a maximal orthogonal family of minimal self-adjoint idempotents in A. By [7, Theorem 16], $x = \sum_\alpha e_\alpha x$, the summation being taken with respect to $\|\cdot\|$, and therefore only a countable number of $e_{\alpha}x \neq 0$; denote those e_{α} for which $e_{\alpha}x \neq 0$ by e_{α_1} , e_{α_2} , ... Clearly $ye_{\alpha_i} \in A$ ($i = 1, 2, \ldots\ldots$). For any two positive integers m, n ($m \leq n$) , Lemma(1.2.3) shows that

$$\left\| \sum_{i=1}^{n} ye_{\alpha_i}x - \sum_{i=1}^{m} ye_{\alpha_i}x \right\| =$$

$$= \left\| \left(y \sum_{i=m+1}^{n} e_{\alpha_i}\right)\left(\sum_{i=m+1}^{n} e_{\alpha_i}x\right) \right\| \leq$$

$$\leq k \left| y \sum_{i=m+1}^{n} e_{\alpha_i} \right| \left\| \sum_{i=m+1}^{n} e_{\alpha_i}x \right\| \leq$$

$$\leq k |y| \left\| \sum_{i=m+1}^{n} e_{\alpha_i}x \right\| ,$$

where k is a constant. Therefore $\left\{ \sum_{i=1}^{n} ye_{\alpha_i}x \right\}$ is a Cauchy sequence in A and so there exists an element $z \in A$ such that $z = \sum_{i=1}^{\infty} ye_{\alpha_i}x$. Since $x = \sum_{i=1}^{\infty} e_{\alpha_i}x$, we have $yx = \sum_{i=1}^{\infty} ye_{\alpha_i}x$, where both of the summations being taken relative to $|\cdot|$, and so $yx = z \in A$. Since $A = A^*$ and $(xy)^* = y^*x^* \in A$, $xy \in A$. This completes the proof.

Definition(4.1.8) ; A Banach algebra A is said to be completely continuous (c.c.) if the left- and right- multiplication operators of every element of A are completely continuous on A .

Theorem(4.1.9) : A complemented c.c. A*-algebra is dual.

Proof : By Theorem(2.2.3) , A is the direct topological sum of all its minimal closed 2-sided ideals I_λ , each of which is a simple c.c. complemented A*-algebra. Since each I_λ is finite dimensional (see [7, P.23]), it is dual. Therefore, by [11, Theorem(2.8.9)] , A is an annihilator algebra and so by Corollary(4.1.2) A is dual.

§2. Continuous complementors on LC(H)

Let A be a B*-algebra with a complementor p and f a minimal idempotent in A . Since A is dual and every maximal closed right ideal of A is modular, by Lemma(2.1.1), there exists a unique minimal p-projection e in A such that fA = eA .

Definition(4.2.1) : Let A be a B*-algebra with a complementor p . Let E denote the set of all self-adjoint minimal idempotents f and E_p the set of all minimal p-projections of A . For each f $\in$ E , let P(f) be the unique element of E_p such that P(f)A = fA . The complementor p is said to be continuous if P is continuous in the relative topologies of E and E_p induced by the given norm on A.

Remark : [1] Since, by [11, Lemma(4.10.1)] , every minimal right ideal of A is of the form fA with unique f $\in$ E , it follows that P maps E onto E_p.

In the rest of the section, H will denote a fixed complex Hilbert space with inner product $(\, , \,)$, and A the algebra $LC(H)$.

Notation : For every closed subspace S of H let $\mathcal{J}(S) = \{ a \in A : a(H) \subset S \}$. For every closed right ideal R of A let $\mathcal{S}(R)$ be the smallest closed subspace of H that contains the range $a(H)$ of each operator a in R.

Lemma(4.2.2) : For every closed right ideal R of A, $R = \mathcal{J}(\mathcal{S}(R))$; and for every closed subspace S of H, $\mathcal{J}(S)$ is a closed right ideal and $S = \mathcal{S}(\mathcal{J}(S))$.

Proof : [1] . Let R be a closed right ideal of A. Since A is dual, by [7, Lemma(2.8.24)], $R = \mathcal{J}(\mathcal{S}(R))$. Let S be a closed subspace of H. We show that $\mathcal{J}(S)$ is a closed right ideal of A. Clearly $\mathcal{J}(S)$ is a right ideal of A. Let P be the orthogonal projection on S, $T \in cl(\mathcal{J}(S))$ and $\{ T_n \}$ a sequence in $\mathcal{J}(S)$ converging to T. Since $ET_n = T_n$ for all n, we obtain $T = ET$, which shows that $T(H) \subset S$ and hence $T \in \mathcal{J}(S)$. Therefore $\mathcal{J}(S)$ is a closed right ideal of A. Clearly $\mathcal{S}(\mathcal{J}(S)) \subset S$. If $m \in S$, then $m \otimes m \in \mathcal{J}(S)$ and so $m \in \mathcal{S}(\mathcal{J}(S))$ which shows that $S \subset \mathcal{S}(\mathcal{J}(S))$. Hence $\mathcal{S}(\mathcal{J}(S)) = S$.

Remark : [1] Lemma(4.2.2) shows that $R \longrightarrow \mathcal{S}(R)$ defines a one-to-one correspondence between the closed right ideals of A and the closed subspaces of H. Moreover if

p is a complementor on A, $S \longrightarrow S^{p'} = \mathcal{S}(\mathcal{J}(S)^{p})$ defines a complementor on the closed subspaces S of H in the sense of [3, Theorem 1]. In fact, let $x, y \in H$, $\|y\| = 1$ and let $T = x \otimes y$. Write $T = T_1 + T_2$ with $T_1 \in \mathcal{J}(S)$ and $T_2 \in \mathcal{J}(S)^{p}$. Then $T_1 y \in \mathcal{S}(\mathcal{J}(S)) = S$ and, since $\mathcal{J}(S)^{p} = \mathcal{J}(S^{p'})$, $T_2 y \in \mathcal{S}(\mathcal{J}(S^{p'})) = S^{p'}$. Since $x = Ty = T_1 y + T_2 y$, $H = S + S^{p'}$. Clearly $(S^{p'})^{p'} = S$, $S^{p'} \cap S = (0)$. If $S_1 \supset S_2$, then $S_1^{p'} \subset S_2^{p'}$. We shall say $S \longrightarrow S^{p'}$ is the complementor on the closed subspaces of H induced by the complementor p on A.

Conversely every complementor $S \longrightarrow S^{p'}$ on the closed subspaces S of H induces a complementor p on A given by the relation $R^{p} = \mathcal{J}(\mathcal{S}(R)^{p'})$ for every closed right ideal R of A. In fact, let P be the projection on $\mathcal{S}(R)$ along $\mathcal{S}(R)^{p'}$ and P' be the projection on $\mathcal{S}(R)^{p'}$ along $\mathcal{S}(R)$. By [13, Theorem 4.8 - D], P and P' are in $L(H)$. Since $\mathcal{S}(R) + \mathcal{S}(R)^{p'} = H$, $P + P' = 1$, the identity operator on H. Let $a \in A$. Clearly $Pa \in \mathcal{J}(\mathcal{S}(R)) = R$ and $P'a \in \mathcal{J}(\mathcal{S}(R)^{p'}) = R^{p}$. Since $a = Pa + P'a$, $A = R + R^{p}$. It is clear that $R^{pp} = R$, and $R \cap R^{p} = (0)$. If $R_1 \supset R_2$, then $R_1^{p} \subset R_2^{p}$. Therefore $R \longrightarrow R^{p}$ is a complementor on A.

<u>Notation</u> : For any non-zero element $x \in H$, $[x]$ will denote the subspace of H spanned by x, e_x the unique minimal p-projection in the minimal right ideal $\mathcal{J}([x])$

and f_x the orthogonal projection $\frac{x \otimes x}{(x,x)}$. By [1, Lemma 5.1], $e_x = x \otimes y$ for some non-zero $y \in H$.

Definition(4.2.3) : Let T be a semi-linear mapping of H onto itself (i.e. , for all $x, y \in H$ and $\alpha \in C$, $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha' Tx$, where $\alpha \longrightarrow \alpha'$ is an automorphism on C) . T is called a p-representing operator if it has the following properties :

(i) $e_x = \frac{x \otimes Tx}{(x, Tx)}$ for all non-zero $x \in H$;

(ii) $(x_o, Tx_o) = 1$ for all non-zero $x_o \in H$.

If the dimension of H is at least three, then, by [1, Theorem 5.3] , there exists a p-representing operator T for each complementor p on A . If, moreover, p is continuous then T is a continuous positive linear operator on H with continuous inverse which is unique to within a multiplicative positive constant. (see [1, Theorem 6.4 and Lemma 6.9]) . Thus, in the rest of this section, we assume that the dimension of H is at least three. p will denote a given complementor on A and T a p-representing operator of p . Since T is one-one and onto , T^{-1} exists. As before, the mapping $S \longrightarrow S^{p'}$ will denote the complementor on H induced by p .

Theorem(4.2.4) : The mapping $q' : S \longrightarrow S^{q'} = T([T^{-1}(S)]^{p'})$ $(= T(\mathcal{S}[\mathcal{J}(T^{-1}(S))^{p}]))$ is a complementor on the set of all closed subspaces S of H .

Proof : We show that $T^{-1}(S)$ and $S^{q'}$ are closed subspaces of H . In fact, this is clear if H is finite dimensional ([13, Theorem 3.12 - B]). If H is infinite dimensional, then , by [1, Theorem 6.8] , p is continuous and so, by [1, Theorem 6.4] , T and T^{-1} are continuous. Hence $T^{-1}(S)$ and $S^{q'}$ are closed. For each $x \in H$, write $T^{-1}x = y + z$ with $y \in T^{-1}(S)$ and $z \in [T^{-1}(S)]^{p'}$. Then $x = Ty + Tz$ with $Ty \in S$ and $Tz \in S^{q'}$ which shows $S + S^{q'} = H$. If $x \in S \cap S^{q'}$, then $T^{-1}x \in T^{-1}(S) \cap [T^{-1}(S)]^{p'} = (0)$ and so $x = 0$. Hence $S \cap S^{q'} = (0)$. Clearly $(S^{q'})^{q'} = S$ and if $S_1 \supset S_2$, then $S_1{}^{q'} \subset S_2{}^{q'}$. Therefore q' is a complementor on H and this completes the proof.

By the remark following Lemma(4.2.2), the mapping $q : \mathcal{J}(S) \longrightarrow \mathcal{J}(T([T^{-1}(S)]^{p'}))$ is a complementor on A such that q induces q' on H . Therefore p and q coincide if and only if p' and q' coincide.

Corollary(4.2.5) : p' and q' (or p and q) coincide if and only if $T(S)^{p'} = T(S^{p'})$ for all closed subspaces S of H .

Proof : Let $M = T(S)$. By the argument in the proof of Theorem(4.2.4) , M is a closed subspace of H . Since each closed subspace M of H is of this form, p' and q' coincide if and only if $M^{p'} = M^{q'}$, or $T(S)^{p'} = T(S^{p'})$.

Lemma(4.2.6) : $T(S^{p'}) = T(S)^{p'}$ for all $S \subset H$ if and only if $T([x]^{p'}) = (T[x])^{p'}$ for all $x \in H$.

Proof : Suppose $T([x]^{p'}) = (T[x])^{p'}$ for all $x \in H$. We show that $S^{p'} = \bigcap \{ [y]^{p'} : y \in S \}$. Let $M = \bigcap \{ [y]^{p'} : y \in S \}$. Since $[y] \subset M^{p'}$, $S = \bigcup [y] \subset M^{p'}$, which shows $S^{p'} \supset M$. Since $S^{p'} \subset [y]^{p'}$, for all $y \in S$, $S^{p'} \subset M$ and so $S^{p'} = M$. Therefore

$$T(S)^{p'} = \bigcap \{ [Ty]^{p'} : y \in S \} =$$

$$= \bigcap \{ T([y]^{p'}) : y \in S \} =$$

$$= T(\bigcap \{ [y]^{p'} : y \in S \}) = T(S^{p'}) .$$

The converse of the lemma is clear and this completes the proof .

Corollary(4.2.7) : Let q' be the complementor on H given in Theorem(4.2.4). p' and q' (or p and q) coincide if and only if $T([x]^{p'}) = (T[x])^{p'}$ for all $x \in H$.

Proof : Follows from Corollary(4.2.5) and Lemma(4.2.6).

Lemma(4.2.8) : Let p be a continuous complementor on A, and T a p-representing operator. For each non-zero element $x \in A$, we have $T(\mathcal{S}(e_x A)) = \mathcal{S}(e_x^* A)$ and $T(\mathcal{S}(1 - e_x)A) = \mathcal{S}((1 - e_x^*)A)$.

Proof : Since p is continuous, by [1, Theorem 6.4], T is a positive continuous linear operator on H. For each

$h \in H$, we have

$$T(e_x h) = T(\frac{x \otimes Tx}{(x, Tx)} h) = \frac{(h, Tx)}{(x, Tx)} Tx = \frac{(Th, x)}{(Tx, x)} Tx =$$

$$= \frac{Tx \otimes x}{(Tx, x)} Th = e_x^* Th \quad .$$

Since $T(\mathcal{S}(e_x A))$ and $\mathcal{S}(e_x^* A)$ are one dimensional subspaces of H and $A(H) = H$, it is easy to see that $T(\mathcal{S}(e_x A)) = \mathcal{S}(e_x^* A)$. For all $a \in A$, $h \in H$, we have

$$(1 - e_x^*)(ah) = (1 - e_x^*)(T(T^{-1}(ah))) = T(1 - e_x)\frac{y \otimes y}{(y, y)} y \ ,$$

where $y = T^{-1}(ah)$. Thus, by the continunity of T , $T(\mathcal{S}((1 - e_x)A)) \supset \mathcal{S}((1 - e_x^*)A)$. Similarly we can show $T(\mathcal{S}((1 - e_x)A)) \subset \mathcal{S}((1 - e_x^*)A)$ and so they are equal. This completes the proof.

Corollary(4.2.9) : $[Tx]^{p'} = \mathcal{S}((1 - e_{Tx})A)$ and $T([x]^{p'}) = \mathcal{S}((1 - e_x^*)A)$.

Proof : Since $\mathcal{J}([x]) = e_x A$, $[x]^{p'} = \mathcal{S}((1 - e_x)A)$. Similarly we have $[Tx]^{p'} = \mathcal{S}((1 - e_{Tx})A)$. By Lemma(4.2.8), $T([x]^{p'}) = \mathcal{S}((1 - e_x^*)A)$.

Theorem(4.2.10) : Let p be a continuous complementor on A and q' the complementor on H given in Theorem(4.2.4). Then the following statements are equivalent :

(i) p' and q' coincide.

(ii) $e_x^* = e_{Tx}$ for all non-zero $x \in H$.

(iii) $T = \alpha 1$, scalar multiple of the identity operator on H with $\alpha > 0$.

(iv) $R^p = \ell(R)^*$ for all closed right ideal R of A .

<u>Proof</u> : (i) $\Longrightarrow$ (ii). Suppose (i) holds. Let x be a non-zero element in H . Since, by Corollary(4.2.7), $T([x]^{p'}) = [Tx]^{p'}$, Lemma(4.2.2) and Corollary(4.2.9) give

$$(1) \qquad (1 - e_x^*)A = (1 - e_{Tx})A \ .$$

Since $\mathcal{S}((1 - e_x)A)^{p'} = \mathcal{S}(e_xA)$, Lemma(4.2.8) shows that $T((\mathcal{S}(1 - e_x)A)^{p'}) = \mathcal{S}(e_x^*A)$. By Corollary(4.2.5) and Lemma(4.2.8) and (1), we have

$$\mathcal{S}(e_{Tx}A) = \mathcal{S}((1 - e_x^*)A)^{p'} = T((\mathcal{S}(1 - e_x)A)^{p'}) = \mathcal{S}(e_x^*A) \ .$$

Since e_x^* and e_{Tx} are idempotents, by (1) and Lemma(2.1.1), $e_x^* = e_{Tx}$.

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds, and let $x \neq 0$; clearly $Tx \neq 0$. Since $e_x^* = e_{Tx}$, we have $Tx \otimes x =$

$= k(x)(Tx \otimes T^2x)$, where $k(x) = \dfrac{(Tx,\ x)}{(Tx,\ T^2x)}$ is a positive-valued function of x . Thus $Tx \otimes (x - k(x)T^2x) = 0$. Since $Tx \neq 0$, $x = k(x)T^2x$. We show that $k(x)$ is a constant. For all $x, y \in H$, we have

$$x + y = k(x + y)(T^2x + T^2y) \ ,$$

$$x + y = k(x)T^2x + k(y)T^2y \ .$$

If x, y are linearly independent, by [1, Lemma 5.1] , T^2x ,

T^2y are also so, and hence $k(x) = k(y) = k(x + y)$. In particular, for a fixed $x_0 \in H$, $k(\alpha x_0) = k(x')$ for all non-zero scalars $\alpha \in C$ and all non-zero $x' \in [x_0]^{\perp}$, the orthogonal complement of $[x_0]$ in H . Let $k(0) = k(x_0)$. Since every $x \in H$ can be written in the form $x = \alpha x_0 + x'$ with $x' \in [x_0]^{\perp}$ and $\alpha \in C$, we have

$$k(x)T^2x = x = \alpha x_0 + x' = k(\alpha x_0)T^2(\alpha x_0) + k(x')T^2(x')$$

$$= k(x_0)(T^2(\alpha x_0 + x')) = k(x_0)T^2x \quad .$$

Therefore k is a positive constant and so $x = kT^2x$. Thus $T^2 = k1$. Since T^2 has a unique positive self-adjoint square root and T is positive, $T = (T^2)^{\frac{1}{2}} = k^{\frac{1}{2}}1 = \alpha 1$, where $\alpha = k^{\frac{1}{2}} > 0$.

(iii) $\Longrightarrow$ (iv) . Suppose (iii) holds. By [1, Corollary 6.10] , we may assume that $T = 1$. We use the notation of [1, Lemma 6.12] . Since $T = 1$, $\langle x, y \rangle = (x, Ty) = (x, y)$, and so, by [1, Corollary 4.3] , $R^p = \ell(R)^{*\langle\cdot\rangle} = \ell(R)^*$,for all closed right ideals R of A .

(iv) $\Longrightarrow$ (ii). Suppose (iv) holds. Then, by [1. Corollary 4.4] , $e_x^* = e_x$. Let $f_x = \dfrac{x \otimes x}{(x, x)}$. Since $\mathcal{S}(f_xA) = \mathcal{S}(e_xA)$, $f_xA = e_xA$. Since f_x and e_x are self-adjoint, by [11, Lemma(4.10.1)] , $f_x = e_x$, which gives $Tx = kx$, for some constant k (depending on x) . Hence $e_{Tx} = e_x^*$

for all non-zero $x \in H$.

(ii) $\Longrightarrow$ (i). Suppose (ii) holds. By Corollary(4.2.7), it suffices to show $T([x]^{p'}) = [Tx]^{p'}$, for all non-zero $x \in H$. By Corollary(4.2.9), we have

$$T([x]^{p'}) = \mathcal{S}((1 - e_x^*)A) = \mathcal{S}((1 - e_{Tx})A) =$$

$$= [Tx]^{p'} .$$

This completes the proof of the theorem.

§3. Complementors on B*-algebras

In this section, we shall prove two results which will be used in the next section.

Let A be a Banach algebra with a complementor p, and I a closed right ideal of A. Let P_I deonte the projection on I along I^p. Since $I^p = \{x \in A : P_I(x) = 0\}$, by [13, Theorem 4.8 - D], P_I is continuous.

Theorem(4.3.1) : Let A be a B*-algebra with a complementor p and let $\{ I_\beta : \beta \in \Delta \}$ be the set of all minimal right ideals of A. The following statements are equivalent :

(i) $\{ |e| : e \in E_p \}$ is bounded, where E_p denotes the set of all minimal p-projections in A.

(ii) $\{ |P_{I_\beta}| : \beta \in \Delta \}$ is bounded, where $|P_{I_\beta}|$ denotes the operator bound of P_{I_β}.

(iii) There exists a constant k such that $k|a_1 + a_2| \geq$ $\geq |a_i|$ ($i = 1, 2$) for all $a_1 \in I_\beta$, $a_2 \in I_\beta^p$ ($\beta \in \Delta$) .

Proof : (i) $\Longrightarrow$ (ii). Suppose $\sup\{|e| : e \in E_p\} \leq c$, where c is a constant. Let I be a minimal right ideal of A . Then there exists an $e \in E_p$ such that $I = eA$, and $I^p = (1 - e)A$. Let $a \in A$. Since $a = ea + (1 - e)a$, $|P_I(a)| = |ea| \leq c|a|$ and so $|P_I| \leq c$. Thus $\{|P_{I_\beta}| : \beta \in \Delta\}$ is bounded.

(ii) $\Longrightarrow$ (iii). Suppose $\sup\{|P_{I_\beta}| : \beta \in \Delta\} \leq k$, where k is a constant. Then, for all $a_1 \in I_\beta$, $a_2 \in I_\beta^p$ and all β ,

$$|a_1| = |P_{I_\beta}(a_1 + a_2)| \leq |P_{I_\beta}|\,|a_1 + a_2|$$
$$\leq k|a_1 + a_2| .$$

Similarly we can show that $|a_2| \leq k|a_1 + a_2|$.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. Let $a \longrightarrow T_a$ be the left regular representation of A . Since A has an approximate identity, $|T_a| = |a|$ for all $a \in A$, where $|T_a|$ denotes the operator bound of T_a . Let $e \in E_p$. Since $a = (1 - e)a + ea$, by (iii), $k|a| \geq |ea|$ which shows $|e| = |T_e| \leq k$. Thus $\{|e| : e \in E_p\}$ is bounded. This completes the proof of the theorem.

Theorem(4.3.2) : Let A be a B*-algebra which has no minimal left ideals of dimension less than three. Let p be a continuous complementor on A and let E_p be the set of all minimal p-projections on A. Then p is uniformly continuous if and only if the set $\{ |e| : e \in E_p \}$ is bounded.

Proof : Suppose p is uniformly continuous. By [1, Theorem 7.4] , there exists an involution $*'$ in A for which $R^p = \ell(R)^{*'}$, for every closed right ideal R of A, and an equivalent norm $|\cdot|'$ on A satisfying the B*-condition for $*'$. Since, by [1, Corollary 4.4], $e^{*'} = e$ and hence $|e|' = 1$, it follows that $\{ |e| : e \in E_p \}$ is bounded.

Conversely, suppose $\sup \{ |e| : e \in E_p \} \leq k$, for some constant $k > 0$. We use the notation of the proof of [1, Theorem 7.4] . Let $\{ T_\lambda \}$ be the family of all p_λ-representing operators such that $\| T_\lambda^{-1} \| = 1$ for all λ . Then the set $\{ \| T_\lambda \| \}$ is bounded ; for if not, by the proof of [1. Theorem 7.4] , there would exists a sequence $\{ H_{\lambda_n} \} \subset \{ H_\lambda \}$ and elements $x_n, y_n \in H_{\lambda_n}$ ($n = 1, 2, \ldots\ldots$) such that $| e_{y_n} - e_{x_n} | \longrightarrow \infty$ and $| f_{y_n} - f_{x_n} | \longrightarrow 0$ as $n \longrightarrow \infty$, which would contradict the fact that $| e_{y_n} - e_{x_n} | \leq 2k$. It follows now from the proof of [1, Theorem 7.4] that p is uniformly continuous. This completes the proof.

§4. Induced complementors

Throughout this section we shall use the notation introduced in Lemma(3.2.1).

Let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$. Let p be a complementor on $\mathfrak{A}$ and q a complementor on A. In this section we are going to give conditions on A, $\mathfrak{A}$ and the complementors p and q such that:

(a) The mapping $I \longrightarrow \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A.

(b) The mapping $R \longrightarrow \mathrm{cl}(R \cap A)^q$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$.

We shall say that the complementor q is induced on A by p and the complementor p is induced on $\mathfrak{A}$ by q.

Lemma(4.4.1) : Let A be a dual A^*-algebra which is a dense 2-sided ideal of the B^*-algebra $LC(H)$. Then, for every complementor p on $LC(H)$, the mapping $I \longrightarrow \mathrm{cl}(I)^p \cap A$ on the closed right ideals I of A is a complementor on A.

Proof : Let p be a complementor on $LC(H)$ and q the mapping $I \longrightarrow \mathrm{cl}(I)^p \cap A$. If the dimension of H is finite, then $A = LC(H)$ and therefore $q = p$; so that q is a complementor on A. Now suppose that the dimension of H is infinite. Then, by [1, Theorem 6.8], p

is continuous and hence, by [1, Theorem 6.11], there exists an involution $*'$ on $LC(H)$ such that $R^p = \ell(R)^{*'}$, for every closed right ideal R of $LC(H)$. This means, by [1, Corollary 6.14], that there exists a positive operator $Q \in L(H)$ with continuous inverse Q^{-1} such that $a^{*'} = Q^{-1}a^*Q$ for all $a \in LC(H)$. Now, from Theorem(4.1.7), we know that A is a 2-sided ideal of $L(H)$. Hence $a^{*'} \in A$ for all $a \in A$ and therefore A is an A*-algebra under the involution $*'$ (and an auxiliary norm $|\cdot|'$ equivalent to $|\cdot|$). Since A is dual, $I \rightarrow \ell_A(I)^{*'}$ is a complementor on A (see proof of Theorem(4.1.4)) and we have

$$I^q = cl(I)^p \cap A = \ell(cl(I))^{*'} \cap A =$$

$$= (\ell(cl(I)) \cap A)^{*'} = \ell_A(I)^{*'} .$$

Thus q is a complementor on A and this completes the proof.

<u>Theorem(4.4.2)</u> : Let A be a dual A*-algebra which is a dense 2-sided ideal of a B*-algebra $\mathfrak{A}$. Suppose that $\mathfrak{A}$ has no minimal left ideals of dimension less than three. Then, for every uniformly continuous complementor p on $\mathfrak{A}$, the mapping $I \longrightarrow cl(I)^p \cap A$ on the closed right ideals I of A is a complementor on A.

<u>Proof</u> : Let p be a uniformly continuous complementor on $\mathfrak{A}$ and let q be the mapping $I \longrightarrow cl(I)^p \cap A$. Since

A is dual, A is the direct topological sum of all its minimal closed 2-sided ideals I_λ ($\lambda \in \Lambda$) . It is easy to see that cl(I_λ) is a minimal closed 2-sided ideal of $\mathfrak{A}$ and hence *-isomorphic to LC(H_λ) , for some Hilbert space H_λ ($\lambda \in \Lambda$) . Clearly $\mathfrak{A}$ is *-isomorphic to $(\sum_\lambda LC(H_\lambda))_0$. In the rest of the proof we identify $\mathfrak{A}$ with $(\sum LC(H_\lambda))_0$. For each λ , let p_λ be the complementor on LC(H_λ) induced by p . Then, by [1, Theorem 3.7] , each p_λ is continuous on LC(H_λ) . Therefore each p_λ gives rise to an involution $*'_\lambda$ on LC(H_λ) and a positive operator $Q_\lambda \in L(H_\lambda)$ with continuous inverse Q_λ^{-1} such that $a^{*'_\lambda} = Q_\lambda^{-1} a_\lambda^* Q_\lambda$ for all $a_\lambda \in$ LC(H_λ) (see proof of Lemma(4.4.1)) ; we may clearly take $| Q_\lambda | = 1$, for all λ . By the proof of [1 , Theorem 7.4] , $a \longrightarrow a^{*'} = (a_\lambda^{*'_\lambda})$ is an involution on $\mathfrak{A}$ under which $\mathfrak{A}$ is a B*-algebra and $R^p = \ell(R)^{*'}$, for all closed right ideals R of $\mathfrak{A}$. We show that A is closed under the involution *' . Let $H = \bigoplus_\lambda H_\lambda$, the Hilbert direct sum of H_λ and $Q = (Q_\lambda)$. Then Q is a positive operator in L(H) with bounded inverse such that $a^{*'} = Q^{-1}a^*Q$ for all $a \in \mathfrak{A}$. (see [1, proof of Theorem 7.4]). Let $\{ e_\alpha \}$ be a maximal orthogonal family of minimal self-adjoint idempotents in A . By the proof of Theorem(4.1.4) , every $x \in A$ can be written as $x = \sum e_\alpha x$, the summation being taken with respect to $\|\cdot\|$. Let e_{α_1} , e_{α_2} , ..., be those e_α's for which $e\,x \neq 0$. Since e_α belongs to some I_λ and $QI_\lambda = Q_\lambda I_\lambda \subset I_\lambda$ (by Theorem(4.1.7)), it follows that each $Qe_{\alpha_i} \in A$ and so $\sum_{i=1}^{n} Qe_{\alpha_i} x \in A$ for $n = 1, 2, \ldots\ldots$ (we identify A as a subalgebra of $L(H)$). It follows now from the proof of Theorem(4.1.7) that $\{ \sum_{i=1}^{n} Q\, e_{\alpha_i} x \}$ is a Cauchy sequence in A and that it converges to Qx (in the norm $\|\cdot\|$), i.e., $Qx = \sum_{i=1}^{\infty} Qe_{\alpha_i} x$. Hence $Qx \in A$ and so $x^*Q = (Qx)^* \in A$; similarly $Q^{-1}x \in A$. Therefore $x^{*'} = Q^{-1}x^*Q \in A$, for all $x \in A$. Thus $*'$ is an involution on A and therefore, since A is dual, $I \longrightarrow \ell_A(I)^{*'}$ is a complementor on A. Now, applying the argument in the proof of Lemma(4.1.1), we obtain that

$$I^q = cl(I)^p \cap A = \ell_A(I)^{*'} ,$$

which shows that q is a complementor on A . This completes tha proof.

<u>Corollary(4.4.3)</u> : Let A , $\mathfrak{A}$ and p be as in Theorem (4.4.2) . Then there exists an involution $*'$ in A such

that $I^q = \ell_A(I)^{*'}$ for every closed right ideal I of A.

Notation : Let A be an algebra of operators on a normed space X . For every closed subspace S of X , let $\mathcal{J}_A(S) = \{ a \in A : a(X) \subset S \}$. For every closed right ideal I of A , let $\mathcal{S}_A(I)$ be the smallest closed subspace of X that contains the range $a(X)$ of each operator a in I .

Lemma(4.4.4) : Let A be a dual A^*-algebra which is a dense subalgebra of $LC(H)$. Then, for every closed right ideal I of A , $I = \mathcal{J}_A(\mathcal{S}_A(I))$ and, for every closed subspace S of H , $\mathcal{J}_A(S)$ is a closed right ideal of A and $S = \mathcal{S}_A(\mathcal{J}_A(S))$.

Proof : It is easy to see that A is simple. Let I be a minimal left ideal of A . Then, by the weak (β_k) property, I is a Hilbert space under the inner product norm $| \cdot |_o$. By the proof of Theorem(3.1.6) , A is a dense subalgebra of $LC(I)$ and contains the set of all operators of finite rank on I . By Lemma(3.2.1), $LC(H)$ and $LC(I)$ are $*$-isomorphic. It is easy to see that, under this isomorphism, each minimal left ideal of $LC(H)$ maps onto a minimal left ideal of $LC(I)$ and so the socle of $LC(H)$ maps onto the socle of $LC(I)$. Thus A contains the set F of all operators of finite rank on H . Since F is a 2-sided ideal of A and A is simple, $cl_A(F) = A$. The proof can now be complete by using the argument (with obvious modifications) given in the proof of Lemma(4.2.2).

Remark : Lemma(4.4.4) shows that $I \longrightarrow \mathcal{S}_A(I)$ defines a one-to-one correspondense between the closed right ideals of A and the closed subspaces of A . Moreover, if q is a complementor on A , then $S \longrightarrow S^{q'} = \mathcal{S}_A(\mathcal{J}_A(S)^q)$ defines a complementor on the closed subspaces S of H . (The proof of this remark is the same as that given in the remark following Lemma(4.2.2)).

Lemma(4.4.5) : Let A be a dual A*-algebra which is a dense subalgebra of $LC(H)$. Then, for every complementor q on A the mapping $R \longrightarrow cl((R \cap A)^q)$ on the closed right ideals R of $LC(H)$ is a complementor on $LC(H)$.

Proof : It is clear that A is simple. Let q be a complementor on A . Then, by the remark above, the mapping $S \longrightarrow S^{q'} = \mathcal{S}_A(\mathcal{J}_A(S)^q)$ defines a complementor on the closed subspace S of H . By the remark following Lemma(4.2.2) the mapping $S \longrightarrow S^{q'}$ induces a complementor p on $LC(H)$ given by the relation $R^p = \mathcal{J}(\mathcal{S}(R)^{q'})$, for every closed right ideal R of $LC(H)$. It is easy to see that $cl(R \cap A) = R$. In fact, let $\mathfrak{A} = LC(H)$ and let $\{ e_\alpha \}$ be the family of all minimal self-adjoint idempotents in R . Then, by [7, Lemma 6], $R = cl(\sum e_\alpha \mathfrak{A})$. But from Lemma(3.2.1) we have $e_\alpha \mathfrak{A} \subset R \cap A$ for all α , hence $R = cl(R \cap A)$. Similarly

$R^p = \mathrm{cl}(R^p \cap A)$. Now $\mathcal{J}_A(S) = \mathcal{J}(S) \cap A$ and by Lemma(4.2.2),

$$\mathcal{J}_A(\mathcal{S}(R)) = \mathcal{J}(\mathcal{S}(R)) \cap A = R \cap A = I .$$

Therefore

$$\begin{aligned} R^p \cap A &= \mathcal{J}(\mathcal{S}(R)^{q'}) \cap A = \\ &= \mathcal{J}(\mathcal{S}_A[\mathcal{J}_A(\mathcal{S}(R))]^q) \cap A = \\ &= \mathcal{J}(\mathcal{S}_A(I^q)) \cap A = \mathcal{J}_A(\mathcal{S}_A(I^q)) = \\ &= I^q = (R \cap A)^q \quad . \end{aligned}$$

Hence

$$R^p = \mathrm{cl}(R^p \cap A) = \mathrm{cl}((R \cap A)^q) ,$$

so that $R \longrightarrow \mathrm{cl}((R \cap A)^q)$ is a complementor on $\mathfrak{A}$. This completes the proof.

Now let A be a dual A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$, and let $\{ I_\lambda : \lambda \in \Lambda \}$ be the family of all minimal closed 2-sided ideals of A . Clearly each $\mathrm{cl}(I_\lambda)$ is a minimal closed 2-sided ideal of $\mathfrak{A}$ and hence *-isomorphic to $LC(H_\lambda)$, for some Hilbert space H_λ . Suppose q is a complementor on A and, for each $\lambda \in \Lambda$, let q_λ be the complementor on I_λ induced by q . Identifying I_λ as a subalgebra of $LC(H_\lambda)$, q_λ induces the complementor p_λ on $LC(H_\lambda)$ (Lemma

(4.4.5)) . For each closed right ideal R_λ of $LC(H_\lambda)$, let P_{R_λ} be the projection on R_λ along R^{p_λ} . Then P_{R_λ} is a bounded linear operator on $LC(H_\lambda)$ whose operator bound we denote by $|P_{R_\lambda}|$. Let

$$m_\lambda = \sup \{ |P_{R_\lambda}| : R_\lambda \subset LC(H_\lambda) \}$$

and let

$$m = \sup \{ m_\lambda : \lambda \in \Lambda \} ;$$

m may be finite or infinite.

Lemma(4.4.6) : If I is a closed right ideal of A , then $I^q \cap I_\lambda = (I \cap I_\lambda)^{q_\lambda}$, for every $\lambda \in \Lambda$.

Proof : Since $I \cap I_\lambda \subset I$, we have $I^q \subset (I \cap I_\lambda)^q$ and hence $I^q \cap I_\lambda \subset (I \cap I_\lambda)^{q_\lambda}$. Now, by Lemma (1.2.1), $cl(I + I_\lambda^q) = I^q \cap I_\lambda$; hence

$$\begin{aligned} cl(I + I_\lambda^q) \cap I_\lambda &= (I^q \cap I_\lambda)^q \cap I_\lambda = \\ &= (I^q \cap I_\lambda)^{q_\lambda} . \end{aligned}$$

Let $x \in (I^q \cap I_\lambda)^{q_\lambda}$. Then $x \in I_\lambda$ and $x = \lim_n x_n$, where $x_n = y_n + z_n$ with $y_n \in I$ and $z_n \in I_\lambda^q$ ($n = 1, 2, ...$) . Since, by [14, Lemma 1], $I_\lambda^q = \ell(I_\lambda)$ and since $x^* \in I_\lambda$, we obtain that

$$xx^* = \lim_n x_n x^* = \lim_n y_n x^* \in I .$$

But, by Theorem (3.2.2), this means that $x \in I$ and therefore $x \in I \cap I_\lambda$. Hence $(I^q \cap I_\lambda)^{q_\lambda} \subset I \cap I_\lambda$ and consequently $I^q \cap I_\lambda = (I \cap I_\lambda)^{q_\lambda}$.

Theorem(4.4.7) : Let A be a dual A*-algebra which is a dense subalgebra of a B*-algebra $\mathfrak{A}$. Then, for every complementor q on A for which m is finite, the mapping $p : R \longrightarrow cl((R \cap A)^q)$ on the closed right ideals R of $\mathfrak{A}$ is a complementor on $\mathfrak{A}$. If, moreover, $\mathfrak{A}$ has no minimal left ideals of dimension less then three and p is continuous, then there exists an involution $*'$ on $\mathfrak{A}$ such that $R^p = \ell(R)^{*'}$.

Proof : We use the notation of the paragraph preceding Lemma(4.4.6). It is clear that $\mathfrak{A}$ is *-isomorphic to $(\sum_\lambda LC(H_\lambda))_o$. In what follows we identify $\mathfrak{A}$ with $(\sum_\lambda LC(H_\lambda))_o$. Let q be a complementor on A for which m is finite. Let R be a closed right ideal of $\mathfrak{A}$ and, for each $\lambda \in \Lambda$, let $R_\lambda = R \cap LC(H_\lambda)$. Then, by [1, Lemma 7.1], $R = (\sum_\lambda R_\lambda)_o$. Define

$$R' = (\sum_\lambda [R \cap LC(H_\lambda)]^{p_\lambda})_o ,$$

where p_λ is the complementor on $LC(H_\lambda)$ induced by q_λ.

Clearly R' is a closed right ideal of $\mathfrak{A}$ and $R' \cap LC(H_\lambda) = R^{p_\lambda}$. Hence

$$(R')' = (\sum_\lambda [\, R' \cap LC(H_\lambda)]^{p_\lambda})_0 =$$

$$= (\sum_\lambda R_\lambda)_0 = R .$$

Moreover, it is easy to see that $R \cap R' = (0)$ and that if R_1 and R_2 are closed right ideals of $\mathfrak{A}$, $R_1 \subset R_2$, then $R_2' \subset R_1'$. Thus if we can show that $R + R' = \mathfrak{A}$, it will follow that $R \longrightarrow R'$ is a complementor on $\mathfrak{A}$. Let $x = (x_\lambda) \in \mathfrak{A}$ and write $x_\lambda = y_\lambda + z_\lambda$, $y_\lambda \in R_\lambda$ and $z_\lambda \in R_\lambda^{p_\lambda}$. Now $|y_\lambda| = |P_{R_\lambda} x_\lambda| \le m|x_\lambda|$ $(\lambda \in \Lambda)$; similarly $|z_\lambda| \le m|x_\lambda|$ $(\lambda \in \Lambda)$. Hence, since m is finite, $(y_\lambda) \in (\sum R_\lambda)_0 = R$ and $(z_\lambda) \in (\sum R_\lambda^{p_\lambda})_0 = R'$. Thus $R + R' = \mathfrak{A}$ and consequently $R \longrightarrow R'$ is a complementor on $\mathfrak{A}$.

We show next that $R' = cl((R \cap A)^q) = R^p$. Let $I = R \cap A$. Since, by [1, Theorem 7.1], we have $cl(I^q) = (\sum [cl(I^q) \cap LC(H_\lambda)])_0$, it suffices to show that $R_\lambda^{p_\lambda} = cl(I^q) \cap LC(H_\lambda)$ $(\lambda \in \Lambda)$. Now, by the duality of $\mathfrak{A}$, we have $cl(I^q) \cap A = I^q$ (Lemma (3.2.1)) and hence

$$[\mathrm{cl}(I^q) \cap LC(H_\lambda)] \cap I_\lambda = [\mathrm{cl}(I^q) \cap A] \cap I_\lambda$$
$$= I^q \cap I_\lambda = (I \cap I_\lambda)^{q_\lambda} ,$$

where the last equality is given by Lemma(4.4.6) . Therefore using the duality of I_λ , we obtain

$$\mathrm{cl}(I^q) \cap LC(H_\lambda) = \mathrm{cl}([\mathrm{cl}(I^q) \cap LC(H_\lambda)] \cap I_\lambda)$$
$$= \mathrm{cl}((I \cap I_\lambda)^{q_\lambda}) = \mathrm{cl}((R \cap A \cap I_\lambda)^{q_\lambda}) =$$
$$= \mathrm{cl}((R_\lambda \cap I_\lambda)^{q_\lambda}) = R_\lambda^{p_\lambda} .$$

To prove the second part of the theorem, by [1, Theorem 7.4] and Theorem(4.3.2), it suffices to show that $\{ |e| : e \in E_p \}$ is bounded. Let $e \in E_p$. It is clear that $e\mathfrak{A} \subset LC(H_\lambda)$ for some $\lambda \in \Lambda$. Therefore $|P_{e\mathfrak{A}}| \leq m$. Since each minimal right ideal of $\mathfrak{A}$ is of the form $e\mathfrak{A}$, by Theorem(4.3.1) , $\{ |e| : e \in E_p \}$ is bounded. This completes the proof of the theorem.

§5 Examples

As an immediate example of a complemented A*-algebra we have an H*-algebra (see [14]). We shall now give another example.

Let H be a Hilbert space and $\tau c(H)$ the trace class operators on H with the trace norm $||\cdot||$. $\tau c(H)$ is an A*-algebra which is a dense 2-sided ideal of LC(H) and, as a Banach space, it is isometrically isomorphic to the conjugate space of LC(H) (see [12, Theorem 3, P.47]). Clearly $\tau c(H)$ contains all operators of finite rank as a dense subset and hence is an annihilator algebra, in fact it is dual as we shall see.

Now let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of Hilbert spaces H_λ and let $(\sum_\lambda \tau c(H_\lambda))_1$ denote the family of all functions f defined on Λ such that $f(\lambda) \in \tau c(H_\lambda)$ for each λ and such that $\sum ||f(\lambda)|| < \infty$. It follows that $(\sum \tau c(H_\lambda))_1$ is a Banach algebra under the norm $||f|| = \sum ||f(\lambda)||$ and the usual operators for functions. $(\sum \tau c(H_\lambda))_1$ is clearly a subalgebra of $(\sum_\lambda LC(H_\lambda))_0$ and an A*-algebra under the involution $f \longrightarrow f^*$, where $f^*(\lambda) = f(\lambda)^{*_\lambda}$ ($*_\lambda$ being the adjoint operation in $\tau c(H_\lambda)$ $(\lambda \in \Lambda)$).

Theorem(4.5.1) : As a Banach space $(\sum \tau c(H_\lambda))_1$ is isometrically isomorphic to the conjugate space of

$(\sum LC(H_\lambda))_0$ . .

Proof : Let $A = (\sum \tau c(H_\lambda))_1$, $\mathfrak{A} = (\sum LC(H_\lambda))_0$ and let $\mathfrak{A}'$ be the conjugate space of $\mathfrak{A}$. Let $f \in A$, and write f_λ for $f(\lambda)$ $(\lambda \in \Lambda)$. By [12, Theorem 3, P.48] , each f_λ can be identified (isometrically) with an element of $LC(H_\lambda)'$, the conjugate space of $LC(H_\lambda)$. Considering now f_λ as an element of $LC(H_\lambda)'$, define the functional f' on $\mathfrak{A}$ by the relation

$$f'(x) = \sum_\lambda f_\lambda(x_\lambda) \qquad (x = (x_\lambda) \in \mathfrak{A}) .$$

Since

$$|f'(x)| \leq \sum |f_\lambda(x_\lambda)| \leq \sum ||f_\lambda|| \, |x_\lambda|$$

$$\leq \sup_\lambda |x_\lambda| \; (\sum ||f_\lambda||) = ||f|| \, |x| \quad ,$$

f' is a continuous linear functional on $\mathfrak{A}$ and $||f'|| \leq ||f||$, where $||f'||$ denotes the operator bound of f' . Since $\sum ||f_\lambda|| < \infty$, only a denumeralbe number of $f_\lambda \neq 0$, denote those non-zero f_λ by $f_{\lambda_1}, f_{\lambda_2}, \ldots$ For given $\varepsilon > 0$, choose a positive integer N such that $\sum_{n=N}^\infty ||f_{\lambda_n}|| < \frac{1}{2}\varepsilon$. Since $f_{\lambda_n} \in LC(H_{\lambda_n})'$, there exists an $x_{\lambda_n} \in LC(H_{\lambda_n})$ such that $|x_{\lambda_n}| = 1$ and

$$f_{\lambda_n}(x_{\lambda_n}) \geq ||f_{\lambda_n}|| - \frac{\varepsilon}{n^2} \qquad (n = 1, 2, \ldots, N).$$

Let $x = (x_\lambda)$, where $x_\lambda = x_{\lambda_n}$ when $\lambda = \lambda_n$ $(n = 1, \ldots, N)$ and

$x_\lambda = 0$ otherwise. Then $|x| = 1$ and

$$f'(x) = \sum_{n=1}^{N} f_{\lambda_n}(x_{\lambda_n}) \geq \sum_{n=1}^{N} \|f_{\lambda_n}\| - \mathcal{E}\left(\sum_{n=1}^{N} \frac{1}{n^2}\right)$$

$$\geq \sum_{n=1}^{\infty} \|f_{\lambda_n}\| - \mathcal{E}\left(\tfrac{1}{2} + \sum_{n=1}^{N} \frac{1}{n^2}\right)$$

$$\geq \sum_{n=1}^{\infty} \|f_{\lambda_n}\| - \mathcal{E}\left(\tfrac{1}{2} + \frac{\pi^2}{6}\right) .$$

Since $\mathcal{E}$ is arbitrary, we have $\|f\| \leq \|f'\|$ and so $\|f\| = \|f'\|$.

Let $f' \in \mathfrak{A}$ and let f_λ be the restriction of f' to $LC(H_\lambda)$ $(\lambda \in \Lambda)$ (We identify $LC(H_\lambda)$ as a subalgebra of $\mathfrak{A}$). Then each $f_\lambda \in LC(H_\lambda)'$. Considering now f_λ as an element of $\tau c(H_\lambda)$, let f be the mapping on Λ such that $f(\lambda) = f_\lambda \in \tau c(H_\lambda)$. We show that $f \in A$. Let $\{\lambda_n : n = 1, 2, \ldots, k\}$ be a finite subset of Λ . For given $\mathcal{E} > 0$, choose $x_{\lambda_n} \in LC(H_{\lambda_n})$ such that $|x_{\lambda_n}| = 1$ and

$$f_{\lambda_n}(x_{\lambda_n}) \geq \|f_{\lambda_n}\| - \frac{\mathcal{E}}{n^2} \qquad (n = 1, \ldots, k) .$$

Let $x = (x_\lambda)$, where $x_\lambda = x_{\lambda_n}$, when $\lambda = \lambda_n$ $(n=1,\ldots,k)$, and $x_\lambda = 0$ otherwise. Then

$$f'(x) = f'(x_{\lambda_1}) + \ldots + f'(x_{\lambda_k}) = \sum_{n=1}^{k} f_{\lambda_n}(x_{\lambda_n}) \geq$$

$$\geq \sum_{n=1}^{k} \|f_{\lambda_n}\| - \varepsilon\left(\sum_{n=1}^{k} \frac{1}{n^2}\right) \geq$$

$$\geq \sum_{n=1}^{k} \|f_{\lambda_n}\| - \frac{\varepsilon \pi^2}{6} \quad .$$

Thus $\sum_{n=1}^{k} \|f_{\lambda_n}\| \leq \|f'\|$ and so $\sum_{\lambda} \|f_\lambda\| \leq \|f'\|$. Therefore $f \in A$ and $\|f\| = \sum \|f_\lambda\| = \|f'\|$ (by above argument). Since clearly $f \longrightarrow f'$ is linear, it follows that $f \longrightarrow f'$ is an isometric isomorphism of A onto $\mathfrak{A}'$.

Theorem(4.5.2) : $(\sum \tau c(H_\lambda))_1$ is a dense 2-sided ideal of $(\sum LC(H_\lambda))_0$.

Proof : Let $A = (\sum \tau c(H_\lambda))_1$ and $\mathfrak{A} = (\sum LC(H_\lambda))_0$. Since $\tau c(H_\lambda)$ is dense in $LC(H_\lambda)$, considering $LC(H_\lambda)$ as a subalgebra of $\mathfrak{A}$ and $\tau c(H_\lambda)$ as a subalgebra of A, we have $LC(H_\lambda) \subset cl(A)$. Hence $\mathfrak{A} \subset cl(A) \subset \mathfrak{A}$ and so $\mathfrak{A} = cl(A)$. Now, let $x = (x_\lambda) \in A$ and $y = (y_\lambda) \in \mathfrak{A}$. Since $\tau c(H_\lambda)$ is a 2-sided ideal of $LC(H_\lambda)$, $x_\lambda y_\lambda \in \tau c(H_\lambda)$ for all λ. By [12, Lemma 8, P.39], $\|x_\lambda y_\lambda\| \leq \|x_\lambda\| \, |y_\lambda|$ and so

$$\|xy\| = \sum_{\lambda} \|x_\lambda y_\lambda\| \leq (\sup_{\lambda} |y_\lambda|)(\sum_{\lambda} \|x_\lambda\|) < \infty ,$$

which gives $xy \in A$. Similarly $yx \in A$. Thus A is a dense 2-sided ideal of $\mathfrak{A}$.

Corollary(4.5.3) : $(\sum_{\lambda} \tau c(H_\lambda))_1$ has weak (β_k) property.

Proof : Follows from Theorem(4.5.2) and Corollary(3.1.3).

Lemma(4.5.4) : $\tau c(H)$ is a dual A*-algebra, and the mapping $I \longrightarrow \ell(I)^*$ on the closed right ideals I is a complementor on $\tau c(H)$.

Proof : Let $A = \tau c(H)$ and let I be a closed right ideal of A . We show that $\mathcal{J}_A(\mathcal{S}_A(I)) = I$. Clearly $I \subset \mathcal{J}_A(\mathcal{S}_A(I))$. Let $T \in \mathcal{J}_A(\mathcal{S}_A(I))$ and $\{T_n\}$ a sequence of operators of finite rank on H such that $||T_n - T|| \longrightarrow 0$ as $n \longrightarrow \infty$. Let P be the orthogonal projection on $\mathcal{S}_A(I)$. Since PT_n is finite dimensional with range in $\mathcal{S}_A(I)$, [11, Theorem(2.4.18)] shows that $PT_n \in I$ for all $n = 1, 2, \ldots\ldots$. Clearly $PT = T$. By [12, Lemma 8 , P. 39] we have

$$||PT_n - T|| = ||PT_n - TP|| \leq |P|\,||T_n - T|| \quad ,$$

so that $||PT_n - T|| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $T \in I$ and consequently $\mathcal{J}_A(\mathcal{S}_A(I)) = I$. Thus, by [11, Lemma(2.8.24)] , and the continuity of the involution, A is dual. Let $T \in A$. Then $T = PT + P'T$ where $P' = 1 - P$, and, since $PT \in I$ and $P'T \in \ell(I)^*$, we have $I + \ell(I)^* = A$. It is now easy to see that the mapping $I \longrightarrow \ell(I)^*$ is a complementor on A .

Lemma(4.5.5) : Let I be a closed right ideal of $(\sum \tau c(H))_1$, and for each $\lambda \in \Lambda$, let

$$I = \{ x_\lambda \in \tau c(H_\lambda) : x_\lambda = x(\lambda) \text{ for some } x \in I \}.$$

Then

(i) I_λ is a closed right ideal of $\tau c(H_\lambda)$ for each $\lambda \in \Lambda$.

(ii) $I = (\sum I_\lambda)_1$.

(iii) $\ell(I) = (\sum \ell_\lambda(I_\lambda))_1$, where $\ell_\lambda(I_\lambda)$ denotes the left annihilator of I_λ in $\tau c(H_\lambda)$ for each $\lambda \in \Lambda$.

Proof : (i) Let $A = (\sum \tau c(H_\lambda))_1$. It is clear that I_λ is a subalgebra of $\tau c(H_\lambda)$. In what follows we identify $\tau c(H_\lambda)$ as a subalgebra of A (this is clearly possible). Let $x_\lambda \in I_\lambda$ and $y_\lambda \in \tau c(H_\lambda)$. Then there exists an $x \in I$ such that $x(\lambda) = x_\lambda$. Since $xy_\lambda \in I$ and $(xy_\lambda)(\lambda) = x_\lambda y_\lambda$, $x_\lambda y_\lambda \in I_\lambda$ and so I_λ is a right ideal

of $\tau c(H_\lambda)$. By the duality of $\tau c(H_\lambda)$, $x_\lambda \in cl(x_\lambda \tau c(H_\lambda)) = cl(x \tau c(H_\lambda)) \subset I$. Hence $I_\lambda \subset I$. If $z_\lambda \in cl(I_\lambda)$, then $z_\lambda \in I$ and $z_\lambda = z_\lambda(\lambda) \in I_\lambda$ and so I_λ is closed. Thus I_λ is a closed right ideal of $\tau c(H_\lambda)$.

(ii) Clearly $I \subset (\sum I_\lambda)_1$. We also have $(\sum I_\lambda)_1 \subset I$. In fact, let $x = (x_\lambda) \in (\sum I_\lambda)_1$. As $||x|| = \sum ||x_\lambda|| < \infty$, only a countable number of $x_\lambda \neq 0$; say $x_{\lambda_1}, x_{\lambda_2}, \ldots$. Let $x_n = \sum_{i=1}^{n} x_{\lambda_i}$ $(n = 1, 2, \ldots)$. Since $x_{\lambda_i} \in I_{\lambda_i} \subset I$, we have $x_n \in I$ for all n and,

since

$$||x - x_n|| = \sum_{i=n}^{\infty} ||x_{\lambda_i}|| \longrightarrow 0,$$

as $n \longrightarrow \infty$ it follows that $x \in I$. Thus $(\sum' I_\lambda)_1 \subset I$ and hence $(\sum' I_\lambda)_1 = I$.

(iii) $x = (x_\lambda) \in \ell(I) \Longleftrightarrow xI = (0) \Longleftrightarrow x(\sum' I_\lambda)_1 = (0) \Longleftrightarrow x_\lambda I_\lambda = (0) \Longleftrightarrow x_\lambda \in \ell_\lambda(I_\lambda) \Longleftrightarrow x \in (\sum' \ell_\lambda(I_\lambda))_1$. This completes the proof.

<u>Theorem(4.5.6)</u> : The mapping $I \longrightarrow \ell(I)^*$ on the closed right ideals I is a complementor on $(\sum' \tau c(H_\lambda))_1$.

<u>Proof</u> : Let $A = (\sum' \tau c(H_\lambda))_1$ and let I be a closed right ideal of A . By Lemma(4.5.5)(i),

$$I_\lambda = \{x_\lambda \in \tau c(H_\lambda) : x_\lambda = x(\lambda) \text{ for some } x \in I\}$$

is a closed right ideal of $\tau c(H_\lambda)$ $(\lambda \in \Lambda)$. Let $x = (x_\lambda) \in A$. By Lemma(4.5.4), $x_\lambda = y_\lambda + z_\lambda$ with $y_\lambda \in I_\lambda$ and $z_\lambda \in \ell_\lambda(I_\lambda)^*$, where $\ell_\lambda(I_\lambda)$ denotes the left annihilator of I_λ in $\tau c(H_\lambda)$; clearly $x_\lambda^* x_\lambda = y_\lambda^* y_\lambda + z_\lambda^* z_\lambda$. Since $x_\lambda^* x_\lambda - y_\lambda^* y_\lambda = z_\lambda^* z_\lambda \geq 0$, by [9, Theorem] , $(x_\lambda^* x_\lambda)^{\frac{1}{2}} \geq (y_\lambda^* y_\lambda)^{\frac{1}{2}}$. Let $\{\varphi_\gamma^\lambda : \gamma \in \Gamma\}$ be an orthonomal basis of H_λ . Then

$$\|x_\lambda\| = \sum_\gamma ((x_\lambda^* x_\lambda)^{\frac{1}{2}} \varphi_\gamma^\lambda, \varphi_\gamma^\lambda)$$

$$\geq \sum_\gamma ((y_\lambda^* y_\lambda)^{\frac{1}{2}} \varphi_\gamma^\lambda, \varphi_\gamma^\lambda) = \|y_\lambda\| \quad ,$$

and so $\sum \|y_\lambda\| \leq \sum \|x_\lambda\| = \|x\|$. Similarly $\sum \|z_\lambda\| \leq \|x\|$. Thus, by Lemma(4.5.5), $(y_\lambda) \in (\sum I_\lambda)_1 = I$ and $(z_\lambda) \in (\sum \ell_\lambda(I_\lambda)^*)_1 = \ell(I)^*$. Since $x = (y_\lambda) + (z_\lambda)$, $A = I + \ell(I)^*$. Clearly $\ell(I)^* \cap I = (0)$ and if $I_1 \supset I_2$, then $\ell(I_1)^* \subset \ell(I_2)^*$. Since $\ell(\ell(I)^*)^* = r(\ell(I)) \supset I$, it is easy to show that $I = \ell(\ell(I)^*)^*$ (see [1]) . Thus the mapping $I \longrightarrow \ell(I)^*$ is a complementor on A .

<u>Corollary(4.5.7)</u> : $(\sum \tau c(H_\lambda))_1$ is dual.

<u>Proof</u> : Since $I = \ell(\ell(I)^*)^* = r(\ell(I))$, by the continuity of the involution, $(\sum \tau c(H_\lambda))_1$ is dual.

<u>Corollary(4.5.8)</u> : $(\sum \tau c(H_\lambda))_1$ is w.c.c.

<u>Proof</u> : Follows from Corollary(4.5.7) and Theorem(3.3.3).

We do not know of an example of a complemented A*-algebra which is not a dense 2-sided ideal of a B*-algebra. Also we do not know if every dual A*-algebra is complemented, and conversely if every complemented A*-algebra is dual.

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