APPLICATION OF THREE-DIMENSIONAL ELASTICITY, SHELL AND BOUNDARY LAYER SOLUTIONS TO AXIALLY COMPRESSED CYLINDERS

By

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ABSTRACT

Solutions based on three-dimensional elasticity and shell and boundary layer theories have been obtained in the past for axially compressed cylinders. Comparative study based on different methods of solution for the problem of axially compressed cylinders has been carried out. This thesis also contains a discussion on the range of validity of various types of solutions.

Three-dimensional elasticity solution is based on Multiple Fourier method. Stresses and displacements are expressed in terms of Galerkin Vector function which satisfies the biharmonic equation. Satisfaction of boundary conditions reduces the problem to an infinite system of linear simultaneous equations. An approximate numerical solution is obtained by the method of reduction. Shell and boundary layer solution is obtained by asymptotic integration of the equations of three-dimensional theory of elasticity. The solution is superimposed over the elementary solution. Major contribution to stresses and displacements is from the elementary and interior solutions. Boundary layer solution is found to be predominant in the narrow edge zones only.

The longitudinal stress and the radial displacement at the outer surface, given by the two solutions, are compared with the experimental results. Three-dimensional elasticity solution gives better results for thick and short cylinders while for thin and long cylinders, shell and boundary layer solution should be preferred.

CONTENTS

NOMENCLATURE

a, b	Inner and outer radii of the cylinder
c	Half length of the cylinder
F, Q	Conversion factor for determining stresses and displacement from experimental strain data
P	Axial compressive load
r, θ, z	Cylindrical coordinates
u, w	Components of displacement
a_1	Mid-surface radius of the cylinder
$a_n, \beta_n^*, A_2, B_2, C_2, D_2, U, V, k_n, p_n$	Complex constants used in the boundary layer solution (section III-3b)
$A_\nu, B_\nu, C_\nu, R, S, T, \gamma$	Constants in terms of Poisson's ratio (section III-3)
$s_r, s_\theta, s_z, s_{rz}$	Non-dimensional stress components in the shell and boundary layer solution
$S_r^{(i)}(\xi), S_z^{(i)}(\xi)$	Middle surface stresses defined in the interior solution
v_r, v_z	Non-dimensional displacements in the shell and boundary layer solution
$V_r^{(i)}(\xi), V_z^{(i)}(\xi)$	Middle surface displacements defined in the interior solution
β_{n1}	Roots of $\sin(2\beta_n^*) + 2\beta_n^* = 0$
β_{n2}	Roots of $\sin(2\beta_n^*) - 2\beta_n^* = 0$
$\epsilon_r, \epsilon_\theta, \epsilon_z, \gamma_{rz}$	Components of the strain tensor ϵ_{ij}

$\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$	Components of the stress tensor τ_{ij}
$\sigma = \frac{P}{\pi(b^2-a^2)}$	Average compressive stress in the cylinder
ξ, η, ρ	Non-dimensional coordinates for shell and boundary layer solution

CHAPTER I

INTRODUCTION

The study of hollow circular cylinders has been of considerable interest and great practical importance because of their numerous uses in the industry. Depending on its applications, a hollow cylinder could be loaded over the curved surfaces or over its ends. Solutions to the problems of solid cylinders under various types of loading could be found in the literature and are based on the three-dimensional theory of elasticity. Some of these solutions have been extended to the problems of hollow cylinders. Shell theory has also been used for problems dealing with hollow cylinders. No work has been done on the range of validity of these solutions. Also, only a very small number of research papers deal with the comparison of the solutions based on these theories. Sensenig [15] has compared the shell theory with the three dimensional theory of elasticity for isotropic shells. In his paper, it has been shown that a solution based on the shell theory is an approximation to the three-dimensional theory of elasticity in the regions away from the edge of the shell provided the displacement gradients, strains, loading and thickness are sufficiently small. This has been done with the help of certain mathematical theorems. But no conclusions have been drawn regarding the application of the two theories. In this thesis, numerical results of the solutions based on these theories have been compared with each other. A detailed study of the applicability of these results along with the recommendations regarding the choice of the method of analysis is also presented. For the present study, a hollow circular cylinder under axi-symmetric compression has been considered. Experiments have also been conducted on the cylinders and the experimental data obtained provides further justification to the conclusions about the transition limits. The loading and the boundary conditions for the cylinder are those produced by the ordinary testing machines. The hollow cylinder is subjected to compressive loads at its ends which are fixed but are allowed a relative axial displacement. The curved surfaces are free from traction. The condition of nonexpansion of the ends in the radial direction gives rise to nonuniformity in the stress distribution at the ends. The loading at the end is considered in the statically equivalent sense.

The study of solid cylinders was first presented in the ninteenth century by Pochhammer, Chree and Steklov. A complete history of the associated problems is given in refrence [8]. A detailed study of solid cylinders under practical systems of loading was reported by Filon [3] in 1902. The paper also deals with the problem of axi-symmetric compression of a solid cylinder by expressing the displacements in terms of Fourier-Bessel series. This method was improved by Picket [11] in 1944, who introduced the Multiple Fourier Method in solving the problem. In the Multiple Fourier Method, two or more series of particular solutions are chosen. Each of these series is selected so as to give a Fourier-series for one component of stress or of displacement at a boundary. Valov [17] and Blair and Veeder [2] have solved the problems of finite solid cylinders in terms of displacement functions. The problems of finite hollow cylinders can also be solved by similar considerations. The solutions, however, involve great deal of labor and computational difficulties. Solution to the problem of axi-symmetric compression of hollow cylinders using Multiple Fourier method was given by Widera and Mirza [18]. This solution, based on three-dimensional theory of elasticity, gives stresses and displacements in terms of the component of Galerkin Vector Function which satisfies the biharmonic equation. Three particular solutions to the biharmonic equation are superimposed to give Galerkin Vector Function. A Fourier analysis is undertaken for satisfying the boundary conditions to yield the desired series expressions at the boundaries and also to eliminate the contributions from the other series. With this analysis, the problem is reduced to the solution of an infinite system of linear simultaneous equations. A numerical solution to this problem has been obtained in this work. The problems of finite hollow cylinders have also been considered by Shibahara and Oda [16]. They have considered various types of loadings on the curved surfaces and the ends of the cylinders and claim that these problems have never been investigated before. It is interesting to note that they have chosen similar types of series expressions for the Galerkin Vector Function as in [18].

Theory of shells was introduced by Love in 1888 and was improved later on by Galerkin, Lur'e, Gol'denweizer, Vlasov etc. Johnson and Reissner [5], in 1957, founded a theory of thin elastic cylindrical shells in which the solution is obtained by asymptotic integration of the equations of three-dimensional theory of elasticity. The solution is given in terms of a dimensionless parameter. The first terms in these expressions represent the thin shell theory. The second and higher order terms give thickness corrections. This solution was given for a semi-infinite thin cylindrical shell. Reiss [12] showed that the solution obtained from the thin shell theory of Johnson and Reissner represents the exact solution in the regions away from the edge,i.e., in the "interior" of the shell. This solution called the interior solution is not valid for the narrow edge zone recognized as boundary layer. A boundary layer is defined as the region in which the expressions for the interior of the cylinder deviate very rapidly from the exact solution. Reiss obtained the expressions for the boundary layer stresses and displacements by asymptotic expansion of the equations of three-dimensional theory of elasticity. Widera and Wu [19], solved the problem of axi-symmetric compression of hollow cylinders by superimposing the shell and boundary layer solutions on the elementary solution. The method of solution differs slightly from that given by Reiss. Reference [12] gives the boundary layer solution from asymptotic integration of the equilibrium and compatibility equations of three-dimensional theory of elasticity while in [19], they are obtained from equilibrium equations and Hooke's law. The shell and boundary solutions are obtained by considering the cylinder to be very long. The solution in [19] is given for orthotropic cylinders and the expressions for the boundary layer cannot be used for isotropic cylinders. Boundary layer stresses and displacements for isotropic cylinders are dealt in this thesis.

A comparison of the solutions from these theories can be made by plotting the stresses and displacements from the numerical results. This results in a very large number of graphs, as such only a few have been included in this thesis. The variation of the longitudinal stress and the radial displacement along the outer curved surface of the cylinder obtained from experiments and theory have been plotted.

The approximate elasticity solution is obtained from Prokopov's solution for solid cylinders [8]. Only first terms of the series expansions are considered. The radial stress is assumed to have a very small value at the curved surfaces and the longitudinal stress is uniform at the ends.

These two assumptions restrict the use of the solutions only to the regions away from the ends and the curved surfaces. This solution has been attempted with a view to get a very simple solution to the complex problem of axi-symmetric compression of hollow cylinders. This has been achieved by reducing the number of boundary conditions and introducing certain approximations. This solution is a slight improvement on the elementary solution.

CHAPTER II

FUNDAMENTAL RELATIONS AND GENERAL FORMULATION

II-1 We consider the axi-symmetric deformations of a hollow circular cylinder subjected to the compressive edge loads P. In terms of the cylindrical coordinates, the cylinder is defined as an elastic body bounded by the co-axial surfaces r=a,b and the planes z=±c. Thus the cylinder under consideration is of constant thickness b-a.

In the shell and boundary layer analysis of the cylinder, mid-surface radius of the cylinder is defined as a_1 and thickness as 2h. Thus the cylinder is bounded by the co-axial surfaces $r=a_1 \pm h$.

II-2 Fundamental Relations

Equilibrium of the cylinder is governed by the following fundamental equations in the cylindrical coordinates.

(a) Equilibrium Equations: With no body forces, equations of equilibrium for the cylinder can be written in the following form.

$$\frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial}{\partial z}(r\tau_{rz}) - \sigma_\theta = 0$$

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) = 0 \qquad (2.1)$$

(b) Strain-Displacement Equations: Considering small displacements, strain-displacement relationships are as below.

$$\epsilon_r = \frac{\partial u}{\partial r}$$

$$\epsilon_z = \frac{\partial \omega}{\partial z}$$

$$\epsilon_\theta = \frac{u}{r}$$

and

$$\gamma_{rz} = \frac{\partial \omega}{\partial r} + \frac{\partial u}{\partial z} \qquad (2.2)$$

(c) Constitutive Equations: The stress-strain relations can be written from generalised Hooke's law.

$$\epsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right]$$

$$\epsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_z + \sigma_r\right)\right]$$

and
$$\gamma_{rz} = \frac{1}{G}\tau_{rz} \qquad (2.3)$$

(d) Compatibility Equations: Using strain-displacement relations along with the equilibrium and constitutive equations, compatibility conditions can be obtained.

$$\nabla^2\sigma_r - \frac{2}{r^2}\left(\sigma_r - \sigma_\theta\right) + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r^2} = 0$$

$$\nabla^2\sigma_\theta + \frac{2}{r^2}\left(\sigma_r - \sigma_\theta\right) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial\theta}{\partial r} = 0$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial^2 z} = 0$$

$$\nabla^2\tau_{rz} - \frac{1}{r^2}\tau_{rz} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r\,\partial z} = 0 \qquad (2.4)$$

where, the Laplacian,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and the stress invariant,

$$\Theta = \sigma_r + \sigma_\theta + \sigma_z$$

In terms of a function $\phi = \phi(r,z)$, stresses and displacements in the three-dimensional elasticity solution can be written as under.

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right]\phi$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right]\phi$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu) - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi$$

$$u = -\frac{1}{2G}\frac{\partial^2\phi}{\partial r\,\partial z}$$

$$w = \frac{1}{2G}\left[2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \qquad (2.5)$$

Here, ϕ is a component of Galerkin's Vector.

Stresses defined by eqs.(2.5) satisfy eqs.(2.1)-(2.3) provided the function ϕ is a biharmonic function,i.e.,

$$\nabla^2\nabla^2\phi = 0 \qquad (2.6)$$

(e) Additional Equations: For the approximate elasticity solution, explained later in chapter III, displacements are expressed in terms of P.F.Papkovich functions so that equilibrium equations are satisfied [8].

$$u = \alpha \phi_1 - \frac{\partial}{\partial r}\left[r\phi_1 + z\phi_2 + \phi_0 \right]$$
$$w = \alpha \phi_2 - \frac{\partial}{\partial z}\left[r\phi_1 + z\phi_2 + \phi_0 \right] \qquad (2.7)$$

where;

$$\alpha = 4(1-\nu)$$

and functions ϕ_0, ϕ_1 and ϕ_2 are such that

$$\nabla^2 \phi_0 = 0$$

and $$\frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_i}{\partial r} - \frac{1}{r^2}\phi_i + \frac{\partial^2 \phi_i}{\partial z^2} = 0 \quad (i=1,2) \qquad (2.8)$$

II-3 Boundary Conditions

Boundary conditions for the three-dimensional elasticity and shell and boundary layer solutions are written separately. This is necessary as the co-ordinates are shifted to the mid-surface in the shell and boundary layer case, while for the elasticity solution, origin is midway along the axis of the cylinder.

(a) Three Dimensional Elasticity Solution: The curved surfaces are tractionfree and ends have a constant axial displacement and are not allowed to displace in the radial direction.
Thus,

$$\sigma_r = \tau_{rz} = 0 \qquad \text{at } r = a, b$$
$$u = 0 \; ; \quad w = \mp k \text{ at } z = \pm c \qquad (2.9)$$

Here the constant k is chosen such that,

$$\int_a^b \sigma_z \Big|_{z=c} 2\pi r \, dr = -P \qquad (2.10)$$

(b) Shell and Boundary Layer Solution: Boundary conditions for the shell and boundary layer solution can be written as in (a).

$$\sigma_r = \tau_{rz} = 0 \quad \text{at} \quad r = a_1 \pm h$$

$$w = \mp K \ , \ u = 0 \quad \text{at} \quad z = \pm c \qquad (2.11)$$

Here, K is a constant which is chosen so that,

$$\int_{a_1-h}^{a_1+h} \sigma_z (c, r) \, 2\pi r \, dr = -P \qquad (2.12)$$

II-4

After formulating the boundary value problem, the three solutions are now introduced.

(a) Three-Dimensional Elasticity Solution: The problem has been solved by the Multiple Fourier Method. Solution is taken to consist of a number of particular solutions in the form of Fourier-Bessel series with proper consideration to the boundary conditions. Thus, function ϕ is taken as [18],

$$\phi = \phi_1 + \phi_2 + \phi_3$$

where,

$$\phi_1 = -L z^3$$

$$\phi_2 = \sum_{m=1}^{\infty} \frac{1}{\alpha_m^2}\left[\alpha_m z \cosh \alpha_m z - (1+\alpha_m c \tanh \alpha_m c) \times \sinh \alpha_m z\right] \times \left[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\right]$$

$$\phi_3 = \sum_{n=1}^{\infty} \frac{\sin \beta_n z}{\beta_n^3}\left[A_n \beta_n i r J_1(i\beta_n r) + B_n J_0(i\beta_n r) + C_n \beta_n r H_1^1(i\beta_n r) + D_n i H_0^1(i\beta_n r)\right] \qquad (2.13)$$

Here, J represents Bessel function of the first kind, Y, Bessel function of the second kind and H^1, Hankel function of the first kind.

With ϕ given by eqs.(2.13), expressions for stresses and displacements can be written from eqs.(2.5) and problem is solved by the satisfaction of boundary conditions (2.9) and (2.10).

(b) Approximate Elasticity Solution: A particular solution of eqs.(2.8) is used to give approximate values of stresses and displacements which are in turn superimposed over the values obtained from the elementary solution.

Functions ϕ_0, ϕ_1, and ϕ_2 are taken as, [8]

$$\phi_0 = b \; g(r) \cos \beta z$$

$$\phi_1 = f(r) \cos \beta z$$

and

$$\phi_2 = 0 \qquad (2.14)$$

Here, g(r) and f(r) are functions of r and are obtained by substituting eqs.(2.14) in to eqs.(2.8). Thus the expressions for displacements and stresses can be obtained from eqs.(2.7) and stress-displacement relations.

(c) Shell and Boundary Layer Solution: The state of stress is obtained by the superimposition of two solutions. First one is the elementary solution. In this solution, the cylinder is uniformly loaded on its ends and all the stresses other than the longitudinal stress are zero, i.e.,

$$\sigma_r^{I} = \sigma_\theta^{I} = \tau_{rz}^{I} = 0 \qquad \sigma_z^{I} = -K\frac{E}{c}$$
$$w^{I} = -K\frac{z}{c} \qquad u^{I} = K\nu\frac{r}{c} \tag{2.15}$$

Here, the superscript I refers to the elementary solution and,

$$K = \frac{Pc}{4\pi a_1 h E}$$

In the second solution, the cylinder is considered to be semi-infinite in terms of a new dimensionless variable,

$$Z_1 = \frac{z+c}{L_1} \tag{2.16}$$

and origin of the co-ordinate system is shifted to z=-c. Here, L_1 is a small but as yet undetermined length scale in the axial direction.

Thus for the second solution, following boundary conditions are defined,

$$w^{II} = 0\,, \quad u^{II} = -u^{I} \qquad \text{at} \quad Z_1 = 0$$

$$\text{Stresses (or displacements)} \rightarrow 0 \quad \text{as} \quad Z_1 \rightarrow \infty \tag{2.17}$$

$$\int_{a_1-h}^{a_1+h} \sigma_z^{II}(0,r)\, 2\pi r\, dr = 0 \tag{2.18}$$

The second solution is achieved by the asymptotic integration of eqs.(2.1) to (2.3). Additional non-dimensional coordinates, displacements and stresses are introduced as follows.

$$\rho = \frac{r - a_1}{h}$$

$$u = \frac{a_1}{E}(1-\nu^2)\sigma v_r$$

$$w = \frac{a_1}{E}(1-\nu^2)\sigma v_z$$

$$\sigma_r = \sigma s_r$$

$$\sigma_z = \sigma s_z$$

$$\sigma_\theta = \sigma s_\theta$$

$$\tau_{rz} = \sigma s_{rz} \tag{2.19}$$

In terms of these variables, eqs. (2.1)-(2.3) are rewritten as

$$(1-\nu^2)\dot{v}_r = \lambda\left[s_r - \nu(s_z + s_\theta)\right]$$

$$(1-\nu^2)\dot{v}_z = -\frac{\lambda}{\mu}(1-\nu^2)v_r' + 2\lambda(1+\nu)s_{rz}$$

$$\frac{\lambda}{\mu}(1-\nu^2)v_z' = \lambda\left[s_z - \nu s_\theta - \nu s_r\right]$$

$$(1-\nu^2)\frac{v_r}{1+\lambda\rho} = s_\theta - \nu s_z - \nu s_r$$

$$\left[(1+\lambda\rho)\, s_{rz}\right]^{\bullet} = -\frac{\lambda}{\mu}\left[(1+\lambda\rho)\, s_z\right]'$$

$$\left[(1+\lambda\rho)\, s_r\right]^{\bullet} = -\frac{\lambda}{\mu}\left[(1+\lambda\rho)\, s_{rz}\right]' + \lambda s_\theta \tag{2.20}$$

Here,

$$\lambda = \frac{h}{a_1} \quad , \quad \mu = \frac{L_1}{a_1} \; , \tag{2.21}$$

$(\;)^{\bullet}$ denotes differentiation with respect to ρ and $(\;)'$ differentiation with respect to z_1. For sufficiently small values of λ, the stress and displacement components can be expanded in a power series in $\lambda^{\frac{1}{2}}$. Two possible choices relating λ and μ are [12],

(i) $\quad \mu = \lambda^{\frac{1}{2}}$ (2.22)

(ii) $\quad \mu = \lambda$ (2.23)

First choice together with system (2.20) is used to determine the stresses and displacements in the interior of the cylinder and is termed the Inerior Problem. The second choice with the same system gives the boundary layer system discussed in the next chapter.

CHAPTER III

THEORETICAL SOLUTIONS

III-1 Three-Dimensional Elasticity Solution

Theoretically, the three-dimensional elasticity solution of the problem is an exact solution because the stresses and displacements are expressed by series and should become exact in the limit as more and more terms are used.

For the axi-symmetric compression of hollow cylinders, displacements and stresses are obtained by using the expressions for ϕ given by (2.13).

$$2GU = \sum_{n=1}^{\infty} \frac{\cos\beta_n z}{\beta_n}\Big[A_n \beta_n r\, J_0(i\beta_n r) + B_n i\, J_1(i\beta_n r)$$

$$- C_n i\beta_n r\, H_0^1(i\beta_n r) - D_n H_1^1(i\beta_n r)\Big]$$

$$+ \sum_{m=1}^{\infty} \frac{1}{\alpha_m}\Big[\alpha_m z \sinh\alpha_m z - \alpha_m c \tanh\alpha_m c . \cosh\alpha_m z\Big]$$

$$\times\Big[E_m J_1(\alpha_m r) + F_m Y_1(\alpha_m r)\Big] \qquad (3.1)\text{a}$$

$$2GW = -(1-2\nu)6LZ + \sum_{n=1}^{\infty} \frac{\sin\beta_n z}{\beta_n}\Big[A_n\Big\{-4(1-\nu)\, J_0(i\beta_n r)$$

$$+ i\beta_n r\, J_1(i\beta_n r)\Big\} + B_n J_0(i\beta_n r) + C_n\Big\{4(1-\nu)\, i\, H_0^1(i\beta_n r)$$

$$+ \beta_n r\, H_1^1(i\beta_n r)\Big\} + D_n i\, H_0^1(i\beta_n r)\Big]$$

$$- \sum_{m=1}^{\infty} \frac{1}{\alpha_m}\Big[\alpha_m z \cosh\alpha_m z - (3-4\nu + \alpha_m c \tanh\alpha_m c)$$

$$\times \sinh\alpha_m z\Big] \times\Big[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\Big] \qquad (3.1)\text{b}$$

$$\sigma_r = -6\nu L + \sum_{n=1}^{\infty} \cos\beta_n z \Big[A_n\Big\{(1-2\nu) J_0(i\beta_n r) - i\beta_n r\, J_1(i\beta_n r)\Big\} - B_n\Big\{J_0(i\beta_n r) + \frac{i J_1(i\beta_n r)}{\beta_n r}\Big\}$$

$$- C_n\Big\{(1-2\nu)\, i\, H_0'(i\beta_n r) + \beta_n r\, H_1'(i\beta_n r)\Big\} - D_n\Big\{ i H_0'(i\beta_n r) - \frac{H_1'(i\beta_n r)}{\beta_n r}\Big\}\Big]$$

$$+ \sum_{m=1}^{\infty} \Big(2\nu \cosh\alpha_m z\big[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\big] + \big[\alpha_m z \sinh\alpha_m z - \alpha_m c \tanh\alpha_m c . \cosh\alpha_m z\big]$$

$$\times \Big[E_m\Big\{ J_0(\alpha_m r) - \frac{J_1(\alpha_m r)}{\alpha_m r}\Big\} + F_m\Big\{Y_0(\alpha_m r) - \frac{Y_1(\alpha_m r)}{\alpha_m r}\Big\}\Big]\Big) \tag{3.1)c}$$

$$\sigma_\theta = -6\nu L + \sum_{n=1}^{\infty} \cos\beta_n z \Big[A_n (1-2\nu) J_0(i\beta_n r) + B_n\, i\frac{J_1(i\beta_n r)}{\beta_n r} - C_n(1-2\nu)\, i\, H_0'(i\beta_n r) - D_n \frac{H_1'(i\beta_n r)}{\beta_n r}\Big]$$

$$+ \sum_{m=1}^{\infty} \Big[2\nu \cosh\alpha_m z\Big\{ E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\Big\} + \Big\{\alpha_m z \sinh\alpha_m z - \alpha_m c \tanh\alpha_m c \cosh\alpha_m z\Big\}$$

$$\times \left\{ E_m \frac{J_1(\alpha_m r)}{\alpha_m r} + F_m \frac{Y_1(\alpha_m r)}{\alpha_m r} \right\} \Bigg] \tag{3.1)d$$

$$\sigma_z = -(1-\nu)\, 6L + \sum_{n=1}^{\infty} \cos \beta_n z \Big[A_n \big\{ -2(2-\nu) J_0(i\beta_n r) + i\beta_n r\, J_1(i\beta_n r) \big\} + B_n J_0(i\beta_n r) + C_n \big\{ 2(2-\nu)\, i H_0'(i\beta_n r) + \beta_n r\, H_1'(i\beta_n r) \big\} + D_n\, i\, H_0'(i\beta_n r) \Big] + \sum_{m=1}^{\infty} \Big[-\alpha_m z \sinh \alpha_m z + (2-2\nu+\alpha_m c \tanh \alpha_m c) \times \cosh \alpha_m z \Big] \times \Big[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r) \Big] \tag{3.1)e$$

$$\tau_{rz} = \sum_{n=1}^{\infty} \sin \beta_n z \Big[A_n \big\{ 2(1-2\nu)\, i J_1(i\beta_n r) - \beta_n r\, J_0(i\beta_n r) \big\} - B_n\, i J_1(i\beta_n r) + C_n \big\{ 2(1-\nu) H_1'(i\beta_n r) + i\beta_n r\, H_0'(i\beta_n r) \big\} + D_n H_1'(i\beta_n r) \Big] + \sum_{m=1}^{\infty} \Big[\alpha_m z \cosh \alpha_m z + (-1+2\nu-\alpha_m c \tanh \alpha_m c) \times \sinh \alpha_m z \Big] \times \Big[E_m J_1(\alpha_m r) + F_m Y_1(\alpha_m r) \Big] \tag{3.1)f$$

The unknown constants A_n, B_n, C_n, D_n, E_m, F_m, α_m, β_n and L are evaluated from the satisfaction of boundary conditions (2.9) and (2.10).

Equations (3.2) given below, are obtained in [18] by satisfying the boundary conditions (2.9) and (2.10). A part of eq.(3.2) has been derived in appendix 1 as the expressions for N_1 and N_2 were found to be incorrect in ref.[18].

$$\cos \beta_n c = 0 \tag{3.2)a}$$

$$J_1(\alpha_m b)\, Y_1(\alpha_m a) - J_1(\alpha_m a)\, Y_1(\alpha_m b) = 0 \tag{3.2)b}$$

$$L = \frac{P}{6\pi(1-\nu)(b^2-a^2)} \tag{3.2)c}$$

$$E_m J_1(\alpha_m b) + F_m Y_1(\alpha_m b) = 0 \tag{3.2)d}$$

$$A_n\left[2(1-\nu)\, i J_1(i\beta_n b) - \beta_n b\, J_0(i\beta_n b)\right] - B_n\, i J_1(i\beta_n b) + C_n\left[2(1-\nu) H_1^1(i\beta_n b) + i\beta_n b\, H_0^1(i\beta_n b)\right] + D_n H_1^1(i\beta_n b) = 0 \tag{3.2)e}$$

$$A_n\left[2(1-\nu)\, i J_1(i\beta_n a) - \beta_n a\, J_0(i\beta_n a)\right] - B_n\, i J_1(i\beta_n a) + C_n\left[2(1-\nu) H_1^1(i\beta_n a) + i\beta_n a\, H_0^1(i\beta_n a)\right] + D_n H_1^1(i\beta_n a) = 0 \tag{3.2)f}$$

$$
\begin{aligned}
&c\Big[A_n\Big\{(1-2\nu)J_0(i\beta_n b) - i\beta_n b\, J_1(i\beta_n b)\Big\} \\
&-B_n\Big\{J_0(i\beta_n b) + i\,\frac{J_1(i\beta_n b)}{\beta_n b}\Big\} - C_n\Big\{(1-2\nu)\,i\,H_0^1(i\beta_n b) \\
&+\beta_n b\, H_1^1(i\beta_n b)\Big\} - D_n\Big\{i\,H_0^1(i\beta_n b) - \frac{H_1^1(i\beta_n b)}{\beta_n b}\Big\}\Big] \\
&+\sum_{m=1}^{\infty} \frac{4\beta_n \cosh\alpha_m c \cdot \sinh\beta_n c}{\beta_n^2+\alpha_m^2}\Big(\nu\Big\{E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\Big\} \\
&-\frac{\alpha_m^2}{\beta_n^2+\alpha_m^2}\Big[E_m\Big\{J_0(\alpha_m b) - \frac{J_1(\alpha_m b)}{\alpha_m b}\Big\} + F_m\Big\{Y_0(\alpha_m b) \\
&-\frac{Y_1(\alpha_m b)}{\alpha_m b}\Big\}\Big]\Big) = \frac{12\nu L \sin\beta_n c}{\beta_n}
\end{aligned}
\tag{3.2)g}
$$

$$
\begin{aligned}
&c\Big[A_n\Big\{(1-2\nu)J_0(i\beta_n a) - i\beta_n a\, J_1(i\beta_n a)\Big\} \\
&-B_n\Big\{J_0(i\beta_n a) + i\,\frac{J_1(i\beta_n a)}{\beta_n a}\Big\} - C_n\Big\{(1-2\nu)\,i\,H_0^1(i\beta_n a) \\
&+\beta_n a\, H_1^1(i\beta_n a)\Big\} - D_n\Big\{i\,H_0^1(i\beta_n a) - \frac{H_1^1(i\beta_n a)}{\beta_n a}\Big\}\Big] \\
&+\sum_{m=1}^{\infty} \frac{4\beta_n \cosh\alpha_m c\ \sinh\alpha_m c}{\beta_n^2+\alpha_m^2}\Big(\nu\Big\{E_m J_0(\alpha_m a) + F_m Y_0(\alpha_m a)\Big\}
\end{aligned}
$$

$$- \frac{\alpha_m^2}{\beta_n^2+\alpha_m^2}\left[E_m\left\{J_0(\alpha_m a) - \frac{J_1(\alpha_m a)}{\alpha_m a}\right\} + F_m\left\{Y_0(\alpha_m a) - \frac{Y_1(\alpha_m a)}{\alpha_m a}\right\}\right]\Bigg) = \frac{12\nu L \sin\beta_n c}{\beta_n} \tag{3.2)h}$$

$$-\frac{1}{\alpha_m}\left[\alpha_m c\cosh\alpha_m c - (3-4\nu+\alpha_m c\tanh\alpha_m c)\sinh\alpha_m c\right] \times \left[E_m N_1 + F_m N_2\right]$$
$$+ \sum_{n=1}^{\infty} \frac{\sin\beta_n c}{\beta_n}\Bigg(A_n\Bigg[\left\{\frac{-4(1-\nu)i\beta_n}{\beta_n^2+\alpha_m^2} + \frac{2i\beta_n^3}{(\beta_n^2+\alpha_m^2)^2}\right\}N_3$$
$$+ \frac{\beta_n^2}{\beta_n^2+\alpha_m^2}N_5\Bigg] + B_n\left[\frac{i\beta_n}{\beta_n^2+\alpha_m^2}N_3\right] + C_n\Bigg[\left\{\frac{4(1-\nu)\beta_n}{\beta_n^2+\alpha_m^2}\right.$$
$$\left.- \frac{2\beta_n^3}{(\beta_n^2+\alpha_m^2)^2}\right\}N_4 - \frac{\beta_n^2}{\beta_n^2+\alpha_m^2}N_6\Bigg] + D_n\left[\frac{\beta_n}{\beta_n^2+\alpha_m^2}N_4\right]\Bigg) = 0 \tag{3.2)i}$$

Here,

$$N_1 = \frac{1}{2}\left\{b^2 J_0^2(\alpha_m b) - a^2 J_0^2(\alpha_m a)\right\} - \frac{J_1(\alpha_m a)}{2Y_1(\alpha_m a)} \times \left\{b^2 J_0(\alpha_m b)Y_0(\alpha_m b) - a^2 J_0(\alpha_m a)\,Y_0(\alpha_m a)\right\} \tag{3.3)a}$$

$$N_2 = \frac{1}{2}\left\{ b^2 J_0(\alpha_m b) Y_0(\alpha_m b) - a^2 J_0(\alpha_m a) Y_0(\alpha_m a) \right\} - \frac{J_1(\alpha_m a)}{2Y_1(\alpha_m a)}\left\{ b^2 Y_0^2(\alpha_m b) - a^2 Y_0^2(\alpha_m a) \right\} \tag{3.3b}$$

$$N_3 = b\, J_1(i\beta_n b)\left[\frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} Y_0(\alpha_m b) - J_0(\alpha_m b) \right] - a\, J_1(i\beta_n a)\left[\frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m a) - J_0(\alpha_m a) \right] \tag{3.3c}$$

$$N_4 = -b\, H_1^1(i\beta_n b)\left[\frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} Y_0(\alpha_m b) - J_0(\alpha_m b) \right] + a\, H_1^1(i\beta_n a)\left[\frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m a) - J_0(\alpha_m a) \right] \tag{3.3d}$$

$$N_5 = b^2 J_0(i\beta_n b)\left[\frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} Y_0(\alpha_m b) - J_0(\alpha_m b) \right] - a^2 J_0(i\beta_n a)\left[\frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m a) - J_0(\alpha_m a) \right] \tag{3.3e}$$

$$N_6 = b^2 i\, H_0^1(i\beta_n b)\left[\frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} Y_0(\alpha_m b) - J_0(\alpha_m b) \right] - a^2 i\, H_0^1(i\beta_n a)\left[\frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m a) - J_0(\alpha_m a) \right] \tag{3.3f}$$

Roots of eqs.(3.2) a and b give the values of β_n and α_m for values of n and m=0,∞. Equations (3.2)d-i form an infinite system of equations with an infinite number of unknowns. This system can be solved by reducing it to a finite system with finite number of unknowns. The number of unknowns should decide the final accuracy of the solution. Once all the constants are determined, the problem is completely solved.

III-2 Approximate Elasticity Solution

It will be seen in chapter VI that the three-dimensional elasticity solution does not exactly satisfy the conditions at ends of the cylinder. Similarly, cylindrical surfaces are not completely stress free. The radial stress has a small residual value(for explanation refer section VI-1 a). Besides this, convergence of the series is not guaranteed and the solution involves lengthy computations and hence time. All these problems lead one to think of a solution which may not exactly satisfy some of the boundary conditions but still be very simple and sufficiently accurate.

For the approximate solution, given in this section, it is assumed that longitudinal stress is uniform at the ends and radial stress instead of being zero at the curved surfaces varies as an even function of z and has a small magnitude. The second assumption though questionable, makes the solution very simple.

The solution is obtained by super-imposing a particular solution on elementary solution. The elementary solution is given as,

$$\sigma_r = \sigma_\theta = \tau_{rz} = 0$$

$$\sigma_z = -\frac{P}{\pi(b^2-a^2)}$$

$$u = \frac{\nu\sigma r}{E}$$

$$w = -\frac{\sigma z}{E} \qquad (3.4)$$

For the particular solution, eqs(2.7) are used along with eqs(2.14). This yields following expressions for the displacements.

$$u = \left[(\alpha - 1) f(r) - \left\{ r f_{,r}(r) + b\, g_{,r}(r) \right\} \right] \cos \beta z$$

$$w = \left[r f(r) + b\, g(r) \right] \beta \sin \beta z$$

The components of the strain tensor are

$$\epsilon_r = \left[(\alpha - 2) f_{,r}(r) - \left\{ r f_{,rr}(r) + b\, g_{,rr}(r) \right\} \cos \beta z \right.$$

$$\epsilon_\theta = \frac{\alpha - 1}{r} f(r) - \left\{ f_{,r}(r) + \frac{b}{r} g_{,r}(r) \right\} \cos \beta z$$

$$\epsilon_z = \left[r f(r) + b\, g(r) \right] \beta^2 \cos \beta z \qquad (3.5)$$

Volumetric dilation is given by

$$e = (\alpha - 2) \left[f_{,r}(r) + \frac{1}{r} f(r) \right] \cos \beta z \qquad (3.6)$$

Expressions for stresses can now be obtained from the constitutive equations (2.3);eqs (3.5) and (3.6).

Thus,

$$\sigma_r = 2G\Big[(3-2\nu)\, f_{,r}(r) - \Big\{\beta^2 r + \frac{(1-2\nu)}{r}\Big\} f(r) + \frac{b}{r}\, g_{,r}(r) - b\beta^2 g(r)\Big] \cos\beta z$$

$$\sigma_\theta = 2G\Big[\frac{(3-2\nu)}{r} f(r) - (1-2\nu)\, f_{,r}(r) - \frac{b}{r}\, g_{,r}(r)\Big] \cos\beta z$$

$$\sigma_z = 2G\Big[\Big\{\frac{2\nu}{r} + \beta^2 r\Big\} f(r) + 2\nu\, f_{,r}(r) + \beta^2 b\, g(r)\Big] \cos\beta z$$

$$\tau_{rz} = 2G\Big[r\, f_{,r}(r) - (1-2\nu) f(r) + b\, g_{,r}(r)\Big] \beta \sin\beta z$$

Functions f(r) and g(r) are shown in appendix 2 to have the following form,

$$f(r) = A\, I_1(\beta r) + B\, K_1(\beta r)$$

$$g(r) = C\, I_0(\beta r) + D\, K_0(\beta r)$$

Here $I_n(x)$ and $K_n(x)$ are modified Bessel functions of the first and second kinds and of order n. The unknown constants A, B, C and D are evaluated from the satisfaction of boundary conditions.

If new constants A_1, B_1, C_1 and D_1 are defined as

$$A_1 = 2GA$$
$$B_1 = 2GB$$
$$C_1 = 2GC$$
$$D_1 = 2GD,$$

displacements and stresses can be written in the form given below.

$$2GU = \Big[A_1\Big\{4(1-\nu)\,I_1(\beta r) - \beta r\, I_0(\beta r)\Big\} + B_1\Big\{4(1-\nu)\,K_1(\beta r) + \beta r\, K_0(\beta r)\Big\} - C_1\beta b\, I_1(\beta r) + D_1 \beta b\, K_1(\beta r)\Big]\cos\beta z$$

$$2GW = \Big[A_1 r\, I_1(\beta r) + B_1 r\, K_1(\beta r) + C_1 b\, I_0(\beta r) + D_1 b K_0(\beta r)\Big] \times \beta\sin\beta z$$

$$\sigma_r = \Big[A_1\Big\{(3-2\nu)\beta I_0(\beta r) - \Big(\frac{4(1-\nu)}{r} + \beta^2 r\Big) I_1(\beta r)\Big\} - B_1\Big\{(3-2\nu)\beta K_0(\beta r) + \Big(\frac{4(1-\nu)}{r} + \beta^2 r\Big) K_1(\beta r)\Big\} + C_1\Big\{\frac{\beta b}{r} I_1(\beta r) - \beta^2 b\, I_0(\beta r)\Big\} - D_1\Big\{\frac{\beta b}{r} K_1(\beta r) + \beta^2 b K_0(\beta r)\Big\}\Big]\cos\beta z$$

$$\sigma_\theta = \Big[A_1 \Big\{ \frac{4(1-\nu)}{r} I_1(\beta r) - (1-2\nu)\beta I_0(\beta r) \Big\}$$

$$+ B_1 \Big\{ \frac{4(1-\nu)}{r} K_1(\beta r) - (1-2\nu)\beta K_0(\beta r) \Big\}$$

$$- C_1 \frac{\beta b}{r} I_1(\beta r) + D_1 \frac{\beta b}{r} K_1(\beta r) \Big] \cos\beta z$$

$$\sigma_z = \Big[A_1 \Big\{ \beta^2 r I_1(\beta r) + 2\nu\beta I_0(\beta r) \Big\} + B_1 \Big\{ \beta^2 r K_1(\beta r)$$

$$- 2\nu\beta K_0(\beta r) \Big\} + C_1 \beta^2 b I_0(\beta r) + D_1 \beta^2 b K_0(\beta r) \Big] \cos\beta z$$

$$\tau_{rz} = \Big[A_1 \Big\{ r\beta I_0(\beta r) - 2(1-\nu) I_1(\beta r) \Big\} - B_1 \Big\{ r\beta K_0(\beta r)$$

$$+ 2(1-\nu) K_1(\beta r) \Big\} + C_1 \beta b I_1(\beta r) - D_1 \beta b K_1(\beta r) \Big] \beta \sin\beta z$$

(3.7)

From the two assumptions made in this solution, boundary conditions for the stresses are,

$$\sigma_z = 0 \qquad \text{at} \qquad z = \pm c$$

$$\sigma_r = p_0 \cos\beta z \qquad \text{at} \qquad r = a, b$$

and $\tau_{rz} = 0 \qquad r = a, b$

On satisfaction of above boundary conditions, following expressions are obtained.

$$\cos \beta c = 0$$

$$A_1\left[(3-2\nu)\beta I_0(\beta b) - \left\{\frac{4(1-\nu)}{b} + \beta^2 b\right\} I_1(\beta b)\right]$$
$$-B_1\left[(3-2\nu)\beta K_0(\beta b) + \left\{\frac{4(1-\nu)}{b} + \beta^2 b\right\} K_1(\beta b)\right]$$
$$+C_1\left[\beta I_1(\beta b) - \beta^2 b I_0(\beta b)\right] - D_1\left[\beta K_1(\beta b) + \beta^2 b K_0(\beta b)\right] = P_0$$

$$A_1\left[(3-2\nu)\beta I_0(\beta a) - \left\{\frac{4(1-\nu)}{a} + \beta^2 a\right\} I_1(\beta a)\right]$$
$$-B_1\left[(3-2\nu)\beta K_0(\beta a) + \left\{\frac{4(1-\nu)}{a} + \beta^2 a\right\} K_1(\beta a)\right]$$
$$+C_1\left[\frac{\beta b}{a} I_1(\beta a) - \beta^2 b I_0(\beta a)\right] - D_1\left[\frac{\beta b}{a} K_1(\beta a) + \beta^2 b K_0(\beta a)\right] = P_0$$

$$A_1\left[\beta b I_0(\beta b) - 2(1-\nu) I_1(\beta b)\right] - B_1\left[\beta b K_0(\beta b) + 2(1-\nu) K_1(\beta b)\right]$$
$$+C_1\left[\beta b I_1(\beta b)\right] - D_1\left[\beta b K_1(\beta b)\right] = 0$$

$$A_1\left[\beta a\, I_0(\beta a) - 2(1-\nu)\, I_1(\beta a)\right] - B_1\left[\beta a\, K_0(\beta a) + 2(1-\nu) \ldots\right.$$

$$+ C_1\left[\beta b\, I_1(\beta a)\right] - D_1\left[\beta b\, K_1(\beta a)\right] = 0$$

(3.

Equations (3.8)b-e form a set of linear simultaneous equations which solved for A_1, B_1, C_1 and D_1 and thus the solution is completely know

III-3 Shell and Boundary Layer Solution

As a consequence of relations (2.22) and (2.23), non-dimens parameter z_1 is redefined for the interior and boundary layer solutio separately as, [19]

$$z_1 = \frac{z + c}{\sqrt{h a_1}}$$

and

$$z_2 = \frac{z + c}{h}$$

(3.

In order to distinguish these solutions from each other, variable z_1 is replaced by ξ for the interior case and by η for the boundary laye case.

Asymptotic expansions for stress and displacement component for the two cases are written as power series in $\lambda^{\frac{1}{2}}$.

$$s(z_1, \rho; \lambda) \sim \sum_{i=0}^{\infty} s^{(i)}(\xi, \rho)\left[\lambda^{\frac{1}{2}}\right]^i$$

$$\vartheta(z_1, \rho; \lambda) \sim \sum_{i=0}^{\infty} v^{(i)}(\xi, \rho)\left[\lambda^{\frac{1}{2}}\right]^i$$

(3

$$s(z_1, \rho; \lambda) \sim \sum_{i=0}^{\infty} t^{(i)}(\eta, \rho)\left[\lambda^{\frac{1}{2}}\right]^i$$

$$v(z, \rho; \lambda) \sim \sum_{i=0}^{\infty} w^{(i)}(\eta, \rho) \left[\lambda^{\frac{1}{2}} \right]^{i} \tag{3.11}$$

Functions $s^{(i)}(\xi,\rho)$ and $v^{(i)}(\xi,\rho)$ are called the interior stress and displacement coefficients while the coefficients $t^{(i)}(\eta,\rho)$ and $w^{(i)}(\eta,\rho)$ are associated with the boundary layer solution.

On substituting (3.10) in to (2.20), equating the coefficients of λ^{0}, $\lambda^{\frac{1}{2}}$, λ^{1}, $\lambda^{\frac{3}{2}}$, and λ^{2} on both sides of the equation and integrating in a step by step manner, first two non-zero systems are obtained. For the purpose of brevity, functions $s_z^{(0)}(\xi,\rho)$, $v_r^{(0)}(\xi,\rho)$ etc., are written as $s_z^{(0)}$ and $v_r^{(0)}$ while middle surface stresses and displacements $S_z^{(0)}(\xi)$, $V_r^{(0)}(\xi)$ etc., are written in the following as $S_z^{(0)}$ and $V_r^{(0)}$.

First System:

$$\begin{aligned}
v_r^{(0)} &= V_r^{(0)} \\
v_z^{(1)} &= V_z^{(1)} - V_r^{(0)'}\rho \\
s_\theta^{(0)} &= V_r^{(0)} + \nu\left[V_z^{(1)'} - V_r^{(0)''}\rho \right] \\
s_z^{(0)} &= \nu V_r^{(0)} + V_z^{(1)'} - V_r^{(0)''}\rho \\
s_{rz}^{(1)} &= S_{rz}^{(1)} - \left[\nu V_r^{(0)'} + V_z^{(1)''} \right]\rho + V_r^{(0)'''}\frac{\rho^2}{2} \\
s_r^{(2)} &= S_r^{(2)} + \left[- S_{rz}^{(1)'} + \nu V_z^{(1)'} + V_r^{(0)} \right]\rho + V_z^{(1)'''}\frac{\rho^2}{2} - V_r^{(0)IV}\frac{\rho^3}{6}
\end{aligned} \tag{3.12}$$

Second System:

$$v_r^{(2)} = V_r^{(2)} - \frac{\nu}{1-\nu}\left[\left\{ V_r^{(0)} + V_z^{(1)'} \right\} \rho - V_r^{(0)''}\frac{\rho^2}{2} \right]$$

$$v_z^{(3)} = V_z^{(1)} - V_r^{(2)'} \rho + \frac{2}{1-\nu} S_{rz}^{(1)} \rho - \left[\frac{\nu}{1-\nu} V_r^{(0)'} + \frac{2-\nu}{1-\nu} V_z^{(1)''} \right] \frac{\rho^2}{2} + \frac{2-\nu}{1-\nu} V_r^{(0)'''} \frac{\rho^3}{6}$$

$$s_z^{(2)} = \nu V_r^{(2)} + V_z^{(3)'} - V_r^{(2)''} \rho + \frac{\nu}{1-\nu} S_r^{(2)} + \frac{2-\nu}{1-\nu} S_{rz}^{(1)'} \rho - \left[\frac{\nu}{2} V_r^{(0)''} + V_z^{(1)'''} \right] \rho^2 + V_r^{(0)IV} \frac{\rho^3}{3}$$

$$s_\theta^{(2)} = V_r^{(2)} + \nu V_z^{(3)'} - \nu V_r^{(2)''} \rho + \frac{\nu}{1-\nu} S_r^{(2)} + \left[\frac{\nu}{1-\nu} S_{rz}^{(1)'} - \nu V_z^{(1)'} - V_r^{(0)} \right] \rho + \left[\nu V_r^{(0)''} - \nu V_z^{(1)'''} \right] \frac{\rho^2}{2} + \nu V_r^{(0)IV} \frac{\rho^3}{6}$$

$$s_{rz}^{(3)} = S_{rz}^{(3)} - \left[\nu V_r^{(2)'} + V_z^{(3)''} \right] \rho + V_r^{(2)'''} \frac{\rho^2}{2} - \left[S_{rz}^{(1)} + \frac{\nu}{1-\nu} S_r^{(2)'} \right] \rho + \left[\nu V_r^{(0)'} + V_z^{(1)''} - \frac{2-\nu}{1-\nu} S_{rz}^{(1)''} \right] \frac{\rho^2}{2} + \left[V_z^{(1)IV} - \frac{1-\nu}{2} V_r^{(0)'''} \right] \frac{\rho^3}{3} - V_r^{(0)V} \frac{\rho^4}{12}$$

$$s_r^{(4)} = S_r^{(4)} + \left[V_r^{(2)} + \nu V_z^{(3)'} - S_{rz}^{(3)'} - \frac{1-2\nu}{1-\nu} S_r^{(2)} \right] \rho + \left[V_z^{(3)'''} - 3 V_r^{(0)} - 3\nu V_z^{(1)'} + \frac{2-\nu}{1-\nu} S_{rz}^{(1)'} + \frac{\nu}{1-\nu} S_r^{(2)''} \right] \frac{\rho^2}{2} + \left[- V_r^{(2)IV} - (2+\nu) V_z^{(1)'''} + 2\nu V_r^{(0)''} + \frac{2-\nu}{1-\nu} S_{rz}^{(1)'''} \right] \frac{\rho^3}{6} + \left[V_r^{(0)IV} - V_z^{(1)V} \right] \frac{\rho^4}{12} + V_r^{(0)VI} \frac{\rho^5}{60} \qquad (3.13)$$

In the above expressions, superscripts over the middle surface displacement and stress terms indicate the relative orders of magnitude. These expressions are written for isotropic cylinders and are similar to those obtained in [5]. In refrence [5], stress and displacement components are expanded in terms of λ and are for orthotropic cylinders. This accounts for the difference in orders of various quantities.

A similar procedure for the boundary layer solution yields the first system as,

$$t_{rz}^{(0)} = \left(\frac{1-\nu}{2}\right)\left[\omega_z^{(2)^{\bullet}} + \omega_r^{(2)^{\prime}}\right]$$

$$t_r^{(0)} = R\left[(1-\nu)\,\omega_r^{(2)^{\bullet}} + \nu\,\omega_z^{(2)^{\prime}}\right]$$

$$t_z^{(0)} = S\,\omega_z^{(2)^{\prime}} + \nu R\,\omega_r^{(2)^{\bullet}}$$

$$t_\theta^{(0)} = T\,\omega_z^{(2)^{\prime}} + \nu R\,\omega_r^{(2)^{\bullet}}$$

$$S\,\omega_z^{(2)^{\prime\prime}} + \left(\frac{1-\nu}{2}\right)\omega_z^{(2)^{\bullet\bullet}} + \left[R\nu + \left(\frac{1-\nu}{2}\right)\right]\omega_r^{(2)^{\prime\bullet}} = 0$$

$$\left(\frac{1-\nu}{2}\right)\omega_r^{(2)^{\prime\prime}} + R(1-\nu)\,\omega_r^{(2)^{\bullet\bullet}} + \left[R\nu + \left(\frac{1-\nu}{2}\right)\right]\omega_z^{(2)^{\prime\bullet}} = 0 \qquad (3.14)$$

Here,

$$R = \frac{(1-\nu^2)}{1-\nu-2\nu^2}$$

$$S = 1 + \frac{\nu^2 R}{1-\nu}$$

$$T = \nu + \frac{\nu^2 R}{1-\nu}$$

Equations (3.12)a-d are identical to those of the conventional thin shell theory and eqs.(3.12)e-f, (3.13) and (3.14) provide thickness corrections.

Boundary conditions (2.17) and (2.18) are to be satisfied by the interior and boundary layer solutions together.

Thus, for $\xi = \eta = 0$

$$
\begin{aligned}
V_r^{(0)}(0) &= -\frac{\nu}{1-\nu^2} \\
V_r^{(0)'}(0) &= 0 \\
\omega_z^{(2)}(0,\rho) &= 0 \\
\omega_r^{(2)}(0,\rho) &= -\frac{\nu}{1-\nu^2}\rho + \frac{\nu}{1-\nu}\left[\left\{V_r^{(0)}(0) + V_z^{(1)'}(0)\right\}\rho \right. \\
&\qquad \left. - V_r^{(0)''}(0)\frac{\rho^2}{2}\right]
\end{aligned} \tag{3.15}
$$

Without any loss of generality, middle surface displacements $V_r^{(2)}$ and $V_z^{(3)}$, which are arbitrary functions of ξ, can be taken as zero at the end $\xi=0$.

So,
$$V_r^{(2)}(0) = 0 \qquad V_z^{(3)}(0) = 0 \tag{3.16}$$

Solutions of systems (3.12), (3.13) and (3.14) subjected to boundary conditions (3.15) are now considered.

(a) Solution of the Interior Problem: Application of boundary conditions (2.11) to eqs. (3.12) yields the following results [19],

$$
\begin{aligned}
S_{rz}^{(1)} &= -\frac{1}{2} V_r^{(0)'''} \\
\nu V_r^{(0)'} + V_z^{(1)''} &= 0 \\
S_r^{(2)} &= -\frac{1}{2} V_z^{(1)'''} \\
V_r^{(0)^{IV}} + 4\gamma^4 V_r^{(0)} &= 0
\end{aligned} \tag{3.17}
$$

where,

$$V_r^{(0)} = -\frac{\nu}{1-\nu^2}\left(\cos\gamma\xi + \sin\gamma\xi\right) e^{-\gamma\xi}$$

$$V_z^{(1)} = -\frac{\nu^2}{\gamma(1-\nu^2)} \cos\gamma\xi \; e^{-\gamma\xi}$$

and $$\gamma^4 = \frac{3(1-\nu^2)}{4} \tag{3.18}$$

In a similar manner, satisfaction of condition (2.11) by eqs.(3.13) yields the expressions for the middle surface displacements and stresses.

$$\nu V_r^{(2)'} + V_z^{(3)''} = B_\nu V_r^{(0)'''}$$

$$V_r^{(2)} + \nu V_z^{(3)'} + \frac{1}{3} V_r^{(2)IV} = A_\nu V_r^{(0)''}$$

$$S_{rz}^{(3)} = -\frac{1}{2} V_r^{(2)'''} + \frac{(5-2\nu)(1+\nu)}{4} V_r^{(0)'}$$

$$S_r^{(4)} = -\frac{1}{2} V_z^{(3)'''} + \frac{(1-\nu^2)(1-2\nu)}{4} V_r^{(0)} \tag{3.19}$$

On eliminating $V_z^{(3)}$ from eqs.(3.19)a and b, a fourth order differential equation for $V_r^{(2)}$ is obtained. In writing eq.(3.20), rigid body displacements are omitted and from physical conditions V_z decreases exponentially along ξ.

$$V_r^{(2)IV} + 4\gamma^4 V_r^{(2)} = 3(A_\nu - \nu B_\nu) V_r^{(0)''} \tag{3.20}$$

where,

$$A_\nu = \frac{24-14\nu-39\nu^2+14\nu^3}{30(1-\nu)}$$

$$B_\nu = \frac{2-3\nu-2\nu^2}{6(1-\nu)}$$

A general solution of eq.(3.20) can be written in the following form,

$$V_r^{(2)} = \left(A_2 \cos\gamma\xi + B_2 \sin\gamma\xi\right) + \frac{C_\nu \xi}{8\gamma^3} \cos\gamma\xi\, e^{-\gamma\xi}$$

Expression for $V_z^{(3)}$ can be obtained by integrating eq.(3.19)a twice with respect to ξ and setting the constant of integration to zero in order to admit only decay type solution. Final expressions for $V_r^{(2)}$ and $V_z^{(3)}$ can be written so as to satisfy condition (3.16).

$$V_r^{(2)} = -\frac{C_\nu \xi}{24\gamma^2\nu} V_r^{(0)'''}$$

$$V_z^{(3)} = \left(B_\nu - \frac{C_\nu}{24\gamma^2}\right) V_r^{(0)'} + \frac{C_\nu \xi}{24\gamma^2} V_r^{(0)''} \qquad (3.21)$$

Here, $C_\nu = \frac{6\nu\gamma^2}{1-\nu}\left(A_\nu - \nu B_\nu\right)$

Middle surface displacements given by eqs.(3.18) and (3.21) can be substituted in to eqs.(3.17)a,c and (3.19)c,d to give middle surface stresses for the two systems. Thus, the systems given by eqs.(3.12) and (3.13) are completely solved and the interior problem for first two non zero systems is given as,

$$
\begin{aligned}
v_r &= v_r^{(0)} + v_r^{(2)}\lambda \\
v_z &= v_z^{(1)}\lambda^{\frac{1}{2}} + v_z^{(3)}\lambda^{\frac{3}{2}} \\
s_\theta &= s_\theta^{(0)} + s_\theta^{(2)}\lambda \\
s_z &= s_z^{(0)} + s_z^{(2)}\lambda \\
s_r &= s_r^{(2)}\lambda + s_r^{(4)}\lambda^2 \\
s_{rz} &= s_{rz}^{(1)}\lambda^{\frac{1}{2}} + s_{rz}^{(3)}\lambda^{\frac{3}{2}}
\end{aligned}
\tag{3.22}
$$

(b) Solution of the Boundary Layer Problem: Boundary layer stresses for the first non zero system are expressed by eqs. (3.14)a-d in terms of the displacements $w_z^{(2)}(\eta,\rho)$ and $w_r^{(2)}(\eta,\rho)$. The contribution of the boundary layer solution is predominant in the narrow edge zones only. It is, therefore, expected that these solutions are very rapidly decaying along η.

Solution of the system (3.14) is assumed to be in the form of an infinite series.

$$
\begin{aligned}
w_z^{(2)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\, U_1^n(\rho,\beta_n^*)\, e^{-\beta_n^*\eta} \\
w_r^{(2)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\, U_2^n(\rho,\beta_n^*)\, e^{-\beta_n^*\eta}
\end{aligned}
\tag{3.23}
$$

Where a_n and β_n^* are the complex coefficients to be determined from the boundary conditions. System (3.14)a-d can now be written as,

$$t_z^{(o)}(\eta,\rho) = \sum_{n=1}^{\infty} a_n \tau_{11}^n(\rho,\beta_n^*) e^{-\beta_n^*\eta}$$

$$t_r^{(o)}(\eta,\rho) = \sum_{n=1}^{\infty} a_n \tau_{22}^n(\rho,\beta_n^*) e^{-\beta_n^*\eta}$$

$$t_{rz}^{(o)}(\eta,\rho) = \sum_{n=1}^{\infty} a_n \tau_{12}^n(\rho,\beta_n^*) e^{-\beta_n^*\eta}$$

$$t_\theta^{(o)}(\eta,\rho) = \sum_{n=1}^{\infty} a_n \tau_{33}^n(\rho,\beta_n^*) e^{-\beta_n^*\eta} \tag{3.24}$$

Here,

$$\tau_{11}^n = -\beta_n^* S u_1^n - \nu R \dot{u}_2^n$$

$$\tau_{22}^n = R\left[(1-\nu)\dot{u}_2^n - \beta_n^* \nu u_1^n\right]$$

$$\tau_{12}^n = \left(\frac{1-\nu}{2}\right)\left[\dot{u}_1^n - \beta_n^* u_2^n\right]$$

$$\tau_{33}^n = -\beta_n^* T u_1^n + \nu R \dot{u}_2^n \tag{3.25}$$

In the above u_1^n and u_2^n are the short forms of functions $u_1^n(\rho,\beta_n^*)$ and $u_2^n(\rho,\beta_n^*)$. Substitution of eqs.(3.23) in to (3.14)e,f gives the following equations.

$$S\beta_n^{*2} u_1^n + \frac{1-\nu}{2}\ddot{u}_1^n - \left(R\nu + \frac{1-\nu}{2}\right)\beta_n^* \dot{u}_2^n = 0$$

$$\frac{1-\nu}{2}\beta_n^{*2} u_2^n + (1-\nu)R\ddot{u}_2^n - \left(R\nu + \frac{1-\nu}{2}\right)\beta_n^* \dot{u}_1^n = 0$$

These equations can be written as,

$$\ddot{u}_1^n + A^* \beta_n^{*2} u_1^n - B^* \beta_n^* \dot{u}_2^n = 0$$
$$\ddot{u}_2^n + A^{**} \beta_n^{*2} u_2^n - B^{**} \beta_n^* \dot{u}_1^n = 0 \qquad (3.26)$$

where,
$$A^* = \frac{1 + \frac{\nu^2}{1-\nu} R}{\frac{1-\nu}{2}}$$

$$A^{**} = \frac{1}{2R}$$

$$B^* = \frac{R\nu + \frac{1-\nu}{2}}{\frac{1-\nu}{2}}$$

$$B^{**} = \frac{R\nu + \frac{1-\nu}{2}}{(1-\nu) R}$$

Equations (3.26) are now solved to yield a fourth order differential equation in u_1^n.

$$\ddddot{u}_1^n + 2\beta_n^{*2} \ddot{u}_1^n + \beta_n^{*4} u_1^n = 0 \qquad (3.27)$$

This is a simplified form of the equation obtained in [19] and is for isotropic cylinders only. Solution of the equation in [19] is for orthotropic cylinders and cannot be used for an isotropic case. Thus, solution of the equation (3.27) will now be obtained.

Equation [3.27] will have two repeated roots and hence the solution can be taken in the following form.

$$u_1^n = (A_2 + \rho B_2)\cos\beta_n^*\rho + (C_2 + \rho D_2)\sin\beta_n^*\rho$$

Here, A_2, B_2, C_2 and D_2 are as yet unknown constants. Substitution of u_1^n in to eq.(3.26)a, results in the expression for the function u_2^n.

$$u_2^n = (U - \rho D_2)\cos\beta_n^*\rho + (V - \rho C_2)\sin\beta_n^*\rho$$

Where,

$$U = \left(\frac{3-4\nu}{\beta_n^*} B_2 - C_2\right)$$

$$V = \left(\frac{3-4\nu}{\beta_n^*} D_2 + A_2\right)$$

For convenience, u_1^n and u_2^n are separated in to symmetric and anti-symmetric parts [6]. Thus, the symmetric and anti-symmetric parts of u_1^n and u_2^n are obtained in the following form.

$$u_1^n = A_2\cos\beta_n^*\rho + \rho D_2\sin\beta_n^*\rho$$

$$u_2^n = V\sin\beta_n^*\rho - \rho D_2\cos\beta_n^*\rho \qquad (3.28)$$

$$u_1^n = C_2\sin\beta_n^*\rho + \rho B_2\cos\beta_n^*\rho$$

$$u_2^n = U\cos\beta_n^*\rho - \rho B_2\sin\beta_n^*\rho \qquad (3.29)$$

In order to have the cylindrical surfaces tractionfree, boundary conditions are,

$$t_r^{(0)}(\eta, \pm 1) = 0$$
$$t_{rz}^{(0)}(\eta, \pm 1) = 0 \qquad (3.30)$$

Substitution of eqs.(3.28) and (3.29) in to eqs.(3.14)a,b and use of conditions (3.30) yields the characteristic equations for β_n^* (see app. 3).

$$\sin 2\beta_n^* + 2\beta_n^* = 0$$
$$\sin 2\beta_n^* - 2\beta_n^* = 0 \qquad (3.31)$$

Final expressions for u_1^n and u_2^n for symmetric and anti-symmetric solutions respectively have been derived in appendix 3 and are given below.

$$u_1^n = k_n \cos \beta_{n1}\rho + \rho \sin \beta_{n1}\rho$$
$$u_2^n = \left(\frac{3-4\nu}{\beta_{n1}} + k_n\right) \sin \beta_{n1}\rho - \rho \cos \beta_{n1}\rho \qquad (3.32)$$

$$u_1^n = p_n \sin \beta_{n2}\rho + \rho \cos \beta_{n2}\rho$$
$$u_2^n = \left(\frac{3-4\nu}{\beta_{n2}} - p_n\right) \cos \beta_{n2}\rho + \rho \sin \beta_{n2}\rho \qquad (3.33)$$

Here,

$$k_n = -\frac{2(1-\nu) \cos \beta_{n1} + \beta_{n1} \sin \beta_{n1}}{\beta_{n1} \cos \beta_{n1}}$$

$$p_n = \frac{2(1-\nu)\sin\beta_{n2} - \beta_{n2}\cos\beta_{n2}}{\beta_{n2}\sin\beta_{n2}}$$

β_{n1} and β_{n2} are roots of eqs.(3.31)a,b respectively.

Complex constants a_n are now evaluated from the displacement boundary conditions (3.15)c,d. For this, principle of variation of energy due to complex stress field is applied [19,20].

$$\mathrm{Re}\int_{-1}^{+1}\left[\left\{\omega_z^{(2)}(0,\rho) - 0\right\}\delta\overline{t_z^{(0)}(0,\rho)}\right]d\rho$$
$$+\mathrm{Re}\int_{-1}^{+1}\left[\omega_r^{(2)}(0,\rho) - \left\{-\frac{\nu}{1-\nu^2}\rho + \frac{\nu}{1-\nu}\left(\left(V_r^{(0)'}(0) + V_z^{(1)'}(0)\right)\rho\right.\right.\right.$$
$$\left.\left.\left. - V_r^{(0)''}(0)\,\frac{\rho^2}{2}\right)\right\}\delta\overline{t_{rz}^{(0)}(0,\rho)}\right]d\rho = 0$$

A simple substitution shows that a_n must satisfy the following system of equations[19].

$$\sum_{n=1}^{\infty} \ell_{mn}\, a_n = b_m \tag{3.34}$$

where,

$$\ell_{mn} = \int_{-1}^{+1}\left(\overline{\tau_{11}^m}\, u_1^n + \overline{\tau_{12}^m}\, u_2^n\right)d\rho$$

and $$b_m = \int_{-1}^{+1}\omega_r^{(2)}(0,\rho)\,\overline{\tau_{12}^m}\,d\rho \tag{3.35}$$

and $\overline{(\ )}$ indicates the complex conjugate of ( ).
Equations (3.35) have been evaluated in appendix 4 and ℓ_{mn} and b_m are expressed in the following form.

$$\ell_{mn} = X_{mn} + Y_{mn}$$

and
$$b_m = -\frac{4\nu}{\overline{\beta_{m1}}\cos\overline{\beta_{m1}}} - \frac{\nu^2}{1-\nu^2}\gamma^2 I_2 \tag{3.36}$$

Here,

$$\begin{aligned}
X_{mn} = (1-\nu)\Bigg[& \frac{\sin(\overline{\beta_{m1}}+\beta_{n1})}{\overline{\beta_{m1}}+\beta_{n1}}\Bigg\{(2\nu-\overline{k_m}\overline{\beta_{m1}})\left(k_n+\frac{1}{\overline{\beta_{m1}}+\beta_{n1}}\right) \\
& -\overline{\beta_{m1}}\left(\frac{k_n}{\overline{\beta_{m1}}+\beta_{n1}}-1+\frac{2}{(\overline{\beta_{m1}}+\beta_{n1})^2}\right)\Bigg\} \\
& +\frac{\sin(\overline{\beta_{m1}}-\beta_{n1})}{\overline{\beta_{m1}}-\beta_{n1}}\Bigg\{(2\nu-\overline{k_m}\overline{\beta_{m1}})\left(k_n-\frac{1}{\overline{\beta_{m1}}-\beta_{n1}}\right)-\overline{\beta_{m1}}\left(\frac{k_n}{\overline{\beta_{m1}}-\beta_{n1}}+1\right. \\
& \left.-\frac{2}{(\overline{\beta_{m1}}-\beta_{n1})^2}\right)\Bigg\} + \frac{\sin(\overline{\beta_{m2}}+\beta_{n2})}{\overline{\beta_{m2}}+\beta_{n2}}\Bigg\{(2\nu+\overline{p_m}\overline{\beta_{m2}})\left(p_n-\frac{1}{\overline{\beta_{m2}}+\beta_{n2}}\right) \\
& -\overline{\beta_{m2}}\left(\frac{p_n}{\overline{\beta_{m2}}+\beta_{n2}}+1-\frac{2}{(\overline{\beta_{m2}}+\beta_{n2})^2}\right)\Bigg\} + \frac{\sin(\overline{\beta_{m2}}-\beta_{n2})}{\overline{\beta_{m2}}-\beta_{n2}}\Bigg\{-(2\nu+\overline{p_m}\overline{\beta_{m2}}) \\
& \left(p_n+\frac{1}{\overline{\beta_{m2}}-\beta_{n2}}\right)+\overline{\beta_{m2}}\left(\frac{p_n}{\overline{\beta_{m2}}-\beta_{n2}}-1+\frac{2}{(\overline{\beta_{m2}}-\beta_{n2})^2}\right)\Bigg\} \\
& +\frac{\cos(\overline{\beta_{m1}}+\beta_{n1})}{\overline{\beta_{m1}}+\beta_{n1}}\Bigg\{-(2\nu-\overline{k_m}\overline{\beta_{m1}})+\overline{\beta_{m1}}\left(k_n+\frac{2}{\overline{\beta_{m1}}+\beta_{n1}}\right)\Bigg\}
\end{aligned}$$

$$+\frac{\cos(\bar{\beta}_{m1}-\beta_{n1})}{\bar{\beta}_{m1}-\beta_{n1}}\left\{(2\nu-\bar{k}_m\bar{\beta}_{m1})+\bar{\beta}_{m1}\left(k_n-\frac{2}{\bar{\beta}_{m1}-\beta_{n1}}\right)\right\}$$

$$+\frac{\cos(\bar{\beta}_{m2}+\beta_{n2})}{\bar{\beta}_{m2}+\beta_{n2}}\left\{(2\nu+\bar{p}_m\bar{\beta}_{m2})+\bar{\beta}_{m2}\left(p_n-\frac{2}{\bar{\beta}_{m2}+\beta_{n2}}\right)\right\}$$

$$\left.+\frac{\cos(\bar{\beta}_{m2}-\beta_{n2})}{\bar{\beta}_{m2}-\beta_{n2}}\left\{(2\nu+\bar{p}_m\bar{\beta}_{m2})-\bar{\beta}_{m2}\left(p_n+\frac{2}{\bar{\beta}_{m2}-\beta_{n2}}\right)\right\}\right]$$

$$Y_{mn}=(1-\nu)\left[\frac{\sin(\bar{\beta}_{m1}+\beta_{n1})}{\bar{\beta}_{m1}+\beta_{n1}}\left\{(\bar{k}_m\bar{\beta}_{m1}+1-2\nu)\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right.\right.\right.$$

$$\left.\left.+\frac{1}{\bar{\beta}_{m1}+\beta_{n1}}\right)+\bar{\beta}_{m1}\left[\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right)\left(\frac{1}{\bar{\beta}_{m1}+\beta_{n1}}\right)-1+\frac{2}{(\bar{\beta}_{m1}+\beta_{n1})^2}\right]\right\}$$

$$+\frac{\sin(\bar{\beta}_{m1}-\beta_{n1})}{\bar{\beta}_{m1}-\beta_{n1}}\left\{-(\bar{k}_m\bar{\beta}_{m1}+1-2\nu)\left(\frac{3-4\nu}{\beta_{n1}}+k_n-\frac{1}{\bar{\beta}_{m1}-\beta_{n1}}\right)\right.$$

$$\left.+\bar{\beta}_{m1}\left[\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right)\left(-\frac{1}{\bar{\beta}_{m1}-\beta_{n1}}\right)-1+\frac{2}{(\bar{\beta}_{m1}-\beta_{n1})^2}\right]\right\}$$

$$+\frac{\sin(\bar{\beta}_{m2}+\beta_{n2})}{\bar{\beta}_{m2}+\beta_{n2}}\left\{(\bar{p}_m\bar{\beta}_{m2}-1+2\nu)\left(\frac{3-4\nu}{\beta_{n2}}-p_n+\frac{1}{\bar{\beta}_{m2}+\beta_{n2}}\right)\right.$$

$$\left.-\bar{\beta}_{m2}\left[\left(\frac{3-4\nu}{\beta_{n2}}-p_n\right)\left(\frac{1}{\bar{\beta}_{m2}+\beta_{n2}}\right)-1+\frac{2}{(\beta_{m2}+\beta_{n2})^2}\right]\right\}$$

$$+\frac{\sin(\bar{\beta}_{m2}-\beta_{n2})}{\bar{\beta}_{m2}-\beta_{n2}}\left\{(\bar{p}_m\bar{\beta}_{m2}-1+2\nu)\left(\frac{3-4\nu}{\beta_{n2}}-p_n-\frac{1}{\bar{\beta}_{m2}-\beta_{n2}}\right)\right.$$

$$- \overline{\beta_{m2}}\left[\left(\frac{3-4\nu}{\beta_{n2}} - p_n\right)\left(\frac{1}{\overline{\beta_{m2}}-\beta_{n2}}\right) + 1 - \frac{2}{(\overline{\beta_{m2}}-\beta_{n2})^2}\right]\Bigg\}$$

$$+ \frac{\cos(\overline{\beta_{m1}}+\beta_{n1})}{\overline{\beta_{m1}}+\beta_{n1}}\Bigg\{-\left(\overline{k_m}\,\overline{\beta_{m1}}+1-2\nu\right) - \overline{\beta_{m1}}\left(\frac{3-4\nu}{\beta_{n1}} + k_n\right.$$

$$\left. + \frac{2}{\overline{\beta_{m1}}+\beta_{n1}}\right)\Bigg\} + \frac{\cos(\overline{\beta_{m1}}-\beta_{n1})}{\overline{\beta_{m1}}-\beta_{n1}}\Bigg\{-\left(\overline{k_m}\,\overline{\beta_{m1}}+1-2\nu\right)$$

$$+ \overline{\beta_{m1}}\left(\frac{3-4\nu}{\beta_{n1}} + k_n - \frac{2}{\overline{\beta_{m1}}-\beta_{n1}}\right)\Bigg\} + \frac{\cos(\overline{\beta_{m2}}+\beta_{n2})}{\overline{\beta_{m2}}+\beta_{n2}}$$

$$\Bigg\{-\left(\overline{p_m}\,\overline{\beta_{m2}}-1+2\nu\right) + \overline{\beta_{m2}}\left(\frac{3-4\nu}{\beta_{n2}} - p_n + \frac{2}{\overline{\beta_{m2}}+\beta_{n2}}\right)\Bigg\}$$

$$+ \frac{\cos(\overline{\beta_{m2}}-\beta_{n2})}{\overline{\beta_{n2}}-\beta_{n2}}\Bigg\{\left(\overline{p_m}\,\overline{\beta_{m2}}-1+2\nu\right) - \overline{\beta_{m2}}\left(-\frac{3-4\nu}{\beta_{n2}} + p_n\right.$$

$$\left. + \frac{2}{\overline{\beta_{m2}}-\beta_{n2}}\right)\Bigg\}\Bigg]$$

$$I_2 = \sin\overline{\beta_{m2}}\left[\left\{\overline{p_m}\,\overline{\beta_{m2}}-(1-2\nu)\right\}\left(\frac{2}{\overline{\beta_{m2}}} - \frac{4}{(\overline{\beta_{m2}})^3}\right)\right.$$

$$\left. - \overline{\beta_{m2}}\left\{\frac{6}{(\overline{\beta_{m2}})^2} - \frac{12}{(\overline{\beta_{m2}})^4}\right\}\right] + \cos\overline{\beta_{m2}}\left[\frac{4}{(\overline{\beta_{m2}})^2}\left\{\overline{p_m}\,\overline{\beta_{m2}}\right.\right.$$

$$\left.\left. -(1-2\nu)\right\} - \overline{\beta_{m2}}\left\{\frac{12}{(\overline{\beta_{m2}})^3} - \frac{2}{\overline{\beta_{m2}}}\right\}\right]$$

System (3.34) can be solved for a_n by the method of reduction and hence first non zero system for the boundary layer solution is completely given by eqs.(3.23) and (3.24).

CHAPTER IV

COMPUTATIONS

IV-1 Non-Dimensionalization

In this chapter, all the expressions obtained in chapter III are first non-dimensionalized in order to make computations and comparison easier.

The following non-dimensional quantities are introduced.

$$\sigma_r' = \frac{\sigma_r}{\sigma}, \quad \sigma_\theta' = \frac{\sigma_\theta}{\sigma}, \quad \sigma_z' = \frac{\sigma_z}{\sigma}, \quad \tau_{rz}' = \frac{\tau_{rz}}{\sigma}$$

$$u' = \frac{u}{b}, \quad w' = \frac{w}{c}, \quad \alpha_m' = \alpha_m b, \quad \beta_n' = \beta_n c$$

$$A_n' = \frac{A_n}{\sigma}, \quad B_n' = \frac{B_n}{\sigma}, \quad C_n' = \frac{C_n}{\sigma}, \quad D_n' = \frac{D_n}{\sigma}$$

$$E_m' = \frac{E_m}{\sigma}, \quad F_m' = \frac{F_m}{\sigma}, \quad G' = \frac{G}{\sigma} \text{ and } L' = \frac{L}{\sigma}$$

All the expressions are written in terms of non-dimensional geometric parameters $\frac{r}{b}$, $\frac{z}{c}$, $\frac{a}{b}$ and $\frac{c}{b}$.

(a) Three-Dimensional Elasticity Solution: Equations (3.1) on non-dimensionalization and simplification can be written as follows.

$$2G'u' = \sum_{n=1}^{\infty} \frac{\cos\beta_n' \frac{z}{c}}{\beta_n' \frac{b}{c}} \Big[A_n' \beta_n' \frac{r}{b}\cdot\frac{b}{c}\, I_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) - B_n' I_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$- C_n' \beta_n' \frac{r}{b}\cdot\frac{b}{c}\, \frac{2}{\pi} K_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) + D_n' \frac{2}{\pi} K_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) \Big]$$

$$+ \sum_{m=1}^{\infty} \Big[\frac{z}{c}\cdot\frac{c}{b} \sinh\left(\alpha_m' \frac{z}{c}\cdot\frac{c}{b}\right) - \frac{c}{b}\cdot\tanh\left(\alpha_m' \frac{c}{b}\right)$$

$$\times \cosh\left(\alpha'_m \frac{c}{b}\cdot\frac{z}{c}\right)\Big] \times \left[E'_m J_1\left(\alpha'_m \frac{r}{b}\right) + F'_m Y_1\left(\alpha'_m \frac{r}{b}\right)\right]$$

(4.1)a

$$2G'w' = -(1-2\nu)\, 6L' \frac{z}{c} + \sum_{n=1}^{\infty} \frac{\sin \beta'_n \frac{z}{c}}{\beta'_n}\Big[A'_n\Big\{-4(1-\nu)\, I_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$\div \beta'_n \frac{r}{b}\cdot\frac{b}{c}\, I_1\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)\Big\} + B'_n I_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$+ C'_n\Big\{4(1-\nu)\frac{2}{\pi} K_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right) - \beta'_n \frac{r}{b}\cdot\frac{b}{c}\,\frac{2}{\pi} K_1\left(\beta'_n \frac{r}{b}\,\frac{b}{c}\right)\Big\}$$

$$+ D'_n \frac{2}{\pi} K_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)\Big] - \sum_{m=1}^{\infty} \frac{1}{\alpha'_m \frac{c}{b}}\Big[\alpha'_m \frac{z}{c}\cdot\frac{c}{b}\,\cdot$$

$$\cosh\left(\alpha'_m \frac{z}{c}\cdot\frac{c}{b}\right) - \Big\{3-4\nu + \alpha'_m \frac{c}{b}\cdot\tanh\left(\alpha'_m \frac{c}{b}\right)\Big\}$$

$$\times \sinh\left(\alpha'_m \frac{z}{c}\,\frac{c}{b}\right)\Big] \times \left[E'_m J_0(\alpha'_m r) + F'_m Y_0(\alpha'_m r)\right]$$

(4.1)b

$$\sigma'_r = -6\nu L' + \sum_{n=1}^{\infty} \cos \beta'_n \frac{z}{c}\Big[A'_n\Big\{(1-2\nu)\, I_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$+ \beta'_n \frac{r}{b}\cdot\frac{b}{c}\, I_1\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)\Big\} - B'_n\Big\{I_0\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right) - \frac{I_1\left(\beta'_n \frac{r}{b}\cdot\frac{b}{c}\right)}{\beta'_n \frac{r}{b}\cdot\frac{b}{c}}\Big\}$$

$$- C_n' \left\{ (1-2\nu) \frac{2}{\pi} K_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) - \frac{2}{\pi} \beta_n' \frac{r}{b}\cdot\frac{b}{c} K_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) \right\}$$

$$- D_n' \left\{ \frac{2}{\pi} K_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right) + \frac{2}{\pi} \frac{K_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)}{\beta_n' \frac{r}{b}\cdot\frac{b}{c}} \right\} \Bigg]$$

$$+ \sum_{m=1}^{\infty} \Bigg(2\nu \cosh\left(\alpha_m' \frac{z}{c} \frac{c}{b}\right) \left[E_m' J_0\left(\alpha_m' \frac{r}{b}\right) + F_m' Y_0\left(\alpha_m' \frac{r}{b}\right) \right]$$

$$+ \Bigg[\alpha_m' \frac{z}{c}\cdot\frac{c}{b} \sinh\left(\alpha_m' \frac{z}{c} \frac{c}{b}\right) - \alpha_m' \frac{c}{b}\cdot \tanh\left(\alpha_m' \frac{c}{b}\right)$$

$$\times \cosh\left(\alpha_m' \frac{z}{c}\cdot\frac{c}{b}\right) \Bigg] \times \Bigg[E_m' \left\{ J_0\left(\alpha_m' \frac{r}{b}\right) - \frac{J_1\left(\alpha_m' \frac{r}{b}\right)}{\alpha_m' \frac{r}{b}} \right\}$$

$$+ F_m' \left\{ Y_0\left(\alpha_m' \frac{r}{b}\right) - \frac{Y_1\left(\alpha_m' \frac{r}{b}\right)}{\alpha_m' \frac{r}{b}} \right\} \Bigg] \Bigg) \tag{4,1)c}$$

$$\sigma_\theta' = -6\nu L' + \sum_{n=1}^{\infty} \cos\beta_n' \frac{z}{c} \Bigg[A_n' (1-2\nu) I_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$- B_n' \frac{I_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)}{\beta_n' \frac{r}{b}\cdot\frac{b}{c}} - C_n' (1-2\nu) \frac{2}{\pi} K_0\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)$$

$$+ D_n' \frac{2}{\pi} \frac{K_1\left(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\right)}{\beta_n' \frac{r}{b}\cdot\frac{b}{c}} \Bigg] + \sum_{m=1}^{\infty} \Bigg[2\nu \cosh\left(\alpha_m' \frac{z}{c} \frac{c}{b}\right)$$

$$\times \left\{ E_m' J_0\left(\alpha_m' \frac{r}{b}\right) + F_m' Y_0\left(\alpha_m' \frac{r}{b}\right) \right\} + \Bigg\{ \alpha_m' \frac{z}{c} \frac{c}{b} \sinh\left(\alpha_m' \frac{z}{c}\cdot\frac{c}{b}\right)$$

$$- \alpha_m' \frac{c}{b} \tanh\left(\alpha_m' \frac{c}{b}\right) . \cosh\left(\alpha_m' \frac{z}{c}\cdot\frac{c}{b}\right) \Bigg] \times \Bigg[E_m' \frac{J_1\left(\alpha_m' \frac{r}{b}\right)}{\alpha_m' \frac{r}{b}}$$

$$+ F_m' \frac{Y_1\left(\alpha_m' \frac{r}{b}\right)}{\alpha_m' \frac{r}{b}} \Bigg] \tag{4,1)d}$$

$$\sigma_z' = -(1-\nu)\,6L' + \sum_{n=1}^{\infty} \cos\beta_n' \frac{z}{c}\Big[A_n'\Big\{-2(2-\nu)\, I_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)$$

$$-\beta_n' \frac{r}{b}\,\frac{b}{c}\, I_1\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big\} + B_n'\, I_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)$$

$$+ C_n'\Big\{2(2-\nu)\frac{2}{\pi}\, K_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big) - \beta_n' \frac{r}{b}\cdot\frac{b}{c}\cdot\frac{2}{\pi} K_1\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big\}$$

$$+ D_n' \frac{2}{\pi}\, K_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big] + \sum_{m=1}^{\infty}\Big[-\alpha_m' \frac{z}{c}\,\frac{c}{b}\,\sinh\Big(\alpha_m' \frac{z}{c}\,\frac{c}{b}\Big)$$

$$+\Big(2-2\nu+\alpha_m' \frac{c}{b}\tanh\alpha_m' \frac{c}{b}\Big)\cosh\Big(\alpha_m' \frac{z}{c}\,\frac{c}{b}\Big)\Big]$$

$$\times\Big[E_m' J_0\Big(\alpha_m' \frac{r}{b}\Big) + F_m' Y_0\Big(\alpha_m' \frac{r}{b}\Big)\Big] \qquad (4.1)e$$

$$\tau_{rz}' = \sum_{n=1}^{\infty} \sin\beta_n' \frac{z}{c}\Big[A_n'\Big\{-2(1-\nu)\, I_1\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)$$

$$-\beta_n' \frac{r}{b}\cdot\frac{b}{c}\, I_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big\} + B_n'\, I_1\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)$$

$$+ C_n'\Big\{-2(1-\nu)\cdot\frac{2}{\pi}\, K_1\Big(\beta_n' \frac{r}{b}\,\frac{b}{c}\Big) + \beta_n' \frac{r}{b}\cdot\frac{b}{c}\,\frac{2}{\pi}\, K_0\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big\}$$

$$- D_n' \frac{2}{\pi}\, K_1\Big(\beta_n' \frac{r}{b}\cdot\frac{b}{c}\Big)\Big] + \sum_{m=1}^{\infty}\Big[\alpha_m' \frac{z}{c}\,\frac{c}{b}\,\cosh\Big(\alpha_m' \frac{z}{c}\,\frac{c}{b}\Big)$$

$$+\Big(-1+2\nu-\alpha_m' \frac{c}{b}\cdot\tanh\alpha_m' \frac{c}{b}\Big)\sinh\Big(\alpha_m' \frac{z}{c}\,\frac{c}{b}\Big)\Big]$$

$$\times\Big[E_m' J_1\Big(\alpha_m' \frac{r}{b}\Big) + F_m' Y_1\Big(\alpha_m' \frac{r}{b}\Big)\Big] \qquad (4.1)f$$

Similarly expressions (3.2) are rewritten in the following form.

$$\cos \beta_n' = 0 \tag{4.2)a}$$

$$J_1(\alpha_m') Y_1(\alpha_m' \tfrac{a}{b}) - J_1(\alpha_m' \tfrac{a}{b}) Y_1(\alpha_m') = 0 \tag{4.2)b}$$

$$L' = \frac{1}{6(1-\nu)} \tag{4.2)c}$$

$$E_m' J_1(\alpha_m') + F_m' Y_1(\alpha_m') = 0 \tag{4.2)d}$$

$$A_n'\left[-2(1-\nu) I_1(\beta_n' \tfrac{b}{c}) - \beta_n' \tfrac{b}{c} I_0(\beta_n' \tfrac{b}{c})\right] + B_n' I_1(\beta_n' \tfrac{b}{c})$$
$$+ C_n'\left[-\tfrac{4}{\pi}(1-\nu) K_1(\beta_n' \tfrac{b}{c}) + \beta_n' \tfrac{b}{c} \cdot \tfrac{2}{\pi} K_0(\beta_n' \tfrac{b}{c})\right]$$
$$+ D_n'\left[-\tfrac{2}{\pi} K_1(\beta_n' \tfrac{b}{c})\right] = 0 \tag{4.2)e}$$

$$A_n'\left[-2(1-\nu) I_1(\beta_n' \tfrac{a}{c}) - \beta_n' \tfrac{a}{c} I_0(\beta_n' \tfrac{a}{c})\right] + B_n' I_1(\beta_n' \tfrac{a}{c})$$
$$+ C_n'\left[-\tfrac{4}{\pi}(1-\nu) K_1(\beta_n' \tfrac{a}{c}) + \beta_n' \tfrac{a}{c} \cdot \tfrac{2}{\pi} K_0(\beta_n' \tfrac{a}{c})\right]$$
$$+ D_n'\left[-\tfrac{2}{\pi} K_1(\beta_n' \tfrac{a}{c})\right] = 0 \tag{4.2)f}$$

$$A_n'\left[(1-2\nu) I_0(\beta_n' \tfrac{b}{c}) + \beta_n' \tfrac{b}{c} I_1(\beta_n' \tfrac{b}{c})\right]$$
$$- B_n'\left[I_0(\beta_n' \tfrac{b}{c}) - \frac{I_1(\beta_n' \tfrac{b}{c})}{\beta_n' \tfrac{b}{c}}\right] - C_n'\left[(1-2\nu)\tfrac{2}{\pi} K_0(\beta_n' \tfrac{b}{c})\right.$$
$$\left. - \beta_n' \tfrac{b}{c} \tfrac{2}{\pi} K_1(\beta_n' \tfrac{b}{c})\right] - D_n'\left[\tfrac{2}{\pi} K_0(\beta_n' \tfrac{b}{c}) + \tfrac{2}{\pi} \frac{K_1(\beta_n' \tfrac{b}{c})}{\beta_n' \tfrac{b}{c}}\right]$$

$$+\sum_{m=1}^{\infty} E_m' \left[\frac{4\beta_n' \cosh(\alpha_m' \frac{c}{b}) \sin\beta_n'}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \left\{ J_0(\alpha_m') \right.\right.$$

$$\left.\left. \times \left(\nu - \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \right) + \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \frac{J_1(\alpha_m')}{\alpha_m'} \right\} \right]$$

$$+\sum_{m=1}^{\infty} F_m' \left[\frac{4\beta_n' \cosh(\alpha_m' \frac{c}{b}) \sin\beta_n'}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \left\{ Y_0(\alpha_m') \right.\right.$$

$$\left.\left. \times \left(\nu - \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \right) + \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \frac{Y_1(\alpha_m')}{\alpha_m'} \right\} \right]$$

$$= \frac{2\nu \sin\beta_n'}{(1-\nu)\,\beta_n'} \tag{4.2)g}$$

$$A_n' \left[(1-2\nu) I_0\left(\beta_n' \frac{a}{c}\right) + \beta_n' \frac{a}{c} I_1\left(\beta_n' \frac{a}{c}\right) \right]$$

$$-B_n' \left[I_0\left(\beta_n' \frac{a}{c}\right) - \frac{I_1(\beta_n' \frac{a}{c})}{\beta_n' \frac{a}{c}} \right] - C_n' \left[(1-2\nu) \frac{2}{\pi} K_0\left(\beta_n' \frac{a}{c}\right) \right.$$

$$\left. -\beta_n' \frac{a}{c} \cdot \frac{2}{\pi} K_1\left(\beta_n' \frac{a}{c}\right) \right] - D_n' \left[\frac{2}{\pi} K_0\left(\beta_n' \frac{a}{c}\right) + \frac{2}{\pi} \frac{K_1(\beta_n' \frac{a}{c})}{\beta_n' \frac{a}{c}} \right]$$

$$+\sum_{m=1}^{\infty} E_m' \left[\frac{4\beta_n' \cosh(\alpha_m' \frac{c}{b}) \sin\beta_n'}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \left\{ J_0\left(\alpha_m' \frac{a}{b}\right) \right.\right.$$

$$\left.\left. \times \left(\nu - \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \right) + \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \frac{J_1(\alpha_m' \frac{a}{b})}{\alpha_m' \frac{a}{b}} \right\} \right]$$

$$+ F_m' \left[\frac{4 \beta_n' \cosh(\alpha_m' \frac{c}{b}) \sin \beta_n'}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \left\{ Y_0(\alpha_m' \frac{a}{b}) \right. \right.$$

$$\times \left(\nu - \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \right) + \frac{(\alpha_m' \frac{c}{b})^2}{(\alpha_m' \frac{c}{b})^2 + (\beta_n')^2} \quad \frac{Y_1(\alpha_m'}{\alpha_m' \frac{a}{b}}$$

$$= \frac{2 \nu \sin \beta_n'}{(1-\nu) \beta_n'}$$

$$\sum_{n=1}^{\infty} A_n' \frac{\sin \beta_n'}{\beta_n' \frac{b}{c}} \left[\left\{ \frac{4(1-\nu) \beta_n' \frac{b}{c}}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} - \frac{2(\beta_n' \frac{b}{c})^3}{\left[(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2 \right]} \right. \right.$$

$$+ \frac{(\beta_n' \frac{b}{c})^2}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} \cdot N_5' \Big] + \sum_{n=1}^{\infty} B_n' \left[- \frac{\sin \beta_n'}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} \right.$$

$$+ \sum_{n=1}^{\infty} C_n' \frac{\sin \beta_n'}{\beta_n' \frac{b}{c}} \left[\left\{ \frac{4(1-\nu) \beta_n' \frac{b}{c}}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} - \frac{2(\beta_n' \frac{b}{c})^3}{\left[(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2 \right]^2} \right. \right.$$

$$- \frac{(\beta_n' \frac{b}{c})^2}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} \quad N_6' \Big] + \sum_{n=1}^{\infty} D_n' \left[\frac{\sin \beta_n'}{(\alpha_m')^2 + (\beta_n' \frac{b}{c})^2} N \right.$$

$$- E_m' N_1' \left[\frac{c}{b} \cosh \alpha_m' \frac{c}{b} - \left\{ 3 - 4\nu + \alpha_m' \frac{c}{b} \tanh \alpha_m' \right. \right.$$

$$\times \frac{\sinh(\alpha_m' \frac{c}{b})}{\alpha_m'} \Big] - F_m' N_2' \left[\frac{c}{b} \cosh \alpha_m' \frac{c}{b} - \left\{ 3 - 4\nu + \alpha \right. \right.$$

$$\times \frac{\sinh(\alpha_m' \frac{c}{b})}{\alpha_m'} \Big] = 0$$

Here,

$$N_1' = \frac{1}{2}\left[\left\{ J_0^2(\alpha_m') - \left(\frac{a}{b}\right)^2 J_0^2\left(\alpha_m' \frac{a}{b}\right)\right\} - \frac{J_1\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} \left\{ J_0(\alpha_m') Y_0(\alpha_m') - \left(\frac{a}{b}\right)^2 J_0\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)\right\}\right]$$

$$N_2' = \frac{1}{2}\left[\left\{ J_0(\alpha_m') Y_0(\alpha_m') - \left(\frac{a}{b}\right)^2 J_0\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)\right\} - \frac{J_1\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} \left\{ Y_0^2(\alpha_m') - \left(\frac{a}{b}\right)^2 Y_0^2\left(\alpha_m' \frac{a}{b}\right)\right\}\right]$$

$$N_3' = I_1\left(\beta_n' \frac{b}{c}\right)\left[\frac{J_1(\alpha_m') Y_0(\alpha_m')}{Y_1(\alpha_m')} - J_0(\alpha_m')\right] - \left(\frac{a}{b}\right) I_1\left(\beta_n' \frac{a}{c}\right)\left[\frac{J_1\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} - J_0\left(\alpha_m' \frac{a}{b}\right)\right]$$

$$N_4' = \frac{2}{\pi} K_1\left(\beta_n' \frac{b}{c}\right)\left[\frac{J_1(\alpha_m') Y_0(\alpha_m')}{Y_1(\alpha_m')} - J_0(\alpha_m')\right] - \left(\frac{a}{b}\right) \frac{2}{\pi} K_1\left(\beta_n' \frac{a}{c}\right)\left[\frac{J_1\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} - J_0\left(\alpha_m' \frac{a}{b}\right)\right]$$

$$N_5' = I_0\left(\beta_n' \frac{b}{c}\right)\left[\frac{J_1(\alpha_m') Y_0(\alpha_m')}{Y_1(\alpha_m')} - J_0(\alpha_m')\right] - \left(\frac{a}{b}\right)^2 I_0\left(\beta_n' \frac{a}{c}\right)\left[\frac{J_1\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} - J_0\left(\alpha_m' \frac{a}{b}\right)\right]$$

$$N_6' = \frac{2}{\pi} K_0\left(\beta_n' \frac{b}{c}\right)\left[\frac{J_1(\alpha_m')\,Y_0(\alpha_m')}{Y_1(\alpha_m')} - J_0(\alpha_m')\right]$$

$$-\left(\frac{a}{b}\right)^2 \frac{2}{\pi} K_0\left(\beta_n' \frac{a}{c}\right)\left[\frac{J_1\left(\alpha_m' \frac{a}{b}\right) Y_0\left(\alpha_m' \frac{a}{b}\right)}{Y_1\left(\alpha_m' \frac{a}{b}\right)} - J_0\left(\alpha_m' \frac{a}{b}\right)\right]$$

(4.3)

(b) Approximate Elasticity Solution: Non-dimensional expressions for the approximate elasticity solution are written from eqs.(3.7) and (3.8).

$$2G'U' = \frac{\nu}{1+\nu}\frac{r}{b} + \left[A_1'\left\{4(1-\nu)\, I_1\left(\beta' \frac{r}{b}\cdot\frac{b}{c}\right) - \beta' \frac{r}{b}\cdot\frac{b}{c} I_0\left(\beta'\frac{r}{b}\cdot\frac{b}{c}\right)\right\}\right.$$

$$+ B_1'\left\{4(1-\nu) K_1\left(\beta'\frac{r}{b}\cdot\frac{b}{c}\right) + \beta'\frac{r}{b}\frac{b}{c} K_0\left(\beta'\frac{r}{b}\cdot\frac{b}{c}\right)\right\}$$

$$\left. - C_1' \beta' \frac{b}{c} I_1\left(\beta'\frac{r}{b}\frac{b}{c}\right) + D_1'\beta'\frac{b}{c} K_1\left(\beta'\frac{r}{b}\frac{b}{c}\right)\right]\cos\beta'\frac{z}{c}$$

$$2G'W' = -\frac{1}{1+\nu}\frac{z}{c} + \left[\frac{r}{b}\cdot\frac{b}{c} A_1' I_1\left(\beta'\frac{r}{b}\frac{b}{c}\right) + \frac{r}{b}\cdot\frac{b}{c} B_1' K_1\left(\beta'\frac{r}{b}\frac{b}{c}\right)\right.$$

$$\left. + \frac{b}{c} C_1' I_0\left(\beta'\frac{r}{b}\frac{b}{c}\right) + \frac{b}{c} D_1' K_0\left(\beta'\frac{r}{b}\frac{b}{c}\right)\right]\left(\beta'\frac{b}{c}\right)\sin\beta'\frac{z}{c}$$

$$\sigma_r' = \left[A_1'\left\{(3-2\nu)\beta'\frac{b}{c} I_0\left(\beta'\frac{r}{b}\frac{b}{c}\right) + \left(\frac{4(1-\nu)}{\frac{r}{b}} + \beta'^2\frac{r}{b}\left(\frac{b}{c}\right)^2\right)\right.\right.$$

$$\left.\times I_1\left(\beta'\frac{r}{b}\frac{b}{c}\right)\right\} - B_1'\left\{(3-2\nu)\beta'\frac{b}{c} K_0\left(\beta'\frac{r}{b}\frac{b}{c}\right) +\right.$$

$$\left[\frac{4(1-\nu)}{\frac{r}{b}} + \beta'^2 \frac{r}{b} \left(\frac{b}{c}\right)^2 \right] K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) \Big\}$$

$$+ C_1' \left\{ \frac{\beta'}{\frac{r}{b}} \frac{b}{c} I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) - \beta'^2 \left(\frac{b}{c}\right)^2 I_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\}$$

$$- D_1' \left\{ \frac{\beta'}{\frac{r}{b}} \frac{b}{c} K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) + \beta'^2 \left(\frac{b}{c}\right)^2 K_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\} \Bigg] \cos \beta' \frac{z}{c}$$

$$\sigma_\theta' = \left[A_1' \left\{ \frac{4(1-\nu)}{\frac{r}{b}} I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) - (1-2\nu) \beta' \frac{b}{c} I_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\} \right.$$

$$+ B_1' \left\{ \frac{4(1-\nu)}{\frac{r}{b}} K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) - (1-2\nu) \beta' \frac{b}{c} K_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\}$$

$$\left. - C_1' \frac{\beta'}{\frac{r}{b}} \frac{b}{c} I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) + D_1' \frac{\beta'}{\frac{r}{b}} \frac{b}{c} K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right] \cos \beta' \frac{z}{c}$$

$$\sigma_z' = -1 + \left[A_1' \left\{ \beta'^2 \frac{r}{b} \left(\frac{b}{c}\right)^2 I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) + 2\nu \beta' \frac{b}{c} I_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\} \right.$$

$$+ B_1' \left\{ \beta'^2 \frac{r}{b} \left(\frac{b}{c}\right)^2 K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) - 2\nu \beta' \frac{b}{c} K_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\}$$

$$\left. + C_1' \beta'^2 \left(\frac{b}{c}\right)^2 I_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) + D_1' \beta'^2 \left(\frac{b}{c}\right)^2 K_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right] \cos \beta' \frac{z}{c}$$

$$\tau_{rz}' = \left[A_1' \left\{ \beta' \frac{r}{b} \frac{b}{c} I_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) - 2(1-\nu) I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\} \right.$$

$$- B_1' \left\{ \beta' \frac{r}{b} \frac{b}{c} K_0\left(\beta' \frac{r}{b} \frac{b}{c}\right) + 2(1-\nu) K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) \right\}$$

$$+ C_1' \beta' \frac{b}{c} I_1\left(\beta' \frac{r}{b} \frac{b}{c}\right) - D_1' \beta' \frac{b}{c} K_1\left(\beta' \frac{r}{b} \frac{b}{c}\right)\Big] \beta' \frac{b}{c} \sin \beta' \frac{z}{c}$$

(4.4)

$$A_1' \left[(3-2\nu) \beta' \frac{b}{c} I_0\left(\beta' \frac{b}{c}\right) - \left\{ 4(1-\nu) + \beta'^2 \left(\frac{b}{c}\right)^2 \right\} I_1\left(\beta' \frac{b}{c}\right) \right]$$

$$- B_1' \left[(3-2\nu) \beta' \frac{b}{c} K_0\left(\beta' \frac{b}{c}\right) + \left\{ 4(1-\nu) + \beta'^2 \left(\frac{b}{c}\right)^2 \right\} K_1\left(\beta' \frac{b}{c}\right) \right]$$

$$+ C_1' \left[\beta' \frac{b}{c} I_1\left(\beta' \frac{b}{c}\right) - \beta'^2 \left(\frac{b}{c}\right)^2 I_0\left(\beta' \frac{b}{c}\right) \right] - D_1' \left[\beta' \frac{b}{c} K_1\left(\beta' \frac{b}{c}\right) \right.$$

$$\left. + \beta'^2 \left(\frac{b}{c}\right)^2 K_0\left(\beta' \frac{b}{c}\right) \right] = p_0'$$

$$A_1' \left[(3-2\nu) \beta' \frac{b}{c} I_0\left(\beta' \frac{a}{c}\right) - \left\{ \frac{4(1-\nu)}{\frac{a}{b}} + \beta'^2 \frac{a}{c} \frac{b}{c} \right\} I_1\left(\beta' \frac{a}{c}\right) \right]$$

$$- B_1' \left[(3-2\nu) \beta' \frac{b}{c} K_0\left(\beta' \frac{a}{c}\right) + \left\{ \frac{4(1-\nu)}{\frac{a}{b}} + \beta'^2 \frac{a}{c} \frac{b}{c} \right\} K_1\left(\beta' \frac{a}{c}\right) \right]$$

$$+ C_1' \left[\frac{\beta'}{\frac{a}{b}} \frac{b}{c} I_1\left(\beta' \frac{a}{c}\right) - \beta'^2 \left(\frac{b}{c}\right)^2 I_0\left(\beta' \frac{a}{c}\right) \right]$$

$$- D_1' \left[\frac{\beta'}{\frac{a}{c}} \frac{b}{c} K_1\left(\beta' \frac{a}{c}\right) - \beta'^2 \left(\frac{b}{c}\right)^2 K_0\left(\beta' \frac{a}{c}\right) \right] = p_0'$$

$$A_1' \left[\beta' \frac{b}{c} I_0\left(\beta' \frac{b}{c}\right) - 2(1-\nu) I_1\left(\beta' \frac{b}{c}\right) \right] - B_1' \left[\beta' \frac{b}{c} K_0\left(\beta' \frac{b}{c}\right) \right.$$

$$\left. + 2(1-\nu) K_1\left(\beta' \frac{b}{c}\right) \right] + C_1' \left[\beta' \frac{b}{c} I_1\left(\beta' \frac{b}{c}\right) \right] - D_1' \left[\beta' \frac{b}{c} K_1\left(\beta' \frac{b}{c}\right) \right] = 0$$

$$A_1'\left[\beta'\frac{a}{c}I_0\left(\beta'\frac{a}{c}\right)-2(1-\nu)I_1\left(\beta'\frac{a}{c}\right)\right]-B_1'\left[\beta'\frac{a}{c}K_0\left(\beta'\frac{a}{c}\right)\right.$$

$$\left.+2(1-\nu)K_1\left(\beta'\frac{a}{c}\right)\right]+C_1'\left[\beta'\frac{b}{c}I_1\left(\beta'\frac{a}{c}\right)\right]-D_1'\left[\beta'\frac{b}{c}K_1\left(\beta'\frac{a}{c}\right)\right]=0 \tag{4.5}$$

The constants A_1, B_1 etc., are non-dimensionalized in the same manner as A_n, B_n etc., in the previous section. Quantity p_0 is divided by σ to make it non-dimensional and has been taken as 4% of σ for computations.

(c) Shell and Boundary Layer Solution: Expressions for the shell and boundary layer solution are written in terms of the non-dimensional quantities ρ, β_n, λ, ξ, and η. For the purpose of comparison, these quantities are expressed in terms of the geomatric parameters mentioned in IV-1(a).

Thus,

$$\rho=\frac{\frac{r}{b}-\frac{1+\frac{a}{b}}{2}}{\frac{1-\frac{a}{b}}{2}}$$

$$\lambda=\frac{1-\frac{a}{b}}{1+\frac{a}{b}}$$

$$\xi=2\left(\frac{c}{b}\right)\frac{\frac{z}{c}+1}{\left[1-\left(\frac{a}{b}\right)^2\right]^{\frac{1}{2}}}$$

$$\eta=2\left(\frac{c}{b}\right)\frac{\frac{z}{c}+1}{1-\left(\frac{a}{b}\right)} \tag{4.6}$$

In terms of these parameters, non-dimensional displacements for the interior solution can be obtained from eqs.(2.19)b,c.

$$2G'u' = (1-\nu)\frac{\left(1+\frac{a}{b}\right)}{2}v_r$$

$$2G'w' = (1-\nu)\frac{\left(1+\frac{a}{b}\right)}{2}v_z \qquad (4.7)$$

Similar expressions can be obtained for the boundary layer solution by replacing v_r and v_z with $w_r^{(2)}(\rho,\beta_n)$ and $w_z^{(2)}(\rho,\beta_n)$ respectively in eqs(4.7).

IV-2 Computational Procedure

The three-dimensional elasticity solution is computed in three steps. The first step finds the roots of eq.(4.2)a. Variation of the function given by left hand side of eq.(4.2)a with α_m' was obtained in order to make an estimate of roots. Finally, roots were calculated by using Mueller's Iteration Method of Successive Bisections and Inverse Parabolic Interpolations.

In the second step, the unknown constants A_n', B_n', C_n', D_n', E_m' and F_m' were calculated by solving the infinite system (4.2)d-i. For this, method of reduction [2] was used which consists of truncating the infinite system after a finite number of equations and a finite number of unknowns N. A computer program was written which, depending on the value of N, calculates the elements of the square matrix of order 6N formed by the coefficients of A_n', B_n' etc. The system is then solved by using a standard program for solving the system of linear simultaneous equations. Third step calculates the stresses and and displacements at different points on the cylinder. Standard programs from the Scientific Subroutine Package were used for the computations of Bessel functions, Mueller's iteration method and solution of system of equations.

In the shell and boundary layer solution, results of the elementary, the interior and the boundary layer solutions were obtained separately. They were then added to give the complete solution. From the symmetry conditions, boundary layer solution at other end of the cylinder can easily be obtained. Appropriate contributions resulting from the change of co-ordinates are also incorporated in the interior solution. It is sufficient to calculate the results for half the portion of the cylinder because of symmetry about the mid-plane.

In the approximate elasticity solution, constants A_1', B_1' etc., are calculated in a subroutine with the main program which calculates the stresses and displacements.

For all the solutions, stresses and displacements were calculated at five points along the thickness as well as the longitudinal axis of hollow cylinders. Variation of the longitudinal stress in the boundary layer case was calculated near the edge zones in order to obtain a correlation between cylinder parameters and boundary layer thickness. Cylinders with different thicknesses and lengths were considered. Ratio of the internal and external radii was given the values $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ and half length to external radius was taken as 1, 2 and 3. Computations were carried out on IBM Computer (360/65) in FORTRAN IV language using double precision for increased accuracy.

```
C
C     PROGRAM FOR FINDING THE ROOTS OF THE TRANSCENDENTAL EQUATION
C     J1(BETA(N))*Y1(BETA(N)*R)-J1(BETA(N)*R)*Y1(BETA(N))=0
C     R IS THE RATIO OF THE INTERNAL RADIUS TO EXT. RADIUS
      IMPLICIT REAL*8(A-H,O-Z)
      DO 45 I=1,6
      GO TO (5,7,10,12,15,20),I
    5 R=1./3.
      GO TO 25
    7 R=1./2.
      GO TO 25
   10 R=2./3.
      GO TO 25
   12 R=3./4.
      GO TO 25
   15 R=5./6.
      GO TO 25
   20 R=7./8.
   25 BL=1.0
      DR=1.0
      BR=BL+DR
      DR1=DR+1.
      DO 45 J=1,80
      IEND=200
      EPS=0.10D-14
      CALL DRTMI(X,F,FCT,BL,BR,EPS,IEND,IER,R)
      WRITE(3,35) IER
   35 FORMAT(/,10X,'THE ERROR PARAMETER FOR THE ROOT IS',I3)
      WRITE(3,40) BL,BR,X,F
   40 FORMAT(//,5X,'THE ROOT LYING BETWEEN',F8.3,'AND',F8.3,'IS',D16.8,
     1/,10X,'THE VALUE OF THE FN. FCT AT THIS ROOT IS',D16.8,////)
      BL=BR
      BR=BR+DR1
   45 CONTINUE
      RETURN
      END
```

```
C      ...................................................................
C      SUBROUTINE FCTN
C         PURPOSE-
C           EVALUATION OF THE FUNCTION FCT
C
C         DESCRIPTION OF PARAMETERS-
C              X--    ARGUMENT OF FCT
C              X--    ARGUMENT OF
C              R--    INNER RADIUS/OUTER RADIUS
C
C      -------------------------------------------------------------------
C
       SUBROUTINE FCTN(FCT,X,R)
       IMPLICITT REAL*8(A-H,O-Z)
       RX=R*X
       WRITE(3,5)R,X
     5 FORMAT(20X,'R=',F10.6,5X,' X=',D16.8)
       D=0.1CD-14
    20 CALL BESJ(X,1,BJ1,D,I1)
       CALL BESJ(RX,1,BJ2,D,I2)
    30 CALL BESY(X,1,BY1,I3)
       CALL BESY(RX,1,BY2,I4)
       FCT=BJ1*BY2-BJ2*BY1
       RETURN
       END
```

```
C
C
C      ---------------------------------------------------------------
C
C
C     PROGRAM FOR EVALUATING THE CONSTANTS AN,BN,CN,DN,EM AND FM
C     TO BE USED IN THE THREE DIMENSIONAL ELASTICITY SOLUTION TO THE
C     PROBLEM OF AXI-SYMMETRIC COMPRESSION OF HOLLOW CYLINDERS
C
C      METHOD-
C         THE INFINITE NO. OF LINEAR SIMULTANEOUS EQUATIONS IS
C        REDUCED TO A FINITE NO. OF EQUATIONS WITH FINITE NO. OF TERMS
C        THE SET OF LINEAR SIMULTANEOUS EQUATIONS SO OBTAINED IS
C        SOLVED BY USING THE SUBROUTINE SIMQ
C
C     DESCRIPTION OF THE PARAMETERS-
C         X-- ARRAY OF THE CONSTANTS TO BE EVALUATED
C         A-- MATRIX OF THE COEFFICIENTS OF CONSTANTS IN X
C           THE ORDER OF MATRIX A IS DETERMINED BY THE NUMBER OF
C            TERMS TAKEN IN THE INFINITE SERIES THAT IS IF MM TERMS
C            ARE TAKEN IN THE SERIES THEN THE ORDER OF A IS 6*MM
C         B-- ARRAY OF THE QUANTITIES ON THE RIGHT HAND SIDE OF THE
C            SIMULTANEOUS LINEAR EQUATIONS
C        R-- RATIO OF THE INNER RADIUS TO THE OUTER RADIUS
C        Y-- RATIO OF THE HALF LENGTH TO THE OUTER RADIUS
C        T(M)-- ALPHA(M)
C        V(M)-- BETA(N)
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(18,18),B(18),T(5),V(5),RT(5),YT(5),SV(5),WV(5),X(3,6)
      MM=3
      PI=4.*DATAN(0.1D01)
      DO 10 M=1,MM
   10 V(M)=PI*(2.*M-1.)/2.
      WRITE(3,14)
   14 FORMAT(//,30X,'THE VALUES OF BETA(N) ARE',/)
      WRITE(3,15)(V(M),M=1,MM)
   15 FORMAT(5D16.8)
      R=1./3.
   44 Y=1.
      READ(1,31)(T(M),M=1,MM)
   31 FORMAT(5D16.8)
```

```
      WRITE(3,30)
   30 FORMAT(/,30X,'THE VALUES OF ALPHA(M) ARE',/)
      WRITE(3,31)(T(M),M=1,MM)
      S=1./Y
      WRITE(3,32)R,Y
   32 FORMAT(////,20X,'R=',F6.4,10X,'Y=',F6.4)
      W=R*S
      MM=3
      IF(R.EQ.5./6. .AND. Y.EQ.3.0) MM=2
      IF(R.EQ.7./8. .AND. Y.EQ.3.0) MM=2
      MMA=6*MM
C
C     ---------------------------------------------------------------
C
C                     GENERATE THE MATRIX A(I,J)
C
C     ---------------------------------------------------------------
      DO 45 I=1,MMA
      DO 45 J=1,MMA
   45 A(I,J)=0.
      DO 50 M=1,MM
      RT(M)=R*T(M)
      YT(M)=Y*T(M)
      SV(M)=S*V(M)
      WV(M)=W*V(M)
C
C     CALL THE SUBROUTINES FOR THE BESSEL FUNCTIONS
C     D IS THE REQUIRED ACCURACY IN THE COMPUTATIONS OF BESSEL FNS.
C
      D=0.10D-14
      CALL BESJ(T(M),0,BJ1,D,I1)
      CALL BESJ(RT(M),0,BJ2,D,I2)
      CALL BESJ(T(M),1,BJ3,D,I3)
      CALL BESJ(RT(M),1,BJ4,D,I4)
      CALL BESY(T(M),0,BY1,I5)
      CALL BESY(RT(M),0,BY2,I6)
      CALL BESY(T(M),1,BY3,I7)
      CALL BESY(RT(M),1,BY4,I8)
      CALL I0(SV(M),RI01)
      CALL I0(WV(M),RI02)
      CALL INUE(SV(M),1,RI01,RI1)
      CALL INUE(WV(M),1,RI02,RI2)
      CALL BESK(SV(M),0,BK1,I9)
```

```
      CALL BESK(WV(M),0,BK2,I10)
      CALL BESK(SV(M),1,BK3,I11)
      CALL BESK(WV(M),1,BK4,I12)
      MI=4*MM+M
      MJ=5*MM+M
      A(M,MI)=BJ4
      A(M,MJ)=BY4
      MI=MM+M
      MJ=2*MM+M
      MK=3*MM+M
      MP=M+MM
C
C     ZNU IS THE POISSON'S RATIO
C
      ZNU=0.3
      A(MP,M)=-2.*(1.-ZNU)*RI1-SV(M)*RI01
      A(MP,MI)=RI1
      A(MP,MJ)=(-4.*(1.-ZNU)*BK3+2.*SV(M)*BK1)/PI
      A(MP,MK)=-2.*BK3/PI
      MQ=M+2*MM
      A(MQ,M)=-2.*(1.-ZNU)*RI2-WV(M)*RI02
      A(MQ,MI)=RI2
      A(MQ,MJ)=(-4.*(1.-ZNU)*BK4+WV(M)*2.*BK2)/PI
      A(MQ,MK)=-2.*BK4/PI
      MP=M+3*MM
      MI=MM+M
      MJ=2*MM+M
      MK=3*MM+M
      A(MP,M)=(1.-2.*ZNU)*RI01+SV(M)*RI1
      A(MP,MI)=-(RI01-(RI1/SV(M)))
      A(MP,MJ)=-2.*((1.-2.*ZNU)*BK1-SV(M)*BK3)/PI
      A(MP,MK)=-2.*(BK1+BK3/SV(M))/PI
      MP=M+4*MM
      A(MP,M)=(1.-2.*ZNU)*RI02+WV(M)*RI2
      A(MP,MI)=-(RI02-(RI2/WV(M)))
      A(MP,MJ)=-((1.-2.*ZNU)*BK2-WV(M)*BK4)*2./PI
      A(MP,MK)=-(BK2+BK4/WV(M))*2./PI
      MP=5*MM+M
      MI=4*MM+M
      MJ=5*MM+M
      ZN1=(((BJ1**2)-R*R*(BJ2**2))-(BJ4/BY4)*(BJ1*BY1-R*R*BJ2*BY2))/2.
      ZN2=((BJ1*BY1-R*R*BJ2*BY2)-(BJ4/BY4)*((BY1**2)-R*R*(BY2**2)))/2.
      AMP=Y*DCOSH(YT(M))-(3.-4.*ZNU+YT(M)*DTANH(YT(M)))*
```

```
      1(DSINH(YT(M)))/T(M)
       A(MP,MI)=-AMP*ZN1
       A(MP,MJ)=-AMP*ZN2
   50  CONTINUE
       MP=3*MM
       MQ=4*MM
       DO 55 N=1,MM
       MR=MP+1
       MP=MP+1
       MQ=MQ+1
       DO 55 M=1,MM
       MS=4*MM+M
       MT=5*MM+M
       AMS=4.*V(N)*DCOSH(YT(M))*DSIN(V(N))/(YT(M)**2+V(N)**2)
       BMS=(YT(M)**2)/(YT(M)**2+V(N)**2)
       CALL BESJ(T(M),0,BJ1,D,I1)
       CALL BESJ(RT(M),0,BJ2,D,I2)
       CALL BESJ(T(M),1,BJ3,D,I3)
       CALL BESJ(RT(M),1,BJ4,D,I4)
       CALL BESY(T(M),0,BY1,I5)
       CALL BESY(RT(M),0,BY2,I6)
       CALL BESY(T(M),1,BY3,I7)
       CALL BESY(RT(M),1,BY4,I8)
       A(MP,MS)=AMS*(BJ1*(ZNU-BMS)+BMS*BJ3/T(M))
       A(MP,MT)=AMS*(BY1*(ZNU-BMS)+BMS*BY3/T(M))
       A(MQ,MS)=AMS*(BJ2*(ZNU-BMS)+BMS*BJ4/RT(M))
       A(MQ,MT)=AMS*(BY2*(ZNU-BMS)+BMS*BY4/RT(M))
   55  CONTINUE
       DO 60 M=1,MM
       MR=5*MM+M
       DO 60 N=1,MM
       MP=MM+N
       MQ=2*MM+N
       MS=3*MM+N
       AMR=T(M)**2+SV(N)**2
       AMRS=(DSIN(V(N))/SV(N))*((4.*(1.-ZNU)*SV(N)/AMR)-(2.*(SV(N)**3)
      1/(AMR)**2))
       ARS=DSIN(V(N))/AMR
       CALL BESJ(T(M),0,BJ1,D,I1)
       CALL BESJ(RT(M),0,BJ2,D,I2)
       CALL BESJ(T(M),1,BJ3,D,I3)
       CALL BESJ(RT(M),1,BJ4,D,I4)
       CALL BESY(T(M),0,BY1,I5)
```

```
      CALL BESY(RT(M),0,BY2,I6)
      CALL BESY(T(M),1,BY3,I7)
      CALL BESY(RT(M),1,BY4,I8)
      CALL I0(SV(N),RI01)
      CALL I0(WV(N),RI02)
      CALL INUE(SV(N),1,RI01,RI1)
      CALL INUE(WV(N),1,RI02,RI2)
      CALL BESK(SV(N),0,BK1,I9)
      CALL BESK(WV(N),0,BK2,I10)
      CALL BESK(SV(N),1,BK3,I11)
      CALL BESK(WV(N),1,BK4,I12)
      ARE=(BJ3*BY1/BY3)-BJ1
      BRE=(BJ4*BY2/BY4)-BJ2
      ZN3=RI1*ARE-R*RI2*BRE
      ZN4=(BK3*ARE-R*BK4*BRE)*2./PI
      ZN5=RI01*ARE-R*R*RI02*BRE
      ZN6=(BK1*ARE-R*R*BK2*BRE)*2./PI
      A(MR,N)=AMRS*ZN3+(SV(N)**2)*ZN5/AMR
      A(MR,MP)=-ARS*ZN3
      A(MR,MQ)=AMRS*ZN4-(SV(N)**2)*ZN6/AMR
      A(MR,MS)=ARS*ZN4
   60 CONTINUE
C
C                    GENERATE THE ARRAY B(J)
C
      DO 65 J=1,MMA
   65 B(J)=0.
      DO 70 M=1,MM
      BEE=2.*ZNU*DSIN(V(M))/(V(M)*(1.-ZNU))
      B(3*MM+M)=BEE
      B(4*MM+M)=BEE
   70 CONTINUE
      WRITE(3,80)
   80 FORMAT(30X,'THE MATRIX A IS')
      WRITE(3,85)((A(I,J),J=1,MMA),I=1,MMA)
   85 FORMAT(6D16.8)
      WRITE(3,95)
   95 FORMAT(1H1,30X,'THE ARRAY B IS')
      WRITE(3,100)(B(M),M=1,MMA)
  100 FORMAT(30X,D16.8)
C
C     CALL THE SUBROUTINE SIMQ FOR THE SOLUTION OF SIMULTANEOUS EQNS.
C
```

```
      CALL SIMQ(A,B,MMA,KS)
      WRITE(3,105) KS
  105 FORMAT(10X,'THE OUTPUT DIGIT IS',I3)
      WRITE(3,110)
  110 FORMAT(//,30X,'THE CONSTANTS X ARE')
      WRITE(3,120)(B(M),M=1,MMA)
  120 FORMAT(32X,D16.8)
      DO 122 I=1,MM
      DO 122 J=1,6
  122 X(I,J)=0.0
      M=1
      DO 125 J=1,6
      DO 125 I=1,MM
      X(I,J)=B(M)
  125 M=M+1
      DO 130 KJ=1,MM
      WRITE(2,123) (X(KJ,J),J=1,6)
  123 FORMAT(5D16.8)
  130 CONTINUE
      Y=Y+1.0
  135 CONTINUE
      RETURN
      END
```

```
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C
C      ------------------------------------------------------------------
C
C
C      PROGRAM FOR THE VARIATION OF STRESSES AND DEFORMATIONS
C      ALONG RADIUS AND AXIS OF THE HOLLOW CYLINDER UNDER AXISYMM.
C      COMPRESSION
C
C       DESCRIPTION OF PARAMETERS--
C           T(M)-- ALPHA(M)
C           V(M)-- BETA(N)
C           IJ --- NO. OF TERMS USED IN THE FOURIER-BESSEL SERIES
C           R ---- RATIO OF THE INNER RADIUS TO THE OUTER RADIUS
C           Y ---- RATIO OF HALF LENGTH TO THE OUTER RADIUS
C           RB --- RADIUS AT ANY POINT ON THE CYLINDER DIVIDED BY
C                  THE OUTER RADIUS
C           ZC --- AXIAL DISTANCE OF ANY POINT ON THE CYLINDER FROM
C                  ITS MID PLANE DIVIDED BY THE HALF LENGTH
C           U ---- NONDIMENSIONAL DEFORMATION IN THE RADIAL DIRECTION
C           W ---- NON DIMENSIONAL DEFORMATION IN THE AXIAL DIRECTION
C           G ---- SHEAR MODULUS OF ELASTICITY
C           GU --- 2*G*U
C           GW --- 2*G*W
C           SIGT - NON DIMENSIONAL HOOP STRESS
C           SIGR - NON DIMENSIONAL RADIAL STRESS
C           TAU - NON DIMENSIONAL SHEAR STRESS
C
C      ------------------------------------------------------------------
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION T(5),V(5),RBT(5),SV(5),RBSV(5),YT(5),ZCYT(5),
      &ZCV(5),X(3,6)
       IJ=3
       PI=4.*DATAN(0.1D01)
       DO 5 M=1,IJ
 5     V(M)=PI*(2.*M-1.)/2.
       WRITE(3,15)
 15    FORMAT(//,40X,'THE VALUES OF BETA(N) ARE')
       WRITE(3,10)(V(M),M=1,IJ)
 10    FORMAT(/,25X,5D16.8)
       DO 106 JJ=1,6
       GO TO (110,115,120,125,130,135),JJ
 110   R=1./3.
```

```
            GO TO 140
      115   R=1./2.
            GO TO 140
      120   R=2./3.
            GO TO 140
      125   R=3./4.
            GO TO 140
      130   R=5./6.
            GO TO 140
      135   R=7./8.
      140   READ(1,40)(T(M),M=1,IJ)
            WRITE(3,40) (T(M),M=1,IJ)
      40    FORMAT(5D16.8)
            Y=1.0
            DO 106 KK=1,3
            S=1./Y
            WRITE(3,45) Y,R
      45    FORMAT(1H1,//,40X,'Y=',F6.3,5X,'R=',F8.5)
            IJ=3
            IF(R.EQ.5./6. .AND. Y.EQ.3.0) IJ=2
            IF(R.EQ.7./8. .AND. Y.EQ.3.0) IJ=2
            DO 30 I=1,IJ
            READ(1,25)(X(I,J),J=1,6)
      25    FORMAT(5D16.8)
      30    CONTINUE
      48    WRITE(3,50)
      50    FORMAT(/,20X,'THE MATRIX OF THE CONSTANTS IS')
            WRITE(3,55)((X(I,J),J=1,6),I=1,IJ)
      55    FORMAT(10X,6D16.8)
            WRITE(3,70)
      70    FORMAT(/,40X,'THE VALUES OF ALPHA(M) ARE')
            WRITE(3,60)(T(M),M=1,IJ)
      60    FORMAT(/,25X,5D16.8)
            WRITE(3,75)
      75    FORMAT(/////,4X,'R/B',5X,'Z/C',4X, 5X,'GU',13X,'GW',13X,'SIGR'
           *,13X,'SIGT',13X,'SIGZ',13X,'TAURZ')
            ZC=-1.0
            DO 105 L=1,5
            DR=(1.-R)/4.
            RB=R
            DO 100 I=1,5
C
C           INITIALISE ALL THE STRESSES AND DEFORMATIONS
```

```
          C
0054            U1=0.
0055            U2=0.
0056            W1=0.
0057            W2=0.
0058            SIGR1=0.
0059            SIGR2=0.
0060            SIGT1=0.
0061            SIGT2=0.
0062            SIGZ1=0.
0063            SIGZ2=0.
0064            TAU1=0.
0065            TAU2=0.
0066            DO 30 M=1,IJ
0067            RBT(M)=RB*T(M)
0068            SV(M)=S*V(M)
0069            RBSV(M)=RB*SV(M)
0070            YT(M)=Y*T(M)
0071            ZCYT(M)=ZC*YT(M)
0072            ZCV(M)=ZC*V(M)
          C
          C     CALL THE SUBROUTINES FOR BESSEL FUNCTIONS
          C     D IS THE REQD. ACCURACY IN THE BESSEL FN. COMPUTATIONS
          C
0073            D=0.10D-14
0074            CALL BESJ(RBT(M),0,BJ1,D,I1)
0075            CALL BESJ(RBT(M),1,BJ2,D,I2)
0076            CALL BESY(RBT(M),0,BY1,I3)
0077            CALL BESY(RBT(M),1,BY2,I4)
0078            CALL I0(RBSV(M),RI0)
0079            CALL INUE(RBSV(M),1,RI0,RI)
0080            CALL BESK(RBSV(M),0,BK1,I5)
0081            CALL BESK(RBSV(M),1,BK2,I6)
0082            A=X(M,5)*BJ2+X(M,6)*BY2
0083            B=X(M,5)*BJ1+X(M,6)*BY1
0084            C=ZCYT(M)*DSINH(ZCYT(M))-YT(M)*DTANH(YT(M))*DCOSH(ZCYT(M))
          C
          C     ZNU IS POISSON'S RATIO
          C
0085            ZNU=0.3
0086            U1=U1+DCOS(ZCV(M))*(X(M,1)*RBSV(M)*RI0-X(M,2)*RI
               1-(X(M,3)*RBSV(M)*BK1+X(M,4)*BK2)*2./PI)/SV(M)
0087            U2=U2+C*A/T(M)
```

```
0088          W1=W1+(DSIN(ZCV(M))/V(M))*(X(M,1)*(-4.*(1.-ZNU)*RIO-RBSV(M)*RI)
             1+X(M,2)*RIO+X(M,3)*(2./PI)*(4.*(1.-ZNU)*BK1-RBSV(M)*BK2)
             2+X(M,4)*(2./PI)*BK1)
0089          W2=W2+(1./YT(M))*(ZCYT(M)*DCOSH(ZCYT(M))
             1-DSINH(ZCYT(M))*((3.-4.*ZNU)+YT(M)*DTANH(YT(M))))*B
0090          SIGR1=SIGR1+DCOS(ZCV(M))*(X(M,1)*((1.-2.*ZNU)*RIO+RBSV(M)*RI)
             1-X(M,2)*(RIO-(RI/RBSV(M)))-X(M,3)*(2./PI)*((1.-2.*ZNU)*BK1
             2-RBSV(M)*BK2)-X(M,4)*(2./PI)*(BK1+BK2/RBSV(M)))
0091          SIGR2=SIGR2+(2.*ZNU*DCOSH(ZCYT(M)))*B+C*(B-A/RBT(M))
0092          SIGT1=SIGT1+DCOS(ZCV(M))*(X(M,1)*(1.-2.*ZNU)*RIO-X(M,2)*
             1RI/RBSV(M)-(2./PI)*(X(M,3)*(1.-2.*ZNU)*BK1+X(M,4)*BK2/RBSV(M)))
0093          SIGT2=SIGT2+(2.*ZNU*DCOSH(ZCYT(M)))*B
             1+C*A/RBT(M)
0094          SIGZ1=SIGZ1+DCOS(ZCV(M))*(X(M,1)*(-2.*(2.-ZNU)*RIO-RBSV(M)*RI)
             1+X(M,2)*RIO+X(M,3)*(2./PI)*(2.*(2.-ZNU)*BK1-RBSV(M)*BK2)
             2+X(M,4)*BK1*(2./PI))
0095          SIGZ2=SIGZ2+(-ZCYT(M)*DSINH(ZCYT(M))+(2.-2.*ZNU+YT(M)*
             1DTANH(YT(M)))*DCOSH(ZCYT(M)))*B
0096          TAU1=TAU1+DSIN(ZCV(M))*(X(M,1)*(-2.*(1.-ZNU)*RI-RBSV(M)*RIO)
             1+X(M,2)*RI+X(M,3)*(-2.*(1.-ZNU)*BK2+RBSV(M)*BK1)*(2./PI)
             2-X(M,4)*2.*BK2/PI)
0097          TAU2=TAU2+(ZCYT(M)*DCOSH(ZCYT(M))-(1.-2.*ZNU+YT(M)*DTANH(YT(M)))*
             1DSINH(ZCYT(M)))*A
0098    90    CONTINUE
0099          GU=U1+U2
0100          GW=-(1.-2.*ZNU)*ZC/(1.-ZNU)+W1-W2
0101          SIGR=-(ZNU/(1.-ZNU))+SIGR1+SIGR2
0102          SIGT=-(ZNU/(1.-ZNU))+SIGT1+SIGT2
0103          SIGZ=-1.0+SIGZ1+SIGZ2
0104          TAU=TAU1+TAU2
0105          WRITE(3,95)RR,ZC,GU,GW,SIGR,SIGT,SIGZ,TAU
0106    95    FORMAT(F10.4, F6.3,6D16.8)
0107          RR=RR+DR
0108    100   CONTINUE
0109          ZC=ZC+0.5
0110    105   CONTINUE
0111          Y=Y+1.0
0112    106   CONTINUE
0113          RETURN
0114          END
```

```
FORTRAN IV G LEVEL  19              MAIN              DATE = 71224          23/06/30            PAGE 00

      C------------------------------------------------------------------------------
      C
      C      PROGRAM FOR THE VARIATION OF STRESSES AND DEFORMATIONS
      C      WITH THE RADIUS AND AXIS FOR A HOLLOW CYLINDER UNDER
      C      AXISYMMETRIC COMPRESSION (INTERIOR SOLUTION )
      C
      C       DESCRIPTION OF PARAMETERS
      C          RB--     RATIO OF THE RADIUS AT ANY POINT TO THE EXTERNAL RAD.
      C          ZC--     RATIO OF THE AXIAL LENGTH AT ANY POINT TO THE HALF
      C             LENGTH OF THE CYLINDER
      C          R--      RATIO OF THE INTERNAL RADIUS TO THE EXTERNAL RAD.
      C          ROW--    NON DIMENSIONAL FORM OF THE RADIUS AT ANY POINT
      C          ZI--     NON DIMENSIONAL FORM OF THE AXIAL LENGTH OF THE
      C             CYLINDER MEASURED FROM THE LOWER END
      C          U--      NON DIMENSIONAL RADIAL DEFORMATION
      C          W--      NON DIMENSIONAL AXIAL DEFORMATION
      C          GU,GW--  2*G*U,2*G*W, G IS THE SHEAR MODULUS DIVIDED BY THE
      C             AVERAGE VALUE OF THE AXIAL STRESS
      C          SR--     NON DIMENSIONAL RADIAL STRESS
      C          ST--     NON DIMENSIONAL HOOP STRESS
      C          SZ--     NON DIMENSIONAL AXIAL STRESS
      C          SRZ--    NON DIMENSIONAL SHEAR STRESS
      C
      C------------------------------------------------------------------------------
      C
             IMPLICIT REAL*8(A-H,O-Z)
             GAMA=(0.91*0.75)**0.25
             DO 30 JJ=1,6
             GO TO (40,42,45,47,50,55),JJ
       40    R=1./3.
             GO TO 60
       42    R=1./2.
             GO TO 60
       45    R=2./3.
             GO TO 60
       47    R=3./4.
             GO TO 60
       50    R=5./6.
             GO TO 60
       55    R=7./8.
       60    Y=1.
             DO 30 I=1,3
             WRITE(3,8)R,Y
```

```
FORTRAN IV G LEVEL  19                    MAIN              DATE = 71224          23/06/30              PAGE 00

      8    FORMAT('1',20X,'R=',F10.4,5X,'Y=',F5.2)
           WRITE(3,10)
     10    FORMAT(//,5X,'R/B',5X,'Z/C',10X,'GU',13X,'GW',13X,'SIGR',
          *13X,'SIGT',13X,'SIGZ',13X,'TAURZ')
           ZC=-1.0
           DO 25 J=1,5
           DR=(1.-R)/4.
           RB=R
           DO 20 K=1,5
           ZI=2.*Y*(ZC+1.)/((1.-R**2)**0.5)
           GZI=GAMA*ZI
           IF(GZI-50.)11,11,12
     11    E=DEXP(-GZI)
           GO TO 13
     12    E=0.
     13    T=DCOS(GZI)+DSIN(GZI)
           Q=DCOS(GZI)-DSIN(GZI)
           ROW=(RB-(1.+R)/2.)/((1.-R)/2.)
      C
      C    ZNU IS POISSON'S RATIO
      C
           ZNU=0.3
           ZNU1=1.-ZNU**2
           VR0=-(ZNU/ZNU1)*T*E
           D1VR0=2.*ZNU*GAMA*E*DSIN(GZI)/ZNU1
           D2VR0=2.*ZNU*(GAMA**2)*Q*E/ZNU1
           D3VR0=-4.*ZNU*(GAMA**3)*DCOS(GZI)*E/ZNU1
           D4VR0=-4.*(GAMA**4)*VR0
           D5VR0=-4.*(GAMA**4)*D1VR0
           D6VR0=-4.*(GAMA**4)*D2VR0
           D7VR0=-4.*(GAMA**4)*D3VR0
           VZ1=-(ZNU**2)*DCOS(GZI)*E/(ZNU1*GAMA)
           D1VZ1=-ZNU*VR0
           D2VZ1=-ZNU*D1VR0
           D3VZ1=-ZNU*D2VR0
           D4VZ1=-ZNU*D3VR0
           D5VZ1=-ZNU*D4VR0
           SRZ1=-0.5*D3VR0
           D1SRZ1=-0.5*D4VR0
           D2SRZ1=-0.5*D5VR0
           D3SRZ1=-0.5*D6VR0
           SR2=-0.5*D3VZ1
           D1SR2=-0.5*D4VZ1
```

```
0058    D2SR2=-0.5*D5VZ1
0059    ANU=(24.-14.*ZNU-39.*ZNU*ZNU+14.*(ZNU**3))/(30.*(1.-ZNU))
0060    BNU=(2.-3.*ZNU-2.*ZNU*ZNU)/(6.*(1.-ZNU))
0061    CNU=6.*ZNU*(GAMA**2)*(ANU-ZNU*BNU)/ZNU1
0062    DNU=CNU/(24.*ZNU*(GAMA**2))
0063    DNUZI=DNU*ZI
0064    VR2=-DNUZI*D3VR0
0065    D1VR2=-DNUZI*D4VR0-DNU*D3VR0
0066    D2VR2=-DNUZI*D5VR0-2.*DNU*D4VR0
0067    D3VR2=-DNUZI*D6VR0-3.*DNU*D5VR0
0068    D4VR2=-DNUZI*D7VR0-4.*DNU*D6VR0
0069    VZ3=(BNU-ZNU*DNU)*D1VR0+DNUZI*ZNU*D2VR0
0070    D1VZ3=-ZNU*VR2+BNU*D2VR0
0071    D2VZ3=-ZNU*D1VR2+BNU*D3VR0
0072    D3VZ3=-ZNU*D2VR2+BNU*D4VR0
0073    SRZ3=-0.5*D2VR2+(5.-2.*ZNU)*(1.+ZNU)*D1VR0/4.
0074    D1SRZ3=-0.5*D4VR2+(5.-2.*ZNU)*(1.+ZNU)*D2VR0/4.
0075    SR4=-0.5*D3VZ3+ZNU1*(1.-2.*ZNU)*VR0/4.
0076    VZ1A=VZ1-D1VR0*ROW
0077    ST0=VR0+ZNU*(D1VZ1-D2VR0*ROW)
0078    SZ0=ZNU*VR0+D1VZ1-D2VR0*ROW
0079    SRZ1A=SRZ1-(ZNU*D1VR0+D2VZ1)*ROW+D3VR0*(ROW**2)/2.
0080    SR2A=SR2+(-D1SRZ1+ZNU*D1VZ1+VR0)*ROW+D3VZ1*(ROW**2)/2.
       &-D4VR0*(ROW**3)/6.
0081    VR2A=VR2-(ZNU/(1.-ZNU))*((VR0+D1VZ1)*ROW-0.5*D2VR0*(ROW**2))
0082    VZ3A=VZ3-D1VR2*ROW+2.*SRZ1*ROW/(1.-ZNU)-(ZNU*D1VR0+(2.-ZNU)*D2VZ1)
       &*(ROW**2)/(2.*(1.-ZNU))+(2.-ZNU)*D3VR0*(ROW**3)/(6.*(1.-ZNU))
0083    SZ2=ZNU*VR2+D1VZ3-D2VR2*ROW+(ZNU/(1.-ZNU))*SR2+(2.-ZNU)*D1SRZ1*ROW
       &/(1.-ZNU)-(ZNU*D2VR0/2.+D3VZ1)*(ROW**2)+D4VR0*(ROW**3)/3.
0084    ST2=VR2+ZNU*D1VZ3-ZNU*D2VR2*ROW+ZNU*SR2/(1.-ZNU)+((ZNU/(1.-ZNU))*
       &D1SRZ1
       &-ZNU*D1VZ1-VR0)*ROW+(ZNU*D2VR0-ZNU*D3VZ1)*0.5*(ROW**2)+ZNU*D4VR0*
       &(ROW**3)/6.
0085    SRZ3A=SRZ3-(ZNU*D1VR2+D2VZ3)*ROW+D3VR2*(ROW**2)*0.5-
       &(SRZ1+(ZNU/(1.-ZNU))*D1SR2)*ROW+(ZNU*D1VR0+D2VZ1-((2.-ZNU)/
       &(1.-ZNU))*D2SRZ1)*0.5*(ROW**2)+(D4VZ1-0.5*(1.-ZNU)*D3VR0)*(ROW**3)
       &/3.-D5VR0*(ROW**4)/12.
0086    SR4A=SR4+(VR2+ZNU*D1VZ3-D1SRZ3-((1.-2.*ZNU)/(1.-ZNU))*SR2)*ROW
       &+(D3VZ3-2.*VR0-3.*ZNU*D1VZ1+((2.-ZNU)/(1.-ZNU))*D1SRZ1
       &+(ZNU/(1.-ZNU))*D2SR2)*0.5*(ROW**2)+(-D4VR2-(2.+ZNU)*D3VZ1
       &+2.*ZNU*D2VR0+((2.-ZNU)/(1.-ZNU))*D3SRZ1)*(ROW**3)/6.+
       &(D4VR0-D5VZ1)*(ROW**4)/12.+D6VR0*(ROW**5)/60.
0087    HA=(1.-R)/(1.+R)
```

```
      VR=VR0+VR2A*HA
      VZ=VZ1A*(HA**0.5)+VZ3A*(HA**1.5)
      SR=SR2A*HA+SR4A*(HA**2)
      ST=ST0+ST2*HA
      SRZ=SRZ1A*(HA**0.5)+SRZ3A*(HA**1.5)
      SZ=SZ0+SZ2*HA
      XX=(1.-ZNU)*(1.+R)/2.
      GU=XX*VR
      GW=XX*VZ
      WRITE(3,15) RB,ZC,GU,GW,SR,ST,SZ,SRZ
   15 FORMAT(F10.4,2X,F6.3,2X,6D16.8)
      WRITE(2,16)GU,GW,SR,ST,SZ,SRZ
   16 FORMAT(6D13.6)
      RB=RB+DR
   20 CONTINUE
      ZC=ZC+0.5
   25 CONTINUE
   26 Y=Y+1.
   30 CONTINUE
      RETURN
      END
```

```
      C-----------------------------------------------------------------------
      C
      C     THIS IS THE PROGRAM FOR EVALUATING THE CONSTANTS  AN IN THE
      C     BOUNDARY LAYER SOLUTION
      C
      C-----------------------------------------------------------------------
      C
      C        PTOGRAM FOR EVALUATING THE COMPLEX CONSTANTS AN FOR
      C        SHELL AND BOUNDARY LAYER SOLUTION.
      C
      C      METHOD-
      C           INFINITE SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS IS
      C           REDUCED TO A FINITE SYSTEM AND THE CONSTANTS ARE
      C           EVALUATED BY SOLVING THE RESULTING SYSTEM.
      C
      C        DESCRIPTION OF PARAMETERS-
      C           S      -- ROOTS OF THE EQN. SIN(X)+X=0
      C           T      -- ROOTS OF THE EQN. SIN(X)-X=0
      C           AS AND AT -- COMPLEX CONJUGATES OF S AND T
      C           A      -- 10 BY 10 MATRIX
      C           B      -- RIGHT HAND SIDE ARRAY OF THE MATRIX EQN. AX=B
      C                 THE RESULTS ARE STORED IN THE LOCATIONS OF ARRAY B.
      C
0001            IMPLICIT COMPLEX*16(A-H,K,O-Y)
0002            DIMENSION S(10),T(10),AS(10),AT(10),AK(10),AP(10),D(10),E(10),
               1DD(10),EE(10),SK(10),TP(10),B(10),K(10),P(10),PT(10),
               2A(10,10),AA(10,10),C(10,10),AC(10,10),X(10,10),Y(10,10),AL(10,10)
0003            READ(1,5)(S(N),N=1,10)
0004            READ(1,5)(T(N),N=1,10)
0005      5     FORMAT(2F10.7)
0006            WRITE(3,10)
0007      10    FORMAT(17X,'BETA(N)',33X,'BETA"(N)')
0008            WRITE(3,15)(S(N),T(N),N=1,10)
0009      15    FORMAT(10X,2F10.7 ,20X,2F10.7 )
      C
      C         GAMA =   ((1-NU**2)*3/4)**0.25  AND ZGAMA IS=GAMA**2
      C
      C
      C         ZNU IS POISSION'S RATIO
      C
0010            ZNU=0.3
0011            ZGAMA=((1.-ZNU**2)*0.75)**0.5
```

```
      DO 20 N=1,10
      AS(N)=DCONJG(S(N))
      AT(N)=DCONJG(T(N))
      K(N)=-(2.*(1.-ZNU)*CDCOS(S(N))+S(N)*CDSIN(S(N)))/(S(N)*
     1CDCOS(S(N)))
      P(N)=-(T(N)*CDCOS(T(N))-2.*(1.-ZNU)*CDSIN(T(N)))/(T(N)*
     1CDSIN(T(N)))
C
C     AK(N) AND AP(N) ARE THE COMPLEX CONJUGATES OF K(N) AND P(N) RESP.
C
      AK(N)=DCONJG( K(N))
      AP(N)=DCONJG( P(N))
      PT(N)=AP(N)*AT(N)-(1.-2.*ZNU)
      B1=-4.*ZNU/(AS(N)*CDCOS(AS(N)))
      B2=CDSIN(AT(N))*(PT(N)*(2./AT(N)-4./(AT(N)**3))-AT(N)
     &*(6./(AT(N)**2)-12./(AT(N)**4)))
      B3=CDCOS(AT(N))*(PT(N)*4./(AT(N)**2)-AT(N)*(12./(AT(N)**3)-
     12./AT(N)))
      B(N)=B1-(ZNU**2/(1.-ZNU**2))*ZGAMA*(B2+B3)
      WRITE(3,150) B(N)
  150 FORMAT(10X,2D16.8)
      SK(N)=K(N)+(3.-4.*ZNU)/S(N)
      TP(N)=((3.-4.*ZNU)/T(N))-P(N)
   20 CONTINUE
      WRITE(3,25)
   25 FORMAT(///,7X,'M',2X,'N',10X,'REAL L',8X,'IMAG. L')
      DO 35 M=1,10
      D(M)=2.*ZNU-AK(M)*AS(M)
      E(M)=2.*ZNU+AP(M)*AT(M)
      DD(M)=1.-D(M)
      FF(M)=E(M)-1.
      DO 35 N=1,10
      A(M,N)=AS(M)+S(N)
      AA(M,N)=AS(M)-S(N)
      C(M,N)=AT(M)+T(N)
      AC(M,N)=AT(M)-T(N)
      X(M,N)=(CDSIN(A(M,N))/A(M,N))*(D(M)*(K(N)+1./A(M,N))
     1-AS(M)*((K(N)/A(M,N))-1.+2./(A(M,N)**2)))+(CDSIN(AA(M,N))/
     2AA(M,N))*(D(M)*(K(N)-1./AA(M,N))-AS(M)*((K(N)/AA(M,N))+1.
     3-2./(AA(M,N)**2)))+(CDCOS(A(M,N))/A(M,N))*(-D(M)+AS(M)*
     4(K(N)+2./A(M,N)))+(CDCOS(AA(M,N))/AA(M,N))*(D(M)+AS(M)*
     5( K(N)-2./AA(M,N)))+(CDSIN(C(M,N))/C(M,N))*(E(M)*(P(N)-
     61./C(M,N))-AT(M)*((P(N)/C(M,N))+1.-2./(C(M,N)**2)))
```

```
                  7+(CDSIN(AC(M,N))/AC(M,N))*(-F(M)*(P(N)+1./AC(M,N))+AT(M)
                  8 *((P(N)/AC(M,N))-1.+2./(AC(M,N)**2)))+(CDCOS( C(M,N))/ C(M,N))
                  9*(E(M)+AT(M)*(P(N)-2./C(M,N)))+(CDCOS(AC(M,N))/AC(M,N))
                  A*(E(M)-AT(M)*(P(N)+2./AC(M,N)))
0042               Y(M,N)=(CDSIN(A(M,N))/A(M,N))*(DD(M)*(SK(N)+1./A(M,N))
                  1+AS(M)*((SK(N)*1./A(M,N))-1.+2./(A(M,N)**2)))+(CDSIN(AA(M,N))/
                  2AA(M,N))*(-DD(M)*(SK(N)-1./AA(M,N))+AS(M)*(-SK(N)*(1./AA(M,N))
                  3-1.+2./(AA(M,N)**2)))+(CDSIN(C(M,N))/C(M,N))*(EF(M)*(TP(N)+1./
                  4C(M,N))-AT(M)*((TP(N)*1./C(M,N))-1.+2./(C(M,N)**2)))
                  5+(CDSIN(AC(M,N))/AC(M,N))*(EF(M)*(TP(N)-1./AC(M,N))-AT(M)*
                  6((TP(N)*1./AC(M,N))+1.-2./(AC(M,N)**2)))+(CDCOS(A(M,N))/
                  7A(M,N))*(-DD(M)-AS(M)*(SK(N)+2./A(M,N)))+(CDCOS(AA(M,N))/
                  8AA(M,N))*(-DD(M)+AS(M)*( SK(N)-2./AA(M,N)))
                  9+(CDCOS(C(M,N))/C(M,N))*(-EF(M)-AT(M)*(-TP(N)-2./C(M,N)))
                  A+(CDCOS(AC(M,N))/AC(M,N))*( EF(M)-AT(M)*(-TP(N)+2./AC(M,N)))
0043               AL(M,N)=(1.-ZNU)*(X(M,N)+Y(M,N))
0044               WRITE(3,30)(M,N,AL(M,N))
0045      30       FORMAT(5X,2I3,5X,2D16.8)
0046      35       CONTINUE
0047               CALL SIMQ(AL  ,B  ,10,M)
0048               WRITE(3,40)
0049      40       FORMAT(///,30X,'THE CONSTANTS AN ARE')
0050               WRITE(3,45)  ( B(N),N=1,10)
0051      45       FORMAT(30X,2D16.8 )
0052               WRITE(2,50)(B(N),N=1,10)
0053      50       FORMAT(2D16.8)
0054               RETURN
0055               END
```

```
C      ----------------------------------------------------------------
C
C      PROGRAM FOR THE VARIATION OF STRESSES AND DEFORMATIONS IN
C      THE BOUNDARY LAYER SOLUTION
C
C       DESCRIPTION OF PARAMETERS
C            S--     ROOTS OF SIN(2*X)+2*X=0
C            T--     ROOTS OF SIN(2*X)-2*X=0
C            RB--    RATIO OF THE RADIUS AT ANY POINT TO THE EXTERNAL RAD.
C            ZC--    RATIO OF THE AXIAL LENGTH AT ANY POINT TO THE HALF
C                    LENGTH OF THE CYLINDER
C            R--     RATIO OF THE INTERNAL RADIUS TO THE EXTERNAL RAD.
C            ROW--   NON DIMENSIONAL FORM OF THE RADIUS AT ANY POINT
C            ETA--   NON DIMENSIONAL FORM OF THE AXIAL LENGTH OF THE
C                    CYLINDER MEASURED FROM THE LOWER END
C            WR--    NON DIMENSIONAL RADIAL DEFORMATION
C            WZ--    NON DIMENSIONAL AXIAL DEFORMATION
C            SR--    NON DIMENSIONAL RADIAL STRESS
C            ST--    NON DIMENSIONAL HOOP STRESS
C            SZ--    NON DIMENSIONAL AXIAL STRESS
C            SRZ--   NON DIMENSIONAL SHEAR STRESS
C
C      ----------------------------------------------------------------
C
C
       IMPLICIT REAL*8(E,R,X-Z)
       IMPLICIT COMPLEX*16(A-D,G,K,P,S-W)
       DIMENSION A(10),S(10),T(10),SRO(10),SETA(10),TRO(10),
      1TETA(10),K(10),P(10)
       READ(1,5)(S(N),N=1,10)
       READ(1,5)(T(N),N=1,10)
       READ(1,10)(A(N),N=1,10)
     5 FORMAT(2F10.7)
    10 FORMAT(2D16.8)
       DO 35 JJ=1,6
       GO TO (40,42,45,47,50,55),JJ
    40 R=1./3.
       GO TO 60
    42 R=1./2.
       GO TO 60
    45 R=2./3.
       GO TO 60
    47 R=3./4.
```

```
            GO TO 60
     50     P=5./6.
            GO TO 60
     55     P=7./8.
     60     Y=1.
            DO 35 I=1,3
            WRITE(3,12)P,Y
     12     FORMAT('1',15X,'R= ',F6.4,5X,'HALF LENGTH/EXT.RAD.= ',F6.3)
            WRITE(3,19)
     19     FORMAT(/,5X,'P/B',6X,'Z/C',10X,'GU',13X,'GW',13X,'SIGR',
           113X,'SIGT',13X,'SIGZ',13X,'TAURZ')
            ZC=-1.0
            DO 30 J=1,5
            FR=(1.-R)/4.
            RB=R
            DO 25 M=1,5
            ETA=2.*Y*(ZC+1.)/(1.-R)
            ROW=(RB-(1.+R)/2.)/((1.-R)/2.)
            WR=(0.,0.)
            WZ=(0.,0.)
            SR=(0.,0.)
            ST=(0.,0.)
            SZ=(0.,0.)
            SRZ=(0.,0.)
            DO 15 N=1,10
      C
      C     ZNU IS POISSON'S RATIO
      C
            ZNU=0.3
            K(N)=-(2.*(1.-ZNU)*CDCOS(S(N))+S(N)*CDSIN(S(N)))/(S(N)*
           1CDCOS(S(N)))
            P(N)=-(T(N)*CDCOS(T(N))-2.*(1.-ZNU)*CDSIN(T(N)))/(T(N)*
           1CDSIN(T(N)))
            SRO(N)=S(N)*ROW
            SETA(N)=S(N)*ETA
            TRO(N)=T(N)*ROW
            TETA(N)=T(N)*ETA
            IF(CDABS(SETA(N))-50.)65,65,70
     65     SEXP=CDEXP(-SETA(N))
            GO TO 75
     70     SEXP=(0.,0.)
     75     IF(CDABS(TETA(N))-50.)80,80,85
     80     TEXP=CDEXP(-TETA(N))
```

```
          GO TO 100
   85     TFXP=(0.,0.)
C
C         GENERATE THE SYMMETRIC AND ANTI SYMMETRIC PARTS OF U1 AND U2
C
  100     U1S=K(N)*CDCOS(SRO(N))+ROW*CDSIN(SRO(N))
          U1AS=P(N)*CDSIN(TRO(N))+ROW*CDCOS(TRO(N))
          U2S=((3.-4.*ZNU)/S(N)+K(N))*CDSIN(SRO(N))-ROW*CDCOS(SRO(N))
          U2AS=((3.-4.*ZNU)/T(N)-P(N))*CDCOS(TRO(N))+ROW*CDSIN(TRO(N))
C
C         COMPUTE THE DERIVATIVES OF THE PRECEDING QUANTITIES
C
          DU1S=(1.-K(N)*S(N))*CDSIN(SRO(N))+SRO(N)*CDCOS(SRO(N))
          DU1AS=(1.+P(N)*T(N))*CDCOS(TRO(N))-TRO(N)*CDSIN(TRO(N))
          DU2S=(2.*(1.-2.*ZNU)+K(N)*S(N))*CDCOS(SRO(N))+SRO(N)*CDSIN(SRO(N))
          DU2AS=-(2.*(1.-2.*ZNU)-P(N)*T(N))*CDSIN(TRO(N))+TRO(N)
         1*CDCOS(TRO(N))
          Z=(1.-ZNU)/(1.-2.*ZNU)
          T11S=Z*(ZNU*DU2S-(1.-ZNU)*S(N)*U1S)
          T11AS=Z*(ZNU*DU2AS-(1.-ZNU)*T(N)*U1AS)
          T22S=Z*((1.-ZNU)*DU2S-ZNU*S(N)*U1S)
          T22AS=Z*((1.-ZNU)*DU2AS-ZNU*T(N)*U1AS)
          T12S=((1.-ZNU)/2.)*(DU1S-S(N)*U2S)
          T12AS=((1.-ZNU)/2.)*(DU1AS-T(N)*U2AS)
          T33S=ZNU*Z*(-S(N)*U1S+DU2S)
          T33AS=ZNU*Z*(-T(N)*U1AS+DU2AS)
          WR=WR+A(N)*(U2S*SFXP+U2AS*TFXP)
          WZ=WZ+A(N)*(U1S*SFXP+U1AS*TFXP)
          SR=SR+A(N)*(T22S*SFXP+T22AS*TFXP)
          ST=ST+A(N)*(T33S*SFXP+T33AS*TFXP)
          SZ=SZ+A(N)*(T11S*SFXP+T11AS*TFXP)
          SRZ=SRZ+A(N)*(T12S*SFXP+T12AS*TFXP)
   15     CONTINUE
          GVR=(1.-R)*(1.-ZNU)*WR/2.
          GVZ=(1.-R)*(1.-ZNU)*WZ/2.
C
C         TAKE THE REAL PARTS OF THE COMPLEX STRESSES AND DEFORMATIONS
C
          RGVR=REAL(GVR)
          RGVZ=REAL(GVZ)
          RSR=REAL(SR)
          RST=REAL(ST)
          RSZ=REAL(SZ)
```

```
          RSRZ=REAL(SRZ)
          WRITE(3,20)RB,ZC,RGVR,RGVZ,RSR,RST,RSZ,RSRZ
   20     FORMAT(3X,2F7.3,2X,6D16.8)
          RB=RB+FR
   25     CONTINUE
          ZC=ZC+0.5
   30     CONTINUE
   31     Y=Y+1.
   35     CONTINUE
          RETURN
          END
```

```
C
C      --------------------------------------------------------------------
C
C      PROGRAM FOR THE VARIATION OF STRESSES AND DISPLACEMENTS OF
C      A HOLLOW CYLINDER UNDER AXI-SYMMETRIC COMPRESSION BY THE
C      APPROXIMATE ELASTICITY SOLUTION
C
C      METHOD--
C            THE UNKNOWN CONSTANTS ARE EVALUATED FROM THE SYSTEM OF
C            LINEAR SIMULTANEOUS EQUATIONS OBTAINED BY THE
C            BOUNDARY CONDITIONS ON RADIAL AND SHEAR STRESSES.
C            THE SYSTEM OF EQUATIONS IS SOLVED BY THE USE OF THE
C            SIMULTANEOUS EQUATIONS.THESE CONSTANTS ARE SUBSTITUTED IN
C            IN THE STRESS EXPRESSIONS TO GET THE VARIATION OF STRESSES
C            AND DISPLACEMENTS. ELEMENTARY SOLUTION TO THE PROBLEM IS
C            SUPER IMPOSED ON THIS SOLUTION TO GET THE COMPLETE SOLUTION
C
C
C       DESCRIPTION OF PARAMETERS--
C             R--    RATIO OF INNER TO OUTER RADIUS
C             Y--    RATIO OF HALF LENGTH TO OUTER RADIUS
C             ZC--   NON DIMENSIONALISED DISTANCE ALONG THE AXIS OF A
C                    POINT FROM THE AXIS OF REFERENCE
C             RB--   NON DIMENSIONAL RADIAL DISTANCE OF ANY POINT ON THE
C                    CYLINDER FROM ITS AXIS
C             GU--   NON DIMENSIONAL RADIAL DISPLACEMENT MULTIPLIED
C                    BY TWICE THE SHEAR MODULUS
C             GW--   NON DIMENSIONAL AXIAL DISPLACEMENT MULTIPLIED
C                    BY TWICE THE SHEAR MODULUS
C             SIGR--NON DIMENSI/NAL RADIAL STRESS
C             SIGT--NON DIMENSIONAL HOOP STRESS
C             SIGZ--NON DIMENSIONAL AXIAL (NORMAL) STRESS
C             ZNU-- POISSON'S RATIO
C
C      --------------------------------------------------------------------
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION X(4)
       DO 90 I=1,6
       GO TO (10,15,20,25,30,35),I
  10   R=1./3.
       GO TO 40
  15   R=1./2.
```

```
0008            GO TO 40
0009      20    R=2./3.
0010            GO TO 40
0011      25    R=3./4.
0012            GO TO 40
0013      30    R=5./6.
0014            GO TO 40
0015      35    R=7./8.
0016      40    Y=1.0
0017            DO 90 J=1,3
0018            S=1./Y
0019            WRITE(3,50)R,Y
0020      50    FORMAT('1',////////,20X,'A/B=',F7.3,5X,'C/B=',F6.3,///)
0021            V=2.*DATAN(0,1D01)
0022            CALL EVAL(R,Y,V,X,4)
0023            ZC=-1.0
0024            DO 85 K=1,5
0025            RB=R
0026            DR=(1.-R)/4.
0027            DO 80 L=1,5
0028            GU=0.
0029            GW=0.
0030            SIGR=0.
0031            SIGT=0.
0032            SIGZ=0.
0033            TAU=0.
0034            RBSV=RB*S*V
0035            ZCV=ZC*V
0036            SV=S*V
0037            CALL I0(RBSV,RI0)
0038            CALL INUF(RBSV,1,RI0,RI)
0039            CALL BESK(RBSV,0,BK1,I1)
0040            CALL BESK(RBSV,1,BK2,I2)
0041            ZNU=0.3
0042            GU=(ZNU/(ZNU+1.0))*RB+(X(1)*(4.*(1.-ZNU)*RI-RBSV*RI0)+
               &X(2)*(4.*(1.-ZNU)*BK2+RBSV*BK1)-X(3)*SV*RI+X(4)*SV*BK2)*DCOS(ZCV)
0043            GW=-ZC/(1.+ZNU)+(X(1)*RB*S*RI+X(2)*RB*S*BK2+X(3)*S*RI0+X(4)*S*BK1)
               &*SV*DSIN(ZCV)
0044            SIGR=(X(1)*((3.-2.*ZNU)*SV*RI0-((4.*(1.-ZNU)/RB)+(SV**2)*RB)*RI)-
               &X(2)*((3.-2.*ZNU)*SV*BK1+((4.*(1.-ZNU)/RB)+(SV**2)*RB)*BK2)
               &+X(3)*((SV*RI/RB)-(SV**2)*RI0)-X(4)*((SV*BK2/RB)+(SV**2)*BK1))
               &*DCOS(ZCV)
0045            TAU=(X(1)*(RBSV*RI0-2.*(1.-ZNU)*RI)-X(2)*(RBSV*BK1+2.*(1.-ZNU)
```

```
                  &*BK2)+X(3)*SV*RI-X(4)*SV*BK2)*SV*DSIN(ZCV)
0046               SIGT=(X(1)*((4.*(1.-ZNU)/RB)*RI-(1.-2.*ZNU)*SV*RIO)+
                  &X(2)*((4.*(1.-ZNU)/RB)*BK2-(1.-2.*ZNU)*SV*BK1)-X(3)*SV*RI/RB
                  &+X(4)*SV*BK2/RB)*DCOS(ZCV)
0047               SIGZ=-1.0+(X(1)*((RI*(SV**2)*RB)+2.*ZNU*SV*RIO)
                  &+X(2)*((BK2*(SV**2)*RB)-2.*ZNU*SV*BK1)+X(3)*RIO*(SV**2)
                  &+X(4)*BK1*(SV**2))*DCOS(ZCV)
0048               WRITE(3,70)RB,ZC,GU,GW,SIGR,SIGT,SIGZ,TAU
0049        70     FORMAT(2F6.3,3X,6(F9.6,2X))
0050               RB=RB+DR
0051        80     CONTINUE
0052               ZC=ZC+0.5
0053        85     CONTINUE
0054               Y=Y+1.0
0055        90     CONTINUE
0056               RETURN
0057               END
```

```
C
C     ------------------------------------------------------------------
C
C     SUBROUTINE EVAL(R,Y,V,B,N)
C        PURPOSE-
C              EVALUATION OF THE CONSTANTS A,B,C,D USED IN
C              THE EXPRESSIONS FOR DISPLACEMENTS AND STRESSES
C
C       DESCRIPTIN OF PARAMETERS
C          B(I)--MATRIX OF THE CONSTANTS A,B,C,D
C        A(I,J)--MATRIX OF THE COEFFICIENTS OF A,B,C AND D
C
C
C          METHOD--
C           THE SUBROUTINE SIMQ.
C           SOLUTION OF THE LINEAR SIMULTANEOUS EQUATIOS BY USING
C
C     ------------------------------------------------------------------
C
      SUBROUTINE EVAL(R,Y,V,B,N)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(4,4),B(4)
      S=1./Y
      W=R*S
      SV=S*V
      WV=W*V
      CALL  I0(SV,RI01)
      CALL  I0(WV,RI02)
      CALL  INUE(SV,1,RI01,RI1)
      CALL  INUE(WV,1,RI02,RI2)
      CALL  BESK(SV,0,BK1,I1)
      CALL  BESK(WV,0,BK2,I2)
      CALL  BESK(SV,1,BK3,I3)
      CALL  BESK(WV,1,BK4,I4)
      DO 10 I=1,4
      B(I)=0.
      DO 10 J=1,4
      A(I,J)=0.
   10 CONTINUE
      ZNU=0.3
      A(1,1)=(3.-2.*ZNU)*SV*RI01-(4.*(1.-ZNU)+(SV**2))*RI1
      A(1,2)=-((3.-2.*ZNU)*SV*BK1+(4.*(1.-ZNU)+(SV**2))*BK3)
```

```
      A(1,3)=SV*RI1-(SV**2)*RI01
      A(1,4)=-(SV*BK3+(SV**2)*BK1)
      A(2,1)=(3.-2.*ZNU)*SV*RI02-((4.*(1.-ZNU)/R)+W*S*(V**2))*RI2
      A(2,2)=-((3.-2.*ZNU)*SV*BK2+((4.*(1.-ZNU)/R)+W*S*(V**2))*BK4)
      A(2,3)=(SV/R)*RI2-(SV**2)*RI02
      A(2,4)=-((SV/R)*BK4+(SV**2)*BK2)
      A(3,1)=SV*RI01-2.*(1.-ZNU)*RI1
      A(3,2)=-(SV*BK1+2.*(1.-ZNU)*BK3)
      A(3,3)=SV*RI1
      A(3,4)=-(SV*BK3)
      A(4,1)=WV*RI02-2.*(1.-ZNU)*RI2
      A(4,2)=-(WV*BK2+2.*(1.-ZNU)*BK4)
      A(4,3)=SV*RI2
      A(4,4)=-SV*BK4
      B(1)=0.40D-01
      B(2)=B(1)
      CALL SIMQ(A,B,4,KS)
      RETURN
      END
```

CHAPTER V

EXPERIMENTATION

V-1 Experimentation in this project consists of measuring the longitudinal and tangential strains on the outer cylindrical surface of the cylinders. Testing was carried out on the Tinius Olson Universal Testing Machine (Model Super L). A self aligning ball-mounted table was used at the base of the cylinders to apply the compressive load. Resistance type strain gages were used along with the digital strain indicator for measurement of strains. Brief descriptions of strain gages, their installation, strain gage indicator and detailed experimental procedure are given in the following sections.

V-2

(a) Wire Resistance Strain Gages: A change in the length of a wire or strain induced in a resistance strain gage changes its resistance as expressed by the formula given below.

$$\text{Resistance} = \frac{(\text{specific resistance})\ (\text{length})}{(\text{area of cross-section})}$$

Thus if the strain gage wire is taken as one of the arms of an initially balanced wheatstone bridge, application of strain will cause an imbalance in the bridge and its output (current or voltage) will be directly proportional to the strain [10]. Bonded wire resistance strain gages are used for the purpose of strain measurement and are identified by the gage factor and the resistance of the wire. The gage factor k of a strain gage is defined as,

$$k = \frac{\Delta R/R}{\Delta \ell/\ell}$$

Here, R is the gage resistance and ΔR the resistance variation caused by a variation $\Delta \ell$ in a gage length of ℓ. For most of the materials, taking Poisson's ratio equal to 0.3, the gage factor is given as,

$$k = 1.6 + \frac{\Delta\rho/\rho}{\Delta\ell/\ell}$$

where, $(\Delta\rho/\rho)/(\Delta\ell/\ell)$ is the specific change in the resistivity ρ of the wire. The constantan-copper-nickel alloy wires are widely used for strain gages as they have a stable gage factor. Variation in the ambient temperature would give a false indication of strain and thus has to be compensated. This is accomplished either by putting two active gages in the Wheatstone bridge thus giving twice the output or by putting an active and a dummy gage in the bridge circuit. The term "active gage" is used for the gage which is subjected to strains while the "dummy gage" remains unstrained.

In order to find the principal stresses or strains at a point under biaxial system of stresses, three independent strain measurements in three different directions are necessary. But if the directions of the principal stresses are known as in our case, two strain gages mounted at right angles to each other along these lines are sufficient to give the complete state of stress. The strain gages used for the experimentation in this project were SR-4 strain gages with a wire resistance of 120±0.2 ohms and gage factor equal to 2.06±1%.

(b) Digital Strain indicator: It is a portable unit and can be operated on battery or AC supply of 115 volts. Main controls of this instrument shown in fig.1 are described below.

(i) Rebalance Knob and Strain Counter: It is used to bring the null meter to zero position while making strain measurements. Strain is read on the digital strain counter. The rebalance knob has a range of 10,000 micro-inch/inch.

(ii) Range Extender: This knob changes the sign of strain for tension or compression readings. It also extends the range of the rebalance knob in steps of 10,000 micro-inch/inch up to a maximum of 50,000 micro-inch/inch.

(iii) Bridge Selector Switch: When this switch is on full bridge, external bridge is completed by two active and two compensating strain gages For quarter and half bridge operations, this switch puts the internal dummy half bridge in to circuit. In quarter bridge operation, temperature compensation is done by means of an internal dummy gage of the indicator.

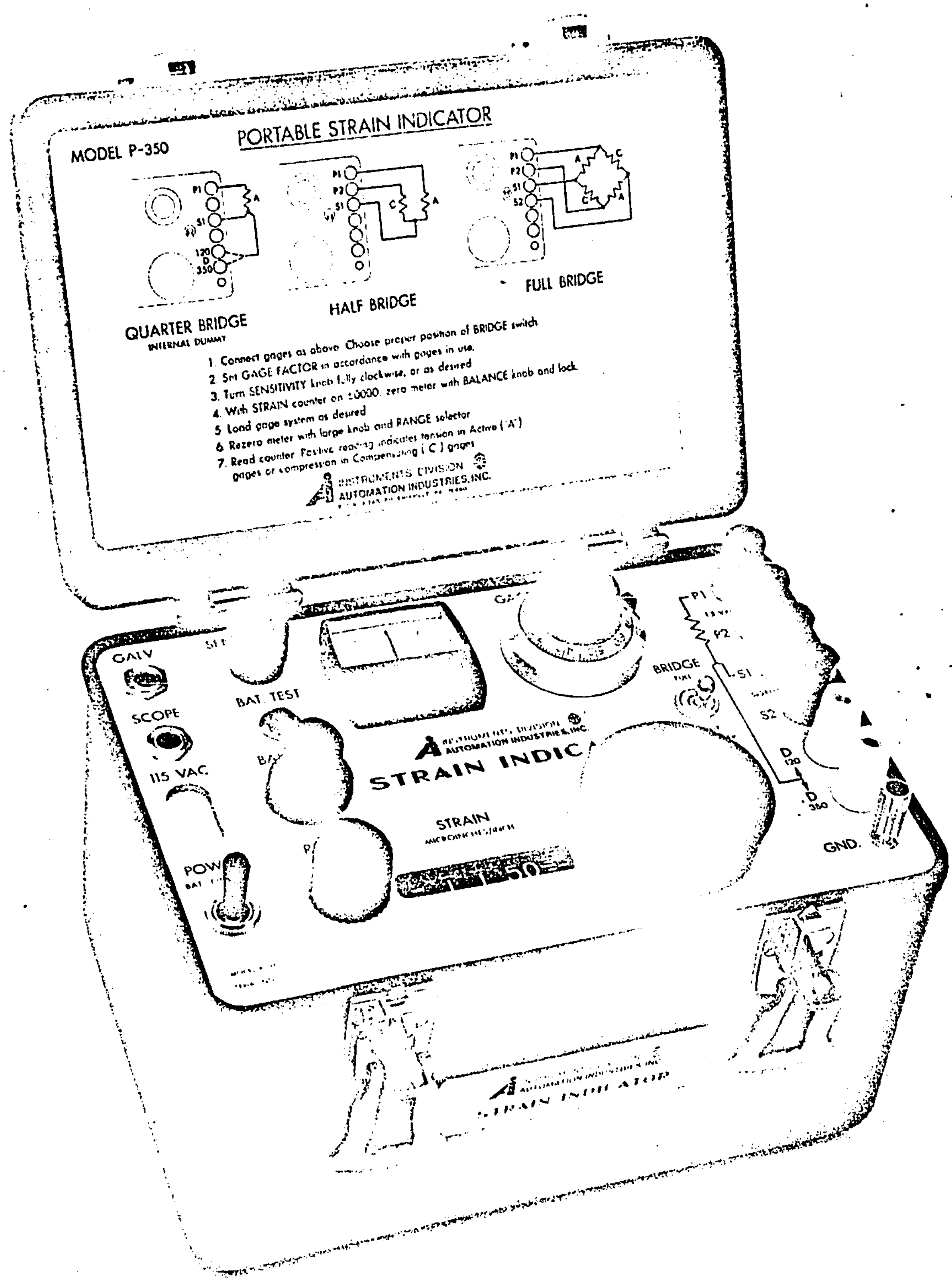


FIGURE 1: DIGITAL STRAIN INDICATOR (PORTABLE) MODEL P-350

(iv) Balance Control: This control is meant for adjusting the initial unbalance in a gage circuit. It is locked after adjusting the gage circuit.

(v) Gage Factor Dial: It is a ten turn locking control to set the appropriate gage factor. This control can be varied from 0.1 to 10.0.

(vi) Null Meter: It is a zero center galvanometer used to determine instrument balance by adjusting the balance and rebalance knobs.

(vii) Sensitivity Control: This control varies the sensitivity of the null meter. When turned fully, the null meter has a sensitivity of approximately 40 micro-inch/inch full scale in either direction.

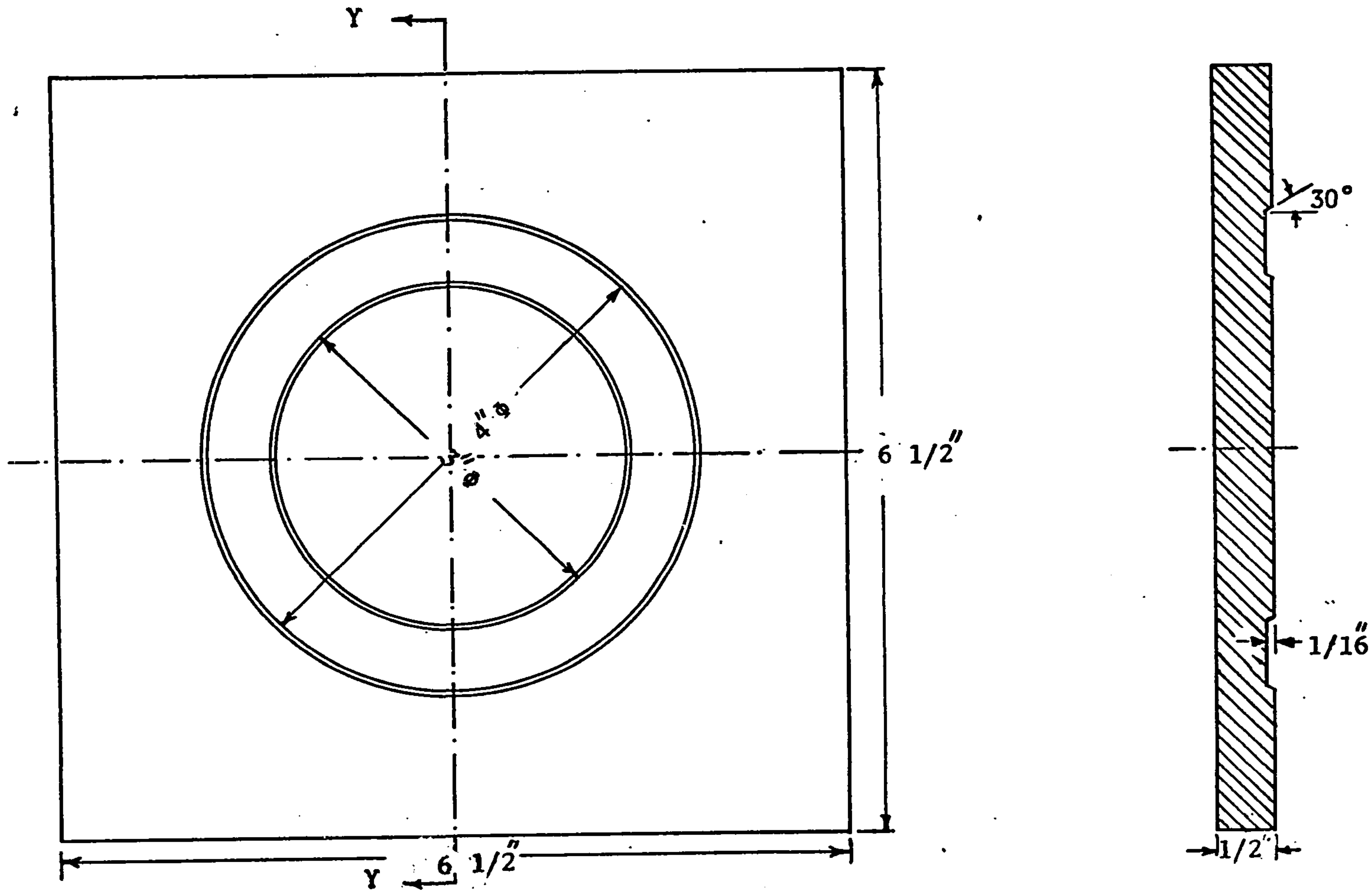
The strain gage indicator was used along with switch and balance unit which facilitates in the systematic reading of multiple gages. This is done by means of a selector knob provided in this unit. This unit can be used for full,half and quarter bridge operations.

V-3 Preparation of Cylinders and End Plates

Cylinders for experimentation were cut from mild steel seamless mechanical tubing. The cylinders were machined at the ends so that they are parallel to each other and normal to the longitudinal axis. The end plates for the cylinders were cut from a half inch thick cold-rolled mild steel plate. Tapered circular grooves 1/16 inch deep were machined on one side of each plate so that the ends of the cylinders fit in to the grooves. Figure 2 shows a plate with the groove for one set of cylinders. This arrangement restricts the ends of the cylinders from expanding sideways. The cylinder and end plate assembly was checked for parallelism and in all the cases it was found to be within ±0.0015 inch.

V-4 Strain Gage Installation

(a) Surface Preparation: Prior to mounting the gages, it is essential that the surface should be smooth and clean. For this purpose, the portions of the cylinder where strain gages were to be mounted were rubbed with medium and smooth silicon carbide papers. Beginning with a coarse paper of 180 grit, smoothening was done by 400 grit paper. Very light lines along which the strain gages were to be mounted were marked on the cylinder.



SECTION AT Y-Y

FIRST QUADRANT PROJECTIONS

FIGURE 2: END PLATE FOR A CYLINDER WITH $\frac{a}{b} = \frac{3}{4}$

The surface of the cylinder was wetted by the GC-4 Neutralizer and cleaned by means of a gauze pad. Application of the neutralizer and cleaning was repeated several times until the pad showed no trace of color or dirt. Duri the process of cleaning, Chlorothene Degreaser was also sprayed.

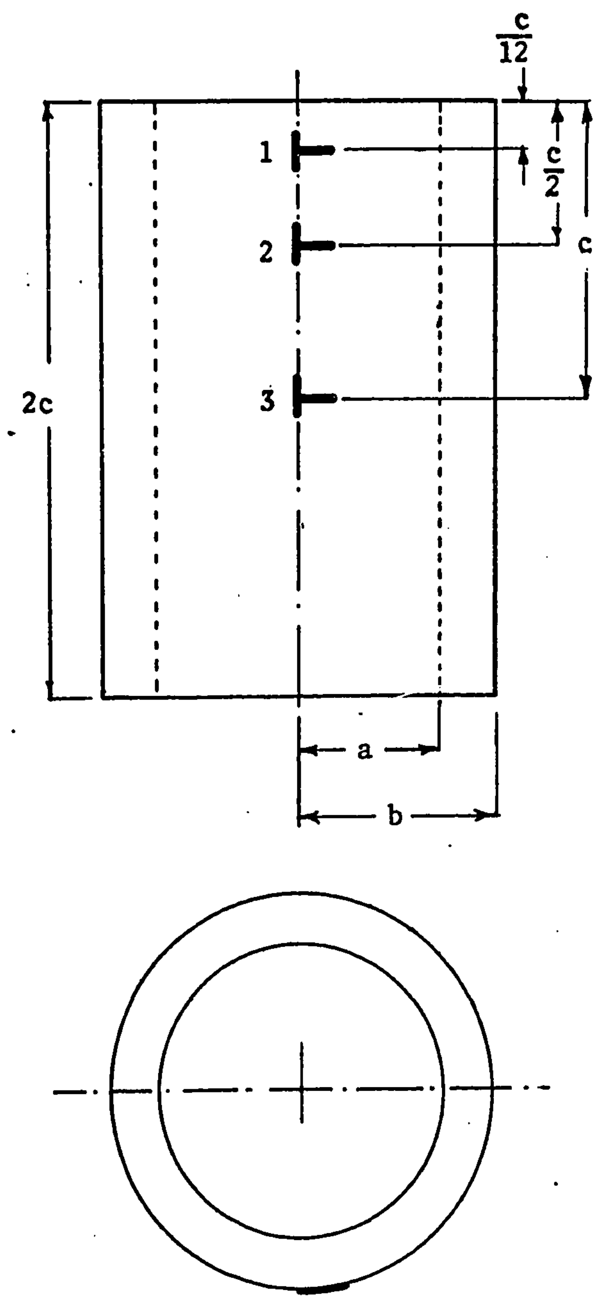
(b) Strain Gage Bonding: The gages were placed on a cleaned surface with the foil side up. A piece of cellophane tape sufficiently long (4" to 5") for convenient handling was placed over the gage. The alignment marks located near the gage foils were extended onto the cellophane tape by using a ball point pen. The tape-and-gage assembly was then placed over the prepar surface of the cylinder in such a manner that the alignment marks on the assembly coincided with the lines marked over the surface of the cylinder. The tape was stuck on the cylinder by using a single wiping motion to avoid wrinkles. The free end of the tape was then lifted by means of tweezers causing the strain gage also to be lifted. The surface, where the strain gage was to be mounted and the back side of the gage were painted with a smooth coat of GA-1 Adhesive by means of the wooden end of a swab. The gage was then pressed in to its position by means of a gauze pad using wiping motion. The gage was kept pressed for about a minute and then the tape was removed carefully. The gage was again pressed by the gauze pad using thumb pressure for four to five minutes to allow for curing of the adhesive. The resistance between the strain gage tab and ground was checked by an Avometer and should be infinity while the resistance of the gage wire should be nearly 120 ohms. Strain gages were mounted only on half the length of the cylinder along a straight line. Depending on the space availab strain gages were installed at two or three points identified with numbers 1, 2 and 3 as shown in figure 3.

(c) Lead Wire Attachment: A piece of Mylar tape, supplied with the Strain Gage Application Kit was placed over the gage exposing only the strain gage solder tabs. The tabs were cleaned by the neutralizer and were tinned by placing the solder on the tab and heating with a clean soldering tip. The lead wire to be soldered to the gage was stripped and tinned separately. The wire was then placed over the tinned tabs by means of tweezers and heated by the soldering tip so that they were connected together taking care that the solder does not flow over the surface of the cylinder. In order to insure that the naked portion of the lead wire does not come in to contact with the surface of the cylinder, a piece of insulating tape was placed under the wire.

(d) Gage Checkout: Prior to experimentation, the gage installation was checked out to insure that the gage was properly bonded. The strain gage was connected to the strain indicator and the null meter was balanced to read zero. A drift of the indicator in excess of 10 micro-inch/inch in five minutes was taken as an indication of a poor bond provided the environmental temperature did not change. The bonding of the gage was also tested by pressing the gage grid slightly with the eraser end of a pencil. For a well bonded gage, the indicator should show some deflection on application of the load and return to the zero position on release of pressure.

V-5 EXPERIMENTAL PROCEDURE

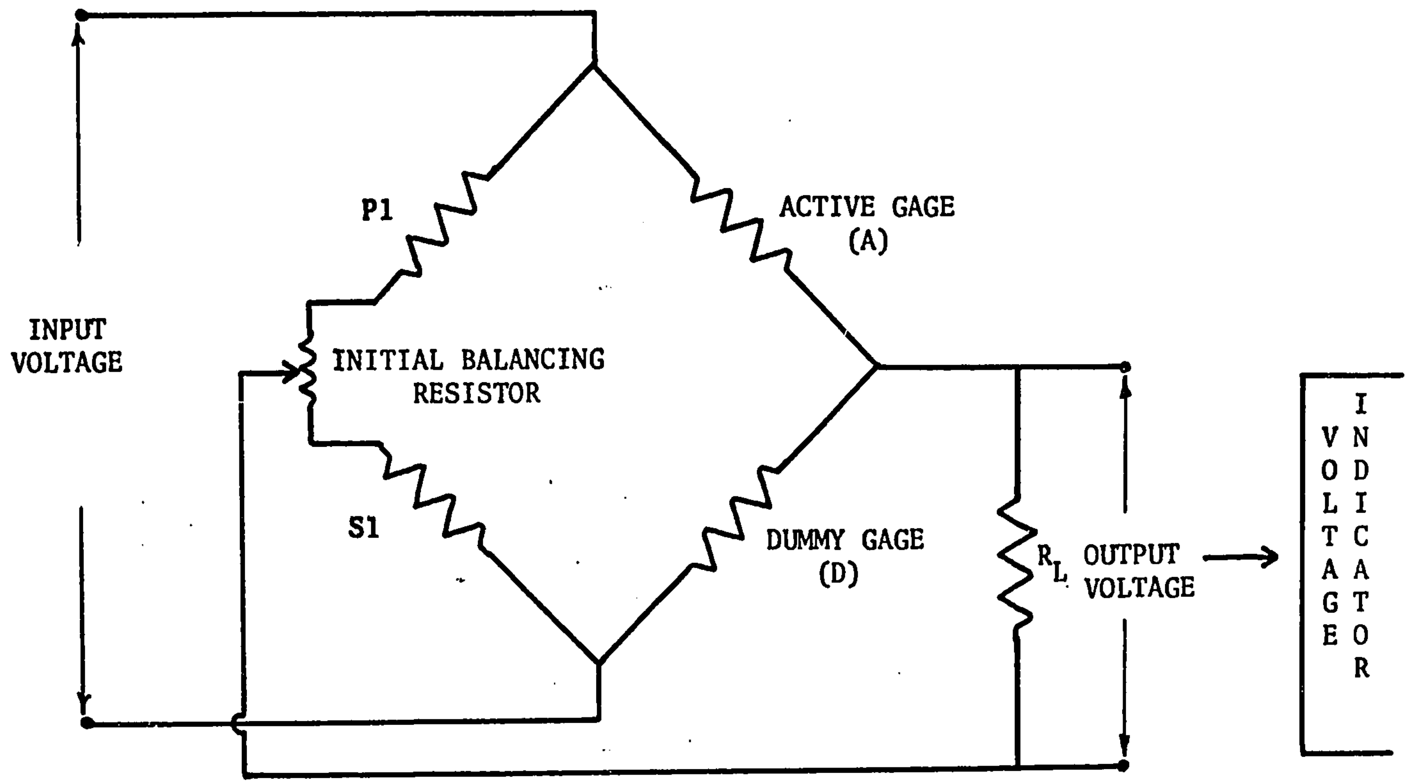
A self aligning ball mounted table was placed over the lower platen of the Tinius Olson machine in such a manner that it exactly covered one of the refrence circles marked over the platen. One of the end plates was placed over this table such that the centers of the table and the circular groove on the end plate were coincident. The cylinder was then placed in the groove and the other end plate was put over it. All the strain gages were connected to the switch and balance unit coupled to the strain indicator. Connections were made for quarter bridge operation using internal dummy. Figure 4 illustrates these connections with only one active gage on channel 10 of the switch and balance unit. The pointer on the load scale of the testing machine was adjusted to zero. One of the gages was put in to circuit by means of the selector switch on the switch and balance unit. The null meter on the strain indicator was balanced to zero and initial reading on the digital counter was noted down. The cylinder was loaded by turning the 'LOAD' handle of the machine in clockwise direction



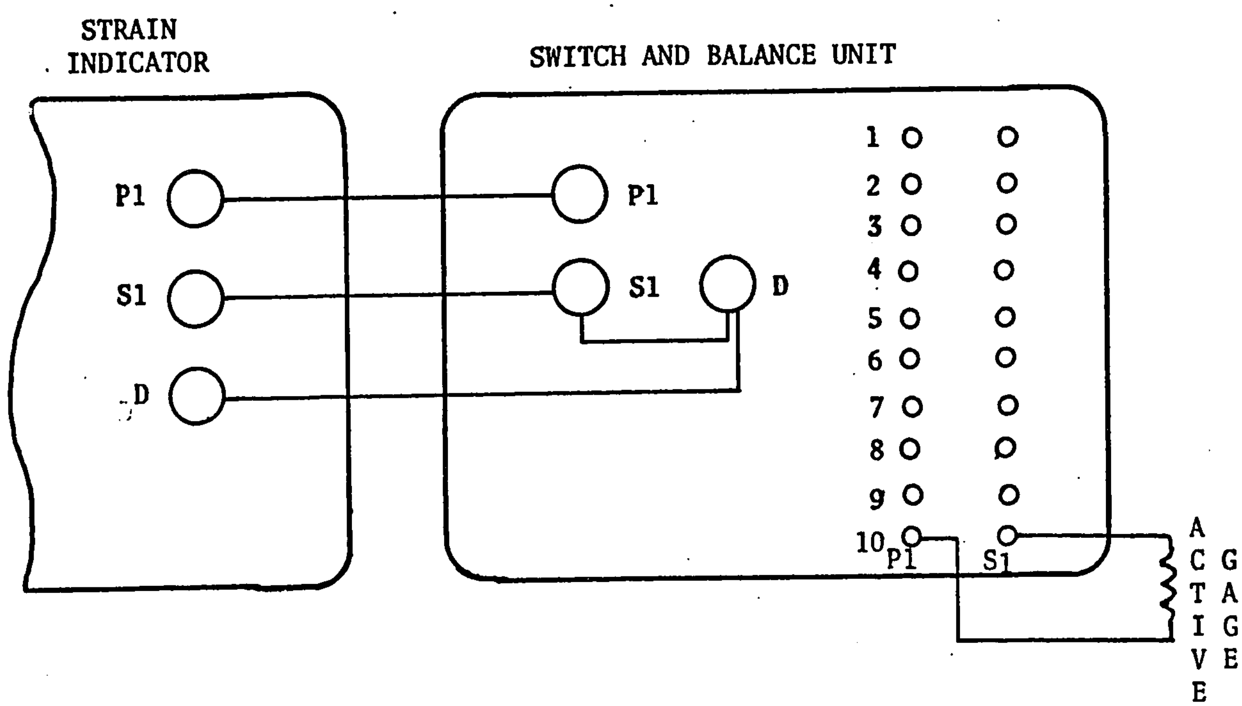
TOP VIEW

FIRST QUADRANT PROJECTIONS

FIGURE 3: LAYOUT OF STRAIN GAGES



(a). BASIC BRIDGE CIRCUIT



(b). CIRCUIT FOR QUARTER BRIDGE OPERATION USING INTERNAL DUMMY ACTIVE GAGE SHOWN IN ONE CHANNEL ONLY

FIGURE 4

and the rate of loading was so adjusted that the pointer on the load scale moved very slowly. The maximum load to be applied on the cylinder was calculated to give an average compressive stress of nearly 2000 psi. Readings of the indicator were taken at half and full or maximum load by rebalancing the null meter. After crossing the maximum load, the 'LOAD' handle was turned in the opposite direction so that the pointer started retracing back showing a reduction in the load. Readings of the indicator were again noted at half and full loads. The cylinder was unloaded by operating the 'UNLOAD' handle. The procedure was repeated by putting all the gages one by one in the circuit. Similar sets of readings were obtained by rotating the upper end plate thrice at right angles.

V-6 REDUCTION OF STRAIN DATA

The two strains measured at a point on the cylinder represent the longitudinal and transverse strains. The strain gages were mounted along the principal stress directions i.e., in the directions of the longitudinal stress σ_z and the circumferential stress σ_θ. As the curved surfaces are stressfree and also because of symmetry, these two stresses are the principal stresses. These stresses can be calculated from the observed strains ε_z and ε_θ using Hooke's law.

$$\sigma_z = \frac{E}{1-\nu^2}\left(\epsilon_z + \nu\epsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\epsilon_\theta + \nu\epsilon_z\right) \qquad (5.1)$$

The radial displacement can be computed from the transverse strain as given below.

$$u = r\,\epsilon_\theta \qquad (5.2)$$

In order to compare these results with those obtained from the two theories, equations (5.1) and (5.2) are non-dimensionalized to give experimental results in the form

$$\sigma_z' = F\left(\epsilon_z + \nu\epsilon_\theta\right)$$

$$\sigma_\theta' = F \left(\epsilon_\theta + \nu \epsilon_z \right)$$

$$2 G' U' = Q \epsilon_\theta \qquad (5.3)$$

Here,

$$F = \frac{1}{1-\nu^2} \frac{EA}{P}$$

$$Q = \frac{1}{1+\nu} \frac{EA}{P}$$

A is the area of cross-section of the cylinder and P is the load applied on it. Young's Modulus (E) was taken as 30,000,000 psi., while ν was assumed to be as 0.3 for mild steel.

CHAPTER VI

DISCUSSION OF RESULTS AND CONCLUSIONS

VI-1 The numerical results for the different solutions were obtained by using maximum possible accuracy in the computer. It is observed that some of the boundary conditions are not exactly satisfied. The results are discussed in details for each of the solutions in the following paragraphs.

(a) Three Dimensional Elasticity Solution: The results show that there is a residual stress σ_r at the curved surfaces r=a,b. This has also been observed by Picket[11] and Sharma [15] in the study of solid cylinders. This is because of the contradiction in the boundary requirements at the curved surfaces and the plane ends z=±c [11]. According to condition (2.9), the radial displacement is zero everywhere at the ends. So the radial and the tangential strains should be zero.

$$\frac{1}{E}\left(\sigma_r - \nu\sigma_\theta - \nu\sigma_z\right) = 0$$

$$\frac{1}{E}\left(\sigma_\theta - \nu\sigma_r - \nu\sigma_z\right) = 0 \tag{6.1}$$

These equations can be solved to give the radial and the tangential stresses at the ends.

Therefore,

$$\sigma_r = \frac{\nu}{1-\nu}\,\sigma_z$$

$$\sigma_\theta = \frac{\nu}{1-\nu}\,\sigma_z \tag{6.2}$$

It is evident from the above expressions that the radial stress does not satisfy the boundary condition (2.9)a at the ends z=±c. This deviation of σ_r from the mathematical model could be the cause of residual stresses at the curved surfaces in the regions other than the ends also. The results for the variation of the stresses σ_r and σ_θ at the ends are in full agreement with eqs.(6.2). This is true for all the cylinders considered in this analysis and Table 1 illustrates the results of three representative cases only.

In order to study the convergence of the solutions, maximum possible number of terms, as permitted by the limitations of the computer, were used. It was observed that the solutions give absurd results on using a larger number of terms in the series. Stresses were found to become tensile in nature as more and more terms were used in the series. These effects were predominant at the ends where stresses were found to have very large magnitudes in comparison to the average stress σ. However, with a larger number of terms used in the series, a slow convergence was observed near the middle of the cylinders. For the thickest and the shortest cylinder considered in this analysis, as many as 15 terms were used in the series. An explanation of what has been mentioned above can be obtained from the analysis of the infinite system of equations (4.2)c-h. This system can be written in the following form.

$$x_i = \sum_{k=1}^{\infty} c_{i,k} x_k + b_i \quad (i = 1, 2, \cdots) \tag{6.3}$$

According to the theory of infinite system of linear simultaneous equations, in the limits, the approximate solution obtained by using the method of reduction converges to the solution of the infinite system provided the system is regular and free terms of which are bounded in a certain manner. An infinite system of the form (6.3) is called regular if the sum of the absolute magnitudes of coefficients of each row is less than unity [7],

$$\sum_{k=1}^{\infty} |c_{i,k}| < 1 \qquad (i = 1, 2, \cdots) \tag{6.4}$$

For a convergent system, the free terms are bounded in the form,

$$|b_i| \leq Q_1\left[1 - \sum_{k=1}^{\infty} |c_{i,k}|\right] \quad (i = 1, 2, \cdots) \tag{6.5}$$

where the constant $Q_1 > 0$.

A rigorous analysis of the infinite system (4.2)c-h could be done by the method used by Blair and Veeder [2]. The system of equations (4.2)c-h can be reduced to two equations by elimination, each equation representing an infinite system of the form (6.3). They can be checked for regularity by substituting asymptotic expansions for Bessel functions and taking the limits i and k $\rightarrow \infty$. This type of theoretical analysis is too complicated and has not been done here. Instead, a numerical procedure was adopted to check the regularity of the infinite system. In the method of reduction, assuming that there are N terms in the series, the system (4.2)c-h reduces to a finite system of 6 by N equations and can be represented by equation (6.3) except that k varies from 1 to N. The problem is then to find the sum of the modulii of off-diagonal elements in each row of the matrix of coefficients. For a regular system, condition (6.4) must be satisfied for sufficiently high values of N. Only one cylinder with $\frac{a}{b} = \frac{1}{3}$ and $\frac{c}{b} = 1$ was checked and N was taken as 15. It was found that condition (6.4) was satisfied by only seven out of ninety rows. It can easily be shown that if (6.4) is not satisfied, condition (6.5) also does not hold good. So the system (4.2)c-h is not regular and hence the convergence of its solution is not guaranteed. This conclusion explains why absurd results were obtained by taking more number of terms. Better results could be expected by taking first two or three terms in the series. For the present study, the value of N was taken as three to solve the system (4.2)c-h and the results were found to be reasonable except at the ends in some cases. Slight variations in the axial displacement at the ends are due to the reasons explained above and also due to the errors introduced in computing Bessel functions and the roots of eqn.(4.2)a. These errors though very small initially may be magnified in subsequent computations. Because of one or all the reasons mentioned above, condition (2.10) on longitudinal stress is satisfied approximately.

(b) Shell and Boundary Layer Solution: The axial displacement instead of being constant is found to vary at the ends. These deviations are caused again by the contradictions in the boundary conditions at the ends and by the errors introduced during computations. According to condition (2.17)a, the interior stress coefficient $v_z^{(1)}(0)$ should be zero thus giving condition (3.15)b. Along with this, the middle surface displacement $V_z^{(1)}(0)$ should also be zero. From (3.17)b, it is seen that the double derivative $V_z^{(1)''}(0)$ is zero. These two conditions lead to different conclusions as evident from expression (3.18)b. The latter condition is fulfilled because with this, the boundary condition for the radial stress is satisfied. Small values of radial displacement at the ends can be due to slight errors in computations in the interior and the boundary layer solutions.

VI-2 BOUNDARY LAYER SOLUTION

The solution suggested in [19] for orthotropic cylinders cannot be used for isotropic cylinders. A similar procedure was adopted to obtain the boundary layer solution for isotropic cylinders. The infinite system (3.34) was solved by reducing it to a 10 by 10 system. The complex constants a_n obtained from the solution of this system are given in Table 3. Matrix ℓ_{mn} of system (3.34) is found to be a hermitian matrix as expected and the system is regular. Thus the solution should be convergent. Use of first ten terms shows an indication of convergence. More number of terms could not be used to get a better convergence because only ten roots (Table 2) of eqs.(3.31), lying in the first quadrant are available [4, 13]. It is seen that boundary layer solution is effective only in the narrow edge zones and decays very rapidly along the length. Variation of the longitudinal stress near the end was plotted and it was found that boundary layer solution gives very small stresses at the outer and middle curved surfaces . The longitudinal stress at the inner surface is 54.9% of the average stress σ at the ends. These variations are given in figs. 5-7 . Boundary layer thickness δ is obtained from the variation of longitudinal stress along the inner curved surface. For this purpose, boundary layer effect is considered up to the point where the longitudinal stress is one percent of the average stress σ. A straight line relation between δ and thickness of the cylinder is observed while length does not have any effect on boundary layer thickness. These results are represented in figs.8 and 9. For thin cylinders, the stress has a larger magnitude at the ends but its effect is carried over a shorter distance from the ends as compared to the thick cylinders for which the stress has a smaller magnitude. Only first non-zero system is considered as each higher system would involve lengthy derivations similar to the first one given in section III-3 (b). It is evident that the contribution of higher order systems is small.

VI-3 COMPARISON OF RESULTS

Longitudinal stress was chosen for comparison of three-dimensional elasticity and shell and boundary layer solutions. Figures 10-27 show the stress σ_z/σ as a function of thickness of the cylinder at different planes along the length according to the two solutions. There is an appreciable difference in the variation of the stress at the ends as obtained from the two solutions. Both the solutions show higher stress at at the outer surface. This is due to the reason that the end planes tend to become concave under loading but the presence of end plates and their rigidity keeps them straight resulting in a higher stress at the outer surface. The stress given by the three-dimensional elasticity solution at the ends has not been plotted for thin and short cylinders. The two solutions give closer results in the regions away from the ends. The distribution of the longitudinal stress is found to be almost uniform near the middle of the cylinder and is in confirmation with Saint-Venant's principle. For longer and thinner cylinders ($\frac{c}{b} \geq 2$ and $\frac{a}{b} \leq \frac{2}{3}$), both the solutions differ very slightly from each other. The variation of the near the middle of the cylinder is similar to that obtained by Picket [11] in the analysis of solid cylinders. The difference in the stress away from the ends, given by the two solutions, is found to be more at the curved surfaces though it becomes smaller for longer and thinner cylinders. Starting with a cylinder of parameters $\frac{a}{b} = \frac{1}{3}$ and $\frac{c}{b} = 1$, the maximum difference in the values of the stress at the outer surface is 27% of the average stress and reduces to 2.5% for a cylinder with parameters $\frac{7}{8}$ and 1 respectively.

The variations of the radial and the axial displacements also behave in a similar manner and are not presented here. The radial, the tangential and the shear stresses have different variations but because of their very small magnitudes in comparison to to the longitudinal stress, they are not considered to be significant for the purpose of comparison. Variations of the longitudinal stress and the radial displacement along the outer curved surface are plotted in figs.28-41. The strain measurements were carried out over half the portion of the cylinders. Thus, the experimental results given in tables 4-10, are plotted for that portion only. The general trend of experimental points gives slightly higher estimation of the longitudinal stress as compared to the theoretical curves.

The experimental points for the radial displacement are slightly lower than the theoretical points. Comparison is done on the basis of how closely the experimental variation is to any of the two solutions and also how close are the experimental points to the theoretical ones. On this basis, it can be concluded that the three-dimensional theory gives better results for thick and short cylinders. While, shell and boundary layer solution should be preferred for thin and long cylinders. For a cylinder with $\frac{a}{b} = \frac{7}{8}$ and $\frac{c}{b} = 1$, only two terms could be used for the series solution in the three-dimensional elasticity case and the variation of the stress at the ends is found to be incorrect. Depending on the availability of the material, experiments were carried out on nine cylinders with parameters $\frac{a}{b} = \frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ and $\frac{c}{b} = 1, 2$ and 3. Results of only seven experiments are plotted as two cases were found to give absurd results due to improperly bonded or damaged gages.

The approximate elasticity solution gives nearly uniform distribution in the regions away from the ends. This solution is a slight improvement on the elementary solution and can be used for thin and long cylinders as the variation of the stress is approximately similar to that for the other two solutions. The purpose of introducing approximate elasticity solution was not to give another solution but it was an attempt to make the complicated problem easier by reducing the number of boundary conditions or approximating them. This solution gives almost negligible shear stress but the tangential and the radial stresses have the same orders of magnitude as those of the other two solutions in the regions away from the ends. Thus the purpose of introducing this solution is solved. Numerical results of the solution are given at the end.

VI-4 CONCLUSIONS

From the interpretation of theoretical and experimental results, following recommendations are made for the use of the three-dimensional elasticity and shell and boundary layer solutions.

(a) For cylinders with radii ratios less than or equal to $\frac{1}{2}$ and length to external diameter ratios less than 3, three-dimensional elasticity solution will give closer results. For very thick cylinders, this solution should be used.

(b) For cylinders with radii ratios more than $\frac{3}{4}$ and length to external diameter ratios more than or equal to 2, shell and boundary layer solution should be used. For very thin cylinders, this solution can be used for length to external diameter ratios of less than 2 also.

(c) Both the solutions give similar results in the regions away from the ends of the cylinders with parameters lying in between the extremes defined in (a) and (b). But looking at the variations of the stresses at the ends and also the time required in computations for the two solutions, shell and boundary layer solution should be preferred.

TABLE 1

VARIATION OF STRESSES AT THE ENDS z=±c IN THREE-DIMENSIONAL ELASTICITY SOLN.

CYLINDER PARAMETERS	$\frac{r}{b}$	$\frac{\sigma_z}{\sigma}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$
$\frac{a}{b} = \frac{1}{3}$, $\frac{c}{b} = 1$	$\frac{1}{3}$	-0.75354	-0.32295	-0.32295
	$\frac{1}{2}$	-1.24086	-0.53180	-0.53180
	$\frac{2}{3}$	-0.96960	-0.41554	-0.41554
	$\frac{5}{6}$	-0.80095	-0.34326	-0.34326
	1	-1.23048	-0.52735	-0.52735
$\frac{a}{b} = \frac{1}{2}$, $\frac{c}{b} = 1$	$\frac{1}{2}$	-0.45280	-0.19406	-0.19406
	$\frac{5}{8}$	-1.08107	-0.46332	-0.46332
	$\frac{3}{4}$	-0.78863	-0.33789	-0.33789
	$\frac{7}{8}$	-0.87839	-0.37645	-0.37645
	1	-1.73285	-0.74265	-0.74265
$\frac{a}{b} = \frac{2}{3}$, $\frac{c}{b} = 1$	$\frac{2}{3}$	0.07270	0.03116	0.03116
	$\frac{3}{4}$	-0.79336	-0.34001	-0.34001
	$\frac{5}{6}$	-0.85718	-0.36736	-0.36736
	$\frac{11}{12}$	-1.11882	-0.47949	-0.47949
	1	-2.07828	-0.89069	-0.89069

TABLE 2

FIRST QUADRANT ROOTS OF EQS. (3.31) [4,13]

n	Re β_{n1}	Im β_{n1}	Re β_{n2}	Im β_{n2}
1	2.1061960	1.1253650	3.7488380	1.3843390
2	5.3562690	1.5515750	6.9499800	1.6761050
3	8.5366830	1.7755440	10.1192590	1.8583840
4	11.6991780	1.9294050	13.2772740	1.9915710
5	14.8540600	2.0468530	16.4298710	2.0966260
6	18.0049330	2.1418910	19.5794090	2.1833980
7	21.1534140	2.2217230	22.7270360	2.2573200
8	24.3003420	2.2905530	25.8733840	2.3217140
9	27.4462030	2.3510480	29.0188310	2.3787580
10	30.5912950	2.0450130	32.1636170	2.4299590

TABLE 3

VALUES OF COMPLEX CONSTANTS a_n FOR SHELL AND BOUNDARY LAYER SOLUTION

n	Re a_n	Im a_n
1	-0.10816669 D-01	-0.48493039 D-01
2	0.83025747 D-04	0.17807225 D-01
3	0.85005298 D-03	-0.98998716 D-02
4	-0.10111889 D-02	0.64630990 D-02
5	0.10548901 D-02	-0.45604208 D-02
6	-0.10876987 D-02	0.33304824 D-02
7	0.11279480 D-02	-0.24246797 D-02
8	-0.11624200 D-02	0.16578505 D-02
9	0.11236178 D-02	-0.89953805 D-03
10	-0.77582778 D-03	0.12208380 D-03

TABLE 4

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{1}{2}$ $\frac{c}{b} = 1$ Area of cross-section=3π sq. in.

Maximum Load= 20,000 lbs. F= 0.01553533 Q=0.01087473

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEM
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
2	-66.25	19.0	-0.94066423	-0.01359341	0.206619
3	-64.875	22.0	-0.90532135	0.03912089	0.239244

TABLE 5

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{1}{2}$ $\frac{c}{b} = 2$ Area of cross-section=3π sq. in.

Maximum Load= 20,000 lbs. F= 0.01553533, Q=0.01087473

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACE
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
1	-61.0	16.25	-0.87192039	-0.03184742	0.176714
2	-62.3	21.5	-0.87075524	0.04272215	0.233806
3	-69.0	20.5	-0.97639549	-0.00310706	0.222951

TABLE 6

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{1}{2}$ $\frac{c}{b} = 3$ Area of cross-section=3π sq. in.

Maximum Load= 20,000 lbs. F= 0.01553533 Q=0.01087473

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEMENT
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
1	-70.0	18.5	-1.0012520	-0.03883832	0.20118250
2	-73.75	21.5	-1.04552770	-0.00970958	0.23380669
3	-73.0	20.75	-1.03737166	-0.01786562	0.22565064

TABLE 7

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{2}{3}$ $\frac{c}{b} = 2$ Area of cross-section= $\frac{5}{4}$ π sq. in.

Maximum Load= 8000 lbs. F= 0.01618264 Q= 0.01132784

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEMENT
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
1	-68.75	12.0	-1.05025333	-0.13957527	0.13593408
2	-64.0	19.0	-0.94344791	-0.00323652	0.21522896
3	-68.0	19.0	-1.00817847	-0.01585897	0.21522896

TABLE 8

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{2}{3}$ $\frac{c}{b} = 3$ Area of cross-section= $\frac{5}{4}\pi$ sq. in.

Maximum Load= 8000 lbs. F=0.01618264 Q=0.01132784

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEMENT
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
1	-72.5	13.5	-1.10770170	0.13350678	0.15292584
2	-70.75	20.25	-1.04661224	-0.01577807	0.22938876
3	-71.5	20.25	-1.058574922	-0.01941916	0.22938876

TABLE 9

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{3}{4}$ $\frac{c}{b} = 2$ Area of cross-section= $\frac{7}{4}\pi$ sq. in.

Maximum Load= 11,000 lbs. F=0.01647687 Q=0.0115338

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEMENT
	ε_z	ε_θ	σ_z	σ_θ	$2\frac{G}{\sigma}\frac{u}{b}$
1	-73.0	13.5	-1.136080	-0.12234075	0.15570630
2	-70.25	20.25	-1.05740313	-0.01359341	0.23355945
3	-72.0	19.0	-1.09241648	-0.04283986	0.21914220

TABLE 10

EXPERIMENTAL RESULTS

$\frac{a}{b} = \frac{3}{4}$ [illegible] $= 3$ Area of cross-section= $\frac{7}{4}\pi$ sq. in.

Maximum Load= 11,000 lbs. F= 0.01647687 Q = 0.0115338

POINT	AVERAGE STRAIN μ-in./in.		STRESSES/σ		DISPLACEMENTS
	ε_z	ε_θ	σ_z	σ_θ	$2\,\frac{G}{\sigma}\,\frac{u}{b}$
1	-54.25	12.5	-0.83208193	0.06220018	0.14417250
2	-65.0	18.25	-0.98078568	0.02059608	0.21049185
3	-66.0	17.5	-1.00096985	0.03789680	0.20184150

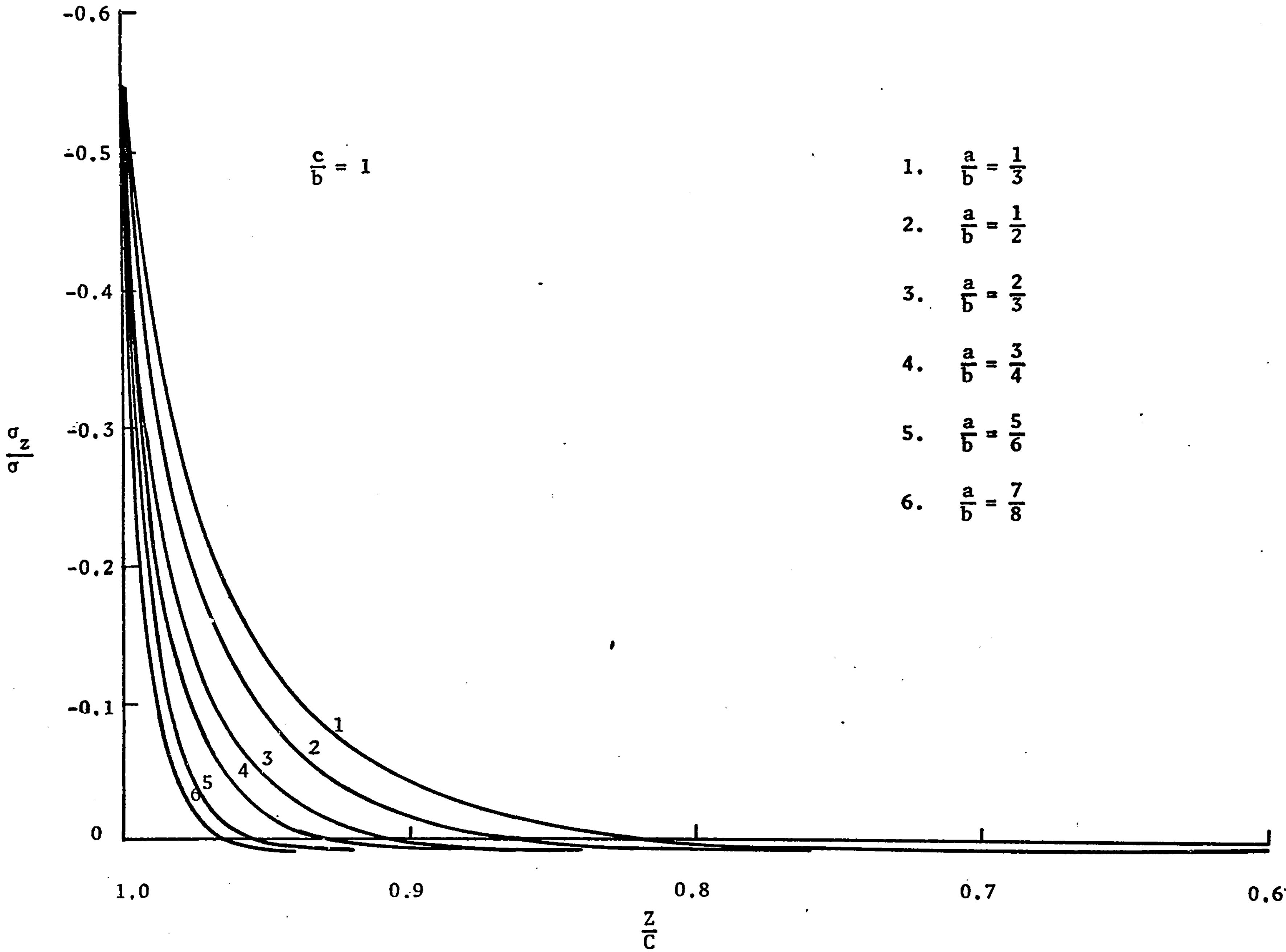


FIGURE 5: VARIATION OF BOUNDARY LAYER STRESS IN THE EDGE ZONES ALONG THE INNER SURFACE

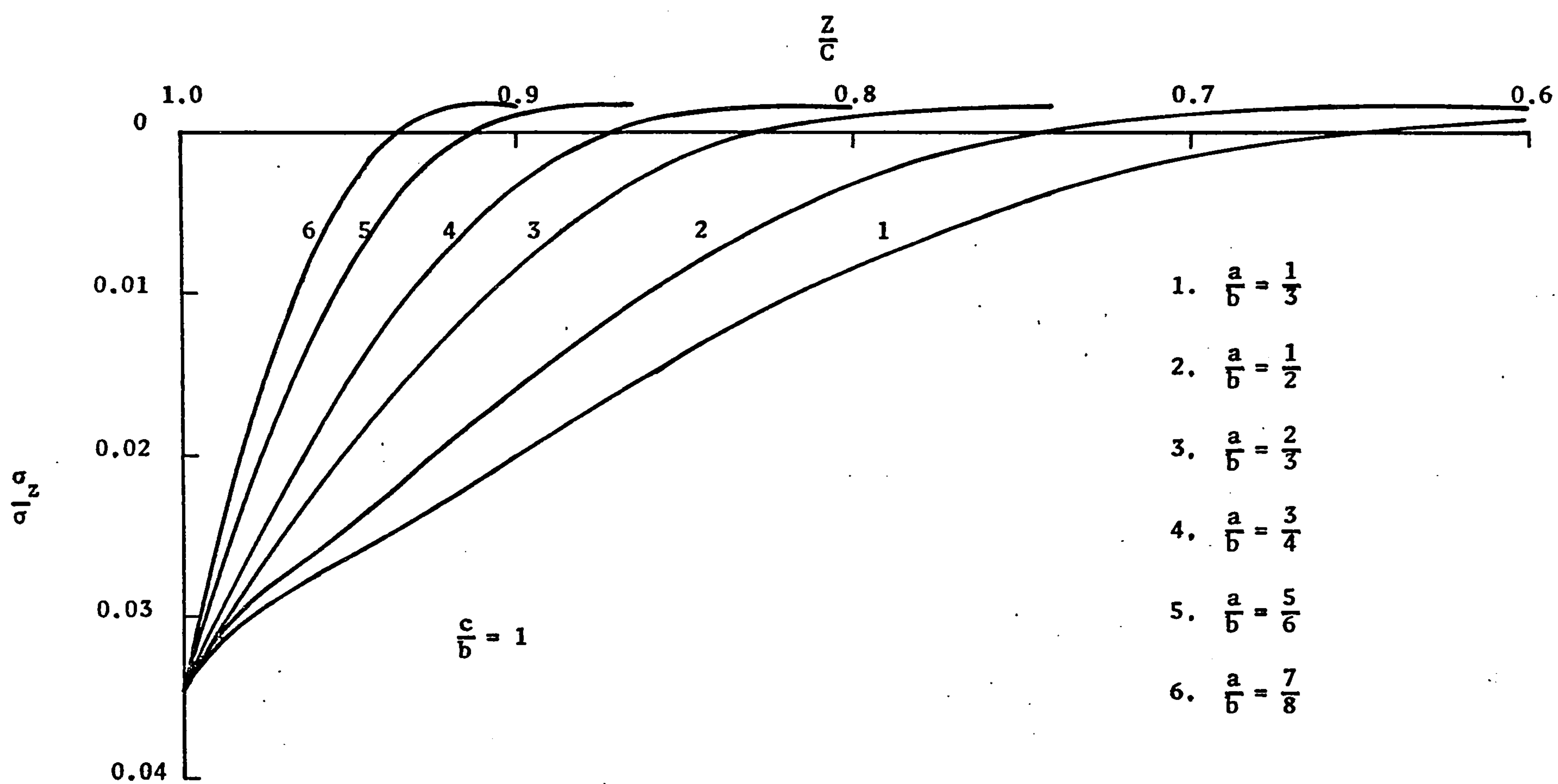


FIGURE 6: VARIATION OF THE BOUNDARY LAYER STRESS IN THE EDGE ZONES ALONG THE MIDDLE SURFACE

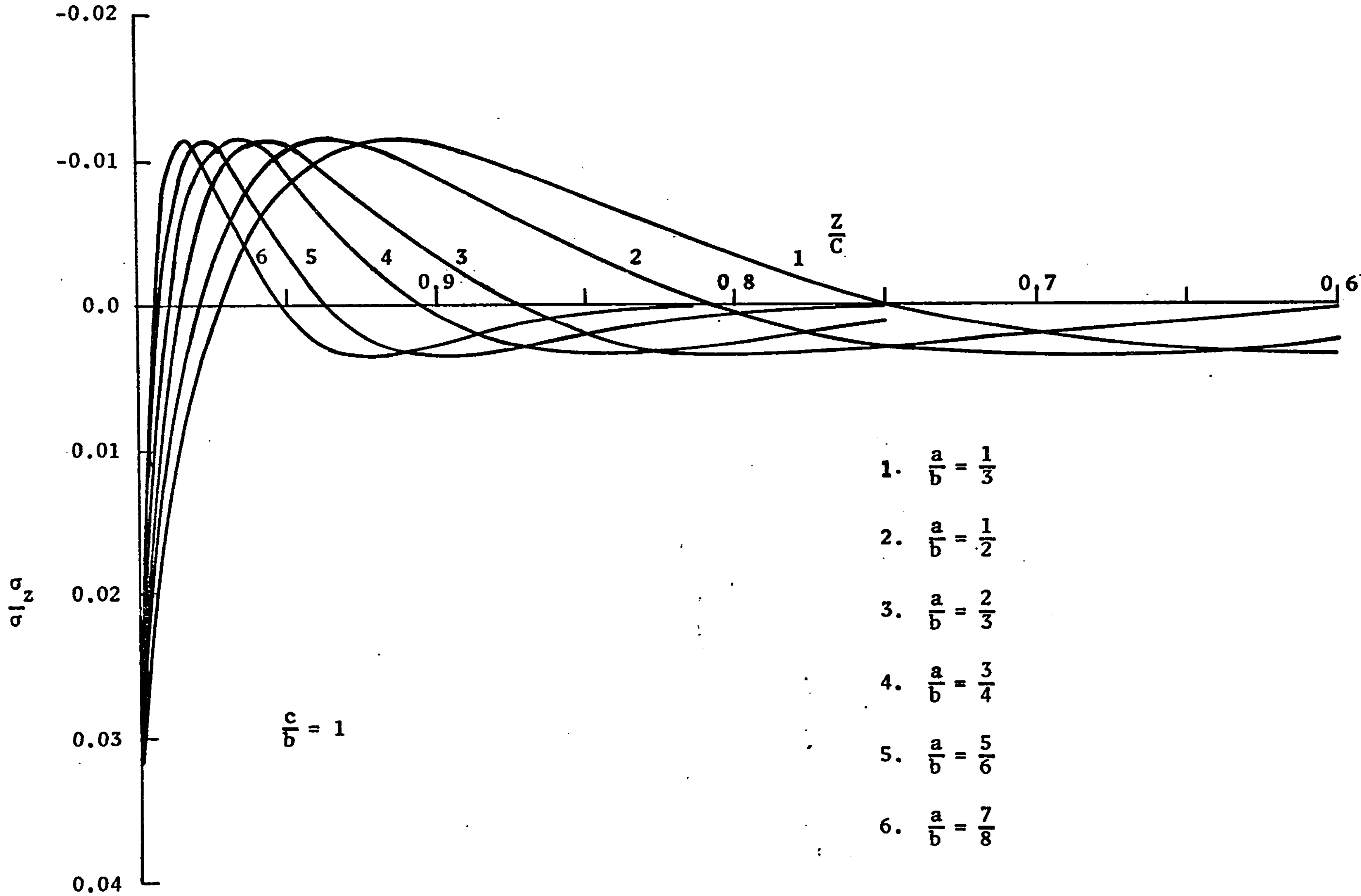


FIGURE 7: VARIATION OF BOUNDARY LAYER STRESS IN THE EDGE ZONES ALONG THE OUTER SURFACE

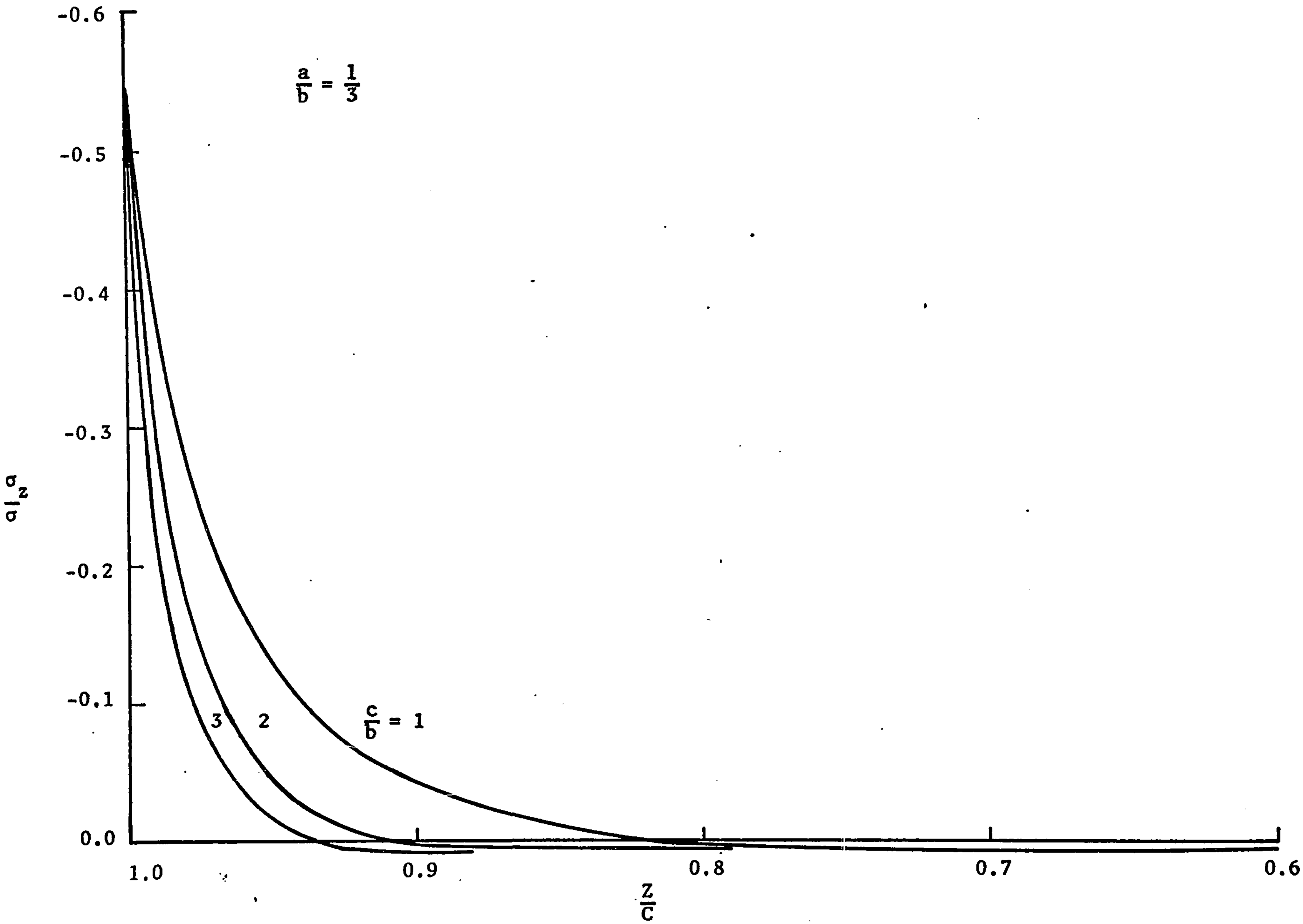


FIGURE 8: VARIATION OF BOUNDARY LAYER STRESS IN THE EDGE ZONES FOR DIFFERENT VALUES OF $\frac{c}{b}$

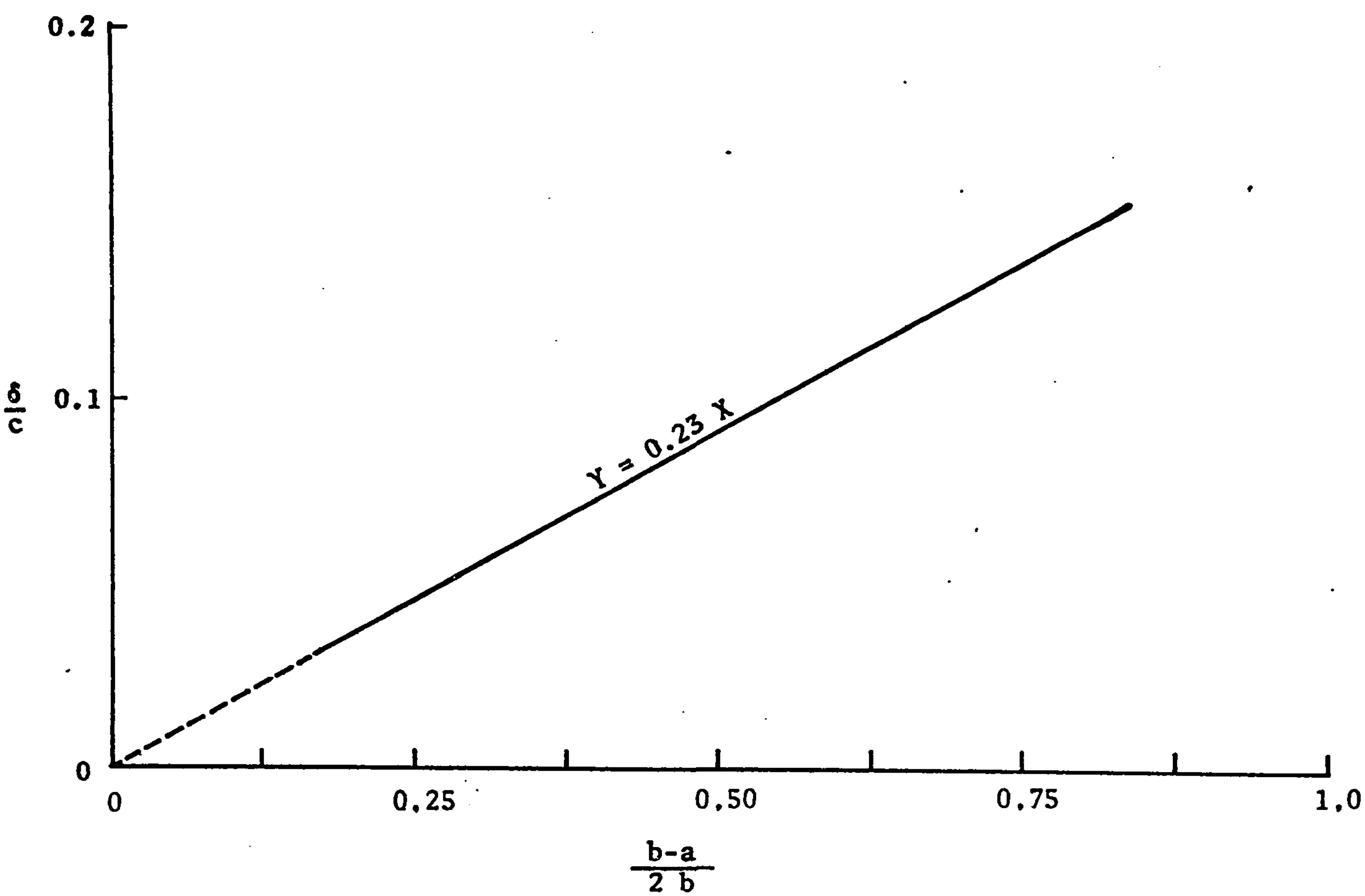
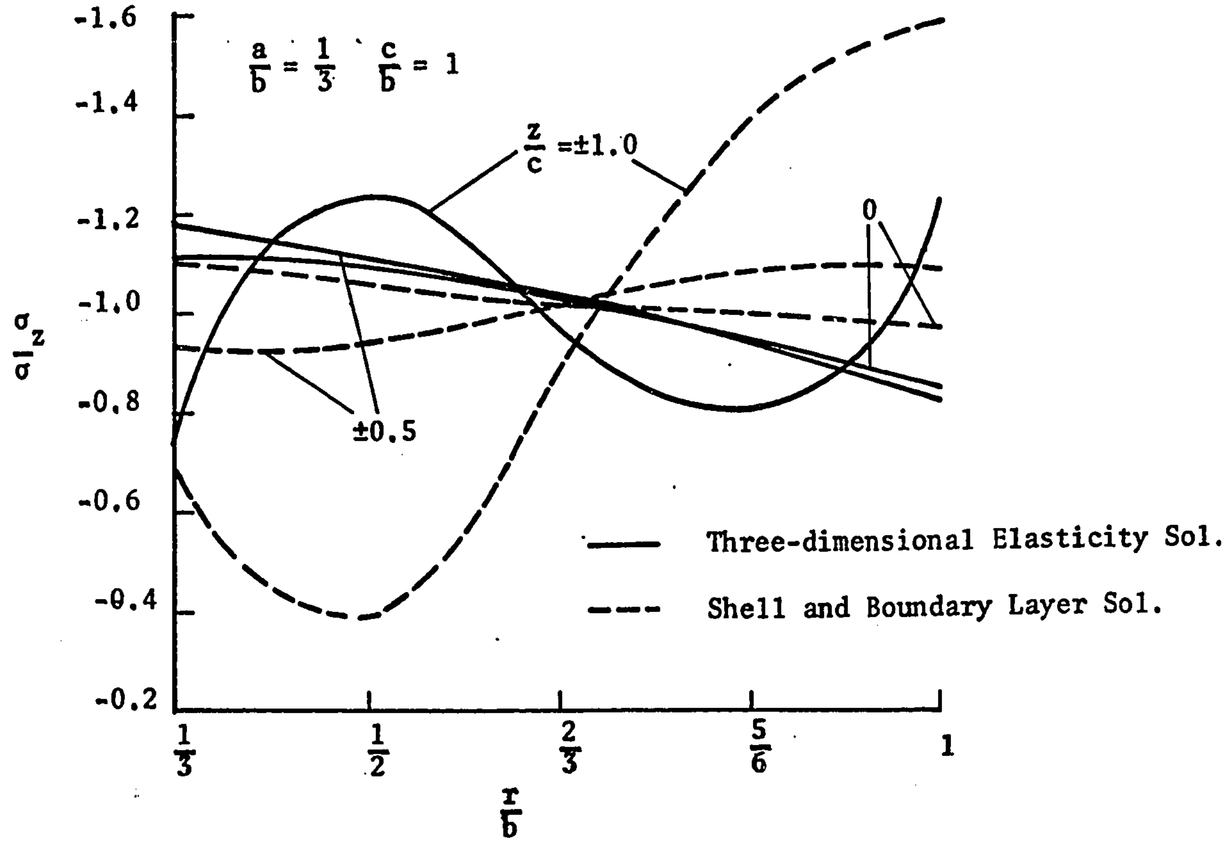
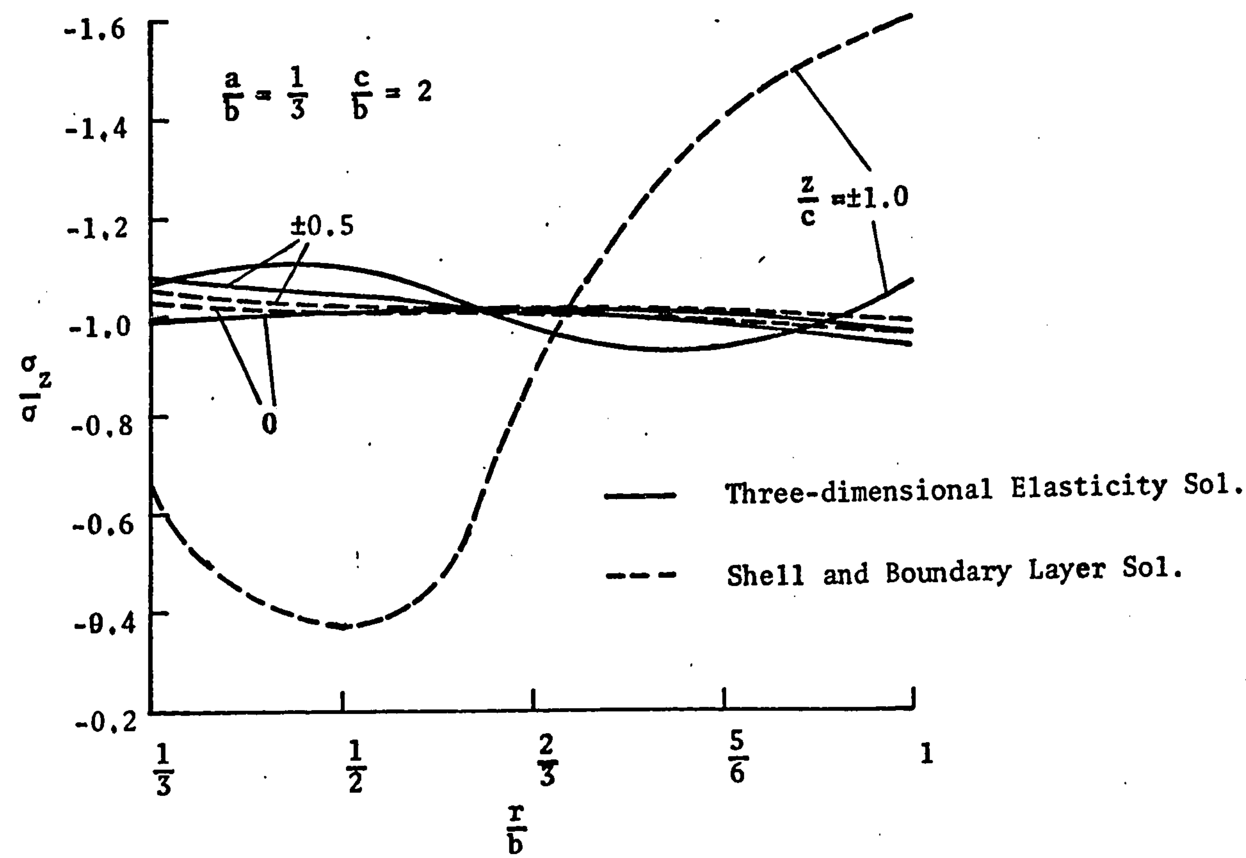
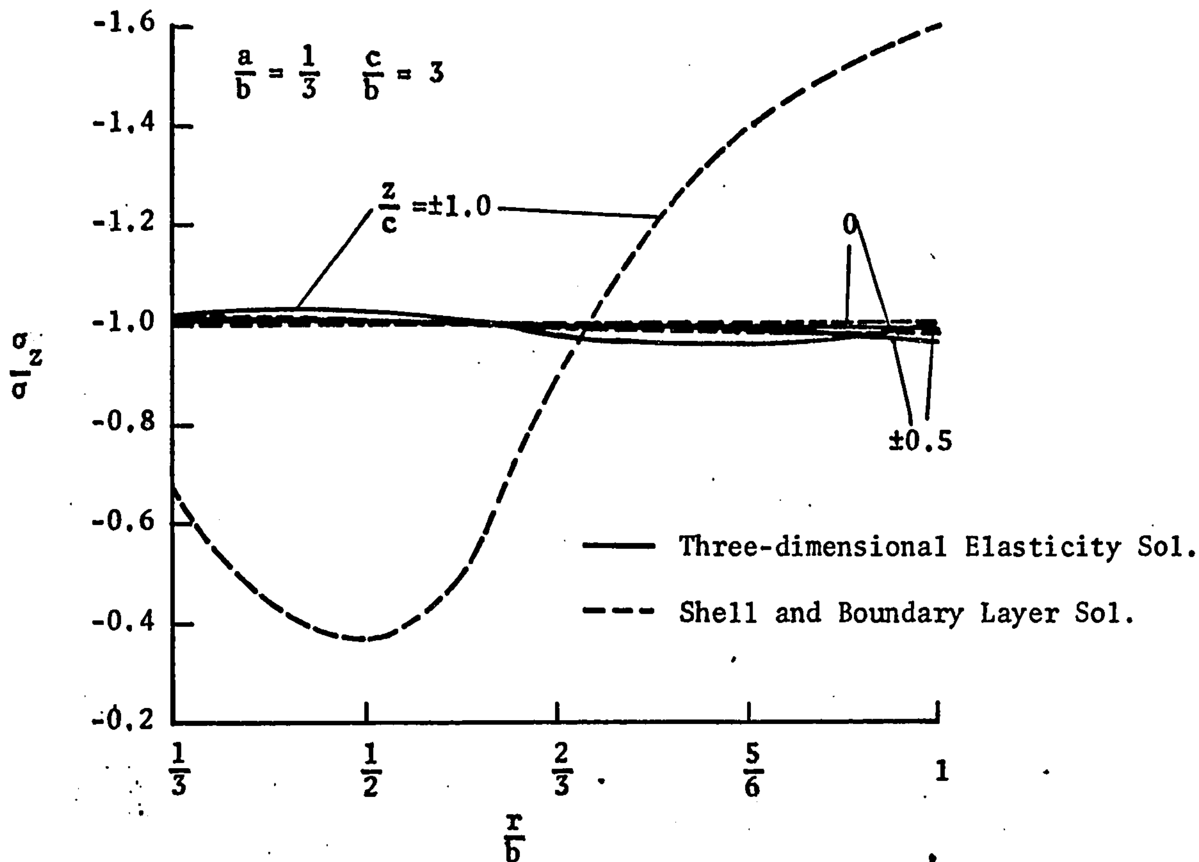


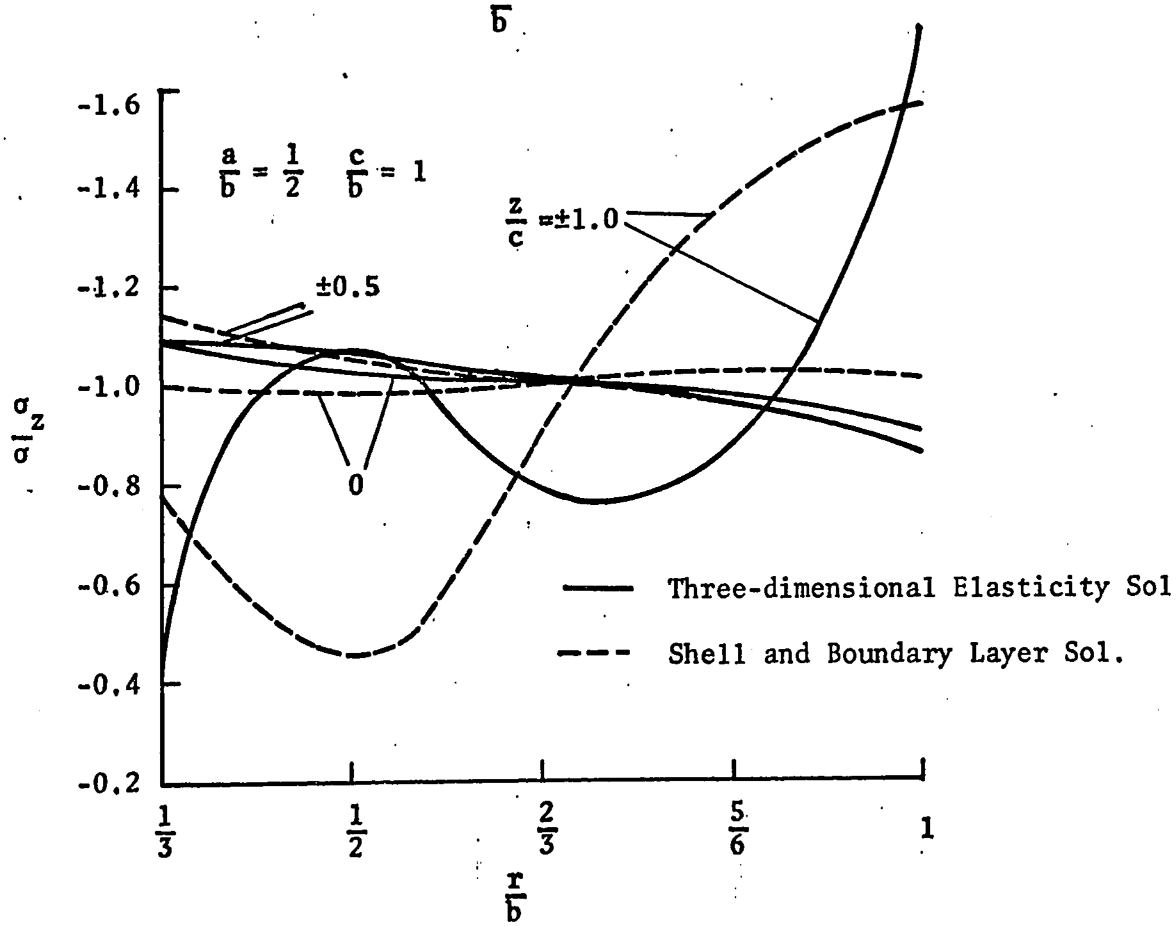
FIGURE 9: CORRELATION BETWEEN BOUNDARY LAYER THICKNESS AND THICKNESS OF CYLINDER



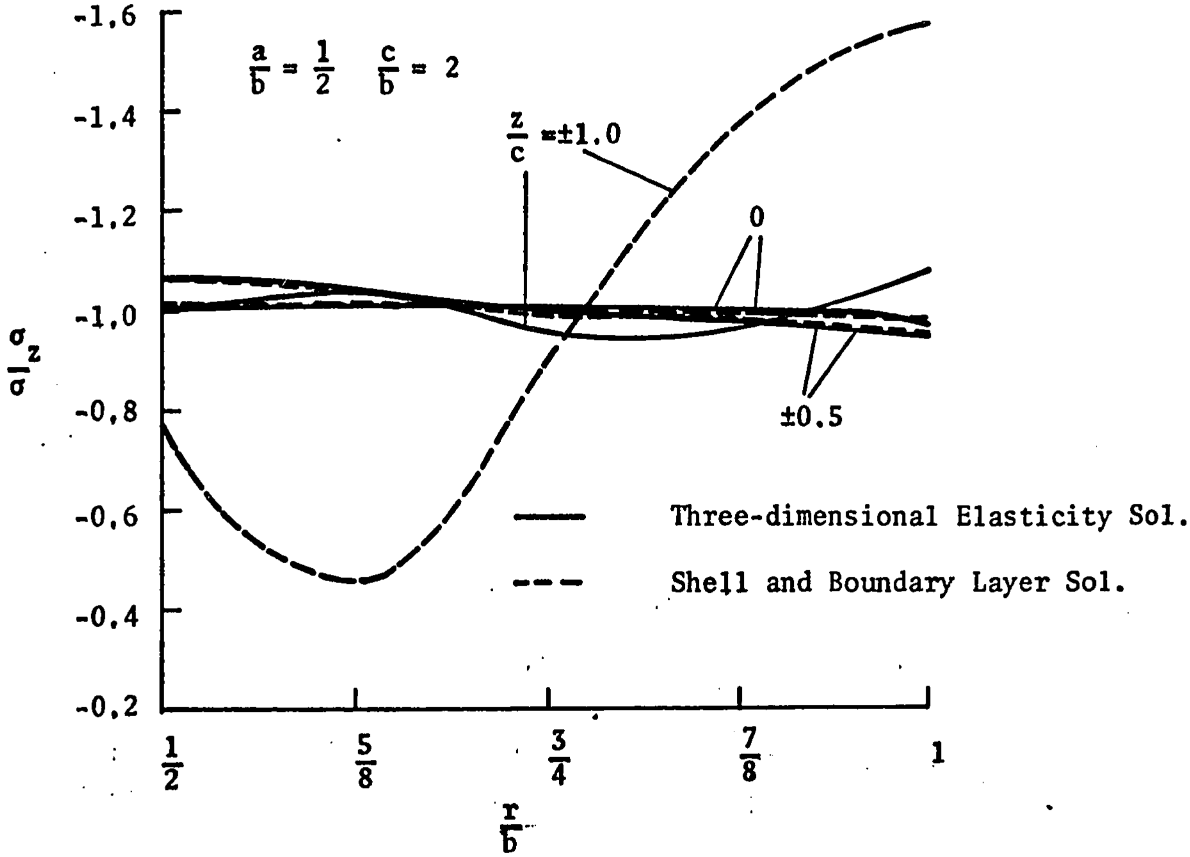


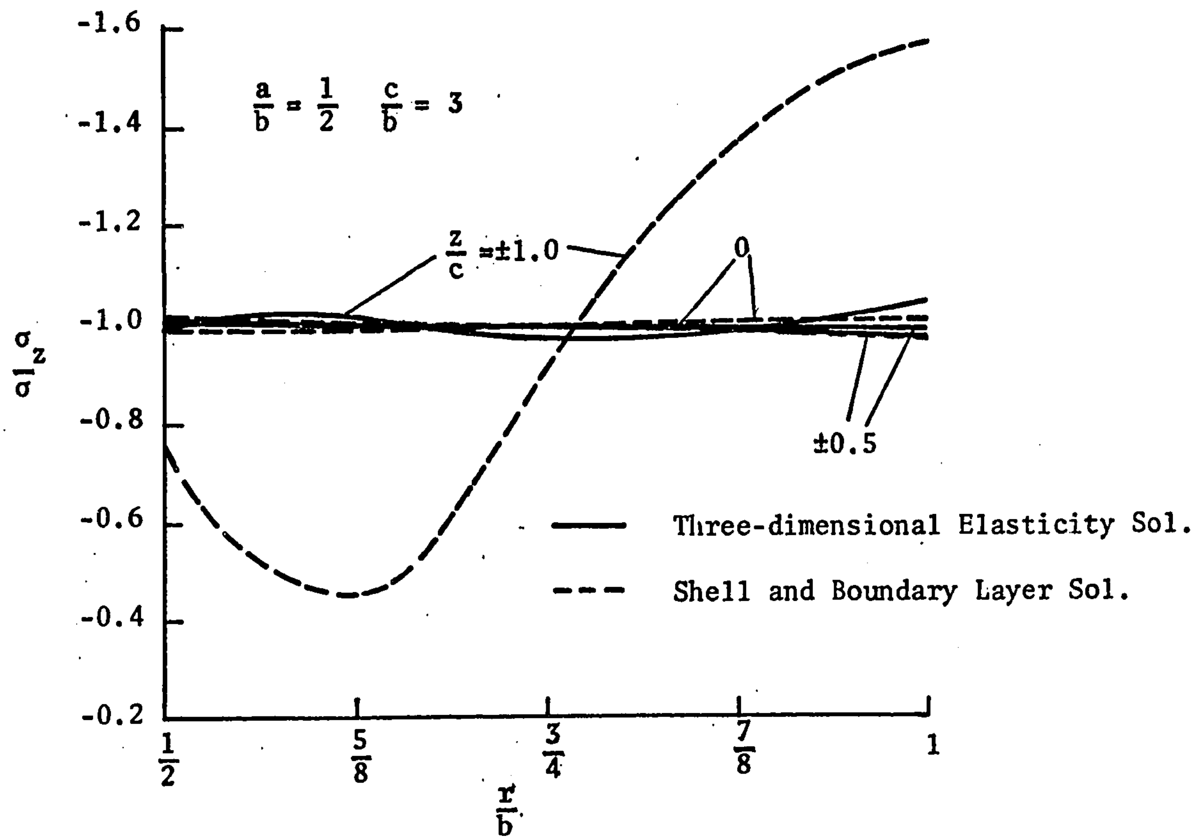
FIGS. 10,11 : DISTRIBUTION OF LONGITUDINAL STRESS



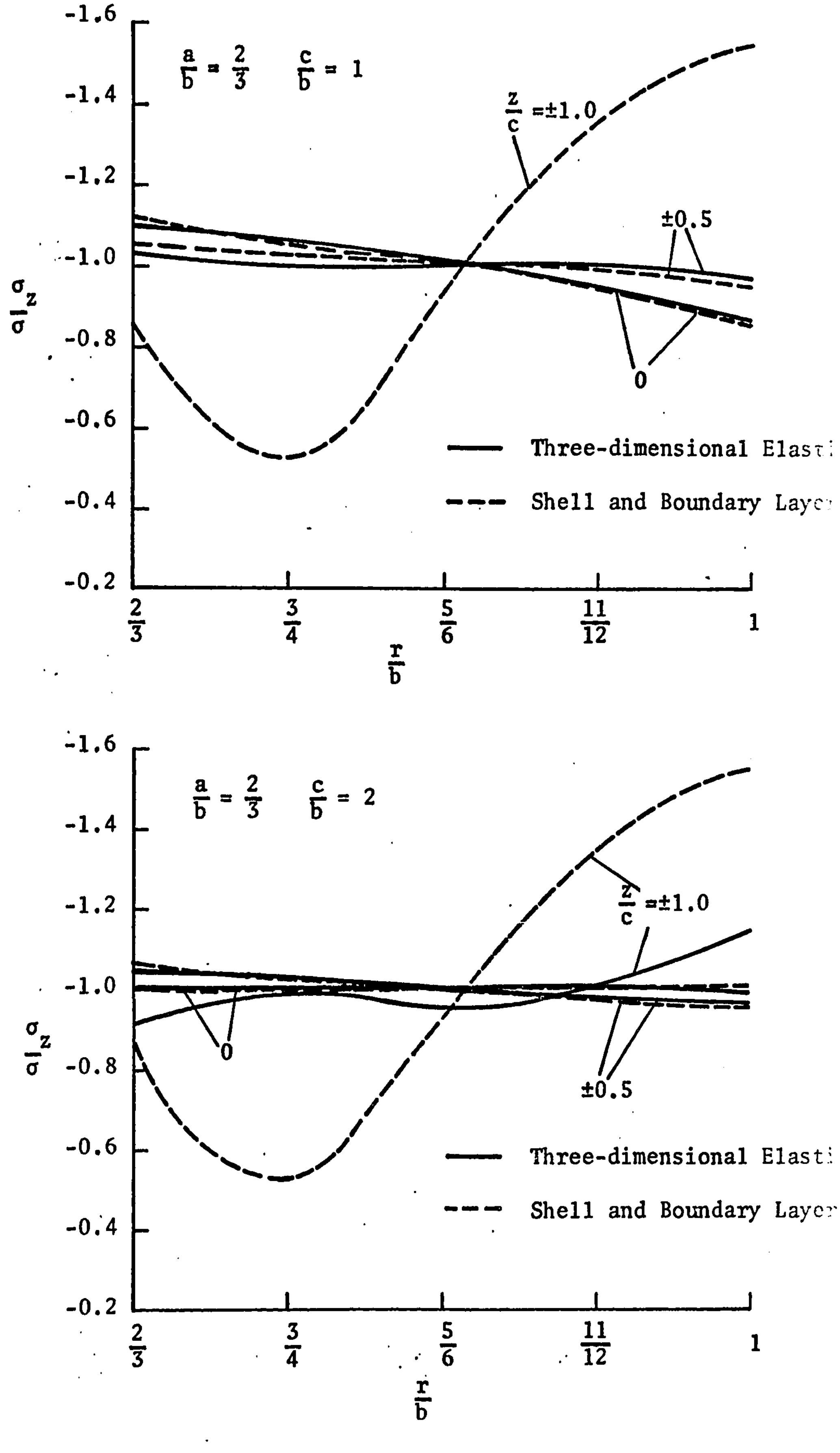


FIGS. 12,13: DISTRIBUTION OF LONGITUDINAL STRESS

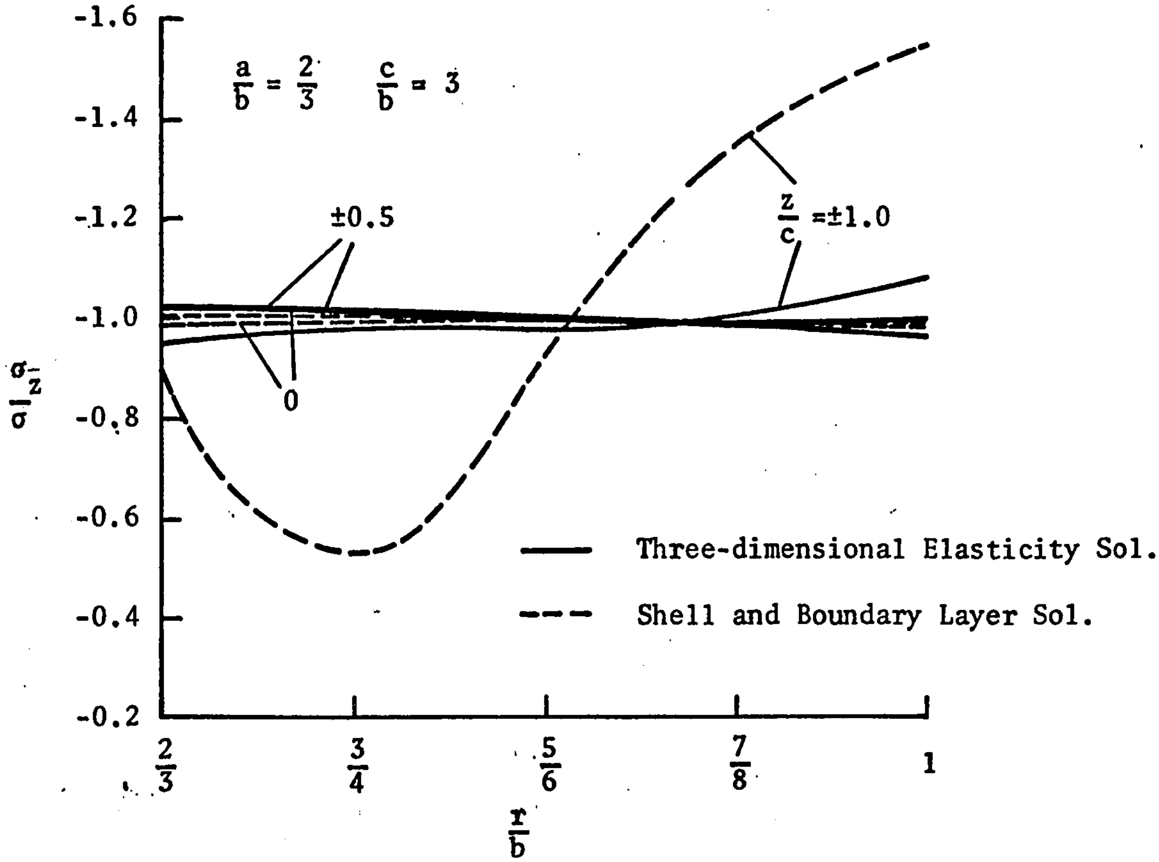


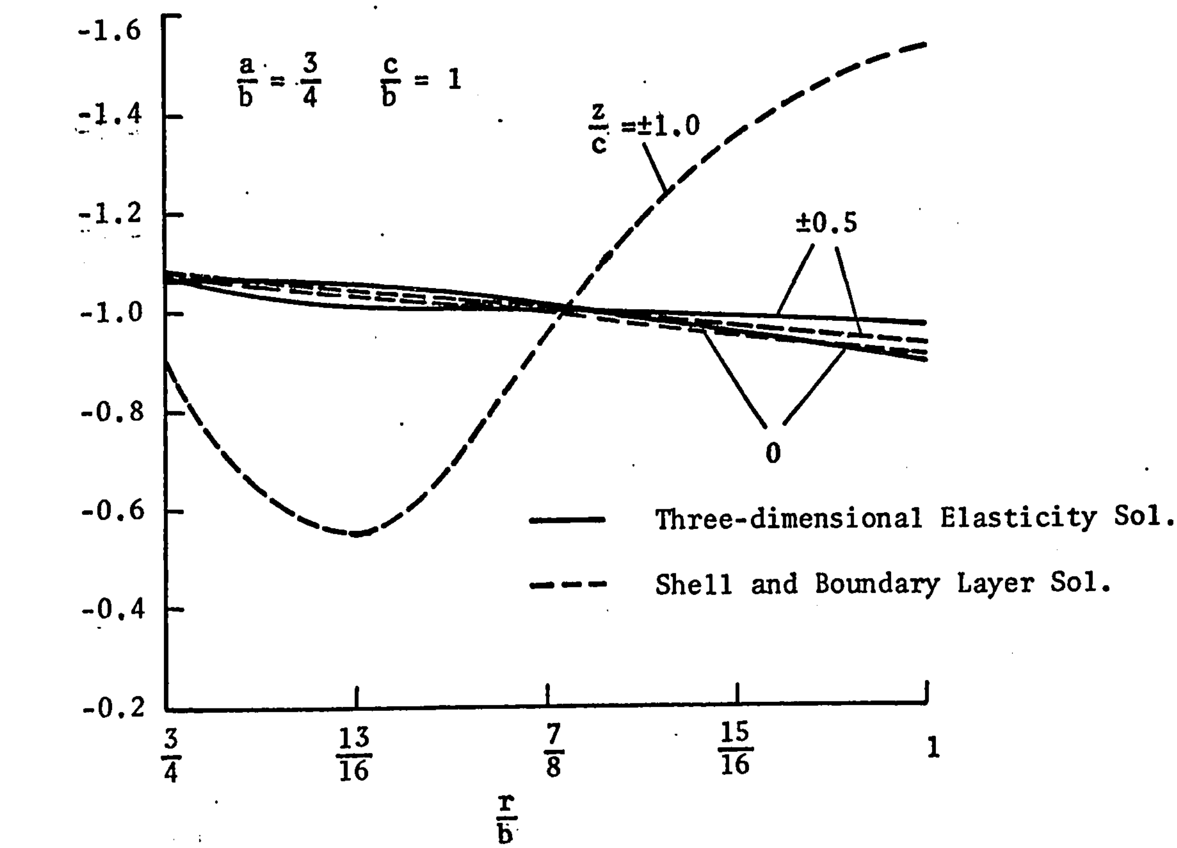


FIGS. 14,15: DISTRIBUTION OF LONGITUDINAL STRESS

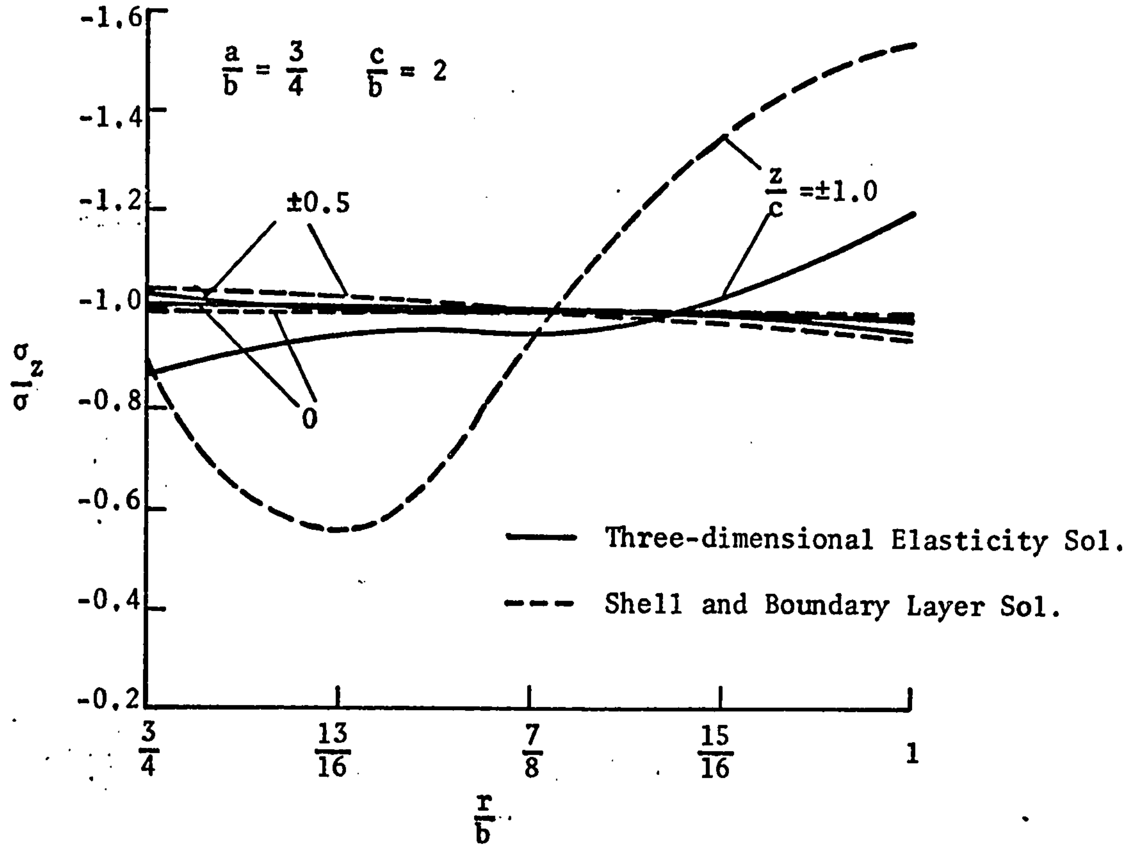


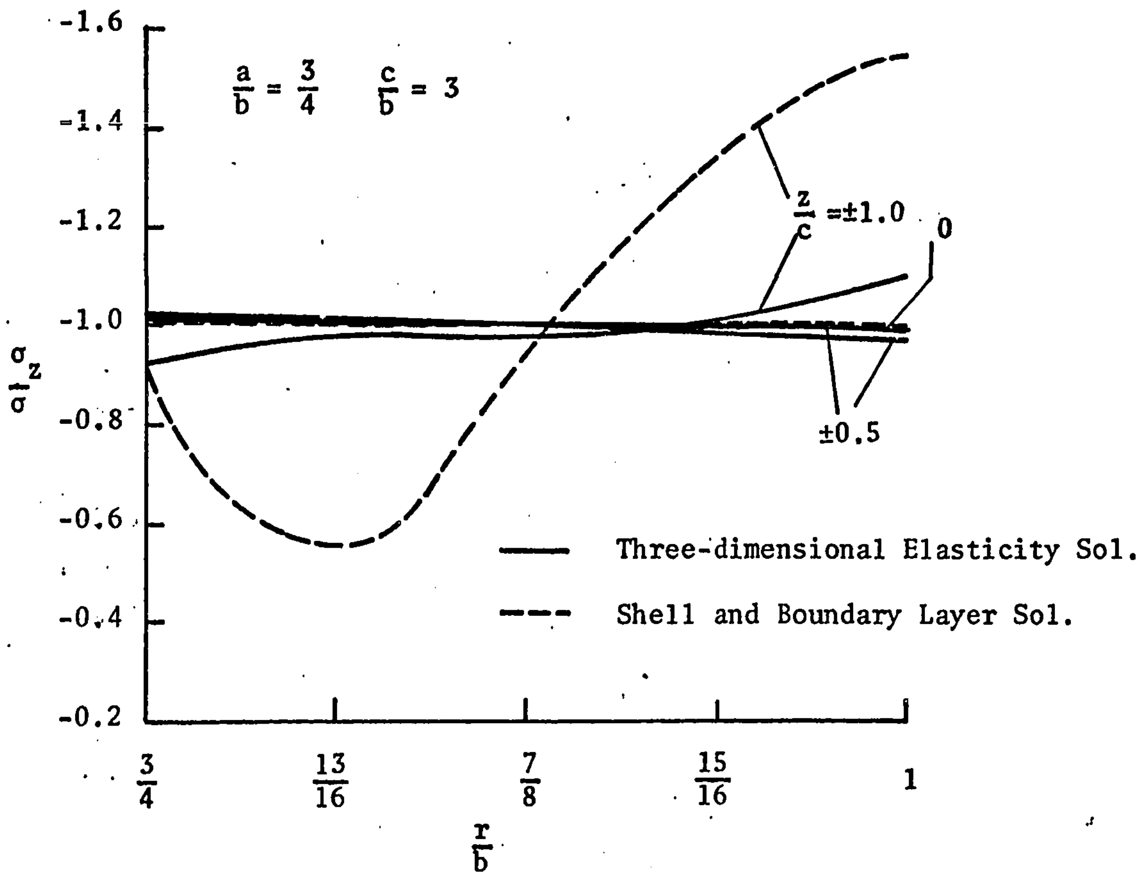
FIGS. 16,17: DISTRIBUTION OF LONGITUDINAL STRESS



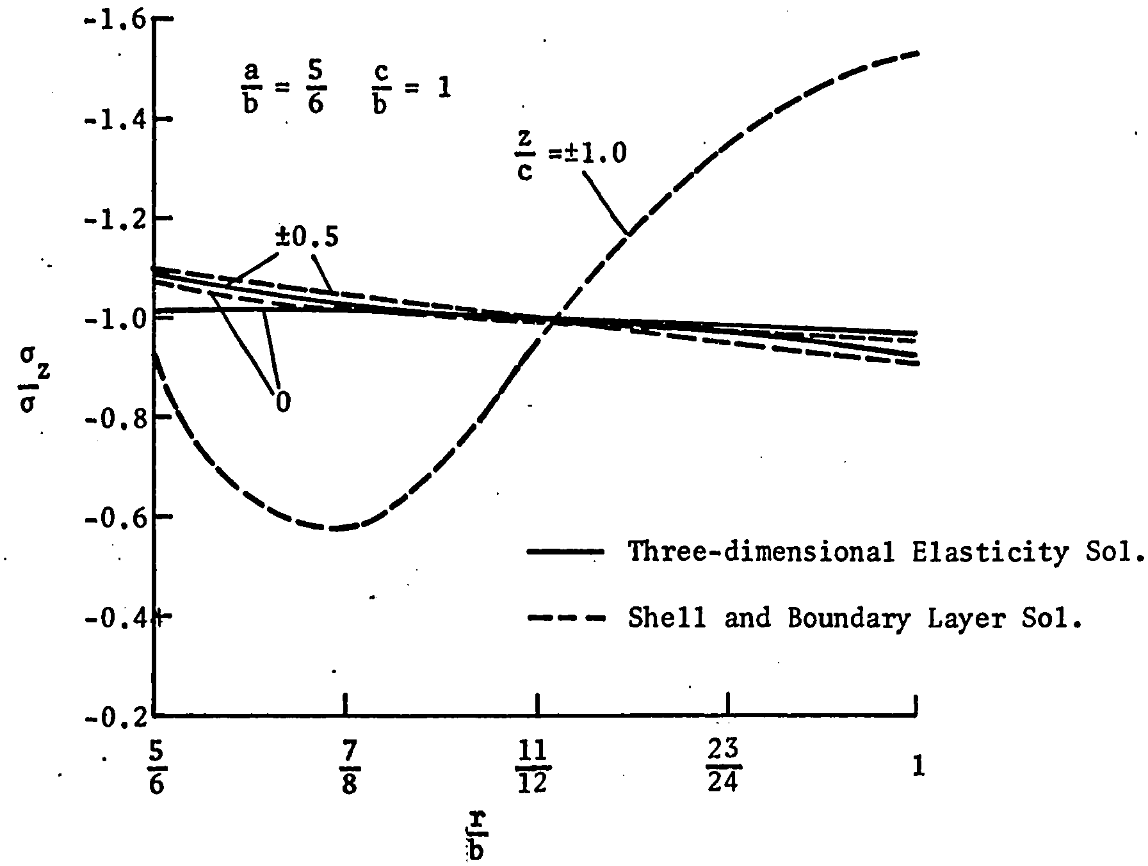


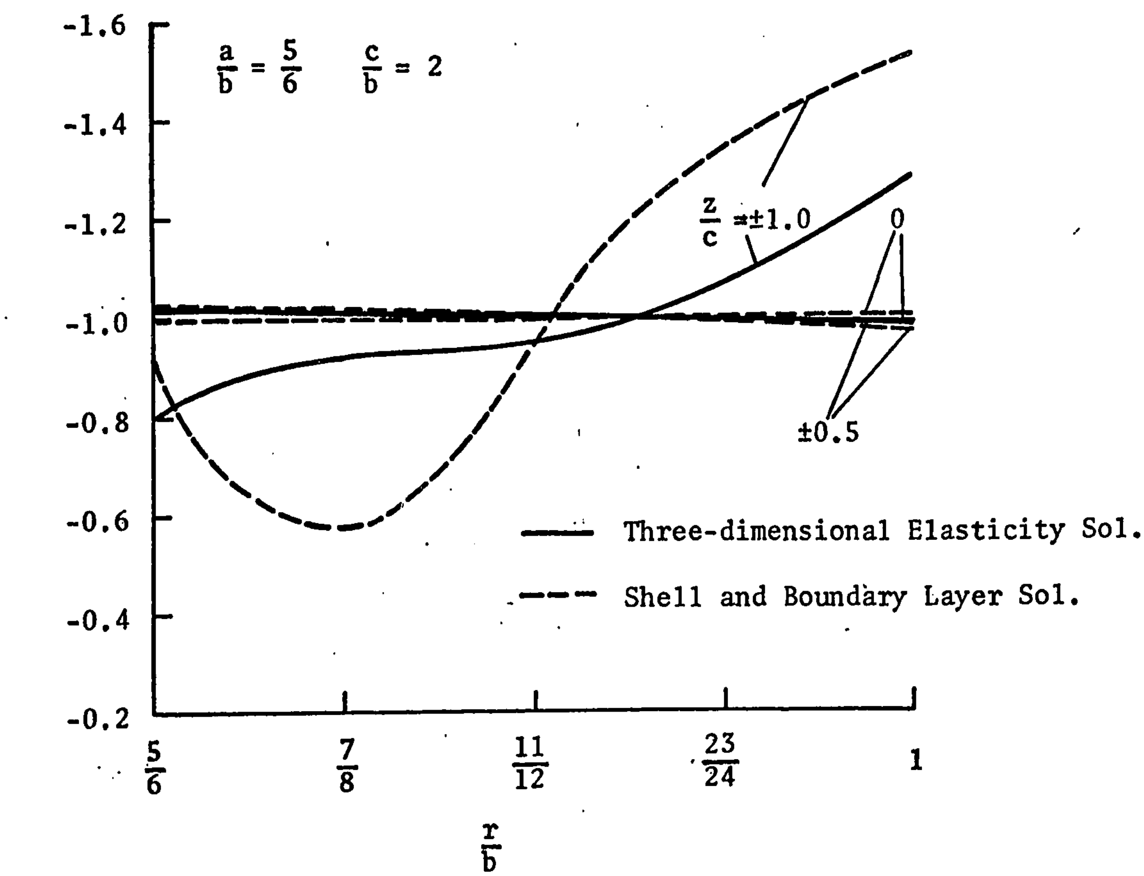
FIGS. 18, 19: DISTRIBUTION OF LONGITUDINAL STRESS



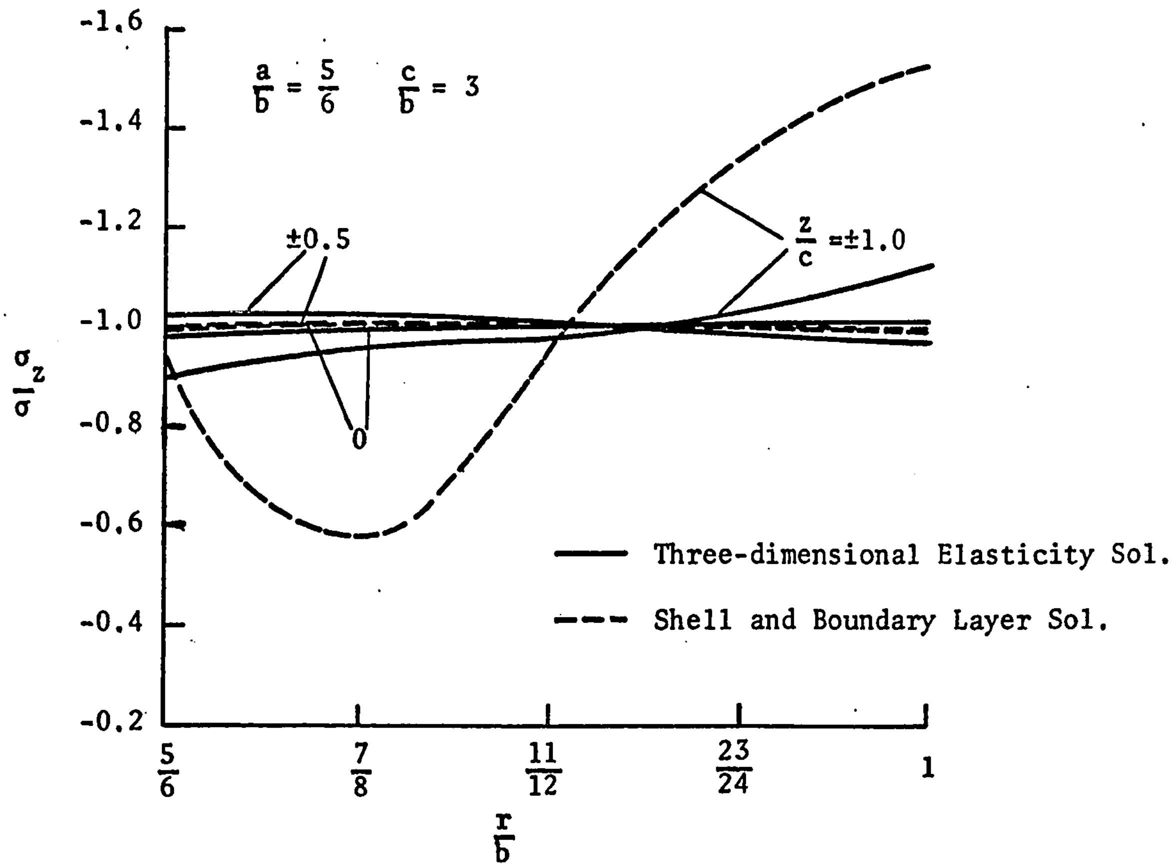


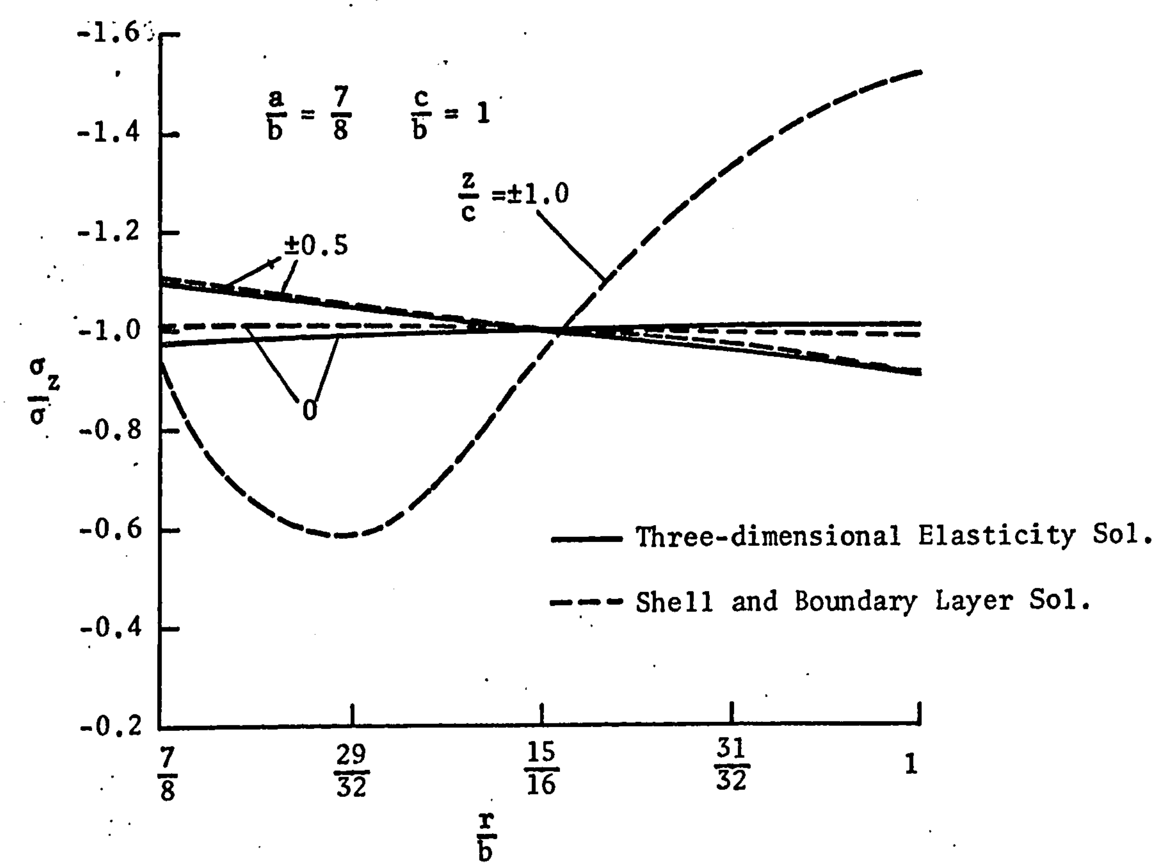
FIGS.20,21: DISTRIBUTION OF LONGITUDINAL STRESS



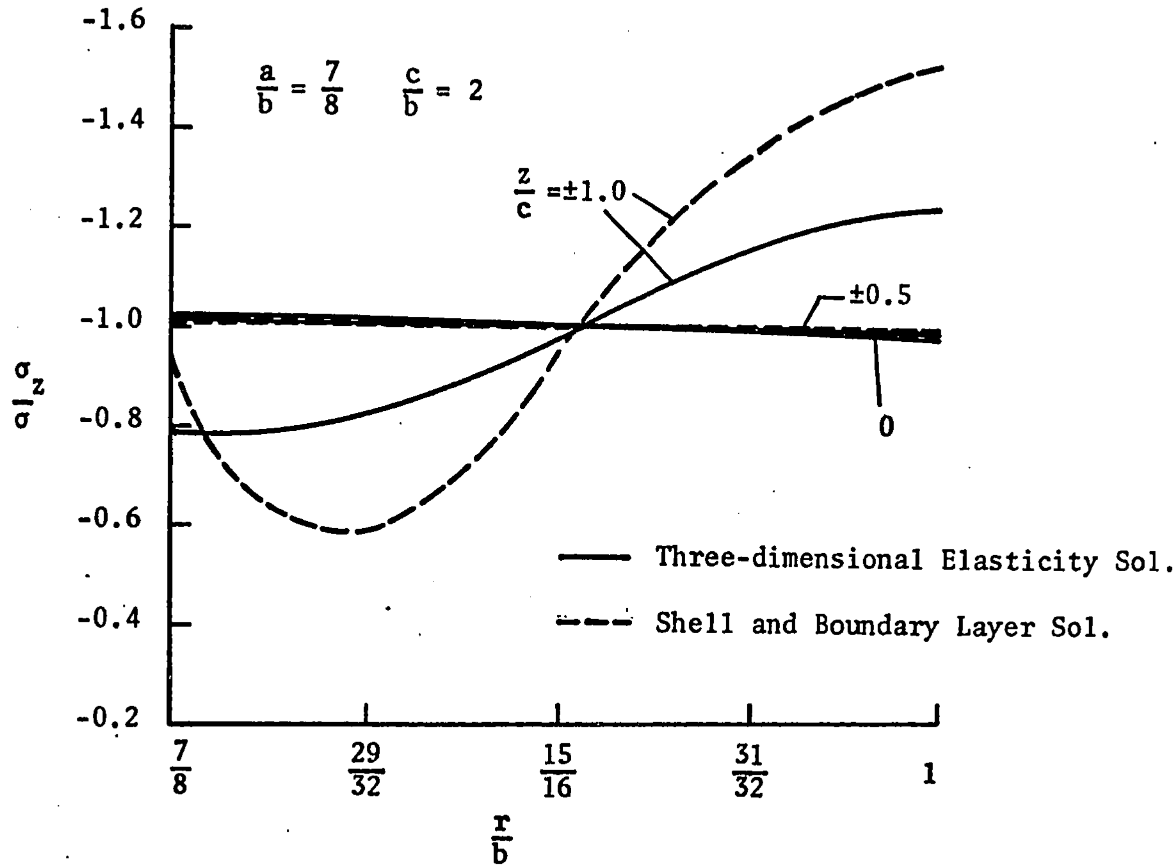


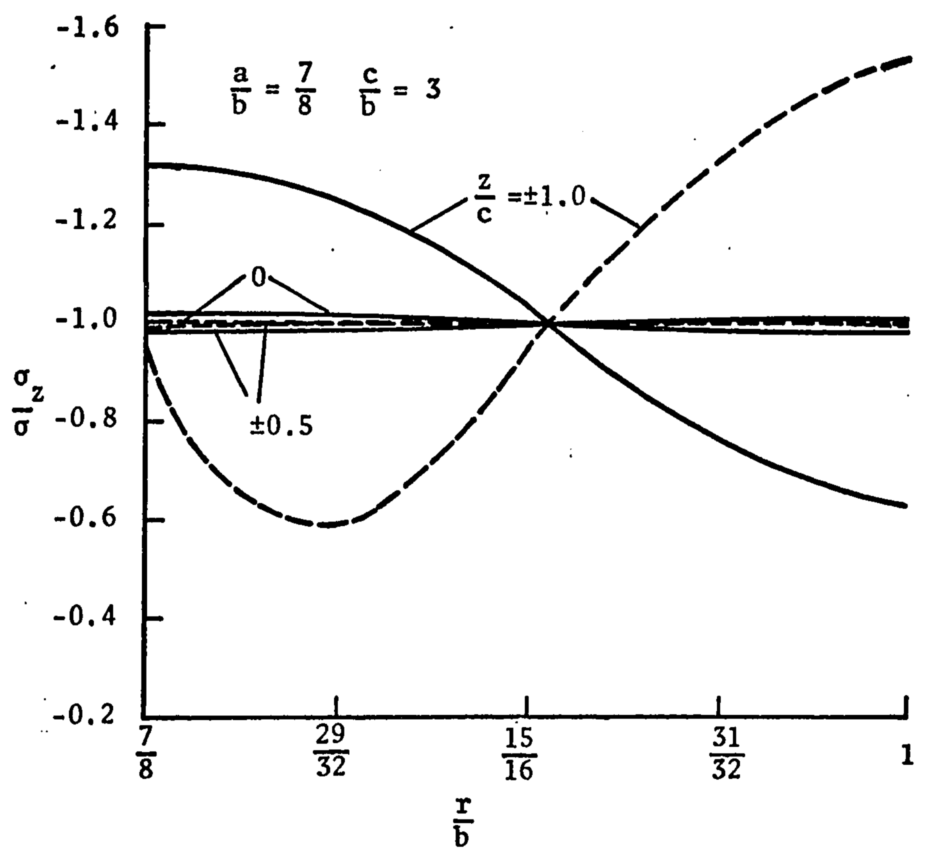
FIGS. 22,23: DISTRIBUTION OF LONGITUDINAL STRESS



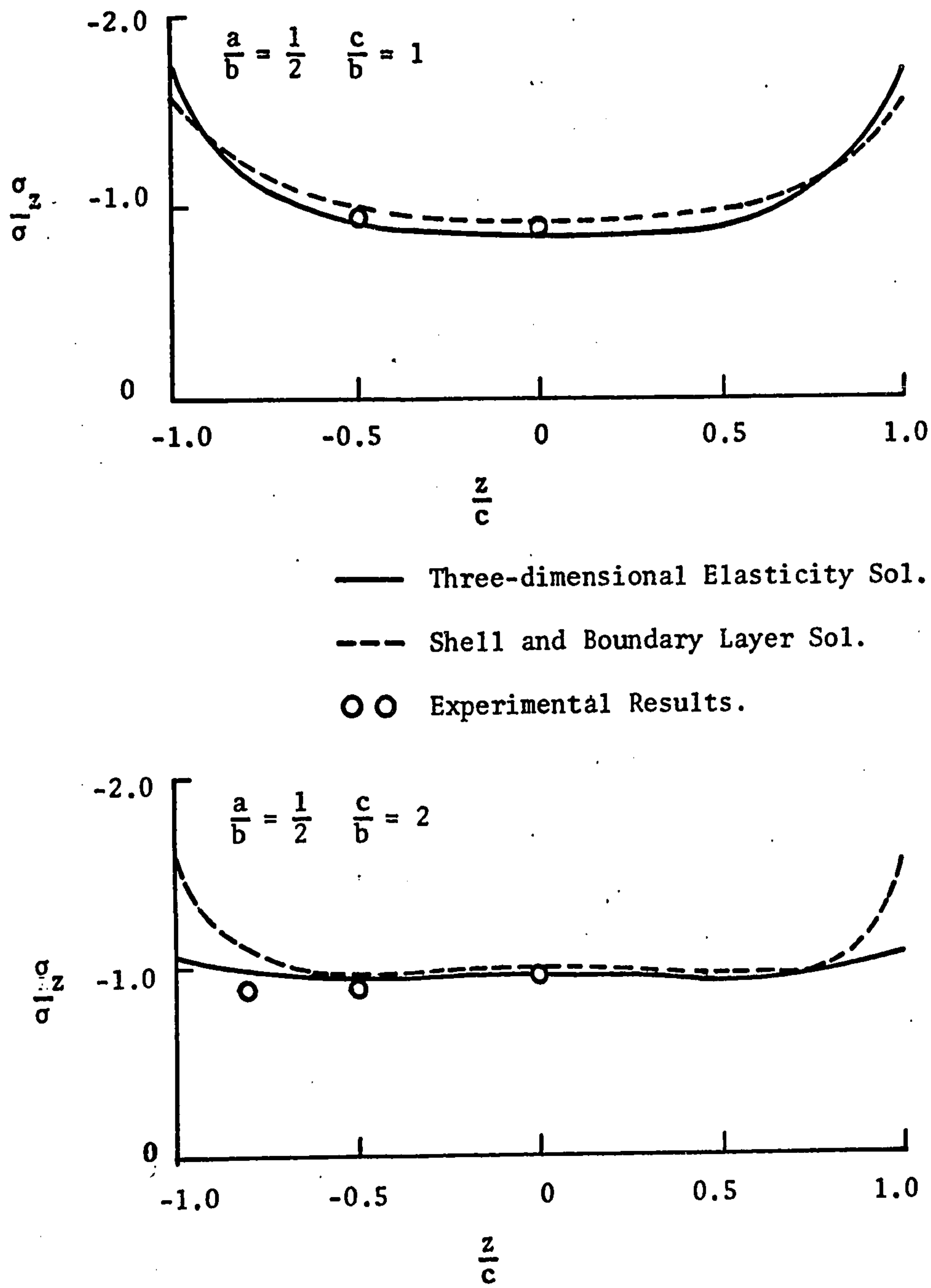


FIGS. 24,25: DISTRIBUTION OF LONGITUDINAL STRESS

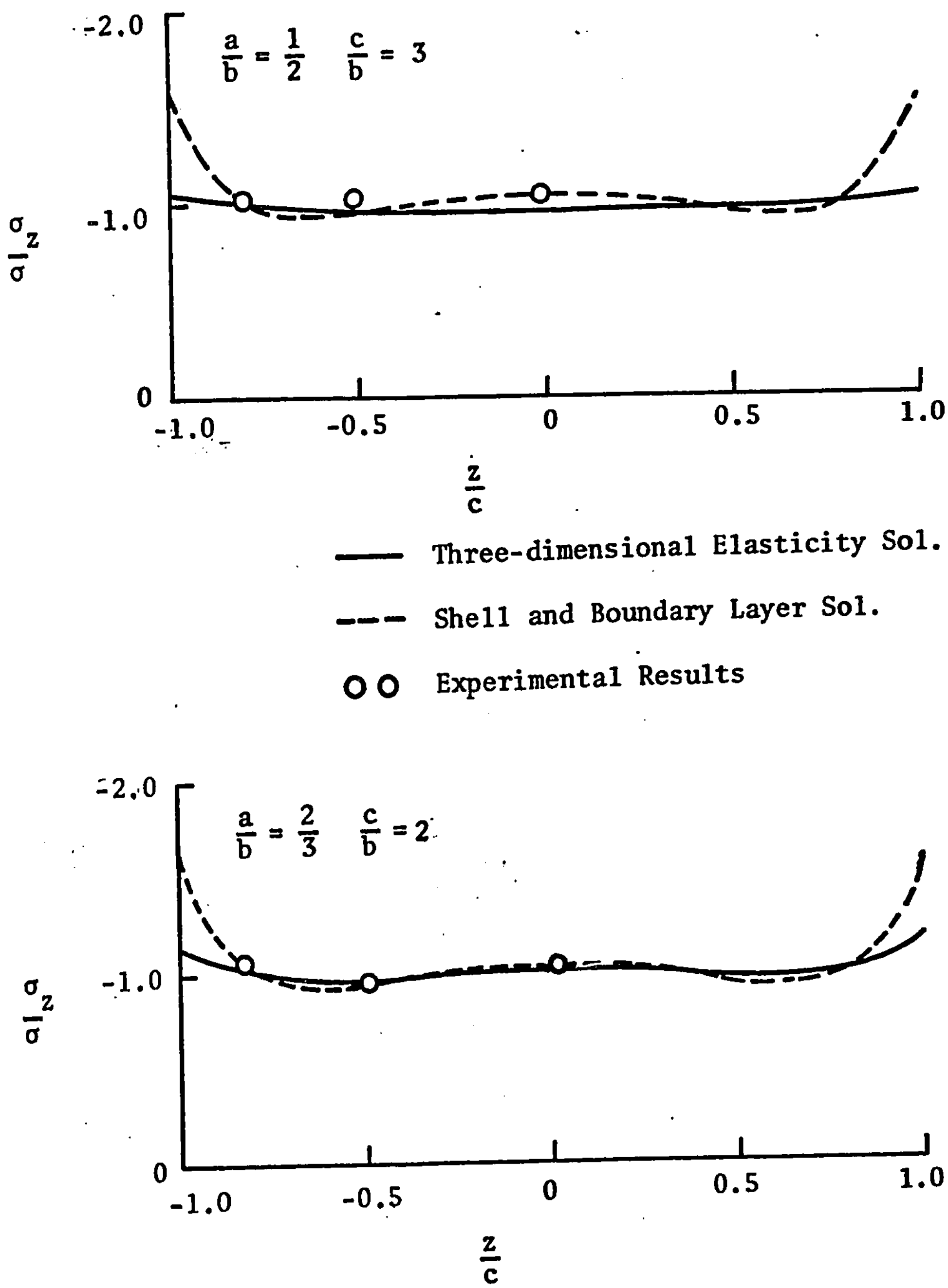




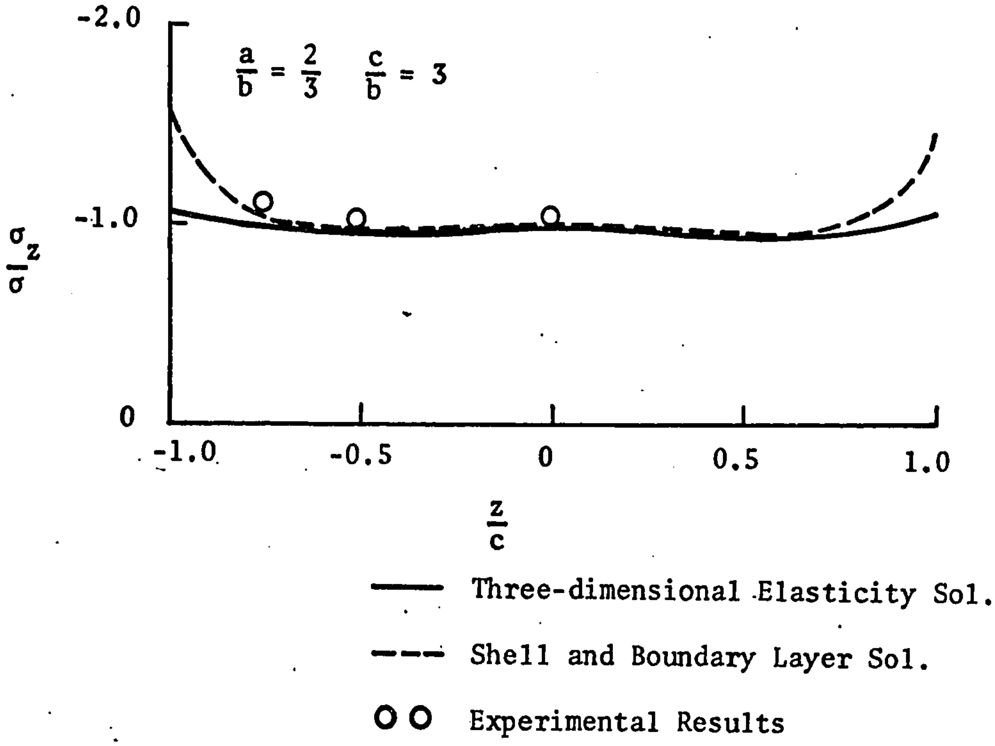
FIGS.26,27: DISTRIBUTION OF LONGITUDINAL STRESS

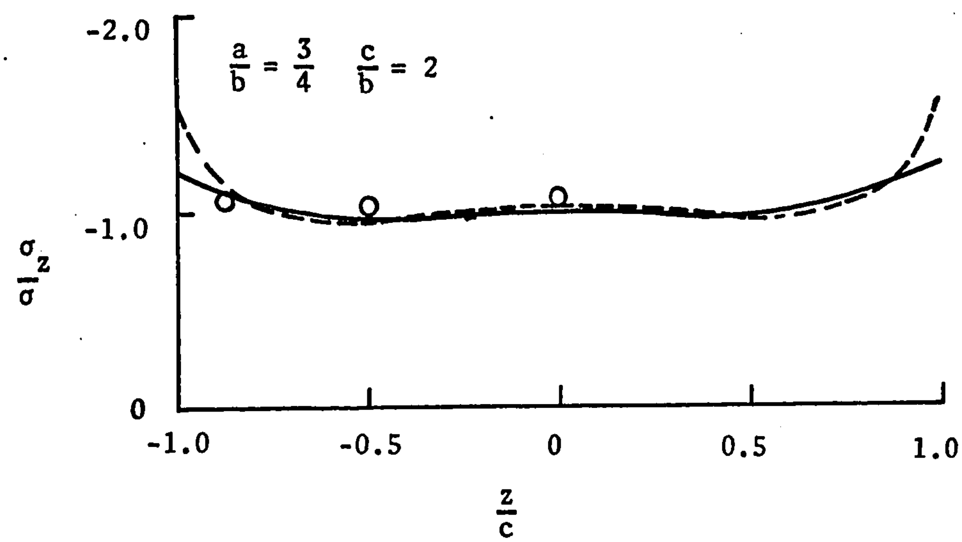


FIGS. 28,29: DISTRIBUTION OF LONGITUDINAL STRESS ALONG THE OUTER SURFACE



FIGS. 30,31: DISTRIBUTION OF LONGITUDINAL STRESS ALONG THE OUTER SURFACE





FIGS. 32,33: DISTRIBUTION OF LONGITUDINAL STRESS ALONG THE OUTER SURFACE

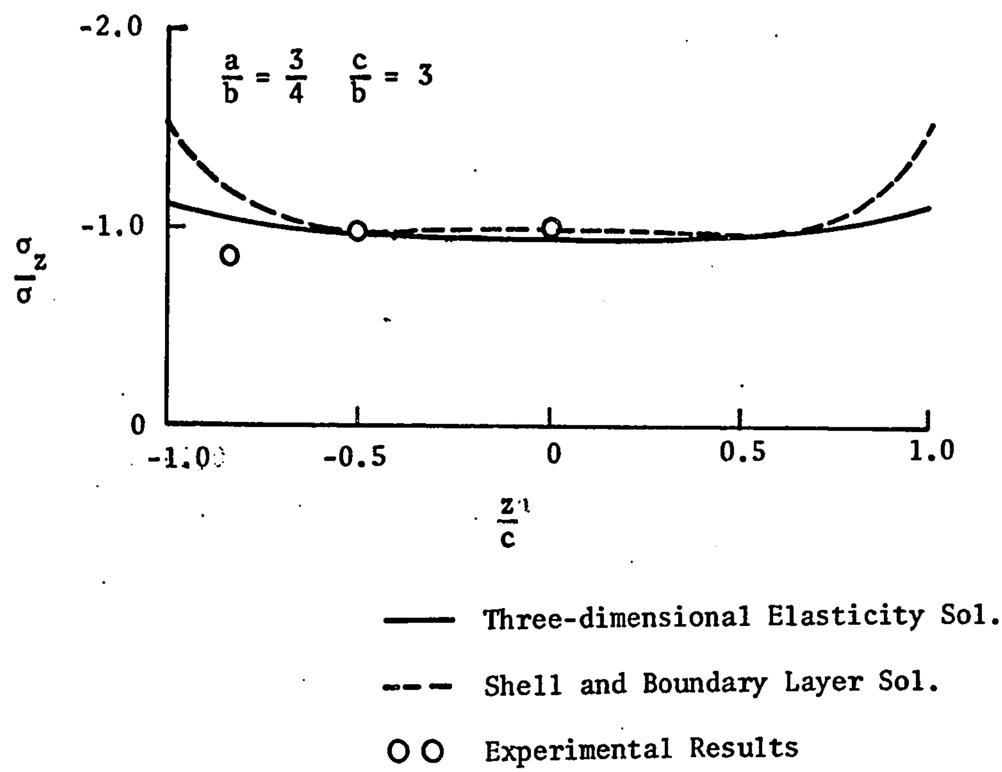
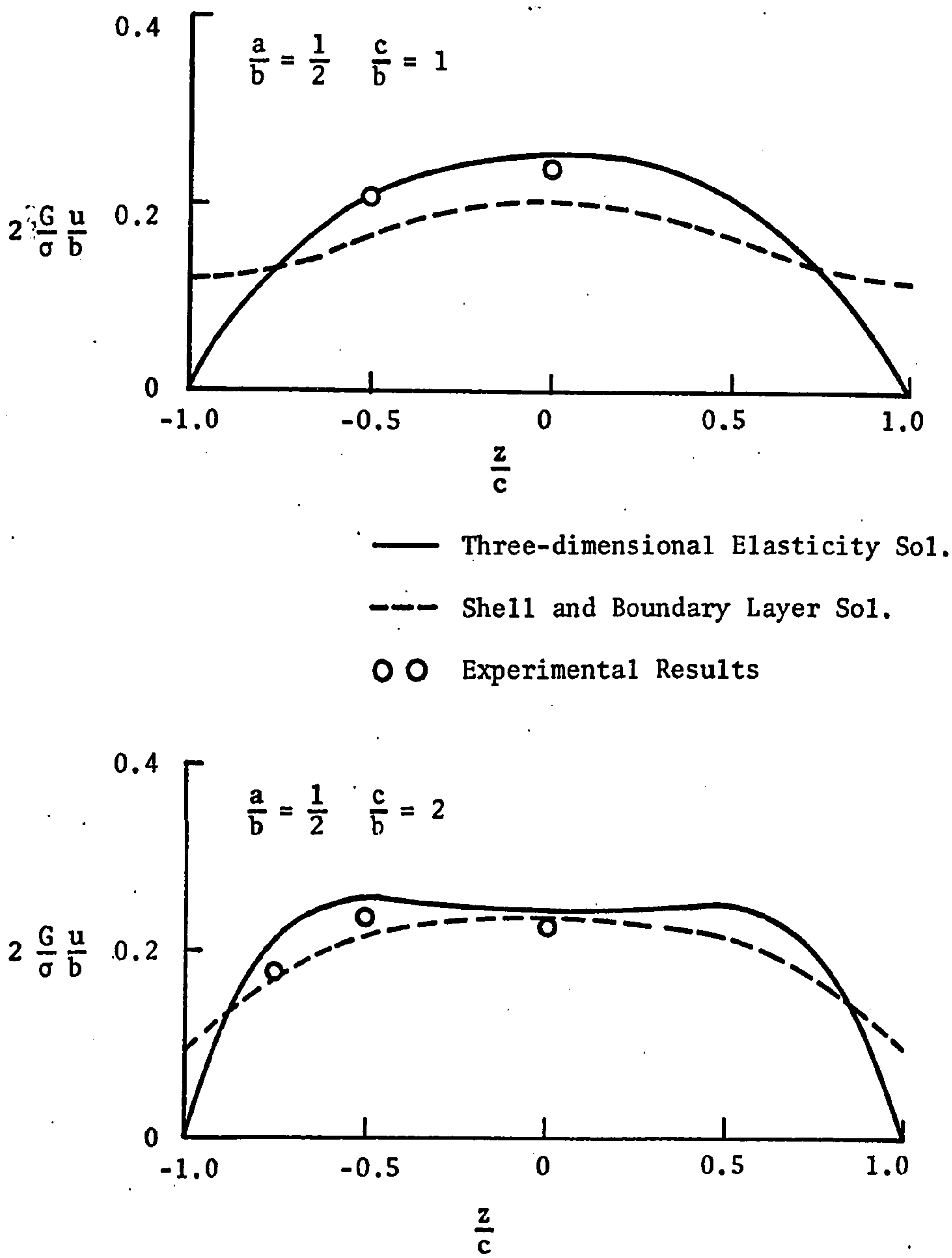
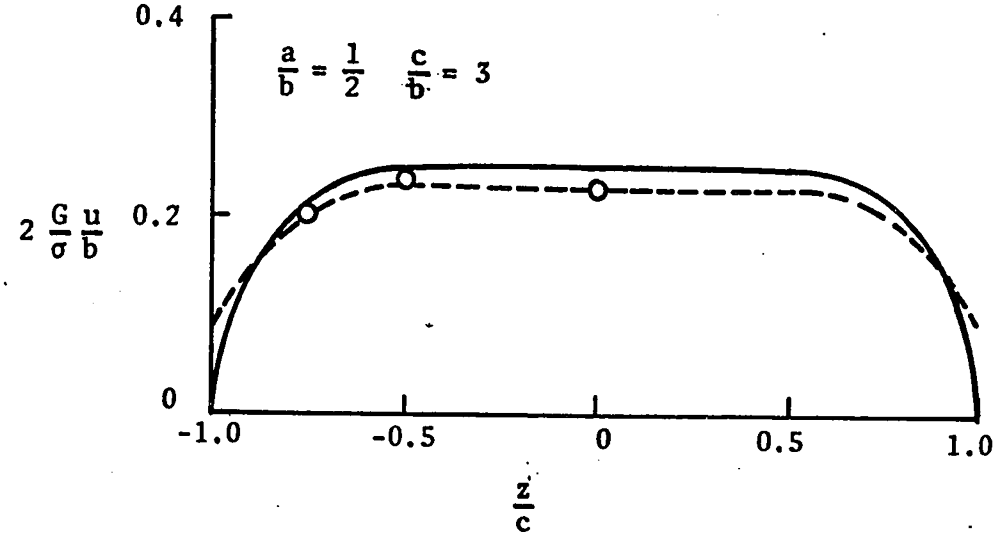


FIG. 34: DISTRIBUTION OF LONGITUDINAL STRESS ALONG THE OUTER SURFACE



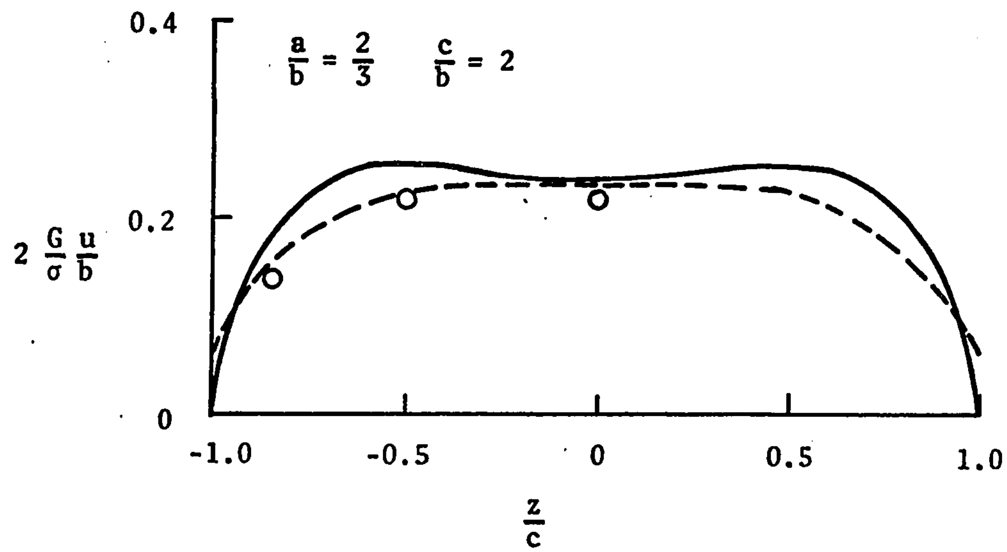
FIGS. 35,36: VARIATION OF RADIAL DISPLACEMENT ALONG THE OUTER SURFACE



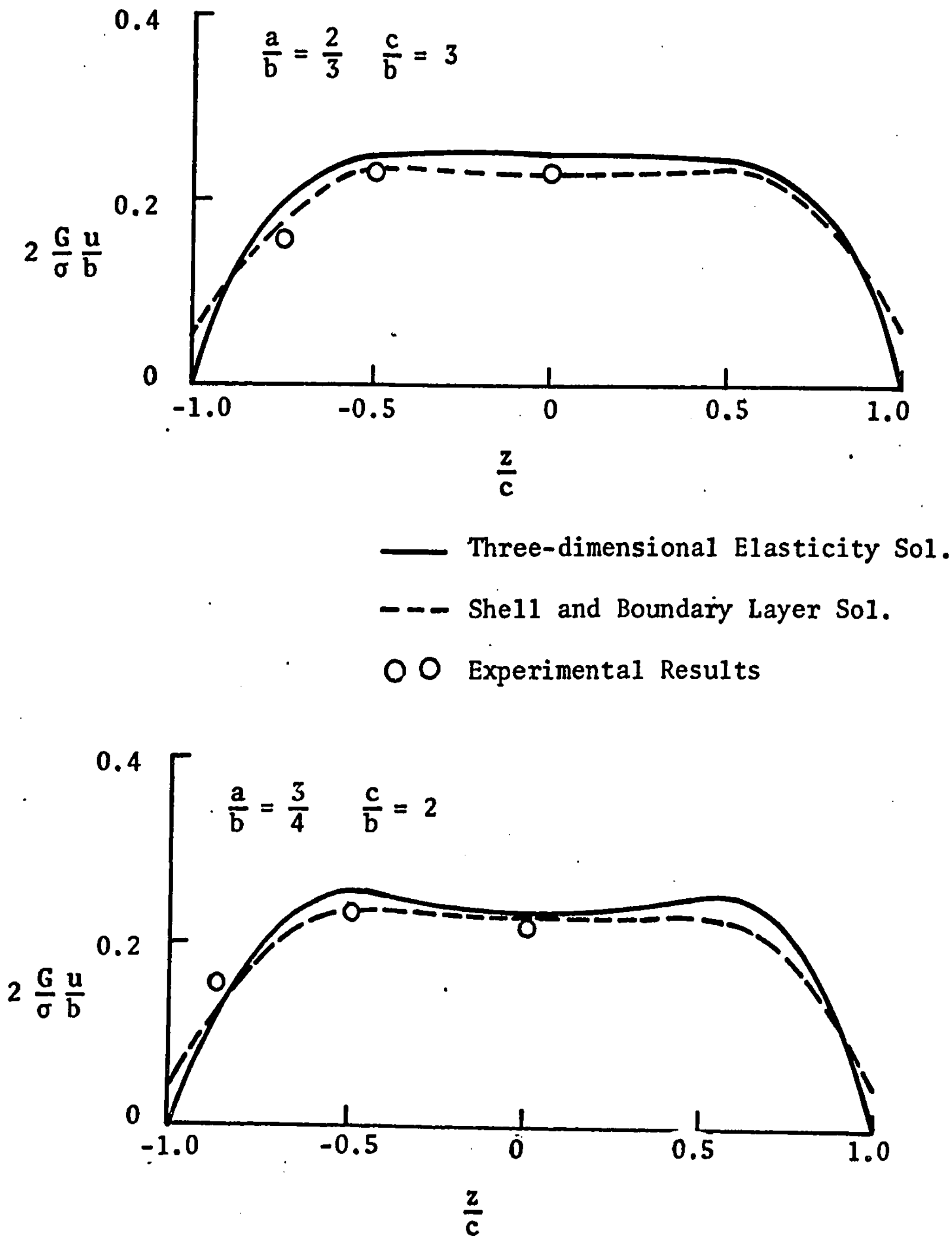
Three-dimensional Elasticity Sol.

Shell and Boundary Layer Sol.

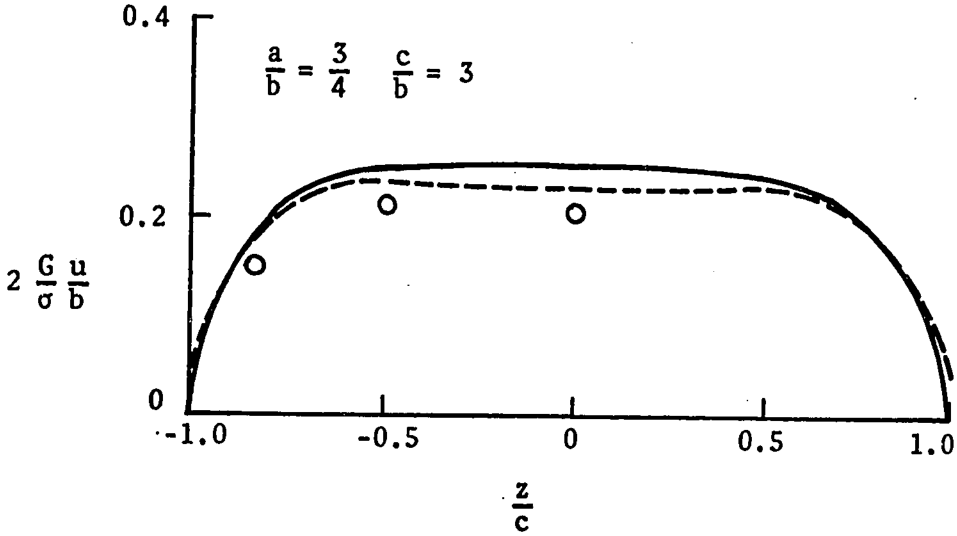
Experimental Results



FIGS. 37,38: VARIATION OF RADIAL DISPLACEMENT ALONG THE OUTER SURFACE



FIGS. 39,40: VARIATION OF RADIAL DISPLACEMENT ALONG THE OUTER SURFACE



——— Three-dimensional Elasticity Sol.

– – – Shell and Boundary Layer Sol.

O O Experimental RESULTS

FIG. 41: VARIATION OF RADIAL DISPLACEMENT
ALONG THE OUTER SURFACE

NUMERICAL RESULTS FOR APPROXIMATE ELASTICITY SOLUTION

A/B= 0.333   C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G}{\sigma}\frac{u}{b}$	$2\frac{G}{\sigma}\frac{w}{c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.333	-1.000	0.076923	0.786080	0.000000	0.000000	-1.000000	-0.000000
0.500	-1.000	0.115385	0.784374	0.000000	0.000000	-1.000000	0.002348
0.667	-1.000	0.153846	0.782165	0.000000	0.000000	-1.000000	0.003181
0.833	-1.000	0.192308	0.779045	0.000000	0.000000	-1.000000	0.002533
1.000	-1.000	0.230769	0.774537	0.000000	0.000000	-1.000000	0.000000
0.333	-0.500	0.081247	0.396530	0.028284	0.022323	-1.009055	-0.000000
0.500	-0.500	0.122604	0.395323	0.027061	0.024816	-1.006242	0.001660
0.667	-0.500	0.163716	0.393761	0.027228	0.026526	-1.002508	0.002250
0.833	-0.500	0.204647	0.391555	0.027780	0.028292	-0.997320	0.001791
1.000	-0.500	0.245340	0.388367	0.028284	0.030344	-0.990053	0.000000
0.333	0.0	0.083038	0.0	0.040000	0.031570	-1.012806	0.0
0.500	0.0	0.125595	0.0	0.038270	0.035095	-1.008828	0.0
0.667	0.0	0.167805	0.0	0.038506	0.037514	-1.003547	0.0
0.833	0.0	0.209758	0.0	0.039287	0.040011	-0.996211	0.0
1.000	0.0	0.251376	0.0	0.040000	0.042913	-0.985932	0.0
0.333	0.500	0.081247	-0.396530	0.028284	0.022323	-1.009055	0.000000
0.500	0.500	0.122604	-0.395323	0.027061	0.024816	-1.006242	-0.001660
0.667	0.500	0.163716	-0.393761	0.027228	0.026526	-1.002508	-0.002250
0.833	0.500	0.204647	-0.391555	0.027780	0.028292	-0.997320	-0.001791
1.000	0.500	0.245340	-0.388367	0.028284	0.030344	-0.990053	-0.000000
0.333	1.000	0.076923	-0.786080	0.000000	0.000000	-1.000000	0.000000
0.500	1.000	0.115385	-0.784374	0.000000	0.000000	-1.000000	-0.002348
0.667	1.000	0.153846	-0.782165	0.000000	0.000000	-1.000000	-0.003181
0.833	1.000	0.192308	-0.779045	0.000000	0.000000	-1.000000	-0.002533
1.000	1.000	0.230769	-0.774537	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.333 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.333	-1.000	0.076923	0.782713	0.000000	0.000000	-1.000000	0.000000
0.500	-1.000	0.115385	0.782160	0.000000	0.000000	-1.000000	0.000366
0.667	-1.000	0.153846	0.781406	0.000000	0.000000	-1.000000	0.000489
0.833	-1.000	0.192308	0.780418	0.000000	0.000000	-1.000000	0.000379
1.000	-1.000	0.230769	0.779161	0.000000	0.000000	-1.000000	-0.000000
0.333	-0.500	0.081942	0.394149	0.028284	0.027155	-1.002818	0.000000
0.500	-0.500	0.123044	0.393758	0.028042	0.027703	-1.001932	0.000259
0.667	-0.500	0.164077	0.393225	0.028056	0.028117	-1.000718	0.000346
0.833	-0.500	0.205049	0.392526	0.028155	0.028555	-0.999132	0.000268
1.000	-0.500	0.245949	0.391637	0.028284	0.029058	-0.997129	-0.000000
0.333	0.0	0.084021	0.0	0.040000	0.038404	-1.003985	0.0
0.500	0.0	0.126217	0.0	0.039658	0.039178	-1.002732	0.0
0.667	0.0	0.168315	0.0	0.039677	0.039763	-1.001015	0.0
0.833	0.0	0.210327	0.0	0.039817	0.040383	-0.998773	0.0
1.000	0.0	0.252237	0.0	0.040000	0.041094	-0.995940	0.0
0.333	0.500	0.081942	-0.394149	0.028284	0.027155	-1.002818	-0.000000
0.500	0.500	0.123044	-0.393758	0.028042	0.027703	-1.001932	-0.000259
0.667	0.500	0.164077	-0.393225	0.028056	0.028117	-1.000718	-0.000346
0.833	0.500	0.205049	-0.392526	0.028155	0.028555	-0.999132	-0.000268
1.000	0.500	0.245949	-0.391637	0.028284	0.029058	-0.997129	0.000000
0.333	1.000	0.076923	-0.782713	0.000000	0.000000	-1.000000	-0.000000
0.500	1.000	0.115385	-0.782160	0.000000	0.000000	-1.000000	-0.000366
0.667	1.000	0.153846	-0.781406	0.000000	0.000000	-1.000000	-0.000489
0.833	1.000	0.192308	-0.780418	0.000000	0.000000	-1.000000	-0.000379
1.000	1.000	0.230769	-0.779161	0.000000	0.000000	-1.000000	0.000000

A/B= 0.333 C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G u}{\sigma b}$	$2\frac{G w}{\sigma c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.333	-1.000	0.076923	0.781790	0.000000	0.000000	-1.000000	-0.000000
0.500	-1.000	0.115385	0.781535	0.000000	0.000000	-1.000000	0.000112
0.667	-1.000	0.153846	0.781183	0.000000	0.000000	-1.000000	0.000150
0.833	-1.000	0.192308	0.780727	0.000000	0.000000	-1.000000	0.000116
1.000	-1.000	0.230769	0.780159	0.000000	0.000000	-1.000000	-0.000000
0.333	-0.500	0.081988	0.393496	0.028284	0.027834	-1.001294	-0.000000
0.500	-0.500	0.123034	0.393316	0.028184	0.028064	-1.000888	0.000079
0.667	-0.500	0.164049	0.393067	0.028187	0.028243	-1.000326	0.000106
0.833	-0.500	0.205035	0.392744	0.028227	0.028435	-0.999598	0.000082
1.000	-0.500	0.245989	0.392343	0.028284	0.028655	-0.998696	-0.000000
0.333	0.0	0.084086	0.0	0.040000	0.039364	-1.001830	0.0
0.500	0.0	0.126202	0.0	0.039859	0.039689	-1.001256	0.0
0.667	0.0	0.168275	0.0	0.039863	0.039942	-1.000461	0.0
0.833	0.0	0.210307	0.0	0.039919	0.040213	-0.999432	0.0
1.000	0.0	0.252293	0.0	0.040000	0.040524	-0.998155	0.0
0.333	0.500	0.081988	-0.393496	0.028284	0.027834	-1.001294	0.000000
0.500	0.500	0.123034	-0.393316	0.028184	0.028064	-1.000888	-0.000079
0.667	0.500	0.164049	-0.393067	0.028187	0.028243	-1.000326	-0.000106
0.833	0.500	0.205035	-0.392744	0.028227	0.028435	-0.999598	-0.000082
1.000	0.500	0.245989	-0.392343	0.028284	0.028655	-0.998696	0.000000
0.333	1.000	0.076923	-0.781790	0.000000	0.000000	-1.000000	0.000000
0.500	1.000	0.115385	-0.781535	0.000000	0.000000	-1.000000	-0.000112
0.667	1.000	0.153846	-0.781183	0.000000	0.000000	-1.000000	-0.000150
0.833	1.000	0.192308	-0.780727	0.000000	0.000000	-1.000000	-0.000116
1.000	1.000	0.230769	-0.780159	0.000000	0.000000	-1.000000	0.000000

A/B= 0.500 C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G u}{\sigma b}$	$2\frac{G w}{\sigma c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.500	-1.000	0.115385	0.785890	0.000000	0.000000	-1.000000	-0.000000
0.625	-1.000	0.144231	0.783914	0.000000	0.000000	-1.000000	0.001699
0.750	-1.000	0.173077	0.781641	0.000000	0.000000	-1.000000	0.002273
0.875	-1.000	0.201923	0.778840	0.000000	0.000000	-1.000000	0.001752
1.000	-1.000	0.230769	0.775265	0.000000	0.000000	-1.000000	-0.000000
0.500	-0.500	0.122257	0.396395	0.028284	0.023164	-1.008422	-0.000000
0.625	-0.500	0.153272	0.394998	0.027695	0.025049	-1.005230	0.001201
0.750	-0.500	0.184120	0.393391	0.027725	0.026632	-1.001500	0.001607
0.875	-0.500	0.214824	0.391410	0.027998	0.028198	-0.996930	0.001239
1.000	-0.500	0.245364	0.388883	0.028284	0.029875	-0.991199	-0.000000
0.500	0.0	0.125104	0.0	0.040000	0.032758	-1.011911	0.0
0.625	0.0	0.157017	0.0	0.039167	0.035425	-1.007396	0.0
0.750	0.0	0.188694	0.0	0.039209	0.037664	-1.002121	0.0
0.875	0.0	0.220167	0.0	0.039595	0.039878	-0.995658	0.0
1.000	0.0	0.251409	0.0	0.040000	0.042249	-0.987553	0.0
0.500	0.500	0.122257	-0.396395	0.028284	0.023164	-1.008422	0.000000
0.625	0.500	0.153272	-0.394998	0.027695	0.025049	-1.005230	-0.001201
0.750	0.500	0.184120	-0.393391	0.027725	0.026632	-1.001500	-0.001607
0.875	0.500	0.214824	-0.391410	0.027998	0.028198	-0.996930	-0.001239
1.000	0.500	0.245364	-0.388883	0.028284	0.029875	-0.991199	0.000000
0.500	1.000	0.115385	-0.785890	0.000000	0.000000	-1.000000	0.000000
0.625	1.000	0.144231	-0.783914	0.000000	0.000000	-1.000000	-0.001699
0.750	1.000	0.173077	-0.781641	0.000000	0.000000	-1.000000	-0.002273
0.875	1.000	0.201923	-0.778840	0.000000	0.000000	-1.000000	-0.001752
1.000	1.000	0.230769	-0.775265	0.000000	0.000000	-1.000000	0.000000

A/B= 0.500 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.500	-1.000	0.115385	0.782490	0.000000	0.000000	-1.000000	0.000000
0.625	-1.000	0.144231	0.781913	0.000000	0.000000	-1.000000	0.000247
0.750	-1.000	0.173077	0.781222	0.000000	0.000000	-1.000000	0.000328
0.875	-1.000	0.201923	0.780398	0.000000	0.000000	-1.000000	0.000249
1.000	-1.000	0.230769	0.779425	0.000000	0.000000	-1.000000	-0.000000
0.500	-0.500	0.122946	0.393991	0.028284	0.027306	-1.002435	0.000000
0.625	-0.500	0.153765	0.393583	0.028162	0.027735	-1.001516	0.000174
0.750	-0.500	0.184539	0.393094	0.028156	0.028114	-1.000410	0.000232
0.875	-0.500	0.215269	0.392512	0.028206	0.028497	-0.999094	0.000176
1.000	-0.500	0.245954	0.391824	0.028284	0.028905	-0.997546	-0.000000
0.500	0.0	0.126078	0.0	0.040000	0.038617	-1.003444	0.0
0.625	0.0	0.157715	0.0	0.039827	0.039223	-1.002144	0.0
0.750	0.0	0.189286	0.0	0.039818	0.039760	-1.000580	0.0
0.875	0.0	0.220797	0.0	0.039890	0.040301	-0.998719	0.0
1.000	0.0	0.252243	0.0	0.040000	0.040878	-0.996530	0.0
0.500	0.500	0.122946	-0.393991	0.028284	0.027306	-1.002435	-0.000000
0.625	0.500	0.153765	-0.393583	0.028162	0.027735	-1.001516	-0.000174
0.750	0.500	0.184539	-0.393094	0.028156	0.028114	-1.000410	-0.000232
0.875	0.500	0.215269	-0.392512	0.028206	0.028497	-0.999094	-0.000176
1.000	0.500	0.245954	-0.391824	0.028284	0.028905	-0.997546	0.000000
0.500	1.000	0.115385	-0.782490	0.000000	0.000000	-1.000000	-0.000000
0.625	1.000	0.144231	-0.781913	0.000000	0.000000	-1.000000	-0.000247
0.750	1.000	0.173077	-0.781222	0.000000	0.000000	-1.000000	-0.000328
0.875	1.000	0.201923	-0.780398	0.000000	0.000000	-1.000000	-0.000249
1.000	1.000	0.230769	-0.779425	0.000000	0.000000	-1.000000	0.000000

A/B= 0.500   C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.500	-1.000	0.115385	0.781673	0.000000	0.000000	-1.000000	0.000000
0.625	-1.000	0.144231	0.781412	0.000000	0.000000	-1.000000	0.000075
0.750	-1.000	0.173077	0.781096	0.000000	0.000000	-1.000000	0.000099
0.875	-1.000	0.201923	0.780721	0.000000	0.000000	-1.000000	0.000075
1.000	-1.000	0.230769	0.780284	0.000000	0.000000	-1.000000	0.000000
0.500	-0.500	0.122989	0.393414	0.028284	0.027896	-1.001103	0.000000
0.625	-0.500	0.153769	0.393229	0.028233	0.028078	-1.000688	0.000053
0.750	-0.500	0.184530	0.393005	0.028229	0.028242	-1.000185	0.000070
0.875	-0.500	0.215270	0.392740	0.028249	0.028409	-0.999588	0.000053
1.000	-0.500	0.245990	0.392431	0.028284	0.028587	-0.998893	0.000000
0.500	0.0	0.126139	0.0	0.040000	0.039459	-1.001560	0.0
0.625	0.0	0.157721	0.0	0.039927	0.039709	-1.000973	0.0
0.750	0.0	0.189274	0.0	0.039921	0.039940	-1.000261	0.0
0.875	0.0	0.220799	0.0	0.039950	0.040176	-0.999417	0.0
1.000	0.0	0.252295	0.0	0.040000	0.040428	-0.998435	0.0
0.500	0.500	0.122989	-0.393414	0.028284	0.027896	-1.001103	-0.000000
0.625	0.500	0.153769	-0.393229	0.028233	0.028078	-1.000688	-0.000053
0.750	0.500	0.184530	-0.393005	0.028229	0.028242	-1.000185	-0.000070
0.875	0.500	0.215270	-0.392740	0.028249	0.028409	-0.999588	-0.000053
1.000	0.500	0.245990	-0.392431	0.028284	0.028587	-0.998893	-0.000000
0.500	1.000	0.115385	-0.781673	0.000000	0.000000	-1.000000	-0.000000
0.625	1.000	0.144231	-0.781412	0.000000	0.000000	-1.000000	-0.000075
0.750	1.000	0.173077	-0.781096	0.000000	0.000000	-1.000000	-0.000099
0.875	1.000	0.201923	-0.780721	0.000000	0.000000	-1.000000	-0.000075
1.000	1.000	0.230769	-0.780284	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.667 C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.667	-1.000	0.153846	0.785073	0.000000	0.000000	-1.000000	0.000000
0.750	-1.000	0.173077	0.783243	0.000000	0.000000	-1.000000	0.000934
0.833	-1.000	0.192308	0.781263	0.000000	0.000000	-1.000000	0.001244
0.917	-1.000	0.211538	0.779046	0.000000	0.000000	-1.000000	0.000942
1.000	-1.000	0.230769	0.776504	0.000000	0.000000	-1.000000	-0.000000
0.667	-0.500	0.163490	0.395818	0.028284	0.024177	-1.006816	0.000000
0.750	-0.500	0.184131	0.394523	0.028059	0.025521	-1.003891	0.000660
0.833	-0.500	0.204686	0.393123	0.028050	0.026767	-1.000704	0.000880
0.917	-0.500	0.225158	0.391556	0.028152	0.027990	-0.997141	0.000666
1.000	-0.500	0.245543	0.389758	0.028284	0.029236	-0.993087	-0.000000
0.667	0.0	0.167484	0.0	0.040000	0.034191	-1.009640	0.0
0.750	0.0	0.188710	0.0	0.039681	0.036091	-1.005503	0.0
0.833	0.0	0.209813	0.0	0.039668	0.037855	-1.000996	0.0
0.917	0.0	0.230799	0.0	0.039813	0.039584	-0.995957	0.0
1.000	0.0	0.251663	0.0	0.040000	0.041345	-0.990224	0.0
0.667	0.500	0.163490	-0.395818	0.028284	0.024177	-1.006816	-0.000000
0.750	0.500	0.184131	-0.394523	0.028059	0.025521	-1.003891	-0.000660
0.833	0.500	0.204686	-0.393123	0.028050	0.026767	-1.000704	-0.000880
0.917	0.500	0.225158	-0.391556	0.028152	0.027990	-0.997141	-0.000666
1.000	0.500	0.245543	-0.389758	0.028284	0.029236	-0.993087	0.000000
0.667	1.000	0.153846	-0.785073	0.000000	0.000000	-1.000000	-0.000000
0.750	1.000	0.173077	-0.783243	0.000000	0.000000	-1.000000	-0.000934
0.833	1.000	0.192308	-0.781263	0.000000	0.000000	-1.000000	-0.001244
0.917	1.000	0.211538	-0.779046	0.000000	0.000000	-1.000000	-0.000942
1.000	1.000	0.230769	-0.776504	0.000000	0.000000	-1.000000	0.000000

A/B= 0.667 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.667	-1.000	0.153846	0.782127	0.000000	0.000000	-1.000000	0.000000
0.750	-1.000	0.173077	0.781634	0.000000	0.000000	-1.000000	0.000126
0.833	-1.000	0.192308	0.781089	0.000000	0.000000	-1.000000	0.000168
0.917	-1.000	0.211538	0.780485	0.000000	0.000000	-1.000000	0.000127
1.000	-1.000	0.230769	0.779816	0.000000	0.000000	-1.000000	-0.000000
0.667	-0.500	0.163965	0.393735	0.028284	0.027505	-1.001836	0.000000
0.750	-0.500	0.184500	0.393386	0.028235	0.027811	-1.001053	0.000089
0.833	-0.500	0.205012	0.393000	0.028228	0.028103	-1.000184	0.000119
0.917	-0.500	0.225502	0.392573	0.028247	0.028394	-0.999222	0.000089
1.000	-0.500	0.245968	0.392100	0.028284	0.028691	-0.998159	-0.000000
0.667	0.0	0.168157	0.0	0.040000	0.038898	-1.002597	0.0
0.750	0.0	0.189232	0.0	0.039930	0.039330	-1.001489	0.0
0.833	0.0	0.210275	0.0	0.039920	0.039743	-1.000261	0.0
0.917	0.0	0.231286	0.0	0.039948	0.040155	-0.998900	0.0
1.000	0.0	0.252264	0.0	0.040000	0.040575	-0.997397	0.0
0.667	0.500	0.163965	-0.393735	0.028284	0.027505	-1.001836	-0.000000
0.750	0.500	0.184500	-0.393386	0.028235	0.027811	-1.001053	-0.000089
0.833	0.500	0.205012	-0.393000	0.028228	0.028103	-1.000184	-0.000119
0.917	0.500	0.225502	-0.392573	0.028247	0.028394	-0.999222	-0.000089
1.000	0.500	0.245968	-0.392100	0.028284	0.028691	-0.998159	0.000000
0.667	1.000	0.153846	-0.782127	0.000000	0.000000	-1.000000	-0.000000
0.750	1.000	0.173077	-0.781634	0.000000	0.000000	-1.000000	-0.000126
0.833	1.000	0.192308	-0.781089	0.000000	0.000000	-1.000000	-0.000168
0.917	1.000	0.211538	-0.780485	0.000000	0.000000	-1.000000	-0.000127
1.000	1.000	0.230769	-0.779816	0.000000	0.000000	-1.000000	0.000000

A/B= 0.667 C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G u}{\sigma b}$	$2\frac{G w}{\sigma c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.667	-1.000	0.153846	0.781500	0.000000	0.000000	-1.000000	0.000000
0.750	-1.000	0.173077	0.781279	0.000000	0.000000	-1.000000	0.000038
0.833	-1.000	0.192308	0.781034	0.000000	0.000000	-1.000000	0.000050
0.917	-1.000	0.211538	0.780762	0.000000	0.000000	-1.000000	0.000038
1.000	-1.000	0.230769	0.780463	0.000000	0.000000	-1.000000	-0.000000
0.667	-0.500	0.163993	0.393291	0.028284	0.027979	-1.000824	0.000000
0.750	-0.500	0.184508	0.393135	0.028263	0.028110	-1.000473	0.000027
0.833	-0.500	0.205013	0.392961	0.028259	0.028236	-1.000082	0.000036
0.917	-0.500	0.225508	0.392769	0.028268	0.028362	-0.999651	0.000027
1.000	-0.500	0.245993	0.392558	0.028284	0.028491	-0.999176	-0.000000
0.667	0.0	0.168195	0.0	0.040000	0.039568	-1.001165	0.0
0.750	0.0	0.189243	0.0	0.039970	0.039754	-1.000669	0.0
0.833	0.0	0.210276	0.0	0.039965	0.039932	-1.000117	0.0
0.917	0.0	0.231295	0.0	0.039976	0.040110	-0.999506	0.0
1.000	0.0	0.252299	0.0	0.040000	0.040293	-0.998834	0.0
0.667	0.500	0.163993	-0.393291	0.028284	0.027979	-1.000824	-0.000000
0.750	0.500	0.184508	-0.393135	0.028263	0.028110	-1.000473	-0.000027
0.833	0.500	0.205013	-0.392961	0.028259	0.028236	-1.000082	-0.000036
0.917	0.500	0.225508	-0.392769	0.028268	0.028362	-0.999651	-0.000027
1.000	0.500	0.245993	-0.392558	0.028284	0.028491	-0.999176	0.000000
0.667	1.000	0.153846	-0.781500	0.000000	0.000000	-1.000000	-0.000000
0.750	1.000	0.173077	-0.781279	0.000000	0.000000	-1.000000	-0.000038
0.833	1.000	0.192308	-0.781034	0.000000	0.000000	-1.000000	-0.000050
0.917	1.000	0.211538	-0.780762	0.000000	0.000000	-1.000000	-0.000038
1.000	1.000	0.230769	-0.780463	0.000000	0.000000	-1.000000	0.000000

A/B= 0.750 C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.750	-1.000	0.173077	0.784368	0.000000	0.000000	-1.000000	-0.000000
0.813	-1.000	0.187500	0.782797	0.000000	0.000000	-1.000000	0.000575
0.875	-1.000	0.201923	0.781134	0.000000	0.000000	-1.000000	0.000766
0.938	-1.000	0.216346	0.779338	0.000000	0.000000	-1.000000	0.000578
1.000	-1.000	0.230769	0.777368	0.000000	0.000000	-1.000000	0.000000
0.750	-0.500	0.184155	0.395319	0.028284	0.024761	-1.005567	-0.000000
0.813	-0.500	0.199615	0.394208	0.028166	0.025815	-1.003065	0.000407
0.875	-0.500	0.215022	0.393032	0.028155	0.026822	-1.000405	0.000542
0.938	-0.500	0.230378	0.391762	0.028207	0.027812	-0.997534	0.000408
1.000	-0.500	0.245682	0.390370	0.028284	0.028805	-0.994399	0.000000
0.750	0.0	0.188743	0.0	0.040000	0.035017	-1.007873	0.0
0.813	0.0	0.204633	0.0	0.039833	0.036508	-1.004335	0.0
0.875	0.0	0.220448	0.0	0.039817	0.037932	-1.000573	0.0
0.938	0.0	0.236191	0.0	0.039890	0.039332	-0.996513	0.0
1.000	0.0	0.251859	0.0	0.040000	0.040737	-0.992079	0.0
0.750	0.500	0.184155	-0.395319	0.028284	0.024761	-1.005567	0.000000
0.813	0.500	0.199615	-0.394208	0.028166	0.025815	-1.003065	-0.000407
0.875	0.500	0.215022	-0.393032	0.028155	0.026822	-1.000405	-0.000542
0.938	0.500	0.230378	-0.391762	0.028207	0.027812	-0.997534	-0.000408
1.000	0.500	0.245682	-0.390370	0.028284	0.028805	-0.994399	-0.000000
0.750	1.000	0.173077	-0.784368	0.000000	0.000000	-1.000000	0.000000
0.813	1.000	0.187500	-0.782797	0.000000	0.000000	-1.000000	-0.000575
0.875	1.000	0.201923	-0.781134	0.000000	0.000000	-1.000000	-0.000766
0.938	1.000	0.216346	-0.779338	0.000000	0.000000	-1.000000	-0.000578
1.000	1.000	0.230769	-0.777368	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.750 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.750	-1.000	0.173077	0.781893	0.000000	0.000000	-1.000000	0.000000
0.813	-1.000	0.187500	0.781483	0.000000	0.000000	-1.000000	0.000075
0.875	-1.000	0.201923	0.781043	0.000000	0.000000	-1.000000	0.000100
0.938	-1.000	0.216346	0.780569	0.000000	0.000000	-1.000000	0.000075
1.000	-1.000	0.230769	0.780060	0.000000	0.000000	-1.000000	-0.000000
0.750	-0.500	0.184477	0.393569	0.028284	0.027622	-1.001456	0.000000
0.813	-0.500	0.199873	0.393279	0.028258	0.027860	-1.000804	0.000053
0.875	-0.500	0.215255	0.392968	0.028253	0.028093	-1.000104	0.000071
0.938	-0.500	0.230623	0.392633	0.028263	0.028324	-0.999352	0.000053
1.000	-0.500	0.245978	0.392273	0.028284	0.028558	-0.998543	-0.000000
0.750	0.0	0.189199	0.0	0.040000	0.039063	-1.002058	0.0
0.813	0.0	0.204998	0.0	0.039962	0.039400	-1.001138	0.0
0.875	0.0	0.220777	0.0	0.039955	0.039729	-1.000147	0.0
0.938	0.0	0.236537	0.0	0.039970	0.040056	-0.999083	0.0
1.000	0.0	0.252278	0.0	0.040000	0.040387	-0.997940	0.0
0.750	0.500	0.184477	-0.393569	0.028284	0.027622	-1.001456	-0.000000
0.813	0.500	0.199873	-0.393279	0.028258	0.027860	-1.000804	-0.000053
0.875	0.500	0.215255	-0.392968	0.028253	0.028093	-1.000104	-0.000071
0.938	0.500	0.230623	-0.392633	0.028263	0.028324	-0.999352	-0.000053
1.000	0.500	0.245978	-0.392273	0.028284	0.028558	-0.998543	0.000000
0.750	1.000	0.173077	-0.781893	0.000000	0.000000	-1.000000	-0.000000
0.813	1.000	0.187500	-0.781483	0.000000	0.000000	-1.000000	-0.000075
0.875	1.000	0.201923	-0.781043	0.000000	0.000000	-1.000000	-0.000100
0.938	1.000	0.216346	-0.780569	0.000000	0.000000	-1.000000	-0.000075
1.000	1.000	0.230769	-0.780060	0.000000	0.000000	-1.000000	0.000000

A/B= 0.750 C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.750	-1.000	0.173077	0.781392	0.000000	0.000000	-1.000000	-0.000000
0.813	-1.000	0.187500	0.781209	0.000000	0.000000	-1.000000	0.000022
0.875	-1.000	0.201923	0.781012	0.000000	0.000000	-1.000000	0.000030
0.938	-1.000	0.216346	0.780800	0.000000	0.000000	-1.000000	0.000022
1.000	-1.000	0.230769	0.780573	0.000000	0.000000	-1.000000	0.000000
0.750	-0.500	0.184495	0.393215	0.028284	0.028029	-1.000650	-0.000000
0.813	-0.500	0.199879	0.393085	0.028273	0.028131	-1.000360	0.000016
0.875	-0.500	0.215257	0.392946	0.028270	0.028232	-1.000046	0.000021
0.938	-0.500	0.230629	0.392796	0.028275	0.028332	-0.999710	0.000016
1.000	-0.500	0.245995	0.392635	0.028284	0.028433	-0.999350	0.000000
0.750	0.0	0.189224	0.0	0.040000	0.039639	-1.000920	0.0
0.813	0.0	0.205007	0.0	0.039984	0.039784	-1.000509	0.0
0.875	0.0	0.220780	0.0	0.039980	0.039925	-1.000066	0.0
0.938	0.0	0.236545	0.0	0.039986	0.040067	-0.999590	0.0
1.000	0.0	0.252302	0.0	0.040000	0.040210	-0.999080	0.0
0.750	0.500	0.184495	-0.393215	0.028284	0.028029	-1.000650	0.000000
0.813	0.500	0.199879	-0.393085	0.028273	0.028131	-1.000360	-0.000016
0.875	0.500	0.215257	-0.392946	0.028270	0.028232	-1.000046	-0.000021
0.938	0.500	0.230629	-0.392796	0.028275	0.028332	-0.999710	-0.000016
1.000	0.500	0.245995	-0.392635	0.028284	0.028433	-0.999350	-0.000000
0.750	1.000	0.173077	-0.781392	0.000000	0.000000	-1.000000	0.000000
0.813	1.000	0.187500	-0.781209	0.000000	0.000000	-1.000000	-0.000022
0.875	1.000	0.201923	-0.781012	0.000000	0.000000	-1.000000	-0.000030
0.938	1.000	0.216346	-0.780800	0.000000	0.000000	-1.000000	-0.000022
1.000	1.000	0.230769	-0.780573	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.833 C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.833	-1.000	0.192308	0.783446	0.000000	0.000000	-1.000000	-0.000000
0.875	-1.000	0.201923	0.782270	0.000000	0.000000	-1.000000	0.000277
0.917	-1.000	0.211538	0.781047	0.000000	0.000000	-1.000000	0.000369
0.958	-1.000	0.221154	0.779766	0.000000	0.000000	-1.000000	0.000277
1.000	-1.000	0.230769	0.778412	0.000000	0.000000	-1.000000	0.000000
0.833	-0.500	0.204819	0.394667	0.028284	0.025390	-1.004000	-0.000000
0.875	-0.500	0.215109	0.393835	0.028236	0.026126	-1.002130	0.000196
0.917	-0.500	0.225373	0.392971	0.028228	0.026845	-1.000183	0.000261
0.958	-0.500	0.235613	0.392065	0.028248	0.027555	-0.998143	0.000196
1.000	-0.500	0.245828	0.391108	0.028284	0.028261	-0.995993	0.000000
0.833	0.0	0.210061	0.0	0.040000	0.035906	-1.005657	0.0
0.875	0.0	0.220570	0.0	0.039931	0.036948	-1.003012	0.0
0.917	0.0	0.231104	0.0	0.039926	0.037965	-1.000259	0.0
0.958	0.0	0.241603	0.0	0.039949	0.038968	-0.997374	0.0
1.000	0.0	0.252066	0.0	0.040000	0.039967	-0.994333	0.0
0.833	0.500	0.204819	-0.394667	0.028284	0.025390	-1.004000	0.000000
0.875	0.500	0.215109	-0.393835	0.028236	0.026126	-1.002130	-0.000196
0.917	0.500	0.225373	-0.392971	0.028228	0.026845	-1.000183	-0.000261
0.958	0.500	0.235613	-0.392065	0.028248	0.027555	-0.998143	-0.000196
1.000	0.500	0.245828	-0.391108	0.028284	0.028261	-0.995993	-0.000000
0.833	1.000	0.192308	-0.783446	0.000000	0.000000	-1.000000	0.000000
0.875	1.000	0.201923	-0.782270	0.000000	0.000000	-1.000000	-0.000277
0.917	1.000	0.211538	-0.781047	0.000000	0.000000	-1.000000	-0.000369
0.958	1.000	0.221154	-0.779766	0.000000	0.000000	-1.000000	-0.000277
1.000	1.000	0.230769	-0.778412	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.833 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.833	-1.000	0.192308	0.781624	0.000000	0.000000	-1.000000	0.000000
0.875	-1.000	0.201923	0.781324	0.000000	0.000000	-1.000000	0.000035
0.917	-1.000	0.211538	0.781010	0.000000	0.000000	-1.000000	0.000047
0.958	-1.000	0.221154	0.780681	0.000000	0.000000	-1.000000	0.000035
1.000	-1.000	0.230769	0.780337	0.000000	0.000000	-1.000000	-0.000000
0.833	-0.500	0.204988	0.393379	0.028284	0.027749	-1.001021	0.000000
0.875	-0.500	0.215247	0.393166	0.028273	0.027915	-1.000545	0.000025
0.917	-0.500	0.225501	0.392944	0.028270	0.028078	-1.000046	0.000033
0.958	-0.500	0.235748	0.392712	0.028274	0.028242	-0.999525	0.000025
1.000	-0.500	0.245988	0.392469	0.028284	0.028405	-0.998978	-0.000000
0.833	0.0	0.210240	0.0	0.040000	0.039243	-1.001444	0.0
0.875	0.0	0.220767	0.0	0.039984	0.039477	-1.000771	0.0
0.917	0.0	0.231284	0.0	0.039980	0.039709	-1.000066	0.0
0.958	0.0	0.241793	0.0	0.039986	0.039940	-0.999328	0.0
1.000	0.0	0.252292	0.0	0.040000	0.040171	-0.998555	0.0
0.833	0.500	0.204988	-0.393379	0.028284	0.027749	-1.001021	-0.000000
0.875	0.500	0.215247	-0.393166	0.028273	0.027915	-1.000545	-0.000025
0.917	0.500	0.225501	-0.392944	0.028270	0.028078	-1.000046	-0.000033
0.958	0.500	0.235748	-0.392712	0.028274	0.028242	-0.999525	-0.000025
1.000	0.500	0.245988	-0.392469	0.028284	0.028405	-0.998978	0.000000
0.833	1.000	0.192308	-0.781624	0.000000	0.000000	-1.000000	-0.000000
0.875	1.000	0.201923	-0.781324	0.000000	0.000000	-1.000000	-0.000035
0.917	1.000	0.211538	-0.781010	0.000000	0.000000	-1.000000	-0.000047
0.958	1.000	0.221154	-0.780681	0.000000	0.000000	-1.000000	-0.000035
1.000	1.000	0.230769	-0.780337	0.000000	0.000000	-1.000000	0.000000

A/B= 0.833 C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.833	-1.000	0.192308	0.781270	0.000000	0.000000	-1.000000	0.000000
0.875	-1.000	0.201923	0.781136	0.000000	0.000000	-1.000000	0.000011
0.917	-1.000	0.211538	0.780996	0.000000	0.000000	-1.000000	0.000014
0.958	-1.000	0.221154	0.780850	0.000000	0.000000	-1.000000	0.000011
1.000	-1.000	0.230769	0.780696	0.000000	0.000000	-1.000000	0.000000
0.833	-0.500	0.204997	0.393128	0.028284	0.028084	-1.000455	0.000009
0.875	-0.500	0.215251	0.393034	0.028279	0.028155	-1.000243	0.000007
0.917	-0.500	0.225503	0.392935	0.028278	0.028226	-1.000021	0.000010
0.958	-0.500	0.235751	0.392831	0.028280	0.028296	-0.999788	0.000007
1.000	-0.500	0.245997	0.392723	0.028284	0.028367	-0.999545	0.000000
0.833	0.0	0.210253	0.0	0.040000	0.039717	-1.000643	0.0
0.875	0.0	0.220772	0.0	0.039993	0.039818	-1.000344	0.0
0.917	0.0	0.231287	0.0	0.039991	0.039917	-1.000029	0.0
0.958	0.0	0.241798	0.0	0.039994	0.040017	-0.999700	0.0
1.000	0.0	0.252305	0.0	0.040000	0.040117	-0.999357	0.0
0.833	0.500	0.204997	-0.393128	0.028284	0.028084	-1.000455	-0.000000
0.875	0.500	0.215251	-0.393034	0.028279	0.028155	-1.000243	-0.000007
0.917	0.500	0.225503	-0.392935	0.028278	0.028226	-1.000021	-0.000010
0.958	0.500	0.235751	-0.392831	0.028280	0.028296	-0.999788	-0.000007
1.000	0.500	0.245997	-0.392723	0.028284	0.028367	-0.999545	-0.000000
0.833	1.000	0.192308	-0.781270	0.000000	0.000000	-1.000000	-0.000000
0.875	1.000	0.201923	-0.781136	0.000000	0.000000	-1.000000	-0.000011
0.917	1.000	0.211538	-0.780996	0.000000	0.000000	-1.000000	-0.000014
0.958	1.000	0.221154	-0.780850	0.000000	0.000000	-1.000000	-0.000011
1.000	1.000	0.230769	-0.780696	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.875 C/B= 1.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.875	-1.000	0.201923	0.782904	0.000000	0.000000	-1.000000	-0.000000
0.906	-1.000	0.209135	0.781975	0.000000	0.000000	-1.000000	0.000161
0.938	-1.000	0.216346	0.781018	0.000000	0.000000	-1.000000	0.000215
0.969	-1.000	0.223558	0.780028	0.000000	0.000000	-1.000000	0.000161
1.000	-1.000	0.230769	0.779000	0.000000	0.000000	-1.000000	0.000000
0.875	-0.500	0.215140	0.394284	0.028284	0.025716	-1.003101	-0.000000
0.906	-0.500	0.222850	0.393627	0.028258	0.026282	-1.001625	0.000114
0.938	-0.500	0.230546	0.392950	0.028253	0.026838	-1.000104	0.000152
0.969	-0.500	0.238227	0.392250	0.028263	0.027389	-0.998530	0.000114
1.000	-0.500	0.245894	0.391523	0.028284	0.027936	-0.996897	0.000000
0.875	0.0	0.220614	0.0	0.040000	0.036368	-1.004385	0.0
0.906	0.0	0.228531	0.0	0.039963	0.037168	-1.002298	0.0
0.938	0.0	0.236428	0.0	0.039955	0.037955	-1.000147	0.0
0.969	0.0	0.244304	0.0	0.039970	0.038733	-0.997922	0.0
1.000	0.0	0.252159	0.0	0.040000	0.039508	-0.995611	0.0
0.875	0.500	0.215140	-0.394284	0.028284	0.025716	-1.003101	0.000000
0.906	0.500	0.222850	-0.393627	0.028258	0.026282	-1.001625	-0.000114
0.938	0.500	0.230546	-0.392950	0.028253	0.026838	-1.000104	-0.000152
0.969	0.500	0.238227	-0.392250	0.028263	0.027389	-0.998530	-0.000114
1.000	0.500	0.245894	-0.391523	0.028284	0.027936	-0.996897	-0.000000
0.875	1.000	0.201923	-0.782904	0.000000	0.000000	-1.000000	0.000000
0.906	1.000	0.209135	-0.781975	0.000000	0.000000	-1.000000	-0.000161
0.938	1.000	0.216346	-0.781018	0.000000	0.000000	-1.000000	-0.000215
0.969	1.000	0.223558	-0.780028	0.000000	0.000000	-1.000000	-0.000161
1.000	1.000	0.230769	-0.779000	0.000000	0.000000	-1.000000	-0.000000

A/B= 0.875 C/B= 2.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G u}{\sigma b}$	$2\frac{G w}{\sigma c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.875	-1.000	0.201923	0.781476	0.000000	0.000000	-1.000000	0.000000
0.906	-1.000	0.209135	0.781241	0.000000	0.000000	-1.000000	0.000020
0.938	-1.000	0.216346	0.780998	0.000000	0.000000	-1.000000	0.000027
0.969	-1.000	0.223558	0.780747	0.000000	0.000000	-1.000000	0.000020
1.000	-1.000	0.230769	0.780487	0.000000	0.000000	-1.000000	-0.000000
0.875	-0.500	0.215242	0.393274	0.028284	0.027817	-1.000785	0.000000
0.906	-0.500	0.222935	0.393108	0.028278	0.027943	-1.000412	0.000014
0.938	-0.500	0.230625	0.392936	0.028276	0.028069	-1.000026	0.000019
0.969	-0.500	0.238311	0.392759	0.028279	0.028195	-0.999627	0.000014
1.000	-0.500	0.245993	0.392575	0.028284	0.028321	-0.999215	-0.000000
0.875	0.0	0.220759	0.0	0.040000	0.039339	-1.001110	0.0
0.906	0.0	0.228652	0.0	0.039991	0.039518	-1.000582	0.0
0.938	0.0	0.236539	0.0	0.039989	0.039696	-1.000037	0.0
0.969	0.0	0.244421	0.0	0.039992	0.039874	-0.999473	0.0
1.000	0.0	0.252298	0.0	0.040000	0.040051	-0.998890	0.0
0.875	0.500	0.215242	-0.393274	0.028284	0.027817	-1.000785	-0.000000
0.906	0.500	0.222935	-0.393108	0.028278	0.027943	-1.000412	-0.000014
0.938	0.500	0.230625	-0.392936	0.028276	0.028069	-1.000026	-0.000019
0.969	0.500	0.238311	-0.392759	0.028279	0.028195	-0.999627	-0.000014
1.000	0.500	0.245993	-0.392575	0.028284	0.028321	-0.999215	0.000000
0.875	1.000	0.201923	-0.781476	0.000000	0.000000	-1.000000	-0.000000
0.906	1.000	0.209135	-0.781241	0.000000	0.000000	-1.000000	-0.000020
0.938	1.000	0.216346	-0.780998	0.000000	0.000000	-1.000000	-0.000027
0.969	1.000	0.223558	-0.780747	0.000000	0.000000	-1.000000	-0.000020
1.000	1.000	0.230769	-0.780487	0.000000	0.000000	-1.000000	0.000000

A/B= 0.875 C/B= 3.000

$\frac{r}{b}$	$\frac{z}{c}$	$2\frac{G\,u}{\sigma\,b}$	$2\frac{G\,w}{\sigma\,c}$	$\frac{\sigma_r}{\sigma}$	$\frac{\sigma_\theta}{\sigma}$	$\frac{\sigma_z}{\sigma}$	$\frac{\tau_{rz}}{\sigma}$
0.875	-1.000	0.201923	0.781203	0.000000	0.000000	-1.000000	0.000000
0.906	-1.000	0.209135	0.781099	0.000000	0.000000	-1.000000	0.000006
0.938	-1.000	0.216346	0.780991	0.000000	0.000000	-1.000000	0.000008
0.969	-1.000	0.223558	0.780879	0.000000	0.000000	-1.000000	0.000006
1.000	-1.000	0.230769	0.780763	0.000000	0.000000	-1.000000	0.000000
0.875	-0.500	0.215248	0.393081	0.028284	0.028114	-1.000349	0.000000
0.906	-0.500	0.222938	0.393007	0.028282	0.028168	-1.000183	0.000004
0.938	-0.500	0.230626	0.392931	0.028281	0.028223	-1.000012	0.000006
0.969	-0.500	0.238313	0.392852	0.028282	0.028277	-0.999834	0.000004
1.000	-0.500	0.245998	0.392770	0.028284	0.028331	-0.999651	0.000000
0.875	0.0	0.220767	0.0	0.040000	0.039759	-1.000494	0.0
0.906	0.0	0.228655	0.0	0.039996	0.039836	-1.000259	0.0
0.938	0.0	0.236541	0.0	0.039995	0.039913	-1.000016	0.0
0.969	0.0	0.244425	0.0	0.039996	0.039989	-0.999765	0.0
1.000	0.0	0.252306	0.0	0.040000	0.040066	-0.999506	0.0
0.875	0.500	0.215248	-0.393081	0.028284	0.028114	-1.000349	-0.000000
0.906	0.500	0.222938	-0.393007	0.028282	0.028168	-1.000183	-0.000004
0.938	0.500	0.230626	-0.392931	0.028281	0.028223	-1.000012	-0.000006
0.969	0.500	0.238313	-0.392852	0.028282	0.028277	-0.999834	-0.000004
1.000	0.500	0.245998	-0.392770	0.028284	0.028331	-0.999651	-0.000000
0.875	1.000	0.201923	-0.781203	0.000000	0.000000	-1.000000	-0.000000
0.906	1.000	0.209135	-0.781099	0.000000	0.000000	-1.000000	-0.000006
0.938	1.000	0.216346	-0.780991	0.000000	0.000000	-1.000000	-0.000008
0.969	1.000	0.223558	-0.780879	0.000000	0.000000	-1.000000	-0.000006
1.000	1.000	0.230769	-0.780763	0.000000	0.000000	-1.000000	-0.000000

APPENDIX 1

Equation (3.2)i is obtained by setting w=+c, multiplying by

$$\left[J_0(\alpha_m r) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m a) \right] r \, dr$$

and integrating with respect to r between the limits a and b. Thus,

$$\int_a^b \Bigg[\sum_{n=1}^{\infty} \frac{\sin \beta_n c}{\beta_n} \Big\{ A_n \Big(-4(1-\nu) J_0(i\beta_n r) + i\beta_n r \, J_1(i\beta_n r) \Big)$$

$$+ B_n J_0(i\beta_n r) + C_n \Big(4(1-\nu) \, i \, H_0^1(i\beta_n r) + \beta_n r \, H_1^1(i\beta_n r) \Big)$$

$$+ D_n \, i \, H_0^1(i\beta_n r) \Big\} - \sum_{m=1}^{\infty} \frac{1}{\alpha_m} \Big\{ \alpha_m c \cosh \alpha_m c$$

$$- (3-4\nu+\alpha_m c \tanh \alpha_m c) \sinh \alpha_m c \Big\} \Big\{ E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r) \Big\} \Bigg]$$

$$\times \left[J_0(\alpha_m r) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0(\alpha_m r) \right] r \, dr = 0$$

$$= \int_a^b \sum_m -\frac{1}{\alpha_m} \left[\alpha_m c \cosh \alpha_m c - (3-4\nu+\alpha_m c \tanh \alpha_m c) \sinh \alpha_m c \right]$$

$$\left[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r) \right] \left[J_0(\alpha_k r) - \frac{J_1(\alpha_k a)}{Y_1(\alpha_k a)} Y_0(\alpha_k r) \right] r \, dr +$$

$$+\int_a^b \left[\sum_{n=1}^{\infty} \frac{\sin\beta_n c}{\beta_n}\left\{A_n\left(-4(1-\nu)\, J_0(i\beta_n r) + i\beta_n r\, J_1(i\beta_n r)\right)\right.\right.$$

$$+ B_n J_0(i\beta_n r) + C_n\left(4(1-\nu)\, i H_0'(i\beta_n r) + \beta_n r\, H_1'(i\beta_n r)\right)$$

$$\left.\left. + D_n i H_0'(i\beta_n r)\right\}\right] \times \left[J_0(\alpha_m r) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)}\, Y_0(\alpha_m r)\right] r\, dr = 0$$

Only first integral of the above expression is evaluated here as the second integral has been evaluated correctly in [18].
First integral can be written in the following form.

$$I = \int_a^b \sum_{m=1}^{\infty} -\frac{1}{\alpha_m}\, G(\alpha_m c)\left[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\right]$$

$$\times \left[J_0(\alpha_k r) - \frac{J_1(\alpha_k a)}{Y_1(\alpha_k a)}\, Y_0(\alpha_k r)\right] r\, dr$$

Here,

$$G(\alpha_m c) = \left[\alpha_m c \cosh\alpha_m c - (3 - 4\nu + \alpha_m c \tanh\alpha_m c)\sinh\alpha_m c\right]$$

$$I = 0 \quad \text{for} \quad k \neq m$$

$$I = -\frac{1}{\alpha_m}\, G(\alpha_m c) \int_a^b \left[E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)\right]\left[J_0(\alpha_m r) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)}\, Y_0(\alpha_m r)\right]$$

$$\times\, r\, dr \quad \text{for} \quad k = m$$

Using Lommel Integrals [9] along with eq.(3.8), this integral can be written as,

$$I = -\frac{1}{\alpha_m} G(\alpha_m c)\Bigg[E_m \Bigg\{ \frac{b^2}{2}\Big(J_0^2(\alpha_m b) - \frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} J_0(\alpha_m b) Y_0(\alpha_m b)\Big)$$

$$- \frac{a^2}{2}\Big(J_0^2(\alpha_m a) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} J_0(\alpha_m a) Y_0(\alpha_m a)\Big)$$

$$+ F_m \Bigg\{ \frac{b^2}{2}\Big(J_0(\alpha_m b) Y_0(\alpha_m b) - \frac{J_1(\alpha_m b)}{Y_1(\alpha_m b)} Y_0^2(\alpha_m b)\Big)$$

$$- \frac{a^2}{2}\Big(J_0(\alpha_m a) Y_0(\alpha_m a) - \frac{J_1(\alpha_m a)}{Y_1(\alpha_m a)} Y_0^2(\alpha_m a)\Big)\Bigg\}\Bigg]$$

This, on rearranging the terms takes the following form,

$$I = -\frac{1}{\alpha_m} G(\alpha_m c)\left[E_m N_1 + F_m N_2 \right]$$

where,

$$N_1 = \frac{1}{2}\left\{ b^2 J_0^2(\alpha_m b) - a^2 J_0^2(\alpha_m a)\right\}$$

$$- \frac{J_1(\alpha_m a)}{2Y_1(\alpha_m a)}\left\{ b^2 J_0(\alpha_m b) Y_0(\alpha_m b) - a^2 J_0(\alpha_m a) Y_0(\alpha_m a)\right\}$$

$$N_2 = \frac{1}{2}\left\{ b^2 J_0(\alpha_m b) Y_0(\alpha_m b) - a^2 J_0(\alpha_m a) Y_0(\alpha_m a)\right\}$$

$$- \frac{J_1(\alpha_m a)}{2 Y_1(\alpha_m a)}\left\{ b^2 Y_0^2(\alpha_m b) - a^2 Y_0^2(\alpha_m a)\right\}$$

APPENDIX 2

Equations (2.8) which are mentioned in chapter II are

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} - \frac{1}{r^2}\phi_1 + \frac{\partial \phi_1}{\partial z^2} = 0$$

$$\frac{\partial^2 \phi_0}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_0}{\partial r} + \frac{\partial^2 \phi_0}{\partial z^2} = 0$$

If ϕ_1 and ϕ_0 are given by eqs.(2.14), then a simple substitution leads to the following differential equations.

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} - \left(\beta^2 + \frac{1}{r^2}\right) f(r) = 0$$

$$\frac{d^2 g(r)}{dr^2} + \frac{1}{r}\frac{dg(r)}{dr} - \beta^2 g(r) = 0$$

These are the Bessel equations and thier roots are given by the modified Beesel functions of first and second kind.

Thus, the solutions can be written as,

$$f(r) = A\, I_1(\beta r) + B\, K_1(\beta r)$$

$$g(r) = C\, I_0(\beta r) + D\, K_0(\beta r)$$

APPENDIX 3

Stress boundary conditions on the curved surfaces are

$$t_r^{(0)}(\eta, \pm 1) = 0$$

$$t_{rz}^{(0)}(\eta, \pm 1) = 0$$

These conditions when substituted in to eqs.(3.14)a,b, give the following expressions.

$$R\left[(1-\nu)\, \dot{\omega}_r^{(2)}(\eta, \pm 1) + \nu\, \omega_z^{(2)'}(\eta, \pm 1)\right] = 0$$

$$\left(\frac{1-\nu}{2}\right)\left[\dot{\omega}_z^{(2)}(\eta, \pm 1) + \omega_r^{(2)'}(\eta, \pm 1)\right] = 0$$

These equations can be simplified as,

$$(1-\nu)\, \dot{u}_2^n(\eta, \pm 1) - \beta_n^* \, \nu\, u_1^n(\eta, \pm 1) = 0 \qquad (1)$$

$$\dot{u}_1^n(\eta, \pm 1) - \beta_n^* \, u_2^n(\eta, \pm 1) = 0 \qquad (2)$$

Expressions for u_n^1 and u_n^2 are now substituted for symmetric and anti-symmetric parts separately.

(a) Symmetric Part: Substituting eqs.(3.28) in to (1) and (2) and simplifying, two expressions are obtained.

$$A_2 \beta_n^* \cos\beta_n^* + D_2\left[2(1-\nu)\cos\beta_n^* + \beta_n^* \sin\beta_n^*\right] = 0 \qquad (3)$$

$$-A_2\beta_n^* \sin\beta_n^* + D_2\left[-(1-2\nu)\sin\beta_n^* + \beta_n^*\cos\beta_n^*\right] = 0$$

For a non-trivial solution of the system formed by eqs.(3) and (4), determinant of the coefficient matrix should vanish. Thus,

$$\begin{vmatrix} \beta_n^* \cos\beta_n^* & 2(1-\nu)\cos\beta_n^* + \beta_n^*\sin\beta_n^* \\ -\beta_n^*\sin\beta_n^* & -(1-2\nu)\sin\beta_n^* + \beta_n^*\cos\beta_n^* \end{vmatrix} = 0$$

This determinant, on expansion gives the following characteristic equation.

$$\sin 2\beta_n^* + 2\beta_n^* = 0$$

Roots of this equation, from now on identified as β_{n1}, have been calculated by Robbins and Smith [13].

From eq.(3), we have,

$$\frac{A_2}{D_2} = -\frac{2(1-\nu)\cos\beta_{n1} + \beta_{n1}\sin\beta_{n1}}{\beta_{n1}\cos\beta_{n1}} = k_n$$

Thus,

$$u_1^n(\rho,\beta_n^*) = D_2\left[k_n\cos\beta_{n1}\rho + \rho\sin\beta_{n1}\rho\right]$$

$$u_2^n(\rho,\beta_n^*) = D_2\left[\left(\frac{3-4\nu}{\beta_{n1}} + k_n\right)\sin\beta_{n1}\rho - \rho\cos\beta_{n1}\rho\right]$$

(c) Anti-Symmetric Part: Substitution of (3.29) in to eqs.(1) and (2) gives the following equations.

$$B_2 \beta_n^* \sin\beta_n^* + C_2\left[\beta_n^* \cos\beta_n^* - 2(1-\nu)\sin\beta_n^*\right] = 0 \qquad (6)$$

$$B_2 \beta_n^* \cos\beta_n^* - C_2\left[(1-2\nu)\cos\beta_n^* + \beta_n^* \sin\beta_n^*\right] = 0 \qquad (7)$$

Again, for a non-trivial solution, determinant of the coefficient matrix of the above system should vanish. As a result, the following characteristic equation is obtained.

$$\sin 2\beta_n^* - 2\beta_n^* = 0$$

Roots of this equation, written as β_{n2}, have been calculated by Hillman and Salzar [4].

From eq.(6), we have

$$\frac{B_2}{C_2} = \frac{2(1-\nu)\sin\beta_{n2} - \beta_{n2}\cos\beta_{n2}}{\beta_{n2}\sin\beta_{n2}} = p_n$$

Thus,

$$U_1^n(\rho, \beta_n^*) = C_2\left[p_n \sin\beta_{n2}\rho + \rho\cos\beta_{n2}\rho\right]$$

$$U_2^n(\rho, \beta_n^*) = C_2\left[\left(\frac{3-4\nu}{\beta_{n2}} - p_n\right)\cos\beta_{n2}\rho + \rho\sin\beta_{n2}\rho\right]$$

In the symmetric and anti-symmetric solutions, constants C_2 and D_2 are arbitrary and without any loss in generality they can be set equal to unity.

Thus, the symmetric and anti-symmetric solutions can be written in the forms given below respectively.

$$u_1^n(\rho, \overset{*}{\beta}_n) = k_n \cos\beta_{n1}\rho + \rho \sin\beta_{n1}\rho$$

$$u_2^n(\rho, \beta_n) = \left(\frac{3-4\nu}{\beta_{n1}} + k_n\right)\sin\beta_{n1}\rho - \rho \cos\beta_{n1}\rho$$

$$u_1^n(\rho, \overset{*}{\beta}_n) = p_n \sin\beta_{n2}\rho + \rho \cos\beta_{n2}\rho$$

$$u_2^n(\rho, \overset{*}{\beta}_n) = \left(\frac{3-4\nu}{\beta_{n2}} - p_n\right)\cos\beta_{n2}\rho + \rho \sin\beta_{n2}\rho$$

APPENDIX 4

Evaluation of ℓ_{mn} and b_m

Substitution of u_1^n, u_2^n, S and R in eqs.(3.25)a,c gives,

$$\tau_{11}^m = (1-\nu)\Big[\Big(\frac{2\nu}{\beta_{m1}} - k_m\Big)\beta_{m1}\cos\beta_{m1}\rho - \Big(\frac{2\nu}{\beta_{m2}} + p_m\Big)\beta_{m2}\sin\beta_{m2}\rho$$

$$-\beta_{m1}\rho\sin\beta_{m1}\rho - \beta_{m2}\rho\cos\beta_{m2}\rho\Big]$$

$$\tau_{12}^n = (1-\nu)\Big[-\Big\{k_m + \frac{1-2\nu}{\beta_{m1}}\Big\}\beta_{m1}\sin\beta_{m1}\rho + \Big\{p_m - \frac{1-2\nu}{\beta_{m2}}\Big\}$$

$$\times\beta_{m2}\cos\beta_{m2}\rho + \beta_{m1}\rho\cos\beta_{m1}\rho - \beta_{m2}\rho\sin\beta_{m2}\rho\Big]$$

In eq.(3,35)a, let,

$$\int_{-1}^{+1} \overline{\tau_{11}^m}\, u_1^n\, d\rho = X_{mn}$$

$$\int_{-1}^{+1} \overline{\tau_{12}^m}\, u_2^n\, d\rho = Y_{mn}$$

therefore,

$$X_{mn} = \int_{-1}^{+1}(1-\nu)\Big[(2\nu - \overline{k_m}\,\overline{\beta_{m1}})\cos\overline{\beta_{m1}}\rho - (2\nu + \overline{p_m}\,\overline{\beta_{m2}})\sin\overline{\beta_{m2}}\rho$$

$$-\overline{\beta_{m1}}\rho\sin\overline{\beta_{m1}}\rho - \overline{\beta_{m2}}\rho\cos\overline{\beta_{m2}}\rho\Big]\Big[k_n\cos\beta_{n1}\rho + \rho\sin\beta_{n1}\rho$$

$$+p_n\sin\beta_{n2}\rho + \rho\cos\beta_{n2}\rho\Big]d\rho$$

$$= (1-\nu)\int_{-1}^{+1}\Big[(2\nu-\overline{k}_m\overline{\beta}_{m1})\Big\{k_n\cos\overline{\beta}_{m1}\rho\cos\beta_{n1}\rho$$

$$+\rho\cos\overline{\beta}_{m1}\rho\sin\beta_{n1}\rho+p_n\cos\overline{\beta}_{m1}\rho\sin\beta_{n2}\rho+\rho\cos\overline{\beta}_{m1}\rho$$

$$\times\cos\beta_{n2}\rho\Big\}-(2\nu+\overline{p}_m\overline{\beta}_{m2})\Big\{p_n\sin\overline{\beta}_{m2}\rho\cos\beta_{n1}\rho$$

$$+\rho\sin\overline{\beta}_{m2}\rho\sin\beta_{n1}\rho+p_n\sin\overline{\beta}_{m2}\rho\sin\beta_{n2}\rho+\rho\cos\beta_{n2}\rho\sin\overline{\beta}_{m2}\rho\Big\}$$

$$-\overline{\beta}_{m1}\Big\{k_n\rho\sin\overline{\beta}_{m1}\rho\cos\beta_{n1}\rho+\rho^2\sin\overline{\beta}_{m1}\rho\sin\beta_{n1}\rho$$

$$+p_n\rho\sin\overline{\beta}_{m1}\rho\sin\beta_{n2}\rho+\rho^2\sin\overline{\beta}_{m1}\rho\cos\beta_{n2}\rho\Big\}$$

$$-\overline{\beta}_{m2}\Big\{k_n\rho\cos\overline{\beta}_{m2}\rho\cos\beta_{n1}\rho+\rho^2\cos\overline{\beta}_{m2}\rho\cos\beta_{n1}\rho$$

$$+p_n\rho\cos\overline{\beta}_{m2}\rho\sin\beta_{n2}\rho+\rho^2\cos\overline{\beta}_{m2}\rho\cos\beta_{n2}\rho\Big\}\Big]d\rho$$

Evaluating the integrals in the above expression and rearranging the terms, we have,

$$X_{mn}=(1-\nu)\Big[\frac{\sin(\overline{\beta}_{m1}+\beta_{n1})}{\overline{\beta}_{m1}+\beta_{n1}}\Big\{(2\nu-\overline{k}_m\overline{\beta}_{m1})$$

$$\times\left(k_n+\frac{1}{\overline{\beta}_{m1}+\beta_{n1}}\right)-\overline{\beta}_{m1}\left(\frac{k_n}{\overline{\beta}_{m1}+\beta_{n1}}-1+\frac{2}{(\overline{\beta}_{m1}+\beta_{n1})^2}\right)\Big\}$$

$$+\frac{\sin(\overline{\beta}_{m1}-\beta_{n1})}{\overline{\beta}_{m1}-\beta_{n1}}\Big\{(2\nu-\overline{k}_m\overline{\beta}_{m1})\left(k_n-\frac{1}{\overline{\beta}_{m1}-\beta_{n1}}\right)$$

$$-\overline{\beta}_m\left(\frac{k_n}{\overline{\beta}_{m1}-\beta_{n1}}+1-\frac{2}{(\overline{\beta}_{m1}-\beta_{n1})^2}\right)\Bigg\}+\frac{\sin(\overline{\beta}_{m2}+\beta_{n2})}{\overline{\beta}_{m2}+\beta_{n2}}$$

$$\Bigg\{(2\nu+\overline{p}_m\overline{\beta}_{m2})\left(p_n-\frac{1}{\overline{\beta}_{m2}+\beta_{n2}}\right)-\overline{\beta}_{m2}\left(\frac{p_n}{\overline{\beta}_{m2}+\beta_{n2}}+1\right.$$

$$\left.-\frac{2}{(\overline{\beta}_{m2}+\beta_{n2})^2}\right)\Bigg\}+\frac{\sin(\overline{\beta}_{m2}-\beta_{n2})}{\overline{\beta}_{m2}-\beta_{n2}}\Bigg\{-(2\nu+\overline{p}_m\overline{\beta}_{m2})$$

$$\times\left(p_n+\frac{1}{\overline{\beta}_{m2}-\beta_{n2}}\right)+\overline{\beta}_{m2}\left(\frac{p_n}{\overline{\beta}_{m2}-\beta_{n2}}-1+\frac{2}{(\overline{\beta}_{m2}-\beta_{n2})^2}\right)\Bigg\}$$

$$+\frac{\cos(\overline{\beta}_{m1}+\beta_{n1})}{\overline{\beta}_{m1}+\beta_{n1}}\Bigg\{-(2\nu-\overline{k}_m\overline{\beta}_{m1})+\overline{\beta}_{m1}\left(k_n+\frac{2}{\overline{\beta}_{m1}+\beta_{n1}}\right)\Bigg\}$$

$$+\frac{\cos(\overline{\beta}_{m1}-\beta_{n1})}{\overline{\beta}_{m1}-\beta_{n1}}\Bigg\{(2\nu-\overline{k}_m\overline{\beta}_{m1})+\overline{\beta}_{m1}\left(k_n-\frac{2}{\overline{\beta}_{m1}-\beta_{n1}}\right)\Bigg\}$$

$$+\frac{\cos(\overline{\beta}_{m2}+\beta_{n2})}{\overline{\beta}_{m2}+\beta_{n2}}\Bigg\{(2\nu+\overline{p}_m\overline{\beta}_{m2})+\overline{\beta}_{m2}\left(p_n-\frac{2}{\overline{\beta}_{m2}+\beta_{n2}}\right)\Bigg\}$$

$$+\frac{\cos(\overline{\beta}_{m2}-\beta_{n2})}{\beta_{m2}-\beta_{n2}}\Bigg\{(2\nu+\overline{p}_m\overline{\beta}_{m2})-\overline{\beta}_{m2}\left(p_n+\frac{2}{\overline{\beta}_{m2}-\beta_{n2}}\right)\Bigg\}\Bigg]$$

Also,

$$Y_{mn}=(1-\nu)\int_{-1}^{+1}\Big[-\left\{\overline{k}_m\overline{\beta}_{m1}+(1-2\nu)\right\}\sin\overline{\beta}_{m1}\rho+\Big\{\overline{p}_m\overline{\beta}_{m2}$$

$$-(1-2\nu)\Big\}\cos\overline{\beta}_{m2}\rho+\overline{\beta}_{m1}\rho\cos\overline{\beta}_{m1}\rho-\overline{\beta}_{m2}\rho\sin\overline{\beta}_{m2}\rho\Big]\times$$

$$\left[\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right)\sin\beta_{n1}\rho-\rho\cos\beta_{n1}\rho+\left(\frac{3-4\nu}{\beta_{n2}}-p_n\right)\right.$$

$$\left.\times\cos\beta_{n2}\rho+\rho\sin\beta_{n2}\rho\right]d\rho$$

Evaluating and rearranging in a similar manner, Y_{mn} takes the form given below.

$$Y_{mn}=(1-\nu)\left[\frac{\sin(\overline{\beta_{m1}}+\beta_{n1})}{\overline{\beta_{m1}}+\beta_{n1}}\left\{\left(\overline{k_m}\,\overline{\beta_{n1}}+1-2\nu\right)\left(\frac{3-4\nu}{\beta_{n1}}\right.\right.\right.$$

$$\left.+k_n+\frac{1}{\overline{\beta_{m1}}+\beta_{n1}}\right)+\overline{\beta_{m1}}\left(\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right)\left(\frac{1}{\overline{\beta_{m1}}+\beta_{n1}}\right)-1\right.$$

$$\left.\left.+\frac{2}{(\overline{\beta_{m1}}+\beta_{n1})^2}\right)\right\}+\frac{\sin(\overline{\beta_{m1}}-\beta_{n1})}{\overline{\beta_{m1}}-\beta_n}\left\{-\left(\overline{k_m}\,\overline{\beta_{m1}}+1-2\nu\right)\right.$$

$$\times\left(\frac{3-4\nu}{\beta_{n1}}+k_n-\frac{1}{\overline{\beta_{m1}}-\beta_{n1}}\right)+\overline{\beta_{m1}}\left(\left(\frac{3-4\nu}{\beta_{n1}}+k_n\right)\right.$$

$$\left.\left.\times\left(\frac{-1}{\overline{\beta_{m1}}-\beta_{n1}}\right)-1+\frac{2}{(\overline{\beta_{m1}}-\beta_{n1})^2}\right)\right\}+\frac{\sin(\overline{\beta_{m2}}+\beta_{n2})}{\overline{\beta_{m2}}+\beta_{n2}}$$

$$\times\left\{\left(\overline{p_m}\,\overline{\beta_{m2}}-1+2\nu\right)\left(\frac{3-4\nu}{\beta_{n2}}-p_n+\frac{1}{\overline{\beta_{m2}}+\beta_{n2}}\right)\right.$$

$$\left.\left.-\overline{\beta_{m2}}\left(\left(\frac{3-4\nu}{\beta_{n2}}-p_n\right)\left(\frac{1}{\overline{\beta_{m2}}+\beta_{n2}}\right)-1+\frac{2}{(\overline{\beta_{m2}}+\beta_{n2})^2}\right)\right\}\right.$$

$$+ \frac{\sin(\overline{\beta_{m2}} - \beta_{n2})}{\overline{\beta_{m2}} - \beta_{n2}} \Bigg\{ \left(\overline{p_m}\, \overline{\beta_{m2}} - 1 + 2\nu \right) \left(\frac{3-4\nu}{\beta_{n2}} - p_n \right.$$

$$\left. - \frac{1}{\overline{\beta_{m2}} - \beta_{n2}} \right) - \overline{\beta_{m2}} \left[\left(\frac{3-4\nu}{\beta_{n2}} - p_n \right) \left(\frac{1}{\overline{\beta_{m2}} - \beta_{n2}} \right) \right.$$

$$\left. \left. + 1 - \frac{2}{(\overline{\beta_{m2}} - \beta_{n2})^2} \right] \right\} + \frac{\cos(\overline{\beta_{m1}} + \beta_{n1})}{\overline{\beta_{m1}} + \beta_{n1}} \Bigg\{ - \left(\overline{k_m}\, \overline{\beta_{m1}} + 1 - 2\nu \right)$$

$$\left. - \overline{\beta_{m1}} \left(\frac{3-4\nu}{\beta_{n1}} + k_{n1} + \frac{2}{\overline{\beta_{m1}} + \beta_{n1}} \right) \right\} + \frac{\cos(\overline{\beta_{m1}} - \beta_{n1})}{\overline{\beta_{m1}} - \beta_{n1}}$$

$$\times \left\{ - \left(\overline{k_m}\, \overline{\beta_{m1}} + 1 - 2\nu \right) + \overline{\beta_{m1}} \left(\frac{3-4\nu}{\beta_{n1}} + k_n - \frac{2}{\overline{\beta_{m1}} - \beta_{n1}} \right) \right\}$$

$$+ \frac{\cos(\overline{\beta_{m2}} + \beta_{n2})}{\overline{\beta_{m2}} + \beta_{n2}} \Bigg\{ - \left(\overline{p_m}\, \overline{\beta_{m2}} - 1 + 2\nu \right) + \overline{\beta_{m2}} \left(\frac{3-4\nu}{\beta_{n2}} \right.$$

$$\left. \left. - p_n + \frac{2}{\overline{\beta_{m2}} + \beta_{n2}} \right) \right\} + \frac{\cos(\overline{\beta_{m2}} - \beta_{n2})}{\overline{\beta_{m2}} - \beta_{n2}} \Bigg\{ \left(\overline{p_m}\, \overline{\beta_{m2}} - 1 \right.$$

$$\left. \left. \left. + 2\nu \right) - \overline{\beta_{m2}} \left(- \frac{3-4\nu}{\beta_{n2}} + p_n + \frac{2}{\overline{\beta_{m2}} - \beta_{n2}} \right) \right\} \right]$$

From the interior solution, $w_r^{(2)}(0,\rho)$ can be written as,

$$w_r^{(2)}(0,\rho) = -\left(\frac{\nu}{1-\nu}\right)\left(1 + \frac{\nu}{1-\nu^2}\gamma^2\rho\right)\rho$$

Substituting $\overline{\tau_{12}^{m}}(\rho,\beta_n)$ and $w_r^{(2)}(0,\rho)$ in to eq.(3.35)b, expression for b_m is obtained.

Thus,

$$b_m = -\nu\int_{-1}^{+1}\Big[-\left\{\overline{k_m}\overline{\beta_{m1}} + (1-2\nu)\right\}\rho\sin\overline{\beta_{m1}}\rho + \left\{\overline{p_m}\overline{\beta_{m2}} - (1-2\nu)\right\}\rho\cos\beta_{m2}\rho + \overline{\beta_{m1}}\rho^2\cos\overline{\beta_{m1}}\rho - \overline{\beta_{m2}}\rho^2\cos\overline{\beta_{m2}}\rho\Big]d\rho - \frac{\nu^2}{1-\nu^2}\gamma^2\int_{-1}^{+1}\Big[-\left\{\overline{k_m}\overline{\beta_{m1}} + (1-2\nu)\right\}\rho^2\sin\overline{\beta_{m1}}\rho + \left\{\overline{p_m}\overline{\beta_{m2}} - (1-2\nu)\right\} \times \rho^2\cos\overline{\beta_{m2}}\rho + \overline{\beta_{m1}}\rho^3\cos\beta_{m1}\rho - \overline{\beta_{m2}}\rho^3\sin\overline{\beta_{m2}}\rho\Big]d\rho$$

Representing the two integrals in this expression by I_1 and I_2, we have,

$$b_m = I_1 + I_2$$

These two integrals are now evaluated and simplified in the following lines.

$$I_1 = \Big[-2\left\{\overline{k_m}\overline{\beta_{m1}} + (1-2\nu)\right\}\left\{\frac{\sin\overline{\beta_{m1}}}{(\overline{\beta_{m1}})^2} - \frac{\cos\overline{\beta_{m1}}}{\overline{\beta_{m1}}}\right\} + 2\overline{\beta_{m1}}\left\{\frac{\sin\overline{\beta_{m1}}}{\overline{\beta_{m1}}} - \frac{2}{\overline{\beta_{m1}}}\left(\frac{\sin\overline{\beta_{m1}}}{\overline{\beta_{m1}}^2} - \frac{\cos\overline{\beta_{m1}}}{\overline{\beta_{m1}}}\right)\right\}\Big]$$

Substituting the expression for $\overline{k}_m$ and using eqn.(3.31)a, I_1 reduces to the form given below.

$$I_1 = \frac{4}{\overline{\beta_{m1}} \cos \overline{\beta_{m1}}}$$

Also,

$$I_2 = \left\{ \overline{P_m}\,\overline{\beta_{m2}} - (1-2\nu) \right\} \left[\frac{2 \sin \overline{\beta_{m2}}}{\overline{\beta_{m2}}} - \frac{4}{\overline{\beta_{m2}}} \times \left\{ \frac{\sin \overline{\beta_{m2}} - \overline{\beta_{m2}} \cos \beta_{m2}}{\overline{\beta_{m2}}^2} \right\} \right] - \overline{\beta_{m2}} \left[- \frac{2 \cos \overline{\beta_{m2}}}{\overline{\beta_{m2}}} + \frac{6 \sin \overline{\beta_{m2}}}{\overline{\beta_{m2}}^2} - \frac{12}{\overline{\beta_{m2}}^2} \left\{ \frac{\sin \overline{\beta_{m2}} - \beta_{m2} \cos \beta_{m2}}{\overline{\beta_{m2}}^2} \right\} \right]$$

Separating sine and cosine terms, final expression for I_2 is obtained.

$$I_2 = \sin \overline{\beta_{m2}} \left[\left\{ \overline{P_m}\,\overline{\beta_{m2}} - (1-2\nu) \right\} \left(\frac{2}{\overline{\beta_{m2}}} - \frac{4}{\overline{\beta_{m2}}^3} \right) - \overline{\beta_{m2}} \left(\frac{6}{\overline{\beta_{m2}}^2} - \frac{12}{\overline{\beta_{m2}}^4} \right) \right] + \cos \overline{\beta_{m2}} \left[\frac{4}{\overline{\beta_{m2}}^2} \times \left\{ \overline{P_m}\,\overline{\beta_{m2}} - (1-2\nu) \right\} - \overline{\beta_{m2}} \left\{ \frac{12}{\overline{\beta_{m2}}^3} - \frac{2}{\overline{\beta_{m2}}} \right\} \right]$$

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