# SYNTHESIS OF LINEAR TIME-VARYING SYSTEMS

by

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## ABSTRACT

The subject of this thesis is an analytical investigation of various methods of synthesis of linear time-varying differential systems.

Investigations have been carried out for methods of synthesis from such system specifications as state equations, state variables, differential equation and system characteristic functions.

Methods of synthesis considered here may be divided into two main categories: feedback configuration and parallel elements configuration.

The models described in the first part have been obtained by the use of methods like analog computer simulation technique and differential equation algebra.

In the second category are included methods of synthesis by means of orthonormal functions, sampling theorem and plane impulse train approximation. The synthesis procedures in this category are carried out in two steps: approximation and realization.

The main results of this investigation is extension of the method of synthesis by means of plane impulse train approximation to develop some models for synthesis. These models have the advantage of simplicity in either the time-varying or the time invariant part of the system.

An example of synthesis has been carried out experimentally using one of these models. Experimental results are presented in graphical form.

## ACKNOWLEDGEMENTS

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## CONTENTS

# INTRODUCTION

The subject of this thesis is synthesis of linear time-varying differential systems. But before going into the details of methods of synthesis, it seems expedient to discuss briefly some aspects of analysis and representation of such systems. Thus a brief discussion of the methods of representation of such systems will follow first.

The systems under consideration are linear, i.e. they possess the property of superposition of causes and their respective effects. More specifically,

if the input pair $\{x_1(t),\ x_2(t)\}$ produces the output pair $\{y_1(t),\ y_2(t)\}$,

then the input $a_1x_1(t) + a_2x_2(t)$ produces the output $a_1y_1(t) + a_2y_2(t)$ for all real constants $a_1$ and $a_2$.

The systems are time varying, i.e. their characteristics do not remain constant under translation in time. More specifically,

if the input $x(t)$ produces output $y_1(t)$ and the delayed input $x(t+\tau)$ produces output $y_2(t)$, then

$$y_2(t) \neq y_1(t+\tau) \text{ for some } t \text{ and } \tau.$$

The systems under consideration can be represented by linear differential equation

$$a_n(t)\,\frac{d^n v(t)}{dt^n} \;+\; a_n(t)\,\frac{d^{n-1}v(t)}{dt^{n-1}} \;+\; \dots + a_o(t)v(t)$$

$$= \quad b_m(t)\,\frac{d^m u(t)}{dt^m} \;+\; b(t)_{m-1}\,\frac{d^{m-1}u(t)}{dt^{m-1}} \;+\; \dots + b_o(t)u(t) \tag{1}$$

$m \leq n$; n and m are integers.

where u(t) and v(t) are the input and output respectively of the system, $a_n(t)$'s and $b_m(t)$'s are continuous functions of time in a certain interval $a < t < b$.

Such a differential system may also be characterized by the canonic state equations [1]

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \tag{2}$$

$$\underline{v}(t) = C(t)\,\underline{x}(t) + D(t)\,\underline{u}(t) \tag{3}$$

where $\underline{u}(t)$ and $\underline{v}(t)$ are the k and m dimensional vector input and output respectively, $\underline{x}(t)$ is the n-dimensional vector known as state vector. A(t) and C(t) are nxn and mxn matrices respectively with time-varying elements. B(t) and D(t) are nxk and mxk matrices respectively with time-varying elements.

The property of linearity implies [2] that if the input u(t) can be resolved into a weighted sum or integral of component functions, then the response of the system to u(t) is identical to a weighted sum or integral of the responses of the system to these component functions. Thus, if the input u(t) may be expressed as

$$u(t) = \int_C k(t, \lambda)\, U(\lambda)\, d\lambda \tag{4}$$

where $\{k(t, \lambda)\}$ is a set of component functions and $U(\lambda)$ is the spectrum of u(t) with respect to $k(t, \lambda)$, then, the output is given by

$$v(t) = \int_C K(t, \lambda)\, U(\lambda)\, d\lambda \tag{5}$$

where $K(t, \lambda)$ is the response of the system to an input $k(t, \lambda)$. $K(t, \lambda)$ is called the characteristic function of the system and may be used as a convenient representation of such systems.

## Particular Cases of Characteristic Functions.

The following particular cases of characteristic functions are considered.

1. Time-domain specification [3].

   (a) $k(t, \lambda) \longrightarrow \delta(t-\tau)$ = delayed unit impulse.

   $K(t, \lambda) \longrightarrow h_1(t, \tau)$ = the impulse response of the system.

In terms of the impulse response response $h_1(t, \tau)$ the response of the system is given by

$$v(t) = \int_{-\infty}^{t} h_1(t, \tau)\, u(\tau)\, d\tau \tag{6}$$

(b) Another important form of the impulse response is $h_3(y, t)$ [4], where $y = t - \tau$ is the age variable.

In terms of $h_3(y,t)$ the output of the system is given by

$$v(t) = \int_0^{\infty} h_3(y,t)\ u(t-y)\ dy \tag{7}$$

2. <u>Mixed Time-frequency Domain Specification.</u>

(a) $k(t,\lambda) \rightarrow \frac{e^{st}}{2\pi j}$; $\lambda \rightarrow s$, complex input frequency

$$K(t,\lambda) \longrightarrow H(s,t)\ \frac{e^{st}}{2\pi j}.$$

$H(s,t)$ is called the H-System function, and in the particular case when $s = j\nu$, then $H(\nu, t)$ is called the frequency response function of the system.

In terms of $H(s,t)$ the output is given by

$$v(t) = \frac{1}{2\pi j} \int_{\mathcal{C}} H(s,t)\ U(s)\ e^{st}\ ds. \tag{8}$$

where $U(s)$ is the spectrum of the input $u(t)$.

(b) $k(t,\lambda) \rightarrow e^{-m\tau}$, $t \rightarrow \tau$ and $\lambda \rightarrow m$, complex output frequency.

$$K(t,\lambda) \longrightarrow G(m,\tau).$$

$G(m,\tau)$ is called the G-System function [5].

In terms of $G(m, \tau)$ the output is given by

$$V(m) = \int_{-\infty}^{\infty} G(m, \tau) \; u(\tau) \; e^{-m\tau} d\tau \tag{9}$$

where V(m) is the output spectrum.

## 3. Frequency Domain Specification [3] .

$k(t, \lambda) \rightarrow \delta(n-s)$; $t \rightarrow n$, complex frequency of system variation; $\lambda \rightarrow s$.

$K(t, \lambda) \rightarrow \Gamma(n, s)$.

$\Gamma(n, s)$ is called the bifrequency system function.

In terms of $\Gamma(n, s)$, the output spectrum of the system is given by.

$$V(n) = \int_c \Gamma(n, s) \; U(s) \; ds \tag{10}$$

This concludes the discussion of various ways of representation of such systems.

Next follows a brief introduction to the problem of synthesis of such systems.

From the above discussion it is evident that the characteristics of linear time-varying systems depend on the absolute times of application of the input and measurement of the output, but not on the relative time, i. e. the difference of time between the application of input and measurement of output.

Consequently, synthesis of time-varying systems involves use of parameters or elements, some or all of which, possess characteristics which vary with time. Thus, synthesis of elements with time-varying characteristics is a problem peculiar to the synthesis of time-varying systems in contrast to the synthesis of time-invariant systems which may be completed with constant parameter elements.

This problem introduces most of the complexities and difficulties in the synthesis of time-varying systems.

Methods of synthesis of linear time-varying systems in the following two forms are considered:

1. Methods of synthesis in feedback configuration,
2. Methods of synthesis in parallel elements configuration.

In general it is more difficult to analyze and synthesize a system in the feedback configuration, but this configuration has the potential advantage of reducing the noise and controlling the sensitivity of the system to component variations.

Another fact of great importance is that the synthesis procedures for time-varying systems are simplified in case of systems which may be divided into a time-invariant part and a time-varying part. Systems for which this separation is possible are known as separable systems. The time-invariant portion possesses a memory T which may approach infinity, but the time-varying part is memoryless.

# PART 1

## METHOD OF SYNTHESIS IN FEEDBACK CONFIGURATION.

This part consists of two chapters.

# CHAPTER 1

## FEEDBACK CONFIGURATIONS USING ANALOG COMPUTER TECHNIQUE.

### 1.1. Introduction.

This chapter deals with methods of synthesis of linear time-varying systems in feedback form using analog computer elements e.g. integrators, adders and time-varying scale factor potentiometers, as elementary building blocks. [6]

Methods of synthesis based on the following three specifications are considered.

1. State equations,
2. State variables,
3. Differential equation.

### 1.2. Method of Synthesis from System State Equations. [1]

In case a system S is described by the general n-th order differential equation

$$L(p,t)\;v(t) \;=\; M(p,t)\;u(t) \tag{1.2.1}$$

where

(i) $$L(p,t) \triangleq \sum_{\ell=0}^{n} \alpha_{n-\ell}(t)\,p^{\ell}$$

$$M(p,t) \triangleq \sum_{\ell=0}^{m} \beta_{m-\ell}(t)\,p^{\ell} \quad \text{with } m \leq n$$

ii) $p \triangleq \frac{d}{dt}$ and $\alpha_o(t) = 1$ for all t.

iii) $\alpha_i(t)$'s are supposed to possess continuous derivatives of order n-i for all i = 0, 1, 2, ... n. Similarly $\beta_i(t)$'s possess continuous derivatives of order m-i for all i = 0, 1, 2, ... m. and

iv) u(t) and v(t) are scalar input and output of the system respectively.

The system can also be described by the canonical state equations,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & 0 & \dots & \dots & 1 \\ -\alpha_n(t), & \cdot & \cdot & \cdot & \cdot & -\alpha_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \vdots \\ \vdots \\ \gamma_n(t) \end{bmatrix} u(t) \tag{1.2.2}$$

and

$$v(t) = x_1(t) + \gamma_o(t)\, u(t) \tag{1.2.3}$$

where $\gamma_i(t)$'s are given by,

$$\gamma_o(t) = \beta_o(t)$$

$$\gamma_j(t) = \beta_j(t) - \sum_{\ell=o}^{j-1} \sum_{s=o}^{j-\ell} \binom{n+s-J}{n-J} \alpha_{j-\ell-s}(t) \quad p^s \gamma_\ell(t) \tag{1.2.4}$$

This can be verified by putting the different derivatives of v(t). i.e. v(t), $v^{(1)}(t)\ldots\ldots v^{(n-1)}(t)$, as obtained from (1.2.2) and (1.2.3) into equation (1.2.1) with the values of $\gamma_j(t)$ given in equation (1.2.4).

Since, equation (1.2.1) describes a general differential system, the problem of synthesis from state equations of such systems is considered.

A system which is described by state equations (1.2.2) and (1.2.3) or whose state equations can be reduced to this form may be realized in a feedback configuration as shown in figure 1.1.

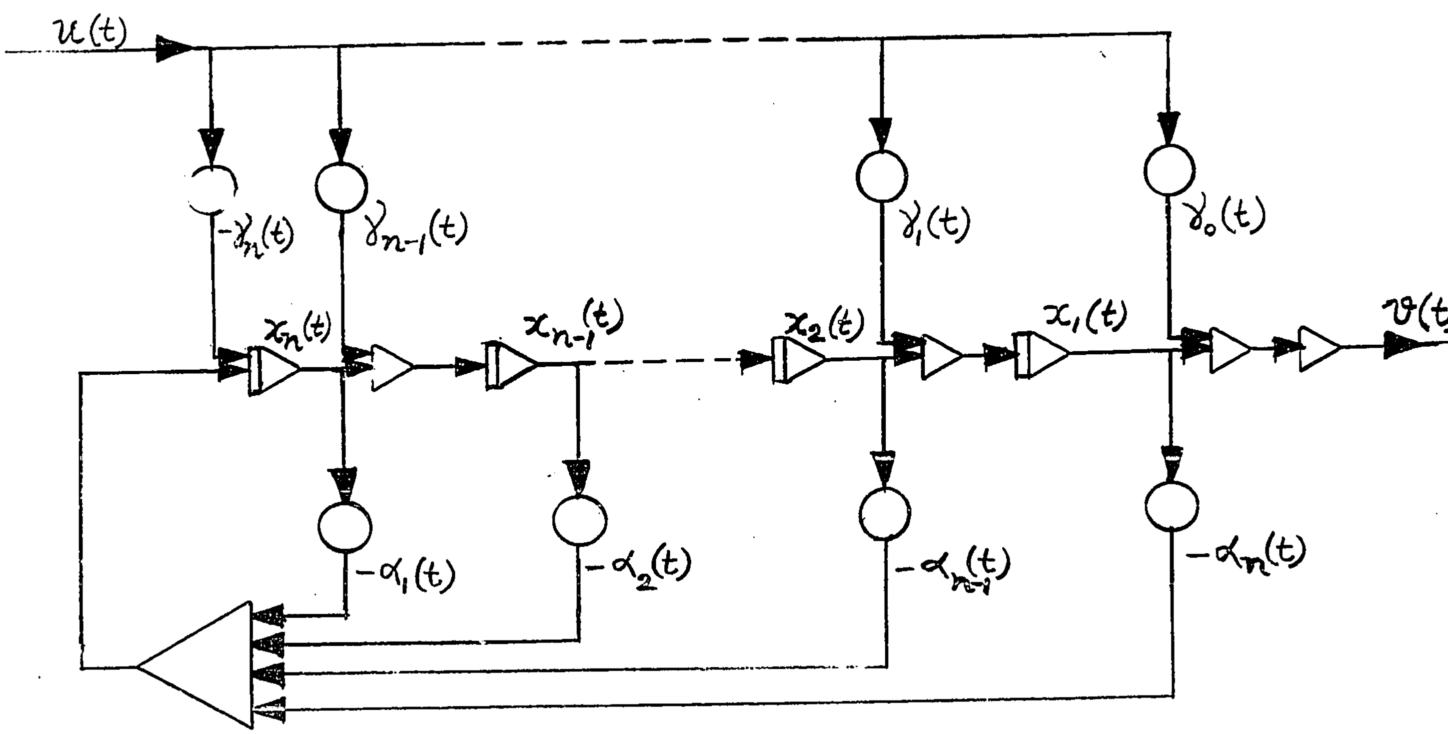
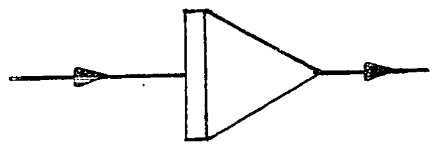
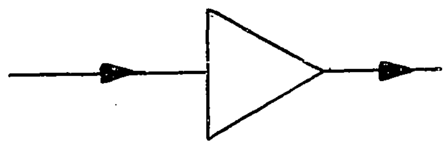



Fig. 1.1. Realization from the State Equations (1.2.2) and (1.2.3).

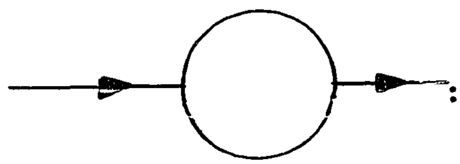
where,



: represents an integrator with sign inversion.



: represents an adder with sign inversion.



: represents a scale factor potentiometer.

## 1.3. Method of Synthesis from State Variables. [1] .

Synthesis of a linear time-varying system from state variables is based mainly on the following two facts.

1. It is always possible to construct a network of 'integrators', 'adders' and 'time-varying scale factor potentiometers' which is equivalent to a given linear time-varying differential system.

2. Given the realization $S_1$ of a linear time-varying system S as a network of 'integrators', adders and 'time-varying potentiometers', the state vector $\underline{x}(t)$ of the system can be identified with a vector $\underline{x}(t) = (x_1(t), x_2(t), \ldots x_n(t))$ . Where the components of $\underline{x}(t)$ are the state vectors of those elements of the configuration $S_1$ which are not memoryless, in this case the integrators.

Further, the state of an integrator at time t is its output, or more generally, a constant times its output at time t. Thus it can be infered that a vector $\underline{x}(t)$ having as components the outputs of the integrators qualifies as the state vector of $S_1$ and equivalently that of the system S. Conversely, given the state vector $\underline{x}(t)$ of the system S its components may be attributed to the outputs of the different integrators in $S_1$.

Thus, given a state vector $\underline{x}(t) = (x_1(t), x_2(t) \ldots x_n(t))$ of a system, it can be realized in a feedback configuration using analog computer elements in the following three steps:

1) An interconnection of integrators, adders and scale factor potentiometers is constructed as shown in figure 1.2 with the number of integrators equal to the number of components of the given state vector, where u(t) and v(t) are the scalar input and output respectively of the system.

2) The outputs of the integrators in figure 1.2 are put equal to the components of the given state vector $\underline{x}(t)$.

3) The scale factor potentiometers are then evaluated from the relations

$$v(t) = -x_1(t) - \gamma_o(t)\, u(t)$$

$$\dot{x}_1(t) = -x_2(t) - \gamma_1(t)\, u(t)$$

$$\dot{x}_2(t) = -x_3(t) - \gamma_2(t)\, u(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\dot{x}_n(t) = \frac{a_{n-1}(t)}{a_n(t)}\, x_n(t) + \frac{a_{n-2}(t)}{a_n(t)}\, x_{n-1}(t) + \ldots$$

$$+ \frac{a_1(t)}{a_n(t)}\, x_2(t) + \frac{a_o(t)}{a_n(t)}\, x_1(t) - \gamma_n(t)\, u(t).$$

with $a_n(t) \neq 0$.

These equations are obtained from figure 1.2.

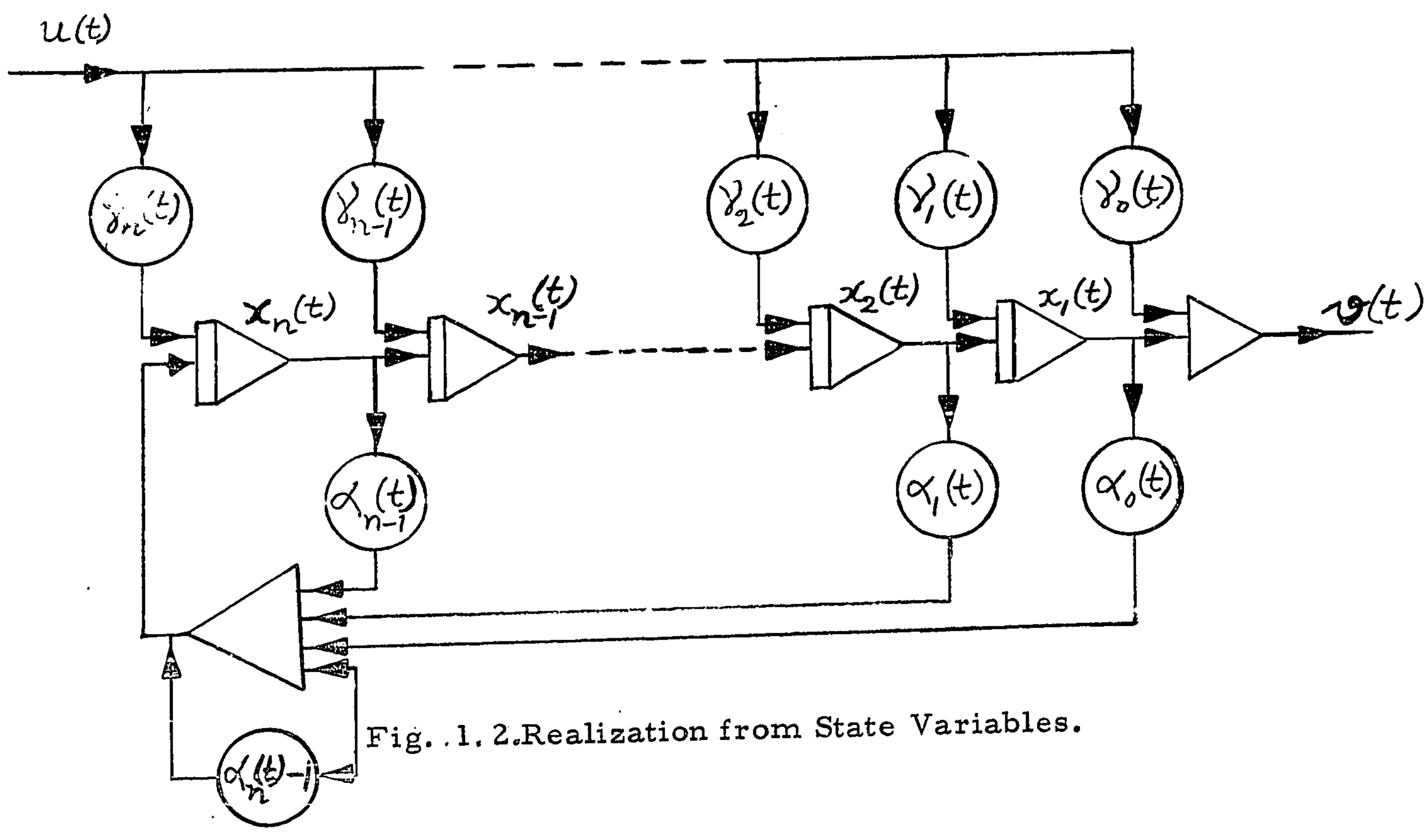



Fig. . 1. 2. Realization from State Variables.

## 1.4. Method of Synthesis from the System Differential Equation. [6].

A linear time-varying system described by equation (1.4.1) is considered.

$$L(p,t)\, v(t) = M(p,t)\, u(t) \tag{1.4.1}$$

where

$$L(p,t) \triangleq \sum_{i=o}^{n} a_i(t)\, p^i \tag{1.4.2}$$

and

$$M(p,t) \triangleq \sum_{i=o}^{m} b_i(t)\, p^i \tag{1.4.3}$$

with

$$p \triangleq \frac{d}{dt} \quad \text{and } m \leq n.$$

A realization of the system in the form of figure 1.3 using integrators, adders and scale factor potentiometers is assumed.

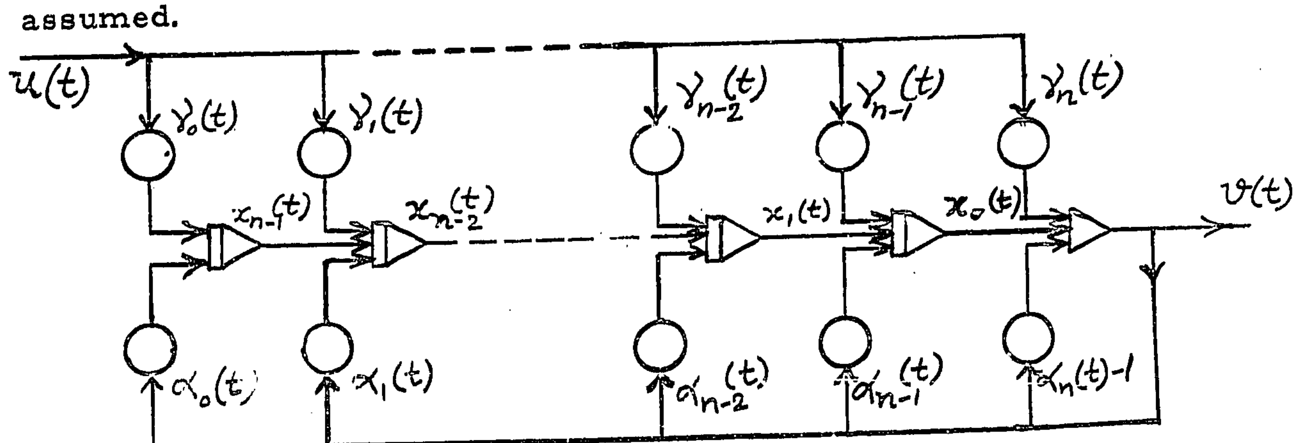



Fig. 1.3. Realization from Differential Equation.

The hitherto undetermined coefficients $\alpha_i(t)$'s and $\gamma_i(t)$'s may be found as follows.

The outputs $x_k(t)$'s; $k = 0, 1, 2, \ldots n-1$ of the integrators in figure 1.3 satisfy the following relations:

$$
\begin{aligned}
x_o(t) &= -\gamma_n(t)\, u(t) - \alpha_n(t)\, v(t) && \text{(a)}\\
-\dot{x}_o(t) &= x_1(t) + \gamma_{n-1}(t)\, u(t) + \alpha_{n-1}(t)\, v(t) && \text{(b)}\\
-\dot{x}_1(t) &= x_2(t) + \gamma_{n-2}(t)\, u(t) + \alpha_{n-2}(t)\, v(t) && \text{(c)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \text{(1.4.4)}\\
-\dot{x}_{k-1}(t) &= x_k(t) + \gamma_{n-k}(t)\, u(t) + \alpha_{n-k}(t)\, v(t) && \text{(e)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
-\dot{x}_{n-1}(t) &= \gamma_o(t)\, u(t) + \alpha_o(t)\, v(t) && \text{(f)}
\end{aligned}
$$

From equation (1.4.4e)

$$x_k(t) = -\dot{x}_{k-1}(t) - \gamma_{n-k}(t)\, u(t) - \alpha_{n-k}(t)\, v(t) \tag{1.4.5}$$

Putting the values of $\dot{x}_{k-1}(t)$, $\dot{x}_{k-2}(t)$ etc as required in equation (1.4.5) from equation (1.4.4), one finally gets,

$$x_k(t) = -\sum_{\ell=0}^{k} (-1)^{k-\ell} \left(\alpha_{n-\ell}(t)\, v(t)\right)^{(k-\ell)} - \sum_{1=0}^{k} (-1)^{k-\ell} \left(\gamma_{n-s}(t)\, u(t)\right)^{(k-\ell)} \tag{1.4.6}$$

Extending to the case when $k = n$, one gets from equations (1.4.5) and (1.4.6)

$$-\dot{x}_{n-1}(t) = x_n(t) + \gamma_o(t)\, u(t) + \alpha_o(t)\, v(t) \tag{1.4.7}$$

and

$$x_n(t) = \sum_{\ell=0}^{n} (-1)^{n-\ell} \left(\alpha_{n-\ell}(t)\, v(t)\right)^{(n-\ell)} - \sum_{\ell=0}^{n} (-1)^{n-\ell} \left(\gamma_{n-s}(t)\, u(t)\right)^{(n-\ell)} \tag{1.4.8}$$

respectively.

Putting values of $\dot{x}_{n-1}(t)$ and $x_n(t)$ into equation (1.4.7) from equation (1.4.4f) and equation (1.4.8) respectively one finally gets,

$$\sum_{\ell=0}^{n} (-1)^{n-\ell} \left(\alpha_{n-\ell}(t)\, v(t)\right)^{(n-\ell)} = \sum_{\ell=0}^{n} (-1)^{n-\ell} \left(\gamma_{n-s}(t)\, u(t)\right)^{(n-\ell)}$$

or, equivalently,

$$\sum_{i=0}^{n} (-1)^{i} \left(\alpha_{i}(t)\ \ v(t)\right)^{(i)} = - \sum_{i=0}^{n} (-1)^{i} \left(\gamma_{i}(t)\ \ u(t)\right)^{(i)} \tag{1.4.9}$$

Thus, the setup in figure 1.3 is also a realization of equation (1.4.9) and consequently, equation (1.4.9) is equivalent to equation (1.4.1).

Now, equation (1.4.9) may be identified with the adjoint of equation (1.4.10).

$$\sum_{i=0}^{n} \alpha_{i}(t) \frac{d^{i}v(t)}{dt^{i}} = - \sum_{i=0}^{n} \gamma_{i}(t) \frac{d^{i}u(t)}{dt^{i}} \tag{1.4.10}$$

Consequently, equation (1.4.10) is also adjoint to equation (1.4.1).

Thus,

$$L^{*}(p,t)\,v(t) = \sum_{i=0}^{n} \alpha_i(t)\,\frac{d^i v(t)}{dt^i} \tag{1.4.11}$$

and

$$M^{*}(p,t)\,u(t) = -\sum_{i=0}^{n} \gamma_i(t)\,\frac{d^i u(t)}{dt^i} \tag{1.4.12}$$

Values of the scale factors $\alpha_n(t)$'s and $\gamma_n(t)$'s may be obtained from equations (1.4.11) and (1.4.12) respectively.

Since, two differential operators which are adjoint to each other are of the same order and the operators $L(p,t)$ and $M(p,t)$ are of orders n and m respectively, therefore, it follows that,

$$\gamma_{m+\ell}(t) = 0 \quad \text{for } \ell = 1, 2, \ldots n-m \tag{1.4.13}$$

further,

$$\gamma_m(t) = (-1)^m b_m(t)$$

$$\gamma_{m-1}(t) = (-1)^m m b_m^{(1)}(t) + (-1)^{m-1} b_{m-1}(t).$$

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and in general

$$\gamma_{m-\ell}(t) = \sum_{i=0}^{\ell} (-1)^{m-i} \frac{(m-i)!}{(m-\ell)!\,(\ell-i)!}\, b_{m-i}^{(\ell-i)}(t) \qquad (1.4.14)$$

for $\ell = 0, 1, 2, \ldots m.$

Likewise the time-varying scale factors $\alpha_n(t)$'s are given by

$$\alpha_{n-\ell}(t) = \sum_{i=0}^{\ell} (-1)^{n-i} \frac{(n-i)!}{(n-\ell)!\,(\ell-i)!}\, a_{n-i}^{(\ell-i)}(t) \qquad (1.4.15)$$

for $\ell = 0, 1, 2, \ldots n.$

Thus once the time-varying scale factors $\alpha_n(t)$'s and $\gamma_n(t)$'s are determined from equations (1.4.15) and (1.4.14) respectively, the realization in the configuration of figure 1.3 is complete.

# CHAPTER 2

## FEEDBACK CONFIGURATIONS USING DIFFERENTIAL EQUATION ALGEBRA AND SOME OTHER CONCEPTS.

### 2.1. Introduction.

Methods of synthesis of linear time-varying systems in feedback configuration from the following three specifications are considered.

1) Impulse response $h_1(t, \tau)$
2) H-system function $H(s, t)$
3) System differential equation.

### 2.2. Method of Synthesis from Impulse Response.

The problem considered in this section is: Given the impulse response $h_1(t, \tau)$ of a system, how to synthesize the system in a feedback configuration.

An arbitrary feedback configuration of figure 2.2.1 is considered

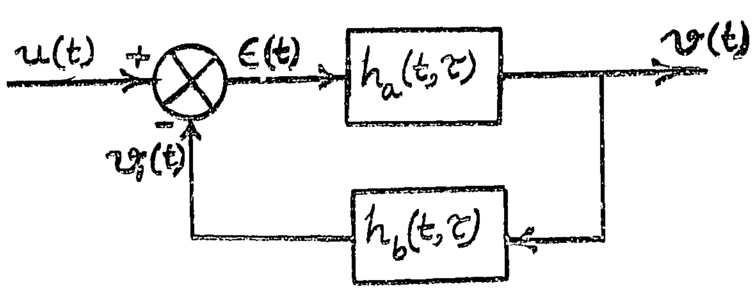


Fig. 2.2.1. Feedback Configuration.

The components in the forward path and the feedback path have the impulse responses $h_a(t, \tau)$ and $h_b(t, \tau)$ respectively. $h_1(t, \tau)$ is assumed to be the overall impulse response of the configuration. First the relation between the impulse responses $h_a(t, \tau)$, $h_b(t, \tau)$ and $h_1(t, \tau)$ is established as follows. If a delta function $\delta(t-\tau)$ is applied at the input of the feedback configuration, then the output is $h_1(t, \tau)$.

Then the output $v_1(t)$ of the block in the feedback path is

$$v_1(t) = \int_{\tau}^{t} h_b(t, \xi)\, h_1(\xi, \tau)\, d\xi \tag{2.2.1}$$

and the input to the block in the forward path is

$$\epsilon(t) = \delta(t-\tau) - \int_{\tau}^{t} h_b(t, \xi)\, h_1(\xi, \tau)\, d\xi \tag{2.2.2}$$

and the output of the block in the forward path is $h_1(t, \tau)$.

So,

$$\begin{aligned} h_1(t, \tau) &= \int_{-\infty}^{t} h_a(t, \rho)\, \epsilon(\rho)\, d\rho \\ &= \int_{\tau}^{t} h_a(t, \rho) \left[\delta(\rho-\tau) - \int_{\tau}^{\rho} h_b(\rho, \xi)\, h_1(\xi, \tau)\, d\xi\right] d\rho \end{aligned} \tag{2.2.3}$$

Making use of the fact that

$$\int_{\tau}^{t} h_a(t, \rho)\, \delta(\rho-\tau)\, d\rho = h_a(t, \tau) \tag{2.2.4}$$

equation (2.2.3) reduces to

$$h_1(t,\tau) = h_a(t,\tau) - \int_{\tau}^{t} h_a(t,\varrho)\, d\varrho \int_{\tau}^{\varrho} h_b(\varrho,\xi)\, h_1(\xi,\tau)\, d\xi \qquad (2.2.5)$$

If the block in the feedback path is an amplifier with unit gain i.e. if $h_b(\varrho,\xi) = \delta(\varrho-\xi)$, then equation (2.2.5) may be put in the form

$$h_1(t,\tau) = h_a(t,\tau) - \int_{\tau}^{t} h_a(t,\varrho)\, h_1(\varrho,\tau)\, d\varrho \qquad (2.2.6)$$

Equation (2.2.5) is the relation expressing the overall system impulse response in terms of the impulse responses of the components in the forward and the feedback paths.

If the synthesis problem does not involve any fixed plant i.e. if neither $h_a(t,\tau)$ nor $h_b(t,\tau)$ are previously fixed then the designer is free to choose any one of them and then solution of equation (2.2.5) specifies the other component.

But generally, in a design problem, the overall required impulse response is specified and some existing parts of the system are also provided, which are either not allowed or not desired to be varied. These parts are called the 'fixed plants' Under these conditions, in order to obtain the required overall characteristics one has to introduce some compensating links either in the forward path or in the feedback path. Thus some components of the system are already fixed and some of them are at the designer's disposal.

If the system is assumed to be synthesized in the form of figure 2.2.1, then two cases may arise.

1) $h_1(t,\tau)$ and $h_b(t,\tau)$ are specified and $h_a(t,\tau)$ is to be determined.

2) $h_1(t,\tau)$ and $h_a(t,\tau)$ are specified and $h_b(t,\tau)$ is to be determined.

These two cases are dealt with below.

Case 1.

It is seen from equation (2.2.5) that to determine $h_a(t,\tau)$ one has to solve the integral equation (2.2.5) for $h_a(t,\tau)$ which is a Volterra integral equation of the second kind with kernel

$$\int_{\tau}^{\rho} h_b(\rho,\xi)\, h_1(\xi,\tau)\, d\xi \tag{2.2.7}$$

Thus the problem reduces to solving a Volterra equation of the second kind, the solution of which may, in most cases, be carried out by the method of successive approximations.

Case 2.

Again from equation (2.2.5) it follows that determination of $h_b(t,\tau)$ may be completed in two steps.

i) Solving equation (2.2.5) to obtain $\int_{\tau}^{\rho} h_b(\rho,\xi)\, h_1(\xi,\tau)\, d\xi$ as a function of $\tau$ and $\rho$.

ii) from the knowledge of the value of this integral to solve the resulting equation for $h_b(\rho,\xi)$.

It is observed that both these steps involve solving Volterra integral equations of the first kind. Solution of Volterra integral equations of the first kind is very difficult and requires computers for solution with sufficient speed and accuracy. Further, Volterra integral equations of the first kind may not have solutions.

Malchikov [7] has shown that the problem of determining $h_b(t, \tau)$ from equation (2. 2. 5) by solving a Volterra integral equation of the first kind can be reduced to a simpler problem of solving a Volterra integral equation of the second kind by making use of the concept of inverse systems.

## 2. 3. Method of Synthesis from H-System Function H(s, t).

Given an H-System function H(s, t), a realization of it in the feedback configuration of figure 2. 3. 1. is assumed.

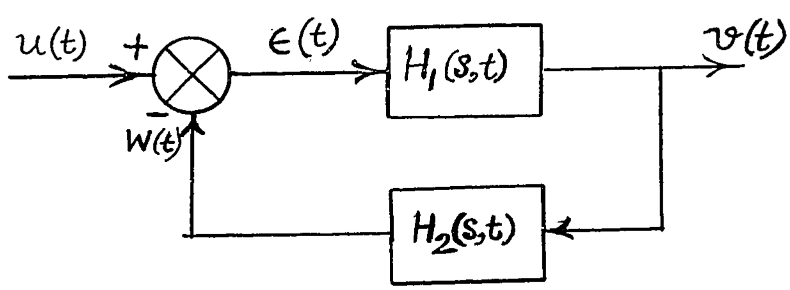


Fig. 2. 3. 1. Realization of H(s, t).

The relation between H(s, t), $H_1(s, t)$ and $H_2(s, t)$ in such a configuration is derived as follows:

In the operational form the output v(t) of a system having system function H(s, t) due to an input u(t) is expressed as

$$v(t) = H(p, t)\, u(t) \triangleq \frac{1}{2\pi j} \int_{\mathcal{C}} H(s, t)\, U(s)\, e^{st} ds \tag{2.3.1}$$

where $p = \frac{d}{dt}$ and U(s) is the spectrum of u(t).

Using this relation, the output of the system in figure 2.3.1, when the input u(t) is equal to $e^{st}$, is:

$$v(t) = H(p, t)\, e^{st} = H(s, t)\, e^{st}$$

Further,

$$\begin{aligned} w(t) &= H_2(p, t)\, v(t) = H_2(p, t) \left[H(s, t)\, e^{st}\right] \\ &= e^{st} H_2(p+s, t) \left[H(s, t)\right]. \end{aligned}$$

$$\text{and } \epsilon(t) = e^{st} - w(t) \tag{2.3.2}$$

Consequently,

$$\begin{aligned} H(s, t)\, e^{st} &= H_1(p, t) \left[\epsilon(t)\right] \\ &= H_1(p, t) \left[e^{st} - w(t)\right] \\ &= e^{st} H_1(s, t) - H_1(p, t) \left[e^{st} H_2(p+s, t)\left[H(s, t)\right]\right] \\ &= e^{st} H_1(s, t) - H_1(p, t) \left[e^{st} H_3(s, t)\right] \end{aligned}$$

where $H_3(s, t) \triangleq H_2(p+s, t)\left[H(s, t)\right]$.

or

$$H(s,t)\, e^{st} = e^{st} H_1(s,t) - e^{st} H_1(p+s,t) \left[ H_3(s,t) \right]$$
$$= e^{st} H_1(s,t) - e^{st} H_1(p+s,t) \left[ H_2(p+s,t) \left[ H(s,t) \right] \right]$$

So,

$$H(s,t) = H_1(s,t) - H_1(p+s,t) \left[ H_2(p+s,t) \left[ H(s,t) \right] \right] \tag{2.3.3}$$

In equation (2.3.3), $H_1(p+s,t)$ and $H_2(p+s,t)$ represent operators as defined in equation (2.3.1).

Equation (2.3.3) gives a relation between the overall system function $H(s,t)$, the forward path system function $H_1(s,t)$ and the feedback path system function $H_2(s,t)$.

Thus, from a knowledge of any two of these system functions, the remaining one may be found from equation (2.3.3) as required by the synthesis problem.

## 2.4. Method of Synthesis from Differential Equation.

### 2.4.1. Introduction.

This section concerns with a method of synthesis of systems in feedback configuration from the knowledge of its differential equation.

This method is due to stubberud [8] and involves the use of a differential operator algebra. This algebra is discussed in details in [8] and [9] . So, here use will be made of certain results of this algebra without proof.

More specifically, the following operations, defined in [8] will be made use of.

1) Multiplication of two differential equations

2) Multiplication of a differential equation and a scalar

3) Addition of two differential equations

Further, use will be made of the following two entities:

1) The zero element

2) Inverse of a given element.

2.4.2. The Synthesis Problem.

The problem of synthesis defined in the introduction 2.4.1 is discussed here in details.

A feedback configuration shown in figure 2.4.1 is assumed, where G, S and H are differential equations of form of equation (2.4.1).

$$\sum_{i=0}^{n} a_i(t) \frac{d^i v(t)}{dt^i} = \sum_{j=0}^{n} b_j(t) \frac{d^j u(t)}{dt^j} \qquad (2.4.1)$$

with u(t) and v(t), the input and output respectively.

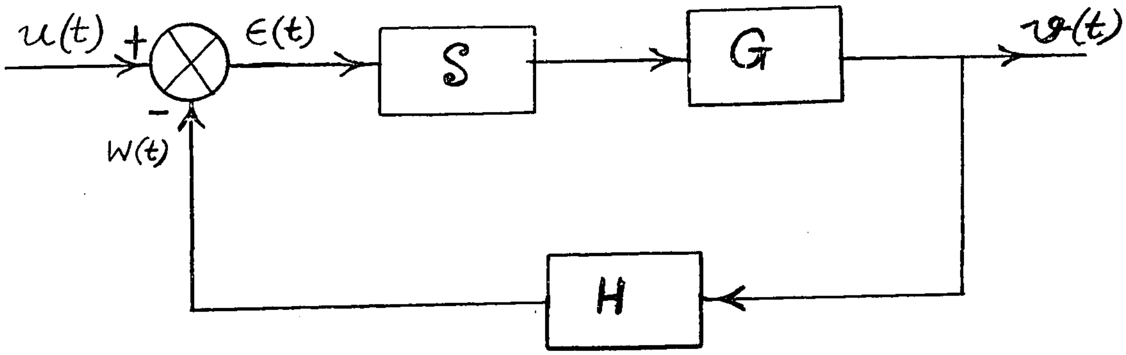


Fig. 2.4.1. General Feedback Configuration.

If M is the overall differential equation of the system, then the input and output are related as

$$v(t) \triangleq M\left[u(t)\right] \tag{2.4.2}$$

where M $\left[u(t)\right]$ symbolises the operation of the differential equation M on u(t).

From figure (2.4.1) it follows that,

$$w(t) = H\left[v(t)\right] \tag{2.4.3}$$

and

$$\epsilon(t) = u(t) - w(t) = u(t) - H\left[v(t)\right] \tag{2.4.4}$$

Since

$$v(t) = GS\left[\epsilon(t)\right] \tag{2.4.5}$$

So,

$$u(t) = I\left[\epsilon(t)\right] + HGS\left[\epsilon(t)\right] = (I+HGS) \quad (t) \tag{2.4.6}$$

as

$$I\left[\epsilon(t)\right] = \epsilon(t) \tag{2.4.7}$$

Multiplying both sides of (2.4.6) by the multiplicative inverse of (I+HGS), one gets

$$(I+HGS)^{-1} \; u(t) = \epsilon(t) \qquad (2.4.8)$$

Substituting for $\epsilon(t)$ from equation (2.4.8) into equation (2.4.5) one gets

$$v(t) = GS\,(I+HGS)^{-1} \left[u(t)\right] \qquad (2.4.9)$$

Then comparison of equations (2.4.2) and (2.4.9) gives

$$M = GS(I+HGS)^{-1} \qquad (2.4.10)$$

Equation (2.4.10) may be considered as the fundamental relationship for figure 2.4.1.

An important special case of the configuration in figure 2.4.1 is the configuration with "unity feedback".

<u>Two cases may arise:</u>

<u>Case 1.</u>

The unity feedback configuration with no fixed plant. (A fixed plant is a part of the system which is in existence and is not desirable to be varied). Under these conditions figure 2.4.1 reduces to figure 2.4.2.

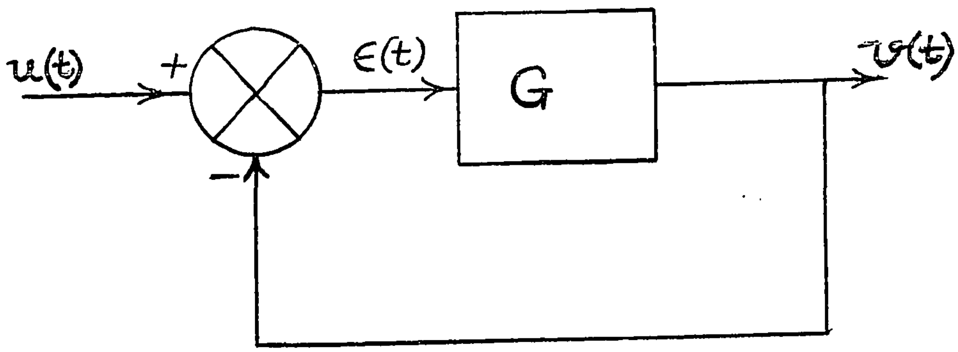


Fig. 2.4.2. Unity Feedback Configuration Without Any Fixed Plant.

Then the problem reduces to finding G in terms of known overall differential equation M.

Putting S = H = I in this case, equation (2.4.10) reduces to

$$M = G(I + G)^{-1} \qquad (2.4.11)$$

or,

$$G = (I - M)^{-1} M \qquad (2.4.12)$$

Then the feedforward element G in figure 2.4.2 is obtained in terms of M from equation (2.4.12).

Case 2.

The unity feedback configuration with fixed plant.

In this case the configuration of figure 2.4.1 reduces to figure 2.4.3.

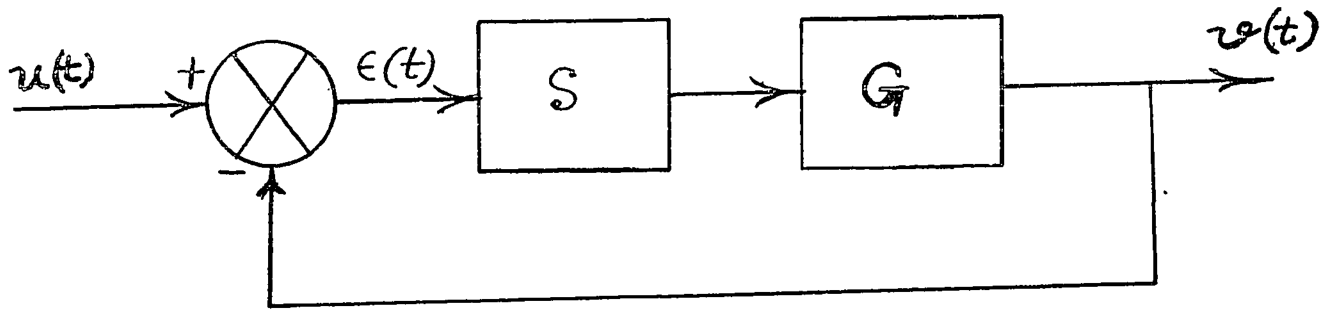


Fig. 2.4.3. Unity Feedback Configuration with a Fixed Plant.

Here the fixed plant is represented by the differential equation G. The problem is to determine the compensating network S so that the overall differential equation becomes equivalent to M.

Putting H = I in this case equation (2.4.10) reduces to

$$M = GS(I+GS)^{-1} \tag{2.2.13}$$

From equation (2.2.13) the compensating element S is given by

$$S = G^{-1}\,(I-M)^{-1}\,M \tag{2.4.14}$$

from which S may be determined from knowledge of the fixed plant differential equation G and the overall differential equation M.

One advantage in this method is that all manipulations can be performed symbolically and the numerical calculations may be done only at the end of the process.

This concludes the discussion on the synthesis of linear time-varying systems in feedback configuration.

# PART II

## METHODS OF SYNTHESIS IN PARALLEL ELEMENTS CONFIGURATION

This part consists of three chapters:

Chapter 3. Methods of Synthesis by Means of Orthonormal Functions.

Chapter 4. Methods of Synthesis by Means of Sampling Theorem.

Chapter 5. Methods of Synthesis by Means of Plane Impulse Train Approximation.

# CHAPTER 3

## METHODS OF SYNTHESIS BY MEANS OF ORTHONORMAL FUNCTIONS.

### 3.1. Introduction.

A set of functions $\{W_m\}$ is said to be orthogonal over the n-dimensional subspace $(a_i \leq x_i \leq b_i;\ i=1, \ldots n)$ of the Euclidean n-space $E^n$. if

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} g(x_1, \ldots x_n)\, W_m(x_1, \ldots x_n)\, \overline{W}_n(x_1, \ldots x_n)\, dx_1 \ldots dx_n.$$

$$\begin{aligned} &= 0 \text{ for } m \neq n \\ &\neq 0 \text{ for } m = n \end{aligned} \qquad (3.1.1)$$

These functions are called orthogonal and normalised (orthonormal) if

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} g(x_1, \ldots x_n)\, W_m(x_1, \ldots x_n)\, \overline{W}_n(x_1, \ldots x_n) dx_1, \ldots dx_n = 1 \quad \text{for } m = n \qquad (3.1.2)$$

where $x_1, \ldots x_n$ represent all the independent variables, the weighting function g is a fixed real positive function of the independent variables and is usually taken equal to unity. The interval (a, b) is the same for all $W_m$'s and '‾' indicates corresponding complex conjugates.

One of the important practical uses of system of orthonormal functions is their use in approximating an arbitrary function.

Thus, if the one-dimensional functions $W_0(x_1)$, $W_1(x_1) \ldots W_n(x_1)$ are integrable in the square and form a set of orthonormal functions defined over the interval (a, b) and $f(x_1, x_2)$ is an arbitrary function defined over the same interval such that

$$\int_a^b \left| f(x_1, x_2) \right|^2 dx_1 < \infty$$

i. e. $f(x_1, x_2)$ is integrable in the square, then one way of approximating the function in terms of the orthonormal functions is to express it as a linear combination

$$a_0(x_2) W_0(x_1) + a_1(x_2) W_1(x_1) + \ldots + a_n(x_2) W_n(x_1) = \sum_{k=0}^{n} a_k(x_2) W_k(x_1) \quad (3.1.3)$$

of the members of the orthonormal set $W_0(x_1) \ldots W_n(x_1)$ in such a way that the integral square error

$$I = \int_a^b \left| f(x_1, x_2) - \sum_{k=0}^{n} a_k(x_2) W_k(x_1) \right|^2 dx_1 \quad (3.1.4)$$

is minimum.

When the constants $a_0, a_1 \ldots a_n$ are determined in such a way that I is minimum, then $f(x_1, x_2)$ is said to be approximated in the mean by the linear combination (3.1.3).

Values of the coefficients $a_k(x_2)$'s can be found for which the integral I in (3.1.4) is minimum. Thus, equating to zero the partial derivative of I with respect to $a_j(x_2)$ one gets,

$$\frac{\partial I}{\partial a_j(x_2)} = -2 \int_a^b f(x_1, x_2)\, \overline{W}_j(x_1)\, dx_1 + 2 \sum_{k=1}^{n} a_k(x_2) \int_a^b W_k(x_1) \overline{W}_j(x_1)\, dx_1 = 0 \quad (3.1.5)$$

Application of the orthonormality property of $W_m$'s to equation (3.1.5), yields

$$a_j(x_2) = \int_a^b f(x_1, x_2)\, \overline{W}_j(x_1)\, dx_1 \qquad (3.1.6)$$

The minimum value of the integral I with $a_j(x_2)$ given by equation (3.1.6) is obtained by evaluating the integral I where possible.

$$I = \int_a^b \left| f(x_1, x_2) - \sum_{k=1}^{n} a_k(x_2)\, W_k(x_1) \right|^2 dx_1$$

$$= \int_a^b \left[ f(x_1, x_2) - \sum_{k=0}^{n} a_k(x_2)\, W_k(x_1) \right] \left[ \overline{f}(x_1, x_2) - \sum_{k=1}^{n} \overline{a}_k(x_2)\, \overline{W}_k(x_1) \right] dx_1$$

$$= \int_a^b \left| f(x_1, x_2) \right|^2 dx_1 - \sum_{k=0}^{n} \int_a^b f(x_1, x_2)\, \overline{a}_k(x_2)\, \overline{W}_k(x_1)\, dx_1$$

$$- \sum_{k=0}^{n} \int_a^b \bar{f}(x_1, x_2)\, a_k(x_2)\, W_k(x_1)\, dx_1 + \sum_{k=0}^{n} \int_a^b \left|a_k(x_2)\right|^2 W_k(x_1)\, \bar{W}_k(x_1)\, dx_1$$

$$= \int_a^b \left|f(x_1, x_2)\right|^2 dx_1 - \sum_{k=0}^{n} \left|a_k(x_2)\right|^2 - \sum_{k=0}^{n} \left|a_k(x_2)\right|^2$$

$$+ \sum_{k=0}^{n} \left|a_k(x_2)\right|^2$$

$$= \int_a^b \left|f(x_1, x_2)\right|^2 dx_1 - \sum_{k=0}^{n} \left|a_k(x_2)\right|^2 \tag{3.1.7}$$

This is the minimum value of I.

The orthonormal set $W_1(x_1)$, $W_2(x_1) \ldots W_n(x_1)$ where

$$\int_a^b W_k^{\,2}(x_1)\, dx_1 < \infty \tag{3.1.8}$$

is said to form a complete set if either of the following conditions holds.

1) No non-zero function $u(x_1, x_2)$ such that

$$\int_a^b \left|u(x_1, x_2)\right|^2 dx_1 < \infty$$

exists such that

$$\int_a^b u(x_1, x_2)\ W_k(x_1)\ dx_1 = 0 \text{ for all } k = 0, 1, 2 \ldots$$

2) Given any piecewise continuous functions $f(x_1, x_2)$ with

$$\int_a^b \left| f(x_1, x_2) \right|^2 dx_1 < \infty \tag{3.1.9}$$

and an arbitrary small preassigned positive quantity $\epsilon$, it is possible to find an integer n and a polynomial

$$\sum_{k=0}^{n} a_k(x_2)\ W_k(x_1) \tag{3.1.10}$$

such that the integral

$$I = \int_a^b \left| f(x_1, x_2) - \sum_{k=0}^{n} a_k(x_2)\ W_k(x_1) \right|^2 dx_1 < \epsilon \tag{3.1.11}$$

Stated in a different way, the $W_k$'s constitute a complete orthonormal set if it is possible to approximate in the mean a function $f(x_1, x_2)$ integrable in the square, as accurately as desired by retaining a sufficient number of terms in the linear combination

$$\sum_{k=0}^{n} a_k(x_2)\; W_k(x_1)$$

Thus it follows, that, for a complete set,

$$\lim_{n\to\infty} \int_a^b \left| f(x_1, x_2) - \sum_{k=0}^{n} a_k(x_2)\; W_k(x_1) \right|^2 dx_1 = 0 \tag{3.1.12}$$

Applying equations (3. 1. 7) to equation (3. 1. 10), one obtains,

$$\int_a^b \left| f(x_1, x_2) \right|^2 dx_1 = \sum_{k=0}^{\infty} \left| a_k(x_2) \right|^2 \tag{3.1.13}$$

The existence of equation (3. 1. 11) with

$$a_k(x_2) = \int_a^b f(x_1, x_2)\; \bar{W}_k(x_1)\; dx_1 \tag{3.1.14}$$

also ensures the completeness of the orthonormal set $W_0(x_1,)\; W_1(x_1) \ldots W_n(x_1)$.

Thus it can be concluded that given a complete orthonormal set $W_0(x_1)$, $W_1(x_1) \ldots W_n(x_1)$ defined over an interval (a, b), any arbitrary function which is absolutely integrable in the square, also defined over the same interval, can be approximated in the mean as closely as desired by the sum $\sum_{k=0}^{n} a_k(x_2)\ W_k(x_1)$ where $a_k(x_2)$ is given by the equation (3.1.6). The series $\sum_{k=0}^{\infty} a_k(x_2)\ W_k(x_1)$ is said to converge in the mean to $f(x_1, x_2)$.

A fact of considerable importance is that given a set of linearly independent functions $\{u_m(x)\}$ it is always possible to form another set of functions $\{W_m(x)\}$, the elements of which are linear combinations of the $u_m$'s, such that $W_m$'s are orthonormal. Of course, when a large number of functions are involved, the process is very laborious.

The method of expansion of functions in terms of orthonormal functions is applied in the subsequent sections for synthesis of linear time-varying systems from the following three specifications:

1) Impulse responses $h_1(t,\tau)$, and $h_3(y,t)$

2) H-System function $H(s,t)$

3) G-System function $G(m,\tau)$.

## 3.2. Synthesis from Impulse Response. [10].

### 3.2.1. Realizability conditions.

Stable systems are considered. A system is stable if the impulse response $h_1(t,\tau)$ satisfies the condition

$$\int_0^{\infty} \left| h_1(t,\tau) \right| \, d\tau < \infty \quad \text{for all } t. \tag{3.2.1}$$

A necessary condition for realizability of such a system is that

$$h_1(t,\tau) \equiv 0 \qquad \text{for } t < \tau \tag{3.2.2}$$

A sufficient condition for realizability of an $h_1(t,\tau)$ as a differential system is that $h_1(t,\tau)$ be separable, i.e. $h_1(t,\tau)$ may be expressed as

$$h_1(t,\tau) = \sum_{k=1}^{N} a_k(t)\, g_k(\tau) \tag{3.2.3}$$

A necessary and sufficient conditition that a given $h_1(t,\tau)$ be realizable as a differential system is that

$$h_1(t,\tau) = \begin{cases} \sum_{k=1}^{N} a_k(t)\, g_k(\tau) & \text{for } t \geqslant \tau \\ 0 & \text{for } t < \tau \end{cases} \tag{3.2.4}$$

A necessary and sufficient condition that a stable system with impulse response $h_3(y,t)$ be realizable as a differential system is that

$$h_3(y,t) = \begin{cases} \sum_{k=1}^{N} a_k(t)\, b_k(y) & \text{for } y \geqslant 0 \\ 0 & \text{for } y < 0 \end{cases} \tag{3.2.5}$$

where $y = t - \tau$.

In order that the system be stable, it is required that

$$\left.\begin{array}{l} \displaystyle\int_0^\infty \left| b_k(y) \right| dy \\ \text{and} \quad \left| a_k(t) \right| < \infty \end{array}\right\} \quad \text{for all } k = 1, 2, \ldots N. \tag{3.2.6}$$

3.2.2. <u>Expansion of Impulse Response in Terms of Orthonormal Functions.</u>

Case 1. <u>Expansion of $h_1(t, \tau)$</u>

Given an $h_1(t, \tau)$ which satisfies the stability condition (3.2.1) and the realizability condition (3.2.4) its realization may be completed in the following steps:

1) A set of linearly independent functions $\{g_k(\tau)\}$ defined over the interval of $\tau$ as $h_1(t, \tau)$, is formed.

2) From the set $g_k(\tau)$ a complete set of orthonormal functions $\{\varphi_k(\tau)\}$ is formed.

3) An approximate expansion $h_1^*(t, \tau)$ for the given $h_1(t, \tau)$ in terms of the $\varphi_k(\tau)$'s is formed with a finite number N of terms.

$$h_1^*(t, \tau) = \sum_{k=1}^{N} a_k(t) \, \varphi_k(\tau) \tag{3.2.7}$$

4) Then the coefficients $a_k(t)$'s are chosen so that the integral

$$I = \int_a^b \left| h_1(t,\tau) - \sum_{k=1}^{N} a_k(t)\, \varphi_k(\tau) \right|^2 d$$

is minimum, for all t.

As shown in section 3.1, (equation 3.1.6) the integral I will be minimum if the coefficients $a_k(t)$'s are chosen to be

$$a_k(t) = \int_a^b h_1(t,\tau)\, \bar{\varphi}_k(\tau)\, d\tau \qquad (3.2.8)$$

which thus defines the values of $a_k(t)$'s.

Once the coefficients $a_k(t)$'s are determined from equation (3.2.8) the system may be realized as shown in figure 3.1 which consists of a parallel connection of n first order linear time-varying subsystems and an adder.

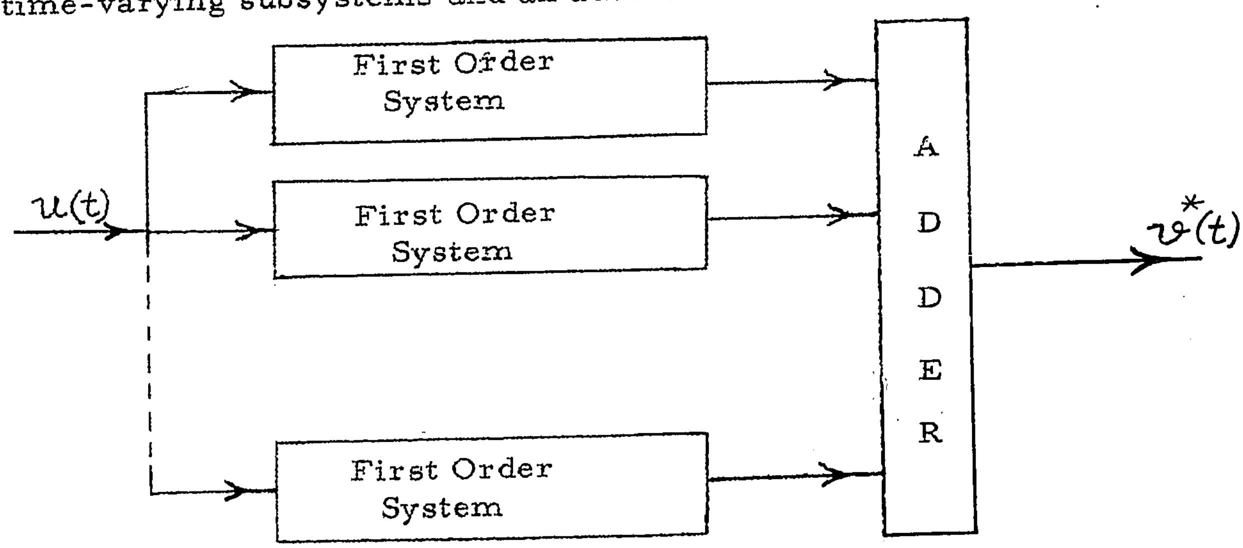


Fig. 3.1. A Realization of $h_1(t, \tau)$ separable in t and $\tau$.

Case 2. Expansion of $h_3(y,t)$.

Given an impulse response $h_3(y,t)$ satisfying the realizability condition (3.2.5) and the stability condition (3.2.6) the realization consists of the following steps.

1) A suitable set of linearly independent functions $\{h_k(y)\}$, each member of which is the impulse response of a fixed system and defined over the same interval of y as $h_3(y,t)$, is formed.

2) From the set $\{h_k(y)\}$ a complete set of orthonormal functions $\{\Psi_k(y)\}$ is formed.

3) An approximate expansion $h_3^*(y,t)$ for the given $h_3(y,t)$ in terms of $\Psi_k(y)$'s is formed with finite number N of terms.

$$h_3^*(y,t) = \sum_{k=1}^{N} b_k(t) \; \Psi_k(y) \tag{3.2.9}$$

4) The coefficients $b_k(t)$'s are chosen as given by the integral

$$b_k(t) = \int_a^b h_3(y,t)\, \overline{\Psi}_k(y)\, dy \tag{3.2.10}$$

When the coefficients $b_k(t)$'s are determined from equation (3.2.10), the system may be realized as shown in figure 3.2.

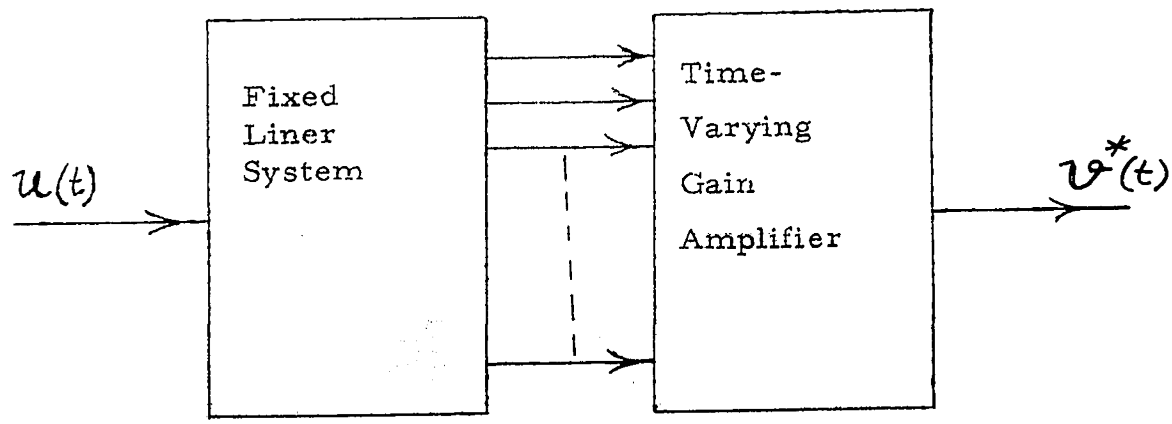


Fig. 3.2. Realization from $h_3(y,t)$.

The realization consists of a multi-output fixed linear system and a multi-input time-varying gain amplifier.

Figure 3.2 may be put into a more convenient form of figure 3.3, consisting of single-input-single-output subsystems.

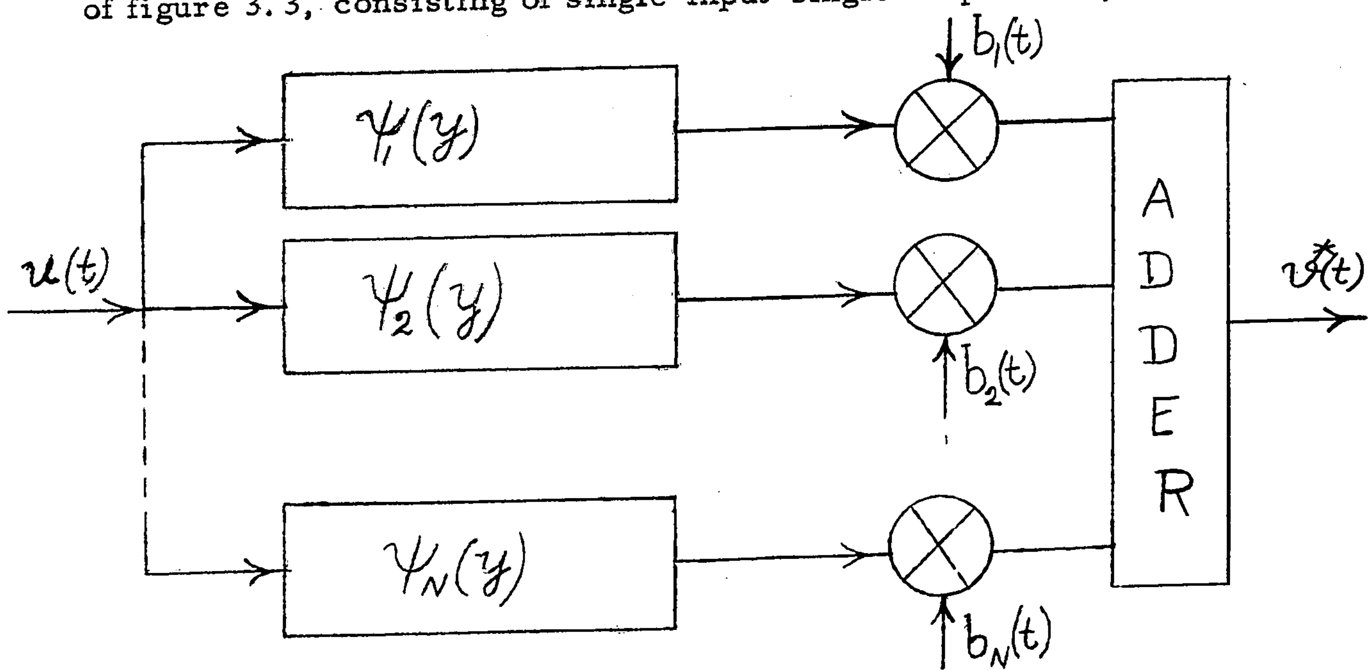


Fig. 3.3. Realization from $h_3(y,t)$.

One feature in which the realizations in figures 3.1 and 3.3 differ is that in the case of figure 3.3, the time-invariant parts and the time-varying parts of the system are separated, which is not the case in figure 3.1.

If $h_3(y,t)$ is expanded in terms of the Laguerre functions [11] $l_n(y)$ then the realization is obtained in the form of figure 3.4 which is a special case of figure 3.3.

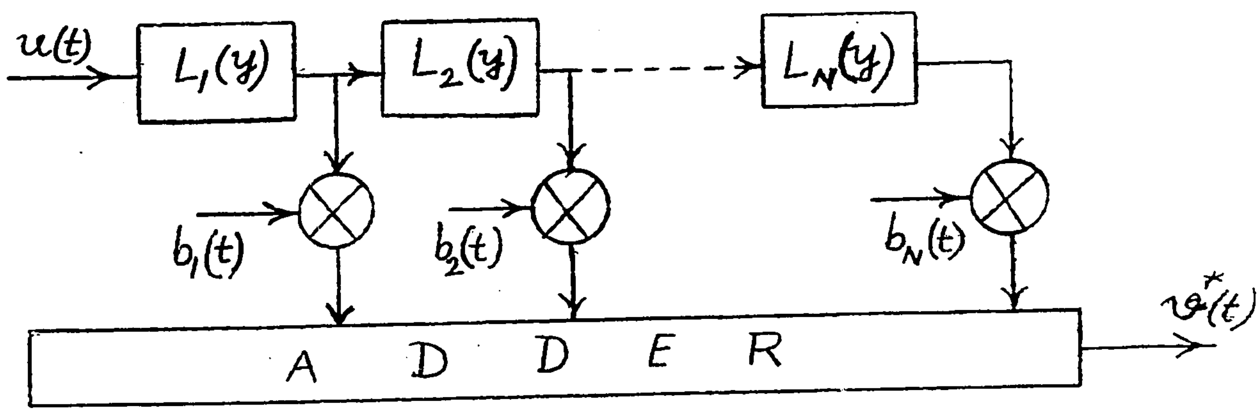


Fig. 3.4. Realization of $h_3(y,t)$ in terms of Laguerre Functions.

In general if the linear time-varying system possesses k input terminals and l output terminals the impulse response is a matrix [12].

$\left[h_l^{ij}(t,y)\right]$ with elements $h_l^{ij}(t,y)$ representing the impulse response at output terminal j due to a unit impulse applied at input terminal i. Such an impulse response matrix may be expanded in terms of orthonormal functions $\psi_{ij}(y)$ as shown in equation (3.2.11).

$$\left[h_l^{*ij}(t,y)\right] = \left[\psi_{ij}(y)\right]_{kxn} \left[b_{ij}(t)\right]_{nxl} \tag{3.2.11}$$

A possible realization of such a system is shown in figure 3. 5.

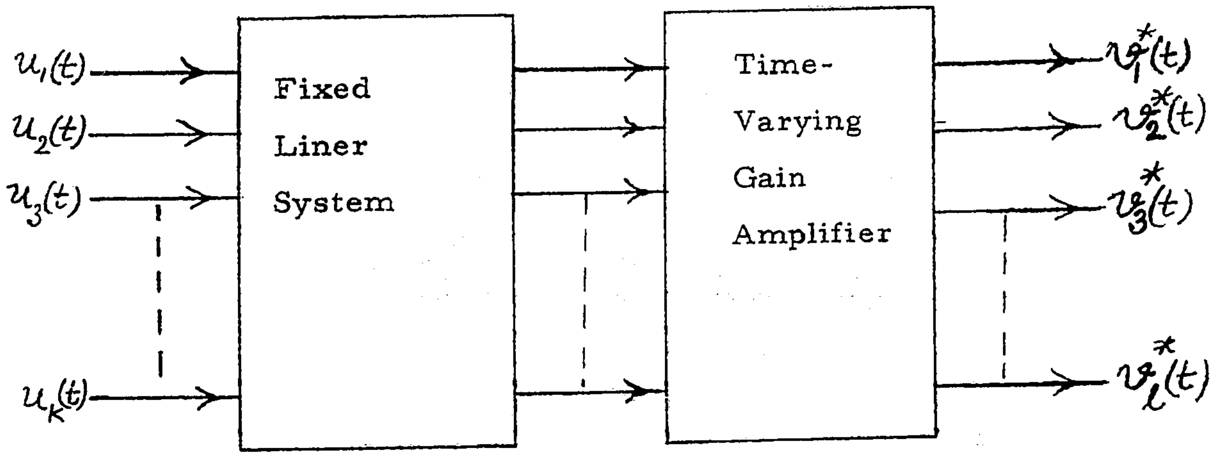


Fig. 3.5. Block Diagram Realization of Multi-input-multi-output System.

## 3. 3. Synthesis from H-System Function H(s, t).

### 3. 3. 1. Realizability conditions.

The necessary and sufficient condition that the system function H(s, t) which is rational in s be exactly realizable as a linear differential system is that all the poles of H(s, t) be time invariant [10] .

### 3. 3. 2. Expansion of H(s, t) in Terms of Orthonormal Functions.

If a given H(s, t) satisfying the above realizability condition is stable and absolutely integrable in the square on a curve C, i. e. $\int_C |H(s,t)|^2 |ds| < \infty$ , then it can be expanded in terms of orthonormal functions analytic in the region enclosed by C and on C and the expansion converges uniformly. This follows because a necessary condition for $H(s,t)$ to be stable function requires that $H(s,t)$ be analytic in the right half of the s-plane and on the $j\nu$ axis for all t.

Under these conditions, the realization of $H(s,t)$ may be completed in the following steps:

1) A suitable set of linearly independent functions $\{G_k(s)\}$, defined over the same domain of s as $H(s,t)$, each member of which is a system function of a fixed system, is formed.

2) An approximate expansion $H^*(s,t)$ for the given $H(s,t)$ in terms of $H_k(s)$'s is formed with a finite number N of terms.

$$H^*(s,t) = \sum_{k=0}^{N} a_k(t)\, H_k(s) \tag{3.3.1}$$

3) Then the coefficients $a_k(t)$'s are chosen so that the integral

$$I = \int_{C} \left| H(s,t) - \sum_{k=0}^{N} a_k(t)\, H_k(s) \right|^2 |ds|$$

is minimised.

As shown in section 3.1 (equation 3.1.6), the integral I is minimum if the coefficients $a_k(t)$'s are chosen to be

$$a_k(t) = \int_C H(s,t)\ \bar{H}_k(s)\ ds \qquad (3.3.2)$$

The realization corresponding to the expansion in equation (3.3.1) is as shown in figure 3.6.

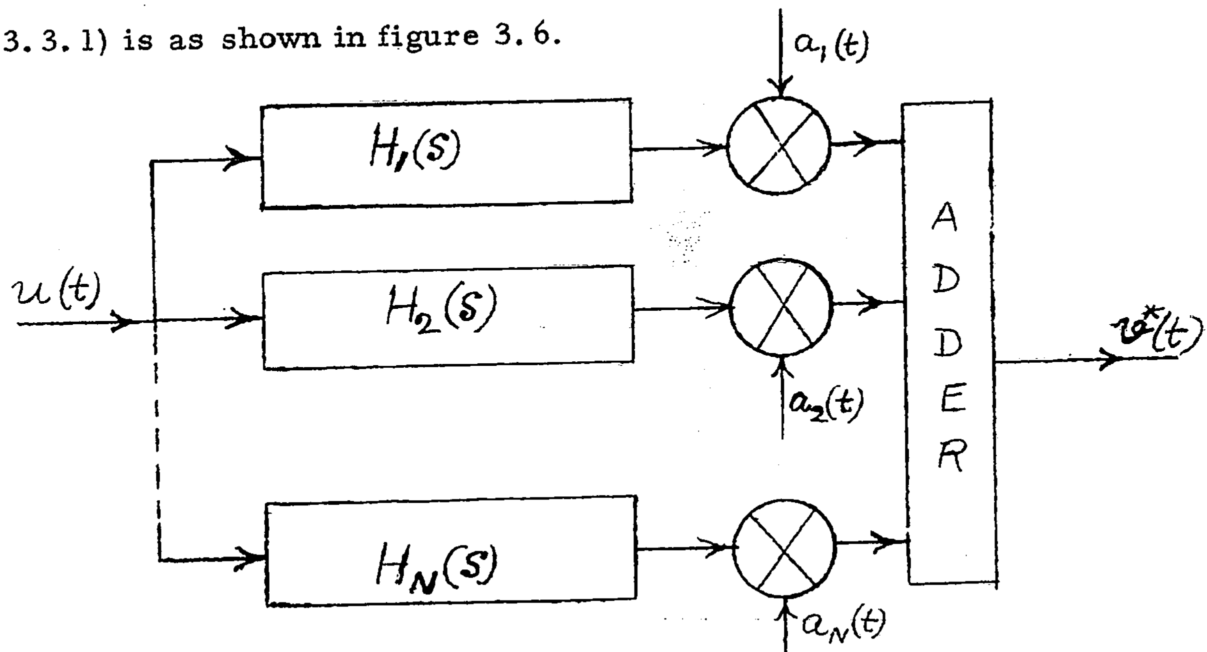


Fig. 3.6. Realization of H(s, t).

## 3.4. Synthesis from G-System Function G(m, $\tau$).

### 3.4.1. Realizability Conditions.

The impulse response $h_2(\xi, \tau)$ defined as the response of the system, due to an impulse input applied at time $\tau$, and measured at time $t = \tau + \xi$, is considered.

A necessary condition for realizability of $h_2(\xi, \tau)$ is that

$$h_2(\xi, \tau) \equiv 0 \text{ for } \xi < 0 \tag{3.4.1}$$

Further, a sufficient condition that $h_2(\xi, \tau)$ be realizable as a linear differential system is that it be separable.

i.e.

$$h_2(\xi, \tau) = \begin{cases} \sum_{k=1}^{N} a_k(\tau)\, h_k(\xi), & \text{for } \xi \geq 0 \\ 0 & \text{for } \xi < 0 \end{cases} \tag{3.4.2}$$

The G-System function $G(m, \tau)$ corresponding to such $h_2(\xi, \tau)$ is given by

$$\begin{aligned} G(m, \tau) &= \int_0^\infty h(\xi, \tau)\, e^{-m\xi}\, d\xi . \\ &= \int_0^\infty \sum_{k=1}^{N} a_k(\tau)\, h_k(\xi)\, e^{-m\xi}\, d\xi \\ &= \sum_{k=1}^{N} a_k(\tau) \int_0^\infty h_k(\xi)\, e^{-m\xi}\, d\xi \end{aligned} \tag{3.4.3}$$

provided the interchange of summation and integration is possible.

The poles of $G(m, \tau)$ are the same as the poles of

$$g_k(m) = \int_0^{\infty} h_k(\xi)\, e^{-m\xi}\, d\xi$$

Now, $g_k(m)$, being a definite integral between limits 0 and $\infty$, is independent of time t. Thus the poles of $g_k(m)$ and consequently those of $G(m, \tau)$ are time invariant.

The foregoing discussion may be summarized in a

Theorem: The necessary and sufficient condition that $G(m, \tau)$ be realizable as a linear differential system is that all the poles of $G(m, \tau)$ be time invariant.

3.4.2. Expansion of $G(m, \tau)$ in terms of orthonormal functions.

If a given $G(m, \tau)$

1) satisfies the above realizability condition,

2) is a stable system function, i.e. is an analytic function in the right half of the m-plane including the imaginary axis for all $\tau$,

3) is absolutely integrable in the square on a curve C,

i.e.
$$\int_C \left| G(m, \tau) \right|^2 |dm| < \infty$$

then the realization of $G(m, \tau)$ in terms of orthonormal functions may be completed in the following steps.

1) A suitable set of linearly independent functions $\{g_k(m)\}$, defined over the same domain of m as for $G(m,\tau)$, each member of which is a realizable system function of a fixed system, is formed.

2) A complete set of orthonormal functions $\{G_k(m)\}$ is formed from the set $\{g_k(m)\}$.

3) The given $G(m,\tau)$ is approximated by $G^*(m,\tau)$ as a polynomial in $G_k(m)$'s with a finite number N of terms.

$$G^*(m,\tau) = \sum_{k=1}^{N} a_k(\tau) \; G_k(m) \tag{3.4.4}$$

4) The coefficients $a_k(\tau)$'s are determined so that the integral

$$\int_C \left| G(m,\tau) - \sum_{k=1}^{N} a_k(\tau) \; G_k(m) \right|^2 |dm| \tag{3.4.5}$$

is minimum.

As shown previously (eqn. 3.1.6), the integral I in equation (3.4.5) is minimum if the coefficients $a_k(\tau)$'s are chosen as

$$a_k(\tau) = \int_C G(m,\tau) \; \bar{G}_k(m) \quad dm \tag{3.4.6}$$

The realization of $G(m,\tau)$ corresponding to the expansion in equation (3.4.4) is shown in figure 3.7.

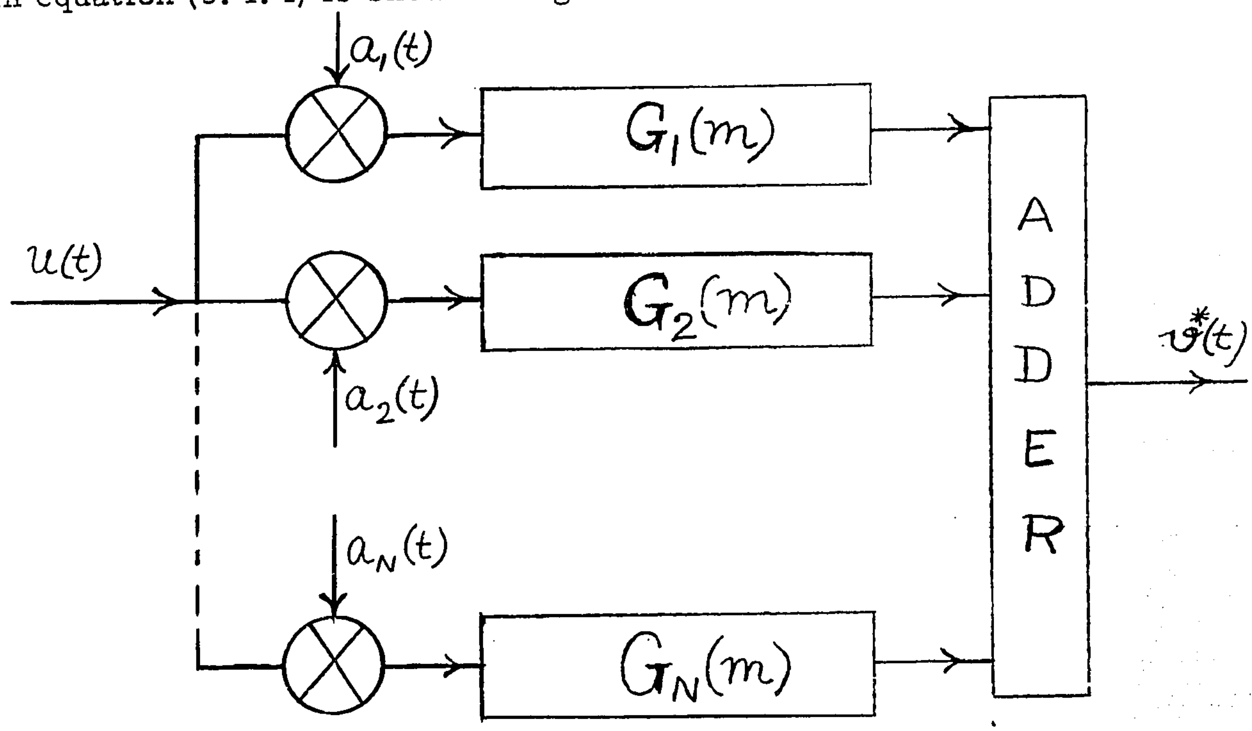


Fig. 3.7. Realization of $G(m,\tau)$.

It may be pointed out that $H(s,t)$ and $G(m,\gamma)$ are dual to each other. This is also true in case of their realizations. Thus, the configurations in figures 3.6 and 3.7 are dual to each other in the sense that in figure 3.6 the input is first filtered through the time invariant filters $H_k(s)$'s and then the result is multiplied by the time varying gain scale factors $a_k(t)$'s while in figure 3.7 the input is first multiplied by the time-varying gain scale factors $a_k(t)$'s and the product is then filtered through the time-invariant systems $G_k(m)$'s.

CHAPTER 4

# METHODS OF SYNTHESIS BY MEANS OF SAMPLING THEOREM.

## 4.1. Introduction.

Before the discussion of sampling theorem [13] proper two functions are defined:

$$1)\quad \text{rect } t = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases} \tag{4.1.1}$$

$$2)\quad \text{Sinc } f = \frac{\sin \pi f}{\pi f} \tag{4.1.2}$$

The sampling theorem has two forms corresponding to the time-domain and frequency domain.

1) The frequency domain sampling theorem states that if a time function $v(t)$ vanishes outside an interval $t_1 - T/2 \leq t \leq T/2 + t_1$ i.e. if

$$v(t) = u(t) \text{ rect } \frac{t-t_1}{T} \tag{4.1.3}$$

where $u(t)$ is a periodic function, then its

Fourier transform V(f) is completely determined by its values at frequencies n/T where n is an integer.

i.e. $$V(f) = \sum_{-\infty}^{\infty} V(\frac{n}{T})\, e^{-j2\pi t_1(f - \frac{n}{T})} \text{Sinc } T(f - \frac{n}{T}) \qquad (4.1.4)$$

2) The time-domain sampling theorem states that if the Fourier transform V(f) of a time function v(t) does not contain any frequencies outside the range $f_0 - \frac{W}{2} \leq f \leq f_o + \frac{W}{2}$, then the time function v(t) can be completely determined by its values at instants $\frac{n}{W}$ where n is an integrer

i.e. $$v(t) = \sum_{-\infty}^{\infty} v(\frac{n}{W})\, e^{j2\pi f_0(t - \frac{n}{W})} \text{ Sinc } W(t - \frac{n}{W}) \qquad (4.1.5)$$

Linear time-varying systems, like all practical systems, may be assumed to possess finite number of degrees of freedom as they are restricted in time-duration, frequency bandwidth, etc. These restrictions suggest and in fact make it possible to apply sampling theorem to the synthesis of linear time-varying systems [4], [14].

Such constraints may be classified as

1) Internal constraint: In this case the relevant system function itself is restricted to non-zero values only within a limited range of the variables like time, frequency, etc.

2) External constraint: In this case the system function itself may be unrestricted but the input or output signals may exist or may be of interest, over only a limited range of the variables, such as time, frequency, etc.

In this connection the following two results are of importance:

1) The system function $H'(\nu, t)$ of a series combination of a time-invariant system with system function $H_1(\nu)$, a time-varying system with system function $H(\nu, t)$ and a time-varying gain amplifier $G_1(t)$ as shown in figure 4.1, is given by

$$H'(\nu, t) = H_1(\nu)\, H(\nu, t)\, G_1(t) \tag{4.1.6}$$

This follows from the input-output relations

$$v(t) = \int_{-\infty}^{\infty} U(\nu)\, H'(\nu, t)\, e^{j2\pi\nu t}\, d\nu \tag{4.1.7}$$

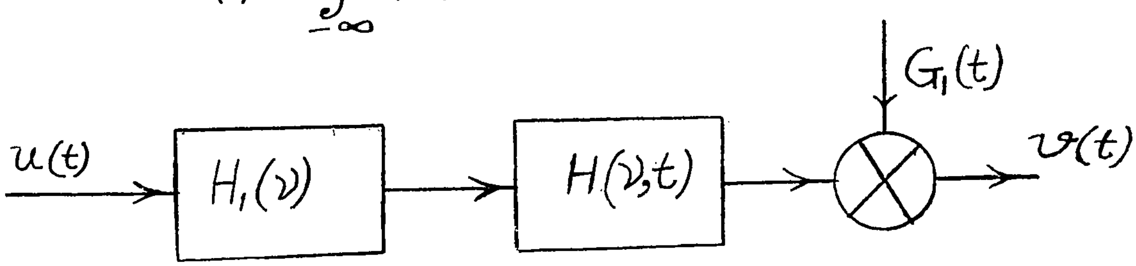


Figure 4.1.

From Fig. 4.1 it follows that preceding the system $H(\nu, t)$ by $H_1(\nu)$ changes the input spectrum $U(\nu)$ to $U(\nu) \cdot H_1(\nu)$ and multiplying $H(\nu, t)$ by $G_1(t)$ simply multiplies the output by the same factor. So, from equation (4.1.7) the result (4.1.6) follows.

2) The system function $G(\omega, \tau)$ of a series combination of a time-varying gain amplifier $G_1(t)$, a time-varying system with system function $G(\omega, \tau)$ and a time-invariant system with system function $H(\omega)$ as shown in figure 4.2, is given by

$$G'(\omega, \tau) = G_1(\tau)\; G(\omega, \tau)\; H(\omega) \tag{4.1.8}$$

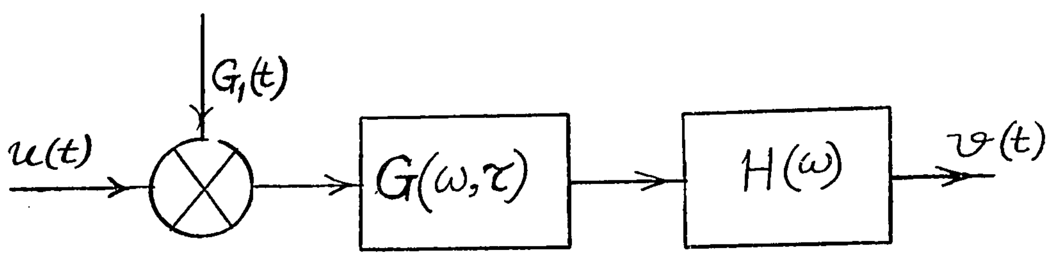


Figure 4.2.

This follows from the input-output relation

$$V(\omega) = \int_{-\infty}^{\infty} u(\tau)\, G'(\omega, \tau)\, e^{-j2\pi\omega\tau}\, d\tau \tag{4.1.9}$$

and the fact that preceding $G(\omega, \tau)$ by $G_1(t)$ changes the input $u(t)$ to $u(t)\ G_1(t)$ and passing the output of $G(\omega, \tau)$ through $H(\omega)$ is equivalent to multiplying both sides of equation (4.1.9) by $H(\omega)$.

In cases of internal constraints the system functions $H(\nu, t)$ and $G(\omega, \tau)$ are themselves assumed restricted in time or frequency and the sampling theorem may be applied to them.

In cases of external constraints the original system functions $H(\nu, t)$ and $G(\omega, \tau)$ are replaced by $H'(\nu, t)$ and $G'(\omega, \tau)$ respectively and the sampling theorem is applied to the latter system functions.

## 4.2. Sampling Models.

Four cases are considered:

1) Input time constraint

2) Input frequency constraint

3) Output time constraint

4) Output frequency constraint

Sampling models for these cases are described in the following sections.

### 4.2.1. Sampling Models Based on Input Time Constraint.

i) Internal constraint:

In such cases $G(\omega, \tau)$ is assumed to vanish outside an interval $t_1 - \frac{T}{2} \leq \tau \leq t_1 + \frac{T}{2}$.

Since $$\Gamma_g(\omega, \eta) = \int_{-\infty}^{\infty} G(\omega, \tau)\, e^{-j2\pi\eta\tau}\, d\tau \qquad (4.2.1)$$

is the Fourier transform of $G(\omega, \tau)$ with respect to $\tau$, the frequency domain sampling theorem may be applied to $\Gamma_g(\omega, \eta)$ and it may be expressed as

$$\Gamma_g(\omega, \tau) = \sum_n \Gamma_g(\omega, \tfrac{n}{T})\, e^{-j2\pi t_1(\eta - \frac{n}{T})} \text{Sinc}\left[T(\eta - \tfrac{n}{T})\right.$$

$$= \sum_n \Gamma_n(\omega)\, e^{-j2\pi t_1(\eta - \frac{n}{T})}\, T\, \text{Sinc}\left[T(\eta - \tfrac{n}{T})\right. \qquad (4.2.2)$$

where $$\Gamma_n(\omega) = \frac{1}{T}\, \Gamma_g(\omega, \tfrac{n}{T}) \qquad (4.2.2a)$$

In terms of $\Gamma_g(\omega, \eta)$ the output spectrum $V(\omega)$ is given by

$$V(\omega) = \int_{-\infty}^{\infty} \Gamma_g(\omega, \eta)\, U(\omega - \eta)\, d\eta \qquad (4.2.3)$$

where $U(\omega)$ is the spectrum of the input.

Replacing $\Gamma_g(\omega, \eta)$ in equation (4.2.3) by its expression in equation (4.2.2) one obtains

$$V(\omega) = \sum_n \Gamma_n(\omega) \int_{-\infty}^{\infty} U(\omega - \eta)\, e^{-j2\pi t_1(\eta - \frac{n}{T})}\, T\, \text{Sinc}\left[T( \quad - \tfrac{n}{T})\right] d\eta \qquad (4.2.4)$$

provided the interchange of summation and integration is possible.

The integral in equation (4.2.4) is a convolution of the input spectrum $U(\omega)$ and $T\, e^{-j2\pi t_1 (\omega - \frac{n}{T})}\, \mathrm{Sinc}\left[T(\omega - \frac{n}{T})\right]$

Since the time function corresponding to $T\, e^{-j2\pi t_1(\omega - \frac{n}{T})}\, \mathrm{Sinc}\ T(\omega - \frac{n}{T})$ is $e^{j2\pi n \frac{t}{T}}\ \mathrm{Rect}\ \frac{t-t_1}{T}$ and a convolution reduces to a product in the transform domain, the time function corresponding to the convolution integral in equation (4.2.4) is

$$u(t)\, e^{j2\pi n \frac{t}{T}}\ \mathrm{Rect}\ \frac{t-t_1}{T}$$

Thus from equation (4.2.4) it follows that the output may be obtained by multiplying the input time function by Rect $\frac{t-t_1}{T}$, frequency shifting the resultant waveforms and then passing it through the filters $\Gamma_n(\omega)$'s. So, the system may be modelled as shown in figure 4.3.

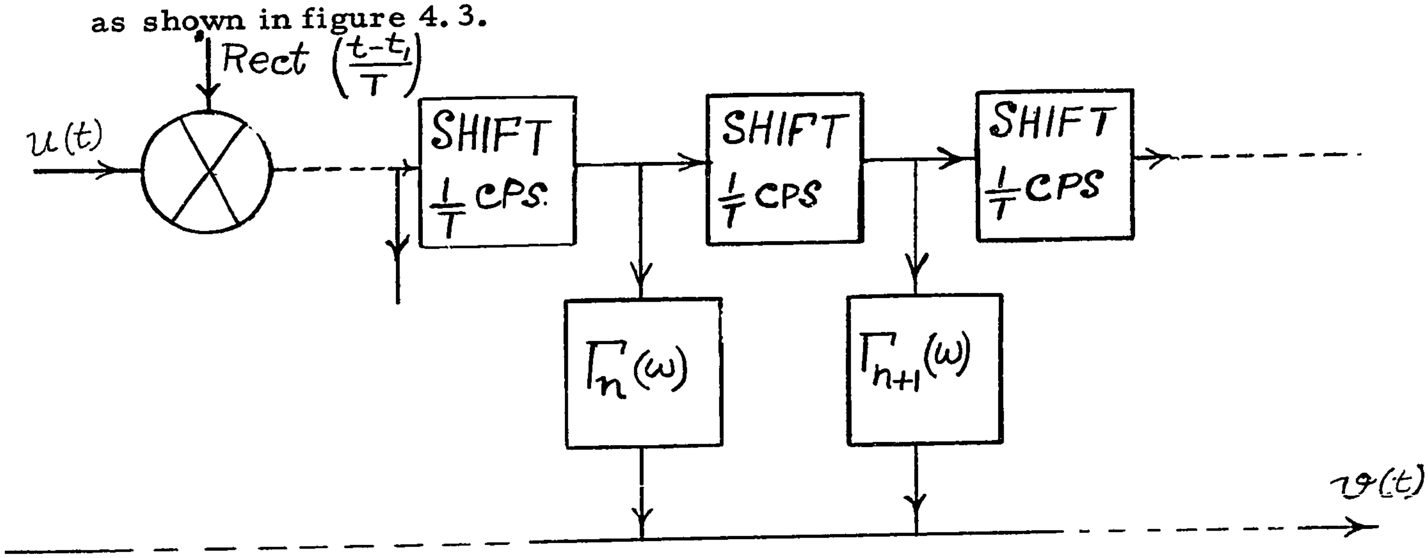


Fig. 4.3. Sampling Model for Input Time Constraint.

ii) External Constraint:

In such cases the system function to be used is

$$G'(\omega,\tau) = G_1(\tau)\, G(\omega,\tau) = \text{Rect}\, \frac{\tau - t_1}{T}\;\; G(\omega,\tau) \qquad (4.2.5)$$

The Fourier transform of $G'(\omega,\tau)$ as given in equation (4.2.5) is

$$\Gamma_g'(\omega,\eta) = \int_{-\infty}^{\infty} e^{-j2\pi t_1(\eta - \mu)}\, T\, \text{Sinc}\left[T(\eta - \mu)\right] \Gamma_g(\omega,\mu)\, d\mu . \qquad (4.2.6)$$

So, from the frequency domain sampling theorem $\Gamma_g'(\omega,\eta)$ may be expanded as

$$\Gamma_g'(\omega,\eta) = \sum_n \Gamma_g'(\omega, \tfrac{n}{T}) e^{-j2\pi t_1(\eta - \frac{n}{T})}, \text{Sinc}\left[T(\eta - \tfrac{n}{T})\right]$$

$$= \sum_n \Gamma_n'(\omega)\, e^{-j2\pi t_1(\eta - \frac{n}{T})}\, T\, \text{Sinc}\left[T(\eta - \tfrac{n}{T})\right] \qquad (4.2.7)$$

where

$$\Gamma_n'(\omega) = \frac{1}{T}\, \Gamma_g'(\omega, \tfrac{n}{T}) \qquad (4.2.8)$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi t_1(\frac{n}{T} - \mu)}\, \text{Sinc}\left[T(\tfrac{n}{T} - \mu)\right. \Gamma_g(\omega,\mu)\, d\mu \qquad (4.2.9)$$

Thus from previous reasonings in (i) it follows that the system may be modelled as in figure 4.3 with $\Gamma_n(\omega)$'s replaced by $\Gamma_n'(\omega)$'s as given by equation (4.2.9).

### 4.2.2. Sampling Model Based on Input Frequency Constraint.

i) Internal constraint:

In such case the proper system function to consider is $H(\nu, t)$ where $\nu$ is the input frequency in cycles per second, subject to the condition that $H(\nu, t)$ vanishes outside an interval $\nu_1 - \frac{W}{2} \leq \nu \leq \nu_1 + \frac{W}{2}$.

Since

$$h_3(y, t) = H(\nu, t)\, e^{-j2\pi\nu y} d\nu \tag{4.2.10}$$

is the inverse Fourier transform of $H(\nu, t)$ with respect to $\nu$, the time domain sampling theorem may be applied and $h_3(y, t)$ may be put in the expansion

$$h_3(y, t) = \sum_n h_3(t, \frac{n}{w}) e^{-j2\pi\nu_1(y - \frac{n}{w})} \operatorname{Sinc}\left[W(y - \frac{n}{W})\right] \tag{4.2.11}$$

In terms of $h_3(y, t)$ the output $v(t)$ due to an input $u(t)$ is given by

$$v(t) = \int_0^\infty u(t-y)\, h_3(y, t)\, dy \tag{4.2.12}$$

Putting $h_3(y, t)$ as given in equation (4.2.11) into equation (4.2.12), one obtains, after interchanging order of summation and integration,

$$v(t) = \sum_n h_n(t) \int_0^\infty u(t-y)\, W \operatorname{Sinc}\left[W(y - \frac{n}{w})\right] e^{j2\pi\nu_1(y - \frac{n}{w})}\, dy \tag{4.2.13}$$

where $$h_n(t) = \frac{1}{W} h_3(t, \frac{n}{W}).$$

From the facts that the integration in equation (4.2.13) is the convolution of the input time function with the time function $We^{j2\pi\nu_1(t-\frac{n}{w})} \cdot \mathrm{Sinc}\left[W(t-\frac{n}{W})\right]$ and that the spectrum of this function is $e^{-j2\pi n\frac{\nu}{W}}$, Rect $\frac{\nu-\nu_1}{W}$, it follows that the spectrum corresponding to the convolution integral in equation (4.2.13) is $U(\nu)$ Rect $\frac{\nu-\nu_1}{W}\, e^{-j2\pi n\frac{\nu}{W}}$. Thus, equation (4.2.13) suggests that the system output may be obtained by

1) band limiting the input by passing it through a filter with transfer function Rect $\frac{\nu-\nu_1}{W}$,

2) delaying the resultant by multiples of $\frac{1}{W}$ seconds,

3) multiplying these delayed functions by gain factors $h_n(t)$'s and then adding up the products. So the system may be modelled as shown in figure 4.4.

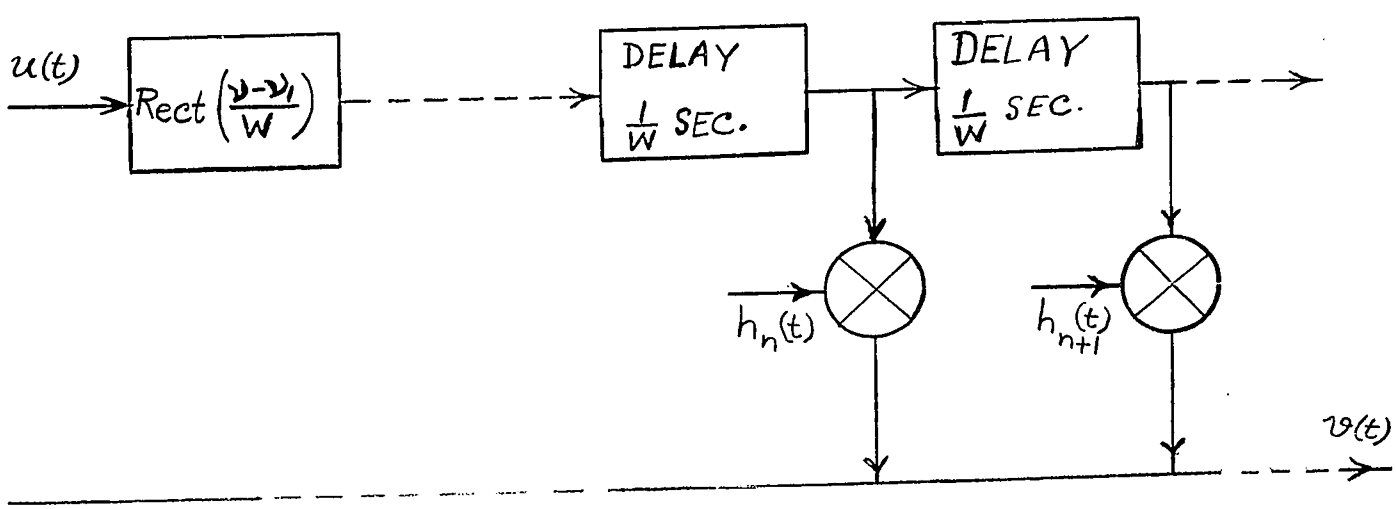


Fig. 4.4. Sampling Model for Input Frequency Constraint.

ii) External constraint:

In such case the system function to be used is

$$H'(\nu, t) = H_1(\nu)\, H(\nu, t) = \text{Rect}\,\frac{\nu - \nu_1}{W}\, H(\nu, t) \tag{4.2.14}$$

The inverse Fourier transform of $H'(\nu, t)$ as given in equation (4.2.14) is

$$h_3'(y,t) = \int_{-\infty}^{\infty} e^{j2\pi \nu_1 (y - \xi)}\, W\, \text{Sinc}\left[W(y-\xi)\right] h_3(t, \quad)\, d \tag{4.2.15}$$

So, from the time domain sampling theorem, $h_3'(y,t)$ may be expanded as

$$\begin{aligned} h_3'(y,t) &= \sum_n h_3(t, \tfrac{n}{W})\, e^{j2\pi \nu_1 (y - \frac{n}{W})}\, \text{Sinc}\left[W\,(y - \tfrac{n}{W})\right] \\ &= \sum_n h_n'(t)\, e^{j2\pi \nu_1 (y - \frac{n}{W})} W\, \text{Sinc}\left[W(y - \tfrac{n}{W})\right] \end{aligned} \tag{4.2.16}$$

where

$$\begin{aligned} h_n'(t) &= \frac{1}{W}\, h_3'(t, \tfrac{n}{W}) \\ &= \int_{0}^{\infty} e^{j2\pi \nu_1 (\frac{n}{W} - \xi)}\, \text{Sinc}\left[W(\tfrac{n}{W} - \xi)\right] h_3(t, \xi)\, d\xi \end{aligned} \tag{4.2.17}$$

Applying to equation (4.2.16) the same reasoning as put forward in connection with equation (4.2.13) in case (i) it is quickly seen that the system may be modeled in the configuration shown in figure 4.4 with the time-varying gain factors $h_n(t)$'s replaced by $h'_n(t)$'s as given in equation (4.2.17).

### 4.2.3. Sampling Model Based on Output Time Constraint.

i) Internal constraint:

In such case $H(\nu, t)$ is assumed to vanish outside an interval $t_1 - \frac{T}{2} \leqslant t \leqslant t_1 + \frac{T}{2}$.

The arguments applied for developing figure 4.3 are applicable to this case. In this case the function

$$\mathcal{H}(\nu, w) = \int_{-\infty}^{\infty} H(\nu, t)\, e^{-j2\pi\omega t} dt \tag{4.2.18}$$

(which is the Fourier transform of $H(\nu, t)$ with respect to t) is expanded by means of frequency domain sampling theorem. The corresponding model is shown in figure 4.5. The filter transfer functions in figure 4.5 are given by

$$\mathcal{H}_n(\nu) = \frac{1}{T}\mathcal{H}(\nu, \frac{n}{T}) \tag{4.2.19}$$

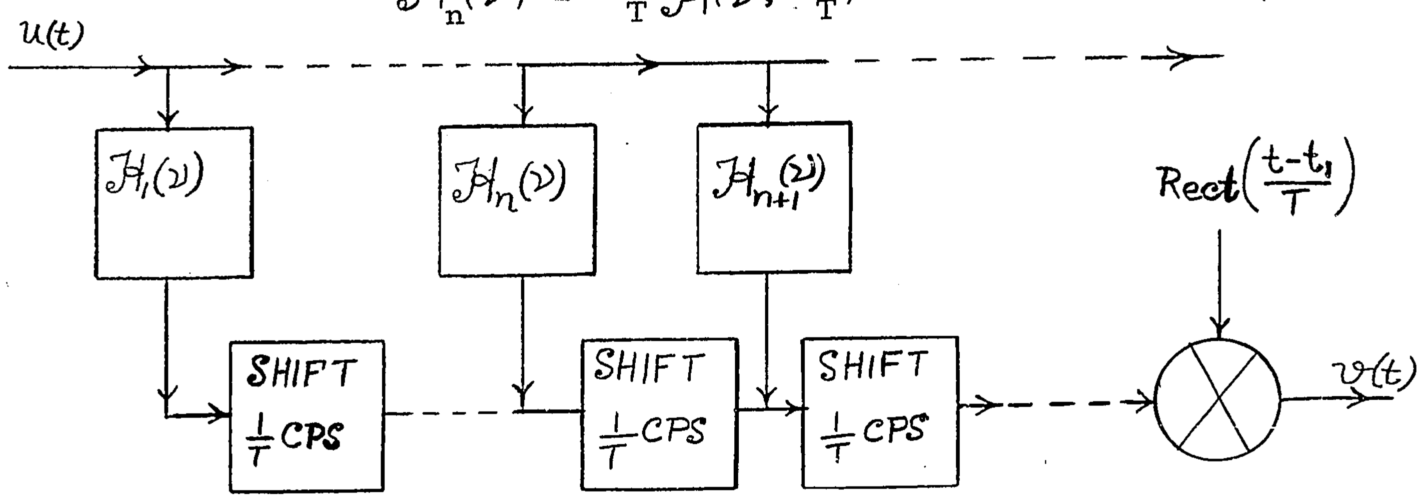


Fig. 4.5. Sampling Model for Output Time Constraint.

ii) External constraint:

In such case the system function to be used is

$$H'(\nu,t) = H(\nu,t)\,G_1(t) = H(\nu,t)\,\mathrm{Rect}\,\frac{t-t_1}{T} \tag{4.2.20}$$

The model in this case may be derived by following the argument in section 4.2.2(ii).

The function

$$\mathcal{H}'(\nu,\omega) = \int_{-\infty}^{\infty} e^{-j2\pi t_1(\omega-\mu)}\,T\,\mathrm{Sinc}\left[T(\omega-\mu)\right] H(\nu,\mu)\,d\mu \tag{4.2.21}$$

which is the Fourier transform of $H'(\nu,t)$ with respect to t, is expanded by means of frequency domain sampling theorem. The corresponding model is the same as that shown, in figure 4.5 with $\mathcal{H}_n(\nu)$'s replaced by $\mathcal{H}'_n(\nu)$'s given by

$$\mathcal{H}'_n(\nu) = \int_{-\infty}^{\infty} e^{j2\pi t_1(\mu-\frac{n}{T})}\,\mathrm{Sinc}\left[T(\mu-\frac{n}{T})\right]\mathcal{H}(\nu,\mu)\,d\mu \tag{4.2.22}$$

### 4.2.4. Sampling Model Based on Output Frequency Constraint.

i) Internal constraint:

In such case the proper system function to consider is $G(\omega,\tau)$ with the condition that $G(\omega,\tau)$ has non-zero value for $\omega$ only over an interval

$$\omega_1 - \frac{W}{2} \leqslant \omega \leqslant \omega_1 + \frac{W}{2}$$

Arguments applied in section 4.2.2 (i) are applicable here.

In this case, the function

$$h(\tau, \xi) = \int_{-\infty}^{\infty} G(\omega, \tau)\, e^{j2\pi\omega\xi} d\omega \qquad (4.2.23)$$

which is the inverse Fourier transform of $G(\omega, \tau)$ with respect to $\omega$, is expanded by means of time domain sampling theorem. Proceeding as in section 4.2.2 (i) the system model is obtained as shown in figure 4.6.

The time-functions $h_n(t)$'s are given by

$$h_n(t) = \frac{1}{W} \quad h(t, \frac{n}{W}) \qquad (4.2.24)$$

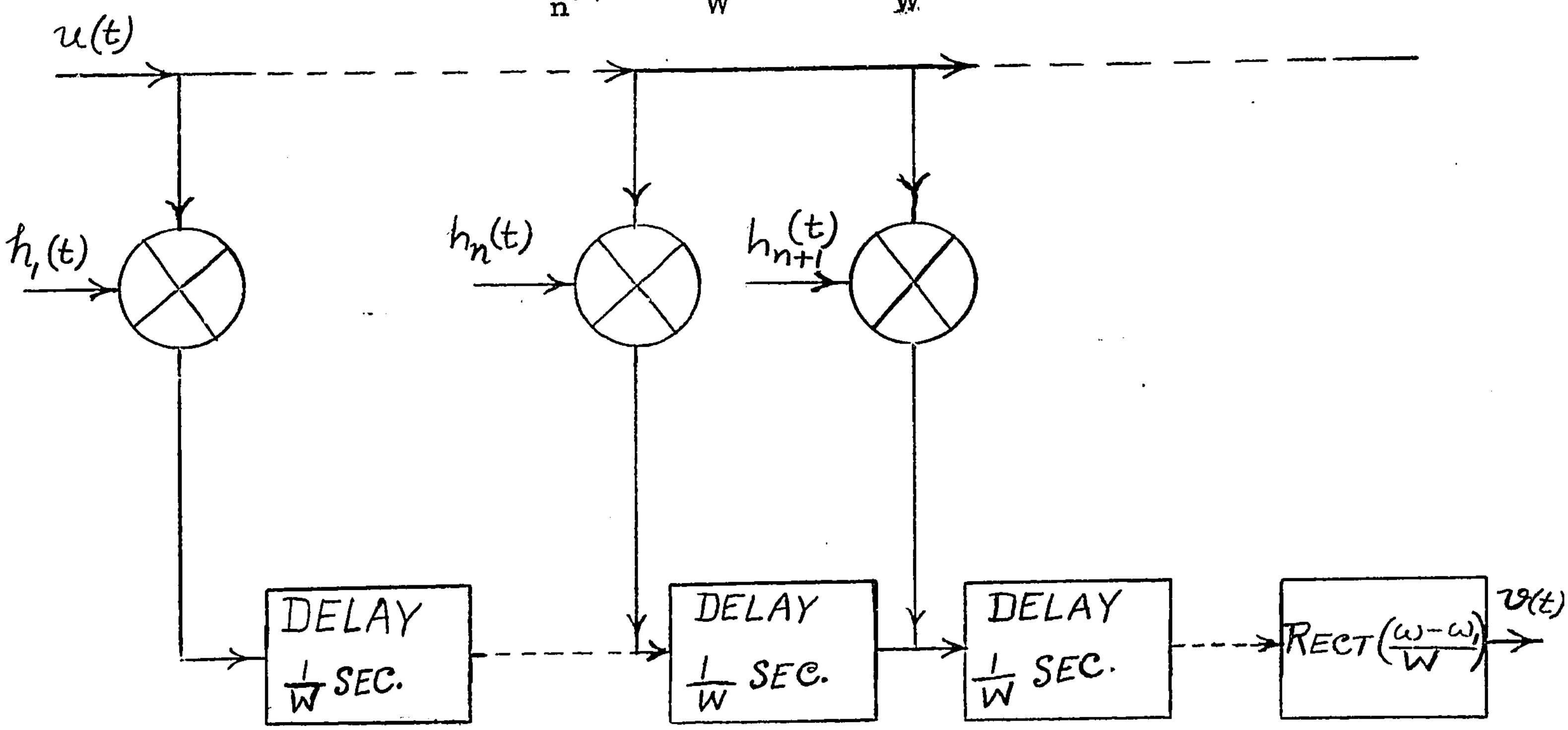


Fig. 4.6. Sampling Model for Output Frequency Constraint.

ii) External constraint:

In such case the system function to be considered is

$$G'(\omega,\tau) = G(\omega,\tau)\, H_1(\omega) = G(\omega,\tau)\ \mathrm{Rect}\,\frac{\omega-\omega_1}{W} \tag{4.2.25}$$

The arguments applied in section 4.2.2 (ii) may be used here.

In this case the function

$$h'(\tau,\xi) = \int_{-\infty}^{\infty} G'(\omega,\tau)\, e^{j2\pi\omega\xi}\, d\omega \tag{4.2.26}$$

which is the Fourier transform of $G(\omega,\tau)$ with respect to $\omega$, is expanded by means of time-domain sampling theorem.

Proceeding as in section 4.2.2 (ii) it is seen that under such conditions the system may be realized as shown in figure 4.6 with the gain factors $h_n(t)$'s replaced by $h_n'(t)$'s given by

$$h_n'(t) = \int_{-\infty}^{\infty} e^{-j2\pi\omega_1(\xi - \frac{n}{W})}\,\mathrm{Sinc}\left[W(\xi - \frac{n}{W})\right] h(\tau,\xi)\, d\xi \tag{4.2.27}$$

Remarks: : It may be observed that the models in figures 4.3, 4.4 and 4.5, 4.6 are dual to each other as they are developed from the dual set of constraints, namely, time constraint and frequency constraint.

It needs mention that these are not the only forms of sampling models obtainable. Applying different other sets of constraints other models may be obtained.

# CHAPTER 5

## METHODS OF SYNTHESIS BY MEANS OF PLANE IMPULSE TRAIN APPROXIMATION.

### 5.1. Introduction.

The method of synthesis described in this chapter is based on the principle of approximation of a given system characteristic function as a weighted sum of delayed impulses.

In the one dimensional case a function f(x), of the independent variable x, is equivalent to the integral

$$f(x) = \int_{-\infty}^{\infty} f(\tau)\, \delta(x - \tau)\, d\tau \tag{5.1.1}$$

where $\delta(x - \tau)$ is a unit impulse at $x = \tau$.

Extension of equation (5.1.1) to the case of two dimensional functions is immediate and a function f(x, y) of x and y, is equivalent to the integral

$$f(x, y) = \int_{-\infty}^{\infty} f(x, \xi)\, \delta(y - \xi)\, d\xi \tag{5.1.2}$$

$$= \int_{-\infty}^{\infty} f(y, \xi)\, \delta(x - \xi)\, d\xi \tag{5.1.3}$$

considering any one, e.g., equation (5.1.2), of the above two equations, it is further seen that the integral may be approximated by the sum in equation (5.1.4).

$$
\begin{aligned}
f(x, y) &= \int_{-\infty}^{\infty} f(x, \xi)\, \delta(y - \xi)\, d\xi \\
&= \sum_{k=1}^{\infty} (\xi_{k+1} - \xi_k)\, f(x, \xi_k)\, \delta(y - \xi_k) \\
&\cong \sum_{k=1}^{N} (\xi_{k+1} - \xi_k)\, f(x, \xi_k)\, \delta(y - \xi_k) \\
&= f^*(x, y)
\end{aligned}
\tag{5.1.4}
$$

where N is a finite integer.

Equation (5.1.4) states that the function f(x, y) may be approximated by a weighted sum of a train of delayed plane impulses. The plane impulse occuring at $y = \xi_k$ has an area $(\xi_{k+1} - \xi_k)\, f(x, \xi_k)$ where $f(x, \xi_k)$ is the value of the function f(x, y) at $y = \xi_k$ and the term $(\xi_{k+1} - \xi_k)$ represents the distance in y direction between two plane impulses at $y = \xi_k$ and $y = \xi_{k+1}$. In the one dimensional case, of course, equation (5.1.4) reduces to the weighted sum of a train of line impulses.

Equation (5.1.4) forms the basis of the method of synthesis of time-varying systems developed in this chapter.

Methods of synthesis based on this principle, from the following system specifications are considered.

1) Impulse response

2) H-System function $H(\nu, t)$

3) G-System function $G(\omega, \tau)$

4) Bifrequency system function $\Gamma(\mu, \nu)$.

## 5.2. Method of Synthesis from Impulse Response.

### 5.2.1. Synthesis from $h_1(t, \tau)$ [15].

The impulse response $h_1(t, \tau)$ is plotted in a three dimensional space in figure 5.1.

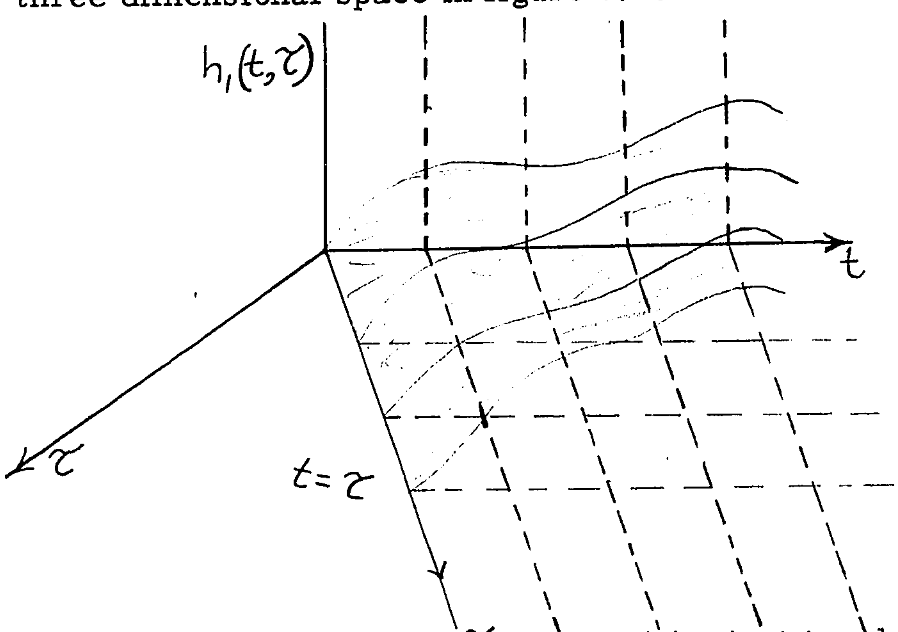


Fig. 5.1. $h_1(t, \tau)$ Plotted Against t and $\tau$.

The impulse response $h_1(t, \tau)$ is expressed as

$$h_1(t, \tau) = \int_{-\infty}^{\infty} h_1(t, t - \tau)\, \delta(t - \tau - \xi)\, d\xi \tag{5.2.1}$$

which, as shown in section 5.1, may be approximated by

$$h_1^*(t,\tau) = \sum_{k=1}^{N} h_1(t,\ t - \xi_k)\,(\xi_{k+1} - \xi_k)\,(t - \tau - \xi_k) \tag{5.2.2}$$

In terms of $h_1(t,\tau)$ the output v(t) of the system to an input u(t) is given by

$$v(t) = \int_{-\infty}^{t} h_1(t,\tau)\,u(\tau)\,d\tau \tag{5.2.3}$$

or for a realizable system

$$v(t) = \int_{-\infty}^{\infty} h_1(t,\tau)\,u(\tau)\,d\tau \tag{5.2.4}$$

Replacing $h_1(t,\tau)$ in equation (5.2.4) by $h_1^*(t,\tau)$ given by equation (5.2.2) one obtains the approximate output as

$$\begin{aligned} v^*(t) &= \int_{-\infty}^{\infty} h_1^*(t,\tau)\,u(\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \sum_{k=1}^{N} h_1(t,\ t - \xi_k)\,(\xi_{k+1} - \xi_k)\,\delta(t - \tau - \xi_k).\ u(\tau)\,d\tau \end{aligned}$$

and interchanging the order of integration and summation,

$$= \sum_{k=1}^{N} h_1(t,\ t - \xi_k)\,(\xi_{k+1} - \xi_k)\int_{-\infty}^{\infty}\delta(t - \tau - \xi_k).\ u(\tau)\,d\tau$$

$$= \sum_{k=1}^{N} a_k(t)\,u(t - \xi_k) \tag{5.2.5}$$

$$\text{where } a_k(t) = h_1(t,\ t - \xi_k)\,(\xi_{k+1} - \xi_k) \tag{5.2.6}$$

Here $h_1(t, t - {}_k)$ is the function of time obtained as the intersection of the surface $h_1(t, \tau)$ with the plane impulse at $t = \tau + \xi_k$ and parallel to the plane $t = \tau$ and $(\xi_{k+1} - \xi_k)$ is the interval in the t direction between the plane impulses at $t = \xi_k$ and $t = \xi_{k+1}$.

Thus it is seen from equation (5.2.5) that the approximate output may be obtained by,

1) delaying the input by appropriate amounts $\xi_k$'s

2) multiplying the delayed signals $u(t - \xi_k)$'s by the corresponding time functions $a_k(t)$'s and

3) then adding them up.

The corresponding model for synthesis is shown in figure 5.2.

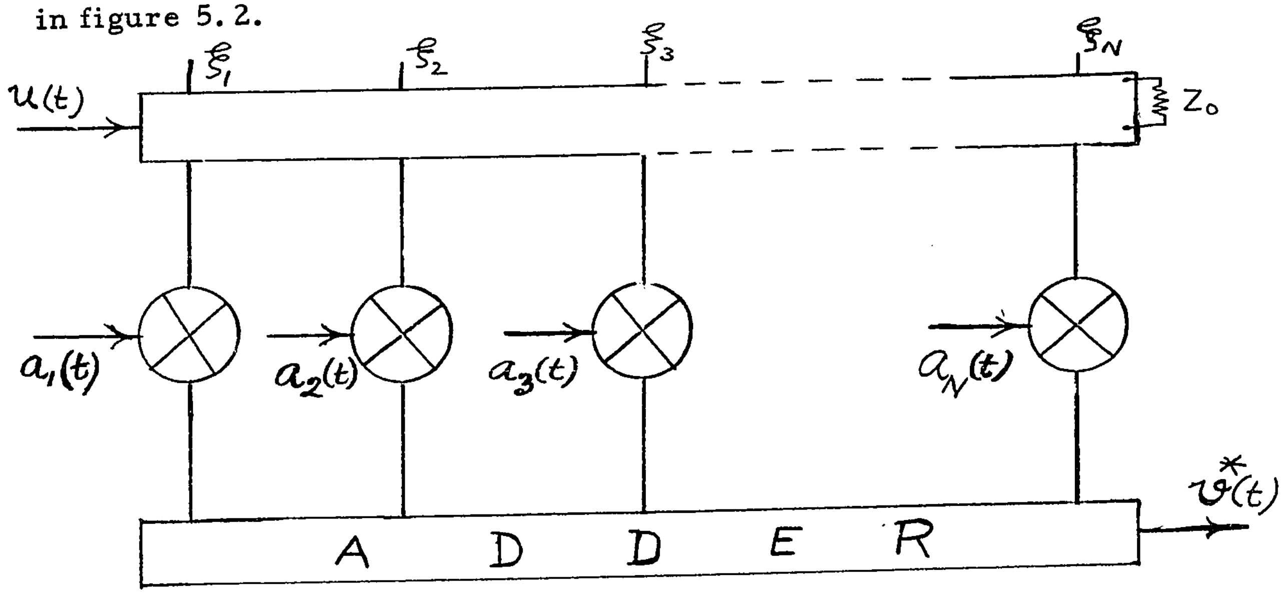


Fig. 5.2. Synthesis From $h_1(t, \tau)$.

It may be mentioned that $h_1(t,\tau)$ may also be expressed as

$$h_1(t,\tau) = \int_{-\infty}^{\infty} h_1(\tau+\xi,\tau)\,\delta(t-\tau-\xi_k)\,d\xi \tag{5.2.7}$$

$$= \sum_{k=1}^{N} h_1(\tau+\xi_k,\tau)\,(\xi_{k+1}-\xi_k)\,\delta(t-\tau-\xi_k) \tag{5.2.8}$$

Following the same procedure outlined in the above case it can be shown that the approximation (5.2.8) also leads to the same configuration as in figure 5.2.

5.2.2. <u>Synthesis From $h_3(y,t)$ Where $y = t - \tau$ is the age variable.</u>

Two cases may arise:

1) $h_3(y,t)$ may be expressed as

$$h_3(y,t) = \int_{-\infty}^{\infty} h_3(y,\xi)\,\delta(t-\xi)\,d\xi\ . \tag{5.2.9}$$

which may be approximated as

$$h_3^*(y,t) = \sum_{k=1}^{N} h_3(y,\xi_k)\,(\xi_{k+1}-\xi_k)\,\delta(t-\xi_k) \tag{5.2.10}$$

The input output relation in terms of $h_3(y,t)$ is given by

$$v(t) = \int_{0}^{\infty} h_3(y,t)\,u(t-y)\,dy \tag{5.2.11}$$

Replacing $h_3(y,t)$ in equation (5.2.11) by $h_3^*(y,t)$ as given in equation (5.2.10) one obtains the approximate output as

$$v^*(t) = \int_0^\infty h_3^*(y,t)\, u(t-y)\, dy \tag{5.2.12}$$

$$= \int_0^\infty \sum_{k=1}^{N} h_3(y, \xi_k)\, (\xi_{k+1} - \xi_k)\, \delta(t - \xi_k)\, u(t-y)\, dy.$$

interchanging order of summation and integration,

$$= \sum_{k=1}^{N} (\xi_{k+1} - \xi_k)\, \delta(t - \xi_k) \int_0^\infty h_3(y, \xi_k)\, u(t-y)\, dy$$

$$= \sum_{k=1}^{N} \Delta\xi_k\, \delta(t - \xi_k)\, v_k(t) \tag{5.2.13}$$

where $\Delta\xi_k = (\xi_{k+1} - \xi_k)$ is the interval in the t direction between two plane impulses at $t = \xi_{k+1}$ and $t = \xi_k$. Thus it is seen that the approximate output $v^*(t)$ may be obtained by,

1) filtering the input through fixed linear system with impulse responses $h_3(y, \xi_k)$'s

2) multiplying the output of the k-th filter by a delta function of area $\Delta\xi_k$ appearing at $t = \xi_k$, and,

3) adding all the resultants.

The last two steps are achieved by means of a sequential switch and a hold circuit as shown in figure 5.3.

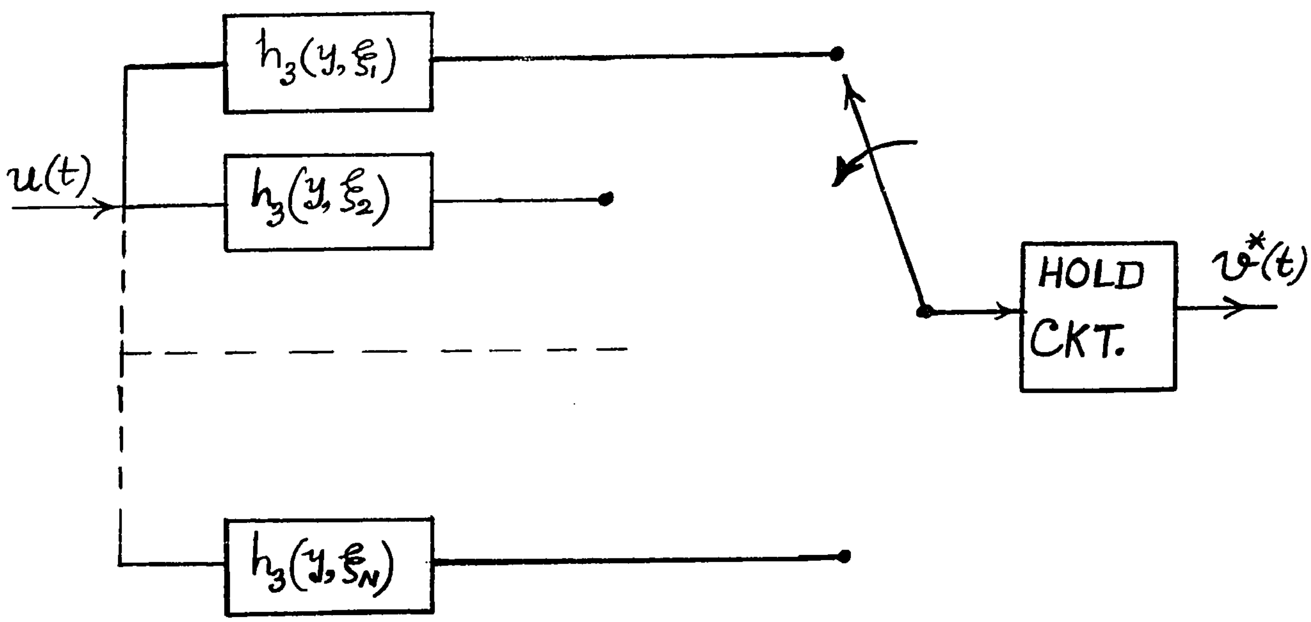


Fig. 5.3. Synthesis From $h_3(y, t)$

The switch is adjusted in such a way that at time $t = \xi_k$ it connects the hold circuit with the output of the k-th filter and the hold circuit maintains this value of output for an interval $\Delta\xi_k$ until the switch connects the next filter. In this way the switch and the hold circuit jointly multiply the output of the k-th filter by an impulse of area $\Delta\xi_k$.

2) $h_3(y, t)$ may also be expressed as

$$h_3(y, t) = \int_{-\infty}^{\infty} h_3(t, \xi) \delta(y - \xi) \, d\xi \tag{5.2.14}$$

which may be approximated by the sum,

$$h_3^*(y, t) = \sum_{k=1}^{N} h_3(t, \xi_k)(\xi_{k+1} - \xi_k) \delta(y - \xi_k) \tag{5.2.15}$$

Replacing $h_3(y,t)$ in equation (5.2.11) by $h_3^*(y,t)$ as given by equation (5.2.15) one obtains the approximate output as

$$
\begin{aligned}
v^*(t) &= \int_0^\infty h_3^*(y,t)\, u(t-y)\, dy \\
&= \int_0^\infty \sum_{k=1}^{N} h_3(t, \xi_k)\,(\xi_{k+1} - \xi_k)\,\delta(y - \xi_k)\, u(t-y)\, dy \\
&= \sum_{k=1}^{N} h_3(t, \xi_k)\,(\xi_{k+1} - \xi_k)\int_0^\infty \delta(y - \xi_k)\, u(t-y)\, dy \\
&= \sum_{k=1}^{N} b_k(t)\, u(t - \xi_k).
\end{aligned}
\qquad (5.2.16)
$$

where

$$
b_k(t) = h_3(t, \xi_k)\,(\xi_{k+1} - \xi_k)
$$

Here $h_3(t, \xi_k)$ is the intersection of the $h_3(t, y)$ surface and the impulse plane parallel to the t axis and at $y = \xi_k$.

Thus, it is seen that the approximate output may be obtained by

1) delaying the input by amounts $\xi_k$'s
2) multiplying the respective delayed signals by corresponding time-functions $b_k(t)$'s, and
3) adding together all the resultants.

The system may be synthesized as shown in figure 5.2 with the time functions $a_k(t)$'s replaced by $b_k(t)$'s.

## 5.3. Synthesis From H-System Function $H(\nu, t)$ where $\nu$ is the input frequency in cps.

Two cases are considered:

i) The system function $H(\nu, t)$ may be expressed as

$$H(\nu, t) = H(\nu, \xi)\, \delta(t - \xi)\, d\xi . \tag{5.3.1}$$

which may be approximated by

$$H^*(\nu, t) = \sum_{k=1}^{N} H(\nu, \xi_k)\, (\xi_{k+1} - \xi_k)\, \delta(t - \xi_k) \tag{5.3.2}$$

The output v(t) of a system with system function $H(\nu, t)$ due to an input u(t) is given by

$$v(t) = \int_{-\infty}^{\infty} H(\nu, t)\, U(\nu)\, e^{j2\pi\nu t}\, d\nu \tag{5.3.3}$$

where $U(\nu)$ is the spectrum of the input u(t).

Replacing $H(\nu, t)$ in equation (5.3.3) by $H^*(\nu, t)$ as given by equation (5.3.2) one obtains the approximate output as

$$v^*(t) = \int_{-\infty}^{\infty} H^*(\nu, t)\, U(\nu)\, e^{j2\pi\nu t}\, d\nu$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{N} H(\nu, \xi_k)\, (\xi_{k+1} - \xi_k) \quad (t - \xi_k)\, U(\nu) e^{j2\pi\nu t} d\nu.$$

and interchanging the order of integration and summation

$$= \sum_{k=1}^{N} (\xi_{k+1} - \xi_k)\, \delta(t - \xi_k) \int_{-\infty}^{\infty} H(\nu, \xi_k)\, U(\nu) e^{j2\pi\nu t} d\nu. \qquad (5.3.4)$$

Equation (5.3.4) indicates that the approximate output $v^*(t)$ may be obtained by:

1) filtering the input through fixed linear filters with system functions $H_k(\nu) = H(\nu, \xi_k)$'s

2) multiplying the output of the k-th filter by an impulse of area $(\xi_{k+1} - \xi_k)$, and

3) then adding all the resultants.

Thus the system may be synthesized in a configuration shown in figure 5.3 with the fixed systems impulse responses $h_3(y, \xi_k)$'s replaced by fixed system functions $H_k(\nu)$'s = $H(\nu, \xi_k)$'s.

ii) $H(\nu, t)$ may also be expressed as

$$H(\nu, t) = \int H(t, \beta)\, \delta(\nu - \beta)\, d\beta \qquad (5.3.5)$$

which may be approximated by

$$H^*(\nu, t) = \sum_{k=1}^{N} H(t, \beta_k)(\beta_{k+1} - \beta_k)\,\delta(\nu - \beta_k) \qquad (5.3.6)$$

Replacing $H(\nu, t)$ in equation (5.3.3) by $H^*(\nu, t)$ as given by equation (5.3.6) one obtains the approximate output as

$$v^*(t) = \int_{-\infty}^{\infty} H^*(\nu, t)\, U(\nu)\, e^{j2\pi\nu t}\, d\nu .$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{N} H(t, \beta_k)(\beta_{k+1} - \beta_k)\delta(\nu - \beta_k)\, U(\nu)\, e^{j2\pi\nu t}\, d\nu$$

and interchanging the order of summation and integration

$$= \sum_{k=1}^{N} H(t, \beta_k) \int_{-\infty}^{\infty} (\beta_{k+1} - \beta_k)\delta(\nu - \beta_k)\, U(\nu)\, e^{j2\pi\nu t}\, d\nu \qquad (5.3.7)$$

Equation (5.3.7) indicates that the approximate output $v^*(t)$ may be obtained by:

1) filtering the input through fixed linear filters with system functions $(\beta_{k+1} - \beta_k)\,\delta(\nu - \beta_k) = H_k(\nu)$'s
2) multiplying the output of the k-th filter by the time function $A_k(t) = H(t, \beta_k)$, and
3) then adding the resultants.

The corresponding model for synthesis of the system is shown in figure 5.4.

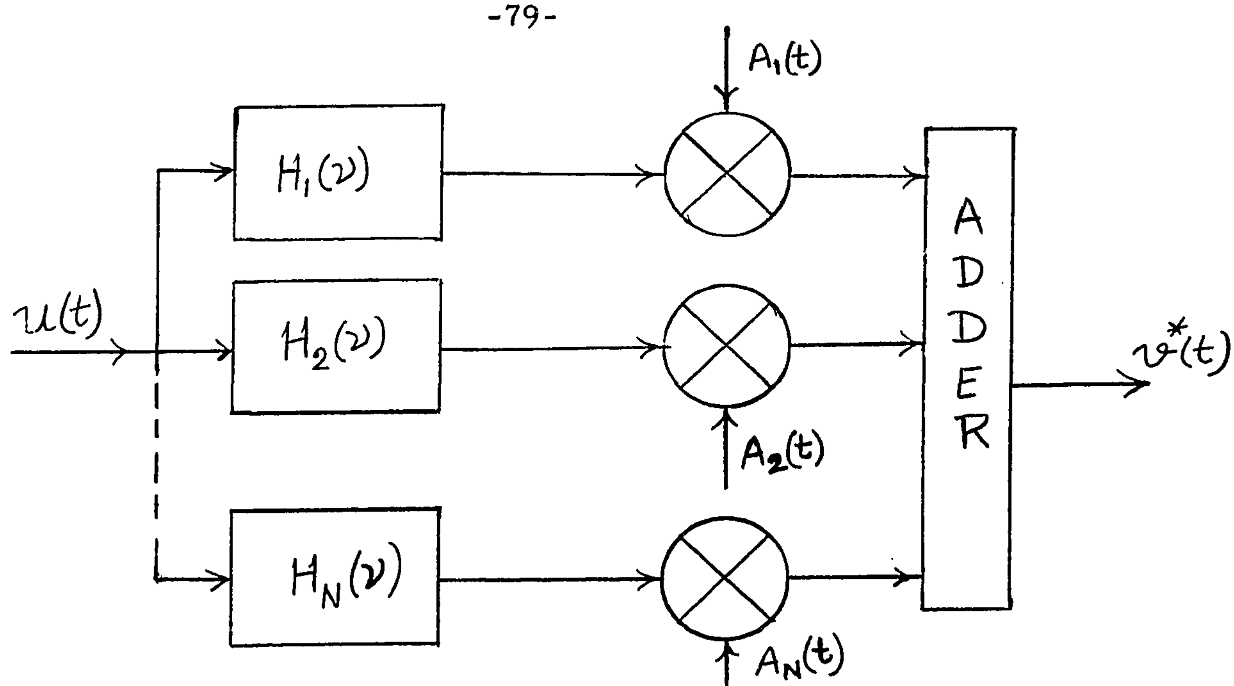


Fig. 5.4. Synthesis from $H(\nu, t)$.

In the configuration of figure 5.4 the operations of multiplication by the time functions $A_k(t)$'s is accomplished by means of multipliers and function generators.

The fixed filters $H_k(\nu)$'s are such that each of them passes signals of frequencies between $\beta_k$ and $\beta_{k+1}$, unattenuated, and stops all other frequency components.

5.4. <u>Method of Synthesis from G-System Function $G(\omega, \tau)$, where $\omega$ is the output frequency in cps.</u>

Two cases are considered:

i) The system function $G(\omega, \tau)$ may be expressed as

$$G(\omega, \tau) = \int_{-\infty}^{\infty} G(\omega, \xi)\, \delta(\tau - \xi)\, d\xi. \tag{5.4.1}$$

which may be approximated as

$$G^*(\omega, \tau) = \sum_{k=1}^{N} G(\omega, \xi_k)(\xi_{k+1} - \xi_k)\,\delta(\tau - \xi_k) \qquad (5.4.2)$$

The output spectrum $V(\omega)$ of a system with system function $G(\omega, \tau)$ to an input u(t) is given by

$$V(\omega) = \int_{-\infty}^{\infty} G(\omega, \tau)\, u(\tau) e^{-j2\pi\omega\tau} d\tau \qquad (5.4.3)$$

Replacing $G(\omega, \tau)$ in equation (5.4.3) by $G^*(\omega, \tau)$ as given by equation (5.4.2) one obtains the approximate output spectrum as

$$V^*(\omega) = \int_{-\infty}^{\infty} G^*(\omega, \tau)\, u(\tau) e^{-j2\pi\omega\tau} d\tau.$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{N} G(\omega, \xi_k)(\xi_{k+1} - \xi_k)\,\delta(\tau - \xi_k)\, u(\tau) e^{-j2\pi\omega\tau} d\tau$$

Interchanging order of integration and summation,

$$= \sum_{k=1}^{N} (\xi_{k+1} - \xi_k)\, G(\omega, \xi_k) \int_{-\infty}^{\infty} \delta(\tau - \xi_k)\, u(\tau) e^{-j2\pi\omega\tau} d\tau$$

$$= \sum_{k=1}^{N} G(\omega, \xi_k)(\xi_{k+1} - \xi_k)\, u(\xi_k) e^{-j2\pi\omega\xi_k}$$

$$= \sum_{k=1}^{N} G_k(\omega)\, X_k(\omega) \qquad (5.4.4)$$

where

$$G_k(\omega) \quad = \quad G(\omega, \xi_k)(\xi_{k+1} - \xi_k) \tag{5.4.5}$$

and

$$X_k(\omega) \quad = \quad u(\xi_k)\, e^{-j2\pi\omega\xi_k} \tag{5.4.6}$$

The time function corresponding to the spectrum $X_k(\omega)$ given in equation (5.4.6) is

$$u(\xi_k)\, \delta(t - \xi_k)$$

Then, from equations (5.4.4), (5.4.5) and (5.4.6) it is seen that the approximate output spectrum $V^*(\omega)$ may be obtained by means of the configuration shown in figure 5.5.

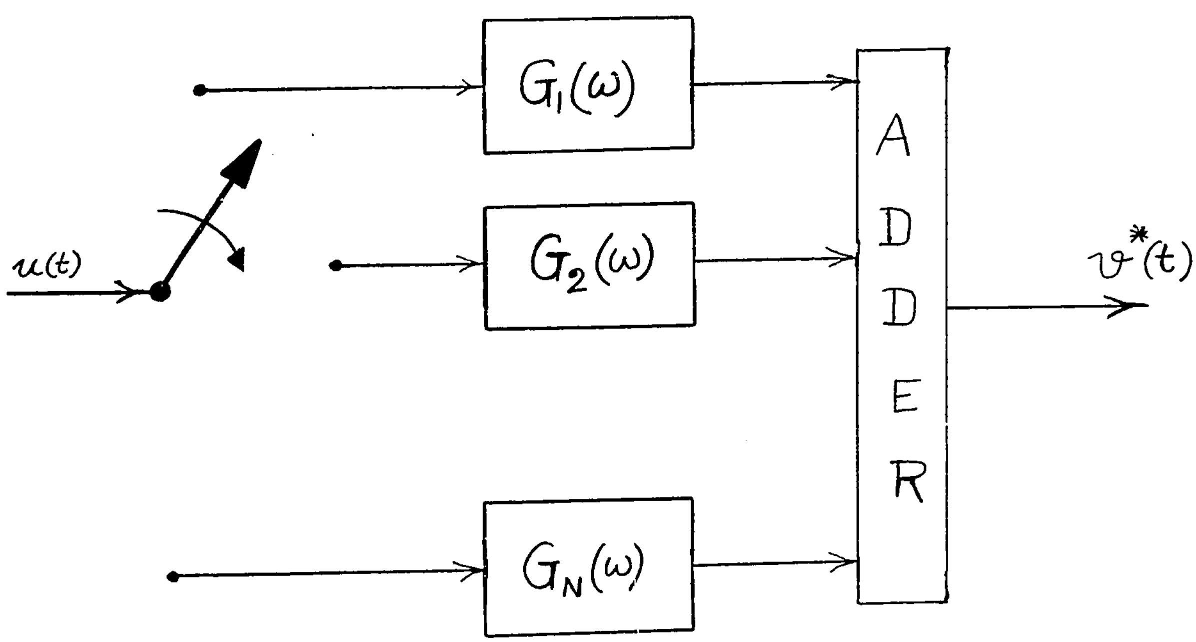


Fig. 5.5. Synthesis from $G(\omega, \tau)$.

In this figure the k-th impulse $u(\xi_k)\,\delta(t-\xi_k)$ is realized by means of the switch. The k-th impulse forms the input to the k-th filter. The corresponding outputs are then added up.

This configuration is dual to the one shown in figure 5.3.

ii) $G(\omega, \tau)$ may also be expressed as

$$G(\omega, \tau) = G(\tau, \beta)\,\delta(\omega-\beta)\,d\beta \tag{5.4.7}$$

which may be approximated as

$$G^*(\omega, \tau) = \sum_{k=1}^{N} G(\tau, \beta_k)\,(\beta_{k+1}-\beta_k)\,\delta(\omega-\beta_k) \tag{5.4.8}$$

Replacing $G(\omega, \tau)$ in the input-output relation (5.4.3) by $G^*(\omega, \tau)$ as given by equation (5.4.8) one obtains the approximate output spectrum as

$$V^*(\omega) = G^*(\omega, \tau)\,u(\tau)\,e^{-j2\pi\omega\tau}\,d\tau \tag{5.4.9}$$

$$= \sum_{k=1}^{N} G(\tau, \beta_k)\,(\beta_{k+1}-\beta_k)\,\delta(\omega-\beta_k)\,u(\tau)\,e^{-j2\pi\omega\tau}\,d\tau$$

Interchanging order of integration and summation

$$= \sum_{k=1}^{N} (\beta_{k+1}-\beta_k)\,\delta(\omega-\beta_k)\int_{-\infty}^{\infty} G(\tau, \beta_k)\,u(\tau)\,e^{-j2\pi\omega\tau}\,d\tau$$

$$= \sum_{k=1}^{N} X_k(\omega)\,Y_k(\omega) \tag{5.4.10}$$

where

$$X_k(\omega) = (\beta_{k+1} - \beta_k)\,\delta\,(\omega - \beta_k) \tag{5.4.11}$$

and

$$Y_k(\omega) = \int_{-\infty}^{\infty} G(\tau, \beta_k)\, u(\tau) e^{-j2\pi\omega\tau} d\tau \tag{5.4.12}$$

From equation (5.4.10) it is seen that the approximate output spectrum is the sum of output spectra of systems with system functions $X_k(\omega)$, $k = 1 \ldots N$, when the spectrum of the input to k-th such system is $Y_k(\omega)$ given by equation (5.4.12).

From equation (5.4.12) it is seen that the spectrum $Y_k(\omega)$ corresponds to a time function $u(t)\ G(t, \beta_k)$ which may be obtained from the input $u(t)$ by multiplying it by the time function $G(t, \beta_k)$ as shown in figure 5.6.

The system function $X_k(\omega)$ may be realized as a system which passes signals of frequencies from $_k$ to $_{k+1}$ unattenuated and stops other frequencies, i.e. a bandpass filter with bandwidth $(\beta_{k+1} - \beta_k)$.

The corresponding system may be realized as shown in figure 5.6.

This configuration is dual to the one in figure 5.4.

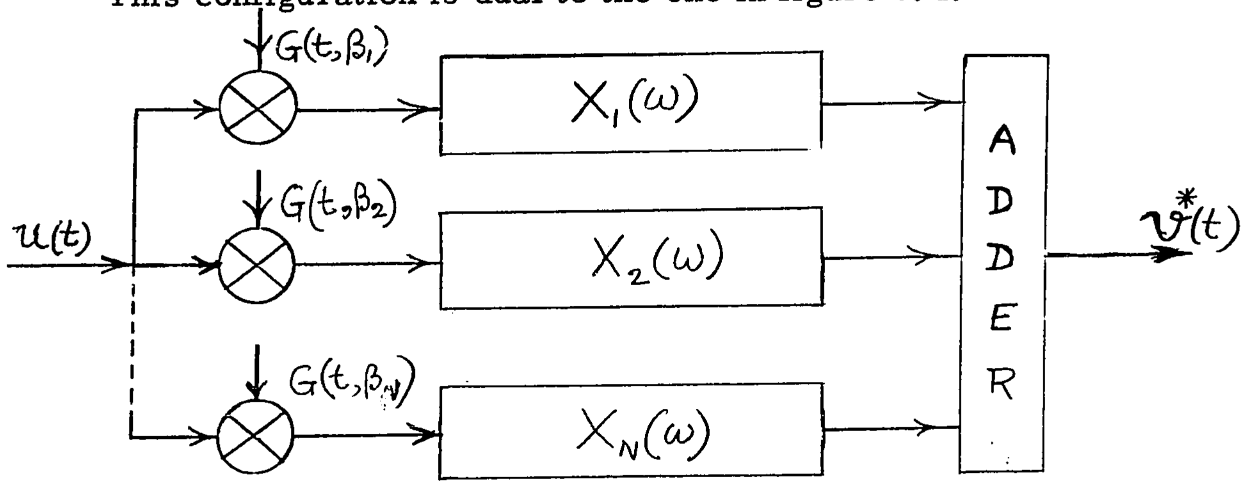


Fig. 5.6. Realization From $G(\omega, \tau)$.

5.5. Method of Synthesis from Bifrequency System Function $\Gamma(\mu, \nu)$ where $\mu$ and $\nu$ are frequencies in cps.

The bifrequency system function $\Gamma(\mu, \nu)$ may be expressed as

$$\Gamma(\mu, \nu) = \int \Gamma(\mu, \beta) \, \delta(\nu - \beta) \, d\beta . \qquad (5.5.1)$$

which may be approximated as

$$\Gamma^*(\mu, \nu) = \sum_{k=1}^{N} \Gamma(\mu, \beta_k) (\beta_{k+1} - \beta_k) \, \delta(\nu - \beta_k) \qquad (5.5.2)$$

In terms of $\Gamma(\mu, \nu)$ the output spectrum of a system is given by

$$V(\mu) = \int_{-\infty}^{\infty} \Gamma(\mu, \nu) \, U(\nu) \, d\nu \qquad (5.5.3)$$

where $U(\nu)$ is the input spectrum.

Replacing $\Gamma(\mu, \nu)$ in equation (5.5.3) by $\Gamma^*(\mu, \nu)$ as given by equation (5.5.2) the approximate output spectrum is obtained as

$$V^*(\mu) = \int_{-\infty}^{\infty} \Gamma^*(\mu, \nu) \, U(\nu) \, d\nu$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{N} \Gamma(\mu, \beta_k) (\beta_{k+1} - \beta_k) \, \delta(\nu - \beta_k) \, U(\nu) \, d\nu$$

Interchanging the order of integration and summation

$$= \sum_{k=1}^{N} \Gamma(\mu, \beta_k) (\beta_{k+1} - \beta_k) \int_{-\infty}^{\infty} \delta(\nu - \beta_k) \, U(\nu) \, d\nu$$

$$= \sum_{k=1}^{N} \Gamma(\mu, \beta_k) (\beta_{k+1} - \beta_k) \, U(\beta_k) \qquad (5.5.4)$$

From equation (5.5.4) it is seen that the output spectrum may be approximated by a sum of component spectra, the k-th one of which is obtained as the output spectrum of a system with system function $\Gamma_k(\mu) = \Gamma(\mu, \beta_k)(\beta_{k+1} - \beta_k)$ when the input spectrum to this system is $U(\beta_k)$, the value of the spectrum of the system input u(t) at a particular frequency $\beta_k$. In its turn, $U(\beta_k)$ may be obtained by passing the input u(t) through a system with system function $H(\beta_k)$, which passes signals of only one frequency $\beta_k$. The resultant model for synthesis is shown in figure 5.7.

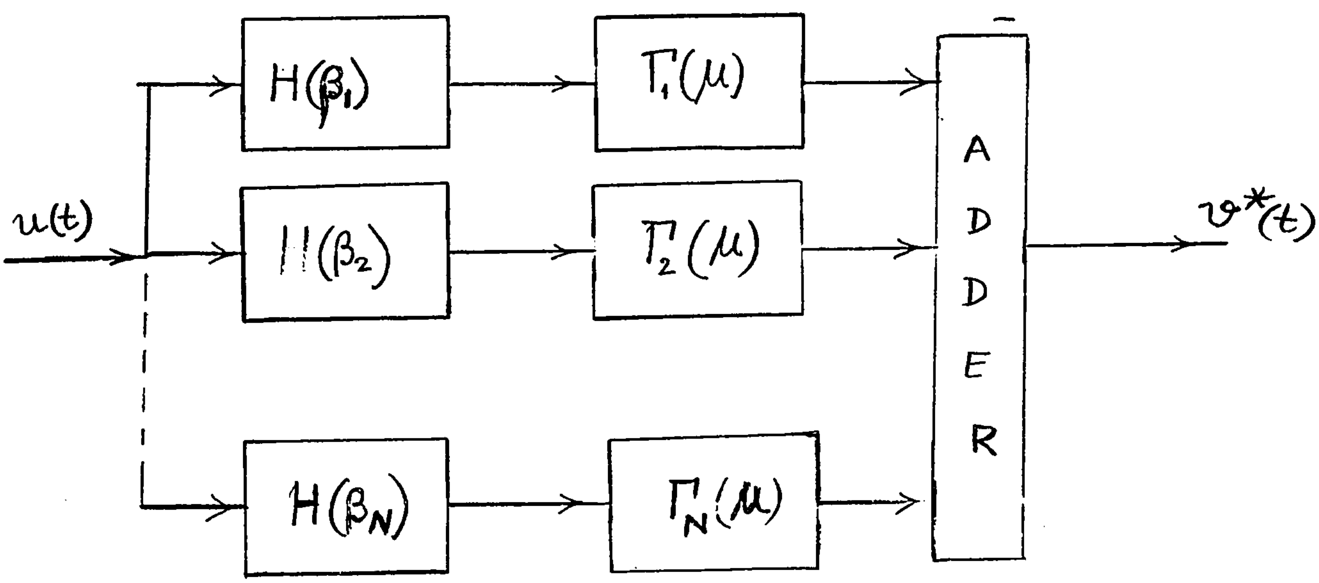


Fig. 5.7. Realization of $\Gamma(\mu, \nu)$.

5.6. General Discussion.

In the derivation of the system models described in this chapter, the aim has been to approximate a given characteristic function e.g., $h_1(t, \tau)$, $H(\nu, t)$ etc. by corresponding approximate characteristic functions $h_1^*(t, \tau)$, $H^*(\nu, t)$ etc. respectively. This purpose has been achieved in the indirect way of approximating the actual output $v(t)$ or $V(\omega)$ of the system by the output $v^*(t)$ or $V^*(\omega)$ of the model for a general class of inputs $u(t)$. Thus considering the case of impulse response $h_1(t, \tau)$ for example, it is seen that a continuous function $h_1(t, \tau)$ is approximated by $h_1^*(t, \tau)$ which consists of a series of delayed plane impulses and consequently $h_1^*(t, \tau)$ is zero almost everywhere and infinite somewhere. So the error involved in approximating $h_1(t, \tau)$ by $h_1^*(t, \tau)$ is obviously large. But for a large class of inputs the error involved in approximating the actual output $v(t)$ of the system by the output $v^*(t)$ of the model is reasonably small. However, a considerable error may result in the case when the input $u(t)$ itself is impulse-like with sharp peaks of short duration unless $h_1(t, \tau)$ also has sharp peaks of short duration. Thus this method is in general useful for inputs which are relatively smooth. But impulse-like or rapidly varing inputs also may be tolerated if $h_1(t, \tau)$ also has sharp peaks of short duration.

A question of considerable importance is what should be the number of plane impulses required for any particular case. More specifically, considering the case of the impulse response $h_1(t, \tau)$ what should be the value of N in the expansion of $h_1^*(t, \tau)$ in equation (5.2.3) so that $h_1^*(t, \tau)$ is a reasonable approximation of $h_1(t, \tau)$. This question may be answered as follows:

The value of N will depend on the amount of allowable error at the output of the system. Considering the impulse response as example, the actual output of the system is given by equation (5.2.4) as

$$v(t) = \int_{-\infty}^{t} h_1(t, \tau)\, u(\tau)\, d\tau \qquad (5.2.4)$$

and the output of the model in figure 5.2 is given by equation (5.2.5) as

$$v^*(t) = \sum_{k=1}^{N} a_k(t)\, u(t - \xi_k) \qquad (5.2.5)$$

Thus the error is

$$\begin{aligned} e(t) &= v(t) - v^*(t) \\ &= \int_{-\infty}^{t} h_1(t,\tau)\, u(\tau)\, d\tau - \sum_{k=1}^{N} a_k(t)\, u(t - \xi_k) \end{aligned} \qquad (5.2.6)$$

Then a suitable error criterion may be chosen and N may be determined such that the error e(t) given by equation (5.2.6) satisfies the error criterion. Thus, if it is required that the integral square error be less than certain preassigned number $\epsilon$, then N should be chosen such that the integral

$$\begin{aligned} I &= \int_{-\infty}^{\infty} /e(t)/^2\, dt \\ &= \int_{-\infty}^{\infty} \Big/ \int_{-\infty}^{t} h_1(t,\tau)\, u(\tau)\, d\tau - \sum_{k=1}^{N} a_k(t)\, u(t - \xi_k) \Big/^2\, dt \end{aligned} \qquad (5.2.7)$$

is less than $\epsilon$.

The remarks so far made in this section with reference to $h_1(t, \tau)$ as an example, very well apply to the synthesis methods from $H(\nu, t)$, $G(\omega, \tau)$ and $\Gamma(\mu, \nu)$.

The main advantage of the models developed in this chapter from the method of plane impulse train approximation, lies in their simplicity.

This section is concluded with a brief discussion of the advantages and disadvantages of the individual models.

The model shown in figure 5.2, first developed by Cruz [15] consists of a tapped delay line, time-varying gain amplifiers and an adder. The N time-varying gain amplifiers may be realized by means of N multipliers and N function generators. Thus this model needs a number of multipliers and function generators. But it has advantage for analog computer simulation as in this case only the time-varying gain amplifiers are to be adjusted.

Other models, developed in this chapter, show specific advantages under different specific conditions.

The model shown in figure 5.3 consists of N fixed systems, a sequential switch and a hold circuit and as such does not require any multiplier and function generator. The fixed system functions obtained in the course of the approximation procedure may not be, in some cases, exactly realizable, but it is assumed that in such cases they may be approximated by other system functions which are realizable and closely equivalent to the original ones.

The model shown in figure 5.4 requires N multipliers and N function generators. But the time invariant parts are systems each of which passes only a specific band of frequencies and are independent of the desired system function being approximated. Thus, the same time-invariant subsystems may be used for modeling of systems which have the same range of input frequencies of interest, but may differ from one another considerably in other respects. This model has advantages in analog computer simulation as in this case only the time functions $A_k(t)$'s need be changed for simulating different systems.

The model shown in figure 5.5 consists of a ganged sequential switch, N fixed systems and an adder. The time-varying portion of the model does not require any multiplier or function generator. This model is dual to the one shown in figure 5.3. Here again the problem of realizability of the fixed system $G_k(\omega)$'s poses itself. It is assumed that, if they are not exactly realizable, they may be approximated arbitrarily closely by realizable system functions.

The model shown in figure 5.6 needs for realization multipliers and function generators and fixed sub-systems. Thus the time varying portion has the disadvantage of requiring multipliers and function generators. But the fixed system functions are such that they are independent of the system to be realized provided only that the output frequencies of interest of the systems are the same. This model has also advantages for simulation on analog computers as only the function generators need be adjusted for simulating different systems.

The model shown in figure 5.7 is the model completely in the frequency domain. The time-varying characteristics of the system are summed up into the system function $\Gamma_k(\mu)$'s where $\mu$ is the frequency of variation of the system.

The derivation of these models does not require any time or frequency constraint on the system itself or the input or output of the system. Thus these models are valid for systems with or without any such restrictions imposed on them. But the models get much simplified if some suitable restrictions are imposed on the system. Thus, for a particular restriction, there is a particular model which proves more advantageous than others.

The following restrictions on the system are discussed:

1) <u>Output time restriction.</u>

Such a situation may arise in two ways.

a) The impulse response $h_3(y,t)$ or system function $H(\nu, t)$ may be restricted in time i.e. may be non-zero over a certain time interval and zero outside this interval.

b) The system function may not be restricted in time but the output may be of interest for a limited interval of time. In this case the original characteristic function is replaced by another which is equivalent to the original one over the interval of interest and zero elsewhere. Thus this restriction also may be reflected as a restriction on the system itself.

Under these conditions the proper characteristic functions to be considered are $h_3(y, t)$ and $H(\nu, t)$ with $t$ limited to an interval 0 to T. The model to be used are shown in figure 5.3. The number of fixed filters needed under these conditions is considerably reduced as the number N of terms required in the approximation formulae (5.2.10) and (5.3.2) is reduced.

An interesting case arises when $h_3(y, t)$ or $H(\nu, t)$ is a periodic function of time. In such case the corresponding function need be approximated only over one period. Further, if the function is symmetrical about the peak value then it is required to approximate it for only half a period. In such cases the set of fixed sub-systems obtained for the first half period may be reused for approximation over the next half period and hence for all time. In some cases it may even be sufficient to approximate the function for only quarter of a period. For the rest of the time the same fixed sub-systems may be used by suitable programming of the switch.

2. Output-frequency restriction.

Such situations may arise in two ways:

a) Output functions of only a limited range of frequencies may be of interest.

b) The system itself may produce outputs of only a limited band of frequencies.

The proper system function to consider in this case is $G(\omega, \tau)$ with the restriction that

$$G(\omega, \tau) \equiv 0 \quad \text{for } \omega_1 \leq \omega \leq \omega_2$$

where $\omega_2 - \omega_1$, is the bandwidth of frequencies of interest.

The model to be used is the one shown in figure 5.6.

Under these conditions the number of terms N in the approximation formula (5.4.8) is limited and the configuration in figure 5.6 also requires fewer fixed sub-systems.

3. Input time restriction.

Such a situation also may arise in two ways.

a) The input functions themselves may be of interest for a limited interval of time.

b) The system accepts inputs only for a limited interval of time.

The system function to consider is $G(\omega, \tau)$ with the restriction that

$$G(\omega, \tau) \equiv 0 \quad \text{for } \tau_1 \leq \tau \leq \tau_2.$$

where $\tau_2 - \tau_1$ is the interval of interest.

Under these conditions the model shown in figure 5.5 is the suitable one.

In this case fewer number of terms N in the approximation formula (5.4.2) are needed and hence the corresponding model shown in figure 5.5 also needs a fewer number of fixed sub-systems.

4. Input frequency restriction.

This restriction also may arise in two ways:

a) The input functions may contain frequencies of only a limited bandwidth.

b) The system may operate on signals of only a limited range of frequencies.

The proper system function to consider is $H(\nu, t)$ with the condition that

$$H(\nu, t) \equiv 0 \qquad \text{for } \nu_1 \leqslant \nu \leqslant \nu_2$$

where $\nu_2 - \nu_1$ is the bandwidth of interest.

In this case the model shown in figure 5.4 is the suitable one.

Under these conditions less terms are required in the approximation formula (5.3.6) and hence the model in figure 5.4 also requires less parallel fixed sub-systems.

## 5.7. An Example of Synthesis.

The problem of synthesis of a system with system function

$$H(s, t) = \frac{1}{s+1+\rho \operatorname{Cos} \omega_0 t} \tag{5.7.1}$$

where s is a complex frequency is considered.

This system is synthesized by the application of the approximation formula (5.7.2) and the corresponding model of figure 5.3.

$$H^{*}(s,t) = \sum_{k=1}^{N} H_k(s)\, \Delta\xi_k\, \delta(t - \xi_k). \tag{5.7.2}$$

First, a study of the behaviour of H(s, t) with time t is made.

Thus,

at

| | |
|---|---|
| $\omega_o t = 0$ radian | $H(s,0) = \dfrac{1}{s+1+\rho}$ |
| $\omega_o t = \dfrac{\pi}{2}$ " | $H(s, \dfrac{\pi}{2\omega_o}) = \dfrac{1}{s+1}$ |
| $\omega_o t = \pi$ " | $H(s, \dfrac{\pi}{\omega_o}) = \dfrac{1}{s+1-\rho}$ |
| $\omega_o t = \dfrac{3\pi}{2}$ " | $H(s, \dfrac{3\pi}{2\omega_o}) = \dfrac{1}{s+1}$ |
| $\omega_o t = 2\pi$ " | $H(s, \dfrac{2\pi}{\omega_o}) = \dfrac{1}{s+1+\rho}$ |
| $\omega_o t = \dfrac{5\pi}{2}$ " | $H(s, \dfrac{5\pi}{2\omega_o}) = \dfrac{1}{s+1}$ |
| $\omega_o t = 3\pi$ " | $H(s, \dfrac{3\pi}{\omega_o}) = \dfrac{1}{s+1-\rho}$ |
| $\omega_o t = 7\dfrac{\pi}{2}$ | $H(s, \dfrac{7\pi}{2\omega_o}) = \dfrac{1}{s+1}$ |
| $\omega_o t = 4\pi$ | $H(s, \dfrac{4\pi}{\omega_o}) = \dfrac{1}{s+1+\rho}$ |

From these considerations it follows that H(s, t) has, among others, the following characteristics.

1) H(s, t) is periodic along t axis with the same period as that of Cos $\omega_o t$.

2) Each period of H(s, t) is symmetric about the peak value.

It further follows that,

1) As H(s, t) is periodic it is necessary to approximate it only for one period of the function.

2) Moreover, as each period of the function is symmetric about the peak value, the approximation for only half a period is sufficient as the filters obtained in this interval may be appropriately used for other half of the period.

Experimental Procedures.

The procedure adopted consists of the following steps:

Sept 1. Responses of the system to delayed step inputs are calculated. The necessary computations are performed with the help of a digital computer.

Step 2. The responses of the system to delayed step inputs are also determined experimentally with the help of an analog computer.

This step consists of the following substeps.

a) Approximating the system function by means of formula (5.7.2).

b) Simulating the system according to equation (5.7.2) on an analog computer.

c) Recording the responses of the system to step inputs delayed by different amounts.

These three substeps are repeated thrice corresponding to the cases of four, seven and ten total terms retained in the approximation formula (5.7.2).

Step 3. The actual responses as obtained by calculation and the responses as obtained experimentally are compared.

These steps are discussed below in rather details.

<u>Step 1.</u>

The response of the system of equation (5.7.1) to delayed unit step $1(t - \tau)$ is given by:

$$v(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{1}{s+1+\rho \operatorname{Cos} \omega_o t} \quad \frac{e^{-s\tau}}{s} e^{st} ds.$$

$$= \frac{1 - e^{-(1+\rho \operatorname{Cos} \omega_o t)(t-\tau)}}{1+\rho \operatorname{Cos} \omega_o t} \tag{5.7.3}$$

The response v(t) as given by equation (5.7.3) is evaluated under the following conditions.

1) $\rho = .5$

2) $\omega_o = 1.047$ radians/sec.

3) $v(t) \equiv 0$ for $t < \tau$

4) $t =$ 0 second to 18 seconds, which is three periods of variation of $H(s,t)$, at equal intervals of .25 seconds.

5) $\tau =$ 0, second, 1.5 seconds and 4.5 seconds.

Step 2.

a) The Approximation Problem.

As explained earlier in this section, it is sufficient to approximate the system for only half a period which is, in this case, 3 seconds.

The system function in (5.7.1) is approximated by the approximation formula (5.7.2).

Three cases are considered:

Case 1. With Four Filters.

In this case the system is approximated for one half of a period, 3 seconds, with only four filters.

As the function $H(s,t)$ is a smooth function, the plane impulses are taken to be equally spaced. Consequently, in this case,

$$\xi_1 = 0 \text{ second}, \quad \xi_2 = 1 \text{ second}, \quad \xi_3 = 2 \text{ seconds}$$

and $\xi_4 = 3$ seconds.

So $\Delta\xi_k = 1$ second for all k.

Thus, the system functions of the required fixed sub-systems are given by,

$$
\begin{aligned}
H_1(s) &= \frac{1}{s+1+\rho\,\mathrm{Cos}0} = \frac{1}{s+1+\rho} = \frac{1}{s+1.5} \\
H_2(s) &= \frac{1}{s+1+\rho\,\mathrm{Cos}\ 1.047} = \frac{1}{s+1+.502\,\rho} = \frac{1}{s+1.251} \\
H_3(s) &= \frac{1}{s+1+\rho\,\mathrm{Cos}\ 2.094} = \frac{1}{s+1-.5\,\rho} = \frac{1}{s+.75} \\
H_4(s) &= \frac{1}{s+1+\rho\,\mathrm{Cos}\ 3.14} = \frac{1}{s+1-\rho} = \frac{1}{s+.5}
\end{aligned}
\qquad (5.7.5)
$$

Thus over one half of a period the system is approximated by

$$H^*(s,t) = \sum_{k=1}^{4} H_k(s)\,\Delta\xi_k\,\delta(t-\xi_k)$$

$$= H_1(s)\,\delta(t) + H_2(s)\delta(t-1) + H_3(s)\delta(t-2) + H_4(s)\delta(t-3)$$

where $H_k(s)$'s are given by equation (5.7.5)

For the remainder of the time these filters are appropriately used as many times as needed.

Case 2. Case with Seven Filters.

In this case seven filters are used to approximate the system over one half of a period. The plane impulses are equally spaced and they occur at intervals of $\Delta\xi_k = .5$ second .

Thus, the system functions of the required fixed systems are given by,

$$
\begin{aligned}
H_1(s) &= \frac{1}{s+1+\varsigma\cos 0} &&= \frac{1}{s+1+\varsigma} &&= \frac{1}{s+1.5} \\
H_2(s) &= \frac{1}{s+1+\varsigma\text{Cos } .524} &&= \frac{1}{s+1+.865\varsigma} &&= \frac{1}{s+1.432} \\
H_3(s) &= \frac{1}{s+1+\varsigma\text{Cos } 1.048} &&= \frac{1}{s+1+.502\varsigma} &&= \frac{1}{s+1.251} \\
H_4(s) &= \frac{1}{s+1+\varsigma\text{Cos } 1.572} &&= \frac{1}{s+1+.06\varsigma} &&= \frac{1}{s+1.003} \\
H_5(s) &= \frac{1}{s+1+\varsigma\text{Cos } 2.096} &&= \frac{1}{s+1-.5\varsigma} &&= \frac{1}{s+.75} \\
H_6(s) &= \frac{1}{s+1+\varsigma\text{Cos } 2.620} &&= \frac{1}{s+1-.867\varsigma} &&= \frac{1}{s+.567} \\
H_7(s) &= \frac{1}{s+1+\varsigma\text{Cos } 3.14} &&= \frac{1}{s+1-\varsigma} &&= \frac{1}{s+.5}
\end{aligned}
\qquad (5.7.6)
$$

Thus over one half of a period, the system is approximated as

$$
\begin{aligned}
H^*(s,t) &= \sum_{k=1}^{7} H_k(s)\ .4\,\delta(t-\xi_k) \\
&= H_1(s)\ .5\,\delta(t) + H_2(s)\ .5\,\delta(t-.5) + \\
&\quad \ldots\ldots\ldots\ H_7(s)\ .5\,\delta(t-3)
\end{aligned}
$$

where $H_k(s)$'s are given by equation (5.7.6). For the remainder of the time these filters are used appropriately.

Case 3. Case with Ten Filters.

In this case the plane impulses are equally spaced at intervals of .333 seconds. The required system functions of the corresponding filters are given by,

$$
\begin{aligned}
H_1(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;0} &&= \frac{1}{s+1+\varrho} &&= \frac{1}{s+1.5} \\
H_2(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;.349} &&= \frac{1}{s+1+939\varrho} &&= \frac{1}{s+1.469} \\
H_3(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;.698} &&= \frac{1}{s+1+.765\varrho} &&= \frac{1}{s+1.382} \\
H_4(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;1.047} &&= \frac{1}{s+1+.502\varrho} &&= \frac{1}{s+1.251} \\
H_5(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;1.396} &&= \frac{1}{s+1+.175\varrho} &&= \frac{1}{s+1.088} \\
H_6(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;1.745} &&= \frac{1}{s+1-.173\varrho} &&= \frac{1}{s+.914} \\
H_7(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;2.094} &&= \frac{1}{s+1-.5\varrho} &&= \frac{1}{s+.75} \\
H_8(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;2.443} &&= \frac{1}{s+1-.765\varrho} &&= \frac{1}{s+.618} \\
H_9(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;2.792} &&= \frac{1}{s+1.939\varrho} &&= \frac{1}{s+.531} \\
H_{10}(s) &= \frac{1}{s+1+\varrho\,\mathrm{Cos}\;3.14} &&= \frac{1}{s+1-\varrho} &&= \frac{1}{s+.5}
\end{aligned}
\tag{5.7.7}
$$

Thus, for one half of a period, the system is approximated by

$$
\begin{aligned}
H^*(s,t) &= \sum_{k=1}^{10} H_k(s)\;.333\,\delta(t-\xi_k) \\
&= H_1(s).333\,\delta(t) + H_2(s)\;.333\,\delta(t-.333) + \\
&\quad \ldots\ldots\ldots H_{10}(s).333\,\delta(t-3)\;.
\end{aligned}
$$

where $H_k(s)$'s are given by equation (5.7.7). For the remainder of the time these filters are used appropriately.

b) Simulation of the System on Analog Computer [16].

Simulation of the whole system is divided into the following two smaller problems.

1) Simulation of the Fixed Linear Systems $H_k(s)$'s

The general purpose analog computer functions in the time domain, so it is possible to simulate directly some functions which are functions of time and/or d/dt. But in this case, it is necessary to simulate system functions $H_k(s)$'s which are functions of s, the complex frequency. In this regard, a result of importance is that if all the initial conditions of the input u(t), output v(t) and their derivatives are assumed to be zero, then it is possible to replace a system function H(s) by an equivalent expression H(p), obtained by replacing s by $p = \frac{d}{dt}$, for simulation on analog computer. This result has been used in simulation of the system functions $H_k(s)$'s. Thus expressions $H_k(p)$'s are derived from $H_k(s)$'s by replacing s with p and then $H_k(p)$'s are simulated.

2) The second problem is to simulate the impulses of area $v_k(t)\,\Delta\xi_k$ where $v_k(t)$ is the output of the k-th filter and $\Delta\xi_k$ is the interval in time between the k-th and (k+1)th impulses and then to add them up as shown in the approximation formula (5.7.2).

The whole operation was simulated by a two-deck ganged sequential switch as shown in figures 5.8, 5.9 and 5.10. One of the switches connects, at time $t = \xi_k$, the output of the k-th filter to the input terminal of the integrator for a very short time and leaves the input terminal disconnected until it connects the next filter output. The second switch, ganged to first one, provides a feedback from the output terminal to the input terminal of the integrator during only the time intervals when the input terminal of the integrator remains connected to the output of a filter. At all other times the feedback is disconnected.

Thus during the time interval for which the input terminal of the integrator is connected to any filter output, the output of the integrator attains that value and holds it for an interval $\Delta\xi_k$ till the next filter is connected. The output of the integrator is an almost smooth curve which is the approximate output of the system to the input u(t).

The simulation set up for the complete system with different number of filters are shown in figures 5.8, 5.9 and 5.10.

c) The outputs of the simulated system are recorded, for step inputs delayed by amounts of 0 second, 1.5 seconds and 4.5 seconds.

A set of typical experimental graphs obtained for different numbers of filters is shown in figures 5.11, 5.12 and 5.13.

Plots of the calculated (ideal) response of the system for these situations are also shown in these figures for comparison.

The symbols used in the following figures are the same as used in Section 4.2.

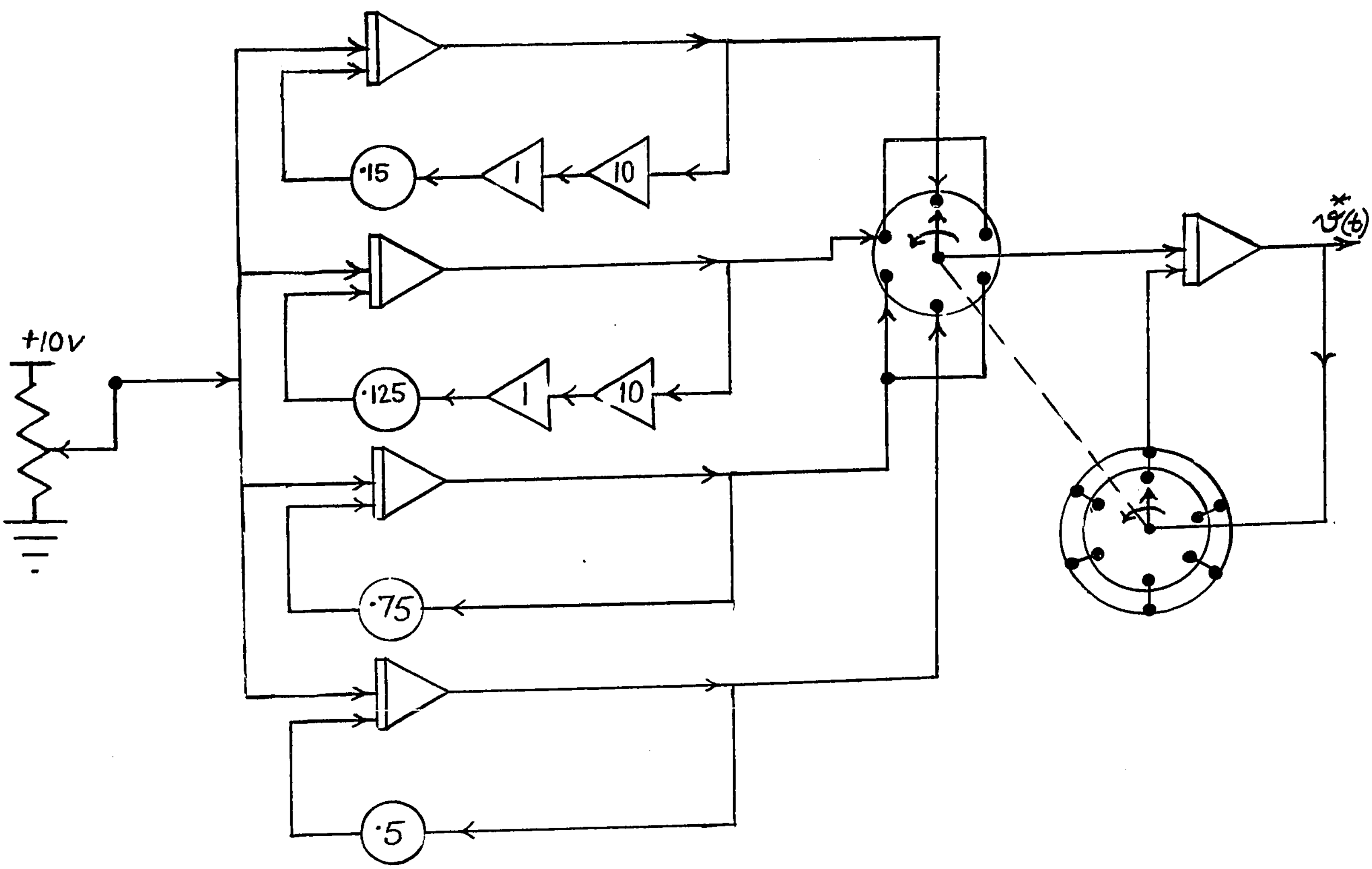


Fig. 5.8. Simulation Set Up With Four Filters.

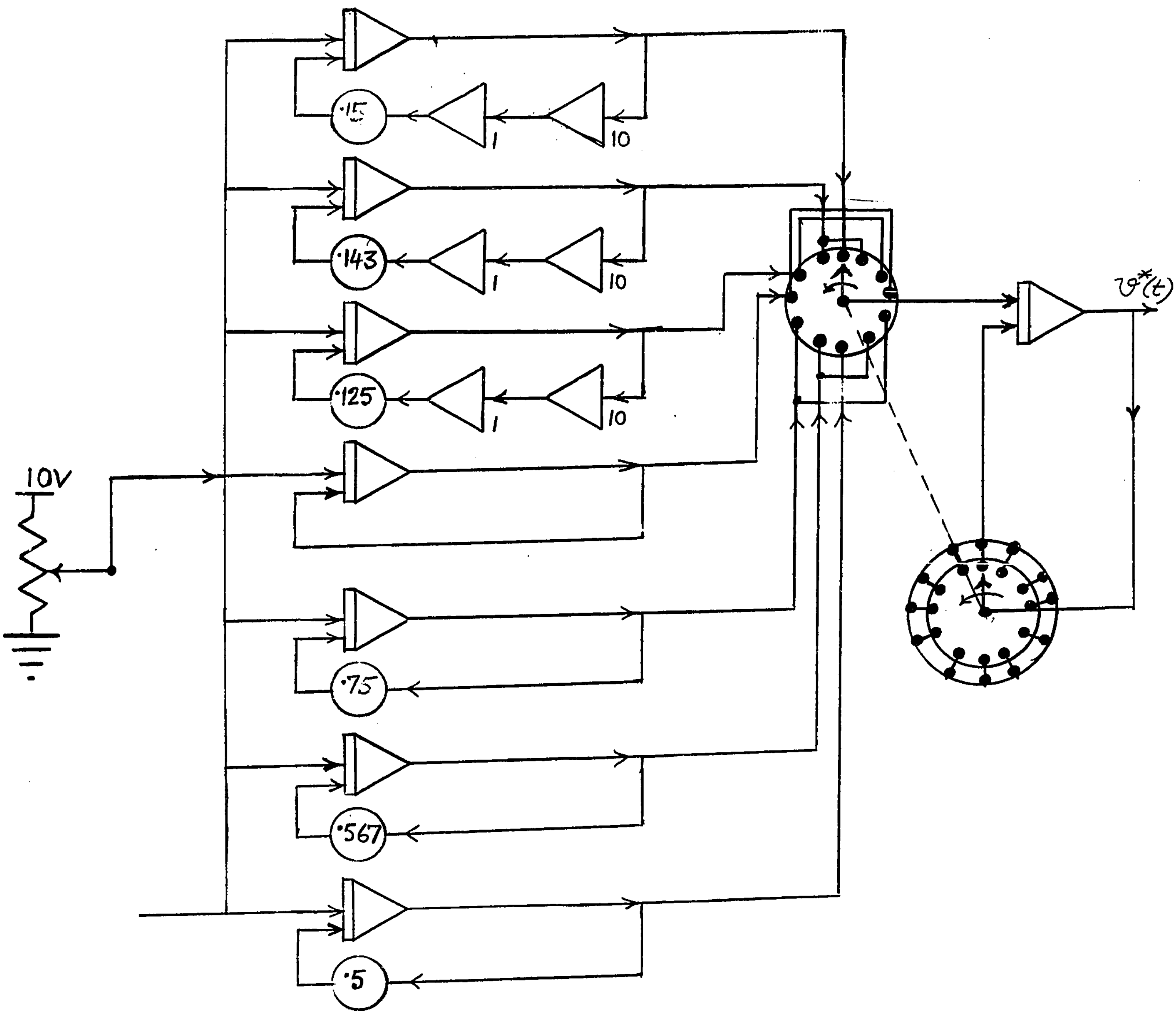


Fig. 5.9. Simulation Set Up With Seven Filters.

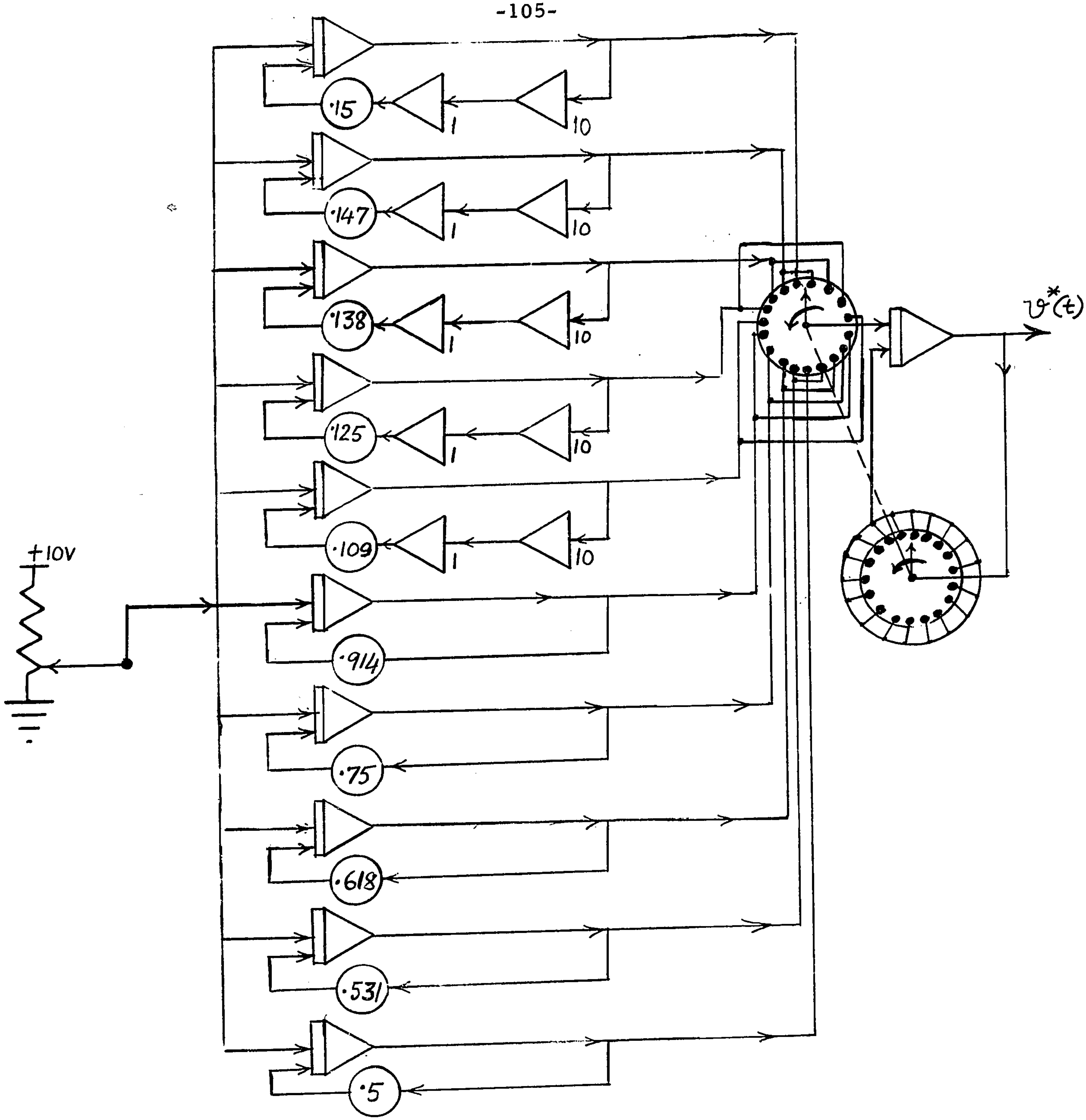


Fig. 5.10. Simulation Set Up with Ten Filters.

$$H(s,t) = \frac{1}{s+1+\rho\cos.\ \omega_0 t} \quad ; \ \rho = 0.5 \ ; \ \tau = 0 \text{ sec.}$$

INPUT : DELAYED UNIT STEP $U(t-\tau)$ ; $\omega_0 = 1.047$ radians /sec.

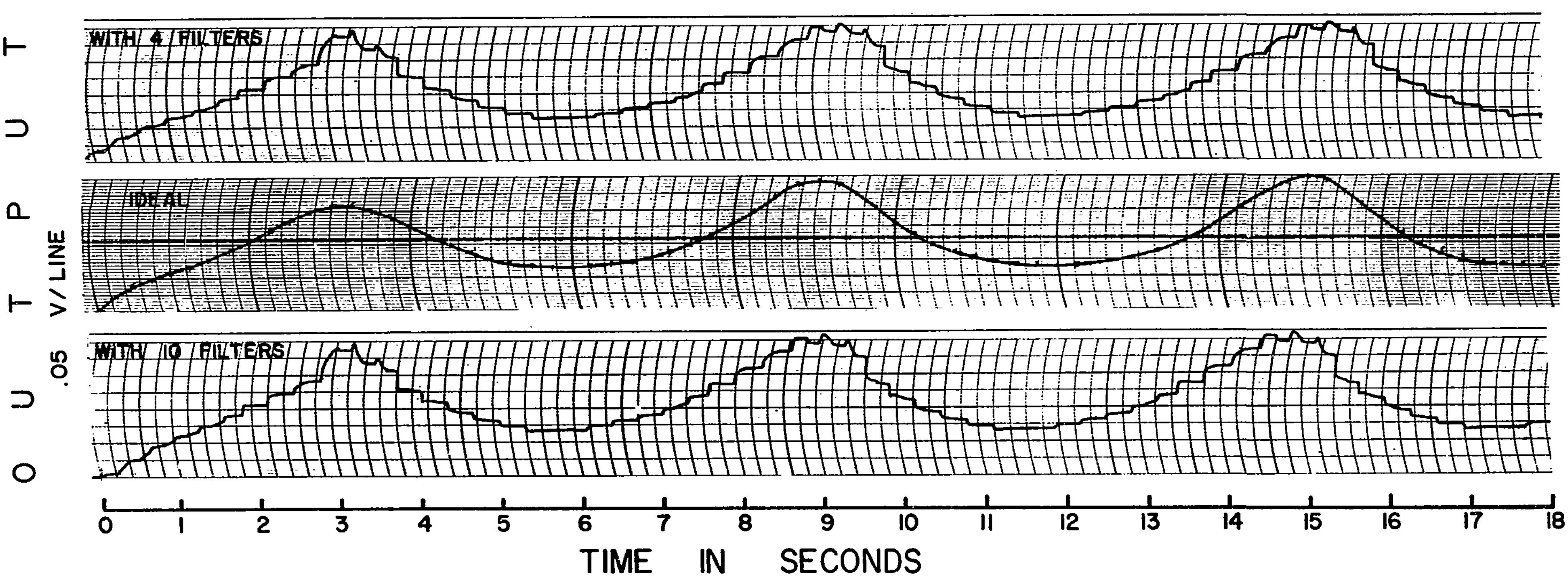


Fig. 5.11 COMPARISON OF IDEAL AND EXPERIMENTAL OUTPUTS

$$H(s,t) = \frac{1}{s+1+\rho \cos. \omega_0 t} \quad ; \rho = 0.5 \quad ; \tau = 1.5 \text{ sec.}$$

INPUT : DELAYED UNIT STEP $U(t-\tau)$ ; $\omega_0 = 1.047$ radians/sec.

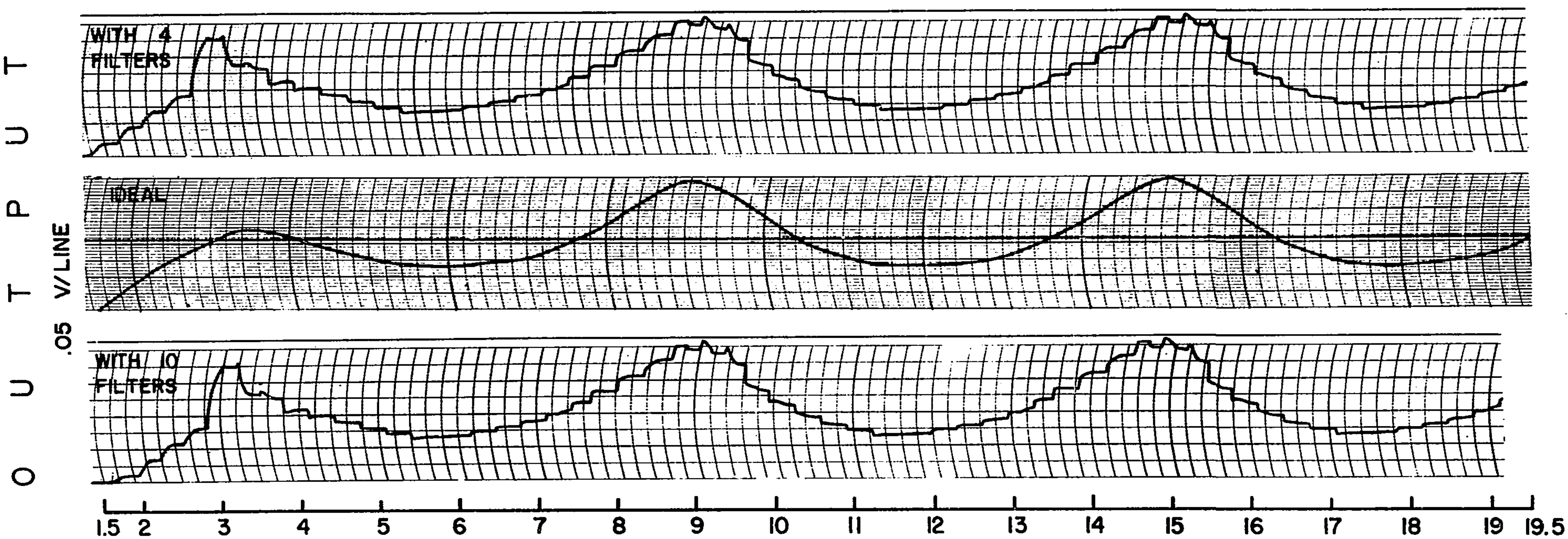


Fig. 5.12 COMPARISON OF IDEAL AND EXPERIMENTAL OUTPUTS

$$H(s,t) = \frac{1}{s+1+\delta \cos. \omega_0 t} \; ; \; \delta = 0.5 \; ; \; \tau = 4.5 \text{ sec.}$$

INPUT : DELAYED UNIT STEP $U(t-\tau)$ ; $\omega_0 = 1.047$ radians/sec.

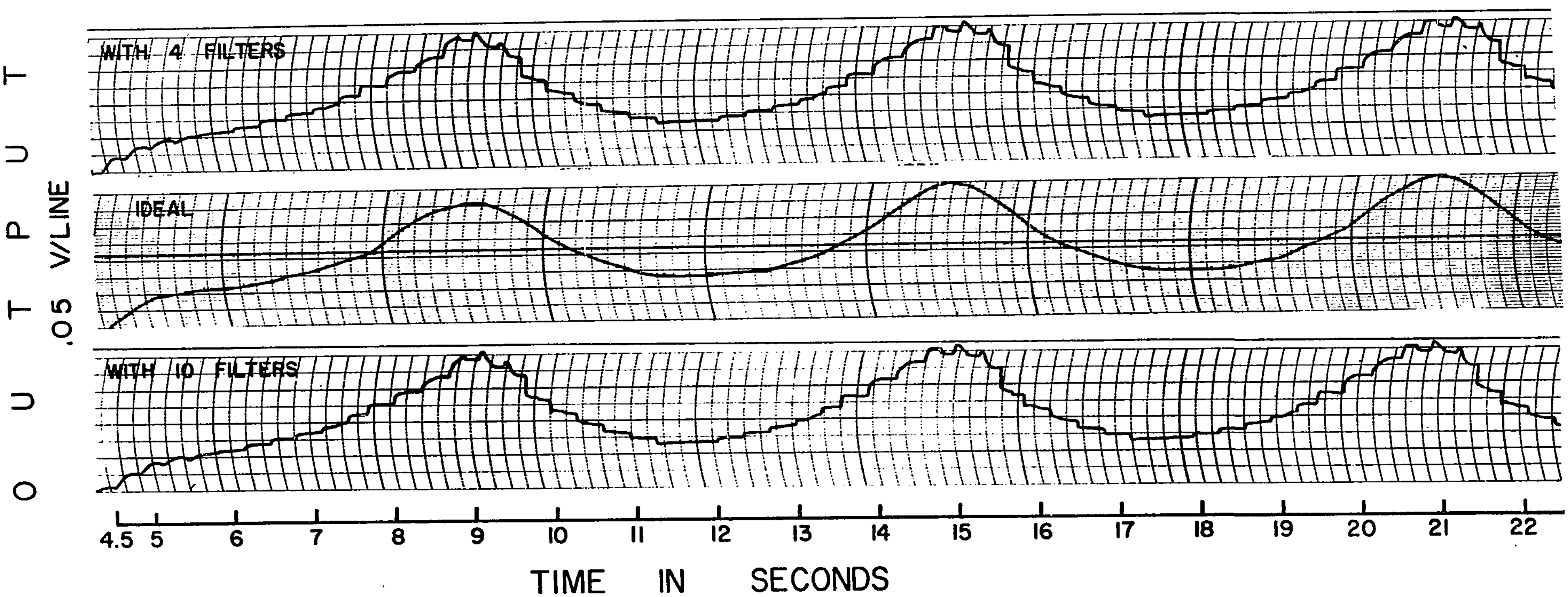


Fig.5.13 COMPARISON OF IDEAL AND EXPERIMENTAL OUTPUTS.

# SUMMARY AND CONCLUSIONS.

This thesis has been mainly concerned with an analytical study of different methods of synthesis of linear time-varying differential systems.

A brief account of the methods of representation of such systems in terms of differential equation, state equations and various characteristic functions has been presented in the introduction.

Methods of synthesis from the points of view of differential equation, state equations and different characteristic functions have been investigated.

Methods of synthesis presented here are divided into two categories, depending on the type of configuration they lead to, i.e. feedback configuration and parallel elements configuration. Accordingly, the material in this thesis has been presented in two parts.

The first part delas with methods of synthesis in feedback configuration from such specifications as differential equation, state equations, state variables and some characteristic functions.

The second part contains the methods of synthesis in parallel elements configuration from system characteristic functions such as impulse response, H-System function, G-System function and bifrequency system function. The models described in this part have been obtained by means of orthonormal functions expansion, sampling theorem and plane impulse train approximation.

The main contribution in this thesis consists in several models obtained by the method of plane impulse train approximation. These models have the advantage of simplicity in either the time-varying or the time invariant portion. A general discussion on the advantages and disadvantages of these models have been presented in section 6.6.

One of these models has been used for simulating a system and experimental results are presented in graphical form.

Further work in this area may be conducted in the following directions:

1) Extension of this method of synthesis, specially the model shown in figure 5.3, to the synthesis of more complex, multi-input-multi-output and probabilistically varying linear time-varying systems.

2) The design of proper switch is a problem needing further study.

3) Study of stability of the models obtained by this method.

Further study in the field of synthesis of linear time-varying systems in general may, of course, be continued.

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