A Polyhedral Approach to Designing Communication Networks

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ABSTRACT

Polytopes Q_{2E}^n and Q_{2N}^n , which are associated with the minimum cost 2-edge-connected subgraph problem and the minimum cost 2-node-connected subgraph problem respectively , are studied in this thesis, and some new classes of facet-inducing inequalities are introduced for these polytopes. These classes of inequalities are related to the so-called clique tree inequalities for the travelling salesman polytope (Q_T^n), and the relationships between Q_T^n and Q_{2E}^n, Q_{2N}^n are exploited in obtaining these new classes of facets.

Due to the use of problem specific facet-inducing inequalities instead of dominant cutting-planes , the linear programming cutting-plane method has proven to be quite successful for solving some NP-hard combinatorial optimization problems. We believe that our new classes of facet-inducing inequalities can be used to further improve the cutting-plane procedure for designing minimum cost survivable communication networks.

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Chapter 1 Introduction and Notation

1.1 Introduction

Communication networks are widely used in our world. They range from local area networks to cross-continent networks. Some typical examples are telephone networks and computer networks. Due to the advances in transmission technology (such as the emergence of fiber optic technology) and computer technology, such networks are becoming increasingly more important.

When designing communication networks , two main issues are economy and survivability. By economy we refer to the construction cost , and by survivability we refer to the ability to restore service when a catastrophic failure of a network component occurs, such as the loss of a link or the failure of a switching node. Service could be restored by routing traffic through other existing network links and nodes, assuming that the design of the network has provided for this additional connectivity. Clearly, a higher level of redundant connectivity results in greater network survivability as well as greater overall network construction cost. Thus one aim of network design is to minimize the construction cost while satisfying given survivability requirements. The structures of star, tree and ring have been used previously. The advent of optical fibre with large bandwidth has introduced some new characteristics to the design of networks. Recent work on methods for designing survivable fibre communication networks in [5] and [20] concludes (1) that survivability is an important issue for fibre networks , (2) that "two-connected" topologies provide a high level of survivability in a cost-effective manner, and (3) that good heuristic methods exist for quickly generating "near-optimal" networks. In particular, it was determined that a network topology should provide for at least two diverse paths between any two offices, thus providing for protection against any single link or single node failure for traffic between the offices. This leads naturally to the problem of designing certain two-connected networks.

We now formalize the network design problems that are being considered in this paper. A set V of nodes is given that represents the locations of the switches (offices) that must be interconnected into a network in order to provide the desired services. A collection E of edges is also specified that represents the possible pairs of nodes between which a direct transmission link can be placed. We let G=(V,E) be the (undirected) graph of possible direct link connections. We study the networks that will survive the loss of any single link or the failure of any switching node. For simplicity, we assume that the total construction cost of a network is the sum of the costs of its links. The cost of a link is typically represented by its length. These problems are known as the 2-edge-connected spanning subgraph problem (TECSP for short), and the 2-node-connected spanning subgraph problem (TNCSP for short) respectively , and are of interest not only practically but theoretically as well. Mathematically, TECSP and TNCSP can be formulated as follows:

TECSP: Given a graph G=(V,E) and vector $c \in R^E$ of edge costs, find a 2-edge-connected spanning subgraph of G having the minimum total edge cost.

TNCSP: Given a graph G=(V,E) and vector $c \in R^E$ of edge costs, find a 2-node-connected spanning subgraph of G having the minimum total edge cost.

Without the loss of generality, we consider in the thesis only the problems where G is the complete graph K_n, since other cases can easily be changed to the above case by assigning sufficiently large cost to the missing edges (with respect to the complete graph) without affecting the optimal solution. For convenience, we will denote the two problems on n nodes by TECSP(n) and TNCSP(n) respectively.

The two-connected problem (both TECSP and TNCSP) is closely related to the well studied (symmetric) Traveling Salesman Problem (TSP in short) which is the problem of finding a minimum cost Hamiltonian cycle in a given weighted complete graph. Like TSP, TECSP and TNCSP are NP-hard, thus it is unlikely that efficient algorithms will be found to solve these problems. Hence heuristic methods or branch-and-bound methods are often used to obtain solutions which may be far from optimal.

In this thesis, we study TECSP(n) and TNCSP(n) from an approach based on polyhedral combinatorics and the linear programming cutting-plane method. This method always guarantees to provide solutions which are optimal or provably close to optimal. First introduced by G.Dantzig, D.Fulkerson and S. Johnson[8], this approach has proven to be quite successful in solving a number of large TSP problems to optimality. In M. Grötschel[10] a 120-city TSP problem is solved to optimality. In H. Crowder et al.[7] 10 TSP problems ranging from 48 cities to 318 cities are solved to optimality. A computation study on 74 TSP problems of size from 15 to 318 cities is conducted in M.Padberg et al.[23], whose results also lend convincing support to the computation value of this method. In M. Padberg[27], a 532-city problem is solved to optimality. In M. Padberg[25], 42 TSP problems of sizes from 48 to 2392 nodes are all solved to optimality in "reasonable" times.

Since we are motivated to use the linear programming cutting-plane method, we first associate with TECSP(n) (or TNCSP(n)) a polytope Q^n_{2E} (or Q^n_{2N}), which is the convex hull of all 0-1 incidence vectors for the edge sets of 2-edge-connected subgraphs (or 2-node-connected subgraphs) of K_n. We then try to find the linear description of the related polytope (i.e. we try to find explicitly the linear constraints for the corresponding linear programming problem). Note that it is unlikely that we could ever find a complete description of Q^n_{2E} (or Q^n_{2N}) since the TECSP (and TNCSP) is NP-hard, and thus we often have only a partial description of the polytope. Next the linear programming cutting-plane method is applied to optimize over the obtained feasible region. Often a commercial linear programming code is used to optimize over the region represented by some subset of the necessary constraints found for Q^n_{2E} (or Q^n_{2N}). If the "optimal solution" obtained corresponds to a solution for TECSP(n) (or TNCSP(n)) , then it is also an optimal solution for TECSP(n) (or TNCSP(n)) . Otherwise it provides a lower bound on the value of an optimal solution of TECSP(n) (or TNCSP(n)). If the lower bound obtained is sufficient for our purpose we can terminate the computation, otherwise we can attempt to identify a constraint in the partial linear description known for Q^n_{2E} (or Q^n_{2N}) which is violated by the "optimal solution", and add this constraint to our current set and repeat the process again to obtain a better lower bound. Sometimes we cannot find such a violated constraint in the partial description, in which case we either accept this lower bound as it is , or we can use some other strategies such as the branch-and-bound method in order to continue our algorithm. (For a more complete description of the linear programming cutting-plane method, see M. Padberg & M.Grötschel [22]).

In any linear description of a polytope, only some of the constraints are necessary , or "facet-inducing" in that description. A great deal of the success of the linear programming cutting-plane method is due to the use of facet-inducing inequalities as cutting-planes, however finding such facet-inducing inequalities is in general difficult.

In M. Grötschel et al. [14] , the linear cutting-plane approach proves to be successful in solving generalized 2-connected spanning subgraph problems of small sizes, from 28 to 39 nodes, where the underlying graph is not complete. In this thesis, we concern ourselves with the search for such facet-inducing inequalities for Q_{2E}^n and Q_{2N}^n , and use innovative new methods introduced in [4] for proving that the inequalities we find are facet-inducing. These methods are based on analyzing facets for a related problem (in this case, the TSP), and extrapolating that information to derive facets for the problems being considered. We hope that our work will facilitate the use of the linear programming cutting-plane approach for large problems. In Chapter 2 of this thesis we describes the relationship between TSP and two-connect problems TECSP and TNCSP. In Chapter 3 and Chapter 4, we introduce two new classes of facet-inducing inequalities for TECSP. In Chapter 5 and Chapter 6, we introduce two new classes of facet-inducing inequalities for TNCSP. In Chapter 7 we address the equivalence problem for these new classes of inequalities. Finally, in Chapter 8, we make some concluding remarks.

We now introduce some basic concepts and terminology in graph theory and polyhedral combinatorics.

1.2 Notation

For any finite set X we let | X | denote the cardinality of X. Given another set Y, we let X\Y denote the members of X which are not members of Y.

For any finite set E we let R^E denote the set of all real vectors indexed by E. If the members in E are ordered, i.e., E={e_1, e_2,......, e_n}, we sometime use R^n instead of R^E. We let I_i denote the unit vector in R^n whose ith entry is 1 and is 0 elsewhere. For any J ∈ E and $x \in R^E$, we let $x(J)$ denote $(\sum x_j : j \in J)$.

For any subset F of E, the *incidence vector* of F is the vector $x \in R^E$ defined by

$$x_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise.} \end{cases}$$

We denote the incidence vector of F by x^F.

A set $X \subseteq R^E$ is *linearly independent* if whenever $\sum_{x \in X}(\alpha_x x | x \in X) = 0$

for some $\alpha \in R^X$ we have $\alpha = 0$. Otherwise, X is *linearly dependent.*

The *linear rank of a set S* $\subseteq R^E$ is the cardinality of a largest linearly independent subset of S and is denoted by r(S). The *linear rank of a matrix A* is the cardinality of a maximal independent subset of the columns of A. We denote the linear rank of A by $r_l(A)$.

Given a matrix $A \in R^{L \times E}$ and subset $J \subseteq E$, we let A_J represent the (|L|×|J|) submatrix of A consisting of those columns of A indexed by J. We abbreviated $A_{\{j\}}$ by A_j.

1.3 Graph Theory

Some standard references can be found in J.A. Bondy et al. [2]. A *graph* G is an ordered pair (V,E), where V is a finite set of members which are called *nodes* and E is a finite set of elements which are called *edges* such that each edge $e \in E$ corresponds to two distinct nodes in V, which are called the *ends* of e. An edge $e \in E$ with ends w and v is sometimes denoted by wv, and we say edge e *joins* node w and node v . Two nodes u and v are said to be *adjacent* if $uv \in E$, and if e=uv, e is said to be *incident* with u and v. Here we will only consider graphs that have no multiple edges, i.e., there is at most one edge joining any two nodes.

Given any graph G , we let V(G) and E(G) denote the node set and edge set of G respectively. For any $S \subseteq V(G)$, we use $\gamma(S)$ to denote the set of edges in E(G) with both ends in S , and we use $\delta(S)$ to denote the set of edges in E(G) with exactly one end in S and one end in V(G)\S. For $V_1, V_2 \subseteq V(G)$, we denote by $[V_1 : V_2]$ the set of edges in E(G) with one end in V_1 and the other end in V_2 . The *degree* of a node $v \in V(G)$ is defined as the number of edges in G incident with node v , and is denoted by $\deg_G(v)$.

A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and every edge has the same ends in H as in G . If V(H)=V(G) , then H is called a *spanning subgraph* of G. Given $E' \subseteq E(G)$ and $V' \subseteq V(G)$, we denote by $G - E' - V'$ the subgraph of G whose node set is $V(G) \setminus V'$ and whose edge set is the set of edges in $E(G) \setminus E'$ with both ends in $V(G) \setminus V'$. In the case $E' = \{e\}$ and $V' = \{v\}$, we write G-e-v.

A graph G is called *complete* if every pair of nodes is joined by exactly one edge. The complete graph on n nodes is denoted by K_n . A *clique* of G is a subset W of the nodes such that any two nodes in W are adjacent in G.

A *path* in a graph G is a finite non-null sequence $v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$ whose terms are alternately nodes and edges such that the nodes are distinct and, for $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i. We say path P joins v_0 and v_k. The *length* of a path is |E(P)|, and a *Hamiltonian path* of G is a path of length |V(G)|-1.

A graph is *connected* if every two nodes in G are joined by a path. A *component* of graph G is any maximal connected subgraph of G. A graph G is called *2-edge-connected* if G-e is connected for any $e \in E(G)$. A graph G is called *2-node-connected* if G-v is connected for any node $v \in V(G)$. A graph G is called *2-connected* if G is 2-edge-connected and 2-node-connected.

A *cycle* C in a graph G is a connected subgraph of G such that every node in V(C) has degree 2 in C. The *length* of C is |E(C)| , and a *Hamiltonian cycle* of G is a cycle in G of length |V(G)| . Sometimes the 0-1 incidence vector of a Hamiltonian cycle is also referred to as a *tour*. A cycle is obviously 2-connected.

A *forest* is a graph containing no cycles and a *tree* is a connected forest. A tree T has the property that |V(T)|=|E(T)|+1.

The *node-edge incidence matrix* of a graph G is a matrix $A \in R^{V(G) \times E(G)}$ such that the entry of A indexed by node v and edge e has value 1 if v is an end of e, otherwise it has value 0.

1.4 Polyhedral combinatorics

In this section we briefly discuss some basic definitions and concepts of polyhedral combinatorics. More detailed treatments of the subject can be found in A. Bachem et al.[1] , G.L. Nemhauser et al.[21] , W.R. Pulleyblank[28].

For any $X \subseteq R^E$, the *convex hull of* X, denoted by conv(X), is the set of all $y \in R^E$ such that y can be expressed as a convex combination of a finite subset of the members of X, i.e., conv(X)={ $y \in R^E \mid y = \sum(\lambda_x x \mid x \in \tilde{X})$ for some finite set $\tilde{X} \subseteq X$ and some $\lambda \in R^{\tilde{X}}$ such that $\sum(\lambda_x \mid x \in \tilde{X}) = 1$, $\lambda_x \geq 0$ for each $x \in \tilde{X}$}.

A *halfspace* is a set $H \subseteq R^E$ of the form {x: $ax \leq a_0$ } and a *hyperplane* is a set $L \subseteq R^E$ of the form { x: $ax = a_0$} for some $a \in R^E$ and $a_0 \in R$. An inequality $ax \leq a_0$ is *valid* for some $S \subseteq R^E$ if S is contained in the halfspace defined by $ax \leq a_0$, i.e. $ay \leq a_0$ for any $y \in S$.

A *polyhedron* $P \subseteq R^E$ is the intersection of finitely many halfspaces. Equivalently, P is the solution set of a finite system of linear equations and inequalities, and can be expressed in the form P={ $x \in R^E \mid Dx=d,\ Ax \leq b$}. A *polytope* is a polyhedron which is bounded . A subset F of polyhedron P is a *face* of P if F is either the empty set or the polyhedron obtained by taking the linear system which defines P and replacing some of the inequalities with the corresponding equations. F is called a *proper* face of P if $F \neq P$. A face F of P is a *facet* of P if F is a proper non-empty maximal face of P.

If $ax \leq a_0$ is a valid inequality for P, then we say the face $F = \{x \in P | ax = a_0\}$ is *induced* by $ax \leq a_0$. Furthermore, $ax \leq a_0$ is said to be *facet-inducing* if the face F induced by $ax \leq a_0$ is a facet of P. Any two valid inequalities $ax \leq a_0$ and $bx \leq b_0$ for P are called *equivalent* with respect to P if they induce the same face, i.e., $\{x \in P | ax = a_0\} = \{x \in P | bx = b_0\}$.

A finite set $X \subseteq R^E$ is *affinely independent* if whenever $\sum_{x \in X}(\alpha_x x | x \in X) = 0$ and $\sum_{x \in X}(\alpha_x | x \in X) = 0$ for some $\alpha \in R^X$, we also have $\alpha = 0$. The *affine rank* of a set $S \subseteq R^E$ is the cardinality of a largest affinely independent subset of S and is denoted by $r_a(S)$. It is well known that for any $S \subseteq R^E$, if $0 \in aff(S)$, then $r_a(S) = r_l(S)+1$, otherwise $r_a(S) = r_l(S)$([28]). For any $X \subseteq R^E$, an *affine basis* of X is a maximal set of affinely independent points in X. The *dimension* of a polyhedron $P \subseteq R^E$ is defined as $r_a(P)-1$ and denoted by dim(P). The affine basis for any facet F of a polyhedron P will consist of dim(P) affinely independent points. We say P is of *full dimension* if dim(P)=|E| or equivalently if there does not exist some linear equation $ax=a_0$ satisfied by all $x \in P$. Given a linear system defining a polyhedron P, the set of constraints which are satisfied with equality by all $x \in P$ is called an *equation system* for P and is related to the dimension of P in the following way:

Theorem 1.1. Let $P \subseteq R^E$ be a non-empty polyhedron and let Ax=b represent an equation system for P. Then we have

$$\dim(P) = |E| - r_l(A).$$

We also have the following result on the equivalence of two inequalities:

Theorem 1.2. Given a polyhedron P with minimal equation system Dx=d, two facet-inducing inequalities $ax \le a_0$ and $bx \le b_0$ are equivalent for P if and only if $b = ra + \lambda D$ and $b_0 = ra_0 + \lambda d$ where r>0 is a scalar and λ is a vector.

Given any polyhedron P, a major problem in polyhedral combinatorics is to find a finite linear system which defines P, hopefully containing as few constraints as possible. To do this often requires finding inequalities which induce facets of P. The following theorem describes the relationship between facets and a minimal defining linear system for a polyhedron P.

Theorem 1.3. Let $P \subseteq R^E$ be a polyhedron and suppose $P=\{x \in R^E \mid Dx=d, Ax \le b\}$. Then this is a minimal linear system sufficient to define P if and only if

i) the rows of D are linearly independent and Dx=d forms an equation system, and

ii) each constraint of $Ax \le b$ induces a distinct facet of P.

Finding a complete minimal linear system which defines a polyhedron P is often a difficult problem, but even a partial description can be useful for combinatorial optimization problems. This is the case when we deal with a NP-hard combinatorial optimization problem, since it is unlikely that we could ever find a complete linear system. Thus the problem of identifying facet-inducing inequalities for P is important. The following provides two basic methods for proving an inequality is facet-inducing.

Theorem 1.4. Let F be a non-empty proper face of $P=\{ x \in R^E \mid Dx=d , Ax \le b\}$, and let Dx=d be an equation system of P and let the rows of D be indexed by a finite set V. Then the following are equivalent:

i) F is a facet of P;

ii) dim(F)=dim(P)-1;

iii) For any $a,b \in R^E$ and $a_0,b_0 \in R$ satisfying $F=\{x \in P \mid ax=a_0\}=\{x \in P \mid bx=b_0\}$. There exists $\lambda \in R^V$ and positive $\gamma \in R$ such that $b=\gamma a+\lambda D$ and $b_0=\gamma a_0+\lambda d$.

In this thesis we will use the method proposed in [4] to prove that a valid inequality is facet-inducing. This method will allow us to exploit the relationship between Q_T^* and Q_{2E}^*, and between Q_T^* and Q_{2N}^*, thus hopefully simplifying our proof.

Chapter 2 Polyhedral Background on the TSP, the TECSP and the TNCSP

Since the two-connected problem (both TECSP and TNCSP) is closely related to the widely studied symmetric Traveling Salesman Problem TSP , we start this section by giving a brief review of the results for TSP.

2.1 The Traveling Salesman Problem Polytope Q_T^n

Let $K_n = (V, E)$ be the complete graph on n nodes. The *Travelling Salesman Polytope* , denoted by Q_T^n , is the convex hull of all incidence vectors of Hamiltonian cycles of K_n, i.e.

$$Q_T^n := conv\{x^F \in R^E \mid (V,F) \text{ is a Hamiltonian cycle of } K_n\} .$$

We give a brief discussion of the currently known facets for Q_T^n in this section .

There has been extensive research on Q_T^n and its facets. Here we only mention the results which are essential for later sections.

Theorem 2.1 (M. Grötschel et al. [16]). The *degree constraints*

$$x(\delta(v)) = 2 \quad \text{for all} \quad v \in V,$$

form a minimal equation system for Q_T^n.

Note that the set of degree constraints for Q_T^n can also be written as $Ax=2$ where A is the node-edge incidence matrix for K_n. A consequence of Theorem 2.1 is the following:

Theorem 2.2 (M. Grötschel et al.[16]). The dimension of Q_T^n is $|E|-n$ for $n \geq 3$.

Since Q_T^n is not full dimensional, we may have inequalities $ax \le a_0$ and $bx \le b_0$ which induce the same facets of Q_T^n and yet look quite different. Sometimes it is difficult to tell if two inequalities are equivalent.

There are several well-known classes of facets for Q_T^n.

Theorem 2.3 (M. Grötschel et al.[16]). For all $n \ge 5$ and all $e \in E$, the nonnegativity constraints $x_e \ge 0$ induce distinct facets of Q_T^n.

These are also sometimes called the *trivial inequalities* of Q_T^n.

Since no tour can contain a subtour, i.e., the 0-1 incidence vector of a cycle of length $\omega < n$, every $x \in Q_T^n$ must satisfy

$$x(\gamma(S)) \le |S|-1 \quad \text{for all } S \subseteq V \quad \text{where } 2 \le |S| \le n-2.$$

These inequalities are termed the *subtour elimination constraints.* Note that the subtour elimination constraints include the facet-inducing inequalities $x_e \le 1$ for $e \in E$ by taking $|S|=2$. If we take any subtour elimination constraint $x(\gamma(S)) \le |S|-1$ and subtract half the sum of all degree constraints for $v \in S$ we obtain $-1/2(x(\delta(S))) \le -1$. Thus the subtour elimination constraints can equivalently be described as

$$x(\delta(S)) \ge 2 \quad \text{for all } S \subseteq V \quad \text{where } 2 \le |S| \le n-2,$$

which is known as the *cut form* of these constraints. It is easy to see that the subtour elimination constraints for S and V\S are equivalent. We have the following theorem describing the facet-inducing properties of subtour elimination constraints.

Theorem 2.4 (M. Grötschel [16]). For all $n \geq 4$ and $S \subseteq V$ satisfying $2 \leq |S| \leq \lfloor n/2 \rfloor$, $x(\delta(S)) \geq 2$ induce distinct facets of Q_T^n.

Another well-known class of facet-inducing inequalities for Q_T^n are the comb constraints , which were first introduced by Chvátal [6] and later generalized by Grötschel and Padberg in [16] .

A *comb* consists of a handle $H \subseteq V$ and mutually disjoint teeth $T_1, T_2, \cdots, T_k \subseteq V (k \geq 3$ and odd) such that

$$T_j \cap H \neq \phi \neq T_j \setminus H \qquad 1 \leq j \leq k.$$

The associated *comb inequality* is

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i| - 1) - \frac{k+1}{2} . \tag{2.1}$$

Note that the coefficients on the left-hand side of this inequality are 0,1 and 2.

Figure 2.1 illustrates the left -hand side coefficients for the comb constraint $ax \leq 8$. Edges with coefficients 0 are not shown.

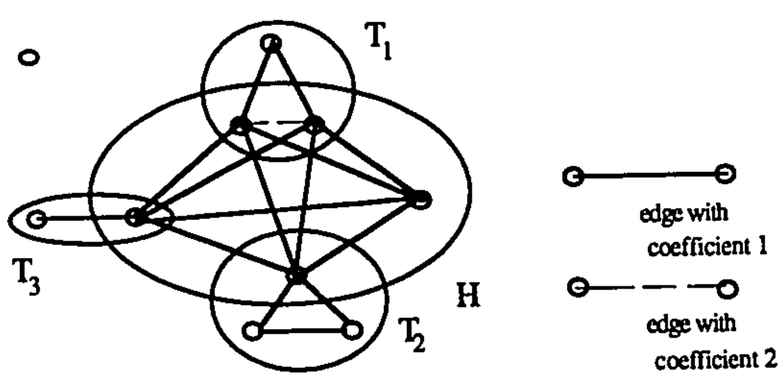


Figure 2.1 Edge coefficients of a comb inequality.

A comb is called *simple* if $|T_i \cap H| = 1$ for i=1,2,…,k, and the associated inequality is called a *simple comb inequality*. In the case $|T_i|=2$ for all $1 \le i \le k$, the comb inequality (2.1) simplifies to

$$x(\gamma(H)) + \sum_{i=1}^{k} x(\gamma(T_i)) \le |H| + \frac{k-1}{2}.$$

These inequalities are called the *2-matching constraints*, and were first introduced by Edmonds [9] .

Theorem 2.5 (M. Grötschel [16,17]). If $\{H, T_1, T_2, \cdots, T_k\}$ is a comb , then so too is $\{V \setminus H, T_1, T_2, \cdots, T_k\}$ and the corresponding inequality induces the same facet of Q_T^n. In all other cases, each comb inequality induces a facet distinct from other comb constraints, subtour elimination constraints and nonnegativity constraints.

Proof. (We include a proof of validity for the comb inequality for completeness.)

To prove the validity of the comb inequality we use the following equations and inequalities which are valid for Q_T^n.

1) $x(\delta(v)) = 2$ for $\forall v \in H$,

2) $x(\gamma(T_i)) \le |T_i| - 1$ $\quad i = 1,2,\ldots,k$,

3) $x(\gamma(T_i \setminus H)) \le |T_i \setminus H| - 1$ $\quad i = 1,2,\ldots,k$,

4) $x(\gamma(T_i \cap H)) \le |T_i \cap H| - 1$ $\quad i = 1,2,\ldots,k$.

By summing these we obtain the following inequality.

$$\sum_{v \in H} x(\delta(v)) + \sum_{1}^{k} \{x(\gamma(T_i)) + x(\gamma(T_i \setminus H)) + x(\gamma(T_i \cap H))\} \le 2|H| + 2\sum_{1}^{k}(|T_i| - 1) - k.$$

Since $x_e \geq 0$ for all $e \in E$, the left hand side of the above inequality is greater than or equal to $2(x(\gamma(H))+\sum_1^k x(\gamma(T_i)))$. Thus

$$2(x(\gamma(H))+\sum_1^k x(\gamma(T_i))) \leq 2|H|+2\sum_1^k (|T_i|-1)-k.$$

Since for every incidence vector of a tour, the left hand side of the above is an even integer, dividing by 2 and rounding down the right hand side gives us

$$x(\gamma(H))+\sum_1^k x(\gamma(T_i)) \leq |H|+\sum_1^k (|T_i|-1)-\frac{k+1}{2}.$$

Thus the comb inequality is valid for Q_T^n. □

A class of facet-inducing inequalities for Q_T^n which are called *clique-tree inequalities* was introduced in Grötschel and Pulleyblank [18] . A *clique tree* is a connected graph C whose maximal cliques partition into two sets, the set of *handles* and the set of *teeth*, which satisfy the following properties:

(1) no two teeth intersect;

(2) no two handles intersect;

(3) Each tooth contains at least two and at most (n-2) nodes, and at least one node belonging to no handle;

(4) The number of teeth intersecting each handle is odd and at least three;

(5) If a tooth T and a handle H have a non-empty intersection , then $H \cap T$ is an articulation set of the clique tree, i.e., $C \setminus (H \cap T)$ is not connected.

Figure 2.2 shows an example of a clique tree, where cliques are indicated by ellipse-shaped figures. Each ellipse containing a '*' is a tooth. The '*' indicates that there must be a node in the respective tooth which does not belong to any handle.

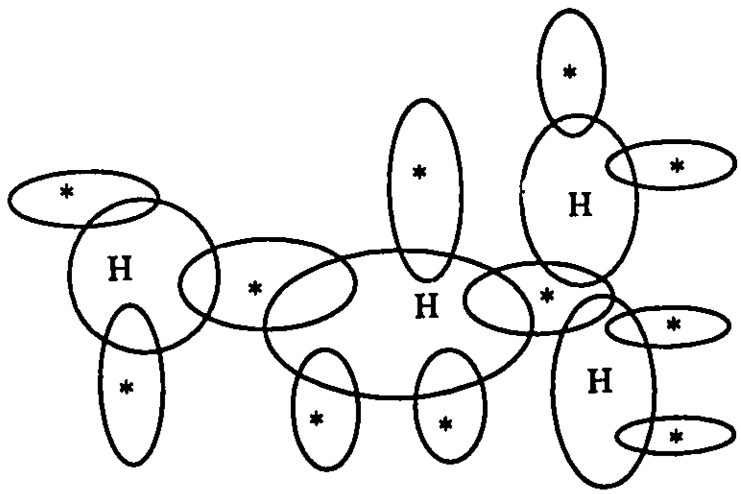


Figure 2.2 A Clique tree.

We call a clique tree *simple* if any handle and any tooth have at most one node in common.

Suppose we have a clique tree C with handles $H_1, H_2, \cdots, H_r$ and teeth $T_1, T_2, \cdots, T_s$. Then the corresponding *clique tree inequality* is

$$\sum_{i=1}^{r} x(\gamma(H_i)) + \sum_{j=1}^{s} x(\gamma(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} = s(C) \tag{2.2}$$

where for every tooth T_j, the integer t_j denotes the number of handles which intersect T_j, j=1,2,⋯, s. The right-hand side s(C) of (2.2) is called the *size* of C.

Note that in the case where there is a tooth T and a handle H with $|H \cap T| \geq 2$, the coefficients on the left-hand side of (2.2) are 0,1 and 2. The inequality (2.2) is a 0-1 inequality only if the clique tree is simple. If W is the set of all nodes of a clique tree, then for simple clique trees, inequality (2.2) can be written as

$$\sum_{i=1}^{r} x(\gamma(H_i)) + \sum_{j=1}^{s} x(\gamma(T_j)) \leq |W| - \frac{s+1}{2}. \tag{2.3}$$

Note that the subtour elimination constraint is a special case of the clique tree inequality in which there is only one tooth and no handle, and the comb inequality is the clique tree inequality having exactly one handle.

Theorem 2.6 ([18]). The clique tree inequality corresponding to the clique tree $\{H_1, H_2, \cdots, H_r, T_1, T_2, \cdots, T_s\}$ with at least two handles induces a facet distinct from all other clique trees, combs, subtour elimination constraints and nonnegativity constraints.

In this paper we use some special classes of clique trees to derive new classes of facet-inducing inequalities for Q_{2E}^n and Q_{2N}^n. First we introduce these special classes.

The first special class of clique trees which we call *Star Clique Trees* and denote by SCT is shown in Figure 2.3.

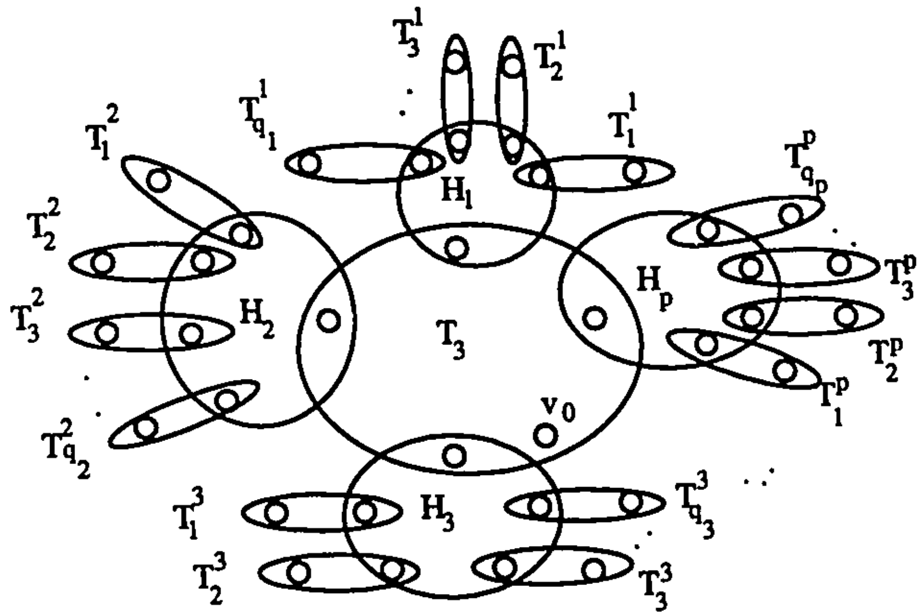


Figure 2.3 A Star Clique Tree.

As we can see from Figure 2.3, a SCT has $p \geq 2$ handles $H_1, H_2, \cdots, H_p$, where each handle H_i contains $q_i + 1$ nodes and intersects q_i 2-matching teeth $T_1^i, T_2^i, \cdots, T_{q_i}^i$, and one non 2-matching tooth T_3. We let v_0 be the unique node in T_3 but not in any handle. Let $q := \sum_{i=1}^{p} q_i$. Note that q is even.

Associated with a SCT is the following clique tree inequality:

$$\sum_{i=1}^{p} x(\gamma(H_i)) + \sum (x(\gamma(T)) : T \textit{ a tooth of SCT}) \leq p + 3q/2$$

or (if we let ax denote the left-hand side)

$$ax \leq p + 3q/2. \tag{2.4}$$

The second special class of clique trees which we call *Path Clique Trees* and denote by PCT is shown in Figure 2.4.

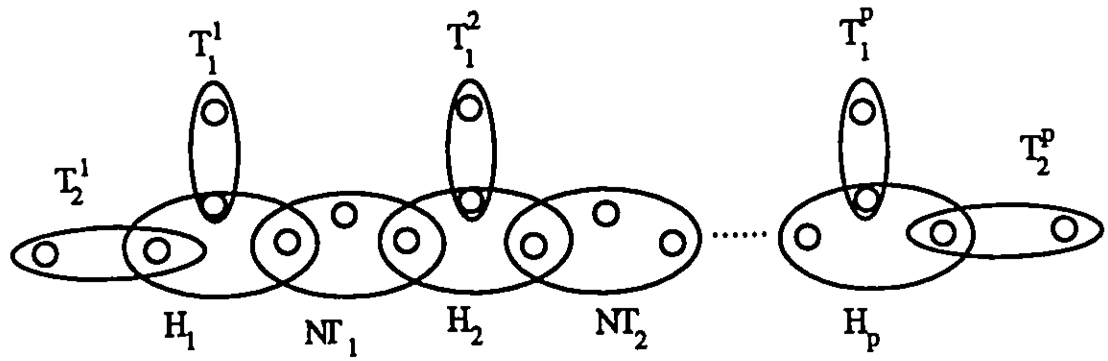


Figure 2.4 A Path Clique Tree .

As we can see from Figure 2.4, a PCT consists of $p \geq 2$ handles $H_1, H_2, \cdots, H_p$ and 2p+1 teeth: p+2 2-matching teeth $T_1^1, T_1^2, \cdots, T_1^p, T_2^1, T_2^p$, and p-1 non 2-matching teeth $NT_1, NT_2, \cdots, NT_{p-1}$. Each handle $H_i, i = 2,3,\cdots,p-1$ intersects the three teeth NT_{i-1}, NT_i and T_1^i, H_1 intersects the three teeth T_2^1, T_1^1, NT_1 , and H_p intersects the three teeth NT_{p-1}, T_1^p, T_2^p.

Associated with a PCT is the following clique tree inequality:

$$\sum_{i=1}^{p} x(\gamma(H_i)) + \sum (x(\gamma(T)) \colon T \text{ a tooth of } PCT) \leq 4p$$

or (if we let ax denote the left-hand side)

$$ax \leq 4p. \tag{2.5}$$

2.2 The 2-Edge-Connected Spanning Subgraph Polytope Q_{2E}^n

The polytope associated with TECSP(n) is

$Q_{2E}^n := conv\{x^F \in R^E | (V,F)$ is a 2-edge-connected spanning subgraph of $K_n\}$

There has been some previous research done on the polytope Q_{2E}^n. In a much more general form, Q_{2E}^n is studied by Grötschel and Monma in [12] and by Grötschel , Monma and Stoer in [13]. It is also studied by Mahjoub in [19] for general graphs G, and by Boyd and Pulleyblank in [4] and Boyd and Hao in [3] for complete graphs. Below we describe all the results pertaining to the facets of Q_{2E}^n from [3,4,12,13,19] .

Theorem 2.7. The dimension of Q_{2E}^n is n(n-1)/2 for n ≥ 4.

Proof.(We include a proof for completeness.)

We will prove the result by constructing (n(n-1)/2)+1 affinely independent points in Q_{2E}^n. Without loss of generality we use $e_1, e_2, \ldots\ldots, e_{|E|}$ to denote the edges in E. For each edge $e_j, j = 1,2,\ldots\ldots,|E|$, we construct a two-edge-connected spanning subgraph T_j which includes all the edges of E except e_j. Also we construct a two-edge-connected spanning subgraph T_0 which includes all the edges of E. We denote the corresponding incidence vectors as

$$x_0 = (1,1,\ldots\ldots,1) \text{ and}$$

$$x_j = (1,1,\ldots,\underset{j}{0},1,\ldots 1) \qquad j = 1,2,\ldots,|E|.$$

Now we show that $x_0, x_1, \ldots\ldots, x_{|E|}$ are affinely independent. This is equivalent to showing that $x_0', x_1', x_2', \ldots, x_{|E|}'$ are linearly independent, where x_j' is the vector obtained from x_j by adding a new component of value 1, for $j = 0,1,2,\ldots\ldots,|E|$. Since

$$x_0' - x_i' = (0,0,\ldots,\underset{i}{1},0,\ldots 0) \qquad i = 1,2,\ldots,|E|,$$

clearly $x_0'-x_1', x_0'-x_2', \ldots, x_0'-x_{|E|}', x_0'$ are linearly independent. Thus $x_0, x_1, \ldots, x_{|E|}$ are affinely independent. Since $|E| = n(n-1)/2$, the result follows. $\square$

Note that by Theorem 2.7, Q^n_{2E} is full dimensional and hence the equivalence problem for Q^n_{2E} is much simpler than for Q^n_T. Two facet-inducing inequalities for Q^n_{2E} are equivalent if and only if one is a positive multiple of the other.

Theorem 2.8. For $e \in E$, $x_e \leq 1$ is facet-inducing for Q^n_{2E} if $n \geq 4$.

Theorem 2.9. For $e \in E$, $x_e \geq 0$ is facet-inducing for Q^n_{2E} if $n \geq 5$.

Theorem 2.10. For all $S \subseteq V$, $\phi \neq S \neq V$, $x(\delta(S)) \geq 2$ is facet-inducing for Q^n_{2E}.

Another class of inequalities called the *lifted 2-cover inequalities* are introduced in M. Grötschel et al.[13] for a more general polytope (of which Q^n_{2E} is a special case), and these are shown to be facet-inducing under certain conditions. These inequalities can be described as follows: Let $H \subseteq V$ be a node set, let $T \subseteq \delta(H)$ be an edge set such that $|T| \geq 3$ and odd, and let $H_1, H_2, \cdots, H_p, p \geq 3$ be a partition of H into nonempty disjoint node sets such that no more than two edges in T intersect any H_i. Then the *lifted 2-cover inequality* is given by

$$x(\gamma(H)) - \sum_{i=1}^{p} x(\gamma(H_i)) + x(\delta(H)) - x(T) \geq p - \frac{|T|-1}{2} \quad . \tag{2.6}$$

It is noted in [13] that the odd wheel inequalities introduced in [19] for the 2-edge connected spanning subgraph polytope of general graphs is a special case of the lifted 2-cover inequality.

The polytope Q_{2E}^n is closely related to the polytope Q_T^n. It is easy to see from a graph theoretic point of view that a 2-edge-connected graph is a Hamiltonian cycle if and only if each node in the graph is of degree 2. Thus $Q_T^n = \{x \in Q_{2E}^n | Ax = 2\}$, where A is the node-edge incidence matrix for K_n, i.e., Q_T^n is a face of Q_{2E}^n. Therefore for every facet-inducing inequality for Q_T^n, there exists an equivalent form (with respect to Q_T^n) of that inequality which is also facet-inducing for Q_{2E}^n. It would be desirable to exploit this relationship and somehow transform a facet-inducing inequality for Q_T^n into an equivalent one which is facet-inducing for Q_{2E}^n. Boyd and Pulleyblank in [4] converted the two-matching constraint into an equivalent form which is facet-inducing for Q_{2E}^n. These inequalities are called *complemented 2-matching constraints*, and have the form

$$ax \geq |\dot{V}| + \frac{k+1}{2} \tag{2.7}$$

where $\dot{V} \subset V$ is the set of nodes not contained in H or any T_i, $1 \leq i \leq k$, and a is defined by

$$a_e = \begin{cases} 0 & \textit{for } e \in \gamma(H) \textit{ or } e \in \gamma(T_i)\ 1 \leq i \leq k, \\ 1 & \textit{otherwise.} \end{cases}$$

Boyd and Hao in [3] converted the comb inequalities (2.1) into a set of equivalent inequalities (with respect to Q_T^n), which are called *complemented comb inequalities*, and showed that these inequalities are facet-inducing for Q_{2E}^n in most cases. These can be defined as follows. Let a *2-matching tooth* be a tooth consisting of only two nodes, and let the nodes of a comb be divided into 6 different classes:

Class 1: Nodes in the handle but not in any tooth;

Class 2: Nodes in the handle as well as in a 2-matching tooth;

Class 3: Nodes in a 2-matching tooth but not in the handle;

Class 4: Nodes in a non-2-matching tooth but not in the handle;

Class 5: Nodes in a non-2-matching tooth as well as in the handle;

Class 6: Nodes not in the handle or in any tooth.

Let C_i, i=1,2,3,4,5,6 denote the node set for each class. Figure 2.5 shows a comb with the different classes of nodes indicated.

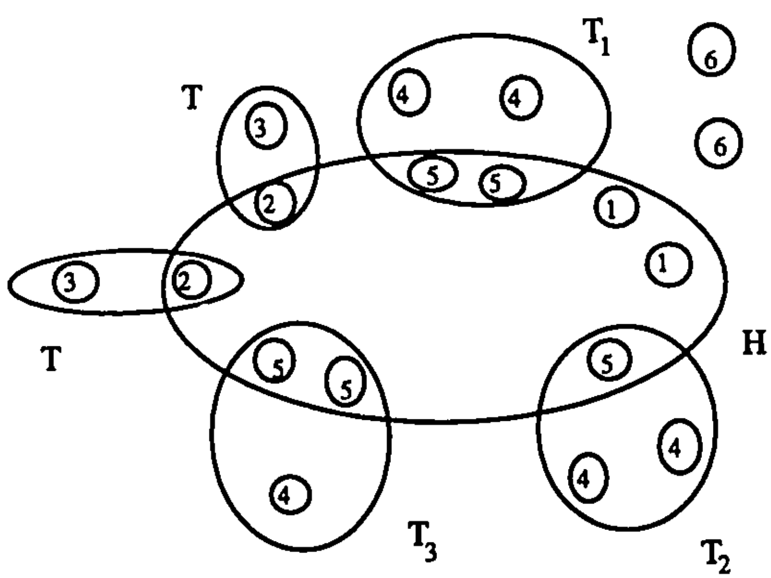


Figure 2.5 A comb with the different classes of nodes labeled

Given a comb C, let s=$|C_1|$, r=$|C_6|$, p be the number of non-2-matching teeth, and q be the number of 2-matching teeth. Assume $T_1, T_2, \cdots, T_p$ are the non-2-matching teeth and $T_{p+1}, T_{p+2}, \cdots, T_k$ are the 2-matching teeth, and let $\underset{o}{T_i} = T_i \cap H, and\ \mathring{T}_i = T_i \setminus H \quad for\ 1 \le i \le k.$

The complemented comb constraint is

$$x(\delta(H))+\sum_{i=1}^{k}x(\delta(T_i))-\sum_{v\in C_2}x(\delta(v))+\sum_{\omega\in C_6}x(\delta(\omega))\geq 3p+2r+q+1. \qquad (2.8)$$

Note that the coefficients on the left-hand side of the inequality are 0,1,2 and 3. We use the notation $bx \geq b_0$ to represent such a constraint for convenience, where

$$b_0 = 3p+2r+q+1$$

and b is defined by

$$b_e=\begin{cases} 0 & if\ e\in\gamma(C_1\cup C_2)\cup\gamma(\overset{0}{T_i})\cup\gamma(\underset{0}{T_i})\cup\gamma(T_j)\ for\ 1\leq i\leq p, p+1\leq j\leq k; \\ 1 & if\ e\in[\underset{0}{T_i}:C_1\cup C_2\cup\overset{0}{T_i}]\ for\ 1\leq i\leq p; \\ 3 & if\ e\in[\underset{0}{T_i}:C_6\cup\overset{0}{T_j}]\ for\ 1\leq i\leq p, 1\leq j\leq k\ and\ i\neq j; \\ 2 & otherwise. \end{cases}$$

Figure 2.6 shows the edge coefficients on the left-hand side for the complemented comb inequality $bx\geq 10$. Here edges e for which the coefficients are 2 are not shown.

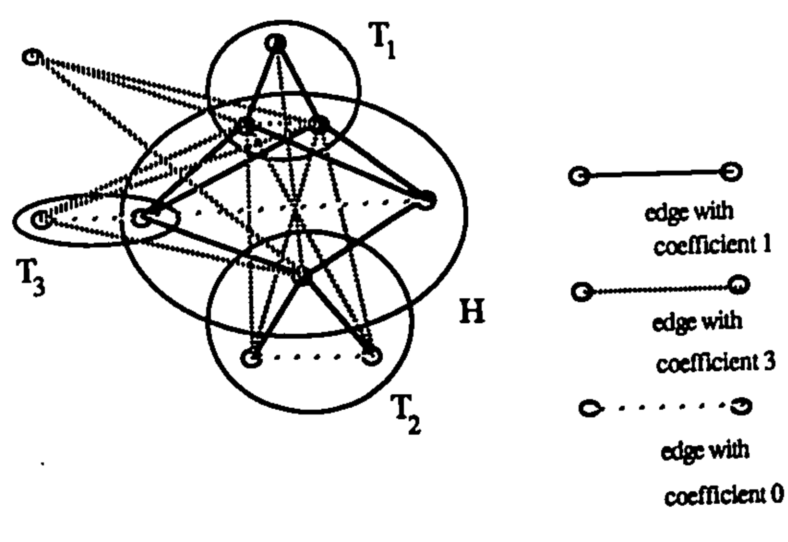


Figure 2.6 Edge coefficients of a complemented comb inequality.

The following theorem found in [3] characterizes the complemented comb constraints which are facet-inducing for Q^n_{2E}.

Theorem 2.11. Given any comb C, the corresponding complemented comb inequality is facet-inducing for Q^n_{2E} if and only if r=0 or (q+s)$\geq$1, where s and r are the number of nodes in node classes C_1 and C_6 in C respectively, and q is the number of 2-matching teeth in C.

2.3 The 2-Node-Connected Spanning Subgraph Polytope Q^n_{2N}

The polytope associated with TNCSP(n) is

Q^n_{2N}:=conv{ $x^F \in R^E$ | (V,F) is a 2-node-connected spanning subgraph of K_n}.

There has been some research done on the polytope Q^n_{2N}. In a much more general form, Q^n_{2N} is studied by Grötschel and Monma in [12] and by Grötschel , Monma and Stoer in [13]. It is also studied by Boyd and Pulleyblank in [4] for complete graphs. Below we describe all the results pertaining to the facets of Q^n_{2N} from [4,12,13] .

Theorem 2.12. The dimension of Q^n_{2N} is n(n-1)/2 for n $\geq$4.

Note that by Theorem 2.12 , the equivalence problem for Q^n_{2N} is much simpler than for Q^n_T. Two facet-inducing inequalities for Q^n_{2N} are equivalent if and only if one is a positive multiple of the other.

Theorem 2.13. For $e \in E$, $x_e \leq 1$ is facet-inducing for Q^n_{2N} if $n \geq 4$.

Theorem 2.14. For $e \in E$, $x_e \geq 0$ is facet-inducing for Q^n_{2N} if $n \geq 5$.

Theorem 2.15. For all $S \subseteq V$, $\phi \neq S \neq V$, $x(\delta(S)) \geq 2$ is facet-inducing for Q^n_{2N}.

Theorem 2.16. For all $S \subseteq V$, $x(S{:}V \setminus (S \cup \{z\})) \geq 1$ is facet-inducing for Q_{2N}^n for any $z \in V$ and $S \subseteq V \setminus \{z\}$.

The polytope Q_{2N}^n is closely related to the polytope Q_T^n. It is easy to see from a graph theoretic point of view that a 2-node-connected graph is a Hamiltonian cycle if and only if each node in the graph is of degree 2. So $Q_T^n = \{x \in Q_{2N}^n | Ax = 2\}$, where A is the node-edge incidence matrix for K_n, i.e., Q_T^n is a face of Q_{2N}^n. Therefore for every facet-inducing inequality for Q_T^n, there exists an equivalent form (with respect to Q_T^n) which is also facet-inducing for Q_{2N}^n. It would be desirable to exploit this relationship and somehow transform a facet-inducing inequality for Q_T^n into an equivalent one which is facet-inducing for Q_{2N}^n. Boyd and Pulleyblank in [4] converted the simple comb inequality into an equivalent form which is facet-inducing for Q_{2N}^n. These inequalities are called *the complemented simple comb inequalities*, and have the form (2.7).

2.4 Relating Facets of a Face to Facets of a Polyhedron

Before ending this chapter we introduce some results given in [3,4]. These results will allow us to exploit the relationship between Q_T^n and Q_{2E}^n, thus hopefully simplifying our proofs.

Given a valid inequality $ax \leq a_0$ for a polyhedron F, let $F_a^=$ represent the face of F induced by $ax \leq a_0$, i.e.,

$$F_a^= := \{x \in F{:}ax = a_0\}.$$

Given a nonempty face F of a full dimensional polytope $P \subseteq R^E$, define a set $D \subseteq R^E$ to be an independent direction set for $F_a^=$ if

(i) For every $d \in D$, there exists $\bar{x}^d \in F_a^=$ such that $x^d := \bar{x}^d + d \in P$;

(ii) For every $d \in D$, $ad = 0$;

(iii) For some minimal equation system $A^F x = b^F$ for F, $\{A^F d : d \in D\}$ is linearly independent.

Theorem 2.17.([4]) Let $ax \le a_0$ be an inequality which is valid for a full dimensional polyhedron $P \in R^E$ and facet-inducing for a nonempty face F of P. Let $A^F x = b^F$ be a minimal equation system for F. If there exists an independent direction set for $F_a^=$ of size $r_l(A^F)$, then $ax \le a_0$ is also facet-inducing for P.

Lemma 2.18.([3]) Let A be the node-edge incidence matrix for the complete graph $K_n = (V,E)$, let $I \in R^{V \times V}$ be the identity matrix whose rows and columns are indexed by V, and let $T = (V_T, E_T)$ be a subgraph of K_n which is a tree. Then the vectors in $\{A_e : e \in E_T\} \cup \{I_v\}$ are linearly independent for any $v \in V$.

Lemma 2.19.([4]) Let A be the node-edge incidence matrix for the complete graph $K_n = (V,E)$, and let $J \subseteq E$. Then J indexes a set of linearly independent columns of A if and only if each component of the graph (V,J) contains no even cycle and at most one odd cycle.

Chapter 3 The Complemented Star Clique Tree Constraints for Q^n_{2E}

3.1 Description of the CSCT1 Constraint for Q^n_{2E}

We now convert (2.4) into an equivalent form (with respect to Q^n_T) which is valid for Q^n_{2E}. The method we use to convert the inequalities (2.4) is as follows:

1. negate the inequality;

2. multiply by 2;

3. add two times the degree constraint for the nodes in handles and one time the degree constraint for v_0.

Let $\dot{V}$ be the set of nodes not in any tooth or handle, let C be all nodes in a 2-matching tooth but not in any handle plus the nodes in $\dot{V}$, let H be the nodes in some handle, and let T be the set of all edges in 2-matching teeth. It is straightforward to check that the resulting inequality converted from (2.4) is

$$bx \geq 2p + q + 2, \tag{3.1}$$

where b is defined as

$$b_e = \begin{cases} 0 & if\ e \in T \cup \gamma(C) \\ 1 & if\ e \in [v_0{:}C] \cup (\delta(v_0) \cap \gamma(T_3)) \\ 2 & if\ e \in (\gamma(H) \cap SCT) \cup ([H{:}C] \cap \overline{SCT}) \\ 3 & if\ e \in [v_0{:}H] \cap \overline{SCT} \\ 4 & if\ e \in \gamma(H) \cap \overline{SCT}. \end{cases}$$

(Note that in the above definition we use SCT to represent the set of edges in the star clique tree.)

We call the inequality (3.1) the *complemented star clique tree constraint 1*, henceforth denoted by the CSCT1 constraint.

Figure 3.1 shows a CSCT1 constraint $bx \geq 12$ with the left hand coefficients for each edge. Note the cofficients for edges not shown are the sum of the weights on their end nodes.

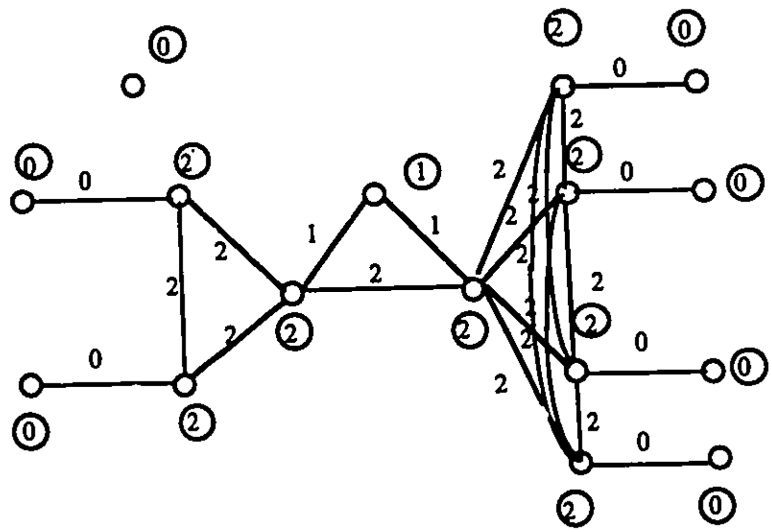


Figure 3.1 A CSCT1 constraint $bx \geq 12$

3.2 Validity of the CSCT1 for Q_{2E}^n

In this section we give the proof of the validity of the CSCT1 constraints for Q_{2E}^n.

Lemma 3.1 For any $x \in Q_{2E}^n (n = |\dot{V}| + p + 2q + 1)$ and star clique tree , the following inequalities are valid:

(i) $x(\delta(v_j)) - x(\gamma(T_j^i)) \geq 1$ for all $v_j = T_j^i \cap H_i$, $i = 1,2,\cdots,p$, $j = 1,2,\cdots,q_i$.

(ii) $x(\delta(v)) \geq 2$ for all $v \in T_3 \setminus v_0$.

(iii) $x_e \geq 0$ for all $e \in [v_0{:}C] \cup ([H{:}C] \cap \overline{SCT})$.

(iv) $\left.\begin{array}{l} x_e \geq 0 \\ x_e \geq 0 \end{array}\right\}$ for all $e \in [v_0 : H] \cap \overline{SCT}$.

(v) $\left.\begin{array}{l} x_e \geq 0 \\ x_e \geq 0 \end{array}\right\}$ for all $e \in \gamma(H) \cap \overline{SCT}$.

Furthermore, either at least two are strict inequalities for any $x \in Q_{2E}^n$ or only one is a strict inequality, and in the latter case we have $x(\delta(v)) > 3$ for some $v \in T_3 \setminus v_0$. (Note that all the sets of nodes are as previously defined in Chapter 2).

Proof. Let $K_n = (V, E)$. For any $x \in Q_{2E}^n$, we have

$0 \leq x_e \leq 1$ for all $e \in E$, and

$x(\delta(S)) \geq 2$ for all $S \subseteq V$, $\phi \neq S \neq V$.

Thus (ii), (iii), (iv) and (v) are all valid for Q_{2E}^n. Inequality (i) can be obtained by adding the two valid constraints $x(\delta(v_j)) \geq 2$ and $-x(\gamma(T_j^i)) \geq -1$, and thus (i) is valid for Q_{2E}^n as well.

Now we prove the second part of the lemma. Without loss of generality, we can assume x is the incidence vector x^F of G=(V,F), a 2-edge-connected spanning subgraph of K_n.

Suppose there exists $x^F \in Q_{2E}^n$ which satisfies all the inequalities (i),(ii), (iii), (iv) and (v) with equality. Note that this implies that $x_e^F = 0$ for all e in $\overline{SCT}$, and that $x(\gamma(T_j^i)) = 1$ for all $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q_i$.

Consider any node $v \in T_3 \setminus v_0$. Suppose the handle containing v is H_i.

Since $H_i \setminus v$ contains an even number of nodes, and each node $r \in H_i \setminus v$ is adjacent to exactly one node in H_i in the 2-edge-connected subgraph G, we have that v is either adjacent to 2 nodes in H_i and none in T_3 in G, or v is adjacent to 2 nodes in T_3 and none in H_i in G. Thus we have only one of the following occurs:

(3.2) $x^F(e_{uv})=0$ for all $u \in T_3 \setminus v$.

(3.3) $x^F(e_{vr})=0$ for all $r \in H_i \setminus v$.

Let $A:=\{v_0\}\cup\{v \in T_3 \setminus v_0$ such that (3.3) occurs$\}$. Then we have $x^F(\delta(A))=0$, contradicting the fact that $x^F(\delta(A))\geq 2$.

Now suppose $x^F \in Q^n_{2E}$ satisfies all but one of the inequalities (i),...,(v) with equality.

Case 1.

$x^F(\delta(v_j))-x^F(\gamma(T^i_j))>1$ for some $v_j \in T^i_j \cap H_i$, $i=1,2,\cdots,p$, $j=1,2,\cdots,q_i$.

Let $w=H_i \cap T_3$. Clearly w is adjacent to 0, 1 or 2 nodes in H_i in G. For any other node $v=H_l \cap T_3, l \neq i$, we have (3.2) or (3.3) occurs, and we let W be the set of such nodes for which (3.3) occurs. Let $A=v_0 \cup W$. If w is adjacent to 0 or 1 node in H_i in G, then add w to A. Then $x^F(\delta(A))\leq 1$, contradicting the fact that $x^F(\delta(A))\geq 2$.

Case 2. $x^F(\delta(v))>2$ for some $v \in T_3 \setminus v_0$.

If $x^F(\delta(v))>3$ we are done. Now suppose $x^F(\delta(v))\leq 3$, *i.e.* $x^F(\delta(v))=3$. Assume v is in handle H_i. Then either (3.3) occurs, or $x^F(\delta(v)\cap\gamma(T_3))=1$. For any other node $v=H_l \cap T_3, l\neq i$, we have (3.2) or (3.3) occurs. Let W be the set of nodes in $T_3 \setminus v_0$ for which (3.3) occurs, and let $A=v_0 \cup W$. Then $x^F(\delta(A))\leq 1$, contradicting the fact that $x^F(\delta(A))\geq 2$.

Case 3. $x_e^F > 0$ for some $e \in [v_0:C] \cup ([H:C] \cap \overline{SCT})$.

Suppose $x_e^F > 0$ for some $e \in [v_0:C]$. For any node $v \in T_3 \setminus v_0$ we have (3.2) or (3.3) occurs. Let $A := \{v | \ v \ is \ v_0$ or $v \in T \setminus v_0$ such that (3.3) occurs$\}$. Then $x^F(\delta(A)) = 1$, contradicting the fact that $x^F(\delta(A)) \geq 2$.

Suppose $x_e^F > 0$ for some $e \in [H:C] \cap \overline{SCT}$. Let e=vw, where $v \in H_i$, $w \in C$.

If $v \in T_3$, then we must have (3.3) occurs. For any other node in $T_3 \setminus v_0$ we have (3.2) or (3.3) occurs. Let $A := \{v | \ v \ is \ v_0$ or $v \in T_3 \setminus v_0$ such that (3.3) occurs$\}$. Then $x^F(\delta(A)) = 1$, contradicting the fact that $x^F(\delta(A)) \geq 2$.

If $v \notin T_3$, then let $u = T_3 \cap H_i$. It follows that $x^F(\delta(u) \cap \gamma(T_3)) = 1$. For any other node in $T_3 \setminus v_0$ we have (3.2) or (3.3) occurs. Let $A := \{v | \ v$ is v_0 or u or $v \in T_3 \setminus v_0$ such that (3.3) occurs$\}$. Then $x^F(\delta(A)) = 1$, contradicting the fact that $x^F(\delta(A)) \geq 2$.

Case4. $\left.\begin{matrix} x_e^F > 0 \\ x_e^F > 0 \end{matrix}\right\}$ for some $e \in [v_0:H] \cap \overline{SCT}$.

This contradicts the fact that only one inequality in (i),...,(v) is a strict inequality.

Case5. $\left.\begin{matrix} x_e^F > 0 \\ x_e^F > 0 \end{matrix}\right\}$ for some $e \in \gamma(H) \cap \overline{SCT}$.

This contradicts the fact that only one inequality in (i),...,(v) is a strict inequality.

This completes our proof. $\square$

Theorem 3.2. The complemented star clique tree constraints (3.1) are valid for Q_{2E}^n $(n=|\overset{0}{V}|+p+2q+1)$.

Proof. Given a star clique tree , let $bx \geq b_0$ be the corresponding complemented star clique tree constraint, and let $\hat{b}x \geq \hat{b}_0$ be the corresponding valid inequality for Q_{2E}^n obtained by adding all the inequalities in (i),(ii),(iii),(iv) and (v) described in Lemma 3.1. By Lemma 3.1, at least two of these inequalities are strict inequalities or else $x(\delta(v)) > 3$ for some $v \in T_3 \setminus v_0$. Thus $\hat{b}x > \hat{b}_0 + 1$ for all $x \in Q_{2E}^n$. Furthermore, since $\hat{b}, \hat{b}_0$ and all the vertices of Q_{2E}^n are integer , we have

$$\hat{b}x \geq \hat{b}_0 + 2 \text{ for all } x \in Q_{2E}^n . \tag{3.4}$$

It is straightforward to check that $b_e = \hat{b}_e$ for all $e \in E(K_n)$. The right-hand side of (3.4) is $2p+q+2 = b_0$, and thus (3.4) gives $bx \geq b_0$ for all $x \in Q_{2E}^n$ as required.

This completes our proof. $\square$

3.3 A Characterization of the Facet-inducing CSCT1 Constraints for Q_{2E}^n

In this section we prove that the complemented star clique tree constraint 1 (3.1) is a facet-inducing inequality for Q_{2E}^n. We will make use of Lemma 3.3 below, which follows directly from the theory discussed in Section 2.4.

Lemma 3.3. For any star clique tree , the corresponding complemented star clique tree inequality 1 $bx \geq b_0$ is facet-inducing for Q_{2E}^n if there exists a set $D = \{d_1, d_2, \cdots, d_n\} \subseteq R^E$ such that:

(P_1) For each $1 \le i \le n$, there exists $\hat{x}_i \in Q_T^n \cap \{x: bx = b_0\}$ such that $x_i := \hat{x}_i + d_i \in Q_{2E}^n$;

(P_2) For each $1 \le i \le n$, $bd_i = 0$;

(P_3) $Ad_1, Ad_2, \cdots, Ad_n$ are linearly independent where A is the node-edge incidence matrix for the complete graph K_n.

Definition 3.4. Given a star clique tree and its corresponding complemented star clique tree inequality 1 $bx \ge b_0 = 2p + q + 2$, we say node $v \in V$ induces a *tight triangle* (v,u,w) if there exists $u, w \in V(K_n)$ and Hamiltonian cycle H of K_n such that

(i) $bx^H = b_0$,

(ii) $b_{uv} + b_{vw} = b_{uw}$,

(iii) H contains uw, but not vw or uv.

Note that if v induces a tight triangle (v,u,w), then $H^1 = (H \setminus uw) \cup \{vw, uv\}$ forms a 2-edge connected graph satisfying $bx^{H^1} = b_0$.

Theorem 3.5. Given any star clique tree , the corresponding complemented star clique tree inequality 1 $bx \ge 2p + q + 2$ is facet-inducing for Q_{2E}^n ($n = |\overset{0}{V}| + p + 2q + 1$).

Proof. Let $G_0 = (V_{G_0}, E_{G_0})$ be the spanning subgraph of $K_n = (V, E)$ whose edge set corresponds to the edges e in K_n for which $b_e = 0$, *i.e.*, $E_{G_0} := \{e \in E: b_e = 0\}$. The graph G_0 consists of p+2 components. To be specific, all nodes not in T_3 are in the same component, and each node in T_3 forms a component by itself.

Consider the component M of G_0 which does not contain any node in T_3, and let J be the edge set of a spanning tree in M. Let $v = T^1_{q_1} \setminus H_1$, and consider the Hamiltanion cycle H shown in Figure 3.2. We have $bx^H = b_0$. Let $w = H_1 \cap T_3$, and $u = T^1_1 \setminus H_1$. Then {v,u,w} is a tight triangle.(See Figure 3.2)

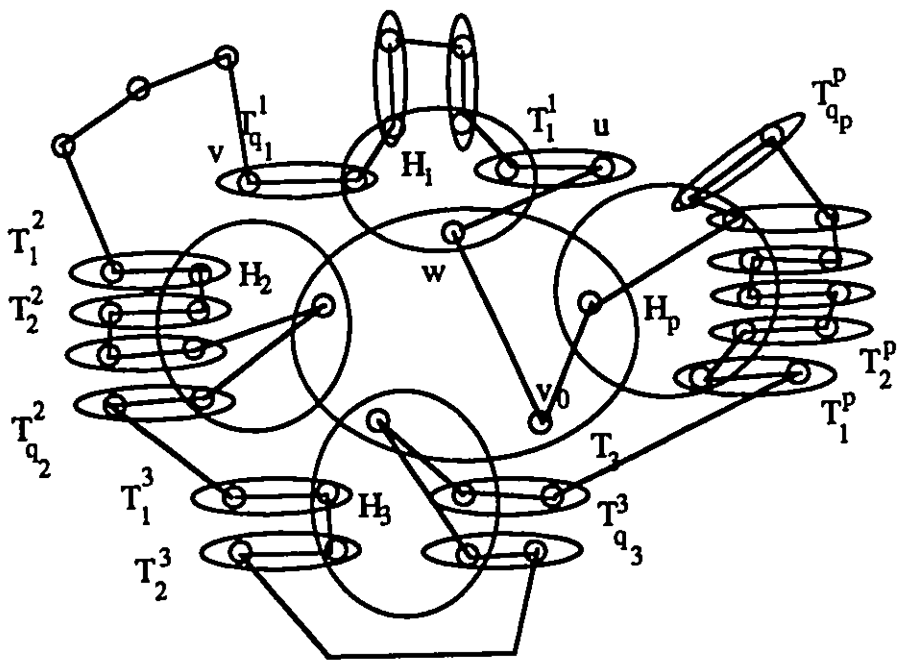


Figure 3.2 Hamiltonian Cycle H and Tight Triangle {v,u,w}.

Let $d^v := x^{vu} + x^{vw} - x^{uw}$, and let $d^e := x^e$ for all $e \in J$. We will show that

$$D^M := \{d^e : e \in J\} \cup \{d^v\}$$

satisfies properties P_1, P_2 and P_3 of Lemma 3.3.

Recall that $bx \geq 2p + q + 2$ is equivalent to a star clique tree constraint for Q^n_T and thus does not induce the facet $x_e = 1$, $e \in E$, in Q^n_T. Hence for each $e \in J$, there exists a Hamiltonian cycle H such that $bx^H = 2p + q + 2$ and H does not contain edge e. Clearly $H \cup \{e\}$ is a 2-edge connected subgraph of K_n, i.e., $x^H + d^e \in Q^n_{2E}$. Furthermore, using the Hamiltonian cycle H shown in

Figure 3.2, we get a subgraph $(H \setminus \{uw\}) \cup \{uv, wv\}$ which is a 2-edge connected subgraph of K_n, *i.e.*, $x^H + d^V \in Q^n_{2E}$. Thus D^M satisfies property P_1 of Lemma 3.3. For each edge $e \in J$, $bd^e = b_e = 0$ by the definition of G_0. Also, $bd^v = b_{uv} + b_{vw} - b_{uw} = 0$, since (v,u,w) is a tight triangle. Thus D^M satisfies property P_2 of Lemma 3.3.

Now consider $\{Ad: d \in D^M\}$, where A is the node-edge incidence matrix of K_n. For each $e \in J, Ad^e = A_e$, and $Ad^v = 2I_v$, where $I \in R^{V \times V}$ is the identity matrix whose rows and columns are indexed by V. Thus by Lemma 2.18, the vectors in $\{Ad: d \in D^M\}$ are linearly independent, and D^M satisfies property P_3 of Lemma 3.3. Thus D^M is an independent direction set of size 2q+ $|\overset{0}{V}|$=n-(p+1)

For node v_0 we define d^{v_0} as follows:

$$d^{v_0} := x^{uv_0} + x^{zv_0} + x^{zy} - x^{yu},$$

where $u = T_3 \cap H_1$, $y = H_1 \cap T^1_{q_1}$ and $z = T^1_{q_1} \setminus H_1$. For d^{v_0} we can find a Hamiltonian cycle H such that $bx^H = 2p + q + 2$, and $x^H + d^{v_0} \in Q^n_{2E}$, as shown in Figure 3.3 .

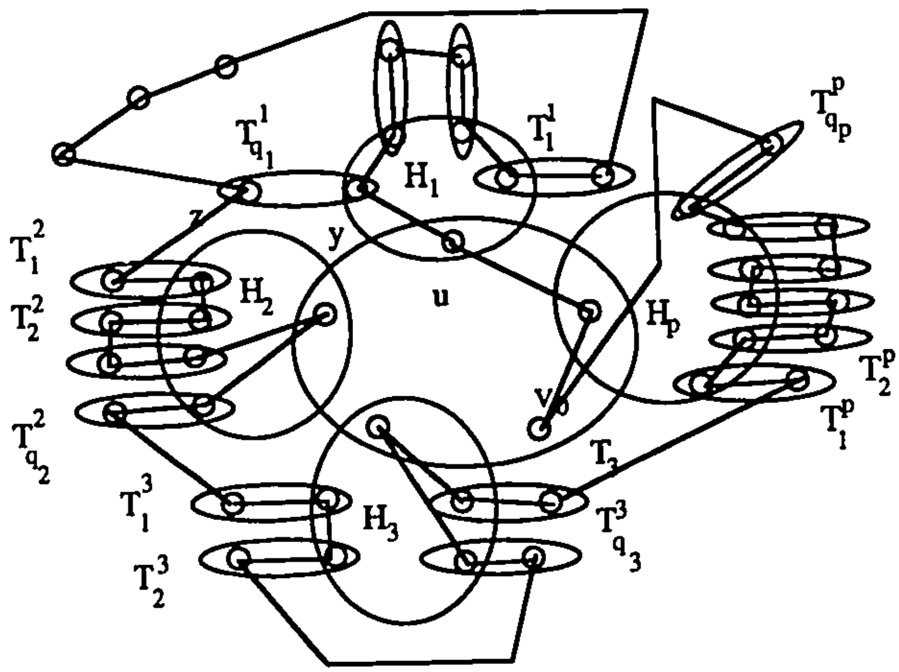


Figure 3.3 Hamiltonian cycle H for v_0.

For each node $v \in T_3 \setminus \{v_0\}$ we define d^v as follows:

$$d^v := x^{vu} + x^{vw} - x^{uw}$$

where u is in the same handle as v, $w \in T_3 \setminus \{v_0, v\}$. For d^v we can find a Hamiltonian cycle H such that $bx^H = 2p+q+2$, and $x^H + d^v \in Q^n_{2E}$, as shown in Figure 3.4.

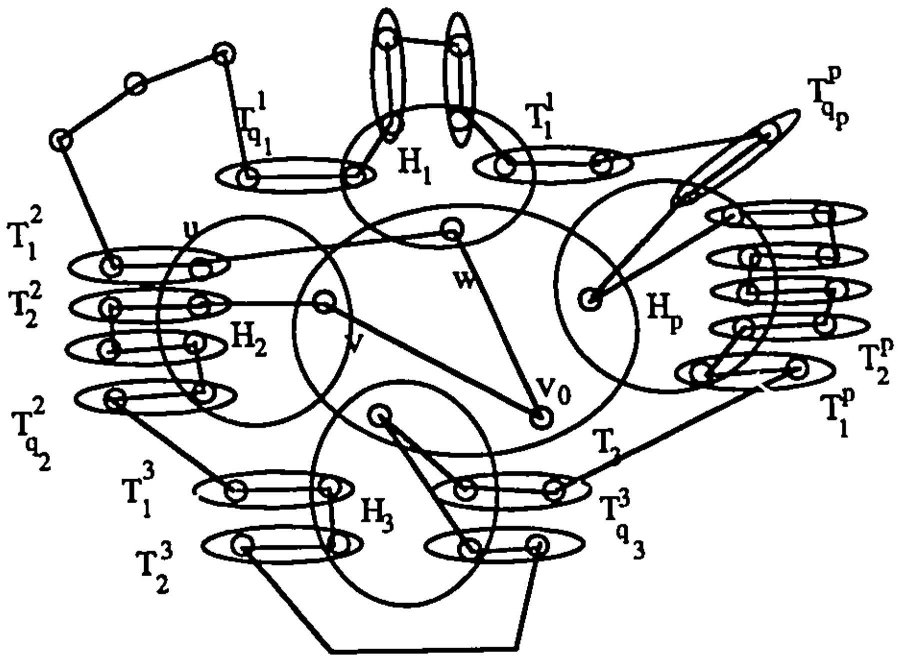


Figure 3.4 Hamiltonian cycle H for v.

We claim $D = D^M \cup \{d^v : v \in T_3 \setminus \{v_0\}\} \cup \{d^{v_0}\}$ is an independent direction set of size ... For the Hamiltonian cycle H shown in Figure 3.3 and d^{v_0}, $(H \setminus \{yu\}) \cup \{uv_0, v_0z, zy\}$ forms a 2-edge-connected subgraph of K_n, thus $x^H + d^{v_0} \in Q^n_{2E}$ and d^{v_0} satisfies property P_1 of Lemma 3.3. For the Hamiltonian cycle H shown in Figure 3.4, $(H \setminus \{uw\}) \cup \{vu, vw\}$ forms a 2-edge-connected subgraph of K_n, thus $x^H + d^v \in Q^n_{2E}$ and d^v satisfies property P_1 of Lemma 3.3 for all $v \in T_3 \setminus \{v_0\}$. Since $bd^{v_0} = b_{uv_0} + b_{v_0z} + b_{zy} - b_{yu} = 0$ and $bd^v = b_{vu} + b_{vw} - b_{uw} = 2 + 2 - 4 = 0$ for all $v \in T_3 \setminus \{v_0\}$, it follows that $\{d^v : v \in T_3 \setminus \{v_0\}\} \cup \{d^{v_0}\}$ satisfies property P_2 of Lemma 3.3.

Now consider $\{Ad : d \in D\}$. We know that the vectors in $\{Ad : d \in D^M\}$ are linearly independent. Furthermore, for v_0, $Ad^{v_0} = 2I_z + 2I_{v_0}$, and for all $v \in T_3 \setminus \{v_0\}$, $Ad^v = 2I_v$. Thus Ad^{v_0} is the only vector in $\{Ad : d \in D\}$ with a non-zero value in its v_0^{th} component and, for each $v \in T_3 \setminus v_0$, Ad^v is the only vector in $\{Ad: d \in D\}$ with a non-zero value in its v^{th} component. Thus the vectors in $\{Ad: d \in D\}$ are linearly independent, and D satisfies property P_3. Since |D|=n, it follows from Lemma 3.3 that $bx \geq 2p+q+2$ is a facet-inducing inequality for Q_{2E}^n.

This completes our proof. □

Chapter 4. The Complemented Path Clique Tree Constraints for Q^n_{2E}

4.1 Description Of the CPCT1 Constraint for Q^n_{2E}

We now convert (2.5) into an equivalent form (with respect to Q^n_T) which is valid for Q^n_{2N}. The method we use to convert the inequality (2.5) is as follows:

1. negate the inequality;
2. multiply by 2;
3. add two times the degree constraint for the nodes in handles and one time the degree constraint for all nodes which are in a non-2-matching tooth but not in any handle.

Let $\dot{V}$ be the set of nodes not in any tooth or handle. let C be all nodes in a 2-matching tooth but not in any handle plus the nodes in $\dot{V}$, let H be the nodes in some handle, let T be the set of all edges in 2-matching teeth, and let NT be the set of all non-2-matching teeth. It is straightforward to check that the resulting inequality obtained from (2.5) is

$$bx \geq 6p-2, \tag{4.1}$$

where b is defined as

$$b_e = \begin{cases} 0 & if\, e \in \gamma(C) \cup T, \\ 1 & if\, e \in ([NT \setminus H{:}H] \cap PCT) \cup ([NT \setminus H{:}C], \\ 2 & if\, e \in (\gamma(H) \cap PCT) \cup ([H{:}C] \cap \overline{PCT}) \cup [NT \setminus H{:}NT \setminus H], \\ 3 & if\, e \in [NT \setminus H{:}H] \cap \overline{PCT}, \\ 4 & if\, e \in \gamma(H) \cap \overline{PCT}. \end{cases}$$

(Note that in the above definition we use PCT to represent the set of edges in the path clique tree.)

We call inequality (4.1) the *complemented path clique tree constraint 1*, henceforth denoted by the CPCT1 constraint.

Figure 4.1 shows a CPCT1 constraint $bx \geq 16$ with the left hand coefficients for each edge. Note that the cofficients for edges not shown are the sum of the weights on their end nodes.

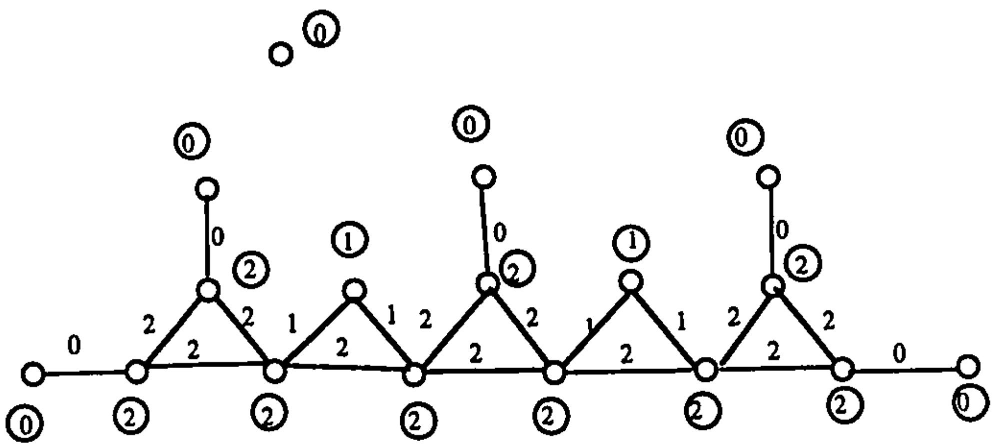


Figure 4.1 A CPCT1 constraint $bx \geq 16$

4.2 Validity of the CPCT1 Constraint for Q^n_{2E}

Consider the comb C as shown in Figure 4.2 .

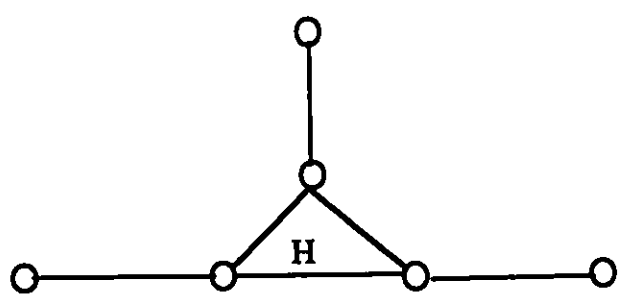


Figure 4.2 A comb consisting of 6 nodes.

The comb inequality in this case is $ax \le 4$. One of its equivalent forms with respect to Q_T^n is $bx \ge 4$, where, letting T be all the edges in 2-matching teeth, b is defined as follows:

$$b_e = \begin{cases} 0 & \text{if } e \in T \cup \gamma(C), \\ 2 & \text{otherwise.} \end{cases}$$

By exchanging H and $\dot{V}$, we see that $bx \ge 4$ is a complemented 2-matching constraint, and is thus facet-inducing for Q_{2E}^n and Q_{2N}^n by results in [4]. Hence we have the following.

Lemma 4.1. $bx \ge 4$ is valid for any $x \in Q_{2E}^n$ and any $x \in Q_{2N}^n$ (n=6+$|\overset{0}{V}|$).

Theorem 4.2. The complemented path clique tree constraint (4.1) is valid for Q_{2E}^n(n=$|\overset{0}{V}|$+5p+1).

Proof. We prove this theorem by induction.

For the purposes of the induction, we will consider the CPCT1 constraint (4.1) for p=1 to be the inequality $bx \ge 4$, which is valid for Q_{2E}^n by Lemma 4.1.

Let p=1. Then the path clique tree becomes the comb shown in Figure 4.2, and by Lemma 4.1 we know that $bx \ge 4$ is a valid inequality for Q_{2E}^n. Assume now that the conclusion is true for any integer p, that is $bx \ge 6p-2$ is a valid inequality for Q_{2E}^n for all $n \ge 5p+1$. We want to show that the conclusion is also true for p+1, that is $bx \ge 6(p+1)-2$ is a valid inequality for Q_{2E}^n for all $n \ge 5(p+1)+1$. Without loss of generality we can assume x is the incidence vector x^F of a 2-edge connected spanning subgraph of K_n with edge set F.

Let the nodes of nonpendent tooth NT1 in the path clique tree be v_A, v_B and v_C, where $v_B \in H_1$, and $v_C \in H_2$. By removing the edges $\gamma(NT1)$, we obtain two subclique trees, one with one handle, and the other with p-1 handles. To the first add pendent tooth $\{v_B, v_C\}$ to obtain path clique tree PCT1, and to the second add pendent tooth $\{v_A, v_C\}$ to obtain path clique tree PCT2. (See Figure 4.3)

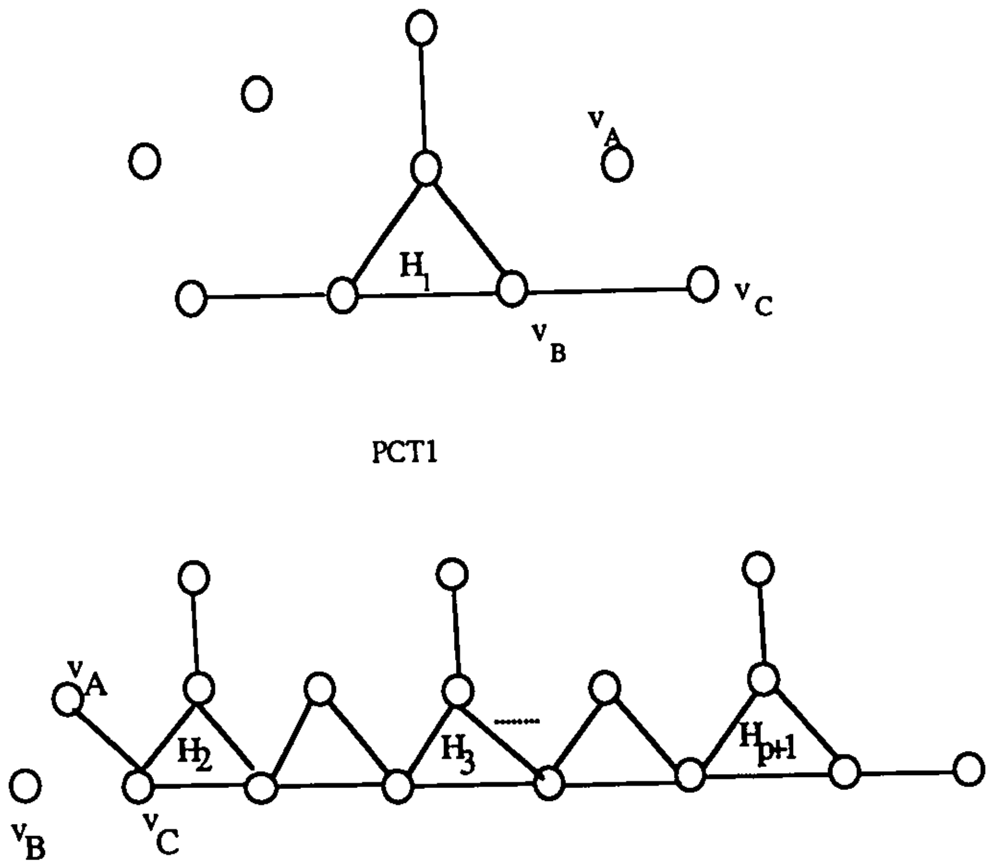


Figure 4.3 Path clique trees PCT1 and PCT2.

Associated with PCT1, we have complemented path clique tree constraint (4.1) $b^1 x^F \geq 4$, and associated with PCT2, we have complemented path clique tree constraint (4.1) $b^2 x^F \geq 6p-2$.

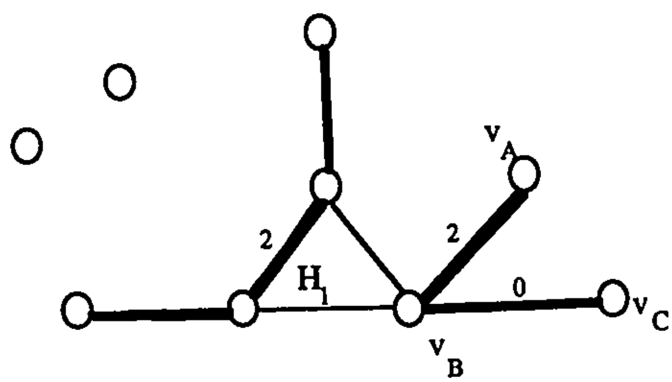
From the definition of b^1 and b^2 we get the following relation:

$$bx^F = b^1x^F + b^2x^F + x^F(\delta(v_A)) - 2x^F(v_Av_B).$$

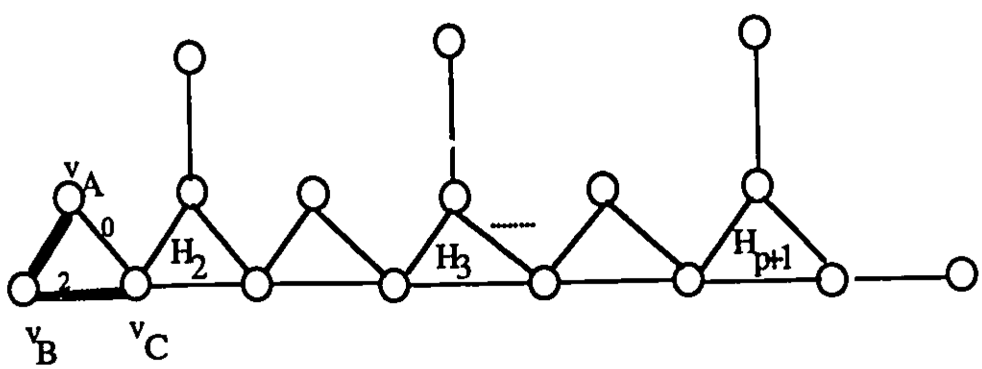
Suppose $x^F(v_Av_B) = 0$. Since $b^1x^F \geq 4$ by Lemma 4.1 and $b^2x^F \geq 6p-2$ by assumption and $x^F(\delta(v_A)) \geq 2$, we have $bx^F \geq 4+6p+2-2 = 6(p+1)-2$.

Suppose $x^F(v_Av_B) = 1$, and $b^1x^F \geq 6$. Since $b^2x^F \geq 6p-2$ by assumption and $x^F(\delta(v_A)) \geq 2$, we have $bx^F \geq 6+6p-2+2-2 = 6(p+1)-2$.

Now we consider the case where $x^F(v_Av_B) = 1$ and $b^1x^F < 6$. The only possibility for x^F passing through PCT1 is as shown in Figure 4.4.



PCT1



PCT2

Figure 4.4 $b^1x^F < 6$ and $x^F(v_Av_B) = 1$.

Case 1: $x^F(v_A v_C)=0$.

Let $F^1=(F\setminus((v_A v_B)\cup(v_B v_C))\cup(v_A v_C)$. We claim that F^1 is a 2-edge connected spanning subgraph of $K_{n-1}=(V_n\setminus\{v_B\},E_{n-1})$. Since among the edges $\delta(v_B)$, F only includes $(v_B v_A)$ and $(v_B v_C)$, *i.e.* $x^F(\delta(v_B)\setminus\{v_B v_A, v_B v_C\})=0$, any two nodes connected by a path passing through node B can now be replaced by a path passing through edge $v_A v_C$. Thus $x^{F^1}\in Q_{2E}^{n-1}$,and by assumption we have $b^2x^{F^1}\geq 6p-2$. Therefore $b^2x^F=b^2x^{F^1}+2\geq 6p-2+2=6p$. Thus $bx^F=b^1x^F+b^2x^F+x^F(\delta(v_A))-2x^F(v_A v_B)\geq 4+6p=6(p+1)-2$.

Case 2. $x^F(v_A v_C)=1$, $x^F(\delta(v_A))\geq 4$.

In this case we have $bx^F\geq 4+6p-2+4-2=6(p+1)-2$.

Case 3. $x^F(v_A v_C)=1$, $x^F(\delta(v_A))\leq 3$.

Suppose $x^F(\delta(v_A))=3$. Let $F^1=F\setminus(v_A v_B, v_B v_C)$. We claim that F^1 is a 2-edge connected spanning subgraph of $K_{n-1}=(V_n\setminus\{v_B\},E_{n-1})$. Since among the edges in $\delta(v_B)$ F only contains $v_B v_A$ and $v_B v_C$, *i.e.*, $x^F(\delta(v_B)\setminus(v_B v_A, v_B v_C))=0$, there exists only one node $v\neq v_B, v_C$ that is connected to v_A by an edge in F, and there exists a path from v to v_C which does not pass through node v_C. Since $x^{F^1}\in Q_{2E}^{n-1}$, by assumption we have $b^2x^{F^1}\geq 6p-2$, therefore $b^2x^F=b^2x^{F^1}+2\geq 6p$. Thus

$$bx^F=b^1x^F+b^2x^F+x^F(\delta(v_A))-2x^F(v_A v_B)\geq 4+6p=6(p+1)-2.$$

Suppose $x^F(\delta(v_A))=2$. Let $v\neq v_B$ be any node not in PCT2. Let $F^1=(F\setminus\{v_A v_B, v_B v_C\})\cup\{v_A v\}$. We claim that F^1 is a 2-edge connected spanning subgraph of $K_{n-1}=(V_n\setminus\{v_B\},E_{n-1})$. Since $x^{F^1}\in Q_{2E}^{n-1}$, by assumption we have $b^2x^{F^1}\geq 6p-2$, therefore $b^2x^F=b^2x^{F^1}+2-0\geq 6p$. Thus

$$bx^F = b^1x^F + b^2x^F + x^F(\delta(v_A)) - 2x^F(v_Av_B) \geq 4 + 6p + 2 - 2 = 6(p+1) - 2.$$

This completes the proof. $\square$

4.3 A Characterization of the Facet-inducing CPCT1 Constraints for Q^n_{2E}

In this section we prove that the complemented path clique tree constraint (4.1) is a facet-inducing inequality for Q^n_{2E}. We will make use of Lemma 4.3 below, which follows directly from the theory discussed in Section 2.4.

Lemma 4.3. For any path clique tree , the corresponding complemented path clique tree constraint 1 $bx \geq b_0$ is facet-inducing for Q^n_{2E} if there exists a set $D = \{d_1, d_2, \cdots, d_n\} \subseteq R^E$ such that:

(P_1) For each $1 \leq i \leq n$, there exists $\hat{x}_i \in Q^n_T \cap \{x: bx = b_0\}$ such that $x_i := \hat{x}_i + d_i \in Q^n_{2E}$;

(P_2) For each $1 \leq i \leq n$, $bd_i = 0$;

(P_3) $Ad_1, Ad_2, \cdots, Ad_n$ are linearly independent, where A is the node-edge incidence matrix for the complete graph K_n.

Definition 4.4. Given a path clique tree and its corresponding complemented path clique tree constraint 1 $bx \geq b_0 = 6p - 2$, we say node $v \in V(K_n)$ induces a *tight triangle* (v,u,w) if there exists $u, w \in V(K_n)$ and Hamiltonian cycle H of K_n such that

(i) $bx^H = b_0$,

(ii) $b_{uv} + b_{vw} = b_{uw}$,

(iii) H contains uw, but not vw or uv.

Note that if v induces a tight triangle (v,u,w) then $H^1 = (H \setminus uw) \cup \{vw, uv\}$ forms a 2-edge forms a 2-edge connected graph satisfying $bx^{H^1} = b_0$.

Theorem 4.5. Given any path clique tree , the corresponding complemented path clique tree constraint 1 $bx \geq 6p-2$ is facet-inducing for Q^n_{2E} (n=$|\overset{0}{V}|$+5p+1).

Proof. Let $G_0 = (V_{G_0}, E_{G_0})$ be the spanning subgraph of $K_n = (V, E)$ whose edge set corresponds to the edges e in K_n for which $b_e = 0$, *i.e.*, $E_{G_0} := \{e \in E: b_e = 0\}$. The graph G_0 consists of 3p-2 components. To be specific, all nodes not in the non-2-matching teeth are in the same component, and each node in a non-2-matching tooth forms a component.

Consider the component M of G_0 which does not contain any node in the non-2-matching teeth, and let J be the edge set of a spanning tree in M. Let $v = T_1^1 \setminus H_1$, and consider the Hamiltonian cycle H shown in Figure 4.5. Clearly $bx^H = b_0$. Let $w = T_1^1 \cap H_1$, and $u \in H_1 \setminus T_1^1$, then (v,u,w) is a tight triangle. (See Figure 4.5)

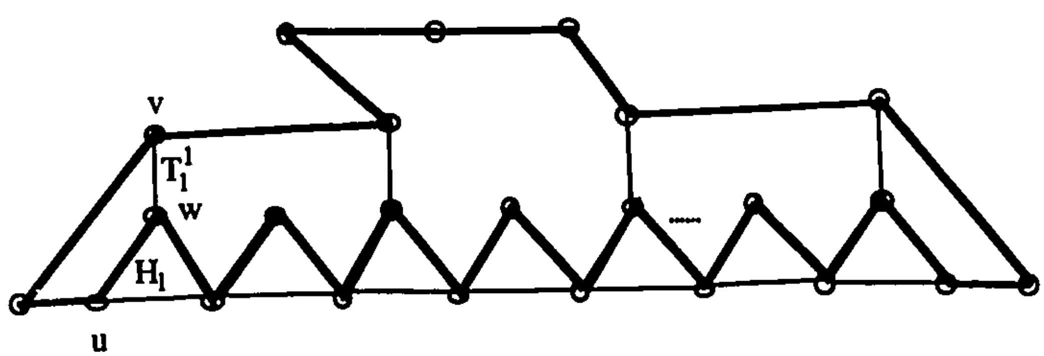


Figure 4.5 Hamiltonian Cycle H and Tight Triangle {v,u,w}.

Let $d^v := x^{uv} + x^{vw} - x^{uw}$, and let $d^e := x^e$ for all $e \in J$. We will show that

$$D^M := \{d^e : e \in J\} \cup \{d^v\}$$

satisfies properties P_1, P_2 and P_3 of Lemma 4.3.

Recall that $bx \geq 6p-2$ is equivalent to a path clique tree constraint for Q_T^n and thus does not induce the facet $x_e = 1, e \in E$, in Q_T^n. Hence for each $e \in J$, there exists a Hamiltonian cycle H such that $bx^H = 6p-2$ and H does not contain edge e. Clearly $H \cup \{e\}$ is a 2-edge connected subgraph of K_n, *i.e.*, $x^H + d^e \in Q_{2E}^n$. Furthermore, using the Hamiltonian cycle H shown in Figure 4.5, we get a subgraph $(H \setminus \{uw\}) \cup \{uv, wv\}$ which is a 2-edge connected subgraph of K_n, *i.e.*, $x^H + d^v \in Q_{2E}^n$. Thus D^M satisfies property P_1 of Lemma 4.3. For each edge $e \in J, bd^e = b_e = 0$ by the definition of G_0. Also, $bd^v = b_{uv} + b_{vw} - b_{uw} = 0$, since (v,u,w) is a tight triangle. Thus D^M satisfies property P_2 of Lemma 4.3.

Now consider $\{Ad : d \in D^M\}$, where A is the node-edge incidence matrix of K_n. For each $e \in J$, $Ad^e = A_e$, and $Ad^v = 2I_v$, where $I \in R^{V \times V}$ is the identity matrix whose rows and columns are indexed by V. Thus by Lemma 2.18, the vectors in $\{Ad : d \in D^M\}$ are linearly independent, and D^M satisfies property P_3 of Lemma 4.3. Thus D^M is an independent direction set of size 2p+ $|\overset{0}{V}|$+4=n-3(p-1).

For each node v in a non-2-matching tooth but not in any handle we define d^v as follows:

$$d^v = x^{vu} + x^{vw} - x^{uw}$$

where u is a node in the same non-2-matching teeth as v, and w is the node in $T_1^2 \setminus H$. For each d^v we can find a Hamiltonian cycle H such that $bx^H = 6p-2$, and $x^H + d^v \in Q_{2E}^n$, as shown in Figure 4.6.

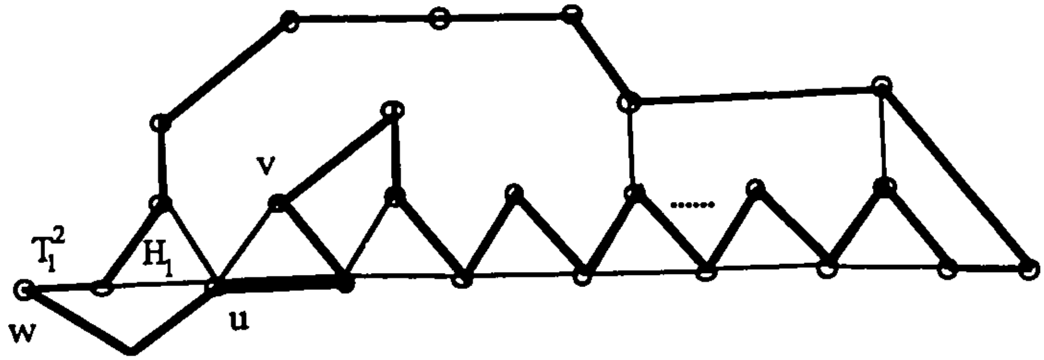


Figure 4.6 Hamiltonian cycle H for v.

For each node v which is in a non-2-matching tooth and in a handle, we define d^v as follows:

$$d^v := x^{vu} + x^{vw} - x^{uw}$$

where u is in the same non-2-matching tooth as v but not in any handle, w is in the same handle as v and in a 2-matching tooth. For each d^v we can find a Hamiltonian cycle H such that $bx^H = 6p-2$, and $x^H + d^v \in Q^n_{2E}$, as shown in Figure 4.7 .

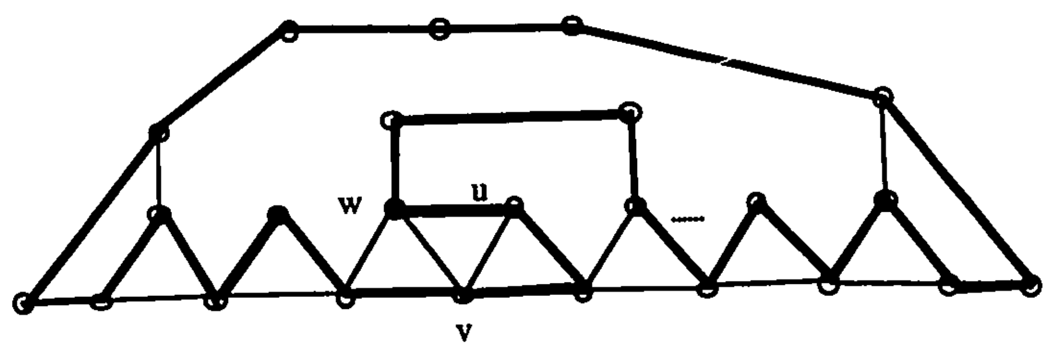


Figure 4.7 Hamiltonian cycle H for v.

Let K={v: v is in a non-2-matching tooth}. We claim $D = D^M \cup \{d^v : v \in K\}$ is an independent direction set of size n. For the Hamiltonian cycles H shown in Figures 4.6 and 4.7, $(H \setminus \{uw\}) \cup \{uv, vw\}$ forms a 2-edge connected subgraph of K_n, thus $x^H + d^v \in Q^n_{2E}$, and d^v satisfies property P_1 of Lemma 4.3 for each $v \in K$. Since $bd^v = b_{uv} + b_{vw} - b_{uw} = 0$, for each $v \in K$, d^v satisfies property P_2 of Lemma 4.3 for each $v \in K$. Now consider $\{Ad : d \in D\}$. We know that the vectors in $\{Ad : d \in D^M\}$ are linearly independent. Furthermore, for $v \in K$, $Ad^v = 2I_v$, and Ad^v is the only vector in $\{Ad : d \in D\}$ with a non-zero value in its v^{th} component. Thus the vectors in $\{Ad : d \in D\}$ are linearly independent, and D satisfies property P_3. Since |D|=n, it follows from Lemma 4.3 that $bx \geq 6p - 2$ is a facet-inducing inequality for Q^n_{2E}.

This completes our proof. □

Chapter 5. The Complemented Star Clique Tree Constraint for Q^n_{2N}

5.1 Description of the CSCT2 Constraint for Q^n_{2N}

We now convert (2.4) into an equivalent form (with respect to Q^n_T) which is valid for Q^n_{2N}. The method we use to convert the inequalities (2.4) is as follows:

1. negate the inequality;
2. multiply by 2;
3. add two times the degree constraint for the nodes in the handles.

Let $\dot{V}$ be the set of nodes not in any tooth or handle, let C be all nodes in a 2-matching tooth but not in any handle plus the nodes in $\dot{V}$, let H be the nodes in some handle, and let T be the set of all the edges in 2-matching teeth. It is straightforward to check that the resulting inequality converted from (2.4) is

$$bx \geq 2p + q \ , \tag{5.1}$$

where b is defined as

$$b_e = \begin{cases} 0 & \text{if } e \in ([v_0{:}H] \cap SCT) \cup \gamma(C \cup v_0) \cup T \\ 2 & \text{if } e \in (\gamma(H) \cap SCT) \cup ([H{:}C] \cap \overline{SCT}) \cup ([v_0{:}H] \cap \overline{SCT}) \\ 4 & \text{if } e \in (\gamma(H) \cap \overline{SCT}). \end{cases}$$

(Note that in the above definition we use SCT to represent the set of edges in the star clique tree.)

We call the inequality (5.1) the *complemented star clique tree constraint 2*, henceforth denoted by the CSCT2 constraint.

Figure 5.1 shows a CSCT2 constraint $bx \geq 10$ with the left hand coefficients for each edge. Note that the cofficients for edges not shown are the sum of the weights on their end nodes.

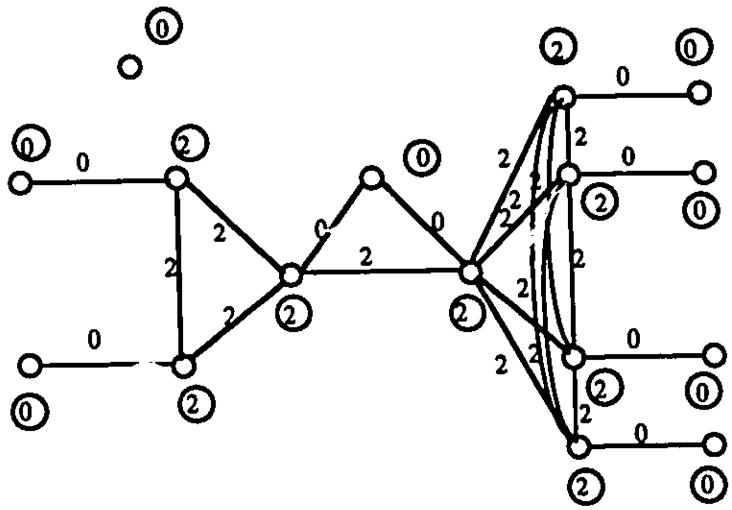


Figure 5.1 A CSCT2 constraint $bx \geq 10$

5.2 Validity of the CSCT2 Constraint for Q_{2N}^n

In this section we prove the validity of the complemented star clique tree constraint 2 (5.1) for Q_{2N}^n.

Lemma 5.1. For any $x \in Q_{2N}^n$ (n=$|\overset{0}{V}|$+p+2q) and star clique tree SCT, we have:

(i) $\quad x(\delta(v_j)) - x(\gamma(T_j^i)) \geq 1 \qquad v_j \in T_j^i \cap H_i \qquad i = 1,2,\cdots,p,\ j = 1,2,\cdots,q_i.$

(ii) $\quad x(\delta(v)) - x(vv_0) \geq 1$ for all $v \in H \cap T_3$.

(iii) $\quad x_e \geq 0 \qquad e \in [H{:}C] \cap \overline{SCT}.$

(iv) $\quad \left.\begin{array}{l} x_e \geq 0 \\ x_e \geq 0 \end{array}\right\} \qquad e \in \gamma(H) \cap \overline{SCT}.$

(Note that all the sets of nodes are as previously defined in Chapter 2).

Proof. Let $K_n = (V,E)$. For any $x \in Q_{2N}^n$, we have

$0 \le x_e \le 1$ for all $e \in E$, and

$x(\delta(S)) \ge 2$ for all $S \subseteq V$, $\phi \ne S \ne V$.

Thus (iii) and (iv) are valid for Q_{2N}^n. Inequalities (i) and (ii) can be obtained by adding $x(\delta(v)) \ge 2$ and $-x(\gamma(T_j^i)) \ge -1$ (or $-x(v:v_0) \ge -1$), and thus (i) and (ii) are valid for Q_{2N}^n as well.

□

Theorem 5.2. The complemented star clique tree constraint 2 (5.1) is valid for Q_{2N}^n ($n=|\overset{0}{V}|+p+2q+1$).

Proof. We prove the theorem by induction.

For the purposes of the induction, we will consider the star clique tree for p=1 to be the comb shown in Figure 5.2.

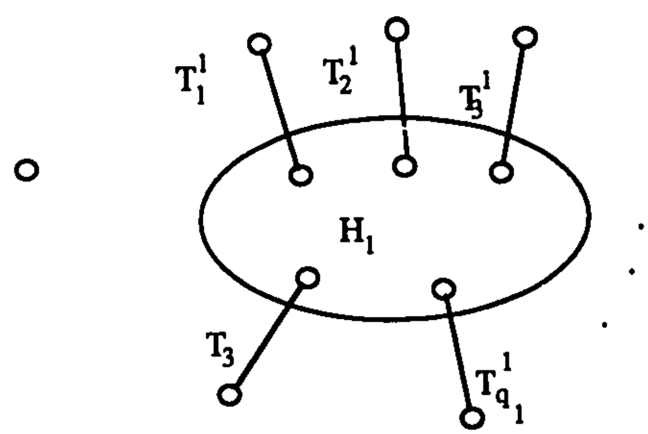


Figure 5.2. Comb for p=1.

By exchanging H and $\overset{0}{V}$, we see that $bx \geq 2+q_1$, where b is defined by

$$b_e = \begin{cases} 2 & \text{if } e \in \gamma(H_1) \cup \delta(H_1) \cap \overline{SCT} \\ 0 & \text{otherwise,} \end{cases}$$

is a complemented 2-matching constraint, and is thus valid for Q_{2N}^n by results in [4].

Assume now that the conclusion is true for any integer p, that is $bx \geq 2p+q$ is a valid inequality for Q_{2N}^n for all $n \geq p+2q+1$. We want to show that the conclusion is also true for p+1, i.e., $bx \geq 2(p+1)+q$ is a valid inequality for Q_{2N}^n for all $n \geq 5(p+1)+2q+1$. Without loss of generality, we can assume x is the incidence vector x^F of a 2-node connected spanning subgraph of K_n with edge set F. We consider the following cases.

Case 1. Assume for the given $x^F \in Q_{2N}^n$, every handle has at least one strict inequality which is of type (i),(ii),(iii),(iv) from Lemma 5.1.

Since $bx^F \geq q+p+1$ can be obtained by adding all the inequalities (i),(ii),(iii),(iv) , and among the inequalities there are at least (p+1) strict inequalities, we have $bx^F \geq 2(p+1)+q$ in this case.

Case 2. There exists at least one handle H_i such that all the inequalities in (i),(ii),(iii),(iv) related to H_i are satisfied by equality.

Let $u = H_i \cap T_3$. We have $x^F(e)=1$ for all $e=\gamma(T_j^i)$, $j=1,2,\cdots,q_i$, $x^F(uv_0)=1$, and $x^F(\delta(v))=2$ for all $v \in H_i$. Also, since q_i is even, we must have the edges in $F \cap \gamma(H_i)$ form a perfect matching of the nodes in $H_i \setminus u$. Thus $x^F[u{:}H_i \setminus u]=0$, and there must exist handle H_l such that $x^F(uu')=1$ for $u' = H_l \cap T_3$.

Each edge $e \in F \cap \gamma(H_i)$ joins two teeth T_j^i and T_k^i. Let e' be the edge joining $T_j^i \setminus H_i$ to $T_k^i \setminus H_i$, and let W be the set of all such edges e' not in F. Let $F' = F \setminus ((F \cap \gamma(H_i)) \cup \{vv_0, uu'\}) \cup W \cup (v'v_0 \colon \text{if } v'v_0 \notin F)$. Then F' forms the edge set of a two-node connected spanning subgraph of $K_{n-|H_i|} = (V_n \setminus H_i, E_{n-|H_i|})$. Consider the SCT obtained by removing H_i and $T_1^i, T_2^i, \cdots, T_{q_i}^i$ from the current SCT, and replacing T_3 by $T_3 \setminus u$. By the induction hypothesis, the corresponding CSCT2 constraint $b^1 x \geq 2p + q - q_i$ is valid for $Q_{2N}^{n-|H_i|}$, and thus $b^1 x^{F'} \geq 2p + q - q_i$. Since $bx^F = 2 + q_i + b^1 x^{F'}$, it follows that $bx^F \geq 2(p+1) + q$.

This completes the proof. □

5.3 A Characterization of the Facet-inducing CSCT2 Constraint for Q_{2N}^n

In this section we prove that the complemented star clique tree constraint 2 (5.1) is a facet-inducing inequality for Q_{2N}^n. We will make use of Lemma 5.3 below, which follows directly from the theory discussed in Section 2.4.

Lemma 5.3. For any star clique tree , the corresponding complemented star clique tree inequality 2 $bx \geq b_0$ is facet-inducing for Q_{2N}^n if there exists a set $D = \{d_1, d_2, \cdots, d_n\} \subseteq R^E$ such that:

(P_1) For each $1 \leq i \leq n$, there exists $\hat{x}_i \in Q_T^n \cap \{x \colon bx = b_0\}$ such that $x_i := \hat{x}_i + d_i \in Q_{2N}^n$;

(P_2) For each $1 \leq i \leq n$, $bd_i = 0$;

(P_3) $Ad_1, Ad_2, \cdots, Ad_n$ are linearly independent where A is the node-edge incidence matrix for the complete graph K_n.

Theorem 5.4. Given any star clique tree , the corresponding complemented star clique tree constraint $2\ bx \geq 2p+q$ is facet-inducing for Q^n_{2N} $(n=|\overset{0}{V}|+p+2q+1)$.

Proof. Let $G_0=(V_{G_0},E_{G_0})$ be the spanning subgraph of $K_n=(V,E)$ whose edge set corresponds to the edge e in K_n for which $b_e=0$, *i.e.*, $E_{G_0}:=\{e\in E{:}b_e=0\}$. The graph G_0 consists of one component. Let $v=T^1_{q_1}\setminus H_1$, let $u=T^1_1\setminus H_1$, and let $w=T^p_{q_p}\setminus H_p$. Let J be the set of edges $(E_{G_0}\cap SCT)\cup(v{:}C)\cup wu\cup vv_0$.(See Figure 5.3)

Note $J\subseteq E_{G_0}$ and $|J|=n$.

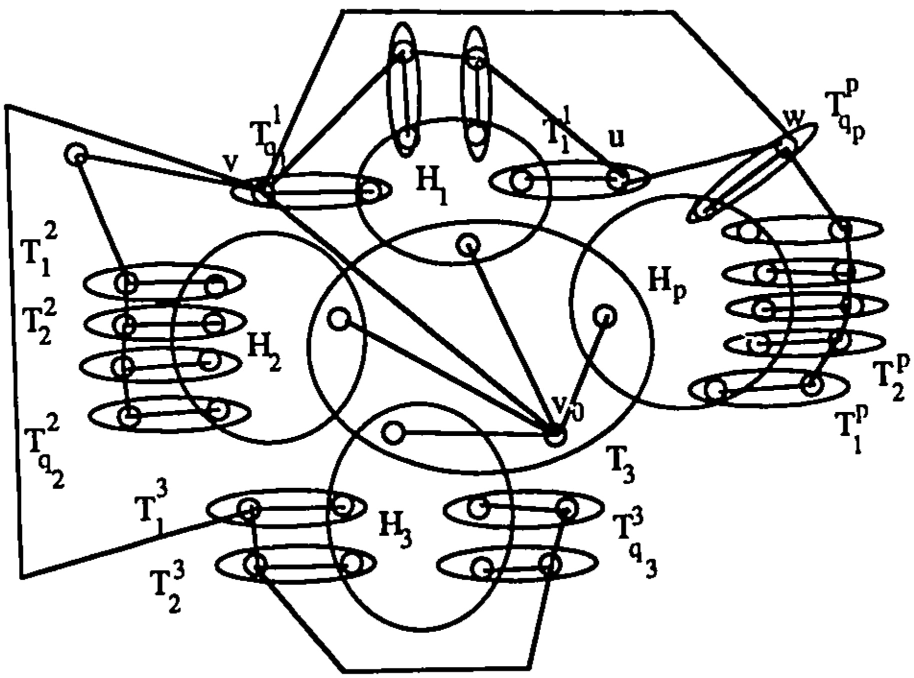


Figure 5.3 Edges in J.

We will show that

$$D:=\{d^e{:}e\in J\}$$

satisfies properties P_1,P_2 and P_3 of Lemma 5.3.

Recall that $bx \geq 2p+q$ is equivalent to a star clique tree constraint for Q_T^n and thus does not induce the facet $x_e = 1$, $e \in E$, in Q_T^n. Hence for each $e \in J$, there exists a Hamiltonian cycle H such that $bx^H = 2p+q$ and H does not contain edge e. Clearly $H \cup \{e\}$ is a two-node connected subgraph of K_n, i.e., $x^H + d^e \in Q_{2N}^n$. Thus D satisfies P_1 of Lemma 5.3. For each edge $e \in J$, $bd^e = b_e = 0$, since $J \subseteq E_{G_0}$. Thus D satisfies P_2 of Lemma 5.3.

Now consider $\{Ad: d \in D\}$, where A is the node-edge incidence matrix of K_n . For each $e \in J$, $Ad^e = A_e$. Thus by Lemma 2.19, the vectors in $\{Ad: d \in D\}$ are linearly independent, and D satisfies P_3 of Lemma 5.3. Thus D is an independent direction set of size n, and it follows from Lemma 5.3 that $bx \geq 2p+q$ is facet-inducing for Q_{2N}^n. This completes our proof.

□

Chapter 6. The Complemented Path Clique Tree Constraints for Q^n_{2N}

6.1 Description of the CPCT2 Constraint for Q^n_{2N}

We now convert each of the inequalities (2.5) into an equivalent form (with respect to Q^n_T) which is valid for Q^n_{2N}. The method we use to convert the inequalities (2.5) is as follows:

1. negate the inequality;
2. multiply by 2;
3. add two times the degree constraint for the nodes in the handles.

Let $\dot{V}$ be the set of nodes not in any tooth or handle, let C be all nodes in a tooth but not in any handle plus the nodes in $\dot{V}$, let H be the nodes in some handle, let T be the set of all edges in 2-matching teeth, and let NT be the set of all non-2-matching teeth. It is a straightforward to check that the resulting inequality obtained from (2.5) is

$$bx \geq 4p \,, \tag{6.1}$$

where b is defined as

$$b_e = \begin{cases} 0 & if\, e \in T \cup \gamma(C \cup NT \setminus H) \cup [NT \setminus H{:}H] \cap PCT \\ 2 & if\, e \in ([H{:}C] \cap \overline{PCT}) \cup (\gamma(H) \cap PCT) \\ 4 & if\, e \in \gamma(H) \cap \overline{PCT}. \end{cases}$$

(Note that in the above definition we use PCT to represent the set of edges in the path clique tree.)

We call the inequality (6.1) the *complemented path clique tree constraint 2*, henceforth denoted by the CPCT2 constraint.

Figure 6.1 shows a CPCT2 constraint $bx \geq 12$ with the left hand coefficients for each edge. Note the cofficients for edges not shown are the sum of the weights on their end nodes.

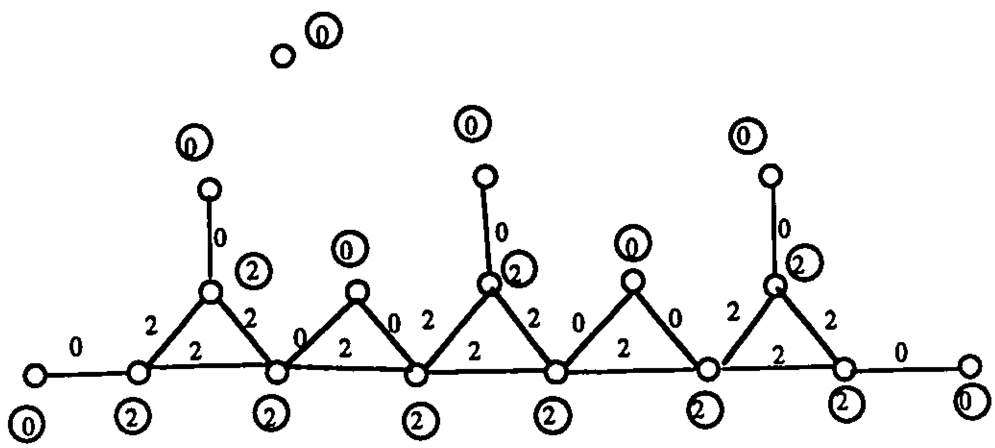


Figure 6.1 A CPCT2 constraint $bx \geq 12$

6.2 Validity of the CPCT2 Constraint for Q^n_{2N}

Theorem 6.1. The complemented path clique tree constraint 2 (6.1) is valid for Q^n_{2N} ($n = |\overset{0}{V}| + 5p + 1$).

Proof. For the purposes of the induction, we will consider the path clique tree for p=1 to be the comb shown in Figure 4.1. By Lemma 4.1 we know that $bx \geq 4$ is a valid inequality for Q^n_{2N}.

Assume now that the conclusion is true for any integer p, i.e., $bx \geq 4p$ is a valid inequality for Q^n_{2N} for all $n \geq 5p+1$. We want to show that the conclusion is also true for p+1, i.e., $bx \geq 4(p+1)$ is a valid inequality for Q^n_{2N} for all $n \geq 5(p+1)+1$. Without loss of generality we can assume x is the incidence vector x^F of a 2-node connected spanning subgraph of K_n with edge set F.

Let the nodes of nonpendent tooth NT1 in the path clique tree be v_A, v_B and v_C, where $v_B \in H_1$, and $v_C \in H_2$. By removing the edges $\gamma(NT1)$, we obtain two subclique trees, one with one handle, and the other with p-1 handles. To the first add pendent tooth $\{v_B, v_A\}$ to obtain path clique tree PCT1, and to the second add pendent tooth $\{v_A, v_C\}$ to obtain path clique tree PCT2.(See Figure 6.2)

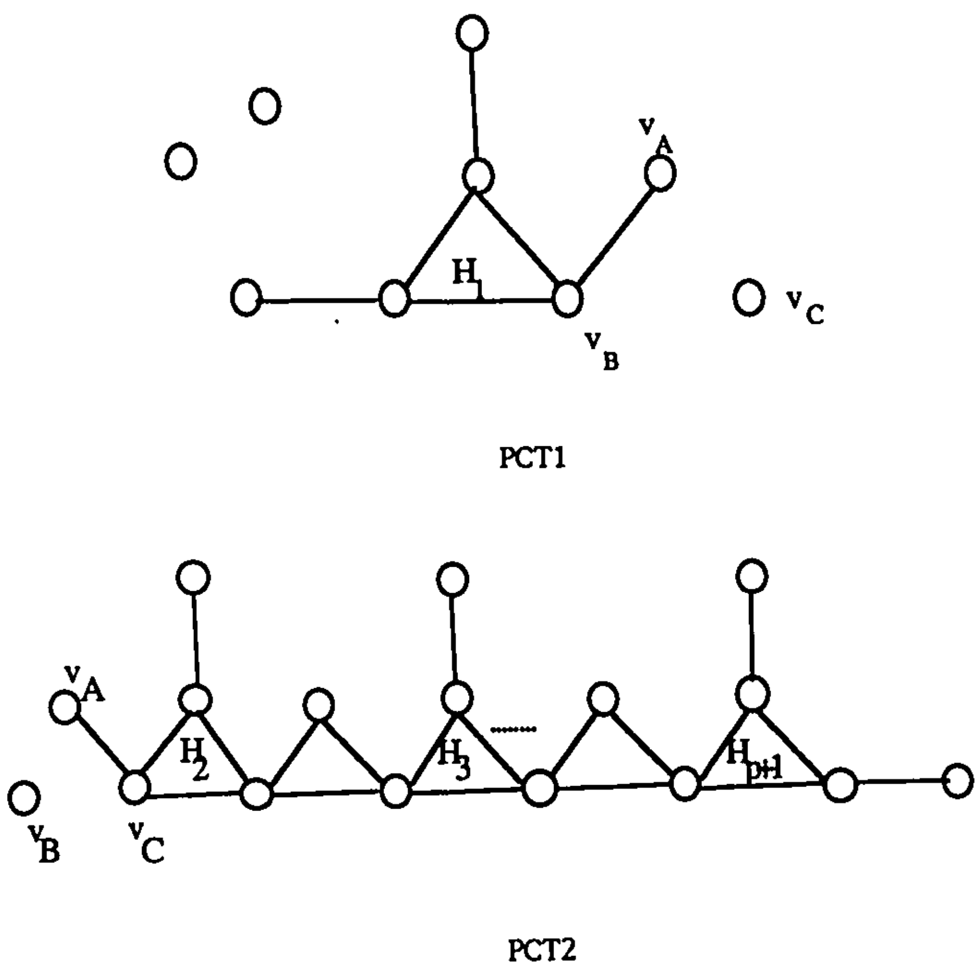


Figure 6.2 Path clique trees PCT1 and PCT2.

Associated with PCT1, we have CPCT2 constraint (6.1) $b^1 x^F \geq 4$, and associated with PCT2, we have CPCT2 constraint (6.1) $b^2 x^F \geq 4p$. From the definition of b^1 and b^2 we have

$$bx^F = b^1x^F + b^2x^F - 2x^F(v_Bv_C).$$

Suppose $x^F(v_Bv_C) = 0$. Then since $b^2x^F \geq 4p$ by assumption, $bx^F \geq 4 + 4p = 4(p+1)$.

Now suppose $x^F(v_Bv_C) = 1$, and $b^1x^F \geq 6$. Since $b^2x^F \geq 4p$ by assumption, we have $bx^F \geq 4(p+1)$.

Now we consider the case where $x^F(v_Bv_C) = 1$, and $b^1x^F < 6$. Then the only possibility for x^F passing through PCT1 is as shown in Figure 6.3.

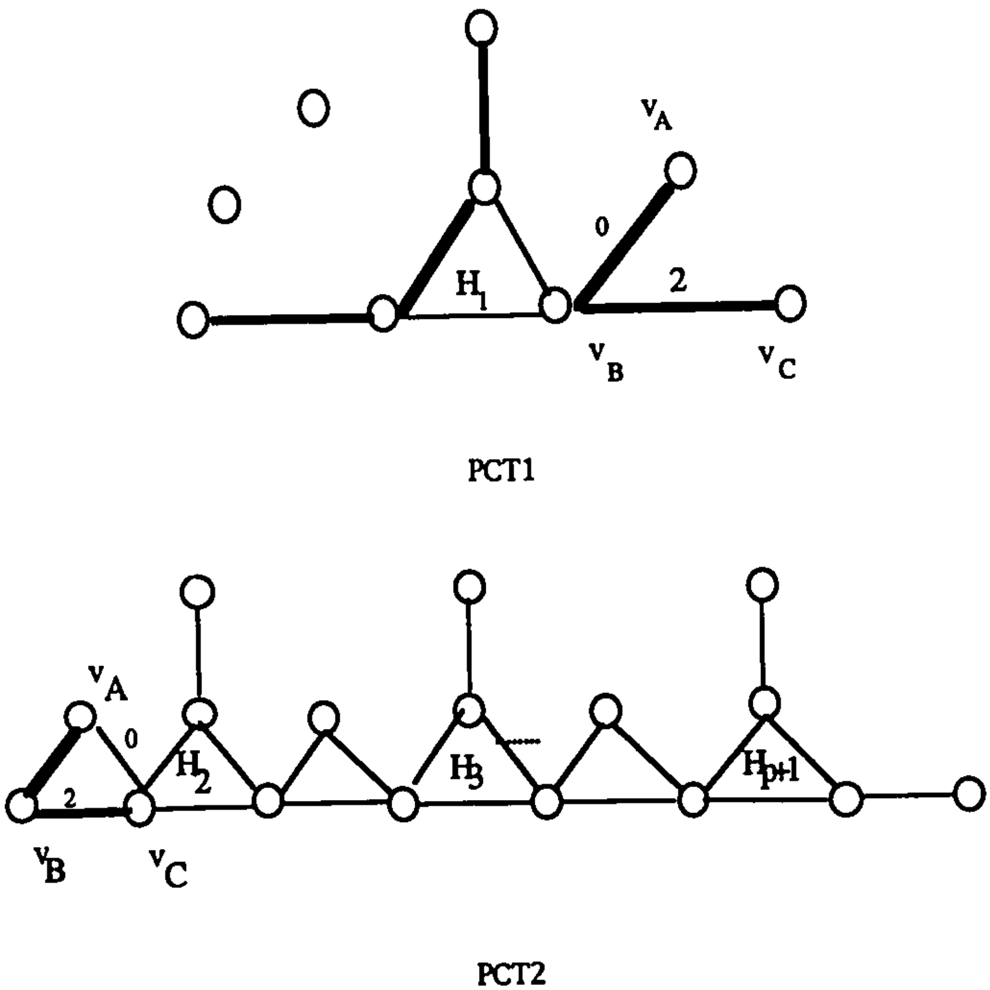


Figure 6.3 $b^1x^F < 6$ and $x^F(v_Bv_C) = 1$.

Case 1. $x^F(v_A v_C)=0$.

Let $F^1 = F \cup (v_A v_C) \setminus \{v_A v_B, v_B v_C\}$. Clearly F^1 is a 2-node connected spanning subgraph of $K_{n-1} = (V_n \setminus \{v_B\}, E_{n-1})$. Since $b^2 x^{F^1} \geq 4p$ by assumption, it follows that $b^2 x^F = b^2 x^{F^1} + 2$, and thus $bx^F \geq 4 + 4p + 2 - 2 = 4(p+1)$.

Case 2. $x^F(v_A v_C)=1$.

Let $F^1 = F \setminus \{v_A v_B, v_B v_C\}$. Clearly F^1 is a 2-node connected spanning subgraph of $K_{n-1} = (V_n \setminus \{v_B\}, E_{n-1})$. Thus $b^2 x^{F^1} \geq 4p$ by assumption, and $b^2 x^F \geq 4p+2$. Therefore $bx^F \geq 4 + 4p + 2 - 2 = 4(p+1)$. This completes our proof. □

6.3 A Characterization of the Facet-inducing CPCT2 Constraints for Q_{2N}^n

In this section we prove that the complemented path clique tree constraint 2 (6.1) is a facet-inducing inequality for Q_{2N}^n. We will make use of Lemma 6.2 below, which follows directly from the theory discussed in Section 2.4.

Lemma 6.2. For any path clique tree , the corresponding complemented path clique tree inequality 2 $bx \geq b_0$ is facet-inducing for Q_{2N}^n if there exists a set $D = \{d_1, d_2, \cdots, d_n\} \subseteq R^E$ such that:

(P_1) For each $1 \leq i \leq n$, there exists $\hat{x}_i \in Q_T^n \cap \{x: bx = b_0\}$ such that $x_i := \hat{x}_i + d_i \in Q_{2N}^n$;

(P_2) For each $1 \leq i \leq n$, $bd_i = 0$;

(P_3) $Ad_1, Ad_2, \cdots, Ad_n$ are linearly independent where A is the node-edge incidence matrix for the complete graph K_n.

Theorem 6.3. Given any path clique tree , the corresponding complemented path clique tree constraint 2 $bx \geq 4p$ is facet-inducing for Q^n_{2N} $(n=5p+|\overset{0}{V}|+1)$.

Proof. Let $G_0=(V_{G_0},E_{G_0})$ be the spanning subgraph of $K_n=(V,E)$ whose edge set corresponds to the edge e in K_n for which $b_e=0$, *i.e.*, $E_{G_0}:=\{e \in E: b_e=0\}$. The graph G_0 consists of one component. Let v be the node $T_1^1 \setminus H_1$, let $u=T_1^2 \setminus H_1$ and let $w=T_2^2 \setminus H_p$. Let J be the set of edges $(E_{G_0} \cap PCT) \cup [v{:}C] \cup wu$. (See Figure 6.4). Note that $J \subseteq E_{G_0}$, and $|J|=n$.

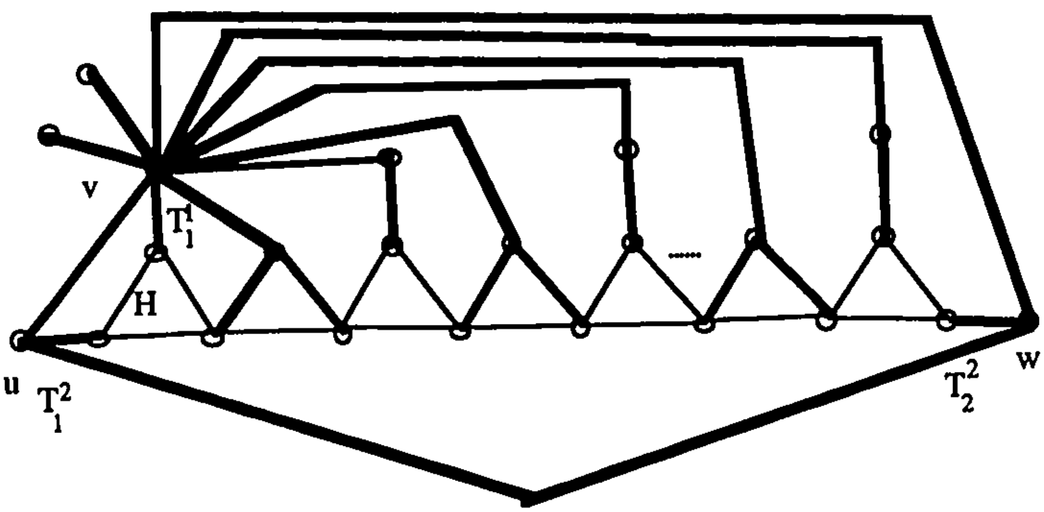


Figure 6.4 Edges in J.

Let $d^e := x^e$ for all $e \in J$. We will show that

$D:=\{d^e : e \in J\}$

satisfies properties P_1, P_2 and P_3 of Lemma 6.2.

Recall that $bx \geq 4p$ is equivalent to a path clique tree constraint for Q^n_T and thus does not induce the facet $x_e=1, e \in E$, in Q^n_T. Hence for each $e \in J$, there exists a Hamiltonian cycle H such that bx^H=4p and H does not contain edge e. Clearly $H \cup \{e\}$ is a 2-node connected subgraph of K_n, *i.e.*, $x^H + d^e \in Q^n_{2N}$. Thus

D satisfies P_1 of Lemma 6.2. For each edge $e \in J, bd^e = b_e = 0$, since $J \subseteq E_{G_e}$. Thus D satisfies P_2 of Lemma 6.2.

Now consider $\{Ad: d \in D\}$, where A is the node-edge incidence matrix of K_n . For each $e \in J, Ad^e = A_e$. Thus by Lemma 2.19, the vectors in $\{Ad: d \in D\}$ are linearly independent, and D satisfies P_3 of Lemma 6.2. Thus D is an independent direction set of size n, and it follows from Lemma 6.2 that $bx \geq 4$p is facet-inducing for Q_{2N}^n. This completes our proof. □

Chapter 7. Equivalence

In the following we investigate the equivalence of the complemented clique tree constraints introduced in the previous chapters to the other known classes of facet-inducing inequalities for Q^n_{2E} and Q^n_{2N} discussed in Chapter 2.

Lemma 7.1 ([4]). Let F be a nonempty face of a polyhedron P. Suppose $ax \le a_0$ and $\hat{a}x \le \hat{a}_0$ are both valid for F and P. If $ax \le a_0$ and $\hat{a}x \le \hat{a}_0$ are not equivalent with respect to F then they are not equivalent with respect to P.

Corollary 7.2 ([4]). Two inequalities, both valid for Q^n_T and Q^n_{2E}, are not equivalent with respect to Q^n_{2E} if they are not equivalent with respect to Q^n_T.

Corollary 7.3. Two inequalities, both valid for Q^n_T and Q^n_{2N}, are not equivalent with respect to Q^n_{2N} if they are not equivalent with respect to Q^n_T.

Theorem 7.4. The complemented star clique tree constraint 1 (3.1) is not equivalent to

(i) $x_e \le 1$ *for* $e \in E$ or

(ii) $x_e \ge 0$ *for* $e \in E$ or

(iii) $x(\delta(S)) \ge 2$ *for* $1 \le |S| \le |V| - 1$ or

(iv) complemented comb inequality (2.8)

with respect to Q^n_{2E}.

Proof. This follows directly from Theorem 2.6 and Corollary 7.2. ▯

Comment: The same conclusion holds for the complemented path clique tree constraint 1 (4.1).

Theorem 7.5. Complemented clique tree constraints (3.1) and (4.1) induce the same facets of Q^{n}_{2E} if and only if p=2 and $q_1 = q_2 = 2$ in (3.1).

Proof. This can be proven by Theorem 2.6 and by observing that a SCT and PCT are the same if and only if p=2 and $q_1 = q_2 = 2$.

□

Theorem 7.6. The complemented star clique tree constraint 2 (5.1) $bx \geq 2p + q + 2$ is not equivalent to

(i) $x_e \leq 1$ *for* $e \in E$ or

(ii) $x_e \geq 0$ *for* $e \in E$ or

(iii) $x(\delta(S)) \geq 2$ *for* $1 \leq |S| \leq |v| - 1$ or

(iv) complemented simple comb inequality (2.7)

with respect to Q^{n}_{2N}.

Proof. This follows directly from Theorem 2.6 and Corollary 7.3.

□

Comment: The same conclusion holds for inequality (6.1).

Theorem 7.7. Complemented clique tree constraints (5.1) and (6.1) induce the same facets of Q^{n}_{2N} if and only if $p = 2$ and $q_1 = q_2 = 2$ in (5.1).

Proof. This can be proven by Theorem 2.6 and by observing that a SCT and PCT are the same if and only if p=2 and $q_1 = q_2 = 2$.

□

The linear programming cutting-plane method has proven to be quite successful in solving large TSP problems. Typically we start with the equation set Ax=2 and inequality set $0 \leq x \leq 1$ as the initial set of constraints, then apply linear programming techniques and identify and add in constraints (facet-inducing inequalities for Q_T^n) as needed. In many cases subtour elimination and comb constraints alone were sufficient for solving large problems to optimality. Considering the close relationship between Q_T^n and 2-connected problems, we thus expect good results in solving TECSP and TNCSP using the linear programming cutting-plane method.

In order to use the linear programming cutting-plane method for TECSP(n) (and TNCSP(n)), we need to be able to either find a constraint known for Q_{2E}^n (or Q_{2N}^n) which is violated by an intermediate solution or prove such constraint does not exist in the set of constraints known for Q_{2E}^n (or Q_{2N}^n). This problem is the so called separation problem, which is NP-hard in general and can be formally stated as follows:

Separation Problem. Given a point $x \in R^E$ and a family Γ of inequalities in R^E, identify one or more inequalities in Γ that is violated by x or prove that no such inequality exists in Γ.

Separation problems for Q_T^n have been studied extensively. It has been shown that the separation problem for the subtour elimination inequalities can be solved in $O(n^4)$ time by using the Gomory-Hu algorithm [11] for the determination of a minimum weighted cut in an undirected graph, and the separation problem for the 2-matching inequalities can also be solved in $O(n^4)$ time ([22], [24]). At present no polynomial algorithm is known for solving the separation problem for the general comb and clique tree inequalities. In [26], heuristics for comb inequalities and special clique tree(called basic) are proposed.

We have restricted ourselves to study the facets of Q_{2E}^n and Q_{2N}^n in this paper and have not stressed the separation problem for the various complemented clique tree inequalities introduced. Since the true success of using linear programming cutting-plane method for optimizing over Q_{2E}^n and Q_{2N}^n depends highly on how successfully the related separation problems will be solved, we believe that the separation problem for the various complemented clique tree inequalities is a good research topic for us to consider in the future. We hope that some of the algorithms and heuristics for solving the separation problems for Q_T^n can be adapted for the various complemented clique tree inequalities.

In this paper we have derived some complemented clique tree inequalities based on some specific classes of clique trees and have proved that these complemented clique tree inequalities are facet-inducing for Q_{2E}^n and Q_{2N}^n respectively. We conjecture that these two inequalities can be combined in the obvious way to form a more general class of facet-inducing inequalities for Q_{2E}^n and Q_{2N}^n. However, we were unable to find a proof of validity for this more general class. Part of the problem stems from the fact that the inductive proof technique used for the path clique trees cannot be used for the more general form. Thus a good research topic for us to consider further is to find a general form of facet-inducing complemented clique tree inequalities for Q_{2E}^n and Q_{2N}^n respectively.

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