Cartesian Linear Bicategories

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Abstract

In this thesis, we extend the theory of cartesian bicategories [14, 13] to linear bicategories [15] and introduce the concept of cartesian linear bicategories for locally ordered linear bicategories. We demonstrate that the linear bicategory **Rel** of sets and relations, along with two other examples, fits within this framework. In our initial structure, called Cyclic cartesian linear bicategories, we believed that by taking the original definition of cartesian bicategories, adding a corresponding cartesian structure for the second horizontal composition, and replacing the adjunctions in bicategories with cyclic linear adjoints, we would achieve a proper cartesian structure on a locally ordered linear bicategory $(\mathcal{B}, \otimes, \top, \oplus, \bot)$. This approach was expected to make the tensor product of the cyclic cartesian structure a linear bicategorical product when restricted to the linear sub-bicategory of cyclic linear adjoints. However, we were surprised to discover that, in our main example, although the linear bicategory **Rel** is cyclic cartesian, the linear bicategorical product of the linear sub-bicategory **CMap**(**Rel**) does not coincide with the monoidal product.

Consequently, we refined our approach to accommodate the dual structures of tensor and par, which are linked in linear settings. By extending the theory of locally ordered cartesian linear bicategories, we introduce a characterization theorem for these structures, which ultimately leads us to a more general definition of cartesian linear bicategories that can be applied beyond the locally ordered case. Additionally, we explore the linear bicategory $\mathbf{Mat}(\mathbb{X})$, where $\mathbb{X}$ is a $\star$-autonomous linearly distributive category with linear products and coproducts [18], as an example of cartesian linear bicategories in the non-locally ordered case.

After studying the theory of cartesian linear bicategories, we introduce knowledge representation in linear bicategories of relations, inspired by Patterson's work in [46]. This concept bridges categorical frameworks and logical systems, providing some applications of our work in databases and machine learning.

Dedications

In memory of Pieter Hofstra (1975-2022)

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Contents

List of Symbols

$(-)^c$	Complement of a relation
$\bot$	Identity of the par composition
$\square$	Linear tensor product
$\boxplus$	Par product
$\boxtimes$	Tensor product
$-\!\!\bullet\!\!\!\rightarrow$	Quantale-valued relation
$\cap$, $\bigcap$	Set intersection
$\|\!\!\rightarrow\!\!\|$	Cyclic linear adjoint
$\coprod$	Coproduct
$\cup$, $\bigcup$	Set union
Δ_X	Comultiplication for a comonoid
$\emptyset$	Empty set
ϵ_X	Unit for a monoid
$\exists$	There exists
$\forall$	For all
$\dashv\!\|$	Linear adjoint
$\Leftrightarrow$	If and only if
∇_X	Multiplication for a monoid
$\oplus$	Par composition
$\oplus_Q$	Comultiplication operation in quntale Q
$\otimes$	Tensor composition
$\otimes_Q$	Multiplication operation in quntale Q
$\prod$	Product
$\mathbf{1}$	The top elememt
$\mathbf{0}$	The bottom elememt
$\sqcup$	Disjoint union of sets
$\top$	Identity of the tensor composition
$\vee$, $\bigvee$	Suprema
$\wedge$, $\bigwedge$	Infima
$\{*\}$	Singleton set
C_L	Constant pseudofunctor at 0-cell L in a bicategory $\mathcal{B}$
t_X	Counit for a comonoid

$;$	Usual composition

Introduction

Cartesian categories, i.e. categories with finite products, are fundamental to most branches of mathematics which make use of category theory. For example, one can define internal groups in any category with finite products. This leads, for example, to the definition of a topological group, i.e. a group in the category of topological spaces and continuous maps.

Bicategories (also called weak 2-categories) extend the notion of categories in two ways. First, they allow for 2-cells, i.e. morphisms between morphisms, so hom-sets become hom-categories. Then, once one has 2-cells, it is possible to talk about composition only being associative up to a specified isomorphism, which must satisfy coherence conditions. See [5]. Bicategories have had applications in many branches of science, from homotopy theory to theoretical computer science to linguistics [36].

The notion of cartesian bicategory was first introduced by Carboni and Walters in 1987 [14], extending the concept of cartesian categories to bicategories. Their initial exploration focused on locally ordered bicategories. These are bicategories for which each hom-category is a partially ordered set. Such bicategories are genuine bicategories, but the 2-cell structure is straightforward enough not to get bogged down in coherence issues. A cartesian structure on a (locally ordered) bicategory consists of a bicategorical tensor product for which every object is equipped with a cocommutative comonoid structure, every 1-cell is a lax comonoid homomorphism, and all of the comonoid structure maps have right adjoints. Details will appear below. Carboni and Walters demonstrated that the symmetric monoidal structure becomes a categorical product when restricted to the full sub-bicategory of 1-cells with right adjoints, i.e. the maps.

The primary example of locally ordered bicategories is the bicategory **Rel** of sets and relations. In this example, **Rel** has a symmetric monoidal structure given by the cartesian product of sets. Since the cartesian product is not the categorical product in **Rel**, this symmetric monoidal structure is not cartesian monoidal. However, the cartesian product is restricted to the locally discrete sub-bicategory **Set** of sets and functions, where it is the categorical product. Another example which they characterized as a cartesian bicategory is the locally ordered bicategory $\mathbf{Ord}(\mathcal{E})$ of ordered objects and ordered ideals [14].

Another example we consider in this thesis is Q-Rel, the bicategory of quantale-valued relations [25]. While the bicategory Q-Rel is abstraction of the bicategory of sets and relations, it does not, in general, form a cartesian bicategory. We begin with the obvious observation that when the quantale is, in fact, a locale, Q-Rel is a cartesian bicategory.

A crucial theorem in the original cartesian bicategories paper, [14, Theorem 1.6], gives an equivalent characterization of the definition. This is significant because it allows for the extension of the definition of cartesian bicategory beyond the locally ordered case. This was accomplished two decades later by Carboni, Kelly, Walters, and Wood in their 2008 paper [13]. According to their definition, a cartesian bicategory $\mathcal{B}$ must be precartesian, and the obtained lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$ from precartesian structure on $\mathcal{B}$ must be invertible. That is, $\mathcal{B}$ must have locally finite products, and its full sub-bicategory of left adjoints must have finite bicategorical products and obtained lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$ are pseudofunctors. This generalized definition characterizes examples of cartesian bicategories, such as the bicategory of spans, which is not locally ordered [35].

The other relevant structure in this thesis is the notion of linear bicategories as introduced by Cockett, Koslowski and Seely [15]. These are a bicategorical variant of the notion of linearly distributive categories [8]. Linearly distributive categories form the categorical framework for modelling the tensor/par fragment of Girard's linear logic [21]. Just as a linearly distributive category has two monoidal structures related by a natural transformation called a linear distributivity, a linear bicategory has two horizontal compositions, making it a bicategory in two ways. These compositions are similarly linked by linear distributions. Their work began with the realization that the usual category of sets and relations has a second composition (dual of the usual composition), making it a bicategory in a second way, and furthermore, these two compositions make the category of sets and relations a linear bicategory.

The theory of linear bicategories was not much considered beyond the initial paper since there seemed to be a lack of examples. But this changed with the work of Blute, Kudzman-Blais and Niefield [9]. They were inspired by the theory of monoidal topology [25] to consider categories of quantale-valued relations, denoted Q-Rel. The theory of quantales can be found in the text of Rosenthal [50] or the article of Niefield and Rosenthal in which the relationship between quantales and locales is examined [41]. The category of quantale-valued relations is studied in [25], which is also an excellent source for some of the applications of this fascinating theory. While it is not the case in general that Q-Rel will have a linearly distributive structure for a general quantale Q, the paper [9] demonstrates that Q-Rel is a linear bicategory when Q is a Girard quantale [50], or more generally an LD-quantale, a notion introduced in [9].

One of the key concepts of linear bicategories discussed in this thesis is the linear functors introduced by Cockett, Koslowski and Seely [15]. Linear functors are extensions of the morphisms between linearly distributive categories introduced by Cockett and Seely [18, 7]. These functors can be easily extended to morphisms of linear bicategories [15]. A linear functor between linear bicategories consists of two coherently linked morphisms that agree on the 0-cells. One of these morphisms is lax with respect to tensor composition, while the other is colax with respect to par composition.

Another key concept in linear bicategories is linear adjunctions. We refer to 1-cells that share a common left and right linear adjoint as cyclic linear adjoints, which form the base of the concept of linear monads. These linear monads can also be viewed as linear functors from the terminal linear bicategory $\mathbb{1}$ into a specified linear bicategory, similar to how a monad in a standard bicategory can be considered a lax functor with a terminal domain. The authors in [18] demonstrated that the collection of cyclic linear adjoints forms a linear bicategory.

Similar to how cartesian structure in bicategories characterizes essential examples of bicategories, this thesis aims to characterize important examples of linear bicategories, such as the linear bicategory **Rel** of sets and relations. This is achieved by extending the theory of cartesian bicategories to incorporate the second bicategorical structure of sets and relations.

In our initial attempt, we believed that by taking the original definition of cartesian bicategories and adding a corresponding cartesian structure for the second horizontal composition and replacing the adjunctions in bicategories with cyclic linear adjoints, we would obtain a cartesian structure on a locally ordered linear bicategory $(\mathcal{B}, \otimes, \top, \oplus, \bot)$. Then, similar to cartesian bicategories, the tensor product of cartesian structure becomes the linear bicategorical product restricted to linear sub-bicategory of cyclic linear adjoints. However, to our surprise, we discovered that while our main example, the linear bicategory **Rel** of sets and relations, is cartesian in this sense, the linear bicategorical product of the linear sub-bicategory **CMap**(**Rel**) do not coincide with the monoidal product. So, we had to change our definition in an appropriate way.

As in all concepts related to linear bicategories, which involve two linked structures for each horizontal composition, we introduce a cartesian structure in a locally ordered linear bicategory $(\mathcal{B}, \otimes, \oplus, \top, \bot)$ using two symmetric monoidal structures. One structure with respect to tensor and the other with respect to par, with both linked by linear distributors of a linear pseudofunctor. In other words, a cartesian structure in a locally ordered linear bicategory $(\mathcal{B}, \otimes, \oplus, \top, \bot)$ includes a cartesian structure on the bicategory $(\mathcal{B}, \otimes, \top)$ and a cocartesian structure on $(\mathcal{B}, \oplus, \bot)$ connected by linear distributions of a linear pseudofunctor. This appropriate structure leads us to a significant characterization theorem for locally ordered cartesian linear bicategories, akin to the Carboni and Walters characterization theorem for locally ordered bicategories [14]. This theorem then enables us to define cartesian linear bicategories in general and explore non-locally ordered cartesian linear bicategories.

Finally, we introduce the concept of knowledge representation in linear bicategories of relations, inspired by Patterson's work on knowledge representation in bicategories of relations [46]. Knowledge representation is one of the most successful applications of category theory [54, 53, 47, 27].

Patterson in [46] demonstrated a strong correspondence between bicategories of relations and certain fragments of first-order logic. For example, regular logic (a fragment of first-order logic) can be effectively modeled using bicategories of relations. This correspondence allows for the transfer of tools and techniques between categorical frameworks and logical systems, enhancing the theoretical foundations and practical applications of both fields [46].

From a logical view, our definition of a linear bicategory of relations includes two bicategories of relations, $(\mathcal{B}, \otimes, \top)$ and $(\mathcal{B}^{co}, \oplus, \perp)$. These structures correspond to two complementary components of first-order logic (FOL): $(\mathcal{B}, \otimes, \top)$ represents the existential conjunctive fragment, while $(\mathcal{B}^{co}, \oplus, \perp)$ represents the universal disjunctive fragment.

We assume that the reader is familiar with the basic theory of categories as in [39]. The outline of the thesis is as follows:

In Chapter 1, we review the appropriate preliminary materials of bicategories and specifically consider Q-Rel, the bicategory of quantale-valued relations.

In Chapter 2, first we briefly review the notion of locally ordered cartesian bicategories from the paper [14]. Then we focus on Q-Rel, the bicategory of quantale-valued relations and show why it is not cartesian in general and how we can fix it just by focusing on a specific type of quantales called locales. We then briefly review the definition of cartesian bicategories in general.

In Chapter 3, we provide an overview of essential concepts in linear bicategories. Additionally, in the final section of this chapter, we revisit the construction of the linear bicategory of Q-Rel as discussed in [9].

In Chapter 4, after reviewing the concepts of limits and colimits in bicategories from [28], we introduce the notion of a linear bicategorical product for linear bicategories by utilizing the concept of limits in linear bicategories. We then present our initial definition of a cartesian structure in locally ordered linear bicategories, which we refer to as a locally ordered cyclic cartesian linear bicategory. This structure is introduced by taking the original definition of locally ordered cartesian bicategories and adding a corresponding cartesian structure for the second horizontal composition and replacing the adjunction axiom with the cyclic linear adjunction. We demonstrate that the linear bicategory **Rel** of sets and relations is cyclic cartesian. Next, we aim to establish a similar characterization theorem for cyclic cartesian structures in a locally ordered linear bicategory. However, we discovered that in the cyclic cartesian linear bicategory **Rel**, the linear bicategorical product of the linear sub-bicategory **CMap**(**Rel**) does not coincide with the monoidal product. This leads us to our second definition, which is provided in next chapter.

In Chapter 5, we establish the appropriate cartesian structure for locally ordered linear bicategories. This structure includes a cartesian stucture on a locally ordered bicategory $\mathcal{B}_\otimes$ and a cocartesian structure on a locally ordered bicategory $\mathcal{B}_\oplus$, connected through linear distributions of a linear pseudofunctor. We then introduce the linear bicategory **Rel** of sets and relations as our primary example, along with additional examples. Furthermore, we prove a characterization theorem for locally ordered cartesian linear bicategories, which leads us to a general definition of cartesian linear bicategories in the subsequent chapter.

In Chapter 6, by using the characterization theorem from the previous chapter we define precaretsian linear bicategories. In a precartesian structure we require bicategorical products on the full sub-bicategory of left adjoints in $\mathcal{B}_\otimes$ and bicategorical coproducts on the full sub-bicategory of left adjoints in $\mathcal{B}_\oplus^{co}$, as well as local products in the hom-category $\mathcal{B}_\otimes(X, Y)$ and local coproducts in the hom-category $\mathcal{B}_\oplus^{co}(X, Y)$. We then demonstrate that any precartesian linear bicategory provides a canonical lax tensor product on $\mathcal{B}_\otimes$ and a colax cotensor product on $\mathcal{B}_\oplus$, which coincide on 0-cells. We proceed to define a precartesian linear bicategory as cartesian when these tensor and cotensor products are components of a linear pseudofunctor. Finally, we finish the chapter by presenting examples of cartesian linear bicategories.

In Chapter 7, after briefly reviewing the concept of knowledge representation for bicategories of relations from [46], we introduce the notion of knowledge representation for linear bicategories of relations. We explain that this concept offers a well-structured, human-oriented framework for machines and databases. Subsequently, we present first-order logic as the corresponding logic for linear bicategories of relations.

Finally, in Chapter 8, we summarize our study presented in this thesis and propose several potential questions and projects for future research.

Chapter 1

Bicategories

The notion of bicategories was introduced in 1967 by Jean Bénabou [5]. Bicategories can be viewed as a weakening of the definition of 2-categories.

In this chapter, we provide a brief overview of the definitions of bicategories, lax functors, lax and oplax transformations, modifications, dualities, adjunctions, and mates. In the final section, we discuss the bicategory Q-Rel of quantale valued relations, where Q is a quantale. It is important to note that in this chapter and throughout the rest of this thesis, we occasionally use pasting diagrams instead of commutative diagrams in the context of bicategories. For more detailed information, we refer the reader to [61, 28].

1.1 Bicategories

Definition 1.1.1. *[28, Definition 2.1.3]* A *bicategory* is a tuple $(\mathcal{B}, I, \otimes, a, l, r)$ consists of the following data:

- $\mathcal{B}$ is equipped with a class $\mathrm{Ob}(\mathcal{B}) = \mathcal{B}_0$, whose elements are called *objects or* 0-*cells* in $\mathcal{B}$. If $X \in \mathcal{B}_0$, we also write $X \in \mathcal{B}$.
- For each pair of 0-cells $X, Y \in \mathcal{B}$, a hom-category $\mathcal{B}(X, Y)$ with:
  - Its objects are called 1-*cells* in $\mathcal{B}$ denoted by $X \xrightarrow{f} Y$. The collection of all the 1-cells in $\mathcal{B}$ is denoted by $\mathcal{B}_1$,
  - Its (morphisms) arrows are called 2-*cells* in $\mathcal{B}$ denoted by $$\begin{tikzcd} X \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "g"', ""{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} ,$$
    The collection of all the 2-cells in $\mathcal{B}$ is denoted by $\mathcal{B}_2$,
  - Composition and identity morphisms in the category $\mathcal{B}(X, Y)$ are called *vertical composition* and *identity* 2-*cells*, respectively.

  - For a 1-cell f, its identity 2-cell is denoted by 1_f.

- A *horizontal composition (bi)functor*:

$$\otimes_{X,Y,Z} : \mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \to \mathcal{B}(X,Z)$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \mapsto \ f \otimes g : X \longrightarrow Z$$

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\alpha\Downarrow"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left, "f\otimes g"] \arrow[r, bend right, "f'\otimes g'"'] \arrow[r, phantom, "\Downarrow\alpha\otimes\beta"] & Z
\end{tikzcd}$$

- For each 0-cell $X \in \mathcal{B}$, an identity functor

$$I_X : \mathbb{1} \to \mathcal{B}(X,X).$$

  We identify the functor I_X with the 1-cell $I_X(*) \in \mathcal{B}(X,X)$, called the *identity 1-cell* of X.

- Natural isomorphisms:

  1. **Associators**:

$$a : (\otimes \times 1);\otimes \Longrightarrow (1 \times \otimes);\otimes$$

$$\begin{tikzcd}
\mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \times \mathcal{B}(Z,W) \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & \mathcal{B}(X,Z) \times \mathcal{B}(Z,W) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, shorten=2em, "a_\otimes"'] \\
\mathcal{B}(X,Y) \times \mathcal{B}(Y,W) \arrow[r, "\otimes"'] & \mathcal{B}(X,W)
\end{tikzcd}$$

  2. **Unitors**:

$$l : 1 \Longrightarrow (I_X \times 1);\otimes \qquad r : 1 \Longrightarrow (1 \times I_Y);\otimes$$

$$\begin{tikzcd}
\mathcal{B}(X,X) \times \mathcal{B}(X,Y) \arrow[dr, "\otimes"'] & \mathcal{B}(X,Y) \arrow[l, "{(I_X,1)}"'] \arrow[r, "{(1,I_Y)}"] \arrow[d, "1"] \arrow[dl, Rightarrow, shorten=1.5em, "l"] \arrow[dr, Rightarrow, shorten=1.5em, "r"'] & \mathcal{B}(X,Y) \times \mathcal{B}(Y,Y) \arrow[dl, "\otimes"] \\
 & \mathcal{B}(X,Y) &
\end{tikzcd}$$

such that the following diagrams commute:

$$\begin{tikzcd}
& ((f \otimes g) \otimes h) \otimes k & \\
(f \otimes g) \otimes (h \otimes k) \arrow[ur, "a"] & & (f \otimes (g \otimes h)) \otimes k \arrow[ul, "a\otimes 1"'] \\
f \otimes (g \otimes (h \otimes k)) \arrow[u, "a"] \arrow[rr, "1\otimes a"'] & & f \otimes ((g \otimes h) \otimes k) \arrow[u, "a"']
\end{tikzcd}$$

$$\begin{tikzcd}
& f \otimes g & \\
f \otimes (I \otimes g) \arrow[ur, "1\otimes r"] \arrow[rr, "a"'] & & (f \otimes I) \otimes g \arrow[ul, "l\otimes 1"']
\end{tikzcd}$$

Example 1.1.2. *The bicategory* ***Rel*** *of sets and relations consists of:*

- 0*-cells are sets:* $X, Y, Z, ...$
- *Each hom-category* $\mathcal{B}(X,Y)$ *is a power set* $\mathcal{P}(X \times Y)$*, which is a partially ordered set poset) and*

  - 1*-cells are relations* $R : X \to Y$*.*
  - 2*-cells are inclusions* $\alpha : R \subseteq R'$*.*

- *The composition (bi)functor is the ordinary composition between relations:*

$$\otimes_{X,Y,Z} : \mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \to \mathcal{B}(X,Z)$$

$$X \xrightarrow{R} Y \xrightarrow{S} Z \mapsto R \otimes S := \{(x,z) | \ \exists y \in Y \ \ (x,y) \in R \text{ and } (y,z) \in S\}$$

$$\begin{tikzcd}
X \arrow[r, bend left, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, "\alpha\Downarrow"] & Y \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left, "R\otimes S"] \arrow[r, bend right, "R'\otimes S'"'] \arrow[r, phantom, "\Downarrow\alpha\otimes\beta"] & Z & = R \otimes S \subseteq R' \otimes S'
\end{tikzcd}$$

- *For any* 0*-cell* $X \in \mathcal{B}$*, we have an identity relation* $I_X : X \to X := \{(x,x') \mid x = x'\}$*.*

Definition 1.1.3. *[28, Definition 2.1.14]*

- A bicategory $\mathcal{B}$ is *locally discrete* if each hom-category is discrete.
- A bicategory $\mathcal{B}$ is *locally partially ordered or locally ordered or locally posetal* if each hom-category is a partially ordered set regarded as a small category.

1.2 Dualities

Definition 1.2.1. *[28, Definition 2.6.2]* Let $\mathcal{B}$ be a bicategory. The opposite bicategory $\mathcal{B}^{op}$ is given by:

- **0-cells**: It has the same 0-cells as $\mathcal{B}$.
- **1-cells**: For 0-cells $X, Y \in \mathcal{B}^{op}$, its hom-category is the hom-category in $\mathcal{B}$,
$$\mathcal{B}^{op}(X, Y) = \mathcal{B}(Y, X),$$
- **Identity**: $I_X^{op} := I_X \in \mathcal{B}(X, X) = \mathcal{B}^{op}(X, X)$.
- **Composition**: Its horizontal composition is the composite
$$\begin{tikzcd}
\mathcal{B}^{op}(X,Y) \times \mathcal{B}^{op}(Y,Z) \arrow[r, "\otimes^{op}_{XYZ}"] \arrow[d, equal] & \mathcal{B}^{op}(X,Z) = \mathcal{B}(Z,X) \\
\mathcal{B}(Y,X) \times \mathcal{B}(Z,Y) \arrow[r, "\cong"'] & \mathcal{B}(Z,Y) \times \mathcal{B}(Y,X) \arrow[u, "\otimes_{ZYX}"']
\end{tikzcd}$$
$$f \otimes^{op} g := g \otimes f \qquad \text{(for 1-cells } f \text{ and } g\text{)},$$
$$\alpha \otimes^{op} \beta := \beta \otimes \alpha \qquad \text{(for 2-cells } \alpha \text{ and } \beta\text{)}.$$
- **Associator**: for 1-cells $f \in \mathcal{B}^{op}(W, Z), g \in \mathcal{B}^{op}(Z, Y)$, and $h \in \mathcal{B}^{op}(Y, X)$, the component of the associator $a^{op}_{h,g,f}$ is the invertible 2-cell $a^{-1}_{f,g,h}$ in $\mathcal{B}(X, W) = \mathcal{B}^{op}(W, X)$.
- **Unitors**: for 1-cell $f \in \mathcal{B}^{op}(W, X) = \mathcal{B}(X, W)$ the unitors are given by $l_f^{op} = r_f$ and $r_f^{op} = l_f$

Next, we define the bicategory in which the 2-cells in $\mathcal{B}$ are reversed.

Definition 1.2.2. The co-bicategory $\mathcal{B}^{co}$ is given by:

- **0-cells**: it has the same 0-cells as $\mathcal{B}$.
- **1-cells**: For 0-cells $X, Y \in \mathcal{B}^{co}$ its hom-category is the hom-category in $\mathcal{B}$,
$$\mathcal{B}^{co}(X, Y) = \big(\mathcal{B}(X, Y)\big)^{op},$$
- **Identity**: $1_X^{co} := (1_X)^{op} \in \mathcal{B}(X, X)^{op} = \mathcal{B}^{co}(X, X)$.
- **Composition**: Its horizontal composition is the composite

$$\begin{tikzcd}
\mathcal{B}^{co}(X,Y) \times \mathcal{B}^{co}(Y,Z) \arrow[r, "\otimes^{co}_{XYZ}"] \arrow[d, equal] & \mathcal{B}^{co}(X,Z) = \mathcal{B}(X,Z)^{op} \\
\mathcal{B}(X,Y)^{op} \times \mathcal{B}(Y,Z)^{op} \arrow[r, "\cong"'] & {[\mathcal{B}(X,Y) \times \mathcal{B}(Y,Z)]^{op}} \arrow[u, "\otimes^{op}_{ZYX}"']
\end{tikzcd}$$

$$\alpha \otimes^{co} \beta := \beta \otimes \alpha (\text{for 2-cells } \alpha \text{ and } \beta).$$

- **Associator**: for 1-cells $f \in \mathcal{B}^{op}(W,Z), g \in \mathcal{B}^{op}(Z,Y)$, and $h \in \mathcal{B}^{op}(Y,X)$, the component of the associator $a^{co}_{h,g,f}$ is the invertible 2-cell $(a^{-1}_{f,g,h})^{op}$ in $\mathcal{B}(Z,W) = \mathcal{B}^{op}(X,W)$.
- **Unitors**: for 1-cell $f \in \mathcal{B}^{co}(W,X) = \mathcal{B}(W,X)^{op}$ the unitors are given by $l^{co}_f = (r^{-1}_f)^{op}$ and $r^{co}_f = (l^{-1}_f)^{op}$

Definition 1.2.3. *[28]* Suppose $\mathcal{B}$ is a bicategory. Define

$$\mathcal{B}^{coop} = (\mathcal{B}^{co})^{op}.$$

Lemma 1.2.4. *[28, Lemma 2.6.5] For a given bicategory $\mathcal{B}$, the following statements hold.*

1. $\mathcal{B}^{op}, \mathcal{B}^{op}$*, and* $\mathcal{B}^{coop}$ *are well-defined bicategories.*
2. $\mathcal{B}^{coop} = (\mathcal{B}^{op})^{co}$.
3. $(\mathcal{B}^{op})^{op} = \mathcal{B} = (\mathcal{B}^{co})^{co}$.

1.3 Lax Functors (Morphisms) of Bicategories

Definition 1.3.1. *[28, Definition 4.1.2]* Let $(\mathcal{B}, I, \otimes, a, l, r)$ and $(\mathcal{B}', I', \otimes', a', l', r')$ be two bicategories. A *lax functor (morphism)* $(F, F^2, F^0) : \mathcal{B} \to \mathcal{B}'$ consists of the following data:

- A function $F : \mathcal{B}_0 \to \mathcal{B}'_0$ mapping 0-cells of $\mathcal{B}$ into 0-cells of $\mathcal{B}'$.
- For each pair (X, Y) of 0-cells, a functor $F_{X,Y} : \mathcal{B}(X,Y) \to \mathcal{B}'(FX, FY)$.
- For each triple X, Y and Z of 0-cells in $\mathcal{B}$, a natural transformation called the *lax functoriality constraint*:

$$F^2_{f,g} = Ff \otimes' Fg \Rightarrow F(f \otimes g)$$

$$\begin{tikzcd}
\mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \arrow[rr, "\otimes"] \arrow[d, "F_{X,Y} \times F_{Y,Z}"'] & & \mathcal{B}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathcal{B}'(FX,FY) \times \mathcal{B}'(FY,FZ) \arrow[rr, "\otimes'"'] \arrow[urr, phantom, ""] & {} \arrow[ur, Rightarrow, "F^2", shorten=4mm] & \mathcal{B}'(FX,FZ)
\end{tikzcd}$$

- For each 0-cell $X \in \mathcal{B}$, a natural transformation called the *lax unity constraint*:

$$F^0 \colon I'_{FX} \Longrightarrow F(I_X)$$

$$\begin{tikzcd}
1 \arrow[r, "I_X"] \arrow[d, equal] & \mathcal{B}(X,X) \arrow[d, "F_{X,X}"] \\
1 \arrow[r, "I'_{FX}"'] \arrow[ur, Rightarrow, "F^0_X", shorten=4mm] & \mathcal{B}'(FX,FX)
\end{tikzcd}$$

satisfying the following coherence axioms:

- **Lax Associativity:**

$$\begin{tikzcd}
Ff \otimes' (Fg \otimes' Fh) \arrow[r, "1_{Ff} \otimes' F^2"] \arrow[d, "a'"'] & Ff \otimes' F(g \otimes h) \arrow[r, "F^2"] & F(f \otimes (g \otimes h)) \arrow[d, "Fa"] \\
(Ff \otimes' Fg) \otimes' Fh \arrow[r, "F^2 \otimes' 1_{Fh}"'] & F(f \otimes g) \otimes' Fh \arrow[r, "F^2"'] & F((f \otimes g) \otimes h)
\end{tikzcd}$$

- **Lax Left and Right Unity:**

$$\begin{tikzcd}
I'_{FX} \otimes' Ff \arrow[r, "F^0 \otimes' 1_{Ff}"] \arrow[d, "r'"'] & FI_X \otimes' Ff \arrow[r, "F^2"] & F(I_X \otimes' f) \arrow[d, "Fr"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}$$

$$\begin{tikzcd}
Ff \otimes' I'_{FB} \arrow[r, "1_{Ff} \otimes' F^0"] \arrow[d, "l'"'] & Ff \otimes' FI_B \arrow[r, "F^2"] & F(f \otimes I_B) \arrow[d, "Fl"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}$$

Definition 1.3.2. *[28, Definition 4.1.2]*

- If F^2 and F^0 are natural isomorphism, then we call F a *pseudofunctor* (*homomorphism*).
- If F^2 and F^0 are identities, then F is called a *strict functor* (*strict homomorphism*).
- A *colax functor* from $\mathcal{B}$ to $\mathcal{B}'$ is a lax functor from $\mathcal{B}^{co}$ to $\mathcal{B}'^{co}$, in which $\mathcal{B}^{co}$ and $\mathcal{B}'^{co}$ are the co-bicategories of $\mathcal{B}$ and $\mathcal{B}'$.
- A strict functor between two *2*-categories is called a 2-*functor*.

Remark 1.3.3. [59]. Bénabou [5] originally used the terms "morphisms" and "homomorphisms" for what we now refer to as "lax functors" and "pseudofunctors." The term "lax functor" is likely attributed to Street [59].

Example 1.3.4. [28, Example 4.1.10] Any lax functor $(F, F^2, F^0)\colon \mathcal{B} \to \mathcal{B}'$ uniquely specifies a lax functor on opposite bicategories

$$(F^{op}, (F^{op})^2, (F^{op})^0)\colon \mathcal{B}^{op} \to \mathcal{B}'^{op}$$

with the following data, in which $\mathcal{B}^{op}$ and $\mathcal{B}'^{op}$ are the opposite bicategories in Definition 1.2.1.

- $F^{op} = F$ on 0-cells.
- For 0-cells $X, Y \in \mathcal{B}$, it is equipped with the functor

  $$F^{op} = F : \mathcal{B}^{op}(X, Y) = \mathcal{B}(Y, X) \to \mathcal{B}'(FY, FX) = \mathcal{B}'^{op}(F^{op}X, F^{op}Y).$$

- For 1-cells $f, g \in \mathcal{B}(Y, X) \times \mathcal{B}(Z, Y)$, $(F^{op})^2_{f,g}$ is the 2-cell

  $$Fg \otimes' Ff \xrightarrow{(F^{op})^2_{f,g} := F^2_{g,f}} F(g \otimes f) \quad \text{in} \quad \mathcal{B}'(Z, X) = \mathcal{B}'^{op}(X, Z).$$

  The lax associativity axiom and the lax unity axioms for F^{op} follow from those for F. We call F^{op} the *opposite lax functor* of F, and similarly if F is a pseudofunctor or a strict functor.

Definition 1.3.5. *[28, Definition 4.5.1]*

Suppose $f \in \mathcal{B}(X, Y)$ is a 1-cell in a bicategory $\mathcal{B}$, and Z is a 0-cell in $\mathcal{B}$.

1. We define the *pre-composition functor* by

$$\begin{aligned} &f^*\colon \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \\ &f^*(h) = f \otimes h \quad \text{for each 1-cell} \quad h \in \mathcal{B}(Y, Z). \\ &f^*(\alpha) = 1_f \otimes \alpha \quad \text{for each 2-cell} \quad \alpha \in \mathcal{B}(Y, Z). \end{aligned}$$

2. We define the *post-composition functor* by

$$\begin{aligned} f_*\colon \mathcal{B}(Z,X) &\to \mathcal{B}(Z,Y) \\ f_*(g) &= g \otimes f \quad \text{for each 1-cell} \quad g \in \mathcal{B}(Z,X). \\ f_*(\beta) &= \beta \otimes 1_f \quad \text{for each 2-cell} \quad \beta \in \mathcal{B}(Z,X). \end{aligned}$$

3. For $f \in \mathcal{B}(X,Y)$ and $g \in \mathcal{B}(Z,W)$, define the *hom-functor* by:

$$\begin{aligned} \mathcal{B}(f,g)\colon \mathcal{B}(Y,Z) &\to \mathcal{B}(X,W) \\ \mathcal{B}(f,g)(h) &= f \otimes h \otimes g \quad \text{for each 1-cell} \quad h \in \mathcal{B}(Y,Z). \\ \mathcal{B}(f,g)(\beta) &= 1_f \otimes \beta \otimes 1_g \quad \text{for each 2-cell} \quad \beta \in \mathcal{B}(Y,Z). \end{aligned}$$

The functoriality of these functors follows from that of the horizontal composition in $\mathcal{B}$.

Proposition 1.3.6. *[28, Proposition 4.5.2] Each* 0*-cell* X *in a bicategory* $\mathcal{B}$ *induces a pseudofunctor*

$$\mathcal{B}(-,X)\colon \mathcal{B}^{op} \to \mathbf{CAT}.$$

Corollary 1.3.7. *[28, Proposition 4.5.3] Each* 0*-cell* X *in a bicategory* $\mathcal{B}$ *induces a pseudofunctor*

$$\mathcal{B}(X,-)\colon \mathcal{B} \to \mathbf{CAT}.$$

1.4 Lax and Oplax Transformations

Definition 1.4.1. *[28, Definition 4.2.1]* Let $(F, F^2, F^0), (G, G^2, G^0) : \mathcal{B} \to \mathcal{B}'$ be two lax functors (morphisms) between bicategories $\mathcal{B}$ and $\mathcal{B}'$. A *lax transformation* $\alpha : F \Rightarrow G$ consists of the following data:

- **Components**: It is equipped with a component 1-cell $\alpha_X \in \mathcal{B}'(FX, GX)$ for each 0-cell $X \in \mathcal{B}$;

- **Lax Naturality Constraints**: For each pair of 0-cells X, Y in $\mathcal{B}$, it is equipped with a natural transformation α. We use the notation $(\alpha_X)^* : \mathcal{B}'(GX, GY) \to \mathcal{B}'(FX, GY)$ for the functor induced by a 1-cell $\alpha_X \in \mathcal{B}'(FX, GX)$ of a bicategory $\mathcal{B}$, and similarly $(\alpha_Y)_* : \mathcal{B}'(FX, FY) \to \mathcal{B}'(FX, GY)$.

$$\begin{tikzcd}
\mathcal{B}(X,Y) \arrow[r, "F_{X,Y}"] \arrow[d, "G_{X,Y}"'] & \mathcal{B}'(FX,FY) \arrow[d, "(\alpha_Y)_*"] \\
\mathcal{B}'(GX,GY) \arrow[r, "(\alpha_X)^*"'] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{B}'(FX,GY)
\end{tikzcd}$$

which is for each 1-cell $f \in \mathcal{B}(X,Y)$ a component 2-cell:

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha X"'] & FY \arrow[d, "\alpha Y"] \\
GX \arrow[r, "Gf"'] \arrow[ur, Rightarrow, shorten=4mm, "\alpha_f"] & GY
\end{tikzcd}$$

satisfying the following coherence conditions:

- **Lax Naturality:**

$$\begin{tikzcd}
(Ff \otimes \alpha_Y) \otimes Gg \arrow[r, "a^{-1}"] & Ff \otimes (\alpha_Y \otimes Gg) \arrow[r, "1_{Ff} \otimes \alpha_g"] & Ff \otimes (Fg \otimes \alpha_Z) \arrow[d, "a"] \\
(\alpha_X \otimes Gf) \otimes Gg \arrow[u, "\alpha_f \otimes 1_{Gg}"] & & (Ff \otimes Fg) \otimes \alpha_Z \arrow[d, "F^2 \otimes 1_{\alpha_Z}"] \\
\alpha_X \otimes (Gf \otimes Gg) \arrow[u, "a"] \arrow[dr, "1_{\alpha_X} \otimes F^2"'] & & F(f \otimes g) \otimes \alpha_Z \\
& \alpha_X \otimes G(f \otimes g) \arrow[ur, "\alpha_{f \otimes g}"'] &
\end{tikzcd}$$

- **Lax Unity:**

$$\begin{tikzcd}
\alpha_X \otimes I_{GX} \arrow[r, "l"] \arrow[d, "1_{\alpha_X} \otimes G^0"'] & \alpha_X \arrow[r, "r^{-1}"] & 1_{FX} \otimes \alpha_X \arrow[d, "F^0 \otimes 1_{\alpha_X}"] \\
\alpha_X \otimes GI_X \arrow[rr, "\alpha_{I_X}"'] & & FI_X \otimes \alpha_X
\end{tikzcd}$$

Definition 1.4.2. *[28, Definition 4.2.1]*

- A *strong transformation* is a lax transformation in which every component 2-cell is invertible.

- A *strict transformation* is a lax transformation in which every component 2-cell α_f is an identity.

Definition 1.4.3. *[28, Definition 4.2.1]* Let $(F, F^2, F^0), (G, G^2, G^0) : \mathcal{B} \to \mathcal{B}'$ be two lax functors (morphisms) between bicategories $\mathcal{B}$ and $\mathcal{B}'$. An *oplax transformations* $\alpha : F \Rightarrow G$ consists of the following data:

- **Components**: It is equipped with a component 1-cell $\alpha_X \in \mathcal{B}'(FX, GX)$ for each 0-cell $X \in \mathcal{B}$;
- **Oplax Naturality Constraints**: For each pair of 0-cells X, Y in $\mathcal{B}$, it is equipped with a natural transformation. We use the notation $(\alpha_X)^* : \mathcal{B}'(GX, GY) \to \mathcal{B}'(FX, GY)$ for the functor induced by a 1-cell $\alpha_X \in \mathcal{B}'(FX, GX)$ of a bicategory $\mathcal{B}$, and similarly $(\alpha_Y)_* : \mathcal{B}'(FX, FY) \to \mathcal{B}'(FX, GY)$.

$$\alpha \colon F; (\alpha_Y)_* \to G; (\alpha_X)^* \colon \mathcal{B}(X, Y) \to \mathcal{B}'(FX, GY),$$

$$\begin{tikzcd}
\mathcal{B}(X,Y) \arrow[r, "F_{X,Y}"] \arrow[d, "G_{X,Y}"'] & \mathcal{B}'(FX,FY) \arrow[d, "(\alpha_Y)_*"] \arrow[dl, Rightarrow, shorten=10pt, "\alpha"'] \\
\mathcal{B}'(GX,GY) \arrow[r, "(\alpha_X)^*"'] & \mathcal{B}'(FX,GY)
\end{tikzcd}$$

whose component 2-cell at $f \in \mathcal{B}(X, Y)$ is:

$$\alpha_f : (Ff) \otimes \alpha_Y \Rightarrow \alpha_X \otimes (Gf)$$

.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha X"'] & FY \arrow[d, "\alpha Y"] \arrow[dl, Rightarrow, shorten=10pt, "\alpha_f"'] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

satisfying the following coherence conditions:

- **Oplax Naturality:**

$$\begin{tikzcd}
(Ff \otimes \alpha_Y) \otimes (Gg) \arrow[d, "\alpha_f \otimes 1_{Gg}"'] & (Ff) \otimes (\alpha_Y \otimes Gg) \arrow[l, "a"'] & Ff \otimes (Fg \otimes \alpha_Z) \arrow[l, "1_{Ff} \otimes \alpha_g"'] \\
(\alpha_X \otimes Gf) \otimes Gg & & (Ff \otimes Fg) \otimes \alpha_Z \arrow[u, "a^{-1}"'] \arrow[d, "F^2 \otimes 1_{\alpha_Z}"] \\
\alpha_X \otimes (Gf \otimes Gg) \arrow[u, "a"] \arrow[dr, "1_{\alpha_X} \otimes G^2"'] & & F(f \otimes g) \otimes \alpha_Z \arrow[dl, "\alpha_{f \otimes g}"] \\
& \alpha_X \otimes G(f \otimes g) &
\end{tikzcd}$$

- **Oplax Unity:**

$$\begin{tikzcd}
I_{FX} \otimes \alpha_X \arrow[r, "r"] \arrow[d, "F^0 \otimes 1_{\alpha_X}"'] & \alpha_X \arrow[r, "l^{-1}"] & \alpha_X \otimes 1_{GX} \arrow[d, "1_{\alpha_X} \otimes G^0"] \\
(FI_X) \otimes \alpha_X \arrow[rr, "\alpha_{I_X}"'] & & \alpha_X \otimes (GI_X)
\end{tikzcd}$$

Lemma 1.4.4. *[28, Lemma 4.3.9] Suppose $\alpha\colon F \Rightarrow G$ is an oplax transformation between lax functors $F, G\colon B \to B'$. Then:*

- *α is uniquely determined by a lax transformation $\alpha^{op}\colon G^{op} \Rightarrow F^{op}$ with F^{op} and G^{op} the opposite lax functors $\mathcal{B}^{op} \to \mathcal{B}'^{op}$.*
- *Each component 2-cell of α is invertible if and only if α defines a strong transformation $\alpha'\colon F \Rightarrow G$ with component 1-cells and $\alpha'_f = \alpha_f^{-1}$ for each 1-cell f.*
- *Each component 2-cell of α is an identity if and only if α is a strict transformation.*

1.5 Modifications

Definition 1.5.1. *[28, Definition 4.4.1]* Let $\alpha, \beta\colon F \Rightarrow G$ be two lax transformations between two lax functors $F, G\colon \mathcal{B} \to \mathcal{B}'$. A *modification* $\gamma\colon \alpha \Rrightarrow \beta$ consists of a component 2-cell $\Gamma_X\colon \alpha_X \Rightarrow \beta_X$ in $\mathcal{B}'(FX, GX)$ for each 0-cell $X \in \mathcal{B}$, such that the following diagram commutes:

$$\begin{tikzcd}
\alpha_X \otimes (Gf) \arrow[r, Rightarrow, "\Gamma_X \otimes 1_{Gf}"] \arrow[d, Rightarrow, "\alpha_f"'] & \beta_X \otimes (Gf) \arrow[d, Rightarrow, "\beta_f"] \\
(Ff) \otimes \alpha_Y \arrow[r, Rightarrow, "1_{Ff} \otimes \Gamma_Y"'] & (Ff) \otimes \beta_Y
\end{tikzcd}$$

for each 1-cell $f \in \mathcal{B}(X, Y)$. A modification is invertible if each component Γ_X is an invertible 2-cell.

Proposition 1.5.2. *[28, Definition 4.4.10, Theorem 4.4.11] Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bicategories. Then* Bicat$(\mathcal{B}, \mathcal{B}')$ *is a bicategory with*

- *0-cells are lax functors $\mathcal{B} \to \mathcal{B}'$.*
- *1-cells are lax transformations between such pseudofunctors.*

- 2-*cells are modifications between such lax transformations.*

Corollary 1.5.3. *[28, Corollary 4.4.13] Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bicategories. Then* $\mathrm{Bicat}(\mathcal{B}, \mathcal{B}')$ *contains a sub-bicategory* $\mathrm{Bicat}^{ps}(\mathcal{B}, \mathcal{B}')$ *with*

- 0-*cells are pseudofunctors $\mathcal{B} \to \mathcal{B}'$.*
- 1-*cells are strong transformations between such pseudofunctors.*
- 2-*cells are modifications between such strong transformations.*

1.6 Adjunctions and Mates

Definition 1.6.1. *[28, Definition 6.1.1]* For a given bicategory $\mathcal{B}$, an *adjunction* $(\eta, \epsilon) : f \dashv g : X \to Y$ in $\mathcal{B}$ consist of:

- 1-cells $f : X \to Y$ and $g : Y \to X$;
- 2-cells $\eta : I_X \Rightarrow f \otimes g$ and $\epsilon : g \otimes f \Rightarrow I_Y$. Then we say f has *right adjoint* g or f is the *left adjoint.*

such that the following diagrams commute:

$$\begin{tikzcd}
I_X \otimes f \arrow[r, "\eta \otimes 1_f"] \arrow[rrdd, "r_f"'] & (f \otimes g) \otimes f \arrow[r, "a^{-1}_{f,g,f}"] & f \otimes (g \otimes f) \arrow[d, "1_f \otimes \epsilon"] \\
 & & f \otimes I_Y \arrow[d, "l_f"] \\
 & & f
\end{tikzcd}$$

$$\begin{tikzcd}
g \otimes I_X \arrow[r, "1_g \otimes \eta"] \arrow[rrdd, "l_g"'] & g \otimes (f \otimes g) \arrow[r, "a^{-1}_{f,g,f}"] & (g \otimes f) \otimes g \arrow[d, "\epsilon \otimes 1_g"] \\
 & & I_Y \otimes g \arrow[d, "r_g"] \\
 & & g
\end{tikzcd}$$

Remark 1.6.2. In this thesis, we use an alternative notation for an adjunction in a bicategory $\mathcal{B}$, denoted as (f, g, η, ϵ) instead of $(\eta, \epsilon) : f \dashv g : X \to Y$ in $\mathcal{B}$ defined in Definition 1.6.1.

Definition 1.6.3. *[14, Definition 1.5]* In a bicategory $\mathcal{B}$ we call left adjoints *maps*, and we will use $\mathbf{Map}(\mathcal{B})$ to represent the full sub-bicategory consisting of all maps.

Theorem 1.6.4. *In the bicategory* **Rel** *of sets and relations maps are precisely function.*

Proof: Assume $f\colon X \to Y$ is a map in **Rel**. That is, $(\eta, \epsilon) : f \dashv g : X \to Y$, where $\eta : I_X \subseteq f \otimes g$ and $\epsilon : g \otimes f \subseteq I_Y$ (since **Rel** is a locally ordered bicategory, so 2-cells are inclusions). By 2-cell $\eta : I_X \subseteq f \otimes g$, we get:

$$(x, x) \in I_X \Rightarrow (x, x) \in f \otimes g = \{(x, x') \,|\, \exists y \in Y \ (x, y) \in f \text{ and } (y, x') \in g\}.$$

Then this implies:

$$\forall x \in X \ \ \exists y \in Y \ \ (x, y) \in f \text{ and } (y, x) \in g,$$

which implies that f is total. Next, by 2-cell $\epsilon : g \otimes f \subseteq I_Y$, we get:

$$\{(y, y') \,|\, \exists x \in X \ (y, x) \in g \text{ and } (x, y') \in f\} \subseteq \{(y, y) \,|\, y \in Y\}.$$

Now consider $x \in X$. Since f is total, there exists $y \in Y$ such that $(x, y) \in f$. For $y'' \in Y$, if we have $(x, y'') \in f$, then by $(y, x) \in g$ we get $(y, y'') \in g \otimes f$. Thus f is a function. ∎

Remark 1.6.5. [28, Example 6.1.11] An adjunction (f, g, η, ϵ) in a bicategory $\mathcal{B}$ induces:

- An adjunction $(g^{op}, f^{op}, \eta, \epsilon)$ in $\mathcal{B}^{op}$.
- An adjunction (g, f, η, ϵ) in $\mathcal{B}^{co}$.

Definition 1.6.6. *[28, Definition 6.2.1]* We call an adjunction (f, g, η, ϵ) with $f : X \to Y$ and $g : Y \to X$ in a bicategory $\mathcal{B}$, an *internal equivalence* or *adjoint equivalence* if η and ϵ are isomorphisms. And, we say the 1-cell $f : X \to Y$ is an *equivalence* in this case.

Definition 1.6.7. *[28, Definition 6.2.8]* For given bicategories $\mathcal{B}$ and $\mathcal{C}$, we say a pseudofunctor $F : \mathcal{B} \to \mathcal{C}$ is a *biequivalence* if there exists a pseudofunctor $G : \mathcal{C} \to \mathcal{B}$ together with the following internal equivalences

$$\mathrm{Id}_{\mathcal{B}} \cong F; G \quad \text{and} \quad G; F \cong \mathrm{Id}_{\mathcal{C}}$$

in $\mathrm{Bicat}^{\mathrm{ps}}(\mathcal{B}, \mathcal{B})$ and $\mathrm{Bicat}^{\mathrm{ps}}(\mathcal{C}, \mathcal{C})$, respectively.

Definition 1.6.8. *[28, Lemma 6.1.13]* For a given bicategory $\mathcal{B}$, a pair of adjunctions $(\eta, \epsilon) : f \dashv g : X \to Y$ and $(\eta', \epsilon') : f' \dashv g' : X' \to Y'$, and 1-cells $a : X \to X'$ and $b : Y \to Y'$, there is a bijection

$$\mathcal{B}(X, Y')(a \otimes f', f \otimes b) \cong \mathcal{B}(Y, X')(g \otimes a, b \otimes g')$$

given by pasting with the unit of one adjunction and the counit of the other:

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \arrow[dl, Rightarrow, "\lambda"'] & & Y \arrow[r, "g"] \arrow[d, equal] & X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dl, Rightarrow, "\epsilon"'] & X' \arrow[r, equal] \arrow[d, "f'"] \arrow[dl, Rightarrow, "\lambda"'] & X' \arrow[d, equal] \arrow[dl, Rightarrow, "\eta'"] \\
Y \arrow[r, "b"'] & Y' & \mapsto & Y \arrow[r, equal] & Y \arrow[r, "b"'] & Y' \arrow[r, "g'"'] & X'
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[r, "b"] \arrow[d, "g"'] & Y' \arrow[d, "g'"] & & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d, "g"'] & Y' \arrow[r, equal] \arrow[d, "g'"] & Y' \arrow[d, equal] \\
X \arrow[r, "a"'] \arrow[ur, Rightarrow, "\mu"] & X' & \mapsto & X \arrow[r, equal] \arrow[ur, Rightarrow, "\eta"] & X \arrow[r, "b"'] \arrow[ur, Rightarrow, "\mu"] & X' \arrow[r, "f'"'] \arrow[ur, Rightarrow, "\epsilon"'] & Y'
\end{tikzcd}$$

The 2-cells λ and μ in the above diagrams are called *mates* with respect to the adjunctions $(\eta, \epsilon) : f \dashv g : X \to Y$ and $(\eta', \epsilon') : f' \dashv g' : X' \to Y'$ and to the 1-cells a, b.

1.7 The Bicategory Q-Rel

In this section, we present the bicategory Q-Rel, which extends the concept of a relation over a quantale Q. However, before proceeding, we will briefly review the notion of quantales.

1.7.1 Quantales

A key foundational concept in monoidal topology [25] is the notion of a quantale, which arises in the category Q-Rel of Q-valued relations. Quantales were initially introduced by Mulvey in [24], aiming to establish a lattice-theoretic framework for C^*-algebras and provide a constructive basis for quantum mechanics.

Definition 1.7.1. *[25, Section II.1.10]* A *quantale* is a complete lattice Q with an associative operation $\otimes_Q \colon Q \times Q \to Q$ and unit $\top$ such that $\otimes_Q$ preserves suprema denoted with $\bigvee$ on the left and the right. That is, for all subsets $P \subseteq Q$ and all elements $q \in Q$, we have

$$\left(\bigvee P\right) \otimes_Q q = \bigvee_{p \in P} p \otimes_Q q \qquad \text{and} \qquad q \otimes_Q \left(\bigvee P\right) = \bigvee_{p \in P} q \otimes_Q p. \tag{1.7.1}$$

Remark 1.7.2. Note that for all $q \in Q$ we have $q \otimes_Q \mathbf{0} = \mathbf{0} = \mathbf{0} \otimes_Q q$, where $\mathbf{0}$ is the bottom element.

Definition 1.7.3. *[25, Section II.1.10]* For all $q \in Q$, the operations $(-) \otimes q$ and $q \otimes_Q (-)$ in the equations 1.7.1 preserve all supremas. That is, they have right adjoints. We denote the right adjoints to $(-) \otimes_Q p$ and $p \otimes_Q (-)$ by $(-) \circ\!\!- p : Q \to Q$ and $p -\!\!\circ (-) : Q \to Q$ respectively which is uniquely determined by:

$$p \otimes_Q q \leq r \quad \Leftrightarrow \quad q \leq p -\!\!\circ r$$

all for $q, r \in Q$ where

$$p -\!\!\circ r = \bigvee \{q \in Q \mid p \otimes_Q q \leq r\} \qquad r \circ\!\!- p = \bigvee \{q \in Q \mid q \otimes_Q p \leq r\}$$

Definition 1.7.4. *[50, Definitions 6.1.1]* A *cyclic dualizing element* $\perp \in Q$ is an element of Q such that for all $q \in Q$, we have

$$\perp \circ\!\!- q = q -\!\!\circ \perp \qquad \text{and} \qquad (q -\!\!\circ \perp) -\!\!\circ \perp = q$$

Definition 1.7.5. *[50, Definitions 6.1.2]* A *Girard quantale* is a unital quantale Q with a cyclic dualizing object $\perp$. We denote $q -\!\!\circ \perp$ as $q^{\perp}$. Note that $\top^{\perp} = \perp$.

Remark 1.7.6. [50, Definition 6.1.3] Any Girard quantale Q carries a second multiplication operation defined by de Morgan duality in linear logic.

Lemma 1.7.7. *[50, Proposition 6.1.3] If $(Q, \otimes_Q, \perp)$ be a Girard quantale, then the operation $(-)^{\perp}$ is a contravariant isomorphism. So Q^{op} is also a quantale. We will denote its multiplication by*

$$p \oplus_Q q = (q^{\perp} \otimes_Q p^{\perp})^{\perp}$$

Evidently this operation satisfies:

$$(\bigwedge P) \oplus_Q q = \bigwedge_{p \in P} p \oplus_Q q \qquad \textit{and} \qquad q \otimes_Q (\bigwedge P) = \bigwedge_{p \in P} q \oplus_Q p$$

Definition 1.7.8. *[41]*

- A quantale Q is said to be *right-sided* if for any $q \in Q, q \otimes_Q \mathbf{1} \leq q$, Where $\mathbf{1}$ is the top element. Similarly, the *left-sided* quantale will be defined by duality.
- A quantale Q is said to be *idempotent* if all elements of Q are idempotent. That is, for any $q \in Q$, $q \otimes_Q q = q$

- A quantale Q is said to be *lean* if for all $p, q \in Q$ we have:

$$\big(p \vee q = \top \quad \text{and} \quad p \otimes_Q q = \bot\big) \implies (p = \top \quad \text{or} \quad q = \top) \tag{1.7.2}$$

- A unital quantale Q is said to be *integral* if $\mathbf{1} = \top$ where $\mathbf{1}$ is the unit element under multiplication and $\top$ is the top element.

Lemma 1.7.9. *[49, Proposition 8.2.6] Let Q be a unital quantale. The following are equivalent:*

- $\top = \mathbf{1}$.
- *Q is left-sided.*
- *Q is right-sided.*
- *Q is 2-sided.*

Definition 1.7.10. *[41]* An idempotent, integral and commutative quantale is called a locale.

1.7.2 The Bicategory Q-Rel

Definition 1.7.11. Let Q be a quantale, we can form the category Q-Rel whose objects are sets and arrows $f \colon X \mathrel{-\!\!\bullet\!\!\!\to} Y$ are functions $f \colon X \times Y \to Q$. Given $f \colon X \mathrel{-\!\!\bullet\!\!\!\to} Y$ and $g \colon Y \mathrel{-\!\!\bullet\!\!\!\to} Z$, composition is defined by

$$(f \otimes g)(x, z) = \bigvee_{y \in Y} f(x, y) \otimes_Q g(y, z)$$

Note that the use of $\otimes_Q$ on the left refers to composition and on the right refers to multiplication in Q.

Identities are given by

$$id_X(x, x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases}$$

See [25] for further details.

If Q is a Girard quantale, there is a second categorical structure, using the fact that Q^{op} is also a quantale. Given $f \colon X \mathrel{-\!\!\bullet\!\!\!\to} Y$ and $g \colon Y \mathrel{-\!\!\bullet\!\!\!\to} Z$, composition is given by

$$(f \oplus g)(x, z) = \bigwedge_{y \in Y} f(x, y) \oplus_Q g(y, z)$$

and obvious identities.

Lemma 1.7.12. *If Q is a quantale, then Q-Rel is a locally ordered bicategory under pointwise order. If Q is a Girard quantale, it has two locally ordered bicategory structures as described above.*

Proof: The bicategory $(Q\text{-}\mathsf{Rel}, \otimes, \top_A)$ is given by:

- 0-cells are sets $A, B, C, ...$
- 1-cells are Q-relations $R : A \times B \to Q$
- 2-cells are inclusions $\alpha : R \subseteq R'$
- Composition (bi)functor is defined by:

$$(R \otimes S)(a,c) := \bigvee_{b \in B} R(a,b) \otimes_Q S(b,c)$$
$$(R \subseteq R') \otimes (S \subseteq S') := (R \otimes S)(a,c) \subseteq (R' \otimes S')(a',c')$$

- Identity $\top_A : \mathbb{1} \to \mathcal{B}(A,A)$ where $\top_A : A \to A$ is

$$\top(a,a') = \begin{cases} \top & \text{if } a = a' \\ \mathbf{0} & \text{if } a \neq a' \end{cases}$$

Similarly we define the bicategory $(Q\text{-}\mathsf{Rel}, \oplus, \bot)$ by using the second composition and dual of the identity in 1.7.11. ∎

Chapter 2

Cartesian Bicategories

2.1 Locally Ordered Cartesian Bicategories

In this chapter, we review the notion of cartesian bicategories from [14]. Cartesian bicategories were originally introduced by Carboni and Walters [14] in 1987 for locally ordered bicategories. In their definition, Carboni and Walters characterized a cartesian bicategory by a symmetric monoidal bicategory where each 0-cell has a comonoid structure, and each 1-cell is specified as a colax homomorphism between the corresponding comonoid structures.

Definition 2.1.1. *[14, Definition 1.1]* A *tensor product* in a locally ordered bicategory $(\mathcal{B}, \otimes, I, a, l, r)$ is a pseudofunctor (homomorphism) of bicategories $\boxtimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ together with 0-cell I, called the identity 0-cell, and the following natural isomorphisms for all 0-cells $X, Y, Z \in \mathcal{B}$

$$\begin{aligned}
&\alpha : (X \boxtimes Y) \boxtimes Z \to X \boxtimes (Y \boxtimes Z)\\
&\rho : X \to X \boxtimes I\\
&\lambda : X \to I \boxtimes X\\
&\gamma : X \boxtimes Y \to Y \boxtimes X
\end{aligned}$$

satisfying the following coherence conditions:

1. **Pentagon law:**

$$\begin{tikzcd}
 & (X \boxtimes Y) \boxtimes (Z \boxtimes W) & \\
((X \boxtimes Y) \boxtimes Z) \boxtimes W \arrow[ur, "\alpha"] \arrow[d, "\alpha \boxtimes 1_W"'] & & X \boxtimes (Y \boxtimes (Z \boxtimes W)) \arrow[ul, "\alpha"'] \\
(X \boxtimes (Y \boxtimes Z)) \boxtimes W \arrow[rr, "\alpha"'] & & X \boxtimes ((Y \boxtimes Z) \boxtimes W) \arrow[u, "1_X \boxtimes \alpha"']
\end{tikzcd}$$

2. **Unit law:**

$$\begin{tikzcd}
& X \boxtimes Y \arrow[dl, "\rho\boxtimes 1_Y"'] \arrow[dr, "1_X\boxtimes\lambda"] & \\
(X \boxtimes I) \boxtimes Y \arrow[rr, "\alpha"'] & & X \boxtimes (I \boxtimes Y)
\end{tikzcd}$$

3. **Hexagon law:**

$$\begin{tikzcd}
(X \boxtimes Y) \boxtimes Z \arrow[r, "\alpha"] \arrow[d, "\gamma\boxtimes 1_Z"'] & X \boxtimes (Y \boxtimes Z) \arrow[r, "\gamma"] & (Y \boxtimes Z) \boxtimes X \arrow[d, "\alpha"] \\
(Y \boxtimes X) \boxtimes Z \arrow[r, "\alpha"] & Y \boxtimes (X \boxtimes Z) \arrow[r, "1_Y\boxtimes\gamma"'] & Y \boxtimes (Z \boxtimes X)
\end{tikzcd}$$

4. **Inverse law:**

$$\begin{tikzcd}
& Y \boxtimes X \arrow[dl, "\gamma"'] \arrow[dr, "\gamma"] & \\
X \boxtimes Y \arrow[rr, "1_{X\boxtimes Y}"'] & & X \boxtimes Y
\end{tikzcd}$$

That is, $(\mathcal{B}, \boxtimes, I)$ carries a *symmetric monoidal* structure.

Definition 2.1.2. A *comonoid* (X, Δ_X, t_X) in a monoidal category $(\mathcal{C}, \boxtimes, I)$ is an object X together with two morphisms

$$\Delta_X : X \to X \boxtimes X, \qquad t_X : X \to I.$$

such that the following diagrams commute:

1. **Associator**:

$$\begin{tikzcd}
X \arrow[rr, "\Delta_X"] \arrow[d, "\Delta_X"'] & & X \boxtimes X \arrow[d, "1\boxtimes\Delta_X"] \\
X \boxtimes X \arrow[r, "\Delta_X\boxtimes 1"'] & (X \boxtimes X) \boxtimes X \arrow[r, "\alpha"'] & X \boxtimes (X \boxtimes X)
\end{tikzcd}$$

2. **Unitors**:

$$\begin{tikzcd}
& X \arrow[dl, "\rho"'] \arrow[d, "\Delta_X"] \arrow[dr, "\lambda"] & \\
X \boxtimes I & X \boxtimes X \arrow[l, "1_X\boxtimes t_X"] \arrow[r, "t_X\boxtimes 1_X"'] & I \boxtimes X
\end{tikzcd}$$

3. **Commutativity law**:

$$\begin{tikzcd} & X \arrow[dl, "\Delta_X"'] \arrow[dr, "\Delta_X"] & \\ X \boxtimes X \arrow[rr, "\gamma"'] & & X \boxtimes X \end{tikzcd}$$

Definition 2.1.3. Given two comonoids (X, Δ_X, t_X) and (Y, Δ_Y, t_Y) in a monoidal category $(\mathcal{C}, \boxtimes, I)$, a morphism $r : X \to Y$ is a *comonoid morphism* when the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "r"'] & X \boxtimes X \arrow[d, "r \boxtimes r"] \\ Y \arrow[r, "\Delta_Y"'] & Y \boxtimes Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "r"'] \arrow[dr, "t_X"] & \\ Y \arrow[r, "t_Y"'] & I \end{tikzcd}$$

Definition 2.1.4. A *monoid* $(X, \nabla_X, \epsilon_X)$ in a monoidal category $(\mathcal{C}, \boxtimes, I)$ is a comonoid (X, Δ_X, t_X) in the opposite category $\mathcal{C}^{op}$.

Definition 2.1.5. *[14, Definition 1.2]* A *cartesian* structure on a bicategory $\mathcal{B}$ consists of

- A tensor product on $\mathcal{B}$
- every 0-cell $X \in \mathcal{B}$, carries a cocommutative comonoid structure. That is, there are 1-cells

$$\Delta_X : X \to X \boxtimes X, \qquad t_X : X \to I.$$

Then the above data must satisfy the following axioms:
(U) Each 1-cell $r : X \to Y$ is a colax comonoid homomorphism. That is

$$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X$$

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\leq"] & X \boxtimes X \arrow[d, "r \boxtimes r"] \\ Y \arrow[r, "\Delta_Y"'] & Y \boxtimes Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "r"'] \arrow[dr, "t_X"] \arrow[d, phantom, shift left=4, "\leq"] & \\ Y \arrow[r, "t_Y"'] & I \end{tikzcd}$$

(M) Comultiplication Δ_X and counit t_X have right adjoints Δ_X^* and t_X^*. The only cocommutative comonoid structure on X, with structure 1-cells having right adjoints, is (X, Δ_X, t_X).

Remark 2.1.6. [14, Remark 1.3(i)] If $\mathcal{B}$ is cartesian, then the opposite bicategory $\mathcal{B}^{op}$ is cartesian.

Theorem 2.1.7. *The bicategory* **Rel** *of sets and relations is cartesian.*

Proof:

- Define $\boxtimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ on 0-cells $X \boxtimes Y := X \times Y$
- Define $\boxtimes$ on morphisms by $R \boxtimes S : X \boxtimes X' \to Y \boxtimes Y'$ where $(x, x')(R \boxtimes S)(y, y')$ iff $(x, y) \in R \wedge (x', y') \in S$. That is, $R \boxtimes S$ is the image of $R \times S \subseteq (X \times Y) \times (X' \times Y') \cong (X \times X') \times (Y \times Y')$
- The unit 0-cell is $I = \{*\}$
- Associator $\alpha_{\otimes} : (X \times Y) \times Z \to X \times (Y \times X)$, where
$$\alpha_{\otimes} = \{\big(((x, y), z), (x', (y', z'))\big) \mid x = x' \text{ and } y = y' \text{ and } z = z' \ \}$$
- Right unitor $\rho_{\otimes} : X \to I \times X$, where
$$\rho_{\otimes} = \{\big(x, (*, x')\big) \mid x = x'\}$$
- Left unitor $\lambda_{\otimes} : X \to X \times I$, where
$$\lambda_{\otimes} = \{\big(x, (x', *)\big) \mid x = x'\}$$
- Braiding $\gamma_{\otimes} : X \times Y \to Y \times X$, where
$$\gamma_{\otimes} = \{\big((x, y), (y', x')\big) \mid x = x' \text{ and } y = y'\}$$
- Composition $\otimes$ is functorial with respect to $\subseteq$ on both variables.
- Interchange law, that is for any morphisms $f : X \to Y$, $g : Y \to Z$, $h : U \to W$ and $j : W \to V$ we have:
$$(f \otimes g) \boxtimes (h \otimes j) = (f \boxtimes h) \otimes (g \boxtimes j).$$

  Since $\boxtimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ is a pseudofunctor.
- For every 0-cell X, we have a cocommutative comonoid by defining 1-cells $\Delta_X : X \to X \times X := \{(x, (x, x)) \mid x \in X\}$ and $t_X : X \to I := \{(x, *) \mid x \in X\}$.

- To show the inequalities

  $$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X,$$

  note that the LHS of the first inequality is computed as:

  $r \otimes \Delta_Y = \{(x,(y,y')) \mid \exists y'' \in Y\ (x,y'') \in r \text{ and } (y'',(y,y')) \in \Delta_Y\} = \{(x,(y,y)) \mid (x,y) \in r\}.$

  And the RHS of the first inequality is computed as:

  $$\Delta_X \otimes (r \boxtimes r) = \{(x,(y,y')) \mid \exists (x',x'') \in X \times X\ (x,(x',x'')) \in \Delta_X \text{ and } ((x',x''),(y,y')) \in r \boxtimes r\}$$
  $$= \{(x,(y,y')) \mid (x,y) \in r \text{ and } (x,y') \in r\}.$$

  Thus, $LHS \subseteq RHS$. For the second inequality, we compute the LHS as:

  $$r \otimes t_Y = \{(x,*) \mid \exists y \in Y\ (x,y) \in r\}$$

  which is obviously subset of $t_X : X \to I := \{(x,*) \mid x \in X\}$.

- Comultiplication Δ_X and counit t_X are maps since they are functions 1.6.4.

∎

More examples of locally ordered cartesian bicategories are provided in [14, 1] as follows:

Example 2.1.8. [14]

(i) The bicategory **Rel**($\mathcal{E}$), of relations over a regular category. For more details see [1, Example 2.1.3].

(ii) The bicategory **Ord**($\mathcal{E}$), whose 0-cells are ordered objects in a regular category, and whose 1-cells are ideals. For more details see [1, Example 2.3.2].

Theorem 2.1.9. *[14, Remark 1.3 (iii)] Let $\mathcal{B}$ be a locally ordered bicategory with tensor product. Then the tensor product is the bicategorical product in $\mathcal{B}$ (see Definition 2.3.1) if and only if every* 0*-cell has a cocomutative comonoid structure (X, Δ_X, t_X) and every* 1*-cell $f : X \to Y$ is a comonoid homomorphism.*

Lemma 2.1.10. *[1, Lemma 2.1.5] If $\mathcal{B}$ is a locally ordered cartesian bicategory then the full sub-bicategory* **Map**($\mathcal{B}$) *in Definition 1.6.3 has finite products.*

The following corollary originally is a component of the definition of cartesian bicategories in [14]. But it can be independently proven as an outcome of the Definition 2.1.5.

Corollary 2.1.11. *[1, Corollary 2.1.6] If $\mathcal{B}$ is a locally ordered cartesian bicategory then the only comonoid structure on X with structure 1-cells having right adjoints is (X, Δ_X, t_X).*

Theorem 2.1.12. *[14, Theorem 1.6] Let $\mathcal{B}$ be a locally ordered bicategory. If $\mathcal{B}$ has a cartesian structure, then*

- **Map**$(\mathcal{B})$ *has finite bicategorical products.*
- *Each hom-category $\mathcal{B}(X, Y)$ has finite products which are denoted by $\wedge$.*
- *For any pair of 1-cells r and s we have the following formula in $\mathcal{B}$:*

$$r \boxtimes s = (p \otimes r \otimes p^*) \wedge (q \otimes s \otimes q^*) \qquad \textit{(p and q are appropriate projections)}$$

Conversely, if a bicategory $\mathcal{B}$ satisfies (i) and (ii) and the formula in (iii) defines a tensor product on $\mathcal{B}$, then $\mathcal{B}$ has a cartesian structure.

Next, we review discreteness axiom for locally ordered cartesian bicategories, now known as the Frobenius axiom [14].

Definition 2.1.13. *[14, Definition 2.1(i)]* A 0-cell X in a locally ordered cartesian bicategory $\mathcal{B}$ is *discrete* when the comultiplication Δ_X satisfies the following equation

$$\Delta_X^* \otimes \Delta_X = (\top_X \boxtimes \Delta_X) \otimes (\Delta_X^* \boxtimes \top_X)$$

Definition 2.1.14. *[14, Definition 2.1(ii)]* A locally ordered cartesian bicategory is called a *bicategory of relations* if every 0-cell is discrete.

Example 2.1.15. [14, Example 2.3(i)] The bicategory **Rel** of sets and relations is a bicategory of relations.

2.2 The Bicategory Q-Rel

While cartesian bicategories were originally developed as an axiomatization of the category of relations, the category Q-Rel, for a given quantale Q, is not in general a cartesian bicategory. The problem lies in the inequalities requiring that every 1-cell must be a lax comonoid homomorphism. However, the following proposition addresses this issue.

Proposition 2.2.1. *If Q is a locale, i.e. a commutative, integral, idempotent quantale, then every 1-cell in Q-Rel is a lax comonoid homomorphism.*

Proof: Define the tensor product $\boxtimes : Q\text{-}\mathsf{Rel} \times Q\text{-}\mathsf{Rel} \to Q\text{-}\mathsf{Rel}$ on 0-cells as $X \boxtimes Y := X \times Y$ which is the cartesian product of sets and define the tensor product $\boxtimes$ for two 1-cells $R : X \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} Y, S : X' \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} Y'$ as $R \boxtimes S : X \times X' \to Y \times Y'$, where $(R \boxtimes S)((x, x'), (y, y')) = R(x, y) \otimes_Q S(x', y')$ where $\otimes_Q$ is the quantale multiplication. Now consider these inequalities. First we have the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "R"'] \arrow[dr, phantom, "\leq"] & X \boxtimes X \arrow[d, "R \boxtimes R"] \\
Y \arrow[r, "\Delta_Y"'] & Y \boxtimes Y
\end{tikzcd}$$

For every 0-cell Y, Define Q-relations $\Delta_Y : Y \mathrel{-\!\!\!\bullet\!\!\!\rightarrow} Y \times Y$ by $\Delta_Y(y, (y', y'')) := \begin{cases} \top & \text{if } y = y' = y'' \\ \mathbf{0} & \text{otherwise} \end{cases}$. Now by calculating the left hand side of the diagram, we get:

$$LHS(x, y', y'') = \bigvee_{y \in Y} R(x, y) \otimes_Q \Delta(y, (y', y'')) = \begin{cases} \mathbf{0} & \text{if } y' \neq y'' \\ R(x, y') & \text{if } y' = y'' \end{cases}$$

whereas the right hand side yields:

$$RHS(x, y', y'') = \bigvee_{(x', x'') \in X \boxtimes X} \Delta(x, (x', x'')) \otimes_Q \big(R(x', y') \otimes_Q R(x'', y'')\big) = R(x, y') \otimes_Q R(x, y'')$$

For the inequality $LHS \leq RHS$ we would need $q \leq q^2$, for all $q \in Q$ which does not hold in general. Of course, it does hold in an idempotent quantale.

The second lax comonoid inequality is:

$$\begin{tikzcd}
X \arrow[d, "R"'] \arrow[dr, "t_X"] \arrow[dr, phantom, "\leq", shift right=3ex] & \\
Y \arrow[r, "t_Y"'] & \top
\end{tikzcd}$$

For every 0-cell Y, define $t_Y(y, *) = \top$. Then the left hand side of this inequality is calculated as:

$$LHS(x, *) = \bigvee_{y \in Y} R(x, y) \otimes_Q t(y, *) = \bigvee_{y \in Y} R(x, y) \otimes_Q \top = \bigvee_{y \in Y} R(x, y)$$

The right hand side is of course just $RHS(x,*) = \top$. In general, we won't have the inequality $LHS(x,*) \leq RHS(x,*)$ unless $\top = \mathbf{1}$, i.e. the quantale is integral. $\blacksquare$

This leads us to the following Theorem. While the result is straightforward, it has not been published anywhere, as far as we know.

Definition 2.2.2. *[25, Section III.1.2]* Given a function $f : X \to Y$, one can define $f_\circ : X \mathrel{-\!\!\!\bullet\!\!\!\to} Y$, the *$Q$-valued graph of f*, defined as

$$f_\circ(x,y) := \begin{cases} \top & \text{if } y = f(x), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Proposition 2.2.3. *[25, Proposition 1.2.1] Given a unitale quantale and Q-Rel, the Q-relations with right adjoints are uniquely the Q-valued graphs if and only if Q is lean.*

Theorem 2.2.4. *If Q is a locale, i.e. a commutative, integral, idempotent quantale, then Q-Rel is a cartesian bicategory.*

Proof:

- $(Q\text{-}\mathsf{Rel}, \boxtimes, \otimes, I)$ is a symmetric monoidal category:
  - Define the tensor product $\boxtimes : Q\text{-}\mathsf{Rel} \times Q\text{-}\mathsf{Rel} \to Q\text{-}\mathsf{Rel}$ on 0-cells as $X \boxtimes Y := X \times Y$ which is the cartesian product of sets,
  - Define the tensor product $\boxtimes$ for two 1-cells $R : X \mathrel{-\!\!\!\bullet\!\!\!\to} Y, S : X' \mathrel{-\!\!\!\bullet\!\!\!\to} Y'$ as $R \boxtimes S : X \times X' \to Y \times Y'$, where $(R \boxtimes S)((x,x'),(y,y')) = R(x,y) \otimes_Q S(x',y')$,
  - Identity 0-cell is the singleton set $I = \{*\}$
  - Associator $\alpha : X \times (Y \times Z) \mathrel{-\!\!\!\bullet\!\!\!\to} (X \times Y) \times Z$, where

$$\alpha((x,(y,z)),((x',y'),z')) = \begin{cases} \top & \text{if } x = x' \wedge y = y' \wedge z = z' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

    ,

  - Unitor $\rho : X \mathrel{-\!\!\!\bullet\!\!\!\to} X \times I$, where

$$\rho(x,(x',*)) = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

    ,

  - Braiding $\gamma : X \times Y \mathrel{-\!\!\bullet\!\!\!\rightarrow} Y \times X$, where

$$\gamma((x,y),(y',x')) = \begin{cases} \top & \text{if } x = x' \wedge y = y' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

,

- Each Q-relation is a lax comonoid homomorphism by Proposition 2.2.1.
- Every 0-cell X carries a cocommutative comonoid structure (X, Δ_X, t_X). Define Q-relations $\Delta_X : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} X \times X$ by $\Delta_X(x,(x',x'')) := \begin{cases} \top & \text{if } x = x' = x'' \\ \mathbf{0} & \text{otherwise} \end{cases}$ and $t_X : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} I$ where $t_X(x,*) = \top$.
  - For every 0-cell X we have

$$\Delta_X \otimes (1_X \boxtimes \Delta_X) \otimes \alpha = \Delta_X \otimes (\Delta_X \boxtimes 1_X) : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} (X \times X) \times X$$

where

$$LHS(x_1,((x_2,x_3),x_4)) = \begin{cases} \top & \text{if } x_1 = x_2 = x_3 = x_4 \\ \mathbf{0} & \text{otherwise} \end{cases} = RHS(x_1,((x_2,x_3),x_4))$$

,

  - For every 0-cell X we have

$$\Delta_X \otimes (\top_X \boxtimes t_X) = \rho : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} X \times I$$

where $LHS(x,(x',*)) = \top(x,x') = RHS(x,(x',*))$

  - For every 0-cell X we have

$$\Delta_X \otimes (t_X \boxtimes 1_X) = \rho \otimes \gamma : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} I \times X$$

where $LHS(x,(*,x')) = \top(x,x') = RHS(x,(*,x'))$,

  - For every 0-cell X we have

$$\Delta_X \otimes \gamma = \Delta_X : X \mathrel{-\!\!\bullet\!\!\!\rightarrow} X \times X$$

where

$$LHS(x_1,((x_2,x_3)) = \begin{cases} \top & \text{if } x_1 = x_2 = x_3 \\ \mathbf{0} & \text{otherwise} \end{cases} = RHS(x_1,((x_2,x_3))$$

.

- The Q-relations Δ_X and t_X have right adjoints denoted $\Delta^* : X \times X \mathrel{-\!\!\bullet\!\!\!\rightarrow} X$ and $t^*_X : I \mathrel{-\!\!\bullet\!\!\!\rightarrow} X$, respectively, as their duals, since Q is a lean locale 2.2.3.

∎

Remark 2.2.5. To ensure the symmetry of the tensor product $\boxtimes$ in Q, a commutativity condition is required. Given that ρ is a natural transformation, it satisfies the equation $(R \boxtimes S) \otimes \gamma = \gamma \otimes (R \boxtimes S)$. That is,

$$\begin{aligned}((R \boxtimes S) \otimes \gamma)((x, x'), (y, y')) &= R(x, y) \otimes_Q S(x', y') \\ &= S(x', y') \otimes_Q R(x, y) \qquad \text{(by commutativity of } Q\text{)} \\ &= (\gamma \otimes (R \boxtimes S))((x, x'), (y, y'))\end{aligned}$$

2.3 Cartesian Bicategories

In this section, we review cartesian bicategories in general from [13]. The authors in [13], introduce the notion of precartesian bicategories before defining cartesian bicategories. These precartesian bicategories are taken from the main Theorem 2.1.12 in locally ordered cartesian bicategories. Precartesian bicategories do not require finite bicategorical products throughout the entire bicategory but only within the full sub-bicategory consisting of left adjoints, which they refer to as $\mathbf{Map}(\mathcal{B})$. They demonstrate that a canonical lax tensor product can be constructed in any precartesian bicategory. The bicategory is then termed cartesian if this tensor product is pseudo. The discussion begins with a review of bicategorical products:

Definition 2.3.1. *[1]* Consider a bicategory $\mathcal{B}$ and 0-cells X, Y in $\mathcal{B}$. A 0-cell $X \times Y$ together with projections $p_{X,Y} : X \times Y \to X$ and $r_{X,Y} : X \times Y \to Y$ is the bicategorical product of X and Y in $\mathcal{B}$, if $\forall Z \in \mathcal{B}$, the following functor is an equivalence:

$$\begin{aligned}\Gamma_{Z,X\times Y} : \mathcal{B}(Z, X \times Y) &\to \mathcal{B}(Z, X) \times \mathcal{B}(Z, Y) \\ (f : Z \to X \times Y) &\mapsto (f \otimes p_{X,Y}, f \otimes r_{X,Y}) \\ (\alpha : f \Rightarrow g) &\mapsto (\alpha \otimes 1_{p_{X,Y}}, \alpha \otimes 1_{r_{X,Y}})\end{aligned}$$

Or equivalently, if the functors $\Gamma_{Z,X\times Y}$, are fully faithful and essentially surjective for all $X, Y, Z \in \mathcal{B}$.

Definition 2.3.2. *[1]* Consider a bicategory $\mathcal{B}$. A *terminal* 0-*cell* in $\mathcal{B}$ is a 0-cell I such that $\mathcal{B}(A, I)$ is equivalent to the terminal category $\mathbb{1}$. That is, for each 0-cell $X \in \mathcal{B}$, there is a unique 1-cell $X \to I$.

To explain a little more what a bicategorical product is, $\Gamma_{Z,X\times Y}$ is an equivalence of categories if there exists a functor

$$\begin{aligned}\langle -,-\rangle_{Z,X\times Y} &: \mathcal{B}(Z,X)\times\mathcal{B}(Z,Y)\to\mathcal{B}(Z,X\times Y)\\ &(h:Z\to X, k:Z\to Y)\mapsto\langle h,k\rangle : Z\to X\times Y\\ &(\beta:h\Rightarrow h',\gamma:k\Rightarrow k')\mapsto\langle\beta,\gamma\rangle:\langle h,k\rangle\Rightarrow\langle h',k'\rangle\end{aligned}$$

such that $\Gamma_{Z,X\times Y};\langle -,-\rangle_{Z,X\times Y}\cong 1_{\mathcal{B}(Z,X\times Y)}$ and $\langle -,-\rangle_{Z,X\times Y};\Gamma_{Z,X\times Y}\cong 1_{\mathcal{B}(Z,X)\times\mathcal{B}(Z,Y)}$. Specifically, there are natural isomorphisms with component 2-cells $\mu_f : \langle f\otimes p_{X,Y}, f\otimes r_{X,Y}\rangle\cong f$ in $\mathcal{B}(Z,X\times Y)$ and $\nu_{h,k}=(\nu_h,\nu_k):(\langle h,k\rangle\otimes p_{X,Y},\langle h,k\rangle\otimes r_{X,Y})\cong(h,k)$ in $\mathcal{B}(Z,X)\times\mathcal{B}(Z,Y)$.

Proposition 2.3.3. *Consider a bicategory $(\mathcal{B},\otimes,\top_X)$ with bicategorical products. That is, for any pair of 0-cells $(X,Y)\in\mathcal{B}$, a bicategorical product $X\times Y$ exists and it induces a pseudofunctor:*

$$\begin{aligned}-\times- :&\mathcal{B}\times\mathcal{B}\to\mathcal{B}\\ &(X,Y)\mapsto X\times Y\\ &(f:X\to X', g:Y\to Y')\mapsto(f\times g)=\langle p_{X,Y}\otimes f, r_{X,Y}\otimes g\rangle : X\times Y\to X'\times Y'\\ &(\alpha:f\Rightarrow f',\beta:g\Rightarrow g')\mapsto\alpha\times\beta=\langle 1_{p_{X,Y}}\otimes\alpha, 1_{r_{X,Y}}\otimes\beta\rangle : f\times g\Rightarrow f'\times g'\end{aligned}$$

Proof: Connsider $(X,Y),(X',Y'),(X'',Y'')\in\mathcal{B}\times\mathcal{B}$, then there is natural isomorphism

$$\begin{aligned}&F^2 : \big((-)\times(-)\big)\times\big((-)\times(-)\big);\otimes\Rightarrow\otimes\times\otimes;(-)\times(-)\\ &:(\mathcal{B}\times\mathcal{B})\big((X,Y),(X',Y')\big)\times(\mathcal{B}\times\mathcal{B})\big((X',Y'),(X'',Y'')\big)\to(\mathcal{B}\times\mathcal{B})\big((X,Y),(X'',Y'')\big)\end{aligned}$$

with component invertible 2-cells $F^2_{(f,g),(h,k)} : (f\times g)\otimes(h\times k)\Rightarrow(f\otimes h)\times(g\otimes k) : X\times Y\to X''\times Y''$ for $(f:X\to X', g:Y\to Y'),(h:X'\to X'', k:Y'\to Y'')$, which are defined as follows.

Recall that $\forall h,k:Z\to X\times Y,\Gamma_{Z,X\times Y}(h,k)$ is bijective. In particular, $\Gamma_{X\times Y,X''\times Y''}\big((f\times g)\otimes(h\times k),(f\otimes h)\times(g\otimes k)\big)$ is bijective, that is if there exists a pair of 2-cells:

$$\begin{aligned}\alpha &: (f\times g)\otimes(h\times k)\otimes p_{X'',Y''}\Rightarrow(f\otimes h)\times(g\otimes k)\otimes p_{X'',Y''},\\ \beta &: (f\times g)\otimes(h\times k)\otimes r_{X'',Y''}\Rightarrow(f\otimes h)\times(g\otimes k)\otimes r_{X'',Y''}\end{aligned}$$

then there exists a unique 2-cell $\gamma:(f\times g)\otimes(h\times k)\Rightarrow(f\otimes h)\times(g\otimes k)$ such that

$(\gamma \otimes 1_{p_{X'',Y''}}, \gamma \otimes 1_{r_{X'',Y''}}) = (\alpha, \beta)$. Now, let α be the following invertible 2-cell:

$$\begin{aligned}
(f \times g) \otimes (h \times k) \otimes p_{X'',Y''} &= \langle p_{X,Y} \otimes f, r_{X,Y} \otimes g\rangle \otimes \langle p_{X',Y'} \otimes h, r_{X',Y'} \otimes k\rangle \otimes p_{X'',Y''} \\
&\overset{1\otimes\nu_{p_{X',Y'}}\otimes h}{\cong} \quad \langle p_{X,Y} \otimes f, r_{X,Y} \otimes g\rangle \otimes p_{X',Y'} \otimes h \\
&\overset{\nu_{p_{X,Y}}\otimes f\otimes 1}{\cong} \quad p_{X,Y} \otimes f \otimes h \\
&\overset{\nu^{-1}_{p_{X,Y}}\otimes f\otimes h}{\cong} \quad \langle p_{X,Y} \otimes f \otimes h, r_{X,Y} \otimes g \otimes k\rangle \otimes p_{X'',Y''} \\
&= (f \otimes h) \times (g \otimes k) \otimes p_{X'',Y''}.
\end{aligned}$$

Similarly, let β be the following invertible 2-cell:

$$\begin{aligned}
(f \times g) \otimes (h \times k) \otimes r_{X'',Y''} &= \langle p_{X,Y} \otimes f, r_{X,Y} \otimes g\rangle \otimes \langle p_{X',Y'} \otimes h, r_{X',Y'} \otimes k\rangle \otimes r_{X'',Y''} \\
&\overset{1\otimes\nu_{p_{X',Y'}}\otimes k}{\cong} \quad \langle p_{X,Y} \otimes f, r_{X,Y} \otimes g\rangle \otimes r_{X',Y'} \otimes k \\
&\overset{\nu_{r_{X,Y}}\otimes g\otimes 1}{\cong} \quad r_{X,Y} \otimes g \otimes k \\
&\overset{\nu^{-1}_{r_{X,Y}}\otimes g\otimes k}{\cong} \quad \langle p_{X,Y} \otimes f \otimes h, r_{X,Y} \otimes g \otimes k\rangle \otimes r_{X'',Y''} \\
&= (f \otimes h) \times (g \otimes k) \otimes r_{X'',Y''}.
\end{aligned}$$

So, $F^2_{(f,g),(h,k)} : (f \times g) \otimes (h \times k) \Rightarrow (f \otimes h) \times (g \otimes k)$ is defined as the pre-image of (α, β) under $\Gamma_{X\times Y, X''\times Y''}\big((f \times g) \otimes (h \times k), (f \otimes h) \times (g \otimes k)\big)$.

Moreover, $\forall (X, Y) \in \mathcal{B} \times \mathcal{B}$, there is an invertible 2-cell $F^0_{X,Y} : \top_{X\times Y} \Rightarrow \top_X \times \top_Y$ defined as follows. $\Gamma_{X\times Y, X\times Y}(\top_{X\times Y}, \top_X \times \top_Y)$ is bijective as above, so defined α' and β' respectively as:

$$\top_{X\times Y} \otimes p_{X,Y} \cong p_{X,Y} \cong p_{X,Y} \otimes \top_X \cong \langle p_{X,Y} \otimes \top_X, r_{X,Y} \otimes \top_Y\rangle \otimes p_{X,Y} \cong \top_X \times \top_Y \otimes p_{X,Y};$$

$$\top_{X\times Y} \otimes r_{X,Y} \cong r_{X,Y} \cong r_{X,Y} \otimes \top_Y \cong \langle p_{X,Y} \otimes \top_X, r_{X,Y} \otimes \top_Y\rangle \otimes r_{X,Y} \cong \top_X \times \top_Y \otimes r_{X,Y}.$$

So, $F^0_{X,Y} : \top_{X\times Y} \Rightarrow \top_X \times \top_Y$ is the pre-image of (α', β') under $\Gamma_{X\times Y, X\times Y}(\top_{X\times Y}, \top_X \times \top_Y)$. ∎

Definition 2.3.4. *[13, Definition 3.1]* Consider a bicategory $\mathcal{B}$. We say $\mathcal{B}$ is *pre-cartesian* if

1. The full sub-bicategory $\mathbf{Map}(\mathcal{B})$ has finite bicategorical products.

2. Each hom-category $\mathcal{B}(X,Y)$ has finite products.

Then for any bicategory $\mathcal{B}$ we consider the pseudofunctor:

$$\mathbf{Map}(\mathcal{B})^{op} \times \mathbf{Map}(\mathcal{B}) \xrightarrow{i^{op}\times i} \mathcal{B}^{op} \times \mathcal{B} \xrightarrow{\mathcal{B}(-,-)} \mathbf{CAT}$$

where i is the inclusion functor, and $\mathcal{B}(-,-)$ is the hom-pseudofunctor and **CAT** is the 2-category of categories. The two-sided Grothendieck construction of **G** is the bicategory given by:

- A 0-cell of **G** is a triple of $(X, R : X \to A, A)$, where $R \in \mathcal{B}(X,A)$.
- A 1-cell in **G** is a triple of $(f,\alpha,g) : (X,R,A) \to (Y,S,B)$ where $f : X \to Y, g : A \to B \in \mathbf{Map}(\mathcal{B})$ and $\alpha : R \otimes g \Rightarrow f \otimes S$ is a 2-cell in $\mathcal{B}$. We call this form of 1-cells *primary form.*

  Remark 2.3.5. *Equivalently we can define* 1*-cells of the Grothendieck construction of* **G** *by a triple* $(f,\beta,g) : (X,R,A) \to (Y,S,B)$ *such that the* 2*-cell* $\beta : R \Rightarrow f \otimes S \otimes g^*$ *is the mate of* 2*-cell* $\alpha : R \otimes g \Rightarrow f \otimes S$ *where* $g \dashv g^*$*. We call this form of* 1*-cells secondary form.*

- For given primary form of 1-cells $(\phi,\psi) : (f,\alpha,g),(f',\alpha',g') \in \mathbf{G}$, define a 2-cell in **G** as $(f,\alpha,g) \Rightarrow (f',\alpha',g')$ where $\phi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ are 2-cells in $\mathbf{Map}(\mathcal{B})$ such that the following diagram commutes.

$$\begin{tikzcd}
f \otimes S \arrow[r, Rightarrow, "\phi\otimes 1_S"] & f' \otimes S \\
R \otimes g \arrow[u, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "1_R\otimes\psi"'] & R \otimes g' \arrow[u, Rightarrow, "\alpha'"']
\end{tikzcd} \tag{2.3.1}$$

Remark 2.3.6. Equivalently, For given secondary form of 1-cells $(f,\beta,g),(f',\beta',g') \in \mathbf{G}$, define 2-cell in **G** as $(f,\beta,g) \Rightarrow (f',\beta',g')$ where $\phi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ are 2-cells in $\mathbf{Map}(\mathcal{B})$ by commutativity of the following diagram

$$\begin{tikzcd}
 & f \otimes S \otimes g^* \arrow[dr, Rightarrow, "\phi\otimes 1_S\otimes 1_{g^*}"] & \\
R \arrow[ur, Rightarrow, "\beta"] \arrow[dr, Rightarrow, "\beta'"'] & & f' \otimes S \otimes g^* \\
 & f' \otimes S \otimes g'^* \arrow[ur, Rightarrow, "1_{f'}\otimes 1_S\otimes\psi^*"'] &
\end{tikzcd} \tag{2.3.2}$$

where $\psi^* : g'^* \Rightarrow g^*$ is the mate of $\psi : g \Rightarrow g'$ with respect the adjunctions $g \dashv g^*$ and $g' \dashv g'^*$.

All the details of this construction are available in Street's paper [60] and in Verity's PhD thesis [61]. We also review this construction in more detail in chapter 6 since we intend to extend a cartesian structure on bicategories to linear bicategories.

Proposition 2.3.7. *[13, Theorem 3.9] Let $\mathcal{B}$ be a precartesian bicategory, then the bicategory $\mathbf{G}$ has finite bicategorical products.*

Proposition 2.3.8. *[13, Theorem 3.15] Let $\mathcal{B}$ be a precartesian bicategory, then there is a lax functor $\boxtimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$.*

Definition 2.3.9. *[13, Definition 4.1]* A precartesian bicategory is said to be *cartesian* if the lax functors $\boxtimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$ are pseudofunctors.

Proposition 2.3.10. *[13, Proposition 4.2] If $\mathcal{B}$ is a cartesian bicategory, then the pseudofunctors*

$$\mathcal{B} \times \mathcal{B} \xrightarrow{\boxtimes} \mathcal{B} \xleftarrow{I} \mathbb{1}$$

1. *restrict to $\mathbf{Map}(\mathcal{B})$ gives the right adjoints:*

   $$\Delta \dashv \boxtimes \qquad \textit{and} \qquad ! \dashv I,$$

   where $\Delta : \mathbf{Map}(\mathcal{B}) \to \mathbf{Map}(\mathcal{B}) \times \mathbf{Map}(\mathcal{B})$ is the diagonal pseudofunctor and $! : \mathbf{Map}(\mathcal{B}) \to \mathbb{1}$ is the unique pseudofunctor;

2. *the composites:*

   $$\mathcal{B}(X,Y) \times \mathcal{B}(X,Y) \xrightarrow{\boxtimes} \mathcal{B}(X \boxtimes X, Y \boxtimes Y) \xrightarrow{\mathcal{B}(\delta_X, \delta_Y^*)} \mathcal{B}(X,Y)$$

   $$\mathcal{B} \xleftarrow{\mathcal{B}(t_x, t_Y^*)} \mathcal{B}(I,I) \xleftarrow{\top} \mathbb{1}$$

   provide right adjoints to:

   $$\mathcal{B}(X,Y) \xrightarrow{\Delta} \mathcal{B}(X,Y) \times \mathcal{B}(X,Y)$$

   $$\mathcal{B}(X,Y) \xrightarrow{!} \mathbb{1}.$$

   Moreover, the pseudofunctors $\boxtimes$ and I which satisfy the above conditions are unique. Thus we can define a cartesian bicategory as a bicategory $\mathcal{B}$ with pseudofunctors $\boxtimes$ and I which satisfy conditions 1 and 2.

2.4 Symmetric Monoidal Bicategories

In this section, we briefly review the definition of symmetric monoidal bicategories from [52, 58].

Definition 2.4.1. *[52, Definition 2.3]* A *symmetric monoidal bicategory* consists of a bicategory $\mathbf{M}$ together with the following data:

- ♣ A distinguished 0-cell $1 \in \mathbf{M}$
- ♣ A pseudofunctor (homomorphism) $\otimes = (\otimes, \phi^{\otimes}_{(f,f')}, \phi^{\otimes}_{(a,a')}) : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$
- • Transformations:

$$♣\begin{cases} \alpha & = (\alpha_{abc}, \alpha_{fgh}) : (a \otimes b) \otimes c \to a \otimes (b \otimes c) \\ l & = (l_a, l_f) : 1 \otimes a \to a \\ r & = (r_a, r_f) : a \to a \otimes 1 \end{cases}$$

$$\diamondsuit \begin{cases} \beta = (\beta_{ab}, \beta_{fg}) : a \otimes b \to b \otimes a \end{cases}$$

  which are adjoint equivalences. We also choose adjoint inverses α^*, l^* and r^* and their associated adjunction data, which we will not name.

- ♣ Invertible modifications π, μ, λ and ρ as in the following diagrams:

$$\begin{tikzcd}
 & & (a \otimes b) \otimes (c \otimes d) \arrow[drr, "\alpha"] & & \\
(a \otimes b) \otimes c) \otimes d \arrow[urr, "\alpha"] \arrow[d, "\alpha \otimes I"'] & & \Uparrow \pi & & a \otimes (b \otimes (c \otimes d)) \\
(a \otimes (b \otimes c)) \otimes d \arrow[rrrr, "\alpha"'] & & & & a \otimes ((b \otimes c) \otimes d) \arrow[u, "I \otimes \alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(a \otimes 1) \otimes b \arrow[r, "\alpha"] & a \otimes (1 \otimes b) \arrow[d, "I \otimes l"] \\
a \otimes b \arrow[u, "r \otimes I"] \arrow[r, "I_{a \otimes b}"'] \arrow[ur, phantom, "\Downarrow \mu"] & a \otimes b
\end{tikzcd}$$

$$\begin{tikzcd}
(1 \otimes a) \otimes b \arrow[rr, "l \otimes I"] \arrow[dr, "\alpha"'] & \Downarrow \lambda & a \otimes b \\
 & 1 \otimes (a \otimes b) \arrow[ur, "l"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes b \arrow[rr, "I\otimes r"] \arrow[dr, "r"'] & & a \otimes (b \otimes 1) \\
& (a \otimes b) \otimes 1 \arrow[ur, "\alpha"'] \arrow[u, phantom, "\Downarrow \rho"] &
\end{tikzcd}$$

- ♢ Invertible modifications R and S, as in the following diagrams:

$$\begin{tikzcd}
& a \otimes (b \otimes c) \arrow[r, "\beta"] & (b \otimes c) \otimes a \arrow[dr, "\alpha"] & \\
(a \otimes b) \otimes c \arrow[ur, "\alpha"] \arrow[dr, "\beta\otimes I"'] & & \Downarrow R & b \otimes (c \otimes a) \\
& (b \otimes a) \otimes c \arrow[r, "\alpha"'] & b \otimes (a \otimes c) \arrow[ur, "I\otimes\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (a \otimes b) \otimes c \arrow[r, "\beta"] & c \otimes (a \otimes b) \arrow[dr, "\alpha^*"] & \\
a \otimes (b \otimes c) \arrow[ur, "\alpha^*"] \arrow[dr, "I\otimes\beta"'] & & \Downarrow S & (c \otimes a) \otimes b \\
& a \otimes (c \otimes b) \arrow[r, "\alpha^*"'] & (a \otimes c) \otimes b \arrow[ur, "\beta\otimes I"'] &
\end{tikzcd}$$

- ♡ An invertible modification σ, in the following diagram:

$$\begin{tikzcd}
a \otimes b \arrow[rr, "I_{a\otimes b}"] \arrow[dr, "\beta"'] & & a \otimes b \\
& b \otimes a \arrow[ur, "\beta"'] \arrow[u, phantom, "\Downarrow \sigma"] &
\end{tikzcd}$$

such that the following axioms are satisfied:

- ♣ The equations (SM1), (SM2.i), and (SM2.ii) from appendix A (depicted in Figures A.1 through A.4) are satisfied, using the data $(\mathbf{M}, 1, \otimes, \alpha, l, r, \pi, \mu, \lambda, \rho)$.
- ♢ The equations (SM3.i), (SM3.ii), (SM4), and (SM5) from appendix A (presented in Figures A.7 through A.14) are satisfied, using the above data and also (β, R, S).

- ♡ The equations (SM6.i) and (SM6.ii) from appendix A (depicted in Figures A.15 and A.16) are satisfied, using the above data and σ.
- ♠ Furthermore, equation (SM7) from appendix A (depicted in Figure A.17) is satisfied.

A bicategory equipped with the data ♣, ♢, ♡ but only satisfying axioms ♣, ♢ and ♡ is a *sylleptic monoidal bicategory*. A bicategory equipped with the data ♣ and ♢ and satisfying the axioms ♣ and ♢ is a *braided monoidal bicategory*, and lastly a bicategory equipped only with data ♣ satisfying the axioms ♣ is a *monoidal bicategory*.

Theorem 2.4.2. *[13, Theorem 2.15] Given a bicategory $\mathcal{B}$ with bicategorical product $\times$ and a terminal 0-cell I, it forms a symmetric monoidal bicategory, where $\times$ serves as the tensor product and I acts as the unit object.*

Chapter 3

Linear Bicategories

In this chapter, we recall briefly some standard notions of linear bicategories from [15]. Linear bicategories, as introduced by Cockett, Koslowski and Seely [15] in 2000 present a generalization of ordinary bicategories. Within this framework, two horizontal (1-cell) compositions are established, corresponding to the operations of "tensor" and "par" found in linear logic [21].

3.1 Linear Bicategories

Informally, a linear bicategory having two bicategory structures linked by linear distributors.

Definition 3.1.1. *[15, Definition 2.1]* A *linear bicategory* $\mathcal{B}$ consists of

- A collection of 0-cells: $X, Y, Z, \ldots,$
- For each pair of 0-cells (X, Y), a hom-category $\mathcal{B}(X, Y)$
- Two bicategory structures $(\otimes\,, \top)$ and $(\oplus\,, \bot)$ with
    1. **Associators**:

$$a_\otimes : (\otimes \times 1); \otimes \Longrightarrow (1 \times \otimes); \otimes \qquad a_\oplus : (1 \times \oplus); \oplus \Longrightarrow (\oplus \times 1); \oplus.$$

$$\begin{tikzcd}
\mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \times \mathcal{B}(Z,W) \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] & \mathcal{B}(X,Z) \times \mathcal{B}(Z,W) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "a_\otimes"] \\
\mathcal{B}(X,Y) \times \mathcal{B}(Y,W) \arrow[r, "\otimes"'] & \mathcal{B}(X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}(X,Y)\times\mathcal{B}(Y,Z)\times\mathcal{B}(Z,W) \arrow[r, "\oplus\times 1"] \arrow[d, "1\times\oplus"'] & \mathcal{B}(X,Z)\times\mathcal{B}(Z,W) \arrow[d, "\oplus"] \\
\mathcal{B}(X,Y)\times\mathcal{B}(Y,W) \arrow[r, "\oplus"'] \arrow[ur, Rightarrow, shorten=10mm, "a_\oplus"] & \mathcal{B}(X,W)
\end{tikzcd}$$

2. **Unitors**:

$$l_\otimes : 1 \Longrightarrow (\top_X \times 1);\otimes \qquad\qquad l_\oplus : (\bot_X \times 1);\oplus \Longrightarrow 1$$
$$r_\otimes : 1 \Longrightarrow (1 \times \top_Y);\otimes \qquad\qquad r_\oplus : (1 \times \bot_Y);\oplus \Longrightarrow 1$$

$$\begin{tikzcd}
\mathcal{B}(X,X)\times\mathcal{B}(X,Y) \arrow[dr, "\otimes"', ""{name=L}] & \mathcal{B}(X,Y) \arrow[l, "{(\top_X,1)}"'] \arrow[r, "{(1,\top_Y)}"] \arrow[d, "1", ""{name=M}] & \mathcal{B}(X,Y)\times\mathcal{B}(Y,Y) \arrow[dl, "\otimes", ""'{name=R}] \\
& \mathcal{B}(X,Y) &
\arrow[Rightarrow, from=M, to=L, "u^L_\otimes"', shorten=2mm]
\arrow[Rightarrow, from=M, to=R, "u^R_\otimes", shorten=2mm]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}(X,X)\times\mathcal{B}(X,Y) \arrow[dr, "\oplus"', ""{name=L}] & \mathcal{B}(X,Y) \arrow[l, "{(\bot_X,1)}"'] \arrow[r, "{(1,\bot_Y)}"] \arrow[d, "1", ""{name=M}] & \mathcal{B}(X,Y)\times\mathcal{B}(Y,Y) \arrow[dl, "\oplus", ""'{name=R}] \\
& \mathcal{B}(X,Y) &
\arrow[Rightarrow, from=L, to=M, "u^L_\oplus", shorten=2mm]
\arrow[Rightarrow, from=R, to=M, "u^R_\oplus"', shorten=2mm]
\end{tikzcd}$$

- Natural transformations δ_L and δ_R, called *linear distributivities*

$$\begin{tikzcd}
\mathcal{B}(X,Y)\times\mathcal{B}(Y,Z)\times\mathcal{B}(Z,W) \arrow[r, "Id\times\oplus"] \arrow[d, "\otimes\times Id"'] & \mathcal{B}(X,Y)\times\mathcal{B}(Y,W) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, shorten=10mm, "\delta_L"] \\
\mathcal{B}(X,Z)\times\mathcal{B}(Z,W) \arrow[r, "\oplus"'] & \mathcal{B}(X,W)
\end{tikzcd}$$

and,

$$\begin{tikzcd}
\mathcal{B}(X,Y)\times\mathcal{B}(Y,Z)\times\mathcal{B}(Z,W) \arrow[r, "\oplus\times Id"] \arrow[d, "Id\times\otimes"'] & \mathcal{B}(X,Z)\times\mathcal{B}(Z,W) \arrow[d, "\otimes"] \arrow[dl, Rightarrow, shorten=10mm, "\delta_R"] \\
\mathcal{B}(X,Y)\times\mathcal{B}(Y,W) \arrow[r, "\oplus"'] & \mathcal{B}(X,W)
\end{tikzcd}$$

These must satisfy several coherence conditions:

- The coherence diagrams regarding bicategory $(\mathcal{B}, \otimes)$. See Definition 1.1.1.
- The coherence diagrams regarding bicategory $(\mathcal{B}, \oplus)$. See Definition 1.1.1.
- The coherence diagrams related to the linear distributivity of $\otimes$, $\oplus$

$$\begin{tikzcd}
A \otimes (B \oplus \bot) \arrow[r, Rightarrow, "1 \otimes u^R_\oplus"] \arrow[d, Rightarrow, "\delta_L"'] & A \otimes B \\
(A \otimes B) \oplus \bot \arrow[ur, Rightarrow, "u^R_\oplus"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes (C \oplus D) \arrow[r, Rightarrow, "a_\otimes"] \arrow[dd, Rightarrow, "\delta_L"'] & A \otimes (B \otimes (C \oplus D)) \arrow[d, Rightarrow, "1 \otimes \delta_L"] \\
 & A \otimes ((B \otimes C) \oplus D) \arrow[d, Rightarrow, "\delta_L"] \\
((A \otimes B) \otimes C) \oplus D \arrow[r, Rightarrow, "a_\otimes \oplus 1"'] & (A \otimes (B \otimes C)) \oplus D
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes (B \oplus C)) \otimes D \arrow[rr, Rightarrow, "a_\otimes"] \arrow[d, Rightarrow, "\delta_L \otimes 1"'] & & A \otimes ((B \oplus C) \otimes D) \arrow[d, Rightarrow, "1 \otimes \delta_L"] \\
((A \otimes B) \oplus C) \otimes D \arrow[dr, Rightarrow, "\delta_R"'] & & A \otimes (B \oplus (C \otimes D)) \arrow[dl, Rightarrow, "\delta_L"] \\
 & (A \otimes B) \oplus (C \otimes D) &
\end{tikzcd}$$

Dualities provide the other diagrams

Example 3.1.2. [15, Example 2.3]

(I) The bicategory **Rel** of sets and relations, where the first composition is the usual relational composition and the second composition is its dual. That is, If we have $R\colon X \to Y$ and $S\colon Y \to Z$ then define:

$$(x, z) \in R \otimes S \quad \text{if and only if } \exists y \quad (x, y) \in R \text{ and } (y, z) \in S$$

with the unit $\top_X : X \to X = \{(x, x') \mid x = x'\}$. And the second composition is given by dual of the first compostion:

$$(x, z) \in R \oplus S \quad \text{if and only if } \forall y \in Y \text{ such that } \quad (x, y) \in R \text{ or } (y, z) \in S$$

with the unit $\bot_X : X \to X = \{(x, x') \mid x \neq x'\}$.

(II) Any bicategory $\mathcal{B}$ is a linear bicategory with $\otimes = \oplus$.

3.2 Linear Adjunctions and Closed Linear Bicategories

Definition 3.2.1. *[15, Definition 3.1]* Given a linear bicategory $\mathcal{B}$, a *linear adjunction* $(\tau, \gamma) : A \dashv\!\!\| B : X \to Y$ consists of:

- 1-cells $A : X \to Y$ and $B : Y \to X$
- 2-cells $\tau : \top_X \Rightarrow A \oplus B$ and $\gamma : B \otimes A \Rightarrow \bot_Y$. Then we say A has the *right linear adjoint* B, satisfying the following equations:

$$\begin{aligned} A \xRightarrow{1_A} A = A &\xRightarrow{u^L_\otimes} \top_X \otimes A \\ &\xRightarrow{\tau \otimes 1_A} (A \oplus B) \otimes A \\ &\xRightarrow{\delta_R} A \oplus (B \otimes A) \\ &\xRightarrow{1_A \oplus \gamma} A \oplus \bot_Y \\ &\xRightarrow{u^R_\oplus} A \end{aligned}$$

$$\begin{aligned} B \xRightarrow{1_B} B = B &\xRightarrow{u^R_\otimes} B \otimes \top_X \\ &\xRightarrow{1_B \oplus \tau} B \otimes (A \oplus B) \\ &\xRightarrow{\delta_L} (B \otimes A) \oplus B \\ &\xRightarrow{\gamma \oplus 1_B} \bot_Y \oplus B \\ &\xRightarrow{u^L_\oplus} B \end{aligned}$$

Example 3.2.2. In the linear bicategory **Rel** of sets and relations, each relation $R : X \to Y$ is left linear adjoint to $R^\perp$ where $R^\perp$ is the inverse complement of R defined as follows:

$$(y, x) \in R^\perp \Leftrightarrow (x, y) \notin R.$$

For the unit of the adjunction, assume $(x, x) \in \top_X$, then we have $(x, x) \in R \oplus R^\perp \Leftrightarrow \forall y \in Y,\ (x, y) \in R$ or $(y, x) \in R^\perp \Leftrightarrow \forall y \in Y,\ (x, y) \in R$ or $(x, y) \notin R$. Thus,

$\top_X \leq R \oplus R^{\perp}$. For the counit of the adjunction, assume $(y, y') \in R^{\perp} \otimes R \Leftrightarrow \exists x \in X,\ (x, y) \in R^{\perp}$ and $(x, y') \in R \Leftrightarrow \exists x \in X,\ (y, x) \notin R$ and $(x, y') \in R$ which implies $(y, y') \in \{(y, y') \,|\, y \neq y'\} = \perp_Y$. Since if $y = y'$, then we get $\{(y, y') \mid \exists x \in X,\ (x, y) \notin R \text{ and } (x, y') \in R\} = \emptyset$.

Definition 3.2.3. *[15, Definition 4.1]* A linear adjoint $A \dashv\!\!\| B$, such that A is both left and right linear adjoint of B is called a *cyclic linear adjoint* denoted by $A \|\!\!\rightarrow\!\!\| B$.

Definition 3.2.4. *[15]* Let $\mathcal{B}$ be a linear bicategory, then $\mathbf{CMap}(\mathcal{B})$ is the linear sub-bicategory in which each 1-cell has a cyclic linear adjoint.

Example 3.2.5. In the linear bicategory **Rel** of sets and relations, every 1-cell is a cyclic linear adjoint. That is, **CMap**(**Rel**)= **Rel**. Since any relation $R : X \to Y$ is cyclic linear adjoint to $R^{\perp} := (R^*)^c$ which is the complement inverse of R. Moreover, note that $(R^{\perp})^{\perp} = R$.

Definition 3.2.6. *[15, Proposition 3.10]* A linear bicategory $\mathcal{B}$ in which every 1-cell has both a linear right adjoint and a linear left adjoint is called a *closed linear bicategory.* Closed linear bicategories are essentially $\star$-*autonomous* [3]. That is, we can take tensor composition and negation as primitive and then define the par by de Morgan duality:

$$B \oplus A \cong (A^{\perp} \otimes B^{\perp})^{\perp},$$
$$B \otimes A \cong (A^{\perp} \oplus B^{\perp})^{\perp}.$$

Example 3.2.7. [15] The linear bicategory **Rel** of sets and relations is closed and essentially $\star$-autonomous. The second composition $\oplus$ and its unit by de Morgan duality as follows:

$$\perp := \top^{\perp} = (\top)^{*c} \quad \text{and} \quad R \oplus S := (S^{\perp} \otimes R^{\perp})^{\perp},$$

where $(S^{\perp} \otimes R^{\perp})^{\perp} = (S^{*c} \otimes R^{*c})^{*c}$. We used $(-)^*$ for the inverse of relations and $(-)^c$ for the complement of relations.

Lemma 3.2.8. *[15, Lemma 3.2] Let $A : X \to Y, B : Y \to X$ be 1-cells in a linear bicategory $\mathcal{B}$. The following are equivalent.*

1. *A is left linear adjoint to B.*

2. *For all 0-cells Z, the functor $(-) \otimes A : Hom(Z, X) \to Hom(Z, Y)$ is left adjoint to the functor $(-) \oplus B$.*

3. *For all 0-cells Z, the functor $B \otimes (-) : Hom(X, Z) \to Hom(Y, Z)$ is left adjoint to the functor $A \oplus (-)$.*

Definition 3.2.9. *[15, Lemma 3.3]* Given a linear bicategory $\mathcal{B}$, a pair of adjunctions $(\tau, \gamma) : A \dashv\!\!\| B : X \to Y$ and $(\tau', \gamma') : A' \dashv\!\!\| B' : X' \to Y'$, and 1-cells $C : X' \to X$ and $D : Y' \to Y$, there is a bijection between 2-cells $a : C \otimes A \Longrightarrow A' \oplus D$ and 2-cells $b : B' \otimes C \Longrightarrow D \oplus B$. That is:

$$\mathcal{B}(X', Y)(C \otimes A, A' \oplus D) \cong \mathcal{B}(Y', X)(B' \otimes C, D \oplus B).$$

The 2-cells a and b above are called *linear mates* with respect to the adjunctions $(\tau, \gamma) : A \dashv B : X \to Y$ and $(\tau', \gamma') : A' \dashv B' : X' \to Y'$ and to the 1-cells C, D.

Definition 3.2.10. *[15, Definition 4.9.]* A closed linear bicategory $\mathcal{B}$ is $\star$*-linear* if it satisfies the following conditions:

- For every linear adjoint $v : A \dashv\!\!\| B$ there is a linear adjoint $v^* : B \dashv\!\!\| A$. For any other $v' : A' \dashv\!\!\| B'$ and $a : A \Rightarrow B$, the two constructed mates for a are equal.
- For any linear adjoint $v = (v^*)^*$.
- If $\nu : A \dashv\!\!\| B$ and $w : A' \dashv\!\!\| B'$, then $(v \otimes w)^* = w^* \oplus v^*$ and $(v \oplus w)^* = w^* \otimes v^*$. If $u_R : \top \dashv\!\!\| \bot$ and $u_L : \bot \dashv\!\!\| \top$ are adjunctions, then $u_R^* = u_L$.

3.3 Linear Functors

In this section, we review the notion of homomorphisms between linear bicategories from [15] called linear functors. Linear functors between linear bicategories are a generalization of monoidal functors between $\star$-autonomous categories.

Definition 3.3.1. *[15, Definition 2.4]* Suppose $\mathcal{B}$ and $\mathcal{B}'$ are two linear bicategories. A *linear functor* $F : \mathcal{B} \to \mathcal{B}'$ consists of

- A function $F : \mathcal{B} \to \mathcal{B}'$ on 0-cells.
- For each pair (X, Y) of 0-cells, two functors $F_\otimes, F_\oplus : \mathcal{B}(X, Y) \to \mathcal{B}'(FX, FY)$.
- 2-cells $m_\top : \top_{F_\otimes(X)} \Longrightarrow F_\otimes(\top_X)$. and $n_\bot : F_\oplus(\bot_X) \Longrightarrow \bot_{F_\oplus(X)}$
- Natural transformations which with $m_\top$ and $n_\bot$ make $F_\otimes$ monoidal (lax) with respect to $\otimes$ and $F_\oplus$ comonoidal (colax) with respect to $\oplus$

$$m_\otimes : F_\otimes(A) \otimes F_\otimes(B) \Longrightarrow F_\otimes(A \otimes B)$$
$$n_\oplus : F_\oplus(A \oplus B) \Longrightarrow F_\oplus(A) \oplus F_\oplus(B)$$

- Natural transformations (called *linear strengths*)

$$\begin{aligned}
\nu_{\otimes}^{R} &: F_{\otimes}(A \oplus B) \Longrightarrow F_{\oplus}(A) \oplus F_{\otimes}(B)\\
\nu_{\otimes}^{L} &: F_{\otimes}(A \oplus B) \Longrightarrow F_{\otimes}(A) \oplus F_{\oplus}(B)\\
\nu_{\oplus}^{R} &: F_{\otimes}(A) \otimes F_{\oplus}(B) \Longrightarrow F_{\oplus}(A \otimes B)\\
\nu_{\oplus}^{L} &: F_{\oplus}(A) \otimes F_{\otimes}(B) \Longrightarrow F_{\oplus}(A \otimes B)
\end{aligned}$$

satisfying the following coherence axioms:

[LF.1] (a) $F_{\otimes}(u_{\oplus}^{L}) = \nu_{\otimes}^{R}; (n_{\perp} \oplus 1); u_{\oplus}^{L}$

$$\begin{tikzcd}
F_{\otimes}(\perp \oplus A) \arrow[rr, Rightarrow, "F_{\otimes}(u_{\oplus}^{L})"] \arrow[d, Rightarrow, "\nu_{\otimes}^{R}"'] & & F_{\otimes}(A) \\
F_{\oplus}(\perp) \oplus F_{\otimes}(A) \arrow[rr, Rightarrow, "n_{\perp} \oplus 1"'] & & \perp \oplus F_{\otimes}(A) \arrow[u, Rightarrow, "u_{\oplus}^{L}"']
\end{tikzcd}$$

(b) $F_{\otimes}(u_{\oplus}^{R}) = \nu_{\otimes}^{L}; (1 \oplus n_{\perp}); u_{\oplus}^{R}$

(c) $F_{\oplus}(u_{\otimes}^{L})^{-1} = (u_{\otimes}^{L})^{-1}; (m_{\top} \otimes 1); \nu_{\oplus}^{R}$

(d) $F_{\oplus}(u_{\otimes}^{R})^{-1} = (u_{\otimes}^{R})^{-1}; (m_{\top} \otimes 1); \nu_{\oplus}^{L}$

[LF.2] (a) $F_{\otimes}(a_{\oplus}); \nu_{\otimes}^{R}; (1 \oplus \nu_{\otimes}^{R}) = \nu_{\otimes}^{R}; (n_{\oplus} \oplus 1); a_{\oplus}$

$$\begin{tikzcd}
F_{\otimes}((A \oplus B) \oplus C) \arrow[r, Rightarrow, "F_{\otimes}(a_{\oplus})"] \arrow[d, Rightarrow, "\nu_{\otimes}^{R}"'] & F_{\otimes}(A \oplus (B \oplus C)) \arrow[d, Rightarrow, "\nu_{\otimes}^{R}"] \\
F_{\otimes}(A \oplus B) \oplus F_{\otimes}(C) \arrow[d, Rightarrow, "n_{\oplus} \oplus 1"'] & F_{\otimes}(A) \oplus F_{\otimes}(B \oplus C) \arrow[d, Rightarrow, "1 \oplus \nu_{\otimes}^{R}"] \\
(F_{\oplus}(A) \oplus F_{\oplus}(B)) \oplus F_{\otimes}(C) \arrow[r, Rightarrow, "a_{\oplus}"'] & F_{\otimes}(A) \oplus (F_{\otimes}(B) \oplus F_{\otimes}(C))
\end{tikzcd}$$

(b) $F_{\otimes}(a_{\oplus}); \nu_{\otimes}^{L}; (1 \oplus n_{\oplus}) = \nu_{\oplus}^{L}; (\nu_{\oplus}^{L} \oplus 1); a_{\oplus}$

(c) $F_{\otimes}(a_{\oplus}); \nu_{\otimes}^{L}; (1 \oplus n_{\oplus}) = a_{\otimes}; (1 \otimes \nu_{\oplus}^{R}); (\nu_{\oplus}^{L} \oplus 1);$

(d) $F_{\otimes}(a_{\oplus}); \nu_{\otimes}^{R}; (1 \oplus \nu_{\otimes}^{R}) = \nu_{\otimes}^{R}; (n_{\oplus} \oplus 1); a_{\oplus}$

[LF.3] (a) $F_\otimes(a_\oplus); \nu^R_\otimes; (1 \oplus \nu^R_\otimes) = \nu^L_\otimes; (\nu^R_\otimes \oplus 1); a_\oplus$

$$\begin{tikzcd}
F_\otimes((A \oplus B) \oplus C) \arrow[r, Rightarrow, "F_\otimes(a_\oplus)"] \arrow[d, Rightarrow, "\nu^L_\otimes"'] & F_\otimes(A \oplus (B \oplus C)) \arrow[d, Rightarrow, "\nu^R_\otimes"] \\
F_\otimes(A \oplus B) \oplus F_\oplus(C) \arrow[d, Rightarrow, "\nu^R_\otimes \oplus 1"'] & F_\oplus(A) \oplus F_\otimes(B \oplus C) \arrow[d, Rightarrow, "1 \oplus \nu^R_\otimes"] \\
(F_\oplus(A) \oplus F_\otimes(B)) \oplus F_\oplus(C) \arrow[r, Rightarrow, "a_\oplus"'] & F_\oplus(A) \oplus (F_\otimes(B) \oplus F_\oplus(C))
\end{tikzcd}$$

(b) $(\nu^R_\oplus \otimes 1); \nu^L_\oplus; F_\oplus(a_\otimes) = a_\otimes; (1 \otimes \nu^L_\oplus); \nu^R_\oplus$

[LF.4] (a) $(1 \otimes \nu^R_\otimes); \delta^L; (\nu^R_\oplus \oplus 1) = m_\otimes; F_\otimes(\delta^L); \nu^R_\otimes$

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, Rightarrow, "1 \otimes \nu^R_\otimes"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A) \otimes (F_\oplus(B) \oplus F_\otimes(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A) \otimes F_\oplus(B)) \oplus F_\otimes(C) \arrow[d, Rightarrow, "\nu^R_\otimes \oplus 1"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, Rightarrow, "\nu^R_\otimes"'] & F_\oplus(A \otimes B) \oplus F_\otimes(C)
\end{tikzcd}$$

(b) $(\nu^L_\otimes \otimes 1); \delta^R; (1 \oplus \nu^L_\oplus) = m_\otimes; F_\otimes(\delta^R); \nu^L_\otimes$

(c) $(1 \otimes \nu^L_\otimes); \delta^L; (\nu^L_\oplus \oplus 1) = \nu^L_\oplus; F_\oplus(\delta^L); n^R_\oplus$

(d) $(n_\oplus \otimes 1); \delta^R; (1 \oplus \nu^L_\oplus) = \nu^L_\oplus F_\oplus(\delta^R); n_\oplus$

[LF.5] (a) $(1 \otimes \nu^L_\otimes); \delta^L; (m_\otimes \oplus 1) = m_\otimes; F_\otimes(\delta^L); \nu^L_\otimes$

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, Rightarrow, "1 \otimes \nu^L_\otimes"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A) \otimes (F_\otimes(B) \oplus F_\oplus(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A) \otimes F_\otimes(B)) \oplus F_\oplus(C) \arrow[d, Rightarrow, "m_\otimes \oplus 1"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, Rightarrow, "\nu^L_\otimes"'] & F_\otimes(A \otimes B) \oplus F_\oplus(C)
\end{tikzcd}$$

(b) $(\nu_{\otimes}^{R} \otimes 1); \delta^{R}; (1 \oplus m_{\otimes}) = m_{\otimes}; F_{\otimes}(\delta^{L}); \nu_{\otimes}^{R}$

(c) $(1 \otimes n_{\oplus}); \delta^{L}; (\nu_{\oplus}^{R} \oplus 1) = \nu_{\oplus}^{R}; F_{\oplus}(\delta^{L}); n_{\oplus}$

(d) $(n_{\oplus} \otimes 1); \delta^{R}; (1 \oplus \nu_{\oplus}^{L}) = \nu_{\oplus}^{L}; F_{\oplus}(\delta^{R}); n_{\oplus}$

Remark 3.3.2. *In other words, a linear functor $F : \mathcal{B} \to \mathcal{B}'$ between linear bicategories consists of a pair $(F_{\otimes}, F_{\oplus})$, where $F_{\otimes} : \mathcal{B}_{\otimes} \to \mathcal{B}'_{\otimes}$ is a lax functor with respect to $\otimes$ composition, and $F_{\oplus} : \mathcal{B}_{\oplus} \to \mathcal{B}_{\oplus}$ is a colax functor with respect to $\oplus$ composition. Both components share the same underlying function on 0-cells and satisfy the four linear strengths outlined in 3.3.1.*

Definition 3.3.3. If in the definition of a linear functor $F : \mathcal{B} \to \mathcal{B}'$, $m_{\top}$, $m_{\otimes}$, $n_{\perp}$ and $n_{\oplus}$ are all natural isomorphism, then we call F a *linear pseudofunctor.*

Proposition 3.3.4. *[15] Linear functors preserve linear adjoints. That is, if $F : \mathcal{B} \to \mathcal{C}$ is a linear functor between linear bicategories $\mathcal{B}$ and $\mathcal{C}$, and $(\tau, \gamma) : A \dashv\!\!\| B : X \to Y$ in $\mathcal{B}$, then $F_{\otimes}(A) \dashv\!\!\| F_{\oplus}(B)$ and $F_{\oplus}(A) \dashv\!\!\| F_{\otimes}(B)$*

Proof: The unit and counit of the linear adjunction $(\tau', \gamma') : F_{\otimes}(A) \dashv\!\!\| F_{\oplus}(A)$ are given by:

$$\tau' := \top \xrightarrow{m_{\top}} F_{\otimes}(\top) \xrightarrow{F_{\otimes}(\tau)} F_{\otimes}(A \oplus B) \xrightarrow{\nu_{\otimes}^{L}} F_{\otimes}(A) \oplus F_{\oplus}(B),$$

$$\gamma' := F_{\oplus}(B) \otimes F_{\otimes}(A) \xrightarrow{\nu_{\oplus}^{L}} F_{\oplus}(A \oplus B) \xrightarrow{F_{\oplus}(\gamma)} F_{\oplus}(\perp) \xrightarrow{n_{\perp}} \perp.$$

The unit and counit of the other linear adjunction are given similarly. ∎

Proposition 3.3.5. *Let $\mathcal{B}$ and $\mathcal{C}$ be $\star$-autonomous linear bicategories and $F : \mathcal{B} \to \mathcal{C}$ be a linear functor containing a lax functor $F_{\otimes} : \mathcal{B}_{\otimes} \to \mathcal{C}_{\otimes}$ and a colax functor $F_{\oplus} : \mathcal{B}_{\oplus} \to \mathcal{C}_{\oplus}$. Then the colax component $F_{\oplus}$ is given by $(F_{\otimes}(-)^{\perp})^{\perp}$.*

Proof: Let $F_{\otimes} := F$ and $F_{\oplus} := (F(-)^{\perp})^{\perp}$. Then $F_{\oplus}$ is a colax functor with respect to $\oplus$ and we get a linear strength $\nu_{\otimes}^{R} : F_{\otimes}(A \oplus B) \Longrightarrow F_{\oplus}(A) \oplus F_{\otimes}(B)$ as follows:

$$\begin{aligned}\nu_{\otimes}^{R} : F_{\otimes}(A \oplus B) &\xRightarrow{u_{\otimes}^{L}} \top \otimes F(A \oplus B)\\ &\xRightarrow{\tau\otimes 1} ((F(A)^{\perp})^{\perp} \oplus F(A^{\perp})) \otimes F(A \oplus B)\\ &\xRightarrow{\delta_R} (F(A)^{\perp})^{\perp} \oplus (F(A^{\perp}) \otimes F(A \oplus B))\\ &\xRightarrow{1\oplus m_{\otimes}} (F(A)^{\perp})^{\perp} \oplus (F(A^{\perp} \otimes (A \oplus B)))\\ &\xRightarrow{1\oplus F(\delta_L)} (F(A)^{\perp})^{\perp} \oplus (F(A^{\perp} \otimes A) \oplus B)))\\ &\xRightarrow{1\oplus F(\gamma\oplus 1)} (F(A)^{\perp})^{\perp} \oplus (F(\perp) \oplus B)))\\ &\xRightarrow{1\oplus F(\gamma\oplus 1)} (F(A)^{\perp})^{\perp} \oplus F(B)\\ &= F_{\oplus}(A) \oplus F_{\otimes}(B)\end{aligned}$$

Similarly, we get the other linear strengths. ∎

3.4 Linear Transformations and Modifications

Definition 3.4.1. *[16, Section 5.1]* A *linear natural transformation* $\omega : F \Longrightarrow G$ between two linear functors F and G consists of the following data:

- A lax natural transformation $\omega^{\otimes} : F^{\otimes} \Rightarrow G^{\otimes}$ and an opcolax natural transformation $\omega^{\oplus} : G^{\oplus} \Rightarrow F^{\oplus}$.
- For every 0-cell $X \in \mathcal{B}$, the 1-cells $\omega^{\otimes}X : FX \to GX$ and $\omega^{\oplus}X : GX \to FX$ are cyclic linear adjoints
- For every 1-cell $f : X \to Y$, 2-cells $\omega^{\otimes}f : \omega^{\otimes}X \otimes G^{\otimes}f \Longrightarrow F^{\otimes}f \otimes \omega^{\otimes}Y$ and $\omega^{\oplus}f : \omega^{\oplus}X \oplus F^{\oplus}f \Longrightarrow G^{\oplus}f \oplus \omega^{\oplus}Y$

which satisfy several coherence axioms. See Appendix B.

The following proposition is a generalization Proposition 4 in [18] for linear bicategories.

Proposition 3.4.2. *Locally ordered linear bicategories, linear functors, and linear transformations form a 2-category, which we denote by* **LiBicat**.

Proof: We define the horizontal composition of 1-cells as

$$\begin{aligned}
&\mathbf{;} : \mathbf{LiBicat}(\mathcal{B},\mathcal{C}) \times \mathbf{LiBicat}(\mathcal{C},\mathcal{D}) \to \mathbf{LiBicat}(\mathcal{B},\mathcal{D}) \\
&(F;G)(X) := G(F(X)) \qquad \text{(for any 0-cell } X\text{)} \\
&(F;G)_\otimes(A) := G_\otimes(F_\otimes(A)) \quad \text{and} \\
&(F;G)_\oplus(A) := G_\oplus(F_\oplus(A)) \quad \text{(for any 1-cell } A\text{)}
\end{aligned} \tag{3.4.1}$$

- Its 0-cells are linear bicategories.
- For two linear bicategories $\mathcal{B}$ and $\mathcal{C}$, a hom-category $\mathbf{LiBicat}(\mathcal{B},\mathcal{C})$ of 1-cells as linear functors and 2-cells as linear transformations.
- An identity 1-cell is an identity linear functor $1_{\mathcal{B}} \in \mathbf{LiBicat}(\mathcal{B},\mathcal{B})$, for each linear bicategory $\mathcal{B}$.
- An identity 2-cell is an identity linear transformation $1_F \in \mathbf{LiBicat}(\mathcal{B},\mathcal{C})(F,F)$, for each 1-cell $F \in \mathbf{LiBicat}(\mathcal{B},\mathcal{C})$ and each pair of linear bicategory $\mathcal{B},\mathcal{C}$.
- First, we show that $F;G$ is a linear functor. Let $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be linear functors between linear bicategories $\mathcal{B},\mathcal{C}$ and $\mathcal{D}$.

$$\begin{aligned}
{}^{(F;G)}m_\otimes : (F;G)_\otimes(A) \otimes (F;G)_\otimes(B) &= G_\otimes(F_\otimes(A)) \otimes G_\otimes(F_\otimes(B)) \\
&\overset{m_\otimes}{\Longrightarrow} G_\otimes(F_\otimes(A) \otimes F_\otimes(B)) \\
&\overset{m_\otimes}{\Longrightarrow} G_\otimes(F_\otimes(A \otimes B)) \\
&= (F;G)_\otimes(A \otimes B)
\end{aligned}$$

and similarly

$$\begin{aligned}
{}^{(F;G)}n_\oplus : (F;G)_\oplus(A \oplus B) &= G_\oplus(F_\oplus(A \oplus B)) \\
&\overset{n_\oplus}{\Longrightarrow} G_\oplus(F_\oplus(A) \oplus F_\oplus(B)) \\
&\overset{n_\oplus}{\Longrightarrow} G_\oplus(F_\oplus(A)) \oplus G_\oplus(F_\oplus(B)) \\
&= (F;G)_\oplus(A) \otimes (F;G)_\oplus(B)
\end{aligned}$$

Moreover,

$$\begin{aligned}
{}^{(F;G)}m_\top : \top_{(F;G)_\otimes(X)} &= \top_{(G_\otimes(F_\otimes(X))} \\
&\overset{m_\top}{\Longrightarrow} G_\otimes(\top_{F_\otimes(X)}) \\
&\overset{m_\top}{\Longrightarrow} G_\otimes(F_\otimes(\top_X)) \\
&= (F;G)_\otimes(\top_X)
\end{aligned}$$

Similarly,

$$\begin{aligned} {}^{(F;G)}n_\perp : (F;G)_\oplus(\perp_X) &\xRightarrow{n_\perp} G_\oplus(F_\oplus(\perp_X)) \\ &\xRightarrow{n_\perp} G_\oplus(\perp_{F_\oplus(X)}) \\ &\xRightarrow{n_\perp} \perp_{(G_\oplus(F_\oplus(X))} \\ &= \perp_{(F;G)_\oplus(X)} \end{aligned}$$

It remains to show that $F;G$ satisfies the linear strengths.

$$\begin{aligned} {}^{(F;G)}\nu_\otimes^R : (F;G)_\otimes(A \oplus B) &= G_\otimes(F_\otimes(A \oplus B)) \\ &\xRightarrow{\nu_\otimes^R} G_\otimes(F_\oplus(A) \oplus F_\otimes(B)) \\ &\xRightarrow{\nu_\otimes^R} G_\oplus(F_\oplus(A)) \oplus G_\otimes(F_\otimes(B)) \\ &= (F;G)_\oplus(A) \oplus (F;G)_\otimes(B) \end{aligned}$$

The linear strength ${}^{(F;G)}\nu_\otimes^L$ follows similarly. Next we get ${}^{(F;G)}\nu_\oplus^R$ as follows:

$$\begin{aligned} {}^{(F;G)}\nu_\otimes^R : (F;G)_\otimes(A) \otimes (F;G)_\oplus(B) &= G_\otimes(F_\otimes(A)) \otimes G_\oplus(F_\oplus(B)) \\ &\xRightarrow{\nu_\oplus^R} G_\oplus(F_\otimes(A) \otimes F_\oplus(B)) \\ &\xRightarrow{\nu_\oplus^R} G_\oplus(F_\oplus(A \otimes B)) \\ &= (F;G)_\oplus(A \otimes B) \end{aligned}$$

And similarly we get the linear strength ${}^{(F;G)}\nu_\oplus^L$. Thus, we conclude $F;G$ is a linear functor.

- **Associator:** The horizontal composition "**;**" of 1-cells is associative, in the sense that for 1-cells $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$ which are linear functors between linear bicategories $\mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$. Then for any 0-cell $X \in \mathcal{B}$ we have

$$\begin{aligned} (F;G);H(X) &= H(F;G(X)) \\ &= H(G(F(X))) \\ &= G;H(F(X)) \\ &= F;(G;H)(X) \end{aligned}$$

  Similarly for any 1-cell $A \in \mathcal{B}(X,Y)$ we have

$$\begin{aligned} ((F;G);H)_\otimes(A) &= H_\otimes((F;G)_\otimes(A)) \\ &= H_\otimes(G_\otimes(F_\otimes(A))) \\ &= (G;H)_\otimes(F_\otimes(A)) \\ &= (F;(G;H))_\otimes(A) \end{aligned}$$

Similarly we get associativity for $((F;G);H)_{\oplus}(A) = (F;(G;H))_{\oplus}(A)$.

- **Unitors:** The identity 1-cell $1_{\mathcal{B}}$ with respect ";" is the identity linear functor $1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$. Then for any linear functor $F : \mathcal{B} \to \mathcal{C}$ we have
$$\begin{gathered}(F;1_{\mathcal{C}})(X) = 1_{\mathcal{C}}(F(X)) = F(X)\\ (F;1_{\mathcal{C}})_{\otimes}(A) = (1_{\mathcal{C}})_{\otimes}(F_{\otimes}(A)) = F_{\otimes}(A)\\ (1_{\mathcal{B}};F)(X) = F(1_{\mathcal{B}}(X)) = F(X)\\ (1_{\mathcal{B}};F)_{\otimes}(A) = F_{\otimes}((1_{\mathcal{B}})_{\otimes}(A)) = F_{\otimes}(A)\end{gathered}$$
Similarly, we get units for $\oplus$.

- The vertical composition of 2-cells: Let $F, G, H : \mathcal{B} \to \mathcal{C}$, $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$. Then we define the vertical composition of α and β as follows:
$$\begin{tikzcd}
\mathcal{B} \arrow[rr, bend left=60, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=60, "H"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\beta"] & & \mathcal{C}
\end{tikzcd}$$
with
$$(\alpha * \beta)_{\otimes} = \alpha_{\otimes} \otimes \beta_{\otimes} \quad \text{and} \quad (\alpha * \beta)_{\oplus} = \beta_{\oplus} \oplus \alpha_{\oplus} \tag{3.4.2}$$

- The horizontal composition of 2-cells: Let $F, G : \mathcal{B} \to \mathcal{C}$ and $H, K : \mathcal{C} \to \mathcal{D}$, $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow K$. Then we define the horizontal composition of α and β as follows:
$$\begin{tikzcd}
\mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{D} \arrow[r, phantom, ":="] & \mathcal{B} \arrow[rr, bend left, "F;H"] \arrow[rr, bend right, "G;K"'] \arrow[rr, phantom, "\alpha;\beta\Downarrow"] & & \mathcal{D}
\end{tikzcd} \tag{3.4.3}$$
with
$$(\alpha;\beta)_{\otimes} = \beta_{\otimes}; K(\alpha_{\otimes}) \quad \text{and} \quad (\alpha;\beta)_{\oplus} = K(\alpha_{\oplus}); \beta_{\oplus} \tag{3.4.4}$$

∎

Definition 3.4.3. *[55]* A linear adjunction $F {-}\!\| G$ between linear functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ is an adjunction in the 2-category **LiBicat** of linear bicategories, linear functors, and linear transformations. In particular, this adjunction contains a $\otimes$-lax adjunction $F_{\otimes} \dashv G_{\otimes}$ and a $\oplus$-colax adjunction $G_{\oplus} \dashv F_{\oplus}$.

3.5 The Linear Bicategory Q-Rel

In this section, we review the construction of the linear bicategory of Q-Rel by using Girard quantale and LD-quantales, as described in [9]. Specifically, if Q is a Girard quantale, then Q-Rel is a linear bicategory. It is furthermore closed in the sense that each 1-cell has a linear adjoint. We also review LD-quantales which are similar to linearly distributive categories. We observe that all Girard quantales are LD, and LD-quantales lead to linear bicategories.

Let Q be a Girard quantale (see Definition 1.7.5). Then we demonstrate that Q-Rel is a linear bicategory. To achieve that we need two notions of composition:

$$\otimes, \oplus \colon \mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \longrightarrow \mathcal{B}(X,Z)$$

Given $f \colon X \nrightarrow Y$ and $g \colon Y \nrightarrow Z$, we define

$$f \otimes g(x,z) = \max_{y\in Y}(f(x,y) \otimes_Q g(y,z)) \qquad\qquad f \oplus g(x,z) = \min_{y\in Y}(f(x,y) \oplus_Q g(y,z))$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases} \qquad\qquad \bot_X(x,x') = \begin{cases} \mathbf{1} & \text{if } x \neq x' \\ \bot & \text{if } x = x' \end{cases}$$

Note that we are using symbols $\otimes_Q$ and $\oplus_Q$ for the multiplication and the compultiplication in Q.

In summary, we have:

Theorem 3.5.1. *[9, Theorem 4.1] The above determines Q-Rel as a linear bicategory.*

Next one can generalize the notion of Girard quantale to what the authors of [9] refer to as LD-quantales.

Definition 3.5.2. *[9, Definition 4.3]* An LD-*quantale* is a complete lattice Q with operations $*$ and $+$ and elements $\top$ and $\bot$ such that

- $(Q, *, \top)$ and $(Q^{op}, +, \bot)$ are quantales.
- $a * (b + c) \leq (a * b) + c$ and $(b + c) * a \leq b + (c * a)$

Suppose $(Q, *, +)$ is an LD-quantale, and $X \nrightarrow Y \nrightarrow Z$ are Q-valued relations. Define

$$R \otimes S(x,z) = \sup_{y\in Y}(R(x,y) * S(y,z)) \qquad R \oplus S(x,z) = \inf_{y\in Y}(R(x,y) + S(y,z)) \quad (3.5.1)$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases} \qquad\qquad \bot_X(x,x') = \begin{cases} \mathbf{1} & \text{if } x \neq x' \\ \bot & \text{if } x = x' \end{cases}$$

Theorem 3.5.3. *[9, Theorem 4.6]* $(Q, *, +)$ *is an* LD*-quantale if and only if* $(Q$-$\mathsf{Rel}, \otimes, \oplus)$*, where* $\otimes$ *and* $\oplus$ *are defined as in 3.5.1, is a linear bicategory.*

Chapter 4

Locally Ordered Cyclic Cartesian Linear Bicategories

4.1 Bi(co)limits in Bicategories

In this section, first, we review definitions of limits in the context of bicategories.

Definition 4.1.1. *[28, Definition 5.1.2]*

1. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a lax functor between bicategories. For a 0-cell L in $\mathcal{B}$, define the category $\mathrm{Cone}^{\mathrm{lax}}(C_L, F) = \mathrm{Bicat}(\mathcal{A}, \mathcal{B})(C_L, F)$ such that $C_L : \mathcal{A} \to \mathcal{B}$ is the constant pseudofunctor at $L \in \mathcal{B}$, with:

   - Objects are lax transformations $C_L \Rightarrow F$.
   - Morphisms are modifications between such lax transformations.
   - The identity morphisms are the identity modifications.

   An object in the pseudo-Cone(C_L, F) is called a *lax cone of* L *over* F.

2. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a pseudofunctor between bicategories. For a 0-cell L in $\mathcal{B}$, define the category pseudo-Cone$(C_L, F) = \mathrm{Bicat}^{\mathrm{ps}}(\mathcal{A}, \mathcal{B})(C_L, F)$ with

   - Objects are pseudo-natural transformations $C_L \Rightarrow F$.
   - Morphisms are pseudo-modifications between such pseudo-natural transformations.
   - The identity morphisms are the identity modifications.

   An object in pseudo-Cone(C_L, F) is called a *pseudo-cone of* L *over* F.

Definition 4.1.2. *[28, Definition 5.1.11]*

1. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a lax functor between bicategories. A *lax bilimit* of F is a pair (L, π) with
   - L is a 0-cell in $\mathcal{B}$ and,
   - $\pi : C_L \Rightarrow F$ is a lax cone of L over F

   such that, for each object X in $\mathcal{B}$, the functor

   $$\mathcal{B}(X, L) \underset{\simeq}{\xrightarrow{\pi_*}} \mathrm{Cone}^{\mathrm{lax}}(C_X, F)$$

   where π_* is an equivalence of categories.
2. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a lax functor between bicategories. A *lax limit* of F is a pair (L, π) as in the previous item such that π_* is an isomorphism of categories.
3. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a pseudofunctor between bicategories. A *pseudo bilimit* of a pseudofunctor F is a pair (L, π) with
   - L a 0-cell in $\mathcal{B}$ and
   - $\pi : C_L \Rightarrow F$ a pseudocone of L over F

   such that, for each object X in $\mathcal{B}$, the functor

   $$\mathcal{B}(X, L) \underset{\simeq}{\xrightarrow{\pi_*}} \text{pseudo-Cone}(C_X, F)$$

   where π_* is an equivalence of categories.
4. A *pseudo limit* of F is a pair (L, π) as in the previous item such that π_* is an isomorphism of categories.

Definition 4.1.3. *[28, Definition 5.2.2]* Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a lax functor between bicategories with Suppose $F^{op}\colon \mathcal{A}^{op} \to \mathcal{B}^{op}$ is the opposite lax functor. Then:

- For each 0-cell $L \in \mathcal{B}$, define the category $\mathrm{Cone}^{\mathrm{op}}(F, C_L) = \mathrm{Bicat}(\mathcal{A}^{op}, \mathcal{B}^{op})(C_L, F^{op})$ such that $C_L : \mathcal{A}^{op} \to \mathcal{B}^{op}$ in the right hand side is the constant pseudofunctor at $L \in \mathcal{B}^{op}$. An object in this category is called an *oplax cone of L under F*.
- A *lax bicolimit* of F is a lax bilimit of F^{op}.
- A *lax colimit* of F is a lax limit of F^{op}.
- If F is a pseudofunctor, then a *pseudo bicolimit* of F is a pseudo bilimit of F^{op}. And *pseudo colimit* of F is a pseudo limit of F^{op}.

Remark 4.1.4. [28, Explanation 5.1.3] As an explanation of the lax cone/pseudo-cone in the above definition, for an object L in $\mathcal{B}$, a lax cone/pseudo-cone $\pi : C_L \Rightarrow F$ is a lax transformation/ pseudotransformation which is determined by a component 1-cell $\pi_A \in \mathcal{B}(L, FA)$ for each object $A \in \mathcal{A}$ such that the following are satisfied:

1. **Lax Naturality Constraints:** For each 1-cell $f \in \mathcal{A}\ (A, A')$, it has a component 2-cell

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "\pi_A"'] & L \arrow[d, "\pi_{A'}"] \\
FA \arrow[r, "Ff"'] \arrow[ur, Rightarrow, shorten=4mm, "\pi_f"] & FA'
\end{tikzcd}$$

in $\mathcal{B}(L, FA')$.

2. **Naturality of π_f:** For each 2-cell $\Theta : f \Rightarrow g$ in $\mathcal{A}\ (A, A')$, the equality

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "\pi_A"'] & L \arrow[d, "\pi_{A'}"] \\
FA \arrow[r, "Ff"'] \arrow[ur, Rightarrow, shorten=4mm, "\pi_f"] & FA'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
L \arrow[rr, "1_L"] \arrow[dd, "\pi_A"'] & & L \arrow[dd, "\pi_{A'}"] \\
& |[alias=G]| {} \arrow[ur, Rightarrow, "\pi_g"] & \\
FA \arrow[rr, bend left, "Fg"] \arrow[rr, bend right, "Ff"'] & \Uparrow F\Theta & FA'
\end{tikzcd}$$

of pasting diagrams holds in $\mathcal{B}(L, FA')$.

3. **Lax Unity:** For each object $A \in \mathcal{A}$, the pasting diagram equality:

$$\begin{tikzcd}
L \arrow[rr, "1_L"] \arrow[dd, "\pi_{A'}"'] & & L \arrow[dd, "\pi_A"] \\
& {} \arrow[ur, Rightarrow, "\pi_{1_A}"] & \\
FA \arrow[rr, bend left, "F1_A"] \arrow[rr, bend right, "1_{FA}"'] & F^0 \Uparrow & FA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
L \arrow[rr, "1_L"] \arrow[dd, "\pi_A"'] \arrow[rrdd, "\pi_f" description] & & L \arrow[dd, "\pi_A"] \\
& {} \arrow[ur, Rightarrow, "r^{-1}"] & \\
FA \arrow[rr, "Ff"'] \arrow[ur, Rightarrow, "l"] & & FA
\end{tikzcd}$$

holds in $\mathcal{B}(L, FA)$.

4. **Lax Naturality:** For 1-cells $f \in \mathcal{A}\ (A, A')$ and $g \in A(A', A'')$, the equality

$$\begin{tikzcd}
L \arrow[rr, "1_L"] \arrow[d, "\pi_{A'}"'] & & L \arrow[d, "\pi_{A''}"] \\
FA \arrow[rr, "F(f\otimes g)"] \arrow[dr, bend right, "Ff"'] \arrow[urr, Rightarrow, shorten=20pt, "\pi_{f\otimes g}"] & & FA'' \\
& FA' \arrow[ur, bend right, "Fg"'] \arrow[u, Rightarrow, shorten=6pt, "F^2"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
L \arrow[r, bend right, "1_L"'] \arrow[rr, bend left=40, "1_L"] \arrow[d, "\pi_A"'] & L \arrow[d, "\pi_{A'}"] \arrow[r, bend right, "1_L"'] \arrow[u, Rightarrow, shorten=4pt, "l"] & L \arrow[d, "\pi_{A''}"] \\
FA \arrow[r, "Ff"'] \arrow[ur, Rightarrow, shorten=12pt, "\pi_f"] & FA \arrow[r, "Fg"'] \arrow[ur, Rightarrow, shorten=12pt, "\pi_g"] & FA''
\end{tikzcd}$$

of pasting diagrams holds in $\mathcal{B}(L, FA'')$.

Remark 4.1.5. [28, Explanation 5.1.4] If F is a pseudofunctor, then a pseudocone $\pi\colon C_L \Rightarrow F$ is a strong transformation, which is as above with each component 2-cell π_f is invertible.

A morphism of lax cones

$$\Gamma : \pi \Rightarrow \phi \in \mathrm{Cone}^{\mathrm{lax}}(C_L, F)$$

is a modification between lax transformations with a component 2-cell

$$\Gamma_A : \pi_A \to \phi_A \quad \text{in} \;\; \mathcal{B}(L, FA),$$

such that for each object $A \in \mathcal{A}$, it satisfies the modification axiom:

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "\pi_A"'] & L \arrow[d, bend right, "\pi_{A'}"'{name=P}] \arrow[d, bend left, "\phi_{A'}"{name=Q}] \arrow[from=P, to=Q, Rightarrow, "\Gamma_{A'}"'] \\
FA \arrow[r, "Ff"'] \arrow[ur, Rightarrow, shorten=12pt, "\pi_f"] & FA'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, bend right, "\pi_A"'{name=P}] \arrow[d, bend left, "\phi_A"{name=Q}] \arrow[from=P, to=Q, Rightarrow, "\Gamma_A"'] & L \arrow[d, "\phi_{A'}"] \\
FA \arrow[r, "Ff"'] \arrow[ur, Rightarrow, shorten=12pt, "\phi_f"] & FA'
\end{tikzcd}$$

for each 1-cell $f \in \mathcal{A}\ (A, A')$. A morphism of pseudocones is a modification between strong transformations, which is described in exactly the same way.

Example 4.1.6. [28, Example 5.3.9] Let the category $\mathcal{C}$ with three objects and two non-identity morphisms be as follows:

$$\begin{tikzcd}
C_1 \arrow[r, "c_1"'] & C_0 & C_2 \arrow[l, "c_2"]
\end{tikzcd}$$

We consider $\mathcal{C}$ as a locally discrete 2-category, That is, a 2-category with no non-identity 2-cells. Suppose $F \colon \mathcal{C} \to \mathcal{A}$ is a 2-functor, which is uniquely determined by:

$$\begin{tikzcd}
FC_1 \arrow[r, "Fc_1"'] & FC_0 & FC_2 \arrow[l, "Fc_2"]
\end{tikzcd}$$

Then a lax cone $\pi \colon C_L \Rightarrow F$ is uniquely determined by

- Component 1-cells $\pi_i in \mathcal{A}\ (L, FC_i)$ for $i \in \{1, 2, 3\}$.
- Two component 2-cells:

$$\begin{tikzcd}
& L \arrow[dl, "\pi_1"', ""{name=A}] \arrow[d, "\pi_0" description, ""{name=B}] \arrow[dr, "\pi_2", ""'{name=C}] & \\
FC_1 \arrow[r, "Fc_1"'] & FC_0 & FC_2 \arrow[l, "Fc_2"]
\arrow[Rightarrow, from=A, to=B, "\pi_{c_1}"']
\arrow[Rightarrow, from=C, to=B, "\pi_{c_2}"]
\end{tikzcd}$$

in $\mathcal{A}\ (L, FC_0)$. A lax bilimit of F, which is also called a *lax bi-pullback*, is such a pair (L, π) such that the two conditions in 4.1.2 are satisfied for each object $X \in \mathcal{A}$.

4.2 Categorical Product in Bicategories

4.2.1 Bicategorical Product

Definition 4.2.1. *[5]* Consider a bicategory $(\mathcal{B}, \otimes, \top_X)$ and 0-cells $X, Y \in \mathcal{B}$. A 0-cell $X \times Y$, together with 1-cells $p_{X,Y} \colon X \times Y \to X$ and $r_{X,Y} \colon X \times Y \to Y$ is *the bicategorical product* of X and Y if the following functor is an equivalence $\forall Z \in \mathcal{B}$:

$$\begin{aligned}
\Gamma_{Z,X \times Y} : \mathcal{B}_{\otimes}(Z, X \times Y) &\to \mathcal{B}_{\otimes}(Z, X) \times \mathcal{B}_{\otimes}(Z, Y) \\
(f : Z \to X \times Y) &\mapsto (f \otimes p_{X,Y}, f \otimes r_{X,Y}) \\
(\alpha : f \Rightarrow g) &\mapsto (\alpha \otimes 1_{p_{X,Y}}, \alpha \otimes 1_{r_{X,Y}})
\end{aligned}$$

Equivalently, if the functor $\Gamma_{Z,X\times Y}$ is fully faithful and essentially surjective. A *terminal* 0*-cell* in $\mathcal{B}_\otimes$ is a 0-cell $\top$ such that $\mathcal{B}_\otimes(X,\top)$ is equivalent to the terminal category $\mathbb{1}$. In other words, for each $X \in \mathcal{B}$, there is a unique 1-cell $X \to \top$.

Proposition 4.2.2. *Consider a bicategory* $(\mathcal{B},\otimes,\top_X)$ *with all bicategorical products. That is, for any pair of* 0*-cells* $(X,Y)\in\mathcal{B}_\otimes$*, the bicategorical product* $X\times Y$ *exists. Then they induce a pseudofunctor:*

$$\begin{aligned} - \times - :&\mathcal{B}_\otimes \times \mathcal{B}_\otimes \to \mathcal{B}_\otimes \\ &(X,Y) \mapsto X\times Y \\ &(f : X \to X', g : Y \to Y') \mapsto (f\times g) = \langle p_{X,Y}\otimes f, r_{X,Y}\otimes g\rangle : X\times Y \to X'\times Y' \\ &(\alpha : f \Rightarrow f', \beta : g \Rightarrow g') \mapsto \alpha\times\beta = \langle 1_{p_{X,Y}}\otimes\alpha, 1_{r_{X,Y}}\otimes\beta\rangle : f\times g \Rightarrow f'\times g' \end{aligned}$$

Proof: See proof of Proposition 2.3.3. ∎

4.3 Linear Bi(co)limits in Linear Bicategories

In this section, we introduce the definition of limits in linear bicategories by adding a proper linear setting to the limits in bicategories. Since linear bicategories are a generalization of bicategories, one of our guiding principles in developing concepts for linear bicategories is that, within a closed linear bicategory, these concepts should be simplified to those found in standard bicategories. This implies that we need to incorporate sufficient duality to handle the two types of composition.

Definition 4.3.1. A *linear pseudo bilimit* of a linear pseudofunctor $F = (F_\otimes, F_\oplus) : \mathcal{A}\ \to \mathcal{B}$ is a pair (L,π) with

- L is a 0-cell in $\mathcal{B}$
- $\pi : C_L \Rightarrow F$ is a *linear pseudocone* of L over F

such that for every $X \in \mathcal{B}$ there are equivalences of categories

$$\mathcal{B}_\otimes(X,L) \xrightarrow[\simeq]{\pi_*^\otimes} \mathrm{Bicat}^{\mathrm{ps}}(\mathcal{A}\ ,\mathcal{B})(C_X^\otimes, F_\otimes).$$

and

$$\mathcal{B}_\oplus(L,X) \xrightarrow[\simeq]{\pi_*^\oplus} \mathrm{Bicat}^{\mathrm{ps}}(\mathcal{A}\ ,\mathcal{B})(F_\oplus, C_X^\oplus).$$

Remark 4.3.2. A linear pseudo-cone $\pi : C_L \Rightarrow F$ of L over F in the above definition, is a linear pseudonatural transformation consists of a pair $(\pi^\otimes : C_L^\otimes \to F^\otimes, \pi^\oplus : F^\oplus \to C_L^\oplus)$ of an invertible lax natural transformation $\pi^\otimes$ and an invertible opcolax natural transformation $\pi^\oplus$ determined by components $\pi_A^\otimes \in \mathcal{B}(L,FA)$ and $\pi_A^\otimes \in \mathcal{B}(FA,L)$ for each 0-cell $A \in \mathcal{A}\ $ such that $\pi_A^\otimes$ and $\pi_A^\oplus$ are cyclic linear adjoints $\pi_A^\otimes \mathrel{\|\!\!\rightarrowtail\!\!\|} \pi_A^\oplus$.

4.4 Categorical Product in Linear Bicategories

4.4.1 Linear Bicategorical Product

Definition 4.4.1. Let $\mathcal{B}$ be a linear bicategory. A *linear bicategorical product* of a pair of 0-cells $(X, Y) \in \mathcal{B}$ is a 0-cell $X \Box Y$ equipped with cyclic linear projections $p_{X,Y} \| \!\!\rightarrowtail\!\! \| q_{X,Y} : X \Box Y \to X$ and $r_{X,Y} \| \!\!\rightarrowtail\!\! \| s_{X,Y} : X \Box Y \to Y$ such that the following functors are equivalences:

$$\begin{aligned}
\Gamma_{Z,X\Box Y} : \mathcal{B}_{\otimes}(Z, X\Box Y) &\to \mathcal{B}_{\otimes}(Z, X) \times \mathcal{B}_{\otimes}(Z, Y) \\
(f : Z \to X\Box Y) &\mapsto (f \otimes p_{X,Y}, f \otimes r_{X,Y}) \\
(\alpha : f \Rightarrow g) &\mapsto (\alpha \otimes 1_{p_{X,Y}}, \alpha \otimes 1_{r_{X,Y}})
\end{aligned}$$

And,

$$\begin{aligned}
\Theta_{X\Box Y,Z} : \mathcal{B}_{\oplus}(X\Box Y, Z) &\to \mathcal{B}_{\oplus}(X, Z) \times \mathcal{B}_{\oplus}(Y, Z) \\
(h : X\Box Y \to Z) &\mapsto (q_{X,Y} \oplus h, s_{X,Y} \oplus h) \\
(\eta : h \Rightarrow k) &\mapsto (1_{q_{X,Y}} \oplus \eta, 1_{s_{X,Y}} \oplus \eta)
\end{aligned}$$

Or equivalently, if functors $\Gamma_{Z,X\Box Y}, \Theta_{X\Box Y,Z}$ are fully faithful and essentially surjective.

Definition 4.4.2. A linear terminal 0-cell in a linear bicategory $\mathcal{B}$ is a 0-cell I such that $\mathcal{B}_{\otimes}(X, I)$ is equivalent to the category $\mathbb{1}_{\otimes}$ and $\mathcal{B}_{\oplus}(I, X)$ is equivalent to the terminal category $\mathbb{1}_{\oplus}$. That is, there are adjunctions $(\eta^{\otimes}, \epsilon^{\otimes}) : I^{\otimes} \dashv !^{\otimes} : \mathcal{B}_{\otimes} \to \mathbb{1}_{\otimes}$ where $!^{\otimes} : \mathbb{1}_{\otimes} \to \mathcal{B}_{\otimes}$ with the unit $\eta^{\otimes}(X) : X \to I$ and $(\eta^{\oplus}, \epsilon^{\oplus}) : !^{\oplus} \dashv I^{\oplus} : \mathbb{1}_{\oplus} \to \mathcal{B}_{\oplus}$ with the counit $\epsilon^{\oplus}(X) : I \to X$ such that $\eta^{\otimes}(X) \| \!\!\rightarrowtail\!\! \| \epsilon^{\oplus}(X)$.

Example 4.4.3. The linear terminal 0-cell in the linear bicategory **Rel** is the empty set $\emptyset$. Since for any 0-cell $X \in \mathbf{Rel}_{\otimes}$, we have a unique 1-cell $\eta^{\otimes}(X) : X \to \emptyset$. Moreover, since **Rel** is a $\star$-autonomous, $\epsilon^{\oplus}_{X} = (\emptyset)^{*c} = \emptyset$. Additionally, we get $\emptyset \| \!\!\rightarrowtail\!\! \| \emptyset$.

4.4.2 Linear Bicategorical Product in Rel

Proposition 4.4.4. *The linear bicategory* **Rel** *of sets and relations has linear bicategorical products.*

Proof: We claim that disjoint union is the linear bicategorical product in **Rel**. Suppose X, Y are two sets in **Rel** and $X \amalg Y := \{(x, 1) | x \in X\} \cup \{(y, 2) | y \in Y\}$ is the disjoint union of X and Y. We define cyclic linear projections $p_{X,Y} \| \!\!\rightarrowtail\!\! \| q_{X,Y} : X \amalg Y \to X$ and $r_{X,Y} \| \!\!\rightarrowtail\!\! \| s_{X,Y} : X \amalg Y \to Y$ as the following:

$$p_{X,Y}(t,i) = \begin{cases} t \in X, & \text{if } i = 1 \\ \text{undefined}, & \text{if } i = 2 \end{cases} = \{((t,1),t), \ |, \ t \in X\}$$

And, $q_{X,Y} = (p^*_{X,Y})^c$. Similarly,

$$r_{X,Y}(t,i) = \begin{cases} \text{undefined}, & \text{if } i = 1 \\ t \in Y, & \text{if } i = 2 \end{cases} = \{((t,2),t), \ |, \ t \in Y\}$$

And, $s_{X,Y} = (r^*_{X,Y})^c$. To get disjoint union as the linear bicategorical product, we must show the following equivalences $\forall Z \in \textsf{Rel}$.

$$\begin{aligned} \Gamma_{Z,X\amalg Y} : \textsf{Rel}(Z, X \amalg Y) &\to \textsf{Rel}(Z,X) \times \textsf{Rel}(Z,Y) \\ (f : Z \to X \amalg Y) &\mapsto (f \otimes p_{X,Y}, f \otimes r_{X,Y}) \\ (\alpha : f \Rightarrow g) &\mapsto (\alpha \otimes 1_{p_{X,Y}}, \alpha \otimes 1_{r_{X,Y}}) \end{aligned} \tag{4.4.1}$$

And,

$$\begin{aligned} \Theta_{X\amalg Y,Z} : \textsf{Rel}(X \amalg Y, Z) &\to \textsf{Rel}(X,Z) \times \textsf{Rel}(Y,Z) \\ (h : X \amalg Y \to Z) &\mapsto (q_{X,Y} \oplus h, s_{X,Y} \oplus h) \\ (\eta : h \Rightarrow k) &\mapsto (1_{q_{X,Y}} \oplus \eta, 1_{s_{X,Y}} \oplus \eta) \end{aligned} \tag{4.4.2}$$

Equivalently, $\Gamma_{Z,X\amalg Y}$ and $\Theta_{X\amalg Y,Z}$ must be fully faithful and essentially surjective. For surjectivity of $\Gamma_{Z,X\amalg Y}$, consider 1-cells $l_X : Z \to X$ and $l_Y : Z \to Y$. Then there exists a 1-cell $l : Z \to X \amalg Y$ such that

$$\begin{tikzcd} & Z \arrow[d, "l"] \arrow[ddl, bend right, "l_Y"'] \arrow[ddr, bend left, "l_X"] & \\ & X \amalg Y \arrow[dl, "r_{X,Y}"] \arrow[dr, "p_{X,Y}"'] & \\ Y & & X \end{tikzcd}$$

$$l \otimes p_{X,Y} = l_X$$
$$l \otimes r_{X,Y} = l_Y$$

where $l : Z \to X \amalg Y$ and

$$(z,(s,j)) \in l \ \text{ iff } \ \begin{cases} (z,s) = (z,x) \in l_X, & \text{if } j = 1 \\ (z,s) = (z,y) \in l_Y, & \text{if } j = 2 \end{cases}$$

Since the disjoint union is the categorical product in $\textsf{Rel}$ under ordinary composition:

$$\begin{aligned}\textsf{Rel}(Z, X \amalg Y) &= \mathcal{P}(Z \times (X \amalg Y))\\ &\cong \mathcal{P}((Z \times X) \amalg (Z \times Y))\\ &\cong \mathcal{P}(Z \times X) \times \mathcal{P}(Z \times Y)\\ &\cong \textsf{Rel}(Z, X) \times \textsf{Rel}(Z, Y)\end{aligned}$$

To show $\Gamma_{Z,X\amalg Y}$ is fully faithful, we must show that the following function is a bijection.

$$\Gamma_{l,m} : \textsf{Rel}(Z, X \amalg Y)(l, m) \to (\textsf{Rel}(Z, X) \times \textsf{Rel}(Z, Y))(\Gamma(l), \Gamma(m))$$

That is, we must show that for every 2-cell $(\beta, \gamma) = (l \otimes p_{X,Y} \Rightarrow m \otimes p_{X,Y}, l \otimes r_{X,Y} \Rightarrow m \otimes r_{X,Y})$ in $\textsf{Rel}(Z, X) \times \textsf{Rel}(Z, Y)$, there exists a unique $\alpha : l \Rightarrow m$ such that

$$\Gamma_{l,m}(\alpha) = (\beta, \gamma).$$

But, $(\beta, \gamma) = (l \otimes p_{X,Y} \Rightarrow m \otimes p_{X,Y}, l \otimes r_{X,Y} \Rightarrow m \otimes r_{X,Y}) = ((l \Rightarrow m) \otimes 1_{p_{X,Y}}, (l \Rightarrow m) \otimes 1_{r_{X,Y}})$. Now let $\alpha : l \Rightarrow m$. Moreover, the uniqueness follows from the fact that 2-cells in **Rel** are inclusions.

Next, we show $\Theta_{X\amalg Y,Z}$ is a surjective and fully faithful functor.

$$\begin{aligned}\Theta_{X\amalg Y,Z} : \textsf{Rel}(X \amalg Y, Z) &\to \textsf{Rel}(X, Z) \times \textsf{Rel}(Y, Z)\\ (h : X \amalg Y \to Z) &\mapsto (q_{X,Y} \oplus h, s_{X,Y} \oplus h)\\ (\eta : h \Rightarrow k) &\mapsto (1_{q_{X,Y}} \oplus \eta, 1_{s_{X,Y}} \oplus \eta)\end{aligned}$$

Specifically, for 1-cells $h_X : X \to Z$ and $h_Y : Y \to Z$, there exists a 1-cell $h : X \amalg Y \to Z$ such that

$$\begin{tikzcd}
 & X \amalg Y \arrow[d, "h"] & \\
 & Z & \\
Y \arrow[uur, bend left, "s_{X,Y}"] \arrow[ur, "h_Y"'] & & X \arrow[uul, bend right, "q_{X,Y}"'] \arrow[ul, "h_X"]
\end{tikzcd}$$

$$q_{X,Y} \oplus h = h_X$$
$$s_{X,Y} \oplus h = h_Y$$

for the equation $q_{X,Y} \oplus h = h_X$, since **Rel** is $\star$-autonomous, equivalently, it suffices to show $(h^{*c} \otimes (q^*_{X,Y})^c)^{*c} = h_X$. But, for $h^{*c} \in \textsf{Rel}_\otimes(Z, X \amalg Y)$,

$$(z, (s, j)) \in h^{*c} \quad \text{iff} \quad \begin{cases}(z, s) \neq (z, x) \in h_X^{*\,c}, & \text{if } j = 1\\ (z, s) \neq (z, y) \in h_Y^{*\,c}, & \text{if } j = 2\end{cases}$$

we have:

$$\begin{aligned}(h^{*c} \otimes (q^{*}_{X,Y})^{c})^{*c} &= (h^{*c} \otimes p_{X,Y})^{*c} \qquad &&(\text{since } q_{X,Y} = (p^{*}_{X,Y})^{c}) \\ &= (h^{*}_{X}{}^{c})^{*c} = h_X. &&(\text{by the equivalence 4.4.1})\end{aligned}$$

Similarly, we get the equation $s_{X,Y} \oplus h = h_Y$. To show $\Theta_{X\amalg Y,Z}$ is fully and faithful, we must show the following function is bijection.

$$\Theta_{h,k} : \mathsf{Rel}(X \amalg Y, Z)(h,k) \to (\mathsf{Rel}(X,Z) \times \mathsf{Rel}(Y,Z))(\Theta(h), \Theta(k))$$

That is, for every 2-cells $(\sigma', \sigma'') = (q_{X,Y} \oplus h \Rightarrow q_{X,Y} \oplus k, s_{X,Y} \oplus h \Rightarrow s_{X,Y} \oplus k)$ in $\mathsf{Rel}(X,Z) \times \mathsf{Rel}(Y,Z)$, there exists a unique 2-cell $\sigma : h \Rightarrow k$ such that

$$\Theta_{h,k}(\sigma) = (\sigma', \sigma'').$$

But, the 2-cell $\sigma' : q_{X,Y} \oplus h \Rightarrow q_{X,Y} \oplus k$ is equivalent to $(h^{*c} \otimes p_{X,Y})^{*c} \Rightarrow (k^{*c} \otimes p_{X,Y})^{*c}$, or equivalently, provides 2-cell $k^{*c} \otimes p_{X,Y} \Rightarrow h^{*c} \otimes p_{X,Y}$ in $\mathsf{Rel}_{\otimes}(Z,Y)$. and similarly, we get $k^{*c} \otimes r_{X,Y} \Rightarrow h^{*c} \otimes r_{X,Y}$ in $\mathsf{Rel}_{\otimes}(Z,X)$. But, then the equivalence 4.4.1 provides a unique 2-cell $k^{*c} \Rightarrow h^{*c}$ in $\mathsf{Rel}_{\otimes}(Z, X \amalg Y)$, or equivalently, a unique 2-cell $h \Rightarrow k$ in $\mathsf{Rel}_{\oplus}(X \amalg Y, Z)$. ∎

Corollary 4.4.5. *In the linear bicategory of* **Rel**, *the linear sub-bicategory* **CMap**(**Rel**) *of cyclic linear adjoints has linear bicategorical products.*

Proof: Since **CMap**(**Rel**)=**Rel**, as demostrated in 3.2.5, and **Rel** has linear bicategorical products. Thus **CMap**(**Rel**) has linear bicategorical products. ∎

4.5 Cyclic Cartesian Linear Bicategories

In this section and the next chapter, We are going to take a slightly unconventional approach, which we want to clarify first. Essentially, we will present two distinct (and not equivalent) definitions of a locally ordered cartesian linear bicategory. We believe both are worth exploring.

The original idea was that the definition of a (locally ordered) cartesian linear bicategory could be formed by modifying the original Carboni-Walters definition 2.1.5 to incorporate linear structure throughout. In more detail, our first idea was that one could take the original definition of cartesian bicategory, add a corresponding structure for the second tensor and replace the adjunctions of axiom (M) with cyclic linear adjoints.

This approach does give a sensible definition, which we refer to as a *cyclic cartesian linear bicategory.* The question, however, is whether this definition is useful. To verify its usefulness, we applied two criteria. First, the linear bicategory **Rel** had to serve as an example. Second, a linear version of Carboni-Walters Theorem in [14], Theorem 2.1.12, needed to hold. While the first condition is met, i.e., **Rel** is indeed an example, the second is not. The linear sub-bicategory **CMap**(**Rel**) does have linear bicategorical products, as demonstrated in 4.4.4, since the linear sub-bicategory **CMap**(**Rel**) is all of **Rel**. However, these products are not derived from the linear tensor product of the monoidal structure.

This prompted us to propose an alternative definition, which we present in Chapter 5.

Here is the first definition we tried:

Definition 4.5.1. A *cyclic linear tensor product* on a locally ordered linear bicategory $\mathcal{B}$ consists of a linear pseudofunctor $\Box : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ together with an identity (unit) 0-cell I. Like any linear functor, $\Box$ provides two pseudofunctors:

$$(\Box_{\otimes} := \boxtimes) : \mathcal{B}_{\otimes} \times \mathcal{B}_{\otimes} \to \mathcal{B}_{\otimes} \quad \text{and} \quad (\Box_{\oplus} := \boxplus) : \mathcal{B}_{\oplus}^{co} \times \mathcal{B}_{\oplus}^{co} \to \mathcal{B}_{\oplus}^{co},$$

where they share same product on 0-cells and they are linked by the linear strengths in 3.3.1. In addition, $(\mathcal{B}, \Box_{\otimes}, \otimes, I)$ forms a symmetric monoidal structure with respect composition $\otimes$ and $(\mathcal{B}, \Box_{\oplus}, \oplus, I)$ forms a symmetric monoidal structure with respect composition $\oplus$. Specifically we have the following data:

- A symmetric monoidal structure on $(\mathcal{B}, \Box_{\otimes}, \otimes, I)$ consists of:
  - A natural isomorphism:

    $$\alpha_{\otimes} : (X \Box_{\otimes} Y) \Box_{\otimes} Z \to X \Box_{\otimes} (Y \Box_{\otimes} X)$$

    called the *associator.*
  - A natural isomorphism

    $$\rho_{\otimes} : X \to (X \Box_{\otimes} I)$$

    called the *left unitor.*
  - A natural isomorphism

    $$\lambda_{\otimes} : X \to (I \Box_{\otimes} X)$$

    called the *right unitor.*

  - A natural isomorphism

$$\gamma_{\otimes} : X\square_{\otimes}Y \to Y\square_{\otimes}X$$

    called the *braiding*.

  satisfying monoidal equations in 2.1.1 with respect to $\otimes$ composition.

- A symmetric monoidal structure on $(\mathcal{B}, \square_{\oplus}, \oplus, I)$ consists of:

  - A natural isomorphism:

$$\alpha_{\oplus} : X\square_{\oplus}(Y\square_{\oplus}Z) \to (X\square_{\oplus}Y)\square_{\oplus}Z$$

    called the *associator*.

  - A natural isomorphism

$$\rho_{\oplus} : (X\square_{\oplus}I) \to X$$

    called the *left unitor*.

  - A natural isomorphism

$$\lambda_{\oplus} : (I\square_{\oplus}X) \to X$$

    called the *right unitor*.

  - A natural isomorphism

$$\gamma_{\oplus} : Y\square_{\oplus}X \to X\square_{\oplus}Y$$

    called the *braiding*.

  satisfying monoidal equations in 2.1.1 with respect to $\oplus$ composition.

such that these two monoidal structures are also connected by cyclic linear adjoints: $\alpha_{\otimes}\|{\mapsto}\!\!\!\dashv\|\alpha_{\oplus}$, $\rho_{\otimes}\|{\mapsto}\!\!\!\dashv\|\rho_{\oplus}$, $\lambda_{\otimes}\|{\mapsto}\!\!\!\dashv\|\lambda_{\oplus}$, and $\gamma_{\otimes}\|{\mapsto}\!\!\!\dashv\|\gamma_{\oplus}$.

Definition 4.5.2. A *cyclic cartesian* structure on a locally ordered linear bicategory $\mathcal{B}$ consists of

(i) A cyclic linear tensor product $\square : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, as described in 4.5.1.

(ii) On every 0-cell $X \in \mathcal{B}$, a cocommutative $\otimes$-comonoid structure $(X, \Delta_X : X \to X\square X, t_X : X \to I)$, and a commutative $\oplus$-monoid structure $(X, \nabla_X : X\square X \to X, \epsilon_X : I \to X)$.

satisfy the following axioms:

(U$_1$) Each 1-cell $r : X \to Y$ is a colax $\otimes$-comonoid homomorphism. That is

$$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X$$

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\leq"] & X\Box X \arrow[d, "r\boxtimes r"] \\
Y \arrow[r, "\Delta_Y"'] & Y\Box Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "r"'] \arrow[dr, "t_X"] \arrow[dr, phantom, "\leq"', near end, shift right=2] & \\
Y \arrow[r, "t_Y"'] & I
\end{tikzcd}$$

(U$_2$) Each 1-cell $r : X \to Y$ is a lax $\oplus$-monoid homomorphism. That is,

$$(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r \qquad \text{and} \qquad \epsilon_Y \leq \epsilon_X \oplus r$$

$$\begin{tikzcd}
Y & Y\Box Y \arrow[l, "\nabla_Y"'] \\
X \arrow[u, "r"] \arrow[ur, phantom, "\geq"] & X\Box X \arrow[l, "\nabla_X"] \arrow[u, "r\boxplus r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y & \\
X \arrow[u, "r"] & I \arrow[l, "\epsilon_X"] \arrow[ul, "\epsilon_Y"'] \arrow[ul, phantom, "\geq", near end, shift left=2]
\end{tikzcd}$$

(M) The comultiplication $\Delta_X \|\!\!\vdash\!\!\!\dashv\!\| \nabla_X$ and the counit $t_X \|\!\!\vdash\!\!\!\dashv\!\| \epsilon_X$ are cyclic linear adjoints. The only cocommutative comonoid structure on X, with structure 1-cells having cyclic linear adjoints, is (X, Δ_X, t_X). And similarly the only commutative monoid structure on X, with structure 1-cells having cyclic linear adjoints, is $(X, \nabla_X, \epsilon_X)$.

In the following theorem, we demonstrate that the linear bicategory Rel of sets and relations is cyclic cartesian.

Theorem 4.5.3. *The linear bicategory of sets and relations* Rel *is cyclic cartesian.*

Proof: Let $R : X \to Y$ and $S : Y \to Z$ are two relations. Then:

- Compositions $R \otimes S := \{(x, z) |\ \exists y \in Y\ \ (x, y) \in R\ \wedge\ (y, z) \in S\}$ and $R \oplus S := \{(x, z) |\ \forall y \in Y\ \ (x, y) \in R\ \vee\ (y, z) \in S\}$.
- $(\mathbf{Rel}, \Box_\otimes, \otimes, I)$ is a symmetric monoidal category:
  - Define $\Box_\otimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ on objects $X \Box_\otimes Y := X \times Y$
  - Define $\Box_\otimes$ on morphisms by $R \Box_\otimes S : X \Box_\otimes X' \to Y \Box_\otimes Y'$ where $(x, x')(R\Box_\otimes S)(y, y')$ iff $(x, y) \in R \wedge (x', y') \in S$. That is, $R\Box_\otimes S$ is the image of $R \times S \subseteq (X \times Y) \times (X' \times Y')$ under the isomorphism

$$(X \times Y) \times (X' \times Y') \cong (X \times X') \times (Y \times Y').$$

  - Unit object is $I = \{*\}$
  - Associator $\alpha_\otimes : (X \times Y) \times Z \to X \times (Y \times Z)$, where

$$\alpha_\otimes = \{\big(((x,y),z),(x',(y',z'))\big) \mid x = x' \text{ and } y = y' \text{ and } z = z' \ \}$$

  - Right unitor $\rho_\otimes : X \to I \times X$, where

$$\rho_\otimes = \{\big(x,(*,x')\big) \mid x = x'\}$$

  - Left unitor $\lambda_\otimes : X \to X \times I$, where

$$\lambda_\otimes = \{\big(x,(x',*)\big) \mid x = x'\}$$

  - Braiding $\gamma_\otimes : X \times Y \to Y \times X$, where

$$\gamma_\otimes = \{\big((x,y),(y',x')\big) \mid x = x' \text{ and } y = y'\}$$

  - Composition $\otimes$ is functorial with respect to $\subseteq$ on both variables.
  - Interchange law, that is for any morphisms $f : X \to Y$, $g : Y \to Z$, $h : U \to W$ and $j : W \to V$ we have:

$$(f \otimes g)\Box_\otimes(h \otimes j) = (f\Box_\otimes h) \otimes (g\Box_\otimes j).$$

    Since $\Box : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ is a linear pseudofunctor.

- $(\mathbf{Rel}, \Box_\oplus, \oplus, I)$ is a symmetric monoidal category
  - Define $\Box_\oplus$ on objects $X\Box_\oplus Y := X \times Y$
  - Define $\Box_\oplus$ on morphisms by $R\Box_\oplus S : X \times X' \to Y \times Y'$ where $(x,x')(R\Box_\oplus S)(y,y')$ iff $(x,y) \in R \vee (x',y') \in S$
  - Unit object is $I = \{*\}$
  - Associator $\alpha_\oplus : X \times (Y \times X) \to (X \times Y) \times Z$, where

$$\alpha_\oplus = (\alpha_\otimes^*)^c = \{\big((x',(y',z')),((x,y),z)\big) \mid x \neq x' \text{ or } y \neq y' \text{ or } \ z \neq z' \ \}$$

  - Right unitor $\rho_\oplus : I \times X \to X$, where

$$\rho_\otimes = (\rho_\otimes^*)^c = \{\big((*,x),x'\big) \mid x \neq x'\}$$

  - Left unitor $\lambda_\oplus : X \times I \to X$, where

$$\lambda_\oplus = (\lambda_\otimes^*)^c = \{\big((x,*),x'\big) \mid x \neq x'\}$$

- Braiding $\gamma_{\oplus} : Y \times X \to X \times Y$, where

  $$\gamma_{\oplus} = \{\big((y,x),(x',y')\big) \mid x \neq x' \text{ or } y \neq y'\}$$

- Composition $\oplus$ is functorial with respect to $\subseteq$ on both variables.
- Interchange law, that is for any morphisms $f : X \to Y$, $g : Y \to Z$, $h : U \to W$ and $j : W \to V$ we have:

  $$(f \oplus g)\Box_{\oplus}(h \oplus j) = (f\Box_{\oplus}h) \oplus (g\Box_{\oplus}j).$$

  Since **Rel** is a closed linear bicategory. And, $\Box : \textbf{Rel} \times \textbf{Rel} \to \textbf{Rel}$ is a linear pseudofunctor.

- The cyclic linear tensor product $\Box : \textbf{Rel} \times \textbf{Rel} \to \textbf{Rel}$ contains components $\Box_{\otimes} := \boxtimes$ and $\Box_{\oplus} := \boxplus = (\boxtimes((\_)^{\perp}))^{\perp}$ provides a linear pseudofunctor.
- For every 0-cell $X \in \mathcal{B}$ we have a cocommutative comonoid with respect to "$\otimes$" and a commutative monoid structure with respect "$\oplus$". That is, arrows $\Delta_X : X \to X \times X := \{(x,(x,x)) \mid x \in X\}$ and $t_X : X \to I := \{(x,*) \mid x \in X\}$ and their duals $\nabla_X : X \times X \to X := \{((x,x'),x'') \mid x \neq x' \text{ or } x \neq x'' \text{ or } x' \neq x''\}$ and $\epsilon_X : I \to X := \emptyset$
- Inequalities

  $$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X$$

  $$(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r \qquad \text{and} \qquad \epsilon_Y \leq \epsilon_X \oplus r$$

  To show $(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r$ by contrapostive we must show if $(x,x',y) \notin \nabla_X \oplus r$ then $(x,x',y) \notin (r \boxplus r) \oplus \nabla_Y$. Then we get

  $$(x,x',y) \notin \nabla_X \oplus r \iff (x,x',y) \in \{(x,x',y) \,|\, (x,y) \notin r \text{ and } (x',y) \notin r \text{ and } x = x'\}$$

  And,

  $$(x,x',y) \notin (r \boxplus r) \oplus \nabla_Y \iff (x,x',y) \in \{(x,x',y) \,|\, (x,y) \notin r \text{ and } (x',y) \notin r\}$$

  So, we get the inequality $(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r$. Now, we consider the second inequality $\epsilon_Y \leq \epsilon_X \oplus r$. By calculating the right hand side of the inequality we get:

  $$RHS(*,y) = \epsilon_X \oplus r = \emptyset \oplus r = \{(*,y) \,|\, \forall x \in X \;\; (x,y) \in r\}$$

  And, in left-hand side we have always $\emptyset$. Thus we always have LHS $\leq$ RHS.

- We have cyclic linear adjoint $\Delta_X \| \!\!\mapsto\!\! \| \nabla_X$ and $t_X \| \!\!\mapsto\!\! \| \epsilon_X$ since $\nabla_X = (\Delta_X)^\perp = ((\Delta_X^*)^c)$ and similarly $\epsilon_X = (t_X)^\perp = (t_X^*)^c$ and conversely.

▮

Remark 4.5.4. Observation in Proposition 4.4.4 and the early discussion in the beginning of Section 4.5, made us to shift our study in another direction, provided in Chapter 5, where we can characterize **Rel** as an example, and have a characterization theorem similar to cartesian bicategories in [14].

Chapter 5

Locally Ordered Cartesian Linear Bicategories

In this chapter, we expand the concept of cartesian bicategories to linear bicategories and define a cartesian structure within a locally ordered linear bicategory. We then establish a fundamental theorem, which serves as the main result of this chapter and forms the foundation for defining cartesian linear bicategories in general in the next chapter. Additionally, we will present numerous examples of locally ordered cartesian linear bicategories.

Remark 5.0.1. In this chapter, $\mathcal{B}$ will denote a locally ordered linear bicategory. That is, each hom-category in $\mathcal{B}$ is a partially ordered set.

5.1 Locally Ordered Cartesian Linear Bicategories

In the following, we present a *linear tensor product* for locally ordered cartesian linear bicategories. This tensor product differs from the one outlined in Definition 4.5.1 in Chapter 4. In the previous definition, cyclic linear adjoints and a linear pseudofunctor were employed to connect two given monoidal structures. However, in the following definition, the tensor product introduces two monoidal structures that are connected by a linear pseudofunctor.

Definition 5.1.1. A *linear tensor product* on a locally ordered linear bicategory $\mathcal{B}$ consists of a linear pseudofunctor $\square : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ together with an identity 0-cell I. Like any linear functor, $\square$ provides two pseudofunctors:

$$(\square_\otimes := \boxtimes) : \mathcal{B}_\otimes \times \mathcal{B}_\otimes \to \mathcal{B}_\otimes \quad \text{and} \quad (\square_\oplus := \boxplus) : \mathcal{B}_\oplus^{co} \times \mathcal{B}_\oplus^{co} \to \mathcal{B}_\oplus^{co},$$

where they share the same products on 0-cells and they are linked by the linear strengths in 3.3.1. In addition, $(\mathcal{B}, \square_\otimes, \otimes, I)$ forms a symmetric monoidal structure with respect to $\otimes$ and $(\mathcal{B}, \Box_\oplus, \oplus, I)$ forms a symmetric monoidal structure with respect to $\oplus$. Specifically we have the following data:

- A symmetric monoidal structure on $(\mathcal{B}, \Box_\otimes, \otimes, I)$ consists of:
  - A natural isomorphism:

    $$\alpha_\otimes : (X \Box_\otimes Y) \Box_\otimes Z \to X \Box_\otimes (Y \Box_\otimes X)$$

    called the *associator*.
  - A natural isomorphism

    $$\rho_\otimes : X \to (X \Box_\otimes I)$$

    called the *left unitor*.
  - A natural isomorphism

    $$\lambda_\otimes : X \to (I \Box_\otimes X)$$

    called the *right unitor*.
  - A natural isomorphism

    $$\gamma_\otimes : X \Box_\otimes Y \to Y \Box_\otimes X$$

    called the *braiding*.

  satisfying monoidal equations in 2.1.1 with respect to $\otimes$ composition.
- A symmetric monoidal structure on $(\mathcal{B}, \Box_\oplus, \oplus, I)$ consists of:
  - A natural isomorphism:

    $$\alpha_\oplus : (X \Box_\oplus Y) \Box_\oplus Z \to X \Box_\oplus (Y \Box_\oplus Z)$$

    called the *associator*.
  - A natural isomorphism

    $$\rho_\oplus : X \to (X \Box_\oplus I)$$

    called the *left unitor*.
  - A natural isomorphism

    $$\lambda_\oplus : X \to (I \Box_\oplus X)$$

    called the *right unitor*.

- A natural isomorphism

$$\gamma_\oplus : X\Box_\oplus Y \to Y\Box_\oplus X$$

  called the *braiding*.

satisfying monoidal equations in 2.1.1 with respect to $\oplus$ composition.

Definition 5.1.2. A *cartesian* structure on a locally ordered linear bicategory $\mathcal{B}$ consists of

(i) A linear tensor product $\Box : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, as described in 5.1.1.

(ii) On every 0-cell $X \in \mathcal{B}$, a cocommutative $\otimes$-comonoid structure $(X, \Delta_X : X \to X\Box X, t_X : X \to I)$, and a commutative $\oplus$-monoid structure $(X, \nabla_X : X\Box X \to X, \epsilon_X : I \to X)$.

satisfy the following axioms:

(U$_1$) Each 1-cell $r : X \to Y$ is a colax $\otimes$-comonoid homomorphism. That is

$$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X$$

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\leq"] & X\Box X \arrow[d, "r\boxtimes r"] \\
Y \arrow[r, "\Delta_Y"'] & Y\Box Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "r"'] \arrow[dr, "t_X"] \arrow[d, phantom, shift left=4ex, "\leq"] & \\
Y \arrow[r, "t_Y"'] & I
\end{tikzcd}$$

(U$_2$) Each 1-cell $r : X \to Y$ is a lax $\oplus$-monoid homomorphism. That is,

$$(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r \qquad \text{and} \qquad \epsilon_Y \leq \epsilon_X \oplus r$$

$$\begin{tikzcd}
Y \arrow[dr, phantom, "\geq"] & Y\Box Y \arrow[l, "\nabla_Y"'] \\
X \arrow[u, "r"] & X\Box X \arrow[l, "\nabla_X"] \arrow[u, "r\boxplus r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, phantom, shift left=4ex, "\geq"] & \\
X \arrow[u, "r"] & I \arrow[l, "\epsilon_X"] \arrow[ul, "\epsilon_Y"']
\end{tikzcd}$$

(M$_1$) The comultiplication Δ_X and the counit t_X have right adjoints with respect to $\otimes$. The only cocommutative comonoid structure on X, with structure 1-cells having right adjoints, is (X, Δ_X, t_X).

(M_2) The multiplication ∇_X and the unit ϵ_X have right adjoints with respect to $\oplus$. The only commutative monoid structure on X, with structure 1-cells having right adjoints, is $(X, \nabla_X, \epsilon_X)$.

The Definition 5.1.2 for locally ordered cartesian bicategories differs from the Definition 4.5.2 for locally ordered cyclic cartesian linear bicategories in two primary aspects. Firstly, they use different tensor products. Secondly, the formulation of axioms "M_1" and "M_2" in the Definition 5.1.2 differs, with the cyclic version utilizing axiom "M". In the cyclic setting, as outlined in the Definition 4.5.2, axiom "M" uses cyclic linear adjoints $\Delta_X \|\!\mapsto\!\| \nabla_X$ and $t_X \|\!\mapsto\!\| \epsilon_X$. However, in the above definition, axioms "M_1" and "M_2" use the adjunctions in bicategories $\mathcal{B}_\otimes$ and $\mathcal{B}_\oplus^{co}$, respectively. Specifically, we have $\Delta_X \dashv_\otimes \Delta_X^*$ and $t_X \dashv_\otimes t_X^*$ in $\mathcal{B}_\otimes$, and $\nabla_X \dashv_\oplus \nabla_X^*$ and $\epsilon_X \dashv_\oplus \epsilon_X^*$ in $\mathcal{B}_\oplus^{co}$.

Remark 5.1.3. The definition of cartesian linear bicategories induces a cartesian structure on a bicategory $\mathcal{B}_\otimes$ and a cartesian structure on a bicategory $\mathcal{B}_\oplus^{co}$ or equivalently a cocartesian structure on $\mathcal{B}_\oplus$, where these two cartesian structures are linked through the linear strengths of the linear tensor product in 3.3.1.

Definition 5.1.4. A 1-cell $f : X \to Y$ in a linear bicategory $(\mathcal{B}, \otimes, \oplus, \top, \bot)$ is called a $\otimes$-map if it has a right adjoint $f^* : Y \to X$ in $\mathcal{B}_\otimes$. We denote set of all $\otimes$-maps by the full sub-bicategory $\mathbf{Map}(\mathcal{B}_\otimes)$.

Definition 5.1.5. A 1-cell $f : X \to Y$ in a linear bicategory $(\mathcal{B}, \otimes, \oplus, \top, \bot)$ is called a $\oplus$-map if it has right adjoint $f^* : Y \to X$ in $\mathcal{B}_\oplus^{co}$. We denote set of all $\oplus$-maps by the full sub-bicategory $\mathbf{Map}(\mathcal{B}_\oplus^{co})$.

Proposition 5.1.6. *In the linear bicategory* $(\mathsf{Rel}^{co}, \otimes, \oplus, \top, \bot)$ *of sets and relations,* $\oplus$*-maps are precisely the complement inverse of functions.*

Proof: Suppose $f : X \to Y$ has a right adjoint $f^* : Y \to X$ in $(\mathsf{Rel}^{co}, \oplus, \bot)$. Then we have:

$$\bot_X \leq f \oplus f^*, \qquad f^* \oplus f \leq \bot_Y.$$

Since Rel is closed, by equations $f \oplus f^* = ((f^*)^{*c} \otimes f^{*c})^c = (f^c \otimes f^{*c})^c$ and $f^* \oplus f = (f^{*c} \otimes (f^*)^{*c})^c = (f^{*c} \otimes f^c)^c$ in Rel we have

$$\bot_X \leq (f^c \otimes f^{*c})^c, \qquad (f^{*c} \otimes f^c)^c \leq \bot_Y$$

Then we have

$$f^c \otimes f^{*c} \leq \top_X, \qquad \top_Y \leq f^{*c} \otimes f^c.$$

But we have the above inequalities when f^{*c} is a function in $\mathsf{Rel}_\otimes$. Thus, f is a complement inverse of a function. ∎

5.2 The Linear Bicategory of Sets and Relations

In this section, we show that the linear bicategory **Rel** of sets and relations is cartesian.

Theorem 5.2.1. *The linear bicategory of sets and relations* **Rel** is cartesian.

Proof:

- Compositions $R \otimes S := \{(x,z) |\ \exists y \in Y\ \ (x,y) \in R\ \wedge\ (y,z) \in S\}$ and $R \oplus S := \{(x,z) |\ \forall y \in Y\ \ (x,y) \in R\ \vee\ (y,z) \in S\}$
- The symmetric monoidal structure of $(\mathbf{Rel}, \Box_\otimes, \otimes, I)$ is exactly same as the given symmetric monoidal structure of $(\mathbf{Rel}, \Box_\otimes, \otimes, I)$ in 4.5.3.
- $(\mathbf{Rel}, \Box_\oplus, \oplus, I)$ is a symmetric monoidal category
  - Define $\Box_\oplus$ on objects $X \Box_\oplus Y := X \times Y$
  - Define $\Box_\oplus$ on morphisms by $R \Box_\oplus S : X \times X' \to Y \times Y'$ where $(x,x')(R \Box_\oplus S)(y,y')$ iff $(x,y) \in R \vee (x',y') \in S$
  - Unit object is $I = \{*\}$
  - Associator $\alpha_\oplus : X \times (Y \times X) \to (X \times Y) \times Z$, where

$$\alpha_\oplus = (\alpha_\otimes^*)^c = \{\big((x',(y',z')),((x,y),z)\big) \mid x \neq x' \text{ or } y \neq y' \text{ or } \ z \neq z'\ \}$$

  - Right unitor $\rho_\oplus : I \times X \to X$, where

$$\rho_\otimes = (\rho_\otimes^*)^c = \{\big((*,x),x'\big) \mid x \neq x'\}$$

  - Left unitor $\lambda_\oplus : X \times I \to X$, where

$$\lambda_\oplus = (\lambda_\otimes^*)^c = \{\big((x,*),x'\big) \mid x \neq x'\}$$

  - Braiding $\gamma_\oplus : Y \times X \to X \times Y$, where

$$\gamma_\oplus = \{\big((y,x),(x',y')\big) \mid x \neq x' \text{ or } y \neq y'\}$$

  - Composition $\oplus$ is functorial with respect to $\subseteq$ on both variables.
  - Interchange law, that is for any morphisms $f : X \to Y$, $g : Y \to Z$, $h : U \to W$ and $j : W \to V$ we have:

$$(f \oplus g) \Box_\oplus (h \oplus j) = (f \Box_\oplus h) \oplus (g \Box_\oplus j).$$

  Since **Rel** is a closed linear bicategory.

- The linear tensor product $\Box : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ contains components $\Box_{\otimes} := \boxtimes$ and $\Box_{\oplus} := \boxplus = (\boxtimes((\_)^{\perp}))^{\perp}$, which provides a linear pseudofunctor.
- The proof for every 0-cell $X \in \mathcal{B}$, we have a cocommutative comonoid with respect to "$\otimes$" and a commutative monoid structure with respect "$\oplus$" is given in 4.5.3.
- The proofs for the inequalities

  $$r \otimes \Delta_Y \leq \Delta_X \otimes (r \boxtimes r) \qquad \text{and} \qquad r \otimes t_Y \leq t_X$$

  $$(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r \qquad \text{and} \qquad \epsilon_Y \leq \epsilon_X \oplus r$$

  are same as the proofs provided in 4.5.3.
- Comultiplication Δ_X and counit t_X are $\otimes$-maps since they are functions under "$\otimes$".
- Multiplication ∇_X and unit ϵ_X are $\oplus$-maps since they are complement inverse of functions under "$\oplus$", see the Proposition 5.1.6.

∎

Remark 5.2.2. (1) Recall the Theorem 2.1.9 from chapter 2. In a bicategory $\mathcal{B}$ with a tensor product, the tensor product is the bicategorical product if and only if every 0-cell has a cocommutative comonoid structure and every 1-cell is a comonoid homomorphism. Moreover, by its dual statement, the tensor product is the bicategorical coproduct if and only if every 0-cell has a commutative monoid structure and every 1-cell is a monoid homomorphism.

(2) The 1-cells

$$\nabla_X^* : X \to X \Box X \qquad\qquad \epsilon_X^* : X \to I$$

which are right adjoints to ∇_X and ϵ_X respectively, provide each 0-cell X with a cocommutative $\otimes$-comonoid structure which satisfy axiom U_2. In fact, $\mathcal{B}_{\oplus}^{coop}$ has a cartesian structure induced from $\mathcal{B}_{\oplus}^{co}$.

Lemma 5.2.3. *If $\mathcal{B}$ is a locally ordered cartesian linear bicategory then* $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$ has finite coproducts.

Proof: Since ∇_X and ϵ_X are $\oplus$-maps, then by the dual of the Fox Theorem for bicategories 5.2.2, $\boxplus$ will be the bicategorical coproduct in $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$ if and only if every $\oplus$-map is a monoid homomorphism. Let $f : X \to Y$ be a $\oplus$-map in $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$.

We know that by axiom (U_2) each 1-cell is a lax $\oplus$-monoid homomorphism. So $f : X \to Y$ and $f^* : Y \to X$ are lax $\oplus$-monoid homomorphisms. That is,

$$(f \boxplus f)^* \oplus \nabla_X \leq \nabla_Y \oplus f^* \qquad \text{and} \qquad \epsilon_X \leq \epsilon_Y \oplus f^*.$$

Then by $\oplus$-adjunction we have:

$$\begin{aligned}(f \boxplus f)^* \oplus \nabla_X \leq \nabla_Y \oplus f^* &\iff \nabla_X \leq (f \boxplus f) \oplus \nabla_Y \oplus f^* \\ &\iff \nabla_X \oplus f \leq (f \boxplus f) \oplus \nabla_Y \\ &\iff \nabla_X \oplus f \leq (f \boxplus f) \oplus \nabla_Y.\end{aligned}$$

and similarly we have:

$$\epsilon_X \leq \epsilon_Y \oplus f^* \iff \epsilon_X \oplus f \leq \epsilon_Y.$$

So, $\nabla_X \oplus f = (f \boxplus f) \oplus \nabla_Y$ and $\epsilon_X \oplus f = \epsilon_Y$, i.e. f is a monoid homomorphism. ∎

The above lemma leads to the main theorem of this chapter, which is the linear version of the Carboni-Walters Theorem presented in [14]. Similar to the second paper on cartesian bicategories [13], in Chapter 6, we will demonstrate how this theorem serves as a definition for cartesian linear bicategories in general.

Theorem 5.2.4. *Let $\mathcal{B}$ be a locally ordered linear bicategory. If $\mathcal{B}$ has a cartesian structure, then*

1. $\mathbf{Map}(\mathcal{B}_\otimes)$ *and* $\mathbf{Map}(\mathcal{B}_\oplus^{co})$ *have finite bicategorical products and coproducts respectively, and they share the same* 0*-cell.*
2. *Hom-categories* $\mathcal{B}_\otimes(X,Y)$ *and* $\mathcal{B}_\oplus^{co}(X,Y)$ *have finite products and coproducts respectively, which are denoted by* $\wedge$ *and* $\vee$ *respectively.*
3. *For any pair of* 1*-cells* F *and* G *we have the following formulas in* $\mathcal{B}$*:*

   $F \boxtimes G = (p \otimes F \otimes p^*) \wedge (q \otimes G \otimes q^*)$ *(p and q are appropriate projections)*

   $F \boxplus G = (i^* \oplus F \oplus i) \vee (j^* \oplus G \oplus j)$ *(i and j are appropriate coprojections)*

   such that $\boxtimes$ *and* $\boxplus$ *provide two components of a linear pseudofunctor where they share the same* 0*-cell in* $\mathbf{Map}(\mathcal{B}_\otimes)$ *and* $\mathbf{Map}(\mathcal{B}_\oplus^{co})$*.*

Conversely, if any locally ordered linear bicategory $\mathcal{B}$ satisfies the above conditions then $\mathcal{B}$ is cartesian.

Proof: For the first item, see [1, Lemma 2.1.5] and Lemma 5.2.3. For hom-categories $\mathcal{B}_{\otimes}(X,Y)$ has finite products, see [14, Theorem 1.6], for hom-categories $\mathcal{B}_{\oplus}^{co}(X,Y)$, let $r,s : X \to Y \in \mathcal{B}_{\oplus}^{co}(X,Y) = \mathcal{B}_{\oplus}(X,Y)^{op}$ be two 1-cells. Then for a 1-cell $r \boxplus s : X\Box X \to Y\Box Y$, we show that $r \leq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y$ and $s \leq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y$. Since each 1-cell is a lax $\oplus$-monoid homomorphism we have $\epsilon_Y \leq \epsilon_X \oplus s$. But then by the right adjoint of ϵ_Y with respect to $\oplus$, we get $\epsilon_X^* \oplus \epsilon_Y \leq s$. Then we have:

$$\begin{aligned}
\nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y &\geq \nabla_X^* \oplus (r \boxplus (\epsilon_X^* \oplus \epsilon_Y)) \oplus \nabla_Y && \text{(by replacing } \epsilon_X^* \oplus \epsilon_Y \leq s\text{)}\\
&= \nabla_X^* \oplus (1 \boxplus \epsilon_X^*) \oplus (r \boxplus 1) \oplus (1 \boxplus \epsilon_Y) \oplus \nabla_Y && \text{(by the interchange law in } \mathcal{B}_{\oplus}^{co}\text{)}\\
&= \rho_X^{-1} \oplus (r \boxplus 1) \oplus \rho_Y && \text{(by naturality of } \rho_{\oplus}\text{)}\\
&= r
\end{aligned}$$

Thus, we get $r \leq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y$. Similarly, we have $s \leq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y$. Therefore, $r \vee s \leq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y$.

For the reverse direction, If $r \leq v$ and $s \leq v$, then we have:

$$v = \bot_X \oplus v \geq \nabla_X^* \oplus \nabla_X \oplus v \geq \nabla_X^* \oplus (v \boxplus v) \oplus \nabla_Y \geq \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y.$$

Now by replacing $v = r \vee s$ we get:

$$r \vee s = \nabla_X^* \oplus (r \boxplus s) \oplus \nabla_Y.$$

Moreover, the initial object in $\mathcal{B}_{\oplus}^{co}(X,Y)$ is given by the following

$$\bot := \bot_{X,Y} = \epsilon_X^* \oplus \epsilon_Y.$$

To prove item (3), suppose $F : X \to A$ and $G : Y \to B$ are 1-cells in $\mathcal{B}$, then we compute $(p \otimes F \otimes p^*) \wedge (q \otimes G \otimes q^*)$ as $F \boxtimes G$ as follows:

$$\begin{aligned}
(p \otimes F \otimes p^*) \wedge (q \otimes G \otimes q^*) &= \Delta_{X\Box Y} \otimes ((p \otimes F \otimes p^*) \boxtimes (q \otimes G \otimes q^*)) \otimes \Delta_{A\Box B}^*\\
&= \Delta_{X\Box Y} \otimes ((p \otimes (F \otimes p^*)) \boxtimes (q \otimes (G \otimes q^*))) \otimes \Delta_{A\Box B}^*\\
&= \Delta_{X\Box Y} \otimes ((p \boxtimes q) \otimes ((F \otimes p^*)) \boxtimes (G \otimes q^*)) \otimes \Delta_{A\Box B}^*\\
&= \Delta_{X\Box Y} \otimes ((p \boxtimes q) \otimes (F \boxtimes G) \otimes (p^* \boxtimes q^*)) \otimes \Delta_{A\Box B}^*\\
&= \Delta_{X\Box Y} \otimes (p \boxtimes q) \otimes (F \boxtimes G) \otimes (p^* \boxtimes q^*) \otimes \Delta_{A\Box B}^*\\
&= F \boxtimes G
\end{aligned}$$

Similarly, we compute $(i^* \oplus F \oplus i) \vee (j^* \oplus G \oplus j)$ as $F \boxplus G$ as follows:

$$\begin{aligned}
(i^* \oplus F \oplus i) \vee (j^* \oplus G \oplus j) &= \nabla^*_{X\Box Y} \oplus ((i^* \oplus F \oplus i) \boxplus (j^* \oplus G \oplus j)) \oplus \nabla_{A\Box B} \\
&= \nabla^*_{X\Box Y} \oplus ((i^* \oplus (F \oplus i)) \boxplus (j^* \oplus (G \oplus j))) \oplus \nabla_{A\Box B} \\
&= \nabla^*_{X\Box Y} \oplus ((i^* \boxplus j^*) \oplus ((F \oplus i) \boxplus (G \oplus j))) \oplus \nabla_{A\Box B} \\
&= \nabla^*_{X\Box Y} \oplus ((i^* \boxplus j^*) \oplus ((F \boxplus G) \oplus (i \boxplus j))) \oplus \nabla_{A\Box B} \\
&= \nabla^*_{X\Box Y} \oplus (i^* \boxplus j^*) \oplus (F \boxplus G) \oplus (i \boxplus j) \oplus \nabla_{A\Box B} \\
&= F \boxplus G
\end{aligned}$$

Conversely, if a locally ordered linear bicategory $\mathcal{B}$ satisfies the above condition, then

- Define $\Box_\otimes, \Box_\oplus : \mathcal{B}_1 \times \mathcal{B}_1 \to \mathcal{B}_1$ by the formulas in (3) on 1-cells and by the shared (co)products in (1) on 0-cells.
- By the first item and Remark 5.2.2(i), we get a cocommutative $\otimes$-comonoid structure on every 0-cell $X \in \mathcal{B}_\otimes$ and a commutative $\oplus$-monoid structure on every 0-cell $X \in \mathcal{B}^{co}_\oplus$.
- For axiom condition (U$_1$), see Theorem 1.6 in [14].
- For axiom (U$_2$), note that by using $\oplus$-adjunction on local unions we get:

  $$\perp_X \oplus (r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus (\nabla^*_X \oplus (r \boxplus r) \oplus \nabla_Y) = \nabla_X \oplus r.$$

  Thus this gives us

  $$(r \boxplus r) \oplus \nabla_Y \leq \nabla_X \oplus r.$$

  Moreover, by using local initials we get:

  $$\perp = \epsilon^*_X \oplus \epsilon_Y \leq r$$

  and applying $\oplus$-adjunction, we get $\epsilon_Y \leq \epsilon_X \oplus r$ and this completes the proof.
- For axiom (M$_1$), 1-cells Δ_X and t_X have right adjoints since they are $\otimes$-comonoid homomorphisms. And the proof for the uniqueness of cocommutative $\otimes$-comonoid structure on X with 1-cells having right adjoints is (X, Δ_X, t_X) can be found in Corollary 2.1.6 in [1].
- For axiom (M$_2$), 1-cells ∇_X and ϵ_X have right adjoints since they are $\oplus$-monoid homomorphisms. And the uniqueness of commutative $\oplus$-monoid structure on X with 1-cells having right adjoints is $(X, \nabla_X, \epsilon_X)$ follows similarly from dual of the proof for the Corollary 2.1.6 in [1].

∎

5.3 Discrete Cartesian Linear Bicategories

In this section, we introduce a discreteness axiom for cartesian linear bicategories, now known as the Frobenius axioms.

Definition 5.3.1. A 0-cell X in a locally ordered cartesian linear bicategory $\mathcal{B}$ is *discrete* when multiplication Δ_X and comultiplication ∇_X satisfy the following equations

$$\Delta_X^* \otimes \Delta_X = (\top_X \boxtimes \Delta_X) \otimes (\Delta_X^* \boxtimes \top_X)$$
$$\nabla_X \oplus \nabla_X^* = (\bot_X \boxplus \nabla_X^*) \oplus (\nabla_X \boxplus \bot_X)$$

together with the cyclic linear adjoints $\Delta_X \| \mapsto \| \nabla_X$ and $t_X \| \mapsto \| \epsilon_X$.

Definition 5.3.2. A locally ordered closed cartesian linear bicategory $\mathcal{B}$ is called a *linear bicategory of relations* (**LinBiRel**) if every 0-cell $X \in \mathcal{B}$ is discrete. In other words, this definition provides us two bicategories of relations $(\mathcal{B}, \otimes, \top)$ and $(\mathcal{B}^{co}, \oplus, \bot)$ which are linked by linear strengths.

Example 5.3.3. The linear bicategory of sets and relations **Rel** is a linear bicategory of relations.

5.4 The Linear Bicategory Q-Rel

While LD-quantales give linear bicategories, they do not generally give cartesian linear bicategories. Again one must use localic structures rather than the more general quantales.

Theorem 5.4.1. *If Q is a Girard locale, then Q-Rel is a cartesian linear bicategory.*

Proof: By Theorem 2.2.4 and since Q is Girard, Q-Rel is a closed linear bicategory [9], we get a cartesian structure on linear bicategory of Q-Rel. ∎

5.5 The Module Linear Bicategory of Locally Ordered Closed Cartesian Linear Bicategories

This section extends the bicategory **Mod**$\mathcal{B}$ for locally ordered cartesian bicategories in [1, Sections 2.2, 2.3] to linear bicategories. First, we define module linear bicategories for locally ordered closed linear bicategories. Then we prove that the module linear bicategory of a locally ordered closed cartesian linear bicategory is cartesian.

Definition 5.5.1. For a locally ordered closed linear bicategory $\mathcal{B}$, we define its *module linear bicategory of* $\mathcal{B}$ to be the linear bicategory $\mathsf{Mod}\mathcal{B}$ that consists of the following data:

- 0-cells in $\mathsf{Mod}\mathcal{B}$ are pairs (A, a_1), where A is a 0-cell in $\mathcal{B}$ and a_1 is an idempotent on A with respect $\otimes$ with $\top_A \leq a_1$. That is, 1-cells $a_1 : A \to A$ in $\mathcal{B}$ such that $a_1 \otimes a_1 = a_1$ with $\top_A \leq a_1$. Such a 0-cell will be called a *monad*.
- If (X, x_1) and (A, a_1) are two monads, then a 1-cell $R : (X, x_1) \to (A, a_1)$ between them is a 1-cell $R : X \to A$ in $\mathcal{B}$ such that $x_1 \otimes R \otimes a_1 = R$. We call such a 1-cell a *module* between monads.
- 2-cells in $\mathsf{Mod}\mathcal{B}$ are inequalities, as in $\mathcal{B}$.
- For a monad (X, x_1), two units $\top_{(X,x_1)} := x_1 : X \to X$ and $\bot_{(X,x_1)} := x_2 := (x_1)^\perp : X \to X$.

  Remark 5.5.2. Equivalently, we can define 1-cells by the equations $x_1 \otimes R = x_1 \otimes R \otimes a_1 = R \otimes a_1$, since if $x_1 \otimes R \otimes a_1 = R$ then $x_1 \otimes R = x_1 \otimes x_1 \otimes R \otimes a_1 = x_1 \otimes R \otimes a_1 = R = x_1 \otimes R \otimes a_1 = x_1 \otimes R \otimes a_1 \otimes a_1 = R \otimes a_1$.

- The compositions of 1-cells are the compositions of 1-cells in $\mathcal{B}$. That is, for any modules $R : (X, x_1) \to (Y, y_1)$ and $S : (Y, y_1) \to (Z, z_1)$, $R \otimes S$ and $R \oplus S$ are modules. Since

$$\begin{aligned} (R \oplus S) \otimes z_1 &\subseteq R \oplus (S \otimes z_1) && \text{(by } \delta_R \text{ in the Dfinition 3.1.1.)} \\ &= R \oplus S && \text{(since } S \text{ is a module.)} \end{aligned} \tag{5.5.1}$$

  And conversely,

$$R \oplus S \overset{u_\otimes^R}{=} (R \oplus S) \otimes \top_Z \subseteq (R \oplus S) \otimes z_1.$$

  Thus, $R \oplus S$ is a module. And, similarly, $R \otimes S$ is a module.

Remark 5.5.3.

1. Since $\mathcal{B}$ is a closed linear bicategory, then we have $x_2 \oplus R = (R^\perp \otimes x_2^\perp)^\perp = (R^\perp \otimes x_1)^\perp = (R^\perp)^\perp = R$. And similarly $R \oplus a_2 = R$ and $x_2 \oplus R \oplus a_2 = R$.
2. similar to Remark 5.5.2, if $x_2 \oplus R \oplus a_2 = R$ then $a_2 \oplus a_2 \oplus R \oplus x_2 = a_2 \oplus R \oplus x_2 = R_\oplus = a_2 \oplus R \oplus x_2 = a_2 \oplus R \oplus x_2 \oplus x_2 = R \oplus x_2$ and conversely, if $x_1 \otimes R = R = R \otimes a_1$ then $x_1 \otimes R \otimes a_1 = R$ and similarly if $a_2 \oplus R = R = R \oplus x_2$ then $a_2 \oplus R \oplus x_2 = R$.

3. If (X, x_1) and (A, a_1) are two monads in $\mathsf{Mod}\mathcal{B}$, then every 1-cell $F : X \to A \in \mathcal{B}$ provides modules $x_1 \otimes F \otimes a_1 : (X, x_1) \to (A, a_1)$ and $x_2 \oplus F \oplus a_2 : (X, x_1) \to (A, a_1)$ between monads, since $x_1 \otimes x_1 \otimes F \otimes a_1 \otimes a_1 = x_1 \otimes F \otimes a_1$ and $x_2 \oplus x_2 \oplus F \oplus a_2 \oplus a_2 = x_2 \oplus F \oplus a_2$.

Example 5.5.4. Consider the locally ordered linear bicategory Rel. Then $\mathsf{Mod}(\mathsf{Rel})$ consists of:

- 0-cells are sets with a reflexive transitive relation (preordered sets). i.e, $(X, \leq)$ such that $\leq : X \to X$ is a transitive relation which means for all $x, y, z \in X$

  $$\text{if} \quad (x \leq y \quad \text{and} \quad y \leq z) \Rightarrow x \leq z.$$

  A reflexive transitive relation $\leq_X : X \to X$ is an idempotent with $\top_X \subseteq \leq_X$. Since for $\leq_X : X \to X$ we have

  $$\top_X = \{(x, x') \,|\, x = x'\} \subseteq \{(x, x') \,|\, x \leq x'\} = \leq_X .$$

  And,

  $$\begin{aligned} (x, x') \in \ (\leq_X \otimes \leq_X) &\Longrightarrow \exists x'' \in X \ \ \text{s.t} \ \ x \leq x'' \ \ \text{and} \ \ x'' \leq x' \\ &\Longrightarrow \exists x'' \in X \quad x \leq x' \quad (\text{since} \leq \text{is transitive.}) \\ &\Longrightarrow \quad (x, x') \in \leq_X \end{aligned} \tag{5.5.2}$$

  conversely, since $\top_X \subseteq \leq_X$ then we have:

  $$\leq_X = \top_X \otimes \leq_X \subseteq \leq_X \otimes \leq_X .$$

- A module $R : (X, \leq_X) \to (Y, \leq_Y,)$ is a relation $R : X \to Y$ such that $R \otimes \leq_Y = R = \leq_X \otimes R$. That is, $(x, y) \in R$ if and only if $(\exists x')(x \leq x'$ and $(x', y) \in R)$ if and only if $(\exists y')((x, y') \in R$ and $y' \leq y)$. Since Rel is a closed linear bicategory, we have $R \oplus \ngeq_Y = ((\ngeq_Y)^\perp \otimes R^\perp)^\perp = (\leq_Y \otimes R^\perp)^\perp = (R^\perp)^\perp = R$. And similarly, $\ngeq_X \oplus R = R$ and $\ngeq_X \oplus R \oplus \ngeq_Y = R$. So, $(x, y) \in R$ iff $(\forall x')((x, x') \in \ngeq_X$ or $(x', y) \in R$ iff $(\forall y')(x, y') \in R$ or $(y', y) \in \ngeq_Y)$.

- 2-cells are given by inclusion as usual.

- Composition $\otimes$ is the usual composition in Rel, and composition $\oplus$ is its dual. That is for any modules $R : (X, \leq_X) \to (Y, \leq_Y)$ and $S : (Y, \leq_Y) \to (Z, \leq_Z)$, $R \otimes S$ and $R \oplus S$ are modules. Since,

  $$\begin{aligned} (R \oplus S) \otimes \leq_Z &\subseteq R \oplus (S \otimes \leq_Z) \\ &= R \oplus S \end{aligned} \tag{5.5.3}$$

And conversely,

$$R \oplus S \overset{u^R_\otimes}{=} (R \oplus S) \otimes \top_Z \subseteq (R \oplus S) \otimes \leq_Z .$$

Thus $R \oplus S$ is a module.

- The units are $\top_{(X,\leq_X)} := \leq : X \to X$ and $\bot_{(X,\leq_X)} := \not\geq_X : X \to X$ is dual of $\leq_X$.

Lemma 5.5.5. *1. If (X, x_1) and (A, a_1) are monads and $F \dashv F^* : X \to A$ is an adjunction in $(\mathcal{B}, \otimes)$ such that $x_1 \otimes F \leq F \otimes a_1$, then we have an adjunction $x_1 \otimes F \otimes a_1 \dashv a_1 \otimes F^* \otimes x_1$ in $(\mathsf{Mod}\mathcal{B}, \otimes)$.*

2. Similarly if (X, x_1) and (A, a_1) are monads and $G \dashv G^* : A \to X$ is an adjunction in $(\mathcal{B}^{co}, \oplus)$ such that $a_2 \oplus G \leq G \oplus x_2$, then we have an adjunction $a_2 \oplus G \oplus x_2 \dashv x_2 \oplus G^* \oplus a_2$ in $(\mathsf{Mod}\mathcal{B}^{co}, \oplus)$.

Proof: For the first adjunction with respect to $\otimes$ see Lemma 2.2.4 in [1]. For the second by adjunction $G \dashv G^*$ in $(\mathcal{B}^{co}, \oplus)$, we get $\bot_A \leq G \oplus G^*$ and $G^* \oplus G \leq \bot_X$. The unit and counit of the adjunction $a_2 \oplus G \oplus x_2 \dashv x_2 \oplus G^* \oplus a_2$ will be given by the following inequalities:

$$\bot_{(A,a_1)} = a_2 = a_2 \oplus \bot_A \oplus a_2 \leq a_2 \oplus G \oplus G^* \oplus a_2 \leq a_2 \oplus G \oplus x_2 \oplus G^* \oplus a_2 = (a_2 \oplus G \oplus x_2) \oplus (x_2 \oplus G^* \oplus a_2),$$

$$\begin{aligned} (x_2 \oplus G^* \oplus a_2) \oplus (a_2 \oplus G \oplus x_2) &= x_2 \oplus G^* \oplus a_2 \oplus G \oplus x_2 \\ &\leq x_2 \oplus G^* \oplus G \oplus x_2 \oplus x_2 \\ &\leq x_2 \oplus \bot_A \oplus x_2 \\ &= x_2 = \bot_{(X,x_1)}. \end{aligned}$$

∎

Lemma 5.5.6. *Let (X, x_1) and (A, a_1) be two monads.*

1. If $R : X \to A$ is an isomorphism in $(\mathcal{B}, \otimes)$, then $x_1 \otimes R \otimes a_1 : (X, x_1) \to (A, a_1)$ is an isomorphism in $(\mathsf{Mod}\mathcal{B}, \otimes)$.

2. If $R : A \to X$ is an isomorphism in $(\mathcal{B}^{co}, \oplus)$, then $a_2 \oplus R \oplus x_2 : (A, a_1) \to (X, x_1)$ is an isomorphism in $(\mathsf{Mod}\mathcal{B}^{co}, \oplus)$.

Proof: For the first item, see the proof of [1, Lemma 2.2.5]. For the second item similarly we can find the inverse of $a_2 \oplus R \oplus x_2$ is $x_2 \oplus R^{-1} \oplus a_2$:

$$\begin{aligned} (a_2 \oplus R \oplus x_2) \oplus (x_2 \oplus R^{-1} \oplus a_2) &= a_2 \oplus R \oplus x_2 \oplus x_2 \oplus R^{-1} \oplus a_2 \\ &= a_2 \oplus R \oplus x_2 \oplus R^{-1} \oplus a_2 \\ &= a_2 \oplus a_2 \oplus R \oplus R^{-1} \oplus a_2 \\ &= a_2 \oplus \bot_A \oplus a_2 \\ &= a_2 = \bot_{(A,a_1)}. \end{aligned}$$

And, similarly we get

$$(x_2 \oplus R^{-1} \oplus a_2) \oplus (a_2 \oplus R \oplus x_2) = x_2 = \bot_{(X,x_1)}.$$

∎

Definition 5.5.7. Locally ordered linear bicategories, linear pseudofunctors, and linear pseudonatural transformations form a 2-category, which we denote it by **LoLBicat**. See Proposition 3.4.2.

Next, we demonstrate that the module envelope of a locally ordered closed cartesian linear bicategory is cartesian. To show that we construct a 2-functor from **LoLBicat** to itself. We begin by proving the following Lemma:

Lemma 5.5.8. *For locally ordered closed linear bicategories $\mathcal{B}$ and $\mathcal{D}$, consider a linear pseudofunctors $F : \mathcal{B} \to \mathcal{D}$. Then the following mapping provides a linear pseudofunctor $F_\# : \mathbf{Mod}\mathcal{B} \to \mathbf{Mod}\mathcal{D}$ consists of pair of functors $(F_\#^\otimes, F_\#^\oplus)$ such that*

$$F_\#(A, a_1) = (FA, Fa_1), \quad \textit{for a monad } (A, a_1) \textit{ and}$$

$$F_\#^\otimes\Big((X, x_1) \xrightarrow{R} (A, a_1)\Big) = (FX, Fx_1) \xrightarrow{F(R)} (FA, Fa_1), \textit{ for a module } R.$$

$$F_\#^\oplus = \Big((X, x_1) \xrightarrow{R} (A, a_1)\Big) = (FX, (F((x_1)^\bot))^\bot) \xrightarrow{F^\oplus(R)} (FA, (F((a_1)^\bot))^\bot), \textit{ for a module } R.$$

Proof: By Lemma 2.2.7 in [1], pseudofunctor $F : \mathcal{B}_\otimes \to \mathcal{D}_\otimes$ induces a pseudofunctor $F_\# : \mathbf{Mod}\mathcal{B}_\otimes \to \mathbf{Mod}\mathcal{D}_\otimes$. But, since $\mathcal{B}, \mathcal{D}$ are locally ordered closed linear bicategories then F induces a linear pseudofunctor with $F^\otimes = F : \mathcal{B}_\otimes \to \mathcal{B}_\otimes$ and $F^\oplus = (F((\_)^\bot))^\bot : \mathcal{B}_\oplus \to \mathcal{B}_\oplus$. Moreover, $F_\#$ is well defined since $F^\otimes a_1 \otimes F^\otimes a_1 = Fa_1 \otimes Fa_1 = F(a_1 \otimes a_1) = Fa_1 = F^\otimes a_1$, and

$$\begin{aligned}
F^\oplus a_2 \oplus F^\oplus a_2 &= \big(F((a_1)^\bot)\big)^\bot \oplus \big(F((a_1)^\bot)\big)^\bot \\
&= \Big(\big((F((a_1)^\bot))^\bot\big)^\bot \otimes \big((F((a_1)^\bot))^\bot\big)^\bot\Big)^\bot \\
&= \big((F(a_1)^\bot) \otimes (F(a_1)^\bot\big)^\bot \\
&= (F(a_1^\bot \otimes a_1^\bot))^\bot \\
&= ((F(a_1)^\bot)^\bot \\
&= F^\oplus a_2.
\end{aligned} \tag{5.5.4}$$

And $F_\#(\top_{(A,a_1)}) = F_\#(\top_A) = F(a_1) = \top_{(FA,Fa_1)} = \top_{F_\#(A,a_1)}$, which means (FA, Fa_1) is a monad. Moreover, $(F^\otimes x_1) \otimes (F^\otimes R) \otimes (F^\otimes a_1) = F^\otimes(x_1 \otimes R \otimes a_1) = F^\otimes R$ and

$$\begin{aligned}
F^{\oplus}x_2 \oplus F^{\oplus}R \oplus F^{\oplus}a_2 &= (F((x_2)^{\perp}))^{\perp} \oplus (F((R)^{\perp}))^{\perp} \oplus (F((a_2)^{\perp}))^{\perp} \\
&= [((F((R)^{\perp}))^{\perp})^{\perp} \otimes ((F((x_2)^{\perp}))^{\perp})^{\perp}]^{\perp} \oplus (F((a_2)^{\perp}))^{\perp} \\
&= (F(R^{\perp}) \otimes F(x_2^{\perp}))^{\perp} \oplus (F((a_2)^{\perp}))^{\perp} \\
&= [((F((a_2)^{\perp}))^{\perp})^{\perp} \otimes ((F(R^{\perp}) \otimes F(x_2^{\perp}))^{\perp})^{\perp}]^{\perp} \\
&= [(F(a_2)^{\perp}) \otimes F(R^{\perp}) \otimes F(x_2^{\perp})]^{\perp} \\
&= [(F(a_1)) \otimes F(R^{\perp}) \otimes F(x_1)]^{\perp} \\
&= [F(R^{\perp})]^{\perp} \\
&= F^{\oplus}R.
\end{aligned} \tag{5.5.5}$$

which shows that $F^{\otimes}R$ and $F^{\oplus}R$ are modules between monads. Then since F is a linear pseudofunctor, it induces $F_{\#}$ is a linear pseudofunctor. ∎

Lemma 5.5.9. *For two locally ordered closed linear bicategories $\mathcal{B}$ and $\mathcal{D}$, consider a linear pseudonatural transformation $\phi : F \Rightarrow G : \mathcal{B} \to \mathcal{D}$. Then we can define a linear pseudonatural transformation $\phi_{\#} : F_{\#} \Rightarrow G_{\#} : \mathsf{Mod}\mathcal{B} \to \mathsf{Mod}\mathcal{D}$ with components $(\phi^{\otimes}_{\#}, \phi^{\oplus}_{\#})$ where $\phi^{\otimes}_{\#}(A, a_1) = F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1$ and $\phi^{\oplus}_{\#}(A, a_1) = G^{\oplus}a_2 \oplus \phi^{\oplus}A \oplus F^{\oplus}a_2$ where $(\phi^{\otimes}, \phi^{\oplus})$ are components of ϕ. Moreover, $\phi_{\#}$ is an isomorphism if ϕ is an isomorphism.*

Proof: By Lemma 2.2.8 in [1], a pseudonatural transformation $\phi : F^{\otimes} \Rightarrow G^{\otimes} : \mathcal{B}_{\otimes} \to \mathcal{D}_{\otimes}$ induces a pseudonatural transformation $\phi^{\otimes}_{\#} : F^{\otimes}_{\#} \Rightarrow G^{\otimes}_{\#} : \mathsf{Mod}\mathcal{B}_{\otimes} \to \mathsf{Mod}\mathcal{D}_{\otimes}$ with components $\phi^{\otimes}_{\#}(A, a_1) = F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1$. But, since $\mathcal{B}, \mathcal{D}$ are locally ordered closed linear bicategories then $\phi^{\otimes}$ induces a linear pseudonatural transformation $\phi = (\phi^{\otimes}, \phi^{\oplus})$, where $\phi^{\oplus} = (\phi^{\otimes}(-)^{\perp})^{\perp} : (G^{\otimes}(-)^{\perp})^{\perp} \to (F^{\otimes}(-)^{\perp})^{\perp} := G^{\oplus} \Rightarrow F^{\oplus}$. By Lemma 2.2.8 in [1] $\phi^{\otimes}_{\#} : F^{\otimes}_{\#} \Rightarrow G^{\otimes}_{\#}$ is a pseudonatural transformation. Then it induces an opcolax pseudonatural transformation $\phi^{\oplus}_{\#} : G^{\oplus}_{\#} \Rightarrow F^{\oplus}_{\#}$. The components $\phi^{\otimes}_{\#}(A, a_1) : (FA, Fa_1) \to (GA, Ga_1)$ are modules since

$$F^{\otimes}a_1 \otimes (F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1) \otimes G^{\otimes}a_1 = F^{\otimes}(a_1 \otimes a_1)\phi^{\otimes}A \otimes G^{\otimes}(a_1 \otimes a_1) = F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1$$

and similarly for $\phi^{\oplus}_{\#}(A, a_2) = (\phi^{\otimes}_{\#}(A, a_1)^{\perp})^{\perp} : (GA, (G^{\otimes}(a_1)^{\perp})^{\perp}) \to (FA, (F^{\otimes}(a_1)^{\perp})^{\perp})$ we get

$$G^{\oplus}a_2 \oplus (G^{\oplus}a_2 \oplus \phi^{\oplus}A \oplus F^{\oplus}a_2) \oplus F^{\oplus}a_2 = G^{\oplus}(a_2 \oplus a_2)\phi^{\oplus}A \oplus F^{\oplus}(a_2 \oplus a_2) = G^{\oplus}a_2 \oplus \phi^{\oplus}A \oplus F^{\oplus}a_2$$

and

Also, $\phi^{\otimes}_{\#} : F^{\otimes}_{\#} \Rightarrow G^{\otimes}_{\#}$ becomes a pseudonatural transformation since

$$\begin{aligned}
\phi^{\otimes}_{\#}(X, x_1) \otimes G^{\otimes}_{\#}R &= F^{\otimes}x_1 \otimes \phi^{\otimes}X \otimes G^{\otimes}x_1 \otimes G^{\otimes}R \\
&= F^{\otimes}(x_1) \otimes \phi^{\otimes}X \otimes G^{\otimes}(x_1 \otimes R) \\
&= F^{\otimes}(x_1) \otimes \phi^{\otimes}X \otimes G^{\otimes}(R \otimes a_1) \\
&= F^{\otimes}(x_1) \otimes \phi^{\otimes}X \otimes G^{\otimes}(R) \otimes G^{\otimes}(a_1) \\
&\leq F^{\otimes}(x_1) \otimes F^{\otimes}(R) \otimes \phi^{\otimes}A \otimes G^{\otimes}(a_1) \\
&= F^{\otimes}(x_1 \otimes R) \otimes \phi^{\otimes}A \otimes G^{\otimes}(a_1) \\
&= F^{\otimes}(R \otimes a_1) \otimes \phi^{\otimes}A \otimes G^{\otimes}(a_1) \\
&= F^{\otimes}(R) \otimes F^{\otimes}(a_1) \otimes \phi^{\otimes}A \otimes G^{\otimes}(a_1) \\
&= F^{\otimes}_{\#}(R) \otimes \phi^{\otimes}_{\#}(A, a_1)
\end{aligned}$$

Similarly we can show that $\phi^{\oplus}_{\#} : G^{\oplus}_{\#} \Rightarrow F^{\oplus}_{\#}$ is an opcolax pseudonatural transformation

$$\begin{aligned}
\phi^{\oplus}_{\#}(A, a_1) \oplus F^{\oplus}_{\#}(R) &= G^{\oplus}(a_2) \oplus \phi^{\oplus}A \oplus F^{\oplus}(a_2) \oplus F^{\oplus}(R) \\
&= G^{\oplus}(a_2) \oplus \phi^{\oplus}A \oplus F^{\oplus}(a_2 \oplus R) \\
&= G^{\oplus}(a_2) \oplus \phi^{\oplus}A \oplus F^{\oplus}(R \oplus x_2) \\
&= G^{\oplus}(a_2) \oplus \phi^{\oplus}A \oplus F^{\oplus}(R) \oplus F^{\oplus}(x_2) \\
&\leq G^{\oplus}(a_2) \oplus G^{\oplus}(R) \oplus \phi^{\oplus}X \oplus F^{\oplus}(x_2) \\
&= G^{\oplus}(a_2 \oplus R) \oplus \phi^{\oplus}X \oplus F^{\oplus}(x_2) \\
&= G^{\oplus}(R \oplus x_2) \oplus \phi^{\oplus}X \oplus F^{\oplus}(x_2) \\
&= G^{\oplus}(R) \oplus G^{\oplus}(x_2) \oplus \phi^{\oplus}X \oplus F^{\oplus}(x_2) \\
&= G^{\oplus}_{\#}R \oplus \phi^{\oplus}_{\#}(X, x_1)
\end{aligned}$$

Next, we must show that for every 0-cell (A, a_1) in $\mathbf{Mod}\mathcal{B}$ the 1-cells $\phi^{\otimes}_{\#}(A, a_1) : F^{\otimes}_{\#}(A, a_1) \to G^{\otimes}_{\#}(A, a_1)$ and $\phi^{\oplus}_{\#}(A, a_1) : G^{\oplus}_{\#}(A, a_1) \to F^{\oplus}_{\#}(A, a_1)$ are cyclic linear adjoints. However, these are derived from cyclic linear adjoints $\phi^{\otimes}(A) \| \!\!\mapsto\!\! \| \phi^{\oplus}(A)$ as components of linear natural transformation ϕ.

Now, if $\phi = (\phi^{\otimes}, \phi^{\oplus})$ is an isomorphism, then by the naturality of $\phi^{\otimes}$ we have $\phi^{\otimes}A \otimes G^{\otimes}a_1 \leq F^{\otimes}a_1 \otimes \phi^{\otimes}A$ and $F^{\otimes}a_1 \otimes \phi^{\otimes}A = \phi^{\otimes}A \otimes (\phi^{\otimes})^{-1}(A) \otimes F^{\otimes}a_1 \otimes \phi^{\otimes}A \leq \phi^{\otimes}A \otimes G^{\otimes}a_1 \otimes (\phi^{\otimes})^{-1}(A) \otimes \phi^{\otimes}A = \phi^{\otimes}A \otimes G^{\otimes}a_1$. Similarly, we have $\phi^{\oplus}A \oplus F^{\oplus}a_2 \leq G^{\oplus}a_2 \oplus \phi^{\oplus}A$ and $G^{\oplus}a_2 \oplus \phi^{\oplus}A = \phi^{\oplus}A \oplus (\phi^{\oplus})^{-1}(A) \oplus G^{\oplus}a_2 \oplus \phi^{\oplus}A \leq \phi^{\oplus}A \oplus \oplus F^{\oplus}a_2 \oplus (\phi^{\oplus})^{-1}(A) \oplus \phi^{\oplus}A = \phi^{\oplus}A \oplus F^{\oplus}a_2$. So, by Lemma 5.5.6 $\phi^{-1}_{\#}(A, a_1)$ is an inverse of $\phi_{\#}(A, a_1)$. ∎

Proposition 5.5.10. *There is a (strict) 2-functor* $(-)_{\#}$*:* $\mathsf{LoLBicat} \to \mathsf{LoLBicat}$ *that maps each locally ordered closed linear bicategory* $\mathcal{B}$ *to its module linear bicategory* $\mathcal{B}_{\#} := \mathsf{Mod}\mathcal{B}$*, each linear pseudofunctor functor* F *to* $F_{\#}$ *and each linear pseudonatural transformation* ϕ *to* $\phi_{\#}$*.*

Proof: let $\mathcal{B}$ and $\mathcal{D}$ be two locally ordered closed linear bicategories. Then the mapping $F \mapsto F_{\#}$ and $\phi \mapsto \phi_{\#}$ gives a functor $(-)_{\#} : \mathsf{LoLBicat}(\mathcal{B},\mathcal{D}) \to \mathsf{LoLBicat}(\mathcal{B},\mathcal{D})$. Indeed, if $F : \mathcal{B} \to \mathcal{D}$ is a linear pseudofunctor, then we get the components of $(1_F)_{\#} = ((1_F)^{\otimes}_{\#}, (1_F)^{\oplus}_{\#})$ on a monad (A, a_1) as $F^{\otimes}(a_1) \otimes 1^{\otimes}_{F}(A) \otimes F^{\otimes}(a_1) = F^{\otimes}(a_1) \otimes 1_{F^{\otimes}A} \otimes F^{\otimes}(a_1) = F^{\otimes}(a_1) \otimes F^{\otimes}(a_1) = F^{\otimes}(a_1)$ and similarly, $F^{\oplus}(a_2) \oplus 1^{\oplus}_{F}(A) \oplus F^{\oplus}(a_2) = F^{\oplus}(a_2) \oplus 1_{F^{\oplus}A} \oplus F^{\oplus}(a_2) = F^{\oplus}(a_2) \oplus F^{\oplus}(a_2) = F^{\oplus}(a_2)$ and the components of $1_{F_{\#}}$ are $1^{\otimes}_{F_{\#}(A,a_1)} = \top_{(FA,F^{\otimes}a_1)} = F^{\otimes}a_1$ and by duality we get $1^{\oplus}_{F_{\#}(A,a_1)} = \perp_{(FA,(F^{\otimes}(a_1)^{\perp})^{\perp})} = F^{\oplus}a_2$. So, $1_{F_{\#}} = (1_F)_{\#}$. Moreover, for the following linear pseudonatural transformation

$$\begin{tikzcd}[column sep=large]
\mathcal{B} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=L}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "\phi"]
\arrow[Rightarrow, from=M, to=L, "\psi"]
\end{tikzcd}$$

and for a monad (A, a_1), we have

$$\begin{aligned}
(\phi_{\#};\psi_{\#})^{\otimes} = \phi^{\otimes}_{\#} \otimes \psi^{\otimes}_{\#}(A,a_1) &= \phi^{\otimes}_{\#}(A,a_1) \otimes \psi^{\otimes}_{\#}(A,a_1) \\
&= F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1 \otimes G^{\otimes}a_1 \otimes \psi^{\otimes}A \otimes H^{\otimes}a_1 \\
&= F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes G^{\otimes}a_1 \otimes \psi^{\otimes}A \otimes H^{\otimes}a_1.
\end{aligned}$$

Similarly for $(\phi_{\#};\psi_{\#})^{\oplus} = \psi^{\oplus}_{\#} \oplus \phi^{\oplus}_{\#}$ we get

$$\psi^{\oplus}_{\#} \oplus \phi^{\oplus}_{\#}(A,a_1) = H^{\oplus}a_2 \oplus \phi^{\oplus}A \oplus G^{\oplus}a_2 \oplus \psi^{\oplus}A \oplus F^{\oplus}a_2.$$

And,

$$(\phi^{\otimes} \otimes \psi^{\otimes})_{\#}(A,a_1) = F^{\otimes}a_1 \otimes (\phi^{\otimes} \otimes \psi^{\otimes}) \otimes H^{\otimes}a_1 = F^{\otimes}a_1 \otimes \phi^{\otimes}A \otimes \psi^{\otimes}A \otimes H^{\otimes}a_1.$$

Similarly,

$$(\psi^{\oplus} \oplus \phi^{\oplus})_{\#}(A,a_1) = H^{\oplus}a_2 \oplus \phi^{\oplus}A \oplus \psi^{\oplus}A \oplus F^{\oplus}a_2.$$

However, by naturalities $\phi^{\otimes}A \otimes G^{\otimes}a_1 = F^{\otimes}a_1 \otimes \phi^{\otimes}A$ and $\phi^{\oplus}A \oplus F^{\oplus}a_2 = G^{\oplus}a_2 \oplus \phi^{\oplus}A$, we get:

$$(\phi^{\otimes} \otimes \psi^{\otimes})_{\#}(A,a_1) = \phi^{\otimes}_{\#} \otimes \psi^{\otimes}_{\#}(A,a_1)$$

and

$$(\psi^{\oplus} \oplus \phi^{\oplus})_{\#}(A,a_1) = \psi^{\oplus}_{\#} \oplus \phi^{\oplus}_{\#}(A,a_1)$$

Moreover, by definition of $(-)_\#$ on linear bicategories $\mathcal{B} \mapsto (\mathcal{B})_\#$ and linear pseudofunctors $F \mapsto (F)_\#$ we get $(1_{\mathcal{B}})_\# = 1_{B_\#}$ and $(F;G)_\# = F_\#;G_\#$. Thus $(-)_\# :$ **LoLBicat** → **LoLBicat** is a 2-functor. ∎

Lemma 5.5.11. *[1, Lemma 2.2.10] Let $\mathcal{B}$ and $\mathcal{D}$ be two locally ordered closed linear bicategories. Then* $\textsf{Mod}\mathcal{B}\times$ $\textsf{Mod}\mathcal{D}$ $\cong \textsf{Mod}(\mathcal{B}\times\mathcal{D})$*.*

Proof: First, note that a pair $((A,B),(a_1,b_1))$ in $\mathcal{B}\times\mathcal{D}$ is a monad if and only if (A,a_1) and (B,b_1) are monads in $\mathcal{B}$ and $\mathcal{D}$ respectively. Since $(a_1,b_1)\otimes(a_1,b_1) = (a_1\otimes a_1, b_1\otimes b_1) = (a_1,b_1)$ if and only if $a_1\otimes a_1 = a_1$ and $b_1\otimes b_1 = b_1$ with with $\top_{(A,B)} \leq (a_1,b_1)$ if and only if $(\top_A,\top_B) \leq (a_1,b_1)$ if and only if $\top_A \leq a_1$ and $\top_B \leq b_1$. Moreover, 1-cell $(S,T) : \big((X,Y),(x_1,y_1)\big) \to \big((A,B),(a_1,b_1)\big)$ is a module between monads $((X,Y),(x_1,y_1))$ and $((A,B),(a_1,b_1))$ in $\mathcal{B}\times\mathcal{D}$ if and only if $S : (X,x_1) \to (A,a_1)$ and $T : (Y,y_1) \to (B,b_1)$ are 1-cells between monads. Since $(x_1,y_1)\otimes(S,T)\otimes(a_1,b_1) = (S,T)$ if and only if $(x_1\otimes S\otimes a_1, y_1\otimes T\otimes b_1) = (S,T)$ if and only if $x_1\otimes S\otimes a_1 = S$ and $y_1\otimes T\otimes b_1 = T$. Since we defined everything componentwise, so the functor $(p_1,p_2)_\# : \textsf{Mod}(\mathcal{B}\times\mathcal{D}) \to \textsf{Mod}\mathcal{B}\times\textsf{Mod}\mathcal{D}$, where p_1 and p_2 are the projections of $\mathcal{B}$ and $\mathcal{D}$, is an equivalence. ∎

Lemma 5.5.12. *For a locally ordered closed cartesian linear bicategory $\mathcal{B}$*

- *If each $\mathcal{B}_\otimes(X,A)$ has finite products, then each $\textsf{Mod}\mathcal{B}_\otimes((X,x_1),(A,a_1))$ has finite products.*
- *If each $\mathcal{B}^{co}_\oplus(X,A)$ has finite coproducts, then each $\textsf{Mod}\mathcal{B}^{co}_\oplus((X,x_1),(A,a_1))$ has finite coproducts.*

Proof: For the first item, see Lemma 2.2.11 in [1]. For the second item, let (X,x_1) and (A,a_1) be monads and $\hat{I}_{X,A} : X \to A$ is the initial object in $\mathcal{B}^{co}_\oplus(X,A)$. Then 1-cell $x_2\oplus\hat{I}_{X,A}\oplus a_2 : (X,x_1)\to(A,a_1)$ is the initial object in $\textsf{Mod}\mathcal{B}^{co}_\oplus((X,x_1),(A,a_1))$. Consider a 1-cell $F : (X,x_1)\to(A,a_1)$ between monads, then $x_2\oplus\hat{I}_{X,A}\oplus a_2 \leq F = x_2\oplus F\oplus a_2$.

Now consider modules $F,G : (X,x_1)\to(A,a_1)$. Then we want to show that $x_2\oplus(F\vee G)\oplus a_2$ is the coproduct of F,G in $\textsf{Mod}\mathcal{B}^{co}_\oplus((X,x_1),(A,a_1))$. By the coprojections we have $F = x_2\oplus F\oplus a_2 \leq x_2\oplus(F\vee G)\oplus a_2$ and $G = x_2\oplus G\oplus a_2 \leq x_2\oplus(F\vee G)\oplus a_2$. If $H : (X,x_1)\to(A,a_1)$ is a 1-cell in $\textsf{Mod}\mathcal{B}^{co}_\oplus$ such that $F\leq H$ and $G\leq H$, then $F\vee G\leq H$ in $\mathcal{B}^{co}_\oplus(X,A)$ and so, $H = x_2\oplus H\oplus a_2 \geq x_2\oplus(F\vee G)\oplus a_2$. Thus we get the coproduct of F,G in $\textsf{Mod}\mathcal{B}^{co}_\oplus((X,x_1),(A,a_1))$ by $F\sqcup G = x_2\oplus(F\vee G)\oplus a_2$. ∎

Lemma 5.5.13. *Let $\mathcal{B}$ be a locally ordered closed cartesian linear bicategory, then*

- $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\otimes})$ *has a terminal* 0*-cell.*
- $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\oplus}^{co})$ *has an initial* 0*-cell.*

Proof: For the first item, see Lemma 2.2.12 in [1]. For the second item let I be the initial 0-cell in $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$. Then $\bot_I = \bot_{I,I}$ is the local initial object in $\mathcal{B}_{\oplus}^{co}(I, I)$. We want to show $(I, \bot_I)$ is the initial object in $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\oplus}^{co})$. Let (A, a_1) be a 0-cell in $\mathsf{Mod}\mathcal{B}_{\oplus}^{co}$. Since $\mathcal{B}$ is cartesian, we have a unique map $\epsilon_A : I \to A$. If $\epsilon_A^* : A \to I$ is its right adjoint in $\mathcal{B}_{\oplus}^{co}$, then we have $\epsilon^* \oplus \epsilon \leq \bot_A$ and $\bot_I \leq \epsilon \oplus \epsilon^*$. The 1-cells $\epsilon_A \oplus a_2 : I \to A$ and $a_2 \oplus \epsilon_A^* : A \to I$ are modules between monads (A, a_1) and $(I, \bot_I)$ and $\epsilon_A \oplus a_2 \dashv a_2 \oplus \epsilon_A^*$ in $\mathsf{Mod}\mathcal{B}_{\oplus}^{co}$, since $(\epsilon_A \oplus a_2) \oplus (a_2 \oplus \epsilon_A^*) = \epsilon_A \oplus a_2 \oplus a_2 \oplus \epsilon_A^* = \epsilon_A \oplus a_2 \oplus \epsilon_A^* \geq \bot_I$ since $\mathcal{B}$ is cartesian and $\epsilon_A \oplus a_2 \oplus \epsilon_A^*$ is an object in $\mathcal{B}_{\oplus}^{co}(I, I)$. Moreover, $a_2 \oplus \epsilon^* \oplus \epsilon \oplus a_2 \leq a_2 \oplus \bot_A \oplus a_2 = a_2 \oplus a_2 = a_2$. Suppose now $R : (I, \bot_I) \to (A, a_1)$ is a map in $\mathsf{Mod}\mathcal{B}_{\oplus}^{co}$, with right adjoin $S : (A, a_1) \to (I, \bot_I)$. Then by the second item of the Remark 5.5.3, $a_2 \oplus S = S$ and $R \oplus a_2 = R$. By $\oplus$-adjunction, we have $S \oplus R \leq \bot_{(A,a_1)} = a_2$ and $\bot_{(I,\bot_I)} \leq R \oplus S$. We want to show that $R = \epsilon_A \oplus a_2$. First, $\epsilon_A = \bot_I \oplus \epsilon_A = \epsilon_I^* \oplus \epsilon_A = \bot_{I,A} \leq R$, since $\mathcal{B}$ is cartesian and $\bot_{I,A}$ is the initial object in $\mathcal{B}_{\oplus}^{co}(I, A)$. So $R = R \oplus a_2 \geq \bot_{I,A} \oplus a_2 = \epsilon_A \oplus a_2$. On the other hand, $\epsilon_A \oplus a_2 \geq \epsilon_A \oplus S \oplus R$. But $\epsilon_A \oplus S \geq \bot_{I,I}$, so $\epsilon_A \oplus a_2 \geq \epsilon_A \oplus S \oplus R \geq \bot_{I,I} \oplus R = R$. ∎

Proposition 5.5.14. *If $\mathcal{B}$ is a locally ordered closed cartesian linear bicategory, the tensor product $\square : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ induces a tensor product $\square_{\#} : \mathsf{Mod}\mathcal{B} \times \mathsf{Mod}\mathcal{B} \to \mathsf{Mod}\mathcal{B}$ consists of two pseufofunctors $\square_{\#}^{\otimes} := \boxtimes_{\#} : \mathsf{Mod}\mathcal{B}_{\otimes} \times \mathsf{Mod}\mathcal{B}_{\otimes} \to \mathsf{Mod}\mathcal{B}_{\otimes}$ and $\square_{\#}^{\oplus} := \boxplus_{\#} : \mathsf{Mod}\mathcal{B}_{\oplus}^{co} \times \mathsf{Mod}\mathcal{B}_{\oplus}^{co} \to \mathsf{Mod}\mathcal{B}_{\oplus}^{co}$ where they share the same product on* 0*-cells and each forms a symmetric monoidal structure.*

Proof: For pseudofunctor $\square_{\otimes}$ see Proposition 2.2.13 in [1] and pseudofunctor $\square_{\oplus}$ follows similarly with respect to composition $\oplus$. ∎

Proposition 5.5.15. *Let $\mathcal{B}$ be a locally ordered closed cartesian linear bicategory, then $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\otimes})$ and $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\oplus}^{co})$ have finite products and coproducts respectively and they share the same (co)product on* 0*-cells.*

Proof: By the restriction of the components of the tensor product $\square_{\#}$ in Proposition 5.5.14 on $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\otimes})$ and $\mathbf{Map}(\mathsf{Mod}\mathcal{B}_{\oplus}^{co})$, we need to show the pseudoadjuction between pseudofunctors $\Delta^{\otimes} \dashv \square_{\#}^{\otimes}$ and the pseudoadjuction between pseudofunctors $\square_{\#}^{\oplus} \dashv \Delta^{\oplus}$. For the first pseudoadjuction with respect to $\otimes$ see [1, Lemma

2.2.14]. For the second pseudoadjuction, we must show $(i_\# \boxplus j_\#) \oplus \nabla_\# \leq a_2 \boxplus b_2$ and $a_2 \boxplus b_2 \leq \nabla_\# \oplus (i_\# \boxplus j_\#)$ where $i : A \to A\Box B$, $j : B \to A\Box B$, $i_\# := a_2 \oplus i \oplus (a_2 \boxplus b_2)$ and $j_\# := b_2 \oplus j \oplus (a_2 \boxplus b_2)$. For the former, we see in the following diagram that $(i_\# \boxplus j_\#) \oplus \nabla_\# \leq a_2 \boxplus b_2$:

$$\begin{tikzcd}
A\Box B & A\Box B \arrow[l, "a_2\boxplus b_2"'] & (A\Box B)\Box(A\Box B) \arrow[l, "\nabla_{A\Box B}"'] & (A\Box B)\Box(A\Box B) \arrow[l, "(a_2\boxplus b_2)\boxplus(a_2\boxplus b_2)"'] \\
 & & A\Box B \arrow[ul, bend left, "a_2\boxplus b_2"] \arrow[u, phantom, "\geq"] & (A\Box B)\Box(A\Box B) \arrow[l, "\nabla_{A\Box B}"'] \arrow[ul, "(a_2\boxplus b_2)\boxplus(a_2\boxplus b_2)"] \arrow[u, "(a_2\boxplus b_2)\boxplus(a_2\boxplus b_2)"'] \arrow[ul, phantom, "=", near start] \\
 & & & A\Box B \arrow[ul, bend left, "\perp_{A\Box B}"] \arrow[u, "i\boxplus j"'] \arrow[ul, phantom, "="] \\
 & & & A\Box B \arrow[uuulll, bend left, "a_2\boxplus b_2"] \arrow[u, "a_2\boxplus b_2"']
\end{tikzcd}$$

For the opposite inequality we have

$$\begin{aligned}
\nabla_\# = (a_2 \boxplus b_2) \boxplus (a_2 \boxplus b_2) \oplus \nabla_{A\Box B} \oplus (a_2 \boxplus b_2) &\geq (a_2 \boxplus b_2) \boxplus (a_2 \boxplus b_2) \oplus (i^* \boxplus j^*) \oplus (a_2 \boxplus b_2) \\
&= ((a_2 \boxplus b_2) \oplus i^* \oplus a_2) \boxplus ((a_2 \boxplus b_2) \oplus j^* \oplus b_2) \\
&= (i_\# \boxplus_\# j_\#)^*,
\end{aligned}$$

So, $\nabla_\# \geq (i_\# \boxplus_\# j_\#)^* \oplus (a_2 \boxplus b_2)$ which gives us $(i_\# \boxplus_\# j_\#) \oplus \nabla_\# \geq (a_2 \boxplus b_2)$. For the second identity, by the following diagram we get $x_2 \geq i_\# \oplus (\nabla_X)_\#$. But we also have

$$\begin{aligned}
\nabla_X \geq i^* \Rightarrow (x_2 \boxplus x_2) \oplus \nabla_X \oplus x_2 \geq (x_2 \boxplus x_2) \oplus i^* \oplus x_2 \Rightarrow \\
(\nabla_X)_\# \geq (i_\#)^* = (i_\#)^* \oplus x_2 \Rightarrow (i_\#) \oplus (\nabla_X)_\# \geq x_2
\end{aligned}$$

So, $(i_\#) \oplus (\nabla_X)_\# = x_2$ and similarly $j_\# \oplus \nabla_X = x_2$.

$$\begin{tikzcd}
X & X \arrow[l, "1_X"'] & X\Box X \arrow[l, "\nabla_X"'] & X\Box X \arrow[l, "x_2\boxplus x_2"'] \\
 & & X \arrow[ul, bend left, "x_2"] \arrow[u, phantom, "\geq"] & X\Box X \arrow[l, "\nabla_X"'] \arrow[ul, "x_2\boxplus x_2"] \arrow[u, "x_2\boxplus x_2"'] \arrow[ul, phantom, "=", near start] \\
 & & & X \arrow[ul, bend left, "\perp_X"] \arrow[u, "i"'] \arrow[ul, phantom, "="] \\
 & & & X \arrow[uuulll, bend left, "x_2"] \arrow[u, "x_2"']
\end{tikzcd}$$

∎

Theorem 5.5.16. *If $\mathcal{B}$ is a locally ordered closed cartesian linear bicategory, then* $\mathbf{Mod}\mathcal{B}$ *is a locally ordered cartesian linear bicategory.*

Proof: By Theorem 5.2.4, Lemma 5.5.12, Proposition 5.5.15 and [1, Theorem 2.2.15], it remains to show that for any 1-cells $F : (X, x_1) \to (A, a_1)$ and $G : (Y, y_1) \to (B, b_1)$ in $\mathbf{Mod}\mathcal{B}$, we have the following formulas:

$$F \boxtimes_{\#} G = F \boxtimes G = (p_{\#} \otimes F \otimes p^*_{\#}) \wedge (q_{\#} \otimes G \otimes q^*_{\#}),$$

$$F \boxplus_{\#} G = F \boxplus G = (i^*_{\#} \oplus F \oplus i_{\#}) \vee (j^*_{\#} \oplus G \oplus j_{\#}).$$

For the first formula with respect to $\otimes$, see [1, Theorem 2.215]. For the second formula, since $\mathcal{B}$ is a locally ordered cartesian closed linear bicategory, we have $F \boxplus G = (i^* \oplus F \oplus i) \vee (j^* \oplus G \oplus j)$. So $F \boxplus G \geq i^* \oplus F \oplus i$ and $F \boxplus G \geq j^* \oplus G \oplus j$. Then we get

$$\begin{aligned} F \boxplus G = (x_2 \boxplus y_2) \oplus (F \boxplus G) \oplus (a_2 \boxplus b_2) &\geq (x_2 \boxplus y_2) \oplus (i^* \oplus F \oplus i) \oplus (a_2 \boxplus b_2) \\ &= (x_2 \boxplus y_2) \oplus (i^* \oplus (x_2 \oplus F \oplus a_2) \oplus i) \oplus (a_2 \boxplus b_2) \\ &= i^*_{\#} \oplus F \oplus i_{\#} \end{aligned}$$

Similarly, $F \boxplus G \geq j^*_{\#} \oplus G \oplus j_{\#}$. So

$$F \boxplus G \geq (i^*_{\#} \oplus F \oplus i_{\#}) \vee (j^*_{\#} \oplus G \oplus j_{\#})$$

On the other hand, notice that

$$a_2 \boxplus b_2 = (i^* \oplus a_2 \oplus i) \vee (j^* \oplus b_2 \oplus j) \Rightarrow a_2 \boxplus b_2 \geq i^* \oplus a_2 \oplus i \Rightarrow i \oplus (a_2 \boxplus b_2) \geq a_2 \oplus i$$

and similarly

$$x_2 \boxplus y_2 = (i^* \oplus x_2 \oplus i) \vee (j^* \oplus y_2 \oplus j) \Rightarrow x_2 \boxplus y_2 \geq i^* \oplus x_2 \oplus i \Rightarrow (x_2 \boxplus y_2) \oplus i^* \geq i^* \oplus x_2$$

So, we have

$$\begin{aligned} i^*_{\#} \oplus F \oplus i_{\#} = ((x_2 \boxplus y_2) \oplus i^* \oplus x_2) \oplus F \oplus (a_2 \oplus i \oplus (a_2 \boxplus a_2)) &\geq i^* \oplus x_2 \oplus x_2 \oplus F \oplus a_2 \oplus a_2 \oplus i \\ &= i^* \oplus F \oplus i \end{aligned}$$

Similarly, $j^*_{\#} \oplus G \oplus j_{\#} \geq j^* \oplus G \oplus j$. So,

$$F \boxplus G \leq (i^*_{\#} \oplus F \oplus i_{\#}) \vee (j^*_{\#} \oplus G \oplus j_{\#}),$$

Which implies that

$$\begin{aligned}(i^*_{\#} \oplus F \oplus i_{\#}) \vee (j^*_{\#} \oplus G \oplus j_{\#}) &= (x_2 \boxplus y_2) \oplus ((i^* \oplus F \oplus i) \vee (j^* \oplus G \oplus j)) \oplus (a_2 \boxplus b_2)\\ &\geq (x_2 \boxplus y_2) \oplus (F \boxplus G) \oplus (a_2 \boxplus b_2)\\ &= F \boxplus G\end{aligned}$$

∎

Chapter 6

Cartesian Linear Bicategories

6.1 Precartesian Linear Bicategories

Similarly to Carboni, Walters and Wood approach in [13], we begin by briefly introducing precartesian linear bicategories using Theorem 5.2.4 before defining cartesian linear bicategories. As we observed in chapter 5, for locally ordered linear bicategories, the definition of cartesian linear bicategories provides cartesian structures on the bicategories $\mathcal{B}_\otimes$ and $\mathcal{B}_\oplus^{co}$. To define cartesian linear bicategories in general, we utilize a precartesian structure on a linear bicategory $\mathcal{B}$, which establishes two precartesian structures on both $\mathcal{B}_\otimes$ and $\mathcal{B}_\oplus^{co}$.

Definition 6.1.1. A linear bicategory $\mathcal{B}$ is said to be *precartesian* if

1. The full sub-bicategories $\mathbf{Map}(\mathcal{B}_\otimes)$ and $\mathbf{Map}(\mathcal{B}_\oplus^{co})$ have finite bicategorical products and coproducts respectively, and they share the same (co)products on 0-cells.
2. Hom-categories $\mathcal{B}_\otimes(X, Y)$ and $\mathcal{B}_\oplus^{co}(X, Y)$ have finite products and coproducts respectively, which are denoted by $\wedge$ and $\vee$ respectively.

Example 6.1.2. Any cartesian linear bicategory is precartesian.

Now for any linear bicategory $\mathcal{B}$, we can establish two Grothendieck constructions $(\mathbf{G}, \otimes)$ and $(\widetilde{\mathbf{G}}, \oplus)$ for the following pseudofunctors, introduced in [60, Section 1.10].

$$(\mathbf{Map}(\mathcal{B}_\otimes))^{op} \times \mathbf{Map}(\mathcal{B}_\otimes) \xrightarrow{i^{op}\times i} \mathcal{B}_\otimes^{op} \times \mathcal{B}_\otimes \xrightarrow{\mathcal{B}_\otimes(-,-)} \mathbf{CAT} \tag{6.1.1}$$

$$(\mathbf{Map}(\mathcal{B}_\oplus^{co}))^{op} \times \mathbf{Map}(\mathcal{B}_\oplus^{co}) \xrightarrow{i^{op}\times i} (\mathcal{B}_\oplus^{co})^{op} \times \mathcal{B}_\oplus^{co} \xrightarrow{\mathcal{B}_\oplus(-,-)} \mathbf{CAT} \tag{6.1.2}$$

where i is the inclusion functor, and $\mathcal{B}_\otimes(-,-)$ and $\mathcal{B}_\oplus(-,-)$ are the hom-pseudofunctors and $\mathbf{CAT}$ is the 2-category of categories.

Lemma 6.1.3. *For maps $f \dashv f^* : X \to Y$ and $g \dashv g^* : A \to B$ in $\mathbf{Map}(\mathcal{B}_\otimes)$, the functor $\mathcal{B}_\otimes(f, g^*)$ is the right adjoint of $\mathcal{B}_\otimes(f^*, g)$.*

Proof: For maps $f : X \to Y$ and $g : A \to B$ in $\mathbf{Map}(\mathcal{B}_\otimes)$, the functor $\mathcal{B}_\otimes(f^*, g) : \mathcal{B}_\otimes(X, A) \to \mathcal{B}_\otimes(Y, B)$ is given by $\mathcal{B}_\otimes(f^*, g)(k) = f^* \otimes k \otimes g$, for any 1-cell $k \in \mathcal{B}_\otimes(X, A)$ and $\mathcal{B}_\otimes(f^*, g)(\alpha) = 1_{f^*} \otimes \alpha \otimes 1_g$, for any 2-cell $\alpha \in \mathcal{B}_\otimes(X, A)$. Now consider the functor $\mathcal{B}_\otimes(f, g^*) : \mathcal{B}_\otimes(Y, B) \to \mathcal{B}_\otimes(X, A)$ where $\mathcal{B}_\otimes(f, g^*)(l) = f \otimes l \otimes g^*$,$\forall l \in \mathcal{B}_\otimes(Y, B)$ and $\mathcal{B}_\otimes(f, g^*)(\beta) = 1_f \otimes \beta \otimes 1_{g^*}$, $\forall \beta \in \mathcal{B}_\otimes(Y, B)$. Then the above information provides us $\forall k \in \mathcal{B}_\otimes(X, A)$:

$$\begin{aligned} k = \top_X \otimes k \otimes \top_A &\Rightarrow (f \otimes f^*) \otimes k \otimes (g \otimes g^*) \\ &\cong f \otimes (f^* \otimes k \otimes g) \otimes g^* \\ &= \mathcal{B}_\otimes(f, g^*)(f^* \otimes k \otimes g) \\ &= \mathcal{B}_\otimes(f, g^*) \circ \mathcal{B}_\otimes(f^*, g)(k) \end{aligned}$$

And $\forall l \in \mathcal{B}_\otimes(Y, B)$

$$\begin{aligned} \mathcal{B}_\otimes(f^*, g) \circ \mathcal{B}_\otimes(f, g^*)(l) &= f^* \otimes (f \otimes l \otimes g^*) \otimes g \\ &\cong (f^* \otimes f) \otimes l \otimes (g^* \otimes g) \\ &\Rightarrow \top_Y \otimes l \otimes \top_B \\ &= l \end{aligned}$$

So the above equations provide us the unit and counit natural transformations $\eta : 1_{\mathcal{B}_\otimes(X,A)} \Rightarrow \mathcal{B}_\otimes(f, g^*) \circ \mathcal{B}_\otimes(f^*, g)$ and $\epsilon : \mathcal{B}_\otimes(f^*, g) \circ \mathcal{B}_\otimes(f, g^*) \Rightarrow 1_{\mathcal{B}_\otimes(Y,B)}$ respectively. Now we will show they satisfy the triangle identities of the adjunction $\mathcal{B}_\otimes(f^*, g) \dashv \mathcal{B}_\otimes(f, g^*)$. For all $k \in \mathcal{B}_\otimes(X, A)$ we have:

$$\begin{aligned} \mathcal{B}_\otimes(f^*, g)(k) &\xLeftarrow{\epsilon(\mathcal{B}_\otimes(f^*,g)(k))} \mathcal{B}_\otimes(f^*, g)\Big(\mathcal{B}_\otimes(f, g^*)\big(\mathcal{B}_\otimes(f^*, g)(k)\big)\Big) \\ &= \mathcal{B}_\otimes(f^*, g)\Big(\mathcal{B}_\otimes(f, g^*)\big(f^* \otimes k \otimes g\big)\Big) \\ &= \mathcal{B}_\otimes(f^*, g)\Big(f \otimes (f^* \otimes k \otimes g) \otimes g^*\Big) \\ &\cong \mathcal{B}_\otimes(f^*, g)\Big((f \otimes f^*) \otimes k \otimes (g \otimes g^*)\Big) \\ &\xLeftarrow{\mathcal{B}_\otimes(f^*,g)(\eta(k))} \mathcal{B}_\otimes(f^*, g)\Big(\top_X \otimes k \otimes \top_A\Big) \\ &= \mathcal{B}_\otimes(f^*, g)(k) \end{aligned}$$

And similarly we get the other triangle identity. ∎

Now, we recall the construction of the bicategory $(\mathbf{G}, \otimes)$ based on the two-sided Grothendieck construction for the pseudofunctor 6.1.1 form [13, 60] as following:

- A 0-cell of $(\mathbf{G}, \otimes)$ is a triple of $(X, R : X \to A, A)$, where $R \in \mathcal{B}_{\otimes}(X, A)$.
- A 1-cell in $(\mathbf{G}, \otimes)$ is a triple of $(f, \alpha, g) : (X, R, A) \to (Y, S, B)$ where $f : X \to Y, g : A \to B \in \mathbf{Map}(\mathcal{B}_{\otimes})$ and $\alpha : R \otimes g \Rightarrow f \otimes S$ is a 2-cell in $\mathcal{B}_{\otimes}$. We call this form of 1-cells *primary form.*

  Remark 6.1.4. *Equivalently we can define* 1*-cells of Grothendieck construction of* $(\mathbf{G}, \otimes)$ *by triple of* $(f, \beta, g) : (X, R, A) \to (Y, S, B)$ *such that* 2*-cell* $\beta : R \Rightarrow f \otimes S \otimes g^* \in \mathcal{B}(X, A)(R, f \otimes S \otimes g^*)$ *is the mate of* 2*-cell* $\alpha : R \otimes g \Rightarrow f \otimes S$ *with respect to the adjunction* $g \dashv_{\otimes} g^*$*. We call this form of* 1*-cells secondary form.*

- For given primary forms of 1-cells $(f, \alpha, g), (f', \alpha', g') \in (\mathbf{G}, \otimes)$, define a 2-cell in $(\mathbf{G}, \otimes)$ as $(\phi, \psi) : (f, \alpha, g) \Rightarrow (f', \alpha', g')$ where $\phi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ are 2-cells in $\mathbf{Map}(\mathcal{B}_{\otimes})$ such that the following diagram commutes.

$$\begin{tikzcd}
f \otimes S \arrow[r, Rightarrow, "\phi\otimes 1_S"] & f' \otimes S \\
R \otimes g \arrow[u, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "1_R\otimes\psi"'] & R \otimes g' \arrow[u, Rightarrow, "\alpha'"']
\end{tikzcd} \tag{6.1.3}$$

  Remark 6.1.5. Equivalentely, for given secondary form of 1-cells $(f, \beta, g), (f', \beta', g') \in (\mathbf{G}, \otimes)$, define 2-cell in $(\mathbf{G}, \otimes)$ as $(\phi, \psi) : (f, \beta, g) \Rightarrow (f', \beta', g')$ where $\phi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ are 2-cells in $\mathbf{Map}(\mathcal{B}_{\otimes})$ by commutativity of the following diagram

$$\begin{tikzcd}
 & f \otimes S \otimes g^* \arrow[dr, Rightarrow, "\phi\otimes 1_S\otimes 1_{g^*}"] & \\
R \arrow[ur, Rightarrow, "\beta"] \arrow[dr, Rightarrow, "\beta'"'] & & f' \otimes S \otimes g^* \\
 & f' \otimes S \otimes g'^* \arrow[ur, Rightarrow, "1_{f'}\otimes 1_S\otimes\psi^*"'] &
\end{tikzcd} \tag{6.1.4}$$

  where $\psi^* : g'^* \Rightarrow g^*$ is the mate of $\psi : g \Rightarrow g'$ with respect to the adjunctions $g \dashv_{\otimes} g^*$ and $g' \dashv_{\otimes} g'^*$.

- The horizontal composition of two 1-cells $(f, \alpha, g) : (X, R, A) \to (Y, S, B)$ and $(h, \gamma, k) : (Y, S, B) \to (Z, T, C)$ in $\mathbf{G}$ is given by $(f, \alpha, g) \otimes (h, \gamma, k) := (f \otimes h, \alpha \otimes \gamma, g \otimes k)$ where $f \otimes h$ and $g \otimes k$ are horizontal compositions of 1-cells in $\mathbf{Map}(\mathcal{B}_\otimes)$ and $\alpha \otimes \gamma$ is given by:

$$\begin{aligned}
\alpha \otimes \gamma &= R \otimes (g \otimes k) \\
&\cong (R \otimes g) \otimes k \\
&\overset{\alpha}{\Rightarrow} (f \otimes S) \otimes k \\
&\cong f \otimes (S \otimes k) \\
&\overset{\gamma}{\Rightarrow} f \otimes (h \otimes T) \\
&\cong (f \otimes h) \otimes T
\end{aligned}$$

Remark 6.1.6. Note that in the construction of $(\mathbf{G}, \otimes)$, every 1-cell $(f, \beta, g) : (X, R, A) \to (Y, S, B) \in (\mathbf{G}, \otimes)$ with $f : X \to Y, g : A \to B \in \mathbf{Map}(\mathcal{B}_\otimes)$ provides an opposite 1-cell $(f^*, \overset{\sim}{\beta}, g^*) : (Y, S, B) \to (X, R, A)$ where $f^* : Y \to X, g^* : B \to A \in \mathbf{Map}(\mathcal{B}^{co}_\otimes)$ and $\overset{\sim}{\beta} : f \otimes S \otimes g^* \Rightarrow R$ is a 2-cell in $\mathcal{B}^{co}_\otimes$. Moreover, every 2-cell $(\phi, \psi) : (f, \beta, g) \Rightarrow (f', \beta', g') \in (\mathbf{G}, \otimes)$ with $\phi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ are 2-cells in $\mathbf{Map}(\mathcal{B}_\otimes)$ provides a 2-cell $(f^*, \overset{\sim}{\beta}, g^*) \Rightarrow (f'^*, \overset{\sim}{\beta}', g'^*)$ such that $f^* \Rightarrow f'^*$ and $g^* \Rightarrow g'^*$ are 2-cells in $\mathbf{Map}(\mathcal{B}^{co}_\otimes)$. Thus, the construction of $(\mathbf{G}, \otimes)$ induces a bicategory $(\mathbf{G}^{op}, \otimes)$ based on the two-sided Grothendieck construction for the pseudofunctor 6.1.2 with respect to $\otimes$.

Proposition 6.1.7. *[13, Proposition 3.4] A 1-cell $(f, \alpha, g) : (X, R, A) \to (Y, S, B)$ in $(\mathbf{G}, \otimes)$ is an equivalence if and only if f, g are equivalences in $\mathbf{Map}(\mathcal{B}_\otimes)$ and α is invertible in $\mathcal{B}_\otimes(X, B)$.*

Proof: Assume $(f, \alpha, g) : (X, R, A) \to (Y, S, B)$ is an equivalence in $\mathbf{G}$. Then there exists a 1-cell $(h, \beta, k) : (Y, S, B) \to (X, R, A)$ together with isomorphisms:

$$\eta : \top_{(X,R,A)} \cong (f, \alpha, g) \otimes (h, \beta, k) \quad \text{and} \quad \mu : (h, \beta, k) \otimes (f, \alpha, g) \cong \top_{(Y,S,B)}.$$

which gives us:

$$\eta : (\top_X, 1_R, \top_A) \cong (f \otimes h, \alpha \otimes \beta, g \otimes k) \quad \text{and} \quad \mu : (h \otimes f, \beta \otimes \alpha, k \otimes g) \cong (\top_Y, 1_S, \top_B).$$

Then the above isomorphisms provide us isomorphisms $\top_X \cong f \otimes h$, $h \otimes f \cong \top_Y$ $\top_A \cong g \otimes k$ and $k \otimes g \cong \top_B$ in $\mathbf{Map}(\mathcal{B}_\otimes)$. So, f, g are equivalences in $\mathbf{Map}(\mathcal{B}_\otimes)$. Moreover,

$$\alpha \otimes \beta = R \otimes (g \otimes k) \Rightarrow (f \otimes h) \otimes R \cong R \otimes \top_A \Rightarrow \top_X \otimes R \cong R \Rightarrow R = 1_R.$$

And similarly $\beta \otimes \alpha \cong 1_S$. Thus α in invertible in $\mathcal{B}_{\otimes}(X, B)$.

Conversely, assume $f : X \to Y, g : A \to B$ are equivalences in $\mathbf{Map}(\mathcal{B}_{\otimes})$. So there exist 1-cells $h : Y \to X$ and $k : B \to A$ together isomorphisms $\top_X \cong f \otimes h$, $h \otimes f \cong \top_Y$ $\top_A \cong g \otimes k$ and $k \otimes g \cong \top_B$. Invertibility of α in $\mathcal{B}_{\otimes}(X, B)$ also provide us isomorphisms $1_R \cong \alpha \otimes \beta$ and $\beta \otimes \alpha \cong 1_S$. So these information provide us

$$\eta : (\top_X, 1_R, \top_A) \cong (f \otimes h, \alpha \otimes \beta, g \otimes k) \quad \text{and} \quad \mu : (h \otimes f, \beta \otimes \alpha, k \otimes g) \cong (\top_Y, 1_S, \top_B).$$

Thus $(f, \alpha, g) : (X, R, A) \to (Y, S, B)$ is an equivalence in $(\mathbf{G}, \otimes)$. $\blacksquare$

Now, By replacing $\otimes$ with $\oplus$ in Remark 6.1.6, we provide the bicategory $(\widetilde{\mathbf{G}} := \mathbf{G}^{op}, \oplus)$ based on the two-sided Grothendieck construction to the pseudofunctor in 6.1.2 as follows:

- A 0-cell of $(\widetilde{\mathbf{G}}, \oplus)$ is a triple of (X, R, A), where $R \in \mathcal{B}_{\oplus}(X, A)$.
- A 1-cell in $(\widetilde{\mathbf{G}}, \oplus)$ is a triple of $(m, \gamma, l) : (Y, S, B) \to (X, R, A)$ where $m : Y \to X, l : B \to A \in \mathbf{Map}(\mathcal{B}_{\oplus}^{co})$ and $\gamma : S \oplus l \Rightarrow m \oplus R$ is a 2-cell in $\mathcal{B}_{\oplus}^{co}$. We call this form of 1-cells *primary form.*

  Remark 6.1.8. *Equivalently we can define* 1*-cells of Grothendieck construction of* $(\widetilde{\mathbf{G}}, \oplus)$ *by triple of* $(m, \gamma, l) : (Y, S, B) \to (X, R, A)$ *such that* 2*-cell* $\theta : m^* \oplus S \oplus l \Rightarrow R \in \mathcal{B}_{\oplus}^{co}(X, A)(R, m^* \oplus S \oplus l)$ *is the mate of* 2*-cell* $\gamma : S \oplus l \Rightarrow m \oplus R$ *in* $\mathcal{B}^{co}$*. We call this form of* 1*-cells secondary form.*

- For given primary forms of 1-cells $(m, \gamma, l), (m', \gamma', l') \in (\widetilde{\mathbf{G}}, \oplus)$, define a 2-cell in $(\widetilde{\mathbf{G}}, \oplus)$ as $(\widetilde{\phi}, \widetilde{\psi}) : (m, \gamma, l) \Rightarrow (m', \gamma', l')$ where $\widetilde{\phi} : m \Rightarrow m'$ and $\widetilde{\psi} : l \Rightarrow l'$ are 2-cells in $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$ such that the following diagram commutes.

$$\begin{tikzcd}
S \oplus l' \arrow[d, Rightarrow, "\gamma'"'] & S \oplus l \arrow[l, Rightarrow, "1_S \oplus \widetilde{\psi}"'] \arrow[d, Rightarrow, "\gamma"] \\
m' \oplus R & m \oplus R \arrow[l, Rightarrow, "\widetilde{\phi} \oplus 1_R"]
\end{tikzcd} \tag{6.1.5}$$

Remark 6.1.9. Equivalentely, for given secondary form of 1-cells $(m,\gamma,l),(m',\gamma',l') \in (\widetilde{\mathbf{G}},\oplus)$, define 2-cell in $(\widetilde{\mathbf{G}},\oplus)$ as $(m,\gamma,l) \Rightarrow (m',\gamma',l')$ where $\widetilde{\phi} : m \Rightarrow m'$ and $\widetilde{\psi} : l \Rightarrow l'$ are 2-cells in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ by commutativity of the following diagram

$$\begin{tikzcd}
& m^* \oplus S \oplus l \arrow[dl, Rightarrow, "\theta"'] & \\
R & & m'^* \oplus S \oplus l \arrow[ul, Rightarrow, "\widetilde{\phi}^* \oplus 1_S \oplus 1_l"'] \arrow[dl, Rightarrow, "1_{m'^*} \oplus 1_S \oplus \widetilde{\psi}"] \\
& m'^* \oplus S \oplus l' \arrow[ul, Rightarrow, "\theta'"] &
\end{tikzcd} \tag{6.1.6}$$

where $(\widetilde{\phi})^* : m'^* \Rightarrow m^*$ is the mate of $\widetilde{\phi} : m \Rightarrow m'$ with respect to the adjunctions in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$.

- The horizontal composition of two 1-cells $(m,\gamma,l) : (Y,S,B) \to (X,R,A)$ and $(m',\gamma',l') : (X,R,A) \to (Z,T,C)$ in $\mathbf{G}$ is given by $(m,\gamma,l) \oplus (m',\gamma',l') := (m \oplus m', \gamma \oplus \gamma', l \oplus l') : (Y,S,B) \to (Z,T,C)$ where $m \oplus m'$ and $l \oplus l'$ are horizontal compositions of 1-cells in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ and $\gamma \oplus \gamma'$ is given by:

$$\begin{aligned}
\gamma \otimes \gamma' &= S \oplus (l \oplus l') \\
&\cong (S \oplus l) \oplus l' \\
&\overset{\gamma}{\Rightarrow} (m \oplus R) \oplus l' \\
&\cong m \oplus (R \oplus l') \\
&\overset{\gamma'}{\Longrightarrow} m \oplus (m' \oplus T) \\
&\cong (m \oplus m') \oplus T
\end{aligned}$$

Proposition 6.1.10. *A 1-cell $(m,\gamma,l) : (Y,S,B) \to (X,R,A)$ in $(\widetilde{\mathbf{G}},\oplus)$ is an equivalence if and only if m,l are equivalences in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ and γ is invertible in $\mathcal{B}^{co}_{\oplus}(Y,A)$.*

Proof: Similar to the provided proof for 6.1.7 and replacing $\otimes$ with $\oplus$. ∎

Proposition 6.1.11. *[13, Lemma 3.6, Lemma 3.8] Let $\mathcal{B}$ be a precartesian linear bicategory, then bicategory $(\mathbf{G},\otimes)$ has finite bicategorical products, where the binary product is denoted by $\widetilde{\boxtimes}$.*

Remark 6.1.12. We denote $R \mathbin{\widetilde{\boxtimes}} S$ for the product in $(\mathbf{G}, \otimes)$ and $R \mathbin{\widetilde{\boxplus}} S$ for the coproduct in $(\widetilde{\mathbf{G}}, \oplus)$, to separate the product $\widetilde{\boxtimes}$ in $(\mathbf{G}, \otimes)$ and the coproduct $\widetilde{\boxplus}$ in $(\widetilde{\mathbf{G}}, \oplus)$ from the product $\boxtimes$ in $\mathbf{Map}(\mathcal{B}_{\otimes})$ and the coproduct $\boxplus$ in $\mathbf{Map}(\mathcal{B}_{\oplus}^{co})$ respectively.

Definition 6.1.13. For given 0-cells $(X, R : X \to A, A)$, $(Y, S : Y \to B, B)$ in $(\widetilde{\mathbf{G}}, \oplus)$, consider 0-cell $(X \Box Y, R \mathbin{\widetilde{\boxplus}} S, A \Box B) \in (\widetilde{\mathbf{G}}, \oplus)$ where $R \mathbin{\widetilde{\boxplus}} S : X \Box Y \to A \Box B$ is defined by

$$R \mathbin{\widetilde{\boxplus}} S = (i^*_{X,Y} \oplus R \oplus i_{A,B}) \vee (j^*_{X,Y} \oplus S \oplus j_{A,B})$$

And we define coprojections $i_{R,S} : R \to R \mathbin{\widetilde{\boxplus}} S$ and $j_{R,S} : S \to R \mathbin{\widetilde{\boxplus}} S$ of $\widetilde{\mathbf{G}}$, in their secondary forms, by

$$i_{R,S} = (i_{X,Y}, \iota, i_{A,B}) \quad \text{and} \quad j_{R,S} = (j_{X,Y}, \kappa, j_{A,B})$$

as the following diagram

$$\begin{tikzcd}
X \arrow[d, "R"'] & X\Box Y \arrow[l, "i^*_{X,Y}"'] \arrow[r, "j^*_{X,Y}"] \arrow[d, "R\widetilde{\boxplus}S"] & Y \arrow[d, "S"] \\
A \arrow[r, "i_{A,B}"'] & A\Box B & B \arrow[l, "j_{A,B}"]
\arrow[from=1-1, to=2-2, phantom, "\overset{\iota}{\Longrightarrow}"]
\arrow[from=1-3, to=2-2, phantom, "\overset{\kappa}{\Longleftarrow}"]
\end{tikzcd} \tag{6.1.7}$$

where ι, κ are the coprojections of the coproduct $\vee$ in $\mathcal{B}_{\oplus}^{co}(X \Box Y, A \Box B)$. The primary form of $i_{R,S}$ and $j_{R,S}$ are

$$\widetilde{i}_{R,S} = (i_{X,Y}, \widetilde{\iota}, i_{A,B}) \quad \text{and} \quad \widetilde{j}_{R,S} = (j_{X,Y}, \widetilde{\kappa}, j_{A,B})$$

as the following diagram

$$\begin{tikzcd}
X \arrow[d, "R"'] \arrow[r, "i_{X,Y}"] & X\Box Y \arrow[d, "R\widetilde{\boxplus}S"] & Y \arrow[l, "j_{X,Y}"'] \arrow[d, "S"] \\
A \arrow[r, "i_{A,B}"'] & A\Box B & B \arrow[l, "j_{A,B}"]
\arrow[from=1-1, to=2-2, phantom, "\overset{\widetilde{\iota}}{\Longrightarrow}"]
\arrow[from=1-3, to=2-2, phantom, "\overset{\widetilde{\kappa}}{\Longleftarrow}"]
\end{tikzcd} \tag{6.1.8}$$

where $\widetilde{\iota}$ is the mate

$$R \oplus i \xrightarrow{\eta_i \oplus 1_R \oplus 1_i} i \oplus i^* \oplus R \oplus i \xrightarrow{1_i \oplus \iota} i \oplus (i^* \oplus R \oplus i) \vee (j^* \oplus S \oplus j)$$

of ι and similarly for $\widetilde{j}_{R,S}$.

Lemma 6.1.14. *For any* 0*-cell* (Z,T,C) *in* $(\widetilde{\mathbf{G}}, \oplus)$*, the functor*

$$\Theta_{R \widetilde{\boxplus} S, T} : \widetilde{\mathbf{G}}(R \widetilde{\boxplus} S, T) \to \widetilde{\mathbf{G}}(R,T) \times \widetilde{\mathbf{G}}(S,T)$$

is essentially surjective on 0*-cells. Moreover, for any*

$$(f,\alpha,u) : (X,R,A) \to (Z,T,C) \in \widetilde{\mathbf{G}}(R,T) \quad \text{and} \quad (g,\beta,v) : (Y,S,B) \to (Z,T,C) \in \widetilde{\mathbf{G}}(S,T)$$

and for any

$$(h,w) : (X \Box Y, A \Box B) \to (Z,C)$$

with invertible 2*-cells*

$$(\mu_0,\mu_1) : (f,u) \Rightarrow (i_{X,Y}, i_{A,B}) \oplus (h,w) \quad \text{and} \quad (\nu_0,\nu_1) : (g,v) \Rightarrow (j_{X,Y}, j_{A,B}) \oplus (h,w)$$

provided by the bicategorical coproduct in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus}) \times \mathbf{Map}(\mathcal{B}^{co}_{\oplus})$*, there is a unique* ρ *making* (h,ρ,w) *a* 1*-cell* $R \boxplus S \to T$ *in* $\widetilde{\mathbf{G}}$ *with invertible* 2*-cells*

$$(\mu_0,\mu_1) : (f,\alpha,u) \Rightarrow i_{R,S} \oplus (h,\rho,w) \quad \text{and} \quad (\nu_0,\nu_1) : (g,\beta,v) \Rightarrow j_{R,S} \oplus (h,\rho,w).$$

Proof: By using secondary forms of 1-cells in $(\widetilde{\mathbf{G}}, \oplus)$, we can lift the coproduct

$$i^* \oplus R \oplus i \xrightarrow{\iota} (i^* \oplus R \oplus i) \vee (j^* \oplus S \oplus j) \xleftarrow{\kappa} j^* \oplus S \oplus j$$

in $\mathcal{B}^{co}_{\oplus}(X \Box Y, A \Box B)$ to

$$\begin{tikzcd}
h^* \oplus (i^* \oplus R \oplus i) \oplus w \arrow[r, "1_{h^*} \oplus \iota \oplus 1_w"] & h^* \oplus (i^* \oplus R \oplus i) \vee (j^* \oplus S \oplus j) \oplus w & h^* \oplus (j^* \oplus S \oplus j) \oplus w \arrow[l, "1_{h^*} \oplus \kappa \oplus 1_w"'] \\
h^* \oplus i^* \oplus R \oplus u \arrow[u, "1_{h^*} \oplus 1_{i^*} \oplus R \oplus \mu_1"] & & h^* \oplus j^* \oplus S \oplus v \arrow[u, "1_{h^*} \oplus 1_{j^*} \oplus S \oplus \nu_1"']
\end{tikzcd}$$

in $\mathcal{B}^{co}_{\oplus}(Z,C)$, since $\mathcal{B}_{\oplus}(h^*, w)$ is the left adjoint of $\mathcal{B}_{\oplus}(h, w^*)$ (see Lemma 6.1.3) and left adjoints preserve the coproducts and μ_1 and ν_1 are invertible. Now the following diagram

$$\begin{tikzcd}
h^*\oplus(i^*\oplus R\oplus i)\oplus w \arrow[rr, "h^*\oplus\iota\oplus w"] & & h^*\oplus(i^*\oplus R\oplus i)\vee(j^*\oplus S\oplus j)\oplus w \arrow[dd, dashed, "\rho"] & & h^*\oplus(j^*\oplus S\oplus j)\oplus w \arrow[ll, "h^*\oplus\kappa\oplus w"'] \\
h^*\oplus i^*\oplus R\oplus u \arrow[u, "h^*\oplus i^*\oplus R\oplus\mu_1"] \arrow[d, "\mu_0^*\oplus R\oplus f"'] & \circlearrowleft & & \circlearrowleft & h^*\oplus j^*\oplus S\oplus v \arrow[u, "h^*\oplus j^*\oplus S\oplus\nu_1"'] \arrow[d, "\nu_0^*\oplus S\oplus g"] \\
f^*\oplus R\oplus u \arrow[rr, "\alpha"'] & & T & & g^*\oplus S\oplus v \arrow[ll, "\beta"]
\end{tikzcd}$$

provides a 1-cell $(h,\rho,w) : R\boxplus S\to T$ in $(\widetilde{\mathbf{G}},\oplus)$, and the commutativity of the left square in the above diagram provides a 2-cell $(\mu_0,\mu_1):(f,\alpha,u)\Rightarrow i_{R,S}\oplus(h,\rho,w)$ in $(\widetilde{\mathbf{G}},\oplus)$ and similarly $(\nu_0,\nu_1):(g,\beta,v)\Rightarrow j_{R,S}\oplus(h,\rho,w)$ is another 2-cell in $(\widetilde{\mathbf{G}},\oplus)$. Moreover ρ is unique since the coproduct in $\mathcal{B}^{co}_{\oplus}(X\Box Y, A\Box B)$ is unique.

∎

Corollary 6.1.15. *If any pairs of* 1*-cells* $(h,\delta,w),(h,\gamma,w):(X\Box Y, R\,\widetilde{\boxplus}\,S, A\Box B)\to(Z,T,C)$ *in* $(\widetilde{\mathbf{G}},\oplus)$ *satisfy*

$$i_{R,S}\oplus(h,\delta,w)=i_{R,S}\oplus(h,\gamma,w)\quad\textit{and}\quad j_{R,S}\oplus(h,\delta,w)=j_{R,S}\oplus(h,\gamma,w).$$

Then $\delta=\gamma$ *and consequentely* $(h,\delta,w)=(h,\gamma,w)$.

Proof: By applying Lemma 6.1.14 and take $(f,\alpha,u)=i_{R,S}\oplus(h,\delta,w)$, $(g,\beta,v)=j_{R,S}\oplus(h,\gamma,w)$, and μ_0,μ_1,ν_0 and ν_1 equal identities. ∎

Lemma 6.1.16. *For any* 0*-cell* (Z,T,C) *in* $(\widetilde{\mathbf{G}},\oplus)$*, the following functor*

$$\Theta_{R\widetilde{\boxplus}S,T}:\widetilde{\mathbf{G}}(R\,\widetilde{\boxplus}\,S,T)\to\widetilde{\mathbf{G}}(R,T)\times\widetilde{\mathbf{G}}(S,T)$$

is fully faithful.

Proof: We must show

$$\widetilde{\mathbf{G}}(R\,\widetilde{\boxplus}\,S,T)\big((\widetilde{h},\widetilde{\gamma},\widetilde{w}),(\widetilde{k},\widetilde{\delta},\widetilde{x})\big)\to\big(\widetilde{\mathbf{G}}(R,T)\times\widetilde{\mathbf{G}}(S,T)\big)\big((\widetilde{h},\widetilde{\gamma},\widetilde{w}),(\widetilde{k},\widetilde{\delta},\widetilde{x})\big)$$
$$\big((\widetilde{\lambda},\widetilde{\xi}):(\widetilde{h},\widetilde{\gamma},\widetilde{w})\Rightarrow(\widetilde{k},\widetilde{\delta},\widetilde{x})\big)\mapsto(1_{i_{R,S}}\oplus(\widetilde{\lambda},\widetilde{\xi}),1_{j_{R,S}}\oplus(\widetilde{\lambda},\widetilde{\xi}))$$

is bijective. Suppose $(\widetilde{h},\widetilde{\gamma},\widetilde{w}),(\widetilde{k},\widetilde{\delta},\widetilde{x})\colon (X\square Y, R\mathbin{\widetilde{\boxplus}} S, A\square B)\to(Z,T,C)$ are 1-cells in $(\widetilde{\mathbf{G}},\oplus)$ in their primary forms. Then $(1_{i_{R,S}}\oplus(\widetilde{\lambda},\widetilde{\xi}),1_{j_{R,S}}\oplus(\widetilde{\lambda},\widetilde{\xi}))\in\big(\widetilde{\mathbf{G}}(R,T)\times\widetilde{\mathbf{G}}(S,T)\big)\big((\widetilde{h},\widetilde{\gamma},\widetilde{w}),(\widetilde{k},\widetilde{\delta},\widetilde{x})\big)$ provides the following 2-cells:

$$(\widetilde{\phi},\widetilde{\psi})\colon i_{R,S}\oplus(\widetilde{h},\widetilde{\gamma},\widetilde{w})\Rightarrow i_{R,S}\oplus(\widetilde{k},\widetilde{\delta},\widetilde{x})\quad\text{and}\quad(\widetilde{\chi},\widetilde{\omega})\colon j_{R,S}\oplus(\widetilde{h},\widetilde{\gamma},\widetilde{w})\Rightarrow j_{R,S}\oplus(\widetilde{k},\widetilde{\delta},\widetilde{x}).$$

But the above 2-cells provide us 2-cells $(\widetilde{\phi},\widetilde{\psi})\colon(i_{X,Y},i_{A,B})\oplus(\widetilde{h},\widetilde{w})\Rightarrow(i_{X,Y},i_{A,B})\oplus(\widetilde{k},\widetilde{x})$ and $(\widetilde{\chi},\widetilde{\omega})\colon(j_{X,Y},j_{A,B})\oplus(\widetilde{h},\widetilde{w})\Rightarrow(j_{X,Y},j_{A,B})\oplus(\widetilde{k},\widetilde{x})$ in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})\times\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ and since $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})\times\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ has finite bicategorical coproducts, then there are unique 2-cells $[\widetilde{\phi},\widetilde{\chi}]\colon\widetilde{h}\Rightarrow\widetilde{k}$ and $[\widetilde{\psi},\widetilde{\omega}]\colon\widetilde{w}\Rightarrow\widetilde{x}$ in $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ such that

$$(i_{X,Y},i_{A,B})\oplus([\widetilde{\phi},\widetilde{\chi}],[\widetilde{\psi},\widetilde{\omega}])=(\widetilde{\phi},\widetilde{\psi})\quad\text{and}\quad(j_{X,Y},j_{A,B})\oplus([\widetilde{\phi},\widetilde{\chi}],[\widetilde{\psi},\widetilde{\omega}])=(\widetilde{\chi},\widetilde{\omega}).$$

Next, we must prove that $([\widetilde{\phi},\widetilde{\chi}],[\widetilde{\psi},\widetilde{\omega}])$ provides a 2-cell $(\widetilde{h},\widetilde{\gamma},\widetilde{w})\Rightarrow(\widetilde{k},\widetilde{\delta},\widetilde{x})$ in $(\widetilde{\mathbf{G}},\oplus)$. That is, we must prove the equality of the following diagrams:

$$\begin{tikzcd}
X\square Y \arrow[rrr, bend left, "\widetilde{k}"] \arrow[rrr, bend right, "\widetilde{h}"'] \arrow[rrr, phantom, "{[\widetilde{\phi},\widetilde{\chi}]}\Uparrow"] \arrow[dd, "R\widetilde{\boxplus}S"'] & & & Z \arrow[dd, "T"] & & X\square Y \arrow[rrr, bend left, "\widetilde{h}"] \arrow[dd, "R\widetilde{\boxplus}S"'] & & & Z \arrow[dd, "T"] \\
& {} \arrow[r, Rightarrow, "\widetilde{\delta}"'] & {} & & = & & {} \arrow[r, Rightarrow, "\widetilde{\gamma}"] & {} & \\
A\square B \arrow[rrr, bend right, "\widetilde{x}"'] & & & C & & A\square B \arrow[rrr, bend left, "\widetilde{x}"] \arrow[rrr, bend right, "\widetilde{w}"'] \arrow[rrr, phantom, "{[\widetilde{\psi},\widetilde{\omega}]}\Uparrow"] & & & C
\end{tikzcd}$$

The above pasting composition is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
(R\mathbin{\widetilde{\boxplus}}S)\oplus\widetilde{x} \arrow[r, Rightarrow, "\widetilde{\delta}"] & \widetilde{k}\oplus T \\
(R\mathbin{\widetilde{\boxplus}}S)\oplus\widetilde{w} \arrow[u, Rightarrow, "1_{R\widetilde{\boxplus}S}\oplus{[\widetilde{\psi},\widetilde{\omega}]}"] \arrow[r, Rightarrow, "\widetilde{\gamma}"'] & \widetilde{h}\oplus T \arrow[u, Rightarrow, "{[\widetilde{\phi},\widetilde{\chi}]}\oplus 1_T"']
\end{tikzcd}$$

Then since both of the above diagrams provide a 1-cell $(\widetilde{h},\widetilde{x})\colon R\,\widetilde{\boxplus}\,S \to T$ in $(\widetilde{\mathbf{G}},\oplus)$, by using the Corollary 6.1.15 and since $(\widetilde{\phi},\widetilde{\psi})$ is a 2-cell in $(\widetilde{\mathbf{G}},\oplus)$, we get the following compositions are equal:

$$\begin{tikzcd}
& & A\Box B \arrow[dll, "\widetilde{x}"'] & & \\
C & & A\Box B \arrow[ll, "\widetilde{w}"'] \arrow[u, Rightarrow, "\widetilde{\psi}"'] & & A \arrow[ll, "i_{A,B}"'] \arrow[ull, "i_{A,B}"'] \\
& \Longleftarrow \widetilde{\gamma} = & & \Longleftarrow \widetilde{i}_{R,S} = & \\
Z \arrow[uu, "T"] & & X\Box Y \arrow[ll, "\widetilde{h}"] \arrow[uu, "R\widetilde{\boxplus}S" description] & & X \arrow[ll, "i_{X,Y}"] \arrow[uu, "R"']
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
C & & A\Box B \arrow[ll, "\widetilde{x}"'] & & A \arrow[ll, "i_{A,B}"'] \\
& \Longleftarrow \widetilde{\delta} = & & \Longleftarrow \widetilde{i}_{R,S} = & \\
Z \arrow[uu, "T"] & & X\Box Y \arrow[ll, "\widetilde{k}"] \arrow[uu, "R\widetilde{\boxplus}S" description] & & X \arrow[ll, "i_{X,Y}"] \arrow[uu, "R"'] \arrow[dll, "i_{X,Y}"] \\
& & X\Box Y \arrow[ull, "\widetilde{h}"] \arrow[u, Rightarrow, "\widetilde{\phi}"] & &
\end{tikzcd}$$

The above pasting compositions is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
R\oplus i_{A,B}\oplus\widetilde{x} \arrow[r, Rightarrow, "\widetilde{i}_{R,S}\oplus 1_{\widetilde{x}}"] & i_{X,Y}\oplus(R\,\widetilde{\boxplus}\,S)\oplus\widetilde{x} \arrow[r, Rightarrow, "1_{i_{A,B}}\oplus\widetilde{\delta}"] & i_{X,Y}\oplus\widetilde{k}\oplus T \\
R\oplus i_{A,B}\oplus\widetilde{w} \arrow[u, Rightarrow, "1_R\oplus\widetilde{\psi}"'] \arrow[dr, Rightarrow, "\widetilde{i}_{R,S}\oplus 1_{\widetilde{w}}"'] & & i_{X,Y}\oplus\widetilde{h}\oplus T \arrow[u, Rightarrow, "\widetilde{\phi}\oplus 1_T"] \\
& i_{X,Y}\oplus(R\,\widetilde{\boxplus}\,S)\oplus\widetilde{w} \arrow[ur, Rightarrow, "1_{i_{X,Y}}\oplus\widetilde{\gamma}"'] &
\end{tikzcd}$$

Similarly, by applying composition with $j_{R,S}$ in Corollary 6.1.15 we get $(\widetilde{h}, \widetilde{\gamma}, \widetilde{w}) \Rightarrow (\widetilde{k}, \widetilde{\delta}, \widetilde{x})$ in $(\widetilde{\mathbf{G}}, \oplus)$ is a 2-cell in $(\widetilde{\mathbf{G}}, \oplus)$. ▮

Proposition 6.1.17. *Let $\mathcal{B}$ be a precartesian linear bicategory, then bicategory $(\widetilde{\mathbf{G}}, \oplus)$ has finite bicategorical coproducts.*

Proof: See Lemmas 6.1.14 and 6.1.16. ▮

Theorem 6.1.18. $(\widetilde{\mathbf{G}}, \oplus)$ *has finite coproducts.*

Proof: Lemmas 6.1.14 and 6.1.16 provide the binary coproducts . We cliam that $(I, (\bot := \bot_{I,I} : I \to I), I)$ provides the initial 0-cell in $(\widetilde{\mathbf{G}}, \oplus)$. We must show that for any 0-cell $(X, R, A) \in (\widetilde{\mathbf{G}}, \oplus)$, there exists a unique 1-cell $(I, \bot_{I,I}, I) \to (X, R, A)$. The existence of this 1-cell is given by the existence of maps $\epsilon_X : I \to X$ and $\epsilon_A : I \to A$ in $\mathbf{Map}(\mathcal{B}_\oplus^{co})$ and since I is the initial 0-cell in $\mathbf{Map}(\mathcal{B}_\oplus^{co})$. Since $\mathcal{B}_\oplus(\epsilon_X^*, \epsilon_A) \dashv \mathcal{B}_\oplus(\epsilon_X, \epsilon_A^*)$ and left adjoints preserves the coproducts, so $\mathcal{B}_\oplus(\epsilon_X^*, \epsilon_A) : \mathcal{B}_\oplus^{co}(I, I) \to \mathcal{B}_\oplus^{co}(X, A)$ sends the initial object $\bot_{I,I}$ in $\mathcal{B}_\oplus^{co}(I, I)$ to initial object $\bot_{X,A} = \epsilon_X^* \oplus \epsilon_A = \epsilon_X^* \oplus \bot_{I,I} \oplus \epsilon_A$ in $\mathcal{B}_\oplus^{co}(X, A)$. Since $\bot_{X,A} = \epsilon_X^* \oplus \epsilon_A = \epsilon_X^* \oplus \bot_{I,I} \oplus \epsilon_A$ is the initial object in $\mathcal{B}_\oplus^{co}(X, A)$, there exists a unique 2-cell $\tau : \bot_{X,A} = \epsilon_X^* \oplus \bot_{I,I} \oplus \epsilon_A \Rightarrow R$. But this 2-cell is the secondary form of $\bot_{I,I} \oplus \epsilon_A \Rightarrow \epsilon_X \oplus R$ which provides us a 1-cell $\epsilon_R : \bot \to R$ in $(\widetilde{\mathbf{G}}, \oplus)$. Specifically, $(\epsilon_X, \epsilon_R, \epsilon_A) : (I, \bot_{I,I}, I) \to (X, R, A)$ is a 1-cell in $(\widetilde{\mathbf{G}}, \oplus)$. The uniqueness of 1-cell $(I, \bot_{I,I}, I) \to (X, R, A)$ is given by the uniqueness of 1-cells $\epsilon_X : I \to X$ and $\epsilon_A : I \to A$ and the uniqueness of τ. Thus, for any 1-cell $(m, \gamma, l) : (Y, S, B) \to (X, R, A)$ we get a unique 2-cell $\epsilon_R \Rightarrow \epsilon_S \oplus (m, \gamma, l)$ in $(\widetilde{\mathbf{G}}, \oplus)$, and it is invertible. ▮

Remark 6.1.19. We denote the 1-cell $\bot_{I,I} \oplus \epsilon_A \Rightarrow \epsilon_X \oplus R$ with $(\epsilon_X, \widetilde{\epsilon_R}, \epsilon_A)$ which is the primary form of $(\epsilon_X, \epsilon_R, \epsilon_A)$. Additionally, $(\epsilon_X, \widetilde{\epsilon_R}, \epsilon_A)$ is the component of a pseudonatural transformation $\epsilon : I^\oplus ; !^\oplus \Rightarrow 1_{\widetilde{\mathbf{G}}_\oplus}$ which is the unit for the pseudoadjunction $I^\oplus \dashv !^\oplus : \mathbb{1}_\oplus \to \widetilde{\mathbf{G}}_\oplus$, where $!^\oplus : \widetilde{\mathbf{G}}_\oplus \to \mathbb{1}_\oplus$ is the unique pseudofunctor and $\mathbb{1}_\oplus$ is the terminal bicategory.

Remark 6.1.20. If we put $A = B$ and $X = Y$ in the diagram 6.1.8, then we get $\nabla^* \oplus (R \mathbin{\widetilde{\boxplus}} S) \oplus \nabla = \nabla^* \oplus (i^* \oplus R \oplus i) \vee (j^* \oplus S \oplus j) \oplus \nabla \cong \nabla^* \oplus i^* \oplus R \oplus i \oplus \nabla \vee \nabla^* \oplus j^* \oplus S \oplus j \oplus \nabla \cong R \vee S$. So, we get $R \vee S = \nabla^* \oplus (R \mathbin{\widetilde{\boxplus}} S) \oplus \nabla$. Moreover, if $R = S$ then we get the formula $R \vee R \cong \nabla^* \oplus (R \mathbin{\widetilde{\boxplus}} R) \oplus \nabla$. Then by composing this isomorphism with $\widetilde{\nabla} : R \vee R \Rightarrow R$ we get 2-cell $\nabla_R : \nabla^*_A \oplus (R \boxplus R) \oplus \nabla_X \Rightarrow R$ which is the secondary form of $\widetilde{\nabla}_R : R \boxplus R \Rightarrow R$. Thus we get $\widetilde{\nabla}_R$ as components of the unit for the pseudoadjunction $\boxplus \dashv \Delta^{\oplus} : \widetilde{\mathbf{G}}_{\oplus} \times \widetilde{\mathbf{G}}_{\oplus} \to \widetilde{\mathbf{G}}_{\oplus}$.

Proposition 6.1.21. *For a precartesian linear bicategory $\mathcal{B}$, the coproduct $\widetilde{\boxplus}$ in bicategory $\widetilde{\mathbf{G}}_{\oplus}$ provides a pseudofunctor $\widetilde{\boxplus} : \widetilde{\mathbf{G}}_{\oplus} \times \widetilde{\mathbf{G}}_{\oplus} \to \widetilde{\mathbf{G}}_{\oplus}$.*

Proof: Proof is similar to the provided proof in 2.3.3 in chapter 4. ▮

Remark 6.1.22. Each 2-cell $\alpha : S \Rightarrow R : X \to A$ in $\mathcal{B}^{co}_{\oplus}$ provides a 1-cell $(\perp_X, \alpha, \perp_A)$ in $(\widetilde{\mathbf{G}}, \oplus)$ as

$$\begin{tikzcd}
X \arrow[dd, "R"'] & & X \arrow[ll, "\perp_X"'] \arrow[dd, "S"] \\
& {} & {} \arrow[l, Rightarrow, "\alpha"'] \\
A & & A \arrow[ll, "\perp_A"]
\end{tikzcd}$$

Moreover, If $\gamma : R \Rightarrow T : X \to A$ is another 2-cell, the composition of 1-cells $(\perp_X, \alpha, \perp_A), (\perp_X, \gamma, \perp_A)$ is

$$(\perp_X, \alpha, \perp_A) \oplus (\perp_X, \gamma, \perp_A) = (\perp_X, \alpha \oplus \gamma, \perp_A).$$

$$\begin{tikzcd}
X \arrow[dd, "T"'] & & X \arrow[ll, "\perp_X"'] \arrow[dd, "R"'] & & X \arrow[ll, "\perp_X"'] \arrow[dd, "S"] \\
& {} & {} \arrow[l, Rightarrow, "\gamma"'] & {} & {} \arrow[l, Rightarrow, "\alpha"'] \\
A & & A \arrow[ll, "\perp_A"] & & A \arrow[ll, "\perp_A"]
\end{tikzcd}$$

Now, consider the coproduct $(\perp_X, \alpha, \perp_A) \mathbin{\widetilde{\boxplus}} (\perp_Y, \beta, \perp_B)$ in $(\widetilde{\mathbf{G}}, \oplus)$, where $\alpha : R' \Rightarrow R : X \to A$ and $\beta : S' \Rightarrow S : Y \to B$. Since the coproduct $\widetilde{\boxplus}$ in $(\widetilde{\mathbf{G}}, \oplus)$ provides a pesudofunctor, provided by the Proposition 6.1.21, then there exist invertible 2-cells

$$i_{R',S'} \oplus (\bot_X, \alpha, \bot_A) \,\widetilde{\boxplus}\, (\bot_Y, \beta, \bot_B) \cong (\bot_X, \alpha, \bot_A) \oplus i_{R,S}$$
$$j_{R',S'} \oplus (\bot_X, \alpha, \bot_A) \,\widetilde{\boxplus}\, (\bot_Y, \beta, \bot_B) \cong (\bot_Y, \beta, \bot_B) \oplus j_{R,S}.$$

Then by using the secondary forms of $i_{R',S'} = (i_{X,Y}, \iota, i_{A,B})$ and $j_{R',S'} = (j_{X,Y}, \kappa, j_{A,B})$ we get:

$$(i_{X,Y}, i_{A,B}) \oplus (\bot_{X\Box Y}, \bot_{A\Box B}) = (\bot_X, \bot_A) \oplus (i_{X,Y}, i_{A,B})$$
$$(j_{X,Y}, j_{A,B}) \oplus (\bot_{X\Box Y}, \bot_{A\Box B}) = (\bot_Y, \bot_B) \oplus (i_{X,Y}, i_{A,B})$$

then Lemma 6.1.14 provides a unique 1-cell $\widetilde{\phi} : R' \,\widetilde{\boxplus}\, S' \to R \,\widetilde{\boxplus}\, S$ such that

$$i_{R',S'} \oplus (\bot_{X\Box Y}, \widetilde{\phi}, \bot_{A\Box B}) = (\bot_X, \alpha, \bot_A) \oplus i_{R,S}$$
$$j_{R',S'} \oplus (\bot_{X\Box Y}, \widetilde{\phi}, \bot_{A\Box B}) = (\bot_Y, \beta, \bot_B) \oplus j_{R,S}.$$

Next, by replacing $\widetilde{\phi}$ by $\alpha \,\widetilde{\boxplus}\, \beta$ and replacing $i_{R',S'}$ and $j_{R',S'}$ with their values in the above equations we get:

$$\iota \oplus (\alpha \,\widetilde{\boxplus}\, \beta) = (i^* \oplus \alpha \oplus i) \oplus \iota$$
$$\kappa \oplus (\alpha \,\widetilde{\boxplus}\, \beta) = (j^* \oplus \beta \oplus j) \oplus \kappa.$$

which provide us
$$\alpha \,\widetilde{\boxplus}\, \beta = (i^* \oplus \alpha \oplus i) \vee (j^* \oplus \beta \oplus j).$$

Thus, we extended the formula $R \,\widetilde{\boxplus}\, S = (i^* \oplus R \oplus i) \vee (j^* \oplus S \oplus j)$ to 2-cells. And, it provides us a functor

$$\widetilde{\boxplus}_{(X,A),(Y,B)} : \mathcal{B}_{\oplus}^{co} \times \mathcal{B}_{\oplus}^{co}\big((X,A),(Y,B)\big) = \mathcal{B}_{\oplus}^{co}(X,Y) \times \mathcal{B}_{\oplus}^{co}(A,B) \to \mathcal{B}_{\oplus}^{co}(X\Box Y, A\Box B)$$

by the following compositions:

$$\mathcal{B}_{\oplus}^{co}(X,A) \times \mathcal{B}_{\oplus}^{co}(Y,B) \xrightarrow{\mathcal{B}_{\oplus}(i^*,i)\times\mathcal{B}_{\oplus}(j^*,j)} \mathcal{B}_{\oplus}^{co}(X\Box Y, A\Box B) \times \mathcal{B}_{\oplus}^{co}(X\Box Y, A\Box B) \overset{\vee}{\to}$$
$$\mathcal{B}_{\oplus}^{co}(X\Box Y, A\Box B).$$

Theorem 6.1.23. *Let $\mathcal{B}$ be a precartesian linear bicategory, then if we define $\widetilde{\boxplus}$ on 0-cells by setting $X \widetilde{\boxplus} Y = X \Box Y$. Then the product $\widetilde{\boxplus} : \mathcal{B}^{co}_{\oplus}(X, A) \times \mathcal{B}^{co}_{\oplus}(Y, B) \to \mathcal{B}^{co}_{\oplus}(X \Box Y, A \Box B)$ provides a lax functor $\mathcal{B}^{co}_{\oplus} \times \mathcal{B}^{co}_{\oplus} \to \mathcal{B}^{co}_{\oplus}$.*

Proof: Let

$$(\widetilde{\boxplus}, \widetilde{\boxplus}^{2}, \widetilde{\boxplus}^{0}) : \mathcal{B}^{co}_{\oplus} \times \mathcal{B}^{co}_{\oplus} \to \mathcal{B}^{co}_{\oplus}$$

First, for 0-cells $X, Y \in \mathcal{B}^{co}_{\oplus}$, by Lemma 6.1.14 there is a unique $\widetilde{\boxplus}^{0} : \bot_X \widetilde{\boxplus} \bot_Y \to \bot_{X \widetilde{\boxplus} Y}$ satisfying:

$$\begin{tikzcd}
X \widetilde{\boxplus} Y \arrow[dd, "\bot_{X \widetilde{\boxplus} Y}"'] & X \widetilde{\boxplus} Y \arrow[l, "\bot_{X \widetilde{\boxplus} Y}"'] \arrow[dd, "\bot_X \widetilde{\boxplus} \bot_Y"] & X \arrow[l, "i_{X,Y}"'] \arrow[dd, "\bot_X"] & & X \widetilde{\boxplus} Y \arrow[dd, "\bot_{X \widetilde{\boxplus} Y}"'] & X \arrow[l, "i_{X,Y}"'] \arrow[dd, "\bot_X"] \\
{} \arrow[r, phantom, "\overset{\widetilde{\boxplus}^{0}}{\Longleftarrow}"] & {} \arrow[r, phantom, "\overset{\widetilde{\iota}_{\bot,\bot}}{\Longleftarrow}"] & {} & = & & \\
X \widetilde{\boxplus} Y & X \widetilde{\boxplus} Y \arrow[l, "\bot_{X \widetilde{\boxplus} Y}"] & X \arrow[l, "i_{X,Y}"] & & X \widetilde{\boxplus} Y & X \arrow[l, "i_{X,Y}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \widetilde{\boxplus} Y \arrow[dd, "\bot_{X \widetilde{\boxplus} Y}"'] & X \widetilde{\boxplus} Y \arrow[l, "\bot_{X \widetilde{\boxplus} Y}"'] \arrow[dd, "\bot_X \widetilde{\boxplus} \bot_Y"] & Y \arrow[l, "j_{X,Y}"'] \arrow[dd, "\bot_Y"] & & X \widetilde{\boxplus} Y \arrow[dd, "\bot_{X \widetilde{\boxplus} Y}"'] & Y \arrow[l, "j_{X,Y}"'] \arrow[dd, "\bot_Y"] \\
{} \arrow[r, phantom, "\overset{\widetilde{\boxplus}^{0}}{\Longleftarrow}"] & {} \arrow[r, phantom, "\overset{\widetilde{\kappa}_{\bot,\bot}}{\Longleftarrow}"] & {} & = & & \\
X \widetilde{\boxplus} Y & X \widetilde{\boxplus} Y \arrow[l, "\bot_{X \widetilde{\boxplus} Y}"] & Y \arrow[l, "j_{X,Y}"] & & X \widetilde{\boxplus} Y & Y \arrow[l, "j_{X,Y}"]
\end{tikzcd}$$

Next, for given 1-cells $R : X \to A$, $S : Y \to B$, $T : A \to L$ and $U : B \to M$, the pasting of the following diagram

$$\begin{tikzcd}
X \arrow[r, "i_{X,Y}"] \arrow[d, "R"'] \arrow[dr, phantom, "= \tilde{\iota}_{R,S} \Longrightarrow"] & X \tilde{\boxplus} Y \arrow[d, "R\boxplus S"] & Y \arrow[l, "j_{X,Y}"'] \arrow[d, "S"] \arrow[dl, phantom, "\Longleftarrow \tilde{\kappa}_{R,S} ="] \\
A \arrow[r, "i_{A,B}"'] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\tilde{\iota}_{T,U}}{\Longrightarrow}"] & A \tilde{\boxplus} B \arrow[d, "T\tilde{\boxplus}U"] & B \arrow[l, "j_{A,B}"] \arrow[d, "U"] \arrow[dl, phantom, "\overset{\tilde{\kappa}_{T,U}}{\Longleftarrow}"] \\
L \arrow[r, "i_{L,M}"'] & L \tilde{\boxplus} M & M \arrow[l, "j_{L,M}"]
\end{tikzcd}$$

gives a 1-cell $\tilde{\boxplus}^2 : (R \oplus T) \tilde{\boxplus} (S \oplus U) \to (R \tilde{\boxplus} S) \oplus (T \tilde{\boxplus} U)$ in $(\tilde{\mathbf{G}}, \oplus)$ which is unique by Lemma 6.1.14. Then we must show the above data satisfy the associativity and unitary coherence conditions. For associativity we must show the two following diagrams are equal for all 1-cells $R : X \to A$, $S : Y \to B$, $T : A \to L$ and $U : B \to M$, $V : L \to C$ and $W : M \to D$ in $\mathcal{B}_{\oplus}^{co}$:

$$\begin{tikzcd}
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "\perp"'] \arrow[dd, "(R\oplus T)\tilde{\boxplus}(S\oplus U)"] & \bullet \arrow[l, "\perp"'] \arrow[ddd, "(R\oplus T\oplus V)\tilde{\boxplus}(S\oplus U\oplus W)"] & & \bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "\perp"'] \arrow[d, "R\tilde{\boxplus}S"] & \bullet \arrow[l, "\perp"'] \arrow[ddd, "(R\oplus T\oplus V)\tilde{\boxplus}(S\oplus U\oplus W)"] \\
\bullet \arrow[d, "T\tilde{\boxplus}U"'] \arrow[r, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & {} \arrow[r, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & & = & \bullet \arrow[d, "T\tilde{\boxplus}U"'] & \bullet \arrow[l, "\perp"'] \arrow[dd, "(T\oplus V)\tilde{\boxplus}(U\oplus W)"] \arrow[r, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & {} \\
\bullet \arrow[d, "V\tilde{\boxplus}W"'] & \bullet \arrow[l, "\perp"] \arrow[d, "V\tilde{\boxplus}W"] & & & \bullet \arrow[d, "V\tilde{\boxplus}W"'] \arrow[r, phantom, "\underset{\tilde{\boxplus}^2}{\Longleftarrow}"] & {} & \\
\bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "\perp"] & & \bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "\perp"]
\end{tikzcd}$$

The equality of the above diagrams follows by applying Corollary 6.1.15, and considering the effect of pasting $i_{(R\oplus T\oplus V),(S\oplus U\oplus W)}$ and $j_{(R\oplus T\oplus V),(S\oplus U\oplus W)}$ on the left and right diagrams in the above, respectively. For example by pasting $i_{(R\oplus T\oplus V),(S\oplus U\oplus W)}$

on the left side diagram we get:

$$\begin{tikzcd}
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "\perp"'] \arrow[dd, dotted, "(R\oplus T)\tilde{\boxplus}(S\oplus U)"] \arrow[ddl, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & \bullet \arrow[l, "\perp"'] \arrow[ddd, dotted, "(R\oplus T\oplus V)\tilde{\boxplus}(S\oplus U\oplus W)"] \arrow[ddl, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & \bullet \arrow[l, "i"'] \arrow[ddd, dashed, "R\oplus T\oplus V"] \arrow[dddl, phantom, "\overset{\tilde{i}_{(R\oplus T\oplus V),(S\oplus U\oplus W)}}{\Longleftarrow}"] & & \bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "\perp"'] \arrow[dd, dotted, "(R\oplus T)\tilde{\boxplus}(S\oplus U)"] \arrow[ddl, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & \bullet \arrow[l, "i"'] \arrow[dd, "R\oplus T"] \arrow[ddl, phantom, "\overset{\tilde{i}_{V,W}}{\Longleftarrow}"] & \\
\bullet \arrow[d, "T\tilde{\boxplus}U"'] & & & & = & \bullet \arrow[d, "T\tilde{\boxplus}U"'] & & & = \\
\bullet \arrow[d, "V\tilde{\boxplus}W"'] & \bullet \arrow[l, "\perp"] \arrow[d, "V\tilde{\boxplus}W"] & & & & \bullet \arrow[d, "V\tilde{\boxplus}W"'] & \bullet \arrow[l, "\perp"] \arrow[d, "V\tilde{\boxplus}W"'] & \bullet \arrow[l, "i"] \arrow[d, "V"] \arrow[dl, phantom, "\underset{\tilde{i}_{V,W}}{\Longleftarrow}"] & \\
\bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "i"] & & \bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "i"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "i"'] \arrow[dl, phantom, "\overset{\tilde{i}_{R,S}}{\Longleftarrow}"] \\
\bullet \arrow[d, "T\tilde{\boxplus}U"'] & \bullet \arrow[l, "i"] \arrow[u, "R"'] \arrow[dl, phantom, "\overset{\tilde{i}_{T,U}}{\Longleftarrow}"] \\
\bullet \arrow[d, "V\tilde{\boxplus}W"'] & \bullet \arrow[l, "i"] \arrow[u, "T"'] \arrow[dl, phantom, "\overset{\tilde{i}_{V,W}}{\Longleftarrow}"] \\
\bullet & \bullet \arrow[l, "i"] \arrow[u, "V"']
\end{tikzcd}$$

Similarly, for the unitary coherence conditions we must show the equality of the following diagrams:

$$\begin{tikzcd}
X\tilde{\boxplus}Y \arrow[d, "\perp X\tilde{\boxplus}Y"'] & X\tilde{\boxplus}Y \arrow[l, "\perp"'] \arrow[d, "\perp_X\tilde{\boxplus}\perp Y"] \arrow[dl, phantom, "\overset{\tilde{\boxplus}^0}{\Longleftarrow}"] & X\tilde{\boxplus}Y \arrow[l, "\perp"'] \arrow[dd, "R\tilde{\boxplus}S"] \arrow[ddl, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & = & X\tilde{\boxplus}Y \arrow[d, "R\tilde{\boxplus}S"'] & X\tilde{\boxplus}Y \arrow[l, "\perp"'] \arrow[d, "R\tilde{\boxplus}S"] & = \\
X\tilde{\boxplus}Y \arrow[d, "R\tilde{\boxplus}S"'] & X\tilde{\boxplus}Y \arrow[l, "\perp"] \arrow[d, "R\tilde{\boxplus}S"] & & & A\tilde{\boxplus}B & A\tilde{\boxplus}B \arrow[l, "\perp"] & \\
A\tilde{\boxplus}B & A\tilde{\boxplus}B \arrow[l, "\perp"] & A\tilde{\boxplus}B \arrow[l, "\perp"] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \tilde{\boxplus} Y \arrow[d, "\perp X\tilde{\boxplus}Y"'] & X \tilde{\boxplus} Y \arrow[l, "\perp"'] \arrow[d, "R\tilde{\boxplus}S"] & X \tilde{\boxplus} Y \arrow[l, "\perp"'] \arrow[dd, "R\tilde{\boxplus}S"] \\
X \tilde{\boxplus} Y \arrow[r, "\perp"] \arrow[d, "R\tilde{\boxplus}S"'] \arrow[dr, phantom, "\overset{\tilde{\boxplus}^0}{\Longleftarrow}"] & X \tilde{\boxplus} Y \arrow[d, "\perp_X\tilde{\boxplus}\perp_Y"] \arrow[r, phantom, "\overset{\tilde{\boxplus}^2}{\Longleftarrow}"] & {} \\
A \tilde{\boxplus} B & A \tilde{\boxplus} B \arrow[l, "\perp"] & A \tilde{\boxplus} B \arrow[l, "\perp"]
\end{tikzcd}$$

To show this equality we apply again Corollary 6.1.15 and consider the effect of pasting i and j on the left and right diagrams:

$$\begin{tikzcd}
\bullet \arrow[d, "\perp X\tilde{\boxplus}Y"'] & \bullet \arrow[l, "\perp"'] \arrow[d, "\perp_X\tilde{\boxplus}\perp Y"] \arrow[dl, phantom, "\overset{\tilde{\boxplus}^0}{\Longleftarrow}"] & \bullet \arrow[l, "\perp"'] \arrow[dd, dotted, "R\tilde{\boxplus}S" description] & \bullet \arrow[l, "i"'] \arrow[dd, "R"] \\
\bullet \arrow[r, "\perp"'] \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[d, "R\tilde{\boxplus}S"'] \arrow[r, phantom, "\underset{\tilde{\boxplus}^2}{\Longleftarrow}"] & {} \arrow[r, phantom, "\underset{\tilde{i}_{R,S}}{\Longleftarrow}"] & {} \\
\bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "\perp"] & \bullet \arrow[l]
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[d, "\perp_{X\square Y}"'] & \bullet \arrow[l, "\perp"'] \arrow[d, dotted, "\perp_X\tilde{\boxplus}\perp_Y" description] \arrow[dl, phantom, "\overset{\tilde{\boxplus}^0}{\Longleftarrow}\wr"] & \bullet \arrow[l, "i"'] \arrow[d, "\perp"] \arrow[dl, phantom, "\overset{\tilde{i}_{\perp_X,\perp_Y}}{\Longleftarrow}"] \\
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "\perp"] \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "i"] \arrow[d, "R"] \arrow[dl, phantom, "\underset{\tilde{i}_{R,S}}{\Longleftarrow}"] \\
\bullet & \bullet \arrow[l, "\perp"] & \bullet \arrow[l, "i"]
\end{tikzcd}
=$$

$$\begin{tikzcd}
\bullet \arrow[d, "\perp_{X\tilde{\boxplus}Y}"'] & \bullet \arrow[l, "i"'] \arrow[d, "\perp_X"] \\
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "i"] \arrow[d, "R"] \\
\bullet & \bullet \arrow[l, "i"]
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[d, "R\tilde{\boxplus}S"'] & \bullet \arrow[l, "i"'] \arrow[d, "R"] \arrow[dl, phantom, "\underset{\tilde{i}_{R,S}}{\Longleftarrow}"] \\
\bullet & \bullet \arrow[l, "i"]
\end{tikzcd}$$

▮

Proposition 6.1.24. *[13, Proposition 3.18] If $\mathcal{B}$ is a precartesian linear bicategory, then the 0-cell $I \in \mathcal{B}$, the 1-cell $\top_\otimes : I \to I$, and the 2-cells $(\tilde{I})^\otimes : \top_\otimes \otimes \top_\otimes \Rightarrow \top_\otimes$ and $(I^0)^\otimes : 1_I \Rightarrow \top_\otimes$ construct a lax functor $I^\otimes : \mathbb{1}_\otimes \to \mathcal{B}_\otimes$. That is, $I^\otimes : \mathbb{1}_\otimes \to \mathcal{B}_\otimes$ is a $\otimes$-monad.*

Proof: See [13, section 3.17] and [28, Explanation 6.4.3]. ▮

Theorem 6.1.25. *[13, Theorem 3.15] Let $\mathcal{B}$ be a precartesian linear bicategory, then there are lax functors $\tilde{\boxtimes} : \mathcal{B}_\otimes \times \mathcal{B}_\otimes \to \mathcal{B}_\otimes$ and $I^\otimes : \mathbb{1}_\otimes \to \mathcal{B}_\otimes$.*

Proposition 6.1.26. *If $\mathcal{B}$ is a precartesian linear bicategory, then there is a* 0*-cell* $I \in \mathcal{B}$*, such that for a* 1*-cell* $\perp_{\oplus} : I \to I$*, the* 2*-cells* $((I)^{\oplus})^2 : \perp_{\oplus} \Rightarrow \perp_{\oplus} \oplus \perp_{\oplus}$ *and* $(I^0)^{\oplus} : \perp_{\oplus} \Rightarrow 1_I$ *construct a colax functor* $I^{\oplus} : \mathbb{1}_{\oplus} \to \mathcal{B}_{\oplus}$*. Essentially,* $I^{\oplus} : \mathbb{1}_{\oplus} \to \mathcal{B}_{\oplus}$ *is a* $\oplus$*-comonad.*

Proof: We define the colax functor $I^{\oplus} : \mathbb{1}_{\oplus} \to \mathcal{B}_{\oplus}$ by the following:

- It sends the unique 0-cell $\star \in \mathbb{1}_{\oplus}$ to $I^{\oplus}(\star) = I \in \mathbf{Map}(\mathcal{B}^{co}_{\oplus})$.
- It sends single 1-cell $\perp_{\star} : \star \to \star \in \mathbb{1}_{\oplus}$ to $I^{\oplus}(\perp_{\star}) : I^{\oplus}(\star) \to I^{\oplus}(\star)$ which gives us a 1-cell $\perp_{I,I} = I^{\oplus}(\perp_{\star}) : I \to I$ in $\mathcal{B}_{\oplus}$. But this 1-cell is the initial object in $\mathcal{B}^{co}_{\oplus}(I, I)$.
- It has a unique 2-cell $\delta = (I^{\oplus}_{\perp_{\star},\perp_{\star}})^2 : I^{\oplus}(\perp_{\star} \oplus \perp_{\star}) \Rightarrow I^{\oplus}(\perp_{\star}) \oplus I^{\oplus}(\perp_{\star})$ which gives us a unique 2-cell $\delta : \perp_{I,I} \Rightarrow \perp_{I,I} \oplus \perp_{I,I}$. This 2-cell is unique since it is an arrow from the initial object in $\mathcal{B}^{co}_{\oplus}(I, I)$.
- It has a unique 2-cell $\epsilon = (I^{\oplus})^0 : I^{\oplus}(\perp_{\star}) \Rightarrow 1_{I^{\oplus}(\star)}$ which gives us a unique 2-cell $\epsilon : \perp_{I,I} \Rightarrow 1_I$. Again 2-cell ϵ is unique since it is an arrow from the initial object in $\mathcal{B}^{co}_{\oplus}(I, I)$.

Then we can easily verify that the above data easily satisfy a comonad equations. Thus, $(I, \perp_{I,I}, \delta, \epsilon)$ provides a comonad in $\mathcal{B}_{\oplus}$.

∎

Theorem 6.1.27. *Let $\mathcal{B}$ be a precartesian linear bicategory, then there are lax functors* $\widetilde{\boxtimes} : \mathcal{B}_{\otimes} \times \mathcal{B}_{\otimes} \to \mathcal{B}_{\otimes}$ *and* $\widetilde{\boxplus} : \mathcal{B}^{co}_{\oplus} \times \mathcal{B}^{co}_{\oplus} \to \mathcal{B}^{co}_{\oplus}$ *where they share the same* 0*-cells denoted by* $X \square Y$*, for every pair of* 0*-cells* $X, Y \in \mathcal{B}$*.*

Proof: See Theorem 6.1.23 and Theorem 6.1.25. ∎

Theorem 6.1.28. *Let $\mathcal{B}$ be a precartesian linear bicategory, then there are lax functors* $I^{\otimes} : \mathbb{1}_{\otimes} \to \mathcal{B}_{\otimes}$ *and* $I^{\oplus} : \mathbb{1}^{co}_{\oplus} \to \mathcal{B}^{co}_{\oplus}$ *where they share the same* 0*-cells.*

Proof: See Proposition 6.1.26 and Theorem 6.1.25. ∎

6.2 Cartesian Linear Bicategories

Definition 6.2.1. A precartesian linear bicategory $\mathcal{B}$ is called *cartesian* if the lax functors $(\widetilde{\boxtimes}, \widetilde{\boxplus})$, provided in Theorem 6.1.27 construct a linear pseudofunctor, denoted by $\square : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, and the lax functors $(I^{\otimes}, I^{\oplus})$, demonstrated in Theorem 6.1.28 construct the linear pseudofunctor $I : \mathbb{1} \to \mathcal{B}$.

Example 6.2.2. *Let $\mathcal{B}$ be a $\star$-autonomous linear bicategory. If $\mathcal{B}_{\otimes}$ is cartesian, then $\mathcal{B}$ is cartesian. It follows from the fact that if $\mathcal{B}$ is $\star$-autonomous and $\mathcal{B}_{\otimes}$ is cartesian, then we can define a linear pseudofunctor using the tensor product $\square_{\otimes} := \boxtimes$ of $\mathcal{B}_{\otimes}$, with the second component $\square_{\oplus} := \boxplus = (\boxtimes((-)^{\perp}))^{\perp}$ on $\mathcal{B}^{co}_{\oplus}$. Additionally by utilizing the $\star$-autonomous structure, we get a cocartesian structure on $\mathcal{B}_{\oplus}$.*

Proposition 6.2.3. *If $\mathcal{B}$ is a cartesian linear bicategory, then the linear pseudofunctors*

$$\mathcal{B} \times \mathcal{B} \xrightarrow{\square} \mathcal{B} \xleftarrow{I} \mathbb{1}$$

1. *resstric to $\mathbf{Map}(\mathcal{B}_{\otimes})$ gives the right adjoints:*

$$\Delta^{\otimes} \dashv \square_{\otimes} \qquad and \qquad !^{\otimes} \dashv I^{\otimes}$$

   where $\Delta^{\otimes} : \mathbf{Map}(\mathcal{B}_{\otimes}) \to \mathbf{Map}(\mathcal{B}_{\otimes}) \times \mathbf{Map}(\mathcal{B}_{\otimes})$ is the diagonal pseudofunctor and $!^{\otimes} : \mathbf{Map}(\mathcal{B}_{\otimes}) \to \mathbb{1}_{\otimes}$ is the unique pseudofunctor;

2. *restric to $\mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ gives the adjunctions:*

$$\square_{\oplus} \dashv \Delta^{\oplus} \qquad and \qquad I^{\oplus} \dashv !^{\oplus}$$

   where $\Delta^{\oplus} : \mathbf{Map}(\mathcal{B}^{co}_{\oplus}) \to \mathbf{Map}(\mathcal{B}^{co}_{\oplus}) \times \mathbf{Map}(\mathcal{B}^{co}_{\oplus})$ is the diagonal pseudofunctor and $!^{\oplus} : \mathbf{Map}(\mathcal{B}^{co}_{\oplus}) \to \mathbb{1}^{co}_{\oplus}$ is the unique pseudofunctor;

3. *the composites:*

$$\mathcal{B}_{\otimes}(X,Y) \times \mathcal{B}_{\otimes}(X,Y) \xrightarrow{\square_{\otimes}} \mathcal{B}_{\otimes}(X \square X, Y \square Y) \xrightarrow{\mathcal{B}_{\otimes}(\Delta_X, \Delta^*_Y)} \mathcal{B}_{\otimes}(X,Y)$$

$$\mathcal{B}_{\otimes} \xleftarrow{\mathcal{B}_{\otimes}(t_x, t^*_Y)} \mathcal{B}_{\otimes}(I,I) \xleftarrow{\top} \mathbb{1}_{\otimes}$$

   provide right adjoints to:

$$\mathcal{B}_{\otimes}(X,Y) \xrightarrow{\Delta^{\otimes}} \mathcal{B}_{\otimes}(X,Y) \times \mathcal{B}_{\otimes}(X,Y)$$

$$\mathcal{B}_{\otimes}(X,Y) \xrightarrow{!^{\otimes}} \mathbb{1}_{\otimes}$$

4. *the composites:*

$$\mathcal{B}_{\oplus}^{co}(X,Y) \times \mathcal{B}_{\oplus}^{co}(X,Y) \xrightarrow{\Box_{\oplus}} \mathcal{B}_{\oplus}^{co}(X\Box X, Y\Box Y) \xrightarrow{\mathcal{B}_{\oplus}^{co}(\nabla_X^*, \nabla_Y)} \mathcal{B}_{\oplus}^{co}(X,Y)$$

$$\mathcal{B}_{\oplus}^{co} \xleftarrow{\mathcal{B}_{\oplus}(\epsilon_X^*, \epsilon_Y)} \mathcal{B}_{\oplus}(I,I) \xleftarrow{\top} \mathbb{1}_{\oplus}$$

provide left adjoints to:

$$\mathcal{B}_{\oplus}^{co}(X,Y) \xrightarrow{\Delta^{\oplus}} \mathcal{B}_{\otimes}(X,Y) \times \mathcal{B}_{\otimes}(X,Y)$$

$$\mathcal{B}_{\oplus}^{co}(X,Y) \xrightarrow{!^{\oplus}} \mathbb{1}_{\oplus}$$

Moreover, the pseudofunctors $\Box$ and I which satisfy the above conditions are unique. Thus we can define a cartesian linear bicategory as a linear bicategory $\mathcal{B}$ with pseudofunctors $\Box$ and I which satisfy conditions 1-4.

Proof: Proofs for the items 1 and 3 are provided in [13]. The proofs for 2 and 4 also follow from the Remarks 6.1.19 and 6.1.20. ∎

Theorem 6.2.4. *A cartesian linear bicategory $(\mathcal{B}, \Box, I)$ provides two symmetric monoidal bicategories which are linked in linear settings.*

Proof: By Theorem 4.6 in [13], we get a symmetric monoidal bicategory $(\mathcal{B}, \Box_{\otimes}, I)$ and similarly we get another monoidal bicategory $(\mathcal{B}, \Box_{\oplus}, I)$ such that they are related by a linear pseudofunctor $\Box$.

∎

6.3 Matrices

In this section, we explore the linear bicategory of matrices of a $\star$-autonomous linearly distributive category equipped with linear products and coproducts [18]. Before exploring this specific linear bicategory, it is essential to briefly review the relevant notions from linearly distributive categories.

6.3.1 Linearly Distributive Categories (LDCs)

Linearly distributive categories (LDCs) provide a categorical semantics for Multiplicative Linear Logic (MLL). The two tensor products in an LDC correspond to the multiplicative conjunction and multiplicative disjunction in linear logic respectively. Informally, a linearly distributive category is a with two monoidal structures linked by a linear distributor. In this section we recall the Definition of LDCs.

Definition 6.3.1. *[17]* A *linearly distributive category*, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a category $\mathbb{X}$ consisting of:

- A monoidal structure $(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)$.
- A monoidal structure $(\oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$.
- The tensor $\otimes$ and the par (cotensor) $\oplus$ are linked by the following natural transformations which are called the left and the right linear *distributors* respectively:

$$\delta^l : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$
$$\delta^r : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

satisfying the following coherence conditions:

- The assosicators and the unitors for the $\otimes$ and the $\oplus$ satisfy pentagon diagram and unit diagram.
- Coherence conditions for unit natural transformations and linear distributors:

  (a) $\delta^l; (u^l_\otimes \oplus 1_B) = u^l_\otimes$

$$\begin{tikzcd}
\top \otimes (A \oplus B) \arrow[d, "\delta^l"'] \arrow[dr, "u^l_\otimes"] & \\
(\top \otimes A) \oplus B \arrow[r, "u^l_\otimes \oplus 1_B"'] & A \oplus B
\end{tikzcd}$$

  (b) $u^r_\otimes = \delta^r; (1 \oplus u^l_\otimes)$

  (c) $\delta^r; u^l_\oplus = u^l_\oplus \otimes 1_B$

  (d) $1 \otimes u^r_\oplus = \delta^l; u^l_\oplus$

$$\begin{tikzcd}
(\bot \oplus A) \otimes B \arrow[r, "\delta^r"] \arrow[dr, "u^l_\oplus \otimes 1_B"'] & \bot \oplus (A \otimes B) \arrow[d, "u^l_\oplus"] \\
& A \otimes B
\end{tikzcd}$$

- Coherences for associativity natural transformations and the distributors:

  (a) $a_\otimes; (1_A \otimes \delta^l); \delta^l = \delta^l; (a_\otimes \oplus 1_D)$

$$\begin{tikzcd}
(A \otimes B) \otimes (C \oplus D) \arrow[r, "a_\otimes"] \arrow[dd, "\delta^l"'] & A \otimes (B \otimes (C \oplus D)) \arrow[d, "1_A \otimes \delta^l"] \\
& A \otimes ((B \otimes C) \oplus D) \arrow[d, "\delta^l"] \\
((A \otimes B) \otimes C) \oplus D \arrow[r, "a_\otimes \oplus 1_D"'] & (A \otimes (B \otimes C)) \oplus D
\end{tikzcd}$$

(b) $\delta^l; (a_\otimes \oplus 1) = a_\otimes; (1 \otimes \delta^l); \delta^l$

(c) $\delta^r; a_\oplus = (a_\oplus \otimes 1_D); \delta^r; (1_A \otimes \delta^r)$

(d) $(1 \otimes a_\oplus); \delta^l = \delta^l; (1 \oplus \delta^l); a_\oplus$

- Coherences between the left and the right linear distributors:

(a) $\delta^r; (1_A \oplus \delta^l) = \delta^l; (\delta^r \oplus 1_D); a_\otimes$

$$\begin{tikzcd}
& (A \oplus B) \otimes (C \oplus D) \arrow[dl, "\delta^l"'] \arrow[dr, "\delta^r"] & \\
((A \oplus B) \otimes C) \oplus D) \arrow[d, "\delta^r \oplus 1_D"'] & & A \oplus (B \otimes (C \oplus D)) \arrow[d, "1_A \oplus \delta^l"] \\
(A \oplus (B \otimes C)) \oplus D \arrow[rr, "a_\otimes"'] & & A \oplus ((B \otimes C) \oplus D)
\end{tikzcd}$$

(b) $a_\otimes; (1 \otimes \delta^r); \delta^l = (\delta^l \otimes 1); \delta^r$

Definition 6.3.2. *[17]* A *symmetric* LDC is an LDC in which both the tensor products are symmetric, with symmetry maps $c_\otimes$ and $c_\oplus$, such that $\delta^R = c_\otimes; (1 \otimes c_\oplus); \delta^L; (c_\otimes \oplus 1); c_\oplus$. For a symmetric LDC, the left linear distributor determines the right linear distributor and vice versa.

[55, Defnition 2.8]

Definition 6.3.3. Suppose $\mathbb{X}$ is an LDC and $A, B \in \mathbb{X}$, then B is *left dual* (or left linear adjoint) to A – or A is *right dual* (right linear adjoint) to B – written $(\eta, \epsilon) : B \dashv\!\!\dashv A$, if there exists a unit map, $\eta : \top \to B \oplus A$ and a counit map, $\epsilon : A \otimes B \to \bot$ such that the following diagrams commute:

$$\begin{tikzcd}
B \arrow[r, "(u_\otimes^L)^{-1}"] \arrow[d, equal] & \top \otimes B \arrow[r, "\eta \otimes 1"] & (B \oplus A) \otimes B \arrow[d, "\delta_R"] \\
B & B \oplus \bot \arrow[l, "u_\oplus^R"] & B \oplus (A \otimes B) \arrow[l, "1 \oplus \epsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(u_\otimes^R)^{-1}"] \arrow[d, equal] & A \otimes \top \arrow[r, "1 \otimes \eta"] & A \otimes (B \oplus A) \arrow[d, "\delta_L"] \\
A & \bot \oplus A \arrow[l, "u_\oplus^L"] & (A \otimes B) \oplus A \arrow[l, "\epsilon \oplus 1"]
\end{tikzcd}$$

A dual, $(\eta, \epsilon) : A \dashv\!\!\mid B$ such that A is both left and right dual of B, is called a *cyclic dual*. In a symmetric LDC, every dual $(\eta, \epsilon) : A \dashv\!\!\mid B$ gives another dual $(\eta c_\oplus, c_\otimes \epsilon) : B \dashv\!\!\mid A$, which is obtained by twisting the wires using the symmetry map. Thus, in a symmetric LDC, every dual is a cyclic dual.

Definition 6.3.4. *[55, Defnition 2.11]* An LDC in which every object has a chosen left and right dual, respectively $(\eta *, \epsilon *) : A^* \dashv\!\!\mid A$ and $(*\eta, *\epsilon) : A \dashv\!\!\mid {}^*A$, is a $*$-*autonomous category*.

Remark 6.3.5. In the symmetric case a left dual gives a right dual using the symmetry.

Next, we revisit the definition of linear functors in LDCs. It is important to note that this definition is similar to linear functors between linear bicategories, as given in Definition 3.3.1. The key distinction lies in the fact that a linear functor between LDCs is defined on objects, with $\otimes$ and $\oplus$ representing monoidal products. In contrast, a linear functor between linear bicategories is defined on both 0-cells and 1-cells, where $\otimes$ and $\oplus$ represent the compositions of two bicategories.

Definition 6.3.6. *[18, Definition 1]* Let $\mathbb{X}$ and $\mathbb{Y}$ be LDCs. A *linear functor* $F : \mathbb{X} \to \mathbb{Y}$ consists of

(i) A pair of functors $F = (F_\otimes, F_\oplus)$; where $(F_\otimes, m_\otimes, m_\top) : \mathbb{X} \to \mathbb{Y}$ is monoidal with respect to $\otimes$ and $(F_\oplus, n_\oplus, n_\bot) : \mathbb{X} \to \mathbb{Y}$ is comonoidal with respect to $\oplus$.

(ii) natural transformations:

$$\begin{aligned}
\nu_\otimes^R &: F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B) \\
\nu_\otimes^L &: F_\otimes(A \oplus B) \to F_\otimes(A) \oplus F_\oplus(B) \\
\nu_\oplus^R &: F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B) \\
\nu_\oplus^L &: F_\oplus(A) \otimes F_\otimes(B) \to F_\oplus(A \otimes B)
\end{aligned}$$

such that the following coherence conditions hold:

[LF.1] (a) $F_\otimes(u_\oplus^L) = \nu_\otimes^R ; (n_\bot \oplus 1) ; u_\oplus^L$

$$\begin{tikzcd}
F_\otimes(\bot \oplus A) \arrow[rr, Rightarrow, "F_\otimes(u_\oplus^L)"] \arrow[d, Rightarrow, "\nu_\otimes^R"'] & & F_\otimes(A) \\
F_\oplus(\bot) \oplus F_\otimes(A) \arrow[rr, Rightarrow, "n_\bot \oplus 1"'] & & \bot \oplus F_\otimes(A) \arrow[u, Rightarrow, "u_\oplus^L"']
\end{tikzcd}$$

(b) $F_\otimes(u^R_\oplus) = \nu^L_\otimes; (1 \oplus n_\perp); u^R_\oplus$

(c) $F_\oplus(u^L_\otimes)^{-1} = (u^L_\otimes)^{-1}; (m_\top \otimes 1); \nu^R_\oplus$

(d) $F_\oplus(u^R_\otimes)^{-1} = (u^R_\otimes)^{-1}; (m_\top \otimes 1); \nu^L_\oplus$

[LF.2] (a) $F_\otimes(a_\oplus); \nu^R_\otimes; (1 \oplus \nu^R_\otimes) = \nu^R_\otimes; (n_\oplus \oplus 1); a_\oplus$

$$\begin{tikzcd}
F_\otimes((A \oplus B) \oplus C) \arrow[r, Rightarrow, "F_\otimes(a_\oplus)"] \arrow[d, Rightarrow, swap, "\nu^R_\otimes"] & F_\otimes(A \oplus (B \oplus C)) \arrow[d, Rightarrow, "\nu^R_\otimes"] \\
F_\otimes(A \oplus B) \oplus F_\otimes(C) \arrow[d, Rightarrow, swap, "n_\oplus \oplus 1"] & F_\otimes(A) \oplus F_\otimes(B \oplus C) \arrow[d, Rightarrow, "1 \oplus \nu^R_\otimes"] \\
(F_\oplus(A) \oplus F_\oplus(B)) \oplus F_\otimes(C) \arrow[r, Rightarrow, swap, "a_\oplus"] & F_\otimes(A) \oplus (F_\otimes(B) \oplus F_\otimes(C))
\end{tikzcd}$$

(b) $F_\otimes(a_\oplus); \nu^L_\otimes; (1 \oplus n_\oplus) = \nu^L_\oplus; (\nu^L_\oplus \oplus 1); a_\oplus$

(c) $F_\otimes(a_\oplus); \nu^L_\otimes; (1 \oplus n_\oplus) = a_\otimes; (1 \otimes \nu^R_\oplus); (\nu^L_\oplus \oplus 1);$

(d) $F_\otimes(a_\oplus); \nu^R_\otimes; (1 \oplus \nu^R_\otimes) = \nu^R_\otimes; (n_\oplus \oplus 1); a_\oplus$

[LF.3] (a) $F_\otimes(a_\oplus); \nu^R_\otimes; (1 \oplus \nu^R_\otimes) = \nu^L_\otimes; (\nu^R_\otimes \oplus 1); a_\oplus$

$$\begin{tikzcd}
F_\otimes((A \oplus B) \oplus C) \arrow[r, Rightarrow, "F_\otimes(a_\oplus)"] \arrow[d, Rightarrow, swap, "\nu^L_\otimes"] & F_\otimes(A \oplus (B \oplus C)) \arrow[d, Rightarrow, "\nu^R_\otimes"] \\
F_\otimes(A \oplus B) \oplus F_\oplus(C) \arrow[d, Rightarrow, swap, "\nu^R_\otimes \oplus 1"] & F_\oplus(A) \oplus F_\otimes(B \oplus C) \arrow[d, Rightarrow, "1 \oplus \nu^R_\otimes"] \\
(F_\oplus(A) \oplus F_\otimes(B)) \oplus F_\oplus(C) \arrow[r, Rightarrow, swap, "a_\oplus"] & F_\oplus(A) \oplus (F_\otimes(B) \oplus F_\oplus(C))
\end{tikzcd}$$

(b) $(\nu^R_\oplus \otimes 1); \nu^L_\oplus; F_\oplus(a_\otimes) = a_\otimes; (1 \otimes \nu^L_\oplus); \nu^R_\oplus$

[LF.4] (a) $(1 \otimes \nu^R_\otimes); \delta^L; (\nu^R_\oplus \oplus 1) = m_\otimes; F_\otimes(\delta^L); \nu^R_\otimes$

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, Rightarrow, "1\otimes\nu_\otimes^R"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A) \otimes (F_\oplus(B) \oplus F_\otimes(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A) \otimes F_\oplus(B)) \oplus F_\otimes(C) \arrow[d, Rightarrow, "\nu_\otimes^R\oplus 1"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, Rightarrow, "\nu_\otimes^R"'] & F_\oplus(A \otimes B) \oplus F_\otimes(C)
\end{tikzcd}$$

(b) $(\nu_\otimes^L \otimes 1); \delta^R; (1 \oplus \nu_\oplus^L) = m_\otimes; F_\otimes(\delta^R); \nu_\otimes^L$

(c) $(1 \otimes \nu_\otimes^L); \delta^L; (\nu_\oplus^L \oplus 1) = \nu_\oplus^L; F_\oplus(\delta^L); n_\oplus^R$

(d) $(n_\oplus \otimes 1); \delta^R; (1 \oplus \nu_\oplus^L) = \nu_\oplus^L F_\oplus(\delta^R); n_\oplus$

[LF.5] (a) $(1 \otimes \nu_\otimes^L); \delta^L; (m_\otimes \oplus 1) = m_\otimes; F_\otimes(\delta^L); \nu_\otimes^L$

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, Rightarrow, "1\otimes\nu_\otimes^L"] \arrow[d, Rightarrow, "m_\otimes"'] & F_\otimes(A) \otimes (F_\otimes(B) \oplus F_\oplus(C)) \arrow[d, Rightarrow, "\delta_L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, Rightarrow, "F_\otimes(\delta_L)"'] & (F_\otimes(A) \otimes F_\otimes(B)) \oplus F_\oplus(C) \arrow[d, Rightarrow, "m_\otimes\oplus 1"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, Rightarrow, "\nu_\otimes^L"'] & F_\otimes(A \otimes B) \oplus F_\oplus(C)
\end{tikzcd}$$

(b) $(\nu_\otimes^R \otimes 1); \delta^R; (1 \oplus m_\otimes) = m_\otimes; F_\otimes(\delta^L); \nu_\otimes^R$

(c) $(1 \otimes n_\oplus); \delta^L; (\nu_\oplus^R \oplus 1) = \nu_\oplus^R; F_\oplus(\delta^L); n_\oplus$

(d) $(n_\oplus \otimes 1); \delta^R; (1 \oplus \nu_\oplus^L) = \nu_\oplus^L; F_\oplus(\delta^R); n_\oplus$

Definition 6.3.7. *[18, Definition 3]* A *linear transformation* $\alpha : F \to G$, between linear functors $F, G : \mathbb{X} \to \mathbb{Y}$ consists of a pair of natural transformations $\alpha = (\alpha_\otimes, \alpha_\oplus)$ such that $\alpha_\otimes : F_\otimes \to G_\otimes$ is a monoidal transformation and $\alpha_\oplus : G_\oplus \to F_\oplus$ is a comonoidal transformation satisfying the following coherence conditions:

(i) $a_\otimes; \nu^R_\otimes; (a_\oplus \oplus 1) = \nu^R_\otimes; (1 \oplus a_\otimes)$

$$\begin{tikzcd}
F_\otimes(A \oplus B) \arrow[rr, "\alpha_\otimes"] \arrow[d, "\nu^R_\otimes"'] & & G_\otimes(A \oplus B) \arrow[d, "\nu^R_\otimes"] \\
F_\oplus(A) \oplus F_\otimes(B) \arrow[dr, "1 \oplus \alpha_\otimes"'] & & G_\oplus(A) \oplus G_\otimes(B) \arrow[dl, "\alpha_\oplus \oplus 1"] \\
& F_\oplus(A) \oplus G_\otimes(B) &
\end{tikzcd}$$

(ii) $\alpha_\otimes; \nu^L_\otimes; (1 \oplus \alpha_\oplus) = \nu^L_\otimes; (\alpha_\otimes \oplus 1)$

(iii) $(1 \otimes \alpha_\otimes); \nu^L_\oplus; (\alpha_\oplus) = (\alpha_\oplus \otimes 1); \nu^L_\oplus$

(iv) $(\alpha_\otimes \otimes 1); \nu^R_\oplus; \alpha_\oplus = (1 \otimes \alpha_\oplus); \nu^R_\oplus$

Definition 6.3.8. *[18, Definition 17]* An LDC $\mathbb{X}$ has *linear terminal* object if there exist a linear constant functor $\mathbb{1} : \mathbf{1} \to \mathbb{X}$, with components $\mathbb{1} = (1, 0)$ and a linear transformation $! : id_{\mathbb{X}} \Rightarrow \mathbb{1}' : \mathbb{X} \to \mathbb{X}$ with $! = (!^\otimes, !^\oplus)$. These must satisfy the usual equation for a terminal object: $!_{\mathbb{1}} = id_{\mathbb{1}} : \mathbb{1} \Rightarrow \mathbb{1} : \mathbf{1} \to \mathbb{X}$ where $\mathbb{1}'$ is the following composition:

$$\mathbb{X} \to \mathbf{1} \xrightarrow{\mathbb{1}} \mathbb{X}.$$

Lemma 6.3.9. *[18, Lemma 18] If an* LDC *has a linear terminal object, then it has a terminal object and an initial object in the usual sense, given by $\mathbb{1}_\otimes, \mathbb{1}_\oplus$ respectively. For an object $A \in \mathbb{X}$, the unique maps*

$$A \xrightarrow{!^\otimes_A} \mathbf{1} \quad \textit{and} \quad \mathbf{0} \xrightarrow{!^\oplus_A} A$$

are given by $!^\otimes(A)$ and $!^\oplus(A)$ respectively.

Proposition 6.3.10. *[18, Proposition 19] For an* LDC*, the following are equivalent:*

(i) $\mathbb{X}$ has linear terminal object $\mathbf{1}$.

(ii) $\mathbb{X}$ has linear terminal object $\mathbf{1}$, an initial object $\mathbf{0}$, and these are distributive. That is, 1 is preserved by $\oplus$ and 0 is preserved by $\otimes$:

$$\mathbf{0} \xrightarrow{\cong} A \otimes \mathbf{0} \qquad A \oplus \mathbf{1} \xrightarrow{\cong} \mathbf{1}$$

are isomorphisms for any $A \in \mathbb{X}$.

Definition 6.3.11. *[18, Definition 21]* An LDC $\mathbb{X}$ has *linear binary products* if there exists a linear functor $\mathbb{x} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ with components $\mathbb{x}_\otimes = \times$ and $\mathbb{x}_\oplus = +$ and a linear transformations $\mathbb{\Delta} : Id_{\mathbb{X}} \Rightarrow \Delta_X ; \mathbb{x}_\otimes : \mathbb{X} \to \mathbb{X}$, with $\mathbb{\Delta} = (\Delta, \nabla)$, and $\pi_i : \mathbb{x} \Rightarrow \pi_i^{\mathbb{X}^2} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ with $\pi_i = (p_i, b_i)$ for $(i = 0, 1)$. These these must satisfy the universal property equations for cartesian products.

$$\mathbb{\Delta}; (\pi_0 \times \pi_1) = id_{\mathbb{x}} : \mathbb{x} \to \mathbb{x} : \mathbb{X} \times \mathbb{X} \to \mathbb{X},$$
$$\mathbb{\Delta}; \pi_i = id : id \to id : \mathbb{X} \to \mathbb{X} \quad (i = 0, 1)$$

Lemma 6.3.12. *[18, Lemma 22] If an* LDC $\mathbb{X}$ *has linear products; then it has cartesian products and coproducts in the usual sense; given by* $\mathbb{x}_\otimes$ *and* $\mathbb{x}_\oplus$ *respectively. For the products; the diagonal and the projections are given by* $\mathbb{\Delta}_\otimes$ *and* $(\pi_i)_\otimes$ *respectively; and for the coproducts the codiagonal and the injections are given by* $\mathbb{\Delta}_\oplus$ *and* $(\pi_i)_\oplus$*; respectively.*

6.3.2 The Linear Bicategory of Matrices of a $\star$-autonomous LDC

Consider a linearly distributive category $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ with linear products and coproducts. Then we can form the linear bicategory $\mathbf{Mat}(\mathbb{X})$ as follows.

(1) 0-cells are small sets $I, J, K, \cdots$

(2) For each pair of small sets (I, J) we define $M \in \mathbf{Mat}(\mathbb{X})(I, J)$ as generalized relation $M : I \to J$ which is a functor $M : J \times I \to \mathbb{X}$. That is, 1-cells are family $(M(j, i))_{j \in J, i \in I}$ of objects of $\mathbb{X}$, We call these 1-cells matrices of objects of $\mathbb{X}$.

(3) For matrices $M : I \to J = (M(j, i))_{j \in J, i \in I}$ and $N : I \to J = (N(j, i))_{j \in J, i \in I}$, 2-cells are natural transformations $\alpha : M \Rightarrow N$ where its components are family $(\alpha(j, i) : M(j, i) \to N(j, i))_{j \in J, i \in I}$ of morphisms of $\mathbb{X}$,

(4) For matrices $M : I \to J$ and $N : J \to K$ we define two compositions as follows:

$$M \otimes N(k, i) = \coprod_{j \in J} M(j, i) \otimes N(k, j) \quad \text{where } \otimes \text{ in the RHS is the monoidal product in } \mathbb{X}$$

$$M \oplus N(k, i) = \prod_{y \in Y} M(j, i) \oplus N(k, j) \quad \text{where } \oplus \text{ in the RHS is the monoidal product in } \mathbb{X}$$

(5) Identities $\top_\otimes$ and $\bot_\oplus$ are defined as follows:

$$\top_X(i, i') = \begin{cases} \top & \text{if } i = i' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$\perp_X(i,i') = \begin{cases} \perp & \text{if } i = i' \\ \mathbf{1} & \text{otherwise} \end{cases}$$

where $\mathbf{0}$ and $\mathbf{1}$ are initial and terminal objects of linear coproduct and linear product in $\mathbb{X}$ respectively. And $\top$ and $\perp$ in the right hand sides are monoidal identities in $\mathbb{X}$ with respect to $\otimes$ and $\oplus$ respectively.

Proposition 6.3.13. *Show* $\mathbf{Mat}(\mathbb{X})$ *is a linear bicategory.*

Proof:

(6) The vertical composition of 2-cells is the usual vertical composition of natural transformations.

(7) The horizontal composition of 2-cells is defined as follows:

$$\begin{tikzcd} I \arrow[r, bend left, "M"{name=U}] \arrow[r, bend right, "M'"'{name=D}] & J \arrow[r, bend left, "N"{name=U2}] \arrow[r, bend right, "N'"'{name=D2}] & K \arrow[from=U, to=D, Rightarrow, "\alpha"'] \arrow[from=U2, to=D2, Rightarrow, "\beta"] \end{tikzcd}$$

For any $j \in J$, $\alpha(j,i) = M(j,i) \to M'(j,i)$ and $\beta(k,i) : N(k,j) \to N'(k,j)$ we define

$$\alpha \otimes \beta(k,i) := \coprod_{j \in J} [\alpha(j,i) \otimes \beta(k,j)]$$

and similarly

$$\alpha \oplus \beta(k,i) := \prod_{j \in J} [\alpha(j,i) \oplus \beta(k,j)]$$

(8) **Associator:** we want to show matrices $M : I \to J$, $N : J \to K$ and $P : K \to L$ and $\forall (l,i) \in L \times I$ we have the following equality:

$$M \otimes (N \otimes P)(l,i) \cong (M \otimes N) \otimes P(l,i).$$

$$\begin{aligned} M \otimes (N \otimes P)(l,i) &= \coprod_{j \in J} M(j,i) \otimes \big(\coprod_{k \in K} N(k,j) \otimes P(l,k) \big) \\ &\cong \coprod_{j \in J} \coprod_{k \in K} \Big(M(j,i) \otimes \big(N(k,j) \otimes P(l,k) \big) \Big) \\ &\cong \coprod_{k \in K} \coprod_{j \in J} \Big(\big(M(j,i) \otimes N(k,j) \big) \otimes P(l,k) \Big) \\ &= \coprod_{k \in K} \Big(\coprod_{j \in J} \big(M(j,i) \otimes N(k,j) \big) \Big) \otimes P(l,k) \\ &= (M \otimes N) \otimes P(l,i) \end{aligned}$$

Similarly, we get the associative law for the second composition $\oplus$:

$$\begin{aligned}
M \oplus (N \oplus P)(l,i) &= \prod_{j\in J} M(j,i) \oplus \big(\prod_{k\in K} N(k,j) \oplus P(l,k)\big)\\
&\cong \prod_{j\in J}\prod_{k\in K}\Big(M(j,i) \oplus \big(N(k,j) \oplus P(l,k)\big)\Big)\\
&\cong \prod_{k\in K}\prod_{j\in J}\Big(\big(M(j,i) \oplus N(k,j)\big) \oplus P(l,k)\Big)\\
&= \prod_{k\in K}\Big(\prod_{j\in J}\big(M(j,i) \oplus N(k,j)\big)\Big) \oplus P(l,k)\\
&= (M \oplus N) \oplus P(l,i)
\end{aligned}$$

(9) **Unitors:** we want to show for matrix $M : I \to J$ and identity $\top_I : I \to I$ and $\forall (j,i) \in J \times I$ we have:

$$\top_I \otimes M(j,i) = M(j,i)$$

$$\begin{aligned}
\top_I \otimes M(j,i) &= \coprod_{i'\in I}\big(\top_I(i',i) \otimes M(j,i')\big)\\
&= (\top \otimes M(j,i)) \coprod_{i'\in I, i\neq i'} (\mathbf{0} \otimes M(j,i'))\\
&\cong M(j,i) \coprod \mathbf{0}\\
&\cong M(j,i)
\end{aligned}$$

and similarly, we get $M(j,i) \otimes \top_J = M(j,i)$. For unitors of the other composition we have:

$$\begin{aligned}
\bot_I \oplus M(j,i) &= \prod_{i'\in I}\big(\bot_I(i',i) \oplus M(j,i')\big)\\
&= (\bot \oplus M(j,i)) \prod_{i'\in I, i\neq i'} (\mathbf{1} \oplus M(j,i'))\\
&\cong M(j,i) \prod \mathbf{1}\\
&\cong M(j,i)
\end{aligned}$$

(10) **Distributors:** we want to show matrices $M : I \to J$, $N : J \to K$ and $P : K \to L$ and $\forall (l,i) \in L \times I$ we have the following equality:

$$M \otimes (N \oplus P)(l,i) \cong (M \otimes N) \oplus P(l,i).$$

$$\begin{aligned} M \otimes (N \oplus P)(l,i) &= \coprod_{j \in J} M(j,i) \otimes \big(\prod_{k \in K} N(k,j) \oplus P(l,k)\big) \\ &\cong \coprod_{j \in J} \prod_{k \in K} \Big(M(j,i) \otimes \big(N(k,j) \oplus P(l,k)\big)\Big) \\ &\Rightarrow \prod_{k \in K} \coprod_{j \in J} \Big(\big(M(j,i) \otimes N(k,j)\big) \oplus P(l,k)\Big) \\ &= \coprod_{k \in K} \Big(\coprod_{j \in J} \big(M(j,i) \otimes N(k,j)\big)\Big) \oplus P(l,k) \\ &= (M \otimes N) \oplus P(l,i) \end{aligned}$$

And similarly we get the other distributor.

∎

Proposition 6.3.14. *Let $\mathbb{X}$ be a symmetric $\star$-autonomous* LDC*. Then the linear bicategory* $\mathbf{Mat}(\mathbb{X})$ *has linear adjoints and it is $\star$-autonomous.*

Proof: It follows from the fact that every object in $\mathbb{X}$ has a left and right dual, since $\mathbb{X}$ is symmetric $\star$-autonomous category. ∎

In the following lemma, we demonstrate that the bicategory of maps in **Rel** is embedded in the bicategory of maps in $\mathbf{Mat}_{\otimes}(\mathbb{X})$.

Lemma 6.3.15. *[1, Lemma 2.6.2] For $\mathbb{X}$ a linearly distributive category with linear finite products and coproducts, there is a locally faithful pseudofunctor $M_{\otimes}$: $\mathbf{Map}(\mathbf{Rel}_{\otimes}) \to \mathbf{Map}(\mathbf{Mat}_{\otimes}(\mathbb{X}))$.*

Proof: We define

$$\begin{aligned} M_{\otimes} : \mathbf{Map}(\mathbf{Rel}_{\otimes}) &\to \mathbf{Map}(\mathbf{Mat}_{\otimes}(\mathbb{X})) \\ I &\mapsto I \\ (f : I \to J) &\mapsto (Mf : J \times I \to \mathbb{X}) := Mf(j,i) = \begin{cases} \mathbf{1}, & \text{if } f(i) = j \\ \mathbf{0}, & \text{otherwise} \end{cases} \end{aligned}$$

And $M1_f(j,s) = \begin{cases} 1_{\mathbf{1}}, & \text{if } f(i) = j \\ 1_{\mathbf{0}}, & \text{otherwise} \end{cases}$ since bicategory of $\mathbf{Map}(\mathsf{\mathbf{Rel}}_{\otimes}) = \mathbf{Set}_{\otimes}$ is a discrete bicategory. Next we prove that Mf is a $\otimes$-map. Define $Mf^* : J \to I$ with $Mf^*(i,j) = Mf(j,i)$. We must show $\forall (i,i') \in I \times I$, $\top_I(i,i') \to Mf \otimes Mf^*(i,i')$.

- If $i = i'$, then $\top_I(i,i') = \mathbf{1}$ and $Mf \otimes Mf^*(i,i) = \coprod_{j \in J} Mf(j,i) \otimes Mf^*(i,j) \cong Mf(f(i),i) \otimes Mf^*(i,f(i)) \cong \mathbf{1} \otimes \mathbf{1} \cong \mathbf{1}$.
- If $i \neq i'$, then $\top_I(i,i') = \mathbf{0}$ and $Mf \otimes Mf^*(i,i') = \coprod_{j \in J} Mf(j,i) \otimes Mf^*(i',j) \cong [\mathbf{0} \otimes Mf(f(i),i)] \coprod [Mf^*(i',f(i')) \otimes \mathbf{0}] \cong \mathbf{0}$.

Then we define: $\eta(i,i') = \begin{cases} 1_{\mathbf{1}}, & \text{if } i = i' \\ 1_{\mathbf{0}}, & \text{otherwise} \end{cases}$. For the count $\epsilon \colon Mf^* \otimes Mf \to I_J$, let $(j,j') \in J \times J$.

- Then if $j = j'$, $I_J(j,j) = 1$. If there exists an $i \in I$ such that $f(i) = j$, then $Mf^* \otimes Mf(j,j) \cong \mathbf{1}$, but, if there is no such i, $Mf^* \otimes Mf(j,j) \cong \mathbf{0}$.
- If $j \neq j'$, $Mf^* \otimes Mf(j,j') = \coprod_{i \in I} Mf^*(i,j) \otimes Mf(j',i) \cong \mathbf{0} \;\; \forall i \in I$

So we define

$$\epsilon(j,j') = \begin{cases} 1_{\mathbf{1}}, & \text{if } \; j = j' = f(i) \text{ for some } \; i \in I \\ ! : \mathbf{0} \to \mathbf{1} & \text{if } \; j = j' = f(i) \text{ and } \; \forall i, f(i) \neq j \\ 1_{\mathbf{0}}, & \text{otherwise} \end{cases}.$$

For the triangle identities we have :

$$\begin{aligned} \eta(j,i) \otimes Mf &= \coprod_{i' \in I} \eta(i',i) \otimes 1_{Mf}(j,i') \cong 1_{Mf}(j,i) \otimes 1_1 \cong 1_{Mf(j,i)} \\ Mf \otimes \epsilon(j,i) &= \coprod_{j' \in J} \epsilon(j,j') \otimes 1_{Mf}(j',i) \cong \epsilon(j,j) \otimes 1_{Mf}(j,i) \otimes 1_1 \cong 1_{Mf(j,i)} \\ &\cong \begin{cases} 1_{\mathbf{1}} \otimes 1_{\mathbf{1}}, & \text{if } f(i) = j \\ 1_{\mathbf{1}} \otimes 1_{\mathbf{0}} & \text{if } \exists i' \in I, i' \neq i \text{ with } \; f(i') = j \\ ! \otimes 1_{\mathbf{0}}, & \text{otherwise} \end{cases} \\ &\cong \begin{cases} 1_{\mathbf{1}}, & \text{if } f(i) = j \\ 1_{\mathbf{0}}, & \text{otherwise} \end{cases} \\ &\cong 1_{Mf(j,i)}. \end{aligned}$$

So $(Mf \otimes \epsilon) \otimes (\eta \otimes Mf) = 1_{Mf}$ and similarly $(\epsilon \otimes Mf^*) \otimes (Mf^* \otimes \eta) = 1_{Mf^*}$.
We now show that $M_\otimes : \mathbf{Map}(\mathsf{Rel}_\otimes) \to \mathbf{Map}(\mathbf{Mat}_\otimes(\mathbb{X}))$ is fully faithful on 2-cells: for a function $f : I \to J$ and the matrix $Mf(j,i) = \begin{cases} \mathbf{1}, & \text{if } f(i) = j \\ \mathbf{0}, & \text{otherwise} \end{cases}$ if $\gamma : Mf \to Mf$, then $\gamma(j,i) = \begin{cases} 1_{\mathbf{1}}, & \text{if } f(i) = j \\ 1_{\mathbf{0}}, & \text{otherwise} \end{cases} = M1_f$. ∎

Lemma 6.3.16. *For $\mathbb{X}$ a linearly distributive category with linear finite products and coproducts, there is a locally faithful pseudofunctor $M_\oplus : \mathbf{Map}(\mathsf{Rel}_\oplus^{co}) \to \mathbf{Map}(\mathbf{Mat}_\oplus^{co}(\mathbb{X}))$.*

Proof: Since **Rel** and $\mathbf{Mat}(\mathbb{X})$ are closed linear bicategories, and essentially $\star$-autonomous, then we can define $M_\oplus := (M_\otimes((\_)^\perp))^\perp$ which will be a locally faithful pseudofunctor. ∎

Proposition 6.3.17. *[1, Proposition 2.6.3] For $\mathbb{X}$ a linearly distributive category with linear finite products and coproducts, the hom-category $\mathbf{Mat}_\otimes(\mathbb{X})(I,J)$ has finite products.*

Proof: We must show $\mathbf{Mat}_\otimes(\mathbb{X})(I,J)$ has products and terminal object. Define $\top_{I,J} : I \to J$ with $\top_{I,J}(j,i) = \mathbf{1}$ for any $(j,i) \in J \times I$. Then for any matrix $M : I \to J$ and for any $(j,i) \in J \times I$ there is a unique arrow $\alpha(j,i)\colon M(j,i) \to \mathbf{1}$ where are components of a unique 2-cell $\alpha : M \Rightarrow \top_{I,J}$. Next, for any pair of matrices $(M : I \to J, N : I \to J)$ we define $M \boxtimes N : I \to J$ as $M \boxtimes N(j,i) := M(j,i) \times N(j,i)$ $\forall (j,i) \in J \times I$ where the product $\times$ in the right hand side is the linear product in $\mathbb{X}$. Moreover, we define the projections $\pi : M \boxtimes N \Rightarrow M$ and $\rho : M \boxtimes N \Rightarrow N$ by the projections $\pi(j,i) : M(j,i) \boxtimes N(j,i) \to M(j,i)$ and $\rho(j,i) : M(j,i) \boxtimes N(j,i) \to N(j,i)$ $\forall (j,i) \in J \times I$ in $\mathbb{X}$. ∎

Proposition 6.3.18. *For $\mathbb{X}$ an LDC with linear finite products and coproducts, the hom-category $\mathbf{Mat}_\oplus^{co}(\mathbb{X})(I,J)$ has finite products.*

Proof: we must show that $\mathbf{Mat}_\oplus^{co}(\mathbb{X})(J,I)$ has binary coproducts and the initial object. Define $\perp_{J,I} : J \to I$ with $\perp_{I,L}(j,i) = \mathbf{0}$ for any $(j,i) \in J \times I$. Then for any matrix $M : I \to J$ and for any $(j,i) \in J \times I$ there is a unique arrow $\beta(j,i)\colon \mathbf{0} \to M(j,i)$ where are components of a unique 2-cell $\beta : \perp_{I,IJ} \Rightarrow M$. Next, for any pair of matrices $(M : I \to J, N : I \to J)$ we define $M \boxplus N : I \to J$ as $M \boxplus N(j,i) := M(j,i) + N(j,i)$ $\forall (j,i) \in I \times I$ where the coproduct $+$ in the right hand side is the linear coproduct in $\mathbb{X}$.

Moreover, we define the coprojections $\iota : M \Rightarrow M \boxplus N$ and $\kappa : N \Rightarrow M \boxplus N$ by the coprojections $\iota(j,i) : M(j,i) \to M(j,i) \boxplus N(j,i)$ and $\kappa(j,i) : N(j,i) \to M(j,i) \boxplus N(j,i)$ $\forall (j,i) \in J \times I$ in $\mathbb{X}$. ▮

In summary, We found that every hom-category in $\mathbf{Mat}_{\otimes}(\mathbb{X})$ and $\mathbf{Mat}^{co}_{\oplus}(\mathbb{X})$ has finite products and coproducts, respectively. But, we don't know if $\mathbf{Map}(\mathbf{Mat}_{\otimes}(\mathbb{X}))$ and $\mathbf{Map}(\mathbf{Mat}^{co}_{\oplus}(\mathbb{X}))$ have finite bicategorical products and finite bicategorical coproducts, respectively, and for this to be true we may need more conditions than $\mathbb{X}$ being cartesian. For example if the pseudofunctors $M_{\otimes}$ and $M_{\oplus}$ in 6.3.15 is moreover locally essential surjective, then this would imply that $\mathbf{Map}(\mathsf{Rel}_{\otimes}) \cong \mathbf{Map}(\mathbf{Mat}_{\otimes}(\mathbb{X}))$ and $\mathbf{Map}(\mathsf{Rel}^{co}_{\oplus}) \cong \mathbf{Map}(\mathbf{Mat}^{co}_{\oplus}(\mathbb{X}))$ and so $\mathbf{Map}(\mathbf{Mat}_{\otimes}(\mathbb{X}))$ and $(\mathbf{Mat}^{co}_{\oplus}(\mathbb{X}))$ would have finite bicategorical products and finite bicategorical coproducts, respectively.

Chapter 7

Knowledge Representation in The Linear Bicategory of Relations

Knowledge representation (KR) is a branch of artificial intelligence that focuses on creating computer models to represent information about the world and solve complex problems. A key aspect of KR is the use of ontologies, which provide a structured framework for defining entities, concepts, and relationships within a specific domain. This structured organization enables computers to process, share, and reason about complex information effectively. Various languages are used in KR, including SQL for databases, RDF and OWL for building ontologies, and Semantic Nets (as discussed in [11]). Another notable language is the *ontology log*, or *olog*, introduced by Spivak and Kent in 2011. An olog uses category theory to model real-world situations in a structured way, further aiding in the organization and understanding of intricate information.

A basic *olog* is a category where objects and arrows are labeled with real-world English-language expressions to show their meanings. Objects represent *types* of things, arrows show functional relationships (also known as *aspects*, *attributes*, or *observables*), and commutative diagrams represent *facts*. Ologs offer a formal, precise, and user-friendly way to create and modify knowledge structures, providing more flexibility than traditional database schemas. They help in forming conceptual worldviews and can be extended or linked through functors to create larger networks, enabling both local and global information integration [54].

A key challenge in designing knowledge representation formalisms is balancing expressivity and tractability [38]. Expressivity is about how well a language can represent detailed and complex information, while tractability is about how efficiently that information can be processed and used. For example, first-order logic (FOL) is highly expressive because it can represent a wide range of mathematical and logical concepts, including relations, functions, and quantifiers. making it a benchmark for evaluating the expressiveness of different knowledge representation languages.

In this chapter, we first introduce an expressive olog called the *linear relational olog*, a knowledge representation framework based on linear bicategory of relations (see Definition 5.3.2). This work is inspired by the work of Evan Patterson for knowledge representation in bicategories of relations [46]. Then, we will demonstrate a correspondence between this olog and first-order logic. That is, first-order logic becomes an internal language of linear bicategory of relations. A similar study is also presented in [10], but it differs from our approach to knowledge representation.

7.1 Linear Relational Olog

In this section, after reviewing how constructing relational ologs based on bicategories of relations in 2.1.14 by Patterson [46], we will extend this structure to linear bicategories of relations. Recall that a bicategory of relations is a cartesian bicategory that satisfies the Frobenius equations in 2.1.14. Patterson in [46] provides a general recipe for constructing a categorical knowledge base. The procedure is as follows:

1. **Select a doctrine**: Choose a doctrine [44] corresponding to the concepts or processes being represented. A doctrine is a collection of categories or higher-order categories equipped with additional structures [46]. For example, in relational olog, the selected doctrine is bicategories of relations 2.1.14.

2. **Establish a finitary specification language**: Develop a finitary specification language for the chosen doctrine. That is, if the selected doctrine has as its internal language some well-known logical system, that system could serve as a specification language. For example, in relational olog, Patterson proved that the internal language corresponding to the bicategory of relations is regular logic as a fragment of first-order logic (see [46, Section 8, Appendix A]).

3. **Define an ontology**: Use the internal language to define an ontology. Mathematically, this ontology is a finitely generated category within the chosen doctrine. In relational ologs, the given olog is:

   Definition 7.1.1. *[46]* A relational ontology log (or relational olog) is a finitely presented bicategory of relations 2.1.14. In more detail, a relational olog is a bicategory of relations $\mathcal{B}$ presented by:

   - A finite set of basic types or 0-cells generators.
   - A finite set of basic relations or 1-cells generators of form $R : X \to Y$, where X, Y are object expressions.
   - A finite set of subsumption axioms or 2-cells generators of form $R \Rightarrow S$, where R, S are well-formed morphism expressions with the same domain and codomain.

We first recall the definition of a linear bicategory of relations, as outlined in 5.3.2 in Chapter 5. A linear bicategory of relations is a closed cartesian linear bicategory that satisfies the Frobenius equations 5.3.1.

Definition 7.1.2. A locally ordered closed cartesian linear bicategory is called a *linear bicategory of relations* (**LinBiRel**), if every 0-cell is discrete. In other words, this definition provides us two bicategories of relations $(\mathcal{B}, \otimes, \top)$ and $(\mathcal{B}^{co}, \oplus, \bot)$ where they are linked by linear setting.

Based on the top recipe, we select the doctrine of the linear bicategories of relations 5.3.2. Then, we will see that our definition of a linear bicategory of relations includes two bicategories of relations, $(\mathcal{B}, \otimes, \top)$ and $(\mathcal{B}^{co}, \oplus, \bot)$. These structures correspond to two complementary parts of FOL: $(\mathcal{B}, \otimes, \top)$ represents the existential conjunctive fragment, while $(\mathcal{B}^{co}, \oplus, \bot)$ represents the universal disjunctive fragment. So, as we will prove in the next section, the internal language corresponding to a linear bicategory of relations will be first-order logic. Finally, we define the *linear relational ontology log* as follows.

Definition 7.1.3. A *linear relational ontology log* (or *linear relational olog*) is a finitely presented linear bicategory of relations. In more detail, a linear relational olog is a linear bicategory of relations $\mathcal{B}$ presented by

- a finite set of basic types or 0-cells generators;
- a finite set of basic relations or 1-cells generators of form $R : X \to Y$, where X, Y are object expressions;
- a finite set of subsumption axioms or 2-cells generators of form $R \Rightarrow S$, where R, S are well-formed morphism expressions with the same domain and codomain.

7.1.1 Example

As Patterson introduced the framework of relational ologs based on the bicategories of relations in [46], he mentioned that there are certain inherent limitations in expressing specific constraints within the current relational ologs. For example, we were to take our ontology more rigorously. In that case, we might want to eliminate the concept of "frenemies" by stating that the "friend of" and "enemy of" relations are mutually exclusive. This was not feasible because we could not represent the empty relation. Additionally, we could not express unions, which prevents us from declaring that the "parent of" relation is a combination of the "mother of" and "father of" relations.

To overcome these limitations, he introduced a more expressive relational olog called distributive relational ologs, which is based on the *union bicategory of relations*

[46, Section 9.2]. In this framework, the empty relation, union relation, and negation are expressible, corresponding to the coherent logic as a fragment of first-order logic.

By utilizing a linear relational olog, we aim to demonstrate an alternative method to express negation and union.

In this section, we aim to present the features of linear ologs in a manner that is accessible and driven by examples. In a borrowed example, we demonstrate that the abstract structure of a linear bicategory of relations provides an expressive language for representing knowledge. The examples we will present are based on the example of “Friend of a friend” from [46].

The “Friend of a friend” (FOAF) ontology is a conceptual framework used to describe relationships between people and organizations.

- **Basic Types and Relations**: The ontology consists of the following basic types:
  - **Person**
  - **Organization**
  - **Number**
  - **String**

  Here are the essential relations defined in the ontology:
  - **knows**: A relation between two persons ($Person \to Person$)
  - **member of**: A relation between a person and an organization ($Person \to Organization$)
  - **friend of**: A more specific relation indicating a friendship between two persons ($Person \to Person$)
  - **works at**: A relation indicating employment between a person and an organization ($Person \to Organization$)
- **Composition of Relations**: The FOAF ontology introduces a composite relation, “friend of a friend,” which is formed by chaining two “friend of” relations:
  - **friend of friend of**: This relation signifies that if Alice is a friend of Bob, and Bob is a friend of Carol, then Alice is a friend of a friend of Carol.

  Moreover, the relation “grandparent” or “grandchild” are examples of compositions “parent of parent of” and “child of child of” respectively. And dually we get the other composition.
- **Subsumption and Symmetry**: Some relations are subsumed by others. For example:

  - **friend of** subsumes **knows**: If Alice is a friend of Bob, then Alice knows Bob.

    $$\text{friend of} \Rightarrow \text{knows}$$

  - The **knows** relation is declared symmetric: If Alice knows Bob, then Bob knows Alice.

    $$\text{knows} = \text{knows}^{\circ}$$

- **Additional Relations**:

  - **age**: Associates a person with a number representing their age ($Person \rightarrow Number$)
  - **family name** and **given name**: Associate a person with their last and first names respectively ($Person \rightarrow String$)

  These relations are treated as total functions, meaning they assign exactly one value to each person.

- **Enemy and Frenemy Relations**: To add more complexity, the ontology includes:

  - **enemy of**: A relation indicating enmity between two persons ($Person \rightarrow Person$)
  - **frenemy of**: A composite relation representing the intersection of "friend of" and "enemy of":

    $$\text{frenemy of} := \text{friend of} \wedge \text{enemy of}$$

- **Family Relations**: The ontology models basic family relationships:

  - **child of**: A relation indicating that one person is the child of another ($Person \rightarrow Person$)
  - **parent of**: The inverse of "child of":

    $$\text{parent of} := (\text{child of})^{*}$$

  - **ancestor of**: A transitive relation indicating that one person is an ancestor of another. This relation is:
    - *Reflexive*: Every person is their own ancestor.
    - *Antisymmetric*: If two people are both ancestors and descendants of each other, they are the same person.

      $$\text{ancestor of ancestor of} \Rightarrow (\text{ancestor of})$$

- **Salary**: In the FOAF ontology, the concept of a **salary** is modeled as a ternary relation with the signature:

  $$\text{salary} : \text{Person} \times \text{Organization} \to \text{Number}$$

  This relation is a partial function. Its domain is characterized by the equation:

  $$\text{salary} = \text{works at}$$

  This implies that a person works at an organization if and only if they draw a salary from that organization. This definition can also be interpreted inversely, where the "works at" relation is defined by the existence of a salary.

- **Organization**: The ontology includes the relation **works at**, which links a person to an organization. This is denoted as:

  $$\text{works at} : \text{Person} \to \text{Organization}$$

  This relation can be further elaborated by the association with the salary relation, indicating that the employment status is confirmed by the drawing of a salary.

- **Colleague**: A **colleague** in the FOAF ontology is defined as a person you know and with whom you share membership in an organization. The formal definition uses the relations "knows" and "member of":

  $$\text{colleague of} := \text{knows} \wedge (\text{member of})$$

  Given the symmetry of the "knows" relation, it can be proven that the "colleague of" relation is also symmetric:

  $$\text{knows}(x, y) \Rightarrow \text{knows}(y, x)$$

  Thus, if Alice is a colleague of Bob, Bob is also a colleague of Alice, provided they both are members of the same organization.

- **Negation**: We know that negation of any relation $R : X \to Y$ in the linear bicategory **Rel** is complement ineverse $R^{\perp} : Y \to X := (R^{*})^{c}$. So, as an expressive example of negation in linear linear relational olog consider "biological parenting" relation between two persons which is a function. Then its negation is:

  $$(\text{parent of})^{\perp} := \text{not child of}$$

  which is not a function.

- **Union**: The relation that indicates a person is "parent of" another person is disjunction or union of the relations indicating this person is "father of" or "mother of" another person.

The FOAF ontology is a powerful and intuitive framework for representing social relationships and attributes in a structured manner.

7.2 First-Order Logic and The Linear Bicategory of Relations

Patterson in [46] demonstrated that there is a correspondence between bicategories of relations and specific fragments of first-order logic. Regular logic (a fragment of first-order logic) can be effectively modelled using bicategories of relations. This correspondence enables the transfer of tools and techniques between categorical frameworks and logical systems, enriching the theoretical foundations and practical applications of both fields [46].

From a logical perspective, our definition of a linear bicategory of relations includes two bicategory of relations, $(\mathcal{B}, \otimes, \top)$ and $(\mathcal{B}^{co}, \oplus, \bot)$. These structures correspond to two complementary components of FOL: $(\mathcal{B}, \otimes, \top)$ represents the existential conjunctive fragment (regular logic), while $(\mathcal{B}^{co}, \oplus, \bot)$ represents the universal disjunctive fragment (coregular logic).

We begin by defining a formal system for first-order logic, utilizing the syntax and proof system from [2, 29]

Definition 7.2.1. *[29, Definition 1.1.1].* A *(first-order) signature* Σ consists of the following data

a A set Σ-Sort of *sorts* which we write generically as $A, B, A_1, B_1, \cdots$

b A set Σ-Fun of *function symbols*, together with a map assigning to each $f \in \Sigma - \text{Fun}$ its *type*, which consists of a finite non-empty list of sorts: we write

$$f : A_1 \cdots A_n \to B$$

to indicate that f has type $(A_1, \cdots, A_n, B)$ (if $n = 0$, f is called a constant of sort B).

c A set $\Sigma - \textsf{Rel}$ of relation symbols, together with a map assigning to each $\Sigma - \textsf{Rel}$ its type, which consists of a finite list of sorts: we write:

$$R \rightarrowtail A_1 \cdots A_n$$

to indicate that R has type $(A_1 \cdots A_n)$. Similar to Patterson in [46], we use the vector notation $A := (A_1 \cdots A_n)$. To indicate that the relation symbol R has the types A, we denote it as $R : A$ or $R : (A_1, \cdots, A_n)$.

Definition 7.2.2. *[29, Definition 1.1.2]* Let Σ be a signature. The collection of *terms* over Σ is defined recursively by the clauses below; simultaneously, we define the sort of each term and write $t : A$ to denote that t is a term of sort A

- $x : A$, if x is a variable of sort A.
- $f(t_1, \cdots, t_n) : B$ if $f : A_1 \cdots A_n \to B$ is a function symbol and $t_1 : A_1, \cdots, t_n : A_n$.

Definition 7.2.3. *[29, Definition 1.1.3]* Consider the following formation rules for recursively building classes of *formulae* F over Σ, together with, for each formula ϕ, the (finite) set $FV(\phi)$ of free variables of ϕ.

1. **Relations**: $R(t_1, \cdots, t_n)$ is in F, if $R \rightarrowtail A_1 \cdots A_n$ is a relation symbol and $(t_1 : A_1, \cdots, t_n : A_n)$ are terms; the free variables of this formula are all the variables occurring in some t_i.
2. **Equality**: $(s = t)$ is in F if s and t are terms of the same sort; $FV(s = t)$ is the set of variables occurring in s or t (or both).
3. **Truth**: $\top$ is in F; $FV(\top) = \emptyset$.
4. **Binary conjunction**: $(\phi \wedge \psi)$ is in F, if ϕ and ψ are in F; $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$.
5. **Falsity**: $\bot$ is in F; $FV(\bot) = \emptyset$.
6. **Binary disjunction**: $(\phi \vee \psi)$ is in F, if ϕ and ψ are in F; $FV(\phi \vee \psi) = FV(\phi) \cup FV(\psi)$.
7. **Implication**: $(\phi \Rightarrow \psi)$ is in F, if ϕ and ψ are in F; $FV(\phi \Rightarrow \psi) = FV(\phi) \cup FV(\psi)$.
8. **Negation**: $\neg\phi$ is in F, if ϕ is in F; $FV(\neg\phi) = FV(\phi)$
9. **Existential quantification**: $(\exists x)\phi$ is in F, if ϕ is in F and x is a variable; $FV((\exists x)\phi) = FV(\phi)\ x$.
10. **Universal quantification**: $(\forall x)\phi$ is in F, if ϕ is in F and x is a variable; $FV((\forall x)\phi) = FV(\phi)\ x$.
11. **Infinitary disjunction**:$\bigvee_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and $\bigcup_{i \in I} FV(\phi_i)$ is finite. in which case the later set is $FV(\bigvee_{i \in I} \phi_i)$.
12. **Infinitary conjunction**:$\bigwedge_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and $\bigcup_{i \in I} FV(\phi_i)$ is finite. in which case the later set is $FV(\bigwedge_{i \in I} \phi_i)$.

Definition 7.2.4 (**Deduction systems (structural rules) for first-order logic**). *[29]* The structural rules consist of:

1. Identity:
$$\phi \vdash \phi$$
2. Substitution:
$$\frac{x : A \mid \phi \vdash \psi \qquad \Gamma \mid t : A}{\Gamma \mid \phi[t/x] \vdash \psi[t/x]}$$
3. Equality:
$$\phi \vdash x = x \quad (x = y) \wedge \phi \vdash \phi[y/x]$$
4. Truth:
$$\phi \vdash \top$$
5. Falsity:
$$\bot \vdash \phi$$
6. Conjunction (finite):
$$\frac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$$
$$\phi \wedge \psi \vdash \phi \quad \phi \wedge \psi \vdash \psi$$
7. Disjunction (finite):
$$\frac{\phi \vdash \chi \qquad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$$
$$\phi \vdash \phi \vee \psi \quad \psi \vdash \phi \vee \psi$$
8. Existential quantifier:
$$\frac{\Gamma, x : A \mid \phi \vdash \psi}{\overline{\Gamma \mid (\exists x : A.\phi) \vdash \psi}}$$
9. Universal quantifier:
$$\frac{\Gamma, x : A \mid \phi \vdash \psi}{\overline{\Gamma \mid \ \vdash (\forall x : A.\phi)\psi}}$$

10. Implication:

$$\frac{\phi \wedge \psi \vdash \chi}{\phi \vdash (\psi \Rightarrow \chi)}$$

11. Distributivity:

$$\phi \wedge (\psi \vee \chi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)$$

12. Frobenious:

$$\Gamma \mid \phi \wedge (\exists : \ A.\, \psi) \vdash \exists x : \ A.\ (\phi \wedge \psi) \quad [x \notin \Gamma]$$

Definition 7.2.5. *[29, Definition 1.1.4]* A *context* is a finite list $\Gamma = x : A = (x_1 : A_1, x_2 : A_2, \cdots x_n : A_n)$ of distinct variables (types). In the case where $n = 0$, we have an *empty context* ( ).

Definition 7.2.6. *[29, Definition 1.1.4]* A *sequent* over a signature Σ is a formal expression of the form $(\phi \vdash_\Gamma \psi)$ (or equivalently $\Gamma \,|\, \phi \vdash \psi$), where ϕ and ψ are formulae over Σ and Γ is a context suitable for both of them. The intended interpretation of this expression is that ψ is a logical consequence of ϕ in the context Γ, i.e. that any assignment of individual values to the variables in Γ which makes ϕ true will also make ψ true.

Definition 7.2.7. *[29, Definition 1.1.6]* A *first-order theory* is defined by a set $\mathbb{T}$ of sequents over Σ, whose elements are called the (non-logical) axioms of $\mathbb{T}$ and written as $\phi \dashv_{\mathbb{T}} \psi$. a formula ϕ *entails* ψ, written $\Gamma \,|\, \phi \vdash_{\mathbb{T}} \psi$, if the sequent $\Gamma \,|\, \phi \vdash \psi$ is deducible from the axioms of $\mathbb{T}$ using the inference rules of first-order logic. In this case we say that the sequent $\Gamma \,|\, \phi \vdash \psi$ is an *entailment* or *theorem* of $\mathbb{T}$.

We will now begin to explore the relationship between linear bicategories of relations and first-order logic by constructing the classifying category for a first-order theory. A classifying category for a first-order theory is a categorical framework that captures the logical structure of the theory. It acts as a link between the syntactic elements of first-order logic, such as formulas and proofs, and their corresponding interpretations in category theory.

Definition 7.2.8. The *classifying category* of a first-order theory $\mathbb{T}$, denoted $Cl(\mathbb{T})$, is the linear bicategory of relations whose objects are finite lists of basic types $A : (A_1, \cdots, A_n)$.

Given a context $\Gamma = x : A$, we also write $[\Gamma] := A$. Its morphisms $A \to B$ are equivalence classes of formulas in context,

$$[x : A; y : B \,|\, \phi]$$

where the equivalence relation $\sim$ is defined by

$$(x : A, y : B \,|\, \phi') \sim (x' : A, y' : B \,|\, \phi') \quad \text{iff} \quad x : A, y : B \,|\, \phi \dashv\vdash_T \phi'[x/x', y/y'].$$

whose 2-cells are the theorems of $\mathbb{T}$. In other words, the 2-cells in $Cl(\mathbb{T})$ are the entailments of $\mathbb{T}$:

$$[\Gamma; \Gamma' \,|\, \phi] \Rightarrow [\Gamma; \Gamma' \,|\, \phi] \Leftrightarrow \Gamma, \Gamma' \,|\, \phi \vdash_{\mathbb{T}} \psi.$$

In the context $(\Gamma; \Gamma')$, the semicolon acts as an extralogical marker, separating the context into the domain $[\Gamma]$ and the codomain $[\Gamma']$ of the morphism $[\Gamma; \Gamma', |, \phi]$.

Lemma 7.2.9. *The classifying category $Cl(\mathbb{T})$ of a first-order theory $\mathbb{T}$ is a linear bicategory of relations.*

Proof:

- Compositions of morphisms:

$$[x : A; y : B \mid \phi] \otimes [y : B; z : C \mid \psi] := [x : A; z : C \mid \exists y : B.\ \phi \wedge \psi]$$
$$[x : A; y : B \mid \phi] \oplus [y : B; z : C \mid \psi] := [x : A; z : C \mid \forall y : B.\ \phi \vee \psi]$$

- Identities:

$$\top_A := [x : A; x' : A \mid x = x']$$
$$\bot_A := [x : A; x' : A \mid x \neq x']$$

- Tensor product with respect $\otimes$ is defined on objects by $A \Box B := (A, B)$ and on morphisms by:

$$[\Gamma; \Delta \mid \phi] \,\Box_{\otimes}\, [\Gamma'; \Delta' \mid \phi] := [\Gamma, \Gamma'; \Delta, \Delta' \mid \phi \wedge \psi]$$

- Cotensor product with respect $\oplus$ is defined on objects again by $A \Box B := (A, B)$ and on morphisms by:

$$[\Gamma; \Delta \mid \phi] \Box_{\oplus} [\Gamma'; \Delta' \mid \phi] := [\Gamma, \Gamma'; \Delta, \Delta' | \phi \vee \psi]$$

- Unit: The empty list

$$I := ()$$

- Braidings:

$$\sigma^{\otimes}_{A,B} := [x : A; y : B; y' : B, x' : A \mid (x = x') \wedge (y = y')]$$

$$\sigma^{\oplus}_{A,B} := [x : A; y : B; y' : B, x' : A \mid (x \neq x') \vee (y \neq y')]$$

- Diagonals with respect to $\otimes$:

$$\Delta_A := [x : A, x' : A, x'' : A \mid (x = x') \wedge (x = x'')]$$

$$t_A := [x : A \mid \top]$$

- Diagonals with respect to $\oplus$:

$$\nabla_A := [x : A, x' : A, x'' : A \mid (x \neq x') \vee (x \neq x'')]$$

$$\epsilon_A := [\bot \mid x : A]$$

Now, we must show that $(Cl(\mathbb{T}), \otimes, \Box_\otimes, I)$ forms a locally ordered cartesian bicategory, as established in [46, Appendix A]. Additionally, we must show that $(Cl(\mathbb{T})^{co}, \oplus, \Box_\oplus, I)$ also constitutes a cartesian locally ordered bicategory, which can be derived as the dual of the cartesian structure on $(Cl(\mathbb{T}), \otimes, \Box_\otimes, I)$ through the use of negation. Finally, we get the distributivity of compositions like the distributivity of compositions in **Rel** and similarly for products. ▮

Next, we construct the internal language of a linear bicategory of relations.

Definition 7.2.10. An *interpretation* or *model* of a signature in a linear bicategory of relations $\mathcal{B}$ is specified by

- For every basic type $[\![A]\!]$, an object of $\mathcal{B}$.
- For every function symbol $f : A \to B$ is interpreted as a morphism $[\![f]\!] : [\![A]\!] \to [\![B]\!]$ of $\mathbf{Map}(\mathcal{B})$.
- For every relation symbol $R : (A_1, A_2, \cdots, A_n)$, a morphism $[\![R]\!] : [\![A]\!] \to I$ of $\mathcal{B}$, where we define $[\![A]\!] := [\![A_1]\!] \Box [\![A_2]\!] \cdots \Box [\![A_n]\!]$.

If $\Gamma = x : A$ is a context, we also write $[\![\Gamma]\!] := [\![A]\!]$.

Lemma 7.2.11. *Let $[\![.]\!]$ be an interpretation of a first-order theory $\mathbb{T}$ in a linear bicategory of relations $\mathcal{B}$. For every theorem*

$$\Gamma \vdash_{\mathbb{T}} \phi \quad \textit{of} \quad \mathbb{T},$$

there is a 2-cell

$$[\![\Gamma|\phi]\!] \Rightarrow [\![\Gamma|\psi]\!] \quad \textit{in} \quad \mathcal{B}.$$

Proof: The proof proceeds by induction on the derivation of a theorem in $\mathbb{T}$. According to the definition of an interpretation, every axiom of $\mathbb{T}$ holds in $\mathcal{B}$. Thus, it is sufficient to demonstrate that each inference rule of first-oder logic holds. Proofs of the identity, substitution, equality, truth, existential quantifier, frobenius and conjunction are given in [46, Appendix A]. The remaining rules are given by:

- Disjunction: If $[\![\Gamma;\Gamma' \mid \phi]\!] \Rightarrow [\![\Gamma;\Gamma' \mid \psi]\!]$ and $[\![\Gamma;\Gamma' \mid \phi]\!] \Rightarrow [\![\Gamma;\Gamma' \mid \chi]\!]$, then

$$[\![\Gamma;\Gamma' \mid \psi \vee \chi]\!] := \nabla^* \oplus ([\![\Gamma;\Gamma' \mid \psi]\!] \boxplus ([\![\Gamma;\Gamma' \mid \chi]\!]) \oplus \nabla$$

- Falsity:

$$[\![\Gamma;\Gamma' \mid \bot]\!] := \epsilon^*_{[\![\Gamma]\!]} \oplus \epsilon_{[\![\Gamma']\!]}$$

- Universal quantifier: Dual to existential quantifier.

∎

Definition 7.2.12. The *internal language* of a small linear bicategory of relations $\mathcal{B}$ is the first-order theory Lang($\mathcal{B}$) defined as follows. Its signature consists of:

- For every 0-cell $A \in \mathcal{B}$, a basic type A.
- A basic function symbol $f : A \to B$, for every pair of types (A, B) and every a morphism $[\![f]\!] : [\![A]\!] \to [\![B]\!]$ of $\mathbf{Map}(\mathcal{B})$,
- A relation symbol $R : (A_1, \cdots A_n)$ for every morphism $R : A_1 \square \cdots \square A_n \to I$ of $\mathcal{B}$.

A sequent $\Gamma \,|\, \phi \vdash \psi$ is considered an axiom of Lang($\mathcal{B}$) if and only if $[\![\Gamma|\phi]\!] \Rightarrow [\![\Gamma|\psi]\!]$ holds in $\mathcal{B}$, where $[\![.]\!]$ denotes the natural interpretation of the signature of Lang($\mathcal{B}$) in $\mathcal{B}$.

Remark 7.2.13. *The expressiveness of the internal language remains unchanged by adding function symbols for the maps since each map is already considered as a relation symbol.*

Theorem 7.2.14. *For every small linear bicategory of relations $\mathcal{B}$, there is an equivalence of categories*

$$Cl(Lang(\mathcal{B})) \simeq \mathcal{B} \quad in \quad \mathbf{LinBiRel}$$

Proof: To establish the equivalence, it is enough to construct a structure-preserving functor $F : \mathrm{Cl}(\mathrm{Lang}(\mathcal{B})) \to \mathcal{B}$ that is full, faithful, and essentially surjective on objects. Define the functor F on objects by

$$F(A) := [\![A]\!] = [\![A_1]\!] \,\square\, [\![A_2]\!] \,\square \cdots \,\square\, [\![A_n]\!],$$

where $A = (A_1, \cdots, A_n)$and each A_i is a basic type of $\mathrm{Lang}(\mathcal{B})$. If $\Gamma = x : A$ is a context, we also write $F(\Gamma) := [\![\Gamma]\!]$. And on morphisms $[\Gamma; \Gamma' \,|\, \phi] : [\Gamma] \to [\Gamma']$ define F by:

$$F([\Gamma; \Gamma' \,|\, \phi]) := [\![\Gamma; \Gamma' \,|\, \phi]\!] : F(\Gamma) \to F(\Gamma').$$

The proof that F is well-defined and possesses the necessary properties follows the sketched proof in [46, Appendix A] and their corresponding dualities. ∎

Chapter 8

Conclusion

In this thesis, we started a journey from bicategories to develop the notion of cartesian linear bicategories. We have explored the structures and notions surrounding cartesian bicategories and linear bicategories. We began by examining the foundational work on cartesian bicategories by Carboni and Walters in [14], who extended the concept of cartesian categories to bicategories and illustrated their symmetric monoidal structure. A pivotal aspect of our study on cartesian bicategories involved applying these theories to Q-Rel, the bicategory of quantale-valued relations. Although Q-Rel generally does not form cartesian bicategories, under specific conditions, such as when the quantale is a locale, it does. This discovery, together with the work of Blute, Kudzman-Blais and Niefield in [9] where they proved Q-Rel is a linear bicategory if Q is a Girard quantale led us later to present Q-Rel as a cartesian linear bicategory when Q is a Girard locale.

Our first study on cartesian linear bicategories focused on using cyclic linear adjoints and linear bicategorical products in our initial structure. In our proposed structure, we realized that this structure does not characterize our primary example, the linear bicategory Rel of sets and relations and consequently, we did not have a characterization theorem similar to Carboni and Walters in [14]. So, this made us shift our study in another direction where we can characterize Rel and have a characterization theorem similar to cartesian bicategories in [14]. This study characterized by two symmetric monoidal structures linked by linear distributions of a linear pseudofunctor. This structure, similar to the Carboni and Walters characterization for locally ordered bicategories, allows for exploring non-locally ordered cartesian linear bicategories. Finally, we introduced the concept of knowledge representation in linear bicategories of relations, drawing connections between categorical frameworks and logical systems, thereby enhancing both theoretical and practical applications.

We discovered that, like cartesian bicategories, cartesian linear bicategories inherit the complexities of the cartesian structure of bicategories. To summarize the definition of cartesian linear bicategories, a linear bicategory $\mathcal{B}$ is said to be cartesian if:

(1) $\mathbf{Map}(\mathcal{B}_\otimes)$ and $\mathbf{Map}(\mathcal{B}_\oplus^{co})$ have finite bicategorical products and coproducts, respectively, and they share the same (co)product on 0-cells.

(2) Hom-categories $\mathcal{B}_\otimes(X, Y)$ and $\mathcal{B}_\oplus^{co}(X, Y)$ have finite products and coproducts, respectively, which are denoted by $\wedge$ and $\vee$, respectively.

(3) Certain derived lax functors $\boxtimes : \mathcal{B}_\otimes \times \mathcal{B}_\otimes \to \mathcal{B}_\otimes$ and $I : \mathbb{1}_\otimes \to \mathcal{B}_\otimes$ and derived colax functors $\boxplus : \mathcal{B}_\oplus \times \mathcal{B}_\oplus \to \mathcal{B}_\oplus$ and $I : \mathbb{1}_\oplus \to \mathcal{B}_\oplus$, extending the shared 0-cell in $\mathbf{Map}(\mathcal{B}_\otimes)$ and $\mathbf{Map}(\mathcal{B}_\oplus^{co})$, are components of linear pseudofunctors.

At first sight, it appears that we duplicated the structure of cartesian bicategories by adding a corresponding cartesian structure with respect to the composition $\oplus$. However, this is not entirely accurate. We want to emphasize an essential property of this definition that might be overlooked. In the first item of the definition of cartesian linear bicategories, we require $\mathbf{Map}(\mathcal{B}_\otimes)$ and $\mathbf{Map}(\mathcal{B}_\oplus^{co})$ to have finite bicategorical products and coproducts, respectively, and these two full sub-bicategories share the same (co)products on 0-cells. This reflects the definition of cartesian bicategories and allows us to derive the lax and colax functors mentioned in item (3) of the definition of cartesian linear bicategories. These functors also share the same values on 0-cells, enabling us to define them as components of a linear pseudofunctor, since in linear functors, the two components share the same value on 0-cells.

Our exploration of cartesian linear bicategories has highlighted several key challenges and areas for further research. The development of more examples, mainly non-locally ordered ones, remains an open question. Additionally, as discussed in Chapter 6, the requirement for finite bicategorical products in $\mathbf{Map}(\mathcal{B}_\otimes)$ and finite bicategorical coproducts in $\mathbf{Map}(\mathcal{B}_\oplus^{co})$, in the first item of the definition of cartesian linear bicategories presents significant complexity. As observed in Chapter 6, verifying these finite bicategorical products in $\mathbf{Map}(\mathbf{Mat}_\otimes(\mathbb{X}))$ and coproducts in $\mathbf{Map}(\mathbf{Mat}_\oplus^{co}(\mathbb{X}))$ was not straightforward, and understanding (co)products in the bicategorical sense is far from trivial. So, finding more examples in non-locally cases can be an issue.

One of the questions that will arise after studying cartesian linear bicategories is whether applying a linear construction to a degenerate structure (where $\otimes = \oplus$), such as Frobenius algebras, will result in cartesian linear Frobenius algebras.

Another possible project involves developing the theory of linear double categories and cartesian linear double categories. First, we can focus on bicategories that can be viewed as (pseudo) double categories with all identity arrows, then extend it to general double categories. A crucial tool for this development is the definition of linear pullbacks, which can be derived from the notion of linear limits discussed in Chapter 6.

In conclusion, the theory of linear bicategories still needs to be developed more than that of bicategories. Over the past years, category theorists have predominantly favoured 2-categories, bicategories and double categories, for developing category theory in higher dimensions. However, linear bicategories have recently garnered attention in [9] by finding more interesting models for them. We hope that the work presented in this thesis will serve as a valuable resource in this context.

Appendix A

Monoidal Bicategories

This appendix provides an overview of symmetric monoidal bicategories as defined in [58]. We are thankful to Michael Stay for graciously granting permission to reproduce his work, including his beautifully crafted diagrams. Furthermore, in Chapter 3, we extend this structure to linear contexts by introducing the concept of symmetric monoidal structures in linear bicategories.

A.1 Monoidal Bicategory

for a given morphism f, any two choices of data (g, e, i) making f an adjoint equivalence are canonically isomorphic, so any choice is as good as any other. When f, g form an adjoint equivalence, we write $g = f^{\bullet}$. Any equivalence can be improved to an adjoint equivalence.

We can often take a 2-morphism and "reverse" one of its edges. Given objects A, B, C and D, morphisms $f : A \to C, g : C \to D, h : D \to B, j : A \to B$ such that h is an adjoint equivalence, and a 2-morphism

$$\begin{tikzcd}
A \arrow[rrr, bend left, "j"] \arrow[dr, "f"'] & & & B \\
& C \arrow[r, "g"'] & D \arrow[ur, "h"'] &
\end{tikzcd}$$

with $\Downarrow \alpha$ between j and $h \circ g \circ f$,

we can get a new 2-morphism

$$(\mathring{r}(g) \circ f)(e_h \circ g \circ f)(h^\bullet \circ \alpha) : h^\bullet \circ j \Rightarrow g \circ f,$$

$$\begin{tikzcd}
A \arrow[rrr, bend left, "j"] \arrow[dr, "f"'] & & \Downarrow \alpha & B \arrow[dr, "h^\bullet"] \arrow[d, phantom, "\Downarrow e_h"] & & = & A \arrow[rrr, bend left, "j"] \arrow[dr, "f"'] & & \Downarrow \alpha_1 & B \arrow[dl, "h^\bullet"] \\
& C \arrow[r, "g"'] \arrow[rrr, bend right, "g"'] & D \arrow[ur, "h"] \arrow[rr, "1_D"] \arrow[rr, phantom, shift right=3ex, "\Downarrow \mathring{r}(g)"] & {} & D & & & C \arrow[r, "g"'] & D &
\end{tikzcd}$$

where $e_h : h^\bullet \circ h \Rightarrow 1$ is the 2-morphism from the equivalence. We denote such variations of a 2-morphism by adding numeric subscripts; the number simply records the order in which we introduce them, not any information about the particular variation.

Definition A.1.1. A *monoidal bicategory* M is a bicategory in which we can "multiply" objects. It consists of the following:

- A bicategory M.
- A tensor product functor $\otimes : M \times M \to M$. This functor involves an invertible "tensorator" 2-morphism $(f \otimes g) \circ (f' \otimes g') \Rightarrow (f \circ f') \otimes (g \circ g')$ which we elide in most of the coherence equations below. The coherence theorem for monoidal bicategories implies that any 2-morphism involving the tensorator is the same no matter how it is inserted [22], Remark 3.1.6, so like the associator for composition of 1-morphisms, we leave it out.

  The *Stasheff polytopes* [56, 57] are a series of geometric figures whose vertices enumerate the ways to parenthesize the tensor product of n objects, so the number of vertices is given by the Catalan numbers; for each polytope, we have a corresponding $(n-2)$-morphism of the same shape with directed edges and faces:

  1. The tensor product of one object A is the one object A itself.
  2. The tensor product of two objects A and B is the one object $A \otimes B$.
  3. There are two ways to parenthesize the product of three objects, so we have an associator adjoint equivalence pseudonatural in A, B, C

     $$\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

     for moving parentheses from the left pair, where $(A \otimes B) \otimes C$ is denotes the functor

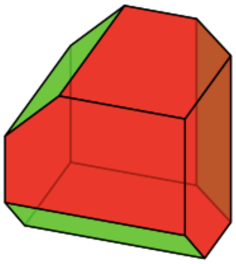
     $$\otimes \circ (\otimes \times 1) : M^3 \to M,$$

     and similarly for $A \otimes (B \otimes C)$.

4. There are five ways to parenthesize the product of four objects, so we have a *pentagonator* invertible modification π relating the two different ways of moving parentheses from being clustered at the left to being clustered at the right. (Mnemonic: Pink Pentagonator.)

$$\begin{tikzcd}
 & (AB)(CD) \arrow[ddr, "a"] & \\
((AB)C)D \arrow[ur, "a"] \arrow[dd, "aD"'] & & \\
 & \Uparrow \pi & A(B(CD)) \\
(A(BC))D \arrow[dr, "a"'] & & \\
 & A((BC)D) \arrow[uur, "Aa"'] &
\end{tikzcd}$$

5. There are fourteen ways to parenthesize the product of five objects, so we have an *associahedron* equation of modifications with fourteen vertices relating the various ways of getting from the parentheses clustered at the left to clustered at the right.



The associahedron is a cube with three of its edges bevelled, and yields the equation (SM1):

$$(SM1.a) = (SM1.b)$$

where the pasting diagram (SM1.a) and (SM1.b) are depicted in Figures 1 and 2, respectively. This holds in the bicategory M, where we have used juxtaposition instead of $\otimes$ for brevity and the unmarked 2-morphisms are instances of pseudonaturality invertible modification for the association. (Mnemonic for the rectangular invertible modifications: GReen conGRuences.)

$$\begin{tikzcd}
 & & A((B(CD))E) \arrow[rr, "Aa"] & & A(B((CD)E)) \arrow[dr, "A(Ba)"] & \\
 & (A(B(CD)))E \arrow[ur, "a"] & & & & A(B(C(DE))) \\
 & A(((BC)D)E) \arrow[uur, "A(aE)"'] \arrow[rrr, "Aa"] \arrow[rrrru, phantom, "\Downarrow A\pi"] & & & A((BC)(DE)) \arrow[ur, "Aa"] & \\
(A((BC)D))E \arrow[uur, "(Aa)E"] \arrow[ur, "a"'] \arrow[uurr, phantom, "\cong"] & & & & & \\
((A(BC))D)E \arrow[u, "aE"] \arrow[drrrr, "a"] \arrow[uurrrr, phantom, "\Downarrow \pi"] & & & & & (AB)(C(DE)) \arrow[uuu, "a"'] \\
(((AB)C)D)E \arrow[u, "(aD)E"] \arrow[drrrr, "a"'] \arrow[rrrr, phantom, "\cong"] & & & & (A(BC))(DE) \arrow[uuu, "a"] \arrow[uuuur, phantom, "\Downarrow \pi"] & \\
 & & & & ((AB)C)(DE) \arrow[u, "a(DE)"] \arrow[uur, "a"'] &
\end{tikzcd}$$

Figure A.1: Pasting diagram for axiom SM1.a

- Just as in any monoid there is an identity element 1, in every monoidal bicategory there is a monoidal unit object I. Associated to the monoidal unit are a series of morphisms—one of each dimension—that express how

$$\begin{tikzcd}
 & & A((B(CD))E) \arrow[r, "Aa"] & A(B((CD)E)) \arrow[dr, "A(Ba)"] & \\
 & (A(B(CD)))E \arrow[ur, "a"] & & & A(B(C(DE))) \\
 & & \Searrow \ \pi & & \cong \\
(A((BC)D))E \arrow[uur, "(Aa)E"] & \Rightarrow \pi E & & & \\
 & ((AB)(CD))E \arrow[uuu, "aE"] \arrow[rr, "a"] & & (AB)((CD)E) \arrow[uuuu, "a"] \arrow[ddr, "(AB)a"'] & \\
((A(BC))D)E \arrow[uu, "aE"] & & \Downarrow \ \pi & & \\
(((AB)C)D)E \arrow[u, "(aD)E"] \arrow[uur, "aE"'] \arrow[drrr, "a"'] & & & & (AB)(C(DE)) \arrow[uuuuu, "a"'] \\
 & & & ((AB)C)(DE) \arrow[ur, "a"'] &
\end{tikzcd}$$

Figure A.2: Pasting diagram for axiom SM1.b

to "cancel" the unit in a product. Each morphism of dimension $n > 0$ has two Stasheff polytopes of dimensions $n-1$ as "subcells", one for parenthesizing $n+1$ objects and the other for parenthesizing the n objects left over after cancellation. There are $n+1$ ways to insert I into n objects, so there are $n+1$ morphisms of dimension n.

1. There is one monoidal unit object I.
2. There are two unitors adjoint equivalences l ad r that are pseudonatural in A. The Stasheff polytopes for two objects and for one object are both

points, so the unitors are line segments joining them.

$$l : I \otimes A \to A \quad \text{and} \quad r : A \otimes I \to A$$

3. There are three 2-unitor invertible modification λ, μ and ρ. The Stashef polytope for three objects is a line segment, and the Stasheff polytope for two objects is a point. so these modifications are triangles. (Mnemonic: Umber Unitor.)

$$\begin{tikzcd}
(IA)B \arrow[rd, "a"] \arrow[dd, "lB"'] & \\
\Rightarrow \lambda & I(AB) \arrow[ld, "l"] \\
AB &
\end{tikzcd}$$

$$\begin{tikzcd}
(AI)B \arrow[rd, "a"] \arrow[dd, "rB"'] & \\
\Rightarrow \mu & A(IB) \arrow[ld, "Al"] \\
AB &
\end{tikzcd}$$

$$\begin{tikzcd}
(AB)I \arrow[rd, "a"] \arrow[dd, "r"'] & \\
\Rightarrow \rho & A(BI) \arrow[ld, "Ar"] \\
AB &
\end{tikzcd}$$

4. There are four equations of modifications: (SM2.i),(SM2.ii), (SM2.iii), and (SM.iv), which are depicted in Figures C.3 through C.6. The Stasheff polytope for four objects is a pentagon. The Stasheff polytope for three objects is a line segment, so these equations are irregular prisms with seven vertices. For monoidal bicategories equations (SM2.iii) and (SM2.iv) are redundant, being implied by the notion, such as monoidal triacategories. These equations would become isomorphism, which is what happens in Trimble's notion of tetracategoriy.

$$\begin{tikzcd}
((AI)C)D \arrow[r, "a"] \arrow[dd, "aD"'] & (AI)(CD) \arrow[d, "a"'] \arrow[dr, "r(CD)"] & & & ((AI)C)D \arrow[rr, "a"] \arrow[dr, "(rC)D"] \arrow[dd, "aD"'] & & (AI)(CD) \arrow[dr, "r(CD)"] & \\
\arrow[r, phantom, "\Uparrow \pi"] & A(I(CD)) \arrow[r, "Al"] \arrow[ur, phantom, "\Nearrow \mu^{-1}"] & A(CD) & = & \arrow[r, phantom, "\Uparrow \mu^{-1}D"] & (AC)D \arrow[rr, "a"] \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & & A(CD) \\
(A(IC))D \arrow[r, "a"'] & A((IC)D) \arrow[u, "Aa"] \arrow[ur, "A(lD)"'] \arrow[u, phantom, "\Nwarrow A\lambda"'] & & & (A(IC))D \arrow[ur, "(Al)D"'] \arrow[rr, "a"'] & & A((IC)D) \arrow[ur, "A(lD)"'] &
\end{tikzcd}$$

Figure A.3: Axiom SM2.i

$$\begin{tikzcd}
(AB)(ID) \arrow[r, "a"] & A(B(ID)) \arrow[dr, "A(Bl)"] & & & (AB)(ID) \arrow[rr, "a"] \arrow[dr, "(AB)l"] & & A(B(ID)) \arrow[dr, "A(Bl)"] & \\
((AB)I)D \arrow[u, "a"] \arrow[d, "aD"'] \arrow[r, phantom, "\Uparrow \pi"] & \arrow[r, phantom, "\Uparrow A\mu"] & A(BD) & = & ((AB)I)D \arrow[u, "a"] \arrow[d, "aD"'] \arrow[r, "rD"] \arrow[ur, phantom, "\Uparrow \mu"] \arrow[dr, phantom, "\Uparrow \rho^{-1}D"] & (AB)D \arrow[rr, "a"] \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & & A(BD) \\
(A(BI))D \arrow[r, "a"'] & A((BI)D) \arrow[uu, "Aa"] \arrow[ur, "A(rD)"'] & & & (A(BI))D \arrow[ur, "(Ar)D"'] \arrow[rr, "a"'] & & A((BI)D) \arrow[ur, "A(rD)"'] &
\end{tikzcd}$$

Figure A.4: Axiom SM2.ii

$$\begin{tikzcd}
((IB)C)D \arrow[r, "a"] \arrow[d, "aD"'] & (IB)(CD) \arrow[dr, "l(CD)"] \arrow[dd, "a"'] & & & ((IB)C)D \arrow[rr, "a"] \arrow[d, "aD"'] \arrow[dr, "(lC)D"] & & (IB)(CD) \arrow[dr, "(lC)D"] \arrow[d, phantom, "\cong"] & \\
(I(BC))D \arrow[d, "a"'] \arrow[r, phantom, "\Uparrow \pi"] & \arrow[r, phantom, "\Uparrow \lambda^{-1}"] & B(CD) & = & (I(BC))D \arrow[r, "lD"] \arrow[d, "a"'] \arrow[ur, phantom, "\Uparrow \lambda^{-1}D"] \arrow[dr, phantom, "\Uparrow \lambda^{-1}"] & (BC)D \arrow[rr, "a"] & & B(CD) \\
I((BC)D) \arrow[r, "Ia"'] & I(B(CD)) \arrow[ur, "l"'] & & & I((BC)D) \arrow[rr, "Ia"'] \arrow[ur, "l"'] & & I(B(CD)) \arrow[ur, "l"'] \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

Figure A.5: Axiom SM2.iii

$$\begin{tikzcd}
(AB)(CI) \arrow[r, "a"] & A(B(CI)) \arrow[dr, "A(Br)"] & & & (AB)(CI) \arrow[rr, "a"] \arrow[dr, "(AB)r"] & & A(B(CI)) \arrow[dr, "A(Br)"] \arrow[d, phantom, "\cong"] & \\
\arrow[r, phantom, "\Uparrow \pi"] & A((BC)I) \arrow[u, "Aa"] \arrow[r, "Ar"] \arrow[ur, phantom, "\Uparrow A\rho"] \arrow[dr, phantom, "\Nwarrow \rho"] & A(BC) & = & \arrow[r, phantom, "\Uparrow \rho"] & (AB)C \arrow[rr, "a"] & & A(BC) \\
((AB)C)I \arrow[uu, "a"] \arrow[r, "aI"'] & (A(BC))I \arrow[u, "a"] \arrow[ur, "r"'] & & & ((AB)C)I \arrow[uu, "a"] \arrow[rr, "aI"'] \arrow[ur, "r"'] & & (A(BC))I \arrow[ur, "r"'] \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

Figure A.6: Axiom SM2.iv

A.2 Braided Monoidal Bicategory

Definition A.2.1. A *braided* monoidal bicategory M is a monoidal bicategory in which objects can be moved past each other. A braided monoidal bicategory consists of the following:

- A monoidal bicategory M
- A series of morphisms for "shuffling".

Definition A.2.2. A *shuffle* of a list $\mathcal{A} = (A_1, \dots, A_n)$ into a list a list $\mathcal{B} = (B_1, \dots, B_k)$ inserts each element of $\mathcal{A}$ into $\mathcal{B}$ such that if $0 < i < j < n+1$ then A_i appears to the left of A_j.

An “(n,k)-shuffle polytope” is an n-dimensional polytope whose vertices are all the different shuffles of an n-element list into k-element list; there are $\binom{n+k}{k}$ ways to do this. General shuffle polytopes were defined by Kapronov and Voevodsky [32]. As with the Stasheff polytopes, we have morphisms of the same shape as (n,k)-shuffle polytopes with directed edges and faces.

- $(n=1, k=1)$: $\binom{1+1}{1} = 2$, so this polytope has two vertices, (A,B) and (B,A). It has a single edge, which we call a “braiding”, which encodes how A moves past B. It is an adjoint equivalence pseudonatural in A, B.

$$b : AB \to BA$$

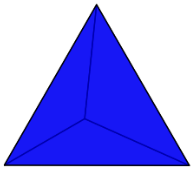
- $(n=1, k=2)$: $\binom{1+2}{1} = \binom{2+1}{1} = 3$, so so whenever the associator is the identity–e.g. in a braided strictly monoidal bicategory–these polytopes are triangles, invertible modifications whose edges are the directed $(1,1)$ polytope, the braiding.

$$\begin{tikzcd}
ABC & & BCA \\
& \Downarrow R & \\
& BAC &
\arrow[from=1-1, to=1-3, "b"]
\arrow[from=1-1, to=3-2, "bC"']
\arrow[from=3-2, to=1-3, "Bb"']
\end{tikzcd}
\qquad
\begin{tikzcd}
ABC & & CAB \\
& \Downarrow S & \\
& ACB &
\arrow[from=1-1, to=1-3, "b"]
\arrow[from=1-1, to=3-2, "Ab"']
\arrow[from=3-2, to=1-3, "bB"']
\end{tikzcd}$$

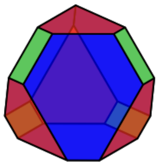
When the associator is not the identity, the triangles’ vertices get replaced with associators, effectively truncating them, and we are left with hexagon invertible modifications. (Mnemonic: Blue Braiding.)

$$\begin{tikzcd}
& A(BC) & (BC)A & \\
(AB)C & \Downarrow R & & B(CA) \\
& (BA)C & B(AC) &
\arrow[from=2-1, to=1-2, "a"]
\arrow[from=1-2, to=1-3, "b"]
\arrow[from=1-3, to=2-4, "a"]
\arrow[from=2-1, to=3-2, "bC"']
\arrow[from=3-2, to=3-3, "a"']
\arrow[from=3-3, to=2-4, "Bb"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& (AB)C & C(AB) & \\
A(BC) & \Downarrow S & & (CA)B \\
& A(CB) & (AC)B &
\arrow[from=2-1, to=1-2, "a^{\bullet}"]
\arrow[from=1-2, to=1-3, "b"]
\arrow[from=1-3, to=2-4, "a^{\bullet}"]
\arrow[from=2-1, to=3-2, "Ab"']
\arrow[from=3-2, to=3-3, "a^{\bullet}"']
\arrow[from=3-3, to=2-4, "bB"']
\end{tikzcd}$$

- $(n = 3, k = 1)$ and $(n = 1, k = 3)$: $\binom{3+1}{1} = \binom{1+3}{1} = 4$, so in a braided strictly monoidal bicategory, these polytopes are tetrahedra whose faces are the $(2,1)$ polytope.



Again, when the associator is not the identity, the vertices get truncated, this time being replaced by pentagonators; as a side-effect, four of the six edges are also beveled.



Equation (SM3.i) governs shuffling one object A into three objects B, C, D:

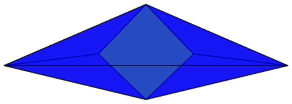
$$(SM3.i.a) = (SM3.i.b)$$

where the pasting diagrams (SM3.i.a) and (SM3.i.b) are depicted in Figures 7 and 8, respectively. Equation (SM3.ii) governs shuffling three objects A, B, C into one object D:

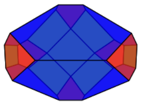
$$(SM3.ii.a) = (SM3.ii.b)$$

where the pasting diagrams (SM3.ii.a) and (SM3.ii.b) are depicted in Figures 9 and 10, respectively.

- $(n = 2, k = 2)$: $\binom{2+2}{2} = 6$; in a braided strictly monoidal bicategory, this polytope is composed mostly of $(2,1)$ triangles, but there is a pair of braidings that commute, so one face is a square.



When the associator is not the identity, the six vertices get truncated and six of the edges get beveled.



Equation (SM4) governs shuffling two objects A, B into two objects C, D:

$$(SM4.a) = (SM4.b)$$

where pasting diagrams (SM4.a) and (SM4.b) are depicted in Figures 11 and 12, respectively.

- The Breen polytope. In a braided monoidal category, the Yang-Baxter equations hold; there are two fundamentally distinct proofs of this fact.

$$\begin{tikzcd}
 & BCA \arrow[dr, "bA"] & \\
BAC \arrow[ur, "Bb"] & & CBA \\
ABC \arrow[u, "bC"] \arrow[uur, "b"'] \arrow[dr, "Ab"'] & & CAB \arrow[u, "Cb"'] \\
 & ACB \arrow[ur, "bB"'] \arrow[uur, "b"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & BCA \arrow[dr, "bA"] & \\
BAC \arrow[ur, "Bb"] \arrow[rr, "b"'] & & CBA \\
ABC \arrow[u, "bC"] \arrow[rr, "b"] \arrow[dr, "Ab"'] & & CAB \arrow[u, "Cb"'] \\
 & ACB \arrow[ur, "bB"'] &
\end{tikzcd}$$

In a braided strictly monoidal bicategory, the two proofs become the front and back face of another coherence law (SM5) governing the interaction of the $(2, 1)$-shuffle polytopes; when the associator is nontrivial, the vertices get truncated:

$$(SM5.a) = (SM5.b)$$

where pasting diagrams (SM5.a) and (SM5.b) are depicted in Figures 13 and 14, respectively.

That the coherence law is necessary was something of a surprise: Kapranov and Voevodsky did not include it in their definition of braided semistrict monoidal 2-categories; Breen [12] corrected the definition.

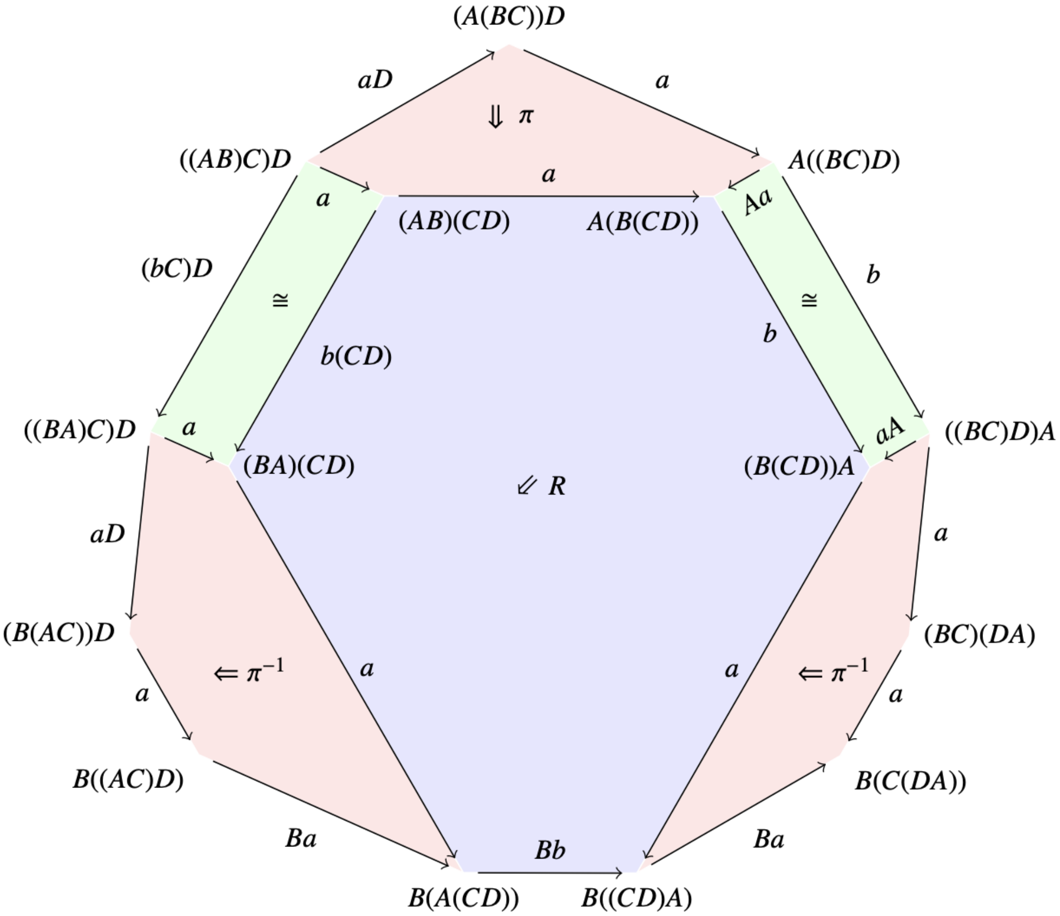


Figure A.7: Axiom SM3.i.a

$$\begin{tikzcd}
 & & & & (A(BC))D \arrow[d, "bD"] \arrow[drr, "a"] & & & & \\
 & & ((AB)C)D \arrow[urr, "aD"] \arrow[dll, "(bC)D"'] & & ((BC)A)D \arrow[ddll, "aD"'] \arrow[ddrr, "a"] & & A((BC)D) \arrow[drr, "b"] & & \\
((BA)C)D \arrow[dd, "aD"'] & & \Swarrow RD & & & & \Leftarrow R & & ((BC)D)A \arrow[dd, "a"] \\
 & & (B(CA))D \arrow[dr, "a"] & & \Swarrow \ \pi^{-1} & & (BC)(AD) \arrow[dl, "a"'] \arrow[drr, "(BC)b"] & & \\
(B(AC))D \arrow[urr, "(Bb)D"] \arrow[dr, "a"'] & \cong & & B((CA)D) \arrow[rr, "Ba"'] & & B(C(AD)) \arrow[drr, "B(Cb)"'] & & \cong & (BC)(DA) \arrow[dl, "a"] \\
 & B((AC)D) \arrow[urr, "B(bD)"'] \arrow[drr, "Ba"'] & & & \Downarrow \ BR^{-1} & & & B(C(DA)) & \\
 & & & B(A(CD)) \arrow[rr, "Bb"] & & B((CD)A) \arrow[urr, "Ba"'] & & &
\end{tikzcd}$$

Figure A.8: Axiom SM3.i.b

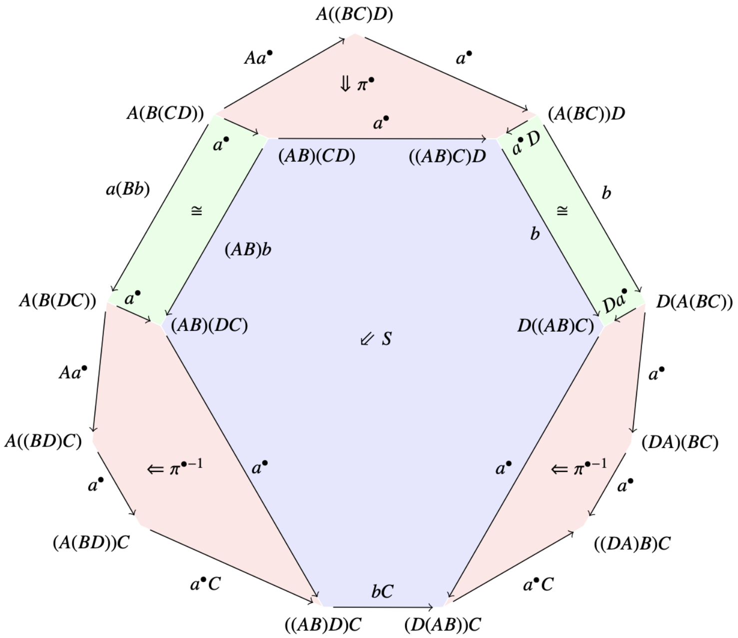


Figure A.9: Axiom SM3.ii.a

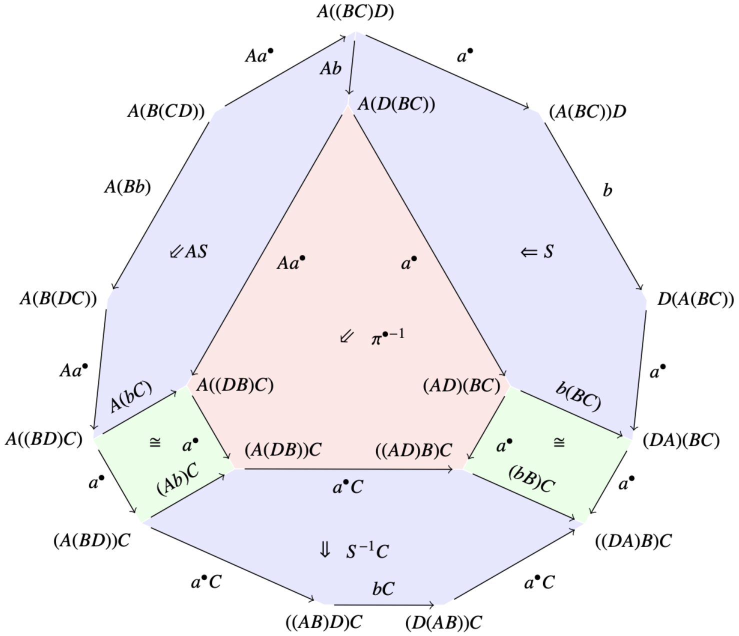


Figure A.10: Axiom SM3.ii.b

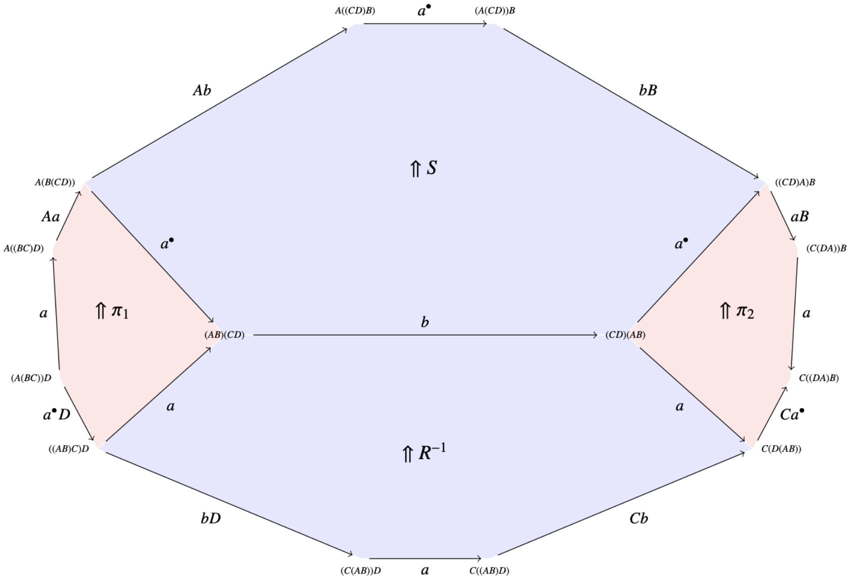


Figure A.11: Axiom SM4.a

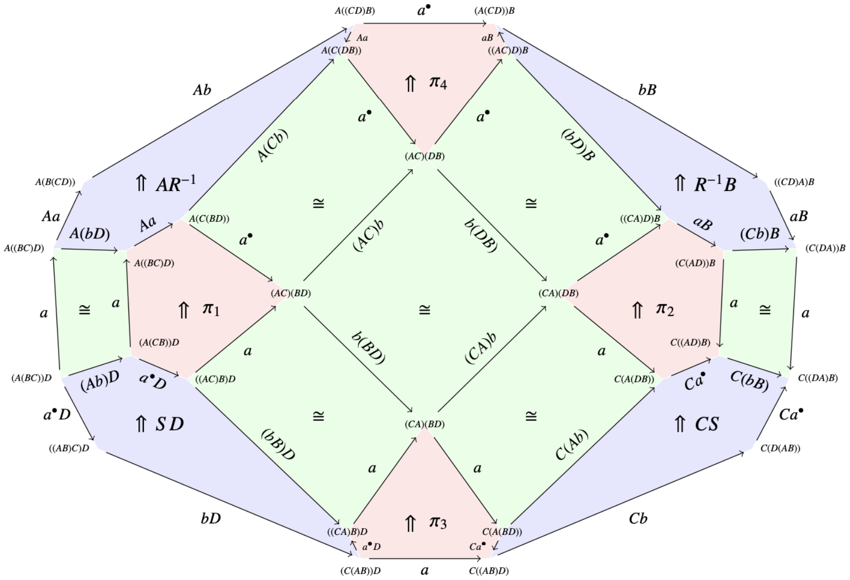


Figure A.12: Axiom SM4.b

We therefore call the following coherence law the "Breen polytope". In retrospect, we can see that this is the start of a series of polytopes whose faces are "permutohedra" [22]; the need for the Breen polytope and the rest of the series became clear in Batanin's approach to weak n-categories [4].

A.3 Sylleptic and Symmetric Monoidal Bicategories

Definition A.3.1. n a *sylleptic* monoidal bicategory, a full twist is not necessarily equal to the identity, but may be only isomorphic to it; this isomorphism is called a syllepsis. A *sylleptic* monoidal bicategory M is a braided monoidal bicategory equipped with a syllepsis subject to the following axioms.

- An invertible modification (Mnemonic: Salmon Syllepsis)

$$\begin{tikzcd} AB \arrow[r, bend left=50, "b"] \arrow[r, bend right=50, "b^\bullet"'] \arrow[r, phantom, "\Downarrow v"] & BA \end{tikzcd}$$

- Equation (SM6.i) holds, depicted in Figure 15, which governs the interaction of the syllepsis with the $(n = 1, k = 2)$ braiding:

$$\begin{tikzcd}
 & A(BC) \arrow[r, bend left, "b"] & (BC)A \arrow[dr, "a"] & & & & A(BC) \arrow[r, bend left=50, "b"] \arrow[r, bend right=50, "b^\bullet"'] \arrow[r, phantom, "\Downarrow v"] & (BC)A \arrow[dr, "a"] & \\
(AB)C \arrow[ur, "a"] \arrow[dr, bend right, "bC"'] & \Downarrow R & & B(CA) & = & (AB)C \arrow[ur, "a"] \arrow[dr, bend right=50, "bC"'] \arrow[dr, bend left=20, "b^\bullet C"] \arrow[dr, phantom, "\Downarrow v^{-1}C"] & \Downarrow S^\bullet & & B(CA) \\
 & (BA)C \arrow[r, "a"'] & B(AC) \arrow[ur, bend right, "Bb"'] & & & & (BA)C \arrow[r, "a"'] & B(AC) \arrow[ur, bend right=50, "Bb"'] \arrow[ur, bend left=20, "Bb^\bullet"] \arrow[ur, phantom, "\Downarrow Bv^{-1}"] &
\end{tikzcd}$$

Figure A.13: Axiom SM6.i

- Equation (SM6.ii) holds, depicted in Figure 16, which governs the interaction of the syllepsis with the $(n = 2, k = 1)$ braiding:

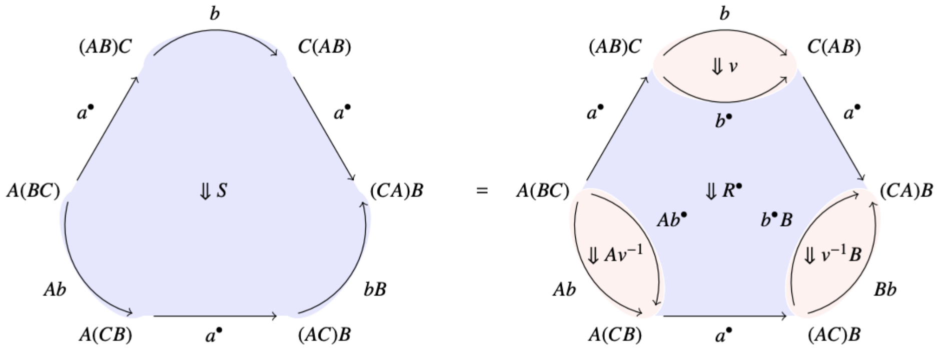


Figure A.14: Axiom SM6.ii

Definition A.3.2. A *symmetric* monoidal bicategory is a sylleptic monoidal bicategory subject to axiom (SM7), depicted in Figure 17, where the unlabeled green cells are identities.

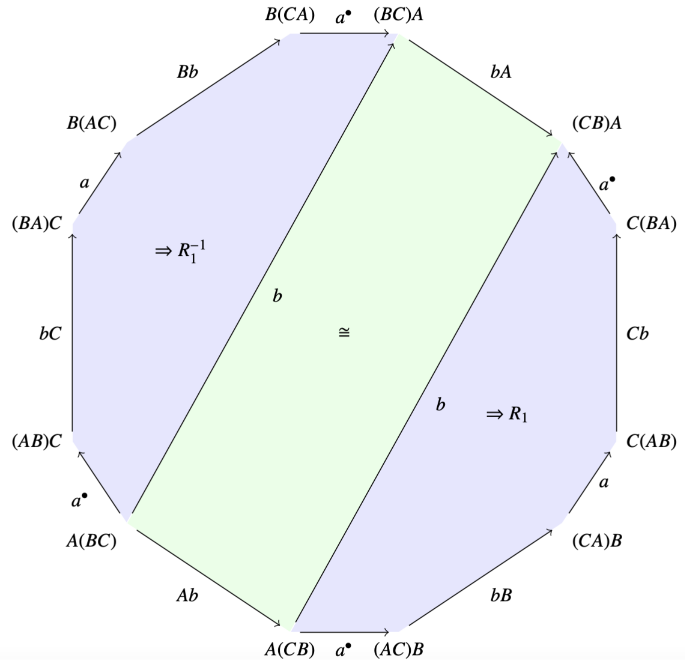


Figure A.15: Axiom SM5.a

$$\begin{tikzcd}
 & & B(CA) \arrow[r, "a^\bullet"] & (BC)A \arrow[dr, "bA"] & & \\
 & B(AC) \arrow[ur, "Bb"] \arrow[rrr, phantom, "\Downarrow S_1^{-1}"] & & & (CB)A & \\
(BA)C \arrow[ur, "a"] \arrow[rrrrr, "b"'] & & & & & C(BA) \arrow[ul, "a^\bullet"'] \\
(AB)C \arrow[u, "bC"] \arrow[rrrrr, "b"] \arrow[urrrrr, phantom, "\cong"] & & & & & C(AB) \arrow[u, "Cb"'] \\
 & A(BC) \arrow[ul, "a^\bullet"] \arrow[dr, "Ab"'] \arrow[rrr, phantom, "\Downarrow S_1"] & & & (CA)B \arrow[ur, "a"'] & \\
 & & A(CB) \arrow[r, "a^\bullet"'] & (AC)B \arrow[ur, "bB"'] & &
\end{tikzcd}$$

Figure A.16: Axiom SM5.b

$$\begin{tikzcd}
AB \arrow[r, "b"] \arrow[d, "b"'] \arrow[dr, "1"] \arrow[dr, phantom, shift left=4ex, "\Downarrow"] \arrow[dr, phantom, shift right=4ex, "\Downarrow v_1"'] & BA \\
BA \arrow[r, "b"'] & AB \arrow[u, "b"']
\end{tikzcd}
=
\begin{tikzcd}
AB \arrow[r, "b"] \arrow[d, "b"'] \arrow[d, phantom, shift left=4ex, "\Downarrow"] & BA \\
BA \arrow[r, "b"'] \arrow[ur, "1"] \arrow[r, phantom, shift left=4ex, "\Downarrow v_1"] & AB \arrow[u, "b"']
\end{tikzcd}$$

Figure A.17: Axiom SM7

Appendix B

Coherence Conditions for Linear Natural Transformations

Linear natural transformations satisfy numerous coherence conditions. we provide a comprehensive list of the coherence conditions as follows from [16]:

- Lax natural transformation of $\omega^{\otimes}$:

$$\begin{aligned}
F^{\otimes}(A) \otimes F^{\otimes}(B) \otimes \omega^{\otimes}(Z) &\xRightarrow{m_{\otimes} \otimes 1} F^{\otimes}(A \otimes B) \otimes \omega^{\otimes}(Z) \\
&\xRightarrow{\omega^{\otimes} \otimes B} \omega^{\otimes} X \otimes G^{\otimes}(A \otimes B) \\
&= F^{\otimes}(A) \otimes F^{\otimes}(B) \otimes \omega^{\otimes} Z \\
&\xRightarrow{1 \otimes \omega^{\otimes} B} F^{\otimes}(A) \otimes \omega^{\otimes} Y \otimes G^{\otimes}(B) \\
&\xRightarrow{\omega^{\otimes} 1} \omega^{\otimes} X \otimes G^{\otimes}(A) \otimes \otimes G^{\otimes}(B) \\
&\xRightarrow{1 \otimes n^{\otimes}} \omega^{\otimes} X \otimes G^{\otimes}(A \otimes B) \\
&: F(X) \to G(Z)
\end{aligned}$$

$$
\begin{aligned}
\omega^{\otimes} X &\xRightarrow{u_L^{\otimes}} \top_{F^{\otimes}(X)} \otimes \omega^{\otimes} X \\
&\xRightarrow{m^{\top} \otimes 1} F^{\otimes}(\top_X) \otimes \omega^{\otimes} X \\
&\xRightarrow{\omega^{\otimes} \top_X} \omega^{\otimes} X \otimes G^{\otimes}(\top_X) \\
&= \omega^{\otimes} X \\
&\xRightarrow{u_R^{\otimes}} \omega^{\otimes} X \otimes \top_{G^{\otimes}(X)} \\
&\xRightarrow{1 \otimes n^{\top}} \omega^{\otimes} X \otimes G^{\otimes}(\top_X) \\
&: F(X) \to G(X)
\end{aligned}
$$

The equations for $\omega^{\oplus}$ follows from the dual of the above equations.

- Naturality equations:

$$
\begin{aligned}
&F^{\otimes}(A \oplus B) \otimes \omega^{\otimes} Z \\
&\qquad \xRightarrow{\nu_R^{\otimes} \otimes 1} (F^{\otimes}(A) \oplus F^{\otimes}(B)) \otimes \omega^{\otimes} Z \\
&\qquad \xRightarrow{\delta_R} F^{\otimes}(A) \oplus (F^{\otimes}(B) \otimes \omega^{\otimes} Z) \\
&\qquad \xRightarrow{1 \oplus \omega^{\otimes} B} F^{\otimes}(A) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B))
\end{aligned}
\tag{B.0.1}
$$

$$
\begin{aligned}
= F^{\otimes}(A \oplus B) \otimes \omega^{\otimes} Z &\xRightarrow{\omega^{\otimes} A \oplus B} \omega^{\otimes} X \otimes G^{\otimes}(A \oplus B) \\
&\xRightarrow{1 \otimes \nu_R^{\otimes}} \omega^{\otimes} X \otimes (G^{\otimes}(A) \oplus G^{\otimes}(B)) \\
&\xRightarrow{1 \otimes 1 \oplus u_L^{\otimes}} \omega^{\otimes} X \otimes (G^{\otimes}(A) \oplus (\top \otimes G^{\otimes}(B))) \\
&\xRightarrow{1 \otimes 1 \oplus \tau \otimes 1} \omega^{\otimes} X \otimes (G^{\otimes}(A) \oplus (\omega^{\oplus} Y \oplus \omega^{\otimes} Y) \otimes G^{\otimes}(B))) \\
&\xRightarrow{1 \oplus \delta_R} \omega^{\otimes} X \otimes ((G^{\otimes}(A) \oplus \omega^{\oplus} Y) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B))) \\
&\xRightarrow{1 \otimes \omega^{\oplus} A \oplus 1} \omega^{\otimes} X \otimes ((\omega^{\oplus} X \oplus F^{\oplus}(A)) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B))) \\
&\xRightarrow{\delta_L \otimes \delta_L \oplus 1} (\omega^{\otimes} X \otimes \omega^{\oplus} X) \oplus F^{\oplus}(A) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B)) \\
&\xRightarrow{\gamma \oplus 1 \oplus 1} \bot \oplus F^{\oplus}(A) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B)) \\
&\xRightarrow{u_L^{\oplus}} F^{\oplus}(A) \oplus (\omega^{\otimes} Y \otimes G^{\otimes}(B)) : F(X) \to G(Z)
\end{aligned}
\tag{B.0.2}
$$

$$\begin{aligned}
&F^{\otimes}(A \oplus B) \\
&\qquad \xRightarrow{\nu_L^{\otimes}} F^{\otimes}(A) \otimes F^{\otimes}(B) \\
&\qquad \xRightarrow{u_R^{\otimes} \oplus 1} (F^{\otimes}(A) \otimes \top) \oplus F^{\otimes}(B) \\
&\qquad \xRightarrow{1 \otimes \tau \oplus 1} (F^{\otimes}(A) \otimes (\omega^{\otimes} Y \oplus \omega^{\oplus} Y)) \oplus F^{\otimes}(B) \\
&\qquad \xRightarrow{\delta_L \oplus 1} ((F^{\otimes}(A) \otimes \omega^{\otimes} Y) \oplus \omega^{\oplus} Y) \oplus F^{\otimes}(B) \\
&\qquad \xRightarrow{\omega^{\otimes} A \oplus 1 \oplus 1} (\omega^{\otimes} X \otimes G^{\otimes}(A)) \oplus \omega^{\oplus} Y \oplus F^{\otimes}(B) \\
= &F^{\otimes}(A \oplus B) \\
&\qquad \xRightarrow{u_R^{\otimes}} F^{\otimes}(A \oplus B) \otimes \top \\
&\qquad \xRightarrow{1 \otimes \tau} F^{\otimes}(A \oplus B) \otimes (\omega^{\otimes} Z \oplus \omega^{\oplus} Z) \\
&\qquad \xRightarrow{\delta_L} (F^{\otimes}(A \oplus B) \otimes \omega^{\otimes} Z) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{\omega^{\otimes} A \oplus B \oplus 1} (\omega^{\otimes} X \otimes G^{\otimes}(A \oplus B)) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{1 \otimes \nu_L^{\otimes} \oplus 1} (\omega^{\otimes} X \otimes (G^{\otimes}(A) \oplus G^{\otimes}(B))) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{\delta_L \oplus 1} (\omega^{\otimes} X \otimes (G^{\otimes}(A)) \oplus G^{\oplus}(B) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{1 \oplus \omega^{\oplus} B} (\omega^{\otimes} X \otimes (G^{\otimes}(A)) \oplus \omega^{\oplus} Y \oplus F^{\oplus}(B) : F(X) \to G(Z)
\end{aligned} \tag{B.0.3}$$

And by duality we get:

$$
\begin{aligned}
&(G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes F^{\otimes}(B) \\
&\qquad \xRightarrow{\omega^{\oplus} A \otimes 1} (\omega^{\oplus} X \oplus F^{\oplus}(A)) \otimes F^{\otimes}(B) \\
&\qquad \xRightarrow{\delta_R} \omega^{\oplus} X \oplus (F^{\oplus}(A) \otimes F^{\otimes}(B)) \\
&\qquad \xRightarrow{1 \oplus \nu_L^{\oplus}} \omega^{\oplus} X \oplus (F^{\oplus}(A \otimes B) \\
&(G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes F^{\otimes}(B) \\
&\qquad \xRightarrow{1 \otimes u_R^{\otimes}} (G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes F^{\otimes}(B)) \otimes \top_Y \\
&\qquad \xRightarrow{1 \otimes 1 \otimes \tau} (G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes F^{\otimes}(B)) \otimes (\omega^{\otimes} Z \oplus \omega^{\oplus} Z) \\
&\qquad \xRightarrow{1 \otimes \delta_L \otimes \delta_R} (G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes (F^{\otimes}(B) \otimes \omega^{\otimes} Z) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{1 \otimes \omega^{\otimes} B \oplus 1} (G^{\oplus}(A) \oplus \omega^{\oplus} Y) \otimes (\omega^{\otimes} Y \otimes G^{\otimes}(B)) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{\delta_R \otimes 1 \oplus 1} ((G^{\oplus}(A) \oplus (\omega^{\oplus} Y \otimes \omega^{\otimes} Y)) \otimes G^{\otimes}(B)) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{1 \oplus \gamma \otimes 1 \oplus 1} ((G^{\oplus}(A) \oplus \perp) \otimes G^{\otimes}(B)) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{u_L^{\oplus} \otimes 1 \oplus 1} ((G^{\oplus}(A) \oplus G^{\otimes}(B)) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{\nu_L^{\oplus} \oplus 1} ((G^{\oplus}(A \oplus B) \oplus \omega^{\oplus} Z \\
&\qquad \xRightarrow{\omega^{\oplus} A \oplus B} \omega^{\oplus} X \oplus F^{\oplus}(A \otimes B) \\
&\qquad\qquad : G(X) \to F(Z)
\end{aligned}
\tag{B.0.4}
$$

$$
\begin{aligned}
&F^{\otimes}(A)\otimes\omega^{\otimes}Y\otimes(G^{\oplus}(B)\oplus\omega^{\oplus}Z)\\
&\qquad\xRightarrow{1\otimes 1\otimes\omega^{\oplus}B} F^{\otimes}(A)\otimes\omega^{\otimes}Y\otimes(\omega^{\oplus}Y\oplus F^{\oplus}(B))\\
&\qquad\xRightarrow{1\otimes\delta_L} F^{\otimes}(A)\otimes((\omega^{\otimes}Y\otimes\omega^{\oplus}Y)\oplus F^{\oplus}(B))\\
&\qquad\xRightarrow{1\otimes\gamma L\oplus 1} F^{\otimes}(A)\otimes(\bot\oplus F^{\oplus}(B))\\
&\qquad\xRightarrow{1\otimes uL^{\oplus}} F^{\otimes}(A)\otimes F^{\oplus}(B)\\
&\qquad\xRightarrow{\nu_R^{\otimes}} F^{\oplus}(A\otimes B)\\
=\,&F^{\otimes}(A)\otimes\omega^{\otimes}Y\otimes(G^{\oplus}(B)\oplus\omega^{\oplus}Z)\\
&\qquad\xRightarrow{1\otimes uL^{\oplus}} F^{\otimes}(A)\otimes F^{\oplus}(B)\\
&\qquad\xRightarrow{\omega^{\otimes}A\otimes 1} \omega^{\otimes}X\otimes G^{\otimes}(A)\otimes(G^{\oplus}(B)\oplus\omega^{\oplus}Z)\\
&\qquad\xRightarrow{1\otimes\delta_L} \omega^{\otimes}X\otimes((G^{\otimes}(A)\otimes(G^{\oplus}(B))\oplus\omega^{\oplus}Z)\\
&\qquad\xRightarrow{1\otimes\nu_R^{\oplus}} \omega^{\otimes}X\otimes((G^{\otimes}(A\otimes B)\oplus\omega^{\oplus}Z)\\
&\qquad\xRightarrow{1\otimes\omega^{\oplus}(A\otimes B)} \omega^{\otimes}X\otimes(\omega^{\oplus}X\oplus F^{\oplus}(A\otimes B))\\
&\qquad\xRightarrow{\delta_L} \omega^{\otimes}X\otimes\omega^{\oplus}X)\oplus F^{\oplus}(A\otimes B)\\
&\qquad\xRightarrow{\gamma\oplus 1} \bot\oplus F^{\oplus}(A\otimes B)\\
&\qquad\xRightarrow{u_L^{\oplus}1} F^{\oplus}(A\otimes B)\\
&\qquad\quad : G(X)\to F(Z)
\end{aligned}
\tag{B.0.5}
$$

$$
\begin{aligned}
&F^{\otimes}(A)\otimes\omega^{\otimes}Y \xRightarrow{F^{\otimes}(f)\otimes 1} F^{\otimes}(B)\otimes\omega^{\otimes}Y\\
&\qquad\xRightarrow{\omega^{\otimes}B} \omega^{\otimes}X\otimes G^{\otimes}(B)\\
=\,&F^{\otimes}(A)\otimes\omega^{\otimes}Y \xRightarrow{\omega^{\otimes}A} \omega^{\otimes}X\otimes G^{\otimes}(A)\\
&\qquad\xRightarrow{1\otimes G\otimes(f)} \omega^{\otimes}X\otimes G^{\otimes}(B)\\
&\qquad\quad : F(X)\to G(Y)
\end{aligned}
\tag{B.0.6}
$$

$$\begin{aligned}
& G^{\oplus}(A) \otimes \omega^{\oplus} Y \xRightarrow{G^{\oplus}(f)\otimes 1} G^{\oplus}(B) \otimes \omega^{\oplus} Y \\
& \qquad \xRightarrow{\omega^{\oplus} B} \omega^{\oplus} X \otimes F^{\oplus}(B) \\
= {} & G^{\oplus}(A) \otimes \omega^{\oplus} Y \xRightarrow{\omega^{\oplus} A} \omega^{\oplus} X \otimes F^{\oplus}(A) \\
& \qquad \xRightarrow{1\otimes F\oplus(f)} \omega^{\oplus} X \otimes F^{\oplus}(B) \\
& \qquad : G(X) \to F(Y)
\end{aligned} \tag{B.0.7}$$

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