THRESHOLD LOGIC FUNCTIONS

by

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ABSTRACT

Threshold logic elements are a new type of multi-input switching components in computer technology. They are more general than "and" or "or" gates. The output of a threshold element is 1, if and only if the weighted sum of its inputs equals or exceeds a threshold, otherwise the output is 0. Boolean functions which are realizable by such an element are called threshold functions. The analysis and synthesis of threshold functions, and the decomposition of threshold functions are the extension of logic theory which deal with the threshold elements in the same way as switching theory treats the conventional gates.

An introduction to threshold logic functions is given in chapter 1. Chapters 2 and 3 which concern with the properties and synthesis of threshold logic functions are the results of a general survey. The method of decomposition of threshold functions given in chapter 4 was developed independently.

TABLE OF CONTENTS

TABLE OF CONTENTS

THRESHOLD LOGIC FUNCTIONS

C.K. Yue

CHAPTER 1: INTRODUCTION TO THRESHOLD LOGIC FUNCTIONS

1.1 Introduction

Since the publication of M. Karnaugh's paper (1) in 1955, threshold functions have been studied by many people under different names such as "Unate Truth Functions" (2), "Linear-input Logic" (3), and "Majority Decision Logic" (4).

In switching theory, there are 2^n configurations of values associated with n binary variables $x_1, \ldots, x_n$. These configurations can be displayed on an n-dimensional Euclidean space (called an n-cube (5)) which contains 2^n vertices. Each configuration can be identified with a vertex of the n-cube. A switching function F of n variables maps the vertices of the n-cube into 1-cube (i.e. either 1 or 0). Threshold functions are the special class of switching functions which will be defined below. This chapter is intended to give a general idea about the characteristics of threshold functions and to point out the various problems that are worth studying.

1.2 Definitions

Switching functions F which can be realized by threshold elements are called threshold functions. A threshold element is a circuit whose output is either 1 or 0 according to the linear inequalities:

$$\text{output} = 1 \quad (\text{or } F=1) \text{ iff } \quad f = \sum_{k=1}^{n} w_k x_k \geq T$$

$$\text{output} = 0 \quad (\text{or } F=0) \text{ iff } \quad f = \sum_{k=1}^{n} w_k x_k < T$$

where $w_1, \ldots, w_n$ are the corresponding weights of the binary inputs $x_1, \ldots, x_n$, and T is the threshold value of the element. $w_1, \ldots, w_n$ and T are real numbers.

f, the algebraic expression, shows the structure of the elements. f = T is called the separating plane equation.

The relationship between the threshold functions of the element and its weights w_k and the threshold value T is best illustrated by a 2-input element. For 2 binary inputs, there are four different input combinations, namely, 00, 01, 10, and 11. We can plot these four points in an Euclidean input space so that each of the four combinations is represented by a point in an Euclidean input space as shown in Fig. 1.

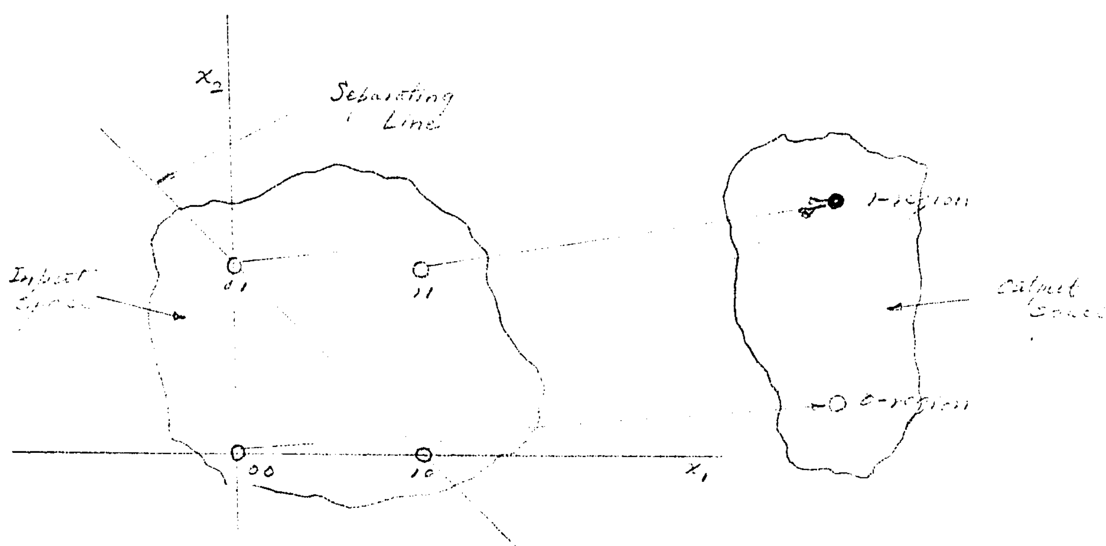

Fig. 1 Relation between $f = w_1 x_1 + w_2 x_2 = 1$ and $F = x_1 + x_2$ where $w_1 = w_2 = 1$.

$F = x_1 + x_2$ and $w_1 = w_2 = 1$. The separating line equation is $f = x_1 + x_2 = 1$. The 2-input element with structure $w_1 = 1$, $w_2 = 1$, and $T = 1$ maps the three points 01, 10, and 11 into 1-region and the point 00 into 0-region in the output space. The threshold function is $F = x_1 + x_2$. The symbol for the element is shown in Fig. 2.

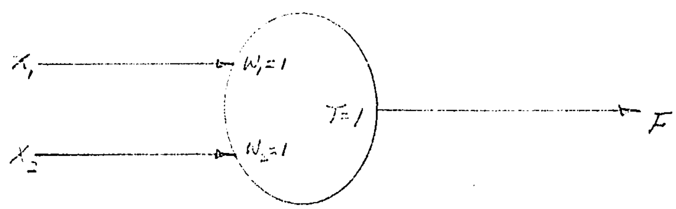

Fig. 2. Threshold element realizing OR function.

1.3 General Description

From Fig. 1 it is easily seen that the weights in the threshold element control the slope of the separating line, and that the threshold T controls the intercepts of the line. Varying the value of T causes the line to move as shown in Fig. 3,

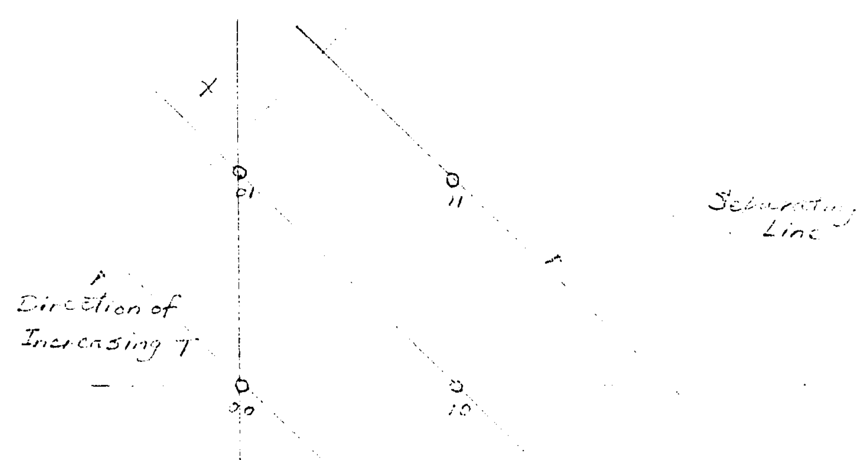

Fig.3 Effect of varying T on the threshold functions.

and varying the value of a weight w_1 causes the line to move as shown in Fig. 4. To each separating line in Fig. 3 and Fig. 4 defines a corresponding threshold function.

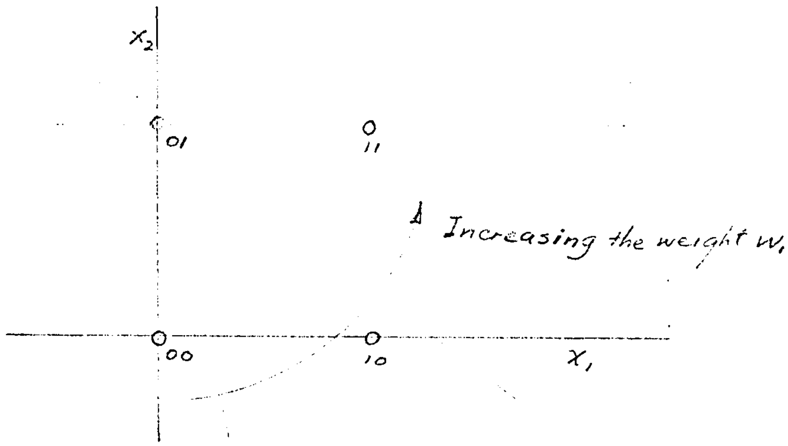

Fig. 4 Effect of varying weight w_1 on the threshold functions.

For 3-input element, $f = x_1 w_1 + x_2 w_2 + x_3 w_3 = T$ is the equation of a separating plane in a 3-cube drawn in 3-dimensional Euclidean space. The changes in position of the plane in the 3-cube depend on the changes of the weights and the threshold of the element. Each position of the separating plane corresponds to some function realized by the element.

Similarly for an n-input element, $f = \sum_{k=1}^{n} w_k x_k = T$ is the equation of the hyperplane in an n-cube. The hyperplane separates vertices which belong to 1-region from the rest of the vertices which belong to the 0-region. The position of the hyperplane depends on the values of the weights and threshold.

1.4 General Survey and Problems in the Study of Threshold Functions.

Threshold functions analysis, synthesis, and decomposition are the study of a special class of Boolean functions which can be realized with a single threshold element. The most general problem in the study is: given a Boolean function, how does one tell whether this function is a threshold functions or not, and if it is, what are the weights and the threshold to realize it. The most fundamental attack on this problem is to devise simple algorithms for testing Boolean functions. These algorithms can be derived from the properties of threshold functions.

One of the necessary properties the threshold functions must have is unateness which was discussed originally by McNaughton (2) and Paull-McCluskey (6). A unate function is a Boolean function representable in a form in which no variable appears both negated and unnegated. Another necessary property the threshold functions must have is complete monotonicity (16). A Boolean function F is complete monotonic when for every two valuations x and y on some common subset of F's variables, F_x and F_y are comparable (Two functions F_x and F_y are said to be comparable if either $F_x \supset F_y$ or $F_y \supset F_x$). The detailed analysis of threshold functions will be found in chapter 2.

A necessary and sufficient condition for a Boolean function to be a threshold function can be stated in the following manner. According to the definition of threshold functions, any Boolean function F of n variables has 2^n linear inequalities corresponding to the 2^n different combinations of n binary variables. F is a threshold function if and only if it has a set of consistent linear inequalities. (A set of linear inequalities is consistent if and only if it has at least one solution). The set of consistent linear inequalities expresses the conditions which the unknown weights in the separating plane equation to be solved must fulfill. Among these 2^n linear inequalities many are redundant and can be deleted. That is to say, these 2^n linear inequalities can be reduced to a manageable set of irredundant linear inequalities which then can be solved by simple hand computation. The detailed discussion of this synthesizing method will be found in chapter 3. Another way of stating the necessary and sufficient condition can be found in Chow (7) and is also discussed in terms of convex sets by Highleyman (15) and Gabelman (12), but is difficult to apply in practice.

1.5 Applications and Advantage of Threshold Elements

Threshold elements can be used to improve the reliability of computer system (11). Linear decision functions can be applied to pattern recognition (13) (14) in the field of artificial intelligence. These basic linear decision functions have turned out to be threshold functions. The switching components used are the neuron-like elements--threshold elements, which are easily controlled.

Threshold elements can be used to simplify the design of computer systems. Their advantage may be seen from the following example.

For conventional design of a function $C = x_1x_2 + x_1x_3 + x_2x_3$ one needs at least four gates and two levels to realize the required function . This is as shown in Fig.5.

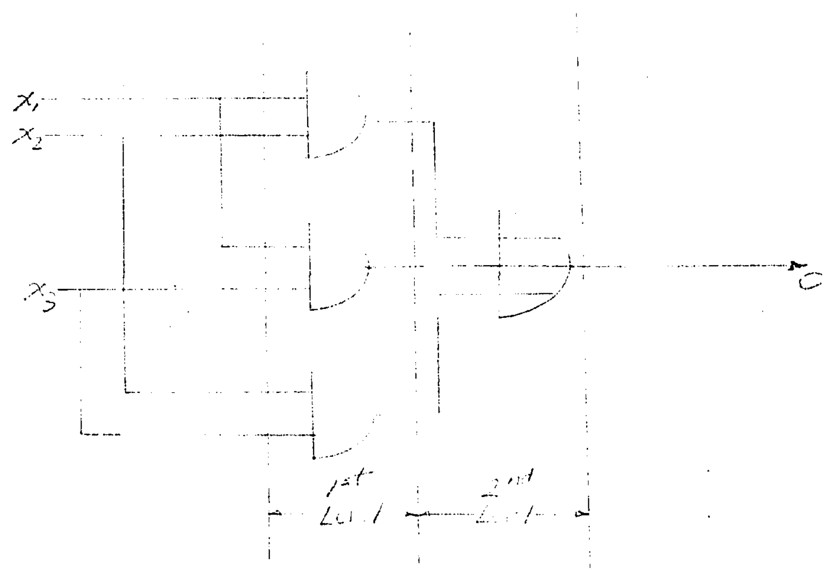

Fig.5 Conventional design circuit to realize C.

But for threshold logic design, we need only one threshold element to realize such function in a single stage as shown in Fig.6.

Fig.6 Threshold logic design circuit to realize C.

CHAPTER 2: ANALYSIS OF THRESHOLD FUNCTIONS

2.1 Introduction

The most general conditions that threshold functions must meet are unateness (2) and inclusion (6) (or monotonicity (16)) which will be discussed below. These two conditions are the necessary but not sufficient conditions for any Boolean to be a threshold function. However, from these two conditions, many properties of threshold functions relating to the "sign" and "size" of weights of input variables can be found. As a result of these properties, the threshold function "tree" and the topological idea of "map" and "gap" to analyse threshold functions can be developed. The necessary and sufficient condition for a Boolean function to be a threshold function can be stated in terms of consistencey of a set of linear inequalities, which will be discussed in chapter 3.

2.2 The General Conditions for Threshold Functions: Unateness and Inclusion

A unate function is a Boolean function representable in a form in which no variable appears both negated and unnegated. For example:

$F = x_1\bar{x}_2 + \bar{x}_3$ is a unate function

$F = x_1\bar{x}_2 + \bar{x}_1x_2$ is not a unate function.

Condition 1. A threshold function must be unate.

Proof: Every switching function can be written in the form

$$F = x_1F_1 + \bar{x}_1F_0$$

where $F_1 = F(x_1 = 1)$

$F_0 = F(x_1 = 0)$

For example:

$$F = x_1x_2 + x_3$$

$$= x_1(x_2 + x_3) + \bar{x}_1 x_3$$

$$= x_1 F_1 + \bar{x}_1 F_0$$

where $F_1 = x_2 + x_3$

$F_0 = x_3$

Let F be the threshold function which can be realized by a threshold element with inputs $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n$. The associated weights of the inputs are $w_1, \bar{w}_1, w_2, \bar{w}_2, \ldots, w_n, \bar{w}_n$.

The structure of the element is as shown in Fig. 6

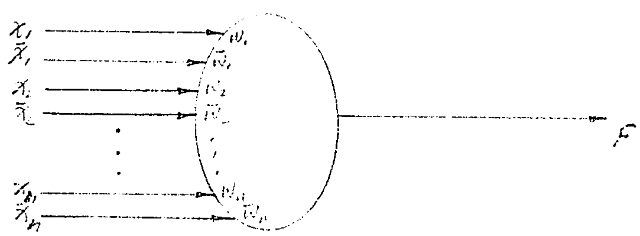

Fig.6 Threshold element realizing F.

The function $F(x_1 = 1) = F_1$ is the function realized by the element with the input $x_1 = 1$ ($\bar{x}_1 = 0$). The input x_1 ($x_1 = 1$) has the effect of changing the threshold. If the original threshold of the element is T, and w_1 is the weight of the input x_1, then the apparent threshold T_1 with $x_1 = 1$ will be $T - w_1$, since an additional threshold of only $T - w_1$ is now necessary to produce the output. Similarly, the function $F(\bar{x}_1 = 1) = F_0$ is the function realized by the element with the input $\bar{x}_1 = 1$ ($x_1 = 0$) and the apparent threshold T_0 with $\bar{x}_1 = 1$ is $T - \bar{w}_1$; i.e.

$$T_1 = T - w_1 x_1$$
$$= T - w_1 \quad \text{(where } x_1 = 1\text{)}$$
$$T_0 = T - \overline{w}_1 \overline{x}_1$$
$$= T - \overline{w}_1 \quad \text{(where } \overline{x}_1 = 1\text{).}$$

By comparing the values of T_1 and T_0 we have 3 possible cases:

(1) $T_1 < T_0$
(2) $T_1 > T_0$
(3) $T_1 = T_0$

In case (1), $T_1 < T_0$, since any combination of weights of inputs which provides enough weighted sum of the inputs to overcome the T_0 threshold must surely overcome the smaller T_1 threshold; i,e.

if $\sum_{k=1}^{n} w_k x_k^* \geq T_0$ and $T_1 < T_0$

then $\sum_{k=1}^{n} w_k x_k^* \geq T_1$

where x_k^* can be either x_k or $\overline{x}_k$

As a result, $F_1 \supset F_0$ (F_1 must be equal or contain F_0)

Similarly in case (2) if $T_1 > T_0$, then $F_0 \supset F_1$ (F_0 must equal or contain F_1).

In case (3), if $T_0 = T_1$, then any combination of weights of the inputs which provides enough weighted sum of the inputs to overcome the threshold T_0, must overcome the equal threshold T_1; i.e.

If $\sum_{k=1}^{n} w_k x_k^* \geq T_1$ and $T_1 = T_0$

then $\sum_{k=1}^{n} w_k x_k^* \geq T_0$

As a result we have:

$F_0 = F_1$.

Again we have 3 cases:

(1) $F_0 = F_1$

(2) $F_1 \supset F_0$

(3) $F_0 \supset F_1$.

Since $F = x_1 F_1 + \bar{x}_1 F_0$, and in case (1) $F_0 = F_1$.

$F = F_1 (x_1 + \bar{x}_1) = F_1 = F_0$;

i.e. the variables x_1 and $\bar{x}_1$ do not exist in F.

In case (2) $F_1 \supset F_0$, F_1 can be written as

$F_1 = F_0 + F_d$

Then F can be written as

$F = x_1 F_1 + \bar{x}_1 F_0$

$= x_1 (F_0 + F_d) + \bar{x}_1 F_0$;

$= F_0 + x_1 F_d$

i.e. the variable $\bar{x}_1$ does not exist in F.

Similarly in case (3) $F_0 \supset F_1$, F_0 can be written as

$F_0 = F_1 + F_e$.

Then F can be written as

$F = x_1 F_1 + \bar{x}_1 F_0$

$= x_1 F_1 + \bar{x}_1 (F_1 + F_e)$

$= F_1 + \bar{x}_1 F_e$;

i.e. the variable x_1 does not exist in F.

Thus if F is a threshold function, it must be unate. A threshold function F will be called unate in variables $x^*_1, x^*_2, \ldots, x^*_n$ where x^*_k represents either $\bar{x}_k$ or x_k. Thus the function $x_1 \bar{x}_2 + \bar{x}_2 x_3$ is unate in variables x_1, $\bar{x}_2$, and x_3.

Condition 2. If F is a threshold function, unate in variables x^*_1, x^*_2, ..

., x_n, then the following 3 possible cases can occur:

(1) $F_{10} = F_{01}$

(2) $F_{10} \supset F_{01}$

(3) $F_{01} \supset F_{10}$

where $F_{10} = F(x_1 = 1, x_2 = 0)$

$F_{01} = F(x_1 = 0, x_2 = 1)$.

Threshold functions F_{10} and F_{01} are said to be comparable if either $F_{10} \supset F_{01}$ or $F_{01} \supset F_{10}$, we have chosen the variables x^*_1 and x^*_2 for the sake of simplicity, and no loss of generality is incurred.

Proof: F_{10} is the threshold function with $x^*_1 = 1$, and $x^*_2 = 0$. The $x^*_1 = 1$ causes the element to appear to the x^*_3, x^*_4, ..., x^*_n inputs as if the element had a threshold $T_1 = T - w_1$. Similarly, F_{01} is a threshold function with $x^*_1 = 0$, and $x^*_2 = 1$, then $x^*_2 = 1$ causes the element to appear to the x^*_3, x^*_4, ..., x^*_n with threshold $T_2 = T - w_2$.

From condition 1

If $T_1 = T_2$, then $F_{10} = F_{01}$

If $T_1 > T_2$, then $F_{01} \supset F_{10}$

If $T_2 > T_1$, then $F_{10} \supset F_{01}$

2.3 Properties of Threshold Functions Relating to the "Sign" of Weights w_k of Input Variables x_k

(i) Let F be the threshold function expressed in the minimum sum of products form (irredundant normal disjunctive form), and let f be the separating plane equation. If an unnegated variable x_k appears in F, its weight w_k in f is positive. If a negated variable $\bar{x}_k$ appears in F, its weight in f is negative.

This property can be obtained by the follwing argument:

Let F = T be the separating plane equation for the threshold function F which is expressed in the minimum sum of product form,* and has the variable x_k as a factor; i.e.

$$F = x_k F_1 + F_2 .$$

Since F is unate, there must be at least one n-tuple say p'_0 whose x_k component is 0 and for which $F_1(p'_0) = 1$, and $F(p'_0) = 0$. If p_0 denote the n-tuple similar to p'_0 except that x_k compnent is 1, then p_0 is the n-tuple such that $F(p_0) = 1$.

$$\text{Since } F(p'_0) = 0, \quad f(p'_0) < T$$

$$\text{Since } F(p_0) = 1, \quad f(p_0) \geq T$$

As a result:

$$f(p_0) = w_k x_k + f(p'_0) \geq T$$

$$\text{Since } f(p'_0) < T \text{ and } x_k = 1$$

$$\text{therefore} \quad w_k > 0 .$$

In a similar manner, it can be shown that if the variable $\bar{x}_k$ appears negated in a minimum sum of products form, than w_k 0; i.e. the sign of the weight w_k is negative. Since $F = \bar{x}_k F_1 + F_2$ and F is unate, there must be at least one n-tuple, say p_0 whose x_k component is 1 and for which $F(p_0) = 0$. Let p'_0 denote the n-tuple similar to p_0 except that $x_k = 0$, such that $F(p'_0) = 1$.

$$\text{Since } F(p_0) = 0, \quad f(p_0) < T$$

$$\text{Since } F(p'_0) = 1, \quad f(p'_0) \geq T$$

*This form consists of the disjunction of all of the function 's prime implicants.

As a result

$$f(p_0) = f(p'_0) + w_k \bar{x}_k < T$$

Since $f(p'_0) \geq T$, and $\bar{x}_k = 1$

therefore $w_k < 0$.

Example,

The separating plane equation for the element in Fig. 7 is

$$f = x_1 + x_2 + 2x_3 = 2$$

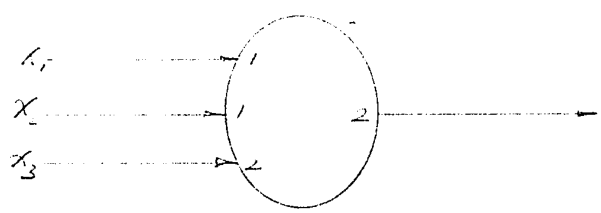

Fig. 7.

The threshold function F is

$$F = \bar{x}_1 \bar{x}_2 x_3 + \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 x_3 + x_1 x_2 \bar{x}_3 + x_1 x_2 x_3$$

The values of F and f are shown in table 1.

p_k	$x_1\ x_2\ x_3$	$f = x_1 + x_2 + 2x_3$	F
p_0	0 0 0	0	0
p_1	0 0 1	2	1
p_2	0 1 0	1	0
p_3	0 1 1	3	1
p_4	1 0 0	1	0
p_5	1 0 1	3	1
p_6	1 1 0	2	1
p_7	1 1 1	4	1

Table 1.

When F is expressed in the minimum sum of products form

$F = x_1x_2 + x_3$ which is unate in x_1, x_2, and x_3,

$F = x_1F_1 + F_2$

where $F_1 = x_2$, and $F_2 = x_3$.

The fundamental product p_2 is ($x_1 = 0$, $x_2 = 1$, $x_3 = 0$).

The fundamental prodcut p_6 is ($x_1 = 1$, $x_2 = 1$, $x_3 = 0$).

Since $F(p_2) = 0$ $f(p_2) = 1 < T$

Since $F(p_6) = 1$ $f(p_6) = 2 = T$

p_2 and p_6 have the same components except x_1

$f(p_6) = w_1x_1 + f(p_2) = T = 2$

Since $f(p_2) = 1$ and $x_1 = 1$

therefore $w_1 = 1$ which is greater than 0.

(ii) If a negated variable $\overline{x}_k$ appears in the minimum sum of products form of a threshold function, the negated variable can be turned into unnegated variable x_k. This can be justified by the following argument:

Let $F(x_1, \ldots, \overline{x}_k, x_{k+1}, \ldots, x_n)$ be the threshold function, and let $-w_k\overline{x}_k + f' = T$ be its separating plane equation, the negative variable $\overline{x}_k$ can be changed into positive x_k so that

$F(x_1, \ldots, \overline{x}_k, \ldots, x_n) \Rightarrow F(x_1, \ldots, x_k, \ldots, x_n)$

Since $-w_k\overline{x}_k + f' = T$ is the separating plane equation for $F(x_1, \ldots, \overline{x}_k, \ldots, x_n)$ and also $-w_k\overline{x}_k = w_kx_k - w_k$, $w_kx_k - w_k + f' = T$ or $w_kx_k + f' = T + w_k$ is the separating plane equation for $F(x_1, \ldots, x_k, \ldots, x_n)$.

In other words, if a negated variable $\overline{x}_k$ is unnegated the sign of its weight $-w_k$ is changed from negative to positive namely w_k, and the magnitude of the weight is added to the threshold.

Example:

The threshold element with separating plane equation $f = x_1 - \bar{x}_2 + 2x_3 = 1$ shown in Fig. 8.

Fig. 8.

is equivalent to the element with separating plane equation $f = x_1 + x_2 + 2x_3 = 2$ as shown in Fig. 9.

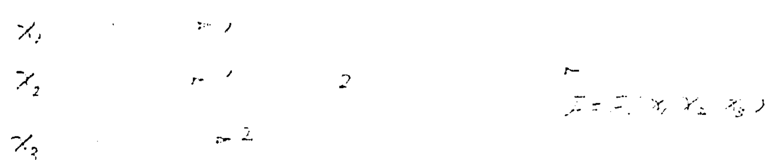

Fig. 9.

They are equivalent because

$$w_2 x_2 = -w_2 \bar{x}_2 + w_2$$

and they realize the same function $F = x_1 x_2 + x_3$

(iii) The dual properties of threshold functions can be stated as follows: If F is a threshold function with separating plane equation

$$f = \sum_{k=1}^{n} w_k x_k \geq T$$

The complement of F or $\overline{F}$ is also a threshold function with separating plane equation f'

$$f' = \sum_{k=1}^{n} -w_k x_k \geq 1 - T$$

This can be shown by the follwoing argument: By definition we have

$$-T + \sum_{k=1}^{n} w_k x_k \geq 0, \qquad F = 1$$

and

$$-T + \sum_{k=1}^{n} w_k x_k < 0, \qquad F = 0$$

Assume that T and w_k are positive integers, By reversing the definition, we have

$$-T + \sum_{k=1}^{n} w_k x_k \leq -1, \qquad \overline{F} = 1$$

$$-T + \sum_{k=1}^{n} w_k x_k > -1, \qquad \overline{F} = 0$$

or

$$T - \sum_{k=1}^{n} w_k x_k \geq 1, \qquad \overline{F} = 1$$

$$T - \sum_{k=1}^{n} w_k x_k < 1, \qquad \overline{F} = 0$$

Therefore the separating equation f' for $\overline{F}$ is

$$f' = \sum_{k=1}^{n} -w_k x_k \geq 1 - T$$

Example:

The threshold element with separating plane equation

$f = x_1 + x_2 + 2x_3 = 2$ as shown in Fig. 10

Fig. 10

$\overline{F}$ can be realize by a threshold element with separating plane equation f' as shown in Fig. 11, where

$$f' = -x_1 + -x_2 + -2x_3 = 1 - 2$$

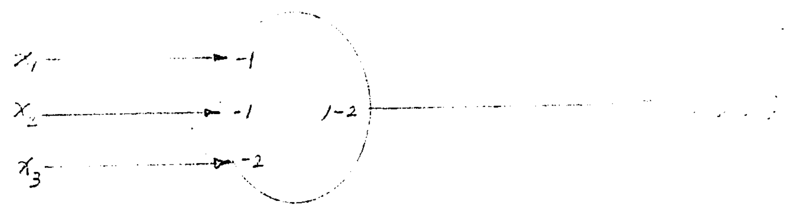

Fig. 11.

which is equivalent to the element shown in Fig. 12.

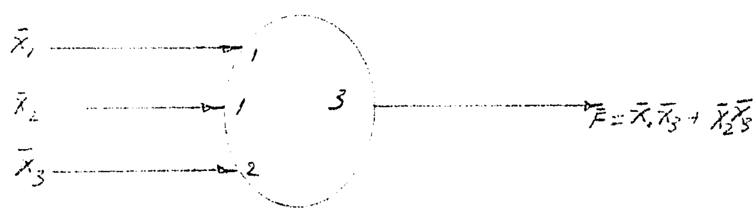

Fig. 12.

In summary, from the above properties concerning the sign of the weights of the variables, every threshold function can be converted to positive (unnegated)-variable function with positive weights. From now on, only functions with positive (unnegated) variables and positive weights will be discussed; of course no loss in generality is incurred

2. 4 Properties of Threshold Functions Relating to the "Size" (7) of Weights w_k

Definition of reduced functions: If we substitute the value of 1 in all the variables in a subset S_1 of a set of input variables $\{x_1, \dots, x_n\}$, and substitute the value 0 in all the variables in a disjoint subset S_2, a new function is formed which is called a re-duced function and denoted by $F(S_1 = 1, S_2 = 0)$, where F is the original function with input variables $x_1, \dots, x_n$.

(i) Let F be the threshold function with a set of input variables $\{x_1, \dots, x_n\}$. Set S_1 and S_2 are any two disjoint subsets of this set of input variables. Let S be the union of S_1 and S_2, and let s_1 and s_2 denote the sums of the weights of variables of S_1 and S_2 respectively. If F is a threshold function with separating plane equation $f = T$, then the reduced function $F(S_1 = 1, S_2 = 0)$ is also a threshold function with separating plane equation

$$f(S_1 = 1, S_2 = 0) = T$$

$$\text{or} \quad f(S = 0) = T - s_1$$

Example:

$F = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3x_4x_5$, the element realizing F is shown in Fig. 13.

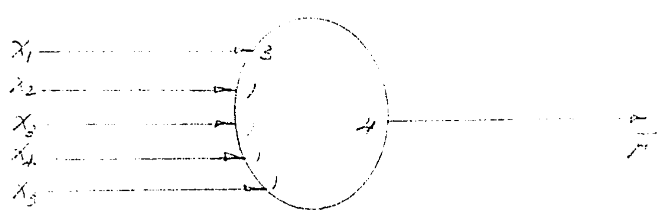

Fig. 13

If $S_1 = \{x_1\}$, $S_2 = \{x_2, x_3\}$, $S = \{x_1, x_2, x_3\}$

$F_1 = F(S_1 = 1, S_2 = 0)$

$\quad = F(x_1 = 1, x_2 = 0, x_3 = 0)$

$= x_4 + x_5$.

$F_2 = F(S_1 = 0, S_2 = 1)$

$= F(x_1 = 0, x_2 = 1, x_3 = 1)$

$= x_4 x_5$.

The separating plane equation for F is

$$f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4.$$

The separating plane equation for F_1 is

$$f_1 = x_4 + x_5 = 4 - 3 = 1.$$

The separating plane equation for F_2 is

$$f_2 = x_4 + x_5 = 4 - (1 + 1) = 2.$$

If either $F_1 \supset F_2$ or $F_2 \supset F_1$, F_1 and F_2 are comparable functions.

(ii) All threshold function which can have the same separating equation $f = T$ are comparable. Let F be a threshold function with separating plane equation $f = T$, and let S, S_1, S_2, s_1, and s_2 be the same as defined in (i), then $F_1 = F(S_1 = 1, S_2 = 0)$ and $F_2 = F(S_1 = 0, S_2 = 1)$ are comparable functions, and $f_1(S = 0) = T - s_1$ is the separating plane equation for F_1, and $f_2(S = 0) = T - s_2$ is the separating plane equation for F_2. Moreover, if

$F(S_1 = 1, S_2 = 0) \supset F(S_1 = 0, S_2 = 1)$, then

$T - s_1 < T - s_2$ and $s_1 > s_2$.

Example:

$$F = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_1 x_5 + x_2 x_3 x_4 x_5$$

The separating plane equation is

$$f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4$$

Let $S_1 = \{x_1\}$, and $S_2 = \{x_2, x_3\}$

$F(S_1 = 1, S_2 = 0) = F(x_1 = 1, x_2 = 0, x_3 = 0) = x_4 + x_5$

$F(S_1 = 0, S_2 = 1) = F(x_1 = 0, x_2 = 1, x_3 = 1) = x_4 x_5$

$s_1 = w_1 = 3;\ s_2 = w_2 + w_3 = 1 + 1 = 2$

Since $s_1 > s_2$, $F(S_1 = 1, S_2 = 0) \supset F(S_1 = 0, S_2 = 1)$;

i.e.

$$x_4 + x_5 \supset x_4 x_5$$

(iii) If F is a threshold function, $F(S_1 = 1, S_2 = 0) \supset F(S_1 = 0, S_2 = 1)$ implies that $s_1 > s_2$ in all the separating plane equations for F. If the sets S_3 and S_4 are such that the union of S_3 and S_4 equals the union of S_1 and S_2, then $F(S_1 = 1, S_2 = 0) \supset F(S_3 = 1, S_4 = 0)$ implies that $s_1 > s_3$ in all the separating plane equations for F.

Example:

$F = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_1 x_5 + x_2 x_3 x_4 x_5$

The separating plane equation is

$f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4.$

Let $S_1 = \{x_1\}$, $S_2 = \{x_2, x_3\}$, $S_3 = \{x_1, x_2\}$, $S_4 = \{x_3\}$

and $S = S_1 \cup S_2 = S_3 \cup S_4 = \{x_1, x_2, x_3\}$

$F_1 = F(S_1 = 1, S_2 = 0) = F(x_1 = 1, x_2 = 0, x_3 = 0) = x_4 + x_5$.

$F_3 = F(S_3 = 1, S_4 = 0) = F(x_1 = 1, x_2 = 1, x_3 = 0) = 1$;

i.e. $F_1 \subset F_3$

Therefore $s_3 > s_1$

where $s_3 = w_1 + w_2 = 3+1 = 4$.

$s_1 = w_1 = 3$.

(iv) If $F(S_1 = 1, S_2 = 0) = F(S_1 = 0, S_2 = 1)$, nothing in general can be said about the comparability of s_1 and s_2

Example:

$F = x_1 x_3 + x_2 x_3$

$F(x_1 = 1, x_2 = 0) = x_3 = F_1$

$F(x_1 = 0, x_2 = 1) = x_3 = F_2$

$F_1 = F_2$

But $w_1 = w_2$ is not necessary true.

The function $F = x_1 x_3 + x_2 x_3$ can be realized by a threshold element with structure shown in Fig. 14 in which $w_2 > w_1$.

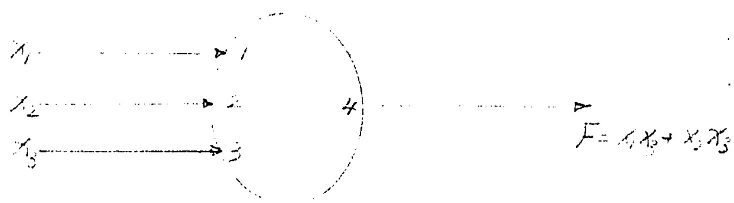

Fig. 14

But this function F can also be realized by a threshold element with structure shown in Fig. 15 in which $w_1 > w_2$

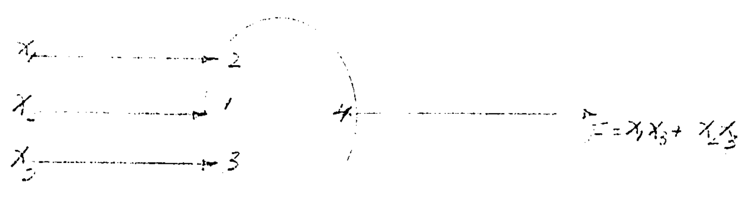

Fig. 15

and again this function can also be realized by the element with structure as shown in Fig. 16 in which $w_1 = w_2$.

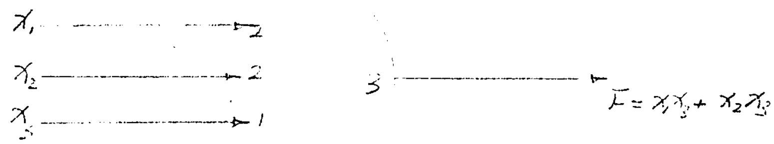

Fig. 16

But if the sets S_1 and S_2 are single variables, say

$S_1 = \{x_i\}$; $S_2 = \{x_k\}$, the equality

$F(x_i = 1, x_k = 0) = F(x_i = 0, x_k = 1)$

implies that the function F can be symmetric in x_i and x_k. If a threshold function is symmetric in m variables, all its weights can be equal in any separating plane equations.

Example:

$F = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3x_4x_5$.

The separating plane equation f for this threshold function is

$f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

$F(x_2 = 1, x_3 = 0) = x_1 + x_1x_4 + x_1x_5 = x_1$.

$F(x_2 = 0, x_3 = 1) = x_1 + x_1x_4 + x_1x_5 = x_1$.

$F(x_2 = 1, x_3 = 0) = F(x_2 = 0, x_3 = 1)$ implies that the fucntion F is symmetric in x_2 and x_3, and the weights of x_2 and x_3 in the separating plane equation $f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4$ are each equal to 1.

(v) Let F be a switching function factored so that $F = x_kF_1 + F_0$, where x_k does not appear in the minimum sum of products form of

F_1 and F_0. If F is a threshold function and if F_1 F_0, F_1 and F_0 are reduced functions of F, and hence, they are threshold functions. Moreover, if $w_k x_k + f'$ is the separating plane equation for F, then f' is the separating plane equation for F_1 and F_0. It is also generally true that if F_1 and F_0 are the threshold function in whose minimum sum of products forms, x_k does not appear, and if F_1 and F_0 can have the same separating plane equation, then $x_k F_1 + F_0$ is a threshold function.

2.5 The Ordering of Weights of Variables of Threshold Functions

From the results of discussing the properties of threshold functions relating to the "size" of the weights, the weights w_k of the variables x_k in the separating plane equation can be ordered in the following steps:

(step 1) Count the number of times each variable appears in the minimum sum of products terms of F which contain exactly m variables where m is the minimum number of variables in any of the minimum sum of products terms. The variable which appears more times than the others has the largest weight. If variables appear the same number of times in the minimum sum of products terms, step 2 must be followed.

(Step 2) Consider those minimum sum of products terms of F that contain exactly m+1 variables. Count the number of appearances of those variables which appear the same number of times in step 1. The variable which appears most, has the larger weight. If the variables appear the same number of times in the m+1 - variable terms, and do not appear in m+2 - variable terms, the weights of those variables can be considered equal.

(Step 3) Repeat the process until the variables are all ordered, and all the terms of the function have been used.

The Procedure of ordering of weights can be easily understood from the following example:

$F = x_1x_2 + x_1x_3x_4 + x_1x_3x_5x_6 + x_2x_3x_4x_5x_6$

Since m (which is the minimum number of variables in any term) is 2, the 2-variable term is x_1x_2. Since x_1 and x_2 appear the same number of times in x_1x_2, but only x_1 appears in the 3-variable term $x_1x_3x_4$, then $w_1 > w_2$.

Since x_2 appears in the 2-variable term x_1x_2, and x_3 does not, then $w_2 > w_3$.

Since x_3 and x_4 appear the same number of time in $x_1x_3x_4$, but x_3 appears twice and x_4 appears once in the 4-variable terms, $x_1x_3x_5x_6$ and $x_2x_3x_4x_5x_6$, then $w_3 > w_4$.

Since x_4 appears in the 3-variable term $x_1x_3x_4$ and x_5 does not appear in 2-variable and 3-variable terms, then $w_4 > w_5$.

Since x_5 and x_6 appear the same number of times in the 5-, 4-variable terms but in no where else, then $w_5 = w_6$.

Summing up: For F, the wieghts can be ordered as follows:

$$w_1 > w_2 > w_3 > w_4 > w_5 = w_6$$

The structure of the element is shown in Fig. 17

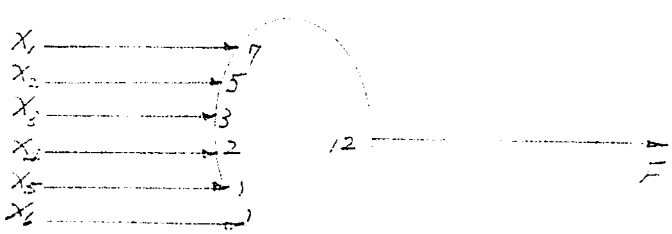

Fig. 17

2.6 Threshold Function "Tree"

Let F be a threshold function, and let us express F in a minimum sum of products form $F = x_k F_1 + F_0$, where x_k is the variable which has the largest weight, and does not appear in F_1 and F_0. This can be obtained by means of weight ordering. Since F is a threshold function, F_1 and F_0 are the reduced functions of F, and $F_1 \supset F_0$. We can display $F = x_k F_1 + F_0$ in a form of a tree-Fig. 18.

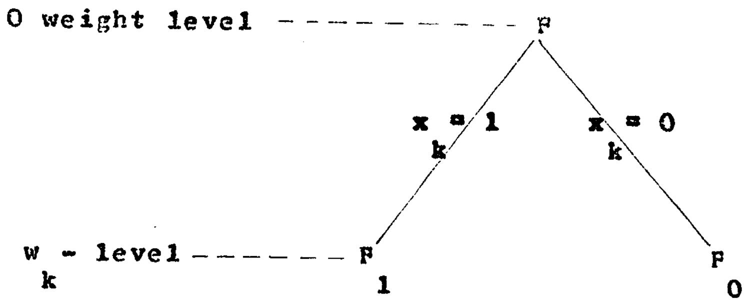

Fig. 18. The uncomplete tree of F.

We call the origin function F 0-weight level function, and F_1 and F_0 w_k-level or x_k-level functions. Since F is a threshold function, then F_1 and F_0 are also threshold functions. Express F_1 and F_0 in the same way as F, i.e.

$$F_1 = x_{k+1} F_{11} + F_{10}$$
$$F_0 = x_{k+1} F_{01} + F_{00}$$

where x_{k+1} is the variable which has the largest weight among the variables in F_1 and F_0, and of course, $w_k > w_{k+1}$.

Again, by displaying F_{11}, F_{10}, F_{01} and F_{00} on the tree, the Fig. 19 is obtained.

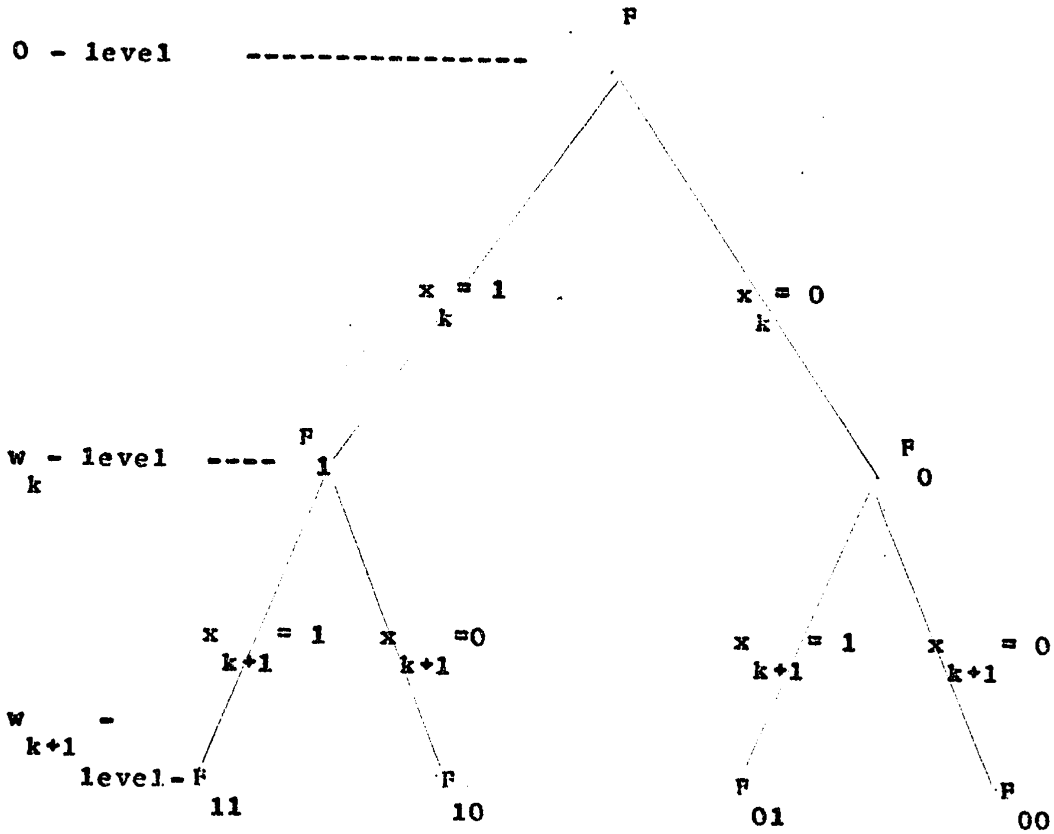

Fig. 19 The uncomplete tree of F.

The splitting of the reduced functions is continued in this way, until the last variable which has the smallest weight in the separating plane equation for F is reached. The final tree or the complete tree is shown in Fig. 20,

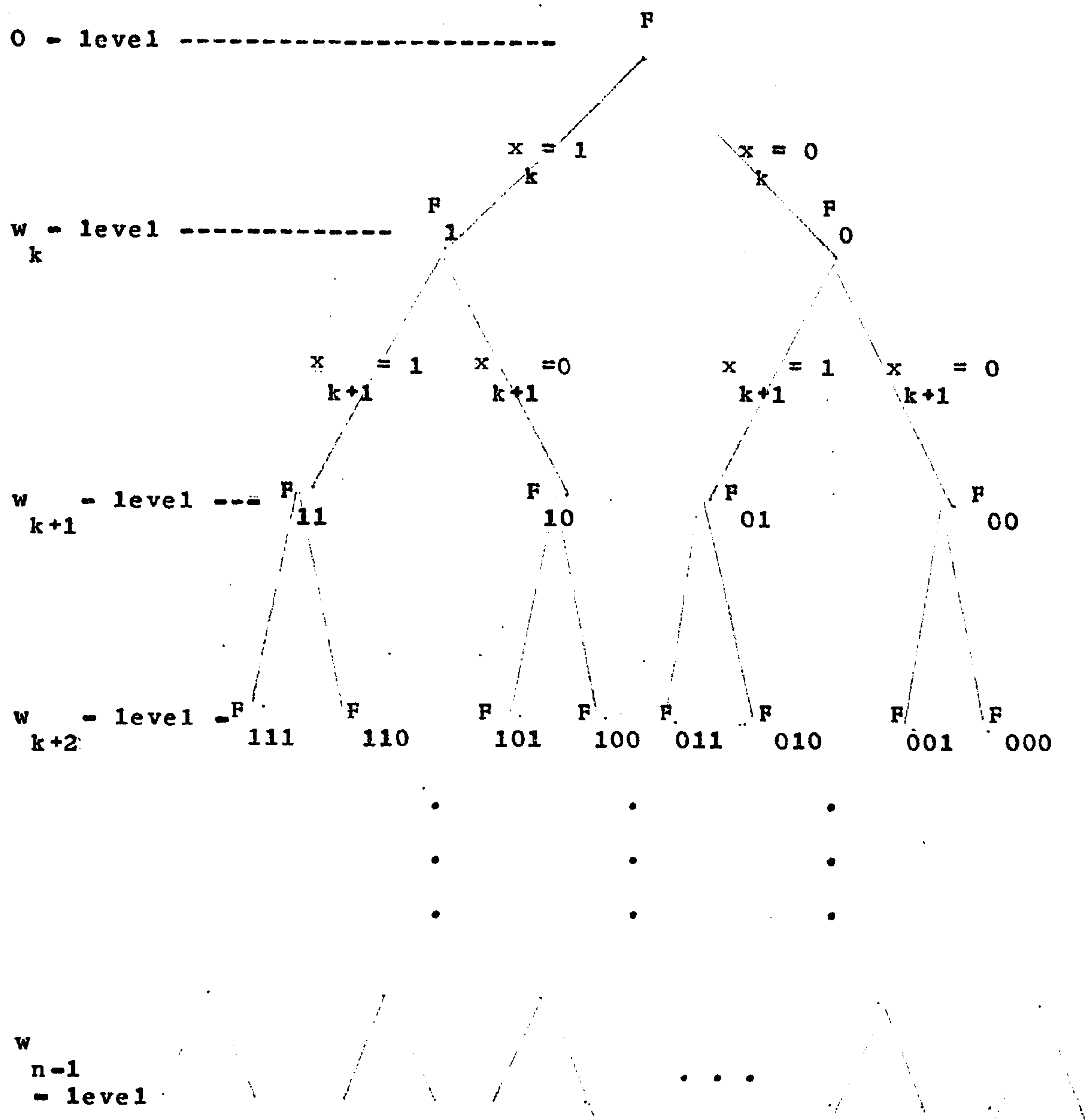

Fig. 20 The complete tree of F

where $w_k \geq w_{k+1} \geq w_{k+2} \geq w_{k+3}, \dots, \geq w_{n-1} \geq w_n$.

Example: Let the threshold function be

$$F = x_1 x_2 + x_1 x_3 x_4 + x_1 x_3 x_5 x_6 + x_2 x_3 x_4 x_5 x_6$$

The complete decomposition of F into a tree is shown in Fig. 21.

By means of weight ordering procedure, we have

$$w_1 > w_2 > w_3 > w_4 > w_5 = w_6$$

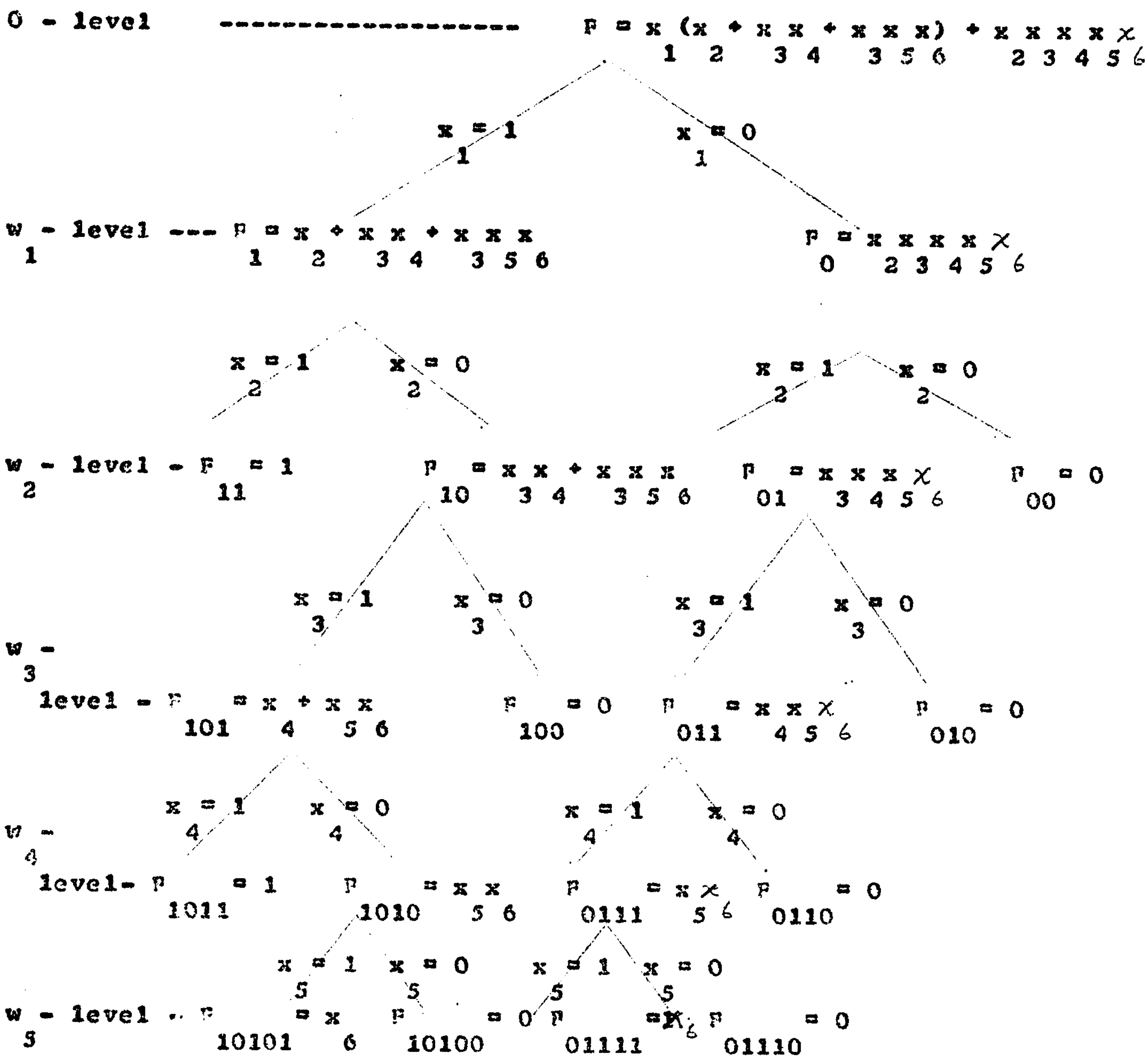

Fig. 21 The tree of F.

The reduced functions which identically equal either 1 or 0 are called constant functions and are also the threshold functions. The function tree consists of n levels, 0-level, w_1-level, w_2-level, and so on. The function on 0-level is the original function F. The functions on w_k-level are reduced functions of F.

Let F^k_q be any one of reduced functions of F on w_k-level. F^k_q can be expressed in terms of its lower pair of reduced functions namely F^{k+1}_{2q+1} and F^{k+1}_{2q} as shown in Fig. 22.

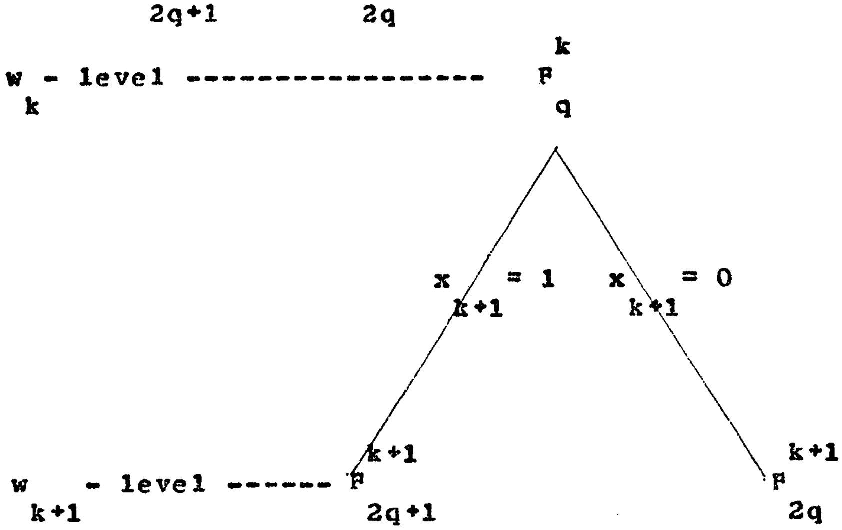

Fig. 22. The lower pair of F^k_q.

That is
$$F^k_q = x_{k+1} F^{k+1}_{2q+1} + F^{k+1}_{2q}$$
Since F^k_q is one of the reduced functions of the original function F, and according to the properties of threshold functions, x_{k+1} must not appear in F^{k+1}_{2q+1} and F^{k+1}_{2q}, and F^{k+1}_{2q+1} must contain F^{k+1}_{2q} (i.e. $F^{k+1}_{2q+1} \supset F^{k+1}_{2q}$). Moreover, if $w_{k+1} + f_{k+1} = f_k$ is the separating plane equation for F^k_q, then f_{k+1} is the separating plane equation for both F^{k+1}_{2q+1} and F^{k+1}_{2q}.

In general each pair of the reduced functions at any w_k - level, $1 \leq k \leq n$, must be comparable. If these conditions are not fulfilled, then F is not a threshold function.

For example:

$$F = x_1 x_2 + x_3 x_4$$

The function is unate and the weights order of this function is

$$w_1 \geq w_2 \geq w_3 \geq w_4$$

The tree of F is shown in Fig. 23

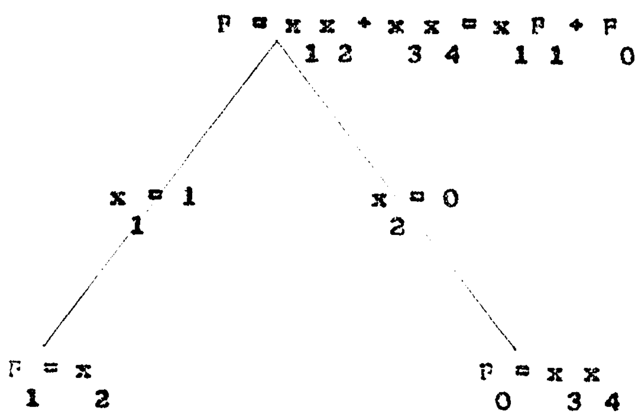

Fig. 23

Since $x_2 \not\geq x_3 x_4$, F is not a threshold function.

2.7 Analysis of Threshold Functions in Terms of "Map" and "Gap"

The threshold function F and the separating plane equation can be related as follows:

$$f(p_k) = \sum_{k=1}^{n} w_k x_k \geq T \quad , \qquad \text{when } F(p_k) = 1$$

$$f(p_k) = \sum_{k=1}^{n} w_k x_k < T \quad , \qquad \text{when } F(p_k) = 0$$

where p_k are the fundamental products.

The separating plane equation and the threshold function F are defined on the set of fundamental products p_k which are equal to 2^n in number, and where n is the number of binary variables. To each fundamental product p_k, there is a corresponding ordered pair $\left[f(p_k), F(p_k) \right]$, where $f(p_k)$ is a positive real number and $F(p_k)$ is either 1 or 0. The complete set of ordered pairs $\left\{ f(p_k)\ F(p_k) \right\}$ defines a "map" of F. (of course, there are 2^n such ordered pairs) The map of F is a representation of the truth table. It displays all the ordered pairs in the truth table on a straight line, and divides the function F into two disjoint subsets called the zero part and the unit part. The zero part contains all points on which F = 0, i.e. $F(p_k) = 0$. The unit part contains all the points on which F = 1, i.e. $F(p_k) = 1$. Example,

$F = x_1 + x_2 x_3$ is a threshold function which can be realized by the following element- Fig. 24

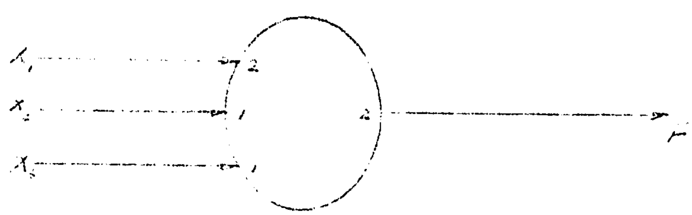

Fig. 24

The separating plane equation is $f = 2x_1 + x_2 + x_3 = 2$. The relation of f and F is shown in the following table 2

p_k	x_1	x_2	x_3	$f = 2x_1 + x_2 + x_3$	F
p_0	0	0	0	0	0
p_1	0	0	1	1	0
p_2	0	1	0	1	0
p_3	0	1	1	2	1
p_4	1	0	0	2	1
p_5	1	0	1	3	1
p_6	1	1	0	3	1
p_7	1	1	1	4	1

Table 2. Values of F and f.

The map of the above table can be displayed as follows:-

Fig. 25.

p_k:	(p_0)	(p_2)	(p_1)	(p_4)	(p_3)	(p_6)	(p_5)	(p_7)
$f(p_k)$:	0	1	1	2	2	3	3	4
	0	0	0	1	1	1	1	1
			l	u				
			← g →					
	← 0 - part →			← 1 - part →				

Fig. 25 The "map" of $F = x_1 + x_2 x_3$.

Of the members of a unit-part, let u be the smallest of $f(p_k)$, of the members of 0-part, let l be the largest of $f(p_k)$, The "gap" of the map is defined as g = u;l. For example the "gap" of the map for $F = x_1 + x_2 x_3$ is

$g = u : l$

$= 2 : 1$

u is called the upper bound of the gap.

l is called the lower bound of the gap.

Threshold functions which are identically 0, i.e. F = 0 or identically 1 are called constant functions.

If F = 0, then it has the gap $= \infty : f_M$;where the upper bound of the gap is ∞ , the lower bound of the gap is f_M , and f_M is the maximum value of $f(p_k)$; i.e.

$$f_M = \sum_{k=1}^{K} w_k$$

If F = 1, then it has the gap $= f_m : -\infty$, where the upper bound of the gap $u = f_m$ which is the minimum value of $f(p_k)$,and the lower bound of the gap $l = -\infty$.

By the definition of gap, if F is a threshold function, f is the separating plane equation such that if

$$f(p_k) = \sum_{k=1}^{n} w_k x_k \geq T, \; F(p_k) = 1$$

then F must have at least one map such that

$u \geq T \geq l$

where u is the upper bound of the gap, l is the lower bound of the gap and T is the threshold.

Let F be a threshold function and is factored as

$F = x_k F_1 + F_0$

where x_k does not appear in F_1 and F_0 . According to the properties of threshold functions, F_1 and F_0 are also threshold functions, and

$F_1 \supset F_0$. If $w_k x_k + f'$ is the separating plane equation for F, then f' is the separating plane equation for F_1 and F_0.

The following argument establishes a useful relationship between the maps of F_1 and F_0 due to the separating plane equation f' and the map of $F = x_k F_1 + F_0$ due to the separating plane equation $f = w_k x_k + f'$ when $F_1 \supset F_0$:

Suppose F_1 has gap $u_1 : l_1$

F_0 has gap $u_0 ; l_0$

Let

$$L(w_k) = l_0 - u_1$$

$$U(w_k) = u_0 - l_1$$

In order that $F = x_k F_1 + F_0$ be threshold function, w_k must lie between $L(w_k)$ and $U(w_k)$, i.e.

$$L(w_k) < w_k < U(w_k)$$

The range of the values of w_k is $R(w_k)$ which is expressed as:

$$R(w_k) = U(w_k) \ast L(w_k)$$

where $L(w_k)$ is called the lower bound of the range

$U(w_k)$ is called the upper bound of the range.

In other words, if f' is the separating plane equation for both F_1 and F_0 where F_1 F_0, then $w_k x_k + f'$ is the separating plane equation for $F = x_k F_1 + F_0$, if and only if, the value of w_k lies in $R(w_k)$. This condition assures that the gap of the map F is the intersection of the gap of the map F_0 and the gap of the displaced map F_1 (after displaced by w_k). If the intersection is empty, the function $F = x_k F_1 + F_0$ is not a threshold function.

Example:

$$F = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3x_4x_5$$

The function is threshold function (given), the separating plane equation is

$$f = 3x_1 + x_2 + x_3 + x_4 + x_5 = 4$$

$$F = x_1(x_2 + x_3 + x_4 + x_5) + x_2x_3x_4x_5$$
$$= x_1F_1 + F_0$$

where

$$F_1 = x_2 + x_3 + x_4 + x_5$$
$$F_0 = x_2x_3x_4x_5$$

The separating plane equation f' for F_1 and F_0 is

$$f' = x_2 + x_3 + x_4 + x_5 = 1$$

The values of F_1, F_0, and f' are shown in Table 3.

x_2	x_3	x_4	x_5	$f' = x_2 + x_3 + x_4 + x_5$	F_1	F_0
0	0	0	0	0	0	0
0	0	0	1	1	1	0
0	0	1	0	1	1	0
0	0	1	1	2	1	0
0	1	0	0	1	1	0
0	1	0	1	2	1	0
0	1	1	0	2	1	0
0	1	1	1	3	1	0
1	0	0	0	1	1	0
1	0	0	1	2	1	0
1	0	1	0	2	1	0
1	0	1	1	3	1	0
1	1	0	0	2	1	0
1	1	0	1	3	1	0
1	1	1	0	3	1	0
1	1	1	1	4	1	1

Table 3 Values of f', F_1 and F_0.

The values of F and f are shown in Table 4:

x_1	x_2	x_3	x_4	x_5	$f = 3x_1 + x_2 + x_3 + x_4 + x_5$	F
0	0	0	0	0	0	0
0	0	0	0	1	1	0
0	0	0	1	0	1	0
0	0	0	1	1	2	0
0	0	1	0	0	1	0
0	0	1	0	1	2	0
0	0	1	1	0	2	0
0	0	1	1	1	3	0
0	1	0	0	0	1	0
0	1	0	0	1	2	0
0	1	0	1	0	2	0
0	1	0	1	1	3	0
0	1	1	0	0	2	0
0	1	1	0	1	3	0
0	1	1	1	0	3	0
0	1	1	1	1	4	1
1	0	0	0	0	3	0
1	0	0	0	1	4	1
1	0	0	1	0	4	1
1	0	0	1	1	5	1
1	0	1	0	0	4	1
1	0	1	0	1	5	1
1	0	1	1	0	5	1
1	0	1	1	1	6	1
1	1	0	0	0	4	1
1	1	0	0	1	5	1
1	1	0	1	0	5	1
1	1	0	1	1	6	1
1	1	1	0	0	5	1
1	1	1	0	1	6	1
1	1	1	1	0	6	1
1	1	1	1	1	7	1

Table 4 Values of f and F.

The maps of F_1, F_0, and F are shown in Fig. 26.

w_1 must lie between $U(w_k)$ and $L(w_k)$; i.e.

$$4 > 3 > 2 .$$

CHAPTER 3: SYNTHESIS OF THRESHOLD FUNCTIONS

3.1 Introduction

As mentioned in chapter 1, a Boolean function F of n binary variables has 2^n linear inequalities, corresponding to the 2^n different combinations of the variables. The set of inequalities can be written according to the following expressions

$$-T + w_k x_k \geq 0 \qquad F(x_1, \ldots, x_n) = 1$$
$$-T + w_k x_k < 0 \qquad F(x_1, \ldots, x_n) = 0$$

For example, the inclusive-or function of 2 variables has the following $2^2 = 4$ inequalities:

$$-T + w_1 + w_2 \geq 0$$
$$-T + w_1 \geq 0$$
$$-T + w_2 \geq 0$$
$$-T < 0$$

The Boolean function F is a threshold function, if and only if its set of linear inequalities is consistent. (A set of linear inequalities is consistent, if and only if it has at least one solution). Among these 2^n linear inequalities, many are redundant and can be deleted. If I is the original set of linear inequalities of of a threshold function F, and I' is the subset of I, containing the inequalities not redundant under any reducing relation, then the solution to I' are the solutions of realization of the given threshold function. This gives a simple test-synthesis procedure. First, derive the reduced set I'. Then check whether the reduced linear inequalities are consistent or not. If they are inconsistent, the function is not threshold function. If they are, solve I'. The solution is the desired realization of the Boolean function.

3.2. Tests for Threshold Functions

A Boolean function F of n variables and its complement $\overline{F}$ can be expressed in the forms of sum of fundamental products. All the fundamental products in F have a value of 1, and all the fundamental products in $\overline{F}$ have a value of 0. Each fundamental product can be represented by an n-tuple of 0's and 1's. F and $\overline{F}$ can be displayed on an n dimensional space, (called an n-cube) which contains 2^n n-tuples, and each n-tuple can be identified with a vertex of the n-cube. Since each fundamental product of F has a value of 1, F constitues the subset of vertices in the n-cube which have a value of 1, and we denote these vertices by $V_1(F)$. Similarly, $\overline{F}$, the complement of F, constitutes the rest of vertices in the n-cube which have a value of 0, and we denote these vertices by $V_0(\overline{F})$. F is a threshold function if and only if there is an (n-1) dimensional hyperplane separating the 1-vertices V_1 from the 0-vertices V_0 so that all the 0-vertices lie on one side of the hyperplane and all the 1-vertices lie on the other side of the hyperplane. The hyperplane equation known as the separating plane equation can be expressed as:

$$\sum_{k=1}^{n} w_k x_k - T = 0.$$

Boolean functions satisfying above condition must be unate and their reduced functions must comparable. We can check the unateness of a given Boolean function by examining its minimum sum of products form. If the variable x_k and its complement $\overline{x}_k$ both appear in the minimum sum of products form, the given function is not a threshold function. After we have checked the unateness of a given function, we test the comparability of its reduced functions. This can be accomplished in the following manner: We write down the minimum sum of products form for F and factor it as $F = x_k F_1 + F_0$ with no x_k appearing in F_1 and F_0. We continue to decompose $F = x_k F_1 + F_0$ in a form of tree as shown in Fig. 20 (page 28). F_1 and F_0 are the reduced functions of F at w_k - level, and F_{11}, F_{10}, F_{01}, and F_{00} are the reduced functions of F at w_{k+1} - level etc. If F is a threshold function, any pair of reduced functions at each level must be comparable, otherwise F is not a threshold function.

3.3. Reduction of 2^n redundant Inequalities

Let V denote the set of all 2^n vertices in the n-cube and let

$$\sum_{k=1}^{n} w_k x_k - T = 0$$

be the separating plane dividing V into two disjoint subsets V_1 and V_0. Then V_1 is the set of vertices corresponding to the set of fundamental products in F, and V_0 is the set of vertices corresponding to the set of fundamental products in $\overline{F}$, the complement of F.

If F is expressed in its minimum sum of products form which is the disjunction of all of the function's prime implicants. Each term P (or each prime implicant) of the minimum sum of products of F can be identified with an n-tuple by designating the k^{th} position as 1 if x_k is present in P, and as 0 if it is not. The set of n-tuples obtained from the prime implicants is called the set of the "lowest vertices" of F in vector space terminology and is denoted by $\{1\}$. In similar manner, let $\overline{F}$ be expressed in its minimum sum of products form, Each term $\overline{P}$ (or each prime implicant) of the minimum sum of products of $\overline{F}$ can be identified with an n-tuple by designating the k^{th} position as a 0 if $\overline{x}_k$ is present, and as 1 if it is not. The set of n-tuples obtained from the prime implicants is called the "highest vertices" of $\bar{F}$ in vector space terminology, and is denoted by $\{h\}$. Example:

$$F = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2x_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3 + x_1x_2x_3$$

The truth table is as follows (Table 5)

x_1	x_2	x_3	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Table 5. Values of F.

$\bar{F} = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3$ (sum of fundamental products)

$F = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2x_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3 + x_1x_2x_3$ (sum of fundamental products)

The set of all vertices V

$$V = \{001,\ 011,\ 101, 110,\ 111,\ 010,\ 100,\ 000\}$$

The set of 1-vertices V_1

$$V_1 = \{001,\ 011,\ 101,\ 110,\ 111\}$$

The set of 0-vertices V_0

$$V_0 = \{000,\ 010,\ 100\}$$

$F = x_1x_2 + x_3$ (minimum sum of products)

$\bar{F} = \bar{x}_1\bar{x}_3 + \bar{x}_2\bar{x}_3$ (minimum sum of products)

The set of lowest vertices obtained from the prime implicants of F is

$$\{1\} = \{110,\ 001\}$$

The set of highest vertices obtained from the minimum sum of products of $\overline{F}$ is

$$\{h\} = \{010,\ 100\}$$

The number of linear inequalities in the set I arising from V_1 and V_0 is 2^n. The number of linear inequalities in the set I' arising from $\{1\}$ and $\{h\}$ is greatly reduced. The solutions to I' are also the solutions of I. This is because the hyperplane which separates $\{1\}$ from $\{h\}$ also separates V_1 from V_0.

By the methods stated in chapter 2, we relabel the variables so that they are in the order of decreasing weights; i.e.

$$w_1 \geq w_2 \geq w_3 \geq, \ldots, \geq w_n > 0$$

The variables associated with these weights are x_1, x_2, x_3, ..., x_n. The ordering of weights can be used to further reduce the number of inequalities in I'. If F is symmetric in variables x_1, ..., x_k, where $\leq 0 \leq k$ n, then the associated weights can be given equal values; i.e.

$$w_1 = w_2 = w_3 =, \ldots, = w_k.$$

The summation $\sum_{k=1}^{n} w_k x_k$ is a measure of distance of the vertex from a separating hyperplane. For example, if the weight ordering is $w_1 > w_2 > w_3 = w_4 > 0$, the distance of the vertex (1, 1, 1, 1) is $w_1 + w_2 + 2w_3$, and the distance of another vertex (1, 1, 0, 1) is $w_1 + w_2 + w_3$. If a vertex t_1 can be determined by the ordering of weights to be further away from the separating hyperplane than the other vertex t_2, then the inequality resulting from t_1 is redundant and can be deleted. The vertex (1, 1, 1, 1) of above example can be deleted. By deleting such vertices from $\{1\}$ and $\{h\}$, the final reduced subsets of vertices $\{L\}$ and $\{H\}$ can be formed. The final set of linear inequalities from which the weights w_k are to be determined are those generated by the vertices of $\{L\}$ and $\{H\}$.

Example:

$$P = x_1x_2 + x_1x_3 + x_1x_4x_5 + x_1x_4x_6 + x_1x_5x_6 + x_2x_4x_5 + x_2x_4x_6 + x_2x_5x_6 + x_2x_5x_6 + x_2x_3x_4 + x_2x_3x_5 + x_2x_3x_6$$

$$\bar{P} = \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_4\bar{x}_5\bar{x}_6 + \bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_6 + \bar{x}_2\bar{x}_3\bar{x}_5\bar{x}_6$$

By means of weights ordering, we have

$$w_1 > w_2 > w_3 > w_4 = w_5 = w_6 > 0$$

The set $\{1\}$ = {11000, 101000, 100110, 100101, 100011, 010110, 010101, 010011, 011100, 011010, 011001}

The inequalities obtained from $\{1\}$ are:

(1) 110000 --- $w_1 + w_2 = I_1$

(2) 101000 --- $w_1 + w_3 = I_2$

(3) 100110 --- $w_1 + w_4 + w_5 = I_3$

(4) 100101 --- $w_1 + w_4 + w_6 = I_4$

(5) 100011 --- $w_1 + w_5 + w_6 = I_5$

(6) 010110 --- $w_2 + w_4 + w_5 = I_6$

(7) 010101 --- $w_2 + w_4 + w_6 = I_7$

(8) 010011 --- $w_2 + w_5 + w_6 = I_8$

(9) 011100 --- $w_2 + w_3 + w_4 = I_9$

(10) 011010 --- $w_2 + w_3 + w_5 = I_{10}$

(11) 011001 --- $w_2 + w_3 + w_6 = I_{11}$

Since $I_2 < I_1$, (because $w_2 > w_3$), I_1 can be deleted.

Since $I_6 < I_3$, (because $w_1 > w_2$; $w_4 = w_5 = w_6$) I_3, I_4, and I_5 can be deleted, because $I_3 = I_4 = I_5$.

Since $I_6 = I_7 = I_8 < I_9 = I_{10} = I_{11}$ (because $w_4 = w_5 = w_6$ and $w_3 > w_4$)

I_7, I_8, I_9, I_{10} and I_{11} can be deleted.

Therefore $\{L\} = \{101000,\ 010110\}$

Similarly the set $\{h\} = \{001111,\ 011000,\ 100100,\ 100010\}$

The inequalities obtained from $\{h\}$ are

(1) 001111 --- $w_3 + w_4 + w_5 + w_6 = I'_1$

(2) 011000 --- $w_2 + w_3 = I'_2$

(3) 100100 --- $w_1 + w_4 = I'_3$

(4) 100010, --- $w_1 + w_5 = I'_4$

Since $I'_3 = I'_4$ (because $w_4 = w_5$), I'_4 can be deleted.

Therefore $\{H\} = \{001111,\ 011000,\ 100100\}$

The final set of linear inequalities are

$(I_1) = w_1 + w_3$

$(I_2) = w_3 + w_4 + w_5 + w_6 = w_3 + 3w_4$

$(I_3) = w_2 + w_4 + w_5 = w_2 + 2w_4$

$(I_4) = w_3 + w_2$

$(I_5) = w_1 + w_4$

It is seen that a set of 2^6 linear inequalities can be reduced to a set of 5 linear inequalities.

3.4 Algorithm for Trial Synthesis

The algorithm can be obtained by the following argument:

The set of reduced linear inequalities can be obtained from the reduced sets of vertices $\{L\}$ and $\{H\}$ according to

$$-T + \sum_{k=1}^{n} w_k x_k \geq 0 \quad \text{for } \{L\} \quad \text{------- (a)}$$

$$-T + \sum_{k=1}^{n} w_k x_k < 0 \quad \text{for } \{H\} \quad \text{------- (b)}$$

By eliminating T in (a) and (b) we have

$$\sum wx > \sum{}' wx \quad \text{-------------------- (c)}$$

where $\sum wx$ denotes $\sum_{k=1}^{n} w_k x_k$ for $\{L\}$ in (a) and $\sum' wx$ denotes $\sum_{k=1}^{n} w_k x_k$ for $\{H\}$ in (b)

Equation (c) shows that if F is a threshold function, every linear combination of weights obtained from $\{L\}$ must be greater than any of the linear combinations of weights obtained from $\{H\}$; i.e. if $\{L\}$ has p vertices, $\{H\}$ has q vertices, for each i and j we must have

$$\sum_i w > \sum_j' w$$

where $1 \leq i \leq p$

$1 \leq j \leq q$

Let us define the multiplicity of the weight w_k, M_r, as follows:

$$M_r = \sum_{k=1}^{r} C_k$$

where $C_k = t$ or 0 corresponding to whether w_k appears in the inequality t or zero times, and where r is

$1 \leq r \leq n$

For each $\sum_i w$, (where $1 \leq i \leq p$), because of the linearity, we have

$$\sum_i w = (w_1 - w_2)M_1 + (w_2 - w_3)M_2 +, \ldots, + (w_{n-1} - w_n)M_{n-1} + w_n M_n$$

$$= \sum_{r=1}^{n} (w_r - w_{r+1}) M_r$$

Similarly, we define

$$M'_r = \sum_{k=1}^{r} C_k$$

and we have

$$\sum_j' w = (w_1 - w_2)M'_1 + (w_2 - w_3)M'_2 +, \ldots, + (w_{n-1} - w_n)M'_{n-1} + w_n M'_n$$

$$= \sum_{r=1}^{n} (w_r - w_{r+1})M'_r$$

Since w_1 w_2 . . . ., w_n 0, the coefficients $(w_r - w_{r+1})$ in $\sum_i w$ and

$\sum_j{}' w$ will never be negative. Thus finding the realization $w = (w_1, \ldots, w_n)$ and T of a threshold function is equivalent to finding its M's such that

$$\left[\sum_{r=1}^{n}(w_r - w_{r+1})M_r\right]_i > \left[\sum_{r=1}^{n}(w_r - w_{r+1})M'_r\right]_j \quad \text{for all i and j.}$$

If we have found a set of M's satisfying the above condition, let each inequality equal to its M. By solving the equalities, we have the realization. The threshold value is then set equal to the minimum M_r. If we cannot find a set of M's satifying the above condition, the set of linear inequalities is not consistent, and therefore the function is not a threshold function.

Thus the method of finding M's can be summarized in a table form (Table 6) and is accomplished in the following steps:

Entry	Vertex	Inequality	M_1	M_2	. . .	M_r	T_1	T_2	. . .	T_s
Top Entry	{L}	I_1 I_2 . . . I_p								
Lower Entry	{H}	I'_1 I'_2 I'_3 . . . I'_q								

Table 6. Format of Trial Synthesis.

(Step 1) Obtain the reduced sets of vertices $\{L\}$ and $\{H\}$ by the method explained in the previous section and list these vertices in the the vertex column.

(Step 2) From these subsets of vertices $\{L\}$ and $\{H\}$ write down the inequalities putting all the inequalities obtained from $\{L\}$ in the top entry, and all the inequalities obtained from $\{H\}$ in the lower entry.

(Step 3) From the inequalities, write down the M's, separating M_k obtained from I_p from M'_k obtained from I'_q.

(Step 4) Sum all the columns M's and check if each of the top entries M_k is larger than any one of the lower entries M'_k.

(Step 5) The first trial T_1 is the sum of these M's obtained in step 4. A solution is obtained if each sum in the top entries is greater than any of the sums in the lower entry. If the above condition is not fulfilled we go to step 6.

(Step 6) Choose a column M_r to be added to T_1 in such a way as to make each sum in the top entry greater than any of the sums in the lower entry in the column T_2. We call T_2 the second trial. If this condition is fulfilled, then a solution is obtained on the second trial. If this condition is not fulfilled, then further trials must be made. The procedure is thus to make successive trials until each of the sums in the top entry is greater than any of the sums in the lower entry. The threshold T is then set equal to the minimum one of the top entry. If we cannot find a set of M's satifying the above condition, the inequalities are inconsistent and cannot be solved, and the function is not a threshold function.

To illustrate the method, consider the following example:

$$F(x_1,\ldots,x_5) = x_1 + x_2(x_3 + x_4 x_5)$$

$$\overline{F}(x_1,\ldots,x_5) = \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_1\bar{x}_3\bar{x}_5$$

The weights can be ordered as:

$$w_1 > w_2 > w_3 > w_4 = w_5$$

The prime implicants of F are x_1, $x_2 x_3$, and $x_2 x_4 x_5$.

Therefore $\{1\} = \{10000,\ 01100,\ 01011\}$

The prime implicants of $\overline{F}$ are $\bar{x}_1\bar{x}_2$, $\bar{x}_1\bar{x}_3\bar{x}_4$, and $\bar{x}_1\bar{x}_3\bar{x}_5$.

Therefore $\{h\} = \{00111,\ 01001,\ 01010\}$

Since $w_4 = w_5$

$$\{L\} = \{10000,\ 01100,\ 01011\}$$

$$\{H\} = \{00111,\ 01010\}$$

Entry	Vertex	Inequality	M_1	M_2	M_3	M_4	T_1	$T_2 = T_1 + M_1$
Top entry $\{L\}$	10000	$I_1 = w_1$	1	1	1	1	4	5
	01100	$I_2 = w_2 + w_3$	0	1	2	2	5	5
	01011	$I_3 = w_2 + 2w_4$	0	1	1	3	5	5
Lower entry $\{H\}$	00111	$I'_1 = w_3 + 2w_4$	0	0	1	3	4	4
	01010	$I'_2 = w_2 + w_4$	0	1	1	2	4	4

A solution to the set of inequalities can be obtained by solving the following linear equalities:

$$w_1 = 5$$
$$w_2 + w_3 = 5$$
$$w_2 + 2w_4 = 5$$
$$w_3 + 2w_4 = 4$$
$$w_2 + w_4 = 4$$

The solution is: $w_1 = 5$, $w_2 = 3$, $w_3 = 2$, $w_4 = w_5 = 1$ and $T = 5$.

The equation of the hyperplane is

$$5x_1 + 3x_2 + 2x_3 + x_4 + x_5 = 5$$

The method described in this chapter is believed to be the best method for the synthesis of threshold functions by hand computation. It always yields a set of positive integral realizations while the method of assignment of weights developed by L.C. Coates, R.B. Kirchner, P.M. Lewis (18) yields only one realization which can be fractional and is more complicated and time consumming.

CHAPTER 4 : DECOMPOSITION OF n-VARIABLE THRESHOLD FUNCTIONS INTO p-VARIABLE THRESHOLD FUNCTIONS, WHERE $p < n$.

4.1. Introduction

By using the properties and synthesis method of threshold functions stated in chapters 2 and 3 any threshold function of n variables can be decomposed into threshold functions of 3 variables. The method of decomposition will be discussed below. As a result of the decomposition, one can construct a network of 3-input threshold elements to realize the n-variable threshold function. Another result is a general formula for decomposing n-variable threshold functions into p-variable threshold functions. By using the fact that reduced functions of a threshold function must be comparable, a more useful decomposition formula for the realization of n-variable threshold functions with p-input threshold elements is obtained.

4.2. Decomposition of n-variable Threshold Functions into 3-variable Threshold Functions

Let F be a threshold function with weights ordering

$$w_1 \geq w_2 \geq , \ldots ., \geq w_n ,$$

corresponding to the input variables $x_1, x_2, \ldots, x_n$. According to the properties of threshold functions, F can be factored into the following form

$$F = x_1 F_1 + F_0 \quad \text{-----------------------(4.1)}$$

where F is expressed in the minimum sum of products form so that x_1 does not appear in F_1 and F_0.

F_1 and F_0 are threshold functions with (n-1) variables, namely

$x_2, x_3, \ldots, x_n$ with associated weights $w_2, w_3, \ldots, w_n$. Equation 4.1 itself is a 3-variable threshold function. Hence F can be considered to be realized by an element with 3 inputs x_1, F_1, and F_0 with weights 1, 1, and 2 respectively. The threshold value of the element is 2.

The n-input threshold element realizing F is shown in Fig.27,

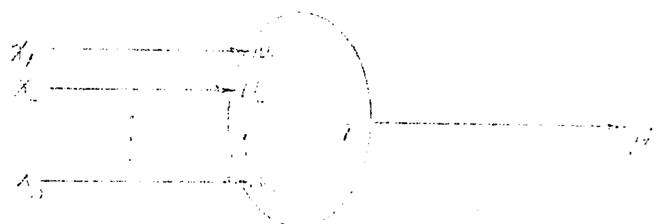

Fig.27. A threshold element with n inputs.

which is equivalent to the threshold element with 3 inputs as shown in Fig. 28.

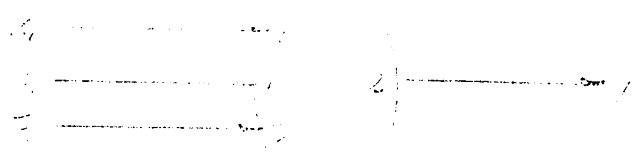

Fig.28. A threshold element with 3 inputs realizing F.

The functions F_1 and F_0 in equation 4.1 are also threshold functions with (n-1) variables $x_2, x_3, \ldots, x_n$ with associated weights $w_2, w_3, \ldots, w_n$. The element which can realize F_1 has the threshold value $T_1 = T - w_1$. This is because

$$w_1x_1 + w_2x_2 + , ..., w_nx_n \geq T$$

and F_1 is constrained by

$$w_2x_2 + w_3x_3 + , ..., w_nx_n \geq T - w_1$$

The element which can realize F_0 has the threshold value T. This is because F_0 is independent of x_1 ; i.e. F_0 is constrained by

$$w_2x_2 + w_3x_3 + , ..., w_nx_n > T$$

The threshold element shown in Fig.28 can be decomposed into a network containing two (n-1)-input elements and one 3-input element; the network is shown in Fig. 29.

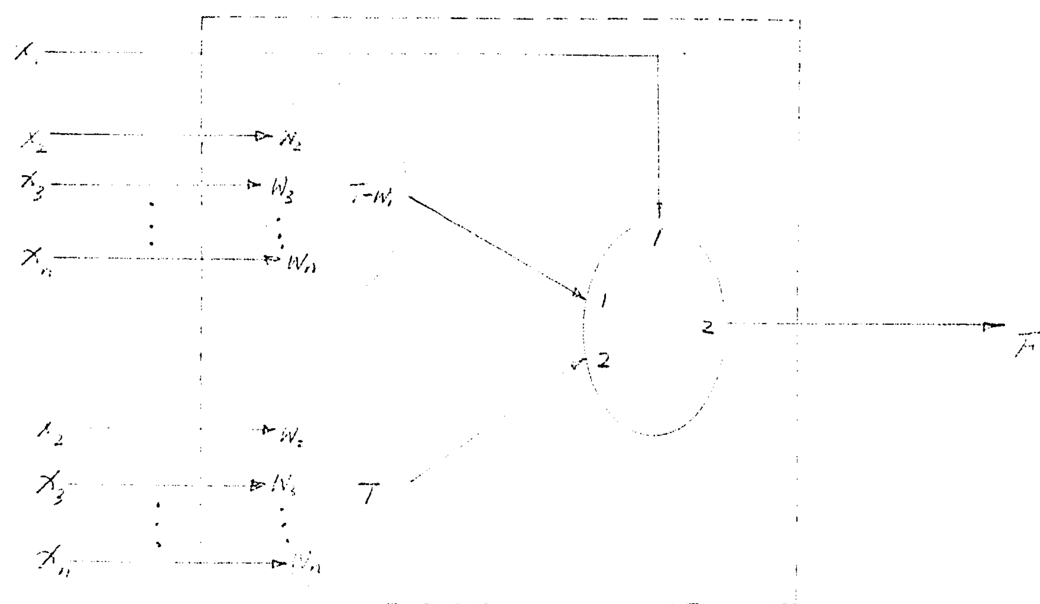

Fig.29. A network of elements realizing F.

4.3 Realization of n-variable Threshold functions With a Network of 3-input Threshold Elements

Since F_1 and F_0 in equation 4.1 are threshold functions, they can be factored again in the same fashion as F, namely

$$F_1 = x_2 F_{11} + F_{10} \qquad \text{--------------------(4.2)}$$

$$F_0 = x_2 F_{01} + F_{00} \qquad \text{--------------------(4.3)}$$

where F_{11}, F_{10}, F_{01}, and F_{00} are threshold functions with n-2 variables, x_3, x_4, . . . , x_n with associated weights w_3, w_4, . . ., w_n.

By following the same argument as in previous section, the threshold value for the element realizing F_{11} is $T - w_1 - w_2$, the threshold value for the element realizing F_{10} is $T - w_1$, the threshold value for the element realizing F_{01} is $T - w_2$, and the threshold value for the element realizing F_{00} is T.

The network of elements realizing F shown in Fig.29 can be now decomposed into a network containing four (n-2) - input elements and three 3-input elements. The network obtained is shown in Fig.30.

Fig.30. A network of elements realizing F.

This process can be repeated until the original function can be realized with a network containing only 3-input elements. The network containing 3-input elements realizing F is shown in Fig.31 .

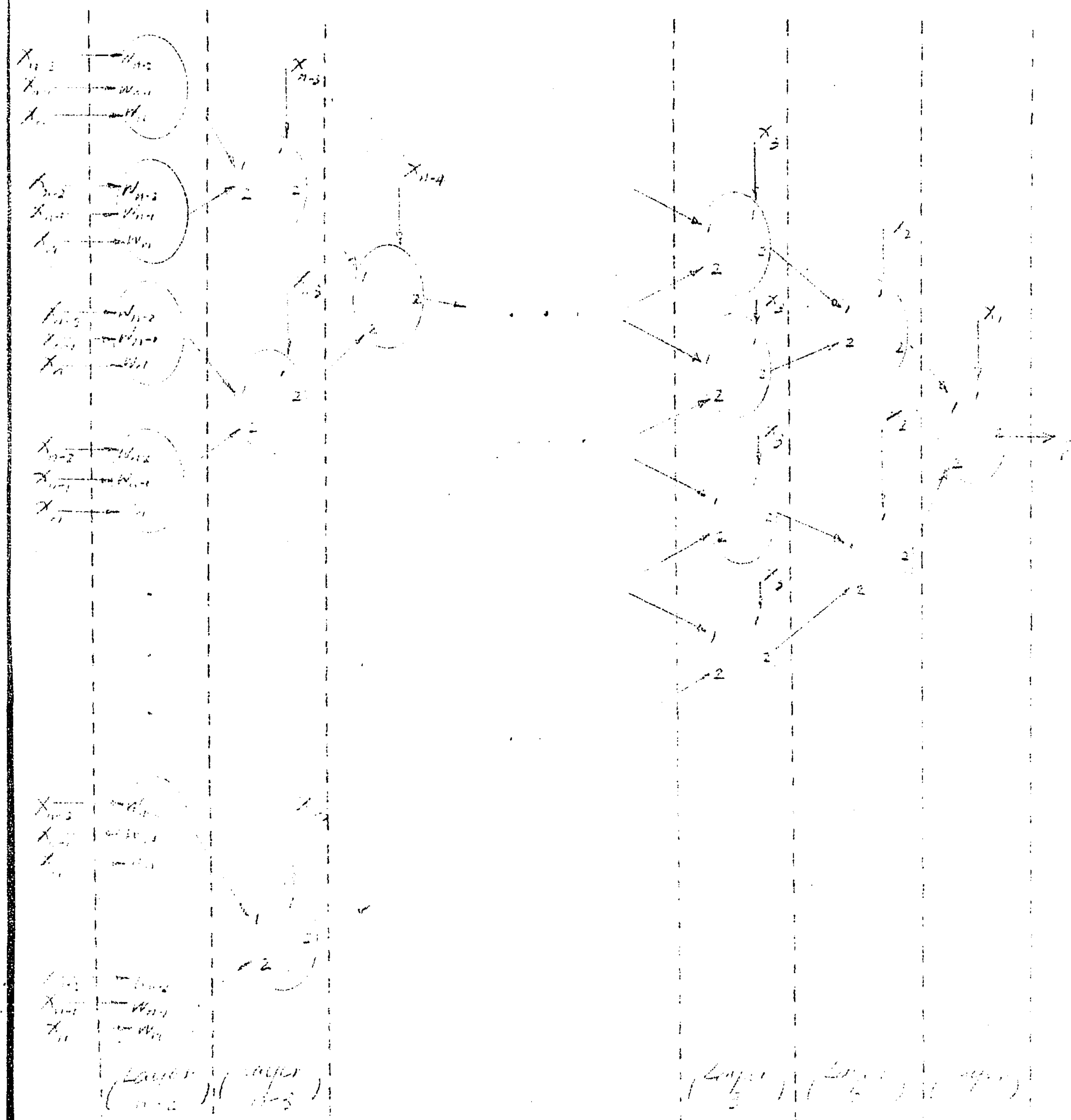

Fig.31. A network Of 3-input elements realizing F.

The network shown in Fig.31 contains 2^{n-3} elements at the $(n-2)^{th}$ layer. Each of these elements has 3 inputs x_{n-2}, x_{n-1}, and x_n associated with weights w_{n-2}, w_{n-1}, and w_n. The only difference between these elements is in the threshold values. Since there are only three variables and according to

$$x_n w_n + x_{n-1} w_{n-1} + x_{n-2} w_{n-2}$$

there are only $2^3 = 8$ different sums corresponding to the 8 possible combinations of the 3 input variables. Thus the $(n-2)^{th}$ layer in Fig.31 is realizing at most 8 different threshold functions of 3 variables. Therefore there are only 8 different threshold values are meaningful. Except for the$(n-2)^{th}$ layer, all the threshold elements in the other layers in the network have fixed weights of 1, 1, and 2, and fixed threshold values of 2.

Example:

$$F = x_1(x_2 + x_3 x_4 + x_3 x_5 x_6) + x_2 x_3 x_4 x_5 .$$

The threshold element realizing F is shown in Fig. 32.

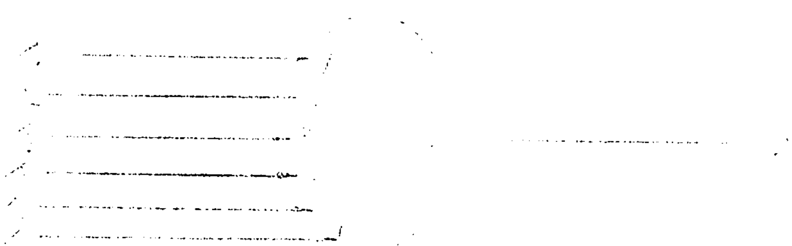

Fig. 32. A threshold element realizing F.

The threshold values of the eight threshold elements in the last layer are:

$$T_1 = T - w_1 - w_2 - w_3 = 12 - 7 - 5 - 3 = -3$$

$$T_2 = T - w_1 - w_2 = 12 - 7 - 5 = 0$$

$T_3 = T - w_1 - w_3 = 12 - 7 - 3 = 2$

$T_4 = T - w_1 = 12 - 7 = 5$

$T_5 = T - w_2 - w_3 = 12 - 5 - 3 = 4$

$T_6 = T - w_3 = 12 - 5 = 7$

$T_7 = T - w_3 = 12 - 3 = 9$

$T_8 = T = 12$

The network of 3-input elements realizing F is shown in Fig.33 on page 58.

4.4. Two General Formulas for Decomposition of n-variable Threshold Functions into p-variable Threshold Functions

4.1. Formula I. The method of decomposition given in the previous section is equivalent to presenting threshold functions in a form of tree called "threshold function tree". Let F be a threshold function which is expressed in the form of the minimum sum of products form namely $F = x_1 F_1 + F_0$ and with weights ordering $w_1 \geq w_2 \geq , \ldots , \geq w_n$. By the method stated in chapter 2, we decompose F until we reach the p_{th} variable where $1 \leq p \leq n$. Then the original F is deomposed into 2^p reduced functions and the final function tree looks like as shown in Fig. 34 on page 59.

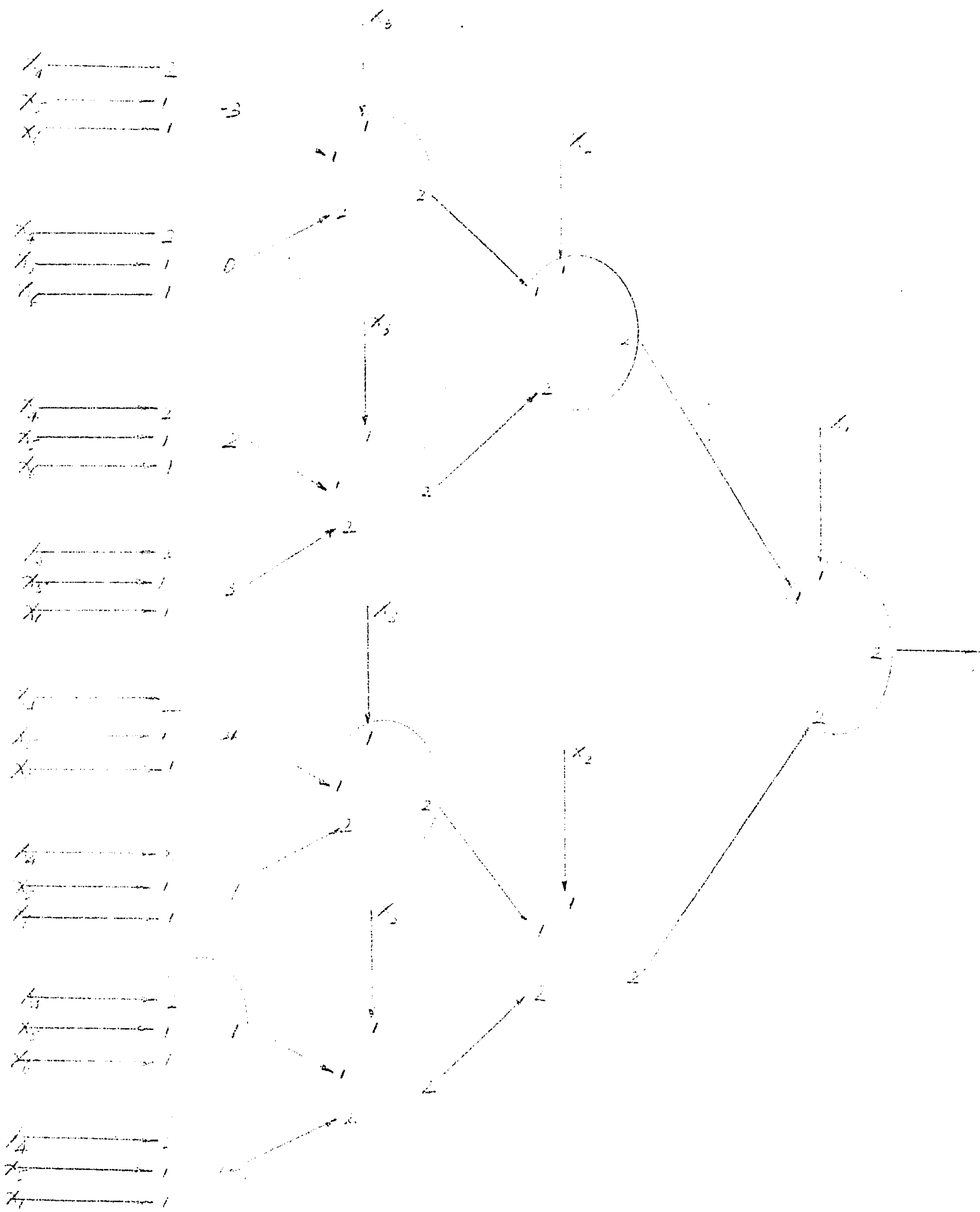

Fig. 33. A network of 3-input elements realizing F.

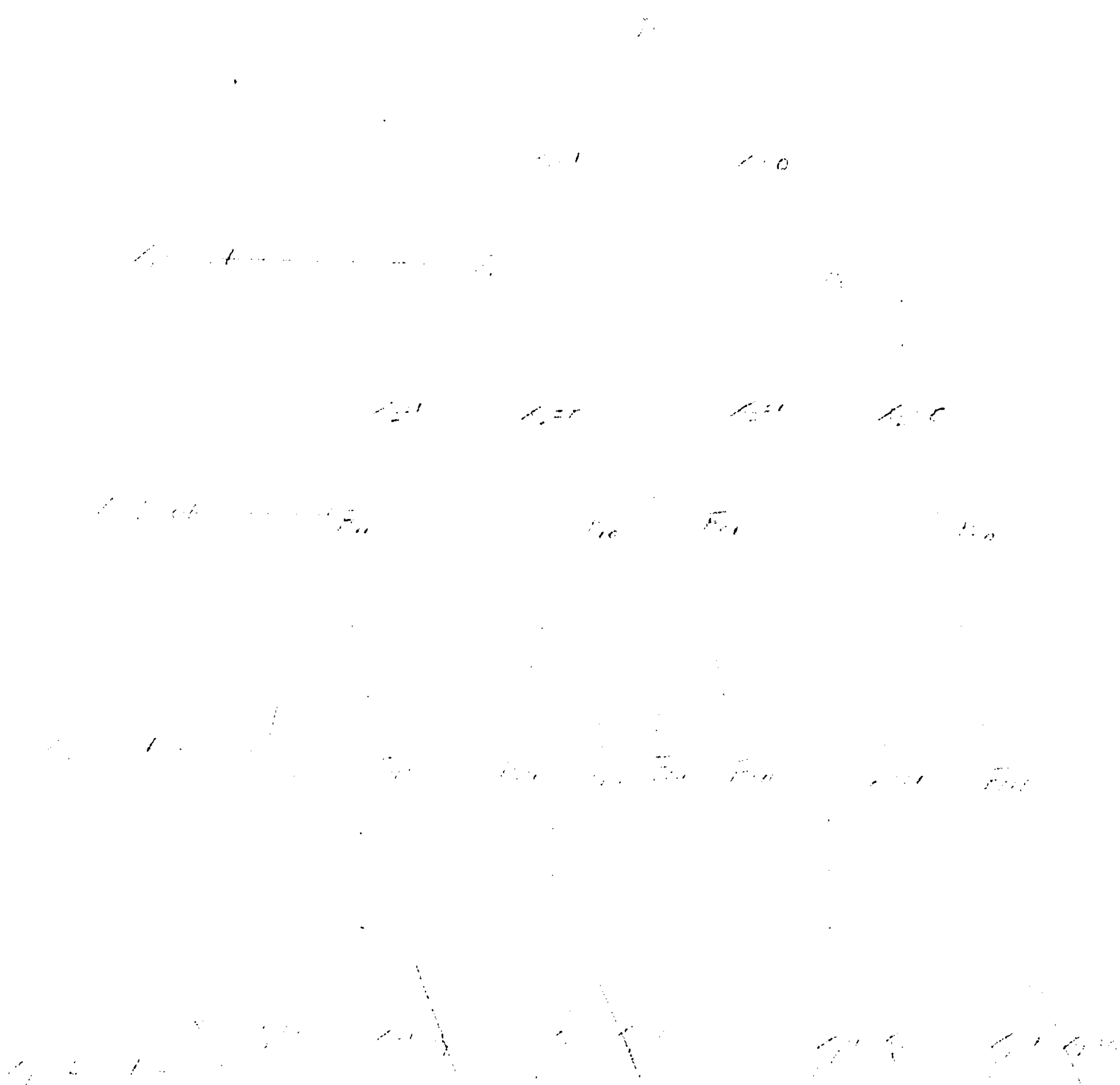

Fig. 34. The function tree of decomposed function F.

In Fig. 34 we call F_1 and F_0 the reduced functions at x_1-level, F_{11}, F_{10}, F_{01}, and F_{00} the reduced functions at x_2-level, etc.

We denote by $\prod_{(2^p - K)}$ the product of variables selected from variables $x_1, x_2, \ldots, x_p$ according to following rules:

x_k is in the product if and only if the binary number

$$(2^p - k) = a_1 2^{p-1} + a_2 2^{p-2} +, \ldots, a_{2^p-1} 2^1 + a_{2^p} 2^0$$

has $a_i = 1$ where $1 \le i \le 2^p$

$1 \le k \le 2^p$

for example, if $p = 3$

$k = 1; \prod_{(2^p-k)} = \prod_7 = \prod_{(111)_{binary}} = x_1 x_2 x_3$

$k = 5 \prod_{(2^p-k)} = \prod_3 = \prod_{(011)_{binary}} = x_2 x_3$

$k = 3 \prod_{(2^p-k)} = \prod_5 = \prod_{(101)_{binary}} = x_1 x_3$

and we define $\prod_0 = \prod_{(000)_{binary}} = 1$

Similarly we denote by $F_{(2^p - k)}$ the reduced functions of the x_p-level of the function tree (Fig. 34). The subscript $(2^p - k)$ is the decimal representation of binary value. For example, if $p = 3$

$F_7 = F_{(111)_{binary}}$

$F_3 = F_{(011)_{binary}}$

$F_5 = F_{(101)_{binary}}$

$F_0 = F_{(000)_{binary}}$

With the notations $\prod_{(2^p-k)}$ and $F_{(2^p-k)}$, F can be expressed in terms of the reduced functions at x_1 - level (Fig. 34) as follows:

$$F = \prod_{(2^1-1)} F_{(2^1-1)} + \prod_{(2^1-2)} F_{(2^1-2)}$$

$$= \prod_1 F_1 + \prod_0 F_0$$

$$= x_1 F_1 + F_0 \qquad \text{------------------(4.4)}$$

$$= \sum_{k=1}^{2^p} \prod_{(2^p-k)} F_{(2^p-k)} \qquad \text{where } p=1$$

F can be expressed in terms of the reduced functions at x_2 - level as follows:

$$F = \prod_{(2^2-1)} F_{(2^2-1)} + \prod_{(2^2-2)} F_{(2^2-2)} + \prod_{(2^2-3)} F_{(2^2-3)}$$

$$+ \prod_{(2^2-4)} F_{(2^2-4)}$$

$$= \prod_{11} F_{11} + \prod_{10} F_{10} + \prod_{01} F_{01} + \prod_{00} F_{00}$$

$$= x_1 x_2 F_{11} + x_1 F_{10} + x_2 F_{01} + F_{00} \qquad \text{----(4.5)}$$

$$= \sum_{k=1}^{2^p} \prod_{(2^p-k)} F_{(2^p-k)} \qquad \text{where } p=2$$

Equation 4.5 can also be obtained by substituting

$$F_1 = x_2 F_{11} + F_{10}$$

$$F_0 = x_2 F_{01} + F_{00}$$

into equation 4.4.

Similarly F can be expressed in terms of the reduced functions at x_3 - level; i.e.

$$F = \sum_{k=1}^{2^3} \prod_{(2^3-k)} F_{(2^3-k)}$$

$$= \prod_{111} F_{111} + \prod_{110} F_{110} + \prod_{101} F_{101} + \prod_{100} F_{100} + \prod_{011} F_{011}$$

$$+ \prod_{010} F_{010} + \prod_{001} F_{001} + \prod_{000} F_{000}$$

$$= x_1 x_2 x_3 F_{111} + x_1 x_2 F_{110} + x_1 x_3 F_{101} + x_1 F_{100} + x_2 x_3 F_{011}$$

$$+ x_2 F_{010} + x_3 F_{001} + F_{000} \quad \text{-----------} \ (4.6)$$

Equation 4.6 can also be obtained by substituting

$$F_{11} = x_3 F_{111} + F_{110}$$

$$F_{10} = x_3 F_{101} + F_{100}$$

$$F_{01} = x_3 F_{011} + F_{010}$$

$$F_{00} = x_3 F_{001} + F_{000}$$

into equation 4.5.

According to the above argument, the general formula for decomposing a threshold function of n variables into threshold functions of n-p variables is

$$F = \sum_{\substack{k=1 \\ \text{Boolean} \\ \text{sum}}}^{2^p} \prod_{(2^p-k)} F_{(2^p-k)} \qquad \text{(Formula I)}$$

The realization of the decomposition formula I with a network of threshold elements of different number of inputs can be shown by the following example :

$$F = \sum_{\substack{k=1 \\ \text{Boolean} \\ \text{sum}}}^{2^p} \prod_{(2^p-k)} F_{(2^p-k)}$$

For $p = 3$

$$F = x_1 x_2 x_3 F_{111} + x_1 x_2 F_{110} + x_1 x_3 F_{101} + x_1 F_{100} + x_2 x_3 F_{011}$$
$$+ x_2 F_{010} + x_3 F_{001} + F_{000}$$
$$= \left\{ x_1 \left[(x_2 x_3 F_{111} + x_2 F_{110} + F_{100}) + x_3 F_{101} \right] + F_{000} \right.$$
$$\left. + x_2 (x_3 F_{011} + F_{010}) \right\} + x_3 F_{001} \qquad \text{--------} \quad (4.7)$$

Let $A = x_2 x_3 F_{111} + x_2 F_{110} + F_{100}$ ------------------ (4.8)

$B = x_2 x_3 F_{011} + x_2 F_{010} + F_{000}$ ------------------ (4.9)

Then, F is equal to

$$F = \left\{ x_1 \left[A + x_3 F_{101} \right] + B \right\} + x_3 F_{001} \qquad \text{-----------} \quad (4.10)$$

Let $C = \left\{ x_1 x_3 F_{101} + x_1 A + B \right\}$ ---------------------- (4.11)

$$F = C + x_3 F_{001} \qquad \text{----------------------------} \quad (4.12)$$

Equation 4.12 can be realized by a 3-input element as shown in Fig. 35.

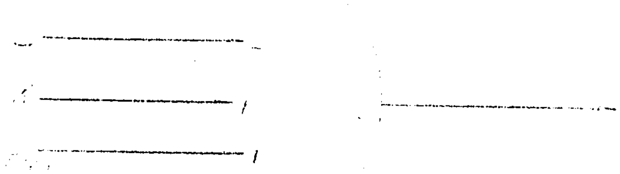

Fig. 35. A threshold element realizing equation 4.12.

Equation 4.11 can be realized by a 5-input element shown in Fig. 36 (Use the method of weight assignment in chapter 3).

Fig. 36. A threshold element realizing equation 4.11.

Equation 4.8 can be realized by a 5-input element shown in Fig.37.

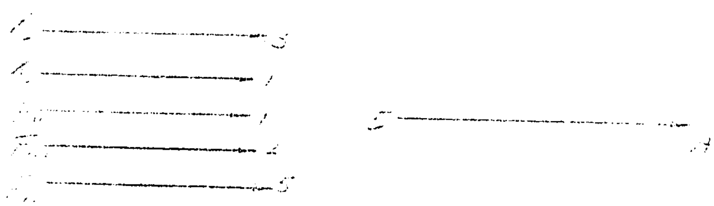

Fig. 37. A threshold element realizing equation 4.8.

Equation 4.9 can be realized by a 5-input element as shown in Fig. 38.

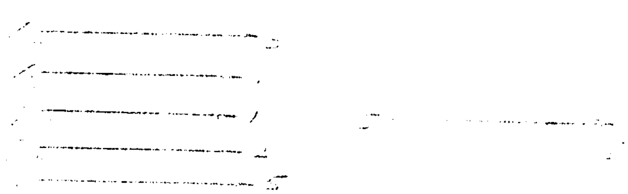

Fig. 38. A threshold element realizing equation 4.9.

By combining these single elements together, we have a network of threshold elements realizing F (Fig. 39.).

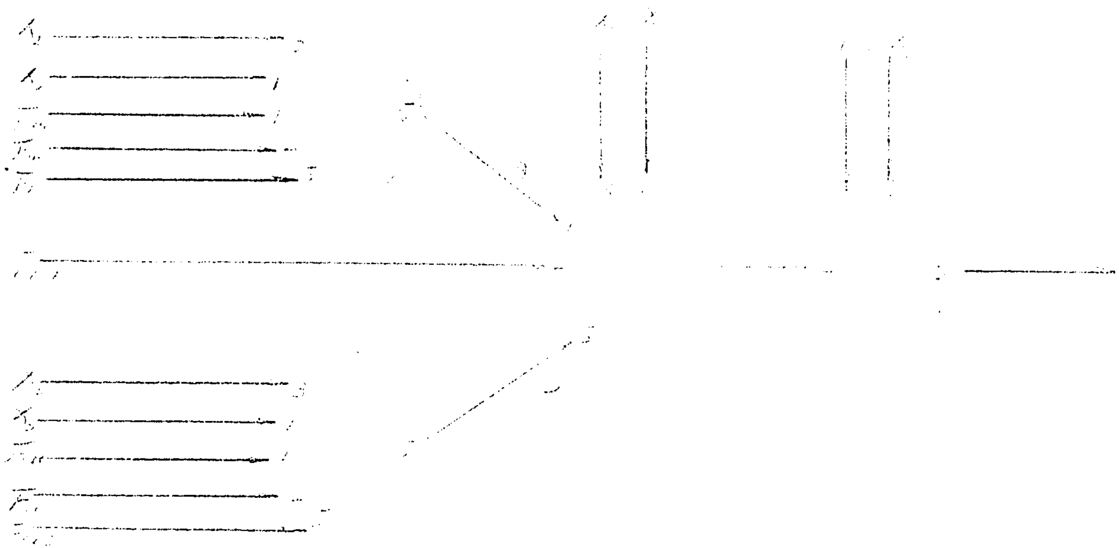

Fig. 39. Result of decomposition.

Formula II. According to the properties of threshold functions the reduced functions at each x_k - level of the function tree (Fig. 34) are comparable. For example, at the x_1 - level, $F_1 \supset F_0$, at the x_2 - level F_{11}, F_{10}, F_{01}, and F_{00} are comparable, etc.

From formula I, F can be written as a function of the binary variables $x_1, x_2, \ldots, x_p$ and reduced functions $F_1, \ldots, F_i, \ldots, F_{2^p}$; i.e.

$$F = F(x_1, x_2, \ldots, x_p, F_1, F_2, \ldots, F_i, \ldots, F_{2^p}).$$

Since the reduced functions $F_1, \ldots, F_i, \ldots, F_{2^p}$ are comparable, then F may be expressed as a symmetric function of these reduced functions. This can be obtained by the following inductive argument:

From Formula I, if p = 1

$$F = x_1 F_1 + F_0 .$$

Since $F_1 \supset F_0$ then

$$F_1 + F_0 = F_1$$

$$F_0 F_1 = F_0 \quad \text{and}$$

$$F = x_1 (F_1 + F_0) + F_1 F_0$$

$$= x_1 F_1 + x_1 F_0 + F_1 F_0$$

If p = 2, then

$$F = x_1 x_2 F_{11} + x_1 F_{10} + x_2 F_{01} + F_{00}$$

Since F_{11}, F_{10}, F_{01}, and F_{00} are comparable, they can be ordered. Let us assume that

$F_{11} \supset F_{10} \supset F_{01} \supset F_{00}$; then we have

$$F_{11} + F_{01} + F_{10} + F_{00} = F_{11}$$

$$F_{11} F_{01} + F_{10} F_{11} + F_{01} F_{00} + F_{10} F_{01} + F_{10} F_{00} + F_{11} F_{00} = F_{10}$$

$$F_{01} F_{10} F_{11} + F_{01} F_{10} F_{00} + F_{01} F_{11} F_{00} + F_{10} F_{11} F_{00} = F_{01}$$

$$F_{11} F_{01} F_{10} F_{00} = F_{00} ,$$

and F can be written as

$$F = x_1 x_2 (F_{11} + F_{01} + F_{10} + F_{00})$$

$$+ x_2 (F_{11} F_{01} + F_{10} F_{11} + F_{01} F_{00} + F_{10} F_{01} + F_{10} F_{00} + F_{11} F_{00})$$

$$+ x_1 (F_{01} F_{10} F_{11} + F_{01} F_{10} F_{00} + F_{01} F_{11} F_{00} + F_{10} F_{11} F_{00})$$

$$+ (F_{01} F_{10} F_{11} F_{00})$$

Assign G_j to each of the reduced functions $F_{(2^p - k)}$ where $1 \leq k \leq 2^p$ such that

$$(G_1 \supset G_2 \supset G_3 \supset, \ldots, \supset G_j, \ldots, \supset G_{2^p})$$

corresponding to

$$(F_{2^p-1} \supset F_j \supset F_k, \ldots, \supset F_f \supset, \ldots, \supset F_0)^{**}$$

where $0 \leq j, k, f, \leq 2^p-1$

For example, if p=2, and $F_{11} \supset F_{01} \supset F_{10} \supset F_{00}$ corresponding to $G_1 \supset G_2 \supset G_3 \supset G_4$, then

$$G_1 = F_{11}, G_2 = F_{01}, G_3 = F_{10}, \text{ and } G_4 = F_{00}.$$

Similarly, if p = 2, and $F_{11} \supset F_{10} \supset F_{01} \supset F_{00}$, then

$$G_1 = F_{11}, G_2 = F_{10}, G_3 = F_{01}, \text{ and } G_4 = F_{00}.$$

G_j also represent the sum of products each of which is formed by taking j reduced functions at a time among the 2^p reduced functions. For example, for p = 2, the reduced functions are F_{11}, F_{10}, F_{01}, and F_{00}, and $G_1 = F_{11}$, $G_2 = F_{10}$, $G_3 = F_{01}$, and $G_4 = F_{00}$; then

**Subscript f of F_f is the decimal representation of the binary value.

$$F_{11} \supset F_{10} \supset F_{01} \supset F_{00}$$

G_1 has $\frac{4!}{1!\,3!} = 4$ products each of which is formed by one reduced function; i.e. $G_1 = F_{11} + F_{10} + F_{01} + F_{00}$. G_2 has $\frac{4!}{2!\,2!} = 6$ products each of which is formed by two reduced function, namely $G_2 = F_{11}F_{01} + F_{10}F_{11} + F_{01}F_{00} + F_{10}F_{01} + F_{10}F_{00} + F_{11}F_{00}$. Similarly $G_3 = F_{01}F_{10}F_{11} + F_{01}F_{10}F_{00} + F_{01}F_{11}F_{00} + F_{10}F_{11}F_{00}$, and $G_4 = F_{01}F_{10}F_{11}F_{00}$

If p = 3, from Formula I

$$F = x_1x_2x_3F_{111} + x_1x_2F_{110} + x_1x_3F_{101} + x_1F_{100} + x_2x_3F_{011}$$
$$+ x_2F_{010} + x_3F_{001} + F_{000}.$$

If $G_1 = F_{111}$, $G_2 = F_{110}$, $G_3 = F_{101}$, $G_4 = F_{100}$, $G_5 = F_{011}$, $G_6 = F_{010}$, $G_7 = F_{001}$, $G_8 = F_{000}$, then

$$F_{111} \supset F_{110} \supset F_{101} \supset F_{100} \supset F_{011} \supset F_{010} \supset F_{001} \supset F_{000}$$

$$F = x_1x_2x_3(F_{111} + F_{110} + F_{101} + F_{100} + F_{011} + F_{010} + F_{001} + F_{000})$$

$+ x_1x_2$(sum of $\frac{8!}{2!\,6!} = 28$ prodcuts each of which formed by 2 out of 8 F_i's) $+ x_1x_3$(sum of $\frac{8!}{3!\,5!} = 56$ products each of which formed by 3 out of 8 F_i's) $+ x_1$(sum of $\frac{8!}{4!\,4!} = 70$ products each formed by 4 out of 8 F_i's)

$+ x_2x_3$(sum of $\frac{8!}{5!\,3!} = 56$ products each formed by 5 out of 8 F_i's) + x_2 (sum of $\frac{8!}{6!2!}$ = 28 products each formed by 6 out of 8 F_i's) + x_3 (sum of $\frac{8!}{7!1!}$ = 8 products each formed by 7 out of 8 F_i's) + $F_{111} F_{110} F_{101} F_{100} F_{011} F_{010} F_{001} F_{000}$,

where F_i's are F_{111}, F_{110}, F_{101}, F_{100}, F_{011}, F_{010}, F_{001}, F_{000}.

In general

$$F = \underset{\substack{\text{Boolean} \\ \text{sum}}}{\sum_{k=1}^{2^p}} \prod_{(2^p - k)} G_j$$

where G_j corresponds to one of the ordered reduced functions $F_{(2^p - k)}$ in Formula I , and $1 \leq j \leq 2^p$, and according to

$$G_1 \supset G_2 \supset G_3 \supset, \ldots, \supset G_j \supset, \ldots, \supset G_{2^p - 1} \supset G_{2^p}.$$

G_j is the symmetric function of the 2^p reduced functions; i.e. it is expressed in the form of the sum of products; each product is formed by j reduced functions out of any 2^p reduced functions; therefore G_j is the sum of $\frac{2^p!}{j!\,(2^p - j)!}$ such products.

The realization of the decomposition of Formula II with threshold element can be shown by the following example:

$$F = \underset{\substack{\text{Boolean} \\ \text{sum}}}{\sum_{k=1}^{2^p}} \prod_{(2^p - k)} G_j$$

For p = 2, the reduced functions are F_{11}, F_{10}, F_{01}, and F_{00}; if $F_{11} \supset F_{10} \supset F_{01} \supset F_{00}$, then $G_1 = F_{11}$, $G_2 = F_{10}$, $G_3 = F_{01}$, and $G_4 = F_{00}$, and

$$F = x_1 x_2 (F_{11} + F_{01} + F_{10} + F_{00}) + x_1 (F_{11} F_{01} + F_{01} F_{11} + F_{01} F_{00} +$$
$$F_{10} F_{01} + F_{10} F_{00} + F_{11} F_{00}) + x_2 (F_{01} F_{10} F_{11} + F_{01} F_{10} F_{00} +$$
$$F_{01} F_{11} F_{00} + F_{10} F_{11} F_{00}) + F_{01} F_{10} F_{11} F_{00} .$$

The weights assignment can be found by the method given in chapter 3. For completeness sake, the structure of the element also can be deduced in the following manner:

1. The function is symmetric in F_{11}, F_{10}, F_{01}, and F_{00}, their weights can be set equal to 1.
2. Since the product $F_{11} F_{10} F_{01} F_{00}$ must produce a one at the output, then $T \leq 4$.
3. Since any product constructed by any 3 out of F_{11}, F_{10}, F_{01}, and F_{00}, and x_2 must produce a one at the output, then $w_2 \geq 1$.
4. Since any product constructed by an 2 out of F_{11}, F_{10}, F_{01} and F_{00}, and x_1 must produce a one at the output, then $w_1 \geq 2$.

Values of $T = 4$, $w_1 = 2$, and $w_2 = 1$ are found to satisfy all the other conditions. Therefore the structure of the element is shown in Fig. 16.

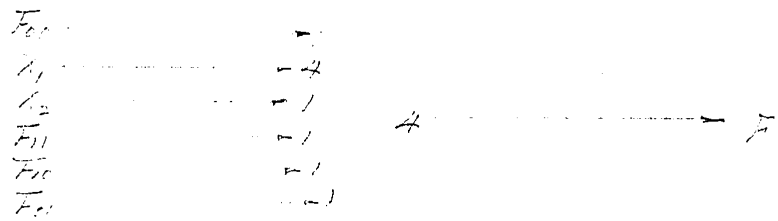

Fig. 40. A threshold element realizing F.

SUMMARY AND CONCLUSIONS

The characteristics and properties of threshold logic functions have been discussed in detail. A method to find the realizations of threshold functions has been presented. Methods for finding the realization of any given Boolean function are developing in the field of "threshold logic". An algorithm to yield a network of minimum number of threshold elements to realize a given Boolean function is still unknown.

The increase of reliability of a threshold element is accomplished by decomposing the element into a network of threshold elements. This network is guaranteed to realize the same number of threshold functions as the original element and each element in the network has less sensitivity to drift in its parameter values. The increase in both the reliability (the sensitivity to parameter change) and the versatility(the number of different possible threshold functions) vs complexity in the network of threshold elements can be found in reference (19).

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