

A thesis submitted

by

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ON THE INTERMEDIATE FLEXIBILITY  
OF SURFACES

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ABSTRACT

N. V. Efimov defines in his article "Qualitative Problems of the Theory of Deformation of Surfaces", which appears in "Differential Geometry and Calculus of Variations", the concept of intermediate flexibility. We shall deal with this subject in this paper.

Let  $x(u, v)$  be the radius vector of a variable point on an orientable surface  $X$  embedded in  $E^3$  with finite connectivity which is closed or bounded by a curve. If there exists a vector field  $y$  defined on  $X$  such that  $[y, dx]$  is locally exact and some but not all of the periods are zero, we say that  $X$  has intermediate flexibility. The periods are the line integral of  $\int_{C_i} [y, dx]$  where each  $C_i$  is a closed curve which belongs to a different basis element of the 1-dim. homology group.

In this thesis we deal primarily with a class  $\tau$  of  $C^2$  tori and require that  $y$  should be  $C^2$ . Our conclusion is a

necessary and sufficient condition for a surface of class  $\tau$  to have intermediate flexibility. Hence, this yields an example of a surface of class  $\tau$  with intermediate flexibility. This is the first known example of a surface with intermediate flexibility.

§ 1 The Definition of Infinitesimal Isometric  
Deformation and An Equivalent Condition To It

Our subject is infinitesimal isometric deformations of the first order of a certain class of surfaces which will be defined later. However, until we reach that definition we will consider a simply connected neighbourhood on a surface embedded in  $E^3$ .

If  $u$  and  $v$  are intrinsic local coordinates of a surface  $X$  we denote by  $x(u, v)$  the radius vector of a variable point on  $X$ .

A deformation is a family of surfaces

$$x^*(u, v, \epsilon) = x(u, v) + \epsilon \cdot z(u, v, \epsilon)$$

so that for each fixed  $\epsilon \geq 0$  we have a surface. We require that the following limit exists:

$$z(u, v) \equiv z(u, v, 0) = \lim_{\epsilon \rightarrow 0} z(u, v, \epsilon)$$

and that  $x$  and  $z$  have partial derivatives of the second order with respect to  $u$  and  $v$ . Also, as is usual in surface theory  $[x_u, x_v] \neq 0$  for any point of  $X^0$ . If we disregard infinitesimals whose order with respect to  $\epsilon$  is higher than the first we get

$$x^*(u, v, \epsilon) = x(u, v) + \epsilon \cdot z(u, v).$$

We are actually interested only in infinitesimal isometric deformations, i. e., we require that the length of each arc on the surface changes only by an infinitesimal of order which is higher than the first with respect to  $\epsilon$ .

Theorem: Let  $x(u, v)$  be the radius vector of a variable point on  $X$ . The infinitesimal deformation

$$x^*(u, v) = x(u, v) + \epsilon z(u, v)$$

is an infinitesimal isometric deformation if and only if

$$dx \cdot dz = 0.$$

• See [1]

Proof: If  $x_\epsilon$  is the surface under the deformation corresponding to  $\epsilon$  the square of the length of a line element on it is

$$ds^{\#2} = ds^2 + 2\epsilon \cdot dx \cdot dz + \epsilon^2 dz^2$$

Now in order that  $ds^{\#} - ds$  be of higher order than the first with respect to  $\epsilon$ , by Taylor's expansion in  $\epsilon$  we get  $dx \cdot dz = 0$ .

Hence for our infinitesimal isometric deformation we obtain the condition

$$dx \cdot dz = 0.$$

Now under the condition  $dx \cdot dz = 0$  we shall see that  $x^{\#}$  is an infinitesimal isometric deformation. In other words, we have to show that  $ds^{\#} - ds$  is of higher order than the first with respect to  $\epsilon$ .

We already had the equation

$$ds^{\#2} = ds^2 + 2\epsilon dx \cdot dz + \epsilon^2 dz^2$$

and it is given that  $dx \cdot dz = 0$ . Hence we conclude

$$ds^{\#2} = ds^2 + \epsilon^2 dz^2,$$

$$ds^* = (ds^2 + \epsilon^2 dz)^{\frac{1}{2}}$$

By Taylor's expansion in  $\epsilon$  around  $\epsilon = 0$  we obtain  $ds^* - ds =$   
{ higher than the first order with respect to  $\epsilon$  }.

We shall proceed with the following theorem which we will use in the sequel.

Theorem: Let  $x(u, v)$  and  $z(u, v)$  be the vector fields defined above.

The condition  $dx \cdot dz = 0$  implies the existence of a vector field  $y$  defined on  $X$  such that  $[y, dx]$  is exact. Conversely, if there exists a vector field  $y$  such that  $[y, dx]$  is locally exact, then locally there exists a vector field  $z$  on  $X$  such that  $dz = [y, dx]$  and  $dx \cdot dz = 0$ .

Proof: We note that

$$dx = x_u du + x_v dv,$$

$$dz = z_u du + z_v dv,$$

hence

$$dx \cdot dz = x_u z_u du^2 + (x_u z_v + x_v z_u) du dv + x_v z_v dv^2 = 0$$

Consequently  $dx \cdot dz = 0$  is equivalent to

$$* \quad \begin{cases} 1^* & x_u z_u = 0 \\ 2^* & x_u z_v + x_v z_u = 0 \\ 3^* & x_v z_v = 0 \end{cases}$$

We have the following

Theorem<sup>o</sup>:

If the field  $z$  satisfies the system  $*$  of equations then

there exists a field  $y$  such that we have the identities

$$** \quad \begin{cases} z_u = [y, x_u] \\ z_v = [y, x_v] \end{cases}$$

For a given field  $z$ , the field  $y$  is uniquely defined, where  $x$ ,  $y$

and  $z$  are vector fields of  $E^3$  defined only on  $X$ .

In other words from  $dx \cdot dz = 0$  we deduce the existence

of a vector field  $y$  such that

$$dz = [y, dx],$$

i. e. ,  $[y, dx]$  is an exact differential.

o See [3].

Proof of the theorem:

Because of 1<sup>#</sup> and 3<sup>#</sup> there exist two vector fields  $y_1$  and  $y_2$  such that

$$*** \begin{cases} z_u = [y_1, x_u] \\ z_v = [y_2, x_v] \end{cases}$$

Substituting these quantities into 2<sup>#</sup> we conclude:

$$x_u \cdot [y_2, x_v] + x_v \cdot [y_1, x_u] = 0,$$

$$-y_2 [x_u, x_v] + y_1 [x_u, x_v] = 0,$$

$$(y_1 - y_2) \cdot [x_u, x_v] = 0,$$

which leads to the existence of two real functions  $\vartheta$  and  $\psi$

such that

$$y_1 - y_2 = \vartheta x_u + \psi x_v.$$

$$\text{Denote } y \equiv y_1 - \vartheta x_u = y_2 + \psi x_v,$$

$$y_1 = y + \vartheta x_u,$$

$$y_2 = y - \psi x_v.$$

Substituting  $y_1$  and  $y_2$  into the two equations denoted by \*\*\* we

derive:

$$z_u = [y, xu] \quad z_v = [y, xv]$$

Since  $[x_u, x_u] = 0$  and  $[x_v, x_v] = 0$ .

For uniqueness suppose that  $y'$  and  $y''$  satisfy \*\*,

$$[(y' - y''), x_u] = 0$$

and

$$[(y' - y''), x_v] = 0,$$

which is only possible if  $y' - y'' = 0$ , for  $[x_u, x_v] \neq 0$ .

To show the second part of the theorem we have to show the following.

Suppose that  $y$  is a smooth vector field ( $\epsilon C^1$ ) defined on a

surface  $X$ . Suppose that  $[y, dx]$  is a locally exact differential.

Because of  $[x, dy] = d[x, y] - [dx, y]$  also  $[x, dy]$  is a locally

exact differential, i. e., locally there exists  $t$  such that:

$$dt = [x, dy].$$

Define:  $z = t + [y, x]$  and

$$x^\# = x + \epsilon \cdot z.$$

Then  $dz = dt + [dy, x] + [y, dx] =$

$$= [x, dy] + [dy, x] + [y, dx] = [y, dx].$$

Using the last equation we get:

$$dx \cdot dz = dx \cdot [y, dx] = 0$$

or  $dx \cdot dz = 0.$

Corollary: Let  $x$  be as defined above. There exists a vector field  $z$  defined locally on  $X$ , such that  $dx \cdot dz = 0$ , if and only if there exists a vector field  $y$  defined locally on  $X$  such that  $[y, dx]$  is exact. Then we have  $dz = [y, dx]$ .

§ 2 The Kinematic Meaning  
of y and t

Let the value of the parameter  $\epsilon$  vary with the time,  
and let  $\delta$  denote the differential with respect to time. Also  
rewrite

$$x^{\#} = x + \epsilon z$$

in the form

$$x^{\#} = x + \delta \epsilon \cdot z .$$

It immediately follows that

$$dx^{\#} = dx + \delta \epsilon \cdot dz .$$

In our notation  $d$  denotes the differential with respect to  
 $u$  and  $v$  and does not depend on time.

We also changed the notation from  $\epsilon$  to  $\delta \epsilon$ , in order to  
get symmetry with the ordinary differential notation in the quantities  
that we will deduce later in our computations.

Also denoting

$$\delta x = x^{\#} - x; \quad \delta dx = dx^{\#} - dx \quad \text{and}$$

we conclude

$$\frac{\delta x}{\delta \epsilon} = z; \quad \frac{\delta dx}{\delta \epsilon} = dz.$$

Thus,  $z$  is the field of velocities of the points of the surface under our deformation, and  $dz$  is the velocity of the change of the line element  $dx$ .

If we substitute  $dz$  we derive:

$$\frac{\delta dx}{\delta \epsilon} = [y, dx].$$

Because  $dx$  does not change in length, and the change of angle between any two line elements at a point of the surface is negligible in our theory, we can look at the deformation locally as a motion of pencil of line elements. Then  $y$  is the instantaneous rotational vector of the motion. Let  $x$  be the radius vector of the point  $M \in X$ , and  $x + dx$  the radius vector of the point  $M' \in X$ .

The velocity of  $M'$  is given by

$$\frac{\delta}{\delta \epsilon} (x + dx) = \frac{\delta x}{\delta \epsilon} + \frac{\delta dx}{\delta \epsilon} = z + [y, dx],$$

and by the definition of  $z$  we get

$$\frac{\delta}{\delta \epsilon} (x + dx) = t + [y, (x + dx)],$$

$$t = \frac{\delta}{\delta \epsilon} (x + dx) - [y, (x + dx)].$$

That is to say,  $t$  is the translational component of the local motion.

The pair of fields  $(y, t)$  is called the screw field of the infinitesimal isometric deformation.

If  $y$  is constant from

$$t = z - [y, x],$$

we obtain

$$dt = dz - [y, dx] = [y, dx] - [y, dx] = 0,$$

i. e.,  $t$  is constant.

That is to say that the deformation is an infinitesimal motion of the surface and we call it the trivial deformation.

§ 3 The  $\tau$  - Class of Surfaces

We are going to define here the  $\tau$  - class of surfaces with which we will deal from now on.

Each surface in that class is generated by revolving a simple closed smooth curve about a fixed axis where the curve does not intersect with the axis.

And more precisely:

Definition: A surface  $X$  embedded in  $E^3 = (x_1, x_2, x_3)$  - space belongs to the class  $\tau$  if it has a parametric representation of the following form

$$x(u, v) = \begin{cases} x_1 = r(v) \cos u \\ x_2 = r(v) \sin u \\ x_3 = h(v) \end{cases}$$

$$0 \leq u \leq 2\pi \quad 0 \leq v \leq L,$$

where  $r(v)$  and  $h(v)$  are 2 functions of class  $C^2$  defining

a simple closed meridian curve  $M$  which does not intersect

the  $x_3$  - axis and for which  $v$  plays the part of the arc length.

Hence  $r$  and  $h$  are periodic of period  $L$  ( $L$  = perimeter of  $M$ ),

$$r(v) > 0 \text{ for all } v, \text{ and } r'^2 + h'^2 = 1 \quad 0 \leq v \leq L$$

$$\left\{ \text{where } r' = \frac{dr}{dv} \quad h' = \frac{dh}{dv} \right\}.$$

§ 4 Rotation Fields for

Class  $\tau$

We shall find now all smooth vector-fields  $y$  in  $E^3$  defined on a surface  $X$  from the class  $\tau$ , such that  $[y, dx]$  is a locally exact differential.

Note that in the case where  $X$  was simply connected the local exactness of  $[y, dx]$  implies global exactness. But in our case for surfaces from class  $\tau$ , the same thing is not implied.

Let us denote the rotational field  $y(u, v)$  and its three components  $y_1, y_2, y_3$ , or

$$y(u, v) = \begin{pmatrix} y_1(u, v) \\ y_2(u, v) \\ y_3(u, v) \end{pmatrix}.$$

We shall have that the  $y_i$  are  $L$  periodic in  $v$  and  $2\pi$  periodic in  $u$  and that  $y_i \in C^2$  for  $i = 1, 2, 3$ .

It is convenient to introduce the following three functions which have the same properties mentioned above.

$$U(u, v) = y_1 \sin u - y_2 \cos u,$$

$$V(u, v) = y_1 \cos u + y_2 \sin u,$$

$$Z(u, v) = y_3. \quad (3)$$

From the above three equations we deduce:

$$y_1 = U \sin u + V \cos u, \quad (1)$$

$$y_2 = -U \cos u + V \sin u. \quad (2)$$

We also know that  $[y, dx]$  is locally exact if and only if we have the following condition.

$$* \quad [y_u, x_v] = [y_v, x_u].$$

For: from the exactness there exists in a neighbourhood of each point a field  $z$  such that:

$$dz = [y, dx],$$

$$z_u du + z_v dv = [y, x_u du] + [y, x_v dv],$$

$$z_u = [y, x_u]$$

$$z_v = [y, x_v].$$

But 
$$z_{uv} = z_{vu}$$

$$[y_v, x_u] + [y, x_{uv}] = [y_u, x_v] + [y, x_{vu}],$$

because of  $x_{uv} = x_{vu}$  ( $x \in C^2$ ),

$$[y_v, x_u] = [y_u, x_v].$$

Now suppose that  $[y_u, x_v] = [y_v, x_u]$ .

We have: 
$$[y, dx] = [y, x_u du] + [y, x_v dv],$$

and by a theorem from advanced calculus we obtain that

$$[y, dx] \text{ is locally exact if and only if } [y, x_u]_v = [y, x_v]_u.$$

I shall prove now the last equation:

$$[y, x_u]_v = [y_v, x_u] + [y, x_{uv}],$$

$$[y, x_v]_u = [y_u, x_v] + [y, x_{vu}].$$

By our assumption  $[y_u, x_v] = [y_v, x_u]$  and by smoothness

of  $X$  we have  $x_{uv} = x_{vu}$ . And immediately we conclude that

$$[y, x_u]_v = [y, x_v]_u \text{ and by the theorem it follows that}$$

$[y, dx]$  is locally exact.

Let us write the condition denoted by \* in components

after substituting x. Then we will conclude:

$$(1) \quad \frac{\partial y_2}{\partial u} h' - \frac{\partial y_3}{\partial u} r' \sin u = - \frac{\partial y_3}{\partial v} r \cos u$$

$$(2) \quad \frac{\partial y_1}{\partial u} h' - \frac{\partial y_3}{\partial u} r' \cos u = \frac{\partial y_3}{\partial v} r \sin u$$

$$(3) \quad \frac{\partial y_1}{\partial u} r' \sin u - \frac{\partial y_2}{\partial u} r' \cos u = \frac{\partial y_1}{\partial v} r \cos u \\ + \frac{\partial y_2}{\partial v} r \sin u.$$

Now we shall substitute the values of  $y_i$  and derive a set of

differential equations:

$$(1) \quad - h' \frac{\partial U}{\partial u} \cos u + h' U \sin u + h' \frac{\partial V}{\partial u} \sin u + \\ + h' V \cos u - \frac{\partial Z}{\partial u} r' \sin u = - \frac{\partial Z}{\partial v} r \cos u$$

$$(2) \quad h' \frac{\partial U}{\partial u} \sin u + h' U \cos u + h' \frac{\partial V}{\partial u} \cos u - h' V \sin u \\ - \frac{\partial Z}{\partial u} r' \cos u = \frac{\partial Z}{\partial v} r \sin u$$

$$\begin{aligned}
 (3) \quad & r' \sin u \frac{\partial U}{\partial u} \sin u + r' \sin u U \cos u + \\
 & r' \sin u \frac{\partial V}{\partial u} \cos u - r' \sin u V \sin u + \\
 & + r' \cos u \frac{\partial U}{\partial u} \cos u - r' \cos u U \sin u \\
 & - r' \cos u \frac{\partial V}{\partial u} \sin u - r' \cos u V \cos u = \\
 & = r \cos u \frac{\partial U}{\partial v} \sin u + r \cos u \frac{\partial V}{\partial v} \cos u \\
 & - r \sin u \frac{\partial U}{\partial v} \cos u + r \sin u \frac{\partial V}{\partial v} \sin u.
 \end{aligned}$$

Rearranging the terms we get:

$$\begin{aligned}
 (1) \quad & \sin u \left( h' U + h' \frac{\partial V}{\partial u} - \frac{\partial Z}{\partial u} r' \right) + \\
 & + \cos u \left( -h' \frac{\partial U}{\partial u} + h' V + r \frac{\partial Z}{\partial v} \right) = 0
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & - \sin u \left( -h' \frac{\partial U}{\partial u} + h' V + \frac{\partial Z}{\partial v} r \right) + \\
 & + \cos u \left( h' U + h' \frac{\partial V}{\partial u} - \frac{\partial Z}{\partial u} r' \right) = 0
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \sin^2 u \left( r' \frac{\partial U}{\partial u} - r' V - r \frac{\partial V}{\partial v} \right) + \\
 & + \sin u \cos u \cdot (0) +
 \end{aligned}$$

$$+ \cos^2 u \left( r' \frac{\partial U}{\partial u} - r' V - r \frac{\partial V}{\partial v} \right) = 0.$$

Multiply (1) by  $\cos u$  we obtain:

$$\begin{aligned} & \cos u \sin u \left( h' U + h' \frac{\partial V}{\partial u} - \frac{\partial Z}{\partial u} r' \right) + \\ & + \cos^2 u \left( -h' \frac{\partial U}{\partial u} + h' V + r \frac{\partial Z}{\partial v} \right) = 0. \end{aligned}$$

Multiply (2) by  $\sin u$  we deduce:

$$\begin{aligned} & \sin u \cos u \left( h' U + h' \frac{\partial V}{\partial u} - \frac{\partial Z}{\partial u} r' \right) = \\ & = \sin^2 u \left( -h' \frac{\partial U}{\partial u} + h' V + \frac{\partial Z}{\partial v} r \right). \end{aligned}$$

From the last two equations we conclude:

$$\begin{aligned} & \sin^2 u \left( -h' \frac{\partial U}{\partial u} + h' V + \frac{\partial Z}{\partial v} r \right) + \\ & + \cos^2 u \left( -h' \frac{\partial U}{\partial u} + h' V + \frac{\partial Z}{\partial v} r \right) = 0. \end{aligned}$$

It immediately follows that

$$\begin{aligned} & -h' \frac{\partial U}{\partial u} + h' V + r \frac{\partial Z}{\partial v} = 0 \\ & \text{or } h' \left( \frac{\partial U}{\partial u} - V \right) = r \frac{\partial Z}{\partial v}. \end{aligned} \tag{4}$$

Similarly from (1) and (2) we derive:

$$h' \left( U + \frac{\partial V}{\partial u} \right) = r' \frac{\partial Z}{\partial u}. \quad (5)$$

Immediately from (3) we get:

$$r' \frac{\partial U}{\partial u} - r' V - r \frac{\partial V}{\partial v} = 0. \quad (6)$$

It is quite obvious that (1), (2) and (3) follow from (4), (5) and (6).

We can derive from (4) and (6) two useful equations in the following way.

Multiply (4) by  $h'$

$$h'^2 \frac{\partial U}{\partial u} = h'^2 V + h' r \frac{\partial Z}{\partial v}.$$

Multiply (6) by  $r'$

$$r'^2 \frac{\partial U}{\partial u} = r'^2 V + r r' \frac{\partial V}{\partial v}.$$

Add together the last two equations and use  $h'^2 + r'^2 = 1$ .

We obtain:

$$\frac{\partial U}{\partial u} = V + r \left( h' \frac{\partial Z}{\partial v} + r' \frac{\partial V}{\partial v} \right). \quad (7)$$

Multiply (4) by  $r'$ . We deduce:

$$r r' \frac{\partial Z}{\partial v} = h' r' \frac{\partial U}{\partial u} - h' r' V.$$

Multiply (6) by  $h'$ . We derive:

$$r h' \frac{\partial V}{\partial v} = h' r' \frac{\partial U}{\partial u} - h' r' V.$$

From the last two equations we conclude:

$$h' \frac{\partial V}{\partial v} = r' \frac{\partial Z}{\partial v}. \quad (8)$$

At this stage I would like to show that (4) and (6) follow

from (5), (7) and (8).

Multiply (7) by  $h'$ . We get:

$$h' \left( \frac{\partial U}{\partial u} - V \right) = r h'^2 \frac{\partial Z}{\partial v} + r r' h' \frac{\partial V}{\partial v}.$$

Using (8) we obtain:

$$h' \left( \frac{\partial U}{\partial u} - V \right) = r h'^2 \frac{\partial Z}{\partial v} + r r'^2 \frac{\partial Z}{\partial v}$$

and (4) follows immediately. Multiply (7) by  $r'$ . We deduce:

$$\begin{aligned} r' \frac{\partial U}{\partial u} &= r' V + r h'^2 \frac{\partial V}{\partial v} + r r'^2 \frac{\partial V}{\partial v} = \\ &= r' V + r \frac{\partial V}{\partial v} \end{aligned}$$

and we get (6).

This proves that (5), (7) and (8) are equivalent to (4), (5), (6).

So, we have proved that the equations (5) (7) and (8) are equivalent to the condition that  $y \times dx$  is locally exact. (Provided that  $x$  and  $y \in C^2$ ). Now we want to find the most general solution for  $U$ ,  $V$  and  $Z$  such that they satisfy the equations (5), (7) and (8), where  $U$ ,  $V$  and  $Z \in C^2$  and are  $L$  periodic in  $v$  and  $2\pi$  periodic in  $u$ .

By our conditions we had  $V$  and  $Z \in C^2$  and are  $2\pi$  periodic in  $u$ .

According to the following theorem we can express  $V$  and  $Z$  by their Fourier Series with respect to  $u$ .

[See [ 2 ]. I simplified the theorem to the case

where  $f$  is smooth and not only sectionally smooth. ]

Theorem: If a function  $f(x)$  is smooth ( $f \in C^1$ ) and periodic,

then its Fourier series converges absolutely and uniformly.

Hence we can write

$$(9) \quad V(u, v) = \frac{1}{2} a_0(v) + \sum_{k=1}^{\infty} [a_k(v) \cos(ku) + b_k(v) \sin(ku)],$$

$$(10) \quad Z(u, v) = \frac{1}{2} c_0(v) + \sum_{k=1}^{\infty} [c_k(v) \cos(ku) + d_k(v) \sin(ku)].$$

where:

$$a_0(v) = \frac{1}{\pi} \int_0^{2\pi} V(u, v) du,$$

$$a_k(v) = \frac{1}{\pi} \int_0^{2\pi} V(u, v) \cos ku du,$$

$$b_k(v) = \frac{1}{\pi} \int_0^{2\pi} V(u, v) \sin ku du,$$

$$c_0(v) = \frac{1}{\pi} \int_0^{2\pi} Z(u, v) du,$$

$$c_k(v) = \frac{1}{\pi} \int_0^{2\pi} Z(u, v) \cos ku du,$$

$$d_k(v) = \frac{1}{\pi} \int_0^{2\pi} Z(u, v) \sin(ku) du.$$

The coefficients  $a_j, b_j, c_j, d_j$  are  $C^2$  functions of  $v$  with period  $L$ , for  $V(u, v)$  and  $Z(u, v)$  are  $v$  periodic with period  $L$ .

Substituting (9) and (10) into (7) and integrating with respect to  $u$  we obtain:

$$(11) \quad U(u, v) = \tilde{U}_0 + \sum_{k=1}^{\infty} \left\{ \frac{1}{k} (a_k + r r' a'_k + r h' c'_k) \sin ku - \frac{1}{k} (b_k + r r' b'_k + r h' d'_k) \cos ku \right\},$$

where  $\tilde{U}_0 = U_0(v) + \frac{1}{2} (a_0 + r h' c'_0 + r r' a'_0) u$ .

In order to justify the operations made above we have to show that we are allowed to derivate  $V$  and  $Z$  term by term in  $v$  in their Fourier expansions and that we are allowed to integrate  $V, \frac{\partial Z}{\partial v}$  and  $\frac{\partial V}{\partial v}$  term by term in  $u$ .

We are given that  $V$  satisfies the conditions of the last

theorem. As well  $V' = \frac{\partial V}{\partial v}$  satisfies them.

So we have for  $V'$  the following expansion:

$$V' = \frac{1}{2} \tilde{a}_0 + \sum_{k=1}^{\infty} \{ \tilde{a}_k \cos ku + \tilde{b}_k \sin ku \}$$

where:

$$\tilde{a}_0(v) = \frac{1}{\pi} \int_0^{2\pi} V' du$$

$$\tilde{a}_u(v) = \frac{1}{\pi} \int_0^{2\pi} V' \cos u \sin du$$

$$\tilde{b}_u(v) = \frac{1}{\pi} \int_0^{2\pi} V' \sin u \sin du.$$

Now according to the rule of Leibnitz we can change the order of differentiating and integrating, for  $V$  is continuous in  $u$  and  $v$  and  $\frac{\partial V}{\partial v}$  exists and is continuous,

$$\tilde{a}'_i = a'_i$$

$$\tilde{b}'_i = b'_i.$$

And we actually deduce that we can derivate term by term.

Similarly we could show the same for  $Z$ .

Previously we also had the Fourier series of  $\frac{\partial V}{\partial v}$  and

$V$  converge uniformly in  $u$ . Hence by a well known theorem

we can integrate each of them term by term in  $u$ .

Since  $U(u, v)$  is periodic in  $u$  we must have  $\tilde{U}_0$  periodic in  $u$ , and this is so if and only if

$$(12) \quad a_0 + r h' c'_0 + r r' a'_0 = 0 ,$$

$$\tilde{U}_0 = U_0(v) .$$

Since  $U$  is periodic in  $v$ ,  $U_0(v)$  must also be periodic in  $v$ .

Using equation (5) we conclude:

$$(13) \quad h' U_0(v) = 0$$

$$(14) \quad h' (a_k + r r' a'_k + r h' c'_k) = k^2 (h' a_k - r' c_k)$$

$$(15) \quad h' (b_k + r r' b'_k + r h' d'_k) = k^2 (h' b_k - r' d_k)$$

$$k = 1, 2, \dots$$

And from equation (8) we derive:

$$(16) \quad h' a'_0 = r' c'_0$$

$$(17) \quad h' a'_k = r' c'_k$$

$$(18) \quad h' b'_k = r' d'_k$$

$$k = 1, 2, \dots$$

Hence the two pairs of functions

$$\left( a_k(v), c_k(v) \right) \quad \text{and} \quad \left( b_k(v), d_k(v) \right)$$

satisfy for  $k = 1, 2, \dots$  the same system of first order linear homogeneous differential equations.

Now we would like to prove that  $a_0 = 0$  and  $c_0 =$  constant. From  $r' a_0 + r a_0' = 0$  which is obtained from (12) and (16) we have:

$$\frac{\partial}{\partial v} (r a_0) = 0 \quad \text{or} \quad r a_0 = \text{const.} = R,$$

$$a_0 = \frac{R}{r} \quad (r \neq 0 \text{ always}).$$

From  $h' a_0 + r c_0' = 0$  which is obtained from (12) and (16)

$$\text{we have } c_0' = -\frac{h' a_0}{r} = -\frac{h'R}{r^2},$$

$$c_0(x) = A - \int_0^x \frac{h'}{r^2} R \, dv,$$

$$c_0(0) = A \quad \text{and} \quad c_0(L) = A,$$

for:  $c_0(v)$  is periodic with period  $L$ .

It follows that

$$\int_0^L \frac{h'}{r^2} R \, dv = 0.$$

And by Stokes' theorem:

$$\int_{c_r} \frac{R}{2} dh = \iint \frac{\partial \frac{R}{2}}{\partial r} ds = -2R \iint \frac{1}{r} ds = 0$$

By the Jordan-Brouwer separation theorem the meridian separates its plane into two open non-empty connected components. And hence, the area enclosed within the meridian is different from zero. Also  $\frac{1}{r}$  is always larger than zero. We derive that the minimum of  $\frac{1}{r}$  is positive on the surface which is bounded by the meridian and which is compact. Denote that minimum by  $m_0$ .

Therefore we have:

$$\iint \frac{1}{r} ds \geq m_0 \cdot \iint ds > 0,$$

$$R = 0,$$

$$a_0 = 0 \text{ and } c_0 = \text{constant.}$$

We also will prove that  $c_1 \equiv 0$ .

We have: (a)  $h' a_1' = r' c_1'$

(b)  $h' r' r a_1' + h'^2 r c_1' = -r' c_1'$

By the last two equations we get

$$r r'^2 c_1' + h'^2 r c_1' = - r' c_1 ,$$

$$(r c_1)' = 0 ,$$

$$r c_1 = Q \text{ (constant) ,}$$

$$c_1 = \frac{Q}{r} .$$

"h" too is periodic, therefore there exists  $\bar{v}$  such that  $h'(\bar{v}) = 0$

and  $r'(\bar{v}) = \pm 1$ .

From (b) and for the above  $\bar{v}$  we obtain:

$$c_1(\bar{v}) = 0 \text{ hence } Q = 0$$

$$c_1 \equiv 0 .$$

By the symmetry of the differential equations we also

have:

$$d_1 \equiv 0 .$$

We obtain the most general rotation field  $y$  on  $X$  by choosing for each  $k$  a periodic solution  $a_k, c_k, b_k, d_k$  from the above equations and substituting them into (9), (10), (11)

and then into (1), (2), (3). Since the above equations are homogeneous there is always, for each  $k$ , at least the zero solution. We deduce:

$$\begin{aligned} y_1 &= U_0(v) \sin u + \sum_{k=1}^{\infty} \left\{ a_k \cos(ku) \cos u + \right. \\ &\quad + b_k \sin(ku) \cos u + \frac{1}{k} \left( a_k + r r' a'_k + r h' c'_k \right) \sin(ku) \sin u \\ &\quad \left. - \frac{1}{k} \left( b_k + r r' b'_k + r h' d'_k \right) \cos(ku) \sin u \right\} \\ y_2 &= -U_0(v) \cos u + \sum_{k=1}^{\infty} \left\{ -\frac{1}{k} \left( a_k + r r' a'_k + r h' c'_k \right) \sin ku \cos u \right. \\ &\quad + \frac{1}{k} \left( b_k + r r' b'_k + r h' d'_k \right) \cos(ku) \cos u \\ &\quad \left. + a_k \cos(ku) \sin u + b_k \sin(ku) \sin u \right\} \\ y_3 &= \frac{1}{2} c_0 + \sum_{k=2}^{\infty} \left\{ c_k(v) \cos(ku) + d_k \sin(ku) \right\} \end{aligned}$$

§ 5 The Periods of  $[y, dx]$

By one of the theorems of De Rham<sup>©</sup>: For a locally exact differential  $[y, dx]$  to be globally exact on a compact orientable manifold it is necessary and sufficient that the periods

$$P_j = \oint_{c_j} [y, dx] \quad j = 1, \dots, b$$

vanish, where  $c_1, \dots, c_b$  are representatives of the elements of a basis of the 1-dimensional homology group of  $X$  and  $b$  is the first Betti member. For the surface  $X \in \tau$  we have  $b = 2$  and we can choose  $c_1$  to be  $x(u, 0) \quad 0 \leq u \leq 2\pi$ , i. e., a parallel and  $c_2$  to be  $x(0, v) \quad 0 \leq v \leq L$  i. e., a meridian.

Theorem: For any rotation field  $y$  on any surface of class  $\tau$  the period  $P_1$  vanishes.

Proof: Denote by  $c_1$  the curve

$$x(u, 0) = \begin{pmatrix} r(0) \cos u \\ r(0) \sin u \\ h(0) \end{pmatrix} \quad 0 \leq u \leq 2\pi$$

© See [4].

and by  $c_2$  the curve

$$x(0, v) = \begin{pmatrix} r(v) \\ 0 \\ h(v) \end{pmatrix} \quad 0 \leq v \leq L.$$

Then we obtain

$$\begin{aligned} P_1 &= e_1 \int_{c_1} (y_2 dx_3 - y_3 dx_2) + e_2 \int_{c_1} y_3 dx_1 - y_1 dx_3 \\ &+ e_3 \int_{c_1} y_1 dx_2 - y_2 dx_1 \\ &= e_1 \int_{c_1} -y_3 dx_2 + e_2 \int_{c_1} y_3 dx_1 + e_3 \int_{c_1} y_1 dx_2 - y_2 dx_1 \end{aligned}$$

If we denote by  $I_1$ ,  $I_2$  and  $I_3$  the 3 components of  $P_1$  we conclude:

$$I_1 = - \int_0^{2\pi} r(0) y_3 \cos u \, du$$

$$I_2 = - \int_0^{2\pi} r(0) y_3 \sin u \, du$$

$$I_3 = \int_0^{2\pi} y_1 r(0) \cos u \, du + \int_0^{2\pi} y_2 r(0) \sin u \, du$$

Substituting  $y_1$ ,  $y_2$  and  $y_3$  and using the orthogonality relations

of the trigonometric functions and the fact that  $a_0 = 0$ ,  $c_0 = \text{const.}$

$$c_1 = 0,$$

$$d_1 = 0,$$

we conclude that  $I_1 = 0$ ,  $I_2 = 0$ ,  $I_3 = 0$ , for:

$$\begin{aligned} I_1 = & \frac{1}{2} c_0 r(0) \int_0^{2\pi} \cos u \, du + \sum_{k=2}^{\infty} c_k(0) r(0) \int_0^{2\pi} \cos ku \cos u \, du \\ & \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\ & \qquad \qquad \qquad 0 \qquad \qquad \qquad 0 \\ & + \sum_{k=1}^{\infty} d_k r(0) \int_0^{2\pi} \sin ku \cos u \, du \\ & \qquad \qquad \qquad \parallel \\ & \qquad \qquad \qquad 0 \\ & + r(0) c_1 \int_0^{2\pi} \cos u \cos u \, du = 0. \\ & \qquad \qquad \qquad \parallel \\ & \qquad \qquad \qquad 0 \end{aligned}$$

$$I_2 = 0 \quad (\text{as above, using } d_1 = 0).$$

$$\begin{aligned} I_3 = & \sum_{k=1}^{\infty} \left\{ r(0) a_k \int \cos ku \cos u \cos u \, du \right. \\ & + r(0) b_k \int \sin ku \cos u \cos u \, du \\ & + r(0) \frac{1}{k} \left( a_k + r r' a'_k + r h' c'_k \right) \int \sin ku \sin u \cos u \, du \\ & \left. - r(0) \frac{1}{k} \left( b_k + r r' b'_k + r h' d'_k \right) \int \cos ku \sin u \cos u \, du \right\} \end{aligned}$$

$$\begin{aligned}
 & - r(0) \frac{1}{k} \left( a_k + r r' a_k' + r h' c_k' \right) \int \sin u \cos ku \sin u \, du \\
 & + r(0) \frac{1}{k} \left( b_k + r r' b_k' + r h' d_k' \right) \int \cos ku \cos u \sin u \, du \\
 & + r(0) a_k \int \cos ku \sin u \sin u \, du \\
 & + r(0) b_k \int \sin ku \sin u \sin u \, du \} \\
 & = \sum_{k=1}^{\infty} r(0) \left\{ a_k \int \left( \cos ku \cos^2 u + \cos ku \sin^2 u \right) du \right. \\
 & \left. + b_k \int \left( \sin ku \cos^2 u + \sin ku \sin^2 u \right) du \right\} \\
 & = \sum_{k=1}^{\infty} r(0) \left\{ a_k \int \cos ku \, du + b_k \int \sin ku \, du \right\} = 0
 \end{aligned}$$

$$P_1 \equiv 0$$

We would like to compute now the second period.

$$\begin{aligned}
 P_2 &= \oint_{c_2} [y, dx] = \\
 &= e_1 \int_0^L h' y_2 \, d + \\
 &+ e_2 \int_0^L (r' y_3 - h' y_1) \, dv \\
 &- e_3 \int_0^L r' y_2 \, dv
 \end{aligned}$$

Let us denote the components of  $P_2$  by  $I'_1$ ,  $I'_2$  and  $I'_3$ .

By using the first of the formulas on page 34 and due to

$h' U_0 = 0$  (formula 13) and  $u = 0$  we get:

$$I'_1 = \sum_{k=1}^{\infty} \frac{1}{k} \int_0^L h' (b_k + r r' b'_k + r h' d'_k) dv.$$

By equation (15) we derive:

$$I'_1 = \sum_{k=1}^{\infty} k \int_0^L (h' b_k - r' d_k) dv.$$

Using equations (15) and (18) we get:

$$k^2 (h' b_k - r' d_k) = h' b_k + r d'_k.$$

Using the equation just mentioned we obtain:

$$(k^2 - 1) (h' b_k - r' d_k) = r d'_k + r' d_k = (r d'_k)'.$$

For  $k > 1$  we deduce:

$$(h' b_k - r' d_k) = \frac{1}{k^2 - 1} (r d'_k)'.$$

Substituting the integrand of  $I'_1$  we conclude:

$$I'_1 = \sum_{k=2}^{\infty} \left\{ \frac{k}{k^2 - 1} \int_0^L (r d'_k)' dv \right\} + \int_0^L (h' b_1 - r' d_1) dv.$$

Since  $d_1 = 0$  and  $\int_0^L (r d_k)' dv = 0$  (both  $r$  and  $d_k$  are  $L$  periodic). We derive:

$$I_1 = \int_0^L h' b_1 dv$$

$$I_2 = \int_0^L (r' y_3 - h' y_1) dv$$

Using the equations on page 34 as well as the facts that  $u = 0$ ,  $c_1 \equiv 0$ ,  $a_0 \equiv 0$  and  $c_0 = \text{constant}$ , we get:

$$I_2 = \frac{c_0}{2} \int_0^L r' dv + \sum_{k=1}^{\infty} \int_0^L r' c_k dv - \sum_{k=1}^{\infty} \int_0^L h' a_k dv .$$

Since  $\int_0^L r' dv = 0$  we obtain:

$$I_2 = \sum_{k=1}^{\infty} \int_0^L (r' c_k - h' a_k) dv .$$

Using (14) and (17) we deduce:

$$(k^2 - 1) (h' a_k - r' c_k) = r c_k' + r' c_k = (r c_k)'$$

For  $k > 1$  we conclude:

$$(r' c_k - h' a_k) = \frac{(r c_k)'}{1 - k^2} .$$

Substituting the integrand of  $I'_2$  we derive:

$$I'_2 = \sum_{k=2}^{\infty} \frac{1}{1-k^2} \int_0^L (r c_k)' dv$$

$$+ \int_0^L (r' c_1 - h' a_1) dv$$

$$= - \int_0^L h' a_1 dv$$

$$I'_3 = - \int_0^L r' y_2 dv .$$

Using the formulas mentioned in the computations of  $I'_1$  and

$I'_2$  we get:

$$I'_3 = \int_0^L r' U_0 dv$$

$$- \sum_{k=1}^{\infty} \frac{1}{k} \int_0^L r' (b_k + r r' b'_k + r h' d'_k) dv .$$

Using equation (18) we obtain:

$$r' (b_k + r r' b'_k + r h' d'_k) = r' b_k + r r'^2 b'_k + r h' r' d'_k =$$

$$= r' b_k + r r'^2 b'_k + r h'^2 b'_k = r' b_k + r b'_k =$$

$$= (r b_k)' ,$$

$$\begin{aligned} I_3 &= \int_0^L r' U_0 \, dv - \sum_{k=1}^{\infty} \frac{1}{k} \int_0^L (r b_u)' \, dv \\ &= \int_0^L U_0 r' \, dv . \end{aligned}$$

Summing up the last computations we deduce:

$$\begin{aligned} P_2 &= e_1 \int_0^L h' b_1 \, dv \\ &\quad - e_2 \int_0^L h' a_1 \, dv \\ &\quad + e_3 \int_0^L U_0 r' \, dv . \end{aligned}$$

§ 6 The Definition of Intermediate Flexibility

Let  $X$  be a compact surface of arbitrary finite connectivity.  $X$  is said to be rigid in the affine sense if there exists no non-trivial rotation field on it; i. e., there exists no  $y$  which is not constant such that  $[y, dx]$  is locally exact. If  $X$  is non-rigid in the affine sense, and  $c_j$  are those curves defined in § 4 and all  $P_j = 0$  then according to the theorem of De Rham  $[y, dx]$  is globally exact so that there exists  $t$  defined on the whole surface such that

$$dt = [y, dx]$$

$(y, t)$  is called the screw field of  $X$ . We also define

$$z = t \div [y, x]$$

and by what is done in § 1, this time,  $x^\# = x + \epsilon z$  is an infinitesimal isometric deformation of the whole surface.

In this case we say that the surface is non-rigid of projective type. We also say that the surface  $X$  is rigid in the projective sense if there exists no non-trivial screw field on it.

We shall define now the intermediate flexible surfaces.

They are non-rigid in the affine sense but not all  $P_j$  vanish.

More precisely we shall say that a surface is non-rigid with

respect to the subgroup  $U$  of the 1-dim. homology group,  $H_1(X)$

if, a) there exists a non-trivial rotation field  $y$  such that  $P(y, C) \equiv$

$\oint_C [y, dx]$  vanishes for all closed curves  $C$  whose homology

class belongs to  $U$ , and b) there exists no non-trivial rotation

field  $y$  having this property for a subgroup of  $H_1(X)$  containing  $U$ .

In the computations of the periods in §4, if we can choose  $h$  and  $r$  such that  $P_2 \neq 0$ , the surface  $X$  has intermediate flexibility, with respect to the subgroup of  $H_1(X)$  generated by the class containing  $c_1$ .

7 The Main Theorem

Theorem:

A surface of class  $\tau$  has intermediate flexibility if and only if its meridian contains a non-vanishing segment perpendicular to the axis of rotation.

Proof:

Denote:

$$H \equiv \{v \mid v \in [0, L], h'(v) = 0\}$$

If  $H$  does not contain an interval we get that

$$H^C \equiv [0, L] - H \text{ is dense in } [0, L].$$

" $b_1$ " satisfies the equation:

$$h' b_1' = 0 \quad (\text{which follows from equation (18),}$$

since  $d_1 \equiv 0$ ).

Denote:  $B = \{ v \mid v \in [0, L], b_1'(v) = 0 \}$ .

Because of the identity  $h' b_1' = 0$  it follows that:

$$H^C \subset B.$$

Since  $H^C$  is dense in  $[0, L]$  also  $B$  is dense in  $[0, L]$ .

We get that  $b_1' = 0$  on a dense subset of  $[0, L]$ . Since

$b_1'$  is continuous it follows that  $b_1' \equiv 0$ . It follows that  $b_1 =$

constant, and

$$\int_0^L h' b_1' dv = b_1 \int_0^L h' dv = 0.$$

Using the same arguments for  $a_1$  which satisfies  $h' a_1' = 0$

(which follows from equation (17) and  $c_1 \equiv 0$ ) we get that

$a_1 =$  constant, and

$$\int_0^L h' a_1' dv = a_1 \int_0^L h' dv = 0.$$

By equation (13),  $h' U_0 = 0$ , and the arguments used above

we get that  $U_0 \equiv 0$  and

$$\int_0^L r' U_0 \, dv = 0.$$

We proved just now that if  $H$  contains no interval then

$$P_2 = 0.$$

Suppose now that  $H$  contains at least one interval.

Denote it by  $W$ . For this interval  $r' = 1$  or  $r' = -1$ .

We can define a  $C^2$  function  $U_0$  which is zero on  $W^C$  and

$$\int_W U_0 \, dv \neq 0.$$

We can always use for  $U_0$  on  $W$  the function

$$f(x) = \begin{cases} \frac{1}{x-1} - \frac{1}{x} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

This proves that if  $H$  contains an interval, we have intermediate flexibility for the surface.

Corollary: If the meridian of a surface  $X$  in class  $\tau$  is analytic then  $X$  has no intermediate flexibility.

Proof: Since  $r'^2 + h'^2 = 1$  the two functions  $r(v)$  and  $h(v)$  can either be simultaneously extended to complex analytic functions or both cannot. By the main theorem  $h'(v)$  vanishes on an open interval if  $X$  has intermediate flexibility. Hence if  $h(v)$  is analytic it would be constant which is impossible under our assumptions.

The values of the vector field  $z$ . The locally defined instantaneous velocity field becomes many valued in the case of intermediate flexibility.

Every smooth closed curve on the surface  $X$  is homologous to the "product" of  $m$  circles of latitude and  $n$  meridians, where  $m, n = 0, 1, 2, \dots$ . Since:  $H_1(X) \approx \mathbb{Z} \times \mathbb{Z}$ , the two generators are the classes containing the meridian and circle of latitude.

In order to study the various determinations of the velocity field on a surface, we will use the following theorem: If two curves are homologous, the values of the line integrals of a closed differential form over these curves coincide. Since  $P_1 \equiv 0$ , it is obvious that the line integral of  $[y, dx]$  over a closed curve homologous to the "product" of  $m$  meridians and  $n$  parallels is  $n \cdot P_2$ .

Our velocity field is now obtained by:

$$z(x_0, C, x) = \int [y, dx],$$

where  $z$  is the line integral from the fixed point  $x_0$  along the curve  $C$  with the end point  $x$ . According to what was mentioned above,  $z(x)$  is defined as a one valued field if we cut the surface along a parallel.

In the same way we conclude that  $z(x)$  is dependent on the number of meridians in the "product" to which the curve is homologous, each meridian in the product homologous to the given

path contributes  $\pm P_2$  to the value of  $z$ . Hence the determinations of  $z$  differ by integer multiples of  $P_2$ .

Corollary: Note, that in case the meridian contains a single straight segment, then the two first components of  $P_2$ , i. e.,  $I'_1$  and  $I'_2$  are equal to zero. The third component can be made equal to any value that we choose.

§ 8 An Example

I shall now give an example of a surface belonging to class  $\tau$ , which has intermediate flexibility.

Let us define the following meridian:

$$h(s) = \begin{cases} 0 & 0 \leq s \leq 1 \\ F(2-s) & 1 \leq s \leq 2 \\ 1 & 2 \leq s \leq 3 \\ F(s-3) & 3 \leq s \leq 4 \end{cases}$$

$$r(s) = \begin{cases} 10 + s & 0 \leq s \leq 1 \\ 11 + \sin(s-1)\pi & 1 \leq s \leq 2 \\ 11 - s + 2 & 2 \leq s \leq 3 \\ 10 - \sin(s-3)\pi & 3 \leq s \leq 4. \end{cases}$$

It is obvious that  $h$  and  $r$  are  $C^2$ . The function  $F$  is

defined as follows:

$$f(x) \equiv \begin{cases} \frac{1}{x-1} - \frac{1}{x} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F(x) \equiv \frac{\int_x^1 f(t) dt}{\int_0^1 f(t) dt} .$$

We define  $U_0$  in the following way:

$$\hat{U}_0(s) = \begin{cases} f(s) & 0 \leq s \leq 1 \\ 0 & 1 \leq s \leq 4 \end{cases}$$

and we derive  $U_0$  by changing the parameter of  $\hat{U}_0$  to the length parameter of the meridian. (In our notation  $v$  is the length parameter of the meridian and  $s$  is any other parameter.) For that function we obtain for the third component of  $P_2$  the following expression:

$$\begin{aligned} \int_0^L r' U_0 dv &= \int_0^{v(1)} f(v) dv = \\ &= \int_0^1 f(s) ds \neq 0. \end{aligned}$$

Hence  $P_2 \neq 0$ .

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