Grid Minors and Products

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Thesis submitted to the University of Ottawa in partial fulfillment of
the Requirements for the Degree of Masters of Computer Science.

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Contents

1 Abstract

Motivated by recent developments regarding the product structure of planar graphs, we study relationships between treewidth, grid minors, and graph products. We show that the Cartesian product of any two graphs, each connected and each having n vertices, contains an $\Omega(\sqrt{n}) \times \Omega(\sqrt{n})$ grid minor. This result is tight: the lexicographic product (which includes the Cartesian product as a subgraph) of a star and any n-vertex tree has no $\omega(\sqrt{n}) \times \omega(\sqrt{n})$ grid minor.

2 Acknowledgements

I would like to dedicate this work to the countless people who have supported me in my academic journey and in my life. Specifically, I would like to thank my supervisors, Dr. Vida Dujmović, Dr. Pat Morin, and Dr. Prosenjit Bose. Without their support, guidance, and mentorship, I would not be the researcher, student, or person I am today. I also wish to thank my partner River, who's unending love and support has made all the long nights possible. Additionally, I want to thank my parents Shelley and Glenn, and my twin sister Deanna, for their consistent support in my studies and for helping me become the person I am today. Finally, this work would not have been possible without the support and camaraderie of my dear friends Daniel Dallaire, Patrick Devaney, and Mia Rondinelli, whose support in academia, and in life, have helped me grow into the person I am today. Thank you all.

Finally, I would like to acknowledge that sometimes the strangest events shape our course. I did not plan to go into computer science initially, and had a co-op position lined up at a processing facility in my hometown. An untimely fire at the facility left me doing a placement in I.T. instead, starting my journey into computer science.

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4 Introduction

A *graph* consists of a set of entities called *vertices* and a set of connections between pairs of vertices, called *edges*. The simplicity of this model makes it exceptionally useful. Graphs are used for modelling complex systems across a wide range of disciplines.

For example, in computer science and engineering, graphs are used to represent networks like the internet; in social sciences, social networks are used to analyze influence and community structure. In such networks, vertices represent individuals and edges represent their relationships (e.g., friendships, collaborations). Similar social networks can be used to identify the important factors in the spread of infectious diseases. Spatial and transportation networks can be derived from urban geospatial data by representing road intersections as vertices and the connecting roads as edges.

Understanding the structure of graphs arising from these and numerous other applications is a key to solving combinatorial and algorithmic problems related to them.

This thesis investigates a structural question concerning graph products, which have played a central role in resolving several long-standing open problems [3, 4, 16, 18–20, 23, 25, 28, 37, 39].

Before introducing and motivating the central question of this thesis, we first provide the necessary background and definitions from graph theory.

5 Preliminaries

A *graph* is a pair $G = (V, E)$ of sets V and E, where elements of E are two-element subsets of V. The elements of V are called the *vertices* of the graph G. The elements of E are called the *edges* of G. We use $E(G)$ and $V(G)$ to refer to the edge set and vertex set of a specific graph G. For an edge $\{v, w\} \in E(G)$, we say that v and w are *endpoints* of the edge. For brevity, we sometimes denote an edge with endpoints v and w as vw.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, we say H is a *spanning subgraph* of G.

5.1 Graph Minors

A graph H is a *minor* of a graph G if H can be obtained from G by a series of edge deletions, vertex deletions, and/or edge contractions. An *edge contraction* is an operation in which an edge is deleted and the two adjacent vertices are merged into one, removing any parallel edges and loops that occur. We use $H \preceq G$ to denote that H is a minor of G.

Graphs arising from a particular application often share a common property. These properties may or may not be maintained by certain graph operations, such as edge deletions and edge contractions. A graph property is *minor-closed* if for every graph G with the property, every minor of G also has the property.

Consider, for example, the property of planarity, which is among the most extensively studied graph properties. A graph is *planar* if it admits a drawing in the plane (or equivalently, on the surface of a sphere) where vertices are represented as points and edges as simple arcs that do not intersect (other than in a common endpoint). Given that we live on the surface of a sphere, many graphs arising from real-world applications are naturally planar. Importantly, planarity is a minor-closed property: it is preserved under vertex and edge deletions, and, with some care, one can verify that it is also preserved under edge contractions.

A graph class $\mathcal{G}$ is *minor-closed* if, for every $G \in \mathcal{G}$, and for every graph $H \preceq G$, we have that $H \in \mathcal{G}$.

Following the example above, we see that the class of all planar graphs is minor-closed. Many classes of graphs arising naturally from applications are minor-closed. Minor-closed classes of graphs have been extensively studied, culminating in one of the most important results in all of graph theory, the *Graph Minor Theorem* by Robertson and Seymour [43]. The theorem, proven in a series of 20 articles, states that every proper[1] minor-closed family of graphs $\mathcal{G}$ has a finite set of *forbidden minors*. Specifically, Robertson and Seymour [43] showed that a proper family of graphs $\mathcal{G}$ is minor-closed if and only if there is a finite set of graphs $\mathcal{H}$ such that, for every $G \in \mathcal{G}$ and every $H \in \mathcal{H}$, H is not a minor of G.

Consider again the class of planar graphs. The graph minor theorem implies that there is a finite set of graphs $\mathcal{H}$ such that, for every planar graph G, and for every graph H in $\mathcal{H}$, G does not contain H as a minor. In fact, by the famous Wagner's theorem [48], for the case of planar graphs we know exactly the graphs in $\mathcal{H}$. They are the complete graph on 5 vertices, K_5 and the complete bipartite graph $K_{3,3}$. The two graphs are depicted in Figure 1. Thus, a graph is planar if and only if it does not have either K_5 nor $K_{3,3}$ as a minor. One can view the Graph Minor theorem as a far-reaching generalization of Wagner's theorem.

[1] A class of graphs is *proper* if it is not the class of all graphs

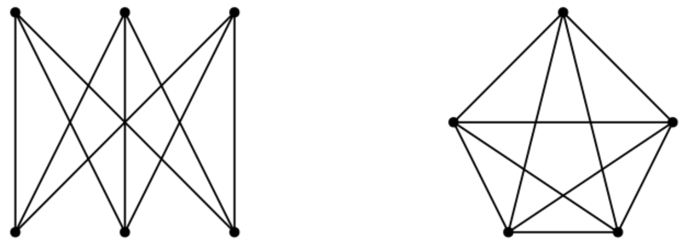

Figure 1: $K_{3,3}$ (left) and K_5 (right), the two forbidden minors for the class of planar graphs

5.2 Treewidth

We now turn our attention towards a particularly useful graph property, the *treewidth*. Treewidth is a ubiquitous parameter in structural graph theory, measuring, in loose terms, how close a graph G is to a tree.

A *tree-decomposition* of a graph G is a collection $(B_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T, such that for every edge $(u, v) \in E(G)$, some bag B_x contains both u and v, and for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T.

The *width* of the tree decomposition $(B_x : x \in V(T))$ is $\max\{|B_x| : x \in V(T)\} - 1$, which is the minimum size of the largest bag in the tree decomposition, minus one. A tree decomposition is not unique. In fact, a tree decomposition of size $|V(G)| - 1$ is possible for every graph by simply putting every vertex into one bag. However, decompositions with smaller bag sizes give a better view of how tree-like G is. The best result comes when this width is minimized. This quantity is called the treewidth.

The *treewidth* of a graph G, denoted by $\text{tw}(G)$, is the minimum width of a tree-decomposition of G.

Given a graph G and a tree-decomposition of G, it is easy (starting with that tree-decomposition) to get a tree-decomposition of smaller or equal width of any minor H of G. Thus, the treewidth of any minor of G is at most the treewidth of G, that is, the treewidth is a minor monotone property.

Treewidth was first introduced as *dimension* by Bertelé and Brioschi [2, pp. 37–38] in 1972, then rediscovered by Halin [32] in 1976. The parameter was popularized when it was once again rediscovered by Robertson and Seymour [44] in 1984 and has since been at the forefront of structural graph theory research.

5.3 Grid Graphs

It is easy to find examples of graphs with low treewidth. For instance, every tree has treewidth equal to one.[2] A slightly harder question is: "What makes a graph have high treewidth?".

As previously stated, treewidth is a minor monotone property. To see this, start with a tree-decomposition of G and note that deleting a vertex or edge cannot increase the size of any bag. Next, consider contracting the edge $uv \in E(G)$ into the vertex w. Remove u and v from every bag of the tree decomposition and add w to any bag that contained u or v. No bag has changed in size, and all adjacencies of u or v are now adjacent to w, maintaining the tree decomposition.

This fact makes the above question relatively simple as well, at least to approximate. We only need to find a family of graphs with high treewidth, then any graph containing a member of said family as a minor would have high treewidth as well. A complete graph on n vertices has treewidth $n-1$, thus providing one example of a high treewidth graph. However, many interesting graph families exclude a complete graph as a minor. For example, as we have discussed already, planar graphs do not have the complete graph on 5 vertices as a minor. However, not all planar graphs have bounded treewidth. An obstruction to low treewidth in planar graphs is a special type of planar graph called the $k \times k$ grid graph, denoted by $\boxplus_k$.

The *$k \times k$ grid graph*, denoted by $\boxplus_k$, is the graph with vertex-set $\{1, \dots, k\} \times \{1, \dots, k\}$ and edge-set $\{(x, y)(x', y') : |x - x'| + |y - y'| = 1, x, y, x', y' \in \{1, \dots, k\}\}$ See Figure 2 for an example.

The treewidth of the $k \times k$ grid graph is equal to k. To see why this is the case, first note that $\text{tw}(\boxplus_k) \leq k$, following from the tree decomposition of Figure 2. The lower bound comes from the concept of a *bramble*. A *bramble* of a graph G is a family of connected subgraphs of G such that for any two disjoint subgraphs there is an edge of G with one endpoint in each subgraph. The order of a bramble is the size of the smallest set of vertices S of G such that S has a non-empty intersection with each subgraph in the bramble. It can be shown (For a full proof, see [33]) that $\boxplus_k$ contains a bramble of order $k+1$. With this, $\text{tw}(\boxplus_k) \geq k$ follows from the *Treewidth Duality Theorem*, which states that if a graph G has a bramble of order n, then $\text{tw}(G) \geq n-1$.

Grid graphs play a central role in structural graph theory due to a found-

[2] Consider a tree T. For each vertex in T, create a bag containing only that vertex. Then, for every edge uv of T, add a new bag (subdividing the edge) that contains both u and v. This results in a decomposition where each bag contains at most two vertices, so the width is 1. It is easy to see that this decomposition satisfies the required properties.

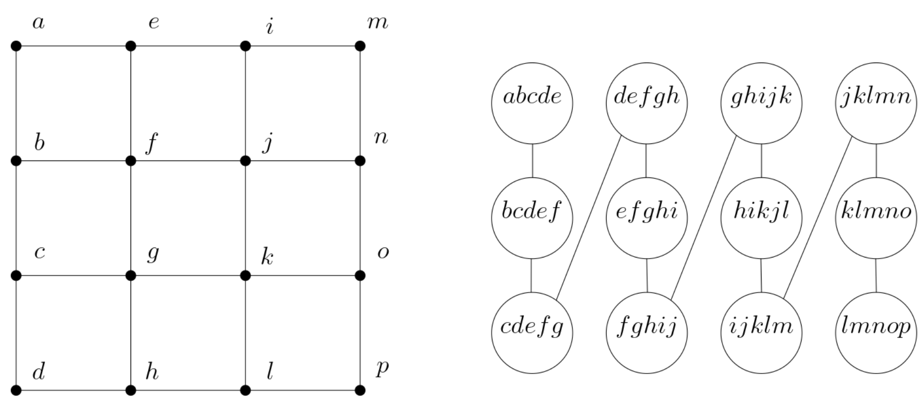

Figure 2: The 4×4 grid graph, denoted $\boxplus_4$ (left), and a tree decomposition of $\boxplus_4$ of width 4 (right)

ational result by Robertson and Seymour, which establishes that grids, as mentioned, are a true obstruction to bounded treewidth. This will be discussed in more detail in the next section.

Due to the importance of grid graphs, a natural question in structural graph theory is to determine the size of the largest grid minor contained in a given graph. Specifically, let $\mathrm{gm}(G)$ be the largest integer k such that $\boxplus_k \preceq G$. Following from the fact that treewidth is minor-monotone and $\mathrm{tw}(\boxplus_k) = k$, we have that for every graph G,

$$\mathrm{tw}(G) \geq \mathrm{gm}(G). \tag{1}$$

5.4 Motivation, Background and Thesis Question

The usefulness of treewidth lies in its powerful implications for algorithm design, particularly in making otherwise intractable problems solvable efficiently. Many computational problems that are NP-hard on general graphs become tractable (i.e., solvable in polynomial time) when restricted to graphs with bounded treewidth. For example, Courcelle's Theorem [7] proves that for many problems that are NP-Hard or NP-Complete on general graphs, they can be solved in linear time on graphs of bounded treewidth. This includes testing Hamiltonicity and testing 3-colorability of a graph.

However, many graph families have unbounded treewidth, meaning the techniques described above do not yield efficient algorithms for these classes.

A cornerstone result in structural graph theory, known as the Grid Minor Theorem (or Excluded Grid Theorem), due to Robertson and Seymour [42], identifies a fundamental obstruction to small treewidth: the presence of a large grid minor. Specifically, Robertson and Seymour [42] proved that there exists a function f such that for every positive integer k, $f(k)$ is the minimum integer such that every graph with treewidth at least $f(k)$ contains a $k \times k$ grid minor.

This result had widespread impact on structural graph theory research and led to further investigation into the best possible bounds for the function f. Robertson and Seymour [42] proved the existence of $f(k)$, which they, along with Thomas, later showed to be in $2^{O(k^5)}$ [45]. Diestel, Jensen, Gorbunov, and Thomassen [13] showed that if G has treewidth $\Omega(k^{4m^2(k+2)})$ where k and m are integers, then G contains either K_m or the $k \times k$ grid as a minor. Leaf and Seymour [40] improved the upper bound to $f \in 2^{O(k \log k)}$. The first polynomial upper bound, stating that $f \in O(k^{98} \log k)$, was found by Chekuri and Chuzhoy [5]. Chuzhoy continued to work towards lowering this exponent, with the current state-of-the-art result by Chuzhoy and Tan [6] showing that $f \in O(k^9 \log^{O(1)} k)$. A lower bound of $f \in \Omega(k^2 \log k)$ was shown by Robertson et al. [45], with Demaine, Hajiaghayi, and Kawarabayashi [12] later conjecturing that $f \in \Theta(k^3)$.

Behind efficient algorithms on graphs on bounded treewidth is a rich theory of parametrized algorithms (see the textbook by Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, and Saurabh [8] for an extensive background). This theory aims to understand NP-hard computational problems where the exponential part of the running time can be isolated to a specific parameter of the input, rather than its overall size. In other words, it focuses on problems that can be solved in $f(k)n^{O(1)}$ time, where n is the size of the input, k is a chosen parameter, and f is some (typically exponential) function. Problems that admit such algorithms are called *fixed-parameter tractable (FPT)*. For extensive background, see the textbook by Cygan et al. [8]. A classic example is the Vertex Cover problem: given a graph G and an integer k, one can determine whether G has a vertex cover of size at most k using a straightforward algorithm with running time $2^k n^{O(1)}$.

However, for certain graph classes, even faster algorithms are possible for NP-complete problems. For instance, on planar graphs, problems such as Vertex Cover, Dominating Set, and Independent Set can be solved in subexponential time—specifically, in $2^{O(\sqrt{k})} n^{O(1)}$ time where k is the size of the desired solution (e.g., the size of the vertex cover, dominating set, or independent set). See Chapter 7 of Cygan et al. [8] for more details and examples. The key to these improvements lies in specific structural properties (that planar graph and some other classes have) — most notably, the linear grid minor property.

A class $\mathcal{G}$ has the *linear grid minor property* if, for some constant c, every graph in $\mathcal{G}$ with treewidth at least ck contains $\boxplus_k$ as a minor. For example, Robertson et al. [45] showed that every planar graph with treewidth at least $6k$ contains $\boxplus_k$ as a minor. Thus the class of planar graphs has the linear grid minor property. More generally, Demaine and Hajiaghayi [10] proved that every proper minor-closed class has the linear grid minor property. The proof used the Graph Minor Structure Theorem, which in turn depends on the Grid Minor Theorem. Kawarabayashi and Kobayashi [38] gave an alternative self-contained proof. In particular, they showed that for any graph H there exists $c \leq |V(H)|^{O(|E(H)|)}$ such that every H-minor-free graph with treewidth at least ck contains $\boxplus_k$ as minor. The linear grid minor property has been used to devise subexponetial exact algorithms (as mentioned above) as well as efficient polynomial time approximation schemes for many NP-hard problems on planar graphs and related graph families (see [9, 11, 12, 27, 30] for examples).

Note that the $\Omega(k^2 \log k)$ lower bound mentioned above shows that general graphs do not have the linear grid minor property.

It turns out that graph classes that slightly outperform this lower bound often admit efficient approximation algorithms for a wide range of problems. In particular, Fomin, Lokshtanov, and Saurabh [31] introduce a unifying framework for designing efficient polynomial-time approximation schemes (EPTASs) on graph classes that satisfy the subquadratic grid minor property. A graph class $\mathcal{G}$ has the subquadratic grid minor property (SQGM) if $\text{tw}(G) = O(\text{gm}(G)^c)$ for all $G \in \mathcal{G}$ and for some $c < 2$. When $c = 1$, this coincides with the linear grid minor property.

Motivated by these developments, the goal of this thesis is to explore the relationship between treewidth and the size of the largest grid minor in graph products, with a particular focus on whether graph products admit the subquadratic grid minor property. While many subfamilies of graph products—such as planar graphs and bounded-genus graphs—do exhibit this property, we will demonstrate that, in general, graph products do not.

Our interest in graph products is inspired by the foundational work of Dujmović et al. [19], which revealed deep connections between graph products and several well-studied graph classes. This framework has led to the resolution of multiple long-standing open problems, as discussed in the next section.

6 Grid Minors, Products and Our Results

6.1 Product Structure Theory

Product Structure Theory aims to express complex graph classes as a product of simpler graphs [19]. We begin by defining the key types of graph products: the Cartesian product, the strong product, and the lexicographic product.

The *Cartesian product* of G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, u_2) and (v_1, v_2) are adjacent iff

- $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or
- $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

The *strong product* of G_1 and G_2, denoted $G_1 \boxtimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, u_2) and (v_1, v_2) are adjacent iff

- $u_1 = v_1$ and $u_2v_2 \in E(G_2)$,
- $u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or
- $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.

The *lexicographic product* of G_1 and G_2, denoted $G_1 \cdot G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, u_2) and (v_1, v_2) are adjacent iff

- $u_1v_1 \in E(G_1)$, or
- $u_1 = v_1$ and $u_2v_2 \in E(G_2)$.

It is important to note that while the Cartesian and strong products are commutative, the lexicographic product is not. An additional important property that follows from the above definitions is that

$$G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \cdot G_2.$$

The relation between grid minors and graph products is very natural due to the fact that $\boxplus_k$ is the Cartesian product of two k-vertex paths.

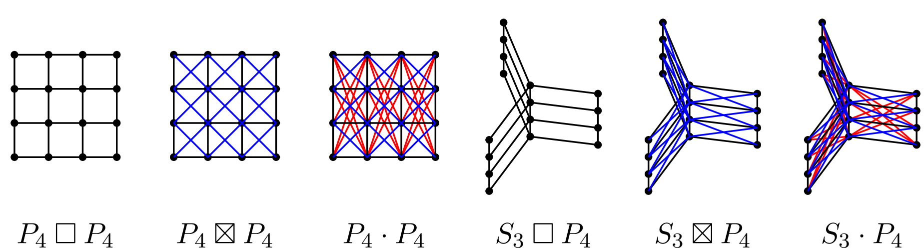

Figure 3: The products of two paths and of the 3 leaf star and a path.

The starting point for recent developments in Graph Product Structure Theory is the *Planar Graph Product Structure Theorem* of Dujmović et al. [19], which states that every planar graph G is isomorphic to a subgraph of the strong product of two very simple graphs, a graph H of treewidth[3] at most 8 and a path P, written as $G \subsetsim H \boxtimes P$. Although $H \boxtimes P$ is a supergraph of the original graph G, it often shares or inherits properties of G, and its rigid structure makes it easier to work with. For example, any induced subgraph of $H \boxtimes P$ of diameter k has treewidth $O(k)$. For a more global example, any n-vertex subgraph of $H \boxtimes P$ has a balanced separator of size $O(\sqrt{n})$ (see [19, Lemma 6] and [17, Lemma 10]). The proofs of both of these facts are considerably simpler than the same results for planar graphs.[4]

Product structure theorems have also been developed for other classes of graphs such as surface-embeddable graphs [14, 19], graphs excluding an apex minor [19, 23, 36], graphs excluding any fixed minor [19, 21], and various non-minor-closed classes [1, 15, 24, 35].

6.2 Our Results

It is known that for all n-vertex connected graphs G_1 and G_2,

$$\text{tw}(G_1 \square G_2) \geq n. \tag{2}$$

To see why, let G_1 and G_2 be connected graphs each with at least n vertices.

For $i \in \{1, 2\}$, let v_i be a leaf of a spanning tree of G_i, and let $G'_i := G_i - v_i$, which is connected. For each $x \in V(G'_1)$, let B_x be the subgraph of $G_1 \square G_2$ induced by $\{x\} \times V(G'_2)$. For each $y \in V(G'_2)$, let B_y be the subgraph of $G_1 \square G_2$ induced by $V(G'_1) \times \{y\}$.

Let B_1 be the subgraph of $G_1 \square G_2$ induced by $\{v_1\} \times V(G_2)$ and let B_2 be the subgraph of $G_1 \square G_2$ induced by $V(G'_1) \times \{v_2\}$.

Let $\mathcal{B} := \{B_x \cup B_y : x \in V(G'_1), y \in V(G'_2)\} \cup \{B_1, B_2\}$. Then it can be seen that $\mathcal{B}$ is a bramble in $G_1 \square G_2$ of order at least $n+1$. By the Treewidth Duality Theorem [46], $\text{tw}(G_1 \square G_2) \geq n$.

This result was extended by Wood [50] who showed that for all k-connected graphs G and H each with at least n vertices, $\text{tw}(G \square H) \geq k(n-2k+2)-1$.

It thus makes sense for a grid minor theorem for graph products to be in terms of n. We show that this is in fact the case by proving the following results:

[3]This treewidth bound was improved to 6 by Ueckerdt, Wood, and Yi [47].

[4]See [26] for discussions on the proof in the planar case, as well as a proof for all bounded-genus graphs, and see [41] for the proof of the Planar Separator Theorem

1. For any two n-vertex connected graphs G_1 and G_2,

   $\mathrm{gm}(G_1 \cdot G_2) \geq \mathrm{gm}(G_1 \boxtimes G_2) \geq \mathrm{gm}(G_1 \square G_2) \in \Omega(\sqrt{n})$ (see Theorem 7).

2. There exists two n-vertex connected graphs G_1 and G_2 (a star and any tree) such that

   $\mathrm{gm}(G_1 \square G_2) \leq \mathrm{gm}(G_1 \boxtimes G_2) \leq \mathrm{gm}(G_1 \cdot G_2) \in O(\sqrt{n})$ (see Theorem 11).

The previous best bound for the product of two n-vertex connected graphs comes from combining (2) with the state-of-the-art Grid Minor Theorem of Chuzhoy and Tan [6], giving $\mathrm{gm}(G_1 \square G_2) \in \Omega(n^{1/9}/\operatorname{polylog}(n))$. The first result above gives an excluded grid theorem for graph products that is much stronger than what could previously be concluded for general graphs.

The second result shows that the first result is tight for the Cartesian, strong, and lexicographic product of two trees. A consequence of the second result and (2) is that there exists two trees whose Cartesian product has treewidth at least n but whose largest grid minor has size $O(\sqrt{n}) \times O(\sqrt{n})$. Thus, even these simple products do not have the linear (or even subquadratic) grid minor property.

6.3 Preliminaries

Recall that in this paper, every graph G is undirected and simple with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is denoted $|G| := |V(G)|$. The following observation follows immediately from the definition of graph minors and the Cartesian product.

Observation 1. *Let G_1, G_2, and H be graphs. If $G_1 \preceq G_2$, then $G_1 \square H \preceq G_2 \square H$.*

A *model* of a graph H in a graph G is a set $\mathcal{M} := \{B_x \subseteq V(G) : x \in V(H)\}$ of subsets of $V(G)$, called *branch sets*, indexed by the vertices of H and such that:

(i) for each distinct pair $x, y \in V(H)$, $B_x \cap B_y = \emptyset$;
(ii) for each $x \in V(H)$, $G[B_x]$ is connected and
(iii) for each $xy \in E(H)$ there exists an edge $vw \in E(G)$ with $v \in B_x$ and $w \in B_y$.

It follows from definitions that $H \preceq G$ if and only if there exists a model of H in G.

For each $n \in \mathbb{N}$, let S_n denote the *n-star*; the rooted tree with n leaves, each of which is adjacent to the root. For each $\ell, p \in \mathbb{N}$, let $S_{\ell,p}$ denote the star with ℓ leaves whose edges have been subdivided $p-1$ times. More formally, $V(S_{\ell,p}) := \{v_0\} \cup \{v_{i,j} : (i,j) \in [\ell] \times [p]\}$ and $E(S_{\ell,p}) := \{v_0 v_{i,1} : i \in [\ell]\} \cup \{v_{i,j} v_{i,j+1} : (i,j) \in [\ell] \times [p-1]\}$. We call $S_{\ell,p}$ a *subdivided star*. Subdivided stars generalize both stars and paths: The n-vertex path P_n is isomorphic to $S_{1,n-1}$ and the n-leaf star S_n is isomorphic to $S_{n,1}$.

Lemma 2. *For any positive integer n and any n-vertex connected graph G, $K_n \preceq G \,\square\, S_n$.*

Note that this lemma is implied by [49, Lemma 5.1]; we include the proof here for the sake of completeness.

Proof. Let y_0 denote the root of S_n, let $y_1, \dots, y_n$ denote the leaves of S_n. Let $V(K_n) = \{1, \dots, n\}$ and let $v_1, \dots, v_n$ denote the vertices of G. We now construct a model $\mathcal{M} := \{B_x : x \in V(K_n)\}$ of K_n in $G \,\square\, S_n$. For each $i \in V(K_n)$, define the branch set

$$B_i := \{(v, y_i) : v \in V(G)\} \cup \{(v_i, y_0)\} \ .$$

We now show that $\mathcal{M}$ is a model of K_n in $G \,\square\, S_n$. The induced graph $(G \square S_n)[B_i]$ is connected because $(G \square S_n)[\{(v, y_i) : v \in V(G)\}]$ is isomorphic to G and (v_i, y_0) is adjacent to (v_i, y_i). For any $1 \le i < j \le n$, B_i and B_j are disjoint because $y_i \ne y_j$ and $v_i \ne v_j$. Furthermore, the vertex $(v_i, y_0) \in B_i$ is adjacent to $(v_i, y_j) \in B_j$. Therefore, there is an edge in $G \,\square\, S_n$ with one endpoint in B_i and one endpoint of B_j for each $1 \le i < j \le n$. $\square$

Note that, for any tree T with n leaves and at least one non-leaf vertex, $S_n \preceq T$. In this case, Lemma 2 and Observation 1 imply that $K_n \preceq G \,\square\, T$.

Finally, we mention the following upper bound on the treewidth of $G_1 \cdot G_2$. This result is stated in [34] for $G_1 \boxtimes G_2$ and is implicit in earlier works; we include the easy proof for completeness.

Lemma 3. *For any graphs G_1 and G_2,*

$$\text{tw}(G_1 \,\square\, G_2) \le \text{tw}(G_1 \boxtimes G_2) \le \text{tw}(G_1 \cdot G_2) \le (\text{tw}(G_1) + 1)|V(G_2)| - 1.$$

Proof. The first two inequalities hold since $G_1 \,\square\, G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \cdot G_2$. For the final inequality, start with a tree-decomposition $(B_x : x \in V(T))$ of G_1 with bags of size at most $\text{tw}(G_1) + 1$. For each $x \in V(T)$ let $B'_x := \{(v, w) : v \in B_x, w \in V(G_2)\}$. Observe that $(B'_x : x \in V(T))$ is a tree-decomposition of $G_1 \cdot G_2$, and $|B'_x| \le |B_x|\,|V(G_2)| \le (\text{tw}(G_1) + 1)|V(G_2)|$ for each $x \in V(T)$. The result follows. $\square$

For any trees T_1 and T_2, we let $m = \min\{|V(T_1)|, |V(T_2)|\}$, then Equation (2) and Lemma 3 imply that,

$$m \leq \operatorname{tw}(T_1 \,\square\, T_2) \leq \operatorname{tw}(T_1 \boxtimes T_2) \leq \operatorname{tw}(T_1 \cdot T_2) \leq 2m - 1. \tag{3}$$

Thus the treewidth of the Cartesian, strong or lexicographic product of two trees is determined (up to a factor of 2) by the number of vertices in the two trees. The remainder of the paper shows that determining the largest grid minor in such a product is more nuanced.

6.4 The Lower Bound

6.4.1 Connected Graphs Contain Large Subdivided Stars

We first state some terminology that will be relevant in the following results. The *length* of a path $v_0, \dots, v_r$ is the number, r, of edges in the path. The *depth* of a vertex v in a rooted tree T is the length of the path, in T, from v to the root of T. A path P in a rooted tree T is *vertical* if, for each $i \in \mathbb{N}$, $V(P)$ contains at most one vertex of depth i. The vertex of minimum depth in a vertical path is its *upper endpoint*, and the vertex of maximum depth in a vertical path is its *lower endpoint*. A vertex v is a *T-ancestor* of a vertex w if the vertical path from w to the root of T contains v. Two vertices of T are *unrelated* if neither is a T-ancestor of the other, otherwise they are *related*. A pair of paths P_1 and P_2 in T is *completely unrelated* if v and w are unrelated, for each $v \in V(P_1)$ and each $w \in V(P_2)$. We say that P_1 and P_2 are *completely related* if v and w are related, for each $v \in V(P_1)$ and each $w \in V(P_2)$. The *height* $h_T(v)$ of a vertex v in T is the maximum order of a vertical path in T whose upper endpoint is v. For each $i \in \mathbb{N}$, let $H_i(T) := \{v \in V(T) : h_T(v) = i\}$ and $n_i(T) := |H_i(T)|$. We have the following observation:

Observation 4. *For any rooted tree T and any $i \in \mathbb{N}$, T contains a set of $n_i(T)$ pairwise completely unrelated vertical paths, each of order i. As a consequence, $S_{n_i(T),i} \preceq T$ for each $i \in \mathbb{N}$.*

Proof. Let $v_1, \dots, v_{n_i(T)} := H_i(T)$ and observe that $v_1, \dots, v_{n_i(T)}$ are pairwise unrelated. For each $j \in \{1, \dots, n_i(T)\}$, let P_j be a path of order i that has v_j as an upper endpoint. (Such a path exists by the definition of $H_i(T)$.) Observe that, for distinct j and k, P_j and P_k are vertex-disjoint, and completely unrelated since v_j and v_k are unrelated. By contracting each edge that has both endpoints of depth less than i into a single vertex x and removing all vertices not in $\{x\} \cup \bigcup_{j \in \{1,\dots,n_i(T)\}} V(P_j)$ we obtain $S_{n_i(T),i}$. Thus $S_{n_i(T),i} \preceq T$. $\square$

We will show that the product $G_1 \square G_2$ of two connected n-vertex graphs G_1 and G_2 contains an $\Omega(\sqrt{n}) \times \Omega(\sqrt{n})$ grid minor by studying the product $T_1 \square T_2$ of two spanning trees of G_1 and G_2, respectively. Lemma 2 allows us to dispense with the case when $n_i(T_b) \in \Omega(n)$ for some i and some $b \in \{1, 2\}$ since, if $n_i(T_b) \in \Omega(n)$, then T_b contains a $S_{\Omega(n)}$-minor, so Lemma 2 implies $K_{\Omega(n)} \preceq T_1 \square T_2$, so $\boxplus_{\Omega(\sqrt{n})} \preceq T_1 \square T_2$. The following lemma will be helpful when this is not possible. (In several places, including the following lemma, we make use of Euler's solution [29] to the Basel Problem: $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$.)

Lemma 5. *Let T be a rooted tree with $n \geq 1$ vertices, and let $p \geq 1$ be an integer such that $n_i(T) \leq \frac{3}{2}n/(\pi i)^2$ for each $i \in \{1, \ldots, p-1\}$. Then T contains pairwise-disjoint vertical paths $P_1, \ldots, P_{\lceil n/4p \rceil}$, each of order p such that, for each $i \neq j$, P_i and P_j are either completely unrelated or completely related.*

Proof. Let $T' := T - (\bigcup_{i=1}^{p-1} H_i(T))$ be the subtree of T induced by vertices of height at least p. Then,

$$|T'| \geq |T| - \sum_{i=1}^{p-1} n_i(T) \geq n - \frac{3n}{2\pi^2} \sum_{i=1}^{p-1} \frac{1}{i^2} \geq n - \frac{n}{4} = \frac{3n}{4} \ .$$

Let L be the set of non-root leaves of T'. Each vertex in L is the upper endpoint of a vertical path in T of order p, as illustrated in Figure 4. Therefore, if $|L| \geq \frac{n}{4p}$ then we are done, so assume that $|L| < \frac{n}{4p}$.

Let S be the set of vertices of T' that have two or more children in T'. In any rooted tree, the number of non-root leaves is greater than the number of non-leaf vertices with at least two children.[5] Therefore, $|S| < |L| < \frac{n}{4p}$.

For each $v \in S \cup L$, let P_v be the vertical path of maximum length whose lower endpoint is v and that does not contain any vertex in $(S \cup L) \setminus \{v\}$. Then $\mathcal{P} := \{P_v : v \in S \cup L\}$ is a partition of $V(T')$ into at most $r := |S| + |L| < \frac{n}{2p}$ parts, each of which induces a vertical path in T'.

Let $\{P_1, \ldots, P_r\} := \mathcal{P}$ and, for each $i \in \{1, \ldots, r\}$, let P'_i be a subpath of P_i of order $p\lfloor |P_i|/p \rfloor$. (So P'_i has order rounded down to a multiple of p.) Then

$$\sum_{i=1}^{r} |P'_i| \geq \sum_{i=1}^{r} (|P_i| - (p-1)) = |T'| - (p-1)r \geq \frac{3n}{4} - \frac{n}{2} = \frac{n}{4}.$$

[5] Let n_i be the number of vertices with i children in a rooted tree T. Thus $\sum_{i \geq 0} i n_i = |E(T)| < |V(T)| = \sum_{i \geq 0} n_i$. Hence, the number of non-root leaves $n_0 > \sum_{i \geq 1} (i-1) n_i \geq \sum_{i \geq 2} n_i$, as claimed.

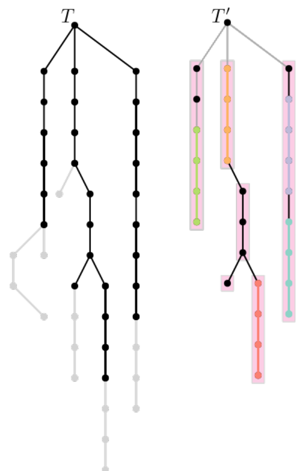

Figure 4: Finding paths in the tree T' induced by vertices of height at least $p = 4$ by partitioning long paths with lower endpoints of height p (in T) into completely related or completely unrelated paths of order p.

For each $i \in \{1, \dots, r\}$, P_i' can be partitioned into exactly $|P_i'|/p$ vertex-disjoint paths, each of order p. The set $\mathcal{P}'$ of paths obtained this way has size $\ell := \sum_{i=1}^{r} |P_i'|/p \geq \frac{n}{4p}$. Therefore T contains ℓ pairwise vertex-disjoint paths, each of order p, where $\ell \geq \frac{n}{4p}$. Except for its lower endpoint, each vertex of a path in $\mathcal{P}'$ has exactly one child in T. This ensures that each path in $\mathcal{P}'$ is either completely related or completely unrelated to any other path in $\mathcal{P}'$. $\square$

6.4.2 The Product of Two Special Trees

Lemma 6. *Let $s, p \geq 1$ be integers, let $\ell := 5s^2$, and let T be a rooted tree that contains s^2 pairwise-disjoint vertical paths, each of order $6p$ such that any pair of these paths is either completely related or completely unrelated. Then $\operatorname{gm}(T \,\square\, S_{\ell,2p}) \geq sp$.*

Proof. Recall that $S_{\ell,2p}$ has vertex set $\{v_0\} \cup \{v_{i,j} : (i,j) \in \{1, \dots, \ell\} \times \{1, \dots, 2p\}\}$. For each $i \in \{1, \dots, \ell\}$, let $A_i = S_{\ell,2p}[\{v_{i,1}, \dots, v_{i,2p}\}]$ denote the ith *arm* of $S_{\ell,2p}$, which is a path of order $2p$. Let $P_1, \dots, P_{s^2}$ be pairwise vertex-disjoint paths in T, each of order $6p$, each pair of which is either completely related or completely unrelated. For each $i \in \{1, \dots, s^2\}$, let $P_i := p_{i,1}, \dots, p_{i,6p}$ where $p_{i,1}$ is the upper endpoint of P_i. Let $T_0 := T \square \{v_0\}$.

For each $i \in \{1, \ldots, \ell\}$ let $T_i := T \,\square\, A_i$, for each $j \in \{1, \ldots, p\}$ let $T_{i,j} := T \square \{v_{i,j}\}$, for each $k \in \{1, \ldots, s^2\}$ let $P_{k,i} := P_k \square A_i$ and $P_{k,i,j} := P_k \square \{v_{i,j}\}$. Note that $P_{k,i}$ is isomorphic to a $6p \times 2p$ grid.

Refer to Figure 5. Consider T_i for some $i \in \{1, \ldots, \ell\}$. For visualization purposes, it is helpful to organize T_i into $2p$ rows $T_{i,1}, \ldots, T_{i,2p}$. For each $j \in \{1, \ldots, 2p-1\}$, $T_{i,j}$ and $T_{i,j+1}$ are 'adjacent' in the sense that each vertex $(a, v_{i,j}) \in V(T_{i,j})$ is adjacent to $(a, v_{i,j+1}) \in V(T_{i,j+1})$. We then organize $T_1, \ldots, T_\ell$ into a sequence of blocks. These blocks are independent in the sense that there is no edge between T_i and T_j for any $i \neq j$. Moreover, there is an additional row T_0 that is adjacent to the first row, $T_{i,1}$, of each block T_i.

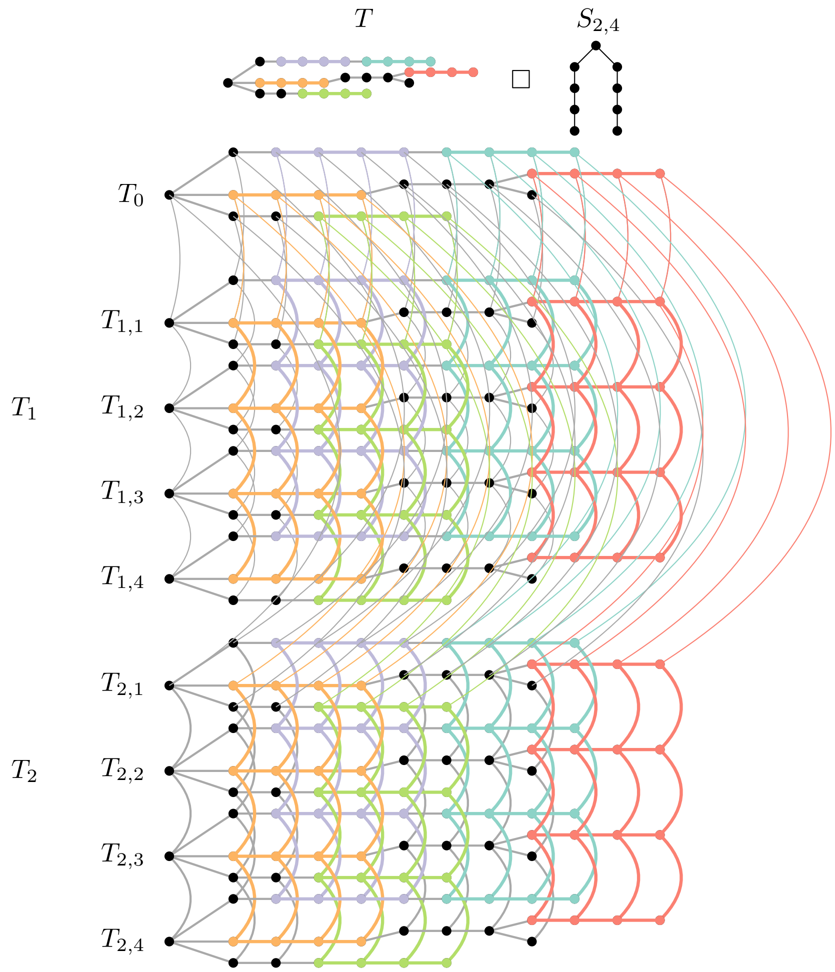

Figure 5: Visualizing the product in Lemma 6

Refer to Figure 6. We will construct a model of $\boxplus_{sp}$. We partition this model into s^2 subgrids each of which is isomorphic to $\boxplus_p$. Therefore, we need s^2 such subgrids $G_1, \ldots, G_{s^2}$. The branch sets of each subgrid G_i will include a $p \times p$ grid within the $6p \times 2p$ grid $P_{i,i}$ (which is contained in the block T_i). The additional row T_0 will allow us to extend the branch sets of the $4p-4$ boundary vertices of the G_i into $T_{i'}$ for any i' and from there they can be extended so that they are adjacent to any other subgrid G_j.

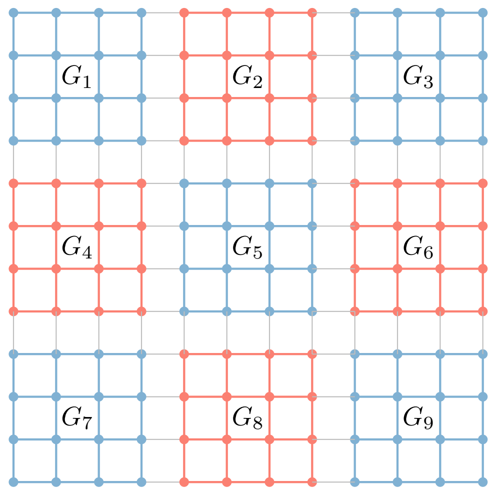

Figure 6: The $sp \times sp$ grid can be partitioned into $(sp/p)^2 = s^2$ subgrids, each of which is a $p \times p$ grid. (The case $sp = 12$ and $p = 4$ is shown here.)

Refer to Figure 7. To construct the branch sets for G_i we start with a $p \times p$ subgrid in $P_{i,i}$ whose top row is $(p_{i,p+1}, v_{i,1}), \ldots, (p_{i,2p}, v_{i,1})$. This subgrid has $4p-4$ 'boundary' vertices, four of which are 'corner' vertices. As illustrated in Figure 7, we create $4p$ disjoint paths in $P_{i,i}$ from the boundary vertices to $P_{i,i,1}$. We then add one vertex of $P_{i,0}$ to each path. In this way, the first $4p$ vertices of the path $P_{i,0}$ are partitioned into four subpaths, each of size p corresponding to edges coming out of the left, top, right, and bottom of boundary of G_i. The final $2p$ vertices of $P_{i,0}$ are not yet used (though we may add them to the branch sets of some boundary vertices later).

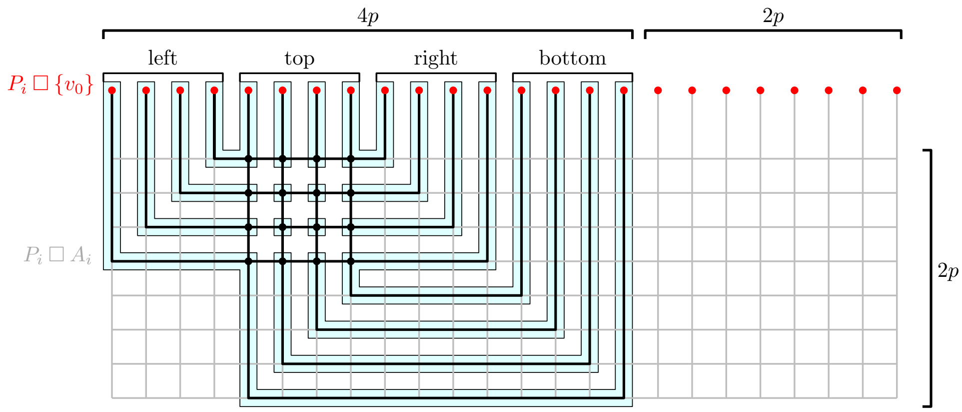

Figure 7: One of the $p \times p$ subgrids used in the proof of Lemma 6.

Our model is not yet complete. At this point, it is a model of a graph that consists of s^2 components, each of which is a $p \times p$ grid. At this point, the vertices in our (not yet-complete) model are all contained in $T_0, T_1, \ldots, T_{s^2}$ and, for each $i \in \{1, \ldots, s^2\}$, the branch sets of vertices in G_i are contained in $P_{i,i} \cup P_{i,0} \subseteq T_i \cup P_{i,0}$. This still leaves the vertices of $T_{s^2+1}, \ldots, T_{5s^2}$ unused. We will use these to extend the branch sets of vertices on the boundary of each subgrid G_i to create the required adjacencies. For each $i \in \{1, \ldots, s^2\}$, the vertices of $T_{s^2+4i-3}, \ldots, T_{s^2+4i}$ will be reserved for the branch sets of G_i.

First, suppose that G_i is a subgrid that is immediately to the right of some subgrid G_j, so that the left boundary of G_i is adjacent to the right boundary of G_j. We will extend the branch sets for vertices on the left boundary of G_i so that they become adjacent to the branch sets for vertices in the right boundary of G_j. Let $x_1, \ldots, x_p$ be the vertices on the left boundary of G_i, ordered from top to bottom and, for each $k \in \{1, \ldots, p\}$, let $x'_k := p_{i,p-k+1}$ so that (x'_k, v_0) is already included in the branch set for x_k. Let $y_0, \ldots, y_p$ be the vertices on the right boundary of G_j, ordered from top to bottom and, for each $k \in \{1, \ldots, p\}$, let $y'_k := p_{j,2p+k}$ so that (y'_k, v_0) is already included in the branch set of y_k. There are two cases to consider. (We strongly urge the reader to refer to Figures 8 and 9.)

- P_i and P_j are completely related (see Figure 8): We will extend the branch sets of $x_1, \ldots, x_p$ into T_{s^2+4i-3}. For each $k \in \{1, \ldots, p\}$, we extend the branch set of x_k by adding the path

$$(x'_k, v_{s^2+4i-3,1}), \ldots, (x'_k, v_{s^2+4i-3,k}) \;,$$

the path in $T_{s^2+4i-3,k}$ from $(x'_k, v_{s^2+4i-3,k})$ to $(y'_k, v_{s^2+4i-3,k})$, and the path

$$(y'_k, v_{s^2+4i-3,k}), \ldots, (y'_k, v_{s^2+4i-3,1}) \ .$$

The first vertex $(x'_k, v_{s^2+4i-3,1})$ of this path is adjacent to (x'_k, v_0), which ensures that the branch set for x_k is connected. The last vertex $(y'_k, v_{s^2+4i-3,1})$ is adjacent to (y'_k, v_0) which ensures that the branch sets for x_k and y_k are adjacent.

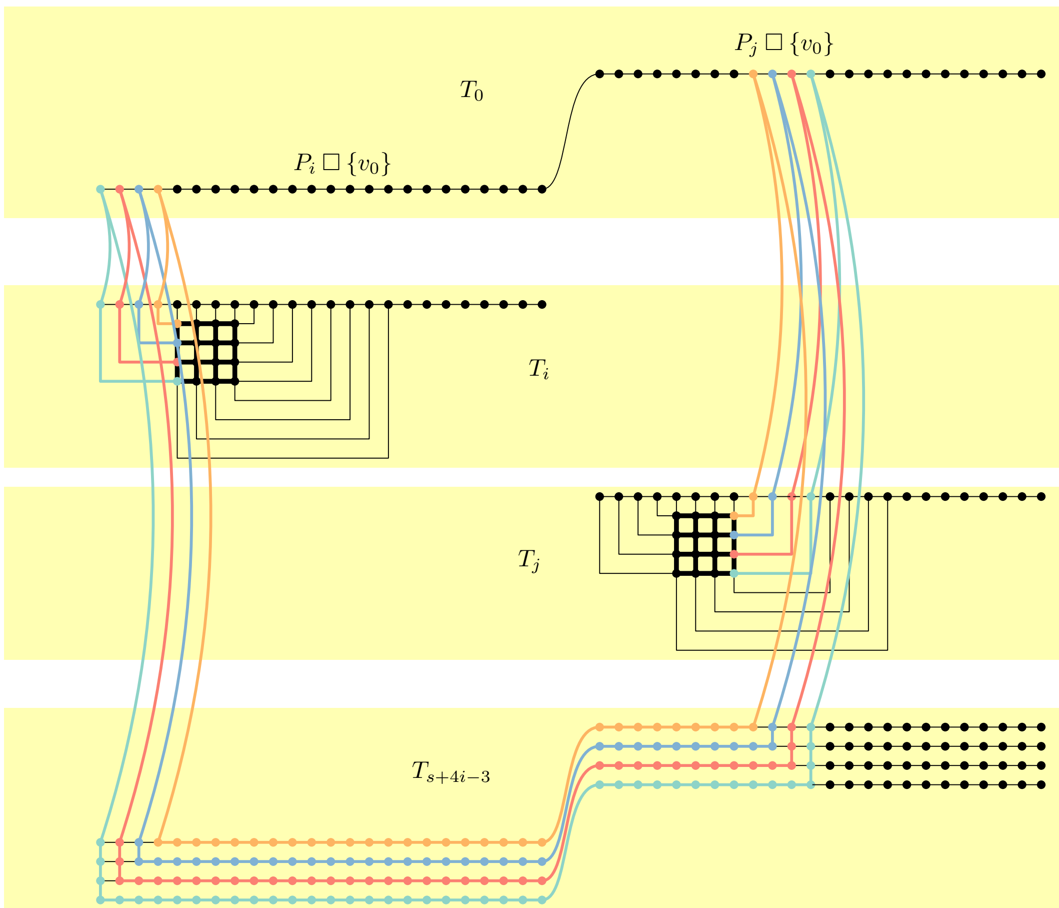

Figure 8: Connecting the left side of G_i to the right side of G_j when P_i and P_j are completely related.

- P_i and P_j are completely unrelated (see Figure 9): We will extend the branch sets of $x_1, \ldots, x_k$ into T_{s^2+4i-3} and T_{s^2+4i-2}. To make the connections between these two blocks we will use an additional p vertices of $P_{i,0}$. The need for a second block in this case is due to the fact that the obvious paths in T_{s^2+4i-3} that were used in the previous case would either intersect each other or reverse the order of connections so that the top-left vertex of G_i would become adjacent to the bottom-right vertex of G_j. Routing these paths through two trees allows us to make the connections in the right order using pairwise vertex-disjoint paths.

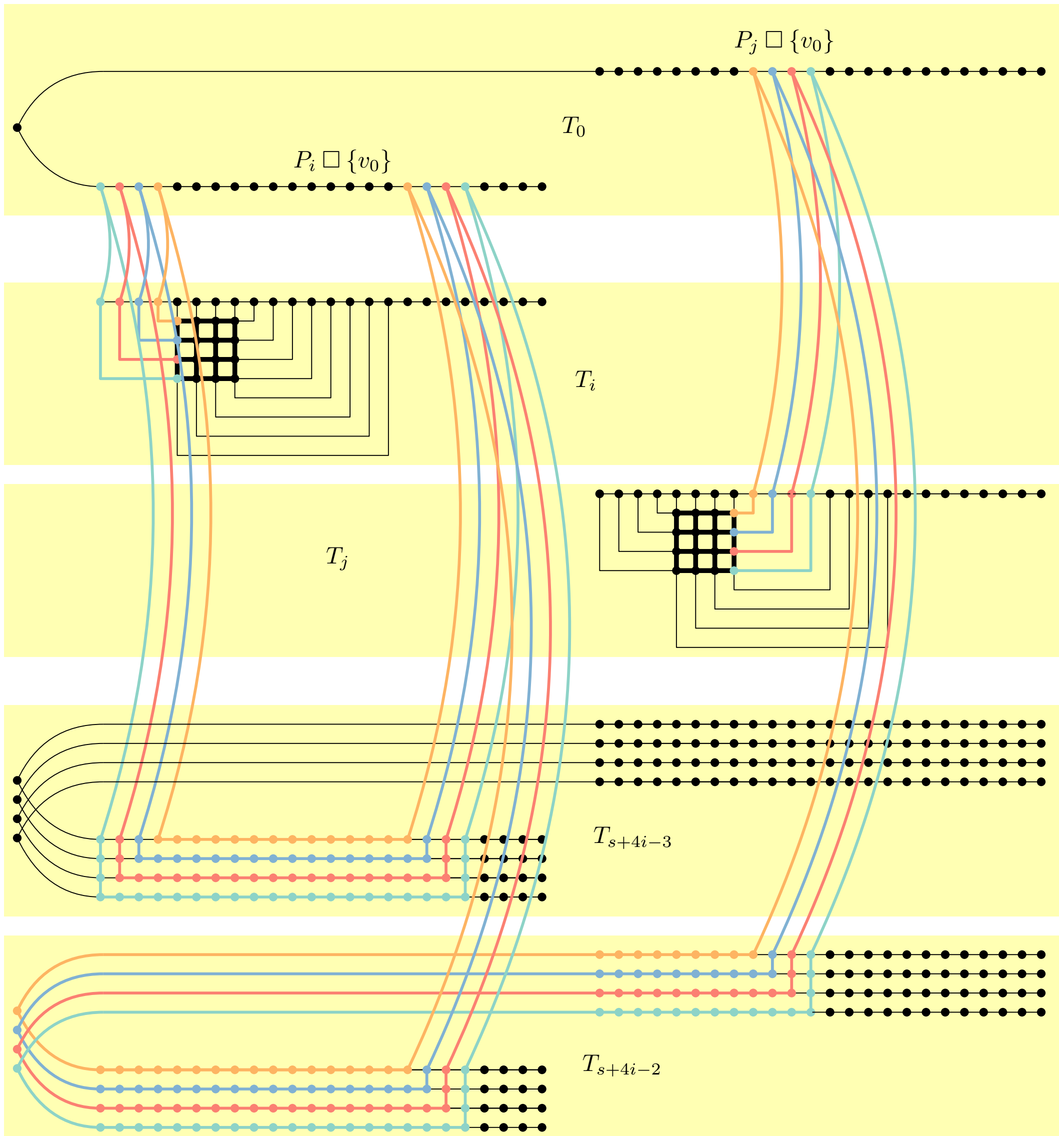

Figure 9: Connecting the left side of G_i to the right side of G_j when P_i and P_j are completely unrelated.

For each $k \in \{1, \ldots, p\}$ we extend the branch set of x_k by adding the path,

$$(x'_k, v_{s^2+4i-3,1}), \dots, (x'_k, v_{s^2+4i-3,k}) \ ,$$

the path in $T_{s^2+4i-3,k}$ from $(x'_k, v_{s^2+4i-3,k})$ to $(p_{i,4p+k}, v_{s^2+4i-3,k})$, the path

$$(p_{i,4p+k}, v_{s^2+4i-3,k}), \dots, (p_{i,4p+k}, v_{s^2+4i-3,0}), (p_{i,4p+k}, v_{s^2+4i-2,1}), \dots, (p_{i,4p+k}, v_{s^2+4i-2,k})$$

the path in $T_{s^2+4i-2,k}$ from $(p_{i,4p+k}, v_{s^2+4i-2,k})$ to $(y'_k, v_{s^2+4i-2,k})$ and finally the path

$$(y'_k, v_{s^2+4i-3,k}), \dots, (y'_k, v_{s^2+4i-3,1}) \ .$$

As in the previous case, the first vertex of this path ensures that the branch set for x_k is connected and the last vertex ensures that the branch sets of x_k and y_k are adjacent.

So far, our model now models every horizontal grid edge but does not yet include the vertical edges between $p \times p$ subgrids. We now sketch how these can be included. Suppose that the subgrid G_i is directly below the subgrid G_j. Let $x_1, \dots, x_p$ be the top boundary of G_i ordered so that x_1 is the the leftmost vertex and x_p is the rightmost. For each $k \in \{1, \dots, p\}$, let $x'_k := (p_{i,p+k})$ so that the branch set of x_k includes x'_k. Let $y_1, \dots, y_p$ be the bottom boundary of G_j ordered so that y_1 is the the leftmost vertex and y_p is the rightmost. For each $k \in \{1, \dots, p\}$, let $y'_k := (p_{j,4p-k+1})$ so that the branch set of y_k includes y'_k. Observe that $x'_1, \dots, x'_p$ occur in order along P_i but $y'_1, \dots, y'_k$ occur in reverse order along P_j. The effect of this is to reverse the two cases that appear above, so that the straightforward case occurs when P_i and P_j are completely unrelated and the more complicated case occurs when they are completely related. Otherwise, the process of growing the branch sets for $x_1, \dots, x_p$ is the same except that the vertices used to grow these new branch sets are contained in T_{s+4i-1}, T_{s+4i}, and $(p_{i,5p+1}, v_0), \dots, (p_{i,6p}, v_0)$. This ensures that these branch sets do not reuse vertices that are used to make G_i adjacent to the neighbour on its left.

Checking that the resulting collection of branch sets is indeed a model of the $sp \times sp$ grid is straightforward; both the disjointedness of the branch sets and the required adjacencies are guaranteed by the construction. □

We now establish our lower bound on the largest grid minor in a Cartesian product:

Theorem 7. *For any connected graphs G_1 and G_2 each having at least $n \geq 1$ vertices,*

$$\mathrm{gm}(G_1 \square G_2) \in \Omega(\sqrt{n}).$$

Proof. For each $b \in \{1,2\}$, let T_b be a tree contained in G_b and having exactly n vertices (which can be constructed by successively deleting leaves starting with a spanning tree of G_b). For each $b \in \{1,2\}$, let $p_b = \min\{i : n_i(T_b) \geq \frac{3n}{2(\pi i)^2}\}$. (This is well-defined since, otherwise $n = \sum_{i=1}^{\infty} n_i(T_b) < \sum_{i=1}^{\infty} \frac{3n}{2(\pi i)^2} = \frac{n}{4}$.) Without loss of generality, assume $p_2 \leq p_1$ and let $\ell := \lceil \frac{3n}{2(\pi p_2)^2} \rceil$. By Observation 4, $S_{\ell,p_2} \preceq T_2 \preceq G_2$. If $p_2 \leq 5$ then $\ell > \frac{3n}{50\pi^2} \in \Omega(n)$ and by Lemma 2 $K_\ell \preceq G_1 \,\square\, S_\ell$. Since $\boxplus_{\sqrt{\ell}} \preceq K_\ell$, this implies that $\mathrm{gm}(G_1 \,\square\, G_2) \geq \sqrt{\ell} = \Omega(\sqrt{n})$ and we are done, so we may assume that $p_2 \geq 6$. Let $p := \lfloor p_2/6 \rfloor \geq 1$.

Since $p_1 \geq p_2$, $n_i(T_1) \leq \frac{3n}{2(\pi i)^2}$ for all $i \in \{1,\ldots,p_2\}$. Therefore, Lemma 5 implies that T_1 contains at least $n/4p_2$ pairwise disjoint paths $P_1,\ldots,P_{\lceil n/4p_2 \rceil}$, each of length $p_2 \geq 6p$, such that each pair of paths is either completely related or completely unrelated. Let

$$s := \lfloor \min\{\sqrt{\ell/5}, \sqrt{n/4p_2}\} \rfloor = \Theta(\sqrt{n}/p)$$

so that $\ell \geq 5s^2$ and $\lceil n/4p_2 \rceil \geq s^2$. By Lemma 6, $\mathrm{gm}(T_1 \,\square\, S_{\ell,6p}) \geq sp = \Theta(\sqrt{n})$. The lemma now follows from Observation 1, the fact that $T_1 \preceq G_1$, and the fact that $S_{\ell,6p} \preceq S_{\ell,p_2} \preceq G_2$.

□

Our next result completes the relationships between grid minors and treewidth in Cartesian and strong products of trees.

Theorem 8. *For any two trees T_1 and T_2,*

$$\mathrm{gm}(T_1 \,\square\, T_2) \leq \mathrm{gm}(T_1 \boxtimes T_2) \leq \mathrm{tw}(T_1 \boxtimes T_2) \in O(\mathrm{gm}(T_1 \,\square\, T_2)^2) \enspace .$$

Proof. First note that $\mathrm{gm}(T_1 \,\square\, T_2) \leq \mathrm{gm}(T_1 \boxtimes T_2)$ since $T_1 \,\square\, T_2 \subseteq T_1 \boxtimes T_2$. Equation (1) shows that $\mathrm{gm}(T_1 \boxtimes T_2) \leq \mathrm{tw}(T_1 \boxtimes T_2)$. It remains to show that $\mathrm{tw}(T_1 \boxtimes T_2) \in O(\mathrm{gm}(T_1 \,\square\, T_2)^2)$.

Let $n_1 := |V(T_1)|$, let $n_2 := |V(T_2)|$, and assume without loss of generality that $n_1 \leq n_2$. By Lemma 3, $\mathrm{tw}(T_1 \boxtimes T_2) \leq 2n_1 - 1$. By Theorem 7, $c\,\mathrm{gm}(T_1 \,\square\, T_2) \geq \sqrt{2n_1}$ for some fixed positive constant c. Therefore,

$$(c\,\mathrm{gm}(T_1 \,\square\, T_2))^2 \geq 2n_1 > \mathrm{tw}(T_1 \boxtimes T_2) \enspace .$$ □

It is worth pointing out that each of the inequalities in Theorem 8 is tight for certain trees T_1 and T_2. The first two inequalities are tight for the product of two paths. Specifically, it is obvious that $\mathrm{gm}(P_n \square P_n) = \mathrm{gm}(P_n \boxtimes P_n) = n$, and $\mathrm{tw}(P_n \boxtimes P_n) < 2n$ by (3).

The last inequality is tight for S_n and P_n since $\mathrm{tw}(S_n \boxtimes P_n) \in \Theta(n)$ by (2) and Lemma 3, and $\mathrm{gm}(S_n \square P_n) \in \Theta(\sqrt{n})$ by Theorem 7 and Theorem 11 below.

6.5 Upper Bound

This section proves the lower bound of Theorem 7 is tight for the Cartesian, strong, and lexicographic products.

Lemma 9. *Fix numbers $\Delta \geq c > 0$. Let $\mathcal{G}$ be a graph class closed under minors and disjoint unions, such that $|E(H)| < c|V(H)|$ for every graph $H \in \mathcal{G}$. Let S be any star and H be any graph in $\mathcal{G}$. Let G be any graph with maximum degree Δ that is a minor of $S \cdot H$. Then*

$$|E(G)| < c|V(G)| + (\Delta - c)\,|V(H)|.$$

Proof. Let $(B_x : x \in V(G))$ be a model of G in $S \cdot H$. Let r be the root of S. Let R be the set of vertices x of G such that $(r, b) \in V(B_x)$ for some $b \in V(H)$. Let Q be the set of vertices x of G such that $V(B_x) \subseteq \{(v, b) : v \in V(S - r), b \in V(H)\}$. Thus $\{R, Q\}$ is a partition of $V(G)$. Moreover, $G[Q]$ is a minor of the disjoint union of n copies of H, implying $G[Q] \in \mathcal{G}$ and $|E(G[Q])| < c|Q|$. The number of edges of G incident to R is at most $\Delta|R|$. Thus $|E(G)| < c|Q| + \Delta|R| = c(|V(G)| - |R|) + \Delta\,|R| = c\,|V(G)| + (\Delta - c)\,|R| \leq c|V(G)| + (\Delta - c)|V(H)|$. □

The class of graphs with treewidth at most t is closed under minors and disjoint unions, and $|E(H)| < t\,|V(H)|$ for every graph H with treewidth at most t. With this, Lemma 9 implies:

Corollary 10. *Fix numbers $\Delta \geq t \geq 1$. Let S be any star and H be any graph with treewidth at most t. Let G be any graph with maximum degree Δ that is a minor of $S \cdot H$. Then*

$$|E(G)| < t|V(G)| + (\Delta - t)\,|V(H)|.$$

The next result completes the proof of the second part of our main theorem stated in Section 6, showing that the lower bound in Theorem 7 is optimal.

Theorem 11. *For any star S and any n-vertex tree T,*

$$\operatorname{gm}(S \,\square\, T) \leq \operatorname{gm}(S \boxtimes T) \leq \operatorname{gm}(S \cdot T) < \sqrt{3n+1} + 1.$$

Proof. The first two inequalities hold by definition. Let $k := \operatorname{gm}(S \cdot T)$. Now apply Corollary 10 with $t = 1$, and with $G := \boxplus_k$ and $\Delta = 4$. Thus

$$2k(k-1) = |E(G)| < |V(G)| + (\Delta - 1)|V(T)| = k^2 + 3n.$$

Thus $k^2 - 2k < 3n$ and $k < \sqrt{3n+1} + 1$. □

7 Conclusion

We have shown that for the Cartesian, strong, and lexicographic product of any two n-vertex graphs, their product contains a grid minor of size $\Theta(\sqrt{n}) \times \Theta(\sqrt{n})$, giving a much stronger, tight bound where previously only an indirect lower bound could be shown. We now present some open problems relating to grid minor theory and graph product structure theory.

7.1 Open Problems

One area of future work is to further investigate the Planar Graph Product Structure Theorem, which we recall states that for every planar graph G, there exists a graph H of bounded treewidth and a path P such that $G \subsetsim H \boxtimes P$. A specific area to investigate is identifying which properties of G can be preserved in $H \boxtimes P$.

Several results of this type are known. For example, in the proof of Dujmović et al. [19], H is a minor of G, and so H is planar. An impossibility result in this area is the following: Even if G is planar and has maximum-degree 5, a result of the form $G \subsetsim H \boxtimes P$ cannot guarantee that H has bounded treewidth and bounded degree [22].

A concrete question that remains open is whether the treewidth of G can be preserved in the product: Is it true that for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \subsetsim H \boxtimes P$ and $\text{tw}(H \boxtimes P) \in O(\text{tw}(G))$? Note that

$$\Omega(\min\{|V(H)|, |V(P)|\}) \leq \text{tw}(H \boxtimes P) \leq O(\min\{|V(H)|, |V(P)|\}).$$

This upper bound follows from Lemma 3 since both H and P have bounded treewidth. The lower bound follows from (2) since we may assume that G, H and P are connected. So this question really asks whether for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \subsetsim H \boxtimes P$ and $\min\{|V(H)|, |V(P)|\} \leq O(\text{tw}(G))$. It is even open whether $\min\{|V(H)|, |V(P)|\} \leq f(\text{tw}(G))$ for some function f, or whether $\min\{|V(H)|, |V(P)|\} \leq O(\sqrt{|V(G)|})$ (which would be implied since $\text{tw}(G) \leq O(\sqrt{|V(G)|})$ for every planar graph G).

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