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GEOMETRIC ASPECTS OF FINITE DIMENSIONAL ALGEBRAS – UNISERIAL REPRESENTATIONS

By

Ahmad Mojiri, M.Sc.

July 2003

A Thesis
submitted to the School of Graduate Studies and Research
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Abstract

K. Bongartz and B. Huisgen-Zimmermann studied uniserial modules over finite dimensional algebras. Given a path $p \notin I$, they associate to it an affine variety V_p, which parameterizes the uniserial modules with mast p. This variety can be calculated algorithmically. It is here shown that these varieties characterize the monomial algebras. An open problem asks for an invariant characterization of algebras isomorphic to monomial algebras. We obtain an algorithmic solution for the open problem for a class of algebras which we call "loosely constricted", and we give a necessary condition for an algebra to be isomorphic to a monomial algebra, which is algorithmic. We then give an analogous version of a theorem of Bardzell and Green which is an invariant characterization of algebras isomorphic to monomial algebras, using uniserial modules. A basis of $\mathrm{Ext}^1_\Lambda(U, V)$ is described, where the quiver has no oriented cycles and the masts of U and V pass through the same vertices, and we get an upper bound for its dimension. Here, each nonzero element of $\mathrm{Ext}^1_\Lambda(U, V)$ represents an indecomposable module. Isomorphism classes of uniserial modules over biserial algebras are described. We study $\alpha(U)$, the number of indecomposable summands of the middle term of an almost split sequence ending in U, where U is a uniserial Λ-module, and give an upper bound for it in the case that Λ is m-multiserial algebra. Irreducible radical embeddings of uniserial modules over triangular multiserial as well as monomial algebras are classified. This confirms a conjecture of A. Boldt in these cases.

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Finally, I wish to thank my family for their encouragement during all of my years of study; especially, my father Jalil Mojiri who passed away on February 11, 2003. I regret that he did not live to see this thesis in print.

Dedication

To my father Jalil Mojiri and

my mother Aghdas Ghadarkhah.

Contents

Chapter 0

Introduction

This dissertation is a study of the representation theory of finite dimensional algebras using *uniserial modules*, i.e., those nonzero modules with a unique composition series. They are the simplest indecomposables, and this makes it interesting to understand their role in the category of finitely generated modules.

Throughout Λ will denote a finite dimensional associative algebra over a field $\mathfrak{K}$ with Jacobson radical J. We will always assume that the algebra Λ is a split basic algebra with the presentation $\Lambda = \mathfrak{K}\Gamma/I$, where Γ is a finite connected quiver and I is an admissible ideal, i.e., $\mathcal{J}^m \subseteq I \subseteq \mathcal{J}^2$, where $\mathcal{J}$ is the ideal in $\mathfrak{K}\Gamma$ generated by the arrows. All modules will be finitely generated left modules.

A theorem of P. Gabriel [18, Section 4] says that any basic finite dimensional algebra over an algebraically closed field is isomorphic to an algebra of the form $\mathfrak{K}\Gamma/I$ for some finite quiver Γ, and some admissible ideal I. Also, recall the classic result in ring theory that any artinian ring is Morita equivalent to a basic one (see [2, Proposition 27.14]). It is then reasonable to restrict our study to algebras of the type $\Lambda = \mathfrak{K}\Gamma/I$.

B. Huisgen-Zimmermann [19, 20], and K. Bongartz and B. Huisgen-Zimmermann [8, 9] studied uniserial representations of finite dimensional algebras using tools of algebraic geometry. To any nonzero path p in $\mathfrak{K}\Gamma/I$, they associate an algebraic variety called V_p which is given explicitly as the vanishing set of certain polynomials which can be calculated algorithmically. Part of this work is the elaboration of their results for specific classes of algebras such as monomial algebras, in Chapter 2, and biserial algebras, in Chapter 4.

In Chapter 2, we focus on *monomial algebras.* An algebra $\Lambda = \mathfrak{K}\Gamma/I$ is monomial if I is generated by some paths. It is well-known that the algebra is monomial if and only if any reduced Gröbner basis of I consist only of paths [15]. First, we give another characterization of these algebras using the geometry of uniserial modules. We show that an algebra Λ is monomial if and only if $\underline{0} \in V_p$ for every path $p \in \mathfrak{K}\Gamma$ where $p \notin I$.

In recent years, there have been significant results on the homological aspects of monomial algebras (such as [21]). A glance at the diagrams for the indecomposable projectives of a monomial algebra (as in [16], and elsewhere) shows that such an algebra is rich in uniserial modules. This suggests that they should play a role in a characterization of algebras isomorphic to monomial algebras.

An open problem in algebra [3, Open problem 5] is to characterize those algebras isomorphic to monomial algebras. (Being "monomial" is a property of its presentation.) We have found a necessary condition, which is algorithmic, for an algebra to be isomorphic to a monomial algebra, using uniserial representations. We call a path $p \in \mathfrak{K}\Gamma$ of length m a *level path,* if $p + I \notin J^{m+1}$, where $J = \mathrm{rad}(\Lambda)$. We say that Λ *satisfies the V_p-condition* if for each level path p in Λ, $V_p \neq \emptyset$. We prove that if Λ is isomorphic to a monomial algebra, then Λ satisfies the V_p-condition. However, in general the converse is not valid.

Next, we introduce a new class of algebras, those we call *loosely constricted algebras* and show that in this class the converse of the above theorem is valid. We say that an algebra $\Lambda = \mathfrak{K}\Gamma/I$ is *loosely constricted* if for each level path p in Λ, p has no detour (see Chapter 2 for details). This class includes constricted algebras, introduced in [4] and weakly constricted algebras, introduced in [5]. We show that if Λ is loosely constricted, then Λ is isomorphic to a monomial algebra if and only if Λ is monomial. This is a generalization of [4, Corollary 4.2] and [5, Corollary 2.5].

In the last part of Chapter 2, we give an analogous version of the main theorem in [4]. This theorem is an invariant characterization of algebras isomorphic to monomial algebras, but one which is not algorithmic since it depends on the existence of a certain grading. The approach in [4] is based on the $H^1(\Lambda)$ and group grading. The use of uniserial representations rather than Hochschild cohomology is a natural alternative approach.

In general, if we have a short exact sequence

$$0 \to V \to X \to U \to 0$$

with uniserial end-terms U and V, then either the middle term X is indecomposable or X is a direct sum of two uniserial modules. However, if in addition Γ has no oriented cycles and $U/JU \cong V/JV$ then the middle term X is indecomposable, if the sequence is not split. Given a uniserial module U with length $l+1$, any path p of length l with $pU \neq 0$ is called a *mast* of U. Throughout Chapter 3, Γ is a finite quiver with no oriented cycles. We focus on $\operatorname{Ext}^1_\Lambda(U,V)$, where U and V are uniserial with masts p and q respectively, where p and q pass through the same sequence of vertices. Under these conditions, each nonzero element of $\operatorname{Ext}^1_\Lambda(U,V)$ gives rise to an indecomposable module. First, we describe a basis for $\operatorname{Ext}^1_\Lambda(U,V)$ and we show that its dimension is bounded by the number of detours on p, in general. Moreover, the dimension of $\operatorname{Ext}^1_\Lambda(U,V)$ is precisely equal to the number of detours on p (resp. the number of detours on p minus 1), when the algebra is hereditary and $U = V$ (resp. $U \neq V$). Then, we find the dimension of $\operatorname{Ext}^1_\Lambda(U,U)$ when Λ is monomial algebra. We give a necessary and sufficient condition for the extension module to have simple top.

An algebra is *biserial* if every indecomposable projective left or right Λ-module P contains uniserial submodules U and V such that $U + V = \operatorname{rad}(P)$ and $U \cap V$ is either zero or simple. In Chapter 4, we characterize the isomorphism classes of uniserial modules over biserial algebras. The two main ingredients in our proof are the Vila-Freyer and Crawley-Boevey's structure theorem of basic biserial algebras and Huisgen-Zimmermann's description of uniserial representations. R. Vila-Freyer and W. Crawley-Boevey [30] describe basic biserial algebras by means of quivers and relations. B. Huisgen-Zimmermann [19] introduced a method for deciding when two uniserial modules over a finite dimensional algebra are isomorphic. Using these tools we characterize the isomorphism classes of uniserial modules over a biserial algebra.

In Chapter 5, we look at almost split sequences with uniserial modules as one of the end terms. A non-split short exact sequence in mod-Λ

$$0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$$

is called an *almost split sequence* if (a) A and C are indecomposable and (b) any morphism $X \to C$ which is not a split epimorphism factors through f. There is a nice relationship between the end terms, given by $A \cong DTrC$ and $C \cong TrDA$ (see Section 1.7). A morphism $g : B \to C$ in mod-Λ is called *irreducible* if g is neither a split monomorphism nor a split epimorphism, and if $g = ts$ for some $s : B \to X$ and $t : X \to C$, then s is a split monomorphism or t is a split epimorphism. Irreducible maps are closely connected to almost split sequences. If we decompose the middle term B, then the corresponding components of f and g are irreducible maps. The number of indecomposable summands in a sum decomposition of B is an important invariant of C, which we denote by $\alpha(C)$. Also, we denote by $\alpha(\Lambda)$ the numerical invariant which is the supremum of the $\alpha(C)$ for C indecomposable and not projective.

Let $0 \to DTrU \xrightarrow{(f_i)_i} \bigsqcup_{i\in I} B_i \xrightarrow{(g_i)_i} U \to 0$ be an almost split sequence with U a non-projective uniserial module with the B_i indecomposable. We are aiming for an upper bound for $\alpha(U)$. First, we note that at most one of the maps $g_i \colon B_i \to U$ is monomorphism and that is $JU \hookrightarrow U$ up to isomorphism. Moreover, the number of epimorphisms is less or equal to $\dim_{\mathfrak{K}} \mathrm{soc}(DTrU)$. An algebra is called *left multiserial* (resp. *m-multiserial*) if, for each primitive idempotent e of Λ, the left ideal Je is a sum of uniserial (resp. m uniserial) Λ-modules. Uniserial representations of left multiserial algebras have been studied by B. Jue in [23]. We prove that if Λ is m-multiserial algebra and $m \geq 2$, then $\alpha(U) \leq m$ for any uniserial module U. Almost split sequences with indecomposable middle term are of interest. (See [3, Chapter XI.5], [12], [24].) It is natural to ask: When does an almost split sequence with a non-projective uniserial module U in its end term have an indecomposable middle term? As a general fact, the number of indecomposable direct summands could be arbitrarily large. (See the example [3, page 152].) We show that in the case of left biserial algebras, the middle term is indecomposable if $Jp = 0$, where p is the mast of U. If the algebra is a left multiserial algebra and p has no detours or nonroutes starting at $e = s(p)$, then $\alpha(U) \leq 2$.

A. Boldt [6] asked the following question in his thesis: For which uniserial modules U over a *triangular algebra* Λ is the radical embedding $JU \hookrightarrow U$ irreducible? We say Λ is a *triangular algebra* when the quiver of Λ has no oriented cycles. We show that if $JU \hookrightarrow U$ is irreducible, where U is a non-simple uniserial module with mast p, where p does not start with an oriented cycle, then all detours on p are inessential and all non-routes are in Jp. This generalizes [6, Conjecture 1.2.1, (1)⇒(2)a] by weakening the assumption that the quiver have no oriented cycles. The method used also differs from that in [6]. A. Boldt proposes necessary and sufficient conditions on the algebra, based on the presentation of the algebra, which make $JU \hookrightarrow U$ irreducible. Here we prove this for a class of algebras including left multiserial algebras and monomial algebras. These confirm a conjecture of A. Boldt [6, Conjecture 1.2.1] for these cases.

Chapter 1

Notation and background material

By Λ, we denote a finite dimensional associative algebra over a field $\mathfrak{K}$ with Jacobson radical J. We will always assume that the algebra is a split basic algebra with presentation $\mathfrak{K}\Gamma/I$, where $\Gamma = (\Gamma_0, \Gamma_1)$ is a finite connected quiver with Γ_0 and Γ_1 the sets of vertices and arrows respectively and I an admissible ideal; i.e., $\mathcal{J}^m \subseteq I \subseteq \mathcal{J}^2$, for some $m \geq 2$, with $\mathcal{J}$ the ideal of $\mathfrak{K}\Gamma$ consisting of all linear combinations of paths of length ≥ 1. For a path p in Γ we often use the notation p for $p + I$, when there is no danger of confusion. If $p + I \neq I$ we often say "p is a nonzero path in Λ" or just "$0 \neq p \in \Lambda$". The set of nonzero elements of $\mathfrak{K}$ is denoted by $\mathfrak{K}^*$. First let us give some definitions.

Definition 1.0.1. *A module U is called* uniserial *if the lattice of submodules forms a chain; i.e., if any two submodules of U are comparable.*

Notice that since Λ is a finite dimensional algebra, the submodules of U are given by J^iU, with $i = 0, \ldots, \text{length}(U)$.

1.1 The geometry of uniserial representations

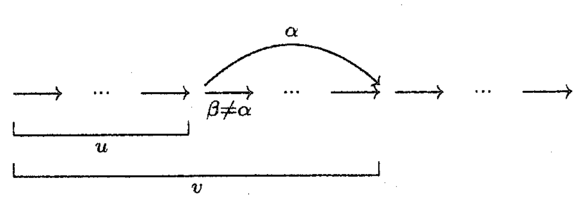
B. Huisgen-Zimmermann [19, 20] and, K. Bongartz and B. Huisgen-Zimmermann [8, 9] studied uniserial representations of finite dimensional algebras using tools of algebraic geometry. To any nonzero path p in $\mathfrak{K}\Gamma/I$, they associate an algebraic variety called V_p which is given explicitly as the vanishing set of certain polynomials which can be calculated algorithmically. Here is their terminology.

Definition 1.1.1. *Given a uniserial module U with length $l+1$, any path p of length l with $pU \neq 0$ is called a* mast *of U.*

A path $q \in \mathfrak{K}\Gamma$ is a right subpath (resp. left subpath) of p if there exists a path r with $p = rq$ (resp. $p = qr$). A proper right subpath of p is a right subpath of p which is strictly shorter than p. It will be convenient to abbreviate the statement "q is a right subpath of p" to the form "$p = \bullet q$" to simplify formulas. If α is an arrow we write $s(\alpha)$ for its starting vertex and $t(\alpha)$ for its terminal vertex.

Definition 1.1.2. *A* detour *on a path p is a pair (α, u) with α an arrow and u a right subpath of p, with the following properties:*

- *$\alpha u \neq 0$ in $\mathfrak{K}\Gamma$*
- *αu is not a right subpath of p, but*
- *There exists a right subpath v of p with* $\text{length}(v) \geq \text{length}(u) + 1$ *such that the terminal point of v coincides with the terminal point of α.*



Following [19] we will abbreviate the statement "(α, u) is a detour on p" by $(\alpha, u) \wr\wr p$.

Definition 1.1.3. *Suppose that p is a path of length l that passes consecutively through the vertices* $e(1), \ldots, e(l+1)$. *A* route *on p is any path in $\mathfrak{K}\Gamma$ which starts in* $e(1)$ *and passes through a subsequence of the sequence* $(e(1), \ldots, e(l+1))$.

Remark 1.1.4. *[19, Remarks (b), page 46] In our work with routes, the following factorization property will be crucial. A path $r \in \mathfrak{K}\Gamma$ is a route on p if and only if it can be written in form*

$$r = r'\alpha_m u_m \cdots \alpha_1 u_1$$

for some $m \geq 0$ ($m = 0$ occurs when $r = r'$) such that there exists a corresponding factorization

$$p = p'w_m \cdots w_1$$

of p with the property that $(\alpha_i, u_i) \wr\wr w_i$ with $t(\alpha_i) = t(w_i)$ for each $i \leq m$, and r' is a right subpath of p'. Note that such a factorization of p corresponding to the given route r need not be unique.

Description of the variety V_p [19]: We start by describing the polynomial ring in which we will be working. Given any detour on p, let

$$V(\alpha, u) = \{v_i(\alpha, u) \mid i \in I(\alpha, u)\}$$

be the family of right subpaths of p in $\mathfrak{K}\Gamma$ which are longer than u and have the same terminal points as α. It is often more convenient to refer to the index set $I(\alpha, u)$ for $V(\alpha, u)$, than to the set $V(\alpha, u)$ itself. Consider the polynomial ring

$$\mathfrak{K}\Gamma[X] = \mathfrak{K}\Gamma[X_i(\alpha, u) \mid i \in I(\alpha, u) \text{ and } (\alpha, u) \wr\wr p]$$

with coefficients in the path algebra $\mathfrak{K}\Gamma$, and independent variables $X_i(\alpha, u)$. Next we introduce an equivalence relation on $\mathfrak{K}\Gamma[X]$ as follows: Let $L(p)$ be the left ideal of $\mathfrak{K}\Gamma[X]$ generated by all the paths q in $\mathfrak{K}\Gamma$ which are not routes on p, together with all the differences

$$\alpha u - \sum_{i \in I(\alpha, u)} X_i(\alpha, u) v_i(\alpha, u)$$

for each detour (α, u) on p. Then, the relation

$$\sigma \mathrel{\widehat{=}} \tau \Leftrightarrow \sigma - \tau \in L(p)$$

for $\sigma, \tau \in \mathfrak{K}\Gamma[X]$ defines a congruence relation relative to addition and left multiplication.

Observation 1.1.5. *[19, Observation 3, page 47] Each element of $\mathfrak{K}\Gamma$ is $\widehat{=}$ congruent to a unique element of the form $\sum_{p=\bullet p'} \tau_{p'}(X)p'$, where the $\tau_{p'}(X)$ are polynomials in*

$$\mathfrak{K}[X] = \mathfrak{K}[X_i(\alpha, u) \mid i \in I(\alpha, u), (\alpha, u) \wr\wr p].$$

To obtain these polynomials $\tau_{p'}(X)$ for a given element $z \in \mathfrak{K}\Gamma$ algorithmically, consider the following substitution equations for p:

(i) $q \widehat{=} 0$ for any path q in $\mathfrak{K}\Gamma$ which is not a route on p.

(ii) $\alpha u \widehat{=} \sum_{i \in I(\alpha,u)} X_i(\alpha, u)v_i(\alpha, u)$ for all detours (α, u) on p.

We will use the phrase "inserting the substitution equations from the right" for the following steps:

$$q''q' \widehat{=} 0 \qquad \text{if } q' \text{ is a nonroute on } p\,;$$

$$q''q' \widehat{=} \sum_{i \in I(\alpha,u)} X_i(\alpha, u) \cdot q''v_i(\alpha, u) \qquad \text{if } q' = \alpha u \text{ with } (\alpha, u) \wr\wr p\,.$$

Note that if q' is not a route on p, then neither is $q''q'$. Inserting the substitution equations from the right into the path occuring in the element $z \in \mathfrak{K}\Gamma$ and repeating this procedure clearly leads to the equations $z \widehat{=} \sum \tau_{p'}(X)p'$ with $\tau_{p'}(X) \in \mathfrak{K}[X]$ after at most d steps, where d is an upper bound on the length of paths involved in z.

Let L be the Loewy length of Λ (i.e., L is minimal with respect to $J^L = 0$), and denote $I^{(L)}$ the $\mathfrak{K}$-subspace of I consisting of all elements which can be written as $\mathfrak{K}$-linear combinations of paths with length at most L. Moreover, choose a finite $\mathfrak{K}$-generating set $t_1, \dots, t_s$ for the space $I^{(L)}$. By the above remarks, there are unique polynomials $\tau_{i,p'}(X) \in \mathfrak{K}[X]$ with the property that $t_i \widehat{=} \sum_{p=\bullet p'} \tau_{i,p'}(X)p'$ for $1 \le i \le s$. We can now define the variety V_p.

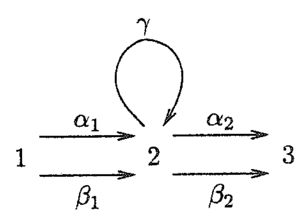
Definition 1.1.6. *Let $V_p = V(\tau_{i,p'}(X) \mid 1 \le i \le s$ and $p = \bullet p')$ be the simultaneous vanishing locus of the polynomials $\tau_{i,p'}(X)$ in affine N-space $\mathbb{A}^N = \mathbb{A}^N(\mathfrak{K})$, where $N = \sum_{(\alpha,u)\wr\wr p} |I(\alpha, u)|$.*

Remarks 1.1.7. *[19, Remarks, page 48]*

1. The variety V_p is independent of the choice of $\mathfrak{K}$-generating set for the $\mathfrak{K}$-space $I^{(L)}$. In fact, V_p is the vanishing locus of all polynomials $\tau_{p'}(X)$ arising in congruences $z \,\widehat{=}\, \sum_{p=\bullet p'} \tau_{p'}(X)p'$ for all elements $z \in I$.

2. Observe that whenever $r_1, \ldots, r_m \in I$ generate I as a left ideal of $\mathfrak{K}\Gamma$, then $V_p = V(\sigma_{i,p'}(X) \mid 1 \leq i \leq m \text{ and } p = \bullet p')$ where $r_i = \sum_{p=\bullet p'} \sigma_{i,p'}(X)p'$ with $\sigma_{i,p'}(X) \in \mathfrak{K}[X]$.

Example 1.1.8. Let $\Lambda = \mathfrak{K}\Gamma/I$, where Γ is the quiver



and I is the (2-sided) ideal of $\mathfrak{K}\Gamma$ generated by the following relations:

$$\gamma^3, \;\; \beta_2\alpha_1, \;\; \beta_2\gamma\alpha_1, \;\; \beta_2\gamma^2\alpha_1 - \alpha_2\gamma^2\alpha_1, \;\; \alpha_2\gamma^2\beta_1 - \alpha_2\alpha_1, \;\; \beta_2\beta_1, \;\; \alpha_2\gamma\alpha_1 - \alpha_2\beta_1$$

Moreover, consider the path $p = \alpha_2\gamma^2\alpha_1$. To compute the variety V_p, observe that the detours on p and the corresponding substitution equations are as follows:

$$\begin{array}{cl}
(\beta_1, e_1) & \beta_1 \,\widehat{=}\, X_1\alpha_1 + X_2\gamma\alpha_1 + X_3\gamma^2\alpha_1 \\
(\alpha_2, \alpha_1) & \alpha_2\alpha_1 \,\widehat{=}\, X_4 p \\
(\beta_2, \alpha_1) & \beta_2\alpha_1 \,\widehat{=}\, X_5 p \\
(\alpha_2, \gamma\alpha_1) & \alpha_2\gamma\alpha_1 \,\widehat{=}\, X_6 p \\
(\beta_2, \gamma\alpha_1) & \beta_2\gamma\alpha_1 \,\widehat{=}\, X_7 p \\
(\beta_2, \gamma^2\alpha_1) & \beta_2\gamma^2\alpha_1 \,\widehat{=}\, X_8 p
\end{array}$$

Observe that the relations listed above, together with the paths $\gamma^3\alpha_1$ and $\gamma^3\beta_1$, generate I as a left ideal of $\mathfrak{K}\Gamma$. By Remarks 1.1.7, we need only to consider these elements of I in determining a generating set of polynomials for V_p. Since these last two paths and γ^3, are non-routes on p, they are $\widehat{=}$-equivalent to 0 and hence do not lead to conditions on the indeterminates X_i. We now insert the substitution equations into the remaining relations. The combination of $\beta_2\alpha_1 \widehat{=} X_5 p$ and $\beta_2\alpha_1 \in I$ implies $X_5 = 0$ on V_p. Analogously, $\beta_2\gamma\alpha_1 \widehat{=} X_7 p$ and $\beta_2\gamma\alpha_1 \in I$ implies $X_7 = 0$. Also $\beta_2\gamma^2\alpha_1 - \alpha_2\gamma^2\alpha_1 \widehat{=} (X_8-1)p$ implies $X_8 - 1 = 0$ on V_p. Moreover, $\beta_2\beta_1 \widehat{=} \beta_2(X_1\alpha_1 + X_2\gamma\alpha_1 + X_3\gamma^2\alpha_1) \widehat{=} (X_1X_5p + X_2X_7p + X_3X_8p)$ yields $X_1X_5 + X_2X_7 + X_3X_8 = 0$ on V_p and consequently $X_3 = 0$. Also $\alpha_2\gamma^2\beta_1 - \alpha_2\alpha_1 \widehat{=} (X_1 - X_4)p$ implies $X_1 - X_4 = 0$ on V_p. Next, $\alpha_2\gamma\alpha_1 - \alpha_2\beta_1 \widehat{=} X_6p - \alpha_2(X_1\alpha_1 + X_2\gamma\alpha_1 + X_3\gamma^2\alpha_1) \widehat{=} (X_6 - X_4X_1 - X_2X_6 + X_3)p$ gives us $X_6 - X_4X_1 - X_2X_6 + X_3 = 0$. Therefore $V_p =$

$$\begin{aligned} &V(X_5, X_7, X_8 - 1, X_1 - X_4, X_1X_5 + X_2X_7 + X_3X_8, X_6 - X_4X_1 - X_2X_6 + X_3) \\ &\cong V(X - Y^2 - XZ). \ \clubsuit \end{aligned}$$

The following theorem contains basic information about the connection between the variety V_p on one hand and uniserial Λ-modules with mast p on the other. Note that V_p does not, in general, determine the isomorphism classes of uniserial modules with mast p.

Theorem 1.1.9. *[19, Theorem A] Suppose that p is a path in $\mathfrak{K}\Gamma$ starting in the vertex $e(1)$.*

(i) There is a surjective map Φ_p from the variety V_p to the set of isomorphism types of uniserial left Λ-modules with mast p. It assigns to each point $\underline{k} = (k_i(\alpha, u))_{i \in I(\alpha,u),\, (\alpha,u) \wr\wr p}$ in V_p the isomorphism type of the module $\Lambda e(1)/U_k$, where

$$U_k = \left(\sum_{(\alpha,u)\wr\wr p} \Lambda \left(\alpha u - \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u) \right) \right) + \left(\sum_{q \text{ non-route on } p} \Lambda q e(1) \right).$$

Alternately, the uniserial module $\Lambda e(1)/U_k$ representing $\Phi_p(k)$ can be described as the unique uniserial quotient module with mast p of the module

$$\Lambda e(1) / \left(\sum_{(\alpha,u)\wr\wr p} \Lambda \left(\alpha u - \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u) \right) \right).$$

(ii) *The variety V_p is nonempty if and only if there exists a uniserial left Λ-module with mast p.*

(iii) *Provided that p does not have a proper right subpath which is an oriented cycle of positive length, the map Φ_p is bijective.*

Definition 1.1.10. *[23] A detour (α, u) on a path p is called* inessential *if*

$$\alpha u = s + \sum_{i \in I(\alpha,u)} k_i v_i(\alpha, u)$$

in Λ, where s is a $\mathfrak{K}$-linear combination of paths, none of which are routes on p, and $k_i \in \mathfrak{K}$ for all $i \in I(\alpha, u)$. A detour is essential *if it is not inessential.*

1.2 The isomorphism problem for uniserial modules

In Chapter 4, we will characterize isomorphism classes of uniserial modules over a biserial algebra, using the following tool from [19, Section 4].

A necessary condition for two uniserial modules to be isomorphic is a joint mast. Huisgen-Zimmermann explicitly describes the equivalence relation on V_p which partitions V_p into the fibres $\Phi_p^{-1}(U)$, where U runs through the uniserial modules in the image $\Phi_p(V_p)$. Theorem 1.1.9, gives us a partial answer: If the path p does not start with an oriented cycle, the map Φ_p is a bijection; in other words, the points on the variety V_p form a complete system of isomorphism classes of uniserial modules with mast p. In general, injectivity of Φ_p may fail. Huisgen-Zimmermann [19] constructed a system of equations $S_p(X, Y, Z)$ in the variables $X_i(\alpha, u)$, $Y_i(\alpha, u)$ (for $(\alpha, u) \wr p$ and $i \in I(\alpha, u)$) and finitely many variables Z_j, which is linear in the Z_j over $\mathfrak{K}[X, Y]$, such that for any pair of points $\underline{k}, \underline{k}' \in V_p$, the linear system $S_p(\underline{k}, \underline{k}', Z)$ is consistent if and only if $\Phi_p(\underline{k}) \cong \Phi_p(\underline{k}')$.

To describe the system $S_p(X, Y, Z)$, suppose that the path $p : e(1) \to \cdots \to e(l+1)$ has precisely t right subpaths of positive length ending in the starting vertex $e(1)$ of

p, say $w_1, \ldots, w_t$. Then our system will have the t linear variables $Z_1, \ldots, Z_t$. Start by considering the following equations $E(\alpha, u)$ in $\mathfrak{K}\Gamma[X, Y, Z]$, one for each detour (α, u) on p:

$$E(\alpha, u): \qquad \alpha u \left(e(1) + \sum_{j=1}^{t} Z_j w_j \right) = \sum_{i \in I(\alpha,u)} X_i(\alpha, u) v_i(\alpha, u) \left(e(1) + \sum_{j=1}^{t} Z_j w_j \right);$$

Now expand both sides of these equations by successively inserting from the right the substitution equations $\beta v \mathrel{\widehat{=}} \sum_{i \in I(\beta,v)} Y_i(\beta, v) v_i(\beta, v)$ for detours (β, v) on p, and the equivalences $q \mathrel{\widehat{=}} 0$ for those paths $q \in \mathfrak{K}\Gamma$ which fail to be routes on p. By Observation 1.1.5, the equation $E(\alpha, u)$ will eventually take on the form

$$\sum_{i \in I(\alpha,u)} a_i(X, Y, Z) v_i(\alpha, u) = \sum_{i \in I(\alpha,u)} b_i(X, Y, Z) v_i(\alpha, u)$$

for suitable polynomials $a_i(X, Y, Z), b_i(X, Y, Z) \in \mathfrak{K}[X, Y, Z]$ which are uniquely determined by the left-hand and right-hand sides of equation $E(\alpha, u)$. Now collect all of the equations of the form $a_i(X, Y, Z) = b_i(X, Y, Z), i \in I(\alpha, u)$, arising in this way for arbitrary detours (α, u) on p, and label the resulting system $S_p(X, Y, Z)$. Observe that this system is polynomial in the X_j and Y_j, and linear in the Z_j.

Theorem 1.2.1. *[19, Theorem B] For $\underline{k}, \underline{k}' \in V_p$, the linear system $S_p(\underline{k}, \underline{k}', Z)$ in $Z = (Z_1, \ldots, Z_t)$ is consistent if and only if $\Phi_p(\underline{k}) \cong \Phi_p(\underline{k}')$.*

1.3 Change of Variable

An important tool that we will use throughout this dissertation is the "change of variable" (see [27]).

An *automorphism of the quiver* Γ is a pair of bijective maps, denoted by the same letter σ, $\Gamma_0 \xrightarrow{\sigma} \Gamma_0$ and $\Gamma_1 \xrightarrow{\sigma} \Gamma_1$, so that if α is an arrow from v to w ($v, w \in \Gamma_0$) then α^σ is an arrow from v^σ to w^σ.

An automorphism of the quiver induces an automorphism of the path algebra $\mathfrak{K}\Gamma$, we denote it also by σ. If now I is an admissible ideal in $\mathfrak{K}\Gamma$, then I^σ is an admissible ideal and σ induces an isomorphism $\mathfrak{K}\Gamma/I \cong \mathfrak{K}\Gamma/I^\sigma$. This will be referred to as an isomorphism *induced from the quiver.* Recall that $\mathcal{J}$ is the ideal in $\mathfrak{K}\Gamma$ generated by arrows.

Definition 1.3.1. *A* change of variables *in $\mathfrak{K}\Gamma$ is an algebra homomorphism $f\colon \mathfrak{K}\Gamma \to \mathfrak{K}\Gamma$ with the following conditions:*

(i) f induces the identity in Γ_0.

(ii) If $v\Gamma_1 w = \{\alpha_1, \dots, \alpha_{m_{vw}}\}$ for $v, w \in \Gamma_0$, then $f(\alpha_l) = \sum_{i=1}^{m_{vw}} k_{il}\alpha_i$ (modulo $\mathcal{J}^2$) for each l, where the matrix of coefficients $M = (k_{il}) \in M_{m_{vw}}(\mathfrak{K})$ is invertible.

Saorín's Theorem 1.3.1. *[27, Theorem 3] Let $\Lambda = \mathfrak{K}\Gamma/I$ be an algebra of Loewy length L, with I admissible. Suppose that $\varphi : \mathfrak{K}\Gamma/I \to \mathfrak{K}\Gamma/I'$ is a homomorphism of $\mathfrak{K}$-algebras. Then the following are equivalent:*

(a) φ is an isomorphism of $\mathfrak{K}$-algebras such that $\varphi((\Gamma_0 + I)/I) = (\Gamma_0 + I')/I'$.

(b) $\mathcal{J}^m \subseteq I'$ for some $m \geq L$, and there are an automorphism σ of $\mathfrak{K}\Gamma$ induced from the quiver and a change of variables f in $\mathfrak{K}\Gamma$ such that $I = \sigma^{-1}(f^{-1}(I'))$ and φ is induced by $f \circ \sigma$.

(c) There are an automorphism σ of $\mathfrak{K}\Gamma$ induced from the quiver and a change of variables f in $\mathfrak{K}\Gamma$ such that $I' = f(I^\sigma) + \mathcal{J}^m$, for some $m \geq L$, and φ induced by $f \circ \sigma$.

1.4 Left multiserial algebras

Uniserial representations of left multiserial algebras have been studied by B. Jue in [23]. Here we recall a result of [23].

Definition 1.4.1. *Let $m \in \mathbb{Z}_+$. An algebra Λ is called* left multiserial *(resp.* m-multiserial*) if, for each primitive idempotent e of Λ, the left ideal Je is a sum of uniserial (resp. m uniserial) Λ-modules.*

We will use the following remark.

Remark 1.4.2. *[23, Remark 2.3]* If Λ is a left multiserial algebra with quiver Γ, then there exists an ideal I of $\mathfrak{K}\Gamma$ generated by a set of relations such that $\Lambda \cong \mathfrak{K}\Gamma/I$, and for every arrow α of Γ, the cyclic $\mathfrak{K}\Gamma/I$-module generated by the residue class of α modulo I is uniserial.

Special biserial algebras and biserial algebras are two important subclasses of left multiserial algebras.

Definition 1.4.3. *An algebra Λ is called a* special biserial algebra *if the following two conditions are satisfied:*

1. *For each vertex $e \in \Gamma_0$, there are at most two arrows beginning in e, and at most two arrows ending in e.*
2. *For each arrow α of Γ_1, there exists at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.*

Recall that the radical $\mathrm{rad}(M)$ of a module M is the intersection of its maximal submodules.

Definition 1.4.4. *An algebra Λ is said to be a* biserial algebra *if every indecomposable projective left or right Λ-module P contains uniserial submodules U and V such that $U + V = \mathrm{rad}(P)$ and $U \cap V$ is either zero or simple.*

We note that every special biserial algebra is biserial, but it is easy to find examples where the converse fails (see [29]).

1.5 Model biserial algebras

In this section, all algebras are assumed to be finite dimensional $\mathfrak{K}$-algebras where $\mathfrak{K}$ is an algebraically closed field. R. Vila-Freyer and W. Crawley-Boevey [30] describe basic biserial algebras by means of quivers and relations. The following is their terminology. Recall that if α is an arrow, we write $s(\alpha)$ for its starting vertex and $t(\alpha)$ for its terminal vertex. If $l \geq 0$, we write $\mathfrak{K}\Gamma_{\leq l}$ for the quotient of $\mathfrak{K}\Gamma$ by the ideal of all paths of length strictly greater than l. A *bisection* of Γ is a pair (σ, τ) of functions from Γ_1 to $\{+1, -1\}$ such that if α and β are distinct arrows with $s(\alpha) = s(\beta)$ (respectively, $t(\alpha) = t(\beta)$), then $\sigma(\alpha) \neq \sigma(\beta)$ (respectively, $\tau(\alpha) \neq \tau(\beta)$). A quiver has a bisection if and only if it is *biserial*, meaning that for every vertex u there are at most two arrows starting at u and at most two arrows terminating at u. Let Γ be a quiver and (σ, τ) a bisection. We say that a path $\alpha_r \ldots \alpha_1$ is a *good path* if $\sigma(\alpha_i) = \tau(\alpha_{i-1})$ for $1 < i \leq r$. Otherwise we say that it is a *bad path*. The paths of length 0 are good.

Definition 1.5.1. *A* model biserial algebra *is an algebra* $\mathfrak{K}\Gamma_{\leq l}/((\alpha - d_{\alpha x})x)$ *where* $l \geq 1$ *is an integer,* Γ *is a biserial quiver with a bisection* (σ, τ)*, and elements* $d_{\alpha x} \in \mathfrak{K}\Gamma$ *are defined for each bad path* αx *of length two, satisfying*

(C1) $d_{\alpha x}$ *is either zero, or of the form* $k\beta_r \cdots \beta_1$ *with* $k \in \mathfrak{K}^*$, $r \geq 1$ *and* $\beta_r \cdots \beta_1 x$ *is a good path with* $t(\beta_r) = t(\alpha), \beta_r \neq \alpha$*, and*

(C2) If $d_{\alpha x} = k\beta$ *and* $d_{\beta y} = k'\alpha$ *with* $k, k' \in \mathfrak{K}^*$*, then* $kk' \neq 1$.

Theorem 1.5.2. *[30, Corollary 3] Model biserial algebras are biserial, and conversely every basic biserial algebra is isomorphic to a quotient of some model biserial algebra.*

1.6 Grading and covering algebras

In Section 2.3, we study monomial algebras using gradings. The following terminology can be found in [4] and [26]. Let Λ be a $\mathfrak{K}$-algebra and G be a group. A family $\{\Lambda_g\}_{g \in G}$ of $\mathfrak{K}$-vector subspaces of Λ, is called a *G-grading on* Λ, if $\Lambda = \bigsqcup_{g \in G} \Lambda_g$ and $\Lambda_g \Lambda_h \subseteq \Lambda_{gh}$, for all $g, h \in G$. We say that a G-grading of Λ *respects simple modules* if simple Λ-modules are gradable. Suppose Λ is G-graded. The *covering algebra*, Λ_G, as a $\mathfrak{K}$-vector space, is $\bigsqcup_{g,h\in G}\Lambda_{gh^{-1}}$. If $x\in\Lambda_{gh^{-1}}$ and $y\in\Lambda_{ab^{-1}}$, then

$$x.y=\begin{cases} xy\in\Lambda_{gb^{-1}} & \text{if } h=a\\ 0 & \text{if } h\neq a\end{cases} \tag{1}$$

A map $W:\Gamma_1\to G$ is called a *weight function.* Then W induces a G-grading on $\mathfrak{K}\Gamma$ as follows: If $p=\alpha_1\cdots\alpha_n$ is a path in $\mathfrak{K}\Gamma$, we define the weight $W(p)$ of p to be $W(\alpha_1)\cdots W(\alpha_n)$. We set $W(v)=1_G$ for each vertex $v\in\Gamma_0$. Let

$$\mathfrak{K}\Gamma_g=\{\sum_{i=1}^{n}k_ip_i \mid k_i\in\mathfrak{K}, p_i \text{ a path}, W(p_i)=g\}\,.$$

Then $\mathfrak{K}\Gamma=\bigsqcup_{g\in G}\mathfrak{K}\Gamma_g$ is a G-grading of $\mathfrak{K}\Gamma$. An element $\sum_{i=1}^{m}k_ip_i\in\mathfrak{K}\Gamma_g$ is called *homogeneous of weight (or degree)* g. If I is a homogeneous ideal in $\mathfrak{K}\Gamma$, i.e., if I is generated by homogeneous elements, then $\Lambda=\mathfrak{K}\Gamma/I$ has an induced G-grading; which is called the *grading induced by the weight function.*

Suppose that $\Lambda=\mathfrak{K}\Gamma/I$ is G-graded by the weight function $W:\Gamma_1\to G$. The following is an alternative description of the covering algebra Λ_G. Let Γ_W be the following quiver. The vertex set is given by $(\Gamma_W)_0=\Gamma_0\times G$ and we write the elements of $(\Gamma_W)_0$ as v_g if $v\in\Gamma_0$ and $g\in G$. The arrow set is given by $(\Gamma_W)_1=\Gamma_1\times G$ where if $\alpha:v\to v'$ in Γ and $g\in G$, then $(\alpha,g):v_g\to v'_{gW(\alpha)}$. Since I is homogeneous, we may generate I by homogeneous elements $f^{(i)}=\sum_j\alpha_j^{(i)}p_j^{(i)}$. We may assume that for each $f^{(i)}$, the paths $p_j^{(i)}$ have the same starting and terminal points for all j. Fix an i. Homogeneity of $f^{(i)}$ implies that if we fix an element $g\in G$ and we lift each $p_j^{(i)}$ to a path with origin $v_g^{(i)}$ then each lifted path will have the same terminus; namely, $u^{(i)}_{gW(p_1)}$. Let I_W be the ideal in $\mathfrak{K}\Gamma_W$ generated by all lifting of the elements $f^{(i)}$. The following theorems can be found in [4] and [26].

Theorem 1.6.1. *Let $W:\Gamma_1\to G$ be a weight function from the arrow set of a quiver Γ to a group G. Let $\Lambda=\mathfrak{K}\Gamma/I$ where I is a homogeneous ideal in $\mathfrak{K}\Gamma$ where $\mathfrak{K}\Gamma$ is given the weight grading. Then Λ is a G-graded algebra with the grading induced from W and the covering algebra Λ_G is isomorphic to $\mathfrak{K}\Gamma_W/I_W$.*

Theorem 1.6.2. *Let Λ be a finite dimensional $\mathfrak{K}$-algebra and G be a group and assume that Λ is G-graded. The following are equivalent.*

1. *The G-grading of Λ respects simple modules.*

2. *If Γ is the quiver of Λ, then there is a presentation $\varphi : \mathfrak{K}\Gamma \longrightarrow \Lambda$ of Λ, and a weight function $W : \Gamma_1 \longrightarrow G$ such that $\ker(\varphi)$ is a homogeneous ideal in the weight grading of $\mathfrak{K}\Gamma$ and such that $\mathfrak{K}\Gamma/\ker(\varphi)$ is isomorphic to Λ as G-graded $\mathfrak{K}$-algebras where $\mathfrak{K}\Gamma/\ker(\varphi)$ is given the G-grading induced by the weight function W.*

1.7 Almost split sequences and irreducible maps

Almost split sequences are a special type of short exact sequences of modules. These play a central role in the representation theory of finite dimensional algebras. In this section, we mention some of the basic definitions and well-known results (see [3]).

A non-split short exact sequence in mod-Λ

$$(*) \qquad 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

is called an *almost split sequence* if (a) A and C are indecomposable and (b) any morphism $X \longrightarrow C$ which is not a split epimorphism factors through f.

Auslander and Reiten proved the following existence and uniqueness theorem of almost split sequences.

Theorem 1.7.1. *Let C be an indecomposable nonprojective module (or A indecomposable noninjective). Then there is an almost split sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$, which is uniquely determined by C (or A, resp.) up to isomorphism.*

It is interesting that in an almost split sequence there is a nice relationship between the end terms, given by $A \cong DTrC$ and $C \cong TrDA$. Here D denotes the ordinary duality, that is $D = \mathrm{Hom}_{\mathfrak{K}}(-, \mathfrak{K})$, and Tr denotes the transpose, that is, if $P_1 \xrightarrow{h} P_0 \longrightarrow C \longrightarrow 0$ is a minimal projective presentation for C in mod-Λ, the TrC is the cokernel of $\mathrm{Hom}_\Lambda(h, \Lambda) : \mathrm{Hom}_\Lambda(P_0, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(P_1, \Lambda)$.

A morphism $g : B \longrightarrow C$ in mod-Λ is called *irreducible* if g is neither a split monomorphism nor a split epimorphism, and if $g = ts$ for some $s : B \longrightarrow X$ and

$t : X \longrightarrow C$, then s is a split monomorphism or t is a split epimorphism. Irreducible maps are closely connected to almost split sequences. Let $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ be an almost split sequence. The end terms A and C are indecomposable, however, the middle term B will usually decompose. If we decompose the middle term $B = \coprod_{i=1}^{m} B_i$, with B_i indecomposable, and we rewrite the above sequence in the form

$$0 \longrightarrow A \xrightarrow{(g_i)_i} \coprod_{i=1}^{m} B_i \xrightarrow{(f_i)_i} C \longrightarrow 0 .$$

Then the f_i and g_i are irreducible maps. The number of indecomposable summands in a sum decomposition of B is an important invariant of C, which we denote by $\alpha(C)$. Also, we denote by $\alpha(\Lambda)$ the numerical invariant which is the supremum of the $\alpha(C)$ for C indecomposable and not projective.

Chapter 2

Uniserial representations of monomial algebras

In this chapter we study geometric aspects of monomial algebras and the algebras isomorphic to them in terms of uniserial representations. In Section 1, we describe monomial algebras using each affine variety V_p corresponding to the uniserial modules with mast p. Then we will give a necessary condition for an algebra to be isomorphic to a monomial algebra. In Section 2, we study a class of algebras containing constricted algebras and we will show that in this class being isomorphic to monomial means being monomial. In Section 3, we prove a theorem analogous to the main theorem of [4, Theorem 2.1], using uniserial modules.

2.1 Monomial algebras

In this section, we will show that an algebra Λ is a monomial algebra if and only if for each nonzero path $p \in \Lambda$, $\underline{0} \in V_p$. Then we will give a necessary condition for an algebra to be isomorphic to a monomial algebra. We will show that if the algebra is isomorphic to a monomial algebra, then for each path in a special basis for the algebra, the corresponding variety is nonempty. We need the following lemmas.

Lemma 2.1.1. *Let $\Lambda = \mathfrak{K}\Gamma/I$ and p be a path nonzero in Λ. If r is a path, which is not a right subpath of p; then $r \mathrel{\widehat{=}} \sum_{p=\bullet p'} \tau_{p'}(X)p'$, where $\tau_{p'}(X)$ has zero constant term.*

Proof. If r is a non-route on p then $r \mathrel{\widehat{=}} 0$ and $\tau_{p'} = 0$. Suppose r is a route on p, then $r = r'\alpha u$ where $(\alpha, u) \wr\wr p$, since r is not a right subpath of p. By substituting the equation $\alpha u \mathrel{\widehat{=}} \sum_{i \in I(\alpha,u)} X_i(\alpha, u)v_i(\alpha, u)$ into r from the right hand side, we get

$$r = r'\alpha u \mathrel{\widehat{=}} \sum_{i \in I(\alpha,u)} X_i(\alpha, u)r'v_i(\alpha, u)\,. \tag{2}$$

We continue to insert the substitution equations into r from the right and repeating this procedure on the paths $r'v_i(\alpha, u)$ which are not right subpaths of p, leads to the equivalence $r \mathrel{\widehat{=}} \sum_{p=\bullet p'} \tau_{p'}(X)p'$. But each $\tau_{p'}(X)$, has factor $X_i(\alpha, u)$ for some $i \in I(\alpha, u)$. ∎

Lemma 2.1.2. *Suppose that $\Lambda = \mathfrak{K}\Gamma/I$ is a monomial algebra. Then I, as a left ideal, is generated by finitely many paths.*

Proof. Let $I = \langle p_1, \ldots, p_m \rangle$, where p_i are paths. Let $\mathcal{J}$ be the ideal of $\mathfrak{K}\Gamma$ generated by the arrows. Since Λ is finite dimensional algebra, $\mathcal{J}^l \subseteq I \subseteq \mathcal{J}^2$ for some $l \geq 2$. Let

$$\mathfrak{A} = \{p_i q \mid q \text{ is path with } p_i q \neq 0 \text{ in } \mathfrak{K}\Gamma \text{ and } l(p_i q) < l\}\,.$$

Then $\mathfrak{A}$ is finite and $I = \Lambda\mathfrak{A} + \mathcal{J}^l$ and so $\mathfrak{A}$ is a left generating set for I. ∎

It is well-known that the algebra is monomial if and only if any reduced Gröbner basis of I consist only of paths [15]. First, we give another characterization of these algebras using the geometry of uniserial modules.

Theorem 2.1.3. *An algebra $\Lambda = \mathfrak{K}\Gamma/I$ is a monomial algebra if and only if $\underline{0} \in V_p$ for every path p in Γ, nonzero in Λ.*

Proof. ($\Rightarrow$) Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a monomial algebra. By Lemma 2.1.2, I is generated, as a left ideal, by finitely many paths, say $r_1, \ldots, r_m$. By Lemma 2.1.1, $r_i \mathrel{\widehat{=}} \sum_{p=\bullet p'} \tau_{i,p'}(X)p'$, where each $\tau_{i,p'}(X)$ has zero constant term. Therefore

$$\underline{0} \in V_p = V(\tau_{i,p'} \mid 1 \leq i \leq m, p = \bullet p').$$

($\Leftarrow$) Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is not monomial. Pick $\lambda_1 p_1 + \cdots + \lambda_n p_n \in I$ with $\lambda_i \in \mathfrak{K}$ and p_i are paths in $\mathfrak{K}\Gamma$ such that $n > 1$ and no linear combination of $p_1, \ldots, p_n$ with fewer than n terms is in I. Suppose p_1 is of minimal length among them. Let us look at V_{p_1}. If all p_i are non-routes on p_1, then $V_{p_1} = \emptyset$. Hence $\lambda_1 p_1 + \cdots + \lambda_n p_n \mathrel{\widehat{=}} \lambda_1 p_1 + \lambda_{1_1} p_{1_1} + \cdots + \lambda_{1_s} p_{1_s}$ where $p_{1_1}, \ldots, p_{1_s}$ are routes on p_1 and $s \geq 1$. Since p_1 is of minimal length, $\text{length}(p_{1_i}) = \text{length}(p_1)$ for all i such that $1 \leq i \leq s$. By Lemma 2.1.1, $p_{1_i} \mathrel{\widehat{=}} \sum_{p_{1_i}=\bullet p'} \tau_{i,p'}(X)p'$, where $\tau_{i,p'}(X)$ has zero constant term. Therefore the coefficient of p_1 in $\lambda_1 p_1 + \lambda_{1_1} p_{1_1} + \cdots + \lambda_{1_s} p_{1_s}$ is $\lambda_1 + f(X)$, where $f(\underline{0}) = 0$. Hence $\underline{0} \notin V_{p_1}$. ∎

An open problem in the theory of representations of algebras [3, Problem 5] is to characterize those algebras isomorphic to monomial algebras. Here we give a necessary condition for an algebra to be isomorphic to a monomial algebra using the uniserial representations. However, it is not sufficient as we will show. See also Sections 2.2 and 2.3. Recall that $J = \text{rad}(\Lambda)$.

The following definition will be used throughout this chapter.

Definition 2.1.4. *A path p of length l in Γ is called a* level path *in Λ, if $p \notin J^{l+1}$.*

Before we state the main results of this section, we study some properties of level paths.

Proposition 2.1.5. *If $\Lambda = \mathfrak{K}\Gamma/I$ and I is length homogeneous, then every path nonzero in Λ is level in Λ.*

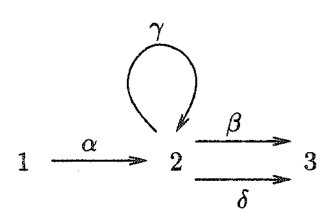
Proof. Suppose that there is a path $p \notin I$ of length m which is not level in Λ. Hence $p \in J^{m+1}$ where $J = \text{rad}(\Lambda)$. Then $p + I = \sum_{i=1}^{t} k_i q_i + I$ where $k_i \in \mathfrak{K}$ and length(q_i) $\geq m+1$. Thus, $p - \sum_{i=1}^{t} k_i q_i \in I$. Therefore, $p \in I$ since I is length homogeneous. ∎

The converse of the above proposition is not true in general as we will show in an example; however, it is true for one particular type of algebra, called a *binomial algebra*, for example in J. K. Sklar [28].

Proposition 2.1.6. *Suppose $\Lambda = \mathfrak{K}\Gamma/I$, I generated by a set of the form $\rho = \{p_i + k_i q_i \mid i = 1, \ldots, l,\ p_i, q_i$ paths, $k_i \in \mathfrak{K}\}$ and every path nonzero in Λ is level in Λ, then I is length homogeneous.*

Proof. Let $p_i + k_i q_i \in \rho$ and suppose $p_i \notin I$. We will prove that length(p_i) = length(q_i). Otherwise, we may assume $m =$ length(p_i) < length(q_i). Then $p_i \in J^{m+1}$ and therefore p_i is not level in Λ. ∎

Example 2.1.7. Let $\Lambda = \mathfrak{K}\Gamma/I$ where Γ is the quiver



and $I = < \gamma^3,\ \delta\alpha + \beta\alpha + \beta\gamma\alpha + \delta\gamma\alpha >$. Then every path in Γ nonzero in Λ is level in Λ, but I is not length homogeneous. ♣

The rest of the chapter will revolve around the concept in the following definition.

Definition 2.1.8. *We say that an algebra $\Lambda = \mathfrak{K}\Gamma/I$* satisfies the V_p-condition *if for each level path p in Λ, $V_p \neq \emptyset$.*

Theorem 2.1.9. *Suppose Λ is isomorphic to a monomial algebra. Then Λ satisfies the V_p-condition.*

Proof. Let $\varphi : \Lambda = \mathfrak{K}\Gamma/I \rightarrow \Lambda' = \mathfrak{K}\Gamma/I'$ be an isomorphism, where Λ' is a monomial algebra and suppose p is a level path in Λ. Let $\mathcal{J}$ be the ideal of $\mathfrak{K}\Gamma$ generated by Γ_1. By Saorín's Theorem (see Section 1.3), there is an automorphism σ of $\mathfrak{K}\Gamma$ induced from the quiver and a change of variable f in $\mathfrak{K}\Gamma$ such that $I = \sigma^{-1}(f^{-1}(I'))$ and $\varphi(r+I) = f\sigma(r) + I'$ for any $r \in \mathfrak{K}\Gamma$. We can assume that σ is the identity, because $\mathfrak{K}\Gamma/\sigma^{-1}(I')$ is also a monomial algebra. Assume, then, that $\Lambda' = \mathfrak{K}\Gamma/\sigma^{-1}(I')$ and so by Theorem 2.1.3, $V_p \neq \emptyset$ for all paths p, nonzero in Λ'.

We can consider any Λ-module M (respectively any Λ'-module M') as a Λ'-module (respectively Λ-module) via $\lambda'.x = \varphi(\lambda')x$ for $\lambda' \in \Lambda'$ and $x \in M$ (respectively $\lambda.y = \varphi^{-1}(\lambda)y$ for $\lambda \in \Lambda, y \in M'$). Then U is a uniserial Λ-module if and only if U is a uniserial Λ'-module. Let $p = \alpha_l \cdots \alpha_1$ be a level path with arrows $\alpha_t : i_t \rightarrow i_{t+1}$ for $1 \leq t \leq l$. Hence

$$f(p) = \lambda_1 p_1 + \cdots + \lambda_m p_m + \sum_{i=1}^{r} t_i q_i, \quad \lambda_j, t_i \in \mathfrak{K},$$

where $p_1, \ldots, p_m$ are paths passing in order through the vertices $(i_1, i_2, \ldots, i_{l+1})$ and $q_1, \ldots, q_r$ are paths with length$(q_i) >$ length(p). Since $p + I \in J^l \backslash J^{l+1}, \varphi(p+I) \in J^l \backslash J^{l+1}$. Therefore $\lambda_i \neq 0$ for some $1 \leq i \leq m$ with $p_i \notin I'$. Suppose $\lambda_1 \neq 0$ and $p_1 \notin I'$. By Theorem 2.1.3, $\underline{0} \in V_{p_1}$. Hence,

$$U = \frac{\Lambda' e_{i_1}}{\sum_{(\alpha,u)\wr p_1} \Lambda' \alpha u + \sum_{q \text{ non-route on } p} \Lambda' q}$$

is a uniserial Λ'-module with mast p_1.

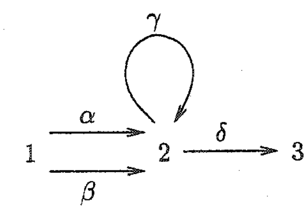
In order to complete the proof, it suffices to show that U, as a Λ-module, is a uniserial module with mast p. We have

$$p.U = \varphi(p)U = (\lambda_1 p_1 + \cdots + \lambda_m p_m + \sum_{i=1}^{t} t_i q_i)U\,.$$

But $p_2, \ldots, p_m \in \sum_{(\alpha,u)\wr p_1} \Lambda' \alpha u$. Hence $p_2 U = \cdots = p_m U = 0$. Therefore $p.U = \varphi(p)U = \lambda_1 p_1 U \neq 0$. ∎

The following example shows that the condition that p is a level path in the above theorem is necessary.

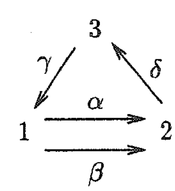
Example 2.1.10. Let Γ be the quiver



and $\Lambda = \mathfrak{K}\Gamma/I$ where $I = \langle \gamma^3, \delta\gamma^2\alpha - \delta\gamma\beta \rangle$. Let $\Lambda' = \mathfrak{K}\Gamma/I'$ with $I' = \langle \gamma^3, \delta\gamma\beta \rangle$. We show that $\Lambda \cong \Lambda'$. Define $f : \mathfrak{K}\Gamma \to \mathfrak{K}\Gamma$, by $f(\alpha) = \alpha + \beta$ and $f(\beta) = \beta + \gamma\alpha + \gamma\beta$, fixing γ and δ. By (1.3.1), f induces an isomorphism $\varphi : \Lambda \cong \Lambda'$. But for $p = \delta\gamma\beta$, which is nonzero in Λ but not a level path in Λ, $V_p = \emptyset$; because $0 = \delta\gamma^2\alpha - \delta\gamma\beta \mathrel{\widehat{=}} -p$. ♣

The following example shows that the converse of Theorem 2.1.9 fails.

Example 2.1.11. Let Γ be the quiver



and let $\Lambda = \mathfrak{K}\Gamma/I$ where $I = \langle \alpha\gamma\delta\beta - \beta\gamma\delta\alpha,\ \text{paths of length } 5 \rangle$. We will show that $V_p \neq \emptyset$ for all p nonzero in Λ, and that Λ is not isomorphic to any monomial algebra. If length$(p) \leq 3$ or $s(p) \neq e_1$ then $V_p \neq \emptyset$ since $r = \alpha\gamma\delta\beta - \beta\gamma\delta\alpha \mathrel{\widehat{=}} 0$. Suppose length$(p) = 4$ and $s(p) = e_1$. Then $p = \alpha\gamma\delta\beta$ or $\beta\gamma\delta\alpha$. Suppose $p = \beta\gamma\delta\alpha$. The detours on p are (β, e_1) and $(\alpha, \gamma\delta\alpha)$. Inserting the substitution equations

$$\beta \mathrel{\widehat{=}} X_1\alpha + X_2 p \quad \text{and} \quad \alpha\gamma\delta\alpha \mathrel{\widehat{=}} X_3 p\,.$$

into the relations yields $V_p = V(X_1X_3 - 1)$.
Suppose $\varphi : \Lambda \to \Lambda'$ is an isomorphism. By (1.3.1),

$$\varphi(\alpha) = k\alpha + k'\beta + w_1$$
$$\varphi(\beta) = t\alpha + t'\beta + w_2\,,$$

where $w_1, w_2 \in J^2$ and $k, k', t, t' \in \mathfrak{K}$ with $kt' - k't \neq 0$. Then

$$0 = \varphi(0) = \varphi(\alpha\gamma\delta\beta - \beta\gamma\delta\alpha) = (kt' - k't)(\alpha\gamma\delta\beta - \beta\gamma\delta\alpha)\,.$$

Therefore Λ' is not a monomial algebra. ♣

The following example shows that Theorem 2.1.9 is a useful negative criterion.

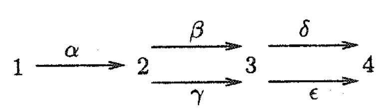
Example 2.1.12. Let Γ be the quiver in Example 2.1.11 and $\Lambda_1 = \mathfrak{K}\Gamma/I_1$ where

$$I_1 = \langle \alpha\gamma\delta\beta - \beta\gamma\delta\alpha, \alpha\gamma\beta\alpha, \text{ paths of length } 5\rangle\ .$$

Let $p = \beta\gamma\delta\alpha$. The detours on p are (β, e_1) and $(\alpha, \gamma\delta\alpha)$ and hence the substitution equations for p are $\beta \mathrel{\widehat{=}} X_1\alpha + X_2p$ and $\alpha\gamma\delta\alpha \mathrel{\widehat{=}} X_3p$ as well as $q \mathrel{\widehat{=}} 0$ whenever q fails to be a route on p, and so the relations $\alpha\gamma\delta\beta - \beta\gamma\delta\alpha$ and $\alpha\gamma\beta\alpha$ yield $X_1X_3 - 1 = 0$ and $X_3 = 0$, whence $V_p = \emptyset$. Because $p \in J^4 \backslash J^5$, by Theorem 2.1.9, Λ_1 is not isomorphic to a monomial algebra. ♣

The following example shows that the V_p-condition is not preserved under isomorphisms.

Example 2.1.13. Let $\Lambda = \mathfrak{K}\Gamma/I$ where $\mathrm{Char}(\mathfrak{K}) \neq 2, 5$ and Γ is the quiver



and $I = \langle \delta\beta\alpha + 3\epsilon\beta\alpha - 3\delta\gamma\alpha - 3\delta\gamma\alpha + \epsilon\gamma\alpha,\ \ \delta\beta\alpha - 2\epsilon\beta\alpha - \delta\gamma\alpha + 2\epsilon\gamma\alpha\rangle$. Define a homomorphism $f : \mathfrak{K}\Gamma \to \mathfrak{K}\Gamma$ by

$$\beta \mapsto (1/2)\beta + (1/2)\gamma \qquad \delta \mapsto (2/5)\delta + (1/5)\epsilon$$
$$\gamma \mapsto -(1/2)\beta + (1/2)\gamma \qquad \epsilon \mapsto (1/5)\delta - (2/5)\epsilon$$

By Saorín's Theorem (see Section 1.3), f induces an isomorphism $\varphi : \Lambda \cong \Lambda' = \mathfrak{K}\Gamma/I'$ with $I' = \langle \delta\beta\alpha - \epsilon\gamma\alpha, \epsilon\beta\alpha\rangle$. We can see that for any path p nonzero in Λ, we have $V_p \neq \emptyset$, but for $q = \delta\beta\alpha$ in Λ', $V_q = \emptyset$. We only show the second part. The detours on p are (γ, α) and $(\epsilon, \beta\alpha)$ and hence the substitution equations for p are $\gamma\alpha \mathrel{\widehat{=}} X_1\beta\alpha$ and $\epsilon\beta\alpha \mathrel{\widehat{=}} X_2p$. Then the relations $\alpha\beta\alpha - \epsilon\gamma\alpha$ and $\epsilon\beta\alpha$ yield $1 - X_1X_2 = 0$ and $X_2 = 0$, whence $V_p = \emptyset$. ♣

2.2 Loosely constricted algebras

In this section, we study a class of algebras containing constricted algebras; we call them *loosely constricted algebras*. We will show that if an algebra is in this class and it is isomorphic to a monomial algebra then the algebra itself is monomial.

An algebra Λ is *square free*([1]) (or *shurian*) in case for every pair e, f of primitive idempotents in Λ, $\dim_{\mathfrak{K}}(e\Lambda f) \leq 1$. Here we are interested in a class of algebras containing square free algebras, the constricted algebras. An algebra $\Lambda = \mathfrak{K}\Gamma/I$ is called *constricted*([4]) if $\dim_{\mathfrak{K}} e\Lambda e = 1$ for each vertex $e \in \Gamma_0$ and $\dim_{\mathfrak{K}} t(\alpha)\Lambda s(\alpha) = 1$ for each arrow $\alpha \in \Gamma_1$. We call an algebra *weakly constricted* if it satisfies the latter condition, i.e., $\dim_{\mathfrak{K}} t(\alpha)\Lambda s(\alpha) = 1$ for each arrow $\alpha \in \Gamma_1$. These algebras have been studied by Bardzell and Marcos [5]. We note that if $\Lambda = \mathfrak{K}\Gamma/I$ is weakly constricted then $\mathfrak{K}\Gamma$ is square free. We say Γ is *square free* if for vertices $e, f \in \Gamma_0$, there exists at most one arrow from e to f in Γ.

Remark 2.2.1. *The above classes of algebras are related as follows:*

$$\mathfrak{K}\Gamma/I \textit{ is square free} \Rightarrow \mathfrak{K}\Gamma/I \textit{ is constricted} \Rightarrow \mathfrak{K}\Gamma/I \textit{ is weakly constricted} \Rightarrow \Gamma \textit{ is square free}$$

We have the following:

Proposition 2.2.2. *If $\Lambda = \mathfrak{K}\Gamma/I$ is weakly constricted algebra then for each path $p \in \Gamma$, nonzero in Λ, p does not have any detours.*

Proof. Suppose $(\alpha, u) \wr p$. Then $p = p'v'u$ with $t(v') = t(\alpha)$. Hence $v', \alpha \in t(\alpha)\Lambda s(\alpha)$. Therefore $\dim_{\mathfrak{K}} t(\alpha)\Lambda s(\alpha) > 1$, since $I \subseteq \mathcal{J}^2$. ∎

Recall that p is a level path in Λ, if $p \in J^m \backslash J^{m+1}$ where $m = \text{length}(p)$.

Definition 2.2.3. *We say that an algebra $\Lambda = \mathfrak{K}\Gamma/I$ is* loosely constricted *if for each level path p in Λ, p has no detours.*

Note that if $\Lambda = \mathfrak{K}\Gamma/I$ is loosely constricted then Γ is square free. This class of algebras lies strictly between the weakly constricted ones and those with Γ square free, as we will see. We will show that being loosely constricted is independent of presentation. We need the following lemma.

Lemma 2.2.4. *Suppose $\Lambda = \mathfrak{K}\Gamma/I \cong \Lambda' = \mathfrak{K}\Gamma/I'$, where Γ is square free. Then p is level in Λ if and only if p is level in Λ'.*

Proof. Let $\varphi : \Lambda \longrightarrow \Lambda'$ be an isomorphism. Then $\varphi^{-1}(p + I') = kp + w + I$, where $0 \neq k \in \mathfrak{K}$ and w is a linear combination of paths longer than p, since Γ is square free. Suppose p is not level in Λ'. Then $p - \sum_{i=1}^{n} k_i p_i \in I'$, where $\text{length}(p_i) \geq m + 1$. Hence $\varphi^{-1}(p - \sum_{i=1}^{n} k_i p_i + I') = 0$. Thus $kp + w + I = \sum_{i=1}^{n} k_i \varphi^{-1}(p_i + I')$. But this implies that p is not level in Λ. ∎

Proposition 2.2.5. *If $\Lambda \cong \Lambda'$ and Λ is loosely constricted, then so is Λ'.*

Proof. Let $\varphi : \Lambda \longrightarrow \Lambda'$ be an isomorphism. Without loss of generality, we can assume that $\Lambda = \mathfrak{K}\Gamma/I$ and $\Lambda' = \mathfrak{K}\Gamma/I'$. We know that Γ is square free. Let p be a level path in Λ' and assume that p has a detour. Then $\varphi^{-1}(p + I') = kp + w + I$ where $k \in \mathfrak{K}$ and w is a $\mathfrak{K}$-linear combination of paths longer than p, since Γ is square free. But the image of p in Λ is zero, otherwise p would have a detour and be level in Λ, by Lemma 2.2.4. Hence $\varphi^{-1}(p + I') = w + I \in J^{m+1}$, which is a contradiction. ∎

Lemma 2.2.6. *If e, f are primitive idempotents, then $e\Lambda f$ is zero or is spanned by level paths.*

Proof. Let $0 \neq p \in e\Lambda f$ be a path of length m. First we show that $p = \sum_{i=1}^{n} k_i p_i$, where at least one of p_i is level in $e\Lambda f$. If p is not level, then $p \in J^{m+1}$. Suppose $p \in J^l \backslash J^{l+1}$. Thus $p = \sum_{i=1}^{n} k_i p_i$, where $k_i \in \mathfrak{K}$ and $\text{length}(p_i) \geq l$. Then at least one of p_i is not in J^{l+1}, otherwise $p \in J^{l+1}$.

Now let us write $p = \sum_{i=1}^{s} k_i p_i + \sum_{i=1}^{n} t_i q_i$ where each p_i is level in $e\Lambda f$ but no q_i is level. Moreover, suppose that s is maximum. Then $n = 0$, because otherwise $q_1 = \sum_{i=1} t_i r_i$ with r_1 a level path. This contradicts the maximality of s. ∎

The next proposition shows the relationship between "constricted" and "loosely constricted".

Proposition 2.2.7. *An algebra Λ is constricted if and only if Λ is loosely constricted and $\dim_{\mathfrak{K}} e\Lambda e = 1$ for each $e \in \Gamma_0$.*

Proof. Suppose that $\Lambda = \mathfrak{K}\Gamma/I$ is loosely constricted and $\dim_{\mathfrak{K}} e\Lambda e = 1$ for each $e \in \Gamma_0$. Suppose that $\alpha : i \longrightarrow j \in \Gamma_1$ and $\dim_{\mathfrak{K}} e_j\Lambda e_i > 1$. By Lemma 2.2.6, there exists a level path, say $p \neq \alpha$, in $e_j\Lambda e_i$. But α is not a right subpath of p, because $\dim_{\mathfrak{K}} e_j\Lambda e_j = 1$ and so (α, e_i) is a detour on p. This is a contradiction, since Λ is loosely constricted. ∎

The following is a generalization of [4, Corollary 4.2] and of [5, Corollary 2.5] and uses the varieties V_p rather than Hochschild cohomology.

Theorem 2.2.8. *Suppose that $\Lambda = \mathfrak{K}\Gamma/I$ is loosely constricted, then the following are equivalent.*

(1) Λ is monomial; i.e., I has a generating set consisting of paths.

(2) Λ is isomorphic to a monomial algebra.

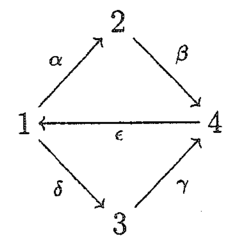
(3) Λ satisfies the V_p-condition.

Proof. We only need to show (3) $\Rightarrow$ (1). Suppose $\Lambda = \mathfrak{K}\Gamma/I$ satisfies the V_p-condition. An element $r = \sum_{i=1}^{m} k_i p_i \in I$ with $k_i \in \mathfrak{K}^*$ and p_i a path, is called non-trivial if no proper subsum is in I. Set

$$\mathcal{A} = \left\{ r = \sum_{i=1}^{m} k_i p_i \in I \mid r \text{ non-trivial and } m > 1 \right\}.$$

If Λ is not monomial then $\mathcal{A} \neq \emptyset$. For $r = \sum_{i=1}^{m} k_i p_i$, we define $\text{length}(r) := \max\{\text{length}(p_i) \mid 1 \leq i \leq m\}$. Pick $r \in \mathcal{A}$ such that $\text{length}(r)$ is maximum. Suppose $r = \sum_{i=1}^{m} k_i p_i$ and $\text{length}(r) = \text{length}(p_1)$. Hence, $p_1 \in J^m \backslash J^{m+1}$, where $m = \text{length}(p_1)$ and so p_1 has no detour. For $2 \leq i \leq m$, if p_i is a route on p_1; then p_i is a right subpath of p_1 by [19, Remark b, page 7] and the fact that p_1 has no detour. Therefore $r \mathrel{\widehat{=}} p_1 + \sum_{j=1}^{l} k_{1_j} p_{1_j}$ where p_{1_j} are right subpaths of p_1. Hence $V_{p_1} = \emptyset$. ∎

Example 2.2.9. Let Γ be the quiver



and let $\Lambda = \mathfrak{K}\Gamma/I$ where $I =< \beta\alpha - \gamma\delta, \alpha\epsilon\beta\alpha, \delta\epsilon\beta\alpha >$. Then Λ is a non-monomial loosely constricted algebra which is not a weakly constricted algebra. Let $\Lambda' = \mathfrak{K}\Gamma/I'$, where $I' =< \alpha\epsilon\beta\alpha, \delta\epsilon\beta\alpha, \epsilon\gamma\delta >$. Then Λ' is a loosely constricted monomial algebra which is not a weakly constricted algebra.

2.3 Monomial algebras and gradings

In this section, we will prove an analogous version of the main theorem in [4] using the results in previous sections. The use of uniserial representations rather than the Hochschild cohomology, an important part of the result in [4], seems more natural. Let Λ be a finite dimensional $\mathfrak{K}$-algebra such that simple modules have $\mathfrak{K}$-dimension 1. (All our algebras satisfy this condition.) Let G be a group and assume that Λ is G-gradable. Then there is a weight function $W : \Gamma_1 \to G$ and a presentation $\Lambda \cong \mathfrak{K}\Gamma/I$, where I is W-homogenous (see Section 1.6). Assume that $\Lambda = \mathfrak{K}\Gamma/I$. Consider the corresponding covering algebra $\Lambda_G = \mathfrak{K}\Gamma_W/I_W$ and the projection map $\pi : \mathfrak{K}\Gamma_W \to \mathfrak{K}\Gamma$ with $\pi(\alpha_g) = \alpha, \pi(e_{i_g}) = e_i$. Let $\rho : \Lambda_G \to \Lambda$ be the corresponding covering map. First, we show that there is a correspondence between the level paths in the algebra and in the corresponding covering algebra. Let $J = \mathrm{rad}(\Lambda)$ and $J_G = \mathrm{rad}(\Lambda_G)$.

Proposition 2.3.1. *p' is level in Λ_G if and only if $\pi(p')$ is level in Λ.*

Proof. Let $\pi(p') = p$ and $\mathrm{length}(p) = \mathrm{length}(p') = m$. Suppose p' is not level in Λ_G. Then $p' \in {J_G}^{m+1}$ and so $p' - \sum_{i=1}^{n} k_i p'_i \in I_W$ where $k_i \in \mathfrak{K}$ and $p'_i \in \mathfrak{K}\Gamma_W$ with $\mathrm{length}(p'_i) \geq m + 1$. Hence $\pi(p' - \sum_{i=1}^{n} k_i p'_i) = p - \sum_{i=1}^{n} k_i \pi(p'_i) \in I$. But length($\pi(p_i')$) = length($p_i'$) $\geq m+1$, implies $p \in J^{m+1}$ and therefore p is not level in Λ.

Now suppose that $\pi(p')$ is not level in Λ. Then $\pi(p') - \sum_{i=1}^{n} k_i p_i \in I$, where $k_i \in \mathfrak{K}$ and $p_i \in \mathfrak{K}\Gamma$ with length(p_i) $\geq m+1$. Thus by the definition of I_W, $p' - \sum_{i+1}^{n} k_i p_i' \in I_W$, with $\pi(p_i') = p_i$. But length(p_i') = length(p_i) $\geq m+1$, and so p' is not level in Λ_G. ∎

Proposition 2.3.2. *If Λ satisfies the V_p-condition, then so does Λ_G.*

Proof. Suppose that p' is level in Λ_G. We need to show that $V_{p'} \neq \emptyset$. Let $p = \pi(p')$. By Proposition 2.3.1, p is level in Λ and so $V_p \neq \emptyset$. Let U be a uniserial Λ-module with mast p. Then U is a Λ_G-module via $\lambda' u = \rho(\lambda')u$, where $\lambda' \in \Lambda_G$, $u \in U$ and $\rho : \Lambda_G \to \Lambda$ is the covering map. Moreover, U is a uniserial Λ_G-module with mast p', because $p'U = \rho(p')U = pU \neq 0$. ∎

Proposition 2.3.3. *Suppose Λ is a G-graded $\mathfrak{K}$-algebra where the grading respects simple modules. If Λ satisfies the V_p-condition and Λ_G is loosely constricted then Λ_G and Λ are monomial.*

Proof. By Proposition 2.3.2, Λ_G satisfies the V_p condition and so is a monomial algebra by Theorem 2.1.9. But this implies that Λ is monomial. ∎

Here, we can prove a version of the theorem in [4, Theorem 2.1] using uniserial representations.

Corollary 2.3.4. *Let Λ be a $\mathfrak{K}$-algebra of dimension d with Jacobson radical J. Then the following statements are equivalent.*

(1) Λ is isomorphic to a monomial algebra.

(2) There is a presentation $\Lambda' \cong \Lambda$ of Λ such that Λ' satisfies the V_p-condition and is G-graded, where $G = \prod_{\alpha \in \Gamma_1} \mathbb{Z}/d$ such that the following conditions hold:

(a) Every simple Λ'-module is G-gradable.

(b) The covering algebra Λ'_G is constricted.

Proof. Suppose $\Lambda \cong \Lambda'$, where Λ' is a monomial algebra. Then Λ' is G-graded, with $G = \prod_{\alpha \in \Gamma_1} \mathbb{Z}/d$ satisfying the properties (a) and (b) [4, Theorem 5.1] and Λ' satisfies V_p-condition by the Theorem 2.1.9.

The converse follows from Proposition 2.3.3. ∎

Note: The V_p-condition on Λ' can be replaced by the V_p-condition on Λ'_G by Theorem 2.2.8.

Chapter 3

Extensions of uniserial modules by uniserial modules

Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a triangular algebra; i.e., Γ is a finite quiver with no oriented cycles and I is an admissible ideal. We will assume that U and V are uniserials with masts p and p', respectively, and p and p' pass through the same sequence of vertices. This will be assumed throughout this chapter except in Proposition 3.2.6.

By a theorem of A. Boldt [6, Proposition 1.1.2] an extension of a uniserial U by a uniserial V is either indecomposable or a direct sum of two uniserial modules. However, if Γ has no oriented cycles and $U/JU \cong V/JV$, then any non-trivial extension of U by V is indecomposable (see Theorem 3.1.1). Hence, each nonzero element of $\mathrm{Ext}^1_\Lambda(U, V)$ give rise to an indecomposable module.

We will give an algorithm for a $\mathfrak{K}$-basis of $\mathrm{Ext}^1_\Lambda(U, V)$. The procedure takes on a simpler form for $\mathrm{Ext}^1_\Lambda(U, U)$ where Λ is a monomial algebra. At the end, we give a necessary and sufficient condition for the extension module to have simple top, i.e., to be a local module.

3.1 Extensions of uniserial modules

Before looking at details, note that in the situation just described Boldt's result takes on a special form.

Theorem 3.1.1. *Let Λ be a triangular algebra. Suppose that there is a nonsplit exact sequence of Λ-modules*

$$0 \to V \xrightarrow{f} M \xrightarrow{g} U \to \tag{3}$$

where U and V are uniserial modules with $U/JU \cong V/JV$. Then the middle term M is indecomposable.

Proof. By [6, Proposition 1.1.2], M is either indecomposable or $M \cong V_1 \sqcup V_2$ where V_1 and V_2 are uniserial modules. Assume that $M = V_1 \sqcup V_2$. Let $f = \binom{f_1}{f_2}$ and $g = (g_1, g_2)$. By tensoring the exact sequence (3) by Λ/J, we obtain:

$$V/JV \to V_1/JV_1 \sqcup V_2/JV_2 \to U/JU \to 0\,.$$

Since V, V_1, V_2 and U are uniserial modules, we have

$$V_1/JV_1 \cong V_2/JV_2 \cong V/JV\,.$$

Because $f(\mathrm{soc}(V)) \neq 0$, at least one of $f_1(\mathrm{soc}(V))$ and $f_2(\mathrm{soc}(V))$ is not zero, say $f_1(\mathrm{soc}(V)) \neq 0$. Thus f_1 is a monomorphism. We first prove that $\bar{f_1} \neq 0$, where $\bar{f_1} : V/JV \to V_1/JV_1$ is the induced map from f_1. Let $x \in V/JV$. If $\bar{f_1} = 0$, then $f_1(x) \in JV_1$. Let m be an integer such that $f_1(x) \in J^mV_1 \backslash J^{m+1}V_1$. Then, f_1 induces a map $\bar{\bar{f_1}} \colon V/JV \to J^mV_1/J^{m+1}V_1$ by $\bar{\bar{f_1}}(x + JV) = f_1(x) + J^{m+1}V_1$. Hence, $V/JV \cong J^mV_1/J^{m+1}V_1$ and therefore $V_1/JV_1 \cong J^mV_1/J^{m+1}V_1$, which contradicts the fact that Λ is a triangular algebra unless $m = 0$. Hence, $\bar{f_1} \neq 0$ and so f_1 is an epimorphism. Therefore f_1 is an isomorphism.

Claim 1: g_2 is a monomorphism. Suppose $g_2(v_2) = 0$ for some $v_2 \in V_2$. Thus, $v_2 \in \ker g = \mathrm{im}\, f$. Write $v_2 = f(v) = f_1(v) + f_2(v)$ for some $v \in V$. Thus, $v_2 = f_2(v)$ and $f_1(v) = 0$. Thus, $v = 0$, since f_1 is a monomorphism. Hence $v_2 = f_2(v) = 0$

Claim 2: g_2 is an epimorphism. Let $u \in U$. Then, $u = g_1(v_1) + g_2(v_2)$ for some $v_1 \in V_1$ and $v_2 \in V_2$. Also, $v_1 = f_1(v)$ for some $v \in V$. Thus,

$$u = g_1 f_1(v) + g_2(v_2) = -g_2 f_2(v) + g_2(v_2) = g_2(-f_2(v) + g_2(v_2)) .$$

■

We fix a triangular algebra Λ and assume $U = \Lambda e/K$ and $V = \Lambda e/L$ are uniserial modules with mast p and p', respectively, where p and p' pass through the same sequence of vertices and

$$K = \sum_{(\alpha,u)\wr p} \Lambda\,(\alpha u - k(\alpha,u)v(\alpha,u)) + \sum_{q \text{ nonroute on } p} \Lambda q ,$$

with $k(\alpha,u) \in \mathfrak{K}$ and $v(\alpha,u)$ is the right subpath of p with $t(v(\alpha,u)) = t(\alpha)$.

We have the exact sequence $0 \longrightarrow K \longrightarrow \Lambda e \longrightarrow U \longrightarrow 0$. Hence,

$$\mathrm{Hom}_\Lambda(\Lambda e, V) \xrightarrow{\ \varphi\ } \mathrm{Hom}_\Lambda(K, V) \longrightarrow \mathrm{Ext}^1_\Lambda(U, V) \longrightarrow 0$$

is exact and therefore $\mathrm{Ext}^1_\Lambda(U,V) \cong \frac{\mathrm{Hom}_\Lambda(K,V)}{\mathrm{im}\,\varphi}$.

Lemma 3.1.2. *With the notation above,* $\mathrm{im}\,\varphi = 0$ *if and only if* $K = L$.

Proof. Let $f : \Lambda e \longrightarrow V$ be given by $e \mapsto ke + L$ where $0 \neq k \in \mathfrak{K}$, because the algebra is triangular. Pick $a \in K$. Then $f(a) = k(a + L)$. Hence $f|_K = 0$ implies $K \subseteq L$. However, U and V have the same $\mathfrak{K}$-dimension and hence, $K = L$. The converse is clear. ■

Lemma 3.1.3. *With the notation above,*

(i) If $K = L$, *then* $\mathrm{Ext}^1_\Lambda(U,V) \cong \mathrm{Hom}_\Lambda(K,V)$.

(ii) If $K \neq L$, *then*

$$\dim_{\mathfrak{K}} \mathrm{Ext}^1_\Lambda(U,V) = \dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(K,V) - 1 .$$

Proof. (i) follows from Lemma 3.1.2.
(ii) If $\operatorname{im}\varphi \neq 0$ then, as in Lemma 3.1.2, for $0 \neq f \in \operatorname{Hom}_\Lambda(\Lambda e, V)$, $f(e) = ke + L, 0 \neq k \in \mathfrak{K}$. Thus $\dim_{\mathfrak{K}} \operatorname{im}\varphi = 1$. Therefore,

$$\dim_{\mathfrak{K}} \operatorname{Ext}^1_\Lambda(U, V) = \dim_{\mathfrak{K}} \operatorname{Hom}_\Lambda(K, V) - 1\,.$$

■

Suppose p passes through $1 \to 2 \to \cdots \to n$. We next pick a generating set for K.

Lemma 3.1.4. *Let* $K' := \sum_{(\alpha,u)\wr p} \Lambda\,(\alpha u - k(\alpha,u)v(\alpha,u)) + \sum_{i=1}^{m} \Lambda q_i$*, where* $q_i = \gamma_i u_i$*, with* u_i *a right subpath of* p*,* $\gamma_i \in \Gamma_1$ *and* $t(\gamma_i) \notin \{1, 2, \dots, n\}$*. Then* $K' = K$*.*

Proof. Let $q \in K$ be a non-route on p. Then by Remark 1.1.4, $q = q'\gamma\alpha_m u_m \cdots \alpha_1 u_1$ for some $m \geq 0$ such that there exists a corresponding factorization $p = p' w_m \cdots w_1$ of p with the property that $(\alpha_i, u_i) \wr w_i$ with $t(\alpha_i) = t(w_i)$ for each $0 \leq i \leq m$ and $t(\gamma) \notin \{1, \dots, n\}$. We prove that $q \in K'$ by induction on m. If $m = 1$,

$$q = q'\gamma\alpha_1 u_1 = q'\gamma\,(\alpha_1 u_1 - k(\alpha_1, u_1)v(\alpha_1, u_1)) + k(\alpha_1, u_1)q'\gamma v(\alpha_1, u_1) \in K'\,.$$

If $m > 1$ and we have the result for $m - 1$ and $q = q'\gamma\alpha_m u_m \cdots \alpha_1 u_1$, then

$$q = q'\gamma\alpha_m u_m \cdots \alpha_2 u_2\,(\alpha_1 u_1 - k(\alpha_1, u_1)v(\alpha_1, u_1)) + k(\alpha_1, u_1)q'\gamma\alpha_m u_m \cdots \alpha_2 u_2 v(\alpha_1, u_1)\,.$$

Therefore $q \in K'$ by the induction hypothesis. ■

Lemma 3.1.5. *With the same notation as in Lemma 3.1.4, if* $\Lambda = \mathfrak{K}\Gamma$*, then*

$$K = \bigoplus_{(\alpha,u)\wr p} \Lambda\,(\alpha u - k(\alpha,u)v(\alpha,u)) \oplus \bigoplus_{i=1}^{m} \Lambda q_i\,.$$

Proof. Note that the algebra is hereditary, i.e., there are no relations. Suppose $\sum_{i=1}^{n} \lambda_i\,(\alpha_i u_i - k(\alpha_i, u_i)v(\alpha_i, u_i)) + \sum_{j=1}^{m} \mu_j q_j = 0$, where $\lambda_i, \mu_j \in \Lambda$, $\lambda_i = \lambda_i e_{t(\alpha_i)}$ and

$\mu_j = \mu_j e_{t(q_j)}$.

Claim: For all i, $\lambda_i = 0$.

Otherwise, choose s such that $\lambda_s \neq 0$ and length(u_s) is minimum. Then

$$-\lambda_s(\alpha_s u_s - k(\alpha_s, u_s)v(\alpha_s, u_s)) = \sum_{i \neq s} \lambda_i \left(\alpha_i u_i - k(\alpha_i, u_i)v(\alpha_i, u_i)\right) + \sum_{j=1}^{m} \mu_j q_j \,. \quad (4)$$

There exists a path q such that $q\alpha_s u_s$ appears non-trivially on the left side of (4). Hence,

Case 1: $q\alpha_s u_s = q'\alpha_i u_i$ with $i \neq s$. But then $u_s = u_i$ and $\alpha_s = \alpha_i$, which is a contradiction.

Case 2: $q\alpha_s u_s = q'v(\alpha_i, u_i)$ with $i \neq s$. Then $l(u_i) < l(v(\alpha_i, u_i)) \leq l(u_s)$, which is in contradiction with the minimality of $l(u_s)$.

Case 3: $q\alpha_s u_s = q'q_i$. This contradicts the choice of q_i. (See Lemma 3.1.4.)

Therefore, since the algebra is hereditary, $\lambda_i = 0$ for all i and thus $\sum_{i=1}^{m} \mu_i q_i = 0$. Among all non-trivial expressions $\sum_{i=1}^{m} \mu_i q_i = 0$, choose one so that $m > 1$ is minimal. We have

$$0 \neq \mu_1 q_1 = \sum_{i=2}^{m} -\mu_i q_i \,. \quad (5)$$

There exists a path q such that qq_1 appears non-trivially on the left of (5) and therefore $qq_1 = q'q_i$ for some $i \neq 1$ and some path q'. But this implies $q_1 = q_i$, by Lemma 3.1.4. This is a contradiction. ∎

The following result is a special case of Proposition 3.2.5; it is included because we will need it.

Proposition 3.1.6. *Let $\Lambda = \mathfrak{K}\Gamma$, where Γ has no oriented cycles, and $U = \Lambda e/K$ and $V = \Lambda e/L$ be uniserial Λ-modules with masts p and p', respectively, where p and p' pass through the same sequence of vertices, then $\dim_{\mathfrak{K}} \operatorname{Hom}_\Lambda(K, V)$ is equal to the number of detours on p.*

Proof. Again suppose p passes through $1 \to 2 \to \cdots \to n$ and let $y = e + L$. By Lemma 3.1.5, we may write

$$K = \bigoplus_{(\alpha,u)\wr p} \Lambda\left(\alpha u - k(\alpha,u)v(\alpha,u)\right) \oplus \bigoplus_{i}^{m} \Lambda q_i\,,$$

where $t(q_i) \notin \{1, 2, \ldots, n\}$. Let $\Delta(\alpha,u) = \alpha u - k(\alpha,u)v(\alpha,u)$. Then

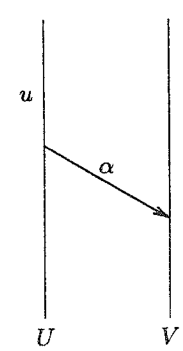
$$\mathrm{Hom}_\Lambda(K,V) = \bigoplus_{(\alpha,u)\wr p} \mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V) \oplus \bigoplus_{i}^{m} \mathrm{Hom}_\Lambda(\Lambda q_i,V)\,.$$

For each i, $\mathrm{Hom}_\Lambda(\Lambda q_i, V) = 0$, since $t(q_i) \notin \{1, 2, \ldots, n\}$. Thus

$$\dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(K,V) = \sum_{(\alpha,u)\wr p} \dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V)\,.$$

It suffices to show that for each $(\alpha,u) \wr p$, $\dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V) = 1$. Define $f_{(\alpha,u)} \in \mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V)$ by $f_{(\alpha,u)}\left(\Delta(\alpha,u)\right) = w(\alpha,u)y$, where $w(\alpha,u)$ is the right subpath of p' ending at $t(\alpha)$. Then $\{f_{(\alpha,u)}\}$ is a $\mathfrak{K}$-basis for $\mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V)$, because given $f \in \mathrm{Hom}_\Lambda(\Lambda\Delta(\alpha,u),V)$, we have $f(\Delta(\alpha,u)) = kw(\alpha,u)y$ with $k \in \mathfrak{K}$ and therefore $f = kf_{(\alpha,u)}$. ∎

The extension associated with $f_{(\alpha,u)}$ in Proposition 3.1.6 could be visualized as follows:



3.2 Description of a basis for $\mathrm{Hom}_\Lambda(K,V)$

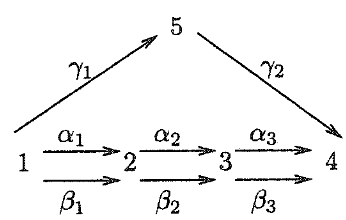
Again fix a triangular algebra $\Lambda = \mathfrak{K}\Gamma/I$ and uniserial modules $U = \Lambda e/K$ and $V = \Lambda e/L$ with masts p and p', respectively, where p and p' pass through the same sequence of vertices. In this section, we will describe a basis for $\mathrm{Hom}_\Lambda(K, V)$ and give an upper bound for its dimension. Let $y = e + L$. We need the following lemma.

Lemma 3.2.1. *Let V be any uniserial with mast p', which is a route on p. If $f \in \mathrm{Hom}_\Lambda(K, V)$ and q is a non-route on p, then $f(q) = 0$.*

Proof. Suppose p passes through $1 \to 2 \to \cdots \to n$. Then $q = q''q'$, where $t(q') \notin \{1, 2, \ldots, n\}$. Thus $q' \in K$ and $f(q) = q''f(q') = 0$, since p' passes through a subsequence of $1 \to 2 \to \cdots \to n$. ∎

Let us first look at an example.

Example 3.2.2. Let $\Lambda = \mathfrak{K}\Gamma/I$, where Γ is the quiver



and I is the ideal in $\mathfrak{K}\Gamma$ generated by $r = \alpha_3\beta_2\beta_1 + \beta_3\alpha_2\beta_1 + \beta_3\beta_2\alpha_1 + \gamma_2\gamma_1$. Consider the paths $p = \alpha_3\alpha_2\alpha_1$ and $p' = \beta_3\beta_2\beta_1$. The detours on p are $(\beta_1, e), (\beta_2, \alpha_1)$ and $(\beta_3, \alpha_2\alpha_1)$. Inserting the corresponding substitution equations

$$\beta_1 \mathrel{\widehat{=}} X_1\alpha_1\,, \quad \beta_2\alpha_1 \mathrel{\widehat{=}} X_2\alpha_2\alpha_1\,, \qquad \beta_3\alpha_2\alpha_1 \mathrel{\widehat{=}} X_3 p\,,$$

as well as $q \mathrel{\widehat{=}} 0$ whenever q is a non-route on p, into the relation r yields $V_p = V(X_1X_2 + X_1X_3 + X_2X_3)$. The detours on p' are $(\alpha_1, e), (\alpha_2, \beta_1)$ and $(\alpha_3, \beta_2\beta_1)$. The corresponding substitution equations are

$$\alpha_1 \mathrel{\widehat{=}} Y_1\beta_1\,, \quad \alpha_2\beta_1 \mathrel{\widehat{=}} Y_2\beta_2\beta_1\,, \quad \alpha_3\beta_2\beta_1 \mathrel{\widehat{=}} Y_3 p'\,.$$

Then $V_{p'} = V(Y_1 + Y_2 + Y_3)$. Suppose $\underline{k} = (k_1, k_2, k_3) \in V_p$, $\underline{l} = (l_1, l_2, l_3) \in V_{p'}$, $U = \Phi_p(\underline{k})$ and $V = \Phi_{p'}(\underline{l})$. Thus $U = \Lambda e_1/K$, where $K = \Lambda(\beta_1 - k_1\alpha_1) + \Lambda(\beta_2\alpha_1 -$

$k_2\alpha_2\alpha_1) + \Lambda(\beta_3\alpha_2\alpha_1 - k_3p) + \Lambda\gamma_1$ and $k_1k_2 + k_1k_3 + k_2k_3 = 0$. Let $f \in \mathrm{Hom}_\Lambda(K, V)$ and let y be a top element of V. Then,

$$\begin{aligned} f(\beta_1 - k_1\alpha_1) &= t_1\beta_1 y\,, \\ f(\beta_2\alpha_1 - k_2\alpha_2\alpha_1) &= t_2\beta_2\beta_1 y\,, \\ f(\beta_3\alpha_2\alpha_1 - k_3p) &= t_3p'y\,, \end{aligned}$$

where t_1, t_2 and $t_3 \in \mathfrak{K}$. Also we have

$$\begin{aligned} \alpha_3\beta_2\beta_1 &= \alpha_3\beta_2(\beta_1 - k_1\alpha_1) + k_1\alpha_3(\beta_2\alpha_1 - k_2\alpha_2\alpha_1) + k_1k_2p\,, \\ \beta_3\alpha_2\beta_1 &= \beta_3\alpha_2(\beta_1 - k_1\alpha_1) + k_1(\beta_3\alpha_2\alpha_1 - k_3p) + k_1k_3p\,, \\ \beta_3\beta_2\alpha_1 &= \beta_3(\beta_2\alpha_1 - k_2\alpha_2\alpha_1) + k_2(\beta_3\alpha_2\alpha_1 - k_3p) + k_2k_3p\,. \end{aligned}$$

Since $k_1k_2 + k_1k_3 + k_2k_3 = 0$, we have

$$\begin{aligned} r = \alpha_3\beta_2(\beta_1 - k_1\alpha_1) &+ k_1\alpha_3(\beta_2\alpha_1 - k_2\alpha_2\alpha_1) \\ &+ \beta_3\alpha_2(\beta_1 - k_1\alpha_1) + k_1(\beta_3\alpha_2\alpha_1 - k_3p) \\ &\quad + \beta_3(\beta_2\alpha_1 - k_2\alpha_2\alpha_1) + k_2(\beta_3\alpha_2\alpha_1 - k_3p) + \gamma_2\gamma_1\,. \end{aligned}$$

Therefore,

$$\begin{aligned} f(r) &= t_1\alpha_3\beta_2\beta_1 y + k_1t_2\alpha_3\beta_2\beta_1 y + t_1\beta_3\alpha_2\beta_1 y + k_1t_3p'y + t_2p'y + k_2t_3p'y \\ &= (t_1l_3 + k_1t_2l_3 + t_2l_2 + k_1t_3 + t_2 + k_2t_3)p'y \\ &= ((l_3)t_1 + (k_1l_3 + l_2 + 1)t_2 + (k_1 + k_2)t_3)p'y\,. \end{aligned}$$

Hence (t_1, t_2, t_3) is a solution of

$$E_r(T) = (l_3)T_1 + (k_1l_3 + l_2 + 1)T_2 + (k_1 + k_2)T_3 = 0\,.$$

Moreover any solution of $E_r(T)$ gives us an element of $\mathrm{Hom}_\Lambda(K, V)$ and we have $\dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(K, V) = 2$. ♣

Now, we describe a basis for $\mathrm{Hom}_\Lambda(K, V)$. Let $S = \{r_1, \ldots r_m\}$ be a left generating set for I and fix $r \in S$. Then

$$r = \sum_{i=1}^{n} l_ip_i + \sum_{i=1}^{n'} l'_iq_i\,,$$

where $l_i, l'_i \in \mathfrak{K}^*$ and p_i are routes on p and q_i are nonroutes on p. By Remark 1.1.4, $p_i = w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{1_i}u_{1_i}$ where $p = p'_i w_{s_i}\cdots w_{1_i}$ with $(\alpha_{m_i}, u_{m_i}) \wr\wr w_{m_i}$ and w_i a right subpath of p'_i. Then,

$$p_i = w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{2_i}u_{2_i}(\alpha_{1_i}u_{1_i} - k_{1_i}v_{1_i}) + k_{1_i}w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{2_i}u_{2_i}v_{1_i}\,.$$

We next look at $(\alpha_{2_i}, u_{2_i}v_{1_i}) \wr\wr p$. Let $k_{2_i} := k(\alpha_{2_i}, u_{2_i}v_{1_i})$ and $v_{2_i} = v(\alpha_{2_i}, u_{2_i}v_{1_i})$. Then,

$$\begin{aligned}p_i = w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{2_i}u_{2_i}\left(\alpha_{1_i}u_{1_i} - k_{1_i}v_{1_i}\right) + k_{1_i}w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{3_i}u_{3_i}\left(\alpha_{2_i}u_{2_i}v_{1_i} - k_{2_i}v_{2_i}\right)\\ + k_{1_i}k_{2_i}w_i\alpha_{s_i}u_{s_i}\ldots\alpha_{3_i}u_{3_i}v_{2_i}\,.\end{aligned}$$

If we continue this process, we get

$$\begin{aligned}p_i = w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{2_i}u_{2_i}\left(\alpha_{1_i}u_{1_i} - k_{1_i}v_{1_i}\right) + k_{1_i}w_i\alpha_{s_i}u_{s_i}\cdots\alpha_{3_i}u_{3_i}\left(\alpha_{2_i}u_{2_i}v_{1_i} - k_{2_i}v_{2_i}\right)\\ + \cdots + k_{1_i}\cdots k_{(s-1)_i}w_i\left(\alpha_{s_i}u_{s_i}v_{(s-1)_i} - k_{s_i}v_{s_i}\right) + k_{1_i}\cdots k_{s_i}w_iv_{s_i}\,, \quad (6)\end{aligned}$$

where $v_{j_i} = v(\alpha_{j_i}, u_{j_i}v_{(j-1)_i})$ and $k_{j_i} = k(\alpha_{j_i}, u_{j_i}v_{(j-1)_i})$. We have that $w_iv_{s_i}$ is a right subpath of p and $t(w_iv_{s_i}) = t(p_i)$. But, $t(p_1) = \cdots = t(p_n)$ implies $w_iv_{s_i} = w_jv_{s_j}$. Rename $w := w_iv_{s_i}$. We have, since all the terms on the right of (4), except the last one, are in K,

$$\begin{aligned}r = \sum_{i=1}^{n} l_ip_i + \sum_{i=1}^{n'} l'_iq_i \;&\widehat{=}\; \textstyle\sum_{i=1}^{n} l_ik_{1_i}\cdots k_{s_i}w_iv_{s_i}\\ &\widehat{=}\; (\textstyle\sum_{i=1}^{n} l_ik_{1_i}\cdots k_{s_i})w\,.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} l_ik_{1_i}\cdots k_{s_i} = 0\,, \quad (7)$$

and then by Equation (6), $\sum_{i=1}^{n} l_ip_i \in K$. Let $f \in \mathrm{Hom}_\Lambda(K, V)$. By Lemma 3.2.1, $f(q_i) = 0$ for each q_i. We have $f(\alpha_{j_i}u_{j_i} - k_{j_i}v_{j_i}) = T_{j_i}v'_{j_i}y$, where $T_{j_i} \in \mathfrak{K}$, v'_{j_i} is the right subpath of p' ending at $t(\alpha_{j_i})$ and $y = e + L$ is a top element of V. Thus

$$\begin{aligned} f(r) &= f\left(\sum_{i=1}^{n} l_i p_i + \sum_{i=1}^{n'} l'_i q_i\right) \\ &= f(\sum_{i=1}^{n} l_i p_i) \\ &= \sum_{i=1}^{n} l_i [T_{1_i} w_i \alpha_{s_i} u_{s_i} \cdots \alpha_{2_i} u_{2_i} v'_{1_i} y + T_{2_i} k_{1_i} w_i \alpha_{s_i} u_{s_i} \cdots \alpha_{3_i} u_{3_i} v'_{2_i} y + \\ &\quad \cdots + T_{s_i} k_{1_i} \cdots k_{s_i} w_i v'_{s_i} y] \\ &= \sum_{i=1}^{n} l_i \left[T_{1_i} c_{1_i} + T_{2_i} k_{1_i} c_{2_i} + \cdots + T_{s_i} k_{1_i} \cdots k_{s_i}\right] w' y\,, \end{aligned}$$

where w' is the right subpath of p' with $t(w') = t(p_i)$ and $c_{j_i} \in \mathfrak{K}$. Note that $t(p_1) = \cdots = t(p_n)$. Therefore,

$$\sum_{i=1}^{n} l_i \left[T_{1_i} c_{1_i} + T_{2_i} k_{1_i} c_{2_i} + \cdots + T_{s_i} k_{1_i} \cdots k_{s_i}\right] = 0\,. \tag{8}$$

Let M be the $m \times d$ matrix of coefficients of these linear equations and $[T(\alpha, u)]_{(\alpha,u)\wr\wr p}$ be a column with d rows, where d is the number of detours on p. Then any element f in $\mathrm{Hom}_\Lambda(K, V)$ gives us a solution for the homogeneous system $M[T(\alpha, u)] = \underline{0}$.

On the other hand, any solution of this system gives us an element of $\mathrm{Hom}_\Lambda(K, V)$. Indeed, assume $T = (t(\alpha, u))_{(\alpha,u)\wr\wr p}$ is a solution. Then, by Lemma 3.1.5, $f_T \in \mathrm{Hom}_{\mathfrak{K}\Gamma}(K, V)$, where $f_T((\alpha, u) - k(\alpha, u)v(\alpha, u)) = t(\alpha, u)v(\alpha, u)$ and $f_T(q) = 0$, with $(\alpha, u)\wr\wr p$ and q is non-route on p. Since $(t(\alpha, u))_{(\alpha,u)\wr\wr p}$ is a solution of the homogeneous system $M[T(\alpha, u)] = \underline{0}$, $f_T(r) = 0$ for all $r \in S$ and thus for all $r \in I$. Therefore f_T induces a Λ-homomorphism, which we again denote by f_T.

Assume that for $1 \leq i \leq l$, each $T_i = (t_i(\alpha, u))_{(\alpha,u)\wr\wr p}$ is a solution of the system and $l_i \in \mathfrak{K}$. We have $\sum_{i=1}^{l} l_i T_i = 0$ if and only if $\sum_{i=1}^{l} l_i f_{T_i} = 0$. Thus, a basis for the null space of the matrix gives us a basis for $\mathrm{Hom}_\Lambda(K, V)$ and vice versa. Therefore we have the following.

Proposition 3.2.3. *Suppose Λ is a triangular algebra and $U = \Lambda e/K$ and $V = \Lambda e/L$ are uniserial modules with masts p and p', respectively, where p and p' pass through* *the same sequence of vertices, and let d be the number of detours on p. Then*

$$\dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(K, V) = \mathrm{nullity}(M) = d - \mathrm{rank}(M)\,,$$

where M is described above.

Corollary 3.2.4. *Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a triangular algebra and U and V are uniserial Λ-modules with masts p and p', respectively, where p and p' pass through the same sequence of vertices. Then,*

$$\dim_{\mathfrak{K}} \mathrm{Ext}^1_\Lambda(U, V) \leq \text{ the number of detours on } p\,.$$

The following is a direct proof of the corollary but one which does not exhibit a basis for $\mathrm{Hom}_\Lambda(K, V)$.

Proof of Corollary 3.2.4 We have $U = \Lambda e/K$ where

$$K = \sum_{(\alpha,u)\wr p} \Lambda\,(\alpha u - k(\alpha, u)v(\alpha, u)) + \sum_{i=1}^{m} \Lambda q_i\,.$$

We can look at U as a $\mathfrak{K}\Gamma$-module using the surjection $\mathfrak{K}\Gamma \to \Lambda$. Let ${}_{\mathfrak{K}\Gamma}U = \Lambda e/L$. Then $L = \sum_{(\alpha,u)\wr p} \mathfrak{K}\Gamma\,(\alpha u - k(\alpha, u)v(\alpha, u)) + \sum_{i=1}^{m} \mathfrak{K}\Gamma q_i$. From [2, Proposition 20.6] (with $M =_{\mathfrak{K}\Gamma} L, N =_\Lambda V, W =_\Lambda \Lambda_{\mathfrak{K}\Gamma}$) we have:

$$\mathrm{Hom}_{\mathfrak{K}\Gamma}(L, \mathrm{Hom}_\Lambda(\Lambda, V)) \cong \mathrm{Hom}_\Lambda((\Lambda \otimes_{\mathfrak{K}\Gamma} L), V)\,.$$

But $\mathrm{Hom}_\Lambda(\Lambda, V) \cong V$. On the other hand we have $\Lambda \otimes L \to K \to 0$ and we know that $\mathrm{Hom}_\Lambda(-, V)$ is a contravariant left exact functor. Therefore

$$0 \to \mathrm{Hom}_\Lambda(K, V) \to \mathrm{Hom}_\Lambda(\Lambda \otimes L, V) \cong \mathrm{Hom}_{\mathfrak{K}\Gamma}(L, V)\,.$$

Hence by Proposition 3.1.6 we have

$$\dim_{\mathfrak{K}} \mathrm{Hom}_\Lambda(K, V) \leq \dim_{\mathfrak{K}} \mathrm{Hom}_{\mathfrak{K}\Gamma}(L, V) = \text{ the number of detours on } p\,.$$

∎

In the case where Λ is a monomial algebra, the formula in Proposition 3.2.3 gives a classification of the self-extensions. By Lemma 2.1.2, I is generated as a left ideal by finitely many paths, say, $p_1, \ldots, p_m$. Let $S = \{p_1, \ldots, p_m\}$, $R = \{\text{routes on } p\}$ and fix $p_i \in S \cap R$. Again, by the same method and keeping the same notation as in Section 3.2, Equation (6) becomes

$$p_i = w_i \alpha_{s_i} u_{s_i} \cdots \alpha_{2_i} u_{2_i} (\alpha_{1_i} u_{1_i} - k_{1_i} v_{1_i}) + k_{1_i} w_i \alpha_{s_i} u_{s_i} \cdots \alpha_{3_i} u_{3_i} (\alpha_{2_i} u_{2_i} v_{1_i} - k_{2_i} v_{2_i}) + \cdots + k_{1_i} \cdots k_{(s-1)_i} w_i (\alpha_{s_i} u_{s_i} v_{(s-1)_i} - k_{s_i} v_{s_i}) + k_{1_i} \cdots k_{s_i} w_i v_{s_i}\,.$$

We call the $(\alpha_{j_i}, u_{j_i} v_{(j-1)_i})$ the *detours on p involved in p_i*. Let

$$B_i = \left\{ (\alpha_{j_i}, u_{j_i} v_{(j-1)_i}) \mid k_{j_i} = 0, \text{ but } k_{l_i} \neq 0 \text{ for all } l \neq j \right\}$$

and let $B = \bigcup_{p_i \in S \cap R} B_i$.

Proposition 3.2.5. *Let $\Lambda = \mathfrak{K}\Gamma/I$ be a triangular monomial algebra and U be a uniserial Λ-module with mast p. Then*

$$\dim_{\mathfrak{K}} \operatorname{Ext}^1_\Lambda(U, U) = \textit{(number of detours on p)} - |B|\,.$$

Proof. In this case Equation (8) becomes

$$l_i \left[T_{1_i} k_{2_i} \cdots k_{s_i} + T_{2_i} k_{1_i} k_{3_i} \cdots k_{s_i} + \cdots + T_{s_i} k_{1_i} \cdots k_{(s-1)_i} \right] = 0\,. \tag{9}$$

By Equation (7), we have $l_i k_{1_i} k_{2_i} \cdots k_{s_i} = 0$. But $l_i \neq 0$, thus $k_{j_i} = 0$ for some $1 \leq j \leq s$. Then Equation (9) becomes

$$T_{j_i} k_{1_i} \cdots k_{(j-1)_i} k_{(j+1)_i} \cdots k_{s_i} = 0\,. \tag{10}$$

Thus $(\alpha_{j_i}, u_{j_i} v_{(j-1)_i}) \in B_i$, if and only if $k_{1_i} \cdots k_{(j-1)_i} k_{(j+1)_i} \cdots k_{s_i} \neq 0$ if and only if $T_{j_i} = 0$ is the only solution of (10). Thus the nullity of the coefficient matrix M is $d - |B|$. ∎

The next result is more general. Here U and V are uniserial Λ-modules, not necessarily with the same mast and the quiver may have oriented cycles. We know that any map φ in $\operatorname{Hom}_\Lambda(K, V)$ gives us an extension of U by V. This extension does not split if φ does not lift to P, where P is a projective cover of U. The next proposition says that the extension module has simple top if and only if φ is onto.

Proposition 3.2.6. *Let $U = \Lambda e/K$ and V be uniserial Λ-modules. Suppose $\varphi \in \mathrm{Hom}_\Lambda(K, V)$ does not lift to Λe. Then the corresponding extension module has simple top if and only if φ is onto.*

Proof. Suppose φ is onto. We have

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & K & \xrightarrow{\iota} & \Lambda e & \longrightarrow & U & \longrightarrow & 0 \\ & & & \varphi\downarrow & & \downarrow & & \| & & \\ \xi: & 0 & \longrightarrow & V & \longrightarrow & M & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

where M is pushout of ι and φ. Thus $M = (\Lambda e \oplus V)/L$ where $L = \{(\iota(k), -\varphi(k)) \mid k \in K\}$. We know that ξ is a non-split extension of U by V, since φ does not lift to Λe. We show that $M = \Lambda((e, 0) + L)$. Let $x \in M$. Then $x = (\lambda e, v) + L$ for some $\lambda \in \Lambda, v \in V$. But since φ is onto, $v = \varphi(k)$ for some $k \in K$. Thus

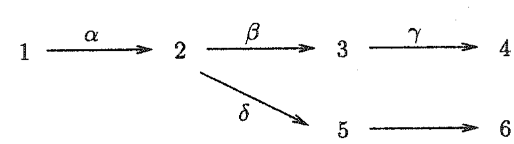
$$(\lambda, v) - (\lambda + \iota(k), 0) = (-\iota(k), \varphi(k)) \in L\,.$$

Hence $x = (\lambda, v) + L = (\lambda + \iota(k), 0) + L$.

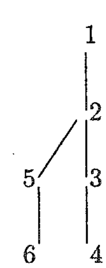
Now suppose that M has a simple top. Hence $JM = (Je \oplus JV + L)/L$. If φ is not onto, then $\varphi(K) \subseteq JV$. Thus $L \subseteq Je \oplus JV$ and so $JM = (Je \oplus JV)/L$. Therefore M does not have a simple top, since $(e, 0) \notin L$ and $(0, x) \notin L$, where $x \in \mathrm{top}(V)$. ∎

The following is a simple illustration of an extension with simple top.

Example 3.2.7. Let $\Lambda = \mathfrak{K}\Gamma$ where Γ is the quiver



Let $U = \Lambda e_1/\Lambda\delta\alpha$, $V = \Lambda e_5$ and $\varphi : \Lambda\delta\alpha \to V$ defined by $\varphi(\delta\alpha) = e_5$. Here $M = \Lambda e_1$.



Chapter 4

Uniserial representations of biserial algebras

B. Huisgen-Zimmermann [19] introduces a method for deciding when two uniserial modules over a finite dimensional algebra are isomorphic. R. Vila-Freyer and W. Crawley-Boevey [30] describe basic biserial algebras by means of quivers and relations. Using these tools we characterize the isomorphism classes of uniserial modules over a basic biserial algebra. B. Jue [23] has obtained the results in this section for the more restricted class of special biserial algebras. The following lemma holds in general.

Lemma 4.0.1. *Let $\Lambda = \mathfrak{K}\Gamma/I$ be an algebra. If for paths p and q, $p - kq \in I$, $k \in \mathfrak{K}^*$, and* $\mathrm{length}(p) < \mathrm{length}(q)$ *then $V_p = \emptyset$.*

Proof. q is a non-route on p and so $p - kq \mathrel{\widehat{=}} p$. ∎

4.1 Model biserial algebra

Suppose Λ is a model biserial algebra (see Section 1.5).

Lemma 4.1.1. *Suppose p is a nonzero path in Λ and $V_p \neq \emptyset$. Then there is a good path q in Λ such that p and q are the masts of the same uniserial modules.*

Proof. Let $p = \alpha_r \cdots \alpha_1$ and i be the smallest integer such that $\alpha_{i+1}\alpha_i$ is a bad path. We have $\alpha_{i+1}\alpha_i \neq 0$ and so $\alpha_{i+1}\alpha_i - k_1 u_{i+1}\alpha_i \in I$, where $u_{i+1}\alpha_i$ is a good path. Thus

$$\alpha_r \cdots \alpha_{i+2}\alpha_{i+1}\alpha_i \cdots \alpha_1 - k_1\alpha_r \cdots \alpha_{i+2}u_{i+1}\alpha_i \cdots \alpha_1 \in I\,.$$

Since $V_p \neq \emptyset$, $k_1 \neq 0$ and, by Lemma 4.0.1, length $u_{i+1} = 1$. We write $\beta_{i+1} := u_{i+1}$. Then

$$p - k_1\alpha_r \cdots \alpha_{i+2}\beta_{i+1}\alpha_i \cdots \alpha_1 \in I\,,$$

where $k_1 \in \mathfrak{K}^*$. Let $q_1 := \alpha_r \cdots \alpha_{i+2}\beta_{i+1}\alpha_i \cdots \alpha_1$. Now, if $\alpha_{i+2}\beta_{i+1}$ is a good path we write $\beta_{i+2} := \alpha_{i+2}$ and if it is a bad path $\alpha_{i+2}\beta_{i+1} - k_2 u_{i+2}\beta_{i+1} \in I$, where $u_{i+2}\beta_{i+1}$ is a good path and $k_2 \in \mathfrak{K}^*$. Then $q_1 - k_2 q_2 \in I$, where $q_2 := \alpha_r \cdots \alpha_{i+3}u_{i+2}\beta_{i+1}\alpha_i \cdots \alpha_1$. We have $p - k_1 k_2 q_2 \in I$. Again by Lemma 4.0.1, $\text{length}(u_{i+2}) = 1$. We write $\beta_{i+2} := u_{i+2}$. By continuing the method we get $p - kq \in I$, where $\text{length}(p) = \text{length}(q)$, q is a good path and $k \in \mathfrak{K}^*$. Then p and q are masts of the same uniserial modules since they have the same length and $p = kq$ in Λ. ∎

Let $(\alpha, u) \wr\wr p$. Recall that $V(\alpha, u) = \{v_i(\alpha, u) \mid i \in I(\alpha, u)\}$. In this chapter, we assume that $\text{length}(v_1(\alpha, u)) < \text{length}(v_2(\alpha, u)) < \cdots < \text{length}(v_n(\alpha, u))$.

Proposition 4.1.2. *Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a quotient of a model biserial algebra and p is a good path starting at e, with $V_p \neq \emptyset$. Then, any detour $(\alpha, u) \wr\wr p$, with $\text{length}(u) \geq 1$ is inessential. Moreover, there is at most one detour (α, e) on p.*

Proof. Suppose $(\alpha, u) \wr\wr p$, with $\text{length}(u) \geq 1$. Then αu is a bad path and so $\alpha u = 0$ or $\alpha u = k\alpha_r \cdots \alpha_1 u$ in Λ, with $k \in \mathfrak{K}^*, r \geq 1$ and $\alpha_r \cdots \alpha_1 u$ a good path with $t(\alpha_r) = t(\alpha)$ and $\alpha_r \neq \alpha$. Thus, (α, u) is an inessential detour on p. Now we assume that there is a detour $(\alpha, e) \wr\wr p$. This detour is the only one, since we have at most two arrows leaving e. ∎

The following remark is a special case of [23, Lemma 2.6 and Theorem 2.8].

Remark 4.1.3. *Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a quotient of a model biserial algebra and p is a path nonzero in Λ. Then $V_p = \emptyset$ or $V_p = \mathbb{A}^m$, for some $m \geq 0$.*

We first state the main result of this section which is a classification of isomorphism classes of uniserial modules with mast p.

Theorem 4.1.4. *Suppose $\Lambda = \mathfrak{K}\Gamma/I$ is a quotient of a model biserial algebra and p nonzero in Λ is a good path with $V_p \neq \emptyset$.*

(i) If there is no detour starting at e, $V_p = \mathbb{A}^0$.

(ii) Let $(\alpha, e) \wr p$.

(a) If $p = \alpha p'$ or $t(p) \neq t(\alpha)$ then for $\underline{k}$ and $\underline{k}' \in V_p$, $\Phi_p(\underline{k}) \cong \Phi_p(\underline{k}')$.

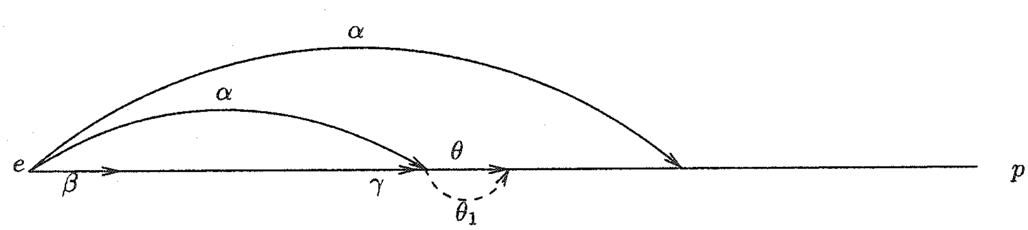
(b) If $p = \gamma p'$ where $t(\gamma) = t(\alpha)$ and $\gamma \neq \alpha$ then $\Phi_p(\underline{k}) \cong \Phi_p(\underline{k}')$ if and only if $k_n(\alpha, e) = k'_n(\alpha, e)$ where $v_n(\alpha, e) = p$.

Proof of (i): (i) follows from Proposition 4.1.2. ∎

In order to prove part (ii) of the above theorem, we need the following lemmas. We assume that $\Lambda = \mathfrak{K}\Gamma/I$ is a quotient of a model biserial algebra and p nonzero in Λ is a good path such that $V_p \neq \emptyset$. Suppose that there is a detour on p starting at e, say $(\alpha, e) \wr p$. Let $\alpha \widehat{=} \sum_{i \in I(\alpha,e)} X_i(\alpha, e) v_i(\alpha, e)$. To simplify notation, we abbreviate $X_i(\alpha, e) = X_i, v_i(\alpha, e) = v_i$, and later $Y_i(\alpha, e) = Y_i$.

Lemma 4.1.5. *Let $\alpha \widehat{=} \sum_{i \in I(\alpha,e)} X_i(\alpha, e) v_i(\alpha, e)$. If $v_i(\alpha, e) = \gamma v'_i$, γ an arrow with $\gamma \neq \alpha$ and $v_i(\alpha, e) \neq p$ then $X_i(\alpha, e) = 0$.*

Proof. Assume the conclusion fails and let s be the smallest i such that $v_i(\alpha, e) = \gamma v'_i \neq p$, $\gamma \neq \alpha$ and $X_i(\alpha, e) \neq 0$.



We know that $\theta\alpha$ is a bad path and so $\theta\alpha = 0$ or $\theta\alpha = k\theta_m \cdots \theta_1\alpha$ in Λ, with $k \in \mathfrak{K}^*$, $m \geq 1$ and $\theta_m \cdots \theta_1\alpha$ is a good path with $t(\theta_m) = t(\theta)$ and $\theta_m \neq \theta$. Let

$$A = \{i \in I(\alpha, e) \mid v_i = \gamma v'_i \neq p\}.$$

If $\theta\alpha = 0$, then

$$0 = \theta\alpha \,\widehat{=} \sum_{i \in I(\alpha,e)} X_i \theta v_i = \sum_{i \in A} X_i \theta v_i \,.$$

For $i \in A$, θv_i is a right subpath of p and so $X_i = 0$ for all $i \in A$. In particular $X_s = 0$, which is a contradiction.

If $\theta\alpha = k\theta_m \cdots \theta_1 \alpha$ in Λ then $r = \theta\alpha - k\theta_m \cdots \theta_1\alpha \in I$. Thus,

$$r \;\widehat{=}\; \sum_{i \in I(\alpha,e)} X_i \theta v_i - \sum_{i \in I(\alpha,e)} k X_i \theta_m \cdots \theta_1 v_i \,.$$

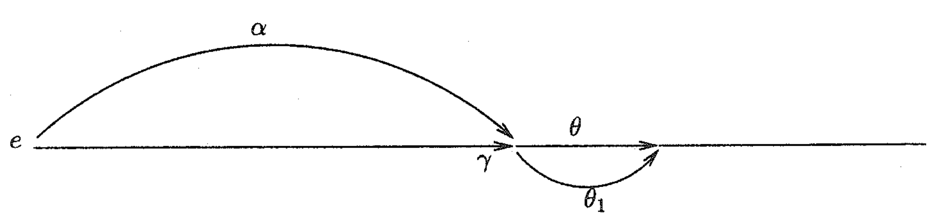
Notice that if $i \notin A$ then the corresponding term in the above sum is

$$\begin{aligned} X_i(\theta v_i - k\theta_m \cdots \theta_1 v_i) &= X_i(\theta \alpha v_i' - k\theta_m \cdots \theta_1 \alpha v_i') \\ &= X_i(k\theta_m \cdots \theta_1 \alpha v_i' - k\theta_m \cdots \theta_1 \alpha v_i') = 0 \,. \end{aligned}$$

Therefore,

$$r \;\widehat{=}\; \sum_{i \in A} X_i \theta v_i - \sum_{i \in A} k X_i \theta_m \cdots \theta_1 v_i \,. \qquad (*)$$

We know that $\theta v_i \not\cong \theta v_s$ and $\theta_m \cdots \theta_1 v_i \not\cong \theta v_s$ for all $i \neq s \in A$, since v_s is the shortest one and for each $i \in A$, θv_i is a right subpath of p. Moreover, θv_s is a right subpath of p. Hence, if $m > 1$, $\theta_m \cdots \theta_1 v_s \not\cong \theta v_s$ and so $X_s = 0$. Therefore $m = 1$ and $\theta\alpha = k\theta_1\alpha$.



$(*)$ simplifies to

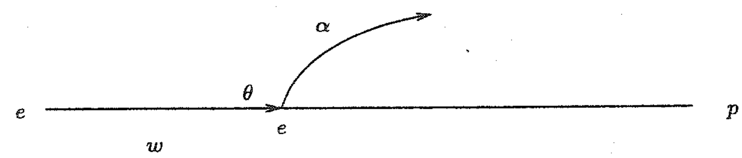
$$r \,\widehat{=} \sum_{i \in A} X_i(\theta v_i - k\theta_1 v_i) \,. \qquad (**)$$

We have that $0 \neq \theta_1\gamma$ is a bad path and so $\theta_1\gamma = 0$ or $\theta_1\gamma - k'\mu_l \cdots \mu_1\gamma \in I$ with $k' \in \mathfrak{K}^*, l \geq 1$ and $\mu_l \cdots \mu_1 \gamma$ is a good path with $t(\mu_l) = t(\theta_1)$ and $\mu_l \neq \theta_1$. But $\theta_1\gamma \neq 0$, otherwise by $(**)$, $r \,\widehat{=} \sum_{i \in A} X_i \theta v_i$ and then $X_i = 0$ for all $i \in A$. Hence,

$\theta_1\gamma - k'\mu_l \cdots \mu_1\gamma \in I$. If $l > 1$, $X_s = 0$ since θv_s is the shortest one. Indeed, $\theta_1 v_s$ reduces to a linear combination of right subpaths of p strictly longer than $\theta_1 v_s$. Hence $l = 1$. We have $\theta\alpha = k\theta_1\alpha, \theta_1\gamma = k'\theta\gamma$ and so $kk' \neq 1$ [30, Corollary 3]. By (**) $X_s - kk'X_s = 0$. Therefore, $X_s = 0$ which is again a contradiction. ∎

Lemma 4.1.6. *Suppose w is a right subpath of p ending at $e = s(p)$. If αw is not a right subpath of p then $\alpha w \mathrel{\widehat{=}} 0$*

Proof. If $w = p$ then $\alpha w \mathrel{\widehat{=}} 0$. If $w \neq p$ then $w = \theta w'$ where $\alpha\theta$ is a bad path.



Since $\alpha\theta$ is a bad path then $\alpha\theta = 0$ or $\alpha\theta = k\alpha_r \cdots \alpha_1\theta$ in Λ where $\alpha_r \neq \alpha$. If $\alpha\theta \neq 0$, then the good path $\alpha_r \cdots \alpha_1 w$ is a non-route on p or a right subpath of p. There are two cases.

Case 1: $p = \alpha p'$ or $t(p) \neq t(\alpha)$. By Lemma 4.1.5, $\alpha \mathrel{\widehat{=}} \sum_{i_j \in I(\alpha,e)} X_{i_j} v_{i_j}$ where $v_{i_j} = \alpha v'_{i_j}$. Therefore, $\alpha w = k\alpha_r \cdots \alpha_1 w \mathrel{\widehat{=}} \sum_{i_j \in I(\alpha,e)} X_{i_j} v_{i_j} w$. Each $v_{i_j} w$ is either longer than p or is a right subpath of p; similarly for $\alpha_r \cdots \alpha_1 w$. However, each $v_{i_j} = \alpha v'_{i_j}$ where $\alpha_r \neq \alpha$. Thus $\alpha w \mathrel{\widehat{=}} 0$.

Case 2: $p = \gamma p'$ where $\gamma \neq \alpha$ and $t(\gamma) = t(\alpha)$. By Lemma 4.1.5, $\alpha \mathrel{\widehat{=}} \sum_{i_j \in I(\alpha,e)} X_{i_j} v_{i_j} + X_n v_n$ where $v_{i_j} = \alpha v'_{i_j}$ and $v_n = p$. As in case 1, we get $\alpha w \mathrel{\widehat{=}} 0$.

∎

Lemma 4.1.7. *Suppose w is a right subpath of p ending at e, αw is a right subpath of p and $v_i = \alpha v'_i$. Then $v_i w \mathrel{\widehat{=}} 0$.*

Proof. Let $v_i = \alpha v'_i = \alpha\mu_s \cdots \mu_1\beta$. We can assume that $v_i w \neq 0$. Hence $0 \neq \beta w$ is a bad path. Thus, $\beta w = k_1\beta_r \cdots \beta_1 w$ with $r \geq 1$, $k_1 \in \mathfrak{K}^*$ and $\beta_r \cdots \beta_1 w$ is a good path with $\beta_r \neq \beta$. We have $0 \neq \mu_1\beta_r$ is a bad path since $\mu_1\beta$ is a good one. Then, $\mu_1\beta_r = k_2\mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r$ with $r_1 \geq 1$, $k_2 \in \mathfrak{K}^*$ and $\mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r$ a good path with $\mu_{r_1}^{(1)} \neq \mu_1$. Thus, $\mu_1\beta w = k_1k_2\mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r \cdots \beta_1 w$ with $\mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r \cdots \beta_1 w$ a good path. But $0 \neq \mu_2\mu_{r_1}^{(1)}$ is a bad path since $\mu_2\mu_1$ is a good one. Then, $\mu_2\mu_{r_1}^{(1)} = k_3\mu_{r_2}^{(2)} \cdots \mu_1^{(2)}\mu_{r_1}^{(1)}$ with $r_2 \geq 1$, $k_3 \in \mathfrak{K}^*$ and $\mu_{r_2}^{(2)} \cdots \mu_1^{(2)}\mu_{r_1}^{(1)}$ a good path with $\mu_{r_2}^{(2)} \neq \mu_2$. Again $0 \neq \mu_3\mu_{r_2}^{(2)}$ is a bad path and if we continue, we get

$$\mu_s \cdots \mu_1\beta w = l\mu_{r_s}^{(s)} \cdots \mu_1^{(s)} \cdots \mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r \cdots \beta_1 w\,.$$

If $v_i w \not\cong 0$, then, since $t(\mu_{r_s}^{(s)}) = t(\mu_s) = e$, $\mu_{r_s}^{(s)} \cdots \mu_1^{(s)} \cdots \mu_{r_1}^{(1)} \cdots \mu_1^{(1)}\beta_r \cdots \beta_1 w$ is a good path of length less than the length of p. It is, then, a right subpath of p ending at e. But $\alpha\mu_{r_s}^{(s)}$ is a bad path since $\alpha\mu_s$ is a good one. This contradicts Lemma 4.1.6. ∎

Now we are ready to prove Theorem 4.1.4(ii).

Proof of Theorem 4.1.4(ii): Suppose that there is a detour on p starting at e, say $(\alpha, e) \wr p$. Let us look at $S_p(X,Y,Z)$ (see Section 1.2). Suppose that the path $p : e(1) = e \to \cdots \to e(l+1)$ has precisely t right subpaths of positive length ending in the starting vertex $e = e(1)$ of p; say $w_1, \ldots, w_t$. The system $S_p(X,Y,Z)$, which is here a single equation, will have the t linear variables $Z_1, \ldots, Z_t$. Consider the following equation $E(\alpha, e)$ in $\mathfrak{K}\Gamma[X,Y,Z]$.

$$E(\alpha,e): \quad \alpha(e + \sum_{j=1}^{t} Z_j w_j) = \sum_{i \in I(\alpha,e)} X_i v_i (e + \sum_{j=1}^{t} Z_j w_j)\,.$$

Now expand both sides of this equation by successively inserting from the right the substitution equation

$$\alpha \mathrel{\widehat{=}} \sum_{i \in I(\alpha,e)} Y_i v_i\,,$$

and the equivalences $q \mathrel{\widehat{=}} 0$ for those paths $q \in \mathfrak{K}\Gamma$ which fail to be routes on p. Then

$$\sum_{i \in I(\alpha,e)} Y_i v_i + \sum_{j=1}^{t} Z_j \alpha w_j = \sum_{i \in I(\alpha,e)} X_i v_i + \sum_{i \in I(\alpha,e)} \sum_{j=1}^{t} X_i Z_j v_i w_j\,. \tag{11}$$

Let

$$J = \{j \mid \alpha w_j \text{ is a right subpath of } p\}\,.$$

By Lemma 4.1.6, $\alpha w_j \mathrel{\widehat{=}} 0$ for $j \notin J$ and by Lemmas 4.1.5 and 4.1.7, $X_i v_i w_j \mathrel{\widehat{=}} 0$ for $j \in J$. Therefore (11) becomes

$$\sum_{i\in I(\alpha,e)} Y_i v_i + \sum_{j\in J} Z_j \alpha w_j = \sum_{i\in I(\alpha,e)} X_i v_i + \sum_{i\in I(\alpha,e)} \sum_{j\notin J} X_i Z_j v_i w_j \,. \tag{12}$$

For $j \in J$, αw_j is a right subpath of p and therefore $\alpha w_j = v_{i_j}$ for some $i_j \in I(\alpha, e)$. On the other hand if $v_i = \alpha v_i'$ then $v_i = \alpha v_i' = \alpha w_s$ for some $s \in J$.

If $j \notin J$ and $v_{i_t} w_j \neq 0$ then $v_{i_t} w_j$ is a good path since βw_j and v_{i_t} are good paths, where β is first arrow of p. Hence $v_{i_t} w_j \mathrel{\widehat{=}} 0$ or $v_{i_t} w_j = v_{(i_t,j)} = \alpha v'_{(i_t,j)}$ is a right subpath of p, with $(i_t, j) \in I(\alpha, e)$.

(a) Suppose $p = \alpha p'$ or $t(p) \neq t(\alpha)$. By Lemma 4.1.5, $X_i = Y_i = 0$ if $v_i = \gamma v_i'$ with $\gamma \neq \alpha$. Therefore Equation (12) reduces to

$$\sum_{j\in J} (Y_{i_j} + Z_j) v_{i_j} = \sum_{j\in J} X_{i_j} v_{i_j} + \sum_{t\in J} \sum_{j\notin J} X_{i_t} Z_j v_{(i_t,j)} \,.$$

We look at the coefficient of a single v_{i_j}, $j \in J$:

$$Y_{i_j} + Z_j = X_{i_j} + \sum_{s,\, m_l} \lambda_{(s,\, m_l)} X_{m_l} Z_s \,,$$

with $s \notin J$, $s < j$ and $\lambda_{(s,\, m_l)} = 0$ or 1. Therefore, for $\underline{k}$, $\underline{k}' \in V_p$, the linear system $S_p(\underline{k}, \underline{k}', Z)$ is consistent.

(b) Now suppose $p = \gamma p'$ where $t(\gamma) = t(\alpha)$ and $\gamma \neq \alpha$. Suppose $I(\alpha, e) = \{1, 2, \ldots, n\}$ and $v_n = p$. By Lemma 4.1.5 we have $X_i = 0$ if $v_i = \gamma v_i'$ and $i \neq n$. Therefore, Equation (12) becomes

$$Y_n v_n + \sum_{j\in J} (Y_{i_j} + Z_j) v_{i_j} = X_n v_n + \sum_{j\in J} X_{i_j} v_{i_j} + \sum_{t\in J} \sum_{j\notin J} X_{i_t} Z_j v(i_t, j) \,.$$

We consider coefficients of v_n and the $v_{i_j}, j \in J$ to get

$$Y_n = X_n$$

$$Y_{i_j} + Z_j = X_{i_j} + \textstyle\sum_{s,\, m_l} \lambda_{(s,\, m_l)} X_{m_l} Z_s \,,$$

with $s \notin J$, $s < j$ and $\lambda_{(s,m_l)} = 0$ or 1. Therefore, for $\underline{k}, \underline{k}' \in V_p$, the linear system $S_p(\underline{k}, \underline{k}', Z)$ is consistent if and only if $k_n(\alpha, e) = k_n'(\alpha, e)$. ∎

4.2 Biserial algebras

Suppose Λ is a biserial $\mathfrak{K}$-algebra, where $\mathfrak{K}$ is an algebraically closed field and let $p \neq 0$ be a path in Λ passing through $S = (S(1), \ldots, S(l+1))$. Then $\Lambda \cong \Lambda'$, where Λ' is a quotient of a model biserial algebra (see 1.5). If $\varphi : \Lambda \to \Lambda'$ is an isomorphism, we can look at a Λ'-module as a Λ-module by

$$\lambda.m = \varphi(\lambda)m \text{ for } \lambda \in \Lambda, m \in M\,.$$

Hence, we have a functor F : Λ'-mod$\to$ Λ-mod. We want to describe the uniserial Λ-modules with mast p using the information available about uniserial Λ'-modules. Suppose $\varphi : \Lambda \to \Lambda'$ is an isomorphism induced by $f : \mathfrak{K}\Gamma \to \mathfrak{K}\Gamma$. By Saorín's Theorem, we have

$$f(p) = k_1 p_1 + \cdots + k_m p_m + \sum_{l(q_i)>l(p)} t_i q_i\,,$$

where $k_i, t_i \in \mathfrak{K}$ and p_i are paths passing through S. There are at most two good paths with a given sequence of simples in a factor of any model biserial algebra. Thus $f(p) = k_1p_1 + k_2p_2 + \sum_{l(q_i')\geq l(p)} t_i'q_i'$ where p_1, p_2 are good paths passing through S and $k_i, t_i' \in \mathfrak{K}$. We need the following definition.

Definition 4.2.1. *[19] Let Λ-uni(p) be the set of isomorphism classes of uniserial Λ-modules with mast p.*

Proposition 4.2.2. *Suppose Λ is a biserial algebra and $S = (S(1), \ldots, S(l+1))$ is a sequence of simple modules. Let $\varphi : \Lambda \to \Lambda'$ be an isomorphism induced by f where Λ' is a factor of a model biserial algebra and p is nonzero in Λ, a path passing through S.*

(a) If $f(p) = k_1p_1 + \sum_{l(q_i')\geq l(p)} t_i'q_i'$ with $k_1 \neq 0$ and p_1 a good path, then Λ-uni$(p) = F(\Lambda'$-uni$(p_1))$.

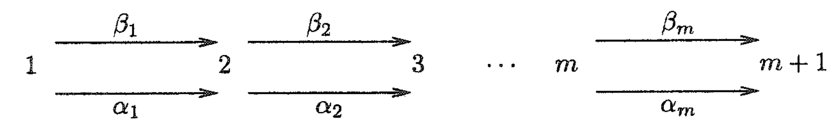
(b) If $f(p) = k_1p_1+k_2p_2+\sum_{l(q_i')\geq l(p)} t_i'q_i'$ with $k_i \neq 0$, p_i good paths and length$(p) \geq 2$, *then Λ-uni$(p) = F(\Lambda'$-uni$(p_1) \bigcup \Lambda'$-uni$(p_2))$. Moreover, the union*

$$\Lambda'\text{-uni}(p_1) \bigcup \Lambda'\text{-uni}(p_2)$$

is disjoint.

Proof. (a) Let $U \in \Lambda'\text{-uni}(p_1)$. Then $p.F(U) = f(p)U = k_1 p_1 U \neq 0$ if $k_1 \neq 0$, since the q_i' are nonroutes on p_1. Therefore $\Lambda\text{-uni}(p) \supseteq F(\Lambda'\text{-uni}(p_1))$. Now let $U \in \Lambda\text{-uni}(p)$. Then $p_1 F^{-1}(U) = \varphi^{-1}(p_1)U = k_1^{-1}pU \neq 0$. Thus $U \in F(\Lambda'\text{-uni}(p_1))$.

(b) We first show that if U is a uniserial Λ'-module with mast p_1, then $p_2 U = 0$; i.e., the union $\Lambda'\text{-uni}(p_1) \bigcup \Lambda'\text{-uni}(p_2)$ is disjoint. Let $p_1 = \alpha_m \cdots \alpha_2\alpha_1$, $p_2 = \beta_m \cdots \beta_2\beta_1$ where $\alpha_i \neq \beta_i$, $m \geq 2$.



$(\beta_1, e_1) \wr p_1$ and $U = \Lambda e_1/K$ where

$$K = \Lambda(\beta_1 - \sum_{i \in I(\beta_1, e_1)} k_i(\beta_1, e_1)v_i(\beta_1, e_1)) + \sum_{r_i \text{ nonroute on } p_1} \Lambda r_i\,.$$

We have $v_1(\beta_1, e_1) = \alpha_1$. Since $m \geq 2$, by Lemma 4.1.5, $k_1(\beta_1, e_1) = 0$. Therefore $p_2 U = \beta_m \cdots \beta_2\beta_1 U = 0$. Now we have

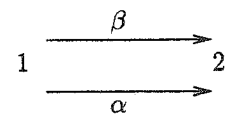
$$p.F(U) = (k_1p_1 + k_2p_2)U = k_1p_1U \neq 0\,.$$

Hence $F(\Lambda'\text{-uni}(p_1) \bigcup \Lambda'\text{-uni}(p_2)) \subseteq \Lambda\text{-uni}(p)$. On the other hand, if $U \in \Lambda\text{-uni}(p)$, then $\varphi(p)F^{-1}(U) = pU \neq 0$ and so $(k_1p_1 + k_2p_2)F^{-1}(U) \neq 0$. Thus $p_1F^{-1}(U) \neq 0$ or $p_2F^{-1}(U) \neq 0$.

∎

The following example shows that in the above theorem, case (b), the condition $\text{length}(p) \geq 2$ is necessary.

Example 4.2.3. Let Γ be the quiver



and $\Lambda = \Lambda' = \mathfrak{K}\Gamma$. Let $\varphi : \Lambda \to \Lambda'$ by $f(\alpha) = \alpha - \beta$ and $f(\beta) = \beta$. Let $p = p_1 = \alpha$ and $p_2 = \beta$. Then $U = \Lambda' e/\Lambda'(\alpha - \beta)$ is a uniserial Λ'-module with masts p_1 and p_2, but $pF(U) = \varphi(p)U = (\alpha - \beta)U = 0$.

Chapter 5

Irreducible maps of uniserials

5.1 Introduction and general results

In Section 1, we quote some of the basic properties of irreducible morphisms and then we get some results about an almost split sequences of the form $0 \to DTrU \to M \to U \to 0$ where U is a uniserial module. Recall [3, page 173] that $\alpha(U)$ is the number of the indecomposable summands in a direct sum decomposition of M. In Section 2, we prove that if U is a non-projective uniserial module over a left m-multiserial algebra then $\alpha(U) \leq m$. In Section 3, we give necessary and sufficient conditions on a class of algebras, including left multiserial algebras and monomial algebras, which make $JU \hookrightarrow U$ irreducible. These confirm a conjecture of A. Boldt [6, Conjecture 1.2.1] for these cases.

Proposition 5.1.1. *If $g\colon B \to C$ is an irreducible morphism in mod Λ, then g is either a monomorphism or an epimorphism.*

Proof. See [3, Lemma V.5.1]. ∎

Proposition 5.1.2. *If $g\colon B \to C$ is an irreducible morphism and B' is a nonzero direct summand of B, then the induced map $B' \to C$ is irreducible.*

Proposition 5.1.3. *Let $0 \to A \to B \to C \to 0$ be an almost split sequence. Then* *(i) If C is not simple, then $0 \to \mathrm{soc}(A) \to \mathrm{soc}(B) \to \mathrm{soc}(C) \to 0$ is exact.*

(ii) If C is simple, then $\mathrm{soc}(A) \cong \mathrm{soc}(B)$.

Proof. For (i) see [3, Lemma V.3.2(d)]. (ii) follows from the fact that soc is left exact, C is simple and $B \to C$ is not split. Indeed, if $0 \to \mathrm{soc}(A) \to \mathrm{soc}(B)$ is not an isomorphism, C is isomorphic to a submodule of B, which is impossible. ∎

Corollary 5.1.4. *Let U be a non-projective uniserial module, then*

$$\alpha(U) \leq \dim_{\mathfrak{K}} \mathrm{soc}(DTrU) + 1\,.$$

Proof. Let $0 \to DTrU \to B \to U \to 0$ be an almost split sequence. Then either $0 \to \mathrm{soc}(DTrU) \to \mathrm{soc}(B) \to \mathrm{soc}(U) \to 0$ is exact, or $\mathrm{soc}(DTrU) \cong \mathrm{soc}(B)$ by Proposition 5.1.3. Therefore

$$\alpha(U) \leq \dim_{\mathfrak{K}} \mathrm{soc}(B) \leq \dim_{\mathfrak{K}} \mathrm{soc}(DTrU) + \dim_{\mathfrak{K}} \mathrm{soc}(U) = \dim_{\mathfrak{K}} \mathrm{soc}(DTrU) + 1\,.$$

∎

Let $0 \to DTrU \xrightarrow{f} \coprod_{i \in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence with U uniserial. We next show that at most one of the induced maps $g_i \colon B_i \to U$ is a monomorphism and the number of epimorphisms is less than or equal to $\dim_{\mathfrak{K}} \mathrm{soc}(DTrU)$.

Lemma 5.1.5. *Let R be a left artinian ring with Jacobson radical J. If $U_1, U_2 \in R$-mod are uniserial, M, N are indecomposable modules and $f \colon M \to U_1$ and $g \colon U_2 \to N$ are irreducible injective and surjective R-linear maps respectively, then*

(1) There exists an isomorphism $\varphi : JU_1 \to M$ so that $f\varphi$ is the natural radical embedding $JU_1 \to U_1$.

(2) There exists an isomorphism $\psi : N \to U_2/\mathrm{soc}(U_2)$ so that ψg is that natural socle factor projection $U_2 \to U_2/\mathrm{soc}(U_2)$.

Proof. (1) Since $\operatorname{im}(f)$ is a proper submodule of U_1, $\operatorname{im}(f) = J^l U_1$ with $l \geq 1$ and $M \cong J^l U_1$ via f. However, if $l > 1$, then $J^l U_1 \to J^{l-1} U_1 \to U_1$ would be a non-trivial factorization of $J^l U_1 \to U_1$ and gives us a factorization of f, which is impossible. The proof of (2) is similar to that of (1). ∎

Let e, f be primitive idempotents in Λ. For a nonzero element $a \in fJe$, the Λ-module $\Lambda e/\Lambda a$ is indecomposable and non-projective. We consider the almost split sequence ending in $\Lambda e/\Lambda a$. We are interested in the case where this module is a uniserial module.

Proposition 5.1.6. *If $U = \Lambda e/\Lambda a$ is a uniserial module, then $\alpha(U) \leq 2$.*

Proof. $\Lambda f \xrightarrow{.a} \Lambda e \to \Lambda e/\Lambda a$ is the start of a minimal projective presentation of $\Lambda e/\Lambda a$, where $.a$ denotes the right multiplication by a. From [3, Proposition V.6.1] we have that the middle term B in the almost split sequence $\delta\colon 0 \to DTrU \to B \to U \to 0$ has a decomposition $B = B' \sqcup B''$ with B' indecomposable and such that if $B'' \neq 0$, the induced morphism $g''\colon B'' \to U$ is an irreducible monomorphism. But, by Lemma 5.1.5, $B'' \cong JU$ is indecomposable and therefore $\alpha(U) \leq 2$. ∎

Proposition 5.1.7. *Let $0 \to DTrU \xrightarrow{f} \bigsqcup_{i\in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence where U is a uniserial module and the B_i are indecomposable, then*

(i) At most one of the induced maps $g_i\colon B_i \to U$ is a monomorphism.

(ii) If $B_i \xrightarrow{g_i} U$ is an epimorphism and $\operatorname{soc} B_i$ is simple, then

$$\operatorname{soc}(B_i) \subseteq f(\operatorname{soc}(DTrU)).$$

(iii) Let $I' = \{i \in I$ such that $g_i\colon B_i \to U$ is an epimorphism $\}$. Then,

$$|I'| \leq dim_{\mathfrak{K}} \operatorname{soc}(DTrU).$$

Proof. (i) Suppose g_1 and g_2 are monomorphisms. Hence $B_1 \cong JU$ and $B_2 \cong JU$. The induced irreducible morphism $B_1 \sqcup B_2 \to U$ cannot be an epimorphism and therefore is a monomorphism and $B_1 \sqcup B_2 \cong JU$ by Lemma 5.1.5, which is a contradiction.

(ii) We have $\mathrm{soc}(B_i) \cap \ker(g_i) \neq 0$ since $\ker(g_i) \neq 0$ and $\mathrm{soc}(B_i)$ is essential in B_i. But $\mathrm{soc}(B_i)$ is simple, so $\mathrm{soc}(B_i) \subseteq \ker(g_i)$. We know that $0 \to \mathrm{soc}(DTrU) \xrightarrow{\bar{f}} \bigsqcup_{i\in I} \mathrm{soc}(B_i) \xrightarrow{\bar{g}} \mathrm{soc}(U)$ is exact. Hence $\mathrm{soc}(B_i) \subseteq \ker \bar{g} = \mathrm{im}\, \bar{f}$. Therefore, $\mathrm{soc}(B_i) \subseteq f(\mathrm{soc}(DTrU))$.

(iii) By (i) we know that there is at most one irreducible monomorphism $g_i\colon B_i \to U$.

Case 1: There is one i such that g_i is a monomorphism. Then

$$|I'| = \alpha(U) - 1 \leq \dim_{\mathfrak{K}} \mathrm{soc}(DTrU)\,.$$

Case 2: For each $i \in I$, g_i is an epimorphism. If U is simple then

$$\alpha(U) \leq \dim_{\mathfrak{K}} \mathrm{soc}(\bigsqcup_{i\in I} B_i) = \dim_{\mathfrak{K}} \mathrm{soc}(DTrU)\,.$$

Otherwise, the exact sequence

$$0 \to \mathrm{soc}(DTrU) \xrightarrow{\bar{f}} \bigsqcup_{i\in I} \mathrm{soc}(B_i) \xrightarrow{\bar{g}} \mathrm{soc}(U) \to 0$$

shows, by (ii), that $\mathrm{soc}\, B_i$ is not simple for some i. Then

$$\alpha(U) \leq \dim_{\mathfrak{K}} \mathrm{soc}(\bigsqcup_{i\in I} B_i) - 1 = \dim_{\mathfrak{K}} \mathrm{soc}(DTrU)\,.$$

∎

We next give a first necessary condition for $JU \hookrightarrow U$ to be irreducible.

Proposition 5.1.8. *Let U be a uniserial Λ-module with mast p.*

(i) If $(\alpha, e) \wr p$, then $JU \hookrightarrow U$ is not irreducible.

(ii) If αe is a nonroute on p, then $JU \hookrightarrow U$ is not irreducible.

Proof. (i) Suppose $p = p'\beta$ with $\beta \in \Gamma_1$ and $U = \Lambda e/K$ where

$$K = \sum_{(\delta,u)\wr p} \Lambda \left(\delta u - \sum_{i \in I(\delta,u)} k_i(\delta,u) v_i(\delta,u) \right) + \sum_{q \text{ nonroute on } p} \Lambda q \,.$$

Let $V = \Lambda e/L$ with

$$L = \sum_{(\delta,u)\wr p,\ (\delta,u)\neq(\alpha,e)} \Lambda \left(\delta u - \sum_{i \in I(\delta,u)} k_i(\delta,u) v_i(\delta,u) \right) + \sum_{q \text{ nonroute on } p} \Lambda q \,.$$

We prove that $JU \hookrightarrow U$ factors nontrivially through V. Indeed

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U \,,$$

where $\varphi(\beta + K) = \beta + L$ and $\psi(e + L) = e + K$. Then $\psi\varphi = id|_{JU}$.
Claim 1: φ is not split monomorphism. Otherwise, suppose $\chi\colon V \to JU$ is a splitting of φ. Then $\chi(e+L) = k_1 w_1 + \cdots + k_n w_n + K$ where $w_1, \ldots, w_n$ are right subpaths of p with $t(w_i) = e$ and $w_i \neq e$ for all i. But we have $\chi\varphi = id$. Thus, $\chi\varphi(\beta + K) = \beta + K$. Therefore $k_1\beta w_1 + \cdots + k_n \beta w_n + K = \beta + K$. Then, $\beta \in J^2U$, which is a contradiction.
Claim 2: ψ is not split epimorphism. Otherwise, we would have $\chi_1\colon U \to V$ such that $\psi\chi_1 = id$. Hence $\chi_1(e + K) = (le + \sum_{i=1}^n l_i w_i) + L$, where $l, l_i \in \mathfrak{K}$ and each w_i is a nontrivial path with $t(w_i) = e$. Then

$$e + K = \psi\chi_1(e+K) = \psi\left((le + \sum_{i=1}^n l_i w_i) + L \right) = (le + \sum_{i=1}^n l_i w_i) + K \,.$$

Therefore $l = 1$ and $\sum_{i=1}^n l_i w_i \in K$. Let $w := \sum_{i=1}^n l_i w_i$, $k_i := k_i(\alpha, e)$ and $v_i := v_i(\alpha, e)$. Then,

$$\chi_1(K) = \chi_1((\alpha - \sum_{i\in I(\alpha,e)} k_i v_i) + K) = \left(\alpha - \sum_{i\in I(\alpha,e)} k_i v_i + (\alpha - \sum_{i\in I(\alpha,e)} k_i v_i) w \right) + L \,.$$

Therefore,

$$\alpha - \sum_{i\in I(\alpha,e)} k_i v_i + (\alpha - \sum_{i\in I(\alpha,e)} k_i v_i) w \in L \,.$$

This is a contradiction, since L is generated by linear combinations of paths of length greater than one or not starting with α.

(ii): By (i), $U = \Lambda e/K$ where $K = \sum_{q \text{ nonroute on } p} \Lambda q$. Let $L = \sum_{\alpha \neq q \text{ nonroute on } p} \Lambda q$ and $V = \Lambda e/L$. Then,

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

where φ and ψ are defined as above. As in the proof of (i), φ is not a split monomorphism. Moreover, ψ is not split epimorphism. Otherwise, we would have $\chi_1 \colon U \to V$ such that $\psi\chi_1 = id$. Then, similarly, $\chi_1(e + K) = e + w$, where $w = \sum_{i=1}^{n} l_i w_i \in K$. Then,

$$\chi_1(K) = \chi_1(\alpha + K) = (\alpha + \alpha w) + L\,.$$

Therefore, $\alpha + \alpha w \in L$, which is again a contradiction.

∎

5.2 On $\alpha(U)$ over left multiserial algebras

Uniserial representations of left multiserial algebras have been studied in [23]. (See Section 1.4.) Here we find an upper bound for $\alpha(U)$ where U is a uniserial module over a left m-multiserial algebra. In particular we show that, when $m \geq 2$, $\alpha(U) \leq m$, where U is a uniserial module over a left m-multiserial algebra.

Theorem 5.2.1. *Let U be a non-projective uniserial module over a left m-multiserial algebra Λ with $m \geq 2$, then $\alpha(U) \leq m$.*

Proof. Suppose p is a mast for U. Put $p = \alpha_l \cdots \alpha_1$. By Remark 1.4.2, we can assume that for every arrow α in Γ_1, $\Lambda\alpha$ is uniserial. Let $\mathcal{A} = \{\Lambda\gamma p \mid \gamma \in \Gamma_1\}$. Any two members of $\mathcal{A}$ are comparable; i.e., for $\gamma_1, \gamma_2 \in \Gamma_1$, either $\Lambda\gamma_1 p \subseteq \Lambda\gamma_2 p$ or $\Lambda\gamma_2 p \subseteq \Lambda\gamma_1 p$, since $\Lambda\alpha_1$ is uniserial. Hence, there exists a maximal element in $\mathcal{A}$, say $\Lambda\gamma p$. Notice that $\Lambda\gamma p$ can be zero. This happens when $Jp = 0$.

Case 1: There is neither a detour nor a nonroute starting at $e := s(p)$. Here, Λe is uniserial and since U is not projective, we have $\gamma p \neq 0$ in Λ and $U = \Lambda e/\Lambda\gamma p$. Therefore $\alpha(U) \leq 2 \leq m$ by Proposition 5.1.6.

Case 2: There are detours $(\alpha_j, e) \wr p$ and nonroutes $\delta_t e$ starting at $e := s(p)$, where $0 \leq j \leq l$ and $l+1 \leq t \leq n$. Notice that $n \leq m-1$, since Λ is m-multiserial. Let $a_j = \alpha_j - \sum_{i \in I(\alpha,e)} k_i(\alpha_j, e) v_i(\alpha_j, e)$ and $a_t = \delta_t$. If $\Lambda\gamma p = 0$, then $U = \Lambda e / \sum_{j=0}^{n} \Lambda a_j$. Otherwise $U = \Lambda e / (\sum_{j=0}^{n} \Lambda a_j + \Lambda \gamma p)$. Let $0 \to DTrU \xrightarrow{f} \bigsqcup_{i \in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence. By Proposition 5.1.8, all the induced irreducible maps $g_i \colon B_i \to U$ are epimorphisms. By Proposition 5.1.7(iii), $\alpha(U) \leq \dim_{\mathfrak{K}} \operatorname{soc} DTrU$. But by [3, Proposition IV.1.11], we know that $\operatorname{soc} DTrU \cong P_1/JP_1$ where $P_1 \to \Lambda e \to U$ is a minimal projective presentation of U. Therefore $\alpha(U) \leq m$.

∎

When Λ is left 1-multiserial, i.e., left serial, and U is uniserial then the proof of Theorem 5.2.1 shows $\alpha(U) \leq 2$.

Corollary 5.2.2. *Suppose U is a non-projective uniserial module with mast p over a left m-multiserial algebra Λ. Then*

(i) If p has no detour or nonroute starting at $e = s(p)$, then $\alpha(U) \leq 2$.

(ii) If $m = 2$, i.e., Λ is a left biserial algebra, and $Jp = 0$, then $\alpha(U) = 1$.

Proof. Part (i) follows from proof of the above theorem. In part (ii), there is either one detour (α, e) or one nonroute δ starting at $e = s(p)$. Then $U = \Lambda e / \Lambda a$, where $a = \alpha - \sum_{i \in I(\alpha,e)} k_i v_i$ or $a = \delta$. Hence $\alpha(U) = 1$ by [3, Proposition V.6.3], because the image of $\Lambda f \xrightarrow{\cdot a} \Lambda e$ is not in $J^2 e$. ∎

5.3 Irreducible radical embeddings of uniserials

Let $0 \to DTrU \xrightarrow{f} \bigsqcup_{i \in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence with U a uniserial module and the B_i indecomposable. Proposition 5.1.7(iii) gives us an upper bound for the number of irreducible epimorphisms $B_i \to U$. Also Proposition 5.1.7(i) says that there is at most one irreducible monomorphism $B_i \to U$; if there is one, it is isomorphic to $JU \hookrightarrow U$. Therefore it is interesting to know whether $JU \hookrightarrow U$ is irreducible or not. In this section, first we give some necessary conditions, in addition to Proposition 5.1.8, for this to happen and then we state a conjecture by A. Boldt [6] which proposes necessary and sufficient conditions for irreducibility in the case where the quiver does not have oriented cycles. We then will show that this conjecture is true for a large class of algebras including left multiserial algebras, and also for monomial algebras.

Recall that a detour (α, u) on a path p is called inessential if

$$\alpha u = s + \sum_{i \in I(\alpha,u)} k_i v_i(\alpha, u)$$

in Λ, where s is a $\mathfrak{K}$-linear combination of paths, none of which is a route on p, and $k_i \in \mathfrak{K}$ for all $i \in I(\alpha, u)$. A detour is *essential* if it is not inessential.

Recall that the set $I(\alpha, u)$ indexes those right subpaths $v_i(\alpha, u)$ of p which are longer than u, and end in the same vertex as α. The following generalizes [6, Conjecture 1.2.1, (1)⇒(2)a], by weakening the assumption that the quiver has no oriented cycles. The method used also differs from that in [6].

Theorem 5.3.1. *Let U be a non-simple uniserial module with mast p, where p does not start with an oriented cycle. If $JU \hookrightarrow U$ is irreducible, then*

(i) All detours on p are inessential.

(ii) All non-routes are in Jp.

In particular, $U = \Lambda e / Jp$, where $e = s(p)$.

Proof. Recall from Proposition 5.1.8 that no detour or non-route starts at $e = s(p)$. Let $p = \alpha_n \cdots \alpha_1$ and suppose $(\delta_i, u_i) \wr\wr p$ for $0 \le i \le m$. Let $N_R = \sum_{q \text{ nonroute on } p} \Lambda q e$ and

$$\Delta_j = \delta_j u_j - \sum_{i \in I(\delta_j,u_j)} k_i(\delta_j, u_j) v_i(\delta_j, u_j)\,.$$

Proof of (i): We first assume that $\mathfrak{K} \neq \mathbb{Z}_2$. Suppose $U = \Lambda e/K$, where $K = \sum_{j=1}^{m} \Lambda\Delta_j + N_R$, with m minimum. If $m > 0$, let $U' = \Lambda e/L$ where $L = \sum_{j=2}^{m} \Lambda\Delta_j + J\Delta_1 + N_R$. Notice that $eJe \subseteq N_R$ by our assumption on p; hence $eU' = \frac{\mathfrak{K}e+L}{L}$. Let

$$V = \frac{U' \sqcup JU'}{H},$$

where $H = \Lambda(p+L, kp+L) + \Lambda(\Delta_1 + L, \Delta_1 + L)$ with $0, 1 \neq k \in \mathfrak{K}$. We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

where $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L) + H$ and $\psi((e + L, 0 + L) + H) = se + K$ and $\psi((0 + L, \alpha_1 + L) + H) = l\alpha_1 + K$, with $s, l \in \mathfrak{K}$ such that $s + l = 1$ and $s + lk = 0$. Note that such elements exist, since $\mathfrak{K} \neq \mathbb{Z}_2$.

1. φ is well-defined:

$$\varphi(K) = \varphi(\Delta_1 + K) = (\Delta_1 + L, \Delta_1 + L) + H = H.$$

2. ψ is well-defined: We have $\psi((p + L, kp + L) + H) = sp + lkp + K = 0 + K$, and $\psi((\Delta_1 + L, \Delta_1 + L) + H) = s\Delta_1 + l\Delta_1 + K = 0 + K$.

3. $\psi\varphi = id|_{JU}$:

$$\psi\varphi(\alpha_1 + K) = \psi((\alpha_1 + L, \alpha_1 + L) + H) = s\alpha_1 + l\alpha_1 + K = \alpha_1 + K.$$

4. φ is not a split monomorphism: Otherwise there would exist $\chi \in \mathrm{Hom}_\Lambda(V, JU)$ such that $\chi\varphi = id$. Then $\chi((e + L, 0 + L) + H) = 0 + K$. Hence,

$$\alpha_1 + K = \chi\varphi(\alpha_1 + K) = \chi((\alpha_1 + L, \alpha_1 + L) + H) = \chi((0 + L, \alpha_1 + L) + H).$$

Then $\chi((0 + L, \alpha_1 + L) + H) = \alpha_1 + K$. Therefore,

$$\begin{aligned} \chi(H) &= \chi((p + L, kp + L) + H) = \chi((p + L, L) + H) + \chi((L, kp + L) + H) \\ &= kp + K \neq K, \end{aligned}$$

which is a contradiction. Therefore, ψ splits; i.e., there exists $\chi_1 \in \text{Hom}_\Lambda(U,V)$ such that $\psi\chi_1 = id$. Hence $\chi_1(e+K) = (s^{-1}e+L, L) + H$ because of the assumption that p does not start with an oriented cycle. Then,

$$\chi_1(K) = \chi_1(\Delta_1 + K) = (s^{-1}\Delta_1 + L, L) + H = H\,.$$

Then, $(s^{-1}\Delta_1 + L, L) \in H$. Hence,

$$(s^{-1}\Delta_1 + L, L) = z(p+L, kp+L) + z'(\Delta_1 + L, \Delta_1 + L)\,,$$

with $z, z' \in \Lambda$. Therefore we have

$$\begin{aligned} s^{-1}\Delta_1 + L &= zp + z'\Delta_1 + L\,, \\ L &= kzp + z'\Delta_1 + L\,. \end{aligned}$$

Then, $s^{-1}\Delta_1 + L = (1-k)zp + L$. Hence $\Delta_1 - s(1-k)zp \in L$. Thus $\Delta_1 - s(1-k)zp = \sum_{i=2}^m \mu_i\Delta_i + \nu\Delta_1 + \omega$ with $\mu_i \in \mathfrak{K}, \nu \in J$ and $\omega \in N_R$. Then $s(1-k)zp \in K$. This implies $zp \in Jp$, since $pU \neq 0$. Hence $\Delta_1 \in L$. This is in contradiction with the minimality of m.

Now suppose $\mathfrak{K} = \mathbb{Z}_2$. With the same notation, let

$$V = \frac{U' \sqcup JU' \sqcup JU'}{H}\,,$$

where $H = \Lambda(0+L, p+L, p+L) + \Lambda(\Delta_1+L, \Delta_1+L, \Delta_1+L)$. Then as in the previous case,

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

is a nontrivial factorization of $JU \hookrightarrow U$ through V, where $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, \alpha_1 + L) + H$ and $\psi((e+L, 0+L, 0+L)+H) = e+K$, $\psi((0+L, \alpha_1+L, 0+L)+H) = \alpha_1 + K$ and $\psi((0+L, 0+L, \alpha_1+L)+H) = \alpha_1 + K$.

Proof of (ii): Again, we first assume that $\mathfrak{K} \neq \mathbb{Z}_2$. By part (i), $U = \Lambda e/K$ where $K = \sum_{i=1}^m \Lambda\beta_i u_i + Jp$, and each $\beta_i u_i$ is nonroute on p with u_i a right subpath of p, $\beta_i \in \Gamma_1$. Assume m is minimum. If $m > 0$, then let $U' = \Lambda e/L$ where $L = (\sum_{i=2}^m \Lambda\beta_i u_i + Jp)$ and

$$V = \frac{U' \sqcup JU'}{H}\,,$$

where $H = \Lambda(p + L, kp + L) + \Lambda(\beta_1 u_1 + L, \beta_1 u_1 + L)$ for some $k \in \mathfrak{K}, k \neq 0, 1$. We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

where $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L)$ and $\psi((e+L, L)+H) = se+K$, and $\psi((L, \alpha_1 + L)+H) = l\alpha_1 + K$ with $s, l \in \mathfrak{K}$ such that $s+l = 1$ and $s+kl = 0$. Similar to (i), we can see that φ, ψ are well-defined, $\psi\varphi = id|_{JU}$, and φ is not a split monomorphism. Therefore ψ splits; i.e., there is a $\chi \in \mathrm{Hom}_\Lambda(U, V)$ such that $\psi\chi = id$. Hence $\chi(e_1 + K) = (s^{-1}e + L, L) + H$. Then

$$\chi(K) = \chi(\beta_1 u_1 + K) = (s^{-1}\beta_1 u_1 + L, L) + H\,.$$

Therefore, $(s^{-1}\beta_1 u_1 + L, L) = w(p+L, kp+L) + w'(\beta_1 u_1 + L, \beta_1 u_1 + L)$ where $w, w' \in \Lambda$. Hence,

$$\begin{aligned} s^{-1}\beta_1 u_1 + L &= wp + w'\beta_1 u_1 + L\,, \\ L &= kwp + w'\beta_1 u_1 + L\,. \end{aligned}$$

Therefore $s^{-1}\beta_1 u_1 + L = (1-k)wp + L$. Hence

$$s^{-1}\beta_1 u_1 + (k-1)wp = vp + \sum_{i=2}^{m} w_i \beta_i u_i\,, \tag{13}$$

where $v \in J$ and $w_i \in \Lambda$. If we multiply equation (13) by $t(\beta_1)$ from the left, we get that $t(\beta_1)wp$ is zero or a nonroute on p, since $t(\beta_1) \neq t(p)$. Then equation (13) is in contradiction with the minimality of m since it expresses $\beta_1 u_1$ as an element of L.

Now suppose that $\mathfrak{K} = \mathbb{Z}_2$. With the same notation, let

$$V = \frac{U' \sqcup JU' \sqcup JU'}{H}\,,$$

where $H = \Lambda(0 + L, p + L, p + L) + \Lambda(\beta_1 u_1 + L, \beta_1 u_1 + L, \beta_1 u_1 + L)$. We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

where $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, \alpha_1 + L)$ and $\psi((e + L, L, L) + H) = e + K$, $\psi((L, \alpha_1 + L, L) + H) = \alpha_1 + K$ and $\psi((L, L, \alpha_1 + L) + H) = \alpha_1 + K$. Similarly, this is a nontrivial factorization of $JU \hookrightarrow U$ through V. ∎

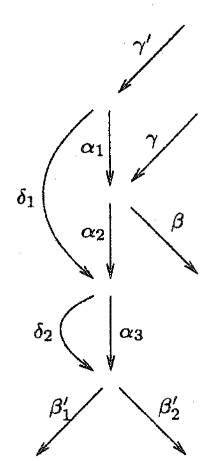
Suppose now that Γ is a finite quiver *without oriented cycles.* To prepare for our analysis in this section, we fix a uniserial left Λ-module U with mast

$$p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \cdots \xrightarrow{\alpha_{n-1}} n\,.$$

We now name all the arrows in Γ that touch p, classifying them according to the type of contact with p. The notation and the following diagram are from [6].

$$\begin{aligned}
B &:= \{\beta \in \Gamma_1 | s(\beta) \in \{1,\dots,n-1\} \text{ and } t(\beta) \notin \{1,\dots,n\}\}\,,\\
B' &:= \{\beta' \in \Gamma_1 | s(\beta') = n\}\,,\\
C &:= \{\gamma \in \Gamma_1 | s(\gamma) \notin \{1,\dots,n\} \text{ and } t(\gamma) \in \{2,\dots,n\}\}\,,\\
C' &:= \{\gamma' \in \Gamma_1 | t(\gamma') = 1\}\,,\\
D &:= \{\delta \in \Gamma_1 | \{s(\delta), t(\delta)\} \subset \{1,\dots,n\} \text{ and } \delta \notin \{\alpha_1,\dots,\alpha_{n-1}\}\}\,.
\end{aligned}$$

We illustrate these definitions with an example. Consider the following quiver Γ, together with the path $p = \alpha_3\alpha_2\alpha_1$:



We then have

$$\begin{aligned} B &= \{\beta\}, \\ B' &= \{\beta'_1, \beta'_2\}, \\ C &= \{\gamma\}, \\ C' &= \{\gamma'\}, \\ D &= \{\delta_1, \delta_2\}. \end{aligned}$$

Conjecture 5.3.2. *[6, Conjecture 1.2.1] The following statements are equivalent:*

(1) The embedding $JU \hookrightarrow U$ is irreducible.

(2) U is not simple and satisfies both (a) and (b) below:

(a) For every $\beta \in B$,

$$\beta\alpha_{s(\beta)-1} \cdots \alpha_1 \in Jp,$$

and for every $\delta \in D$,

$$\delta\alpha_{s(\delta)-1} \cdots \alpha_1 \in \mathfrak{K}\alpha_{t(\delta)-1} \cdots \alpha_1 .$$

(b) There exists a subset $R \subset J$ such that $\{rp + J^2p \mid r \in R\}$ forms a $\mathfrak{K}$-basis for Jp/J^2p and (i) and (ii) both hold:

(i) For every $\gamma \in C$ there exists $w \in pJ$ such that, for every $r \in R$,

$$r\alpha_{n-1} \cdots \alpha_{t(\gamma)}\gamma = rw .$$

(ii) For every $\delta \in D$ and every $r \in R$,

$$r\alpha_{n-1} \cdots \alpha_{t(\delta)}\delta \in \mathfrak{K}r\alpha_{n-1} \cdots \alpha_{s(\delta)} .$$

A. Boldt [6] proved (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2)(a). Theorem 5.3.1 is a generalization of (1) $\Rightarrow$ (2)(a) and our proof also uses a different method from the one in [6]. Moreover, we will show that this conjecture is true for left multiserial algebras as well as monomial algebras.

5.4 The conjecture for a class of algebras including left multiserial algebras

Throughout this section we assume that the quiver Γ has no oriented cycles. Here we prove that the conjecture is true if the algebra in addition has the property that

$$\dim_{\mathfrak{K}}\left(J\alpha_{n-1}/J^2\alpha_{n-1}\right) \leq 1\,.$$

In particular, the conjecture is true for left multiserial algebras with a presentation so that for each $\alpha \in \Gamma_1$, $\Lambda\alpha$ is uniserial (see 1.4.2). The following lemma holds only assuming Γ has no oriented cycles.

Lemma 5.4.1. *Let U be a uniserial module with mast p and β' be an arrow. If $JU \hookrightarrow U$ is irreducible and $\beta' p \neq 0$, then there is a uniserial module with mast $\beta' p$.*

Proof. There is a basis $\{\beta_i' p + J^2 p \mid 1 \leq i \leq m, \beta_i' \in \Gamma_1\}$ for Jp/J^2p, with $\beta_1' = \beta'$. Let

$$p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \;\cdots\; \xrightarrow{\alpha_{n-1}} n\,,$$

and $n+1 := t(\beta')$, $q := \beta' p$ and suppose $(\delta, u) \wr q$. If $t(\delta) \in \{1, 2, \ldots, n\}$, then $(\delta, u) \wr p$ and so by Theorem 5.3.1(i), $\delta u \in \mathfrak{K}\alpha_{t(\delta)-1} \cdots \alpha_1$. If $t(\delta) = n+1$, then by Theorem 5.3.1(ii), $\delta u \in Jp$. Hence,

$$\delta u = l_1 \beta' p + l_2 \beta_2' p + \cdots + l_m \beta_m' p + wp\,, \tag{14}$$

with $w \in J^2, l_i \in \mathfrak{K}$. If for some $\beta \in \Gamma_1$, βu is a nonroute on q, then it is a nonroute on p as well and so $\beta u \in Jp$ and $t(\beta) \notin \{1, \ldots, n+1\}$. Hence, in this case, $\beta u \in \sum_{i=2}^m \mathfrak{K}\beta_i' p + J^2 p$. Define $V = \Lambda e_1/L$, where

$$L := Jq + \sum_{i=2}^{m} \Lambda \beta_i' p + \sum_{(\delta,u)\wr q, t(\delta)=n+1} \Lambda\left(\delta u - l_1 q\right)\,. \tag{15}$$

Thus, V is a uniserial module. We only need to show that $qV \neq 0$. Suppose $qV = 0$. Then, $q \in L$ and by equations (14) and (15), we get $q \in Jq + \sum_{i=2}^m \Lambda\beta_i' p + J^2 p$. Then,

$$q = vq + \sum_{i=2}^{m} \lambda_i \beta_i' p + w' p\,, \tag{16}$$

with $v \in J, \lambda_i \in \Lambda$ and $w' \in J^2$. Multiply equation (16) by $t(\beta')$. Since the quiver does not have oriented cycles, $vq = 0$, which contradicts the choice of the basis of Jp/J^2p.

■

Lemma 5.4.2. *Suppose* $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} = 1$. *Then there exists an arrow* β' *such that* $\mathfrak{K}\beta'\alpha_{n-1} + J\beta'\alpha_{n-1} = J\alpha_{n-1}$.

Proof. By the hypothesis there is some $\beta' \in \Gamma_1$ with $\beta'\alpha_{n-1} \notin J^2\alpha_{n-1}$. We will show that $J^2\alpha_{n-1} = J\beta'\alpha_{n-1}$. For this we only need to show that any path in $J^2\alpha_{n-1}$ is in $J\beta'\alpha_{n-1}$. If not, let q be a longest path in $J^2\alpha_{n-1} \backslash J\beta'\alpha_{n-1}$. Then $q = \gamma_r \cdots \gamma_1\alpha_{n-1}$, where $\gamma_i \in \Gamma_1$ and $\gamma_1\alpha_{n-1} \notin J^2\alpha_{n-1}$, otherwise q could be replaced by a longer path. Hence $\gamma_1\alpha_{n-1} = k\beta'\alpha_{n-1} + w\alpha_{n-1}$, where $0 \neq k \in \mathfrak{K}$ and $w \in J^2$. Therefore,

$$q = \gamma_r \cdots \gamma_1\alpha_{n-1} = k\gamma_r \cdots \gamma_2\beta'\alpha_{n-1} + \gamma_r \cdots \gamma_2 w\alpha_{n-1} \,.$$

Since $\gamma_r \cdots \gamma_2 w\alpha_{n-1}$ is a linear combination of paths in $J^2\alpha_{n-1}$ longer than q, and therefore $\gamma_r \cdots \gamma_2 w\alpha_{n-1} \in J\beta'\alpha_{n-1}$ and so is q. This is a contradiction. ■

Theorem 5.4.3. *Let* Λ *be an algebra where the quiver has no oriented cycles and* U *be a uniserial* Λ*-module with mast* $p = \alpha_{n-1} \cdots \alpha_1$. *If* $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} \leq 1$, *then the following statements are equivalent:*

(1) The embedding $JU \hookrightarrow U$ *is irreducible.*

(2) U *is not simple and satisfies both (a) and (b) below:*

(a) For every $\beta \in B$,

$$\beta\alpha_{s(\beta)-1} \cdots \alpha_1 \in Jp \,,$$

and for every $\delta \in D$,

$$\delta\alpha_{s(\delta)-1} \cdots \alpha_1 \in \mathfrak{K}\alpha_{t(\delta)-1} \cdots \alpha_1 \,.$$

(b) $Jp/J^2p = 0$ *or there is an arrow* β' *such that* $\{\beta'p + J^2p\}$ *forms a* $\mathfrak{K}$*-basis for* Jp/J^2p *and (i) and (ii) both hold:*

(i) For every $\gamma \in C$ there exists $w \in pJ$ such that

$$\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma = \beta' w\,.$$

(ii) For every $\delta \in D$

$$\beta'\alpha_{n-1}\cdots\alpha_{t(\delta)}\delta \in \mathfrak{K}\beta'\alpha_{n-1}\cdots\alpha_{s(\delta)}\,.$$

Proof. Note first that, under the present hypotheses, the conditions (2) are equivalent to those in Conjecture 5.3.2. The conditions 2(a) are identical.

We have that $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} \leq 1$ so that, by Lemma 5.4.2, we can take the set R of Conjecture 5.3.2(2)(b) to be $\{\beta'p + J^2p\}$ or $\emptyset$. Then Conjecture 5.3.2(2)(b)(i) and (ii) reduce to the corresponding parts of this theorem.

Now we get (2) $\Rightarrow$ (1) from [6]. We get (1) $\Rightarrow$ (2)(a) by Proposition 5.3.1, or [6, page 18].

(1) $\Rightarrow$ (2)(b)(i):

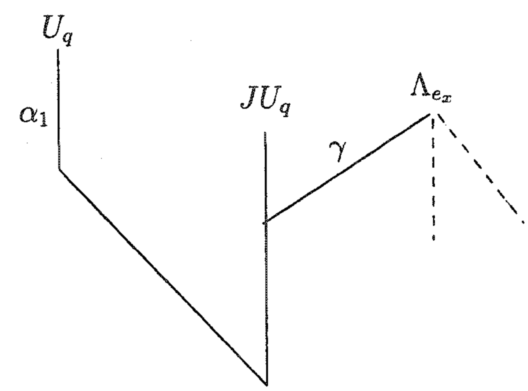
Suppose $Jp/J^2p \neq 0$. Let $\beta'p \in Jp\backslash J^2p$ with $\beta' \in \Gamma_1$. Then $\beta'\alpha_{n-1} \in J\alpha_{n-1}\backslash J^2\alpha_{n-1}$ and $\{\beta'\alpha_{n-1} + J^2\alpha_{n-1}\}$ is a basis for $J\alpha_{n-1}/J^2\alpha_{n-1}$. We will show that for $\gamma \in C$, $\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma \in \beta'pJ$. By (2)(a), we know that $U = \Lambda e_1/Jp$. Let $q = \beta'p$ and $K = Jp$. By Lemma 5.4.1, there exists a uniserial module $U_q = \Lambda e/L$ with mast q, where

$$L = Jq + \sum_{(\delta,u)\wr p,\ t(\delta)=t(q)} (\delta u - l(\delta,u)q)\,.$$

Let

$$V = \frac{U_q \sqcup JU_q \sqcup \Lambda e_x}{H}\,,$$

where $H = \Lambda(q+L, q+L, 0) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma)$ with $e_x = s(\gamma)$.



Notice that $e_1 V = \mathfrak{K}(e_1 + L, 0, z) + H$, where z is a linear combination of paths from $s(\gamma) = e_x$ to e_1. We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

where $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, 0) + H$ and $\psi((e_1 + L, L, 0) + H) = e_1 + K, \psi((L, \alpha_1 + L, 0) + H) = 0$ and $\psi((L, L, e_x) + H) = 0$. Then $\varphi(K) = \varphi(\beta' \alpha_{n-1} \cdots \alpha_1 + K) = (q + L, q + L, 0) + H = 0$ and so φ is well-defined.

Claim: φ is not a split monomorphism; otherwise there would exist $\chi\colon V \to JU$ such that $\chi\varphi = id$. We have $\alpha_1 + K = \chi\psi(\alpha_1 + K) = \chi(\alpha_1 + L, \alpha_1 + L, 0) = \chi(\alpha_1 + L, L, 0) + \chi(L, \alpha_1 + L, 0) = \chi(L, \alpha_1 + L, 0)$, because $\chi(e_1 + L, L, 0) = 0$. Also we have $\chi(L, L, e_x) = 0$. But

$$\chi(H) = \chi((L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_{t(\gamma)}\gamma) + H) = \alpha_n \cdots \alpha_2 \alpha_1 + K \neq K,$$

which is a contradiction. Therefore ψ splits, since $JU \hookrightarrow U$ is irreducible. Hence there exists $\chi_1 : U \to V$, with $\psi\chi_1 = id$. We have $\chi_1(e_1 + K) = ((e_1 + L, L, \sum_{i=1}^m k_i w_i) + H)$, where w_i are the paths from e_x to e_1 and $k_i \in \mathfrak{K}$. But $q \in K$ and so

$$0 = \chi_1(K) = \chi_1(q + K) = (q + L, L, \sum_{i=1}^m k_i q w_i) + H.$$

Hence,

$$(q + L, L, \sum_{i=1}^m k_i q w_i) \in \Lambda(q + L, q + L, 0) + \Lambda(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_{t(\gamma)}\gamma).$$

Then, by Lemma 5.4.2

$$(q+L, L, \sum_{i=1}^{m} k_i q w_i) = k(q+L, q+L, 0) + l\beta'(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma))$$
$$+ \sum_{l(u_i)\geq 1} l_i u_i \beta'(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma)\,,$$

where $k, l, l_i \in \mathfrak{K}$. Therefore $k = 1$ and $l = -1$. Hence,

$$(*)\quad \beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma = -\sum k_i \beta'\alpha_{n-1}\cdots\alpha_1 w_i - \sum_{l(u_i)\geq 1} l_i u_i \beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma\,.$$

If we multiply $(*)$ from the left by $t(\beta')$; using the fact that quiver does not have oriented cycles, $t(\beta')u_i = 0$. Then,

$$\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma = -\sum k_i \beta'\alpha_{n-1}\cdots\alpha_1 w_i \in \beta' p J\,.$$

(1) $\Rightarrow$ (2)(b)(ii):
Suppose $\delta \in D$. We will show that $\beta'\alpha_{n-1}\cdots\alpha_{t(\delta)}\delta \in \mathfrak{K}\beta'\alpha_{n-1}\cdots\alpha_{s(\delta)}$. Let $\delta\colon i \to j$ and $q := \beta'\alpha_{n-1}\cdots\alpha_1$. Again let $U_q = \Lambda e_1/L$ be the uniserial with mast q, with L as above. Let

$$V = \frac{U_q \sqcup JU_q \sqcup \Lambda e_i}{H}\,,$$

where

$$H = \Lambda(q+L, q+L, 0) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_j\delta) + \Lambda(L, L, \alpha_{n-1}\cdots\alpha_i)\,.$$

We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

where $\varphi(\alpha_1+K) = (\alpha_1+L, \alpha_1+L, 0)+H$ and $\psi((e_1+L, L, 0)+H) = e_1+K$, $\psi((L, \alpha_1+L, 0)+H) = 0$ and $\psi((L, L, e_i)+H) = 0$.
Claim: φ is not a split monomorphism; otherwise there would exist $\chi\colon V \to JU$ such that $\chi\varphi = id$. Then we would have $\alpha_1+K = \chi\varphi(\alpha_1+K) = \chi((\alpha_1+L, \alpha_1+L, 0)+H) = \chi((\alpha_1+L, L, 0)+H) + \chi((L, \alpha_1+L, 0)+H) = \chi((L, \alpha_1+L, 0)+H)$, because $\chi((e_1+L, L, 0)+H) = 0$. Also we know that $\chi((L, L, e_i)+H) = k\alpha_{i-1}\cdots\alpha_1+K$,

where $k \in \mathfrak{K}$. Thus, $\chi(H) = \chi((L, L, \alpha_{n-1} \cdots \alpha_i) + H) = k\alpha_{n-1} \cdots \alpha_1 + K$. Therefore, $k = 0$. But

$$\chi(H) = \chi((L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) + H) = \alpha_{n-1} \cdots \alpha_2 \alpha_1 + K \neq K\,,$$

which is a contradiction.

Therefore, ψ splits, since $JU \hookrightarrow U$ is irreducible. Hence there exists $\chi_1 : U \to V$ with $\psi\chi_1 = id$. We have $\chi_1(e_1 + K) = (e_1 + L, L, 0) + H$. Hence, $\chi_1(K) = \chi_1(q + K) = (q_1 + L, L, 0) + H$. Therefore,

$$(q + L, L, 0) \in \Lambda(q + L, q + L, 0) + \Lambda(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) + \Lambda(L, L, \alpha_{n-1} \cdots \alpha_i)\,.$$

Then, by Lemma 5.4.2

$$\begin{aligned}(q + L, L, 0) = k(q + L, q + L, 0) &+ l\beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta)\\ &+ \sum_{l(u_s) \geq 1} l_s u_s \beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta)\\ &+ v(L, L, \alpha_{n-1} \cdots \alpha_i)\,,\end{aligned}$$

where $l, l_s \in \mathfrak{K}$, $u_s \in J$ and $v \in \Lambda$. Hence $k = 1$ and $l = -1$. Therefore,

$$(**)\quad q_3 = \beta' \alpha_{n-1} \cdots \alpha_j \delta = \sum_{l(u_s) \geq 1} l_s u_s \beta' \alpha_{n-1} \cdots \alpha_j \delta + v\alpha_{n-1} \cdots \alpha_i\,,$$

in Λe_i. If we multiply $(**)$ from the left by $t(\beta')$; using the fact that there are no oriented cycles, $t(\beta')u_i = 0$. We get

$$\beta' \alpha_{n-1} \cdots \alpha_j \delta = t(\beta') v \alpha_{n-1} \cdots \alpha_i\,.$$

Then $t(\beta')v\alpha_{n-1} \in J\alpha_{n-1}$. But $J\alpha_{n-1} = \mathfrak{K}\beta'\alpha_{n-1} + J\beta'\alpha_{n-1}$ by Lemma 5.4.2. Therefore,

$$\beta' \alpha_{n-1} \cdots \alpha_j \delta = k\beta' \alpha_{n-1} \cdots \alpha_i + w\beta' \alpha_{n-1} \cdots \alpha_i\,,$$

where $w \in J$. But, $t(\beta')w = 0$, since there are no oriented cycles. Therefore, $\beta' \alpha_{n-1} \cdots \alpha_j \delta = k\beta' \alpha_{n-1} \cdots \alpha_i$. ∎

5.5 The Conjecture for the monomial algebras

In this section we prove that the conjecture is true for monomial algebras.

Theorem 5.5.1. *Suppose Λ is a monomial algebra whose quiver has no oriented cycles and U is a uniserial Λ-module with mast p. Then the following statements are equivalent:*

(1) The embedding $JU \hookrightarrow U$ is irreducible.

(2) U is not simple and satisfies both (a) and (b) below:

(a) (i) For every $\beta \in B$, $\beta\alpha_{s(\beta)-1}\cdots\alpha_1 = 0$,
(ii) For every $\delta \in D$, $\delta\alpha_{s(\delta)-1}\cdots\alpha_1 = 0$.

(b) For every $\beta' \in B'$ such that $\beta' p \neq 0$ we have:
(i) For every $\gamma \in C$, $\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma = 0$,
(ii) For every $\delta \in D$, $\beta'\alpha_{n-1}\cdots\alpha_{t(\delta)}\delta = 0$.

Proof. Note first that, since the algebra is monomial, the conditions (2) are equivalent to the ones in Conjecture 5.3.2.

As in Theorem 5.4.3, we have (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2)(a).
(1) $\Rightarrow$ (2)(b)(i):
Let $p = \alpha_{n-1}\cdots\alpha_1$ and $U = \Lambda e_1/K$. Suppose that there is $\beta' \in B'$ such that $\beta' p \neq 0$ and $\beta'\alpha_{n-1}\cdots\alpha_i\gamma \neq 0$ for some $\gamma \in C$, where $x \xrightarrow{\gamma} i$, with $x \notin \{1, 2, \ldots, n\}$. By condition 2(a), $V_p = \{\underline{0}\}$. Let

$$q_1 := \beta'\alpha_{n-1}\cdots\alpha_1, \; q_2 := \beta'\alpha_{n-1}\cdots\alpha_i\gamma \,.$$

Since Λ is a monomial algebra and $q_i \neq 0$; Proposition 2.1.3 says that $\underline{0} \in V_{q_i}$ for $i = 1$ and 2. Let $U_{q_1} := \Phi_{q_1}(\underline{0}) = \Lambda e_1/L$ and $U_{q_2} := \Phi_{q_2}(\underline{0}) = \Lambda e_x/F$, where $e_x = s(\gamma)$. Let

$$V = \frac{U_{q_1} \sqcup JU_{q_1} \sqcup U_{q_2}}{H} \,,$$

where

$$H = \Lambda(q_1 + L, q_1 + L, F) + \Lambda(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_i \gamma + F).$$

Once again, for $v \in V$, $e_1 v = (ke_1 + L, L, z + F) + H$, where z is a linear combination of paths from $s(\gamma)$ to e_1. However, such a path goes through e_1 and so is a non-route on q_2, i.e., $z \in F$. We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

where $\varphi\colon JU \to V$ by $\alpha_1 + K \mapsto (\alpha_1 + L, \alpha_1 + L, F) + H$ and $\psi\colon V \to U$ by $(e_1 + L, L, F) + H \mapsto e_1 + K, (L, \alpha_1 + L, F) + H \mapsto 0 + K$ and $(L, L, e_x + F) + H \mapsto 0$. Similar to the proof of Theorem 5.4.3, φ and ψ are well-defined, $\psi\varphi = id|_{JU}$ and φ is not a split monomorphism. We will prove that ψ is also not a split epimorphism, which contradicts the irreducibility of $JU \hookrightarrow U$.

Claim: ψ is not a split epimorphism; otherwise there would exist $\chi : U \to V$ with $\psi\chi = id$. We have $\chi(e_1 + K) = (e_1 + L, L, F) + H$. But $q_1 = \beta' \alpha_{n-1} \cdots \alpha_1 \in K$. Hence $\chi(K) = \chi(q_1 + K) = (q_1 + L, L, F) + H$ is zero in V. Then $(q_1 + L, L, 0) = k(q_1 + L, q_1 + L, 0) + l\beta'(\alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_{t(\gamma)}\gamma + L, F)$, where $k, l \in \mathfrak{K}$. Therefore $k = 1,\ k + l = 0,\ l = 0$, which is a contradiction.

(1) $\Rightarrow$ (2)(b)(ii):

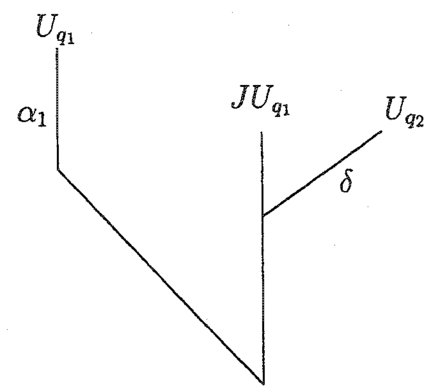
Suppose there is $\beta' \in B'$ such that $\beta' p \neq 0$ and $\beta' \alpha_{n-1} \cdots \alpha_{t(\delta)} \delta \neq 0$ for some $\delta \in D$. Let $\delta\colon i \to j$. By (a)(ii), $s(\delta) = i \neq 1$. Let

$$q_1 := \beta' \alpha_{n-1} \cdots \alpha_1, q_2 := \beta' \alpha_{n-1} \cdots \alpha_j \delta.$$

and let $U_{q_i} = \Lambda e_1 / L_i$ be the uniserial module corresponding to $0 \in V_{q_i}$ for $i = 1, 2$. Let

$$V = \frac{U_{q_1} \sqcup JU_{q_1} \sqcup U_{q_2}}{H},$$

where $H = \Lambda(q_1, q_1, 0) + \Lambda(0, \alpha_{n-1} \cdots \alpha_1, \alpha_{n-1} \cdots \alpha_j \delta)$.



We have

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U\,,$$

where $\varphi\colon JU \to V$ by $\alpha_1 + K \mapsto (\alpha_1 + L, \alpha_1 + L, F) + H$ and $\psi\colon V \to U$ by $(e_1 + L, L, F) + H \mapsto e_1 + K, (L, \alpha_1 + L, F) + H \mapsto K$ and $(L, L, e_i + F) + H \mapsto K$. Similarly, φ and ψ are well-defined.

Claim: φ is not a split monomorphism.

Suppose there exists $\chi\colon V \to JU$ such that $\chi\varphi = id$. Then, we have $\alpha_1 + K = \chi\psi(\alpha_1+K) = \chi(\alpha_1+L, \alpha_1+L, F)+H = \chi((\alpha_1+L, L, F)+H)+\chi((L, \alpha_1+L, F)+H) = \chi((L, \alpha_1 + L, F) + H)$. Also we know that $\chi((L, L, e_i + F) = k\alpha_{i-1}\cdots\alpha_1 + K$, where $k \in \mathfrak{K}$. Then $\chi((L, L, \delta e_i + F) = k\delta\alpha_{i-1}\cdots\alpha_1 + K = 0$, by (a)(ii), and

$$\chi(H) = \chi(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_j\delta + F) = \alpha_{n-1}\cdots\alpha_2\alpha_1 + K \neq K\,,$$

which is a contradiction.

Claim: ψ is not a split epimorphism.

Suppose there exists $\chi_1 : U \to V$ with $\psi\chi_1 = id$. We have $\chi_1(e_1+K) = (e_1+L, L, F)+H$. Hence $\chi_1(K) = \chi_1(q_1+K) = (q_1+L, L, F)+H$. Therefore $(q_1+L, L, F)+H = H$, and so

$$(q_1 + L, L, F) \in \Lambda(q_1 + L, q_1 + L, F) + \Lambda\beta'(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_j\delta + F)\,.$$

Then $(q_1 + L, L, F) = k(q_1 + L, q_1 + L, F) + l\beta'(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_j\delta + F)$, with $k, l \in \mathfrak{K}$. Therefore $k = 1,\ k + l = 0,\ l = 0$, which is a contradiction.

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