SPDEs with infinite-variance Lévy noise

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Abstract

This thesis is devoted to the study of the existence and uniqueness of solutions for stochastic partial differential equations (SPDEs) driven by Lévy noise. The main contributions of this work are contained in the recent publications [32] and [5]. Article [32] focuses on a stochastic wave equation with multiplicative Lévy noise. We establish the existence and uniqueness of a random field solution, relying only on the integrability of the Lévy measure on the region $|z| \leq 1$. Furthermore, we show that this solution has finite moments up to a certain stopping time, which depends on a bounded region of space. Article [5] studies a broader class of SPDEs driven by heavy-tailed Lévy noise, which includes the Parabolic Anderson Model (PAM) and the Hyperbolic Anderson Model (HAM). Specifically, we demonstrate the existence of solutions for SPDEs driven by symmetric α-stable Lévy noise. Using the Lepage representation of the noise and techniques borrowed from the theory of multiple stable integrals, we construct a solution that has a series representation which depends only on the points of the jump measure associated with the noise.

Dedication

This thesis is dedicated to my beloved parents, Elisa and Luis. Your constant love, encouragement, and unconditional support have been the cornerstone of my journey through this PhD and every challenge I have faced throughout my life. Your daily example of hard work and discipline has been a profound inspiration to me, showing me the importance of dedication and resilience in achieving my goals. This work stands as a testament to your love, sacrifices, and unwavering confidence in me. From the bottom of my heart, thank you for everything.

I also dedicate this thesis to the memory of my grandmother, Cenide, whose love and support remain unforgettable. You were a guiding light in my life, and your warmth and encouragement continue to inspire me even in your absence.

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Chapter 1

Introduction

The study of stochastic partial differential equations (SPDEs) is motivated by the need to model systems influenced by random perturbations. These stochastic equations capture the interplay between deterministic dynamics and stochastic effects, providing a framework to understand complex phenomena in various fields. Random perturbations appear in a wide variety of fields, where systems are influenced by unpredictable external or internal forces. For instance, in physics, Brownian motion exemplifies random particle movements due to molecular collisions. In finance, the fluctuations of asset prices are modeled using stochastic differential equations (SDEs) to reflect inherent market uncertainties. Mechanical systems in engineering are often subject to random vibrations caused by environmental noise, while in biology, population dynamics can fluctuate due to random environmental factors. One way to interpret random perturbations is through the concept of noise.

There are several approaches to defining a noise in the literature. In this thesis, we assume that a noise Λ on a set U is a stochastic process $\{\Lambda(A); A \in \mathscr{U}\}$ that satisfies certain conditions, such as σ-additivity and independence over disjoint sets. Here, $\mathscr{U}$ represents a collection of subsets of U that has a specific algebraic structure, e.g. it is a ring. The goal of the thesis is to study the existence and uniqueness of solutions to non-linear SPDEs of the form

$$\mathcal{L}u = \sigma(u)\dot{\Lambda}, \tag{1.0.1}$$

subject to some initial conditions. We assume that $\mathcal{L}$ is a second-order partial differential operator (such as the wave operator, or the heat operator), $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, and Λ represents a noise defined on a domain $U \subset \mathbb{R}^d$. Specifically, this work focuses on the case where Λ is a space-time Lévy noise without Gaussian component, and possibly infinite variance, given by the Lévy Itô decomposition:

$$\Lambda(dt, dx) = b\, dt\, dx + \int_{\{|z|\leq 1\}} z\widehat{J}(dt, dx, dz) + \int_{\{|z|>1\}} zJ(dt, dx, dz),$$

where J is a Poisson random measure with intensity $\mu(dt, dx, dz) = dt\, dx\, \nu(dz)$, where ν is a *Lévy measure* on $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, i.e

$$\int_{\mathbb{R}_0} (1 \wedge z^2)\nu(dz) < +\infty.$$

Here, $\widehat{J}$ is the compensator of J, i.e. $\widehat{J} = J - \mu$.

We say that a process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a *mild solution* of (1.0.1) if it satisfies the stochastic integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda(ds,dy), \tag{1.0.2}$$

where G is the fundamental solution of $\mathcal{L}G = \delta_0$, where δ_0 is the Dirac measure in 0, and w is the deterministic solution of $\mathcal{L}w = 0$ subject to the same initial conditions. We will consider only mild solutions of equation (1.0.1).

The novelty of this thesis lies in the results presented in Chapters 5 and 6, which are included in articles [32] and [5], respectively. These results can be summarized as follows:

(i) In Chapter 5, we generalize the results of [4] by establishing the existence and uniqueness of a solution to the stochastic wave equation with multiplicative Lévy noise, considering only the p-integrability of the intensity in the region of small jumps, i.e., $|z| \leq 1$. Specifically, for a fixed interval $[0,T]$, we show the uniqueness of a random field solution u that satisfies (1.0.2) and

$$\sup_{t\in[0,T]} \sup_{x\in\overline{D}} \mathbb{E}[|u(t,x)|^p \mathbb{1}_{[\![0,\tau_N(D)]\!]}(t)] < +\infty,$$

for all bounded domains D and $N \in \mathbb{N}$, where $\overline{D}$ is the topological closure of D, and $p > 0$ is an exponent satisfying $\int_{|z|\leq 1} |z|^p \nu(dz) < +\infty$, with ν denoting the Lévy measure of the noise. Here, $\tau_N(D)$ is a stopping time associated with the light-cone region corresponding to D.

(ii) In Chapter 6, we prove the existence of a solution for a class of SPDEs of the form $\mathcal{L}u = u\dot{Z}$, where Z is a *symmetric α-stable Lévy noise*. Furthermore, we construct this solution using the Lepage representation (also called Ferguson Klass representation, see also [26]) of the noise along with the multiple stable integrals developed in [50, 51, 52].

SPDEs with Lévy noises provide a mathematical framework to model diffusive or vibrating systems perturbed by heavy-tailed noises, which are useful for modeling a wide variety of discontinuous phenomena across several fields, such as financial crashes, turbulent flows, and certain types of cosmic radiation. The study of discontinuous random phenomena has a long history in physics, dating back to the early 20th century when researchers first began to explore systems with sudden changes. Classical models like Brownian motion were initially used to describe continuous random fluctuations, but they proved inadequate for systems exhibiting large, unpredictable jumps. This led to the development of models incorporating discontinuities, such as Lévy processes, which allowed for the mathematical representation of such phenomena.

Although the study of stochastic differential equations (SDEs) has been a very active area of research since the development of Itô's stochastic calculus, there is still no unified theory for solving SPDEs. To the best of our knowledge, there are two main mathematical approaches to interpret SPDEs:

(i) The *random field approach*, introduced by Walsh in [57], which considers the solutions of SPDEs as real-valued random fields;

(ii) The *infinite-dimensional approach*, developed by Da Prato and Zabczyk in [21], which views the solutions of SPDEs as stochastic processes taking values in a Hilbert space.

In this thesis, we focus exclusively on SPDEs driven by Lévy noises, using the random field approach (i).

We briefly summarize the main developments in the study of SPDEs driven by Lévy noise with infinite variance with respect to the approach (i).

One of the earliest works on SPDEs driven by infinite-variance Lévy noise was the article [49] by Saint-Loubert Bié in 1998, where it was proved that the stochastic heat equation (SHE) on the entire space $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}(t,x) \quad t>0,\ x\in\mathbb{R}^d, \tag{1.0.3}$$

has a unique solution u that satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\left[|u(t,x)|^p\right] < +\infty,$$

provided that σ is Lipschitz, the Lévy measure ν of the noise Λ satisfies

$$\int_{\mathbb{R}} |z|^p \nu(dz) < +\infty \quad \text{for some } p>0 \tag{1.0.4}$$

and

$$p < 1 + \frac{2}{d}. \tag{1.0.5}$$

Condition (1.0.5) comes from the requirement that $\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)dyds < +\infty$, where $G_t(x) = (2\pi t)^{-d/2}\exp(-\frac{|x|^2}{2t})$ is the heat kernel.

Condition (1.0.4) does not hold for the α-stable noise with $\alpha\in(0,2)$, for which the Lévy measure is given by

$$\nu(dz) = \left(c_+\alpha z^{-\alpha-1}\mathbb{1}_{(0,\infty)}(z) + c_-\alpha(-z)^{-\alpha-1}\mathbb{1}_{(-\infty,0)}(z)\right)dz,$$

for some $c_+\geq 0, c_-\geq 0$; see Definition (2.1.9) below.

In [38], Mueller proved the existence of a solution of (SHE) driven by α-stable Lévy noise with $\alpha\in(0,1)$ and a non-Lipschitz function $\sigma(u)=u^\gamma$ (with $\gamma>0$) multiplying the noise. The same problem for the case $\alpha\in(1,2)$ was treated by Mytnik in [39]. In [38], the Laplacian Δ is replaced by the fractional power of the Laplacian $-(-\Delta)^{\beta/2}$ for some $\beta\in(0,2]$.

The problem of existence of a solution of a general SPDE with α-stable Lévy noise and Lipschitz function σ has remained open until 2014, when Balan [2] established the existence of a solution of a large class of SPDEs on bounded domains, driven by an α-stable Lévy noise. The major breakthrough came in 2017, when Chong [17] finally solved the problem for equations on the entire space $\mathbb{R}^d$. More precisely, he proved the existence of a random field solution of (1.0.3), in the case when the driving Lévy noise Λ may have infinite variance. His construction of the solution relied on the condition that the Lévy measure ν of the noise Λ satisfies the following condition:

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) + \int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty \quad \text{for some } 0 < q \leq p. \tag{1.0.6}$$

In particular, condition (1.0.6) is satisfied by the α-stable Lévy noise.

In 2019, Chong, Dalang, and Humeau [18] generalized the results of [2] to a Lévy noise with infinite variance, again focusing only on (SHE). They also proved the regularity of the sample paths of the solution, on a bounded domain and the entire space $\mathbb{R}^d$.

In 2023, Balan [4] studied the stochastic wave equation (SWE) on $\mathbb{R}^d$ with $d \leq 2$:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}(t,x), \quad t > 0,\, x \in \mathbb{R}^d, \tag{1.0.7}$$

and established the existence of a solution using techniques inspired by [17].

Also in 2023, Berger, Chong, and Lacoin [11] proved the uniqueness of the solution of (SHE) on $\mathbb{R}^d$, in the case when

$$\sigma(u) = \beta u \quad \text{for some } \beta > 0 \quad \text{and} \quad \nu(-\infty, 0) = 0.$$

However, the uniqueness of solutions for the non-linear (SHE) with a general Lipschitz function σ remains an open problem.

In 2024, in [32], the author of this thesis proved the uniqueness of the solution to (1.0.7) on $\mathbb{R}^d$, in dimension $d \leq 2$, for any Lipschitz function σ. In the case $d = 2$, this was established under the condition:

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < +\infty, \quad \text{for some} \quad p \in (0,2),$$

while in the case $d = 1$, no conditions on the Lévy measure were needed.

Regarding approach (ii), it is important to note that stochastic evolution equations driven by a *cylindrical α-stable Lévy noise* have been studied extensively by various authors; see [15, 33, 46] and the references therein. In the case of equations driven by Gaussian noise, a direct comparison between the results obtained using approaches (i) and (ii) can be found in [25]. A similar comparison is not available yet in the literature for equations driven by Lévy noise.

We describe below the content of each chapter of the thesis.

In Chapter 2, we introduce the concepts of *Lévy white noise* and *Lévy basis*. In the literature, there are different ways to define a Lévy white noise. In this thesis, we interpret a Lévy noise as a random measure. We then present the theory of stochastic integration with respect to a Lévy basis, using the framework developed in [13, 19, 16], based on the concept of *Daniell mean*. (The stochastic integral which appears on the right-hand side of (1.0.2) is understood in this sense.) This integration theory has been successfully applied to the study of SDEs and SPDEs with heavy-tailed noises in [17, 16, 18, 4]. In addition, we introduce the *local property* of the stochastic integral, which plays a fundamental role in this thesis.

In Chapter 3, we study in detail some auxiliary results that allow us to work "locally in time" when dealing with a Lévy basis which may not have any finite moments. These results are crucial for deriving desirable properties of the stochastic integral and ensuring that the solution has finite moments, locally in time (up to a stopping time). Additionally, we briefly explore the connection between this local property and suitable stopping times, which enable us to obtain the existence of solutions on the entire space $\mathbb{R}^d$. It is worth mentioning that these tools for solving SPDEs are inherited from the techniques developed by Protter in the monograph [43] for SDEs driven by semi-martingales.

In Chapter 4, we study the groundbreaking results of [17]. This was the first article to prove the existence of a solution of an SPDE (namely (SHE)) on the entire space, driven by a Lévy noise with possibly infinite variance. The techniques developed in [17] are also applicable to (SWE), as shown by Balan in [4]. The main focus of this chapter is to delve into the proofs of Theorem 3.1 from [17] and Theorem 2.7 of [4].

In Chapter 5, we present the main results of the recent article [32], written by the author of this thesis, concerning the existence and uniqueness of solutions for (SWE) in spatial dimensions $d \leq 2$, extending the results of [4]. Specifically, we establish the following results:
(i) We first prove the uniqueness of the solution, using a methodology similar to [17, 4], based on the self-map property of the stochastic Volterra operator associated with (SWE), and the *past light-cone property* (PLCP) of the wave equation. This result is obtained under condition (1.0.6). Note that in the case of (SWE) in dimension $d = 1$, we only need to impose the following condition:

$$\int_{|z|>1} |z|^q \nu(dz) < +\infty, \quad \text{for some } q > 0. \tag{1.0.8}$$

(ii) Next, we establish the existence and uniqueness of the solution of (SWE), using techniques different from those in [17, 4]. This approach leverages the PLCP derived from the compact support property of the fundamental solution of the wave operator. Remarkably, this enables us to obtain a solution without requiring condition (1.0.8). Additionally, we derive a novel bound for the moments of the solution, valid up to a stopping time that depends on a spatial region in $\mathbb{R}^d$.

In Chapter 6, we present the results from the recent preprint [5], by the author of this thesis and Balan. In this chapter, we introduce a novel method for solving the parabolic Anderson model (PAM) and the hyperbolic Anderson model (HAM) driven by a symmetric α-stable (SαS) Lévy noise. These models refer to equation (1.0.1) with $\sigma(u) = u$, in the case when $\mathcal{L}$ is the heat operator, respectively the wave operator. For (HAM), we assume that the spatial dimension is $d \leq 2$. This method leads to an explicit series representation of the solution, and does not require the use of stopping times. The main tools used in this chapter are the Lépage representation of the SαS Lévy noises, and the theory of multiple stochastic integrals developed in [50, 51, 52, 53]. We should mention that the existence of a solution of (PAM) driven by Lévy noise with positive jumps was proved in [9] using a different method than in [5]. This method relies on the "continuum directed polymer" developed in [11], and consists in solving the equation driven by the truncated noise with jumps that exceed a fixed value a, and then let $a \to 0$. A different truncation method was used in [16, 17, 18] for (SHE) with general Lévy noise multiplied by a Lipschitz function $\sigma(u)$. This method relies on first solving the equation driven by the noise truncated to the region $\{(x,z); |z| < Kh(x)\}$ for a suitable function $h(x)$, up to a stopping time τ_K, show that the solutions are consistent if $K < K'$, and finally paste together all these solutions. Unfortunately, this method does not yield uniqueness of the solution. The same truncation method was used in [4] to show the existence of a solution of (SWE) on $\mathbb{R}^d$ with $d \leq 2$.

Finally, the appendices provide supplementary material and technical results that are used in the thesis. Appendix A contains some basic results about stochastic integration with respect to a compensated Poisson random measure. Appendix B contains some results from analysis.

We conclude the introduction by specifying the notation used in this thesis.

- $\mathbb{R}_+$ is the set of non-negative real-numbers: $\mathbb{R}_+ = [0, \infty)$.
- $\mathbb{R}_0$ is the set $\mathbb{R}\backslash\{0\}$ equipped with the distance $d(x,y) = |x^{-1} - y^{-1}|$.
- $\mathcal{B}(\mathbb{R}_0)$ is the class of Borel subsets of $\mathbb{R}_0$.
- $\mathcal{B}_b(\mathbb{R}_0)$ is the class of bounded Borel subsets of $\mathbb{R}_0$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.
- L^p is the set of all random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the norm $||X||_p := \mathbb{E}\left[|X|^p\right]$ if $0 < p < 1$ and $||X||_p := \left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}}$ if $p \geq 1$.
- L^0 is the set of all random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the pseudo-norm $||X||_0 := \mathbb{E}[|X| \wedge 1]$.
- L^∞ is the set of all random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the norm $||X||_\infty := \inf\{C \geq 0 : \mathbb{P}(|X| \leq C) = 1\}$.
- λ_d is the Lebesgue measure in $\mathbb{R}^d$.
- $\mathcal{B}(E)$ is the class of Borel sets $A \subset E$, for some Borel set $E \subset \mathbb{R}^d$.
- $\mathcal{B}_b(E)$ is the class of bounded Borel sets $A \subset E$.
- $\mathcal{P}_0$ is the predictable σ-field on $\Omega \times \mathbb{R}_+$.

- $\tilde{\mathcal{P}}$ is the tempo-spatial predictable σ-field on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$.
- $\mathcal{P}$ is the set of all $\tilde{\mathcal{P}}$-measurable maps $\phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$.
- $\tilde{\mathcal{P}}_b$ is the set of all sets $A \in \tilde{\mathcal{P}}$ such that there exists $k \in \mathbb{N}$ with $A \subset \Omega \times [0,k] \times [-k,k]^d$.
- $\mathfrak{B}$ is the class of bounded domains in $\mathbb{R}^d$. (A domain is an open connected set.)
- For $D \in \mathfrak{B}$, $\overline{D}$ is the topological closure of D with respect to the usual topology.
- $[\![R,S]\!] := \{(\omega,t) \in \Omega \times \mathbb{R}_+; R(\omega) \le t \le S(\omega)\}$ for two $\mathcal{F}_t$-stopping times R and S.
- $(\!(R,S]\!] := \{(\omega,t) \in \Omega \times \mathbb{R}_+; R(\omega) < t \le S(\omega)\}$ for two $\mathcal{F}_t$-stopping times R and S.
- For $p \in (0,\infty]$, B^p is the set of all $\phi \in \mathcal{P}$ such that

$$||\phi||_{p,T} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} ||\phi(t,x)||_p < +\infty, \quad \text{for all} \quad T \in \mathbb{R}_+.$$

- For $p \in (0,\infty]$, B^p_{loc} is the set of all $\phi \in \mathcal{P}$ such that

$$||\phi||_{p,T,R} := \sup_{t\in[0,T]} \sup_{|x|\le R} ||\phi(t,x)||_p < +\infty, \quad \text{for all } T,R \in \mathbb{R}_+.$$

- If τ is a $\mathcal{F}$-stopping time, $\phi \in B^p_{\text{loc}}(\tau)$ if $\phi \mathbb{1}_{[\![0,\tau]\!]} \in B^p_{\text{loc}}$, i.e.,

$$\sup_{t\in[0,T]} \sup_{|x|\le R} ||\phi(t,x)\mathbb{1}_{[\![0,\tau]\!]}(t)||_p < +\infty, \quad \text{for all } T,R \in \mathbb{R}_+.$$

- $B_r(x) := \{y \in \mathbb{R}^d;\, |x-y| < r\}$ for $x \in \mathbb{R}^d$ and $r > 0$.
- Δ is the Laplacian operator, i.e.

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}.$$

Chapter 2

Lévy bases

In this chapter, we introduce the concept of Lévy white noise and establish the framework for performing stochastic integration with respect to such noise. Notably, the Lévy noise considered in this thesis may not have finite variance. Since Itô's foundational work, several theories of stochastic integration have emerged, depending on the properties of the noise and its regularity. In this work, we use the theory of stochastic integration with respect to an L^p-random measure developed in [12, 13, 19, 31]. However, this is not the only theoretical framework for developing stochastic calculus, even within the random field approach. For example, if we consider a noise that is a martingale in time, such as the Gaussian white noise, the stochastic integration theory developed in [57] has had a profound impact on the study of SPDEs with Gaussian noises, leading to breakthrough results, such as [22]. On the other hand, if the noise exhibits a heavy-tailed distribution and behaves as a semi-martingale in time, the stochastic integration theory developed in [57] is not well-suited for this case. Hence, we will require using the theory for stochastic integration with respect to a semi-martingale originated in [13], and later extended in [12, 19, 31]. In particular, this theory relies on the concept of the Daniell mean, which allows us to perform stochastic integration in the absence of moments. That is, we can integrate with respect to an L^0-random measure that may have infinite moments of any order.

This chapter is organized as follows. In Section 2.1, we introduce the concept of infinitely divisible (ID) independently scattered random measure. In Section 2.2, we present the construction of a Lévy white noise. Section 2.3 introduces the concept of Lévy basis and reviews the theory of stochastic integration with respect to this object, that will be used throughout this thesis. In Section 2.4, we examine some local properties of the stochastic integral with respect to a Lévy basis.

2.1 ID independently scattered random measures

In this section, we introduce the concept of infinitely divisible (ID) independently scattered random measure, which is a generalization of a classical process with independent increments indexed by the real line. These objects play a crucial role in various fields, such as signal processing and mathematical finance. Many well-known noise processes belong to this class, such as Lévy noises, Gaussian noises, and more general processes used to model discontinuous phenomena. Although the notion of infinitely divisible noise is not recent and has been studied from various perspectives since Lévy's foundational work [36] in 1937, it was not until 1989 that the formal characterization of an ID independently scattered random measures was fully developed. In the foundational work [44], Rajput and Rosinski provided the necessary structure for understanding how these random measures can be used to model both spatial and temporal random effects. In particular, these authors provided the spectral representation of an ID random measure, and provided the necessary and sufficient conditions for integrability with respect to these objects, in the case of non-random integrands. This theory extends classical results from finite-dimensional stochastic processes to more complex infinite-dimensional settings. We note that the case of random integrands is not studied in [44]. To address this case, first one has to embed the ID independently scattered random measure into a Lévy basis, and then use the theory of integration with respect to this object. This will be reviewed in Section 2.3.

Recall that a random variable X has an *infinitely divisible* (ID) distribution if for any $n \geq 1$, there exist some i.i.d. random variables $X_1, \dots, X_n$ such that $X \overset{d}{=} X_1 + \dots + X_n$.

We present now the definition of an ID independently scattered random measure, which was introduced in [44].

Definition 2.1.1. Let E be a Borel-measurable subset of $\mathbb{R}^d$. An *independently scattered random measure* on E is a collection $M = \{M(A); A \in \mathcal{B}_b(E)\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties:

a) For any sequence $\{A_i\}_{i\in\mathbb{N}}$ of disjoint sets in $\mathcal{B}_b(E)$ with $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{B}_b(E)$, we have:

$$M\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} M(A_i) \quad \text{a.s.,} \tag{2.1.1}$$

b) $M(A_1), \dots, M(A_n)$ are independent, for any disjoint sets $A_1, \dots, A_n \in \mathcal{B}_b(E)$.

If in addition, $M(A)$ is infinitely divisible for any $A \in \mathcal{B}_b(E)$, we say that M is an *ID independently scattered random measure*.

Note that since the variables $\{M(A_i)\}_{i\geq 1}$ are independent, by Lévy equivalence theorem (see Theorem 5.3.4 in [20]), (2.1.1) is equivalent to

$$M\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} M(A_i) \quad \text{in} \quad L^0. \tag{2.1.2}$$

Note that convergence in the L^0-norm is equivalent to convergence in probability, since for any $\varepsilon \in (0,1)$,

$$\|X\|_{L^0} \leq \varepsilon + \mathbb{P}(|X| > \varepsilon) \quad \text{and} \quad \mathbb{P}(|X| > \varepsilon) = \mathbb{P}(|X| \wedge 1 > \varepsilon) \leq \frac{1}{\varepsilon}\|X\|_{L^0}.$$

ID independently scattered random measures are characterized using the following class of measures in $\mathbb{R}$.

Definition 2.1.2. We say that ν is a *Lévy measure* on $\mathbb{R}$ if it satisfies the following conditions:

$$\int_{\mathbb{R}_0} (1 \wedge z^2)\, \nu(dz) < +\infty \quad \text{and} \quad \nu(\{0\}) = 0.$$

Let M be an ID independently scattered random measure. Since $M(A)$ is ID for every $A \in \mathcal{B}_b(E)$, we have:

$$\mathbb{E}\left[e^{iuM(A)}\right] = \exp\left\{iuv_0(A) - \frac{u^2 v_1(A)}{2} + \int_{\mathbb{R}} (e^{iuz} - 1 - iu\tau(z))\nu_A(dz)\right\}, \tag{2.1.3}$$

where $v_0(A) \in \mathbb{R}$, $v_1(A) \geq 0$, ν_A is a Lévy measure on $\mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ is the truncation function given by

$$\tau(z) = \begin{cases} z & \text{if} \quad |z| \leq 1, \\ 1 & \text{if} \quad |z| > 1, \end{cases}$$

Moreover, v_0 and v_1 are signed measures on E.

The following result is presented in [44] in a slightly more general form.

Proposition 2.1.3 (Proposition 2.1 in [44])**.** Let E be a Borel-measurable subset of $\mathbb{R}^d$. For any triplet (b, σ, ν) with $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and ν a Lévy measure, there exists an ID independently scattered random measure M on E, such that for any $A \in \mathcal{B}_b(E)$ and $u \in \mathbb{R}$,

$$\mathbb{E}\left[e^{iuM(A)}\right] = \exp\left\{\lambda_d(A)\Big(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iuz} - 1 - iu\tau(z))\nu(dz)\Big)\right\}. \tag{2.1.4}$$

If (2.1.4) holds, we say that M has *triplet* (γ, σ, ν).

We include below a well-known example of an ID independently scattered random measure.

Example 2.1.4. A *Gaussian space-time white noise* on $\mathbb{R}_+ \times \mathbb{R}^d$ is a zero-mean Gaussian process $W = \{W(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance:

$$\mathbb{E}[W(A)W(B)] = \lambda_{d+1}(A \cap B) \quad \text{for any } A, B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d). \tag{2.1.5}$$

For this process,

$$\mathbb{E}\left[e^{iuW(A)}\right] = \exp\left\{-\lambda_{d+1}(A)\frac{u^2}{2}\right\} \quad \text{for all } A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d).$$

Note that W is a random measure that takes values in $L^2(\Omega)$, i.e. $W : \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d) \to L^2(\Omega)$, and for any disjoint sets $\{A_i\}_{i\in\mathbb{N}}$ in $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we have

$$W\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} W(A_i) \quad \text{in } L^2(\Omega).$$

We define $W(1_A) = W(A)$. By linearity and a density argument, we can extend this definition to all functions $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. We say that

$$W(f) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t,x) W(dt,dx)$$

is the *Wiener integral* of $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. Relation (2.1.5) can be extended as follows: for any functions $f, g \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$:

$$\mathbb{E}\left[W(f)W(g)\right] = \langle f, g \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}$$

In the case of random integrands, a theory of stochastic integration with respect to W was developed in [57], using the concept of worthy martingale.

The random field $W(t,x) = W([0,t] \times [0,x])$ is called a *Brownian sheet*. Below is a simulation of this process on $[0,1] \times [0,1]$.

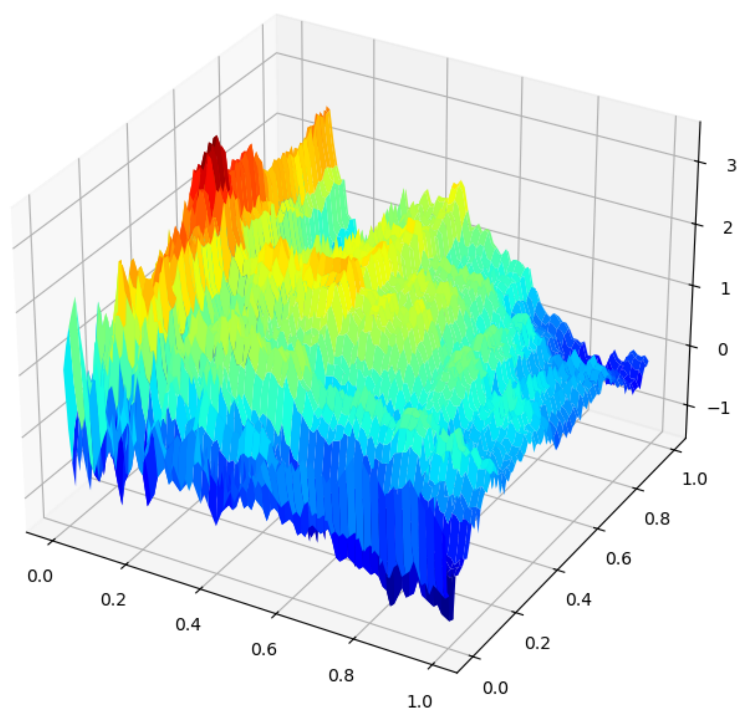


Figure 2.1: Simulation of the Brownian sheet on $[0,1] \times [0,1]$

Note that there exist Gaussian noises which are not independently scattered random measures. For example, the *fractional Gaussian noise* (fGn) does not satisfy condition (b) of Definition 2.1.1. Recall that the fGn with indices $H_0, H_1, \ldots, H_d \in (\frac{1}{2}, 1)$ is a zero-mean Gaussian process $W = \{W(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance

$$\mathbb{E}[W(A)W(B)] = \int_A \int_B \prod_{i=0}^{d} |t_i - s_i|^{2H_i - 2} d\mathbf{t} d\mathbf{s},$$

where $\mathbf{t} = (t_0, t_1, \ldots, t_d), \mathbf{s} = (s_0, s_1, \ldots, s_d) \in \mathbb{R}_+ \times \mathbb{R}^d$. Moreover, the fGn does not induce a semi-martingale in time: for any $A \in \mathcal{B}_b(\mathbb{R}^d)$, the process $\{W_t(A) := W([0,t] \times A); t \geq 0\}$ is a fractional Brownian motion (modulo a constant), which is not a semi-martingale (see [40]). Therefore, the stochastic integration in the Itô sense is not well-suited for fGns, and one must consider the stochastic integral in the Skorokhod sense (see page 40 in [41]).

An important class of ID independently scattered random measures are the Poisson random measures (PRMs). In particular, PRMs are key objects in the theory of stochastic processes: any semi-martingale with càdlàg (i.e. right continuous with left limits) sample paths has a jump part which can be described by a PRM.

Before providing the definition of PRMs, we present some preliminary concepts.

Let E be a Borel subset of $\mathbb{R}^d$ and $U = E \times \mathbb{R}_0$. Let $M_p(U)$ be the set of point measures on U, equipped with the topology of vague convergence. Recall that a *point measure* is a Radon measure with values in the set $\{0, 1, 2, \ldots\}$. (We say that μ is a *Radon measure* on U if $\mu(F) < +\infty$ for any compact set $F \subseteq U$. If $(\mu_n)_{n \geq 1}$ and μ are Radon measures on U, we say that $(\mu_n)_{n \geq 1}$ *converges vaguely* to μ if $\mu_n(F) \to \mu(F)$ for any compact set $F \subseteq U$.) We let $\mathcal{M}_p(U)$ denote the Borel σ-field generated by $M_p(U)$.

Definition 2.1.5. A *Poisson random measure* (PRM) on U of intensity μ is a map $\mathcal{N} : \Omega \to M_p(U)$ which is $\mathcal{M}_p(U)$-measurable and satisfies the following properties:
(i) $\mathcal{N}(F)$ has a Poisson distribution with mean $\mu(F)$[1], for any Borel set F in U;
(ii) $\mathcal{N}(F_1), \ldots, \mathcal{N}(F_n)$ are independent, for any disjoint Borel sets $F_1, \ldots, F_n$ in U.

We denote by $\mathbb{PRM}(\mu)$ a Poisson random measure (PRM) with intensity measure μ.

If $\mathcal{N}$ is a PRM on U of intensity μ, we define the *compensated process* $\widehat{\mathcal{N}}$ by

$$\widehat{\mathcal{N}}(F) = \mathcal{N}(F) - \mu(F)$$

for any Borel set F in U with $\mu(F) < \infty$. Unlike $\mathcal{N}(\omega, \cdot)$, $\widehat{\mathcal{N}}(\omega, \cdot)$ is not a measure for any fixed $\omega \in \Omega$.

A Poisson random measure $\mathcal{N}$ on $(E, \mathcal{B}_b(E))$ with finite intensity measure μ can be expressed as a sum of Dirac delta measures. Specifically, $\mathcal{N}$ is represented as

$$\mathcal{N} = \sum_{i=1}^{\tau} \delta_{X_i},$$

where $\{X_i\}_{i \geq 1}$ are i.i.d. random elements in E with law $\mu/\mu(E)$, and τ is a Poisson-distributed random variable with mean $\mu(E)$. Each point X_i is treated as an atom, and $\delta_{X_i}(A)$ is equal to 1 if $X_i \in A$, and 0 otherwise. Thus, the Poisson random measure can be understood as a collection of point masses, where the measure $\mathcal{N}(A)$ counts the number of random points X_i that fall within the set A, i.e. $\mathcal{N}(A) = \text{card}\{i \geq 1; X_i \in A\}$.

An important component of a space-time Lévy noise is the jump part, which is governed by a PRM on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0$. This PRM controls the frequency and magnitude of jumps,

[1] We use the convention that $\mathcal{N}(F) = +\infty$ a.s. if $\mu(F) = +\infty$

ensuring that the larger jumps occur less frequently, but still contribute significantly to the overall dynamics of the system, without causing singularities or concentrations of large jumps in compact regions of space-time. Meanwhile, the small jumps, which often accumulate near zero intensity, are typically modeled by the compensator of this PRM, ensuring that this part behaves like a local martingale in time. In a bounded region of $\mathbb{R}^d$, a Lévy noise can exhibit an infinite number of jumps. A consequence of this is that a Lévy noise behaves as a semi-martingale with respect to time. A PRM under these conditions is often given by a $\mathbb{PRM}(\mu)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0$, where the intensity measure is given by

$$\mu(dt, dx, dz) = \lambda_{d+1}(dt, dx)\nu(dz).$$

Here, ν is a Lévy measure on $\mathbb{R}$. Note that $\mu(F) < +\infty$ for all $F \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0)$.

On the other hand, if ν is not a finite measure, i.e., there exists $\varepsilon \in (0, 1]$ such that $\nu(\{|z| < \varepsilon\}) = +\infty$, then

$$\mu(A \times \{|z| < \varepsilon\}) = +\infty$$

for any $A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\lambda_{d+1}(A) > 0$. This implies that J has an infinite number of points in the region $\{|z| < \varepsilon\}$.

The notion of a Lévy white noise can be defined in various ways. In this thesis, we define a space-time Lévy white noise as an ID independently scattered random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. For further details on the equivalence and unification of the definitions of Lévy noises, we refer the reader to Chapter 2 of [28].

Definition 2.1.6. A *space-time Lévy white noise* is a process $L = \{L(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ given by:

$$L(A) = b\lambda_{d+1}(A) + \sigma W(A) + L_{\text{small}}(A) + L_{\text{large}}(A) \tag{2.1.6}$$

where $b \in \mathbb{R}$, $\sigma \geq 0$, W is a Gaussian space-time white noise,

$$L_{\text{small}}(A) := \int_{A\times\{|z|\leq 1\}} z\widehat{J}(dt, dx, dz), \quad \text{and} \quad L_{\text{large}}(A) := \int_{A\times\{|z|>1\}} zJ(dt, dx, dz).$$

Here J is a $\mathbb{PRM}$ independent of W with intensity $\mu = \lambda_{d+1} \times \nu$, $\widehat{J} = J - \mu$, and ν is a Lévy measure. We say that (2.1.6) is the *Lévy–Itô decomposition* of L.

Lemma 2.1.7 (Lemma 2 of [2])**.** A process L with representation (2.1.6) is an ID independently scattered random measure on $E = \mathbb{R}_+ \times \mathbb{R}^d$, and has (b, σ, ν).

Remark 2.1.8. (i) By Theorem 6.6 in [42], $L_{\text{small}} = \{L_{\text{small}}(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is a centered process with finite variance

$$\mathbb{E}\left[|L_{\text{small}}(A)|^2\right] = \lambda_{d+1}(A) \int_{\{|z|\leq 1\}} |z|^2\nu(dz) < +\infty.$$

Moreover, the process $t \mapsto L_{\text{small}}([0, t] \times B)$ is a martingale for any fixed $B \in \mathcal{B}_b(\mathbb{R}^d)$.

(ii) On the other hand, since $\mu(A\times\{|z|>1\})<+\infty$, the process $L_{\text{large}}=\{L_{\text{large}}(A);A\in\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)\}$ satisfies

$$|L_{\text{large}}(A)|=\left|\sum_{i\geq 1}1_A(T_i,X_i)Z_i1_{\{|Z_i|>1\}}\right|<+\infty \quad \text{a.s.},$$

where (T_i,X_i,Z_i) are the points of J. Note that the sum above has a finite number of terms that are non-zero. In addition, $t\mapsto L_{\text{large}}([0,t]\times B)$ is a càdlàg processes with trajectories of bounded variation on any finite time interval.

Note that if $\sigma=0$, we say that L is a *pure-jump* space-time Lévy white noise.

The following definition introduces an important class of Lévy white noises. SPDEs driven by this type of noise will be examined in Chapter 6.

Definition 2.1.9. Let $\alpha\in(0,2)$ be arbitrary. A Lévy noise L with triplet (b,σ,ν) is called an *α-stable Lévy noise* if $\sigma=0$ and $\nu=\nu_\alpha$, where

$$\nu_\alpha(dz)=\left(c_+\alpha z^{-\alpha-1}\mathbb{1}_{(0,\infty)}(z)+c_-\alpha(-z)^{-\alpha-1}\mathbb{1}_{(-\infty,0)}(z)\right)dz, \tag{2.1.7}$$

for some $c_+\geq 0, c_-\geq 0$. In particular, if $b=0$ and $c_+=c_-=\frac{1}{2}$, then

$$\nu_\alpha(dz)=\frac{1}{2}\alpha|z|^{-\alpha-1}\mathbb{1}_{\{|z|>0\}}\,dz, \tag{2.1.8}$$

and we say that L is a *symmetric α-stable (SαS) Lévy noise.*

Note that an α-stable Lévy noise is in fact an α-stable random measure, in the sense of Definition 3.1.1 of [53]. We discuss this briefly below. More properties of α-stable random measures can be found in Chapter 3 of [53].

Definition 2.1.10. A random variable X has an *α-stable distribution* with stability index $\alpha\in(0,2)$, location parameter $\mu\in\mathbb{R}$, scale parameter $\sigma\in[0,+\infty)$ and skewness parameter $\beta\in[-1,1]$, if for any $u\in\mathbb{R}$,

$$\mathbb{E}\left(e^{iuX}\right)=\exp\left\{-|u|^\alpha\sigma^\alpha\left(1-i\,\text{sgn}(u)\,\beta\,\tan\frac{\pi\alpha}{2}\right)+iu\mu\right\}, \quad \text{if } \alpha\neq 1,$$

or

$$\mathbb{E}\left(e^{iuX}\right)=\exp\left\{-|u|\sigma\left(1+i\,\text{sgn}(u)\,\beta\frac{2}{\pi}\ln|u|\right)+iu\mu\right\}, \quad \text{if } \alpha=1.$$

We denote this distribution by $S_\alpha(\sigma,\beta,\mu)$.

Definition 2.1.11 (Definition 3.3.1 [53])**.** An ID independently scattered random measure M that satisfies

$$M(A)\sim S_\alpha(m(A)^{1/\alpha},\beta,0)$$

is called an *α-stable random measure on $(E,\mathcal{B}_b(E))$ with control measure m and skewness intensity β.* If $\beta=0$, we say that M is a *symmetric α-stable (SαS) random measure.*

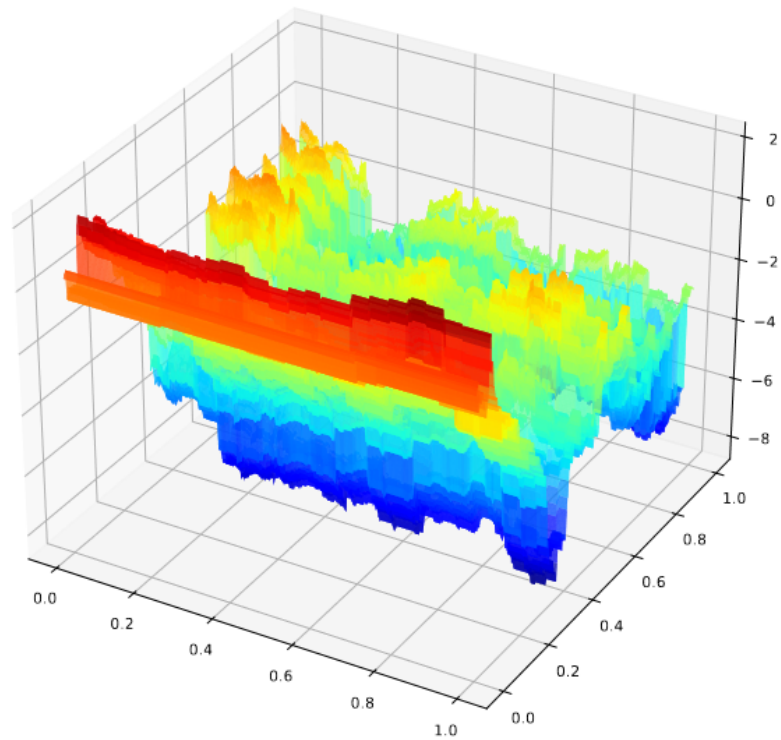


Figure 2.2: Simulation of a symmetric α-stable Lévy sheet on $[0,1]\times[0,1]$, i.e. $L(x,y) = L([0,x]\times[0,y])$ for $x,y\in[0,1]$, with $\alpha = 1.5$.

Lemma 2.1.12 (Lemma 3 of [2])**.** An α-stable Lévy noise on $\mathbb{R}_+\times\mathbb{R}^d$ with drift term given given by

$$b = \begin{cases} \int_{\{|z|\le 1\}} z\nu_\alpha(dz) & \text{if} \quad \alpha < 1, \\ (c_- - c_+)\int_0^\infty (\sin z - z\mathbb{1}_{\{|z|\le 1\}})z^{-2}dz & \text{if} \quad \alpha = 1, \\ -\int_{\{|z|>1\}} z\nu_\alpha(dz) & \text{if} \quad \alpha > 1, \end{cases} \tag{2.1.9}$$

is an α-stable random measure with control measure m given by:

$$m(A) = C_\alpha^{-1}\lambda_{d+1}(A) \quad \text{where} \quad C_\alpha = \left(\int_0^\infty \frac{\sin x}{x^\alpha}dx\right)^{-1}, \tag{2.1.10}$$

for any $A\in\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$, and skewness intensity $\beta = c_+ - c_-$.

By a direct calculation, it can be proved that

$$b = \beta\frac{\alpha}{\alpha-1} \quad \text{if} \quad \alpha\neq 1,$$

where b is the constant given by (2.1.9) and β as in Lemma 2.1.12.

Remark 2.1.13. Let L be an α-stable Lévy noise with representation (2.1.6) with $\sigma = 0$. Then, we have the following decompositions of L:

(i) For any $A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$L(A) = \int_{A\times\mathbb{R}} zJ(ds, dy, dz), \quad \text{if } \alpha < 1, \tag{2.1.11}$$

and

$$L(A) = \int_{A\times\mathbb{R}_0} z\widehat{J}(ds, dy, dz), \quad \text{if } \alpha > 1, \tag{2.1.12}$$

provided b is chosen as in (2.1.9).

(ii) If $\alpha = 1$, $b = 0$, and $c_+ = c_-$, then L satisfies

$$L(A) \overset{\mathbb{P}}{=} \lim_{\varepsilon\to 0} \int_{A\times\{|z|\geq\varepsilon\}} zJ(ds, dy, dz). \tag{2.1.13}$$

Indeed, for $\varepsilon \in (0, 1)$, by the symmetry of the noise,

$$\begin{aligned} X_\varepsilon(A) &:= L(A) - \int_{A\times\{|z|\geq\varepsilon\}} zJ(ds, dy, dz) \\ &= \int_{A\times\{|z|\leq 1\}} z\widehat{J}(ds, dy, dz) - \int_{A\times\{\varepsilon\leq|z|\leq 1\}} zJ(ds, dy, dz) \\ &= \int_{A\times\{|z|<\varepsilon\}} z\widehat{J}(ds, dy, dz). \end{aligned} \tag{2.1.14}$$

Note that $X_\varepsilon(A) \in L^2(\Omega)$ although $L(A)$ and $\int_{A\times\{|z|\geq\varepsilon\}} zJ(ds, dy, dz)$ may not be in $L^2(\Omega)$. Hence, by (2.1.14) and Chebysev's inequality, we have: for any $\delta > 0$,

$$\begin{aligned} \mathbb{P}\Big(|X_\varepsilon(A)| > \delta\Big) &\leq \frac{1}{\delta^2}\mathbb{E}\left[\left|\int_{A\times\{|z|<\epsilon\}} z\widehat{J}(ds, dy, dz)\right|^2\right] \\ &= \frac{1}{\delta^2}\int_{A\times\{|z|<\epsilon\}} |z|^2\nu(dz)dsdy \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Hence, (2.1.13) holds.

In a nutshell, an ID independently scattered random measure on E is a collection of elements of L^0 (indexed by subsets of E), which are independent for disjoint regions of E. Thus, it is natural to develop a stochastic integration theory based on convergence in probability, as in [55], which was the first work to establish a theory of stochastic integration for random integrators. This theory was later used in [44] to characterize the class of (non-random) integrable functions with respect to an ID independently scattered random measure. For Gaussian random measures such as W in Example 2.1.4, Walsh developed a theory of stochastic integration for predictable processes; see [57]. However, for Lévy white noises that lack L^p-integrability conditions, we can only develop a stochastic integration for deterministic integrands, as in [44]. To develop a theory of stochastic integration for predictable processes, it is necessary to embed the ID independently scattered random measure into a Lévy basis, as in [16]. This approach connects the concept of ID independently scattered random measures in [44] with the theory of random measures established in [13], which will be discussed in Section 2.3.

For the remainder of this section, we give the characterization of deterministic functions that are L^0-integrable with respect to an ID independently scattered random measure M, as outlined in [44].

We say that f is a *simple function* if it is given by

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}(x), \tag{2.1.15}$$

where $A_1, \dots, A_n$ are disjoint sets in $\mathcal{B}_b(E)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. We denote by $\overline{\mathcal{S}}$ the collection of simple functions of the form (2.1.15). If f is a simple function of the form (2.1.15), we define the stochastic integral of f with respect to M by:

$$I^M(f) = \int_E f(x)M(dx) = \sum_{i=1}^{n} \alpha_i M(A_i),$$

and stochastic integral of f on the set $A \in \mathcal{B}_b(E)$, with respect to M, by:

$$I_A^M(f) = \int_A f(x)M(dx) = \sum_{i=1}^{n} \alpha_i M(A \cap A_i).$$

Definition 2.1.14. Let M be an ID independently scattered random measure in $E \subset \mathbb{R}^d$. A Borel measurable function $f : E \to \mathbb{R}$ is M*-integrable* if there exists a sequence $\{f_n\}_{n\geq 0}$ in $\overline{\mathcal{S}}$ such that

i) $f_n \to f$ a.s. for $n \to +\infty$,

ii) for every $A \in \mathcal{B}_b(E)$, the sequence $\{I_A^M(f_n)\}_{n\geq 0}$ converges in probability, as $n \to +\infty$.

If f is M-integrable, we set

$$I^M(f) \overset{\mathbb{P}}{=} \lim_{n\to+\infty} I^M(f_n).$$

One of the main goals in [44] is to characterize integrable functions with respect to an ID independently scattered random measure, as follows.

Proposition 2.1.15 (Theorem 2.7 in [44]). Let $E \in \mathcal{B}(\mathbb{R}^d)$ and M be an ID independently scattered random measure on E, with triplet (b, σ, ν). Let $f : E \to \mathbb{R}$ be a Borel measurable function. Then f is M-integrable if and only if the following conditions hold:

i) $\int_E |bf(x) + \int_{\mathbb{R}} zf(x)(\mathbb{1}_{|zf(x)|\leq 1} - \mathbb{1}_{|z|\leq 1})\nu(dz)|dx < +\infty$,

ii) $\int_E |\sigma f(x)|^2 dx < +\infty$,

iii) $\int_{E\times\mathbb{R}}(|zf(x)|^2 \wedge 1)dx\nu(dz) < +\infty$.

Moreover, if f is M-integrable, then

$$\mathbb{E}\left[e^{iuI^M(f)}\right] = \exp\left\{\int_E \Big(ibf(x)u - \frac{\sigma^2 u^2}{2}|f(x)|^2 + \int_{\mathbb{R}}(e^{iuf(x)z} - 1 - iuf(x)\tau(z))\nu(dz)\Big)dx\right\}. \tag{2.1.16}$$

Note that (2.1.16) is the analog of the Lévy-Khintchine formula for classical Lévy processes.

Remark 2.1.16. An extension of Proposition 2.1.15 to the case of random integrands is given by Theorem 4.1 of [19], using stochastic integration techniques developed in [13].

Proposition 2.1.15 gives a full characterization of non-random L^0-integrands with respect to a Lévy white noise L, in terms of the triplet (b, σ, ν). However, if we aim to integrate predictable processes with respect to M without any moment assumptions, we need to modify the structure of the noise L. This issue will be addressed in Section 2.3. Note that the noise L possesses moments of order p depending on the integrability properties of ν. Indeed, for any $p > 0$, we have:

$$\int_{\{|z|>1\}} |z|^p \nu(dz) < +\infty \Leftrightarrow \mathbb{E}\left[|L(A)|^p\right] < +\infty \quad \text{for all } A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d). \tag{2.1.17}$$

Remark 2.1.17 (Pure-jump Lévy white noise with finite variance)**.** Let L be space-time Lévy white noise given by (2.1.6) with $\sigma = 0$. Suppose that the Lévy measure ν satisfies:

$$\int_{\{|z|>1\}} |z|^2 \nu(dz) < +\infty, \tag{2.1.18}$$

and $b = -\int_{\{|z|>1\}} z\nu(dz)$. (Note that (2.1.18) implies that $\int_{\{|z|\geq 1\}} |z|\nu(dz) < +\infty$.) Then for all $A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, and

$$L(A) = \int_{A\times\mathbb{R}_0} z\widehat{J}(dt, dx, dz).$$

Clearly, L is a centered process with finite variance. Hence, for a fixed $B \in \mathcal{B}_b(\mathbb{R}^d)$, the process $\{X_t = L([0,t]\times B); t \geq 0\}$ is a square-integrable martingale (with possibly discontinuous sample paths), making it suitable for theories typically used for L^2-random measures, such as Gaussian noises and Walsh's theory [57]. Additionally, in this case, the isometry property holds:

$$\mathbb{E}\left[\left|\int_{\mathbb{R}_+\times\mathbb{R}^d} f(s,y)L(ds,dy)\right|^2\right] = m_2 \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathbb{E}\left[|f(s,y)|^2\right] dyds,$$

for any predictable function f, where $m_2 = \int_{\mathbb{R}_0} |z|^2\nu(dz)$. Consequently, the existence and uniqueness of solutions to SPDEs with finite-variance Lévy noise can be established similarly to the space-time Gaussian case (see for instance [6, 8]).

2.2 Construction of a space-time Lévy white noise

In this section, we present a construction of the space-time Lévy white noise, as outlined in Section 2 of [2]. Since the continuous part of Lévy noises has been widely studied in the literature, we assume in this construction that the space-time Lévy white noise has no Gaussian component.

Let J be a PRM on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0$ with intensity measure $\mu(dt, dx, dz) = \lambda_{d+1}(dt, dx)\nu(dz)$, where ν is a Lévy measure on $\mathbb{R}_0$. We assume that J has the point representation

$$J = \sum_{i \geq 1} \delta_{(T_i, X_i, Z_i)}.$$

Let $\{\varepsilon_j\}_{j\geq 1}$ be a strictly decreasing sequence of positive real numbers such that $\varepsilon_j \downarrow 0$ as $j \to +\infty$ and $\varepsilon_0 = 1$. Let us define

$$\Gamma_j = \{z \in \mathbb{R}; \varepsilon_j < |z| < \varepsilon_{j-1}\},\ j \geq 1, \quad \text{and} \quad \Gamma_0 = \{z \in \mathbb{R}; |z| > 1\}.$$

For any $A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define

$$L_j(A) = \int_{A\times\Gamma_j} zJ(dt, dx, dz) = \sum_{(T_i, X_i)\in A} Z_i \mathbb{1}_{\{Z_i \in \Gamma_j\}},\ j \geq 0. \tag{2.2.1}$$

Note that $L_j(A)$ is a finite sum for all $j \geq 0$ since Γ_j is bounded away from 0, which implies that $J(A \times \Gamma_j) < +\infty$ a.s. For any $j \geq 0$, the variable $L_j(A)$ has a compound Poisson distribution with jump intensity measure $\lambda_{d+1}(A) \cdot \nu|_{\Gamma_j}$, i.e.,

$$\mathbb{E}[e^{iuL_j(A)}] = \exp\left\{\lambda_{d+1}(A) \int_{\Gamma_j} (e^{iuz} - 1)\nu(dz)\right\},\ u \in \mathbb{R}. \tag{2.2.2}$$

It follows that

$$\mathbb{E}[L_j(B)] = \lambda_{d+1}(A) \int_{\Gamma_j} z\nu(dz), \quad \text{and} \quad \mathrm{Var}[L_j(A)] = \lambda_{d+1}(A) \int_{\Gamma_j} z^2\nu(dz),$$

for any $j \geq 0$. Define

$$Y(A) = \sum_{j\geq 1} \big(L_j(A) - \mathbb{E}[L_j(A)]\big) + L_0(A). \tag{2.2.3}$$

The sum (2.2.3) converges a.s. by Kolmogorov's criterion, since $\{L_j(A) - \mathbb{E}[L_j(A)]\}_{j\geq 1}$ are independent zero-mean random variables with

$$\sum_{j\geq 1} \mathrm{Var}[L_j(A)] = \lambda_{d+1}(A) \int_{|z|\leq 1} z^2\nu(dz) < \infty.$$

By (2.2.2) and (2.2.3), it follows that $Y(A)$ is an infinitely divisible random variable with characteristic function given by the Lévy-Khintchine formula, i.e.,

$$\mathbb{E}[e^{iuY(A)}] = \exp\left\{\lambda_{d+1}(A)\int_{\mathbb{R}}(e^{iuz} - 1 - iuz\mathbb{1}_{\{|z|\leq 1\}})\nu(dz)\right\}, \quad u \in \mathbb{R}. \tag{2.2.4}$$

In particular, the process Y defined in (2.2.3) is a space-time Lévy white noise with decomposition (2.1.6) with $b = 0$ and $\sigma = 0$.

Remark 2.2.1. (i) If $\int_{\{|z|>1\}}|z|\nu(dz) < \infty$ then $\mathbb{E}[Y(A)] = \mathbb{E}[L_0(A)] = \lambda_{d+1}(A)\int_{\{|z|>1\}} z\nu(dz)$ is finite. In this case, if we define

$$L(A) = \sum_{j\geq 0}\big(L_j(A) - \mathbb{E}[L_j(A)]\big),$$

then $\{L(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is space-time Lévy white noise with zero-mean, and characteristic function:

$$\mathbb{E}[e^{iuL(A)}] = \exp\left\{\lambda_{d+1}(A)\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu(dz)\right\}, \quad u \in \mathbb{R}.$$

If in addition, $\int_{\{|z|>1\}}|z|^2\nu(dz) < \infty$, then $\mathrm{Var}[L(A)]$ is finite and is given by:

$$\mathrm{Var}[L(A)] = \lambda_{d+1}(A)\int_{\{|z|>1\}}|z|^2\nu(dz) < \infty.$$

An example of such a process is the *Gamma white noise* for which the measure ν is given by

$$\nu(dz) = \alpha z^{-1}e^{-z/\beta}1_{\{z>0\}}$$

for some $\alpha > 0$ and $\beta > 0$. For this process, $X(A) := L(A) + \alpha\beta\lambda_{d+1}(A)$ has a Gamma distribution with parameters $\alpha\lambda_{d+1}(A)$ and β.

(ii) If $\int_{\{|z|\leq 1\}}|z|\nu(dz) < \infty$, then $\sum_{j\geq 1}\mathbb{E}[L_j(A)] = \int_{\{|z|\leq 1\}} z\nu(dz)$ is finite, and we can define

$$Z(A) = \sum_{j\geq 0} L_j(A).$$

Note that $Z = \{Z(A); A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is space-time Lévy white noise with characteristic function:

$$\mathbb{E}[e^{iuZ(A)}] = \exp\left\{\lambda_{d+1}(A)\int_{\mathbb{R}}(e^{iuz} - 1)\nu(dz)\right\}, \quad u \in \mathbb{R}.$$

If in particular ν is a finite measure, then Z is a compound-Poisson process.

2.3 Lévy bases

In this section, we introduce the definition of a Lévy basis and present the construction and main properties of the stochastic integral with respect to this object, closely following the notation and terminology of [16]. In a nutshell, a Lévy basis in space-time is a random measure in the sense of [13], with its space-time component acting as an ID independently scattered random measure as in [44]. Specifically, the theory developed in [13, 19, 16] enables integration of random processes with respect to a random measure, provided that this random measure induces a semi-martingale in time. Note that in the absence of the temporal component, we cannot define a semi-martingale, which means that results from [13, 19, 16] cannot be applied without time. Therefore, in this section, we work exclusively in a space-time region.

In this section, we assume that $E = I \times \mathbb{R}^d$, where $I = [0,T]$ for some $T > 0$ or $I = \mathbb{R}_+$. For any $t \in I$, we define

$$I_t := \begin{cases} (t,T] & \text{if} \quad I = [0,T], \\ (t,+\infty) & \text{if} \quad I = \mathbb{R}_+. \end{cases}$$

Let $\widetilde{\mathcal{P}}$ be the predictable σ-field on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, i.e.

$$\widetilde{\mathcal{P}} = \sigma(\mathcal{E}),$$

where $\mathcal{E}$ is the class of linear combinations of elementary processes of the form (2.3.4) below.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We assume that this space is *complete*, i.e. if $A \in \mathcal{F}, \mathbb{P}(A) = 0$ and $B \subset A$, then $B \in \mathcal{F}$. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. (Recall that a *filtration* is an increasing collection $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. The filtration is *right-continuous* if $\cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$ for all $t \geq 0$.) Let $\widetilde{\mathcal{P}}_b^E = \{A \in \widetilde{\mathcal{P}}_b ; A \subset E\}$, where $\tilde{\mathcal{P}}_b$ is the set of all sets $A \in \tilde{\mathcal{P}}$ such that there exists $k \in \mathbb{N}$ with $A \subset \Omega \times [0,k] \times [-k,k]^d$.

Definition 2.3.1. A *Lévy basis* on E is a mapping $\Lambda : \widetilde{\mathcal{P}}_b^E \to L^0$ which satisfies the following:
(1) $\Lambda(\emptyset) = 0$ a.s.
(2) For any disjoint sets $(A_i)_{i\geq 1}$ in $\widetilde{\mathcal{P}}_b^E$ with $\bigcup_{i\geq 1} A_i \in \widetilde{\mathcal{P}}_b^E$,

$$\Lambda\Big(\bigcup_{i\geq 1} A_i\Big) = \sum_{i\geq 1} \Lambda(A_i) \quad \text{in} \quad L^0.$$

(3) For all $A \in \widetilde{\mathcal{P}}_b^E$ with $A \subseteq \Omega \times [0,t] \times U$, $\Lambda(A)$ is $\mathcal{F}_t$-measurable.
(4) For all $A \in \widetilde{\mathcal{P}}_b^E$ and $F \in \mathcal{F}_t$ for some $t \in I$,

$$\Lambda\big(A \cap (F \times I_t \times U)\big) = 1_F \Lambda\big(A \cap (\Omega \times I_t \times U)\big) \quad \text{a.s.} \tag{2.3.1}$$

(5) For any disjoint sets $(B_i)_{i\geq 1}$ in $\mathcal{B}_b(E)$, $\{\Lambda(\Omega \times B_i)\}_{i\geq 1}$ are independent. Moreover, if $B \in \mathcal{B}_b(E)$ is such that $B \subseteq I_t \times U$ for some $t \in I$, then $\Lambda(\Omega \times B)$ is independent of $\mathcal{F}_t$.
(6) For any $B \in \mathcal{B}_b(E)$, $\Lambda(\Omega \times B)$ has an infinitely divisible (ID) distribution.
(7) for any $t \in I$ and $k \in \mathbb{N}$, $\Lambda(\Omega \times \{t\} \times [-k,k]^d) = 0$ a.s.

Intuitively, a Lévy basis is a generalization of a Lévy process. But for our purposes, it is important to realize that if Λ is Lévy basis, then the process Z defined by

$$Z(B) = \Lambda(\Omega \times B) \quad \text{for all } B \in \mathcal{B}_b(E) \tag{2.3.2}$$

is an ID independently scattered random measure.

Similarly to the classical Lévy-Itô decomposition of a Lévy process, a Lévy basis on E has a canonical decomposition (see Theorem 3.2 of [19]): for $A \in \widetilde{\mathcal{P}}_b^E$

$$\Lambda(A) = B(A) + \Lambda^C(A) + \int_{E\times\{|z|\le 1\}} 1_A(t,x) z \widehat{N}(dt,dx,dz) + \int_{E\times\{|z|>1\}} 1_A(t,x) z N(dt,dx,dz), \tag{2.3.3}$$

which contains three terms:
(i) a drift term, given by a deterministic signed measure B on E;
(ii) a continuous component, given by a Gaussian random measure Λ^C on E with variance measure C;
(iii) a pure-jump component, characterized by an underlying Poisson random measure N on $\mathcal{U} = E \times \mathbb{R}_0$ of intensity measure μ, whose compensated version is called $\widehat{N}$.

Moreover, B, C and $\mu(\cdot, dz)$ have respective densities $b(t,x)$, $c(t,x)$, and $\nu(t,x,dz)$ with respect to the Lebesque measure. When these densities do not depend on (t,x), we say that Λ is a *homogeneous Lévy basis*. If $C = 0$, we say that Λ *pure-jump*. If $b(t,x) = 0$ and the measure $\nu(t,x,\cdot)$ is symmetric for all (t,x), then we say that Λ is *symmetric*.

Definition 2.3.2. A homogenous Lévy basis Λ with canonical decomposition (2.3.3) with $B = 0$, $\Lambda^C = 0$ and $\nu = \nu_\alpha$ (with ν_α given by (2.1.8)) is called a *SαS Lévy basis*.

Note that if Λ is a SαS Lévy basis, then the process Z given by (2.3.2) is a SαS Lévy noise, as specified by Definition 2.1.9.

We recall now briefly the construction and main properties of the stochastic integral with respect to a Lévy basis Λ, which shares many elements with the classical stochastic integral with respect to semi-martingales.

Definition 2.3.3. (i) A *simple integrand* on E is a linear combination of indicators of the form 1_A with $A \in \widetilde{\mathcal{P}}_b^E$.
(ii) An *elementary process* is a linear combination of processes of the form

$$X(t,x) = Y 1_{(a,b]}(t) 1_B(x), \tag{2.3.4}$$

where Y is $\mathcal{F}_a$-measurable, $0 \le a < b$, $(a,b] \subset I$ and $B \in \mathcal{B}_b(U)$.

We let $\mathcal{S}$ be the set of all simple integrands and $\mathcal{E}$ be the set of all elementary processes.

Recall that $\widetilde{\mathcal{P}} = \sigma(\mathcal{E})$. In particular, a set of the form $A = F \times (a,b] \times B$ with $F \in \mathcal{F}_a$, $0 \le a < b$, $(a,b] \subset I$, and $B \in \mathcal{B}_b(E)$ is in $\widetilde{\mathcal{P}}_b^E$. Moreover, by property (2.3.1) of Λ, we have:

$$\Lambda(F \times (a,b] \times B) = 1_F Z((a,b] \times B). \tag{2.3.5}$$

The next result gives another representation for $\widetilde{\mathcal{P}}$.

Lemma 2.3.4. $\widetilde{\mathcal{P}} = \mathcal{P}_0 \otimes \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}_0$ is the predictable σ-field on $\Omega \times \mathbb{R}_+$.

Proof. Let $\mathcal{E}$ be the set of all linear combinations of processes of form (2.3.4).

(a) We first prove that any process $X \in \mathcal{E}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. This will imply that $\widetilde{\mathcal{P}} \subset \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$, since $\widetilde{\mathcal{P}}$ is the minimal σ-field with respect to which all processes in $\mathcal{E}$ are measurable.

Let $X \in \mathcal{E}$ be arbitrary. Without loss of generality, we may assume that X is of the form (2.3.4). Since $(\omega, t) \mapsto Y(\omega)1_{(a,b]}(t)$ is $\mathcal{P}$-measurable, and $x \mapsto 1_A(x)$ is $\mathcal{B}(\mathbb{R}^d)$-measurable, it follows that $(\omega, t, x) \mapsto Y(\omega)1_{(a,b]}(t)1_A(x)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.

(b) For the reverse inclusion, it is enough to prove that $\mathcal{R} \subset \widetilde{\mathcal{P}}$, where $\mathcal{R}$ is the set of rectangles of the form $F \times B$, with $F \in \mathcal{P}$ and $B \in \mathcal{B}(\mathbb{R}^d)$. For this, we will use a $\lambda-\pi$-class argument. More precisely, let $\mathcal{L}$ be the class of all sets $F \in \mathcal{P}$ such that $F \times B \in \widetilde{\mathcal{P}}$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. Note that $\mathcal{L}$ is a σ-field.

We need to prove that $\mathcal{L} = \mathcal{P}$. For this, we observe that $\mathcal{P} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the π-system consisting of sets of the form

$$F = \bigcap_{i=1}^{k} F_i, \quad \text{with} \quad F_i = \{(\omega, t) \in \Omega \times \mathbb{R}_+; X_i(\omega, t) \in A_i\}, \tag{2.3.6}$$

where $k \geq 1$, $X_1, \ldots, X_k$ are simple processes, and $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R})$. It is enough to prove that

$$\mathcal{A} \subset \mathcal{L}. \tag{2.3.7}$$

This will imply that $\mathcal{P} = \sigma(A) \subset \mathcal{L}$, and hence $\mathcal{P} = \mathcal{L}$. To prove (2.3.7), let F be a set of the form (2.3.6). We have to prove that $F \times B \in \widetilde{\mathcal{P}}$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. Let $B \in \mathcal{B}(\mathbb{R}^d)$ be arbitrary. Then

$$F \times B = \bigcap_{i=1}^{k} (F_i \times B) = \bigcap_{i=1}^{k} \{(\omega, t, x); X_i(\omega, t) \in A_i, x \in B\}.$$

Note that

$$\{(\omega, t, x); X_i(\omega, t)1_B(x) \in A_i\} = \{(\omega, t, x); X_i(\omega, t) \in A_i, x \in B\} \cup \{(\omega, t, x); 0 \in A_i, x \in B^c\}.$$

The last set is $\Omega \times \mathbb{R}_+ \times B^c$ if $0 \in A_i$, or $\emptyset$ if $0 \notin A_i$. Hence

$$\{(\omega, t, x); X_i(\omega, t) \in A_i, x \in B\} = \begin{cases} \{(\omega, t, x); X_i(\omega, t)1_B(x) \in A_i\} \backslash (\Omega \times \mathbb{R}_+ \times B^c) & \text{if } 0 \in A_i \\ \{(\omega, t, x); X_i(\omega, t)1_B(x) \in A_i\} & \text{if } 0 \notin A_i \end{cases}$$

In both cases, $\{(\omega, t, x); X_i(\omega, t) \in A_i, x \in B\} \in \widetilde{\mathcal{P}}$. Hence $F \times B \in \widetilde{\mathcal{P}}$. ∎

The two sets $\mathcal{S}$ and $\mathcal{E}$ are not the same, but every process in $\mathcal{E}$ can be approximated by a process in $\mathcal{S}$, as shown by the next lemma.

Lemma 2.3.5. For any process $X \in \mathcal{E}$, there exists a sequence $(S_n)_{n \geq 1}$ in $\mathcal{S}$ such that for all (t, x), $S_n(t, x) \to X(t, x)$ as $n \to +\infty$ and $|S_n(t, x)| \leq |X(t, x)|$ for all n.

Proof. Without loss of generality, we assume that X is an elementary process of the form (2.3.4). By Theorem 13.5 of [14], there exists a sequence $(Y_n)_{n\geq 1}$ of simple random variables such that $Y_n \to Y$, $|Y_n| \leq |Y|$ for all n, and each Y_n is a linear combination of indicator functions of the form 1_F with $F \in \mathcal{F}_a$. It suffices to take $S_n(t,x) = Y_n 1_{(a,b]}(t) 1_B(x)$. ∎

For any $A \in \widetilde{\mathcal{P}}_b^E$, let $\int 1_A d\Lambda = \Lambda(A)$. By linearity, we extend this definition to $\mathcal{S}$. For any predictable process H and $p \geq 0$, we define the *Daniell mean*:

$$\|H\|_{\Lambda,p} = \sup_{S\in\mathcal{S}, |S|\leq|H|} \left\| \int S d\Lambda \right\|_p. \tag{2.3.8}$$

Recall that $\|\cdot\|_p$ is defined differently for $p=0$, $p\in(0,1)$, and $p\geq 1$. See the notation given in the introduction.

Note that the Daniell mean satisfies the triangular inequality:

$$\|H_1 + H_2\|_{\Lambda,p} \leq \|H_1\|_{\Lambda,p} + \|H_2\|_{\Lambda,p}. \tag{2.3.9}$$

Definition 2.3.6. Let Λ be a Lévy basis and Z be the corresponding ID independently scattered random measure (given by (2.3.2)). Let $p \geq 0$ be arbitrary and H be a predictable process.
(i) We say that H is *p-integrable* with respect to Λ if there exists a sequence $(S_n)_{n\geq1}$ in $\mathcal{S}$ such that $\|S_n - H\|_{\Lambda,p} \to 0$ as $n \to \infty$. If $p = 0$, we simply say that H is *integrable* with respect to Λ.
(ii) We say that H is *p-integrable* with respect to Z if H is p-integrable with respect to Λ.

By simplicity, in the remainder of this thesis, we use the following notation:

$$\|\cdot\|_\Lambda = \|\cdot\|_{\Lambda,0}.$$

We denote by $L^0(\Lambda)$ the class of integrable processes with respect to Λ, which is the closure of $\mathcal{S}$ with respect to $\|\cdot\|_\Lambda$. If Z is the ID independently scattered random measure induced by Λ (via relation (2.3.2)), by abuse of terminology, we let

$$L^0(Z) = L^0(\Lambda) \quad \text{and} \quad \|\cdot\|_Z = \|\cdot\|_\Lambda \tag{2.3.10}$$

For any $S \in \mathcal{S}$, we denote $I^\Lambda(S) = \int S d\Lambda$. Unlike Itô's theory, the map $I^\Lambda : \mathcal{S} \to L^0$ is *not* an isometry! But the fact that this map satisfies the following trivial inequality

$$\|I^\Lambda(S)\|_{L^0} \leq \|S\|_\Lambda \quad \text{for all } S \in \mathcal{S} \tag{2.3.11}$$

is sufficient for extending I^Λ from $\mathcal{S}$ to $L^0(\Lambda)$. More precisely, if $H \in L^0(\Lambda)$ and $(S_n)_{n\geq1}$ is the approximating sequence of simple integrands given by Definition 2.3.6, then $\{I^\Lambda(S_n)\}_{n\geq1}$ is a Cauchy sequence in L^0 since

$$\|I^\Lambda(S_n) - I^\Lambda(S_m)\|_{L^0} \leq \|S_n - S_m\|_\Lambda \leq \|S_n - H\|_\Lambda + \|S_m - H\|_\Lambda \to 0,$$

as $n, m \to \infty$. By definition, we set $I^{\Lambda}(H) = \lim_{n\to\infty} I^{\Lambda}(S_n)$ in L^0, and we say that $I^{\Lambda}(H)$ is the *stochastic integral* of H with respect to Λ (or Z). We will use both notations:

$$I^{\Lambda}(H) = \int H d\Lambda = \int H dZ = I^{Z}(H). \tag{2.3.12}$$

By approximation, it follows that inequality (2.3.11) can be extended to $L^0(\Lambda)$:

$$\|I^{\Lambda}(H)\|_{L^0} \leq \|H\|_{\Lambda} \quad \text{for all } H \in L^0(\Lambda). \tag{2.3.13}$$

In the case that H is p-integrable with respect to Λ, contraction property (2.3.13) holds with respect to $\|\cdot\|_{\Lambda,p}$, i.e.

$$\|I^{\Lambda}(H)\|_p \leq \|H\|_{\Lambda,p}. \tag{2.3.14}$$

The stochastic integral I^{Λ} satisfies the dominated convergence theorem. We include this result below, since we will use it often in the present thesis. This result was stated as relation (2.6) of [13]. The form that we present here corresponds to Theorem A.1 of [18].

Theorem 2.3.7 (Dominated Convergence Theorem for I^{Λ})**.** Let Λ be a homogeneous Lévy basis. Let $(H_n)_{n\geq 1}$ be predictable processes such that $(H_n)_{n\geq 1}$ converges pointwise to H, and $|H_n| \leq |H_0|$ for all n, for some $H_0 \in L^0(\Lambda)$. Then $H_n, H \in L^0(\Lambda)$ and $\|H_n - H\|_{\Lambda} \to 0$. Consequently,

$$I^{\Lambda}(H_n) \to I^{\Lambda}(H) \quad \text{in} \quad L^0.$$

Theorem 4.1 of [19] gives necessary and sufficient conditions for integrability with respect to an arbitrary L^0-*random measure* (see Definition 2.1 of [19]); by Remark 4.4 of [19], a Lévy basis is an orthogonal L^0-random measure. We include the statement of this result below for a homogeneous Lévy basis, which is a generalization Theorem 2.7 of [44], that corresponds to ID independently scattered random measures with deterministic integrands.

Theorem 2.3.8 (Theorem 4.1 of [19] for homogeneous Lévy bases)**.** Let Λ be a pure-jump homogeneous Lévy basis, with canonical decomposition (2.3.3) (with $\Lambda^C = 0$), and H be a predictable process. Then $H \in L^0(\Lambda)$ if and only if

$$\int_E U(H(t,x))dxdt < \infty \text{ a.s.} \quad \text{and} \quad \int_E V_0(H(t,x))dxdt < \infty \text{ a.s.}, \tag{2.3.15}$$

where

$$U(y) = by + \int_{\mathbb{R}} \big(\tau(yz) - y\tau(z)\big)\nu(dz) \quad \text{and} \quad V_0(y) = \int_{\mathbb{R}} (|yz|^2 \wedge 1)\nu(dz),$$

for any $y \in \mathbb{R}$, where $\tau(z) = z1_{\{|z|\leq 1\}}$.

In the particular case of a SαS Lévy basis, $b = 0$ and $\nu = \nu_\alpha$ is given by (2.1.8). So, in this case, $U(y) = 0$ (by symmetry) and $V_0(y) = \frac{2}{2-\alpha}|y|^{\alpha}$ (by direct calculation). This means that condition (2.3.15) is equivalent to $H \in L^{\alpha}(E)$ a.s. More precisely, we have the following corollary.

Corollary 2.3.9. Let Λ be a SαS Lévy basis, and H be a predictable process. Then $H \in L^0(\Lambda)$ if and only if

$$\int_E |H(t,x)|^\alpha dxdt < +\infty \quad \text{a.s.} \tag{2.3.16}$$

The following result shows that for elementary processes, the stochastic integral I^Λ coincides with the Itô integral (defined in the Walsh' sense [57]). We include its proof since we could not find it in the literature.

Theorem 2.3.10. Let X be an elementary process of the form (2.3.4). If Λ is a pure-jump homogeneous Lévy basis, and Z be given by (2.3.2), then $X \in L^0(\Lambda)$ and

$$I^\Lambda(X) = YZ\big((a,b] \times B\big) \quad \text{a.s.} \tag{2.3.17}$$

Proof. Note that $U(X(t,x)) = U(Y)1_{(a,b]}(t)1_B(x)$ and $V_0(X(t,x)) = V_0(Y)1_{(a,b]}(t)1_B(x)$. Hence, condition (2.3.15) holds, and $X \in L^0(\Lambda)$.

Let $(S_n)_{n\geq 1}$ be the sequence of simple integrands given by Lemma 2.3.5.(ii). More precisely, if $Y_n = \sum_{i=1}^{k_n} a_{i,n} 1_{F_{i,n}}$ with $a_{i,n} \in \mathbb{R}$ and $F_{i,n} \in \mathcal{F}_a$ is the sequence of simple random variable such that $Y_n \to Y$ and $|Y_n| \leq |Y|$ for all n, then

$$S_n(t,x) = \sum_{i=1}^{k_n} a_{i,n} 1_{F_{i,n}} 1_{(a,b]}(t) 1_B(x).$$

Using (2.3.5), we obtain that:

$$I^\Lambda(S_n) = \sum_{i=1}^{k_n} a_{i,n} \Lambda(F_{i,n} \times (a,b] \times B) = \sum_{i=1}^{k_n} a_{i,n} 1_{F_{i,n}} Z\big((a,b] \times B\big) = Y_n Z\big((a,b] \times B\big).$$

Relation (2.3.17) follows letting $n \to \infty$, since $I^\Lambda(S_n) \xrightarrow{P} I^\Lambda(X)$, by Theorem 2.3.7.

∎

2.4 Local property of the stochastic integral

In this section, we discuss some local properties (in time) of the stochastic integral with respect to a Lévy basis, as defined by (2.3.12). In particular, these properties are important for constructing solutions of non-linear SPDEs, since they enable us to work up to a suitable stopping time where the stochastic integral has some nice properties.

As in Section 2.3, we assume that $E = I \times \mathbb{R}^d$, and we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$.

We begin by recalling the following classical definitions.

Definition 2.4.1. A random variable $\tau : \Omega \to [0, +\infty]$ is a *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}_t$, for every $t \geq 0$.

Definition 2.4.2. Let $\{X_t\}_{t\geq 0}$ be a stochastic process and $B \in \mathcal{B}(\mathbb{R})$. Define

$$\tau = \inf\{t > 0; X_t \in B\}.$$

Then τ is called a *hitting time of* B for X.

Definition 2.4.3. We say that a process $\{X_t\}_{t\geq 0}$ is *adapted* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable, for any $t \geq 0$.

Theorem 2.4.4 (Theorem 3, Chapter 1 of [43])**.** Let $\{X_t\}_{t\geq 0}$ be a process with càdlàg sample paths, which is adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let B be an open set. If τ is a hitting time of B for X, then τ is a stopping time.

Theorem 2.4.5 (Theorem 4,, Chapter 1 of [43])**.** Let $\{X_t\}_{t\geq 0}$ be an adapted càdlàg stochastic process, and B be a closed subset of $\mathbb{R}$. Then, the random variable

$$\tau = \inf\{t > 0;\ X_t \in B \text{ or } X_{t-} \in B\}$$

is a stopping time.

For the rest of this section, we assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is fixed.

Definition 2.4.6. We say that a stopping time τ is *elementary* if it takes finitely many values, i.e. τ can be expressed as

$$\tau = \sum_{i=1}^{n} t_i \mathbb{1}_{A_i}$$

for some $t_i \geq 0$ and $A_i \in \mathcal{F}_{t_i}$, for any $i = 1, \ldots, n$.

Definition 2.4.7. Let Λ be a Lévy basis on $E = I \times U$. We say that a predictable process H is *locally integrable with respect to* Λ (and we write $H \in L^0_{\text{loc}}(\Lambda)$) if there exists a non-decreasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \uparrow +\infty$ a.s. for $n \to +\infty$, such that

$$H^{\tau_n} \in L^0(\Lambda) \quad \text{for all } n \in \mathbb{N}, \tag{2.4.1}$$

where H^{τ_n} is the *stopped process*, defined by $H^{\tau_n}(t,x) = H(t,x)\mathbb{1}_{[\![0,\tau_n]\!]}(t)$.

Proposition 2.4.8 (Proposition 2.13 of [13])**.** For any Lévy basis Λ, $L^0_{\text{loc}}(\Lambda) \subset L^0(\Lambda)$.

The following local property of the stochastic integral plays an important role in this thesis. While different versions of this property can be found in [13, 12, 31], we include its proof here because we could not find a direct reference that addresses it with respect to a Lévy basis, as defined in Section 2.3.

Proposition 2.4.9. Let Λ be a Lévy basis on E and $H \in L^0_{\text{loc}}(\Lambda)$. If τ is an elementary stopping time, then for any $t \in I$,

$$\mathbb{1}_{[\![0,\tau]\!]}(t) \int_0^t \int_U H(s,y)\Lambda(ds,dy) = \mathbb{1}_{[\![0,\tau]\!]}(t) \int_0^t \int_U H(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy). \tag{2.4.2}$$

Proof. *Step 1.* First, we prove that (2.4.2) holds for a process H given by $H(s,y) = \mathbb{1}_A(s,y)$, with $A \in \tilde{\mathcal{P}}_b^E$. We will prove that

$$\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\Lambda(ds,dy) = \mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy). \tag{2.4.3}$$

Assume that τ takes values $0 = t_0 < t_1 < t_2 < \ldots < t_{N_0+1}$. Then the stochastic interval $[\![0,\tau]\!]$ can be decomposed as

$$[\![0,\tau]\!] = (\{\tau = 0\} \times \{t = 0\}) \cup \bigcup_{n=0}^{N_0} \{\tau \geq t_{n+1}\} \times (t_n, t_{n+1}]. \tag{2.4.4}$$

Hence,

$$\mathbb{1}_{[\![0,\tau]\!]} = \mathbb{1}_{\{t=0\}\times\{\tau=0\}} + \sum_{n=0}^{N_0} \mathbb{1}_{\{\tau\geq t_{n+1}\}\times(t_n,t_{n+1}]}. \tag{2.4.5}$$

Observe that we can re-arrange the sum on (2.4.5) in the following way:

$$\sum_{n=0}^{N_0} \mathbb{1}_{\{\tau\geq t_{n+1}\}\times(t_n,t_{n+1}]} = \sum_{n=0}^{N_0}\sum_{i=n+1}^{N_0} \mathbb{1}_{\{\tau=t_i\}}\mathbb{1}_{(t_n,t_{n+1}]} = \sum_{n=1}^{N_0+1} \mathbb{1}_{\{\tau=t_n\}\times(0,t_n]}. \tag{2.4.6}$$

Then, we can re-write (2.4.5) as

$$\mathbb{1}_{[\![0,\tau]\!]} = \mathbb{1}_{\{t=0\}\times\{\tau=0\}} + \sum_{n=1}^{N_0+1} \mathbb{1}_{\{\tau=t_n\}\times(0,t_n]}. \tag{2.4.7}$$

Hence, by (2), (4), (7) in Definition 2.3.1, we have:

$$\begin{aligned}
&\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy) = \Lambda(A\cap([\![0,\tau]\!]\times U)\cap(\Omega\times[0,t]\times U)) \\
&= \sum_{n=0}^{N_0} \Lambda(A\cap(\{\tau\geq t_{n+1}\}\times(t_n,t_{n+1}]\times U)\cap(\Omega\times[0,t]\times U)) \\
&= \sum_{n=0}^{N_0} \mathbb{1}_{\{\tau\geq t_{n+1}\}}\Lambda(A\cap(\Omega\times(t_n,t_{n+1}]\times U)\cap(\Omega\times[0,t]\times U)) \\
&= \sum_{n=0}^{N_0} \mathbb{1}_{\{\tau\geq t_{n+1}\}}\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(t_n,t_{n+1}]}(s)\Lambda(ds,dy) \\
&= \sum_{n=1}^{N_0+1} \mathbb{1}_{\{\tau=t_n\}}\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(0,t_n]}(s)\Lambda(ds,dy).
\end{aligned} \tag{2.4.8}$$

Using the same argument as in (2.4.6) and the linearity of Λ on $\tilde{\mathcal{P}}_b$, we can re-arrange the sum in the last equality of (2.4.8) as follows:

$$\begin{aligned}&\sum_{n=0}^{N_0} \mathbb{1}_{\{\tau\geq t_{n+1}\}}\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(t_n,t_{n+1}]}(s)\Lambda(ds,dy)\\&=\sum_{n=1}^{N_0+1}\mathbb{1}_{\{\tau=t_n\}}\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(0,t_n]}(s)\Lambda(ds,dy).\end{aligned}\tag{2.4.9}$$

Additionally, for a fixed $i\in\{1,2,\ldots,N_0\}$, it holds:

$$\mathbb{1}_{\{\tau=t_i\}}\mathbb{1}_{[\![0,\tau]\!]}(t)=\mathbb{1}_{\{\tau=t_i\}}\mathbb{1}_{(0,t_i]}(t).\tag{2.4.10}$$

Hence, by (2.4.7), (2.4.8), (2.4.9), and (2.4.10), we have:

$$\begin{aligned}&\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy)\\&=\sum_{n=1}^{N_0+1}\mathbb{1}_{\{\tau=t_n\}}\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(0,t_n]}(s)\Lambda(ds,dy)\\&=\sum_{n=1}^{N_0+1}\mathbb{1}_{\{\tau=t_n\}}\mathbb{1}_{(0,t_n]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\mathbb{1}_{(0,t_n]}(s)\Lambda(ds,dy)\\&=\sum_{n=1}^{N_0+1}\mathbb{1}_{\{\tau=t_n\}}\mathbb{1}_{(0,t_n]}(t)\int_0^t\int_U \mathbb{1}_A(s,y)\Lambda(ds,dy)\\&=\mathbb{1}_{[\![0,\tau]\!]}(t)\int_U \mathbb{1}_A(s,y)\Lambda(ds,dy).\end{aligned}\tag{2.4.11}$$

Step 2: Consider now the case when $H\in L^0_{\text{loc}}(\Lambda)$. By Proposition 2.4.8, $H\in L^0(\Lambda)$. Then, there exists a sequence of simple integrands $\{S_n\}_{n\in\mathbb{N}}$ such that $\|H-S_n\|_{\Lambda,0}\to 0$ as $n\to+\infty$. By the linearity of I^Λ and *Step 1* above, we have:

$$\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U S_n(s,y)\Lambda(ds,dy)=\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U S_n(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy),\tag{2.4.12}$$

for all $n\in\mathbb{N}$. Letting $n\to+\infty$ in (2.4.12), we conclude that (2.4.2) holds. ∎

The next result shows that Proposition 2.4.9 holds for any stopping time τ.

Proposition 2.4.10. Let Λ be a Lévy basis on E and $H\in L^0_{\text{loc}}(\Lambda)$. If τ is a stopping time, then for any $t\in I$,

$$\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U H(s,y)\Lambda(ds,dy)=\mathbb{1}_{[\![0,\tau]\!]}(t)\int_0^t\int_U H(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy).\tag{2.4.13}$$

Proof. Let $\{\tau_n\}_{n\geq 1}$ be a sequence of elementary stopping times such that $\tau_n \to \tau$ as $n \to +\infty$. By Proposition 2.4.8, $H \in L^0(\Lambda)$. By Theorem 2.3.7 (Dominated Convergence Theorem for random measures), $I^\Lambda(H^{\tau_n}) \to I^\Lambda(H)$ in L^0, as $n \to \infty$. Hence, there exists a subsequence $\{n_k\}_{k\geq 1}$ such that

$$\lim_{k\to+\infty} \int_0^t \int_U H(s,y) \mathbb{1}_{[\![0,\tau_{n_k}]\!]}(s)\Lambda(ds,dy) = \int_0^t \int_U H(s,y)\mathbb{1}_{[\![0,\tau]\!]}(s)\Lambda(ds,dy) \quad \text{a.s.}$$

In addition, it is clear that

$$\lim_{k\to+\infty} \mathbb{1}_{[\![0,\tau_{n_k}]\!]}(t) \int_0^t \int_U H(s,y)\Lambda(ds,dy) = \mathbb{1}_{[\![0,\tau]\!]}(t) \int_0^t \int_U H(s,y)\Lambda(ds,dy) \quad \text{a.s.}$$

By Proposition 2.4.9, we have:

$$\mathbb{1}_{[\![0,\tau_{n_k}]\!]}(t) \int_0^t \int_U H(s,y)\Lambda(ds,dy) = \mathbb{1}_{[\![0,\tau_{n_k}]\!]}(t) \int_0^t \int_U H(s,y)\mathbb{1}_{[\![0,\tau_{n_k}]\!]}(s)\Lambda(ds,dy). \quad (2.4.14)$$

Letting $k \to +\infty$ in (2.4.14), we obtain (2.4.13).

∎

Chapter 3

Truncated Lévy noises

In this chapter, we will introduce the *truncated Lévy noise*, which is a fundamental tool for the construction of random-field solutions of SPDEs driven by infinite-variance Lévy noise. The use of a truncated noise for constructions of solutions is a well-known technique for solving stochastic differential equations (SDEs), as outlined in [12, 43]. One of the main challenges in working with SPDEs with heavy-tailed noises is the lack of p-moments and finite variance. This makes approximating a solution a challenging task, as there is no suitable Banach or Hilbert space in which a full solution can be constructed. In particular, working with a truncated noise allows us to derive nice properties of the stochastic integral operator, such as bounded moments and self-mapping properties. Using a pasting argument with an increasing sequence of stopping times, we can then construct a global solution for SPDEs with heavy-tailed noises.

Throughout the study of SPDEs in this thesis, we often omit the continuous part of a Lévy basis, i.e. we assume that $\Lambda^C = 0$ in (2.3.3), since this part has been widely studied. To be precise, we are interested in studying SPDEs with a pure-jump Lévy basis Λ on $E = I \times U$, given by

$$\begin{aligned}\Lambda(A) = b \int_E \mathbb{1}_A(t,x) dt dx + \int_{E\times\{|z|\le 1\}} \mathbb{1}_A(t,x) z \widehat{J}(dt,dx,dz) \\ + \int_{E\times\{|z|>1\}} \mathbb{1}_A(t,x) z J(dt,dx,dz) \quad \text{for all } A \in \widetilde{\mathcal{P}}_b^E,\end{aligned} \tag{3.0.1}$$

where $b \in \mathbb{R}$, J is a Poisson random measure on $E \times \mathbb{R}$ with intensity $m(dt,dx,dz) = dt\,dx\,\nu(dz)$, and $\widehat{J}$ is the compensated Poisson random measure of J, given by $\widehat{J} = J - m$. Here, ν is a Lévy measure defined on $\mathbb{R}$.

We fix $\eta > 0$, and we consider the truncation function

$$h(x) = 1 + |x|^\eta, \quad \text{for all } x \in \mathbb{R}^d.$$

Let $N \in \mathbb{N}$ be arbitrary. The *truncated Lévy basis* Λ_N (corresponding to Λ and h) is defined by:

$$\begin{aligned}\Lambda_N(A) = b\int_E \mathbb{1}_A(t,x)dtdx + \int_{E\times\{|z|\leq 1\}} \mathbb{1}_A(t,x)z\widehat{J}(dt,dx,dz) \\ + \int_{I\times\{(x,z)\in U\times\mathbb{R};1<|z|\leq Nh(x)\}} \mathbb{1}_A(t,x)zJ(dt,dx,dz) \quad \text{for all } A \in \widetilde{\mathcal{P}}_b^E.\end{aligned} \tag{3.0.2}$$

Note that Λ_N is a truncated version of Λ on the region $I \times \{(x,z) \in U \times \mathbb{R}; |z| \leq Nh(x)\}$.

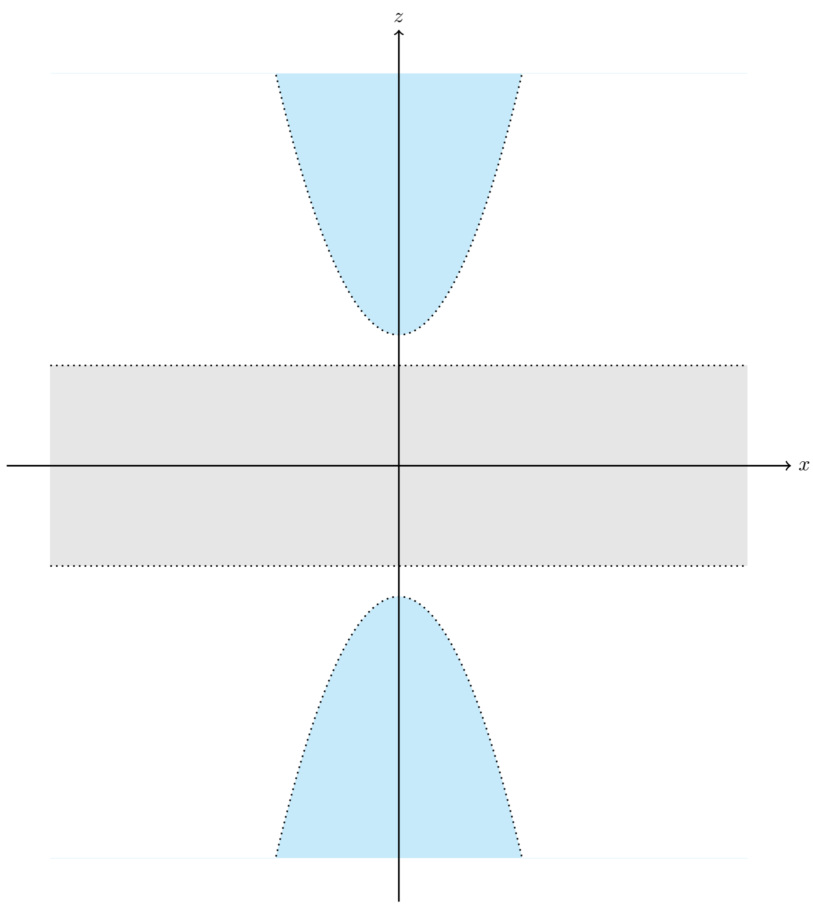


Figure 3.1: Graphical representation of the non-standard truncation region $\{|z| \geq Kh(x)\}$, where $h(x) = K(1 + |x|^\eta)$ with $\eta > 0$ and $K \geq 2$. The shaded gray region represents the band $\{|z| \leq 1\}$.

In Figure 3.1, the blue shaded region corresponds to $\{(x,z) \in U \times \mathbb{R} : |z| \geq Nh(x)\}$, while the gray shaded region represents the small jumps $\{(x,z) \in U \times \mathbb{R} : |z| \leq 1\}$. The classical truncation region corresponds to the case $h(x) = 1$.

We introduce also another truncated version of Λ, which corresponds to the truncation function $h(x) = 1$, and is given by

$$\begin{aligned}\overline{\Lambda}_N(A) = b\int_E \mathbb{1}_A(t,x)dtdx + \int_{E\times\{|z|\le 1\}} \mathbb{1}_A(t,x)z\widehat{J}(dt,dx,dz) \\ + \int_{E\times\{1<|z|\le N\}} \mathbb{1}_A(t,x)zJ(dt,dx,dz) \quad \text{for all } A \in \widetilde{\mathcal{P}}_b^E.\end{aligned} \tag{3.0.3}$$

3.1 Moment inequalities

In this section, we study certain moment inequalities of the stochastic integral with respect to the noises given by (3.0.2) and (3.0.3), under specific integrability conditions on the Lévy measure associated with these noises. For simplicity, we assume that $E = [0,T] \times \mathbb{R}^d$. However, the main results of this section can be easily adapted to $E = [0,T] \times U$, where U is a bounded domain of $\mathbb{R}^d$.

As in [17], we introduce the following assumption.

Assumption 3.1.1. There exists $0 < q \le p$ such that

$$\gamma_1 = \int_{\{|z|\le 1\}} |z|^p \nu(dz) < \infty \quad \text{and} \quad \gamma_2 = \int_{|z|>1} |z|^q \nu(dz) < \infty, \tag{3.1.1}$$

In addition, if $p < 1$, we assume that $b = \int_{\{|z|\le 1\}} z\nu(dz)$.

Under Assumption 3.1.1, we have the following decompositions:

(i) If $p < 1$, then

$$\Lambda_N(A) = \int_{E\times\{|z|\le Nh(x)\}} \mathbb{1}_A(t,x)zJ(dt,dx,dz). \tag{3.1.2}$$

due to the condition $b = \int_{|z|\le 1} z\nu(dz)$.

(ii) If $p \ge 1$, then

$$\Lambda_N(A) = D_N(A) + L_N(A), \tag{3.1.3}$$

where

$$D_N(A) = \int_E \mathbb{1}_A(s,y)\left(b + \int_{1<|z|\le Nh(x)} z\nu(dz)\right)dxdt$$

and

$$L_N(A) = \int_{E\times\{|z|\le Nh(x)\}} \mathbb{1}_A(t,x)\, z\widehat{J}(dt,dx,dz).$$

Proposition 3.1.2. Let $h(x) = 1 + |x|^\eta$ for some $\eta > 0$, and suppose that Assumption 3.1.1 holds. Then, the following inequality hold for any $N \in \mathbb{N}$:

$$\int_{|z|\le Nh(y)} |z|^p \nu(dz) \le N^{p-q}(\gamma_1 + \gamma_2)h(y)^{p-q}. \tag{3.1.4}$$

If $p > 1$, then

$$\left| \int_{1<|z|\leq Nh(y)} z\nu(dz) \right|^p \leq \gamma_2^p (Nh(y))^{p-q}, \tag{3.1.5}$$

and

$$\left| b + \int_{1<|z|\leq Nh(y)} z\nu(dz) \right|^p \leq 2^{p-1}(|b|^p + \gamma_2^p (Nh(y))^{p-q}). \tag{3.1.6}$$

Proof. For (3.1.4), we have:

$$\begin{aligned}
\int_{|z|\leq Nh(y)} |z|^p \nu(dz) &= \int_{|z|\leq 1} |z|^p \nu(dz) + \int_{1<|z|\leq Nh(y)} |z|^p \nu(dz) \\
&= \int_{|z|\leq 1} |z|^p \nu(dz) + \int_{1<|z|\leq Nh(y)} |z|^{p-q}|z|^q \nu(dz) \\
&\leq \int_{|z|\leq 1} |z|^p \nu(dz) + Nh(y)^{p-q} \int_{1<|z|} |z|^q \nu(dz) \\
&\leq N^{p-q}(\gamma_1 + \gamma_2) h(y)^{p-q}.
\end{aligned}$$

For (3.1.5), if $q \in (0,1]$, we get:

$$\begin{aligned}
\left(\int_{\{1<|z|\leq Nh(y)\}} |z|\nu(dz) \right)^p &= \left(\int_{\{1<|z|\leq Nh(y)\}} |z|^q |z|^{1-q} \nu(dz) \right)^p \\
&\leq \left([Nh(y)]^{(1-q)} \int_{\{1<|z|\leq Nh(y)\}} |z|^q \nu(dz) \right)^p \\
&\leq [Nh(y)]^{p(1-q)} \gamma_2^p \leq [Nh(y)]^{p-q} \gamma_2^p.
\end{aligned}$$

Now, assume that $q > 1$: since $N\,h(y) > 1$ and $p \geq q$, we have:

$$\left(\int_{\{1<|z|\leq Nh(y)\}} |z|\nu(dz) \right)^p \leq \gamma_2^p \leq [Nh(y)]^{p-q} \gamma_2^p.$$

Inequality (3.1.6) follows directly from (3.1.5) and the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$. ∎

For the subsequent lemma, we use the following notation.

(i) $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a measurable non-negative function.

(ii) $g_p(t,x) := \mathcal{G}_t^p(x) + \mathcal{G}_t(x) \mathbb{1}_{\{p\geq 1\}}$ for $p > 0$.

(iii) $\mathcal{I}^{(t,x)}(\phi)(s,y) := \mathcal{G}_{t-s}(x-y)\phi(s,y)\mathbb{1}_{\{t>s\}}$, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\phi \in \mathcal{P}$.

(iv) Given $\phi \in \mathcal{P}$, we define the random field $\mathfrak{T}_N(\phi)$ given by

$$\mathfrak{T}_N(\phi)(t,x) := \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\Lambda_N(ds,dy),$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where σ is a Lipschitz function.

Lemma 3.1.3 (Lemma 3.3 in [17]). Let $T > 0$ and $N \in \mathbb{N}$ be fixed. If Λ_N satisfies Assumption 3.1.1 and

$$\int_0^T \int_{\mathbb{R}^d} g_p(t,x)\, dx\, dt < +\infty,$$

then we have the following estimations:

(i) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a constant $C = C(T,N,p) > 0$ such that for all $\phi \in \mathcal{P}$, we have:

$$\mathbb{E}\left[|\mathfrak{T}_N(\phi)(t,x)|^p\right] \le C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\left(1+\mathbb{E}\left[|\phi(s,y)|^p\right]\right) h(y)^{p-q}\, dy\, ds. \quad (3.1.7)$$

(ii) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a constant $C = C(T,N,p) > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and ϕ_1, $\phi_2 \in \mathcal{P}$ with $\mathfrak{T}_N(\phi_1)(t,x)$, $\mathfrak{T}_N(\phi_2)(t,x) < +\infty$ a.s., we have:

$$\begin{aligned}&\mathbb{E}\left[|\mathfrak{T}_N(\phi_1)(t,x) - \mathfrak{T}_N(\phi_2)(t,x)|^p\right] \\ &\le C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|\phi_1(s,y)-\phi_2(s,y)|^p\right] h(y)^{p-q}\, dy\, ds.\end{aligned} \quad (3.1.8)$$

Proof. Case $p < 1$: by decomposition (3.1.2) and (A.1.3), we get:

$$\begin{aligned}\mathbb{E}\left[|\mathfrak{T}_N(\phi)(t,x)|^p\right] &\le \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d \times \{|z|<Nh(y)\}} \mathcal{G}^p_{t-s}(x-y)|\sigma(\phi(s,y))|^p |z|^p \nu(dz)dyds\right] \\ &\le C_T \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left(\int_{|z|\le Nh(y)} |z|^p\nu(dz)\right)dyds \\ &\le C_T \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])h(y)^{p-q}dyds,\end{aligned} \quad (3.1.9)$$

where the last equality is due to (3.1.4).

Case $p \in [1,2)$: using decomposition (3.1.3), we have:

$$\begin{aligned}\mathbb{E}\left[|\mathfrak{T}_N(\phi)(t,x)|^p\right] \le c_p \Bigg\{ &\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))D_N(dy,ds)\right|^p\right] \\ &+ \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))L_N(ds,dy)\right|^p\right]\Bigg\}.\end{aligned} \quad (3.1.10)$$

First, we estimate the integral term with respect to integral D_N in (3.1.10). Indeed, note that

$$\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))D_N(dy,ds)\right|^p\right]\leq C_{T,p,\sigma}\left(\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)dyds\right)^{p-1}$$
$$\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+|\phi(s,y)|^p)\left(b+\int_{1<|z|\leq Nh(y)}z\nu(dz)\right)^p dsdy\right]$$
$$=C_{T,p,\sigma}\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left(b+\int_{1<|z|\leq Nh(y)}z\nu(dz)\right)^p dsdy$$
$$\leq C_{T,p,\sigma}\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])h(y)^{p-q}dsdy, \tag{3.1.11}$$

where the last inequality is due to (3.1.6). Now, by (A.1.2), it holds:

$$\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))L_N(ds,dy)\right|^p\right]\leq$$
$$C_p\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d\times\{|z|\leq Nh(y)\}}\mathcal{G}^p_{t-s}(x-y)|\sigma(\phi(s,y))|^p|z|^p\nu(dz)dyds\right]$$
$$\leq C_{T,p,\sigma}\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left(\int_{|z|\leq Nh(y)}|z|^p\nu(dz)\right)dyds$$
$$\leq C_{T,p,\sigma}\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])h(y)^{p-q}dyds. \tag{3.1.12}$$

Now, we assume that $p\geq 2$. Clearly, inequality (3.1.11) holds for $p\geq 2$. For the term involving L_N, we use the maximal inequality (A.1.1) as it follows:

$$\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))L_N(ds,dy)\right|^p\right]$$
$$\leq C\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d\times\{|z|\leq Nh(y)\}}\mathcal{G}^2_{t-s}(x-y)\sigma(\phi(s,y))|^2|z|^2\nu(dz)dyds\right)^{\frac{p}{2}}\right] \tag{3.1.13}$$
$$+C_p\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d\times\{|z|\leq Nh(y)\}}\mathcal{G}^p_{t-s}(x-y)\sigma(\phi(s,y))|^p|z|^p\nu(dz)dyds\right].$$

Regarding the second term in the right hand side of (3.1.13), we can estimate it in the same way as in (3.1.12). For the first term on (3.1.13), by Hölder's inequality, we have:

$$C\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d\times\{|z|\leq Nh(y)\}} \mathcal{G}^2_{t-s}(x-y)\sigma(\phi(s,y))|^2|z|^2\nu(dz)dyds\right|^{\frac{p}{2}}\right]$$
$$\leq C\Big(\int_0^t \int_{\mathbb{R}^d} \mathcal{G}^2_{t-s}(x-y)dsdy\Big)^{\frac{p}{2}-1}\int_0^t \int_{\mathbb{R}^d} \mathcal{G}^2_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])h(y)^{p-q}dsdy$$
$$C\int_0^t \int_{\mathbb{R}^d} (\mathcal{G}_{t-s}(x-y)+\mathcal{G}^p_{t-s}(x-y))(1+\mathbb{E}\left[|\phi(s,y)|^p\right])h(y)^{p-q}dsdy,$$

where the last inequality is due to $|x|^2 < |x| + |x|^p$ for all $p \geq 2$.

(ii) can be proved in a similar way as in (i). ∎

Remark 3.1.4. Let $||\cdot||_{\Lambda_N,p}$ be the Daniell mean with respect to Λ_N (for $p > 0$), given by (2.3.8).

(i) By Lemma A.2 in [18], and using the similar arguments as in the proof of Lemma 3.1.3, it can be proved that for $\phi \in \mathcal{P}$ and $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\mathbb{E}\left[|\mathfrak{T}_N(\phi)(t,x)|^p\right] \leq ||\mathcal{I}^{(t,x)}(\sigma(\phi))||^{p\vee 1}_{\Lambda_N,p}$$
$$\leq C\int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\left(1+\mathbb{E}\left[|\phi(s,y)|^p\right]\right)h(y)^{p-q}\,dy\,ds,$$

provided that $\mathcal{I}^{(t,x)}(\sigma(\phi))$ is p-integrable with respect to Λ_N. (Note that the first inequality above is due to the contraction property (2.3.14).)

(ii) Similarly, for all $(t,x) \in [0,T]\times\mathbb{R}^d$ and ϕ_1, $\phi_2 \in \mathcal{P}$, such that $\mathcal{I}^{(t,x)}(\sigma(\phi_1)$ and $\mathcal{I}^{(t,x)}(\sigma(\phi_2))$ are p-integrable with respect to Λ_N, we get:

$$\mathbb{E}\left[|\mathfrak{T}_N(\phi_1)(t,x)-\mathfrak{T}_N(\phi_2)(t,x)|^p\right] \leq ||\mathcal{I}^{(t,x)}(\sigma(\phi_1)-\sigma(\phi_2))||^{p\vee 1}_{\Lambda_N,p}$$
$$\leq C\int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|\phi_1(s,y)-\phi_2(s,y)|^p\right]h(y)^{p-q}\,dy\,ds.$$

For the noise $\overline{\Lambda}_N$ defined by (3.0.3), Lemma B.2.6 holds under weaker conditions than those in Assumption 3.1.1. Indeed, if we set:

$$\overline{\mathfrak{T}}_N(\phi)(t,x) := \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{\Lambda}_N(ds,dy),$$

for any $(t,x) \in \mathbb{R}_+\times\mathbb{R}^d$ and $\phi \in \mathcal{P}$, then using the same calculations as in the proof of Lemma B.2.6, we obtain the following result.

Lemma 3.1.5. Let $T > 0$ and $N \in \mathbb{N}$ be fixed. If there exists $p > 0$ such that

$$\int_0^T \int_{\mathbb{R}^d} g_p(t,x)\,dx\,dt < +\infty, \quad \text{and} \quad \int_{|z|\leq 1} |z|^p\nu(dz) < +\infty,$$

then we have the following estimations:

(i) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a constant $C = C(T,N,p) > 0$ such that for all $\phi \in \mathcal{P}$, we have:

$$\mathbb{E}\left[|\overline{\mathfrak{T}}_N(\phi)(t,x)|^p\right] \le C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\left(1+\mathbb{E}\left[|\phi(s,y)|^p\right]\right) dyds. \tag{3.1.14}$$

(ii) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a constant $C = C(T,N,p) > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\phi_1,\ \phi_2 \in \mathcal{P}$ with $\overline{\mathfrak{T}}_N(\phi_1)(t,x) < +\infty$ a.s. and $\overline{\mathfrak{T}}_N(\phi_2)(t,x) < +\infty$ a.s., we have:

$$\begin{aligned} &\mathbb{E}\left[\left|\overline{\mathfrak{T}}_N(\phi_1)(t,x) - \overline{\mathfrak{T}}_N(\phi_2)(t,x)\right|^p\right] \\ &\le C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|\phi_1(s,y)-\phi_2(s,y)|^p\right] dyds. \end{aligned} \tag{3.1.15}$$

Proof. First, note that

$$\begin{aligned} \int_{|z|\le N} |z|^p \nu(dz) &= \int_{|z|\le 1} |z|^p \nu(dz) + \int_{1<|z|\le N} |z|^p \nu(dz) \\ &\le \int_{|z|\le 1} |z|^p \nu(dz) + N^p \nu\{1<|z|\le N\} < +\infty. \end{aligned} \tag{3.1.16}$$

and

$$\left|\int_{1<|z|\le N} z\nu(dz)\right|^p \le N^p(\nu\{1<|z|\le N\})^p < +\infty. \tag{3.1.17}$$

If $p<1$, we can write $\overline{\Lambda}_N$ as

$$\overline{\Lambda}_N(A) = \int_{E\times\{|z|\le N\}} \mathbb{1}_A(t,x) zJ(dt,dx,dz). \tag{3.1.18}$$

due to the condition $b = \int_{|z|\le 1} z\nu(dz)$. If $p \ge 1$, we can write Λ_N as

$$\overline{\Lambda}_N(A) = \overline{D}_N(A) + \overline{L}_N(A), \tag{3.1.19}$$

where

$$\overline{D}_N(A) = \int_E \mathbb{1}_A(s,y)\left(b + \int_{1<|z|\le N} z\nu(dz)\right) dxdt,$$

and

$$\overline{L}_N(A) = \int_{E\times\{|z|\le N\}} \mathbb{1}_A(t,x)\, z\widehat{J}(dt,dx,dz).$$

Case $p<1$: We use the decomposition (3.1.2). So, by (A.1.3), it holds:

$$\begin{aligned}\mathbb{E}\left[|\overline{\mathfrak{T}}_N(\phi)(t,x)|^p\right] &\le \mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d\times\{|z|<N\}}\mathcal{G}^p_{t-s}(x-y)|\sigma(\phi(s,y))|^p|z|^p\nu(dz)dyds\right]\\ &\le C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left(\int_{|z|\le N}|z|^p\nu(dz)\right)dyds\\ &\le C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])dyds,\end{aligned}\tag{3.1.20}$$

where the last equality is due to (3.1.16).

Case $p\in[1,2)$: using decomposition (3.1.3),

$$\begin{aligned}\mathbb{E}\left[|\overline{\mathfrak{T}}_N(\phi)(t,x)|^p\right] \le c_p\Bigg\{&\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{D}_N(dy,ds)\right|^p\right]\\ &+\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{L}_N(ds,dy)\right|^p\right]\Bigg\}.\end{aligned}\tag{3.1.21}$$

Note that

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{D}_N(dy,ds)\right|^p\right]\le D^p_\sigma\left(\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)dyds\right)^{p-1}\\ &\qquad\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+|\phi(s,y)|^p)\left|b+\int_{1<|z|\le N}z\nu(dz)\right|^p dsdy\right]\\ &=C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left|b+\int_{1<|z|\le N}z\nu(dz)\right|^p dsdy\\ &\le C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])dsdy,\end{aligned}\tag{3.1.22}$$

where the last inequality is due to (3.1.17). Now, by (A.1.2), it holds:

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{L}_N(ds,dy)\right|^p\right]\le\\ &C_p\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d\times\{|z|\le N\}}\mathcal{G}^p_{t-s}(x-y)|\sigma(\phi(s,y))|^p|z|^p\nu(dz)dyds\right]\\ &\le C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\left(\int_{|z|\le N}|z|^p\nu(dz)\right)dyds\\ &\le C_T\int_0^t\int_{\mathbb{R}^d}\mathcal{G}^p_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])dyds.\end{aligned}\tag{3.1.23}$$

Case $p \geq 2$: note that inequality (3.1.22) holds for $p \geq 2$. For the term involving $\overline{L}_N$, we use the maximal inequality (A.1.1) as it follows:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}(x-y)\sigma(\phi(s,y))\overline{L}_N(ds,dy)\right|^p\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d\times\{|z|\leq N\}} \mathcal{G}^2_{t-s}(x-y)\sigma(\phi(s,y))|^2|z|^2\nu(dz)dyds\right)^{\frac{p}{2}}\right] \\
&+ C_p\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d\times\{|z|\leq N\}} \mathcal{G}^p_{t-s}(x-y)\sigma(\phi(s,y))|^p|z|^p\nu(dz)dyds\right].
\end{aligned} \tag{3.1.24}$$

The second term in the right hand side of (3.1.24) is bounded in the same way as in (3.1.23). For the first term on (3.1.24), by Hölder's inequality, we have:

$$\begin{aligned}
&C\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d\times\{|z|\leq N\}} \mathcal{G}^2_{t-s}(x-y)\sigma(\phi(s,y))|^2|z|^2\nu(dz)dyds\right|^{\frac{p}{2}}\right] \\
&\leq C\Big(\int_0^t \int_{\mathbb{R}^d} \mathcal{G}^2_{t-s}(x-y)dsdy\Big)^{\frac{p}{2}-1} \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^2_{t-s}(x-y)(1+\mathbb{E}\left[|\phi(s,y)|^p\right])dsdy \\
&C\int_0^t \int_{\mathbb{R}^d} (\mathcal{G}_{t-s}(x-y)+\mathcal{G}^p_{t-s}(x-y))(1+\mathbb{E}\left[|\phi(s,y)|^p\right])dsdy,
\end{aligned}$$

where the last inequality is due to $|x|^2 < |x| + |x|^p$ for all $p \geq 2$. ∎

3.2 Stopping times and local property

In this section, we assume that $E = \mathbb{R}_+ \times \mathbb{R}^d$, and we study the relation between Λ defined by (3.0.1) and Λ_N defined by (3.0.2). Moreover, we show that under suitable conditions, there exists a non-decreasing sequence $\{\tau_N\}_{N\geq 1}$ of stopping times such that

$$\Lambda(A) = \Lambda_N(A) \quad \text{a.s. on } \{t \leq \tau_N\},$$

for all $A \in \widetilde{\mathcal{P}}_b$ with $A \subset \Omega \times [0,t] \times \mathbb{R}^d$.

Lemma 3.2.1 (Lemma 3.2 in [17]). Assume that Assumption 3.1.1 is satisfied. Let $h(x) = 1 + |x|^\eta$ with some $\eta \in \mathbb{R}_+$. For any $N \in \mathbb{N}$, we define

$$\tau_N = \inf\left\{T \in \mathbb{R}_+ : \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}} \mathbb{1}_{\{|z|>Nh(x)\}}\, J(dt,dx,dz) \neq 0\right\}. \tag{3.2.1}$$

If $\eta > d/q$, then $\tau_N > 0$ *a.s.* for all $N \in \mathbb{N}$ and $\tau_N \uparrow \infty$ *a.s.* for $N \uparrow \infty$.

Proof. First, for a fixed $N \in \mathbb{N}$, let us define the set

$$\mathcal{R}^N = \Big\{(x,z) \in \mathbb{R}^d \times \mathbb{R} \,:\, |z| \geq Nh(x)\Big\}.$$

So, note that

$$\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_{\{|z|>Nh(x)\}}\, J(dt,dx,dz) = J([0,T] \times \mathcal{R}^N)$$

Then, we can re-write τ_N as:

$$\tau_N = \inf\Bigg\{T \in \mathbb{R}_+ : J([0,T] \times \mathcal{R}^K) \neq 0\Bigg\}.$$

Note that if we prove that

$$J([0,T] \times \mathcal{R}^N) < \infty \quad \text{a.s.} \tag{3.2.2}$$

then $\tau_N > 0$ a.s for all $N \in \mathbb{N}$. Otherwise, if $\tau_N = 0$, there exists a sequence $\{T_n\}_{n\in\mathbb{N}} \subset [0,T]$ such that $T_n \to 0$ as $n \to \infty$ and $J([0,T_n] \times \mathcal{R}^N) > 0$, then there are infinitely many points of J in $[0,T] \times \mathcal{R}^N$, which is a contradiction with $J([0,T] \times \mathcal{R}^N) < +\infty$ a.s.

Now, we will show (3.2.2). Consider the partition of $\mathbb{R}^d$ defined as $\mathcal{U}_0 = \emptyset$ and

$$\mathcal{U}_n = \{x \in \mathbb{R}^d : |x| \leq \pi^{-1/2}\Gamma(1+d/2)^{1/d} n^{1/2}\} \setminus \mathcal{U}_{n-1}, \quad n \in \mathbb{N}.$$

This partition has the following properties: $\lambda^d(\mathcal{U}_n) = 1$ for all $n \in \mathbb{N}$. For any $x \in \mathcal{U}_n$, we have

$$h(x) \geq C(n-1)^{\eta/d}, \quad n \in \mathbb{N},$$

for some constant C independent of x and n. Note that for any $T > 0$ and $N \in \mathbb{N}$ fixed,

$$[0,T] \times \{(x,z) \in \mathcal{U}_n \times \mathbb{R} \,:\, |z| > Nh(x)\} \subset [0,T] \times \mathcal{U}_n \times \{|z| > a_n\} := E_n,\ n \in \mathbb{N},$$

where $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is the sequence defined as $a_1 = 1$ and $a_n = C\,N(n-1)^{\eta/d}$ for $n \geq 2$. Moreover, we can choose $a_1 = 1$ since $Nh(x) \geq 1$ for all $x \in \mathbb{R}^d$. Then,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_{\{|z|>Nh(x)\}}\, J(dt,dx,dz) &= \sum_{n=1}^\infty \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_{\mathcal{U}_n}(x)\, \mathbb{1}_{\{|z|>Nh(x)\}}(x,z)\, J(dt,dx,dz) \\ &\leq \sum_{n=1}^\infty \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_{\mathcal{U}_n}(x)\, \mathbb{1}_{\{|z|>a_n\}}(z)\, J(dt,dx,dz) \\ &= \sum_{n=1}^\infty J(E_n).\end{aligned} \tag{3.2.3}$$

Note that $J(E_n)$ is a Poisson random variable which is well-defined for each $n \in \mathbb{N}$, since $\{|z| > a_n\}$ is bounded away from 0 for all $n \in \mathbb{N}$. So, if we prove that $J([0,T] \times \mathcal{U}_n \times \{|z| > a_n\}) \neq 0$ a.s for a finite number of values $n \in \mathbb{N}$, then the sum in the last line of (3.2.3) is finite almost surely. Hence, $J([0,T] \times \mathcal{R}^N) < +\infty$ a.s.

To prove that $\sum_{n\geq 1} J(E_n) < +\infty$ a.s is enough to show that

$$\sum_{n=1}^{\infty} \mathbb{P}(\{J(E_n) > 0\}) < +\infty, \tag{3.2.4}$$

Hence, by Borel-Cantelli lemma, we have

$$\mathbb{P}\left(\bigcup_{n\geq 1}\bigcap_{j\geq n}\{J(E_j)=0\}\right) = 1.$$

This means there exists a random integer $n_0(\omega)$ such that $J(E_n) = 0$ a.s for all $n \geq n_0$. Then, we have:

$$J([0,T]\times\mathcal{R}^N) \leq \sum_{n=1}^{\infty} J(E_n) = \sum_{n=1}^{n_0} J(E_n) < \infty \quad \text{a.s.}$$

It remains to show (3.2.4). Indeed, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\mathbb{P}(\{J(E_n) > 0\}) = 1 - \mathbb{P}(\{J(E_n) = 0\}) &= 1 - e^{-m(E_n)} \\
&= 1 - e^{-T\nu(\{|z|>a_n\})} \\
&\leq T\nu(\{|z| > a_n\}) \\
&\leq T\int_{|z|>a_n} 1\nu(dz) \\
&\leq \int_{|z|>a_n} \left(\frac{|z|}{a_n}\right)^q \nu(dz) \\
&\leq T\,a_n^{-q}\left(\int_{|z|>1} |z|^q\nu(dz)\right) = T\,\gamma_2\,a_n^{-q},
\end{aligned}$$

where the equality in the last line is due to Assumption 3.1.1. By $\eta > \frac{d}{q}$, we get:

$$\sum_{n=1}^{\infty} \mathbb{P}(\{J(E_n) > 0\}) \leq T\gamma_2 \sum_{n=1}^{\infty} a_n^{-q} < +\infty.$$

Next, we will show that $\tau_N \uparrow \infty$ as $N \uparrow \infty$. For any $t \in \mathbb{R}_+$ fixed, there exists $N_0(t,\omega) \in \mathbb{N}$ large enough such that $J([0,t]\times\mathcal{R}^{N_0}) = 0$ a.s., since $J([0,t]\times\mathcal{R}^N) < +\infty$ a.s. for all $N \in \mathbb{N}$. This implies that $\mathbb{P}(\{\tau_N > t\}) = 1$ for all $N > N_0$ due to

$$J([0,t]\times\mathcal{R}^N) = 0 \text{ a.s. in } \{\tau_N > t\}. \tag{3.2.5}$$

This proves that $\tau_N \uparrow \infty$ as $N \uparrow \infty$. ∎

The following result shows that Λ and Λ_N coincide up to time τ_N.

Proposition 3.2.2. Under the assumptions of Lemma 3.2.1, for any $N \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we have:

$$\Lambda(A) = \Lambda_N(A) \quad \text{in} \quad \{t \leq \tau_N\},$$

for all $A \in \widetilde{\mathcal{P}}_b$ with $A \subset \Omega \times [0,t] \times \mathbb{R}^d$.

Proof. By Lemma 2.4.2 , we have:

$$\mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} H(s,x)\Lambda(ds,dx) = \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} H(s,x)\mathbb{1}_{[\![0,\tau_N]\!]}(s)\Lambda(ds,dx). \qquad (3.2.6)$$

On the other hand, for any $A \in \tilde{\mathcal{P}}_b$, it follows that Λ can be decompose as

$$\Lambda(A) = \Lambda_N(A) + M^N_{\text{large}}(A), \qquad (3.2.7)$$

where

$$M^N_{\text{large}}(A) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathbb{1}_{\mathcal{R}^N} \mathbb{1}_A(s,y) z J(ds,dx,dz).$$

By Proposition 2.4.10, note that

$$\begin{aligned}
\mathbb{1}_{[\![0,\tau_N]\!]}(t)\Lambda(A \cap (\Omega \times [0,t] \times \mathbb{R}^d)) &= \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau_N]\!](s)}\Lambda(ds,dy) \\
&= \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s)\Lambda_N(ds,dy) \\
&+ \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathbb{1}_{\mathcal{R}^N}(s,y)\mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s) z J(ds,dx,dz) \\
&= \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s)\Lambda_N(ds,dy),
\end{aligned}$$

the last equality above is due to

$$\mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathbb{1}_{\mathcal{R}^N}(s,y)\mathbb{1}_A(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s) z J(ds,dx,dz) = 0.$$

Then,

$$\mathbb{1}_{[\![0,\tau_N]\!]}(t)\Lambda(A \cap (\Omega \times [0,t] \times \mathbb{R}^d)) = \mathbb{1}_{[\![0,\tau_N]\!]}(t)\Lambda_N(A \cap (\Omega \times [0,t] \times \mathbb{R}^d)).$$

∎

The next result gives the local property of the stochastic integral with respect to Λ. Its proof follows directly from Proposition 2.4.10 and Proposition 3.2.2.

Lemma 3.2.3. Let $H \in L^0_{\text{loc}}(\Lambda)$ and τ_N defined by (3.2.1). Then, for $t \geq 0$

$$\mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} H(s,x)\Lambda(ds,dx) = \mathbb{1}_{[\![0,\tau_N]\!]}(t) \int_0^t \int_{\mathbb{R}^d} H(s,x)\mathbb{1}_{[\![0,\tau_N]\!]}(s)\Lambda_N(ds,dx), \qquad (3.2.8)$$

for all $N \in \mathbb{N}$.

For the rest of this section, we consider the case of the truncation functions $h(x) = 1$. We fix $T > 0$, and let $E = [0, T] \times \mathbb{R}^d$.

The following result introduces a new stopping time $\overline{\tau}_N$.

Lemma 3.2.4. Let D be a bounded Borel set in $\mathbb{R}^d$ and $T > 0$. Then, for each $N \in \mathbb{N}$,

$$\overline{\tau}_N(D) := \inf\{t \in [0, T]; J([0, t] \times D \times \{|z| > N\}) > 0\} \tag{3.2.9}$$

is a stopping time for all $N \in \mathbb{N}$. Moreover, $\overline{\tau}_N \leq \overline{\tau}_{N+1}$ a.s. for all $N \in \mathbb{N}$, and $\overline{\tau}_N = +\infty$ a.s. for sufficiently large N.

Proof. Note that J has a finite number of points in the region $[0, t] \times D \times \{|z| > N\}$, which implies $J([0, t] \times D \times \{|z| > N\}) < +\infty$. Hence, we can use the same argument as in the proof of Lemma 3.2.1. ∎

Using the same arguments as in the proof of Lemma 3.2.3, we have the following lemma, which gives the local property of the stochastic integral with respect to Λ, using the stopping time $\overline{\tau}_N$. Its proof is similar to the proof of Lemma 3.2.3.

Lemma 3.2.5. Let D be a bounded Borel set in $\mathbb{R}^d$. Then, for any $H \in L^0_{\text{loc}}(\Lambda)$ and $t \in [0, T]$,

$$\mathbb{1}_{[\![0,\overline{\tau}_N]\!]}(t) \int_0^t \int_D H(s, x)\Lambda(ds, dx) = \mathbb{1}_{[\![0,\overline{\tau}_N]\!]}(t) \int_0^t \int_D H(s, x)\mathbb{1}_{[\![0,\overline{\tau}_N]\!]}(s)\overline{\Lambda}_N(ds, dx), \tag{3.2.10}$$

for all $N \in \mathbb{N}$, where $\overline{\tau}_N(D)$ is the stopping time defined in (3.2.9).

Chapter 4

Stochastic heat equation with Lévy noises

In this chapter, we study the existence of solutions to the stochastic heat equation (SHE) driven by an infinite-variance Lévy noise. Parabolic SPDEs, such as the SHE, play an important role in several sub-fields of physics and mathematics, including polymers, spin glass theory, and particle physics.

In this chapter, we study the SHE with multiplicative noise, i.e.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}(t,x) & t>0,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^d, \end{cases} \tag{4.0.1}$$

where Λ is a Lévy noise in $E = \mathbb{R}_+ \times \mathbb{R}^d$ as defined in (3.0.1), and u_0 is a deterministic bounded function on $\mathbb{R}^d$. A *mild solution* of (4.0.1) is a predictable random field u that satisfies

$$u(t,x) = w_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda(ds,dy), \tag{4.0.2}$$

where $G_t(x)$ is the heat kernel, i.e. $G_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right) \mathbb{1}_{\{t>0\}}$ and

$$w_0(t,x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy.$$

In the remainder of this chapter, we assume that G_t is the heat kernel.

In [17], Chong proved for the first time that the heat equation (4.0.1) driven by Lévy noises with infinite variance, such as the α-stable Lévy noise, has a mild solution. Specifically, Chong demonstrated the existence of a solution $u = \{u(t,x); t\geq 0, x\in\mathbb{R}^d\}$ such that for all $N\in\mathbb{N}$,

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u(t,x)|\,\mathbb{1}_{[\![0,\tau_N]\!]}(t)\right]^p < +\infty, \text{ for all } T,R\in\mathbb{R}_+,$$

where τ_N is the stopping time defined in (3.2.1).

This chapter is organized as follows. In Section 4.1, we present some properties of the heat kernel. In Section 4.2, we present the results of [17] regarding the existence of a mild solution to (4.0.1).

4.1 Some properties of the heat kernel

In this section, we include some properties of the heat kernel that are used in the proof of the main results of Section 4.2 and are also employed in the proof of Proposition 6.5.2 for the verification of condition (6.0.13) in Chapter 6.

We will use the following property of the heat kernel:

$$G_t^p(x) = \overline{K}_{p,d}\, t^{\frac{d(1-p)}{2}} G_{t/p}(x), \quad \text{with} \quad \overline{K}_{p,d} = (2\pi)^{\frac{d(1-p)}{2}} p^{-\frac{d}{2}}. \tag{4.1.1}$$

Lemma 4.1.1. In the case of the heat equation, for any $p > 0$, $t > 0$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} G_t^p(x-y)dy = \overline{K}_{p,d} t^{\frac{d(1-p)}{2}} \quad \text{with} \quad \overline{K}_{p,d} = (2\pi)^{\frac{d(1-p)}{2}} p^{-d/2}.$$

Consequently, if $p < 1 + \frac{2}{d}$, then

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)dyds = K_{p,d} t^{\frac{d(1-p)}{2}+1} \quad \text{with} \quad K_{p,d} = \frac{\overline{K}_{p,d}}{\frac{d(1-p)}{2}+1}.$$

Proof. This follows by direct calculation, using relation (4.1.1). ∎

Lemma 4.1.2. In the case of the heat equation, for any $\gamma > 0$, $p > 0$, $t > 0$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} G_t^p(x-y)|y|^\gamma dy \le \overline{C}_{\gamma,p,d}\, t^{\frac{d(1-p)}{2}} (|x|^\gamma + t^{\gamma/2}),$$

where

$$\overline{C}_{\gamma,p,d} = \overline{K}_{p,d}(2^{\gamma-1} \vee 1)(1 \wedge p)^{-\gamma/2} \left[1 + \frac{2^{\gamma/2}}{\Gamma(d/2)} \Gamma\left(\frac{\gamma+d}{2}\right)\right].$$

Consequently, if $p < 1 + \frac{2}{d}$, then

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y)|y|^\gamma dyds \le C'_{\gamma,p,d}\, t^{\frac{d(1-p)}{2}+1}(|x|^\gamma + t^{\gamma/2}),$$

where

$$C'_{\gamma,p,d} = \frac{\overline{C}_{\gamma,p,d}}{\frac{d(1-p)}{2}+1} = K_{p,d}(2^{\gamma-1} \vee 1)(1 \wedge p)^{-\gamma/2} \left[1 + \frac{2^{\gamma/2}}{\Gamma(d/2)} \Gamma\left(\frac{\gamma+d}{2}\right)\right]. \tag{4.1.2}$$

Proof. Let X be a random vector with a $N_d(0,(t/p)I_d)$ distribution. By (4.1.1), we have:

$$\int_{\mathbb{R}^d} G_t^p(x-y)|y|^\gamma dy = \int_{\mathbb{R}^d} G_t^p(y)|x-y|^\gamma dy = \overline{K}_{p,d}\, t^{\frac{d(1-p)}{2}} \int_{\mathbb{R}^d} G_{t/p}(y)|x-y|^\gamma dy$$
$$= \overline{K}_{p,d}\, t^{\frac{d(1-p)}{2}} \mathbb{E}|x-X|^\gamma \leq \overline{K}_{p,d}\, t^{(1-p)d/2}(2^{\gamma-1}\vee 1)(|x|^\gamma + \mathbb{E}|X|^\gamma).$$

Let $Z = X/\sqrt{t/p}$. Then Z a $N_d(0,I_d)$ distribution, and $\mathbb{E}|X|^\gamma = (t/p)^{\gamma/2} z_\gamma$, where

$$z_\gamma := \mathbb{E}|Z|^\gamma = \frac{2^{\gamma/2}}{\Gamma(d/2)}\Gamma\Big(\frac{\gamma+d}{2}\Big).$$

Hence,

$$|x|^\gamma + \mathbb{E}|X|^\gamma \leq (1+z_\gamma)(|x|^\gamma + p^{-\gamma/2}t^{\gamma/2}) \leq (1+z_\gamma)(1\wedge p)^{-\gamma/2}(|x|^\gamma + t^{\gamma/2}).$$

This proves the first statement. The second statement follows by direct calculation. ∎

The following result is essentially contained in the proof of Theorem 3.1 of [17]. We include its proof since we need the explicit form of the constant $C_{\eta,p,d}$.

Lemma 4.1.3. If G is the fundamental solution of the heat equation, then for any $\eta > 0$ and $0 < p < 1 + \frac{2}{d}$,

$$I_{\eta,p}^{\text{heat}}(t,x) := \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G_{t_{k+1}-t_k}^p (x_{k+1}-x_k)(1+|x_k|^\eta) d\boldsymbol{x} d\boldsymbol{t} \leq$$
$$C_{\eta,p,d}^n \left\{1 + |x|^{n\eta} + t^{n\eta/2}\Gamma\left(\frac{1+n\eta}{2}\right)\right\} \frac{t^{n(\frac{d(1-p)}{2}+1)}}{\Gamma(n(\frac{d(1-p)}{2}+1)+1)},$$

where $t_{n+1} = t$, $x_{n+1} = x$ and

$$C_{\eta,p,d} = 3\overline{K}_{p,d}(2^{\eta-1}\vee 1)d^\eta\Big[\Big(\frac{2}{p}\Big)^{\eta/2}\vee 1\Big]\Gamma\Big(\frac{d(1-p)}{2}+1\Big). \tag{4.1.3}$$

Proof. Let $h(x) = 1 + |x|^\eta$ for $x \in \mathbb{R}^d$. By the generalized Hölder's inequality,

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}-x_k)h(x_k)d\boldsymbol{x} \leq \prod_{i=1}^n \left(\int_{(\mathbb{R}^d)^n} \prod G_{t_{k+1}-t_k}^p(x_{k+1}-x_k)h(x_i)^n d\boldsymbol{x}\right)^{1/n}$$
$$= \prod_{i=1}^n \left(\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_k)h(x - \sum_{j=i}^n x_j)^n d\boldsymbol{x}\right)^{1/n}. \tag{4.1.4}$$

We estimate separately each of the integrals appearing in the product above. Using relation (4.1.1), we write

$$\prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_k) = \overline{K}_{p,d}^n \prod_{k=1}^n (t_{k+1}-t_k)^{\frac{d(1-p)}{2}} \prod_{k=1}^n f_{X_k}(x_k),$$

where $X_1, \dots, X_k$ are independent random variables, $X_k \sim N_d(0, \frac{t_{k+1}-t_k}{p} I_d)$, and f_{X_k} is the density of X_k. Hence,

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G^p_{t_{k+1}-t_k}(x_k) h(x - \sum_{j=i}^n x_j)^n d\boldsymbol{x} = \overline{K}^n_{p,d} \prod_{k=1}^n (t_{k+1}-t_k)^{\frac{d(1-p)}{2}} \mathbb{E}\Big[h\big(x - \sum_{j=i}^n X_j\big)^n\Big].$$

Note that $h(x - \sum_{j=i}^n X_j) \leq (2^{\eta-1} \vee 1)(1 + |x|^\eta + |\sum_{j=i}^n X_j|^\eta)$, and hence

$$\mathbb{E}\Big[h\big(x - \sum_{j=i}^n X_j\big)^n\Big] \leq (2^{\eta-1} \vee 1)^n 3^{n-1}\Big(1 + |x|^{n\eta} + \mathbb{E}\Big|\sum_{j=i}^n X_j\Big|^{n\eta}\Big).$$

Moreover, $|x| = \sqrt{\sum_{\ell=1}^d |x^{(\ell)}|^2} \leq \sum_{\ell=1}^d |x^{(\ell)}|$, for any $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$, and hence,

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=i}^n X_j\Big|^{n\eta} &\leq \mathbb{E}\Big[\Big(\sum_{\ell=1}^d \Big|\sum_{j=i}^n X_j^{(\ell)}\Big|\Big)^{n\eta}\Big] \leq d^{n\eta-1} \sum_{\ell=1}^d \mathbb{E}\Big|\sum_{j=i}^n X_j^{(\ell)}\Big|^{n\eta} \\ &\leq d^{n\eta} \max_{\ell=1,\dots,d} \mathbb{E}\Big|\sum_{j=i}^n X_j^{(\ell)}\Big|^{n\eta}.\end{aligned}$$

Putting together these estimates, we infer that, for any $i = 1, \dots, n$ fixed,

$$\begin{aligned}&\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G^p_{t_{k+1}-t_k}(x_k) h(x - \sum_{j=i}^n x_j)^n d\boldsymbol{x} \leq \\ &\Big(3\overline{K}_{p,d}(2^{\eta-1} \vee 1) d^\eta\Big)^n \prod_{k=1}^n (t_{k+1}-t_k)^{\frac{d(1-p)}{2}} \Big(1 + |x|^{n\eta} + \max_{\ell=1,\dots,d} \mathbb{E}\Big|\sum_{j=i}^n X_j^{(\ell)}\Big|^{n\eta}\Big).\end{aligned} \tag{4.1.5}$$

We will show that this can be bounded by a quantity not depending on i. Note that

$$\sum_{j=i}^n X_j^{(\ell)} \sim N(0, \gamma_i), \quad \text{where} \quad \gamma_i := \sum_{j=i}^n \frac{t_{k+1}-t_k}{p} = \frac{t - t_i}{p}.$$

Recall that if $Z \sim N(0, \sigma^2)$, then $\mathbb{E}|Z|^p = (2\sigma^2)^{p/2} \pi^{-\frac{1}{2}} \Gamma(\frac{1+p}{2})$ for any $p > 0$. Hence,

$$\mathbb{E}\Big|\sum_{j=i}^n X_j^{(\ell)}\Big|^{n\eta} = (2\gamma_i)^{n\eta/2} \pi^{-\frac{1}{2}} \Gamma\left(\frac{1+n\eta}{2}\right) \leq \left(\frac{2t}{p}\right)^{n\eta/2} \Gamma\left(\frac{1+n\eta}{2}\right).$$

Using this estimate in (4.1.5), we obtain the following bound, which does not depend on i:

$$\begin{aligned}&\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n G^p_{t_{k+1}-t_k}(x_k) h(x - \sum_{j=i}^n x_j)^n d\boldsymbol{x} \leq \\ &(C'_{\eta,p,d})^n \prod_{k=1}^n (t_{k+1}-t_k)^{\frac{d(1-p)}{2}} \left(1 + |x|^{n\eta} + t^{n\eta/2} \Gamma\Big(\frac{1+n\eta}{2}\Big)\right),\end{aligned}$$

where $C'_{\eta,p,d} := 3\overline{K}_{p,d}(2^{\eta-1} \vee 1)d^{\eta}\big((\frac{2}{p})^{\eta/2} \vee 1\big)$.

Returning to (4.1.4), and integrating on $T_n(t)$ we get:

$$I_{\eta,p}^{\text{heat}}(t,x) \leq (C'_{\eta,p,d})^n \left(1 + |x|^{n\eta} + t^{n\eta/2}\Gamma\Big(\frac{1+n\eta}{2}\Big)\right) \int_{T_n(t)} \prod_{k=1}^{n} (t_{k+1}-t_k)^{\frac{d(1-p)}{2}} d\boldsymbol{t}.$$

The last integral is equal to $\frac{\Gamma(a+1)^n t^{n(a+1)}}{\Gamma(n(a+1)+1)}$, with $a := \frac{d(1-p)}{2} > -1$. The conclusion follows.

∎

4.2 Existence of solution

In this section, we review the main result of [17] which gives the existence of a solution of the stochastic heat equation (4.0.1). The proof of this result relies on a truncation technique. First, we prove that the following equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}_N(t,x), & t>0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.2.1}$$

has a mild solution $u^{(N)}$ in the space B^p_{loc} for each $N \in \mathbb{N}$, where Λ_N is the truncated noise given by (3.0.1), and p is exponent from Assumption 3.1.1. Recall that u is a solution of (4.2.1) if it satisfies the integral equation

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda_N(ds,dy). \tag{4.2.2}$$

Then, we show that the random field u defined by $u(t,x) := u^{(N)}(t,x)$ on $\{t \leq \tau_N\}$ is a mild solution to (4.0.1), where τ_N is the stopping time given by (3.2.1).

We define the following stochastic-integral operator:

$$\mathcal{J}_N(\phi)(t,x) := w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(\phi(s,y))\Lambda_N(ds,dy), \quad \text{for } \phi \in \mathcal{P}. \tag{4.2.3}$$

Following [17], we introduce the following assumption, which is similar to Assumption 3.1.1, but includes some additional restrictions on exponents p and q.

Assumption 4.2.1. There exist exponents $p, q \in \mathbb{R}$ satisfying $0 < q \leq p < 1 + \frac{2}{d}$ and $\frac{p}{1+(1+\frac{2}{d}-p)} < q$, such that

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) + \int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty.$$

Also, if $p < 1$, we assume that $b = \int_{\{|z|\leq 1\}} z\nu(dz)$.

The following result gives the existence of a solution to (SHE) driven by the truncated noise Λ_N.

Theorem 4.2.2 (Theorem 3.1 in [17])**.** Suppose that Assumption 4.2.1 holds and u_0 is a deterministic bounded function. Then equation (4.2.1) has a solution $u^{(N)} = \{u^{(N)}(t,x); t \geq 0, x \in \mathbb{R}^d\}$ which satisfies: for every $N \in \mathbb{N}$ and $T, R \in \mathbb{R}_+$,

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u^{(N)}(t,x)|^p\right] < +\infty.$$

Proof. We choose $\eta \in \mathbb{R}_+$ such that

$$\eta > \frac{d}{q} \quad \text{and} \quad \eta\frac{(p-q)}{2} < 1 - \frac{d(p-1)}{2}. \tag{4.2.4}$$

To simplify the writing, we drop the index N for the remaining of the proof. (The parameter N will play an important role in the next theorem for constructing the solution of (4.0.1) driven by Λ.)

Step 1: To prove that (4.2.2) has a non-trivial solution, we consider the Picard iteration sequence defined by $u_0(t,x) = 0$ and

$$u_n(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_{n-1}(s,y))\Lambda_N(ds,dy), \text{ for } n \in \mathbb{N}. \tag{4.2.5}$$

By Lemma 6.2 in [16], we can choose a predictable version of $u_n(t,x)$. By induction over n, it can be proved that

$$\mathbb{E}\left[|u_n(t,x)|^p\right] \leq C_{n,t}\left(1 + |x|^{n\eta(p-q)}\right) \tag{4.2.6}$$

for all $n \in \mathbb{N}$. Indeed, if $\mathbb{E}\left[|u_{n-1}(t,x)|^p\right] \leq C_{n,t}\left(1 + |x|^{(n-1)\eta(p-q)}\right)$, then by (3.1.7), it holds:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_{n-1}(s,y))\Lambda_N(ds,dy)\right|^p\right] \leq \\
&C_t \int_0^t \int_{\mathbb{R}^d} \left(G^p_{t-s}(x-y) + G_{t-s}(x-y)\mathbb{1}_{p\geq 1}\right)\left(1 + \mathbb{E}\left[|u_{n-1}(s,y)|^p\right]\right) h(y)^{p-q} dyds \\
&\leq C_{t,n} \int_0^t \int_{\mathbb{R}^d} \left(G^p_{t-s}(x-y) + G_{t-s}(x-y)\mathbb{1}_{p\geq 1}\right)\left(1 + |y|^{(n-1)\eta(p-q)}\right) h(y)^{p-q} dyds \\
&\leq C_{t,n} \int_0^t \int_{\mathbb{R}^d} \left(G^p_{t-s}(x-y) + G_{t-s}(x-y)\mathbb{1}_{p\geq 1}\right)\left(1 + |y|^{n\eta(p-q)}\right) dyds \\
&\leq C_{t,n}\left(1 + |x|^{n\eta(p-q)}\right),
\end{aligned}$$

where the last inequality is due to Lemma 4.1.1. Note that this calculation implies that $u_n \in B^p_{\text{loc}}$, *i.e.*, for any $T > 0$ and $R > 0$,

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u_n(t,x)|^p\right] < \infty.$$

Step 2: Next, we will prove that $\{u_n(t,x)\}_{n\geq 0}$ is a Cauchy sequence in B^p_{loc}. In fact, we will show that

$$\sum_{n\geq 1} \|u_n - u_{n-1}\|_{p,T,R} < \infty. \tag{4.2.7}$$

We abbreviate $g_p(t,x) = G^p_t(x) + G_t(x)\mathbb{1}_{\{p\geq 1\}}$. By Lemma 3.1.3-2, we have:

$$\mathbb{E}\left[|u_n(t,x) - u_{n-1}(t,x)|^p\right] \leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s, x-y)\mathbb{E}\left[|u_{n-1}(s,y) - u_{n-2}(s,y)|^p\right] h(y)^{p-q}\, dy\, ds \tag{4.2.8}$$

for all $n \geq 2$. If we iterate this inequality n times, we obtain:

$$\begin{aligned}\mathbb{E}\left[|u_n(t,x) - u_{n-1}(t,x)|^p\right] \leq C_T^n \int_0^t \int_{\mathbb{R}^d} \cdots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} g_p(t-t_1, x-x_1) \ldots g_p(t_{n-1}-t_n, x_{n-1}-x_n) \\ \times h(x_1)^{p-q} \ldots h(x_n)^{p-q}\, dt_n\, dx_n \ldots dt_1\, dx_1.\end{aligned} \tag{4.2.9}$$

Using Lemma 4.1.3, we get that

$$\mathbb{E}\left[|u_n(t,x) - u_{n-1}(t,x)|^p\right] \leq C_T^n \Gamma\left(\frac{1+n\eta(p-q)}{2}\right) \frac{\Gamma(1-\frac{d}{2}(p-1))^n}{\Gamma(1+(1-\frac{d}{2}(p-1))n)}(1+|x|^{n\eta(p-q)}). \tag{4.2.10}$$

Note that

$$1 - \frac{d}{2}(p-1) - \frac{\eta(p-q)}{2} > 0$$

due to the choice in (4.2.4). Returning to (4.2.8), and using Proposition B.2.7, we have for $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\mathbb{E}\left[|u_n(t,x) - u_{n-1}(t,x)|^p\right] \leq \frac{C_T^n}{(n!)^{1-\frac{d}{2}(p-1)-\frac{\eta(p-q)}{2}}}(1+|x|^{n\eta(p-q)}) := \beta_n(x), \tag{4.2.11}$$

for all $n \geq 2$. Therefore,

$$\begin{aligned}\sum_{n\in\mathbb{N}} \sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u_n(t,x) - u_{n-1}(t,x)|^p\right] &\leq \sum_{n\in\mathbb{N}} \sup_{(t,x)\in[0,T]\times[-R,R]^d} \beta_n(x) \\ &= \sum_{n\in\mathbb{N}} \frac{C_T^n}{(n!)^{1-\frac{d}{2}(p-1)-\frac{\eta(p-q)}{2}}}(1+K_R^{n\eta(p-q)}) < \infty,\end{aligned}$$

where $K_R = \max_{x\in[-R,R]^d} |x|$. This finishes the proof of (4.2.7).

Step 3: Since $\{u_n\}_{n\geq 1}$ is a Cauchy sequence in B^p_{loc}, there exists $u = u^{(N)} \in B^p_{\text{loc}}$ such that

$$\|u_n - u\|_{p,T,R} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.2.12}$$

for every $T, R \in \mathbb{R}_+$.

In this step, we prove that u satisfies the integral equation (4.2.2). For this, we let $n \to \infty$ in (4.2.5). By construction, on the left-hand side, we know that $u_n(t,x) \overset{L^p(\Omega)}{\to} u(t,x)$ for a fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. It remains to show the convergence of the right-hand side, *i.e.*

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_n(s,y))\Lambda_N(ds,dy) \overset{L^p(\Omega)}{\to} \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda_N(ds,dy) \tag{4.2.13}$$

as $n \to \infty$. By (2.3.14), it is enough to prove that $G_{t-\cdot}(x-\cdot)\sigma(u_n)$ converges to $G_{t-\cdot}(x-\cdot)\sigma(u)$ with respect to the Daniell mean $\|\cdot\|_{\Lambda_N,p}$, i.e.

$$\|V_n^{(t,x)} - V^{(t,x)}\|_{\Lambda_N,p} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.2.14}$$

where

$$\begin{cases} V_n^{(t,x)}(s,y) = G_{t-s}(x-y)\sigma(u_n(s,y))\mathbb{1}_{[0,t]}(s), \\ V^{(t,x)}(s,y) = G_{t-s}(x-y)\sigma(u(s,y))\mathbb{1}_{[0,t]}(s). \end{cases}$$

By Remark 3.1.4 and using the same arguments as above, note that $V_n^{(t,x)}$ is a Cauchy sequence with respect to $\|\cdot\|_{\Lambda_N,p}$. So, by (2.7) in [13], $V_n^{(t,x)}$ converges to a limit with $\|\cdot\|_{\Lambda_N,p}$. Moreover, by Remark 3.1.4-(iii), we get that

$$||V_n^{(t,x)} - V^{(t,x)}||^p_{\Lambda_N,p} \leq C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\Big[|u_n(s,y)-u(s,y)|^p\Big]h(y)^{p-q}dyds.$$

Since $\{u_n(t,x)\}_{n\geq 0}$ is a Cauchy sequence with respect to $L^p(\Omega)$ for a fixed $(t,x) \in [0,T]\times\mathbb{R}^d$, there exists a sufficiently large $n \in \mathbb{N}$ such that $\|u_m(t,x)-u_n(t,x)\|_p < 1$ for $m > n$, which implies $\|u_n(t,x)-u_m(t,x)\|_p^p \leq \|u_n(t,x)-u_m(t,x)\|_p$. Hence,

$$\|u_n(t,x)-u(t,x)\|_p^p \leq \|u_n(t,x)-u(t,x)\|_p \leq \sum_{k\geq n+1} \|u_k(t,x)-u_{k-1}(t,x)\|_p, \tag{4.2.15}$$

where the last inequality is due to Proposition B.1.3. So, by (4.2.15) and the monotone convergence theorem, we have:

$$\begin{aligned} ||V_n^{(t,x)} - V^{(t,x)}||^p_{\Lambda_N,p} &\leq C \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y) \sum_{k\geq n+1} \|u_k(s,y)-u_{k-1}(s,y)\|_p h(y)^{p-q}dyds \\ &= C \sum_{k\geq n+1} \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\|u_k(s,y)-u_{k-1}(s,y)\|_p h(y)^{p-q}dyds \\ &\leq C \sum_{k\geq n+1} \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\beta_k^{1/p}(y)h(y)^{p-q}dyds \\ &\leq C \sum_{k\geq n+1} \frac{C_T^k}{(k!)^{\frac{1}{p}(1-\frac{d}{2}(p-1)-\frac{\eta(p-q)}{2})}} \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)(1+|y|^{(k+1)\eta(p-q)})h(y)^{p-q}dyds \\ &\leq C \sum_{k\geq n+1} \frac{C_T^k}{(k!)^{\frac{1}{p}(1-\frac{d}{2}(p-1)-\frac{\eta(p-q)}{2})}}(1+|x|^{(k+1)\eta(p-q)}), \end{aligned}$$

where the last inequality is due to Lemma 4.1.2. Thus, $\|V_n^{(t,x)} - V^{(t,x)}\|_{\Lambda_N,p} \to 0$ as $n \to \infty$. This completes the proof of (4.2.14). ∎

For the following result, we recall that B^p is the set of random functions, whose definition can be found in the Notation part of the Introduction chapter. In particular, if u_0 is a bounded deterministic function, we obtain the uniqueness of the solution of equation (4.2.1).

Lemma 4.2.3. The solution $u^{(N)}$ of (4.2.1) in Theorem 4.2.2 is the unique (up to modifications) solution in B^p_{loc} of the fixed point problem

$$\begin{cases} \mathcal{J}_N(u) = u, \\ u_n = \mathcal{J}_N^{(n)}(u_0) \xrightarrow{B^p_{\text{loc}}} u \quad \text{as} \quad n \to \infty, \end{cases} \tag{4.2.16}$$

for any choice of $u_0 \in B^p$.

Proof. First, let us define the operators:

$$\begin{cases} \mathcal{J}^{(1)}(\phi) = \mathcal{J}(\phi), & \mathcal{J}^{(n)}(\phi) = \mathcal{J}(\mathcal{T}^{(n-1)}(\phi)), \\ \mathcal{J}_N^{(1)}(\phi) = \mathcal{J}_N(\phi), & \mathcal{J}_N^{(n)}(\phi) = \mathcal{J}(\mathcal{J}_N^{(n-1)}(\phi)), \end{cases}$$

for an integer $n \geq 2$ and $\phi \in \mathcal{P}$. We define the composition operators $\mathcal{J}^{(n)}$ and $\mathcal{J}_N^{(n)}$. Setting $\mathcal{J}^{(n)}(\phi)$, $\mathcal{J}_N^{(n)}(\phi) = +\infty$ as soon as $\mathcal{J}^{(n-1)}(\phi)$, $\mathcal{J}_N^{(n-1)}(\phi) = +\infty$.

Observe that the Picard iteration sequence (4.2.5) can be expressed as

$$u_n = \mathcal{J}_N(u_{n-1}) = \mathcal{J}_N(\mathcal{J}_N(u_{n-2})) = \ldots = \mathcal{J}_N^{(n)}(u_0),$$

given an initial condition $u_0 \in B^p$. So, the solution $u = u^{(N)}$ that we found in the proof of Theorem (4.2.2) satisfies the fixed-point problem given by (4.2.16). Assume that there exists a $v \in B^p_{\text{loc}}$ that satisfies (4.2.16) for an initial condition $v_0 \in B^p$ *i.e.*

$$\begin{cases} \mathcal{J}_N(v) = v, \\ v_n = \mathcal{J}_N^{(n)}(v_0) \xrightarrow{B^p_{\text{loc}}} v \quad \text{as} \quad n \to \infty. \end{cases}$$

By Lemma 3.1.3-2, we have:

$$\mathbb{E}\left[|u_n(t,x) - v_n(t,x)|^p\right] \leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s, x-y) \mathbb{E}\left[|u_{n-1}(s,y) - v_{n-1}(s,y)|^p\right] h(y)^{p-q} dy ds. \tag{4.2.17}$$

If we iterate (4.2.17) $n-1$ times, we have:

$$\begin{aligned} &\mathbb{E}\left[|u_n(t,x) - v_n(t,x)|^p\right] \leq \\ &C_T^n \int_0^t \int_{\mathbb{R}^d} \ldots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} g_p(t-t_1, x-x_1) \ldots g_p(t_{n-1}-t_n, x_{n-1}-x_n) \\ &\qquad (1 + \mathbb{E}\left[|u_0(t_n,x_n) - v_0(t_n,x_n)|^p\right]) h(x_1)^{p-q} \ldots h(x_n)^{p-q} dt_n dx_n \ldots dt_1 dx_1 \\ &\leq C_T^n \int_0^t \int_{\mathbb{R}^d} \ldots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} g_p(t-t_1, x_1) h(x-x_1)^{p-q} \ldots g_p(t_{n-1}-t_n, x_n) \\ &\qquad\qquad h(x - x_1 - \ldots - x_n)^{p-q} dx_n \ldots dx_1 \, dt_n \ldots dt_1 \\ &\leq \beta_n(x), \end{aligned} \tag{4.2.18}$$

where the inequality follows from the same steps as in (4.2.11). By (4.2.18), it holds:

$$||u_n - v_n||_{p,T,R} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.2.19}$$

for every $T, R \in \mathbb{R}_+$. By the triangle inequality in B^p_{loc}, we obtain:

$$||u - v||_{p,T,R} \leq ||u - u_n||_{p,T,R} + ||u_n - v_n||_{p,T,R} + ||v_n - v||_{p,T,R} \to 0,$$

as $n \to \infty$, for every $T, R \in \mathbb{R}_+$. Hence, $u = v$. ∎

Finally, we present the result about the existence of a global solution of (4.0.1).

Theorem 4.2.4 (Theorem 3.1 of [17]). Suppose that Assumption 4.2.1 holds, $\eta > d/q$, and u_0 is a deterministic bounded function. Then equation (4.0.1) has a solution $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$. Moreover, for all $N \in \mathbb{N}$ and $T, R \in \mathbb{R}_+$,

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u(t,x)|\, \mathbb{1}_{[\![0,\tau_N]\!]}(t)\right]^p < \infty,$$

where p is the exponent from Assumption 4.2.1.

Proof. We define:

$$u(t,x) = u^{(1)}(t,x)\mathbb{1}_{[\![0,\tau_1]\!]}(t) + \sum_{N=2}^{\infty} u^{(N)}(t,x)\mathbb{1}_{(\!(\tau_{N-1},\tau_N]\!]}(t). \tag{4.2.20}$$

Without loss of generality, we assume that $u_0(x) = 0$. By Lemma 4.2.3,

$$u^{(M)}(t,x)\mathbb{1}_{[\![0,\tau_N]\!]}(t,x) = u^{(N)}(t,x)\mathbb{1}_{[\![0,\tau_N]\!]}(t,x) \quad \text{for all } N, M \in \mathbb{N}, N \leq M.$$

Using the local property of the integral with respect to Λ given by Lemma 3.2.3, we obtain:

$$\begin{aligned}
&\mathbb{1}_{[\![0,\tau_M]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda(ds,dy) \\
&= \mathbb{1}_{[\![0,\tau_M]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\mathbb{1}_{[\![0,\tau_M]\!]}(s)\Lambda(ds,dy) \\
&= \mathbb{1}_{[\![0,\tau_M]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y)\mathbb{1}_{[\![0,\tau_M]\!]}(s))\Lambda(ds,dy) \\
&= \mathbb{1}_{[\![0,\tau_M]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(M)}(s,y))\Lambda_M(ds,dy) \\
&= \mathbb{1}_{[\![0,\tau_M]\!]}(t)u^{(M)}(t,x) = \mathbb{1}_{[\![0,\tau_M]\!]}(t)u(t,x)
\end{aligned}$$

Letting $M \to \infty$, we obtain that u satisfies (4.0.2), using the fact that $\tau_M \to \infty$ a.s. ∎

Remark 4.2.5 (Uniqueness of the solution). (i) The solution u of (4.0.1) defined in (4.2.20) is "unique", in the sense that for any $u_0 \in B^p$ fixed, u is the only solution in $B^p_{\text{loc}}(\tau_N)$ of the fixed-point problem:

$$\begin{cases} \mathcal{J}(u) = u, \\ ||(u - \mathcal{J}^{(n)}(u_0))\mathbb{1}_{[\![0,\tau_N]\!]}||_{p,T,R} \to 0 \quad \text{as} \quad n \to \infty, \end{cases} \qquad \text{for } T, R \in \mathbb{R}_+,\ N \in \mathbb{N}. \tag{4.2.21}$$

Here p is the exponent from Assumption 4.2.1.

(ii) To the best of our knowledge, the uniqueness of the solution of equation (4.0.1) for a globally Lipschitz function σ remains an open problem, with the exception of the case $\sigma(u) = \beta u$, when $\beta > 0$. As mentioned on page 13 in [17], the main issue in finding a unique mild solution to (4.0.1) is that it does not seem possible to find a complete subspace of B^p_{loc} such that the stochastic-integral operator $\mathcal{J}_N$ given by (4.2.3) is a self-map. Consequently, it is not possible to establish the uniqueness of the solution of equation (4.0.1) via the Banach fixed-point theorem. A different strategy was employed in [9], where it was proved that there exists a unique mild solution to (4.0.1), when $\nu(-\infty, 0) = 0$ and $\sigma(u) = \beta u$, with $\beta > 0$.

Chapter 5

Stochastic wave equation with Lévy noises

In this chapter, we study the stochastic wave equation (SWE) driven by the same infinite-variance Lévy noise as in Chapter 3. This chapter is based on the recent article [32] in which the author of this thesis studied the existence of the solution of (SWE) (using techniques similar to Chong's article [17] for (SHE)), as well as the uniqueness of this solution. The existence part is built upon and improves some of the techniques developed in the article [4]. In the last section, we present a new technique for proving the existence and uniqueness of the solution under conditions that are weaker than those in [4, 17]. This approach is based on the past light-cone property, which is inspired by Dalang's work in [24] and is included also in [32].

Recall that we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is endowed with be a right-continuous filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We consider the stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}(t,x), & t>0,\, x\in\mathbb{R}^d\ (d\le 2), \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{5.0.1}$$

where σ is a globally Lipschitz function, u_0 and v_0 are assumed to be non-random measurable functions, and Λ is a pure-jump Lévy basis in $E = \mathbb{R}_+ \times \mathbb{R}^d$ given by (3.0.1).

A predictable random field $u = \{u(t,x)\,;\, t\ge 0, x\in\mathbb{R}^d\}$ is called a *mild solution* of (5.0.1) if it satisfies the following stochastic integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda(ds,dy), \tag{5.0.2}$$

where $G_t(x)$ is the fundamental solution of the wave operator, defined as:

$$G_t(x) = \begin{cases} \frac{1}{2}\mathbb{1}_{\{|x|<t\}} & \text{if} \quad d=1, \\ \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbb{1}_{\{|x|<t\}} & \text{if} \quad d=2, \end{cases} \tag{5.0.3}$$

and w solves the homogeneous wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial conditions matching those of (5.0.1):

$$w(t,x) = (G_t * v_0)(x) + \frac{\partial}{\partial t}(G_t * u_0)(x). \tag{5.0.4}$$

In the remainder of this chapter, we assume that G_t is given by (5.0.3).

We assume the following conditions for u_0 and v_0.

Assumption 5.0.1. u_0 and v_0 are deterministic functions with the following properties.

- For $d = 1$, u_0 is locally bounded and continuous, and v_0 is locally bounded and measurable.
- For $d = 2$, u_0 is continuously differentiable ($C^1(\mathbb{R}^2)$), and v_0 is locally q_0-integrable with exponent $q_0 \in (2, \infty]$, i.e., $v_0 \in L^{q_0}_{\mathrm{loc}}(\mathbb{R}^2)$.

Under Assumption 5.0.1, we obtain,

$$\sup_{t\in[0,T]} \sup_{|x|\le R} |w(t,x)| < +\infty, \qquad \text{for all } T, R \in \mathbb{R}_+. \tag{5.0.5}$$

(5.0.5) can be proved similarly to Lemma 4.2 in [25] (see also Theorem 1.2 in [37]).

Despite the extensive literature on (5.0.1) driven by L^2-random measures, to our knowledge [4] is the only work that addresses the stochastic wave equation (5.0.1) driven by a multiplicative Lévy white noise which may have infinite variance. Specifically, in [4], it was proved the existence of a mild solution to (5.0.1) if ν satisfies the following conditions:

$$\begin{cases} \int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty & \text{if } d = 1, \text{ for some } q \in (0,2), \\ \int_{\{|z|\le 1\}} |z|^p \nu(dz) + \int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty & \text{if } d = 2, \text{ for some } 0 < q \le p < 2. \end{cases} \tag{5.0.6}$$

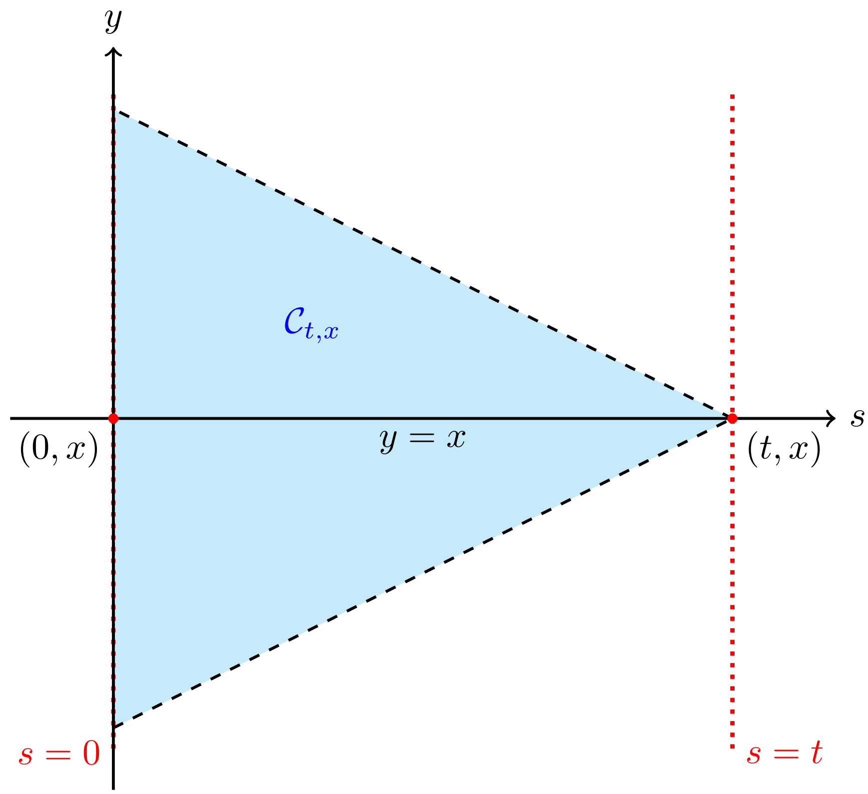
The regularity of the solution paths is also studied in [4].

The goal of this chapter is to establish the existence and uniqueness of solutions for the stochastic wave equation (5.0.1) under conditions which are weaker than (5.0.6).

A core principle used throughout this chapter is the fact that the fundamental solution G_t of the wave operator satisfies the following property: for any given point $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the function $(s,y) \mapsto G_{t-s}(x-y)$ has support in the conic region

$$\mathcal{C}_{t,x} := \{(s,y) \in [0,t] \times \mathbb{R}^d \,;\, |x-y| \le t-s\}. \tag{5.0.7}$$

The region $\mathcal{C}_{t,x}$ is called *the past light-cone* or *the domain of dependence*. In physics, the past light-cone illustrates causality, ensuring that the effects at a point are only due to sources within this cone. This ensures that solutions to the wave equation adhere to the principle of causality, i.e., the information or energy can only travel within the constraint set by the speed of wave propagation (see Theorem 14.1 of [56]).



The *past light-cone property* (PLCP) has also been used in [24] for the study of the stochastic wave equation in dimension $d = 3$, driven by a colored Gaussian noise. Unlike the Gaussian noise, which typically influences the entire random field uniformly, a Lévy noise can introduce abrupt changes or jumps. This makes the analysis of dependencies and influences within the past light-cone crucial for understanding how waves propagate in a heavy-tailed random field.

This chapter is organized as follows.

In Section 5.1, we prove the existence of a unique (up to modifications) mild solution u of (5.0.1) that satisfies $u \in B^p_{\text{loc}}(T_N)$ for all $N \in \mathbb{N}$, where $\{T_N\}_{n\geq 1}$ is an increasing sequence of stopping times with $T_N \to +\infty$ as $N \to +\infty$. The main novelty of this section is the uniqueness of a solution to (5.0.1) for the class of random fields that lie in $B^p_{\text{loc}}(T_N)$ for all $N \in \mathbb{N}$, employing the same techniques and stopping times used in [4, 17]. Furthermore, we extend these results to a broader class of wave equations.

In Section 5.2, we use a different strategy to show the existence and uniqueness of solutions to (5.0.1) in a finite time interval, under conditions which are weaker than (5.0.6). To be precise, by employing the PLCP, in Theorem 5.2.1, we construct a solution to (5.0.1) without imposing the condition $\int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty$ for some $q > 0$. In particular, our results show that:

(i) If $d = 1$, there exists a unique solution to (5.0.1) in the interval $[0, T]$, for a fixed $T > 0$, under Assumption 5.0.1.

(ii) If $d = 2$, there exists a unique solution to (5.0.1) in the interval $[0, T]$, for a fixed $T > 0$, under Assumption 5.0.1 and $\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < +\infty$ for some $p \in (0, 2)$.

Our method for constructing a solution to equation (5.0.1), using the PLCP, differs from the method in Section 5.1. For this, we use similar techniques as in [2, 18] for solving SPDEs on bounded domains. Additionally, we would like to point out that the uniqueness of the solution, using the PLCP approach in Section 5.2, is obtained in a different class of random fields compared to Section 5.1. Hence, it is natural to wonder how these two solutions are related. In Theorem 5.2.3, we prove that these two solutions are identical almost surely for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

5.1 Existence and uniqueness of a solution

In this section, we establish the existence and uniqueness of a solution to (5.0.1) using the same approach as in [4]. The novelty of this section, compared with the results of [4], is the uniqueness of a mild solution to (5.0.1). More precisely, using (5.0.7), we show that (5.0.1) has a unique (up to modification) mild solution that lies in $B^p_{\text{loc}}(\tau_N)$ for each $N \in \mathbb{N}$, where τ_N is given by (3.2.1).

Before presenting the main results of this section, we provide some preliminary results.

Lemma 3.1.3 will be a fundamental tool throughout this section. For the application of Lemma 3.1.3, we have to consider the following assumption on ν. In particular, if $d = 1$, we can extend the constraint $q \in (0, 2)$ in (5.0.6) to $q \in (0, +\infty)$.

Assumption 5.1.1. *(i)* For $d = 1$, there exists $q \in (0, +\infty)$ such that

$$\int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty.$$

(ii) For $d = 2$, there exist $0 < q \leq p < 2$ such that

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) + \int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty. \tag{5.1.1}$$

Additionally, if $p < 1$, we assume $b = \int_{\{|z|\leq 1\}} z\nu(dz)$.

Notice that the fundamental solution $G_t(x)$ of the wave operator given by (5.0.3) satisfies:

$$\int_{\mathbb{R}^d} G^p_t(x)dx = \begin{cases} 2^{1-p}\, t & \text{for any } p > 0 \text{ if } d = 1, \\ \frac{(2\pi)^{1-p}}{2-p}\, t^{2-p} & \text{for any } p \in (0, 2) \text{ if } d = 2, \end{cases} \tag{5.1.2}$$

for all $t \in \mathbb{R}_+$. We denote $g_p(t, x) = G^p_t(x) + G_t(x)\mathbb{1}_{\{p\geq 1\}}$.

In the following remark, we explain why we can extend the value of q to interval $(0, +\infty)$.

Remark 5.1.2. For equation (5.0.1) in dimension $d = 1$, there is no need to impose the p-integrability condition $\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < +\infty$ on the small jumps, since this condition is automatically satisfied for all $p \geq 2$:

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) \leq \int_{\{|z|\leq 1\}} |z|^2 \nu(dz) < +\infty.$$

Moreover, when $d = 1$, then $\int_0^T \int_{\mathbb{R}} G_t^p(x)\, dx\, dt < +\infty$ for all $p > 0$. Thus, for $d = 1$, we can apply Lemma 3.1.3 with $\mathcal{G}_t = G_t$, and choose any value $p \geq 2 \vee q$. On the other hand, for equation (5.0.1) in dimension $d = 2$, we must impose condition (5.1.1), since $\int_0^T \int_{\mathbb{R}^d} G_t^p(x)\, dx\, dt < +\infty$ only holds for $p \in (0, 2)$.

To establish the existence of a mild solution to the stochastic wave equation (5.0.1), we follow a strategy similar to [17] for the stochastic heat equation. More precisely, we first show that the stochastic wave equation driven by Λ_N,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}_N(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{5.1.3}$$

has a unique mild solution $u^{(N)}$ in B^p_{loc} for each $N \in \mathbb{N}$, i.e., $u^{(N)}$ is the only (up to modifications) random field in B^p_{loc} satisfying

$$u^{(N)}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(N)}(s,y))\Lambda_N(ds,dy). \tag{5.1.4}$$

We define the operator $\mathcal{T}_N : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{T}_N(\phi)(t,x) := w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(\phi(s,y))\Lambda_N(ds,dy), \tag{5.1.5}$$

for any $\phi \in \mathcal{P}$. By Lemma 6.6 of [16], the random field $\mathcal{T}_N(\phi)$ admits a predictable modification. We will always work with this modification.

Using the compact support property of G_t, we establish the self-mapping property of $\mathcal{T}_N$ in B^p_{loc}. Therefore, the existence of a unique mild solution for (5.1.3) will be a consequence of the Banach fixed-point theorem applied to the operator $\mathcal{T}_N$. Note that this approach cannot be used to prove the uniqueness of the solution to equation (4.0.1). This limitation arises from the fact that the operator $\mathcal{J}_N$ given by (4.2.3), is well-defined from B^p to B^p_{loc}, but lacks self-mapping attributes in B^p_{loc}, as mentioned in the previous section.

The following result gives the existence and uniqueness of the solution to the stochastic wave equation with truncated noise Λ_N, and is similar to Theorem 4.2.2 (for the stochastic heat equation).

Theorem 5.1.3. Assume that Assumptions 5.0.1 and 5.1.1 are satisfied. Then, for any fixed $N \in \mathbb{N}$, equation (5.1.3) has a unique (up to modifications) mild solution $u^{(N)}$ that satisfies

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|u^{(N)}(t,x)|^p\right] < +\infty,$$

for all $T>0$ and $R>0$, where p is any arbitrary value such that $p \geq q \vee 2$ if $d=1$, and p is the exponent from Assumption 5.1.1 if $d=2$.

Proof. *Step 1 ($\mathcal{T}_N$ is a self-map in B^p_{loc}).* Note that for any $0 \leq s < t \leq T$ and $|x| \leq R$, we have:

$$\operatorname{supp}(G_{t-s}(x-\cdot)) \subseteq \overline{B_T(x)} \subset \overline{B_{T+R}(0)}. \tag{5.1.6}$$

Hence,

$$h(y)^{p-q} < C_T(1+|R|^\gamma) \quad \text{for all } y \in \overline{B_{T+R}(0)}, \tag{5.1.7}$$

where $\gamma = \eta(p-q)$. Then, by Lemma 3.1.3, (5.1.7), and (5.1.6), for any $\phi \in B^p_{\text{loc}}$, we have:

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(\phi(s,y))\Lambda_N(ds,dy)\right|^p\right] \\ &\leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\ (1+\mathbb{E}\left[|\phi(s,y)|^p\right]) h(y)^{p-q} dyds \\ &\leq C_T(1+|R|^\gamma)\int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\,(1+\mathbb{E}\left[|\phi(s,y)|^p\right])\,dyds \\ &\leq C_{p,T,R}\left(1+\sup_{s\in[0,T]}\sup_{|y|\leq R+T}\mathbb{E}\left[|\phi(s,y)|^p\right]\right). \end{aligned} \tag{5.1.8}$$

Therefore, by (5.0.5) and (5.1.8), we get:

$$\sup_{t\in[0,T]}\sup_{|x|\leq R}\mathbb{E}\left[|\mathcal{T}_N(\phi)(t,x)|^p\right] < +\infty, \qquad \text{for all } T, R \in \mathbb{R}_+. \tag{5.1.9}$$

Step 2 (Convergence of the Picard iterations). In this step, we consider the Picard iteration $u^{(N)}_n = \{u^{(N)}_n(t,x)\,;\, t\geq 0,\, x\in\mathbb{R}^d\}$ given by: $u^{(N)}_0(t,x) := \Psi_0(t,x)$, where Ψ_0 is an arbitrary element of B^p_{loc}, and $u^{(N)}_n := \mathcal{T}_N(u^{(N)}_{n-1})$ for all $n\in\mathbb{N}$, i.e.,

$$u^{(N)}_n(t,x) = w(t,x) + \int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(N)}_{n-1}(s,y))\Lambda_N(ds,dy), \text{ for } n\in\mathbb{N}. \tag{5.1.10}$$

By (5.1.9), it follows that $u^{(N)}_n \in B^p_{\text{loc}}$ for all $n\in\mathbb{N}$ by induction over n. Next, we will show that $\{u^{(N)}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B^p_{loc}. By Lemma 3.1.3-(ii), we have:

$$\begin{aligned} &\mathbb{E}\left[|u^{(N)}_n(t,x)-u^{(N)}_{n-1}(t,x)|^p\right] \\ &\leq C_T\int_0^t\int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|u^{(N)}_{n-1}(s,y)-u^{(N)}_{n-2}(s,y)|^p\right] h(y)^{p-q}dyds. \end{aligned} \tag{5.1.11}$$

Iterating (5.1.11), we get:

$$\begin{aligned} &\mathbb{E}\left[|u^{(N)}_n(t,x)-u^{(N)}_{n-1}(t,x)|^p\right] \\ \leq &C_T^n \int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n g_p(t_{i+1}-t_i,x_{i+1}-x_i) \\ &\times\prod_{i=1}^n h(x_i)^{p-q}\mathbb{E}\left[|u^{(N)}_1(t_1,x_1)-\Psi_0(t_1,x_1)|^p\right] d\mathbf{x}d\mathbf{t}, \end{aligned} \tag{5.1.12}$$

where $\mathbf{t} = (t_1, \dots, t_n)$, $\mathbf{x} = (x_1, \dots, x_n)$ and we set $t_{n+1} = t$ and $x_{n+1} = x$. For a fixed $\mathbf{t}$, note that the function $\mathbb{G}_{\mathbf{t}} : (\mathbb{R}^d)^n \to [0, +\infty)$ given by

$$\mathbb{G}_{\mathbf{t}}(\mathbf{x}) := \prod_{i=1}^{n} g_p(t_{i+1} - t_i, x_{i+1} - x_i) \mathbb{1}_{T_n(t)}(\mathbf{t}),$$

has support in the set

$$\{\mathbf{x} \in (\mathbb{R}^d)^n \,;\, |x_{i+1} - x_i| \leq t_{i+1} - t_i, \text{ for } i = 1, \dots, n\}.$$

Hence, if $t \in [0, T]$ and $|x| \leq R$, the integral in (5.1.12) can be restricted to the values $\mathbf{x}$ in the bounded set

$$\left\{\mathbf{x} \in (\mathbb{R}^d)^n \,;\, |x_i| \leq R + T \text{ for } i = 1, \dots, n\right\}, \tag{5.1.13}$$

since

$$|x - x_i| \leq |x - x_n| + \sum_{k=i}^{n-1} |x_{k+1} - x_k| \leq (t - t_n) + \sum_{k=i}^{n-1} (t_{k+1} - t_k) = t - t_i \leq t < T. \tag{5.1.14}$$

Then, by (5.1.13), it follows that

$$\begin{aligned}\mathbb{E}\left[|u_n^{(N)}(t,x) - u_{n-1}^{(N)}(t,x)|^p\right] \leq C_T^n \sup_{s\in[0,T]} \sup_{|y|\leq R+T} \left(\mathbb{E}\left[|u_1^{(N)}(t,x)|^p\right] + \mathbb{E}\left[|\Psi_0(t,x)|^p\right]\right) \\ \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} g_p(t_{i+1} - t_i, x_{i+1} - x_i) \prod_{i=1}^{n} h(x_i)^{p-q} d\mathbf{x} d\mathbf{t}.\end{aligned} \tag{5.1.15}$$

On the restricted set given by (5.1.13), we have:

$$\prod_{i=1}^{n} h(x_i)^{p-q} \leq \prod_{i=1}^{n} \left[1 + (R+T)^\eta\right]^{p-q} \leq C_T^n (1 + |R|^{n\gamma}). \tag{5.1.16}$$

Hence, by (5.1.15) and (5.1.16), we obtain:

$$\begin{aligned}\mathbb{E}\left[|u_n^{(N)}(t,x) - u_{n-1}^{(N)}(t,x)|^p\right] &\leq C_{T,R}^n \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} g_p(t_{i+1} - t_i, x_{i+1} - x_i) d\mathbf{x} d\mathbf{t} \\ &:= C_{T,R}^n A_n^{(p)}(t).\end{aligned} \tag{5.1.17}$$

Note that $A_n^{(p)}(t)$ does not depend on x. If $p < 1$, by (5.1.2) and Lemma B.2.6, we get:

$$A_n^{(p)}(t) = C_p^n \int_{T_n(t)} \prod_{j=1}^{n} (t_{j+1} - t_j)^\alpha d\mathbf{t} = C_p^n \frac{t^{(\alpha+1)n}}{\Gamma((\alpha+1)n + 1)},$$

where C_p is a constant that depends on p, and

$$\alpha = \begin{cases} 1 & \text{if } d = 1, \\ 2 - p & \text{if } d = 2. \end{cases}$$

Hence,

$$\text{if } p<1, \qquad \sup_{t\in[0,T]} A_n^{(p)}(t) \leq \begin{cases} C_p^n \frac{T^{2n}}{(n!)^2} & \text{if } d=1, \\ C_p^n \frac{T^{(3-p)n}}{(n!)^{3-p}} & \text{if } d=2. \end{cases} \tag{5.1.18}$$

Assume that $p \geq 1$. If $d=1$, it holds that $G_t^p(x) = 2^{1-p}G_t(x)$, so we can proceed in the same way as for $p<1$, which implies $\sup_{t\in[0,T]} A_n^{(p)}(t) \leq 2^{n(1-p)}C_p^n \frac{T^{2n}}{(n!)^2}$. If $d=2$, by (5.1.2), we have:

$$\int_{\mathbb{R}^2} g_p(t,x)dx = c_p t^{2-p} + t \leq (c_p + T^{p-1})t^{2-p}, \qquad \text{with} \quad c_p = \frac{(2\pi)^{1-p}}{2-p}.$$

Hence, $\sup_{t\in[0,T]} A_n^{(p)}(t) \leq (c_p+T^{p-1})^n C_p^n \frac{T^{(3-p)n}}{(n!)^{3-p}}$. Thus,

$$\text{if } p\geq 1, \qquad \sup_{t\in[0,T]} A_n^{(p)}(t) \leq \begin{cases} 2^{n(1-p)}C_p^n \frac{T^{2n}}{(n!)^2} & \text{if } d=1, \\ (c_p+T^{p-1})^n C_p^n \frac{T^{(3-p)n}}{(n!)^{3-p}} & \text{if } d=2. \end{cases} \tag{5.1.19}$$

Therefore, for both cases $p<1$ and $p\geq 1$, it holds

$$\sum_{n\geq 1} C_{T,R,p}^n \sup_{t\in[0,T]} A_n^{(p)}(t) < +\infty. \tag{5.1.20}$$

By (5.1.17) and (5.1.20), $\{u_n^{(N)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B_{loc}^p. Hence, there exists an element $u^{(N)} \in B_{\text{loc}}^p$ such that $u_n^{(N)} \xrightarrow{B_{\text{loc}}^p} u^{(N)}$ as $n\to+\infty$.

Step 3 (Existence of the solution). In this step, we verify that $u^{(N)}$ satisfies (5.1.4). First, we apply Lemma 3.1.3-(ii) with $\mathcal{G}_t = G_t$. Then,

$$\begin{aligned} &\mathbb{E}\left[\Big|\int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)(\sigma(u_n^{(N)}(s,y)) - \sigma(u^{(N)}(s,y)))\Lambda_N(ds,dy)\Big|^p\right] \\ &\leq \left\|\mathcal{I}^{(t,x)}(\sigma(u_n^{(N)}) - \sigma(u^{(N)}))\right\|_{\Lambda_N,p}^{p\vee 1} \\ &\leq C_T \int_0^t\int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|u_n^{(N)}(s,y) - u^{(N)}(s,y)|^p\right] h(y)^{p-q}dyds \\ &\leq C_{T,R,p} \sup_{s\in[0,T]}\sup_{|y|\leq T+R} \mathbb{E}\left[|u_n^{(N)}(s,y) - u^{(N)}(s,y)|^p\right] \int_0^t\int_{\mathbb{R}^d} g_p(t-s,x-y)dyds, \end{aligned} \tag{5.1.21}$$

we used (5.1.6) and (5.1.7) in the previous inequality. Now, if we let n approach infinity in (5.1.21), we find that for a fixed pair $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, the expression $\mathcal{I}^{(t,x)}(\sigma(u_n^{(N)}))$ converges to $\mathcal{I}^{(t,x)}(\sigma(u^{(N)}))$ with the semi-norm $||\cdot||_{\Lambda_N,p}^{p\vee 1}$. This convergence implies that

$$\lim_{n\to+\infty} \mathcal{T}_N(u_n^{(N)})(t,x) = \mathcal{T}_N(u^{(N)})(t,x) \quad \text{in } L^p(\Omega).$$

Moreover, we have $u_n^{(N)} = \mathcal{T}_N(u_{n-1}^{(N)})$ for all $n\in\mathbb{N}$, and the sequence $\{u_n^{(N)}\}_{n\in\mathbb{N}}$ converges to $u^{(N)}$ in the space B_{loc}^p as $n\to+\infty$. Therefore, we conclude that $u^{(N)}$ satisfies (5.1.4).

Step 4 (Uniqueness of the solution). Assume that there exists another process $v^{(N)} \in B^p_{\text{loc}}$ that satisfies (5.1.4), i.e., $\mathcal{T}_N(v^{(N)}) = v^{(N)}$. Then, by Lemma 3.1.3-(ii), we have:

$$\begin{aligned}
&\mathbb{E}\left[|u_n^{(N)}(t,x) - v^{(N)}(t,x)|^p\right] \\
&\leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s, x-y)\mathbb{E}\left[|u_{n-1}^{(N)}(s,y) - v^{(N)}(s,y)|^p\right] h(y)^{p-q} dsdy.
\end{aligned}$$

Iterating the inequality above as in (5.1.12), and following the same steps as in (5.1.17), we get:

$$\begin{aligned}
&\mathbb{E}\left[|u_n^{(N)}(t,x) - v^{(N)}(t,x)|^p\right] \\
&\leq C^n_{p,T,R} \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n g_p(t_{i+1}-t_i, x_{i+1}-x_i)\mathbb{E}\left[|u_1^{(N)}(t_1,x_1) - v^{(N)}(t_1,x_1)|^p\right] d\mathbf{x}d\mathbf{t} \\
&\leq \sup_{s\in[0,T]} \sup_{|y|\leq R+T} \mathbb{E}\left[|u_1^{(N)}(s,y) - v^{(N)}(s,y)|^p\right] C^n_{p,T,R} A^p_n(t) \to 0,
\end{aligned}$$

as $n \to +\infty$. Therefore, $u_n^{(N)} \to v^{(N)}$ in B^p_{loc} as $n \to +\infty$. Alternatively, $u_n^{(N)} \to u^{(N)}$ in B^p_{loc} as $n \to +\infty$, which implies $u^{(N)} = v^{(N)}$ in B^p_{loc}. ∎

The following result gives the existence of the global solution to the stochastic wave equation driven by noise Λ. The existence part is similar to Theorem 4.2.4 (for the stochastic heat equation), while the uniqueness part is new.

Theorem 5.1.4. Let τ_N be the stopping time given by (3.2.1). Under the same assumptions as in Theorem 5.1.3 with $\eta > d/q$, equation (5.0.1) has a unique (up to modifications) mild solution u that satisfies

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|u(t,x)|^p \mathbb{1}_{[\![0,\tau_N]\!]}(t)\right] < +\infty,$$

for all $T > 0$ and $R > 0$, where p is any arbitrary value such that $p \geq q \vee 2$ if $d = 1$, and p is the exponent in Assumption 5.1.1 if $d = 2$.

Proof. *Step 1 (Existence of the solution).* The existence of a solution to (5.0.1) follows as in the proofs of Theorem 4.2.4. First, note that $\Lambda(B \cap ([\![0,\tau_N]\!] \times \mathbb{R}^d)) = \Lambda_N(B \cap ([\![0,\tau_N]\!] \times \mathbb{R}^d))$ for all $B \in \tilde{\mathcal{P}}_b$. Therefore, by Lemma 3.2.3, the random field u given by

$$u(t,x) = u^{(1)}(t,x)\mathbb{1}_{[\![0,\tau_1]\!]}(t) + \sum_{N=2}^{\infty} u^{(N)}(t,x)\mathbb{1}_{(\!(\tau_{N-1},\tau_N]\!]}(t),$$

is a mild solution to (5.0.1), where $u^{(N)}$ is the solution to (5.1.3) given by Theorem 5.1.3.

Step 2 (Uniqueness of the solution). Assume that v is another solution of (5.0.1) such that $v \in B^p_{\text{loc}}(\tau_N)$ for all $N \in \mathbb{N}$. By Lemma 3.2.3 and Lemma 3.1.3, we have:

$$\begin{aligned}
&\mathbb{E}\left[|(u_n^{(N)}(t,x) - v(t,x))\mathbb{1}_{[\![0,\tau_N]\!]}(t)|^p\right] \\
&\leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|(u_{n-1}^{(N)}(s,y) - v(s,y))\mathbb{1}_{[\![0,\tau_N]\!]}(s)|^p\right] h(y)^{p-q} dyds.
\end{aligned}$$

Iterating the inequality above and using the same steps as in (5.1.17), for $t \in [0,T]$ and $|x| \leq R$, we obtain that:

$$\begin{aligned}&\mathbb{E}|(u_n^{(N)}(t,x) - v(t,x))\mathbb{1}_{[\![0,\tau_N]\!]}(t)|^p \\ &\leq C_{T,R}^n \Big(\sup_{s\in[0,T]} \sup_{|y|\leq R+T} \Big[\mathbb{E}\left[|u_1^{(N)}(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s)|^p \right] + \mathbb{E}\Big[|v(s,y)\mathbb{1}_{[\![0,\tau_N]\!]}(s)|^p \Big] \Big] \Big) A_n^{(p)}(t),\end{aligned}$$

where $A_n^{(p)}(t)$ is given by (5.1.17). Using the fact that $\sup_{t\in[0,T]} A_n^{(p)}(t) \to 0$ as $n \to +\infty$, we conclude that

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|(u_n^{(N)}(t,x) - v(t,x))\mathbb{1}_{[\![0,\tau_N]\!]}(t)|^p \right] \to 0 \qquad \text{as } n \to +\infty,$$

for all $R, T > 0$. On the other hand, note that $u_n^{(N)} \to u$ in $B_{\text{loc}}^p(\tau_N)$ as $n \to +\infty$ for all $N \in \mathbb{N}$. Then, $u(t,x)\mathbb{1}_{[\![0,\tau_N]\!]}(t) = v(t,x)\mathbb{1}_{[\![0,\tau_N]\!]}(t)$ a.s., and letting $N \to +\infty$, we obtain that $u(t,x) = v(t,x)$ a.s. for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ due to $\tau_N \uparrow +\infty$ a.s. for $N \to +\infty$. ∎

Next, we investigate the stochastic wave equation driven by a more general heavy-tailed noise, which contains also a Gaussian component, in dimension $d = 1$.

Consider $L = \{L(B); B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ given by

$$L(B) = b|B| + aW(B) + \int_{B\times\{|z|\leq 1\}} z\widehat{J}(dt,dx,dz) + \int_{B\times\{|z|>1\}} zJ(dt,dx,dz), \qquad (5.1.22)$$

where $a > 0$ and W is a space-time Gaussian white noise, i.e., $W := \{W(A)\,;\, A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is a zero mean Gaussian process with covariance

$$E[W(A)W(B)] = \lambda_d(A \cap B).$$

In the case $d = 2$, since $\int_0^t \int_{\mathbb{R}^d} G_{t-s}^2(x-y)dsdy = +\infty$, there is no mild solution of (5.0.1) driven by L instead of Λ. Therefore, using the same steps of Theorem 5.1.3 and Theorem 5.1.4, we have the following result.

Corollary 5.1.5. Under Assumptions 5.0.1 and 5.1.1, the stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{L}(t,x), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \qquad (5.1.23)$$

has a unique (up to modifications) mild solution u that satisfies

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|u(t,x)|^p \mathbb{1}_{[\![0,\tau_N]\!]}(t) \right] < +\infty, \quad \text{for all } p \geq 2,$$

for all $T, R \in \mathbb{R}_+$ and $N \in \mathbb{N}$.

Proof. We use the same argument as in the proof of Theorem 5.1.4. For this argument, we apply Lemma 3.3 of [17], which is stated there for a general Lévy noise which contains a Gaussian component. ∎

For the remaining part of this section, we present an extension of Theorem 5.1.4 to a more general class of stochastic wave equations driven by multiplicative noises with a non-linear term $\sigma(t,x,u)\dot{\Lambda}$ and drift $f(t,x,u)$. More precisely, we consider the stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{\Lambda}(t,x), & t>0,\ x\in\mathbb{R}^d\ (d\le 2), \\ u(0,x)=u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x\in\mathbb{R}^d \end{cases} \tag{5.1.24}$$

where u_0 and v_0 are the same initial conditions as in (5.0.1). We impose the following conditions on the processes σ and f.

Assumption 5.1.6. σ and f are functions defined as $\Omega\times\mathbb{R}_+\times\mathbb{R}^{d+1}\to\mathbb{R}$ which are measurable with respect to $\tilde{\mathcal{P}}\times\mathcal{B}(\mathbb{R})$. In addition, we assume there exist positive processes $\mathcal{C}_f$, $\mathcal{C}_\sigma\in B^\infty_{loc}$ such that for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ and l_1, $l_2\in\mathbb{R}$, we have:

$$|\sigma(t,x,l_1)-\sigma(t,x,l_2)| \le \mathcal{C}_\sigma(t,x)\,|l_1-l_2| \quad \text{a.s.}, \tag{5.1.25}$$

and

$$|f(t,x,l_1)-f(t,x,l_2)| \le \mathcal{C}_f(t,x)\,|l_1-l_2| \quad \text{a.s.} \tag{5.1.26}$$

Denote $\sigma_0(t,x)=\sigma(t,x,0)$ and $f_0(t,x)=f(t,x,0)$. A *mild solution* to (5.1.24) is a predictable random field u that satisfies

$$\begin{aligned} u(t,x) = w(t,x) &+ \int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y,u(s,y))\,dy\,ds \\ &+ \int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))\,\Lambda(ds,dy). \end{aligned} \tag{5.1.27}$$

Theorem 5.1.7. Under Assumptions 5.0.1, 5.1.1, and 5.1.6, the following results hold.

(i) For $d=1$, assume that there exists $p\ge 2\vee q$ such that σ_0 and f_0 belong to B^p_{loc}, then (5.1.24) admits a unique (up to modifications) mild solution u satisfying

$$\sup_{t\in[0,T]}\sup_{|x|\le R}\mathbb{E}\left[|u(t,x)|^p\,\mathbb{1}_{[\![0,\tau_N]\!]}(t)\right] < +\infty,$$

for all $T,R>0$ and $N\in\mathbb{N}$.

(ii) For $d=2$, assume that σ_0 and f_0 belong to B^p_{loc}, where p is the exponent in (5.1.1). Additionally, if $p<1$, we impose $f(t,x,l)=0$ a.s. for all $(t,x,l)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}$. Then, equation (5.1.24) admits a unique (up to modifications) mild solution u satisfying

$$\sup_{t\in[0,T]}\sup_{|x|\le R}\mathbb{E}\left[|u(t,x)|^p\,\mathbb{1}_{[\![0,\tau_N]\!]}(t)\right] < +\infty,$$

for all $T,R>0$ and $N\in\mathbb{N}$.

Proof. First, we prove that the operator $\Im_N : \mathcal{P} \to \mathcal{P}$ given by

$$\Im_N(\phi)(t,x) := w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y,\phi(s,y)) dy ds \\ + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,\phi(s,y))\Lambda_N(ds,dy),$$

is a self-map in B^p_{loc}, for each fixed $N \in \mathbb{N}$. Note that $\Im_N$ is well-defined since $\sigma(t,x,\phi(t,x))$ and $f(t,x,\phi(t,x))$ are predictable for all $\phi \in \mathcal{P}$ as a consequence of σ and f being measurable with respect to $\tilde{\mathcal{P}} \otimes \mathcal{B}(\mathbb{R})$. Hence, by Lemma 6.2 in [16], $\Im_N(\phi)(t,x)$ has a predictable modification. Now, by Lemma 3.1.3, (5.1.6), (5.1.25), and Hölder's inequality $||XY||_1 \leq ||X||_\infty ||Y||_1$, for any $t \in [0,T]$ and $|x| \leq R$, we have:

$$\begin{aligned} &\mathbb{E}\left[\Big|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,\phi(s,y))\Lambda_N(ds,dy)\Big|^p\right] \\ &\leq C_{p,T} \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|\sigma(s,y,\phi(s,y))|^p\right] h(y)^{p-q} dy ds \\ &\leq C_{p,T,R}\Big(\sup_{s\in[0,T]} \sup_{|y|\leq T+R} ||\mathcal{C}_\sigma(s,y)||^p_\infty \vee \mathbb{E}\left[|\sigma_0(s,y)|^p\right]\Big) \\ &\quad \times \Big(\sup_{s\in[0,T]} \sup_{|y|\leq T+R} \mathbb{E}\left[|\phi(s,y)|^p\right] + 1\Big). \end{aligned} \tag{5.1.28}$$

Hence, if $\phi \in B^p_{\text{loc}}$, then

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[\Big|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,\phi(s,y))\Lambda_N(ds,dy)\Big|^p\right] < +\infty. \tag{5.1.29}$$

Next, we examine the integral that corresponds to the drift f. Recall that if $p \in (0,1)$, we assume that $f = 0$. Hence, we consider only the case $p \geq 1$. By Hölder's inequality and (5.1.26), for any $t \in [0,T]$ and $|x| \leq R$,

$$\begin{aligned} &\mathbb{E}\left[\Big|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y,\phi(s,y)) dy ds\Big|^p\right] \\ &\leq \Big(\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) dy ds\Big)^{p-1} \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathbb{E}\left[|f(s,y,\phi(s,y))|^p\right] dy ds \\ &\leq C_T\Big(\sup_{s\in[0,T]} \sup_{|y|\leq T+R} ||\mathcal{C}_f(s,y)||^p_\infty \vee \mathbb{E}\left[|f_0(s,y)|^p\right]\Big)\Big(\sup_{s\in[0,T]} \sup_{|y|\leq T+R} \mathbb{E}\left[|\phi(s,y)|^p\right] + 1\Big). \end{aligned} \tag{5.1.30}$$

Then,

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[\Big|\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y,\phi(s,y)) dy ds\Big|^p\right] < +\infty.$$

Therefore, by (5.1.29) and (5.1.30),

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|\Im_N(\phi)(t,x)|^p\right] < +\infty. \tag{5.1.31}$$

Now, consider the Picard's iteration sequence $\{u_n^{(N)}\}_{n\geq 0}$ given by $u_0^{(N)}(t,x) := \Psi_0(t,x)$ where $\Psi_0 \in B^p_{\text{loc}}$, and

$$\begin{aligned} u_{n+1}^{(N)}(t,x) := w(t,x) &+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y,u_n^{(N)}(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u_n^{(N)}(s,y))\Lambda_N(ds,dy). \end{aligned} \tag{5.1.32}$$

By (5.1.31), it follows that

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[|u_n^{(N)}(t,x)|^p\right] < +\infty,$$

for all $R, T > 0$ and $n \in \mathbb{N}$. Similarly to (5.1.29) and (5.1.30), we obtain

$$\begin{aligned} &\mathbb{E}\left[|u_n^{(N)}(t,x) - u_{n-1}^{(N)}(t,x)|^p\right] \\ &\leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s,x-y)\mathbb{E}\left[|u_{n-1}^{(N)}(s,y) - u_{n-2}^{(N)}(s,y)|^p\right] h(y)^{p-q} dsdy. \end{aligned} \tag{5.1.33}$$

Following the same procedure as in the proof of Theorem 5.1.3, by iterating inequality (5.1.33), we obtain that:

$$\mathbb{E}\left[|u_n^{(N)}(t,x) - u_{n-1}^{(N)}(t,x)|^p\right] \leq C_{p,T,R} A_n^{(p)}(t),$$

where $A_n^{(p)}(t)$ is defined as in (5.1.17). Hence, $\{u_n^{(N)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B^p_{loc}. Consequently, there exists a limit $u^{(N)}$ in B^p_{loc}. The existence and uniqueness of a solution u to (5.1.24) follow in the same manner as in the proof of Theorem 5.1.4.

∎

5.2 Past light-cone property

In this section, we fix $T > 0$ and consider the stochastic wave equation in the interval $[0,T]$,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{\Lambda}(t,x), & t \in (0,T],\ x \in \mathbb{R}^d,\ d \leq 2, \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{5.2.1}$$

where σ is a globally Lipschitz function, u_0 and v_0 satisfy Assumption 5.0.1. By exploiting the PLCP of the wave equation, we will show the existence and uniqueness of a solution to (5.2.1) without imposing the assumption of q-integrability over the large jumps: for $d = 1$, no assumptions are required. For $d = 2$, the only assumption that we impose is

$$\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < +\infty, \quad \text{for some } p \in (0,2).$$

The PLCP has been extensively studied for hyperbolic PDEs in the literature. For instance, in [24], it was proved that the solution of the stochastic wave equation, in dimension $d = 3$, driven by a colored Gaussian noise remains invariant in a region of space, if the problem is restricted to that region. As in Section 6 of [24], for a fixed region $D \in \mathfrak{B}$, we define the conic region

$$\mathcal{K}^D(s) := \{y \in \mathbb{R}^d \,;\, d(y, D) \leq T - s\}, \quad \text{for any } s \in [0, T].$$

Clearly, $\mathcal{K}^D(0) = \bigcup_{t\in[0,T]} \mathcal{K}^D(t)$.

As we do not impose the assumption,

$$\int_{\{|z|>1\}} |z|^q \nu(dz) < +\infty, \tag{5.2.2}$$

the stopping time τ_N given by (3.2.1) may not be well-defined. Therefore, we consider another stopping time given by

$$\tau_N(D) := \inf\left\{t \in [0, T]\,;\, J([0, t] \times \mathcal{K}^D(0) \times \{|z| > N\}) > 0\right\}, \quad N \in \mathbb{N},\ D \in \mathfrak{B}. \tag{5.2.3}$$

Note that $\tau_N(D) > 0$ a.s., and $\tau_N(D) < \tau_{N+1}(D)$ a.s. for all $N \in \mathbb{N}$. Moreover, $\tau_N(D) = +\infty$, for large N.

Additionally, for a fixed $(t, x) \in [0, T] \times \overline{D}$, note that $\text{supp}(H^{(t,x)}) \subset [0, t] \times \mathcal{K}^D(0)$, where

$$H^{(t,x)}(s, y) = G_{t-s}(x - y)\mathbb{1}_{\{t\geq s\}},$$

This implies that for a fixed $(t, x) \in [0, T] \times \overline{D}$, the value $u(t, x)$ of the solution of (5.0.2) only depends on the values of Λ on $[0, T] \times \mathcal{K}^D(0)$. Thus, for any $(t, x) \in [0, T] \times \overline{D}$, if ϕ is integrable with respect to Λ,

$$\begin{aligned}
&\mathbb{1}_{[\![0,\tau_N(D)]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(\phi(s, y))\Lambda(ds, dy) \\
&= \mathbb{1}_{[\![0,\tau_N(D)]\!]}(t) \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(\phi(s, y))\mathbb{1}_{[\![0,\tau_N(D)]\!]}(s)\overline{\Lambda}_N(ds, dy),
\end{aligned} \tag{5.2.4}$$

due to Proposition 2.4.10, and the fact that

$$\Lambda([0, t] \times A) = \overline{\Lambda}_N([0, t] \times A) \quad \text{on} \quad \{t \leq \tau_N(D)\},$$

for all $A \in \mathcal{B}_b(\mathbb{R}^d)$, with $A \subset \mathcal{K}^D(0)$.

The stopping times defined in (5.2.3) are analogous to those used in [2, 18] for solving SPDEs driven by Lévy noise in bounded domains in $\mathbb{R}^d$. For example, in [18], it was proved that for any fixed $T > 0$ and $D \in \mathfrak{B}$, the stochastic heat equation on D,

$$\begin{cases}
\dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2}\Delta u(t, x) + \sigma(u(t, x))\dot{\Lambda}(t, x), & t \in (0, T],\ x \in D, \\
u(t, x) = 0, & t \in [0, T],\ x \in \partial D, \\
u(0, x) = u_0(x), & x \in D,
\end{cases} \tag{5.2.5}$$

has a unique solution $u = \{u(t,x)\ ;\ t \in [0,T],\, x \in D\}$ that satisfies

$$\sup_{t\in[0,T]}\sup_{x\in D}\mathbb{E}\left[|u(t,x)|^p\,\mathbb{1}_{[\![0,\tau^\star_N(D)]\!]}(t)\right] < +\infty,$$

for all $N \in \mathbb{N}$. Here, $\overline{\tau}_N(D)$ is the stopping time defined by (3.2.9). However, the main issue associated with the use of $\overline{\tau}_N(D)$ for SPDEs on the entire space (such as (5.0.1) and (4.0.1)) is that if $D = \mathbb{R}^d$, then $\overline{\tau}_N(D) = 0$ almost surely for all $N \in \mathbb{N}$. This happens because the region $[0,t] \times \mathbb{R}^d \times \{|z| > N\}$ may contain infinitely many points of J. By contrast, using the PLCP, we can construct a "local solution" $u^{(D)}$ of (5.2.1) on $[0,T]\times\overline{D}$ using the stopping times given by (5.2.3). As these local solutions are consistent and agree almost surely on the same region in space, we can construct a mild solution to (5.2.1).

The resemblance between $\tau_N(D)$ and $\tau^\star_N(D)$ is not a coincidence. The primary result of this section shows that the solution u to (5.2.1) given by Theorem 5.2.1 satisfies

$$u^{(D)}(t,x) = u(t,x) \quad \text{a.s. for all } (t,x) \in [0,T]\times\overline{D}\ ,$$

where $u^{(D)}$ is a predictable process satisfying the stochastic-integral equation:

$$u^{(D)}(t,x) = w(t,x) + \int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(D)}(s,y))\mathbb{1}_{\mathcal{K}^D(s)}(y)\Lambda(ds,dy). \tag{5.2.6}$$

Note that the integrand of the stochastic integral on the right-hand side of equation (5.2.6) has support on $\mathcal{K}^D(0)$. Hence, we can solve (5.2.6) using $\tau_N(D)$ similarly to solving SPDEs driven by Lévy noise on bounded domains.

Furthermore, since the method that we use to construct a solution to (5.2.6) essentially requires the same integrability condition on ν as for solving SPDEs on bounded domains, it suffices to consider only the assumption on the small jumps: $\int_{\{|z|\le 1\}}|z|^p\nu(dz) < +\infty$ for some $p > 0$. More precisely, we only need this assumption on the small jumps for the wave equation (5.2.1) in dimension $d = 2$, while for dimension $d = 1$, no additional conditions are required.

The following theorem is the main result of this section.

Theorem 5.2.1. (a) If $d = 1$, (5.2.1) has a unique (up to modifications) mild solution u that satisfies

$$\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|u(t,x)|^p\mathbb{1}_{[\![0,\tau_N(D)]\!]}(t)\right] < +\infty, \quad \text{for all } p \ge 2,\ N \in \mathbb{N}, \text{ and } D \in \mathfrak{B}.$$

(b) If $d = 2$, suppose that there exists $p \in (0,2)$ such that

$$\int_{\{|z|\le 1\}}|z|^p\nu(dz) < +\infty, \tag{5.2.7}$$

and if $p < 1$, then $b = \int_{\{|z|\le 1\}} z\nu(dz)$. Then (5.0.1) has a unique (up to modifications) mild solution u that satisfies

$$\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|u(t,x)|^p\,\mathbb{1}_{[\![0,\tau_N(D)]\!]}(t)\right] < +\infty, \quad \text{for all } N \in \mathbb{N} \text{ and } D \in \mathfrak{B}.$$

Proof. As in the previous section, the main distinction between $d = 1$ and $d = 2$ lies in the integrability properties of G_t. If $d = 1$, $\int_0^T \int_{\mathbb{R}^d} G_t^p(x)\, dt\, dx < +\infty$ for all $p > 0$, and $\int_{\{|z|\leq 1\}} |z|^p \nu(dz) < +\infty$ for all $p \geq 2$ by Remark 5.1.2. In contrast, if $d = 2$, $\int_0^T \int_{\mathbb{R}^d} G_t^p(x)\, dt\, dx < +\infty$ holds only for $p \in (0, 2)$, which requires imposing the p-integrability of ν on the small jumps.

Step 1 (Existence and uniqueness of a local solution). Let $D \in \mathfrak{B}$. By employing a similar approach as in the proofs of Theorem 5.1.3 and Theorem 5.1.4, we can establish the existence of a unique solution $u^{(D)}$ to (5.2.6) that satisfies

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[\left|u^{(D)}(t,x)\right|^p \mathbb{1}_{[\![0,\tau_N(D)]\!]}(t)\right] < +\infty, \quad \text{for all } N \in \mathbb{N} \text{ and } R > 0.$$

Additionally,

$$u^{(D)}(t,x) = u^{(D,N)}(t,x), \quad \text{on} \quad \{t \leq \tau_N(D)\},$$

where $u^{(D,N)}(t,x)$, up to modifications, is the unique solution to the truncated problem,

$$u^{(D,N)}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(D,N)}(t,x))\mathbb{1}_{\mathcal{K}^D(s)}(y)\overline{\Lambda}_N(ds,dy), \tag{5.2.8}$$

that satisfies

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[\left|u^{(D,N)}(t,x)\right|^p\right] < +\infty, \quad \text{for all } R > 0. \tag{5.2.9}$$

Here, $\overline{\Lambda}_N$ is the noise Λ_N in (3.0.1) when $h(x) = 1$, i.e.,

$$\overline{\Lambda}_N(dt,dx) = b\, dt\, dx + \int_{\{|z|\leq 1\}} z\widehat{J}(dt,dx,dz) + \int_{\{1<|z|\leq N\}} zJ(dt,dx,dz). \tag{5.2.10}$$

Notice that $u^{(D,N)}(t,x)$ is the limit in B^p_{loc} of the Picard iteration sequence $\{u_n^{(D,N)}\}_{n\geq 0}$, given by $u_0^{(D,N)} = \Psi_0^{(D)}$, with $\Psi_0^{(D)}$ being an arbitrary element of B^p_{loc}, and

$$u_n^{(D,N)}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u_{n-1}^{(D,N)}(s,y))\mathbb{1}_{\mathcal{K}^D(s)}(y)\overline{\Lambda}_N(ds,dy), \tag{5.2.11}$$

for $n \in \mathbb{N}$.

The primary distinction from the proofs of Theorem 5.1.3 and Theorem 5.1.4 is that we can apply Lemma 3.1.5 for $\overline{\Lambda}_N$ (Λ_N when $h(x) = 1$), without condition (5.2.2), i.e., considering only condition (5.2.7); this same observation can be found on page 477 of [4]. This enables us to demonstrate that

$$\sup_{t\in[0,T]} \sup_{|x|\leq R} \mathbb{E}\left[\left|u_n^{(D,N)}(t,x) - u^{(D,N)}(t,x)\right|^p\right] \to 0, \quad \text{for all } R > 0, \tag{5.2.12}$$

as $n \to +\infty$, without (5.2.2). The uniqueness of the solution $u^{(D)}$ can be proven using the same arguments of the proof of Theorem 5.1.4.

The proof for constructing local-truncated solutions $u^{(D,N)}$ for $D \in \mathfrak{B}$ is essentially the same as in Theorem 2.8 of [4]; the only differences are that the initial condition w satisfies

(5.0.5) and we used the compact support property (5.0.7) to obtain the self-map property of the truncated operator related to the fixed-point problem (5.2.8). This implies that we can obtain a unique (up to modifications) mild solution to (5.2.8) that satisfies (5.2.9).

Step 2 (Consistency). For any $A, B \in \mathfrak{B}$ with $A \subset B$, we will show:

$$u^{(A)}(t,x) = u^{(B)}(t,x) \quad \text{a.s.,} \quad \text{for all } (t,x) \in [0,T] \times \overline{A}, \tag{5.2.13}$$

where $u^{(A)}$ (resp. $u^{(B)}$) is the solution of (5.2.6) found in Step 1 with $D = A$ (resp. $D = B$).

Our goal is to demonstrate that

$$\sup_{(t,x)\in[0,T]\times\overline{A}} \mathbb{E}\left[|(u^{(A)}(t,x) - u^{(B)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)\wedge\tau_N(A)]\!]}(t)|^p\right] = 0 \quad \text{for all } N \in \mathbb{N}. \tag{5.2.14}$$

If (5.2.14) holds, then,

$$\mathbb{1}_{[\![0,\tau_N(B)\wedge\tau_N(A)]\!]}(t)u^{(A)}(t,x) = \mathbb{1}_{[\![0,\tau_N(B)\wedge\tau_N(A)]\!]}(t)u^{(B)}(t,x) \quad \text{a.s.} \quad \text{for all } (t,x) \in [0,T] \times \overline{A}.$$

Letting N be sufficiently large, we obtain (5.2.13). Now, for any $(t,x) \in [0,T] \times \overline{A}$, by the triangle inequality, we have:

$$\begin{aligned}
&\mathbb{E}\left[|(u^{(A)}(t,x) - u^{(B)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)\wedge\tau_N(A)]\!]}(t)|^p\right] \\
&\leq c_p\Big[\mathbb{E}\left[|(u^{(A,N)}(t,x) - u_n^{(A,N)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)]\!]}(t)|^p\right] \\
&+ \mathbb{E}\left[|(u_n^{(A,N)}(t,x) - u_n^{(B,N)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)]\!]}(t)|^p\right] \\
&+ \mathbb{E}\left[|(u_n^{(B,N)}(t,x) - u^{(B,N)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)]\!]}(t)|^p\right]\Big],
\end{aligned} \tag{5.2.15}$$

where $u_n^{(A,N)}$ (resp. $u_n^{(B,N)}$) is the sequence defined in (5.2.11) when $D = A$ (resp. $D = B$). In (5.2.15), we used the fact that $\tau_N(B) \leq \tau_N(A)$ a.s. for all $N \in \mathbb{N}$. By (5.2.12), the first and third terms on the right-hand side of (5.2.15) converge to zero. Therefore, to prove (5.2.14), it remains to show that:

$$\sup_{(t,x)\in[0,T]\times\overline{A}} \mathbb{E}\left[|(u_n^{(A,N)}(t,x) - u_n^{(B,N)}(t,x))\mathbb{1}_{[\![0,\tau_N(B)]\!]}(t)|^p\right] \to 0 \quad \text{as } n \to +\infty. \tag{5.2.16}$$

For a fixed $(t,x) \in [0,T] \times \overline{A}$ and fixed $n \in \mathbb{N}$, we define the set $\mathcal{A}_n^{(t,x)}$ as

$$\begin{aligned}
\mathcal{A}_n^{(t,x)} := \Big\{&(t,x,s_1,y_1,\dots,s_n,y_n) \in ([0,T]\times\mathbb{R}^d)^{n+1}; \\
&t > s_1 > s_2 > \ldots > s_n > 0,\ y_k \in B_{s_{k-1}-s_k}(y_{k-1}),\ k = 1,\dots,n\Big\},
\end{aligned}$$

where, $(s_0, y_0) = (t,x)$. In contrast with the previous section, we use the reverse order index for the simplex defined in Lemma B.2.6, i.e., $t > s_1 > s_2 > \ldots > s_n > 0$. We denote

$$\mathbf{s} := (s_1, s_2, \dots, s_n), \quad \text{and} \quad \mathbf{y} := (y_1, y_2, \dots, y_n).$$

Additionally, note that

$$\int_{\tilde{T}_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} g_p(s_{i-1}-s_i, y_{i-1}-y_i)\, d\mathbf{y}\, d\mathbf{s} = A_n^{(p)}(t), \tag{5.2.17}$$

where $\tilde{T}_n(t) := \{(s_1,\dots,s_n) \in (0,t)^n\,;\, s_n < \dots < s_1\}$, and $A_n^{(p)}(t)$ is defined as in (5.1.20).

By the triangle inequality, note that

$$B_{s_{i-1}-s_i}(y_{i-1}) \subset \mathcal{K}^A(s_i) \subset \mathcal{K}^B(s_i), \quad \text{for all } i = 1,\dots,n. \tag{5.2.18}$$

for any $(t,x,s_1,y_1,\dots,s_n,y_n) \in \mathcal{A}_n^{(t,x)}$. Due to (5.2.18), we have:

$$\begin{aligned} u_{n-k+1}^{(A,N)}(s_{k-1},y_{k-1}) &= w(s_{k-1},y_{k-1}) \\ &\quad + \int_0^{s_{k-1}} \int_{\mathbb{R}^d} G_{s_{k-1}-s_k}(y_{k-1}-y_k)\sigma(u_{n-k}^{(A,N)}(s_k,y_k))\overline{\Lambda}_N(ds_k,dy_k) \end{aligned} \tag{5.2.19}$$

and

$$\begin{aligned} u_{n-k+1}^{(B,N)}(s_{k-1},y_{k-1}) &= w(s_{k-1},y_{k-1}) \\ &\quad + \int_0^{s_{k-1}} \int_{\mathbb{R}^d} G_{s_{k-1}-s_k}(y_{k-1}-y_k)\sigma(u_{n-k}^{(B,N)}(s_k,y_k))\overline{\Lambda}_N(ds_k,dy_k). \end{aligned} \tag{5.2.20}$$

for $k = 1,\dots,n$.

Let

$$Y_{n,k}^N(s_k,y_k) := (u_{n-k}^{(A,N)}(s_k,y_k) - u_{n-k}^{(B,N)}(s_k,y_k))\mathbb{1}_{[\![0,\tau_N(B)]\!]}(s_k),$$

for all $k = 0,1,\dots,n$. Then, using (5.2.19), (5.2.20), Proposition 2.4.10, and Lemma 3.1.3, we have:

$$\begin{aligned} &\mathbb{E}\left[|Y_{n,k-1}^N(s_{k-1},y_{k-1})|^p\right] \\ &\leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(s_{k-1}-s_k, y_{k-1}-y_k)\mathbb{E}\left[|Y_{n,k}^N(s_k,y_k)|^p\right]\, dy_k\, ds_k, \end{aligned} \tag{5.2.21}$$

for $k = 1,\dots,n$. Observe that the constant C_T in (5.2.21) differs from the constant in (3.1.8). However, this is not an issue as this constant depends only on N, T, σ, and p.

Now, by (5.2.21), we can iterate the following inequality $n-1$-times:

$$\mathbb{E}\left[|Y_{n,0}^N(t,x)|^p\right] \leq C_T \int_0^t \int_{\mathbb{R}^d} g_p(t-s_1, x-y_1)\mathbb{E}\left[|Y_{n,1}^N(s_1,y_1)|^p\right]\, dy_1\, ds_1, \tag{5.2.22}$$

over $(t,x,s_1,y_1,\dots,s_n,y_n) \in \mathcal{A}^{(t,x)}$. Thus,

$$\mathbb{E}\left[|Y_{n,0}^N(t,x)|^p\right] \leq C_T^n \int_{\tilde{T}_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} g_p(s_{i-1}-s_i, y_{i-1}-y_i)\mathbb{E}\left[|Y_{n,n}^N(s_n,y_n)|^p\right] d\mathbf{y} d\mathbf{s}. \tag{5.2.23}$$

Therefore, by (5.2.23), (5.2.17), (5.1.18) and (5.1.19),

$$
\begin{aligned}
&\mathbb{E}\left[|u_n^{(A,N)}(t,x) - u_n^{(B,N)}(t,x)\mathbb{1}_{[\![0,\tau_N(B)]\!]}(t)|^p\right] \\
&\leq C_T^n \int_{\tilde{T}_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n g_p(s_{i-1}-s_i, y_{i-1}-y_i)\mathbb{E}\left[|u_0^{(A,N)}(t_n,x_n) - u_0^{(B,N)}(s_n,y_n)|^p\right] d\mathbf{y}d\mathbf{s} \\
&\leq \sup_{s\in[0,T]} \sup_{|y|\leq R_A+T} \mathbb{E}\left[|u_0^{(A,N)}(s,y) - u_0^{(B,N)}(s,y)|^p\right] \sup_{t\in[0,T]} C_T^n A_n^{(p)}(t) \to 0, \quad \text{as } n\to+\infty,
\end{aligned}
$$

where $R_A := \sup_{x\in\overline{A}} |x|$.

Step 3 (Global solution). Let $u = \{u(t,x)\,;\, t\geq 0, x\in\mathbb{R}^d\}$ be the random field defined by

$$
u(t,x) := u^{(D_k)}(t,x), \quad \text{if} \quad (t,x)\in[0,T]\times[-k,k]^d, \tag{5.2.24}
$$

where $D_k = (-k,k)^d$ for all $k\in\mathbb{N}$. Here, $u^{(D_k)}(t,x)$ is the solution to (5.2.6) when $D = D_k$. Note that u is well-defined due to (5.2.13). Furthermore, u defined in (5.2.24) is a solution to (5.2.1). To demonstrate this, let us fix $(t,x)\in[0,T]\times\mathbb{R}^d$, and define $V_{T,x} = \{y\in\mathbb{R}^d; |y| < T+|x|\}$. Then, we have:

$$
\begin{aligned}
&w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))\Lambda(ds,dy) \\
&= w(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u^{(V_{T,x})}(s,y))\mathbb{1}_{\mathcal{K}^{V_{T,x}}(s)}(y)\Lambda(ds,dy) \\
&= u^{(V_{T,x})}(t,x) = u(t,x) \quad \text{a.s.}
\end{aligned}
$$

This shows that u is indeed a solution to (5.2.1).

Step 4 (Uniqueness of the solution). We assume that v is a mild solution of (5.2.1) that satisfies

$$
\sup_{t\in[0,T]} \sup_{x\in\overline{D}} \mathbb{E}\left[|v(t,x)|^p \mathbb{1}_{[\![0,\tau_N(D)]\!]}(t)\right] < +\infty, \tag{5.2.25}
$$

for all $N\in\mathbb{N}$ and $D\in\mathfrak{B}$, where $\tau_N(D)$ is given by (3.2.1). We will show that for any D:

$$
u(t,x) = v(t,x) \quad \text{a.s.} \quad \text{for all } t\in[0,T] \text{ and } x\in\overline{D},
$$

where u is the solution to (5.2.1) on the previous step.

First, we define

$$
V_{T,D} := \{y\in\mathbb{R}^d; |y| < T+R_D\}, \quad \text{where } R_D = \sup_{x\in\overline{D}} |x|.
$$

Note that for all $(t,x)\in[0,T]\times\overline{D}$, we have:

$$
u(t,x) = u^{(V_{T,D})}(t,x) \quad \text{a.s.}
$$

Also, recall that $u^{(V_{T,D})}(t,x) = u^{(V_{T,D},N)}(t,x)$ a.s. on the event $\{t \leq \tau_N(V_{T,D})\}$ (see Step 1). Hence, by the triangle inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|(u(t,x)-v(t,x))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)|^p\right] \\
&\leq 2^{p-1}\Bigg(\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|(u^{(V_{T,D},N)}(t,x)-u_n^{(V_{T,D},N)}(t,x))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)|^p\right] \\
&+\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|(u_n^{(V_{T,D},N)}(t,x)-v(t,x))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)|^p\right]\Bigg),
\end{aligned}$$

where $u_n^{(V_{T,D},N)}$ is the sequence in (5.2.11) with respect to $V_{T,D}$. The first term on the right hand side of the inequality above converges to 0 by (5.2.12). It remains to show that:

$$\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|(u_n^{(V_{T,D},N)}(t,x)-v(t,x))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)|^p\right]\to 0 \quad \text{as } n\to+\infty,$$

for a fixed $N \in \mathbb{N}$.

Let $(t,x,s_1,y_1,\ldots,s_n,y_n) \in \mathcal{A}^{(t,x)}$, using the same argument as in (5.2.18), we get:

$$B_{s_{i-1}-s_i}(y_{i-1}) \subset \mathcal{K}^D(s_i) \subset \mathcal{K}^{V_{T,D}}(s_i), \quad \text{for all } i=1,\ldots,n. \tag{5.2.26}$$

Then, by (5.2.26) and (5.2.4), we have:

$$\begin{aligned}
&\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(s_{k-1})\int_0^{s_{k-1}}\int_{\mathbb{R}^d}G_{s_{k-1}-s_k}(y_{k-1}-y_k)\sigma(v(s_k,y_k))\Lambda(ds_k,dy_k) \\
=&\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(s_{k-1}) \\
&\times\int_0^{s_{k-1}}\int_{\mathbb{R}^d}G_{s_{k-1}-s_k}(y_{k-1}-y_k)\sigma(v(s_k,y_k))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(s_k)\overline{\Lambda}_N(ds_k,dy_k),
\end{aligned} \tag{5.2.27}$$

for $k=1,\ldots,n$.

Now, let us define,

$$W_{n,k}^N(s_k,y_k) := (u_{n-k}^{(V_{T,D},N)}(s_k,y_k)-v(s_k,y_k))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(s_k),$$

for $k=1,\ldots,n$. Using (5.2.18), (5.2.27), (5.2.4), and Lemma 3.1.3, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[|W_{n,k-1}^N(s_{k-1},y_{k-1})|^p\right] \\
&\leq C_T\int_0^t\int_{\mathbb{R}^d}g_p(s_{k-1}-s_k,y_{k-1}-y_k)\mathbb{E}\left[|W_{n,k}^N(s_k,y_k)|^p\right]\,dy_k\,ds_k,
\end{aligned} \tag{5.2.28}$$

for $k=1,\ldots,n$. Applying the same reasoning as in Step 2, we get:

$$\mathbb{E}\left[|W_{n,0}^N(t,x)|^p\right] \leq C_T^n\int_{\tilde{T}_n(t)}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^n g_p(s_{i-1}-s_i,y_{i-1}-y_i)\mathbb{E}\left[|W_{n,n}^N(s_n,y_n)|^p\right]d\mathbf{y}d\mathbf{s}. \tag{5.2.29}$$

Therefore, by (5.2.25), (5.2.17), (5.1.18) and (5.1.19),

$$\begin{aligned}&\sup_{t\in[0,T]}\sup_{x\in\overline{D}}\mathbb{E}\left[|(u_n^{(V_{T,D},N)}(t,x)-v(t,x))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)|^p\right]\\&\leq\sup_{s\in[0,T]}\sup_{|y|\leq R_D+T}\mathbb{E}\left[|(u_0^{(V_{T,D},N)}(s,y)-v(s,y))\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(s)|^p\right]\sup_{t\in[0,T]}C_T^nA_n^{(p)}(t)\to 0,\end{aligned}$$

as $n\to+\infty$. Finally, we conclude that

$$u(t,x)\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)=\mathbb{1}_{[\![0,\tau_N(V_{T,D})]\!]}(t)v(t,x)\quad\text{a.s.},\quad\text{for all }(t,x)\in[0,T]\times\overline{D}.$$

For sufficiently large $N\in\mathbb{N}$, we have $\tau_N(V_{T,D})=+\infty$ a.s. Consequently, $u(t,x)=v(t,x)$ a.s., for any $(t,x)\in[0,T]\times\overline{D}$. ∎

The following remark is derived directly from the proof of Theorem 5.2.1.

Remark 5.2.2. Let u be the solution to (5.2.1) given by Theorem 5.2.1. Then

$$u(t,x)=u^{(D)}(t,x)\quad\text{a.s.},\quad\text{for all }(t,x)\in[0,T]\times\overline{D},$$

where $u^{(D)}$ is the solution to (5.2.6) constructed in Step 1 of Theorem 5.2.1.

A natural inquiry at this point is the relationship between the solution obtained in Theorem 5.2.1 for a fixed time interval $[0,T]$, and the solution to (5.0.1) derived in Theorem 5.1.4, under identical initial conditions and same function σ. The following theorem confirms that both solutions are almost surely identical for all $(t,x)\in[0,T]\times\mathbb{R}^d$.

Theorem 5.2.3. Under the same assumptions as in Theorem 5.2.1, if v is a mild solution to (5.2.1) that satisfies

$$\sup_{t\in[0,T]}\sup_{|x|\leq R}\mathbb{E}\left[|v(t,x)|^p\,\mathbb{1}_{[\![0,\tilde{\tau}_N]\!]}(t)\right]<+\infty,\quad\text{for all }N\in\mathbb{N}\text{ and }R>0,$$

where p is the same exponent as in Theorem 5.2.1, and $\{\tilde{\tau}_N\}_{N\geq1}$ is a non-decreasing sequence of stopping times such that $\tilde{\tau}_N\uparrow+\infty$ a.s. for $N\to+\infty$, then

$$v(t,x)=u(t,x)\quad\text{a.s. for all }(t,x)\in[0,T]\times\mathbb{R}^d,$$

where u is the solution to (5.2.1) given by Theorem 5.2.1.

Proof. Let $V_{T,R}:=\{y\in\mathbb{R}^d\,;\,|y|<T+R\}$ for a fixed $R>0$. Note that $u(t,x)=u^{(V_{T,R})}(t,x)$ for any $t\in[0,T]$ and $|x|\leq R$. Hence, by the triangle inequality,

$$\begin{aligned}&\sup_{t\in[0,T]}\sup_{|x|\leq R}\mathbb{E}\left[\left|(u(t,x)-v(t,x))\mathbb{1}_{[\![0,\tilde{\tau}_N\wedge\tau_N(V_{T,R})]\!]}(t)\right|^p\right]\\&\leq 2^{p-1}\Bigg(\sup_{t\in[0,T]}\sup_{|x|\leq R}\mathbb{E}\left[\left|(u^{(V_{T,R},N)}(t,x)-u_n^{(V_{T,R},N)}(t,x))\mathbb{1}_{[\![0,\tilde{\tau}_N\wedge\tau_N(V_{T,R})]\!]}(t)\right|^p\right]\\&\quad+\sup_{t\in[0,T]}\sup_{|x|\leq R}\mathbb{E}\left[\left|(u_n^{(V_{T,R},N)}(t,x)-v(t,x))\mathbb{1}_{[\![0,\tilde{\tau}_N\wedge\tau_N(V_{T,R})]\!]}(t)\right|^p\right]\Bigg).\end{aligned}$$

The first term on the right-hand side of the inequality above converges to 0. For the second term, using the same arguments as in Step 4 of the proof of Theorem 5.2.1, it can be shown that this term also converges to 0. Consequently, $u(t,x) = v(t,x)$ a.s. for all $(t,x) \in [0,T] \times \mathbb{R}^d$. ∎

Remark 5.2.4. Theorem 5.2.3 implies that under Assumptions 5.0.1 and 5.1.1, and with the same initial conditions and function σ, the solution to (5.2.1) given by Theorem 5.2.1 is almost surely identical to the solution to (5.0.1) given by Theorem 5.1.4 on $[0,T] \times \mathbb{R}^d$.

Chapter 6

SPDEs with symmetric α-stable Lévy noise

This chapter is based on the recent preprint [5], which contains some new results regarding the existence of a solution of a general SPDE driven by a symmetric α-stable Lévy noise, and gives a series representation of this solution.

In this chapter we fix $T > 0$ and we consider the equation:

$$\mathcal{L}u(t,x) = u(t,x)\dot{Z}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d \tag{6.0.1}$$

with constant initial condition 1, where $\mathcal{L}$ is a second-order pseudo-differential operator, and Z is a SαS Lévy noise as in Definition 2.1.9.

As examples, we consider the case of the heat operator

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta,$$

and the case of the wave operator

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta \quad \text{with } d \leq 2.$$

These examples are referred in the literature as the *parabolic Anderson model* (PAM), respectively *the hyperbolic Anderson model* (HAM).

Note that by Lemma 2.1.7, Z is an ID independently scattered random measure. Moreover, by Lemma 2.1.12, for any $B \in \mathcal{B}_b$, $Z(B)$ has a $S_\alpha(m(B)^{1/\alpha}, 0, 0)$ distribution with characteristic function

$$\mathbb{E}[e^{iuZ(B)}] = e^{-m(B)|u|^\alpha} \quad \text{for any } u \in \mathbb{R},$$

where m is given by (2.1.10), i.e. Z is a SαS random measure on $[0,T] \times \mathbb{R}^d$.

Recall that $S_\alpha(\mu, \sigma, b)$ denotes the α-stable distribution given by Definition 2.1.10, with stability parameter $\alpha \in (0,2)$, location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$ and skewness parameter $\beta \in [-1,1]$.

By Remark 2.1.13, Z has the following decomposition:

$$Z(B) = \int_{B\times\mathbb{R}_0} zN(dt,dx,dz), \quad \text{if} \quad \alpha < 1,$$

$$Z(B) = \int_{B\times\mathbb{R}_0} z\widehat{N}(dt,dx,dz), \quad \text{if} \quad \alpha > 1,$$

and

$$Z(B) \stackrel{\mathbb{P}}{=} \lim_{\varepsilon\to 0} \int_{B\times\{|z|\geq\varepsilon\}} zN(dt,dx,dz), \quad \text{if } \alpha = 1, \tag{6.0.2}$$

where N is a Poisson random measure N on $[0,T]\times\mathbb{R}^d\times\mathbb{R}_0$ of intensity $\mu(dt,dx,dz) = dtdx\nu_\alpha(dz)$, $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, $\widehat{N}(F) = N(F) - \mu(F)$ is the compensated version of N, and ν_α is given by (2.1.8). Note that (6.0.2) is a particular case of (2.1.13).

On the other hand, note also that Z has the *LePage series representation*: for any $B \in \mathcal{B}_b$,

$$Z(B) = \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\psi(T_i,X_i)} 1_B(T_i,X_i) \quad \text{a.s.}, \tag{6.0.3}$$

provided the points of N are chosen as

$$N = \sum_{i\geq 1} \delta_{(\varepsilon_i\Gamma_i^{-1/\alpha}\psi^{-1}(T_i,X_i),T_i,X_i)},$$

where $(\varepsilon_i)_{i\geq 1}$ are i.i.d. Rademacher random variables, i.e.

$$\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2},$$

$\{\Gamma_i = \sum_{j=1}^i E_j, i \geq 1\}$ are the arrival times of a Poisson process on $\mathbb{R}_+$ of intensity 1 (with $(E_i)_{i\geq 1}$ i.i.d. exponential random variables of mean 1), and $\{(T_i,X_i)\}_{i\geq 1}$ are i.i.d. random variables on $[0,T]\times\mathbb{R}^d$ with law $m_\psi(dt,dx) = \psi^\alpha(t,x)dtdx$, where $\psi : [0,T]\times\mathbb{R}^d \to (0,\infty)$ is an arbitrary measurable function satisfying

$$\int_0^T \int_{\mathbb{R}^d} \psi^\alpha(t,x)dxdt = 1. \tag{6.0.4}$$

The sequences $(\varepsilon_i)_{i\geq 1}$, $(E_i)_{i\geq 1}$ and $\{(T_i,X_i)\}_{i\geq 1}$ are independent. Note that the series on the right-hand side of (6.0.3) converges a.s. since $W_i = \frac{1}{\psi(T_i,X_i)} 1_B(T_i,X_i)$ are i.i.d. random variables with $\mathbb{E}\left[W_i^\alpha\right] < \infty$ (see Theorem 1.4.2 of [53]).

For simplicity, throughout this work we will use a weight function ψ of the form:

$$\psi(t,x) = T^{-1/\alpha}\phi(x) \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^d, \tag{6.0.5}$$

where $\phi : \mathbb{R}^d \to (0,\infty)$ is a measurable function such that $\int_{\mathbb{R}^d} \phi^\alpha(x)dx = 1$. Therefore, $(T_i)_{i\geq 1}$ are i.i.d. random variables with a uniform distribution on $[0,T]$, $(X_i)_{i\geq 1}$ are i.i.d.

random vectors in $\mathbb{R}^d$ with density $\phi^\alpha(x)$, and $(T_i)_{i\geq 1}$ and $(X_i)_{i\geq 1}$ are independent. The LePage series representation (6.0.3) becomes: for any $B \in \mathcal{B}_b$,

$$Z(B) = T^{1/\alpha} \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\phi(X_i)} 1_B(T_i, X_i) \quad \text{a.s.}$$

Recall that a mild solution of (6.0.1) is a predictable random field $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ which satisfies the integral equation:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) u(s,y) Z(ds,dy), \tag{6.0.6}$$

where G is the fundamental solution of the deterministic equation $\mathcal{L}u = \delta_0$. By convention, we let $G_t(x) = 0$ if $t \leq 0$.

To give a meaning to the right-hand side of equation (6.0.6), we need a stochastic integral which can be defined for *random integrands*. This case is not discussed in [53] or [44]. It turns out that the theory of stochastic integration with respect to α-stable random measures developed by the first author in [2] (as a multi-dimensional extension of the theory of [27]) is too restrictive, and the best approach is to use the theory of stochastic integration with respect to L^0-random measures, which was introduced in [13] and was developed further in [34, 35, 12, 19]. This theory was used in the literature for other studies of SPDEs with heavy-tailed noise, such as [16, 17, 18]. To implement this method, we need embed the SαS random measure Z into a more general process Λ (called a *Lévy basis*), indexed by subsets of $\Omega \times [0,T] \times \mathbb{R}^d$. This will unavoidably increase the technical level of the work, but the gain will be substantial. At the same time, we need to preserve the series representation (6.0.3), to define the multiple integrals with respect to Z. This delicate technical issue is addressed in Section 6.1.2 below, where we give an explicit construction of a Lévy basis Λ which allows us to achieve both goals. This construction uses as the source of randomness of the three sequences $(\varepsilon_i)_{i\geq 1}$, $(\Gamma_i)_{i\geq 1}$ and $\{(T_i, X_i)\}_{i\geq 1}$ to define a Poisson random measure N, which in turn is used to define Λ, via its canonical decomposition. Specifically, $u(t,x)$ is represented as a series of *random multilinear forms* that depend only on $(\epsilon_i)_{i\geq 1}$, $(\Gamma_i)_{i\geq 1}$, $\{(T_i, X_i)\}_{i\geq 1}$, and G.

Note that the first integral appearing in series (6.0.8) is $\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) Z(ds,dy)$, and this integral is well-defined if and only if

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^\alpha(x-y) dy ds < \infty. \tag{6.0.7}$$

In the case of the heat equation, (6.0.7) is equivalent to $\alpha < 1 + \frac{2}{d}$, whereas in the case of the wave equation in dimension $d \leq 2$, (6.0.7) holds for all $\alpha \in (0,2)$. In light of this, we introduce the following assumption:

Assumption 6.0.1. The fundamental solution $(t,x) \mapsto G_t(x)$ of the operator $\mathcal{L}$ is a jointly measurable function on $[0,T] \times \mathbb{R}^d$, which satisfies the following condition:

$$\int_0^T \int_{\mathbb{R}^d} G_t^\alpha(x) dx dt < \infty.$$

Note that Assumption 6.0.1 does not hold for the wave operator in dimension $d \geq 3$, since the fundamental solution is a distribution.

In this chapter, we introduce a different method for solving (6.0.1), which is more robust than the method developed in the previous chapters, does not require any truncation, and can be applied to a large class of SPDEs, being inspired by the methodology used in the Gaussian case. We explain this method below.

Writing

$$u(s,y) = 1 + \int_0^s \int_{\mathbb{R}^d} G_{s-r}(y-z)u(r,z)Z(dr,dz),$$

and inserting this into (6.0.6), we obtain:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)Z(ds,dy) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) \left(\int_0^s \int_{\mathbb{R}^d} G_{s-r}(y-z)u(r,z)Z(dr,dz) \right) Z(ds,dy).$$

Intuitively, it should be possible to iterate this procedure, and therefore obtain that the solution has the "stable chaos expansion":

$$u(t,x) = 1 + \sum_{n \geq 1} \int_{([0,T] \times \mathbb{R}^d)^n} f_n(t_1, x_1, \ldots, t_n, x_n, t, x) Z(dt_1, dx_1) \ldots Z(dt_n, dx_n), \tag{6.0.8}$$

where the kernel $f_n(\cdot, t, x)$ is given by:

$$f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = G_{t-t_n}(x-x_n) \ldots G_{t_2-t_1}(x_2-x_1) 1_{\{0<t_1<\ldots<t_n<t\}}, \tag{6.0.9}$$

and the integral is interpreted as a "multiple stable integral", which was introduced and studied in [51, 52], using series representations (originating from the LePage series representation (6.0.3)). The construction and basic properties of the multiple stable integrals are recalled in Section 6.1.1.

We expect that if the series (6.0.8) is well-defined, the corresponding partial sum sequence will coincide with the sequence $(u_n)_{n\geq 0}$ of Picard's iterations, defined by: $u_0(t,x) = 1$,

$$u_{n+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u_n(s,y) Z(ds,dy), \quad n \geq 0. \tag{6.0.10}$$

This procedure is well-established in the case of equations with Gaussian noise or finite variance Lévy noise, using tools from Malliavin calculus (see e.g. [30, 8]). The goal of this chapter is to show that this method can be extended to the case of the SαS noise, using the LePage series representations of the noise and of the associated multiple stable integrals.

Guided again by the intuition gained from the Gaussian framework, we expect this solution to be unique. But uniqueness turns out to be a delicate problem in our framework. The fact that our noise may not have any moments forces us to work in the space L^0 of random variables equipped with the pseudo-norm $\|\cdot\|_0$. Since the topology induced by $\|\cdot\|_{L^0}$ is not locally convex, we do not have access to the same techniques as in a Hilbert space, such as L^2, nor do we have the Banach space structure of L^p for $p > 0$. As a consequence, proving the uniqueness of the solution remains an open problem.

We are now ready to state the main result of this chapter. But first, we need to introduce some notation and assumptions. For any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $p > 0$, we define:

$$K_n^{(p)}(t,x) := \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} f_n^p(t_1, x_1, \dots, t_n, x_n, t, x) \prod_{k=1}^n \phi^{\alpha-p}(x_k) d\boldsymbol{x} d\boldsymbol{t}, \tag{6.0.11}$$

with $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{t} = (t_1, \dots, t_n)$ and $T_n(t) = \{\boldsymbol{t} \in [0,t]^n; t_1 < \ldots < t_n\}$.

Assumption 6.0.2. There exists $p \in (\alpha, 2]$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\sum_{n\geq 1} \left(T^{(\frac{p}{\alpha}-1)n} K_n^{(p)}(t,x) \right)^{1/2} \left(\sum_{j\geq 1} \Gamma_j^{-p/\alpha} \right)^{n/2} < \infty \text{ a.s.} \tag{6.0.12}$$

Assumption 6.0.3. There exists $p \in (\alpha, 2]$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\sum_{n\geq 1} \left(T^{(\frac{p}{\alpha}-1)n} \int_0^t \int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y) K_n^{(p)}(s,y)\, dy\, ds \right)^{\frac{\alpha\wedge 1}{p}} \left(\sum_{j\geq 1} \Gamma_j^{-p/\alpha} \right)^{\frac{n(\alpha\wedge 1)}{p}} < \infty \quad \text{a.s.} \tag{6.0.13}$$

The following theorem is the main result of this chapter.

Theorem 6.0.4. Suppose that the fundamental solution G satisfies Assumption 6.0.1.

(a) If Assumption 6.0.2 holds, then the series on the right-hand side of (6.0.8) converges absolutely almost surely, for any $(t,x) \in [0,T] \times \mathbb{R}^d$.

(b) If Assumptions 6.0.2 and 6.0.3 hold (with possibly different values p), then the process $\{u(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ given by (6.0.8) is a solution of equation (6.0.1). Moreover, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y) |u(s,y)|^{\alpha}\, ds\, dy < \infty \quad \text{a.s.} \tag{6.0.14}$$

and $u(t,x)$ has representation:

$$u(t,x) = 1 + \sum_{n\geq 1} T^{n/\alpha} n! \sum_{j_1 < \ldots < j_n} \prod_{k=1}^n \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \dots, T_{j_n}, X_{j_n}, t, x) \quad \text{a.s.,} \tag{6.0.15}$$

where $\widetilde{f}_n(\cdot, t, x)$ is the symmetrization of $f_n(\cdot, t, x)$.

To verify Assumptions 6.0.2 and 6.0.3, we introduce the following condition on ϕ:

Assumption 6.0.5. There exist some $c_0 > 0$ and $\delta > 0$ such that

$$\frac{1}{\phi(x)} \leq c_0(1 + |x|^{\delta}) \quad \text{for all } x \in \mathbb{R}^d. \tag{6.0.16}$$

Remark 6.0.6. An example of a function $\phi > 0$ which satisfies Assumption 6.0.5 is

$$\phi(x) = c\big(1_{\{|x|\leq 1\}} + |x|^{-\delta} 1_{\{|x|>1\}}\big), \tag{6.0.17}$$

with $\delta > d/\alpha$, and value $c = c(\alpha, \delta) > 0$ chosen such that $\int_{\mathbb{R}^d} \phi^\alpha(x)dx = 1$.

As an application of Theorem 6.0.4, we obtain the following result.

Theorem 6.0.7. Suppose that ϕ satisfies Assumption 6.0.5. If either
(i) $\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$ is the heat operator and $\alpha < 1 + \frac{2}{d}$, or
(ii) $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$ is the wave operator and $d \leq 2$,
then Assumptions 6.0.2 and 6.0.3 are satisfied, and consequently, the conclusion of Theorem 6.0.4.(b) holds.

This chapter is organized as follows. In Section 6.1, we introduce the necessary background material, related to multiple stable integrals and Lévy bases. In Section 6.2 we give the proof of Theorem 6.0.4.(a), i.e. we show that the series (6.0.8) converges absolutely almost surely. In Section 6.3, we show that the partial sum sequence $(u_n)_{n\geq 0}$ associate with series (6.0.8) satisfies the recurrence relation (6.0.10). In Section 6.4, we show that the process $u(t,x)$ defined by (6.0.8) satisfies the integral equation (6.0.6). In Section 6.5, we give the proof of Theorem 6.0.7, i.e. we show that Assumptions 6.0.2 and 6.0.3 are satisfied in the case of the heat and wave equations.

The notation $\phi^{-1}(x)$ is used for $1/\phi(x)$ (not for the inverse function of ϕ).

6.1 Background

In this section, we include the necessary background material, which is substantial, since it covers two topics: multiple stable integrals (Section 6.1.1) and integration with respect to Lévy bases (Section 2.3). In Section 6.1.2, we give the construction of the noise.

6.1.1 Multiple stable integrals

In this section, we review the construction of the multiple stable integral with respect to the SαS random measure Z, following [51, 52]. The results contained in these references are stated for SαS random measures on $\mathbb{R}$, but they can be easily extended to more general spaces, such as $[0,T] \times \mathbb{R}^d$.

Before we begin, we recall that the stochastic integral $I_1(f) = \int f dZ$ of a deterministic function f with respect to Z is constructed using the classical procedure, starting with simple functions, followed by approximation. This is explained in Chapter 3 of [53]. It turns out that the integral $I_1(f)$ is well-defined for any function $f \in L^\alpha([0,T] \times \mathbb{R}^d)$, and the process

$\{I_1(f); f \in L^\alpha([0,T]\times\mathbb{R}^d)\}$ has α-stable finite dimensional distributions. In particular, $I_1(f)$ has a $S_\alpha(\sigma_f, 0, 0)$-distribution, where

$$\sigma_f = \left(\int_{[0,T]\times\mathbb{R}^d} |f(t,x)|^\alpha m(dt,dx)\right)^{1/\alpha}.$$

This property is in fact valid even in the non-symmetric case, when $\nu_\alpha(z,\infty) = pz^{-\alpha}$ and $\nu(-\infty,-z) = q(-z)^{-\alpha}$ for all $z > 0$, for some $p, q \geq 0$. From this perspective, α-stable random measures are similar to isonormal Gaussian processes from Malliavin calculus, in which case the integrals $I_1(f)$ live in the first Wiener chaos. This analogy is further enhanced by the fact that, *in the symmetric case* (when ν_α is given by (2.1.8)), it is possible to define a multiple integral with respect to Z, and this is what we will explain below.

Remark 6.1.1. If $\alpha \in (1,2)$, then $\int_{|z|>1} |z|\nu_\alpha(dz) < \infty$, then by Remark 2.1.13.(i), $Z(B) = \int_{B\times\mathbb{R}_0} z\widehat{N}(dt,dx,dz)$. In view of this, one may try to define the multiple integral with respect to Z of a function $f : ([0,T]\times\mathbb{R}^d)^n \to \mathbb{R}$ by:

$$I_n(f) = \int_{([0,T]\times\mathbb{R}^d\times\mathbb{R}_0)^n} f(t_1,x_1,\ldots,t_n,x_n) z_1\ldots z_n \widehat{N}(dt_1,dx_1,dz_1)\ldots\widehat{N}(dt_n,dx_n,dz_n).$$

This procedure works for finite-variance Lévy noise (see [8]), but fails for the infinite variance noise Z, since the multiple integral with respect to $\widehat{N}$ is defined only for functions which are square-integrable with respect to the measure μ^n (see Section 5.4 of [1]).

A different idea, which has been exploited successfully in [51, 52], is to start with a SαS random measure Z *which has the LePage representation* (6.0.3), and extend this representation to multi-index series. We explain this procedure below. To simplify the writing, we let $E = [0,T]\times\mathbb{R}^d$, $\mathcal{B}$ be the class of Borel sets of E, and ℓ be the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{B}_0 = \{A \in \mathcal{B}; \ell(A) < \infty\}$.

The first step is the construction of the product stable measure $Z^{(n)}$. For this, we recall that a *symmetric rectangle* of E^n is a set of the form

$$B = \bigcup_{\pi\in\Sigma_n} B_{\pi(1)} \times \ldots \times B_{\pi(n)}$$

for some disjoint sets $B_1,\ldots,B_n \in \mathcal{B}_0$, where Σ_n is the set of all permutations of $\{1,\ldots,n\}$. Defining

$$Z^{(n)}(B) := n! Z(B_1)\ldots Z(B_n),$$

and using the Le Page representation (6.0.3) for each B_i, we obtain that:

$$Z^{(n)}(B) = n! \sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \varepsilon_{j_k}\Gamma_{j_k}^{-1/\alpha}\psi^{-1}(T_{j_k},X_{j_k}) 1_B(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n}). \qquad (6.1.1)$$

The following theorem shows that this representation can be extended to more general sets. First, we need to introduce some notation. Let $\mathcal{B}_n^{(s)}$ be the σ-algebra generated by the symmetric rectangles and $\mathcal{B}_{n,0}^{(s)} = \{B \in \mathcal{B}_n^{(s)}; \ell^{(n)}(B) < \infty\}$, where $\ell^{(n)}$ is the Lebesgue measure on E^n.

Theorem 6.1.2. For any set $B \in \mathcal{B}_{n,0}^{(s)}$, the series

$$S_n(B) = n! \sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \psi^{-1}(T_{j_k}, X_{j_k}) 1_B(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n})$$

converges a.s. if and only if

$$\sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \Gamma_{j_k}^{-2/\alpha} \psi^{-2}(T_{j_k}, X_{j_k}) 1_B(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}) < \infty \quad \text{a.s.} \tag{6.1.2}$$

In this case, we let $Z^{(n)}(B) := S_n(B)$ a.s.

We proceed now with the construction of the multiple stable integral. We recall some terminology.

Definition 6.1.3. Let $f : E^n \to \mathbb{R}$ be an arbitrary function. We say that:
a) f is *symmetric* if for any $x_1, \ldots, x_n \in E$ and for any $\pi \in \Sigma_n$,

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)});$$

b) f *vanishes on the diagonals* if $f(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

Lemma 6.1.4. A symmetric $\mathcal{B}^n$-measurable function $f : E^n \to \mathbb{R}$ which vanishes on the diagonals is $\mathcal{B}_n^{(s)}$-measurable.

We say that $f : E^n \to \mathbb{R}$ is a *simple function* if it is the form $f = \sum_{i=1}^{k} a_i 1_{B_i}$, for some $a_1, \ldots, a_k \in \mathbb{R}$ and disjoint sets $B_1, \ldots, B_k \in \mathcal{B}_{n,0}^{(s)}$. If $Z^{(n)}(B_i)$ is well-defined for any $i = 1, \ldots, k$, we say that f is *n-times integrable with respect to Z*, and we let

$$I_n(f) := \sum_{i=1}^{k} a_i Z^{(n)}(B_i).$$

6.1.2 Construction of the noise

In this section, we give the construction of the noise.

Let $(\varepsilon_i)_{i\geq 1}$, $(\Gamma_i)_{i\geq 1}$ and $\{(T_i, X_i)\}_{i\geq 1}$ be the sequences mentioned in the introduction, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that $\left(\varepsilon_i \Gamma_i^{-1/\alpha}\right)_{i\geq 1}$ are the points of a PRM on $\mathbb{R}_0$ of intensity ν_α. Therefore, using a procedure called "augmentation" (see Proposition 3.8 of [45]), the process

$$J_\psi = \sum_{i\geq 1} \delta_{(T_i, X_i, \varepsilon_i \Gamma_i^{-1/\alpha})} \tag{6.1.3}$$

is a PRM on $[0,T]\times\mathbb{R}^d\times\mathbb{R}_0$ of intensity $m_\psi\times\nu_\alpha$. Consider the transformation $T_\psi(t,x,z)=(t,x,\frac{z}{\psi(t,x)})$. By Proposition 3.7 of [45], the process

$$N_\psi = J_\psi\circ T_\psi^{-1}=\sum_{i\geq 1}\delta_{T_\psi(T_i,X_i,\varepsilon_i\Gamma_i^{-1/\alpha})}=\sum_{i\geq 1}\delta_{(T_i,X_i,\varepsilon_i\Gamma_i^{-1/\alpha}\psi^{-1}(T_i,X_i))}$$

is also PRM on $[0,T]\times\mathbb{R}^d\times\mathbb{R}_0$ of intensity

$$(m_\psi\times\nu_\alpha)\circ T_\psi^{-1}=\text{Leb}\times\text{Leb}\times\nu_\alpha.$$

Now, using the PRM N_ψ, we define the homogeneous Lévy basis Λ on $[0,T]\times\mathbb{R}^d$:

$$\Lambda(A)=\int_0^T\int_{\mathbb{R}^d}\int_{\{|z|\leq 1\}}1_A(t,x)z\widehat{N_\psi}(dt,dx,dz)+\int_0^T\int_{\mathbb{R}^d}\int_{\{|z|>1\}}1_A(t,x)zN_\psi(dt,dx,dz), \tag{6.1.4}$$

where $\widehat{N_\psi}$ is the compensated version of N_ψ.

We let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration associated with N_ψ, i.e.

$$\mathcal{F}_t=\sigma\left\{N_\psi([0,s]\times B\times F);s\in[0,t],B\in\mathcal{B}_b,F\in\mathcal{B}_b(\mathbb{R}_0)\right\}. \tag{6.1.5}$$

Remark 6.1.5. If Λ is a SαS Lévy basis with canonical decomposition (2.3.3), then the process Z defined by (2.3.2) is a SαS random measure with representation (2.1.11) if $\alpha\leq 1$ and (2.1.12) if $\alpha>1$.

For any $B\in\mathcal{B}_b$, we define $Z(B)$ by (2.3.2). Then Λ is a SαS Lévy basis (see Definition 2.3.2), and Z is a SαS random measure (see Remark 6.1.5). The next result shows that Z has the series representation (6.0.3).

Lemma 6.1.6. If Λ is the Lévy basis given by (6.1.4), then the process Z given by (2.3.2) has the LePage series representation (6.0.3).

Proof. Let $\varepsilon\in(0,1)$ be arbitrary. By the symmetry of ν_α, $\int_{B\times\{\varepsilon<|z|\leq 1\}}zdtdx\nu_\alpha(dz)=0$ and hence,

$$\int_{B\times\{\varepsilon<|z|\leq 1\}}z\widehat{N_\psi}(dt,dx,dz)=\int_{B\times\{\varepsilon<|z|\leq 1\}}zN_\psi(dt,dx,dz),$$

and

$$Z(B)=\int_{B\times\{0<|z|<\varepsilon\}}z\widehat{N_\psi}(dt,dx,dz)+\int_{B\times\{|z|>\varepsilon\}}zN_\psi(dt,dx,dz)=:Z_\varepsilon(B)+S_\varepsilon(B).$$

Note that $Z_\varepsilon(B)\xrightarrow{L^2}0$ as $\varepsilon\to 0$, since $\mathbb{E}\left[|Z_\varepsilon(B)|^2\right]=\int_{B\times\{\varepsilon<|z|\leq 1\}}z^2dtdx\nu_\alpha\to 0$ as $\varepsilon\to 0$, by the dominated convergence theorem. Next, we prove that:

$$S_\varepsilon(B)\xrightarrow{P}S(B)\quad\text{as }\varepsilon\to 0, \tag{6.1.6}$$

where $S(B)$ is the sum on the right-hand side of (6.0.3). Note that

$$S(B) - S_\varepsilon(B) = \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \psi^{-1}(T_i, X_i) 1_{\{\Gamma_i^{-1/\alpha} \leq \varepsilon \psi(T_i, X_i)\}} 1_B(T_i, X_i) =: \overline{S}_\varepsilon(B).$$

By the Cauchy-Schwarz inequality,

$$\|\overline{S}_\varepsilon(B)\|_0 = \mathbb{E}\big[\min\big(1, |\overline{S}_\varepsilon(B)|\big)\big] \leq \Big\{\mathbb{E}\big[\min\big(1, |\overline{S}_\varepsilon(B)|^2\big)\big]\Big\}^{1/2}.$$

Using the inequality $\mathbb{E}[\min(1,|X|)] \leq \min(1, \mathbb{E}|X|)$ and the orthogonality of $(\varepsilon_i)_{i\geq 1}$,

$$\begin{aligned}
&\mathbb{E}\Big[\min\big(1, |\overline{S}_\varepsilon(B)|^2\big) \mid (\Gamma_i)_i, (T_i)_i, (X_i)_i\Big] \\
&\leq \min\Big(1, \sum_{i\geq 1} \Gamma_i^{-2/\alpha} \psi^{-2}(T_i, X_i) 1_{\{\Gamma_i^{-1/\alpha} \leq \varepsilon \psi(T_i, X_i)\}} 1_B(T_i, X_i)\Big).
\end{aligned}$$

Taking expectation in the above inequality, we obtain:

$$\mathbb{E}\Big[\min\big(1, |\overline{S}_\varepsilon(B)|^2\big)\Big] \leq \mathbb{E}\Big[\min\Big(1, \sum_{i\geq 1} \Gamma_i^{-2/\alpha} \psi^{-2}(T_i, X_i) 1_{\{\Gamma_i^{-1/\alpha} \leq \varepsilon \psi(T_i, X_i)\}} 1_B(T_i, X_i)\Big)\Big].$$

The term on the right-hand side above converges to 0 as $\varepsilon \to 0$, by an application of the dominated convergence theorem, which is justified by the fact that

$$X := \sum_{i\geq 1} \Gamma_i^{-2/\alpha} \psi^{-2}(T_i, X_i) 1_{\{\Gamma_i^{-1/\alpha} \leq \psi(T_i, X_i)\}} 1_B(T_i, X_i) = \int_{B\times\{|z|\leq 1\}} z^2 N_\psi(dt, dx, dz) < \infty \quad \text{a.s.}$$

since $\mathbb{E}[X] = \int_{B\times\{|z|\leq 1\}} z^2 dt dx \nu_\alpha(dz) < \infty$. ∎

The following result gives an alternative representation for Λ. Its proof follows essentially using a change of measure, which allows us to pass not only from N_ψ to J_ψ, but also from $\widehat{N_\psi}$ to $\widehat{J_\psi}$ (although these are not measures), using the symmetry of ν_α.

Proposition 6.1.7. Let Λ be the Lévy basis given by (6.1.4). For any $A \in \widetilde{\mathcal{P}}_b$, we have:

$$\begin{aligned}
\Lambda(A) = &\int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\leq \psi(t,x)\}} 1_A(t,x) \frac{z}{\psi(t,x)} \widehat{J_\psi}(dt, dx, dz) + \\
&\int_0^T \int_{\mathbb{R}^d} \int_{\{|z|> \psi(t,x)\}} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt, dx, dz), \qquad (6.1.7)
\end{aligned}$$

where $\widehat{J_\psi}$ is the compensated version of J_ψ. Moreover, if $\alpha \in (0,1)$, then for any $A \in \widetilde{\mathcal{P}}_b$,

$$\begin{aligned}
\Lambda(A) &= \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}_0} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt, dx, dz) \qquad (6.1.8) \\
&= \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\psi(T_i, X_i)} 1_A(T_i, X_i). \qquad (6.1.9)
\end{aligned}$$

Proof. We first prove (6.1.7). Let $\varepsilon \in (0,1)$ be arbitrary. We write

$$\Lambda(A) = \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\leq\varepsilon\}} 1_A(t,x) z \widehat{N_\psi}(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{\varepsilon<|z|\leq 1\}} 1_A(t,x) z \widehat{N_\psi}(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>1\}} 1_A(t,x) z N_\psi(dt,dx,dz) =: T_1^{(\varepsilon)} + T_2^{(\varepsilon)} + T_3,$$

and then we let $\varepsilon \to 0$. Since $N_\psi = J_\psi \circ T_\psi^{-1}$, using a change of measure, we have

$$T_3 = \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>\psi(t,x)\}} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt,dx,dz).$$

By Itô's isometry, $T_1^{(\varepsilon)} \stackrel{L^2}{\to} 0$ as $\varepsilon \to 0$. For the second term,

$$\begin{aligned} T_2^{(\varepsilon)} &= \int_0^T \int_{\mathbb{R}^d} \int_{\{\varepsilon<|z|\leq 1\}} 1_A(t,x) z N_\psi(dt,dx,dz) \quad \text{(by the symmetry of } \nu_\alpha) \\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\{\varepsilon<\frac{|z|}{\psi(t,x)}\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt,dx,dz) \quad (\text{since } N_\psi = J_\psi \circ T_\psi^{-1}) \\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\{\varepsilon<\frac{|z|}{\psi(t,x)}\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} \widehat{J_\psi}(dt,dx,dz) \quad \text{(by the symmetry of } \nu_\alpha) \\ &\xrightarrow{L^2} \int_0^T \int_{\mathbb{R}^d} \int_{\{\frac{|z|}{\psi(t,x)}\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} \widehat{J_\psi}(dt,dx,dz) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

The last convergence holds by Itô's isometry, using the fact that

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}^d} \int_{\{\frac{|z|}{\psi(t,x)}\leq\varepsilon\}} 1_A(t,x) \frac{z^2}{\psi^2(t,x)} \widehat{J_\psi}(dt,dx,dz)\right|^2\right] \\ &\quad = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A(t,x) \psi^{\alpha-2}(t,x) \left(\int_{\{\frac{|z|}{\psi(t,x)}\leq\varepsilon\}} z^2 \nu_\alpha(dz)\right) dtdx\right] \\ &\quad = \frac{\varepsilon^{2-\alpha}}{2-\alpha} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A(t,x) dtdx\right]. \end{aligned}$$

This proves (6.1.7). Relation (6.1.8) follows directly from (6.1.7), since

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \int_{\{\frac{|z|}{\psi(t,x)}\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} \widehat{J_\psi}(dt,dx,dz) \\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\{\frac{|z|}{\psi(t,x)}\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt,dx,dz). \end{aligned}$$

To see this, note that by Fubini's theorem and the symmetry of ν_α,

$$\int_0^T \int_{\mathbb{R}^d} \int_{\{\frac{|z|}{\psi(t,x)} \leq 1\}} 1_A(t,x) z \psi^{\alpha-1}(t,x) dt dx \nu_\alpha(dz) = 0.$$

To justify the application of Fubini's theorem, we note that since $\alpha < 1$,

$$\int_0^T \int_{\mathbb{R}^d} 1_A(t,x)\psi^{\alpha-1}(t,x) \left(\int_{\{|z| \leq \psi(t,x)\}} |z| \nu_\alpha(dz) \right) dx dt = \frac{1}{1-\alpha} \int_0^T \int_{\mathbb{R}^d} 1_A(t,x) dt dx < \infty.$$

Finally, (6.1.9) follows directly from (6.1.8), using definition (6.1.3) of J_ψ. ∎

Remark 6.1.8. When $\alpha \in (0,1)$, relation (6.1.9) extends the LePage series representation (6.0.3) to sets $A \in \widetilde{\mathcal{P}}_b$. This relation will play an important role in proving the recurrence relation (6.0.10); see the proof of Theorem 6.3.3 below. We do not know if this representation holds for sets $A \in \widetilde{\mathcal{P}}_b$, when $\alpha \in [1,2)$.

For integration purposes, we will use the following representation of Λ.

Proposition 6.1.9. Let Λ be the Lévy basis given by (6.1.4). For any $A \in \widetilde{\mathcal{P}}_b$, we have:

$$\begin{aligned}\Lambda(A) = &\int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\leq 1\}} 1_A(t,x) \frac{z}{\psi(t,x)} \widehat{J_\psi}(dt,dx,dz) + \\ &\int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>1\}} 1_A(t,x) \frac{z}{\psi(t,x)} J_\psi(dt,dx,dz). \end{aligned} \tag{6.1.10}$$

Proof. Consider the following (non-homogeneous) Lévy basis:

$$L_\psi(A) = \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\leq 1\}} 1_A(t,x) z \widehat{J_\psi}(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>1\}} 1_A(t,x) z J_\psi(dt,dx,dz). \tag{6.1.11}$$

This is an orthogonal L^0-random measure which has the canonical decomposition (3.2) of [19], with respect to the truncation function $\tau(z) = z1_{\{|z|\leq 1\}}$, with $b=0$, $C=0$, $\mu = J_\psi$, $\nu(dt,dx,dz) = \psi^\alpha(t,x) dy dx \nu_\alpha(dz)$, i.e. $A(dt,dx) = dt dx$ and $K(t,x,dz) = \psi^\alpha(t,x)\nu_\alpha(dz)$. For this L^0-random measure, we have:

$$U(t,x,y) = b(t,x)y + \int_{\mathbb{R}_0} \big(\tau(yz) - y\tau(z)\big) K(t,x,dz) = 0,$$

$$V_0(t,x,y) = \int_{\mathbb{R}_0} \big(|yz|^2 \wedge 1\big) K(t,x,dz) = \frac{2}{2-\alpha} |y|^\alpha \psi^\alpha(t,x).$$

By Theorem 4.1 of [19], if H is a predictable process, then

$$H \in L^0(L_\psi) \text{ if and only if } \int_0^T \int_{\mathbb{R}^d} \psi^\alpha(t,x) |H(t,x)|^\alpha dx dt < \infty \text{ a.s.} \tag{6.1.12}$$

In particular, $1_A \psi^{-1} \in L^0(L_\psi)$, for any $A \in \widetilde{P}_b$.

By Theorem 3.5 of [19], we can define a null-spatial L^0-random measure $H \cdot L_\psi$ by $(H \cdot L_\psi)(A) = \int 1_A(t)H(t,x)L_\psi(dt,dx)$ for suitable sets $A \in \mathcal{P}$, and this has canonical decomposition:

$$(H \cdot L_\psi)(A) = \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}_0} 1_A(t)\tau(H(t,x)z)\widehat{J}_\psi(dt,dx,dz) + \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}_0} 1_A(t)\big(H(t,x)z - \tau(H(t,x)z)\big)J_\psi(dt,dx,dz).$$

On the other hand, using the canonical decomposition (6.1.11) of L_ψ, we have:

$$(H \cdot L_\psi)(A) = \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}_0} 1_A(t)H(t,x)\tau(z)\widehat{J}_\psi(dt,dx,dz) + \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}_0} 1_A(t)H(t,x)\big(z - \tau(z)\big)J_\psi(dt,dx,dz).$$

Combining these two expressions and writing them for $A = \overline{\Omega} := \Omega \times [0,T]$, we obtain:

$$\begin{aligned}(H \cdot L_\psi)(\overline{\Omega}) &= \int_0^T \int_{\mathbb{R}^d} \int_{\{|H(t,x)z|\le 1\}} H(t,x)z\widehat{J}_\psi(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|H(t,x)z|>1\}} H(t,x)zJ_\psi(dt,dx,dz)\\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\le 1\}} H(t,x)z\widehat{J}_\psi(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>1\}} H(t,x)zJ_\psi(dt,dx,dz).\end{aligned}$$

We apply this for $H = 1_A\psi^{-1}$ with $A \in \widetilde{P}_b$. The conclusion follows by (6.1.7). ∎

Finally, we give the following representation for the integral with respect to Λ.

Proposition 6.1.10. For any $H \in L^0(\Lambda)$, we have:

$$\int_0^T \int_{\mathbb{R}^d} H(t,x)\Lambda(dt,dx) = \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\le 1\}} H(t,x)\frac{z}{\psi(t,x)}\widehat{J}_\psi(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>1\}} H(t,x)\frac{z}{\psi(t,x)}J_\psi(dt,dx,dz). \tag{6.1.13}$$

Proof. By Proposition 6.1.9, relation (6.1.13) holds for $H = 1_A$ with $A \in \widetilde{P}_b$, i.e.

$$\Lambda(A) = \int_0^T \int_{\mathbb{R}^d} 1_A(t,x)\frac{1}{\psi(t,x)}L_\psi(dt,dx), \tag{6.1.14}$$

where L_ψ is the Lévy basis given by (6.1.11). By Corollary 2.3.9, $H \in L^0(\Lambda)$ is equivalent to $\int |H(t,x)|^\alpha dtdx < \infty$ a.s., which in turn is equivalent to $H\psi^{-1} \in L^0(L_\psi)$, by (6.1.12).

By Theorem 13.5 of [14], there exists a sequence $(S_n)_{n\geq 1}$ of simple integrands such that $S_n \to H$ and $|S_n| \leq |H|$ for all n. By Dominated Convergence Theorem (Theorem 2.3.7),

$$I^\Lambda(S_n) \xrightarrow{P} I^\Lambda(H) \quad \text{and} \quad I^{L_\psi}(S_n\psi^{-1}) \xrightarrow{P} I^{L_\psi}(H\psi^{-1}).$$

By (6.1.14), $I^\Lambda(S_n) = I^{L_\psi}(S_n\psi^{-1})$ for all n. By uniqueness of the limit, $I^\Lambda(H) = I^{L_\psi}(H\psi^{-1})$. ∎

The following result will be used in the proof of Lemma 6.3.7 below (in the case $\alpha \geq 1$).

Proposition 6.1.11. Assume that $\alpha \in [1,2)$. For any $H \in L^0(\Lambda)$ and $a \in (0,1)$, we have:

$$\int_0^T \int_{\mathbb{R}^d} H(t,x)\Lambda(dt,dx) = \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|\leq a\}} H(t,x)\frac{z}{\psi(t,x)}\widehat{J}_\psi(dt,dx,dz) + \int_0^T \int_{\mathbb{R}^d} \int_{\{|z|>a\}} H(t,x)\frac{z}{\psi(t,x)}J_\psi(dt,dx,dz).$$

Proof. In (6.1.13), we write the first term on the right hand-side as the sum of two integrals corresponding to sets $\{|z| \leq a\}$ and $\{a < |z| \leq 1\}$. Notice that

$$\int_0^T \int_{\mathbb{R}^d} \int_{\{a<|z|\leq 1\}} H(t,x)\frac{z}{\psi(t,x)}\widehat{J}_\psi(dt,dx,dz) = \int_0^T \int_{\mathbb{R}^d} \int_{\{a<|z|\leq 1\}} H(t,x)\frac{z}{\psi(t,x)}J_\psi(dt,dx,dz).$$

This is because $\int_0^T \int_{\mathbb{R}^d} \int_{\{a<|z|\leq 1\}} H(t,x)z\psi^{\alpha-1}(t,x)dxdt\nu_\alpha(dz) = 0$, by Fubini's theorem and the symmetry of ν_α. To justify the application of Fubini's theorem, we need to prove

$$\int_0^T \int_{\mathbb{R}^d} |H(t,x)|\psi^{\alpha-1}(t,x)\left(\int_{\{a<|z|\leq 1\}} |z|\nu_\alpha(dz)\right)dxdt < \infty,$$

which is equivalent to $\int_0^T \int_{\mathbb{R}^d} |H(t,x)|\psi^{\alpha-1}(t,x)dxdt < \infty$. This last fact is clear when $\alpha = 1$, and can be proved using Hölder's inequality with $p = \alpha$, when $\alpha \in (1,2)$. ∎

Definition 6.1.12. Let $f : E^n \to \mathbb{R}$ be a $\mathcal{B}^n$-measurable symmetric function which vanishes on the diagonals of E^n. We say that f *is n-times integrable with respect to Z* if there exists a sequence $\{f^{(k)}\}_{k\geq 1}$ of simple functions which are n-times integrable with respect to Z such that:
(i) $\{f^{(k)}\}_{k\geq 1}$ converges to f in the measure $\ell^{(n)}$;
(ii) for any $B \in \mathcal{B}_n^{(s)}$, the sequence $\{I_n(f^{(k)}1_B)\}_{k\geq 1}$ converges in probability to a limit denoted by $I_n(f1_B)$.
In this case, we let $I(f1_B)$ be the limit in probability of $\{I(f^{(k)}1_B)\}_{k\geq 1}$.

Recall that the *symmetrization* of a function f is defined by:

$$\widetilde{f}(x_1,\ldots,x_n) = \frac{1}{n!}\sum_{\pi\in\Sigma_n} f(x_{\pi(1)},\ldots,x_{\pi(n)}).$$

A $\mathcal{B}^n$-measurable function f which vanishes on the diagonals is n-times integrable with respect to Z if its symmetrization $\widetilde{f}$ is so; in this case, we let $I_n(f) = I_n(\widetilde{f})$.

We use the notation

$$I_n(f) = \int_{E^n} f(t_1, x_1, \dots, t_n, z_n) Z(dt_1, dx_1) \dots Z(dt_n, dx_n),$$

and we say that I_n is the *multiple integral of order* n of f with respect to Z.

We now present a criterion for integrability.

Theorem 6.1.13. A symmetric $\mathcal{B}^n$-measurable function $f : E^n \to \mathbb{R}$ which vanishes on the diagonals is n-times integrable with respect to Z if and only if

$$\sum_{j_1 < \dots < j_n} \prod_{k=1}^{n} \Gamma_{j_k}^{-2/\alpha} \psi^{-2}(T_{j_k}, X_{j_k}) f^2(T_{j_1}, X_{j_1}, \dots, T_{j_n}, X_{j_n}) < \infty \quad \text{a.s.} \tag{6.1.15}$$

In this case, the series

$$S_n(f) := n! \sum_{j_1 < \dots < j_n} \prod_{k=1}^{n} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \psi^{-1}(T_{j_k}, X_{j_k}) f(T_{j_1}, X_{j_1}, \dots, T_{j_n}, X_{j_n})$$

converges a.s. and $I_n(f) = S_n(f)$ a.s.

A sufficient condition for (6.1.15) is

$$\int_{E^n} |f(t_1, x_1, \dots, t_n, x_n)|^\alpha \left[\ln_+ \frac{|f(t_1, x_1, \dots, t_n, x_n)|}{\psi(t_1, x_1) \dots \psi(t_n, x_n)} \right]^{n-1} dt_1 dx_1 \dots dt_n dx_n < \infty.$$

Under this condition, the multiple integral has the following asymptotic tail behaviour:

$$\lim_{\lambda \to \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} \mathbb{P}(|I_n(f)| > \lambda) = n (n!)^{\alpha - 2} \alpha^{n-1} \|f\|_{\alpha, n}^\alpha, \quad \text{for any } n \geq 3, \tag{6.1.16}$$

where $\|f\|_{\alpha,n}^\alpha = \int_{E^n} |f(t_1, x_1, \dots, t_n, x_n)|^\alpha dt_1 dx_1 \dots dt_n dx_n$. Relation (6.1.16) holds also for $n = 2$ under the additional condition:

$$\int_{E^2} |f(t_1, x_1, t_2, x_2)|^\alpha \ln_+ \frac{|f(t_1, x_1, t_2, x_2)|}{\psi(t_1, x_1) \psi(t_2, x_2)} \ln_+ \left| \ln \frac{|f(t_1, x_1, t_2, x_2)|}{\psi(t_1, x_1) \psi(t_2, x_2)} \right| dt_1 dx_1 dt_2 dx_2 < \infty.$$

This concludes our summary about multiple stable integrals.

6.2 Convergence of the series

In this section, we give the proof of Theorem 6.0.4.(a). First, we show that under a condition weaker than the one given by Assumption 6.0.2, the series on the right-hand side of (6.0.8) converges in probability. Then, we show that under Assumption 6.0.2, this series converges absolutely almost surely. For the rest of the chapter, we will denote this series by $u(t,x)$:

$$u(t,x) := 1 + \sum_{n\geq 1} I_n\big(f_n(\cdot,t,x)\big) = 1 + \sum_{n\geq 1} I_n\big(\widetilde{f}_n(\cdot,t,x)\big). \tag{6.2.1}$$

Recall that $f_n(\cdot,t,x)$ is the kernel given by (6.0.9) and $\widetilde{f}_n(\cdot,t,x)$ is its symmetrization. By Theorem 6.1.13, the multiple stable integral $I_n(\widetilde{f}_n(\cdot,t,x))$ is well-defined if and only if

$$h_n^{(2)}(t,x) := T^{\frac{2n}{\alpha}}(n!)^2 \sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \Gamma_{j_k}^{-2/\alpha}\phi^{-2}(X_{j_k})\widetilde{f}_n^2(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},t,x) < \infty \quad \text{a.s.}$$

and in this case, it has the LePage series representation:

$$I_n(\widetilde{f}_n(\cdot,t,x)) = T^{n/\alpha} n! \sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \varepsilon_{j_k}\Gamma_{j_k}^{-1/\alpha}\phi^{-1}(X_{j_k})\widetilde{f}_n(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},t,x) \quad \text{a.s.} \tag{6.2.2}$$

We have the following result.

Lemma 6.2.1. Let $(\varepsilon_j)_{j\geq 1}$ be i.i.d. with $\mathbb{P}(\varepsilon_j = 1) = \mathbb{P}(\varepsilon_j = -1) = 1/2$. Let $\{a_{j_1,\ldots,j_n}; 1 \leq j_1 < \ldots < j_n, j_i \in \mathbb{N}\}$ be an array of non-negative numbers. The series

$$\sum_{n\geq 1}\sum_{j_1<\ldots<j_n} \varepsilon_{j_1}\ldots\varepsilon_{j_n} a_{j_1,\ldots,j_n} \quad \text{converges in } L^2(\Omega)$$

if and only if $\sum_{n\geq 1}\sum_{j_1<\ldots<j_n} a_{j_1,\ldots,j_n}^2 < \infty$, and in this case,

$$\mathbb{E}\left[\left|\sum_{n\geq 1}\sum_{j_1<\ldots<j_n} \varepsilon_{j_1}\ldots\varepsilon_{j_n} a_{j_1,\ldots,j_n}\right|^2\right] = \sum_{n\geq 1}\sum_{j_1<\ldots<j_n} a_{j_1,\ldots,j_n}^2.$$

Proof. The terms of the series are orthogonal in $L^2(\Omega)$ since

$$\mathbb{E}\left[\varepsilon_{j_1}\ldots\varepsilon_{j_n}\varepsilon_{k_1}\ldots e_{k_m}\right] = \begin{cases} 1 & \text{if } \{j_1,\ldots,j_n\} = \{k_1,\ldots,k_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

∎

Remark 6.2.2. Let $n \in \mathbb{N}$ fixed. If $\sum_{j_1<\ldots<j_n} a_{j_1,\ldots,j_n}^2 < \infty$, then by the generalized Khintchine inequality (Proposition 1 of [51]), the series $\sum_{j_1<\ldots<j_n} \varepsilon_{j_1}\ldots\varepsilon_{j_n} a_{j_1,\ldots,j_n}$ converges a.s. and in $L^p(\Omega)$ for any $p \geq 0$.

Using Lemma 6.2.1, we see that

$$h_n^{(2)}(t,x) = \mathbb{E}\Big[|I_n(\widetilde{f}_n(\cdot,t,x))|^2 \mid (\Gamma_i),(T_i),(X_i)\Big]. \tag{6.2.3}$$

The following basic conditioning fact will be used several times in the sequel: if X and Y are independent random elements with values in measurable spaces $(E,\mathcal{E})$, respectively $(F,\mathcal{F})$, and $f : E\times F \to [0,\infty]$ is a measurable function, then

$$\mathbb{E}[f(X,Y)|X=x] = \mathbb{E}[f(x,Y)]. \tag{6.2.4}$$

Lemma 6.2.3. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. If

$$\mathcal{A}_2(t,x) := \sum_{n\geq 1} h_n^{(2)}(t,x) < \infty \quad \text{a.s.}, \tag{6.2.5}$$

then the series $\sum_{n\geq 1} I_n(\widetilde{f}_n(\cdot,t,x))$ converges in probability.

Proof. Recall that convergence in probability is equivalent with the convergence in L^0 equipped with the norm $\|X\|_0 = \mathbb{E}[\min(1,|X|)]$.

Let $S_n = \sum_{k=1}^n I_k(\widetilde{f}_k(\cdot,t,x))$. By the Cauchy-Schwarz inequality, for any $n > m$,

$$\mathbb{E}[\min(1,|S_n - S_m|)] \leq \left(\mathbb{E}[\min(1,|S_n - S_m|^2)]\right)^{1/2}.$$

Conditioning on $(\Gamma_i),(T_i),(X_i)$, we have:

$$\mathbb{E}[\min(1,|S_n - S_m|^2)] = \mathbb{E}[\mathbb{E}[\min(1,|S_n - S_m|^2)|(\Gamma_i),(T_i),(X_i)]].$$

Using the inequality $\mathbb{E}[\min(1,|X|)] \leq \min(1,E|X|)$, followed by Lemma 6.2.1, we have:

$$\begin{aligned}
&\mathbb{E}[\min(1,|S_n - S_m|^2)|(\Gamma_i)=(\gamma_i),(T_i)=(t_i),(X_i)=(x_i)] \\
&= \mathbb{E}\left[\min\left(1,\left|\sum_{k=m+1}^n T^{k/\alpha}k! \sum_{j_1<\ldots<j_k}\prod_{\ell=1}^k \varepsilon_{j_\ell}\gamma_{j_\ell}^{-1/\alpha}\phi^{-1}(x_{j_\ell})\widetilde{f}_n(t_{j_1},x_{j_1},\ldots,t_{j_k},x_{j_k},t,x)\right|^2\right)\right] \\
&\leq \min\left\{1,\mathbb{E}\left|\sum_{k=m+1}^n T^{k/\alpha}k! \sum_{j_1<\ldots<j_k}\prod_{\ell=1}^k \varepsilon_{j_\ell}\gamma_{j_\ell}^{-1/\alpha}\phi^{-1}(x_{j_\ell})\widetilde{f}_n(t_{j_1},x_{j_1},\ldots,t_{j_k},x_{j_k},t,x)\right|^2\right\} \\
&= \min\left\{1,\sum_{k=m+1}^n T^{2k/\alpha}(k!)^2 \sum_{j_1<\ldots<j_k}\prod_{\ell=1}^k \gamma_{j_\ell}^{-1/\alpha}\phi^{-2}(x_{j_\ell})\widetilde{f}_n^2(t_{j_1},x_{j_1},\ldots,t_{j_k},x_{j_k},t,x)\right\}.
\end{aligned}$$

This means that

$$\mathbb{E}\Big[\min(1,|S_n - S_m|^2)|(\Gamma_i),(T_i),(X_i)\Big] \leq \min\left\{1,\sum_{k=m+1}^n h_k^{(2)}(t,x)\right\}.$$

Hence,

$$\mathbb{E}[\min(1,|S_n - S_m|^2)] \leq \mathbb{E}\left[\min\left(1,\sum_{k=m+1}^n h_k^{(2)}(t,x)\right)\right].$$

By (6.2.5) and the dominated convergence theorem, the last term above converges to 0 as $n,m \to \infty$. This proves that $(S_n)_{n\geq 1}$ is a Cauchy sequence in L^0. ∎

The following result gives a criterion for verifying (6.2.5). Recall that $K_n^{(p)}(t,x)$ is defined by (6.0.11).

Proposition 6.2.4. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be fixed. If there exists $p \in (\alpha, 2]$ such that

$$\sum_{n\geq 1} T^{(\frac{p}{\alpha}-1)n} K_n^{(p)}(t,x)\Big(\sum_{j\geq 1}\Gamma_j^{-p/\alpha}\Big)^n < \infty \quad \text{a.s.}, \tag{6.2.6}$$

then (6.2.5) holds, and consequently, $\sum_{n\geq 1} I_n\big(\widetilde{f}_n(\cdot,t,x)\big)$ converges in probability.

Proof. By the sub-additivity of the function $\varphi(x) = x^{p/2}, x > 0$, we have:

$$\Big(h_n^{(2)}(t,x)\Big)^{p/2} \leq h_n^{(p)}(t,x), \tag{6.2.7}$$

where

$$h_n^{(p)}(t,x) := T^{pn/\alpha}(n!)^p \sum_{j_1<\ldots<j_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t, x).$$

Hence,

$$\mathcal{A}_2^{p/2}(t,x) \leq \sum_{n\geq 1}\Big(h_n^{(2)}(t,x)\Big)^{p/2} \leq \sum_{n\geq 1} h_n^{(p)}(t,x) =: \mathcal{A}_p(t,x).$$

We will prove that $\mathcal{A}_p(t,x) < \infty$ a.s. For this, we apply Remark B.3.1 to $X = (\Gamma_i)_{i\geq 1}$ and $Y = \{(T_i, X_i)\}_{i\geq 1}$. We will show that:

$$\mathbb{E}[\mathcal{A}_p(t,x)|(\Gamma_i)] < \infty \quad \text{a.s.} \tag{6.2.8}$$

By the independence between $(\Gamma_i)_i$ and $\{(T_i, X_i)\}_i$, we have:

$$\mathbb{E}\Big[h_n^{(p)}(t,x)|(\Gamma_i)\Big] = T^{pn/\alpha}(n!)^p \sum_{j_1<\ldots<j_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \mathbb{E}\Big[\prod_{k=1}^n \phi^{-p}(X_{j_k}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t, x)\Big].$$

For any $j_1 < \ldots < j_n$ fixed, we denote

$$A_{j_1,\ldots,j_n}^{(p)}(t,x) := (n!)^p \mathbb{E}\left[\prod_{k=1}^n \phi^{-p}(X_{j_k}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t, x)\right]. \tag{6.2.9}$$

We now use the explicit expression of $A_{j_1,\ldots,j_n}^{(p)}(t,x)$, given by Lemma 6.2.5 below, which in particular, shows that $A_{j_1,\ldots,j_n}^{(p)}(t,x)$ does not depend on $j_1, \ldots, j_n$. It follows that

$$\begin{aligned}\mathbb{E}\Big[h_n^{(p)}(t,x)|(\Gamma_i)\Big] &= T^{(p/\alpha-1)n} K_n^{(p)}(t,x)\, n! \sum_{j_1<\ldots<j_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \\ &\leq T^{(p/\alpha-1)n} K_n^{(p)}(t,x) \left(\sum_{j\geq 1}\Gamma_j^{-p/\alpha}\right)^n,\end{aligned} \tag{6.2.10}$$

where for the last inequality, we used the fact that for any $a_j > 0$,

$$n! \sum_{j_1 < \ldots < j_n} a_{j_1} \ldots a_{j_n} \leq \Big(\sum_{j \geq 1} a_j \Big)^n. \tag{6.2.11}$$

By the strong law of large numbers, $\Gamma_j / j \to 1$ a.s., and hence $\sum_{j \geq 1} \Gamma_j^{-p/\alpha} < \infty$ a.s. Finally, (6.2.8) follows by (6.2.6), since

$$\mathbb{E}[\mathcal{A}_p(t,x)|(\Gamma_i)] = \sum_{n \geq 1} \mathbb{E}\Big[h_n^{(p)}(t,x)|(\Gamma_i)\Big] \leq \sum_{n \geq 1} T^{(p/\alpha - 1)n} K_n^{(p)}(t,x) \left(\sum_{j \geq 1} \Gamma_j^{-p/\alpha} \right)^n.$$

∎

The following lemma gives the explicit expression for $A^{(p)}_{j_1,\ldots,j_n}(t,x)$.

Lemma 6.2.5. For any $t \in [0,T]$, $x \in \mathbb{R}^d$, $p > 0$ and $j_1 < \ldots < j_n$, we have:

$$A^{(p)}_{j_1,\ldots,j_n}(t,x) = \frac{n!}{T^n} K_n^{(p)}(t,x). \tag{6.2.12}$$

Proof. By the definition of $\widetilde{f}_n(\cdot,t,x)$,

$$\begin{aligned} A^{(p)}_{j_1,\ldots,j_n}(t,x) &= \mathbb{E}\left[\prod_{k=1}^n \phi^{-p}(X_{j_k}) \Big(\sum_{\pi \in \Sigma_n} f_n(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, t, x) \Big)^p \right] \\ &= \mathbb{E}\left[\left(\sum_{\pi \in \Sigma_n} \frac{f_n(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, t, x)}{\phi(X_{j_{\pi(1)}}) \ldots \phi(X_{j_{\pi(n)}})} \right)^p \right]. \end{aligned} \tag{6.2.13}$$

We now present two methods leading to the desired relation (6.2.12).

Method 1. (due to G. Samorodnitsky) Recall that $f_n(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, t, x)$ contains the indicator of

$$0 < T_{j_{\pi(1)}} < \ldots < T_{j_{\pi(n)}} < t. \tag{6.2.14}$$

For fixed values $T_{j_1}, \ldots, T_{j_n} \in [0,t]$, there is a unique permutation π for which (6.2.14) holds. Therefore, the sum over all permutations above in fact contains *only one term*, all the other terms vanishing. The vector $(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}})$ corresponding to that term has the same distribution as $(T_{(1)}, X_1, \ldots, T_{(n)}, X_n)$, where $T_{(1)} < \ldots < T_{(n)}$ are the order statistics of $T_1, \ldots, T_n$. Hence,

$$\begin{aligned} A^{(p)}_{j_1,\ldots,j_n}(t,x) &= \mathbb{E}\left[\prod_{k=1}^n \phi^{-p}(X_k) f_n^p(T_{(1)}, X_1, \ldots, T_{(n)}, X_n, t, x) \right] \\ &= \frac{n!}{T^n} \int_{T_n(T)} \int_{(\mathbb{R}^d)^n} f_n^p(t_1, x_1, \ldots, t_n, x_n, t, x) \prod_{k=1}^n \phi^{\alpha - p}(x_k) dx_1 \ldots dx_n dt_1 \ldots dt_n, \end{aligned}$$

using the fact that $(T_{(1)}, \ldots, T_{(n)})$ has a uniform distribution over the simplex $T_n(T)$. The desired relation follows since $f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = 0$ if $t_n > t$.

Method 2. From (6.2.13), since $(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n})$ has density $T^{-n}\prod_{k=1}^{n}\phi^{\alpha}(x_k)$, we obtain:

$$A^{(p)}_{j_1,\ldots,j_n}(t,x) = T^{-n}\int_{[0,T]^n}\int_{(\mathbb{R}^d)^n}\left(\sum_{\pi\in\Sigma_n}\frac{f_n(t_{\pi(1)},x_{\pi(1)},\ldots,t_{\pi(n)},x_{\pi(n)},t,x)}{\phi(x_{\pi(1)})\ldots\phi(x_{\pi(n)})}\right)^p\prod_{k=1}^{n}\phi^{\alpha}(x_k)d\boldsymbol{x}d\boldsymbol{t}$$
$$= T^{-n}\sum_{\rho\in\Sigma_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t}\int_{(\mathbb{R}^d)^n}\left(\sum_{\pi\in\Sigma_n}\frac{f_n(t_{\pi(1)},x_{\pi(1)},\ldots,t_{\pi(n)},x_{\pi(n)},t,x)}{\phi(x_{\pi(1)})\ldots\phi(x_{\pi(n)})}\right)^p\prod_{k=1}^{n}\phi^{\alpha}(x_k)d\boldsymbol{x}d\boldsymbol{t}$$
$$= T^{-n}\sum_{\rho\in\Sigma_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t}\int_{(\mathbb{R}^d)^n}\left(\frac{f_n(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)},t,x)}{\phi(x_{\rho(1)})\ldots\phi(x_{\rho(n)})}\right)^p\prod_{k=1}^{n}\phi^{\alpha}(x_k)d\boldsymbol{x}d\boldsymbol{t}$$
$$= T^{-n}n!\int_{0<t_1<\ldots<t_n<t}\int_{(\mathbb{R}^d)^n}\left(\frac{f_n(t_1,x_1,\ldots,t_n,x_n,t,x)}{\phi(x_1)\ldots\phi(x_n)}\right)^p\prod_{k=1}^{n}\phi^{\alpha}(x_k)d\boldsymbol{x}d\boldsymbol{t}.$$

∎

The following result shows that a condition stronger than (6.2.5) implies the a.s. absolute convergence of the series (6.2.1), and thus proves Theorem 6.0.4.(a). The fact that the series converges *absolutely* will play a crucial role in Section 6.4 when we will show that the process $u(t,x)$ given by (6.2.1) is indeed a solution of equation (6.0.1).

Proposition 6.2.6. Let $t\in[0,T]$ and $x\in\mathbb{R}^d$ be fixed. If there exists $p\in(\alpha,2]$ such that (6.0.12) holds, then $\mathcal{V}(t,x) := \sum_{n\geq 1}|I_n(\widetilde{f}_n(\cdot,t,x))| < \infty$ a.s.

Proof. We apply Remark B.3.2 to $X = \{(\Gamma_i),(T_i),(X_i)\}$ and $Y = (\varepsilon_i)$. We will prove that:

$$\mathbb{E}\Big[\mathcal{V}(t,x)|(\Gamma_i),(T_i),(X_i)\Big] < \infty \quad \text{a.s.}$$

By Jensen's inequality for conditional expectation and (6.2.3), we have

$$\mathbb{E}[|I_n(\widetilde{f}_n(\cdot,t,x))|\,|(\Gamma_i),(T_i),(X_i)] \leq \Big(h_n^{(2)}(t,x)\Big)^{1/2},$$

and hence $\mathbb{E}\Big[\mathcal{V}(t,x)|(\Gamma_i),(T_i),(X_i)\Big] \leq \sum_{n\geq 1}\Big(h_n^{(2)}(t,x)\Big)^{1/2}$. We take power $p/2$. Using sub-additivity of the function $\varphi(x) = x^{p/2}, x>0$ and (6.2.7), we get:

$$\left(\mathbb{E}\Big[\mathcal{V}(t,x)|(\Gamma_i),(T_i),(X_i)\Big]\right)^{p/2} \leq \sum_{n\geq 1}\Big(h_n^{(2)}(t,x)\Big)^{p/4} \leq \sum_{n\geq 1}\Big(h_n^{(p)}(t,x)\Big)^{1/2} =: \mathcal{B}_p(t,x).$$

To prove that $\mathcal{B}_p(t,x)<\infty$ a.s., we use Remark B.3.2 with $X=(\Gamma_i)$ and $Y=\{(T_i),(X_i)\}$. We will show that:

$$\mathbb{E}[\mathcal{B}_p(t,x)|(\Gamma_i)] < \infty \quad \text{a.s.} \tag{6.2.15}$$

We will use the following inequality:

$$\mathbb{E}(X^p|\mathcal{G}) \leq (\mathbb{E}[X|\mathcal{G}])^p \quad \text{if } p\in(0,1) \text{ and } X\geq 0, \tag{6.2.16}$$

which is Jensen's inequality applied to the concave function $\varphi(x) = x^p, x > 0$. Hence,

$$\mathbb{E}[\mathcal{B}_p(t,x)|(\Gamma_i)] = \sum_{n\geq 1} \mathbb{E}\Big[\Big(h_n^{(p)}(t,x)\Big)^{1/2} |(\Gamma_i)\Big] \leq \sum_{n\geq 1} \Big(\mathbb{E}\Big[h_n^{(p)}(t,x)|(\Gamma_i)\Big]\Big)^{1/2}$$
$$\leq \sum_{n\geq 1} \left(T^{(p/\alpha-1)n} K_n^{(p)}(t,x)\right)^{1/2} \Big(\sum_{j\geq 1} \Gamma_j^{-p/\alpha}\Big)^{n/2},$$

where the last inequality is due to (6.2.10). Relation (6.2.15) follows by (6.0.12). ∎

6.3 The recurrence relation

In this section, we prove that the partial sum sequence $(u_n)_{n\geq 0}$ given by

$$u_0(t,x) = 1 \quad \text{and} \quad u_n(t,x) = 1 + \sum_{k=1}^{n} I_k(f_k(\cdot,t,x)),\ n \geq 1 \tag{6.3.1}$$

is indeed the Picard's iteration sequence, i.e. it satisfies the recurrence relation (6.0.10).

By linearity, instead of proving (6.0.10), it is enough to show that for any $n \geq 0$,

$$I_{n+1}(f_{n+1}(\cdot,t,x)) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) I_n(f_n(\cdot,s,y)) Z(ds,dy).$$

where $f_0(s,y) = 1$ and $I_0(x) = x$. For this, we define *the multiple integral process*:

$$X_n^{(t,x)}(s,y) = G_{t-s}(x-y) I_n(f_n(\cdot,s,y)), \quad s \in [0,T], y \in \mathbb{R}^d.$$

For any $n \geq 0$, $t \in [0,T]$ and $x \in \mathbb{R}^d$ fixed, we have to prove that:
(i) the multiple integral process is integrable wr.r.t. Z, i.e.

$$X_n^{(t,x)} \in L^0(Z); \tag{6.3.2}$$

(ii) the integral w.r.t. Z of the multiple integral process coincides with $I_{n+1}(f_{n+1}(\cdot,t,x))$:

$$I^Z(X_n^{(t,x)}) = I_{n+1}(f_{n+1}(\cdot,t,x)). \tag{6.3.3}$$

These facts will be proved separately in the following two sections.

6.3.1 Integrability of the muliple integral process

In this section, we give the proof of (6.3.2). By Corollary 2.3.9, we need to check that

$$\mathcal{I}_n(t,x) := \int_0^t \int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y) |I_n(f_n(\cdot,s,y))|^{\alpha} dy ds < \infty \quad \text{a.s.}, \tag{6.3.4}$$

provided that $X_n^{(t,x)}$ is predictable. The next lemma addresses the issue of predictability.

Lemma 6.3.1. The process $\{I_n(\widetilde{f}_n(\cdot,t,x)); t \in [0,T], x \in \mathbb{R}^d\}$ has a predictable modification. Using this modification, $X_n^{(t,x)}$ is predictable.

Proof. We only need to prove the first statement, since the map $(\omega, s, y) \mapsto G_{t-s}(x-y)$ is clearly predictable. By LePage representation (6.2.2), $I_n(f_n(\cdot,t,x)) = S_n(t,x)$ a.s. where

$$S_n(t,x) = T^{n/\alpha} n! \sum_{j_1<\ldots<j_n} \prod_{k=1}^{n} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t, x).$$

Hence, it is enough to prove that S_n has a predictable modification. For this, we proceed as in the proof of Lemma 6.2 of [16]. For any $x \in \mathbb{R}^d$, let $S_n^P(\cdot,x)$ be the *extended predictable projection* of $S_n(\cdot,x)$, given by Theorem I.2.28 of [31], i.e. $S_n^P(\cdot,x)$ is $\mathcal{P}$-measurable and $S_n^P(\cdot,x) = \mathbb{E}[S_n(t,x)|\mathcal{F}_{t-}]$ a.s. for all $t \in [0,T]$.

By Proposition 3 of [54], there exists a predictable process $\widetilde{S}_n$ such that $\widetilde{S}_n(t,x) = S_n^P(t,x)$ a.s. for all (t,x). Hence, for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\widetilde{S}_n(t,x) = \mathbb{E}[S_n(t,x)|\mathcal{F}_{t-}] = S_n(t,x) \quad \text{a.s.},$$

where the last equality is due to the fact that $S_n(t,x)$ is $\mathcal{F}_{t-}$-measurable, which is true because $S_n(t,x)$ is a function of the points of N_ψ situated in $[0,t) \times \mathbb{R}^d \times \mathbb{R}_0$. (Recall that $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration associated with N_ψ, given by (6.1.5).) ∎

We continue now with the verification of (6.3.4). This will be the consequence of the following more general result, which will be needed for the proof of Theorem 6.4.1 below.

Theorem 6.3.2. Suppose that Assumption 6.0.1 holds. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be fixed. If there exists $p \in (\alpha, 2]$ such that (6.0.13) holds, then

$$\sum_{n\geq 1} \mathcal{I}_n(t,x)^{\frac{1}{\alpha \vee 1}} < \infty \quad \text{a.s.} \tag{6.3.5}$$

Moreover, if there exists $p \in (\alpha, 2]$ such that

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y) K_n^{(p)}(s,y) dy ds < \infty, \tag{6.3.6}$$

then $\mathcal{I}_n(t,x) < \infty$ a.s., and consequently $X_n^{(t,x)} \in L^0(Z)$.

Proof. We only need to prove the first statement. The second statement is proved in the same way, dropping the sum over $n \geq 1$.

We consider first the case $\alpha \leq 1$. We will prove that:

$$\mathcal{C}(t,x) := \sum_{n\geq 1} \mathcal{I}_n(t,x) < \infty \quad \text{a.s.}$$

We apply Remark B.3.2 with $X = (\Gamma_i)$ and $Y = \{(T_i), (X_i), (\varepsilon_i)\}$. We will prove that:

$$\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i)] < \infty \quad \text{a.s.} \tag{6.3.7}$$

We use the fact that $\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i)] = \mathbb{E}[\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i),(T_i),(X_i)]|(\Gamma_i)]$, and we estimate separately the inner conditional expectation. By Jensen's inequality,

$$\begin{aligned}\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i),(T_i),(X_i)] &= \sum_{n\geq 1}\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\mathbb{E}[|I_n(f_n(\cdot,s,y))|^{\alpha}|(\Gamma_i),(T_i),(X_i)]dyds \\ &\leq \sum_{n\geq 1}\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\left(h_n^{(2)}(s,y)\right)^{\alpha/2}dyds.\end{aligned}$$

We now use the following form of Jensen's inequality: if $(E,\mathcal{E},\mu)$ is a finite measure space with $\mu(E)=a$, then for any measurable function $f: E \to \mathbb{R}_+$,

$$\int_E f^p d\mu \leq a^{1-p}\left(\int_E f d\mu\right)^p \quad \text{for any } p\in(0,1]. \tag{6.3.8}$$

We apply this inequality to the finite measure $\mu(ds,dy) = G^{\alpha}_{t-s}(s-y)1_{(0,t)}(s)dsdy$ (whose total mass we denote C_t), and the exponent $p' = \alpha/p$. We obtain:

$$\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\left(h_n^{(2)}(s,y)\right)^{\frac{\alpha}{2}}dyds \leq C_t^{1-\frac{\alpha}{p}}\left(\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\left(h_n^{(2)}(s,y)\right)^{\frac{p}{2}}dyds\right)^{\frac{\alpha}{p}}.$$

It follows that

$$\begin{aligned}\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i)] &\leq C_t^{1-\frac{\alpha}{p}}\sum_{n\geq 1}\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\left(h_n^{(2)}(s,y)\right)^{\frac{p}{2}}dyds\right)^{\frac{\alpha}{p}}\Bigg|(\Gamma_i)\right] \\ &\leq C_t^{1-\frac{\alpha}{p}}\sum_{n\geq 1}\left(\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\mathbb{E}[\left(h_n^{(2)}(s,y)\right)^{p/2}|(\Gamma_i)]dyds\right)^{\frac{\alpha}{p}},\end{aligned}$$

using (6.2.16) (with exponent $p' = \alpha/p$) for the last inequality. We now pass from $(h_n^{(2)}(s,y))^{p/2}$ to $h_n^{(p)}(s,y)$ (using inequality (6.2.7)), and estimate $\mathbb{E}[h_n^{(p)}(s,y)|(\Gamma_i)]$ using (6.2.10). We get:

$$\mathbb{E}[\mathcal{C}(t,x)|(\Gamma_i)] \leq C_t^{1-\frac{\alpha}{p}}\sum_{n\geq 1}\left(T^{(p/\alpha-1)n}\big(\sum_{j\geq 1}\Gamma_j^{-p/\alpha}\big)^n\int_0^t\int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)K_n^{(p)}(s,y)dyds\right)^{\frac{\alpha}{p}}.$$

The last series converges with probability 1, due to (6.0.13). This proves (6.3.7).

Next, we consider the case $\alpha > 1$. We will prove that:

$$\mathcal{C}'(t,x) := \sum_{n\geq 1}\left(\mathcal{I}_n(t,x)\right)^{\frac{1}{\alpha}} < \infty \quad \text{a.s.}$$

By the same application of Remark B.3.2 as above, it suffices to prove that:

$$\mathbb{E}[\mathcal{C}'(t,x)|(\Gamma_i)] < \infty \quad \text{a.s.} \tag{6.3.9}$$

We use again double conditioning relation, and we estimate separately the inner conditional expectation. By Jensen's inequality (6.2.16),

$$\begin{aligned}
\mathbb{E}[\mathcal{C}'(t,x)|(\Gamma_i),(T_i),(X_i)] &= \sum_{n\geq 1} \mathbb{E}\Big[\big(\mathcal{I}_n(t,x)\big)^{\frac{1}{\alpha}}|(\Gamma_i),(T_i),(X_i)\Big] \\
&\leq \sum_{n\geq 1} \Big(\mathbb{E}\Big[\mathcal{I}_n(t,x)|(\Gamma_i),(T_i),(X_i)\Big]\Big)^{\frac{1}{\alpha}} \\
&= \sum_{n\geq 1} \left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\mathbb{E}[|I_n(f_n(\cdot,s,y))|^{\alpha}|(\Gamma_i),(T_i),(X_i)]dyds\right)^{\frac{1}{\alpha}} \\
&\leq \sum_{n\geq 1} \left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\big(h_n^{(2)}(s,y)\big)^{\alpha/2}dyds\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Conditioning on (Γ_i), and using again the conditional Jensen's inequality (6.2.16) to push the power $1/\alpha$ outside the conditional expectation, we get:

$$\begin{aligned}
\mathbb{E}[\mathcal{C}'(t,x)|(\Gamma_i)] &\leq \sum_{n\geq 1} \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\big(h_n^{(2)}(s,y)\big)^{\alpha/2}dyds\right)^{\frac{1}{\alpha}} \middle| (\Gamma_i)\right] \\
&\leq \sum_{n\geq 1} \left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\mathbb{E}[\big(h_n^{(2)}(s,y)\big)^{\alpha/2}|(\Gamma_i)]dyds\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

To arrive to the desired exponent $p/2$ for $h_n^{(2)}(s,y)$, we apply Jensen's inequality $\mathbb{E}(|X||\mathcal{G}) \leq \big(\mathbb{E}[|X|^p|\mathcal{G}]\big)^{1/p}$ for $p \geq 1$. Combining this with bound (6.2.7), we obtain:

$$\mathbb{E}[\big(h_n^{(2)}(s,y)\big)^{\alpha/2}|(\Gamma_i)] \leq \Big(\mathbb{E}[\big(h_n^{(2)}(s,y)\big)^{p/2}|(\Gamma_i)]\Big)^{\frac{\alpha}{p}} \leq \big(\mathbb{E}[h_n^{(p)}(s,y)|(\Gamma_i)]\big)^{\frac{\alpha}{p}}.$$

Finally, we use (6.2.10) to estimate the last conditional expectation. Therefore,

$$\begin{aligned}
\mathbb{E}[\mathcal{C}'(t,x)|(\Gamma_i)] &\leq \sum_{n\geq 1} \left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)\left(\mathbb{E}[h_n^{(p)}(s,y)|(\Gamma_i)]\right)^{\frac{\alpha}{p}} dyds\right)^{\frac{1}{\alpha}} \\
&\leq \sum_{n\geq 1} \left(T^{(p/\alpha-1)n}\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y)K_n^{(p)}(s,y)dyds\right)^{\frac{1}{p}} \Big(\sum_{j\geq 1}\Gamma_j^{-p/\alpha}\Big)^{n/p}.
\end{aligned}$$

Relation (6.3.9) follows by (6.0.13). ∎

6.3.2 The integral of the multiple integral process: $\alpha \in (0,1)$

In this section, we give the proof of (6.3.3) in the case $\alpha < 1$. For this, an essential role is played by LePage representation (6.1.9) of $\Lambda(A)$ for $A \in \widetilde{\mathcal{P}}_b$, obtained in Section 6.1.2.

We start with a preliminary result, which gives the series representation for $I^Z(X_n^{(t,x)})$.

Theorem 6.3.3. Assume that $\alpha \in (0,1)$. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be fixed. Suppose that there exists $p \in (\alpha, 1]$ such that $K_{n+1}^{(p)}(t,x) < \infty$, and $\mathcal{I}_n(t,x) < \infty$ a.s. Then

$$I^Z(X_n^{(t,x)}) = T^{1/\alpha} \sum_{i \geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\phi(X_i)} G_{t-T_i}(x - X_i) I_n\big(f_n(\cdot, T_i, X_i)\big) \quad \text{a.s.} \tag{6.3.10}$$

Proof. Condition $\mathcal{I}_n(t,x) < \infty$ a.s. ensures that $X_n^{(t,x)} \in L^0(Z)$.

Step 1. In this step, we prove that the series on the right hand-side of (6.3.10) converges absolutely a.s., i.e.

$$U := \sum_{i \geq 1} \Gamma_i^{-1/\alpha} \phi^{-1}(X_i) G_{t-T_i}(x - X_i) \left| I_n\big(f_n(\cdot, T_i, X_i)\big) \right| < \infty \quad \text{a.s.}$$

Since $p \leq 1$, by subadditivity,

$$U^p \leq \sum_{i \geq 1} \Gamma_i^{-p/\alpha} \phi^{-p}(X_i) G_{t-T_i}^p(x - X_i) \left| I_n\big(f_n(\cdot, T_i, X_i)\big) \right|^p =: U'.$$

So it is enough to prove that $U' < \infty$ a.s.

Using the LePage representation (6.2.2), the subadditivity of the function $\varphi(x) = |x|^p$, and the fact that $\widetilde{f}_n(t_1, x_1, \ldots, t_n, x_n, t, x) = 0$ if $t_i = t$ for some $i = 1, \ldots, n$, we have:

$$\begin{aligned} |I_n(\widetilde{f}_n(\cdot, T_i, X_i))|^p &\leq T^{pn/\alpha} (n!) \sum_{j_1 < \ldots < j_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_i, X_i) \\ &\leq T^{pn/\alpha} \sum_{\substack{j_1 < \ldots < j_n \\ i \notin \{j_1, \ldots, j_n\}}} \sum_{\pi \in \Sigma_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) f_n^p(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, T_i, X_i). \end{aligned}$$

We multiply this inequality by $\Gamma_i^{-p/\alpha} \phi^{-p}(X_i) G_{t-T_i}^p(x - X_i)$, then we take the sum for all $i \geq 1$. We use the fact that $G_{t-T_i}^p(x - X_i) f_n^p(\cdot, T_i, X_i) = f_{n+1}^p(\cdot, T_i, X_i, t, x)$. We obtain:

$$\begin{aligned} U' \leq T^{pn/\alpha} \sum_{i \geq 1} \Gamma_i^{-p/\alpha} \phi^{-p}(X_i) &\sum_{\substack{j_1 < \ldots < j_n \\ i \notin \{j_1, \ldots, j_n\}}} \sum_{\pi \in \Sigma_n} \prod_{k=1}^n \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) \\ &f_{n+1}^p(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, T_i, X_i, t, x). \end{aligned}$$

We now use the following fact: for any $a_{j_1,\ldots,j_n} \geq 0$,

$$\sum_{j_1<\ldots<j_n} \sum_{\pi \in \Sigma_n} a_{j_{\pi(1)},\ldots,j_{\pi(n)}} = \sum_{j_1,\ldots,j_n \geq 1 \text{ distinct}} a_{j_1,\ldots,j_n}. \tag{6.3.11}$$

Hence,

$$\begin{aligned}
U' &\leq T^{pn/\alpha} \sum_{i\geq 1} \Gamma_i^{-p/\alpha} \phi^{-p}(X_i) \sum_{\substack{j_1,\ldots,j_n \geq 1 \text{ distinct} \\ i \notin \{j_1,\ldots,j_n\}}} \prod_{k=1}^{n} \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) \\
&\qquad f_{n+1}^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_i, X_i, t, x) \\
&= T^{pn/\alpha} \sum_{j_1,\ldots,j_{n+1}\geq 1 \text{ distinct}} \prod_{k=1}^{n+1} \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) f_{n+1}^p(T_{j_1}, X_{j_1}, \ldots, T_{j_{n+1}}, X_{j_{n+1}}, t, x) \\
&= T^{pn/\alpha} \sum_{j_1<\ldots<j_{n+1}} \prod_{k=1}^{n+1} \Gamma_{j_k}^{-p/\alpha} \phi^{-p}(X_{j_k}) \sum_{\pi\in\Sigma_{n+1}} f_{n+1}^p(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n+1)}}, X_{j_{\pi(n+1)}}, t, x) \\
&=: B_{n+1}^{(p)}(t,x).
\end{aligned} \tag{6.3.12}$$

We prove that $B_{n+1}^{(p)}(t,x) < \infty$ a.s. For this, we apply Remark B.3.2 with $X = (\Gamma_i)$ and $Y = \{(T_i, X_i)\}$. We will prove that

$$\mathbb{E}[B_{n+1}^{(p)}(t,x) | (\Gamma_i)] < \infty \quad \text{a.s.}$$

Note that $(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n+1)}}, X_{j_{\pi(n+1)}})$ has density $T^{-(n+1)/\alpha} \prod_{k=1}^{n+1} \phi^{\alpha}(x_k)$. Hence,

$$\mathbb{E}\left[\prod_{k=1}^{n+1} \phi^{-p}(X_{j_k}) f_{n+1}^p(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n+1)}}, X_{j_{\pi(n+1)}}, t, x)\right] = \frac{1}{T^{(n+1)/\alpha}} K_{n+1}^{(p)}(t,x),$$

and

$$\begin{aligned}
\mathbb{E}[B_{n+1}^{(p)}(t,x)|(\Gamma_i)] &= T^{pn/\alpha} \sum_{j_1<\ldots<j_{n+1}} \prod_{k=1}^{n+1} \Gamma_{j_k}^{-p/\alpha} \cdot (n+1)! \frac{1}{T^{(n+1)/\alpha}} K_{n+1}^{(p)}(t,x) \\
&\leq T^{pn/\alpha} \frac{1}{T^{(n+1)/\alpha}} K_{n+1}^{(p)}(t,x) \left(\sum_{j\geq 1} \Gamma_j^{-p/\alpha}\right)^{n+1} < \infty \text{ a.s.}
\end{aligned}$$

Step 2. Since $X_n^{(t,x)}$ is predictable, by Theorem 13.5 of [14], there exists a sequence $(S_k)_{k\geq 1}$ of simple integrands such that $S_k \to X_n^{(t,x)}$ as $k \to \infty$, and $|S_k| \leq |X_n^{(t,x)}|$ for all k. Note that $X_n^{(t,x)} \in L^p(Z)$, since $\mathcal{I}_n(t,x) < \infty$ a.s. By Theorem 2.3.7, $I^Z(S_k) \xrightarrow{P} I^Z(X_n^{(t,x)})$ as $k \to \infty$. This convergence is a.s., along a subsequence.

Since S_k is a linear combination of sets in $\widetilde{\mathcal{P}}_b$, by the LePage representation (6.1.9),

$$I^Z(S_k) = I^{\Lambda}(S_k) = \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\psi(T_i, X_i)} S_k(T_i, X_i).$$

Relation (6.3.10) follows letting $k \to \infty$. On the right hand-side, we use the dominated convergence theorem, whose application is justified by *Step 1*. ∎

Theorem 6.3.4. Assume that $\alpha \in (0,1)$. Let $t > 0$ and $x \in \mathbb{R}^d$ be arbitrary. Suppose that there exists $p \in (\alpha, 1]$ such that $K_{n+1}^{(p)}(t,x) < \infty$ a.s. and $\mathcal{I}_n(t,x) < \infty$ a.s. Then (6.3.3) holds.

Proof. We use (6.3.10), in which we replace $I_n\big(f_n(\cdot, T_i, X_i)\big)$ by its LePage representation (6.2.2). Using the fact that $\widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_i, X_i) = 0$ if $i \in \{j_1, \ldots, j_n\}$, we have:

$$I^Z(X_n^{(t,x)}) = T^{\frac{n+1}{\alpha}} \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \phi^{-1}(X_i) G_{t-T_i}(x - X_i) \sum_{\substack{j_1<\ldots<j_n \\ i\notin\{j_1,\ldots,j_n\}}} \prod_{k=1}^{n} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k})$$
$$\sum_{\pi\in S_n} f_n(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, T_i, X_i).$$

We now use the fact that $G_{t-T_i}(x - X_i) f_n(\cdot, T_i, X_i) = f_{n+1}(\cdot, T_i, X_i, t, x)$, and we apply (6.3.11) two times. We obtain:

$$\begin{aligned}
I^Z(X_n^{(t,x)}) &= T^{\frac{n+1}{\alpha}} \sum_{i\geq 1} \varepsilon_i \Gamma_i^{-1/\alpha} \phi^{-1}(X_i) \sum_{\substack{j_1,\ldots,j_n\geq 1 \text{ distinct} \\ i\notin\{j_1,\ldots,j_n\}}} \prod_{k=1}^{n} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) \\
&\quad f_{n+1}(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_i, X_i, t, x) \\
&= T^{\frac{n+1}{\alpha}} \sum_{j_1,\ldots,j_{n+1}\geq 1 \text{ distinct}} \prod_{k=1}^{n+1} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) f_{n+1}(T_{j_1}, X_{j_1}, \ldots, T_{j_{n+1}}, X_{j_{n+1}}, t, x) \\
&= T^{\frac{n+1}{\alpha}} \sum_{j_1<\ldots<j_{n+1}} \sum_{\pi\in\Sigma_{n+1}} \prod_{k=1}^{n+1} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) f_{n+1}(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n+1)}}, X_{j_{\pi(n+1)}}, t, x) \\
&= T^{\frac{n+1}{\alpha}} (n+1)! \sum_{j_1<\ldots<j_{n+1}} \prod_{k=1}^{n+1} \varepsilon_{j_k} \Gamma_{j_k}^{-1/\alpha} \phi^{-1}(X_{j_k}) \widetilde{f}_{n+1}(T_{j_1}, X_{j_1}, \ldots, T_{j_{n+1}}, X_{j_{n+1}}, t, x) \\
&= I_{n+1}\big(\widetilde{f}_{n+1}(\cdot, t, x)\big).
\end{aligned}$$

∎

6.3.3 The integral of the multiple integral process: $\alpha \in [1, 2)$

In this section, we give the proof of (6.3.3) in the case $\alpha \geq 1$. The proof is significantly more involved than in the case $\alpha < 1$, being the most technical part of the chapter. The reason is that in the case $\alpha \geq 1$, we are not able to prove a LePage representation for $\Lambda(A)$ for all $A \in \widetilde{\mathcal{P}}_b$, as it was mentioned in Remark 6.1.8.

We fix $k \geq 1$. In the LePage representation (6.2.2) of $I_n\big(\widetilde{f}_n(\cdot,s,y)\big)$, in front of $\prod_{i=1}^n$, we introduce the factor $1 = 1_{\{\Gamma_{j_n}^{-1/\alpha} > k^{-1}\}} + 1_{\{\Gamma_{j_n}^{-1/\alpha} \leq k^{-1}\}}$. Since $\Gamma_{j_1}^{-1/\alpha} > \ldots > \Gamma_{j_n}^{-1/\alpha}$, we see that $\{\Gamma_{j_n}^{-1/\alpha} > k^{-1}\} = \bigcap_{j=1}^n \{\Gamma_{j_i}^{-1/\alpha} > k^{-1}\}$, We obtain the decomposition:

$$I_n\big(\widetilde{f}_n(\cdot,s,y)\big) = \mathcal{J}_n^{(k)}(s,y) + \mathcal{R}_n^{(k)}(s,y), \tag{6.3.13}$$

where

$$\mathcal{J}_n^{(k)}(s,y) = T^{n/\alpha} n! \sum_{j_1<\ldots<j_n} \prod_{i=1}^n \varepsilon_{j_i} \Gamma_{j_i}^{-1/\alpha} 1_{\{\Gamma_{j_i}^{-1/\alpha} > k^{-1}\}} \phi^{-1}(X_{j_i}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, s, y),$$

$$\mathcal{R}_n^{(k)}(s,y) = T^{n/\alpha} n! \sum_{j_1<\ldots<j_n} 1_{\{\Gamma_{j_n}^{-1/\alpha} \leq k^{-1}\}} \prod_{i=1}^n \varepsilon_{j_i} \Gamma_{j_i}^{-1/\alpha} \phi^{-1}(X_{j_i}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, s, y).$$

The following result shows that $\mathcal{R}_n^{(k)}(t,x)$ is asymptotically negligible in probability, when $k \to \infty$.

Lemma 6.3.5. Assume that $\alpha \in [1,2)$. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. If there exists $p \in (\alpha, 2]$ such that $K_n^{(p)}(t,x) < \infty$, then $\mathcal{R}_n^{(k)}(t,x) \xrightarrow{P} 0$ as $k \to \infty$.

Proof. We argue as in the proof of Lemma 6.2.3. By the Cauchy-Schwarz inequality,

$$\|\mathcal{R}_n^{(k)}(t,x)\|_{L^0} = \mathbb{E}\big[\min\big(1, |\mathcal{R}_n^k(t,x)|\big)\big] \leq \big(\mathbb{E}\big[\min\big(1, |\mathcal{R}_n^k(s,y)|^2\big)\big]\big)^{1/2}.$$

By Lemma 6.2.1 and the inequality $\mathbb{E}[\min(1,|X|)] \leq \min(1, \mathbb{E}|X|)$, we have:

$$\mathbb{E}\big[\min\big(1, |\mathcal{R}_n^k(s,y)|^2\big) \,\big|\, (\Gamma_i), (T_i), (X_i)\big] \leq$$
$$\min\Big(1, T^{2n/\alpha} (n!)^2 \sum_{j_1<\ldots<j_n} 1_{\{\Gamma_{j_n}^{-1/\alpha} \leq k^{-1}\}} \prod_{i=1}^n \Gamma_{j_i}^{-2/\alpha} \phi^{-2}(X_{j_i}) \widetilde{f}_n^2(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t, x)\Big).$$

The last expression converges to 0 as $k \to \infty$, by the dominated convergence theorem. To justify the application of this theorem, we bound the indicator above by 1, which leads to $\min(1, h_n^{(2)}(t,x))$. To see that $h_n^{(2)}(t,x) < \infty$ a.s., we apply Remark B.3.2 to $X = (\Gamma_i)$ and $Y = \{(T_i), (X_i)\}$, noting that, by (6.2.7) and (6.2.10),

$$\mathbb{E}[\big(h_n^{(2)}(t,x)\big)^{p/2} | (\Gamma_i)] \leq \mathbb{E}[h_n^{(p)}(t,x) | (\Gamma_i)] \leq T^{(p/\alpha - 1)n} K_n^{(p)}(t,x) \left(\sum_{j\geq 1} \Gamma_j^{-p/\alpha}\right)^n < \infty \quad \text{a.s.}$$

Finally, another application of dominated convergence theorem shows that

$$\mathbb{E}[\min(1, |\mathcal{R}_n^{(k)}(t,x)|^2)] = \mathbb{E}\big[\mathbb{E}\big[\min\big(1, |\mathcal{R}_n^{(k)}(t,x)|^2\big) \,|\, (\Gamma_i), (T_i), (X_i)\big]\big] \to 0.$$

∎

Note that, if there exists $p \in (\alpha, 2]$ such that (6.3.6) holds, then the same argument as in the proof of Theorem 6.3.2 shows that

$$\int_0^t \int_{\mathbb{R}^2} G_{t-s}^{\alpha}(x-y) |\mathcal{J}_n^{(k)}(s,y)|^{\alpha} dyds < \infty \quad \text{a.s.}$$

and

$$\int_0^t \int_{\mathbb{R}^2} G_{t-s}^{\alpha}(x-y) |\mathcal{R}_n^{(k)}(s,y)|^{\alpha} dyds < \infty \quad \text{a.s.}$$

Choosing predictable modifications for the processes $Y_{n,k}^{(t,x)}(s,y) = G_{t-s}(x-y)\mathcal{J}_n^{(k)}(s,y)$ and $Z_{n,k}^{(t,x)}(s,y) = G_{t-s}(x-y)\mathcal{R}_n^{(k)}(s,y)$, we infer that these processes are in $L^0(Z)$, by Corollary 2.3.9. By decomposition (6.3.13),

$$I^Z(X_n^{(t,x)}) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathcal{J}_n^{(k)}(s,y) Z(ds,dy) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathcal{R}_n^{(k)}(s,y) Z(ds,dy).$$

Hence, to prove that $I^Z(X_n^{(t,x)}) = I_{n+1}\big(f_{n+1}(\cdot,t,x)\big)$, it suffices to show that the first term converges in probability to $I_{n+1}\big(f_{n+1}(\cdot,t,x)\big)$ as $k \to \infty$, and the second one is negligible. This will be achieved by the following two lemmas.

Lemma 6.3.6. Assume that $\alpha \in [1,2)$ and Assumption 6.0.1 holds. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. If there exists $p \in (\alpha, 2]$ such that $K_{n+1}^{(p)}(t,x) < \infty$ and (6.3.6) holds, then

$$A_k := \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathcal{R}_n^{(k)}(s,y) Z(ds,dy) \xrightarrow{P} 0 \quad \text{as } k \to \infty.$$

Proof. By Proposition 6.1.10, we have the decomposition:

$$A_k = \int_0^t \int_{\mathbb{R}^d} \int_{\{|z| \le 1\}} G_{t-s}(x-y)\mathcal{R}_n^{(k)}(s,y) \frac{z}{\psi(s,y)} \widehat{J}_{\psi}(ds,dy,dz) +$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\{|z| > 1\}} G_{t-s}(x-y)\mathcal{R}_n^{(k)}(s,y) \frac{z}{\psi(s,y)} J_{\psi}(ds,dy,dz) =: T_1^{(k)} + T_2^{(k)}.$$

We treat separately the two terms. Recall that $\psi(s,y) = T^{-1/\alpha}\phi(y)$.

Step 1. First, we treat $T_2^{(k)}$. Since J_ψ has points $\{(T_j, X_j, \varepsilon_j \Gamma_j^{-1/\alpha})\}_{j \ge 1}$, we have

$$T_2^{(k)} = \sum_{j \ge 1} \varepsilon_j \Gamma_j^{-1/\alpha} 1_{\{\Gamma_j^{-1/\alpha} > 1\}} W_{n,j}^{(k)},$$

with

$$W_{n,j}^{(k)} := T^{1/\alpha} G_{t-T_j}(x - X_j)\mathcal{R}_n^{(k)}(T_j, X_j)\phi^{-1}(X_j). \tag{6.3.14}$$

Using the fact that $\Gamma_j^{-1/\alpha} \le \Gamma_j^{-2/\alpha}$ on the event $\{\Gamma_j^{-1/\alpha} > 1\}$, we obtain:

$$\big|T_2^{(k)}\big| \le \sum_{j \ge 1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha} > 1\}} \big|W_{n,j}^{(k)}\big| \le \sum_{j \ge 1} \Gamma_j^{-2/\alpha} \big|W_{n,j}^{(k)}\big|.$$

We use the following fact for any random variable X and sub-σ-field $\mathcal{G}$,

$$\|X\|_{L^0} \leq \big\|\mathbb{E}[|X|\,|\mathcal{G}]\big\|_{L^0}. \tag{6.3.15}$$

This leads to the following inequality:

$$\big\|T_2^{(k)}\big\|_{L^0} \leq \Big\| \sum_{j\geq 1} \Gamma_j^{-2/\alpha} \big|W_{n,j}^{(k)}\big| \,\Big\|_{L^0} \leq \Big\| \sum_{j\geq 1} \Gamma_j^{-2/\alpha} \mathbb{E}\Big[\big|W_{n,j}^{(k)}\big| \,\Big|(\Gamma_i)_i\Big]\Big\|_{L^0}. \tag{6.3.16}$$

We fix $j \geq 1$. By Hölder's inequality for conditional expectation,

$$\mathbb{E}\Big[\big|W_{n,j}^{(k)}\big| \,\Big|(\Gamma_i)_i\Big] \leq \Big(\mathbb{E}\Big[\big|W_{n,j}^{(k)}\big|^\alpha \,\Big|(\Gamma_i)_i\Big]\Big)^{1/\alpha}.$$

Note that $\mathcal{R}_n^{(k)}(T_j, X_j)$ is a series depending on multi-indices $j_1 < \ldots < j_n$ and each term in this series contains the factor $\widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j)$, which vanishes if $j \in \{j_1, \ldots, j_n\}$. So, we can assume that $j \notin \{j_1, \ldots, j_n\}$. Moreover, $\mathcal{R}_n^{(k)}(T_j, X_j)$ is a function of the sequences $(\varepsilon_i)_{i\geq 1}, (\Gamma_i)_{i\geq 1}, (T_i)_{i\geq 1}, (X_i)_{i\geq 1}$.

We apply the basic conditioning fact (6.2.4) to $X = (\Gamma_i)_i$ and $Y = \{(\varepsilon_i)_i, (T_i)_i, (X_i)_i\}$. Using the definition of $\mathcal{R}_n^{(k)}(T_j, X_j)$, we obtain:

$$\mathbb{E}\Big[\big|W_{j,n}^{(k)}\big|^\alpha \,\Big|(\Gamma_i)_i = (\gamma_i)_i\Big] = T\mathbb{E}\Big[G_{t-T_j}^\alpha(x - X_j)\frac{1}{\phi^\alpha(X_j)}$$

$$\Big|T^{n/\alpha} n! \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\gamma_{j_n}^{-1/\alpha} \leq k^{-1}\}} \prod_{i=1}^n \varepsilon_{j_i} \gamma_{j_i}^{-1/\alpha} \phi^{-1}(X_{j_i}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j)\Big|^\alpha\Big].$$

We use the fact that for independent random elements X and Y with values in measurable spaces $(E, \mathcal{E})$, respectively $(F, \mathcal{F})$, and a measurable function $f : E \times F \to [0, \infty]$,

$$\mathbb{E}[f(X, Y)] = \int_E \mathbb{E}[f(x, Y)]\mathbb{P}_X(dx) = \mathbb{E}\left[\int_E f(x, Y)\mathbb{P}_X(dx)\right],$$

where $\mathbb{P}_X$ denotes the law of X. We will use this fact with $X = (T_j, X_j)$ (which has law $T^{-1}\phi^\alpha(y)dsdy$) and $Y = \{(\varepsilon_i)_{i\geq 1}, (T_i, X_i)_{i\neq j}\}$, to compute the previous expectation:

$$\mathbb{E}\Big[\big|W_{j,n}^{(k)}\big|^\alpha \,\Big|(\Gamma_i)_i = (\gamma_i)_i\Big] = T\,\mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^d} G_{t-s}^\alpha(x - y)\frac{1}{\phi^\alpha(y)}$$

$$\Big|T^{\frac{n}{\alpha}} n! \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\gamma_{j_n}^{-1/\alpha} \leq k^{-1}\}} \prod_{i=1}^n \varepsilon_{j_i} \gamma_{j_i}^{-1/\alpha} \phi^{-1}(X_{j_i}) \widetilde{f}_n(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, s, y)\Big|^\alpha T^{-1}\phi^\alpha(y)dyds\Bigg].$$

Using again the basic conditioning fact (6.2.4) for the expectation above, we infer that

$$\mathbb{E}\Big[\big|W_{j,n}^{(k)}\big|^\alpha \,\Big|(\Gamma_i)_i = (\gamma_i)_i\Big] = \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^d} G_{t-s}^\alpha(x - y)$$

$$\left.\left|T^{\frac{n}{\alpha}}n! \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \varepsilon_{j_i}\Gamma_{j_i}^{-1/\alpha}\phi^{-1}(X_{j_i})\widetilde{f}_n(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},s,y)\right|^{\alpha} \right|(\Gamma_i)_i=(\gamma_i)_i\Bigg].$$

In summary, we have proved the following non-trivial fact:

$$A_{n,j}^{(k)} := \mathbb{E}\Big[|W_{j,n}^{(k)}|^{\alpha}\,\Big|(\Gamma_i)_i\Big] = \mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)|\mathcal{R}_{n,j}^{(k)}(s,y)|^{\alpha}dyds\,\Big|(\Gamma_i)_i\right], \tag{6.3.17}$$

where

$$\mathcal{R}_{n,j}^{(k)}(s,y) = T^{n/\alpha}n! \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \varepsilon_{j_i}\Gamma_{j_i}^{-1/\alpha}\phi^{-1}(X_{j_i})\widetilde{f}_n(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},s,y).$$

We continue with the estimation of $A_{n,j}^{(k)}$. Using a double conditioning argument for the term on the right hand side of (6.3.17), we see that $A_{n,j}^{(k)} = \mathbb{E}\big[Q_{n,j}^{(k)}\,\big|(\Gamma_i)_i\big]$, where

$$Q_{n,j}^{(k)} := \int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)\mathbb{E}\left[|\mathcal{R}_{n,j}^{(k)}(s,y)|^{\alpha}\,\Big|(\Gamma_i)_i,(T_i)_i,(X_i)_i\right]dyds.$$

Using Hölder's inequality for conditional expectation, we have:

$$\mathbb{E}\left[|\mathcal{R}_{n,j}^{(k)}(s,y)|^{\alpha}\,\Big|(\Gamma_i)_i,(T_i)_i,(X_i)_i\right] \leq \left(\mathbb{E}\left[|\mathcal{R}_{n,j}^{(k)}(s,y)|^{2}\,\Big|(\Gamma_i)_i,(T_i)_i,(X_i)_i\right]\right)^{\frac{\alpha}{2}} =: \left(h_{n,j}^{(k)}(s,y)\right)^{\frac{\alpha}{2}}.$$

We obtain that:

$$\begin{aligned} Q_{n,j}^{(k)} &\leq \int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)\left(h_{n,j}^{(k)}(s,y)\right)^{\frac{\alpha}{2}}dyds \\ &\leq C_t^{1-\frac{\alpha}{p}}\left(\int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)\big(h_{n,j}^{(k)}(s,y)\big)^{p/2}dyds\right)^{\frac{\alpha}{p}}, \end{aligned}$$

where for the last line we applied Jensen's inequality (6.3.8), exactly as before. Hence,

$$\begin{aligned} A_{n,j}^{(k)} &\leq C_t^{1-\frac{\alpha}{p}}\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)\big(h_{n,j}^{(k)}(s,y)\big)^{p/2}dyds\right)^{\frac{\alpha}{p}}\,\Bigg|(\Gamma_i)_i\right] \\ &\leq C_t^{1-\frac{\alpha}{p}}\left(\int_0^t\int_{\mathbb{R}^d} G_{t-s}^{\alpha}(x-y)\mathbb{E}\big[\big(h_{n,j}^{(k)}(s,y)\big)^{p/2}\,\big|(\Gamma_i)_i\big]dyds\right)^{\alpha/p}, \end{aligned} \tag{6.3.18}$$

where for the last line we used Jensen's inequality (6.2.16), and we switched the $dyds$ integral with the conditional expectation.

To continue the previous estimation, we need to evaluate $h_{n,j}^{(k)}(s,y)$, which is in fact very similar to $h_n^{(2)}(s,y)$ (see (6.2.3)):

$$h_{n,j}^{(k)}(s.y) = T^{2n/\alpha}(n!)^2 \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha}\phi^{-2}(X_{j_i})\widetilde{f}_n^2(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},s,y).$$

We take power $p/2$, which we then move inside the sum, by sub-additivity. Dropping the restriction $j \notin \{j_1, \ldots, j_n\}$, and recalling definition (6.2.9) of $A^{(p)}_{j_1,\ldots,j_n}(s,y)$, we obtain:

$$\mathbb{E}\big[\big(h^{(k)}_{n,j}(s,y)\big)^{p/2}\,\big|(\Gamma_i)\big] \leq T^{pn/\alpha} \sum_{j_1<\ldots<j_n} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \Gamma_{j_i}^{-p/\alpha} A^{(p)}_{j_1,\ldots,j_n}(s,y)$$
$$= n! T^{(p/\alpha-1)n} \sum_{j_1<\ldots<j_n} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \Gamma_{j_i}^{-p/\alpha} K^{(p)}_n(s,y),$$

where for the last line, we used the expression of $A^{(p)}_{j_1,\ldots,j_n}(s,y)$ given by Lemma 6.2.5. Plugging this into (6.3.18), we obtain:

$$A^{(k)}_{n,j} \leq C_t^{1-\frac{\alpha}{p}} \left(T^{(p/\alpha-1)n} n! \sum_{j_1<\ldots<j_n} 1_{\{\Gamma_{j_n}^{-1/\alpha}\leq k^{-1}\}} \prod_{i=1}^{n} \Gamma_{j_i}^{-p/\alpha} \right)^{\frac{\alpha}{p}} \left(\int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y) K^{(p)}_n(s,y) dy ds \right)^{\frac{\alpha}{p}}.$$

By the dominated convergence theorem, the first factor converges to 0 a.s. as $k \to \infty$. The second factor is a constant, that we denote C_t'. Therefore $A^{(k)}_{n,j} \to 0$ a.s. as $k \to \infty$. Moreover, $A^{(k)}_{n,j} \leq C_t^{1-\frac{\alpha}{p}} \big[T^{(p/\alpha-1)n}\big(\sum_{j\geq 1} \Gamma_j^{-p/\alpha}\big)^n\big]^{\alpha/p} C_t'$, due to inequality (6.2.11). By another application of the dominated convergence theorem,

$$\sum_{j\geq 1} \Gamma_j^{-2/\alpha} A^{(k)}_{n,j} \to 0 \quad \text{a.s.} \quad \text{as } k \to \infty.$$

In particular, the previous sum converges in probability to 0, as $k \to \infty$. Recalling inequality (6.3.16), we infer that $T_2^{(k)} \xrightarrow{P} 0$ as $k \to \infty$, which concludes *Step 1.*

Step 2. Next, we treat $T_1^{(k)}$. Note that for any (t_0, x_0) fixed, the process

$$M_t^{(t_0,x_0)} := \int_0^t \int_{\mathbb{R}^d} \int_{\{|z|\leq 1\}} G_{t_0-s}(x_0-y) \mathcal{R}^{(k)}_n(s,y) \frac{z}{\psi(s,y)} \widehat{J}_{\psi}(ds,dy,dz), \quad t \in [0,T],$$

is a local martingale. By Lenglart's inequality, for any $\varepsilon > 0$ and $\eta > 0$,

$$\mathbb{P}(|M_t^{(t_0,x_0)}| > \varepsilon) \leq \frac{\eta}{\varepsilon^2} + \mathbb{P}\left(\int_0^t \int_{\mathbb{R}^d} \int_{\{|z|\leq 1\}} G^2_{t_0-s}(x_0-y) |\mathcal{R}^{(k)}_n(s,y)|^2 \frac{z^2}{\psi^2(s,y)} J_{\psi}(ds,dy,dz) > \varepsilon \right).$$

We apply this inequality to $(t_0, x_0) = (t, x)$. Using the points of J_ψ, we obtain:

$$\mathbb{P}(|T_1^{(k)}| > \varepsilon) \leq \frac{\eta}{\varepsilon^2} + \mathbb{P}\left(\sum_{j\geq 1} G^2_{t-T_j}(x-X_j) |\mathcal{R}^{(k)}_{n,j}(T_j,X_j)|^2 1_{\{\Gamma_j^{-1/\alpha}\leq 1\}} \frac{\Gamma_j^{-2/\alpha}}{T^{-2/\alpha}\phi^2(X_j)} > \varepsilon \right).$$

We will show below that:

$$T_3^{(k)} := \sum_{j\geq 1} G^2_{t-T_j}(x-X_j) |\mathcal{R}^{(k)}_{n,j}(T_j,X_j)|^2 1_{\{\Gamma_j^{-1/\alpha}\leq 1\}} \frac{\Gamma_j^{-2/\alpha}}{T^{-2/\alpha}\phi^2(X_j)} \xrightarrow{P} 0 \quad \text{as} \quad k \to \infty. \tag{6.3.19}$$

It will follow that $\limsup_{k\to\infty} \mathbb{P}(|T_k^{(1)}| > \varepsilon) \le \eta/\varepsilon^2$ for any $\varepsilon > 0$ and $\eta > 0$. Letting $\eta \to 0$, we infer that $\lim_{k\to\infty} \mathbb{P}(|T_k^{(1)}| > \varepsilon) = 0$ for any $\varepsilon > 0$, i.e. $T_k^{(1)} \xrightarrow{P} 0$ as $k \to \infty$.

It remains to prove (6.3.19). We proceed as for $T_1^{(k)}$. Recalling definition (6.3.14) of $W_{n,j}^{(k)}$, we see that

$$T_3^{(k)} = \sum_{j\ge1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}\le 1\}} \big(W_{n,j}^{(k)}\big)^2.$$

We use again inequality (6.3.15), with $\mathcal{G}$ the σ-field generated by $(\Gamma_i)_i, (T_i)_i, (X_i)_i$. We obtain:

$$\|T_3^{(k)}\|_{L^0} \le \Big\| \sum_{j\ge1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}\le 1\}} B_{n,j}^{(k)} \Big\|_{L^0}, \tag{6.3.20}$$

where

$$\begin{aligned} B_{n,j}^{(k)} &:= \mathbb{E}\big[\big(W_{n,j}^{(k)}\big)^2 \,\big|(\Gamma_i)_i, (T_i)_i, (X_i)_i\big] \\ &= T^{2/\alpha} G_{t-T_j}^2(x - X_j)\phi^{-2}(X_j)\, \mathbb{E}\Big[\big|\mathcal{R}_n^{(k)}(T_j, X_j)\big|^2 \,\big|(\Gamma_i)_i, (T_i)_i, (X_i)_i\Big]. \end{aligned}$$

Similarly to (6.2.3),

$$\begin{aligned} &\mathbb{E}\Big[\big|\mathcal{R}_n^{(k)}(T_j, X_j)\big|^2 \,\big|(\Gamma_i)_i, (T_i)_i, (X_i)_i\Big] \\ &\quad = T^{\frac{2n}{\alpha}} (n!)^2 \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} 1_{\{\Gamma_{j_n}^{-1/\alpha}\le k^{-1}\}} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha} \phi^{-2}(X_{j_i}) \widetilde{f}_n^2(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j) \\ &\quad \le h_n^{(2)}(T_j, X_j), \end{aligned}$$

where for the last line, we just bounded $1_{\{\Gamma_{j_n}^{-1/\alpha}\le k^{-1}\}}$ by 1. Therefore,

$$B_{n,j}^{(k)} \le T^{2/\alpha} G_{t-T_j}^2(x - X_j)\phi^{-2}(X_j) h_n^{(2)}(T_j, X_j).$$

We will prove that

$$B_{n,j}^{(k)} \to 0 \quad \text{a.s.} \quad \text{as } k \to \infty, \text{ for any } j \ge 1, \tag{6.3.21}$$

and

$$S := \sum_{j\ge1} \Gamma_j^{-2/\alpha} G_{t-T_j}^2(x - X_j)\phi^{-2}(X_j) h_n^{(2)}(T_j, X_j) < \infty \quad \text{a.s.} \tag{6.3.22}$$

Then, by the dominated convergence theorem, $\sum_{j\ge1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}\le 1\}} B_{n,j}^{(k)} \to 0$ a.s. as $k \to \infty$. Consequently, by (6.3.20), $\|T_3^{(k)}\|_0 \to 0$ as $k \to \infty$. This proves (6.3.19).

We now prove (6.3.21). We fix $j \ge 1$. It is enough to prove that

$$\mathbb{E}\Big[\big|\mathcal{R}_n^{(k)}(T_j, X_j)\big|^2 \,\big|(\Gamma_i)_i, (T_i)_i, (X_i)_i\Big] \to 0 \quad \text{as} \quad k \to \infty.$$

This follows from the dominated convergence theorem, provided that we show that

$$\mathcal{M}_2 := (n!)^2 \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha} \phi^{-2}(X_{j_i}) \widetilde{f}_n^2(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j) < \infty \quad \text{a.s.}$$

For this, we proceed as in the proof of Proposition 6.2.4. By sub-additivity,

$$\mathcal{M}_2^{p/2} \leq (n!)^p \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \Gamma_{j_i}^{-p/\alpha} \phi^{-p}(X_{j_i}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j) =: \mathcal{M}_p.$$

So, it is enough to prove that $\mathcal{M}_p < \infty$ a.s. Note that $\mathcal{M}$ is a measurable function of independent random elements $X = \{(\Gamma_i)_i, T_j, X_i\}$ and $Y = \{(T_i)_{i\neq j}, (X_i)_{i\neq j}\}$. Using Lemma B.3.1, it suffices to prove that

$$\mathbb{E}\big[\mathcal{M}_p \,\big|(\Gamma_i)_i, T_j, X_j\big] < \infty \quad \text{a.s.}$$

Using the basic conditioning fact (6.2.4), we see that

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{M}_p \,\big|(\Gamma_i)_i = (\gamma_i)_i, T_j = t_j, X_j = x_j\big] \\
&\quad = (n!)^p \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \gamma_{j_i}^{-p/\alpha} \mathbb{E}\Big[\prod_{i=1}^{n} \phi^{-1}(X_{j_i}) \widetilde{f}_n^p(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, t_j, x_j)\Big] \\
&\quad = \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \gamma_{j_i}^{-p/\alpha} \cdot \frac{n!}{T^n} K_n^{(p)}(t_j, x_j),
\end{aligned}$$

using Lemma 6.2.5 for the last line. Hence,

$$\begin{aligned}
\mathbb{E}\big[\mathcal{M}_p \,\big|(\Gamma_i)_i, T_j, X_j\big] &= \frac{n!}{T^n} K_n^{(p)}(T_j, X_i) \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \Gamma_{j_i}^{-p/\alpha} \\
&\leq \frac{1}{T^n} K_n^{(p)}(T_j, X_i) \left(\sum_{j\geq 1} \Gamma_j^{-p/\alpha} \right)^n < \infty \quad \text{a.s.}
\end{aligned}$$

Finally, we prove (6.3.22). We denote by $\overline{f}_n^{(2)}(\cdot, t, x)$ the symmetrization of $f_n^{(2)}(\cdot, t, x)$. Note that $\widetilde{f}_n^2(\cdot, t, x) \leq \frac{1}{n!} \overline{f}_n^{(2)}(\cdot, t, x)$. Therefore,

$$h_n^{(2)}(T_j, X_j) \leq T^{\frac{2n}{\alpha}} n! \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha} \phi^{-2}(X_{j_i}) \overline{f}_n^{(2)}(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j)$$

and

$$S \leq T^{\frac{2n}{\alpha}} \sum_{j\geq 1} \Gamma_j^{-2/\alpha} \phi^{-2}(X_j) G_{t-T_j}^2(x - X_j)$$

$$\sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \sum_{\pi\in\Sigma_n} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha}\phi^{-2}(X_{j_i}) f_n^2(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n)}}, X_{j_{\pi(n)}}, T_j, X_j)$$
$$= T^{\frac{2n}{\alpha}} \sum_{j\geq 1} \Gamma_j^{-2/\alpha}\phi^{-2}(X_j) G^2_{t-T_j}(x-X_j)$$
$$\sum_{\substack{j_1,\ldots,j_n\geq 1 \text{ distinct} \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^{n} \Gamma_{j_i}^{-2/\alpha}\phi^{-2}(X_{j_i}) f_n^2(T_{j_1}, X_{j_1}, \ldots, T_{j_n}, X_{j_n}, T_j, X_j),$$

where for the last line we used (6.3.11). We now use the fact that $G^2_{t-T_j}(x-X_j) f_n^2(\cdot, T_j, X_j) = f^2_{n+1}(\cdot, T_j, X_j, t, x)$. Denoting $j = j_{n+1}$, it follows that

$$S \leq T^{\frac{2n}{\alpha}} \sum_{j_1<\ldots<j_{n+1}} \prod_{i=1}^{n+1} \Gamma_{j_i}^{-2/\alpha}\phi^{-2}(X_{j_i}) f^2_{n+1}(T_{j_1}, X_{j_1}, \ldots, T_{j_{n+1}}, X_{j_{n+1}}, t, x)$$
$$= T^{\frac{2n}{\alpha}} \sum_{j_1<\ldots<j_{n+1}} \sum_{\pi\in\Sigma_{n+1}} \prod_{i=1}^{n+1} \Gamma_{j_i}^{-2/\alpha}\phi^{-2}(X_{j_i}) f^2_{n+1}(T_{j_{\pi(1)}}, X_{j_{\pi(1)}}, \ldots, T_{j_{\pi(n+1)}}, X_{j_{\pi(n+1)}}, t, x),$$

where for the last line we used again (6.3.11). Taking power $p/2$, and recalling definition (6.3.12), we obtain that $S^{p/2} \leq B^{(p)}_{n+1}(t,x) < \infty$ a.s. ∎

Lemma 6.3.7. Assume that $\alpha \in [1,2)$. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. If there exists $p \in (\alpha, 2]$ such that $K^{(p)}_{n+1}(t,x) < \infty$ and (6.3.6) holds, then

$$B_k := \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathcal{J}_n^{(k)}(s,y) Z(ds,dy) \xrightarrow{P} I_{n+1}\big(f_{n+1}(\cdot,t,x)\big) \quad \text{as } k\to\infty.$$

Proof. Applying Proposition 6.1.11 with $a = k^{-1}$, we have:

$$B_k = \int_0^t \int_{\mathbb{R}^d} \int_{\{|z|\leq k^{-1}\}} G_{t-s}(x-y)\mathcal{J}_n^{(k)}(s,y) \frac{z}{\psi(s,y)} \widehat{J}_\psi(ds,dy,dz) +$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\{|z|> k^{-1}\}} G_{t-s}(x-y)\mathcal{J}_n^{(k)}(s,y) \frac{z}{\psi(s,y)} J_\psi(ds,dy,dz) =: S_1^{(k)} + S_2^{(k)}.$$

We will prove that:

$$S_1^{(k)} \xrightarrow{P} 0 \quad \text{as} \quad k\to\infty, \tag{6.3.23}$$
$$S_2^{(k)} \xrightarrow{P} I_{n+1}\big(f_{n+1}(\cdot,t,x)\big) \quad \text{as} \quad k\to\infty. \tag{6.3.24}$$

We prove (6.3.23). As in the proof of Lemma 6.3.6 (Step 2), by Lenglart's inequality, it suffices to prove that:

$$S_3^{(k)} := \sum_{j\geq 1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}<k^{-1}\}} G^2_{t-T_j}(x-X_j) |\mathcal{J}_n^{(k)}(s,y)|^2 \phi^{-2}(X_j) \xrightarrow{P} 0 \quad \text{as} \quad k\to\infty. \tag{6.3.25}$$

For this, denote $V_{n,j}^{(k)} := G_{t-T_j}(x-X_j)|\mathcal{J}_{n,j}^{(k)}(T_j,X_j)|\phi^{-2}(X_j)$. Using again inequality (6.3.15), we obtain:

$$\|S_3^{(k)}\|_{L^0} \leq \Big\| \sum_{j\geq 1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}\leq k^{-1}\}} D_{n,j}^{(k)} \Big\|_{L^0},$$

where

$$\begin{aligned} D_{n,j}^{(k)} &:= \mathbb{E}\big[\big(V_{n,j}^{(k)}\big)^2 \,\big| (\Gamma_i)_i, (T_i)_i, (X_i)_i\big] \\ &= G_{t-T_i}^2(x-X_i)\phi^{-2}(X_i)\,\mathbb{E}\big[\big|\mathcal{J}_n^{(k)}(T_j,X_j)\big|^2 \,\big| (\Gamma_i)_i, (T_i)_i, (X_i)_i\big]. \end{aligned}$$

By direct calculation,

$$\begin{aligned} &\mathbb{E}\big[\big|\mathcal{J}_n^{(k)}(T_j,X_j)\big|^2 \,\big| (\Gamma_i)_i, (T_i)_i, (X_i)_i\big] = \\ &(n!)^2 \sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^n \Gamma_{j_i}^{-2/\alpha} 1_{\{\Gamma_{j_i}^{-1/\alpha}>k^{-1}\}} \phi^{-2}(X_{j_i}) \widetilde{f}_n^2(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},T_j,X_j), \end{aligned}$$

which we can bound by $T^{-\frac{2n}{\alpha}} h_n^{(2)}(T_j,X_j)$, using the fact that $1_{\{\Gamma_{j_i}^{-1/\alpha}>k^{-1}\}} \leq 1$. Hence,

$$\|S_3^{(k)}\|_{L^0} \leq \left\| T^{-\frac{2n}{\alpha}} \sum_{j\geq 1} \Gamma_j^{-2/\alpha} 1_{\{\Gamma_j^{-1/\alpha}\leq k^{-1}\}} G_{t-T_j}^2(x-X_j)\phi^{-2}(X_j) h_n^{(2)}(T_j,X_j) \right\|_{L^0}.$$

By the dominated convergence theorem, the last series converges to 0 a.s. as $k\to\infty$. The application of this theorem is justified due to (6.3.22). This proves (6.3.23).

We prove (6.3.24). Using the points of J_ψ, we write:

$$S_2^{(k)} = \sum_{j\geq 1} G_{t-T_j}(x-X_j)\mathcal{J}_n^{(k)}(T_j,X_j) \frac{\varepsilon_j \Gamma_j^{-1/\alpha}}{\psi(T_j,X_j)} 1_{\{\Gamma_j^{-1/\alpha}>k^{-1}\}}.$$

Recalling the definition of $\mathcal{J}_n^{(k)}(s,y)$, we see that

$$\begin{aligned} S_2^{(k)} = n! &\sum_{j\geq 1} G_{t-T_j}(x-X_j)\varepsilon_j \phi^{-1}(X_j)\Gamma_j^{-1/\alpha} 1_{\{\Gamma_j^{-1/\alpha}>k^{-1}\}} \\ &\sum_{\substack{j_1<\ldots<j_n \\ j\notin\{j_1,\ldots,j_n\}}} \prod_{i=1}^n \varepsilon_{j_i} \Gamma_j^{-1/\alpha} 1_{\{\Gamma_{j_i}^{-1/\alpha}>k^{-1}\}} \phi^{-1}(X_{j_i}) \widetilde{f}_n(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},T_j,X_j). \end{aligned}$$

Using the same argument as in the proof of Theorem 6.3.4, with $\Gamma_j^{-1/\alpha}$ replaced by $\Gamma_j^{-1/\alpha} 1_{\{\Gamma_j^{-1/\alpha}>k^{-1}\}}$. We obtain:

$$\begin{aligned} S_2^{(k)} &= (n+1)! T^{\frac{n+1}{\alpha}} \sum_{j_1<\ldots<j_{n+1}} \prod_{i=1}^{n+1} \varepsilon_{j_i} \Gamma_{j_i}^{-1/\alpha} 1_{\{\Gamma_{j_i}^{-1/\alpha}>k^{-1}\}} \phi^{-1}(X_{j_i}) \widetilde{f}_{n+1}(T_{j_1},X_{j_1},\ldots,T_{j_n},X_{j_n},t,x) \\ &= \mathcal{J}_{n+1}^{(k)}(t,x) \xrightarrow{P} I_{n+1}\big(f_{n+1}(\cdot,t,x)\big), \end{aligned}$$

since $I_{n+1}\big(f_{n+1}(\cdot,t,x) - \mathcal{J}_{n+1}^{(k)}(t,x) = \mathcal{R}_{n+1}^{(k)}(t,x) \xrightarrow{P} 0$ as $k\to\infty$, by Lemma 6.3.5. ∎

6.4 Solvability

In this section, we give the proof of Theorem 6.0.4.(b). More precisely, we show that the process $\{u(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ defined by (6.2.1) is indeed a solution of equation (6.0.1). This will be achieved in Theorem 6.4.1, by letting $n \to \infty$ in the recurrence relation (6.0.10).

The last statement of Theorem 6.0.4.(b) does not require any effort: relation (6.0.14) is a restatement of the fact that $v^{(t,x)} \in L^0(Z)$, where

$$v^{(t,x)}(s,y) = G_{t-s}(x-y)u(s,y), \quad s \in [0,T], y \in \mathbb{R}^d,$$

according to Corollary 2.3.9, while the fact that $u(t,x)$ has representation (6.0.15) follows from definition (6.2.1) of $u(t,x)$, combined with the series representation (6.2.2) of $I_n\big(f_n(\cdot,t,x)\big)$.

Theorem 6.4.1. Suppose that Assumption 6.0.1 holds. If Assumptions 6.0.2 and 6.0.3 hold (with possibly different values $p \in (\alpha, 2]$), then for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $v^{(t,x)} \in L^0(Z)$ and

$$u(t,x) = 1 + I^Z(v^{(t,x)}) \quad a.s. \tag{6.4.1}$$

Proof. *Step 1.* Denote $v_n^{(t,x)}(s,y) = G_{t-s}(x-y)u_n(s,y)$. In this step, we show that:

$$v^{(t,x)} \in L^0(Z) \quad \text{and} \quad \|v_n^{(t,x)} - v^{(t,x)}\|_Z \to 0. \tag{6.4.2}$$

By Proposition 6.2.6, for any $(s,y) \in [0,t] \times \mathbb{R}^d$, $\sum_{n \geq 1} |I_n(f_n(\cdot,s,y))| < \infty$ a.s., and so,

$$\begin{aligned} v_n^{(t,x)}(s,y) - v^{(t,x)}(s,y) &= G_{t-s}(x-y)\big(u_n(s,y) - u(s,y)\big) \\ &= G_{t-s}(x-y) \sum_{k \geq n+1} I_k(f_k(\cdot,s,y)) \to 0 \quad \text{a.s.} \end{aligned}$$

Notice that we have the following natural dominator:

$$|v_n^{(t,x)}(s,y)| \leq G_{t-s}(x-y) \sum_{k=1}^{n} |I_k(f_k(\cdot,s,y))| \leq G_{t-s}(x-y) \sum_{n \geq 1} |I_n(f_n(\cdot,s,y))| =: \overline{v}^{(t,x)}(s,y).$$

Recalling our convention (2.3.10), the desired conclusion (6.4.2) will follow by Theorem 2.3.7, provided that we show that $\overline{v}^{(t,x)} \in L^0(Z)$, which is equivalent to: (see Corollary 2.3.9)

$$\int_0^t \int_{\mathbb{R}^d} |\overline{v}^{(t,x)}(s,y)|^\alpha dy ds < \infty \quad \text{a.s.} \tag{6.4.3}$$

To prove (6.4.3), we will use Minkowski's inequality in $L^\alpha([0,t] \times \mathbb{R}^d)$. Recall that for a measure space $(E, \mathcal{E}, \mu)$ and a measurable function $f: E \to \mathbb{R}$,

$$\|f\|_{L^\alpha(E)} = \int_E |f|^\alpha d\mu \quad \text{if } \alpha \leq 1 \quad \text{and} \quad \|f\|_{L^\alpha(E)} = \left(\int_E |f|^\alpha d\mu\right)^{1/\alpha} \quad \text{if } \alpha > 1.$$

Hence,

$$\|\overline{v}^{(t,x)}\|_{L^\alpha([0,t]\times\mathbb{R}^d)} = \left\| \sum_{n\geq 1} G_{t-*}(x-*) I_n\big(f_n(\cdot,*)\big)\right\|_{L^\alpha([0,t]\times\mathbb{R}^d)} \leq \sum_{n\geq 1} |\mathcal{I}_n(t,x)|^{\frac{1}{\alpha\vee 1}}.$$

By Theorem 6.3.2, the last series converges almost surely.

Step 2. In this step, we show that (6.4.1) holds almost surely. For this, we let $n\to\infty$ in (6.0.10). On the left hand-side, $u_{n+1}(t,x)\to u(t,x)$ almost surely, by Proposition 6.2.6. The term on the right hand-side of (6.0.10) is equal to $1+I^Z(v_n^{(t,x)})$, which converges in probability to $1+I^Z(v^{(t,x)})$, due to (6.4.2) and property (2.3.13) of integral I^Z. ∎

6.5 Applications: the heat and wave equations

In this section, we give the proof of Theorem 6.0.7. We consider separately the heat and wave equations.

6.5.1 Heat equation

In this section, we show that Assumptions 6.0.2 and 6.0.3 are satisfied in the case of the heat equation.

The following result gives an estimate for $K_n^{(p)}(t,x)$ in the case of the heat equation.

Lemma 6.5.1. Suppose that ϕ satisfies Assumption 6.0.5. In the case of the heat equation, for any $\alpha<p<1+\frac{2}{d}$, we have:

$$K_n^{(p)}(t,x) \leq c_0^{n(p-\alpha)} C_{\eta,p,d}^n \left\{1+|x|^{n\eta}+t^{n\eta/2}\Gamma\left(\frac{1+n\eta}{2}\right)\right\} \frac{t^{n(\frac{d(1-p)}{2}+1)}}{\Gamma(n(\frac{d(1-p)}{2}+1)+1)},$$

where $\eta=\delta(p-\alpha)$, and $C_{\eta,p,d}$ is given by (4.1.3).

Proof. Recall definition (6.0.11) of $K_n^{(p)}(t,x)$. Using (6.0.16), we see that

$$\phi^{\alpha-p}(x) \leq c_0^{p-\alpha}(1+|x|^{\delta(p-\alpha)}) \quad \text{for any } x\in\mathbb{R}^d. \tag{6.5.1}$$

The conclusion follows by Lemma 4.1.3. ∎

To find upper and lower bounds for the Gamma functions appearing in the above estimate, we use Stirling's formula. For any $a>0$, $\Gamma(an+1)\sim a^{an+1/2}(2\pi n)^{(1-a)/2}(n!)^a$, and hence

$$C_a^{-n}(n!)^a \leq \Gamma(an+1) \leq C_a^n (n!)^a \quad \text{for all } n\geq 1, \tag{6.5.2}$$

where $C_a > 1$ is a constant depending on a. Moreover, for any $a > 0$ and $b \in \mathbb{R}$, $\Gamma(an+1+b) \sim \Gamma(an+1)n^b$, and hence

$$C_{a,b}^{-n}(n!)^a \le \Gamma(an+1+b) \le C_{a,b}^n (n!)^a \quad \text{for all } n \ge 1. \tag{6.5.3}$$

where $C_{a,b} > 1$ is a constant depending on a and b.

The following result shows that Assumptions 6.0.2 and 6.0.3 are satisfied in the case of the heat equation.

Proposition 6.5.2. Suppose that ϕ satisfies Assumption 6.0.5. If $\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$ is the heat operator and $\alpha < 1 + \frac{2}{d}$, then (6.0.12) and (6.0.13) hold for any $(t,x) \in [0,T] \times \mathbb{R}^d$, and for any $\alpha < p < 1 + \frac{2}{d}$ such that

$$0 < \delta(p-\alpha) < d(1-p) + 2. \tag{6.5.4}$$

Proof. We use the estimate for $K_n^{(p)}(t,x)$ given by Lemma 6.5.1. Recall that $\eta = \delta(p-\alpha)$ and $C_{\eta,p,d}$ is given by (4.1.3).

Step 1. We first verify (6.0.12). By (6.5.2) and (6.5.3), there exist some constants $C_{d,p} > 1$ and $C_\eta > 1$ such that for all $n \ge 1$,

$$\Gamma\Big(n\Big(\frac{d(1-p)}{2}+1\Big)+1\Big) \ge C_{d,p}^{-n}(n!)^{\frac{d(1-p)}{2}+1} \quad \text{and} \quad \Gamma\Big(\frac{1+n\eta}{2}\Big) \le C_\eta^n (n!)^{\eta/2}.$$

Hence,

$$K_n^{(p)}(t,x) \le C^n\big(1+|x|^{n\eta}+t^{n\eta/2}\big)\frac{t^{n(\frac{d(1-p)}{2}+1)}}{(n!)^{\frac{d(1-p)-\eta}{2}+1}},$$

with $C = c_0^{p-\alpha}C_{\eta,p,d}C_{d,p}C_\eta$. It follows that

$$\sum_{n\ge1}\Big(T^{(\frac{p}{\alpha}-1)n}K_n^{(p)}(t,x)\Big)^{1/2}\Big(\sum_{j\ge1}\Gamma_j^{-p/\alpha}\Big)^{n/2} \le$$
$$\sum_{n\ge1} C^{n/2}\big(1+|x|^{n\eta}+t^{n\eta/2}\big)^{1/2}\frac{T^{\frac{n}{2}(\frac{d(1-p)}{2}+\frac{p}{\alpha})}}{(n!)^{\frac{1}{2}(\frac{d(1-p)-\eta}{2}+1)}}\Big(\sum_{j\ge1}\Gamma_j^{-p/\alpha}\Big)^{n/2}.$$

The last series converges provided that $0 < \eta < d(1-p)+2$. This proves (6.0.12).

Step 2. Next, we verify (6.0.13). Using Lemma 6.5.1, we have:

$$T^{(\frac{p}{\alpha}-1)n}\int_0^t\int_{\mathbb{R}^d} G_{t-s}^\alpha(x-y)K_n^{(p)}(s,y)dyds \le c_0^{n(p-\alpha)}C_{\eta,p,d}^n\frac{T^{n(\frac{p}{\alpha}+\frac{d(1-p)}{2})}}{\Gamma(n(\frac{d(1-p)}{2}+1)+1)}$$
$$\left\{\int_0^t\int_{\mathbb{R}^d} G_{t-s}^\alpha(x-y)(1+|y|^{n\eta})dyds + t^{n\eta/2}\Gamma\left(\frac{1+n\eta}{2}\right)\int_0^t\int_{\mathbb{R}^d} G_{t-s}^\alpha(x-y)dyds\right\}.$$

We use Lemmas 4.1.1 and 4.1.2 to estimate the two integrals above. We get:

$$T^{(\frac{p}{\alpha}-1)n}\int_0^t\int_{\mathbb{R}^d}G^{\alpha}_{t-s}(x-y)K^{(p)}_n(s,y)dyds \leq c_0^{n(p-\alpha)}C^n_{\eta,p,d}\frac{T^{n(\frac{p}{\alpha}+\frac{d(1-p)}{2})}}{\Gamma(n(\frac{d(1-p)}{2}+1)+1)}$$
$$\left\{K_{\alpha,d}t^{\frac{d(1-\alpha)}{2}+1}+C'_{n\eta,\alpha,d}t^{\frac{d(1-\alpha)}{2}+1}(|x|^{n\eta}+t^{n\eta/2})+K_{\alpha,d}\Gamma\left(\frac{1+n\eta}{2}\right)t^{\frac{n\eta+d(1-\alpha)}{2}+1}\right\}.$$

Recalling definition (4.1.2) of $C'_{\gamma,p,d}$, and denoting

$$C^*_{\gamma,p,d}=\frac{C'_{\gamma,p,d}}{K_{p,d}}=(2^{\gamma-1}\vee 1)(1\wedge p)^{-\gamma/2}\left[1+\frac{2^{\gamma/2}}{\Gamma(d/2)}\Gamma\left(\frac{\gamma+d}{2}\right)\right], \qquad (6.5.5)$$

we obtain:

$$T^{(\frac{p}{\alpha}-1)n}\int_0^t\int_{\mathbb{R}^d}G^{\alpha}_{t-s}(x-y)K^{(p)}_n(s,y)dyds \leq c_0^{n(p-\alpha)}C^n_{\eta,p,d}\frac{T^{n(\frac{p}{\alpha}+\frac{d(1-p)}{2})}}{\Gamma(n(\frac{d(1-p)}{2}+1)+1)}$$
$$K_{\alpha,d}\left\{t^{\frac{d(1-\alpha)}{2}+1}\left[1+C^*_{n\eta,\alpha,d}(|x|^{n\eta}+t^{n\eta/2})\right]+\Gamma\left(\frac{1+n\eta}{2}\right)t^{\frac{n\eta+d(1-\alpha)}{2}+1}\right\}.$$

Using inequalities (6.5.2) and (6.5.3), we obtain the estimates: $C^*_{n\eta,\alpha,d}\leq C^n(n!)^{\eta/2}$,

$$\Gamma\left(n\left(\frac{d(1-p)}{2}+1\right)+1\right)\geq C^{-n}(n!)^{\frac{d(1-p)}{2}+1} \quad \text{and} \quad \Gamma\left(\frac{1+n\eta}{2}\right)\leq C^n(n!)^{\eta/2},$$

where $C>0$ is a constant that depends on (η,d,p). Hence,

$$T^{(\frac{p}{\alpha}-1)n}\int_0^t\int_{\mathbb{R}^d}G^{\alpha}_{t-s}(x-y)K^{(p)}_n(s,y)dyds \leq c_0^{n(p-\alpha)}C^n_{\eta,p,d}\frac{T^{n(\frac{p}{\alpha}+\frac{d(1-p)}{2})}}{(n!)^{\frac{d(1-p)}{2}+1}}C^{2n}$$
$$K_{\alpha,d}\left\{t^{\frac{d(1-\alpha)}{2}+1}\left[1+(n!)^{\eta/2}(|x|^{n\eta}+t^{n\eta/2})\right]+t^{\frac{n\eta+d(1-\alpha)}{2}+1}(n!)^{\eta/2}\right\}$$
$$=c_0^{n(p-\alpha)}C^n_{\eta,p,d}T^{n(\frac{d(1-p)}{2}+\frac{p}{\alpha})}C^{2n}K_{\alpha,d}\left\{\frac{t^{\frac{d(1-\alpha)}{2}+1}}{(n!)^{\frac{d(1-p)}{2}+1}}+\frac{|x|^{n\eta}+t^{n\eta/2}+t^{\frac{n\eta+d(1-\alpha)}{2}+1}}{(n!)^{\frac{d(1-p)-\eta}{2}+1}}\right\}.$$

Using this estimate, it is not difficult to see that condition (6.0.13) holds, since $\frac{d(1-p)-\eta}{2}+1>0$ (due to condition (6.5.4)). ∎

6.5.2 Wave equation

In this section, we show that Assumptions 6.0.2 and 6.0.3 are satisfied in the case of the wave equation.

Lemma 6.5.3. If G is the fundamental solution of the wave equation in dimension $d\leq 2$, then for any $\eta>0$ and for any $p>0$ if $d=1$, respectively $p\in(0,2)$ if $d=2$, we have

$$I^{\text{wave}}_{\eta,p}(t,x):=\int_{T_n(t)}\int_{(\mathbb{R}^d)^n}\prod G^p_{t_{k+1}-t_k}(x_{k+1}-x_k)(1+|x_k|^{\eta})d\boldsymbol{x}d\boldsymbol{t}$$

$$\leq C_{\eta,p,d}^n(1+|x|^{n\eta}+t^{n\eta})\frac{t^{an}}{\Gamma(an+1)},$$

where $t_{n+1}=t$, $x_{n+1}=x$,

$$a=\begin{cases}2 & \text{if } d=1,\\ 3-p & \text{if } d=2,\end{cases} \tag{6.5.6}$$

and the constant $C_{\eta,p,d}$ is given by

$$C_{\eta,p,d}=\begin{cases}3(2^{\eta-1}\vee 1)2^{1-p} & \text{if } d=1,\\ 3(2^{\eta-1}\vee 1)\dfrac{(2\pi)^{1-p}}{2-p}\Gamma(3-p) & \text{if } d=2.\end{cases} \tag{6.5.7}$$

Proof. We use similar arguments to those contained in the proof of Theorem 2.4 of [32]. In both cases $d=1$ and $d=2$, the product $\prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}-x_k)$ contains the indicator of $\{|x_2-x_1|<t_2-t_1,\dots,|x-x_n|<t-t_n\}$. On this set, for any $k=1,\dots,n$,

$$|x-x_k|\leq\sum_{j=k}^n|x_{j+1}-x_j|\leq\sum_{j=k}^n(t_{j+1}-t_j)=t-t_k<t,$$

and $|x_k|\leq|x|+|x_k-x|\leq|x|+t$. Hence $\prod_{k=1}^n(1+|x_k|^\eta)\leq C_\eta^n(1+|x|^{n\eta}+t^{n\eta})$, where $C_\eta=3(2^{\eta-1}\vee 1)$. It follows that

$$\begin{aligned}I_{t,x}^{\text{wave}}&\leq C_\eta^n(1+|x|^{n\eta}+t^{n\eta})\int_{T_n(t)}\int_{(\mathbb{R}^d)^2}\prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}-x_k)d\boldsymbol{x}d\boldsymbol{t}\\&=C_\eta^n(1+|x|^{n\eta}+t^{n\eta})\int_{T_n(t)}\prod_{k=1}^n\left(\int_{\mathbb{R}^d}G_{t_{k+1}-t_k}^p(x_k)dx_k\right)d\boldsymbol{t}.\end{aligned}$$

If $d=1$, $\int_{\mathbb{R}}G_t^p(x)dx=2^{1-p}t$ for any $p>0$, and

$$I_{t,x}^{\text{wave}}\leq(C_\eta 2^{1-p})^n(1+|x|^{n\eta}+t^{n\eta})\int_{T_n(t)}\prod_{k=1}^n(t_{k+1}-t_k)d\boldsymbol{t}=(C_\eta 2^{1-p})^n(1+|x|^{n\eta}+t^{n\eta})\frac{t^{2n}}{(2n)!}.$$

If $d=2$, $\int_{\mathbb{R}^2}G_t^p(x)dx=\frac{(2\pi)^{1-p}}{2-p}t^{2-p}$ for any $p\in(0,2)$, and

$$\begin{aligned}I_{t,x}^{\text{wave}}&\leq\left(C_\eta\frac{(2\pi)^{1-p}}{2-p}\right)^n(1+|x|^{n\eta}+t^{n\eta})\int_{T_n(t)}\prod_{k=1}^n(t_{k+1}-t_k)^{2-p}d\boldsymbol{t}\\&=\left(C_\eta\frac{(2\pi)^{1-p}}{2-p}\right)^n(1+|x|^{n\eta}+t^{n\eta})\cdot\frac{\Gamma(3-p)^n t^{n(3-p)}}{\Gamma((3-p)n+1)}.\end{aligned}$$

∎

Lemma 6.5.4. Suppose that ϕ satisfies Assumption 6.0.5. In the case of the wave equation in dimension $d \leq 2$, for any $p > 0$ if $d = 1$, respectively $p \in (0,2)$ if $d = 2$,

$$K_n^{(p)}(t,x) \leq c_0^{n(p-\alpha)} C_{\eta,p,d}^n (1 + |x|^{n\eta} + t^{n\eta}) \frac{t^{an}}{\Gamma(an+1)},$$

where $\eta = \delta(p-\alpha)$, a is given by (6.5.6), and $C_{\eta,p,d}$ is given by (6.5.7).

Proof. This follows using the definition (6.0.11) of $K_n^{(p)}(t,x)$, the bound (6.5.1) for $\phi^{\alpha-p}(x)$ and Lemma 6.5.3. ∎

Note that for any $p > 0$ if $d = 1$, respectively for any $p \in (0,2)$ if $d = 2$, we have:

$$\int_0^t \int_{\mathbb{R}^d} G_{t-s}^p(x-y) dy ds = C_p t^a, \tag{6.5.8}$$

where a is given by (6.5.6), and $C_p = 2^{-p}$ if $d = 1$, respectively $C_p = \frac{(2\pi)^{1-p}}{(2-p)(3-p)}$ if $d = 2$.

The following result shows that Assumptions 6.0.2 and 6.0.3 are satisfies in the case of the wave equation.

Proposition 6.5.5. Suppose that ϕ satisfies Assumption 6.0.5. If $\mathcal{L} = \frac{\partial}{\partial t^2} - \Delta$ is the wave operator in dimension $d \leq 2$, then (6.0.12) and (6.0.13) hold for any $(t,x) \in [0,T] \times \mathbb{R}^d$, and for any $p > 0$ if $d = 1$, respectively for any $p \in (0,2)$ if $d = 2$.

Proof. We use the estimate for $K_n^{(p)}(t,x)$ given by Lemma 6.5.4. Recall that the constant $C_{\eta,p,d}$ is given by (6.5.7) and a is given by (6.5.6).

Step 1. We first prove that (6.0.12) holds. By (6.5.2), there exists a constant $C_a > 1$ such that $\Gamma(an+1) \geq C_a^{-n}(n!)^a$ for all $n \geq 1$. Hence,

$$K_n^{(p)}(t,x) \leq C^n (1 + |x|^{n\eta} + t^{n\eta}) \frac{t^{an}}{(n!)^a},$$

where $C = c_0^{p-\alpha} C_{\eta,p,d} C_a$. It follows that

$$\sum_{n\geq 1} \Big(T^{(p/\alpha-1)n} K_n^{(p)}(t,x) \Big)^{1/2} \Big(\sum_{j\geq 1} \Gamma_j^{-p/\alpha} \Big)^{n/2} \leq$$
$$\sum_{n\geq 1} C^{n/2} \big(1 + |x|^{n\eta} + t^{n\eta}\big)^{1/2} \frac{T^{\frac{n}{2}(a+\frac{p}{\alpha}-1)}}{(n!)^{\frac{a}{2}}} \Big(\sum_{j\geq 1} \Gamma_j^{-p/\alpha} \Big)^{n/2} < \infty.$$

Step 2. Next, we prove that (6.0.13) holds. Note that $G_{t-s}^{\alpha}(x-y)$ contains the indicator of the set $B_{t,x} := \{s \in (0,t), y \in \mathbb{R}^d; |x-y| < t-s\}$. For any $(s,y) \in B_{t,x}$, we have:

$$1+|y|^{n\eta}+s^{n\eta} \leq 1+(|x|+t)^{n\eta}+s^{n\eta} \leq 1+(2^{n\eta-1}\vee 1)(|x|^{n\eta}+t^{n\eta})+t^{n\eta} \leq C^n(1+|x|^{n\eta}+t^{n\eta}),$$

where $C > 0$ is a constant depending on η. Combing this with (6.5.8), we infer that:

$$T^{(\frac{p}{\alpha}-1)n} \int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y) K^{(p)}_n(s,y) dyds$$
$$\le c_0^{n(p-\alpha)} C^n_{\eta,p,d} \frac{T^{(\frac{p}{\alpha}-1+a)n}}{\Gamma(an+1)} C^n (1+|x|^{n\eta}+t^{n\eta}) \int_0^t \int_{\mathbb{R}^d} G^{\alpha}_{t-s}(x-y) dyds$$
$$= c_0^{n(p-\alpha)} C^n_{\eta,p,d} \frac{T^{(\frac{p}{\alpha}-1+a)n}}{\Gamma(an+1)} C^n (1+|x|^{n\eta}+t^{n\eta}) C_p t^{an}.$$

From this estimate, it is not difficult to see that relation (6.0.13) holds. ∎

6.6 Simulations

In this section, we include some simulations for the profile of the solution of an SPDE with additive noise:

$$\mathcal{L}u(t,x) = u(t,x)\dot{Z}(t,x) \tag{6.6.1}$$

as a function of $(t,x) \in [0,1]^2$. We assume that the initial condition is 1, so that the solution of equation (6.6.1) is given by:

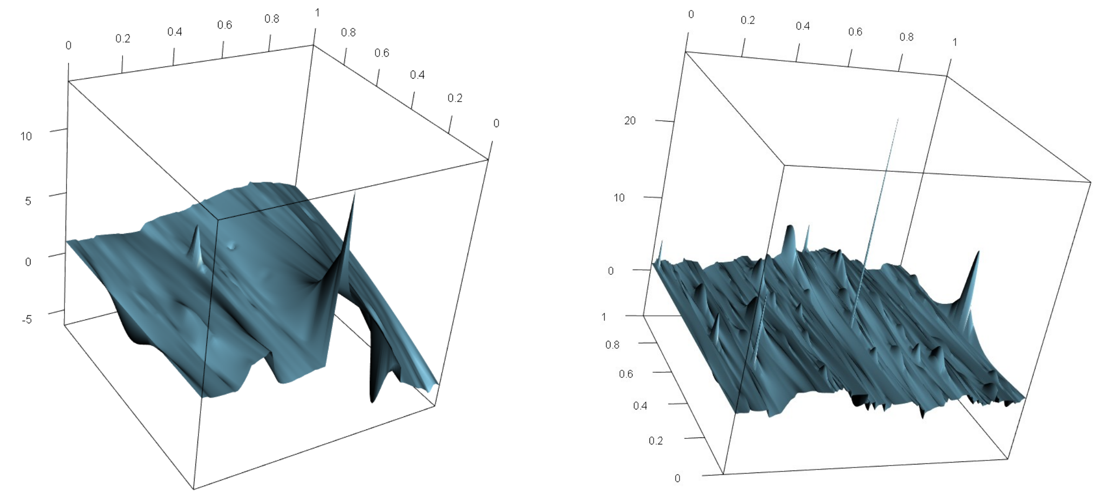
$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) Z(ds,dy), \tag{6.6.2}$$

where G is the fundamental solution of the operator $\mathcal{L}$. Using the LePage representation (6.0.3) of the noise Z, we see that this solution has the series expansion:

$$u(t,x) = 1 + \sum_{i \ge 1} \varepsilon_i \Gamma_i^{-1/\alpha} \frac{1}{\phi(X_i)} G_{t-T_i}(x-X_i).$$

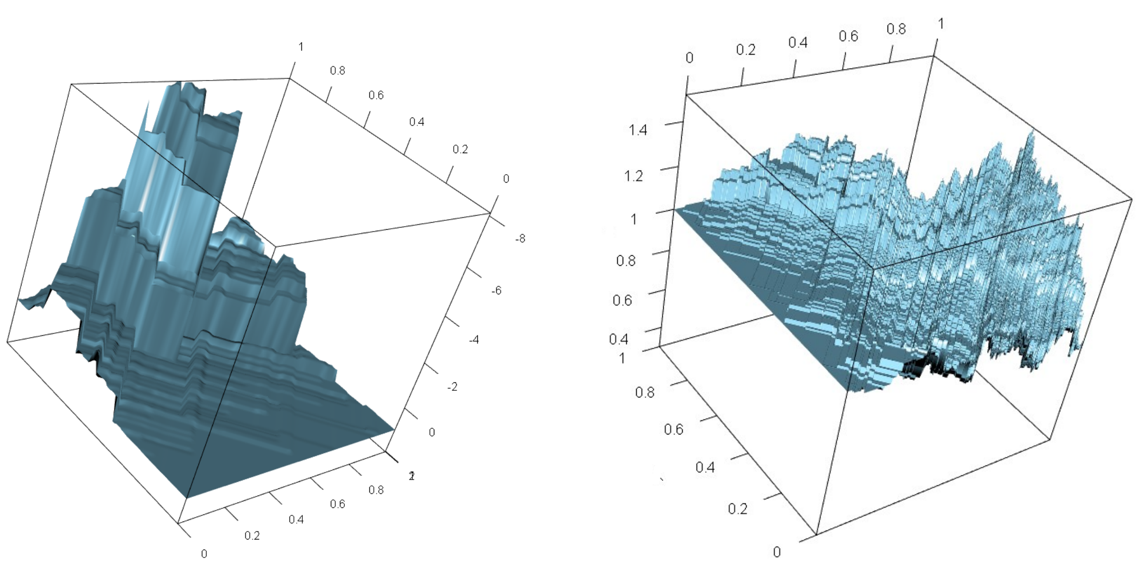
We consider the case when $\mathcal{L}$ is the heat operator or the wave operator, and we simulate separately the solution of (SHE) with additive noise and (SWE) with additive noise on $[0,1] \times \mathbb{R}$, when $\alpha = 0.7$ and $\alpha = 1.5$. We approximate the series (6.6.2) by the partial sum up to $n = 1000$, and we used the function ϕ given by (6.0.17) with $\delta = 1.5$.

Remark 6.6.1. If $d = 1$, the solution of (SHE) is continuous in x for any fixed t, for any $\alpha \in (0,2)$ (by Theorem 3.1 of [18]), and if $\alpha < 1$, it is also continuous in t for any fixed x (by Theorem 3.5 of [18]). However, the solution is not jointly continuous in (t,x); see the comment on page 125 of [18].



(a) $\alpha = 0.7$ (b) $\alpha = 1.5$

Figure 6.1: Simulation of the solution of (SHE) with additive SαS Lévy noise



(a) $\alpha = 0.7$ (b) $\alpha = 1.5$

Figure 6.2: Simulation of the solution of (SWE) with additive SαS Lévy noise

Appendix A

Integration with respect to PRM

In this chapter, we present some results related to maximal inequalities and stochastic integration with respect to Poisson random measures (PRM) and their compensated versions. Recall that PRMs have been given by Definition 2.1.5.

Throughout this chapter, we let N be a PRM on the space $U = \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0$, of intensity

$$\mu(dt, dx, dz) = dtdx\nu(dz),$$

where ν is a Lévy measure on $\mathbb{R}$, i.e. ν satisfies:

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|z|^2 \wedge 1)\nu(dz) < \infty.$$

The compensated version of N is defined by $\widehat{N}(F) = N(F) - \mu(F)$, for any Borel set F in U with $\mu(F) < \infty$.

A.1 Maximal Inequalities

In this section, we present some maximal inequalities for the stochastic integral with respect to $\widehat{N}$, which are taken from [6]. Recall that "càdlàg" stands for "continue à droite avec limites à gauche": a càdlàg process has right-continuous sample paths with left limits.

Theorem A.1.1. Let $Y = \{Y(t)\}_{t \geq 0}$ be a process given by:

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} H(s, x, z)\widehat{N}(ds, dx, dz).$$

Then, there exists a càdlàg modification of Y such that for any $t > 0$ and $p \geq 2$, it holds:

$$E\Big[|\sup_{s\leq t} Y(s)|^p\Big] \leq B_p\Bigg\{E\Bigg[\bigg(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(s,x,z)|^2 \nu(dz)dxds\bigg)^{p/2}\Bigg] + E\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |H(s,x,z)|^p \nu(dz)dxds\Bigg\}. \tag{A.1.1}$$

Theorem A.1.2. a) For any predictable process H and $p \in [1,2]$,

$$E\left[\sup_{s\le T}\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} H(s,x,z)\widehat{N}(ds,dx,dz)\right|^p\right] \le C_p\, E\left[\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} |H(t,x,z)|^p\nu(dz)dxdt\right], \tag{A.1.2}$$

where $C_p > 0$ is a constant depending on p.
b) For any predictable process H and $p \in (0,1]$,

$$E\left[\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} H(s,x,z)N(ds,dx,dz)\right|^p\right] \le E\left[\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} |H(s,x,z)|^p\nu(dz)dxdt\right]. \tag{A.1.3}$$

A.2 Stochastic integration

In this section, we recall some key components of the integration theory with respect to the compensated process $\widehat{N}$.

The stochastic integral with respect to the compensated process $\widehat{N}$ is defined similarly to the Itô integral, as explained for instance in Chapter 4 of [1]. More precisely, for any $\widetilde{\mathcal{P}} \times \mathcal{B}(\mathbb{R}_0)$-measurable process H with $\mathbb{E}\int_U H^2 d\mu < \infty$, the stochastic integral $I^{\widehat{N}}(H) = \int_U X d\widehat{N}$ is a zero mean random variable with $\mathbb{E}|I^{\widehat{N}}(H)|^2 = \mathbb{E}\int_U X^2 d\mu$, and $\{M_t = I^{\widehat{N}}(1_{[0,t]}H); t \ge 0\}$ is a square-integrable martingale. The definition of the integral can be extended to processes H satisfying $\int_U |H|^2 d\mu < \infty$ a.s., and in this case M is a local martingale, which satisfies:

$$\mathbb{P}(|M_t| > \varepsilon) \le \frac{\eta}{\varepsilon^2} + \mathbb{P}\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0} H^2(s,x,z)\mu(dz,dx,dz) > \eta\right)$$

for any $\varepsilon > 0$ and $\eta > 0$. The process M has a càdlàg modification (denoted also by M), whose jump at time s is given by

$$\Delta M_s = \sum_{i\ge 1} H(T_i, X_i, Z_i)1_{\{T_i = s\}} \quad \text{where } N = \textstyle\sum_{i\ge 1}\delta_{(T_i,X_i,Z_i)}.$$

By Lemma I.4.51 of [31], the quadratic variation of M is

$$[M]_t = \sum_{s\in[0,t]}(\Delta M_s)^2 = \sum_{i\ge 1} H^2(T_i,X_i,Z_i)1_{\{T_i\le s\}} = \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0} H^2(s,x,z)N(ds,dx,dz).$$

By the Burkholder-Davis-Gundy inequality for càdlàg local martingales,

$$\mathbb{E}\left(\sup_{s\le\tau} M_s^2\right) \le \mathbb{E}[M]_\tau,$$

for any stopping time τ, i.e. the process $M_t^* = \sup_{s\le t} M_s^2$ is L-dominated by $[M]$ (in the sense of Definition I.3.29 of [31]. By Lenglart's inequality (Lemma I.3.30 of [31]),

$$\mathbb{P}(\sup_{s\le t}|M_s| > \varepsilon) \le \frac{\eta}{\varepsilon^2} + \mathbb{P}\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0} H^2(s,x,z)N(dz,dx,dz) > \eta\right),$$

for any $\varepsilon > 0$ and $\eta > 0$.

Appendix B

Basic tools from analysis

In this chapter, we present several results from analysis, which have been used in the thesis.

B.1 Banach fixed-point theorem

In this section, we present some results about Banach spaces and fixed-point theorems, which are fundamental to understand the existence and uniqueness of SPDEs.

Theorem B.1.1 (Banach fixed-point theorem)**.** Let $(X, ||\cdot||_X)$ be a Banach space, and $\mathcal{T} : X \to X$ a map such that T is a contraction *i.e.* there exists $c \in (0, 1)$ such that

$$||\mathcal{T}(x) - \mathcal{T}(y)||_X \leq c||x - y||_X \qquad \text{for all } x, y \in X.$$

Then, there exists a unique $u \in X$ for which

$$\mathcal{T}u = u. \tag{B.1.1}$$

Given a map $T : X \to X$, we define the composition operator for $n \in \mathbb{N}$ as

$$\mathcal{T}^{(n)}(x) := \mathcal{T}(\mathcal{T}^{(n-1)}(x))$$

for all $x \in X$.

Theorem B.1.2. Let $(X, ||\cdot||_X)$ be a Banach space, and $\mathcal{T} : X \to X$ a map such that for all x, $y \in X$ we have:

$$\sum_{n=1}^{\infty} ||\mathcal{T}^{(n)}(x) - \mathcal{T}^{(n)}(y)||_X < +\infty \tag{B.1.2}$$

and $\mathcal{T}$ is continuous on $(X, ||\cdot||_X)$. Then, (B.1.1) has a unique solution in X.

Proof. Let $x_0 \in X$ arbitrary, and $\{x_n\}_{n\geq 0}$ the sequence in X given by:

$$x_{n+1} := \mathcal{T}(x_n) \quad n \in \mathbb{N}.$$

For any $n > m$,

$$||x_n - x_m||_X \leq \sum_{k=n}^{m-1} ||x_{k+1} - x_k||_X = \sum_{k=n}^{m-1} ||\mathcal{T}^{(k)}(x_1) - \mathcal{T}^{(k)}(x_0)||_X$$

By (B.1.2), for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n}^{m-1} ||\mathcal{T}^{(k)}(x_1) - \mathcal{T}^{(k)}(x_0)||_X < \varepsilon,$$

for all $n > m > N_\varepsilon$. So, the sequence $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in $(X, ||\cdot||_X)$, which implies there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$ in the norm $||\cdot||_X$. Now, by the continuity of the operator $\mathcal{T}$, x is a solution of (B.1.1). Indeed,

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}(x_n) = \mathcal{T}(x).$$

∎

Proposition B.1.3. Let $(X, ||\cdot||_X)$ be a Banach space. If $\{x_n\}_{n\geq 0}$ is a sequence in X such that

$$\sum_{n\geq 1} ||x_n - x_{n-1}||_X < \infty,$$

then, there exists $x \in X$ for which $||x_n - x||_X \to 0$ as $n \to \infty$. Moreover,

$$||x_n - x||_X \leq \sum_{k\geq n+1} ||x_k - x_{k-1}||_X. \tag{B.1.3}$$

B.2 Classical tools from analysis

In this section, we review some classical tools from analysis, including some basic integral inequalities.

Theorem B.2.1 (Hölder's inequality)**.** Let $(X, \mathcal{X}, \mu)$ be a measure space and let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all measurable real-values functions f and g on S,

$$\int_X |f(x)g(x)|\mu(dx) \leq \left(\int_X |f(x)|^p \mu(dx)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q \mu(dx)\right)^{\frac{1}{q}}. \tag{B.2.1}$$

The following remark gives an immediate consequence of Theorem B.2.1.

Remark B.2.2. If $\mu(X) < \infty$ in Theorem B.2.1, then

$$\left(\int_X |f(x)|\mu(dx)\right)^p \leq [\mu(X)]^{p-1} \left(\int_X |f(x)|^p \mu(dx)\right).$$

Moreover, if $\mu(dx) = \varphi(x)dx$, we can re-write the equation above as

$$\left(\int_X |f(x)|\varphi(x)dx\right)^p \leq \left(\int_X \varphi(x)dx\right)^{p-1}\left(\int_X |f(x)|^p\varphi(x)dx\right). \tag{B.2.2}$$

Theorem B.2.3 (Hölder's inequality for multiple products)**.** Let $(X, \mathcal{X}, \mu)$ be a measure space and assume that $p_1, \ldots, p_n \in (0, \infty]$ such that

$$\sum_{k=1}^{n} \frac{1}{p_k} = 1.$$

Then for all $\mathcal{X}$ -measurable functions $f_1, \ldots, f_n$, we have:

$$\int_X \prod_{k=1}^{n} f_k(x)\mu(dx) \leq \left(\prod_{k=1}^{n}\int_X |f_k(x)|^{p_k}\mu(dx)\right)^{1/p_k}$$

Theorem B.2.4 (Minkowski's inequality)**.** Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be measure spaces such that $(X \times Y, \mathcal{X} \bigotimes \mathcal{Y}, \mu \times \nu)$ is σ-finite. If $f(x, y)$ is a measurable function with respect to this product space, then

$$\left(\int_Y \left(\int_X |f(x,y)|\mu(dx)\right)^p \nu(dy)\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p\nu(dy)\right)^{\frac{1}{p}} \mu(dx).$$

The following result gives a well-known property of the normal distribution.

Lemma B.2.5. If X is an $\mathcal{N}(0, \sigma^2)$-distribution random variable, then for every $p \in (-1, \infty)$ we have

$$\mathbb{E}\left[|X|^p\right] = (2\sigma^2)^{p/2}\pi^{-1/2}\Gamma\left(\frac{1+p}{2}\right)$$

The following lemma can be proved by induction using properties of the Beta function.

Lemma B.2.6. For any $\beta_1 > -1, \ldots, \beta_n > -1$,

$$\int_{T_n(t)} \prod_{j=1}^{n}(t_{j+1} - t_j)^{\beta_j} dt_1 \ldots dt_n = \frac{\prod_{j=1}^{n}\Gamma(\beta_j + 1)}{\Gamma(\sum_{j=1}^{n}\beta_j + n + 1)} t^{\sum_{j=1}^{n}\beta_j + n},$$

where $T_n(t) := \{(t_1, \ldots, t_n) \in (0, t)^n\, ;\, t_1 < \ldots < t_n\}$ and $t = t_{n+1}$.

The following remark gives a property of the Gamma function, which can be proved using Stirling's formula:

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{2\pi x} \quad \text{as } x \to \infty$$

and the fact that $\Gamma(x + b) \sim \Gamma(x)x^b$ as $x \to \infty$, for any $b \in \mathbb{R}$.

Remark B.2.7 (Remark 2.3 in [29])**.** For any $a > 0$ and $b \in \mathbb{R}$, there exists positive constants $C_{a,b}$ and $c_{a,b}$ depending on a and b, such that

$$c_{a,b}^n (n!)^a \leq \Gamma(an + 1 + b) \leq C_{a,b}^n (n!)^a \quad \text{for all } n \in \mathbb{N}.$$

B.3 An application of Fubini's theorem

In this section, we include an application of Fubini's theorem which is used frequently in Chapter 6.

Lemma B.3.1. Let X and Y be independent random variables with values in measurable spaces $(E, \mathcal{E})$, respectively $(F, \mathcal{F})$, and $f : E \times F \to [0, \infty]$ be a measurable function. Let $\mathbb{P}_X$ be the law of X. If $f(x, Y) < \infty$ a.s. for $\mathbb{P}_X$-almost all $x \in E$, then $f(X, Y) < \infty$ a.s. In particular, if $\mathbb{E}[f(x, Y)] < \infty$ for $\mathbb{P}_X$-almost all $x \in E$, then $f(X, Y) < \infty$ a.s.

Proof. We know that $\mathbb{P}(f(x, Y) < \infty) = 1$ for all $x \in N^c$, where $\mathbb{P}_X(N) = 0$. By Fubini's theorem,

$$\begin{aligned}\mathbb{P}(f(X,Y) = \infty) &= \int_E \int_F 1_{\{f(x,y)=\infty\}} \mathbb{P}_X(dx)\mathbb{P}_Y(dy) = \int_{N^c} \left(\int_F 1_{\{f(x,y)=\infty\}} \mathbb{P}_Y(dy) \right) \mathbb{P}_X(dx) \\ &= \int_{N^c} \mathbb{P}(f(x,Y) = \infty)\mathbb{P}_X(dx) = 0.\end{aligned}$$

∎

Remark B.3.2. Note that $h(x) = \mathbb{E}[f(x, Y)] < \infty$ for $\mathbb{P}_X$-almost all $x \in E$ is equivalent to $h(X) < \infty$ a.s. On the other hand, $h(X) = \mathbb{E}[f(X, Y)|X]$ a.s., since X and Y are independent. So the criterion given by Lemma B.3.1 can be stated as follows: for independent random variables X and Y,

$$\text{if } \mathbb{E}[f(X,Y)|X] < \infty \text{ a.s., then } f(X,Y) < \infty \text{ a.s.}$$

Here we use a generalized definition of the conditional expectation $\mathbb{E}[Z|\mathcal{G}]$ of a random variable Z given a σ-field $\mathcal{G}$, for which Z does not have to be integrable.

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