Construction and approximation of stable Lévy motion with values in the Skorohod space

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Abstract

Under an appropriate regular variation condition, the affinely normalized partial sums of a sequence of independent and identically distributed random variables converges weakly to a non-Gaussian stable random variable. A functional version of this is known to be true as well, the limit process being a stable Lévy process. In this thesis, we developed an explicit construction for the α-stable Lévy process motion with values in $\mathbb{D}([0, 1])$, by considering the cases $\alpha < 1$ and $\alpha > 1$. The case $\alpha < 1$ is the simplest since we can work with the uniform topology of the sup-norm on $\mathbb{D}([0, 1])$ and the construction follows more or less by classical techniques. The case $\alpha > 1$ required more work. In particular, we encountered two problems : one was related to the construction of a modification of this process (for all time), which is right-continuous and has left-limit with respect to the J_1 topology. This problem was solved by using the $\widehat{It\hat{o}}$-Nisio theorem. The other problem was more difficult and we only managed to solve it by developing a criterion for tightness of probability measures on the space of cadlag fonction on $[0, T]$ with values in $\mathbb{D}([0, 1])$, equipped with a generalization of Skorohod's J_1 topology.

In parallel with the construction of the infinite-dimensional process Z, we focus on the functional extension of Roueff and Soulier [29]. This part of the thesis was completed using the method of point process, which gave the convergence of the truncated sum. The case $\alpha > 1$ required more work due to the presence of centering. For this case, we developed an ad-hoc result regarding the continuity of the addition for functions on $[0, T]$ with values in $\mathbb{D}([0, 1])$, which was tailored for our problem.

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Dedications

Dedicated
to my beloved parents
and sisters
For their love
endless, support, encouragement
and sacrifices.

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Chapter 1

Introduction

Functional limit theorems for partial-sum processes of i.i.d. random variables have been known for quite some time. These theorems can be divided into two main groups depending on whether the second moments of the underlying random variables are finite or infinite. In probability theory, we are often concerned with models representing the result of a series of small events or influences. It is a remarkable feature of such models that, even when very little is known about the precise nature of these small effects, we can often say a great deal about the overall result. Indeed, it is this universality at large scales which accounts for much of the success of probabilistic methods in the natural and social sciences.

As an illustration, consider the following toy model. Suppose that we have a collection of real-valued random variables $X_1, X_2, \ldots$ which are independent and all have the same distribution, and should model small-scale events. Denote by S_n the sum of the first n of these, that is, $S_n = X_1 + \cdots + X_n$. We are then interested in the behavior of S_n when n is very large. In particular, suppose there exist sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that the rescaled, recentered sums

$$a_n^{-1} S_n - n b_n$$

converge in distribution to a random variable U. What distributions can U possess? It emerges that only very few distributions can arise in this manner; they are called the normal distribution and stable distributions, and they occupy a central position in probability theory.

On the other hand, if we wish to capture in the limit not only the final sum S_n, but also the whole path $(S_1, S_2, \ldots, S_n)$, we require the theory of stochastic processes. A stochastic process X is a collection of random variables $(X_t)_{t \geq 0}$, with each X_t representing the position of X at time t. The natural processes which arise as limits in this context are Brownian motion and stable processes. In fact, the distribution of a Brownian motion at any time is a normal distribution, while the distribution of a stable process at any time is a stable distribution. Brownian motion has been extensively studied for many decades, and there is a huge body of research on a great many aspects of the process. This is due in part to the wide variety of mathematical tools which may be applied to its study, such as analysis via its generator, potential theory and connections to differential equations, excursion theory, martingale theory and stochastic calculus. Stable processes have proven more difficult to analyse by these means, and have therefore historically taken second place to Brownian motion, in spite of their central importance in the theory of stochastic processes. When studying stable processes, we have access to the rich literature on general Lévy processes, but do not restrict ourselves to continuous paths as in the case of Brownian motion. In fact, any stable process has infinite jump activity, in that it is guaranteed to have an infinite number of (mostly very small) jumps in any finite time period. Classical references in the i.i.d. case is the book by Feller [13], while in [24] we can find an elegant probabilistic proof of sufficiency and a nice representation of the limiting distribution. The asymptotic behavior of the processes $S_n(\cdot) = a_n^{-1} S_{[nt]} - [nt]b_n$ as $n \to \infty$ is an extensively studied subject in the probability literature. In our considerations the index of regular variation α will be less than 2, which implies the variance of X_1 is infinite. In the finite-variance case, functional limit theorems differ considerably and have been investigated in greater depth, see for instance [6] and [16]. A very readable proof of the functional limit theorem for i.i.d. sequences of regularly varying random variables with infinite variance can be found in [28]. [11] considered functional limit theorems for dependent random variables in the context of martingale theory, while [23] studied this question in the context of extreme value theory.

Suppose X is a random variable with distribution function F. The outcome of X may be thought of as the measurement of a sea-level, the daily loss from investing in a stock, the total amount of claims faced by an insurance company in one year etc. In such applications it is relevant to compute the probability of a very large (extreme) outcome, for instance the probability that the sea-level exceeds a high barrier, the probability that we make a large loss from an investment in the stock, or the probability that the total amount of claims faced by the insurance company in one year exceeds a high threshold. This means, we would have to compute $1-F(x)$ where x is large. For this reason it is of course important to know what the distribution function F looks like for large x, e.g. at which rate the function $1 - F(x)$ tends to zero as $x \to \infty$. If the decay is fast then the probability mass is concentrated around the center of the distribution. As an example we may consider the standard normal distribution where $1 - F(x) \sim (x\sqrt{2})^{-1} \exp(-\frac{x^2}{2})$ as $x \to \infty$. In this case we say that the distribution has light (right) tail. If, on the other hand, the decay is slow, then there is a significant amount of probability mass far out in the (right) tail of the distribution. The slow decay of the probability distribution as $x \to \infty$ is often referred to as heavy (right) tail. As an example we may consider the Pareto distribution where $1 - F(x) \approx x^{-\alpha}$, as $x \to \infty$ and $\alpha > 0$. Similar considerations can of course be made for the left tail as well. Then we consider the rate at which $F(x)$

tends to zero as $x \to \infty$. There is no precise definition of how fast or slow the decay must be to say that a probability distribution has light or heavy tails. We will work with a more precise concept called regular variation, which specifies the rate at which $1 - F(x)$ tends to zero.

In heavy-tailed distributions large values may occur with non-negligible probability. This is often observed in insurance data, for instance in the so-called catastrophe insurances including fire, wind-storm and flooding insurances. The large claims may lead to large fluctuations in the cash-flow process faced by the insurance company, increasing the risk in such portfolios. The situation is similar in finance where extremely large losses sometimes occur, which indicate heavy tails of the return distribution. The probability of extreme stock-movements has to be accounted when analyzing the risk of a portfolio. Another application is queuing models where extreme service times, modeled by heavy-tailed distributions, result in huge waiting times in the system and large fluctuations in the workload process.

The concept of regular variation can be extended to random vectors leading to multivariate regular variation. The authors of [25] extended this to infinite dimensions, by introducing the concept of regular variation for processes with values in the space $D[0,1]$ of right-continuous functions with left limits. For instance, $X_i(t)$ is the high tide water level at time i and location $t \in [0,1]$ along a coast protected by a dike. In finance, $X_i(t)$ may represent the stock price at time t in day i, and observations of X_i are made continuously during the time interval 9 a.m. to 5 p.m. (identified with the interval $[0,1]$).

The main result of this thesis shows that for a regularly varying sequence, the properly centered partial sum process $(S_n(t))_{t\in[0,1]}$ converges to an α-stable Lévy process with values in the space $\mathbb{D}$ endowed with Skorohods topology J_1. In proving this result we combine some ideas used in the i.i.d. case by Resnick [27, 28] with a new point process convergence result from [29] and some particularities of the J_1 metric on $\mathbb{D}$ that can be deduced from Whitt [34].

The outline of this thesis is as follows: In Chapter 2, we review some properties of the multivariate stable distribution with special emphasis to spectral representation, the different representations of the α-stable random vectors and the stable Central Limit Theorem in $\mathbb{R}^d$.

In Chapter 3, we introduce the spaces $\mathbb{D}([0,1];\mathbb{D})$ and $\mathbb{D}([0,\infty);\mathbb{D})$. We equip this spaces with different metrics and discus the differences between them. We refer to Whitt [34] for a detailed discussion on this concepts. By defining a new topology for $\mathbb{D}([0,1];\mathbb{D})$ (the Skorokhod topology) families of measures on this space can be constructed and sufficient conditions for weak convergence specified.

In Chapter 4, we present the construction of an infinite-dimensional Lévy process with values in the space $\mathbb{D}$ of cádlág functions (equipped with Skorokhod's J_1 topology) based on the $It\widehat{o}$ representation of a Lévy process.

In Chapter 5, we derive an extension of the stable functional central limit theorem (FCLT) to the case of random elements in $\mathbb{D}$.

In Chapter 6, we give a simulation of $\mathbb{D}$-valued α-stable Lévy process using the functional central limit theorem from chapter 5.

Finally, in Chapter 7, we present a summary of the thesis and outline plans for future research problems.

Chapter 2

Multivariate stable distributions

2.1 Introduction

The stable distribution is a natural generalization of the normal distribution with which it shares the property of being stable under addition, a property that is needed in actuarial sciences for continuously compounded returns. Furthermore, stable distributions are capable of capturing excess kurtosis shown by historical stock returns. In this chapter, we give a characterization of stable distributions using Lévy-Khinchine and spectral representations, and prove the Stable Central Limit Theorem in $\mathbb{R}^d$.

2.2 One-dimensional case

In this section, we review some proprieties of the univariate α-stable distribution, using [30].

Definition 2.2.1. *A random variable X has a **stable distribution** if for any positive numbers A and B, there is a positive number C and a real number D such that*

$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX + D \tag{2.2.1}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X, and $\stackrel{d}{=}$ denotes equality in distribution.

Theorem 2.2.2. *For any stable random variable X, there is a number $\alpha \in (0, 2]$ such that the number C in (2.2.1) satisfies:*

$$C^\alpha = A^\alpha + B^\alpha.$$

In this case, we say that X has an α-stable distribution.

Theorem 2.2.3. *A random variable X has a stable distribution if for any $n \geq 2$, there is a positive number C_n and a real number D_n such that*

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} C_n X + D_n$$

where $X_1, X_2, \ldots, X_n$ are independent copies of X.

Theorem 2.2.4. *A random variable X has a stable distribution if it has a domain of attraction, i.e., there is a sequence of i.i.d. random variable $Y_1, Y_2, \ldots$, a sequence $\{d_n\}$ of positive numbers and a sequence $\{a_n\}$ of real numbers, such that*

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \stackrel{d}{\to} X,$$

where $\stackrel{d}{\to}$ denotes convergence in distribution.

Theorem 2.2.5. *A random variable X has a stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $\mu \in \mathbb{R}$ such that the characteristic function of X has the following form:*

1. *if $\alpha \neq 1$, for all $u \in \mathbb{R}$*

$$E(\exp(iuX)) = \exp\Bigg(- \sigma^{\alpha} \left|u\right|^{\alpha} \Big(1 - i\beta sgn(u) \tan\big(\frac{\pi\alpha}{2}\big)\Big) + i\mu u\Bigg) \qquad (2.2.2)$$

2. *if $\alpha = 1$, for all $u \in \mathbb{R}$*

$$E(\exp(iuX)) = \exp\Bigg\{ -\sigma \left|u\right| \Big(1 + i\beta \frac{2}{\pi} sgn\ (u) \ln\left|u\right|\Big) + i\mu u \qquad (2.2.3)$$

Remark 2.2.6. The α-stable distribution requires four parameters for complete description: an index $\alpha \in (0, 2]$ of stability (also called the tail index, tail exponent or characteristic exponent) a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \in \mathbb{R}$. We denote by $S_\alpha(\sigma, \beta, \mu)$ the α-stable distribution with characteristic function given by (2.2.2) if $\alpha \neq 1$ and by (2.2.3) if $\alpha = 1$. We write $X \sim S_\alpha(\sigma, \beta, \mu)$.

The following result shows that the α-stable distribution (with $\alpha < 2$) is a particular case of an infinitely divisible distribution (without the Gaussian component) and gives the explicit form of the parameters (σ, β, μ) based on the Lévy measure $\nu_{\alpha,p}$ and the shift vector μ.

Theorem 2.2.7. *Let X be an infinitely divisible random variable with characteristic function given by: for any $u \in \mathbb{R}$*

$$E(\exp(iuX)) = \exp\left\{iu\gamma + \int_{\mathbb{R}} \Big(\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\Big)\nu_{\alpha,p}(dx)\right\}$$

with

$$\nu_{\alpha,p}(dx) = \left[p\alpha x^{-\alpha-1}\mathbf{1}_{(0,\infty)} + q\alpha(-x)^{-\alpha-1}\mathbf{1}_{(-\infty,0)}\right]dx$$

$\alpha \in (0,2)$, $p \geq 0$ and $p+q=1$. Then X has a $S_\alpha(\sigma,\beta,\mu)$ distribution with: $\sigma^\alpha = C_\alpha^{-1}$, $\beta = p-q$ and

$$\mu = \begin{cases} \gamma + \beta\frac{\alpha}{\alpha-1} & \text{if } \alpha \neq 1 \\ \gamma + \beta a & \text{if } \alpha = 1 \end{cases} \tag{2.2.4}$$

where

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} & \text{if } \alpha = 1 \end{cases} \tag{2.2.5}$$

and

$$a = \int_0^\infty \Big(\sin(r) - r\mathbf{1}_{[0,1]}(r)\Big)r^{-2}dr. \tag{2.2.6}$$

Proof. We consider separately three cases:
Case 1: $0 < \alpha < 1$

$$\begin{aligned} E(\exp(iuX)) &= \exp\left\{iu\gamma + \int_{\mathbb{R}}\Big(\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\Big)\nu_{\alpha,p}(dx)\right\} \\ &= \exp\left\{iu\gamma - iu\int_{\mathbb{R}} x\mathbf{1}_{\{|x|\ leq1\}}\nu_{\alpha,p}(dx) + \int_{\mathbb{R}}\Big(\exp(iux) - 1\Big)\nu_{\alpha,p}(dx)\right\} \\ &= \exp\left(iu\Big(\gamma - \int_{|x|\leq 1} x\nu_{\alpha,p}(dx)\Big) + p\alpha\int_0^\infty\Big(\exp(iux) - 1\Big)x^{-\alpha-1}dx\right. \\ &+ \left. q\alpha\int_{-\infty}^0\Big(\exp(iux) - 1\Big)(-x)^{-\alpha-1}dx\right). \end{aligned}$$

Note that:

$$\begin{aligned} \gamma - \int_{|x|\leq 1} x\nu_{\alpha,p}(dx) &= \gamma - \Big(p\alpha\int_0^1 x^{-\alpha}dx - q\alpha\int_{-1}^0(-x)^{-\alpha}dx\Big) \\ &= \gamma - (p-q)\alpha\int_0^1 x^{-\alpha}dx \\ &= \gamma - (p-q)\frac{\alpha}{1-\alpha} \\ &= \gamma + \beta\frac{\alpha}{\alpha-1} = \mu. \end{aligned}$$

Using Lemma A.0.1 (Appendix A), we can establish the following results: for any $u \in \mathbb{R}$

$$\alpha \int_0^\infty \Big(\exp(iux) - 1\Big) x^{-\alpha-1} dx = -\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(-i\frac{\pi\alpha}{2}\mathrm{sgn}(u)\Big), \quad (2.2.7)$$

and

$$\alpha \int_{-\infty}^0 \Big(\exp(iux) - 1\Big)(-x)^{-\alpha-1} dx = -\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(i\frac{\pi\alpha}{2}\mathrm{sgn}\ (u)\Big).$$

Then using the definition of C_α and β we have,

$$\begin{aligned}
E(\exp(iuX)) &= \exp\Bigg(iu\mu - p\frac{\Gamma(2-\alpha)}{1-\alpha}|u|^\alpha \exp\Big(-i\frac{\pi\alpha}{2}\mathrm{sgn}\ (u)\Big) \\
&- q\frac{\Gamma(2-\alpha)}{1-\alpha}|u|^\alpha \exp\Big(i\frac{\pi\alpha}{2}\mathrm{sgn}\ (u)\Big)\Bigg) \\
&= \exp\Big\{iu\mu - |u|^\alpha \frac{\Gamma(2-\alpha)}{1-\alpha}\cos\Big(\frac{\pi\alpha}{2}\Big)\Big(1 - i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}\ (u)(p-q)\Big)\Big\} \\
&= \exp\Big\{iu\mu - |u|^\alpha C_\alpha^{-1}\Big(1 - i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}\ (u)\beta\Big)\Big\}
\end{aligned}$$

Case 2: $1 < \alpha < 2$

$$\begin{aligned}
E(\exp(iuX)) &= \exp\Big\{iu\gamma + \int_{\mathbb{R}}\Big(\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\le 1\}}\Big)\nu_{\alpha,p}(dx)\Big\} \\
&= \exp\Big\{iu\gamma + iu\int_{|x|>1} x\nu_{\alpha,p}(dx) + \int_{\mathbb{R}}\Big(\exp(iux) - 1 - iux\Big)\nu_{\alpha,p}(dx)\Big\} \\
&= \exp\Bigg(iu\Big(\gamma + \int_{|x|>1} x\nu_{\alpha,p}(dx)\Big) + p\alpha\int_0^\infty\Big(\exp(iux) - 1 - iux\Big)x^{-\alpha-1}dx \\
&+ q\alpha\int_{-\infty}^0\Big(\exp(iux) - 1 - iux\Big)(-x)^{-\alpha-1}dx\Bigg).
\end{aligned}$$

Note that

$$\begin{aligned}
\gamma + \int_{|x|>1} x\nu_{\alpha,p}(dx) &= \gamma + p\alpha\int_1^\infty x^{-\alpha}dx - q\alpha\int_{-\infty}^{-1}(-x)^{-\alpha}dx \\
&= \gamma + (p-q)\alpha\int_1^\infty x^{-\alpha}dx \\
&= \gamma + (p-q)\frac{\alpha}{\alpha-1} \\
&= \gamma + \beta\frac{\alpha}{\alpha-1} = \mu.
\end{aligned}$$

Using Lemma A.0.1 (Appendix A), we can establish the following results: for any $u \in \mathbb{R}$

$$\alpha \int_0^\infty (\exp(iux) - 1 - iux) x^{-\alpha-1} dx = -\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(-i\frac{\pi\alpha}{2} \text{sgn}(u)\Big), \quad (2.2.8)$$

and

$$\alpha \int_{-\infty}^0 (\exp(iux) - 1 - iux)(-x)^{-\alpha-1} dx = -\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(i\frac{\pi\alpha}{2} \text{sgn}(u)\Big).$$

Then using the definition of C_α and β we have,

$$\begin{aligned}
E(\exp(iuX)) &= \exp\Big(iu\mu - p\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(-i\frac{\pi\alpha}{2}\text{sgn}(u)\Big) \\
&- q\frac{\Gamma(2-\alpha)}{1-\alpha} |u|^\alpha \exp\Big(i\frac{\pi\alpha}{2}\text{sgn}(u)\Big)\Big) \\
&= \exp\Big(iu\mu - |u|^\alpha \frac{\Gamma(2-\alpha)}{1-\alpha} \cos\Big(\frac{\pi\alpha}{2}\Big) \\
&\quad \Big(1 - i\tan\Big(\frac{\pi\alpha}{2}\Big)\text{sgn}(u)(p-q)\Big)\Big) \\
&= \exp\Big\{iu\mu - |u|^\alpha C_\alpha^{-1}\Big(1 - i\tan\Big(\frac{\pi\alpha}{2}\Big)\text{sgn}(u)\beta\Big)\Big\}
\end{aligned}$$

Case 3: $\alpha = 1$

$$\begin{aligned}
E(\exp(iuX)) &= \exp\Big\{iu\gamma + \int_{\mathbb{R}} (\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\le 1\}})\nu_{1,p}(dx)\Big\} \\
&= \exp\Big(iu\gamma + p\int_0^\infty (\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\le 1\}})x^{-2}(dx) \\
&+ q\int_{-\infty}^0 (\exp(iux) - 1 - iux\mathbf{1}_{\{|x|\le 1\}})(-x)^{-2}(dx)\Big).
\end{aligned}$$

Using Lemma A.0.1 (Appendix A), we can establish the following results: for any $u \in \mathbb{R}$

$$\int_0^\infty (\exp(iux) - 1 - iux\mathbf{1}_{(0,1]})x^{-2}(dx) = -|u|\Big(\frac{\pi}{2} + i\text{sgn}(u)\ln|u|\Big) + iau \quad (2.2.9)$$

and

$$\int_{-\infty}^0 (\exp(iux) - 1 - iux\mathbf{1}_{[-1,0)})(-x)^{-2}(dx) = -|u|\Big(\frac{\pi}{2} - i\text{sgn}(u)\ln|u|\Big) - iau.$$

Then using the definition of C_1 and β we have,

$$\begin{aligned} E(\exp(iuX)) &= \exp\bigg(iu\gamma + p\Big(-\frac{\pi}{2}\,|u| - iu\mathrm{sgn}\ (u)\ln|u| + iau\Big) \\ &+ q\Big(-\frac{\pi}{2}\,|u| + iu\mathrm{sgn}\ (u)\ln|u| - iau\Big)\bigg) \\ &= \exp\bigg(iu\Big(\gamma + a(p-q)\Big) - \frac{\pi}{2}\,|u|\,\Big(1 + i\frac{2}{\pi}\mathrm{sgn}\ (u)\ln|u|\,\beta\Big)\bigg) \end{aligned}$$

The conclusion follows noting that : $\sigma = C_1^{-1}$, $\beta = p - q$ and $\mu = \gamma + a\beta$. $\square$

Corollary 2.2.8. *Let*

$$\nu(dx) = \big(c_+\alpha x^{-\alpha-1}1_{(x>0)} + c_-\alpha(-x)^{-\alpha-1}1_{(x<0)}\big)\,dx$$

for some $c_+, c_- > 0$ *and* $\alpha \in (0,2)$, $\alpha \neq 1$

$$\delta^\alpha = C_\alpha^{-1}(c_+ + c_-) \ \ and\ \beta = \frac{c_+ - c_-}{c_+ + c_-}$$

where C_α *is given by (2.2.5). Let* $a > 0$ *be arbitrary.*
a) If $\alpha < 1$*, then the variable* Y *with the characteristic function:*

$$E(\exp(iuY)) = \exp\left\{a^\alpha \int_{\mathbb{R}} \big(\exp(iuy) - 1\big)\nu(dy)\right\}\ ,\ u \in \mathbb{R}$$

has a $S_\alpha(a\delta, \beta, 0)$ *distribution.*
b) If $\alpha > 1$*, then the variable* Y *with the characteristic function:*

$$E(\exp(iuY)) = \exp\left\{a^\alpha \int_{\mathbb{R}} \big(\exp(iuy) - 1 - iuy\big)\nu(dy)\right\}\ ,\ u \in \mathbb{R}$$

has a $S_\alpha(a\delta, \beta, 0)$ *distribution.*

Proof. a) We consider first the case $a = 1$. Let X be the random variable in Theorem 2.2.7 with $\gamma = \int_{|x|\leq 1} x\nu_{\alpha,p}(dx)$, where $p = c_+/c$, $q = c_-/c$ and $c = c_+ + c_-$. Note that X has characteristic function :

$$E(\exp(iuX)) = \exp\left\{\int_{\mathbb{R}} \big(\exp(iux) - 1\big)\nu_{\alpha,p}(dx)\right\}.$$

By Theorem 2.2.7, X has a $S_\alpha(\sigma, \beta, 0)$ distribution with $\sigma^\alpha = C_\alpha^{-1}$ and $\beta = p - q$. We have to find the relationship between X and Y. Let $h = c^{1/\alpha}$ and $T_h : \mathbb{R} \to \mathbb{R}$ be given by $T_h(x) = hx$. We have:

$$E(\exp(iuhX)) = \exp\left\{\int_{\mathbb{R}} \big(\exp(iuhx) - 1\big)\nu_{\alpha,p}(dx)\right\}$$

$$\begin{aligned}
&= \exp\left\{\int_{\mathbb{R}} \big(\exp(iuy)-1\big)\nu_{\alpha,p}\circ T_h^{-1}(dy)\right\} \\
&= \exp\left\{h^{\alpha}\int_{\mathbb{R}} \big(\exp(iuy)-1\big)\nu_{\alpha,p}(dy)\right\} \\
&= \exp\left\{c\int_{\mathbb{R}} \big(\exp(iuy)-1\big)\nu_{\alpha,p}(dy)\right\} = E(\exp(iuY)),
\end{aligned}$$

where for the third equality we used the scaling property $\nu_{\alpha,p}\circ T_h^{-1} = h^{\alpha}\nu_{\alpha,p}$, and for the last equality we used the fact that $\nu = c\nu_{\alpha,p}$. It follows that $Y \stackrel{d}{=} hX$. By Property 1.2.3 of [30], hX has a $S_{\alpha}(h\sigma,\beta,0)$ distribution. The conclusion follows since $\delta^{\alpha} = cC_{\alpha}^{-1} = h^{\alpha}\sigma^{\alpha}$ and hence $\delta = h\sigma$. The case $a \neq 1$ follows by a similar argument, using the scaling property of the measure ν.
b) The argument is similar to the case $\alpha > 1$. We omit the details.□
.

Remark 2.2.9. In Theorem 2.2.7, we obtain the following relation :

$$\mu = \begin{cases} \gamma - \int_{|x|\le 1} x\nu_{\alpha,p}(dx) & \text{if } \alpha < 1 \\ \gamma + \int_{|x|>1} x\nu_{\alpha,p}(dx) & \text{if } \alpha > 1 \end{cases}$$

We will see that a similar relation holds for the α-stable distribution in $\mathbb{R}^d$.

2.3 Lévy-Khinchine and Spectral Representations

In this section, we derive the characteristic function of an α-stable random vector in $\mathbb{R}^d$. Similarly to the case $d = 1$, we will see that this characteristic function can be written in two ways, using the Lévy measure ν (the Lévy-Khinchine representation) or the spectral measure Γ (the spectral representation). The references for the results in this section are [31] and [30].

We let $\|x\| = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$ be the Euclidean norm of $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $x\cdot y = \sum_{i=1}^d x_i y_i$ be the scalar product of vectors $x = (x_1,\ldots,x_d)$ and $y = (y_1,\ldots,y_d)$. We denote

$$\nu_{\alpha}(dx) = \alpha x^{-\alpha-1}\mathbf{1}_{(0,\infty)}dx.$$

We define the polar coordinate transformation $T : \mathbb{R}_0^d \longmapsto (0,\infty)\times\mathbb{S}_d$ by:

$$T(x) = \left(\|x\|, \frac{x}{\|x\|}\right)$$

where $\mathbb{R}_0^d = \mathbb{R}^d\setminus\{0\}$ and $\mathbb{S}_d = \left\{x\in\mathbb{R}^d; \|x\| = 1\right\}$ is the unit sphere in $\mathbb{R}^d$.

Definition 2.3.1. *A random vector X in $\mathbb{R}^d$ has an* ***infinitely divisible distribution*** *if for each n:*

$$X \stackrel{d}{=} X_{1,n} + X_{2,n} + \ldots + X_{n,n}$$

where $X_{1,n}, X_{2,n} \ldots X_{n,n}$ are i.i.d random vectors in $\mathbb{R}^d$.

Definition 2.3.2. *A Borel measure ν defined on $\mathbb{R}^d$ is called a* ***Lévy measure*** *if $\nu\{0\} = 0$ and*

$$\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

Theorem 2.3.3. *(Lévy-Khinchine Representation) The characteristic function of an infinitely divisible random vector X in $\mathbb{R}^d$ is given by: for all $u \in \mathbb{R}^d$*

$$E(\exp(iu \cdot X)) = \exp\left(i\mu \cdot u - \frac{1}{2}Au \cdot u + \int_{\mathbb{R}^d} \left(\exp(iu \cdot x) - 1 - iu \cdot x\mathbf{1}_{\{\|x\| \le 1\}}\right)\nu(dx)\right) \tag{2.3.1}$$

where A is a symmetric non-negative-definite $d \times d$ matrix, ν is a Lévy measure on $\mathbb{R}^d$ and $\mu \in \mathbb{R}^d$. Relation (2.3.1) is called the Lévy-Khinchine Representation of the characteristic function of X.

Remark 2.3.4. The triplet (μ, A, ν) is unique and its called the *generating triplet* of X. The notation $X \sim ID(\mu, A, \nu)$ will be used to denote an infinitely divisible random vector X with characteristic function (2.3.1).

Remark 2.3.5. Let $X \sim ID(\mu, A, \nu)$ with a characteristic function given by (2.3.1), then $X \sim ID(\mu_c, A, \nu)$ where the characteristic function is given by:

$$E(\exp(iu \cdot X)) = \exp\left(i\mu_c \cdot u - \frac{1}{2}Au \cdot u + \int_{\mathbb{R}^d} \left(\exp(iu \cdot x) - 1 - iu \cdot x\mathbf{1}_{\{\|x\| \le c\}}\right)\nu(dx)\right) \tag{2.3.2}$$

where $\mu_c = \mu + \int_{\mathbb{R}^d} x\mathbf{1}_{\{1 < \|x\| \le c\}}(x)\nu(dx)$ and $c > 1$

Definition 2.3.6. *A random vector $X = (X_1, ..., X_d)$ in $\mathbb{R}^d$ has a* ***stable distribution*** *if for any positive numbers A and B, there is a positive number C and vector $D \in \mathbb{R}^d$ such that*

$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX + D \tag{2.3.3}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X and $\stackrel{d}{=}$ denotes equality in distribution.

Theorem 2.3.7. *Let $X = (X_1, ..., X_d)$ be a stable random vector in $\mathbb{R}^d$. Then there is a constant $\alpha \in (0, 2]$ such that, in (2.3.3), $C = (A^\alpha + B^\alpha)^{\frac{1}{\alpha}}$. In this case, we say that X has an α-stable distribution.*

Theorem 2.3.8. *Any linear combination of the components of an α-stable random vector X of the type $Y = \sum_{k=1}^{d} b_k X_k$ is an α-stable random variable.*

Theorem 2.3.9. *(Theorem 14.3 of [31]) Let $X \sim ID\,(\mu, 0, \nu)$ be a random vector in $\mathbb{R}^d$ where $\mu \in \mathbb{R}^d$ and ν is a Lévy measure. The following statements are equivalent:*
(i) X has an α-stable distribution;
(ii) there exist $c > 0$ and a probability measure Γ_1 on $\mathbb{S}_d$ such that :

$$\nu \circ T^{-1} = c\nu_\alpha \times \Gamma_1 \ \text{on} \ (0, \infty) \times \mathbb{S}_d \tag{2.3.4}$$

(iii) ν satisfies the following scaling property:

$$\nu\,(sA) = s^{-\alpha}\nu\,(A) \tag{2.3.5}$$

for any $s > 0$ and for any Borel set $A \subset \mathbb{R}^d$.

Remark 2.3.10. Let Z be a random variable with a characteristic function given by (2.3.2). If ν satisfies (2.3.4), then Z has an α-stable distribution (see Remark 2.3.5).

Remark 2.3.11. It follows from (2.3.4) that

$$\int_{\mathbb{R}^d} f(x)\nu(dx) = c\int_{\mathbb{S}_d}\int_0^\infty f(rz)\nu_\alpha(dr)\Gamma_1(dz) \tag{2.3.6}$$

for any function $f : \mathbb{R}^d \longmapsto \mathbb{R}$ which is integrable with respect to ν.

Remark 2.3.12. Note that (2.3.4) implies that ν is a Lévy measure on $\mathbb{R}^d$, since by (2.3.6) we have:

$$\begin{aligned} \int_{\mathbb{R}^d}(\|x\|^2 \wedge 1)\nu(dx) &= c\int_{\mathbb{S}_d}\int_0^\infty(\|rz\|^2 \wedge 1)\alpha r^{-\alpha-1}dr\Gamma_1(dz) \\ &= c\alpha\Big(\int_0^1 r^{1-\alpha}dr + \int_1^\infty r^{-\alpha-1}dr\Big) < \infty. \end{aligned}$$

Lemma 2.3.13. *If ν satisfies (2.3.5) for any $s > 0$ and $I = [y, x]^c$ for $y \le 0 \le x$, then ν satisfies (2.3.5) for any $s > 0$ and for any Borel set $A \subset \mathbb{R}^d$. Here we define $[y, x]$ by*

$$[y, x] = \big\{z = (z_1, \ldots, z_d) \in \mathbb{R}^d, y_i \le z_i \le x_i, i = 1, \ldots, d\big\}$$

Lemma 2.3.14. *(Proposition 14.5 of [31]) Let ν be the Lévy measure of an α-stable random vector in $\mathbb{R}^d$. Then:*
(i) $\alpha < 1$ if and only if $\int_{\|x\|\le 1}\|x\|\,\nu(dx) < \infty$
(ii) $\alpha > 1$ if and only if $\int_{\|x\|>1}\|x\|\,\nu(dx) < \infty$.

Remark 2.3.15. Due to Lemma 2.3.14, we see that if $X \sim ID(\mu, 0, \nu)$ where ν satisfies (2.3.4), then the characteristic function of X has the following form:
(i) If $\alpha < 1$, for any $u \in \mathbb{R}^d$,

$$E(\exp(iu \cdot X)) = \exp\left\{iu \cdot \mu^0 + \int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1\Big)\nu(dx)\right\}, \tag{2.3.7}$$

(ii) if $\alpha > 1$, for any $u \in \mathbb{R}^d$,

$$E(\exp(iu \cdot X)) = \exp\left\{iu \cdot \mu^0 + \int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 - iu \cdot x\Big)\nu(dx)\right\}, \tag{2.3.8}$$

where

$$\mu^0 = \begin{cases} \mu - \int_{\|x\| \le 1} x\nu(dx) & \text{if } \alpha < 1 \\ \mu + \int_{\|x\| > 1} x\nu(dx) & \text{if } \alpha > 1. \end{cases} \tag{2.3.9}$$

The following result is an extension of Theorem 2.2.7 to $\mathbb{R}^d$.

Theorem 2.3.16. *(Spectral Representation) If $X \sim ID(\mu, 0, \nu)$ where $\mu \in \mathbb{R}^d$ and ν satisfies (2.3.4), then the characteristic function of X can be written as follows:*
(i) If $\alpha \neq 1$, for all $u \in \mathbb{R}^d$

$$E(\exp(iu \cdot X)) = \exp\left\{iu \cdot \mu^0 - \int_{\mathbb{S}_d} |u \cdot z|^\alpha \left(1 - isgn\,(u \cdot z)\tan\left(\frac{\pi\alpha}{2}\right)\right)\Gamma(dz)\right\}.$$

(ii) If $\alpha = 1$, for all $u \in \mathbb{R}^d$

$$E(\exp(iu \cdot X)) = \exp\left\{iu \cdot \mu^0 - \int_{\mathbb{S}_d} |u \cdot z| \left(1 + i\frac{2}{\pi} sgn\,(u \cdot z)\ln|u \cdot z|\right)\Gamma(dz)\right\},$$

where Γ is a finite measure on $\mathbb{S}_d$ given by:

$$\Gamma = cC_\alpha^{-1}\Gamma_1 \tag{2.3.10}$$

for a probability measure Γ_1 on $\mathbb{S}_d$, c being the constant appearing in (2.3.4), C_α the constant defined in (2.2.5) and the shift vector μ^0 is given by

$$\mu^0 = \begin{cases} \mu - \int_{\|x\| \le 1} x\nu(dx) & \text{if } \alpha < 1 \\ \mu + \int_{\|x\| > 1} x\nu(dx) & \text{if } \alpha > 1 \\ \mu + ca\mu^1 & \text{if } \alpha = 1 \end{cases} \tag{2.3.11}$$

with a defined by (2.2.6) and the vector μ^1 defined by:

$$\mu_j^1 = \int_{\mathbb{S}_d} z_j \Gamma_1(dz). \tag{2.3.12}$$

Proof. We consider separately three cases:

Case 1: $0 < \alpha < 1$

From Remark 2.3.15, we know that

$$E(\exp(iu\cdot X)) = \exp\left\{iu\cdot\mu^0 + \int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1\Big)\nu(dx)\right\} \qquad (2.3.13)$$

with μ^0 given by (2.3.9). Due to (2.3.6), it follows that

$$\begin{aligned}\int_{\mathbb{R}^d}(\exp(iu\cdot x)-1)\nu(dx)) &= c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(iu\cdot rz)-1\Big)\nu_\alpha(dr)\Gamma_1(dz)\\ &= c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(ir(u\cdot z))-1\Big)\frac{\alpha}{r^{\alpha+1}}dr\Gamma_1(dz).\end{aligned}$$

From (2.2.7), it follows that

$$\alpha\int_0^\infty\Big(\exp(ir(u\cdot z))-1\Big)\frac{1}{r^{\alpha+1}}dr = -\frac{\Gamma(2-\alpha)}{1-\alpha}\,|u\cdot z|^\alpha\exp\Big(-i\frac{\pi\alpha}{2}\mathrm{sgn}(u\cdot z)\Big)$$

Then we obtain,

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1\Big)\nu(dx))\\ =\;& c\int_{\mathbb{S}_d}\Big(-\frac{\Gamma(2-\alpha)}{1-\alpha}\,|u\cdot z|^\alpha\exp\Big(-i\frac{\pi\alpha}{2}\;sgn(u\cdot z)\Big)\Big)\Gamma_1(dz)\\ =\;& c\int_{\mathbb{S}_d}\Big(-\frac{\Gamma(2-\alpha)}{1-\alpha}\,|u\cdot z|^\alpha\cos\Big(\frac{\pi\alpha}{2}\Big)\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\;\mathrm{sgn}(u\cdot z)\Big)\Big)\Gamma_1(dz)\\ =\;& -cC_\alpha^{-1}\int_{\mathbb{S}_d}\Big(|u\cdot z|^\alpha\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}(u\cdot z)\Big)\Big)\Gamma_1(dz)\\ =\;& -\int_{\mathbb{S}_d}\Big(|u\cdot z|^\alpha\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}(u\cdot z)\Big)\Big)cC_\alpha^{-1}\Gamma_1(dz).\end{aligned}$$

The conclusion follows using (2.3.10) and (2.3.13).

Case 2: $1 < \alpha < 2$

From Remark 2.3.15, we know that

$$E(\exp(iu\cdot X)) = \exp\left\{iu\cdot\mu^0 + \int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx)\right\} \qquad (2.3.14)$$

with μ^0 is given by (2.3.9). Due to (2.3.6), it follows that

$$\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx) = c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(iu\cdot rz)-1-iu\cdot rz\Big)$$

$$\begin{aligned}&\nu_\alpha(dr)\Gamma_1(dz)\\ =\ & c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(ir(u\cdot z))-1-ir(u\cdot z)\Big)\frac{\alpha}{r^{\alpha+1}}dr\Gamma_1(dz).\end{aligned}$$

From (2.2.8), it follows that

$$\alpha\int_0^\infty\Big(\exp(ir(u\cdot z))-1-ir(u\cdot z)\Big)\frac{1}{r^{\alpha+1}}dr=-\frac{\Gamma(2-\alpha)}{1-\alpha}\left|u.z\right|^\alpha\exp\Big(-i\frac{\pi\alpha}{2}\mathrm{sgn}(u\cdot z)\Big).$$

Then we obtain,

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx))\\ =\ & c\int_{\mathbb{S}_d}\bigg(-\frac{\Gamma(2-\alpha)}{1-\alpha}\left|u\cdot z\right|^\alpha\exp\Big(-i\frac{\pi\alpha}{2}\mathrm{sgn}(u\cdot z)\Big)\bigg)\Gamma_1(dz)\\ =\ & c\int_{\mathbb{S}_d}\bigg(-\frac{\Gamma(2-\alpha)}{1-\alpha}\left|u\cdot z\right|^\alpha\cos\Big(\frac{\pi\alpha}{2}\Big)\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}(u\cdot z)\Big)\bigg)\Gamma_1(dz)\\ =\ & -cC_\alpha^{-1}\int_{\mathbb{S}_d}\bigg(\left|u\cdot z\right|^\alpha\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\mathrm{sgn}(u\cdot z)\Big)\bigg)\Gamma_1(dz)\\ =\ & -\int_{\mathbb{S}_d}\bigg(\left|u\cdot z\right|^\alpha\Big(1-i\tan\Big(\frac{\pi\alpha}{2}\Big)\ \mathrm{sgn}(u\cdot z)\Big)\bigg)cC_\alpha^{-1}\Gamma_1(dz).\end{aligned}$$

The conclusion follows using (2.3.10) and (2.3.14).
Case 3: $\alpha=1$
Using 2.3.1 and 2.3.6 we have for any $u\in\mathbb{R}^d$:

$$\begin{aligned}&E(\exp(iu\cdot X))=\exp\bigg\{iu\cdot\mu+\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1-iu\cdot x\mathbf{1}_{\{\|x\|\le1\}}(x)\Big)\nu(dx)\bigg\}\\ =\ & \exp\bigg\{iu\cdot\mu+c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(i(u\cdot rz))-1-i(u\cdot rz)\mathbf{1}_{[0,1]}(r)\Big)\frac{dr}{r^2}\Gamma_1(dz)\bigg\}\\ =\ & \exp\bigg\{iu\cdot\mu+c\int_{\mathbb{S}_d}\int_0^\infty\Big(\exp(ir(u\cdot z))-1-ir(u\cdot z)\mathbf{1}_{[0,1]}(r)\Big)\frac{dr}{r^2}\Gamma_1(dz)\bigg\}\end{aligned}$$

From (2.2.9), it follows that:

$$\int_0^\infty\Big(\exp(ir(u\cdot z))-1-ir(u\cdot z)\mathbf{1}_{[0,1]}(r)\Big)\frac{dr}{r^2}=-\frac{\pi}{2}\left|u\cdot z\right|-i\left|u\cdot z\right|\ sgn(u\cdot z)\ln\left|u\cdot z\right|+ia(u\cdot z).$$

Then,

$$E(\exp(iu\cdot X))\ =\ \exp\bigg(-c\frac{\pi}{2}\int_{\mathbb{S}_d}\left|u\cdot z\right|\Big(1-i\frac{2}{\pi}\ \mathrm{sgn}(u\cdot z)\ln\left|u\cdot z\right|\Big)\Gamma_1(dz)$$

$$+ \quad ica \int_{\mathbb{S}_d} u \cdot z \Gamma_1(dz) + iu \cdot \mu \Big).$$

The conclusion follows by the definition (2.3.10) of Γ (recalling that $C_1^{-1} = \frac{\pi}{2}$) and the fact that

$$\mu^0 = \mu + ca \int_{\mathbb{S}_d} z \Gamma_1(dz).$$

This finishes the proof. $\square$

Remark 2.3.17. An α-stable random vector X in $\mathbb{R}^d$ is determined uniquely by the measure Γ (called spectral measure) and the shift vector $\mu^0 \in \mathbb{R}^d$ appearing in Theorem 2.4.2. We say that (Γ, μ^0) is the **spectral pair** of X.

Remark 2.3.18. As a consequence of Theorem 2.3.16 and Remark 2.3.15, we see that an α-stable random vector X with spectral pair $(0, \Gamma)$ has the characteristic function given by: for any $u \in \mathbb{R}^d$

$$E(\exp(iu \cdot X)) = \exp\left\{ \int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 \Big) \nu(dx) \right\} \text{ if } \alpha < 1$$

$$E(\exp(iu \cdot X)) = \exp\left\{ \int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 - iu \cdot x \Big) \nu(dx) \right\} \text{ if } \alpha > 1,$$

and if $\alpha = 1$

$$E(\exp(iu \cdot X)) = \exp\left\{ iu \cdot \mu + \int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 - iu \cdot x \mathbf{1}_{\{\|x\| \leq 1\}}(x) \Big) \nu(dx) \right\}$$

where ν and Γ are related by formulas (2.3.4) and (2.3.10), and $\mu = -ca\mu^1$ with a given by (2.2.6) and μ^1 given by (2.3.12).

2.4 Representations of α-stable random vectors

In this section, we give three representations of an α-stable random vector with spectral pair $(0, \Gamma)$: the series representation, the integral representation and the Poisson representation. We begin by recalling the definition of an α-stable random measure, following Section 3.3 of [30]

Definition 2.4.1. *Let $(E, \mathcal{E}, m)$ be a measure space and $\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}$. Let $\alpha \in (0, 2)$ and $\beta \in [-1, 1]$ be arbitrary. Let $M = \{M(A)\}_{A \in \mathcal{E}_0}$ be a collection of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that:*

- *$M(A_1), \ldots, M(A_k)$ are independent random variables if $A_1, \ldots, A_k \in \mathcal{E}_0$ are disjoint*

- *$M(\bigcup_{k\geq 1} A_k) = \sum_{k\geq 1} M(A_k)$ for any pairwise disjoint sets $(A_k)_k \subset \mathcal{E}_0$ such that $m(\bigcup_{k\geq 1} A_k) < \infty$*

- *For each $A \in \mathcal{E}_0$, $M(A) \sim S_\alpha\Bigg((m(A))^{\frac{1}{\alpha}}, \beta, 0\Bigg)$.*

*We say that M is called an α-**stable random measure** on $(E, \mathcal{E})$ with control measure m and constant skewness intensity β.*

Fix $\beta \in [-1, 1]$. Let $(E, \mathcal{E}, m)$ be a finite measure space and $\mathcal{F}_\alpha(m)$ the linear space of the $\mathcal{E}$-measurable functions such that:

1. if $\alpha \neq 1$,
$$\int_E |f(x)|^\alpha \, m(dx) < \infty$$

2. if $\alpha = 1$,
$$\int_E |f(x)| \, m(dx) < \infty \text{ and } \int_E \beta \, |f(x) \ln |f(x)|| \, m(dx) < \infty$$

For any $f \in \mathcal{F}_\alpha(m)$, one can define the α-stable integral

$$I(f) = \int_E f(x) M(dx)$$

(see Section 3.4 of [30]). The following result gives the finite dimensional distributions of the process $\{I(f)\}_{f\in\mathcal{F}_\alpha(m)}$

Proposition 2.4.2 (Proposition 3.4.3 of [30])**.** *Let M be an α-stable random measure on E with a control measure m and constant skewness intensity β. For any $f_1, f_2, \ldots f_d \in \mathcal{F}_\alpha(m)$, the random vector $(I(f_1), I(f_2), \ldots, I(f_d))$ has an α-stable distribution with spectral pair (μ^0, Γ) constructed as follows. For any Borel set $A \subset \mathbb{S}_d$*

$$\Gamma(A) = pm_1(g^{-1}(A)) + qm_1(g^{-1}(-A)) \tag{2.4.1}$$

where $p = (1 + \beta)/2$, $q = (1 - \beta)/2$, $m_1(dx) = \left(\sum_{j=1}^{d} f_j^2(x)\right)^{\alpha/2} m(dx)$ is a measure on $E_+ = \left\{x \in E, \sum_{j=1}^{d} f_j^2(x) > 0\right\}$ and $g(x) = (g_1(x), g_2(x), \ldots, g_d(x))$ with $g_j(x) =$

$\frac{f_j(x)}{\left(\sum\limits_{j=1}^{d} f_j^2(x)\right)^{1/2}}$, $j = 1, 2, \ldots, d$. *The shift vector* $\mu^0 = (\mu_1^0, \mu_2^0, \ldots, \mu_d^0)$ *has components given by: for any* $j = 1, 2, \ldots, d$,

$$\mu_j^0 = \begin{cases} 0 & \text{if } \alpha \neq 1 \\ -\frac{1}{\pi}\beta \int_{E+} \Big(f_j(x) \ln(\sum\limits_{j=1}^{d} f_j^2(x))\Big) m(dx) & \text{if } \alpha = 1 \end{cases} \tag{2.4.2}$$

For this result, we need the following centering constants, which are encountered also in the series representation of an α-stable random variable: for any $i = 1, 2, \ldots, d$

$$b_i^{(\alpha)} = \begin{cases} 0 & \text{if } \alpha < 1 \\ \\ \frac{\alpha}{\alpha-1}(i^{\frac{\alpha-1}{\alpha}} - (i-1)^{\frac{\alpha-1}{\alpha}}) & \text{if } \alpha > 1. \\ \\ \int_{1/i}^{1/i-1} \frac{\sin x}{x^2} dx & \text{if } \alpha = 1 \end{cases} \tag{2.4.3}$$

Theorem 2.4.3. *Let* $(E, \mathcal{E}, m)$ *be a finite measure space. Let* $\beta \in [-1, 1]$ *and* $\alpha \in (0, 2)$. *Let* $p = (1+\beta)/2$ *and* $q = (1-\beta)/2$. *Consider* $\{\varepsilon_1, \varepsilon_2, \ldots\}$, $\{V_1, V_2, \ldots\}$ *and* $\{\overline{\Gamma_1}, \overline{\Gamma_2}, \ldots\}$ *three independent sequences of random variables satisfying:*

- $P(\varepsilon_i = 1) = p, P(\varepsilon_i = -1) = q$
- $(V_i)_{i\geq 1}$ *are i.i.d with values in* E *with law* $m(\cdot)/m(E)$
- $\overline{\Gamma_i} = \sum\limits_{j=1}^{i} e_j$ *where* $(e_i)_{i\geq 1}$ *are i.i.d exponential random variables with mean 1.*

Let $f_1, f_2, \ldots, f_d \in \mathcal{F}_\alpha(m)$ *be arbitrary. Define* $\eta_{f_1,f_2,\ldots,f_d} = (\eta_{f_1}, \eta_{f_2} \ldots, \eta_{f_d})$ *where:*

$$\eta_f = \begin{cases} 0 & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \ln(\frac{2}{\pi} m(E))\beta \int_E f(x) m(dx) & \text{if } \alpha = 1. \end{cases}$$

For each $i \geq 1$, *let*

$$W_i = (f_1(V_i), f_2(V_i) \ldots, f_d(V_i))$$

Then

$$X \overset{a.s}{=} (C_\alpha m(E))^{1/\alpha} \sum_{i\geq 1} \Big(\varepsilon_i \overline{\Gamma}_i^{-1/\alpha} W_i - b_i^{(\alpha)} \beta E(W_i)\Big) + \eta_{f_1,f_2,\ldots,f_d} \tag{2.4.4}$$

has an α*-stable distribution in* $\mathbb{R}^d$ *with spectral pair* (μ^0, Γ) *specified by (5.2.8) and (2.4.2)*

Proof. For any $f \in \mathcal{F}_\alpha(m)$, let $I(f) = \int_E f(x)M(dx)$ be the α-stable integral with respect to M and

$$S(f) = (C_\alpha m(E))^{1/\alpha} \sum_{i\geq 1} \left(\varepsilon_i \overline{\Gamma}_i^{-1/\alpha} f(V_i) - b_i^{(\alpha)} \beta \mathbb{E} f(V_i)\right) + \eta_f$$

By Theorem 3.10.1 of [30],

$$(I(f_1), I(f_2) \ldots, I(f_d)) \stackrel{d}{=} (S(f_1), S(f_2) \ldots, S(f_d)).$$

By Proposition 2.4.2, the vector $X = (I(f_1), I(f_2) \ldots, I(f_d))$ has an α-stable distribution in $\mathbb{R}^d$ with spectral pair (μ^0, Γ). On the other hand, the vector $Y = (S(f_1), S(f_2) \ldots, S(f_d))$ has the series representation given on the right hand side of (2.4.4). □

Based on the previous result, we obtain the series representation of an α-stable random vector.

Theorem 2.4.4. *(Series Representation) Let X be an α-stable random vector in $\mathbb{R}^d$ with spectral pair $(0, \Gamma)$. Assume that Γ is given by (2.3.10) where $c > 0$, $\alpha \in (0, 2)$, $\overline{\Gamma_1}$ is a probability measure on $\mathbb{S}_d$ and C_α is given by (2.2.5). Then X can be represented as:*

$$X \stackrel{d}{=} c^{1/\alpha} \sum_{i\geq 1} \left(\overline{\Gamma}_i^{-1/\alpha} W_i - b_i^{(\alpha)} E(W_i)\right) + \eta$$

where $b_i^{(\alpha)}$ is given by (2.4.3), $(\overline{\Gamma}_i)_{i\geq 1}$ is the same as in Theorem 2.4.3, $(W_i)_{i\geq 1}$ is a sequence of i.i.d random variables on $\mathbb{S}_d$ with law $\overline{\Gamma_1}$, independent of $\overline{(\Gamma_i)}_{i\geq 1}$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$ has components given by: for any $j = 1, \ldots, d$

$$\eta_j = \begin{cases} 0 & \text{if } \alpha \neq 1 \\ c\ln(c)\mu_j^1 & \text{if } \alpha = 1 \end{cases}$$

Here μ^1 is the vector defined by (2.3.12).

Proof. We apply Theorem 2.4.3 with $E = \mathbb{S}_d, m = \Gamma, \beta = 1$ and the functions $f_j : \mathbb{S}_d \to \mathbb{R}$ given by $f_j(z) = z_j$, for $j = 1, \ldots, d$. Note that in this case, $V_i = W_i$ and

$$C_\alpha m(E) = C_\alpha \Gamma(\mathbb{S}_d) = C_\alpha c C_\alpha^{-1} = c.$$

Moreover $\sum_{j=1}^d f_j^2(z) = \sum z_j^2 = \|z\|^2 = 1$ for all $z \in \mathbb{S}_d$. Hence $g(z) = z$ for all $z \in \mathbb{R}_d$, and $m_1 = m = \Gamma$. When $\alpha = 1$, $C_1 = \frac{2}{\pi}$ and hence for any $j = 1, \ldots, d$

$$\eta_j = C_1 \ln(C_1 m(E)) \int_{\mathbb{S}_d} z_j \Gamma(dz) = C_1 \ln(c) c C_1^{-1} \int_{\mathbb{S}_d} z_j \Gamma_1(dz) = c\ln(c)\mu_j^1.$$

This finishes the proof. □

Theorem 2.4.5. *(Integral Representation) Let $X = (X_1, X_2, \ldots, X_d)$ be an α-stable random vector in $\mathbb{R}^d$ with the spectral pair $(0, \Gamma)$. Then X can be represented as :*

$$X \stackrel{d}{=} \left(\int_{\mathbb{S}_d} z_1 M(dz), \int_{\mathbb{S}_d} z_2 M(dz), \ldots, \int_{\mathbb{S}_d} z_d M(dz) \right) \tag{2.4.5}$$

where M is an α-stable random measure on $\mathbb{S}_d$ with control measure $m = \Gamma$ and constant skewness intensity $\beta = 1$.

Proof. We apply Proposition 2.4.2 with $E = \mathbb{S}_d, m = \Gamma, \beta = 1$ and the functions $f_j : \mathbb{S}_d \to \mathbb{R}$ given by $f_j(z) = z_j$, for $j = 1, \ldots, d$. As in the proof of Theorem 6.2.2, in this case we have $m_1 = m = \Gamma$. The fact that the shift parameter is 0 in the case $\alpha = 1$ follows because in this case μ^0 is given by: for any $j = 1, \ldots, d$

$$\mu_j^0 = -\frac{1}{\pi} \int_{\mathbb{S}_d} z_j \ln(\|z\|^2) m(dz) = 0.$$

This finishes the proof. $\square$

The last representation of an α-stable random vector is based on a Poisson random measure. For this construction, we need the following preliminary result.

Lemma 2.4.6. *Let $N = \sum_{i \geq 1} \delta_{J_i}$ be be a Poisson random measure on $\mathbb{R}^d$ of intensity ν and let $I \subset \mathbb{R}^d$ be a set bounded away from 0. Define :*

$$Z = \int_I x N(dx) = \sum_{i \geq 1} J_i 1_{\{J_i \in I\}}.$$

Then, Z is a compound Poisson random vector with characteristic function given by: for any $u \in \mathbb{R}^d$

$$E(\exp(iu \cdot Z)) = \exp \left\{ \int_I \Big(\exp(iu \cdot x) - 1 \Big) \nu(dx) \right\}.$$

Proof. Note that $N|_I \stackrel{d}{=} \sum_{i=1}^{\tau} \delta_{T_i}$ where $(T_i)_{i \geq 1}$ are i.i.d. random vectors with values in the set I, and distributions $\nu/\nu(I)$, and τ is a Poisson random variable of mean $\nu(I)$, independent of $(T_i)_{i \geq 1}$. Then $Z \stackrel{d}{=} \sum_{i=1}^{\tau} T_i$, and for any $u \in \mathbb{R}^d$ we have :

$$\begin{aligned} E\Big(\exp(iu \cdot Z) \Big) &= E\big(\exp(iu \cdot \sum_{i=1}^{\tau} T_i) \big) = E\big(\exp(i \sum_{i=1}^{\tau} u \cdot T_i) \big) \\ &= E\left[E\Big(\exp(i \sum_{i=1}^{\tau} u \cdot T_i) \Big| \tau \Big) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\geq 1}\Big(E(\exp(iu\cdot T_1))\Big)^k P(\tau=k)\\
&= \exp(-\nu(I))\sum_{k\geq 1}[E(\exp(iu\cdot T_1))\nu(I)]^k\frac{1}{k!}\\
&= \exp\Big(-\nu(I)\Big)\exp\Big(\nu(I)E\big(\exp(iu\cdot T_1)\big)\Big)\\
&= \exp\big\{\nu(I)\big(E(\exp(iu\cdot T_1)-1)\big)\big\}\\
&= \exp\left\{\nu(I)\int_I\big(\exp(iu\cdot x-1)\big)\frac{\nu(dx)}{\nu(I)}\right\}\\
&= \exp\left\{\int_I\big(\exp(iu\cdot x-1)\big)\nu(dx)\right\}.
\end{aligned}
$$

This finishes the proof. □

Theorem 2.4.7. *(Poisson Representation) Let X be an α-stable random vector in $\mathbb{R}^d$ with spectral pair $(0,\Gamma)$, where Γ is given by (2.3.10) for some $c>0$, $\alpha\in(0,2)$ and Γ_1 a probability measure on $\mathbb{S}_d$. Let $N=\sum_{i\geq 1}\delta_{J_i}$ be a Poisson random measure on $\mathbb{R}^d$ of intensity ν given by (2.3.4). Then X can be represented as follows:*
(i) if $\alpha<1$

$$X\overset{d}{=}\lim_{\varepsilon\to 0}\int_{\varepsilon<\|x\|\leq 1}xN(dx)+\int_{\|x\|>1}xN(dx)$$

(ii) if $\alpha>1$

$$X\overset{d}{=}\lim_{\varepsilon\to 0}\left(\int_{\|x\|>\varepsilon}xN(dx)-\int_{\|x\|>\varepsilon}x\nu(dx)\right)$$

(iii) if $\alpha=1$

$$X\overset{d}{=}\lim_{\varepsilon\to 0}\left(\int_{\|x\|>\varepsilon}xN(dx)-\int_{\varepsilon<\|x\|\leq 1}x\nu(dx)\right)+\mu$$

with $\mu=-ca\mu^1$ where μ^1 is given by (2.3.12) and a given by (2.2.6).

Proof. There are two methods for proving this result. The first method relies on the general Poisson representation given by Theorem 3.12.2 of [31]. We present below another method inspired by Section 5.5 of [28]. Let $V_\varepsilon=\big\{x\in\mathbb{R}^d;\|x\|>\varepsilon\big\}$ and consider a sequence $\varepsilon_k\downarrow 0$ such that $\varepsilon_0=1$. Define

$$Z_0=\int_{\|x\|>1}xN(dx)=\sum_{i\geq 1}J_i\mathbf{1}_{\{\|J_i\|>1\}}\tag{2.4.6}$$

and for each $j\geq 1$,

$$Z_j=\int_{\varepsilon_j<\|x\|\leq\varepsilon_{j-1}}xN(dx)=\sum_{i\geq 1}J_i\mathbf{1}_{\{\varepsilon_j<\|J_i\|\leq\varepsilon_{j-1}\}}.$$

By Lemma 2.4.6, Z_j is a compound Poisson random variable with characteristic function

$$E(\exp(iu \cdot Z_j)) = \exp\left\{\int_{I_j}\Big(\exp(iu\cdot x) - 1\Big)\nu(dx)\right\}, u \in \mathbb{R}^d,$$

where $I_j = \left\{x \in \mathbb{R}^d; \varepsilon_j < \|x\| \le \varepsilon_{j-1}\right\}$ for $j \ge 1$ and $I_0 = \left\{x \in \mathbb{R}^d; \|x\| > 1\right\}$. In particular, if $Z_j = \left(Z_j^{(1)}, \dots, Z_j^{(d)}\right)$, then for any $l = 1, \dots, d$

$$E(Z_j^{(l)}) = \int_{I_j} x^{(l)}\nu(dx) \text{ and } \mathrm{Var}(Z_j^{(l)}) = \int_{I_j}\left(x^{(l)}\right)^2\nu(dx).$$

By Lemma A.0.2 the variables $(Z_j)_{j\ge 0}$ are independent (since the sets $(I_j)_{j\ge0}$ are disjoint). Let

$$Z = Z_0 + \sum_{j\ge1}(Z_j - \mathbb{E}(Z_j)) \text{ a.s.} \tag{2.4.7}$$

Note that this series converges by Kolmogorov's theorem since by Remark 2.3.12, for any $l = 1, \dots, d$

$$\sum_{j\ge1}\mathrm{Var}(Z_j^{(l)}) = \int_{\|x\|\le1}\left(x^{(l)}\right)^2\nu(dx) < \int_{\|x\|\le1}\|x\|^2\nu(dx) < \infty.$$

Then,

$$\begin{aligned}
Z &\overset{a.s.}{=} \lim_{k\to\infty}\sum_{j=1}^{k}(Z_j - \mathbb{E}(Z_j)) + Z_0\\
&= \lim_{k\to\infty}\sum_{j=1}^{k}\left(\sum_{i\ge1}J_i 1_{\{\varepsilon_j<\|J_i\|\le\varepsilon_{j-1}\}} - \int_{\varepsilon_j<\|x\|\le\varepsilon_{j-1}} x\nu(dx)\right)\\
&+ \sum_{i\ge1}J_i\mathbf{1}_{\{\|J_i\|>1\}}\\
&= \lim_{k\to\infty}\sum_{i\ge1}\sum_{j=1}^{k}J_i 1_{\{\varepsilon_j<\|J_i\|\le\varepsilon_{j-1}\}} - \sum_{j=1}^{k}\int_{\varepsilon_j<\|x\|\le\varepsilon_{j-1}} x\nu(dx)\\
&+ \sum_{i\ge1}J_i\mathbf{1}_{\{\|J_i\|>1\}}\\
&= \lim_{k\to\infty}\left(\sum_{i\ge1}J_i 1_{\{\|J_i\|>\varepsilon_k\}} - \int_{\varepsilon_k<\|x\|\le1}x\nu(dx)\right) := \lim_{k\to\infty} Z^{(\varepsilon_k)}.
\end{aligned}$$

Clearly

$$Z^{(\varepsilon_k)} \overset{a.s.}{\to} Z, \text{ as } k \to \infty$$

implies that

$$Z^{(\varepsilon_k)} \xrightarrow{d} Z, \text{ as } k \to \infty. \tag{2.4.8}$$

We show that Z is an α-stable random vector. To see this, note that for any $u \in \mathbb{R}^d$

$$\begin{aligned}
E(\exp(iu\cdot Z)) &= E\Bigg(\exp\Bigg\{iu\cdot(Z_0+\sum_{j\geq 1}(Z_j-\mathbb{E}(Z_j)))\Bigg\}\Bigg)\\
= \quad & E\Bigg(\exp\Bigg\{iu\cdot Z_0+\big(iu\cdot\sum_{j\geq 1}Z_j-\mathbb{E}(Z_j)\big)\Bigg\}\Bigg)\\
= \quad & E\Big(\exp(iu\cdot Z_0)\Big)\prod_{j=1}^{\infty}E\Big(\exp\{iu\cdot(Z_j-\mathbb{E}(Z_j))\}\Big)\\
= \quad & \exp\Bigg\{\int_{\|x\|>1}(\exp(iu\cdot x)-1)\nu(dx)\Bigg\}\\
. \quad & \prod_{j=1}^{\infty}\exp\Bigg(\int_{\varepsilon_j<\|x\|\leq\varepsilon_{j-1}}\Big(\exp(iu\cdot x)-1\Big)\nu(dx)-i\int_{\varepsilon_j<\|x\|\leq\varepsilon_{j-1}}u\cdot x\nu(dx)\Bigg)\\
= \quad & \exp\Bigg(\int_{\|x\|>1}(\exp(iu\cdot x)-1)\nu(dx)\\
+ \quad & \sum_{j\geq 1}\Big(\int_{\varepsilon_j<\|x\|\leq\varepsilon_{j-1}}\Big(\exp(iu\cdot x)-1\Big)\nu(dx)-i\int_{\varepsilon_j<\|x\|\leq\varepsilon_{j-1}}u\cdot x\nu(dx)\Big)\Bigg)\\
= \quad & \exp\Bigg\{\int_{\mathbb{R}^d}\Bigg(\exp(iu\cdot x)-1-iu\cdot x\mathbf{1}_{\{\|x\|\leq 1\}}\Bigg)\nu(dx)\Bigg\}.
\end{aligned}$$

This proves that $Z \sim ID(0,0,\nu)$ (see Lévy-Kinchine representation; Theorem 2.3.3). Note that ν is a Lévy measure since it satisfies (2.3.4); see Remark 2.3.10. Using Theorem 2.3.9 and the Spectral Representation (Theorem 2.3.16), we conclude that Z has an α-stable distribution with spectral pair (μ^0,Γ), where the spectral measure Γ is given by (2.3.10) and the shift vector μ^0 is given by

$$\mu^0 = \begin{cases} -\int_{\|x\|\leq 1} x\nu(dx) & \text{if } \alpha<1 \\ \\ \int_{\|x\|>1} x\nu(dx) & \text{if } \alpha>1 \\ \\ ca\mu^1 & \text{if } \alpha=1. \end{cases} \tag{2.4.9}$$

The conclusion follows using (2.4.8), observing that the random vector X can be represented as

$$X = \begin{cases} Z + \int_{\|x\|\le 1} x\nu(dx) & \text{if } \alpha < 1 \\ Z - \int_{\|x\|>1} x\nu(dx) & \text{if } \alpha > 1 \\ Z - c\alpha\mu^1 & \text{if } \alpha = 1. \end{cases} \tag{2.4.10}$$

This finishes the proof. □

2.5 Stable Central Limit Theorem in $\mathbb{R}^d$

In this section, we show that an α-stable random vector in $\mathbb{R}^d$ arises as the limit in distribution of the partial sum of a sequence of i.i.d regularly varying vectors in $\mathbb{R}^d$. This result will be proved using the method of point process convergence (Lemma A.4, Appandix A.2). We let $\overline{\mathbb{R}}_0^d = [-\infty,\infty]^d \setminus \{0\}$ and we denote by $M_p(\overline{\mathbb{R}}_0^d)$ the set of point measures on $\overline{\mathbb{R}}_0^d$. We refer to [28] for more details. Define $x \cdot y = \sum_{j=1}^d x_j y_j$

Theorem 2.5.1. *Let $(X_i)_i$ be i.i.d. regularly varying random vectors in $\mathbb{R}^d$ i.e there exist a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \to \infty$ such that*

$$nP\left(\frac{X_1}{a_n} \in \cdot\right) \xrightarrow{v} \nu \text{ on } \overline{\mathbb{R}}_0^d, \tag{2.5.1}$$

where ν is a non-null Radon measure on $\overline{\mathbb{R}}_0^d$ with $\nu\left(\overline{\mathbb{R}}_0^d \backslash \mathbb{R}^d\right) = 0$. Let $S_n = \sum\limits_{i=1}^n X_i$.
(i) If $\alpha < 1$ then,

$$Z_n := \frac{S_n}{a_n} \xrightarrow{d} Z,$$

where Z is an α-stable random vector in $\mathbb{R}^d$ with characteristic function given by:

$$E(\exp(iu\cdot Z)) = \exp\left\{\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x) - 1\Big)\nu(dx)\right\}. \tag{2.5.2}$$

(ii) If $\alpha > 1$ and for any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\left(\left\|\sum_{i=1}^n \Big(X_i \mathbf{1}_{\{\|X_i\|\le\varepsilon a_n\}} - E\left[X_i\mathbf{1}_{\{\|X_i\|\le\varepsilon a_n\}}\right]\Big)\right\| > a_n\delta\right) = 0, \tag{2.5.3}$$

then,

$$Z_n := \frac{S_n}{a_n} - \frac{n}{a_n}\mathbb{E}\left[X_1\right] \xrightarrow{d} Z$$

where Z is an α-stable random vector in $\mathbb{R}^d$ with characteristic function given by:

$$E(\exp(iu \cdot Z)) = \exp\left\{\int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 - iu \cdot x\Big)\nu(dx)\right\}. \qquad (2.5.4)$$

(iii) If $\alpha = 1$ and if we suppose that (2.5.3) is satisfied then letting $c > 0$ be such that $\nu\Big(\partial\left\{x \in \overline{\mathbb{R}}_0^d; \|x\| = c\right\}\Big) = 0$,

$$Z_n := \frac{S_n}{a_n} - \frac{n}{a_n}\mathbb{E}\left[X_1 \mathbf{1}_{\|X_1\| \le c a_n}\right] \xrightarrow{d} Z$$

where Z is an α-stable random vector in $\mathbb{R}^d$ with characteristic function given by:

$$E(\exp(iu \cdot Z)) = \exp\left\{\int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1 - iu \cdot x\mathbf{1}_{\{\|x\| \le c\}}\Big)\nu(dx)\right\}. \qquad (2.5.5)$$

Proof. The proof is divided into several steps:

Step 1. Using (2.5.1) and Lemma B.0.6, we have:

$$N_n = \sum_{i=1}^{n} \delta_{\frac{X_i}{a_n}} \xrightarrow{d} N = \sum_{i \ge 1} \delta_{J_i} \text{ in } M_p(\overline{\mathbb{R}}_0^d), \qquad (2.5.6)$$

where N is a Poisson random measure on $\overline{\mathbb{R}}_0^d$ of intensity ν.

Step 2. Define the map

$$\chi_\varepsilon : M_p(\overline{\mathbb{R}}_0^d) \longmapsto \mathbb{R}^d$$

by

$$\chi_\varepsilon\left(\sum_{i \ge 1} \delta_{x_i}\right) = \sum_{i \ge 1} x_i 1_{\{\varepsilon < \|x_i\| < \infty\}}.$$

Let Λ_ε be the subset of $M_p(\overline{\mathbb{R}}_0^d)$ given by:

$$\Lambda_\varepsilon = \left\{\mu \in M_p(\overline{\mathbb{R}}_0^d); \mu\left(\partial V_\varepsilon\right) = 0\right\},$$

where $V_\varepsilon = \left\{x \in \overline{\mathbb{R}}_0^d; \|x\| > \varepsilon\right\}$. Let $\varepsilon > 0$ be fixed such that:

$$\nu(\partial V_\varepsilon) = 0. \qquad (2.5.7)$$

Note that such ε exists because the set $D = \{\varepsilon > 0; \nu(\partial V_\varepsilon) > 0\}$ is countable and hence D^c is dense in $(0, \infty)$. We make two claims:

Claim (1). $P\left(N \in \Lambda_\varepsilon\right) = 1$

To see this, note that $\mathbb{E}\left[N(\partial V_\varepsilon)\right] = \nu(\partial V_\varepsilon) = 0$ by (2.5.7). Since $N(\partial V_\varepsilon)$ is a non-negative random variable (it is a Poisson random variable), then we must have $N(\partial V_\varepsilon) = 0$ a.s. which is equivalent to say that $N \in \Lambda_\varepsilon$ a.s.

Claim (2). χ_ε is continuous on the set Λ_ε
To prove this let $(m_n)_n$ and m be point measures on $\overline{\mathbb{R}}_0^d$ such that $m_n \stackrel{v}{\to} m$. Let $K = \overline{V_\varepsilon}$. By Lemma B.0.10 (Appendix A) there exists $N \in \mathbb{N}$ such that for any $n \geq N_k$

$$m_n|_K = \sum_{i=1}^{p} \delta_{x_i^{(n)}} \text{ and } m|_K = \sum_{i=1}^{p} \delta_{x_i}$$

and $x_i^{(n)} \to x_i$, for all $i = 1, 2, \ldots, p$. Then for any $\delta > 0$, there exists $N_\delta > N$ such that for any $n \geq N_\delta$,

$$\|\chi_\varepsilon(m_n) - \chi_\varepsilon(m)\| \;\; = \;\; \left\|\sum_{i=1}^{p} x_i^{(n)} - \sum_{i=1}^{p} x_i\right\| \leq \sum_{i=1}^{p} \left\|x_i^{(n)} - x_i\right\| \leq p\delta.$$

This finishes the proof of Claim (2).

From Claim (2) we infer that $\Lambda_\varepsilon \subset \left\{\mu \in M_p(\mathbb{R}^d); \chi_\varepsilon \text{ is continuous at } \mu\right\}$ and hence $\text{Disc}(\chi_\varepsilon) \subset \Lambda_\varepsilon^c$ where $\text{Disc}(\chi_\varepsilon)$ is the set of discontinuity points of χ_ε. Then

$$P(N \in \text{Disc}(\chi_\varepsilon)) \leq P(N \in \Lambda_\varepsilon^c) = 0.$$

By (2.5.6) and the continuous mapping theorem, it follows that:

$$\chi_\varepsilon(N_n) \stackrel{d}{\to} \chi_\varepsilon(N),$$

which is equivalent to saying that

$$S_n^{>\varepsilon} := \sum_{i=1}^{n} \frac{X_i}{a_n} 1_{\{\|X_i\| > a_n\varepsilon\}} \stackrel{d}{\to} \sum_{i\geq 1} J_i 1_{\{\|J_i\| > \varepsilon\}}, \text{ as } n \to \infty \tag{2.5.8}$$

To summarize, relation (2.5.8) holds for all $\varepsilon > 0$ for which (2.5.7) holds.

Step 3. Let $\mu_n = nP(\frac{X_1}{a_n} \in \cdot)$. Let $(Z_j)_{j\geq 0}$ be the sequence of independent variables defined in the proof of Theorem 2.4.7 corresponding to a sequence $\varepsilon_k \downarrow 0$ with $\varepsilon_0 = c > 0$ such that $\nu(\partial V_{\varepsilon_k}) = 0$ for any $k \geq 1$. Note that in definition (2.4.6) of Z_0, we now replace the value 1 by c. Note that the regularly varying condition (2.5.1) implies that ν satisfy (2.3.4). (This follows from the proof of Theorem 6.1 of [28]).

We consider separately three cases:

Case 1: $\alpha < 1$
Let $S_n^{\le\varepsilon} = \sum_{i=1}^{n} \frac{X_i}{a_n} 1_{\{\|X_i\| \le a_n\varepsilon\}}$. By Markov inequality,

$$\begin{aligned} P(\left\|S_n^{\le\varepsilon}\right\| > \delta) &= P\left(\left\|\sum_{i=1}^{n} \frac{X_i}{a_n} 1_{\{\|X_i\| \le a_n\varepsilon\}}\right\| > \delta\right) \le \frac{1}{\delta} E\left(\left\|\sum_{i=1}^{n} \frac{X_i}{a_n} 1_{\{\|X_i\| \le a_n\varepsilon\}}\right\|\right) \\ &\le \frac{1}{\delta a_n} \sum_{i=1}^{n} E\Big(\left\|X_i 1_{\{\|X_i\| \le a_n\varepsilon\}}\right\| \Big) \\ &= \frac{n}{\delta a_n} E\Big(\left\|X_1 1_{\{\|X_1\| \le a_n\varepsilon\}}\right\| \Big). \end{aligned}$$

We make two claims:

Claim (3). $\|X\|$ is regularly varying of index $-\alpha$
To see this, note that (2.5.1) is equivalent to (Theorem 6.1 of [28]) :

$$nP\left(\frac{\|X_1\|}{a_n} \in \cdot, \frac{X_1}{\|X_1\|} \in \cdot\right) \xrightarrow{v} c\nu_\alpha \times \Gamma_1 \text{ in } (0,\infty] \times \mathbb{S}_d$$

which implies that for any $x > 0$

$$nP\left(\frac{\|X_1\|}{a_n} \ge x, \frac{X_1}{\|X_1\|} \in \mathbb{S}_d\right) \xrightarrow{v} cx^{-\alpha} \times \Gamma_1(\mathbb{S}_d)$$

and finally

$$nP(\|X\| \ge xa_n) \to cx^{-\alpha}.$$

Hence, $\|X\|$ is regularly varying of index $-\alpha$.

Claim (4). We have :

$$E(\|X_1\| \, 1_{\{\|X_1\| \le a_n\varepsilon\}}) \sim \frac{\alpha}{1-\alpha} a_n\varepsilon P(\|X_1\| > a_n\varepsilon), \text{ as } n \to \infty.$$

Recall that $a_n \sim b_n$ means that $a_n/b_n \to 1$ as n$\to \infty$ To prove this, let $Y = \|X_1\| \, 1_{\{\|X_1\| \le a_n\varepsilon\}}$. Then,

$$\begin{aligned} E(\|X_1\| \, 1_{\{\|X_1\| < a_n\varepsilon\}}) &= \int_0^\infty P(Y > y)dy = \int_0^{a_n\varepsilon} P(y < \|X_1\| \le a_n\varepsilon)dy \\ &= \int_0^{a_n\varepsilon} P(\|X_1\| > y)dy - a_n\varepsilon P(\|X_1\| > a_n\varepsilon). \end{aligned}$$

From Claim(3), we know that $\|X\|$ is regularly varying of index $-\alpha$. Then using Karamata's theorem (Theorem 2.1 of [28]) we obtain

$$\int_0^{a_n\varepsilon} P(\|X_1\| > y)dy \sim \frac{1}{1-\alpha} a_n\varepsilon P(\|X_1\| > a_n\varepsilon),$$

which implies that

$$\lim_{n\to\infty} \frac{E(\|X_1\| \, 1_{\{\|X_1\|\le a_n\varepsilon\}})}{a_n\varepsilon P(\|X_1\| > a_n\varepsilon)} = \lim_{n\to\infty} \frac{\int_0^{a_n\varepsilon} P(\|X_1\| > y)dy}{a_n\varepsilon P(\|X_1\| > a_n\varepsilon)} - 1 = \frac{\alpha}{1-\alpha}.$$

This finishes the proof of Claim (4).

From Claim (4), we obtain that, as $n \to \infty$

$$\frac{n}{\delta a_n} E\Big(\left\|X_1 1_{\{\|X_1\|\le a_n\varepsilon\}}\right\| \Big) \sim \frac{\alpha}{1-\alpha}\frac{\epsilon}{\delta} nP(\|X_1\| > a_n\varepsilon) \sim \frac{\alpha}{1-\alpha}\frac{1}{\delta}\varepsilon^{1-\alpha} \tag{2.5.9}$$

Using (2.5.9), we obtain

$$\limsup_{n\to\infty} P(\left\|S_n^{\le\varepsilon}\right\| > \delta) \le \lim_{n\to\infty} \frac{n}{\delta a_n} E\Big(\left\|X_1 1_{\{\|X_1\|\le a_n\varepsilon\}}\right\| \Big) = \frac{\alpha}{1-\alpha}\frac{1}{\delta}\varepsilon^{1-\alpha}$$

and then,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\left(\left\|S_n^{\le\varepsilon}\right\| > \delta \right) = 0. \tag{2.5.10}$$

Note that (2.5.10) is equivalent to the negligibility condition given by : for any $\delta > 0$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\left(\|S_n^{>\varepsilon} - S_n\| > \delta\right) = 0. \tag{2.5.11}$$

Let

$$Z = Z_0 + \sum_{j\ge 1} Z_j$$

where $(Z_j)_{j\ge 0}$ are defined as in the proof of Theorem 2.4.7 We claim that $\sum_{j\ge1} Z_j$ converge absolutely a.s. To see this, note that

$$\begin{aligned}
E(\sum_{j\ge 1}\|Z_j\|) &= E\left(\sum_{j\ge1}\left\|\sum_{i\ge1} J_i 1_{\{\varepsilon_j<\|J_i\|\le\varepsilon_{j-1}\}}\right\|\right) \le E\Big(\sum_{j\ge1}\sum_{i\ge1}\|J_i\| \, 1_{\{\varepsilon_j<\|J_i\|\le\varepsilon_{j-1}\}}\Big) \\
&= E\Big(\sum_{i\ge1}\sum_{j\ge1}\|J_i\| \, 1_{\{\varepsilon_j<\|J_i\|\le\varepsilon_{j-1}\}}\Big) = E\Big(\sum_{i\ge1}\|J_i\| \, 1_{\{0<\|J_i\|\le c\}}\Big) \\
&= E\Big(\int_{0<\|x\|\le c}\|x\| \, N(dx)\Big) = \int_{0<\|x\|\le c}\|x\| \, \nu(dx) < \infty.
\end{aligned}$$

Then

$$Z^{(\varepsilon_k)} \xrightarrow{d} Z, \text{ as } k\to\infty \tag{2.5.12}$$

where

$$Z^{(\varepsilon_k)} = Z_0 + \sum_{i=1}^{k} Z_i = \sum_{i\ge1} J_i 1_{\{\|J_i\|>\varepsilon_k\}}.$$

From (2.5.8), (2.5.11), (2.5.12) and using the second converging together (see Theorem A.0.2), we conclude that

$$Z_n \xrightarrow{d} Z, \text{ as } n \to \infty$$

To complete the proof when $\alpha < 1$ it remains to show that Z is an α-stable random vector with characteristic function given by (2.5.2). To see this, note that for any $u \in \mathbb{R}^d$

$$\begin{aligned} E(\exp(iu \cdot Z)) &= E\left[\exp\Big(iu \cdot (Z_0 + \sum_{j \geq 1} Z_j)\Big)\right] \\ &= E\left[\exp(iu \cdot Z_0)\right] \prod_{j=1}^{\infty} E\left[\exp\left\{iu \cdot Z_j\right\}\right] \\ &= \exp\left\{\int_{\|x\|>1} (\exp(iu \cdot x) - 1)\nu(dx)\right\} \\ &\cdot \prod_{j=1}^{\infty} \exp\left\{\Big(\int_{\varepsilon_j < \|x\| \leq \varepsilon_{j-1}} \Big(\exp(iu \cdot x) - 1\Big)\nu(dx)\Big)\right\} \\ &= \exp\left\{\int_{\mathbb{R}^d} \Big(\exp(iu \cdot x) - 1\Big)\nu(dx)\right\}. \end{aligned}$$

By Theorem 2.3.9 we conclude that Z has an α-stable distribution in $\mathbb{R}^d$.

Case 2: $\alpha > 1$

Let $\epsilon > 0$ be such that (2.5.7) holds. From (2.5.1), Lemma B.0.7 and Lemma B.0.8 with $K = \big\{x \in \mathbb{R}^d; \|x\| > \varepsilon\big\}$ we have:

$$\mu_n(K \cap \cdot) \xrightarrow{w} \nu(K \cap \cdot),$$

as measures on K. Using the continuous mapping theorem (with $f(x) = x^j$, $x \in K$ where $x = (x^1, \ldots, x^d)$)

$$\frac{n}{a_n}\mathbb{E}\left[X_1 \mathbf{1}_{\{\|X_1\| > \varepsilon a_n\}}\right] = n\int_{\{\|x\| > \varepsilon a_n\}} xP \circ \left(\frac{X_1}{a_n}\right)^{-1} dx \to \int_{\|x\| > \varepsilon} x\nu(dx). \quad (2.5.13)$$

From (2.5.8) and (2.5.13), it follows by Slutsky theorem that:

$$Z_n^{(\varepsilon)} \xrightarrow{d} Z^{(\varepsilon)}, \text{ as } n \to \infty \quad (2.5.14)$$

for any $\varepsilon > 0$ which satisfies (2.5.7) where

$$Z_n^{(\varepsilon)} = \sum_{i=1}^{n} \frac{X_i}{a_n} 1_{\{\|X_i\| > \varepsilon a_n\}} - \frac{n}{a_n}\mathbb{E}\left[X_1 \mathbf{1}_{\{\|X_1\| > \varepsilon a_n\}}\right] = S_n^{>\varepsilon} - \mathbb{E}(S_n^{>\varepsilon})$$

and

$$Z^{(\varepsilon)} = \sum_{i\geq 1} J_i 1_{\{\|J_i\|>\varepsilon\}} - \int_{\|x\|>\varepsilon} x\nu(dx).$$

The negligibility condition (2.5.3) is equivalent to say that: for any $\delta > 0$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\left(\left\|Z_n^{(\varepsilon)} - Z_n\right\| > \delta\right) = 0. \tag{2.5.15}$$

Let $\epsilon_k \downarrow 0$ with $\epsilon_0 = c$ be such that (2.5.7) holds for ϵ_k, for all $k \geq 1$. Let

$$Z = \big(Z_0 - E(Z_0)\big) + \sum_{j\geq 1}\big(Z_j - E(Z_j)\big),$$

where $(Z_j)_{j\geq 0}$ are defined as in the proof of Theorem 2.4.7 . (In the proof of Theorem 2.4.7 we showed that the previous series converges a.s.). Note that

$$E\|Z_0\| = \int_{\|x\|>c} \|x\|\,\nu(dx) < \infty$$

by Lemma 2.3.14 since $\alpha > 1$. Then

$$Z^{(\varepsilon_k)} \xrightarrow{d} Z,\ \text{ as } k\to\infty \tag{2.5.16}$$

where

$$Z^{(\varepsilon_k)} = \big(Z_0 - E(Z_0)\big) + \sum_{i=1}^{k}\big(Z_i - E(Z_i)\big) = \sum_{i\geq 1} J_i 1_{\{\|J_i\|>\varepsilon_k\}} - \int_{\|x\|>\varepsilon_k} x\nu(dx).$$

From (2.5.14), (2.5.15), (2.5.16) and using the second converging together (see Theorem A.0.2), we conclude that

$$Z_n \xrightarrow{d} Z,\ \text{ as } n\to\infty.$$

To complete the proof when $\alpha > 1$ it remains to show that Z is an α-stable random vector with characteristic function given by (2.5.4). To see this, note that for any $u \in \mathbb{R}^d$

$$\begin{aligned} E(\exp(iu\cdot Z)) &= E\left[\exp\left\{iu\cdot\left(Z_0 - E(Z_0) + \sum_{j\geq 1}(Z_j - E(Z_j))\right)\right\}\right] \\ &= E\left[\exp\left\{iu\cdot\big(Z_0 - E(Z_0)\big) + iu\cdot\left(\sum_{j\geq 1}(Z_j - E(Z_j))\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[\exp\left(iu\cdot\big(Z_0-E(Z_0)\big)\right)\right]\prod_{j=1}^{\infty}E\left[\exp\left\{iu\cdot\big(Z_j-E(Z_j)\big)\right\}\right]\\
&= \exp\int_{\|x\|>1}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx)\\
&\cdot\ \prod_{j=1}^{\infty}\exp\Big(\int_{\varepsilon_j<\|x\|\le\varepsilon_{j-1}}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx)\Big)\\
&= \exp\left\{\int_{\mathbb{R}^d}\Big(\exp(iu\cdot x)-1-iu\cdot x\Big)\nu(dx)\right\}
\end{aligned}
$$

By Theorem 2.3.9 we conclude that Z has an α-stable distribution in $\mathbb{R}^d$ (since ν satisfies (2.3.4)).

Case 3: $\alpha = 1$
From (2.5.1), Lemma B.0.7 and Lemma B.0.8 with $K = \left\{x \in \mathbb{R}^d; \varepsilon < \|x\| \le c\right\}$ we have:

$$\mu_n(K\cap\cdot)\xrightarrow{w}\nu(K\cap\cdot),$$

for any $0 < c < \varepsilon$ satisfying (2.5.7). By continuous mapping theorem (with $f(x) = x^j$, $x \in K$, where $x = (x^1,\dots,x^d)$)

$$\frac{n}{a_n}\mathbb{E}\left[X_1\mathbf{1}_{\left\{\varepsilon a_n<\|\frac{X_1}{a_n}\|\le c\right\}}\right]=n\int_{\{\varepsilon a_n<\|x\|\le c\}}xP\circ\left(\frac{X_1}{a_n}\right)^{-1}dx\to\int_{\varepsilon<\|x\|\le c}x\nu(dx). \tag{2.5.17}$$

From (2.5.8), (2.5.17) and using Slutsky theorem we have:

$$Z_n^{(\varepsilon)}\xrightarrow{d}Z^{(\varepsilon)},\ \text{as } n\to\infty \tag{2.5.18}$$

for any $0 < c < \varepsilon$ satisfying (2.5.7) where

$$Z_n^{(\varepsilon)}=\sum_{i=1}^{n}\frac{X_i}{a_n}1_{\{\|X_i\|>\varepsilon a_n\}}-\frac{n}{a_n}\mathbb{E}\left[X_1\mathbf{1}_{\{\varepsilon a_n<\|X_1\|\le c a_n\}}\right]$$

$$Z^{(\varepsilon)}=\sum_{i\ge1}J_i1_{\{\|J_i\|>\varepsilon\}}-\int_{\varepsilon<\|x\|\le c}x\nu(dx).$$

The negligibility condition (2.5.3) is equivalent to: for any $\delta > 0$

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}P\left(\left\|Z_n^{(\varepsilon)}-Z_n\right\|>\delta\right)=0. \tag{2.5.19}$$

We let Z be given by relation (2.4.7) in the proof of Theorem 2.4.7. The conclusion follows using the converging together theorem (see Lemma A.0.2). The fact that Z has an α-stable distribution in $\mathbb{R}^d$ follows by Remark 2.3.10. The characteristic function of Z is computed similarly to case 2. $\square$

Chapter 3

Càdlàg functions with values in $\mathbb{D}$

In this chapter, we introduce the spaces $\mathbb{D}([0,1];\mathbb{D})$ and $\mathbb{D}([0,\infty);\mathbb{D})$ of càdlàg functions defined $[0,1]$, respectively $[0,\infty)$, with values in $\mathbb{D}$. These spaces are equipped with the Skorohod distance introduced in [34]. We examine briefly the weak convergence of probability measures on these spaces, a topic which is developed at length in the companion paper [3].

Some of the results presented in this chapter can be extended to the space $\mathbb{D}([0,1];S)$ where S is a complete separable metric space. However, our major result about tightness of probability measures on $\mathbb{D}([0,1];S)$ (Theorem 3.3.8) is specific to $S=\mathbb{D}$. Moreover, the concept of regular variation for random elements with values in S has only been developed in the literature in the case $S=\mathbb{D}$.

3.1 The space $\mathbb{D}$

In this section, we recall briefly from [5] the definition and basic properties of the Skorohod space $\mathbb{D}$.

We denote by $\mathbb{D}=\mathbb{D}\,[0,1]$ the space of càdlàg functions $x:\ [0,1]\to\mathbb{R}$, i.e. functions which are right-continuous and have left limits. Any function $x\in\mathbb{D}$ has countably many discontinuities. We denote by $\mathrm{Disc}(x)$ the set of discontinuity of $x\in\mathbb{D}$. We denote by $\|\cdot\|$ the uniform norm in $\mathbb{D}$:

$$\|x\|=\sup_{t\in[0,1]}|x(t)|\ ,\ x\in\mathbb{D}$$

Note that $(\mathbb{D},\|\cdot\|)$ is a Banach space, but it is not separable. The space $\mathbb{D}$ is equipped with the Skorohod distance d_{J_1} defined by

$$d_{J_1}(x,y)=\inf_{\lambda\in\Lambda}\{\|\lambda-e\|\vee\|x-y\circ\lambda\|\}$$

where e is the identity map on $[0,1]$ and Λ is the set of all continuous strictly increasing functions λ that map $[0,1]$ onto $[0,1]$. We write $x_n\xrightarrow{J_1}x$ if $d_{J_1}(x_n,x)\to 0$. This is equivalent to saying that there exists a sequence $(\lambda_n)_{n\geq 1} \subset \Lambda$ such that $\|\lambda_n - e\| \to 0$ and $\|x_n \circ \lambda_n - x\| \to 0$.

The space $\mathbb{D}$ is separable under d_{J_1}, but is not complete. However, there exists a distance $d^0_{J_1}$ on $\mathbb{D}$ which is equivalent to d_{J_1}, under which $\mathbb{D}$ is separable and complete. This distance is given by

$$d^0_{J_1}(x,y) = \inf_{\lambda\in\Lambda} \left\{ \|\lambda\|^0 \vee \|x - y\circ\lambda\| \right\} \tag{3.1.1}$$

where $\|\lambda\|^0 = \sup_{s<t} \left|\log \frac{\lambda(t)-\lambda(s)}{t-s}\right|$. By taking $\lambda = e$, we obtain that for all $x, y \in \mathbb{D}$

$$d_{J_1}(x,y) \leq \|x-y\| \text{ and } d^0_{J_1}(x,y) \leq \|x-y\|.$$

Note also that

$$d_{J_1}(x,0) = d^0_{J_1}(x,0) = \|x\| \text{ for all } x \in \mathbb{D} \tag{3.1.2}$$

By relation (12.17) of [6],

$$\sup_{s\in[0,1]} |\lambda(s) - s| \leq e^{\|\lambda\|^0} - 1 \quad \text{for all } \lambda \in \Lambda. \tag{3.1.3}$$

Taking $\lambda = e$ in (3.1.1), we obtain:

$$d^0_{J_1}(x,y) \leq \|x-y\| \quad \text{for all} \quad x,y \in \mathbb{D}. \tag{3.1.4}$$

For functions $(x_n)_{n\geq 1}$ and x in $\mathbb{D}$, we write $x_n \xrightarrow{J_1} x$ if $d^0_{J_1}(x_n, x) \to 0$. For any $\delta \in (0,1)$, we consider the following modulus of continuity of a function $x \in \mathbb{D}$:

$$w''(x,\delta) = \sup_{s_1\leq s\leq s_2, s_2-s_1\leq\delta} \big(|x(s) - x(s_1)| \wedge |x(s_2) - x(s)|\big). \tag{3.1.5}$$

We denote by $\mathcal{D}$ the Borel σ-field of $\mathbb{D}$, i.e. the σ-field generated by all open sets (with respect to the J_1 topology). By Theorem 12.5 of [6], $\mathcal{D}$ coincides with the σ-field generated by the projections $\pi_t : \mathbb{D} \to \mathbb{R}$, given by $\pi_t(x) = x(t)$.

In this thesis, we will work with random elements in $\mathbb{D}$. A *random element* in $\mathbb{D}$ is a function $X : \Omega \to \mathbb{D}$ which is $\mathcal{F}/\mathcal{D}$ measurable, where $(\Omega, \mathcal{F}, P)$ is a probability space. This is equivalent to saying that $X(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable for any $t \in [0,1]$. We denote by $P \circ X^{-1}$ the law of X. We say that a sequence $(X_n)_n$ of random elements of $\mathbb{D}$ convergences in distribution to a random element X of $\mathbb{D}$ (and we write $X_n \xrightarrow{d} X$) if $(P \circ X_n^{-1})_{n\geq 1}$ converges weakly to $P \circ X^{-1}$.

The following result plays an important role in the definition of regular variation for random elements in $\mathbb{D}$ (see Section 5.2). We include its proof since we could not find a reference for it.

Lemma 3.1.1. *The supremum norm $\|\cdot\|$ is J_1-continuous on $\mathbb{D}$.*

Proof. Let $(x_n)_{n\geq 1} \subset \mathbb{D}$ and $x \in \mathbb{D}$ be such that $x_n \overset{J_1}{\to} x$. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that $\|\lambda_n - e\| \to 0$ and $\|x_n \circ \lambda_n - x\| \to 0$. Then $\|x_n \circ \lambda_n\| \to \|x\|$ since $|\|x_n \circ \lambda_n\| - \|x\|| \leq \|x_n \circ \lambda_n - x\|$. Note that

$$\|x_n\| = \sup_{t\in[0,1]} |x_n(t)| = \sup_{t\in[0,1]} |x_n(\lambda_n(t))| = \|x_n \circ \lambda_n\|.$$

Hence, $\|x_n\| \to \|x\|$.□

3.2 Basic properties of $\mathbb{D}([0,1];\mathbb{D})$

In this section, we introduce the definition of the space $\mathbb{D}([0,1];\mathbb{D})$ and discuss some of its properties.

We denote by $\mathbb{D}([0,1];\mathbb{D})$ the space of functions $x : [0,1] \to \mathbb{D}$ with the following proprieties:
(i) x is *right continuous* with respect to the J_1 topology, i.e. for any $t \in [0,1]$ and for any sequence $(t_n)_{n\geq 1} \subset [0,T]$ such that $t_n \to t$ and $t_n \geq t$ for all $n \geq 1$, we have $x(t_n) \to x(t)$ in $(\mathbb{D}, J_1)$;
(ii) x has *left limits* with respect to J_1 topology space, i.e. for any $t \in [0,1]$ there exist an element in $\mathbb{D}$ denoted by $x(t-)$ such that for any sequence $(t_n)_{n\geq 1} \subset [0,1]$ such that $t_n \to t$ and $t_n < t$ for all $n \geq 1$, we have $x(t_n) \to x(t-)$ in $(\mathbb{D}, J_1)$.

The *Skorohod distance* on $\mathbb{D}([0,1];\mathbb{D})$ is defined in [34] by :

$$d_{\mathbb{D}}(x,y) = \inf_{\lambda\in\Lambda} \{\|\lambda - e\| \vee \rho_{\mathbb{D}}(x, y\circ\lambda)\}. \tag{3.2.1}$$

for any $x, y \in \mathbb{D}([0,1];\mathbb{D})$ where $\rho_{\mathbb{D}}$ is the *uniform distance* on $\mathbb{D}([0,1];\mathbb{D})$ given by :

$$\rho_{\mathbb{D}}(x,y) = \sup_{t\in[0,1]} d^0_{J_1}(x(t), y(t)). \tag{3.2.2}$$

For any $x \in \mathbb{D}([0,1];\mathbb{D})$, we define the *super-uniform norm* by:

$$\|x\|_{\mathbb{D}} = \sup_{t\in[0,1]} \|x(t)\|. \tag{3.2.3}$$

Hence, $d_{\mathbb{D}}(x_n, x) \to 0$ if and only if there exists a sequence $(\lambda_n)_{n\geq 1} \subset \Lambda$ such that

$$\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0 \quad \text{and} \quad \sup_{t\in[0,1]} d^0_{J_1}(x_n(\lambda_n(t)), x(t)) \to 0. \tag{3.2.4}$$

(By the discussion in small print on page 122 of [6], the set $\{x(t); t \in [0,1]\}$ is relatively compact in $(\mathbb{D}, J_1)$, and hence, $\|x\|_{\mathbb{D}} < \infty$ by Theorem 12.3 of [6].)

By relation (3.1.2), it follows that for any $x \in \mathbb{D}([0,1];\mathbb{D})$,

$$d_{\mathbb{D}}(x,0) = \rho_{\mathbb{D}}(x,0) = \|x\|_{\mathbb{D}}. \tag{3.2.5}$$

Note that for any $x, y \in \mathbb{D}([0,1];\mathbb{D})$, we have:

$$d_{\mathbb{D}}(x,y) \leq \rho_{\mathbb{D}}(x,y) \leq \|x-y\|_{\mathbb{D}}. \tag{3.2.6}$$

Lemma 3.2.1. *a) If $d_{\mathbb{D}}(x_n,x) \to 0$, then $x_n(t) \stackrel{J_1}{\to} x(t)$ for any continuity point t of x (with respect to J_1).*
b) If $d_{\mathbb{D}}(x_n,x) \to 0$ and x is continuous on $[0,1]$ with respect to J_1, then $\rho_{\mathbb{D}}(x_n,x) \to 0$.

Proof: Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (3.2.4) holds. a) Then

$$d^0_{J_1}(x_n(t),x(t)) \leq d^0_{J_1}(x_n(t),x(\lambda_n(t))) + d^0_{J_1}(x(\lambda_n(t)),x(t)) \to 0.$$

b) Since x is continuous on the compact set $[0,1]$, it is also uniformly continuous. Hence

$$\rho_{\mathbb{D}}(x_n,x) \leq \sup_{t\in[0,1]} d^0_{J_1}(x_n(t),x(\lambda_n(x(t)))) + \sup_{t\in[0,1]} d^0_{J_1}(x(\lambda_n(x(t))),x(t)) \to 0.$$

□

The following result shows that the super-uniform norm is continuous on $\mathbb{D}([0,1];\mathbb{D})$.

Lemma 3.2.2. *If $(x_n)_{n\geq 1}$ and x are functions in $\mathbb{D}([0,1];\mathbb{D})$ such that $d_{\mathbb{D}}(x_n,x) \to 0$ as $n \to \infty$, then $\|x_n\|_{\mathbb{D}} \to \|x\|_{\mathbb{D}}$ as $n \to \infty$.*

Proof: Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (3.2.4) holds. By (3.2.5), we have:

$$|\|x_n \circ \lambda_n\|_{\mathbb{D}} - \|x\|_{\mathbb{D}}| = |\rho_{\mathbb{D}}(x_n \circ \lambda_n, 0) - \rho_{\mathbb{D}}(x,0)| \leq \rho_{\mathbb{D}}(x_n \circ \lambda_n, x) \to 0.$$

The conclusion follows since $\|x_n \circ \lambda_n\|_{\mathbb{D}} = \|x_n\|_{\mathbb{D}}$ (because λ_n is a one-to-one map).
□

For any set $T \subset [0,1]$ and for any $x \in \mathbb{D}([0,1];\mathbb{D})$, we let

$$w_{\mathbb{D}}(x,T) = \sup_{t_1,t_2\in T} d^0_{J_1}(x(t_1),x(t_2)).$$

The following result is proved similarly to Lemma 1 (page 122) of [6].

Lemma 3.2.3. *For any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$, there exist $0 = t_0 < t_1 < \ldots < t_v = 1$ such that*

$$w_{\mathbb{D}}(x,[t_{i-1},t_i)) < \varepsilon \quad \textit{for all} \quad i = 1,\ldots,v.$$

A consequence of this result is that for $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$, there can be at most finitely many points $t \in [0,1]$ such that $d^0_{J_1}(x(t), x(t-)) > \varepsilon$. Hence, any function $x \in \mathbb{D}([0,1];\mathbb{D})$ has a countable set of discontinuities with respect to J_1, which we denote by $\mathrm{Disc}(x)$. The maximum jump of x is defined by:

$$j(x) = \sup_{t\in[0,1]} d^0_{J_1}(x(t), x(t-))$$

For any $\delta \in (0,1)$ and $x \in \mathbb{D}([0,1];\mathbb{D})$, we let

$$w'_{\mathbb{D}}(x,\delta) = \inf_{\{t_i\}} \max_{1\le i\le v} w_{\mathbb{D}}(x, [t_{i-1}, t_i)), \tag{3.2.7}$$

where the infimum is taken over all δ-sparse sets $\{t_i\}_{0\le i\le v}$.

Clearly, the function $w'_{\mathbb{D}}(x,\cdot)$ is non-decreasing. The following two results give some further properties of $w'_{\mathbb{D}}(x,\delta)$.

Lemma 3.2.4. *For any $x \in \mathbb{D}([0,1];\mathbb{D})$,*

$$\lim_{\delta\to 0} w'_{\mathbb{D}}(x,\delta) = 0 \tag{3.2.8}$$

$$w'_{\mathbb{D}}(x,\delta) \le w_{\mathbb{D}}(x,2\delta) \quad \textit{for any} \quad \delta \in (0,1/2),$$
$$w_{\mathbb{D}}(x,\delta) \le 2w'_{\mathbb{D}}(x,\delta) + j(x) \quad \textit{for any} \quad \delta \in (0,1).$$

Proof: To prove the first relation, let $\varepsilon > 0$ be arbitrary and $\{t_i\}_{0\le i\le v}$ be the sequence given by Lemma 3.2.3. Pick $0 < \delta_\varepsilon < \min_{0\le i\le v}(t_i - t_{i-1})$. For any $\delta \in (0,\delta_\varepsilon)$, $\{t_i\}_{0\le i\le v}$ is δ-sparse, and hence $w'_{\mathbb{D}}(x,\delta) \le \max_{1\le i\le v} w_{\mathbb{D}}(x,[t_{i-1},t_i)) < \varepsilon$. The last two relations are proved similarly to (12.7) and (12.9) of [6], using the triangle inequality in $(\mathbb{D}, d^0_{J_1})$. We omit the details. $\square$

Lemma 3.2.5. *$w'_{\mathbb{D}}(\cdot,\delta)$ is upper-semicontinuous on $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$.*

Proof: Let $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$ be arbitrary. We have to prove that there exists $\eta > 0$ such that $w'_{\mathbb{D}}(y,\delta) < w'_{\mathbb{D}}(x,\delta) + \varepsilon$ for any $y \in \mathbb{D}([0,1];\mathbb{D})$ such that $d_{\mathbb{D}}(x,y) < \eta$. This follows by the same argument as in Lemma 4 (page 130) of [6], replacing $|y(t) - x(\lambda(t))|$ by $d^0_{J_1}(y(t), x(\lambda(t)))$ and using the triangle inequality in $(\mathbb{D}, d^0_{J_1})$. $\square$

The space $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$ is separable, but it is not complete. Similarly to the distance $d^0_{J_1}$ on $\mathbb{D}$, we consider another distance $d^0_{\mathbb{D}}$ on $\mathbb{D}([0,1];\mathbb{D})$, given by:

$$d^0_{\mathbb{D}}(x,y) = \inf_{\lambda\in\Lambda}\{\|\lambda\|^0 \vee \rho_{\mathbb{D}}(x, y\circ\lambda)\}. \tag{3.2.9}$$

Then $d_{\mathbb{D}}(x,y) \leq e^{d^0_{\mathbb{D}}(x,y)} - 1$ for all $x, y \in \mathbb{D}([0,1];\mathbb{D})$.
Note that if x: $[0,1] \to \mathbb{D}$ and y: $[0,1] \to \mathbb{D}$ are given by $x(t) = x_0$ for all $t \in [0,1]$ and $y(t) = y_0$ for all $t \in [0,1]$, for some fixed functions $x_0, y_0 \in \mathbb{D}$ then

$$d_{\mathbb{D}}(x,y) = d^0_{\mathbb{D}}(x,y) = d^0_{J_1}(x_0, y_0)$$

Similarly to Theorems 12.1 and 12.2 of [6], and using the fact that $\mathbb{D}$ is separable and complete under $d^0_{J_1}$, we obtain the following result. (See also Theorem 2.6 of [34].)

Theorem 3.2.6. *The metrics $d_{\mathbb{D}}$ and $d^0_{\mathbb{D}}$ are equivalent. The space $\mathbb{D}([0,1];\mathbb{D})$ is separable under $d_{\mathbb{D}}$ and $d^0_{\mathbb{D}}$, and is complete under $d^0_{\mathbb{D}}$.*

The following result characterizes the relatively compact subsets of $\mathbb{D}([0,1];\mathbb{D})$, being the analogue of Theorem 12.3 of [6].

Theorem 3.2.7. *A set $A \subset \mathbb{D}([0,1];\mathbb{D})$ is relatively compact with respect to $d_{\mathbb{D}}$ if and only if it satisfies the following three conditions:*
(i) $\sup_{x\in A} \|x\|_{\mathbb{D}} < \infty$;
(ii) $\lim_{\delta\to 0} \sup_{x\in A} \sup_{t\in[0,1]} w'\big(x(t), \delta\big) = 0$;
(iii) $\lim_{\delta\to 0} \sup_{x\in A} w'_{\mathbb{D}}(x,\delta) = 0$.

Proof: Note that conditions *(i)* and *(ii)* are equivalent to saying that the set $U = \{x(t); x \in A, t \in [0,1]\}$ is relatively compact in $(\mathbb{D}, J_1)$ (see Theorem 12.3 of [6]). Suppose that A is relatively compact in $\mathbb{D}([0,1];\mathbb{D})$. We first prove that U is relatively compact in $(\mathbb{D}, J_1)$. Let $\{x_n(t_n)\}_{n\geq 1}$ be an arbitrary sequence in U, with $x_n \in A$ and $t_n \in [0,1]$. Since A is relatively compact, there exists a subsequence $N \subset$ such that $d_{\mathbb{D}}(x_n, x) \to 0$ as $n \to \infty, n \in N$. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ such that (3.2.4) hold as $n \to \infty, n \in N$. The sequence $(t_n)_{n\in N}$ has a monotone convergent sub-sequence $(t_n)_{n\in N'}$ with $N' \subset N$: either $t_n \uparrow t$ or $t_n \downarrow t$ as $n \to \infty, n \in N'$. Since λ_n^{-1} is strictly increasing, either $\lambda_n^{-1}(t_n) \uparrow t$ or $\lambda_n^{-1}(t_n) \downarrow t$ as $n \to \infty, n \in N'$. Therefore, either $x(\lambda_n^{-1}(t_n)) \xrightarrow{J_1} x(t-)$ or $x(\lambda_n^{-1}(t_n)) \xrightarrow{J_1} x(t)$ as $n \to \infty, n \in N'$. In the first case,

$$d^0_{J_1}\big(x_n(t_n), x(t-)\big) \leq d^0_{J_1}\big(x_n(t_n), x(\lambda_n^{-1}(t_n))\big) + d^0_{J_1}\big(x(\lambda_n^{-1}(t_n)), x(t-)\big) \to 0,$$

as $n \to \infty, n \in N'$. In the second case, $d^0_{J_1}\big(x_n(t_n), x(t)\big) \to 0$ as $n \to \infty, n \in N'$. This shows that the sequence $\{x_n(t_n)\}_{n\geq 1}$ has a J_1-convergence subsequence.

To prove *(iii)*, we apply Dini's theorem, as stated in Appendix M8 of [6]. Since $w'_{\mathbb{D}}(\cdot, 1/n)$ is upper semi-continuous for any n, and $w'_{\mathbb{D}}(x, 1/n) \downarrow 0$ for any $x \in \mathbb{D}([0,1];\mathbb{D})$, this convergence is uniform on compact sets. Hence $\sup_{x\in A} w'_{\mathbb{D}}(x, n^{-1}) \to 0$ as $n \to \infty$. Condition *(iii)* follows since $w'_{\mathbb{D}}(x, \cdot)$ is non-decreasing.

Next, suppose that the set A satisfies conditions *(i)*-*(iii)*. Since $\mathbb{D}([0,1];\mathbb{D})$ is complete with respect to $d^0_{\mathbb{D}}$, the closure $\overline{A}$ of A is also complete. To show that $\overline{A}$ is compact, it suffices to show that $\overline{A}$ is totally bounded with respect to $d_{\mathbb{D}}^0$ (see Theorem of Appendix M5 of [6]). This follows as in the sufficiency part of the proof of Theorem 12.3 of [6], by choosing H to be a finite ε-net of the set U in $\mathbb{D}$. □

Similarly to (3.1.5), for any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\delta \in (0,1)$, we consider the following modulus of continuity:

$$w''_{\mathbb{D}}(x,\delta) = \sup_{t_1 \le t \le t_2,\ t_2 - t_1 \le \delta} \big(d^0_{J_1}(x(t),x(t_1)) \wedge d^0_{J_1}(x(t_2),x(t))\big). \tag{3.2.10}$$

The following result will be used in the proof of Theorem 4.3.7 below.

Lemma 3.2.8. *For any $x, y \in \mathbb{D}([0,1];\mathbb{D})$, we have:*

$$w''_{\mathbb{D}}(x+y,\delta) \le w''_{\mathbb{D}}(x,\delta) + 2\|y\|_{\mathbb{D}}.$$

Proof: Let $t_1 \le t \le t_2$ be such that $t_2 - t_1 \le \delta$. By triangle inequality and (3.1.4),

$$\begin{aligned} d^0_{J_1}\big(x(t)+y(t), x(t_1)+y(t_1)\big) &\le d^0_{J1}\big(x(t)+y(t), x(t)\big) + d^0_{J_1}\big(x(t),x(t_1)\big) \\ &\quad + d^0_{J_1}\big(x(t_1), x(t_1)+y(t_1)\big) \\ &\le \|y(t)\| + d^0_{J_1}\big(x(t),x(t_1)\big) + \|y(t_1)\| \\ &\le d^0_{J_1}\big(x(t),x(t_1)\big) + 2\|y\|_{\mathbb{D}}. \end{aligned}$$

Similarly, $d^0_{J_1}\big(x(t)+y(t), x(t_2)+y(t_2)\big) \le d^0_{J_1}\big(x(t),x(t_2)\big) + 2\|y\|_{\mathbb{D}}$. If $a_1, a_2, b_1, b_2, c \in \mathbb{R}$ are such that $a_i \le b_i + c$ for $i = 1, 2$, then it is easy to see that $a_1 \wedge a_2 \le b_1 \wedge b_2 + c$. It follows that $d^0_{j_1}\big(x(t)+y(t), x(t_1)+y(t_1)\big) \wedge d^0_{J_1}\big(x(t)+y(t), x(t_2)+y(t_2)\big)$ is less than

$$d^0_{J_1}\big(x(t),x(t_1)\big) \wedge d^0_{J_1}\big(x(t),x(t_2)\big) + 2\|y\|_{\mathbb{D}} \le w''_{\mathbb{D}}(x,\delta) + 2\|y\|_{\mathbb{D}}.$$

The conclusion follows taking the supremum over all $t_1 \le t \le t_2$ such that $t_2 - t_1 \le \delta$. □

The following result is the analogue of Theorem 12.4 of [6].

Theorem 3.2.9. *A set $A \subset \mathbb{D}([0,1];\mathbb{D})$ is relatively compact with respect to $d_{\mathbb{D}}$ if and only if it satisfies the following three conditions:*
(i) $\sup_{x \in A} \|x\|_{\mathbb{D}} < \infty$;
(ii′)

$$\begin{cases} (a) & \lim_{\delta \to 0} \sup_{t \in [0,1]} w''(x(t),\delta) = 0 \\ (b) & \lim_{\delta \to 0} \sup_{x \in A} \sup_{t \in [0,1]} |x(t,\delta), x(t,0)| = 0 \\ (c) & \lim_{\delta \to 0} \sup_{x \in A} \sup_{t \in [0,1]} |x(t,1-), x(t,1-\delta)| = 0; \end{cases}$$

(iii′)

$$\begin{cases} (a) & \lim_{\delta \to 0} w''_{\mathbb{D}}(x,\delta) = 0 \\ (b) & \lim_{\delta \to 0} \sup_{x \in A} d^0_{J_1}(x(\delta), x(0)) = 0 \\ (c) & \lim_{\delta \to 0} \sup_{x \in A} d^0_{J_1}(x(1-), x(1-\delta)) = 0. \end{cases}$$

Proof: If A is relatively compact, then conditions (i)-(iii) of Theorem 3.2.7 hold. Condition (ii') follows by applying inequality (12.31) of [6] to the function $x(t) \in \mathbb{D}$, for any $t \in [0,1]$. Condition (iii') follows by the following inequality (proved similarly to (12.31) of [6]):

$$w''_{\mathbb{D}}(x,\delta) \vee d^0_{J_1}\big(x(\delta), x(0)\big) \vee d^0_{J_1}\big(x(1-), x(1-\delta)\big) \leq w'_{\mathbb{D}}(x, 2\delta) \tag{3.2.11}$$

Suppose that conditions (i), (ii') and (iii') hold. The fact that A is relatively compact will follow by Theorem 3.2.7, once we show that conditions (ii) and (iii) of this theorem hold. Condition (ii) follows from (ii') by applying inequality (12.32) of [6] to the function $x(t) \in \mathbb{D}$ for any $t \in [0,1]$. Condition (iii') follows by the following inequality

$$w'_{\mathbb{D}}(x,\delta/2) \leq 12\{w''_{\mathbb{D}}(x,\delta) + d^0_{J_1}\big(x(\delta), x(0)\big) + d^0_{J_1}\big(x(1-), x(1-\delta)\big)\}. \tag{3.2.12}$$

This is proved similarly to inequality (12.32) of [6], using the triangle inequality in $\mathbb{D}$ and the fact that $x_n \xrightarrow{J_1} x$ implies that $d^0_{J_1}(x_n, y) \to d^0_{J_1}(x,y)$ for any $y \in \mathbb{D}$. $\square$

We conclude this section with a discussion about measurability and finite-dimensional sets in $\mathbb{D}([0,1];\mathbb{D})$. Let $\mathcal{D}_{\mathbb{D}}$ be the Borel σ-field of $\mathbb{D}([0,1];\mathbb{D})$ with respect to $d_{\mathbb{D}}$. For any $t \in [0,1]$, we let $\pi^{\mathbb{D}}_t : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}$ be the projection given by $\pi^{\mathbb{D}}_t(x) = x(t)$. By Lemma 2.3 of [34], $\pi^{\mathbb{D}}_t$ is $\mathcal{D}_{\mathbb{D}}/\mathcal{D}$-measurable for any $t \in [0,1]$. By Theorem 2.7 of [34], $\mathcal{D}_{\mathbb{D}}$ coincides with the σ-field generated by the projections $\pi^{\mathbb{D}}_t$ for $t \in [0,1]$. Similarly to the classical case, the function $\pi^{\mathbb{D}}_t$ has the following continuity properties.

Lemma 3.2.10. *a) $\pi^{\mathbb{D}}_0$ and $\pi^{\mathbb{D}}_1$ are continuous with respect to $d_{\mathbb{D}}$.*
b) For any $t \in (0,1)$, $\pi^{\mathbb{D}}_t$ is continuous at x with respect to $d_{\mathbb{D}}$ if and only if x is continuous at t with respect to J_1.

Proof: a) Assume that $d_{\mathbb{D}}(x_n, x) \to 0$. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (3.2.4) holds. In particular, since $\lambda_n(0) = 0$, we obtain: $d^0_{J_1}(x_n(0), x(0)) \to 0$. This shows that $\pi^{\mathbb{D}}_0(x_n) \xrightarrow{J_1} \pi_0(x)$. Similarly, $\pi^{\mathbb{D}}_1(x_n) \xrightarrow{J_1} \pi_1(x)$.

b) Suppose that x is continuous at t with respect to J_1. Assume that $d_{\mathbb{D}}(x_n, x) \to 0$. Then $\pi^{\mathbb{D}}_t(x_n) \xrightarrow{J_1} \pi^{\mathbb{D}}_t(x)$, by Lemma 3.2.1.a). Suppose next that x is discontinuous at t with respect to J_1, i.e. $d^0_{J_1}(x(t-), x(t)) > 0$. Let $\lambda_n \in \Lambda$ be such that $\lambda_n(t) = t - 1/n$, and λ is linear on $[0,t]$ and $[t,1]$. Define $x_n(s) = x(\lambda_n(s))$. Then $d_{\mathbb{D}}(x_n, x) \to 0$, and $\pi^{\mathbb{D}}_t(x_n) = x_n(t) = x(\lambda_n(t)) = x(t-1/n) \xrightarrow{J_1} x(t-)$, and so $\pi^{\mathbb{D}}_t(x_n)$ does not converge in J_1 to $x(t)$. This shows that $\pi^{\mathbb{D}}_t$ is discontinuous at x with respect to $d_{\mathbb{D}}$. $\square$

For an arbitrary set $T \subset [0,1]$, we let $\mathcal{D}^{\mathbb{D}}_{f,T}$ be the class of finite-dimensional sets of the form $(\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1}(H)$ for some $0 \leq t_1 < \ldots < t_k \leq 1$, $t_i \in T$, $H \in \mathcal{D}^k$ and $k \geq 1$. Note that the σ-field generated by $\mathcal{D}^{\mathbb{D}}_{f,T}$ coincides with $\sigma\{\pi^{\mathbb{D}}_t; t \in T\}$, the minimal σ-field with respect to which the maps $\pi^{\mathbb{D}}_t, t \in T$ are measurable.

Theorem 3.2.11. *If $T \subset [0,1]$ is such that $1 \in T$ and T is dense in $[0,1]$, then:*
a) $\mathcal{D}_{\mathbb{D}}$ is the σ-field generated by $\mathcal{D}_{f,T}^{\mathbb{D}}$;
b) $\mathcal{D}_{f,T}^{\mathbb{D}}$ is a separating class of $\mathcal{D}_{\mathbb{D}}$, i.e. if P and Q are two probability measures on $(\mathbb{D}, \mathcal{D}_{\mathbb{D}})$ such that $P(A) = Q(A)$ for any $A \in \mathcal{D}_{f,T}^{\mathbb{D}}$, then $P = Q$.

Proof: a) Since $\pi_t^{\mathbb{D}}$ is $\mathcal{D}_{\mathbb{D}}$-measurable, $\sigma\{\pi_t^{\mathbb{D}}; t \in T\} \subset \mathcal{D}_{\mathbb{D}}$. To prove the other inclusion, it suffices to show that the identity $i : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}([0,1];\mathbb{D})$ given by $i(x) = x$ is $\sigma\{\pi_t^{\mathbb{D}}; t \in [0,1]\}/\mathcal{D}_{\mathbb{D}}$-measurable. For this, we use the same argument as in the proof of Theorem 12.5.(iii) of [6]. For any $\sigma = \{t_i\}_{i=0,\ldots,k}$ such that $0 = t_0 < t_1 < \ldots < t_k = 1$, we define the map $A_\sigma : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}([0,1];\mathbb{D})$ by $A_\sigma(x) = \sum_{i=1}^k x(t_{i-1}) 1_{[t_{i-1},t_i)} + x(1) 1_{\{1\}}(t)$. Similarly to Lemma 3 (page 127) of [6], it can be proved that

$$\max_{1 \leq i \leq k}(t_i - t_{i-1}) \leq \delta \text{ implies that } d_{\mathbb{D}}(A_\sigma(x), x) \leq \delta \vee w'_{\mathbb{D}}(x,\delta). \tag{3.2.13}$$

For any σ as above, we consider also the map $V_\sigma : \mathbb{D}^{k+1} \to \mathbb{D}([0,1];\mathbb{D})$ given by $V_\sigma(\alpha) = \sum_{i=1}^k \alpha_{i-1} 1_{[t_{i-1},t_i)}(t) + \alpha_k 1_{\{1\}}(t)$, for $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{D}^{k+1}$.

The function $V_\sigma : \mathbb{D}^{k+1} \to \mathbb{D}([0,1];\mathbb{D})$ is $\rho_{\mathbb{D}}$-continuous (hence $d_{\mathbb{D}}$-continuous), where $\mathbb{D}^{k+1}$ is endowed with the product topology: if $\alpha^n, \alpha \in \mathbb{D}^k$ are such that $\alpha_i^n \xrightarrow{J_1} \alpha_i$ as $n \to \infty$, for $i = 0, \ldots, k$, then

$$\rho_{\mathbb{D}}(V_\sigma(\alpha^n), V_\sigma(\alpha)) = \sup_{t \in [0,1]} d_{J_1}^0\big(V_m(\alpha^n)(t), V_m(\alpha)(t)\big) = \max_{0 \leq i \leq k} d_{J_1}^0(\alpha_i^n, \alpha_i) \to 0.$$

It follows that V_σ is $\mathcal{D}^{k+1}/\mathbb{D}_{\mathbb{D}}$-measurable. If $t_i \in T$ for all i, then A_σ is $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$-measurable, since $A_\sigma = V_\sigma \circ \pi_{t_0,\ldots,t_k}^{\mathbb{D}}$ and $\pi_{t_0,\ldots,t_k}^{\mathbb{D}}$ is $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}^{k+1}$-measurable.

For any $m \geq 1$, choose $\sigma_m = \{t_i^m\}_{i=0,\ldots,k_m}$ such that $t_i^m \in T$ and $\max_i(t_i^m - t_{i-1}^m) < 1/m$. By (3.2.8) and (3.2.13), it follows that $d_{\mathbb{D}}\big(A_{\sigma_m}(x), x\big) \to 0$ as $m \to \infty$. This proves that the identity map i is the pointwise limit (with respect to $d_{\mathbb{D}}$) of the sequence $(A_{\sigma_m})_{m \geq 1}$ of $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$-measurable maps. Since $\mathbb{D}_{\mathbb{D}}$ is the Borel σ-field corresponding to $d_{\mathbb{D}}$, it the map i is also $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$-measurable.

b) This follows by Theorem 3.3 of [7], since $\mathcal{D}_{f,T}^{\mathbb{D}}$ is a π-system generating $\mathcal{D}_{\mathbb{D}}$. $\square$

The characterization of tightness of probability measures on $\mathbb{D}([0,1];\mathbb{D})$ given in Section 3.3 relies on certain events involving the functions $w'_{\mathbb{D}}(\cdot,\delta)$ and $w''_{\mathbb{D}}(\cdot,\delta)$. Measurability of these functions is essential for this purpose. Before establishing this, we need the following simple result (which is valid in any metric space).

Lemma 3.2.12. *The map $\Phi : \mathbb{D} \times \mathbb{D} \to [0,\infty)$ given by $\Phi(x,y) = d_{J_1}^0(x,y)$ is continuous with respect to the product of J_1-topologies on $\mathbb{D} \times \mathbb{D}$.*

Proof: If $x_n \xrightarrow{J_1} x$ and $y_n \xrightarrow{J_1} y$, then $d_{J_1}^0(x_n,y_n) \to d_{J_1}^0(x,y)$ since

$$|d_{J_1}^0(x_n,y_n) - d_{J_1}^0(x,y)| \leq |d_{J_1}^0(x_n,y_n) - d_{J_1}^0(x,y_n)| + |d_{J_1}^0(x,y_n) - d_{J_1}^0(x,y)|$$

$$\leq d^0_{J_1}(x_n, x) + d^0_{J_1}(y_n, y).$$

□

Lemma 3.2.13. *The functions $w'_{\mathbb{D}}(\cdot,\delta)$ and $w''_{\mathbb{D}}(\cdot,\delta)$ are $\mathcal{D}_{\mathbb{D}}$-measurable.*

Proof: The measurability of $w'_{\mathbb{D}}(\cdot,\delta)$ follows by Lemma 3.2.5. For $w''_{\mathbb{D}}(\cdot,\delta)$, note that in the definition (3.2.10) of $w''_{\mathbb{D}}(x,\delta)$, we may take t_1, t, t_2 to be rational numbers. By Lemma 3.2.12, Φ is $\mathcal{D}\times\mathcal{D}$-measurable, and so the map $\Phi\circ\pi^{\mathbb{D}}_{t,t_1}$ given by $x\mapsto d^0_{J_1}(x(t), x(t_1))$ is $\mathcal{D}_{\mathbb{D}}$-measurable, for any $t_1, t \in [0,1]$. Therefore, the map $x \mapsto d^0_{J_1}(x(t), x(t_1)) \wedge d^0_{J_1}(x(t_2), x(t))$ is $\mathcal{D}_{\mathbb{D}}$-measurable, for any rational numbers $t_1, t, t_2 \in [0,1]$ with $t_1 \leq t \leq t_2$. The conclusion follows since the supremum of a countable collection of measurable functions is measurable. □

Finally, we recall the definition of a random element in $\mathbb{D}([0,1];\mathbb{D})$.

Definition 3.2.14. Let $(\Omega,\mathcal{F},P)$ be a probability space. A map $X:\Omega\to\mathbb{D}([0,1];\mathbb{D})$ is called a *random element* in $\mathbb{D}([0,1];\mathbb{D})$ if X is $\mathcal{F}/\mathcal{D}_{\mathbb{D}}$-measurable, i.e. $X(t)$ is $\mathcal{F}/\mathcal{D}$-measurable for any $t\in[0,1]$.

3.3 Weak convergence and tightness

In this section, we study the weak convergence and tightness of probability measures on the space $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$, following the discussion in Section 13 of [6] for probability measures on $(\mathbb{D},\mathcal{D})$.

Recall that if $(P_n)_{n\geq1}$ and P are probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$, we say that $(P_n)_{n\geq1}$ *converges weakly* to P if $\int f dP_n \to \int f dP$ for any $d_{\mathbb{D}}$-continuous bounded function $f:\mathbb{D}([0,1];\mathbb{D})\to\mathbb{R}$. In this case, we write $P_n \stackrel{w}{\to} P$. Since $\mathbb{D}([0,1];\mathbb{D})$ is separable, there is a distance on the set of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ (called the *Prohorov distance*), which gives rise to the topology of weak convergence (see page 72 of [6]).

If $(X_n)_{n\geq1}$ and X are random elements in $\mathbb{D}([0,1];\mathbb{D})$ (possibly defined on different probability spaces) with respective laws denoted by $(P_n)_{n\geq1}$ and P, we say that $(X_n)_{n\geq1}$ *converges in distribution* to X if $P_n \stackrel{w}{\to} P$. In this case, we write $X_n \stackrel{d}{\to} X$.

For any probability measure P on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$, we let T_P be the set of $t\in[0,1]$ for which the projection $\pi^{\mathbb{D}}_t$ is $d_{\mathbb{D}}$-continuous a.s. with respect to P. Note that $0,1\in T_P$. If $t\in(0,1)$, then $t\in T_P$ if and only if $P(J_t)=0$, where $J_t=\{x\in\mathbb{D}([0,1];\mathbb{D}); t\in \mathrm{Disc}(x)\}$.

Using the same argument as in the classical case (page 238 of [6]), it can be shown that $P(J_t)>0$ is possible for at most countably many $t\in(0,1)$. Hence, the complement of T_P in $[0,1]$ is countable. The following result follows by the continuous mapping theorem.

Lemma 3.3.1. *Let $(P_n)_{n\geq 1}$ and P be probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ such that $P_n \xrightarrow{w} P$. Then $P_n \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \xrightarrow{w} P \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1}$ for any $t_1,\ldots,t_k \in T_P$.*

We recall the following definitions.

Definition 3.3.2. A family Π of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ is *tight* if for every $\eta > 0$, there exists a $d_{\mathbb{D}}$-compact set K in $\mathbb{D}([0,1];\mathbb{D})$ such that $P(K) \geq 1-\eta$ for all $P \in \Pi$.

Definition 3.3.3. A family Π of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ is *relatively compact* if for every sequence $(P_n)_{n\geq 1}$ in Π, there exists a subsequence $(P_{n_k})_{k\geq 1}$ which converges weakly to a probability measure Q (which is not necessarily an element of Π).

The following result follows by Prohorov's theorem, since $\mathbb{D}([0,1];\mathbb{D})$ is separable and complete (see Theorems 5.1 and 5.2 of [6]).

Theorem 3.3.4. *A family Π of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ is tight if and only if it is relatively compact.*

The next result is an important tool for proving weak convergence in $\mathbb{D}([0,1];\mathbb{D})$. Its proof is the same as in the classical case (see Theorem 13.1 of [6]). We include it for the sake of completeness.

Theorem 3.3.5. *Let $(P_n)_{n\geq 1}$ and P be probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ such that*

$$P_n \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \xrightarrow{w} P \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \ \text{in } \mathbb{D}^k, \ \text{for any } t_1,\ldots,t_k \in T_P \tag{3.3.1}$$

and $(P_n)_{n\geq 1}$ is tight. Then $P_n \xrightarrow{w} P$.

Proof: It is enough to prove that for any subsequence $(n_k)_{k\geq 1}$, there exists a further sub-subsequence $(k_l)_{l\geq 1}$ such that $P_{n_{k_l}} \xrightarrow{w} P$ as $l \to \infty$ (see e.g. Appendix 5.1.2 of [20]).

Let $(n_k)_{k\geq 1}$ be an arbitrary subsequence. By Theorem 3.3.4, $(P_n)_{n\geq 1}$ is relatively compact. Hence, there exists a sub-subsequence $(k_l)_{l\geq 1}$ such that $P_{n_{k_l}} \xrightarrow{w} Q$ as $l \to \infty$, for some probability measure Q on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$. By hypothesis, $P_{n_{k_l}} \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \xrightarrow{w} P \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1}$ as $l \to \infty$, for any $t_1,\ldots,t_k \in T_P$. By Lemma 3.3.1, $P_{n_{k_l}} \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \xrightarrow{w} Q \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1}$ as $l \to \infty$, for any $t_1,\ldots,t_k \in T_Q$. Uniqueness of the limit implies that:

$$P \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} = Q \circ (\pi^{\mathbb{D}}_{t_1,\ldots,t_k})^{-1} \quad \text{for all } t_1,\ldots,t_k \in T_P \cap T_Q.$$

The set $T = T_P \cap T_Q$ contains 0 and 1, and is dense in $[0,1]$ (since its complement in $[0,1]$ is countable). By Theorem 3.2.11, $\mathcal{D}_{f,T}$ is a separating class of $\mathcal{D}_{\mathbb{D}}$, and hence $P = Q$. $\square$

We continue now with a discussion about tightness. The next result gives a criterion for tightness, being the analogue of Theorem 13.2 of [6] for the space $\mathbb{D}([0,1];\mathbb{D})$. Conditions *(i)* and *(iii)* of this theorem are similar to (13.4) and (13.5) of [6], but *(ii)* is a new condition, due to the space variable s of an element in $\mathbb{D}([0,1];\mathbb{D})$. Recall that $w'\big(x(t),\delta\big)$ is given by (3.2.7), whereas $w'_{\mathbb{D}}(x,\delta)$ is given by (3.2.7), for any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $t \in [0,1]$.

Theorem 3.3.6. *A sequence $(P_n)_{n\geq 1}$ of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}}\big)$ is tight if and only if it satisfies the following three conditions:*
(i) We have:

$$\lim_{a\to\infty} \limsup_{n\to\infty} P_n\big(\{x;\ \|x\|_{\mathbb{D}} \geq a\}\big) = 0. \tag{3.3.2}$$

(ii) For any $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\big(\{x;\ w'\big(x(t),\delta\big) \geq \varepsilon \textit{ for some } t \in [0,1]\}\big) = 0. \tag{3.3.3}$$

(iii) For any $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\big(\{x;\ w'_{\mathbb{D}}(x,\delta) \geq \varepsilon\}\big) = 0. \tag{3.3.4}$$

Proof: We use a similar argument as in the proof of Theorem 13.2 of [6] (see also the proof of Theorem 7.3 of [6]). Suppose that $(P_n)_{n\geq 1}$ is tight. Let $\eta > 0$ and $\varepsilon > 0$ be arbitrary. We have to prove that there exist $a > 0$, $\delta \in (0,1)$ and an integer $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\begin{cases} (a) & P_n\big(\{x;\ \|x\|_{\mathbb{D}} \geq a\}\big) \leq \eta \\ (b) & P_n\big(\{x;\ w'\big(x(t),\delta\big) \geq \varepsilon \text{ for some } t \in [0,1]\}\big) \leq \eta \\ (c) & P_n\big(\{x;\ w'_{\mathbb{D}}(x,\delta) \geq \varepsilon\big) \leq \eta. \end{cases} \tag{3.3.5}$$

We will show that *(a)*-*(c)* hold with $n_0 = 1$. By Theorem 3.3.4, $(P_n)_{n\geq 1}$ is relatively compact. Hence, there exists a compact set K in $\mathbb{D}([0,1];\mathbb{D})$ such that $P_n(K) \geq 1-\eta$ for all $n \geq 1$. The set K is characterized using Theorem 3.2.7. More precisely, we know that:

$$\begin{cases} (a') & \sup_{x\in K} \|x\|_{\mathbb{D}} < \infty \\ (b') & \lim_{\delta\to 0} \sup_{x\in K} \sup_{t\in[0,1]} w'\big(x(t),\delta\big) = 0 \\ (c') & \lim_{\delta\to 0} \sup_{x\in K} w'_{\mathbb{D}}(x,\delta) = 0 \end{cases} \tag{3.3.6}$$

Due to (a'), we can choose $a > \sup_{x\in K} \|x\|_{\mathbb{D}}$ arbitrary. Then $K \subset \{x; \|x\|_{\mathbb{D}} < a\}$ and so,

$$P_n\big(\{x;\ \|x\|_{\mathbb{D}} \geq a\}\big) \leq P_n(K^c) \leq \eta \quad \text{for all } n \geq 1.$$

By (b'), there exists $\delta \in (0,1)$ such that $w'(x(t),\delta) < \varepsilon$ for all $x \in K, t \in [0,1]$. Hence, $K \subset \{x;\, w'(x(t),\delta) < \varepsilon \text{ for all } t \in [0,1]\}$, and so

$$P_n\big(\{x;\, w'\big(x(t),\delta\big) < \varepsilon \text{ for some } t \in [0,1]\}\big) \leq P_n(K^c) \leq \eta \quad \text{for all } n \geq 1.$$

By (c'), there exists $\delta \in (0,1)$ such that $w'_{\mathbb{D}}(x,\delta) < \varepsilon$ for all $x \in K$. Hence, $K \subset \{x;\, w'_{\mathbb{D}}(x,\delta) < \varepsilon\}$, and so

$$P_n\big(\{x;\, w'\big(x,\delta\big) < \varepsilon\}\big) \leq P_n(K^c) \leq \eta \quad \text{for all } n \geq 1.$$

Suppose next that conditions *(i)*-*(iii)* hold. Let $\eta > 0$ and $\varepsilon > 0$ be arbitrary. Then there exist $a' > 0$, $\delta' \in (0,1)$ and an integer $n_0 \geq 1$ such that (3.3.5) holds for all $n \geq n_0$ (with a' and δ' replacing a and δ). We first prove that (3.3.5) actually holds for all $n \geq 1$, for some values a and δ which will be given below. Fix $i \in \{1,\ldots,n_0-1\}$. Since $\mathbb{D}([0,1];\mathbb{D})$ is separable and complete, the single probability measure P_i is tight, and therefore it satisfies conditions *(i)*-*(iii)*. Hence, there exists $a_i > 0$ and $\delta_i \in (0,1)$ such that

$$\begin{cases} P_i\big(\{x;\, \|x\|_{\mathbb{D}} \geq a_i\}\big) \leq \eta \\ P_i\big(\{x;\, w'\big(x(t),\delta_i\big) \geq \varepsilon \text{ for some } t \in [0,1]\}\big) \leq \eta \\ P_i\big(\{x;\, w'_{\mathbb{D}}(x,\delta_i) \geq \varepsilon\big) \leq \eta. \end{cases}$$

Then (3.3.5) holds for all $n \geq 1$, with $a = \max\{a', \max_{i \leq n_0-1} a_i\}$ and $\delta = \min\{\delta', \min_{i \leq n_0-1} \delta_i\}$.

Let $B = \{x; \|x\|_{\mathbb{D}} < a\}$. Then $P_n(B) \geq 1-\eta$ for all $n \geq 1$. By parts *(b)* and *(c)* of (3.3.5) with $\varepsilon = 1/k$ and η replaced by $\eta/2^k$, there exists $\delta_k \in (0,1)$ such that for all $n \geq 1$,

$$P_n(B_k) \geq 1 - \frac{\eta}{2^k} \quad \text{and} \quad P_n(C_k) \geq 1 - \frac{\eta}{2^k},$$

where $B_k = \{x; \sup_{t\in[0,1]} w'(x(t),\delta_k) < 1/k\}$ and $C_k = \{x; w'_{\mathbb{D}}(x,\delta_k) < 1/k\}$. Let $A = B \cap \big(\cap_{k\geq 1} B_k\big) \cap \big(\cap_{k\geq 1} C_k\big)$ and $K = \overline{A}$. For any $n \geq 1$, $P_n(K) \geq P_n(A) \geq 1-3\eta$, since

$$P_n(A^c) \leq P_n(B^c) + \sum_{k\geq 1} P_n(B_k^c) + \sum_{k\geq 1} P_n(C_k^c) \leq \eta + \sum_{k\geq 1} \frac{\eta}{2^k} + \sum_{k\geq 1} \frac{\eta}{2^k} = 3\eta.$$

We show that K is compact in $\mathbb{D}([0,1];\mathbb{D})$. By Theorem 3.2.7, this is equivalent to showing that K satisfies (3.3.6). Since $\|x\|_{\mathbb{D}} < a$ for any $x \in B$ and $A \subset B$, we have $\sup_{x\in A} \|x\|_{\mathbb{D}} < a$. This shows that (a') holds. Note that for any $k \geq 1$, $\sup_{x\in A} \sup_{t\in[0,1]} w'(x(t),\delta_k) < 1/k$ (since $A \subset B_k$), and so (b') holds. Finally, for any $k \geq 1$, $\sup_{x\in A} w'_{\mathbb{D}}(x,\delta_k) < 1/k$ (since $A \subset C_k$), and hence (c') holds. This proves that $(P_n)_{n\geq 1}$ is tight. $\square$

The following result gives a replacement for condition *(i)* in Theorem 3.3.6. This condition is the analogue of (13.6) of [6].

Corollary 3.3.7. *Condition (i) of Theorem 3.3.6 can be replaced by the following condition:*
(i') for each t in a dense subset T of $[0,1]$ which contains 1, we have:

$$\lim_{a\to\infty}\limsup_{n\to\infty} P_n\big(\{x; \|x(t)\| \geq a\}\big) = 0. \tag{3.3.7}$$

Proof: Suppose that condition (i) of Theorem 3.3.6 holds. Then (i') clearly holds, since $\{x; \|x(t)\| \geq a\} \subset \{x; \|x\|_{\mathbb{D}} \geq a\}$ for any $t \in T$.

Suppose next that conditions (i') and (iii) hold. We prove that (i) holds, using a similar argument as in the Corollary on page 140 of [6]. Let $\eta > 0$ be arbitrary. By condition (iii), there exist $\delta \in (0,1)$ and an integer $n_1 \geq 1$ such that

$$P_n\big(\{x; w'_{\mathbb{D}}(x,\delta) \geq 1\}\big) \leq \eta \quad \text{for all } n \geq n_1. \tag{3.3.8}$$

Let $\{t_i\}_{i=1,\ldots,v}$ be a δ-sparse set with $0 = t_0 < t_1 < \ldots < t_v = 1$ such that $w_{\mathbb{D}}(x,[t_{i-1},t_i)) \leq w'_{\mathbb{D}}(x,\delta)+1$ for all $i = 1,\ldots,v$. Choose points $0 = s_0 < s_1 < \ldots < s_k = 1$ such that $s_j \in T$ and $s_j - s_{j-1} < \delta$ for all $k = 1,\ldots,k$. Let $m(x) = \max_{1\leq j\leq k} \|x(s_j)\|$. By (3.3.7), $\lim_{a\to\infty}\limsup_{n\to\infty} P_n\big(\{x; m(x) \geq a\}\big) = 0$. So, there exist $a > 0$ and $n_2 \geq 1$ such that

$$P_n\big(\{x; m(x) \geq a\}\big) \leq \eta \quad \text{for all } n \geq n_2. \tag{3.3.9}$$

We claim that for any $x \in \mathbb{D}([0,1];\mathbb{D})$,

$$\|x\|_{\mathbb{D}} \leq w'_{\mathbb{D}}(x,\delta) + 1 + m(x). \tag{3.3.10}$$

To see this, note that since $\{t_i\}_i$ is δ-sparse, each interval $[t_{i-1},t_i)$ contains at least one point s_j, that we call s_{j_i}. For any $i = 1,\ldots,v$ and for any $t \in [t_{i-1},t_i)$,

$$\|x(t)\| = d^0_{J_1}\big(x(t),0\big) \leq d^0_{J_1}\big(x(t),x(s_{j_i})\big) + d^0_{J_1}\big(x(s_{j_i}),0\big) = d^0_{J_1}\big(x(t),x(s_{j_i})\big) + \|x(s_{j_i})\|.$$

Hence,

$$\sup_{t\in[t_{i-1},t_i)} \|x(t)\| \leq w_{\mathbb{D}}(x,[t_{i-1},t_i)) + \|x(s_{j_i})\| \leq w'_{\mathbb{D}}(x,\delta) + 1 + m(x).$$

Relation (3.3.10) follows since $\|x\|_{\mathbb{D}} = \max\{\max_{1\leq i\leq v}\sup_{t\in[t_{i-1},t_i)}\|x(t)\|, \|x(1)\|\}$.

Let $n_0 = \max(n_1,n_2)$. From (3.3.8), (3.3.9) and (3.3.10), we infer that

$$P_n\big(\{x; \|x\|_{\mathbb{D}} \geq a+2\}\big) \leq P_n\big(\{x; w'_{\mathbb{D}}(x,\delta) + m(x) \geq a+1\}\big) \leq 2\eta \quad \text{for all } n \geq n_0.$$

This concludes the proof of (i). $\square$

The following result is the analogue of relation (13.8) of [6] (or Theorem 15.3 of [5]), and it plays a crucial role in article [3] (see Theorem 2.4 of [3]).

Theorem 3.3.8. *A sequence $(P_n)_{n\geq 1}$ of probability measures on $\big(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}}\big)$ is tight if and only if it satisfies condition (i) of Theorem 3.3.6 and the following two conditions:*
(ii') For any $\varepsilon>0$,

$$\begin{cases} (a) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ w''(x(t),\delta)\geq \varepsilon \text{ for some } t\in[0,1]\})=0;\\ (b) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ |x(t,\delta)-x(t,0)|\geq \varepsilon \text{ for some } t\in[0,1]\})=0;\\ (c) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ |x(t,1-)-x(t,1-\delta)|\geq \varepsilon \text{ for some } t\in[0,1]\})=0.\end{cases}$$

(iii') For any $\varepsilon>0$,

$$\begin{cases} (a) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ w''_{\mathbb{D}}(x,\delta)\geq \varepsilon\})=0;\\ (b) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ d^0_{J_1}\big(x(\delta),x(0)\big)\geq \varepsilon\})=0;\\ (c) & \lim_{\delta\to 0}\limsup_{n\to\infty} P_n(\{x;\ d^0_{J_1}\big(x(1-),x(1-\delta)\big)\geq \varepsilon\})=0.\end{cases}$$

Proof: This follows directly from Theorem 3.3.6. To see this, note that (ii') is equivalent to (ii) of Theorem 3.3.6, due to inequalities (12.31) and (12.32) of [6]), whereas (iii') is equivalent to (iii) of Theorem 3.3.6, due to inequalities (3.2.11) and (3.2.12). $\square$

The following result is the analogue of Theorem 13.3 of [6].

Theorem 3.3.9. *Let $(P_n)_{n\geq 1}$ and P be probability measures on $\mathbb{D}([0,1];\mathbb{D})$ such that (4.3.5) holds, $(P_n)_{n\geq 1}$ satisfies parts (ii') and $(iii'.a)$ of Theorem 3.3.8, and P satisfies*

$$\lim_{\delta\to 0} P\big(\{x;\ d^0_{J_1}\big(x(1),x(1-\delta)\big)\geq \varepsilon\}\big)=0 \quad \text{for all } \varepsilon>0. \tag{3.3.11}$$

Then $P_n \xrightarrow{w} P$.

Proof: By Theorem 13.1, it is enough to prove that $(P_n)_{n\geq 1}$ is tight. For this, we use Theorem 3.3.8. We first check condition (i') given by Corollary 3.3.7, with $T=T_P$. Let $t\in T_P$ be arbitrary. The sequence $\{P_n\circ(\pi_t^{\mathbb{D}})^{-1}\}_{n\geq 1}$ is relatively compact in $\mathbb{D}$ being weakly convergent. By Prohorov theorem, this sequence is tight. Hence, for any $\eta>0$, there exists a compact set K in $\mathbb{D}$ such that $[P_n\circ(\pi_t^{\mathbb{D}})^{-1}](K^c)\leq \eta$ for all $n\geq 1$. By Theorem 12.3 of [6], $M:=\sup_{y\in K}\|y\|<\infty$. For any $a>M$, $\{y\in\mathbb{D};\|y\|\geq a\}\subset K^c$ and

$$P_n\big(\{x;\|x(t)\|\geq a\}\big)\leq [P_n\circ(\pi_t^{\mathbb{D}})^{-1}](K^c)\leq \eta \quad \text{for all } n\geq 1.$$

Next, we check that part (b) of (iii') holds. Let $\varepsilon>0$ and $\eta>0$ be arbitrary. By the right continuity of elements in $\mathbb{D}([0,1];\mathbb{D})$, $P\big(\{x;d^0_{J_1}\big(x(\delta),x(0)\big)\geq\varepsilon\}\big)\to 0$ as $\delta\to 0$. Choose $\delta\in T_P$ small such that $P\big(\{x;d^0_{J_1}\big(x(\delta),x(0)\big)\}\big)<\eta$. By (4.3.5), $P_n\circ(\pi^{\mathbb{D}}_{0,\delta})^{-1}\xrightarrow{w} P\circ(\pi^{\mathbb{D}}_{0,\delta})^{-1}$ in $\mathbb{D}^2$. By Lemma 3.2.12, the set $A=\{(y_1,y_2)\in$

$\mathbb{D}^2; d^0_{J_1}(y_1, y_2) \geq \varepsilon\}$ is closed in $\mathbb{D}^2$ with respect to the product of J_1-topologies. By Portmanteau theorem, it follows that

$$\limsup_{n\to\infty} P_n\big(\{x; d^0_{J_1}\big(x(\delta), x(0)\big)\}\big) \leq P\big(\{x; d^0_{J_1}\big(x(\delta), x(0)\big)\}\big) < \eta.$$

We prove that part (c) of (iii') holds. By the left continuity of elements in $\mathbb{D}([0,1];\mathbb{D})$, $P\big(\{x; d^0_{J_1}\big(x(1-), x(1-\delta)\big) \geq \varepsilon\}\big) \to 0$ as $\delta \to 0$, for any $\varepsilon > 0$. By (3.3.11), it follows that $P\big(\{x; d^0_{J_1}\big(x(1), x(1-)\big) \geq \varepsilon\}\big) = 0$, for any $\varepsilon > 0$. Hence, $P\big(\{x; d^0_{J_1}\big(x(1), x(1-)\big) > 0\}\big) = 0$. The rest of the argument is the same as for part (b). $\square$

The previous theorem can also be stated in terms of random elements, as follows.

Theorem 3.3.10. *Let $(X_n)_{n\geq 1}$ and X be random elements in $\mathbb{D}([0,1];\mathbb{D})$ defined on the same probability space. Let $T_X = \{t \in [0,1]; P(X(t) = X(t-)) = 1\}$. Suppose that:*
a) $\big(X_n(t_1), \ldots, X_n(t_k)\big) \xrightarrow{d} \big(X(t_1), \ldots, X(t_k)\big)$ in $\mathbb{D}^k$, for any $t_1, \ldots, t_k \in T_X$;
b) $d^0_{J_1}\big(X(1), X(1-\delta)\big) \xrightarrow{P} 0$ as $\delta \to 0$;
c) for any $\varepsilon > 0$,

$$\begin{cases} \lim_{\delta\to 0}\limsup_{n\to\infty} P\big(\{w''\big(X_n(t), \delta\big) \geq \varepsilon \text{ for some } t \in [0,1]\}\big) = 0, \\ \lim_{\delta\to 0}\limsup_{n\to\infty} P(|X_n(t,\delta) - X_n(t,0)| \geq \varepsilon \text{ for some } t \in [0,1]) = 0, \\ \lim_{\delta\to 0}\limsup_{n\to\infty} P(|X_n(t,1-) - X_n(t,1-\delta)| \geq \varepsilon \text{ for some } t \in [0,1]) = 0; \end{cases}$$

d) for any $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P(w''_{\mathbb{D}}(X_n, \delta) \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0. \tag{3.3.12}$$

Then $X_n \xrightarrow{d} X$ in $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$.

Remark 3.3.11. Hypothesis *c)* of Theorem 3.3.10 may be difficult to verify in practice. In the proof of Theorem 3.14 of [3], this hypothesis is verified by showing that

$$\inf_{n_0\geq 1}\sup_{n\geq n_0} P(\|X_n - X_{n_0}\|_{\mathbb{D}} \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0. \tag{3.3.13}$$

Since for any $n_0 \geq 1$, the single probability measure $P \circ X_{n_0}^{-1}$ is tight in $\mathbb{D}([0,1];\mathbb{D})$, part ($ii'$) of Theorem 3.3.8 gives:

$$\begin{cases} \lim_{\delta\to 0} P\big(\{w''\big(X_{n_0}(t), \delta\big) \geq \varepsilon \text{ for some } t \in [0,1]\}\big) = 0, \\ \lim_{\delta\to 0} P(|X_{n_0}(t,\delta) - X_{n_0}(t,0)| \geq \varepsilon \text{ for some } t \in [0,1]) = 0, \\ \lim_{\delta\to 0} P(|X_{n_0}(t,1-) - X_{n_0}(t,1-\delta)| \geq \varepsilon \text{ for some } t \in [0,1]) = 0; \end{cases}$$

Hypothesis *c)* then follows from (3.3.13), using the following inequalities:

$$w''\big(X_n(t), \delta\big) \leq w''\big(X_{n_0}(t), \delta\big) + 2\|X_n - X_{n_0}\|_{\mathbb{D}}$$
$$|X_n(t,\delta) - X_n(t,0)| \leq |X_{n_0}(t,\delta) - X_{n_0}(t,0)| + 2\|X_n - X_{n_0}\|_{\mathbb{D}}$$
$$|X_n(t,1-) - X_n(t,1-\delta)| \leq |X_{n_0}(t,1-) - X_{n_0}(t,1-\delta)| + 2\|X_n - X_{n_0}\|_{\mathbb{D}}.$$

3.4 Criteria for existence and convergence

In this section, we give a criterion for weak convergence of random elements in $\mathbb{D}([0,1];\mathbb{D})$, and a criterion for the existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$ based on its finite-dimensional distributions. Both these results rely on some maximal inequalities which are of independent interest.

The following result is the analogue of Theorem 10.3 of [6].

Theorem 3.4.1. *Let T be a Borel set in $[0,1]$ and $\{X(t)\}_{t\in T}$ a collection of random elements in $\mathbb{D}$ defined on the same probability space $(\Omega,\mathcal{F},P)$ such that the map $T \ni t \mapsto X(\omega,t)$ is right-continuous with respect to J_1, for any $\omega \in \Omega$. (If T is finite, this imposes no restriction.) For any $r,s,t \in T$ with $r \leq s \leq t$, let*

$$m_{rst}^{J_1} = d_{J_1}^0\big(X(r),X(s)\big) \wedge d_{J_1}^0\big(X(s),X(t)\big) \tag{3.4.1}$$

and $L_{J_1}(X) = \sup_{r,s,t\in T;\, r\leq s\leq t} m_{rst}^{J_1}$. Suppose that there exist $\alpha > 1/2$, $\beta \geq 0$ and a finite measure μ on T such that for any $\lambda > 0$ and for any $r,s,t \in T$ with $r \leq s \leq t$,

$$P(m_{rst}^{J_1} \geq \lambda) \leq \frac{1}{\lambda^{4\beta}}\{\mu(T\cap(r,t])\}^{2\alpha}. \tag{3.4.2}$$

Then there exists a constant K depending on α and β such that for any $\lambda > 0$,

$$P(L_{J_1}(X) > \lambda) \leq \frac{K}{\lambda^{4\beta}}\mu^{2\alpha}(T). \tag{3.4.3}$$

Proof: We follow the same idea as in the proof of Theorem 10.3 of [6].

Case 1. $T=[0,1]$ and μ is the Lebesgue measure. Let $D_k = \{i/2^k; 0 \leq i \leq 2^k\}$. Define B_k be the maximum of all $m_{t_1t_2t_3}^{J_1}$ for all $t_1,t_2,t_3 \in D_k$ with $t_1 \leq t_2 \leq t_3$ and A_k be the maximum of $m_{t_1t_2t_3}^{J_1}$ with $t_1=(i-1)/2^k$, $t_2=i/2^k$ and $t_3=(i+1)/2^k$, for $i=1,\ldots,2^k-1$. It can be proved that $B_k \leq 2(A_1+\ldots+A_k)$ for any $k \geq 1$. Note that $B_k \leq B_{k+1}$ for all $k \geq 1$. We claim that:

$$L_{J_1}(X) = \lim_{k\to\infty} B_k. \tag{3.4.4}$$

To see this, let $\varepsilon > 0$ be arbitrary. Let $t_1,t_2,t_3 \in T$ be such that $t_1 \leq t_2 \leq t_3$. For each $k \geq 1$, there exist $t_1^k,t_2^k,t_3^k \in D_k$ with $t_1^k \leq t_2^k \leq t_3^k$ such that $t_i^k \downarrow t_i$ as $k\to\infty$, for $i=1,2,3$. Since $t\mapsto X(t)$ is right-continuous with respect to J_1, $X(t_i^k) \xrightarrow{J_1} X(t_i)$ as $k\to\infty$, for $i=1,2,3$. By Lemma 3.2.12, $a_k = d_{J_1}^0\big(X(t_1^k),X(t_2^k)\big) \to a = d_{J_1}^0\big(X(t_1),X(t_2)\big)$ as $k\to\infty$ and $b_k = d_{J_1}^0\big(X(t_2^k),X(t_3^k)\big) \to b = d_{J_1}^0\big(X(t_2),X(t_3)\big)$ as $k\to\infty$. Hence, there exists k_ε such that $a_{k_e} \geq a-\varepsilon$ and $b_{k_\varepsilon} \geq b-\varepsilon$. So, $a\wedge b \leq a_{k_e}\wedge b_{k_e}+\varepsilon \leq B_{k_e}+\varepsilon$. Since t_1,t_2,t_3 were arbitrary, we obtain that $L_{J_1}(X) \leq B_{k_e}+\varepsilon$.

From (3.4.4), it follows that $L_{J_1}(X) \leq 2\sum_{k\geq 1} A_k$. From this, we deduce relation (3.4.3) using (3.4.2) to estimate the tail probability of A_k (see page 110 of [6]).

The other cases follow as in the proof of Theorem 10.3 of [6]. $\square$

The following result is proved exactly as Theorem 10.4 of [6].

Corollary 3.4.2. *If condition* (3.4.2) *of Theorem 3.4.1 only holds for $t - r < 2\delta$, then*

$$P(L_{J_1}(X,\delta) > \lambda) \leq \frac{2K}{\lambda^{4\beta}}\mu(T) \sup_{0\leq t\leq 1-2\delta} \mu^{2\alpha-1}\big(T \cap [t, t+2\delta]\big),$$

where $L_{J_1}(X,\delta)$ is the supremum of $m^{J_1}_{rst}$ for all $r,s,t \in T$ with $r \leq s \leq t$ and $t-r < \delta$, and $m^{J_1}_{rst}$ is given by (3.4.1)*. In particular, if $T = [0,1]$, then $L_{J_1}(X,\delta) = w''_{\mathbb{D}}(X,\delta)$.*

3.5 The space $\mathbb{D}([0,\infty);\mathbb{D})$

We denote by $\mathbb{D}([0,T];\mathbb{D})$ the space of functions $x : [0,T] \to \mathbb{D}$ which are right continuous and have left limits with respect to J_1. For any $T > 0$, we denote by $d_{T,\mathbb{D}}$ the *Skorohod distance* on $\mathbb{D}([0,T];\mathbb{D})$, which is defined similarly to (3.2.1) :

$$d_{T,\mathbb{D}} = \inf_{\lambda\in\Lambda_T} \{\|\lambda - e\|_T \vee \rho_{T,\mathbb{D}}(x, y\circ\lambda)\} \tag{3.5.1}$$

for any $x, y \in \mathbb{D}([0,T];\mathbb{D})$, where $\rho_{T,\mathbb{D}}$ is the *uniform distance* on $\mathbb{D}([0,T];\mathbb{D})$ given by :

$$\rho_{T,\mathbb{D}}(x,y) = \sup_{t\in[0,T]} d^0_{J_1}(x(t), y(t)). \tag{3.5.2}$$

Here Λ_T is the set of continuous strictly increasing function λ that map $[0,T]$ onto $[0,T]$ and $\|\lambda\|_T = \sup_{t\in[0,T]}|\lambda(t)|$ for any $\lambda \in \Lambda_T$. For any $x \in \mathbb{D}([0,T];\mathbb{D})$, we define the super-uniform norm by analogy with (3.2.3) :

$$\|x\|_{T,\mathbb{D}} = \sup_{t\in[0,T]} \|x(t)\| \tag{3.5.3}$$

Note that

$$d_{T,\mathbb{D}}(x,y) \leq \rho_{T,\mathbb{D}}(x,y) \leq \|x - y\|_{T,\mathbb{D}} \tag{3.5.4}$$

In this section, we introduce the space $\mathbb{D}([0,\infty);\mathbb{D})$ and we list some of its properties.

For any fixed $T > 0$, we let $\mathbb{D}([0,T];\mathbb{D})$ be the set of functions $x : [0,T] \to \mathbb{D}$ which are right-continuous and have left-limits with respect to J_1. Let Λ_T be the set of strictly increasing continuous functions from $[0,T]$ onto itself. Similarly to the case $T = 1$, we define the Skorohod distance on $\mathbb{D}([0,T];\mathbb{D})$ by:

$$d_{T,\mathbb{D}}(x,y) = \inf_{\lambda\in\Lambda_T} \{\|\lambda - e\|_T \wedge \rho_{T,\mathbb{D}}(x, y\circ\lambda)\}, \tag{3.5.5}$$

where $\|\cdot\|_T$ is the supremum norm on Λ_T, e is the identity function on $[0,T]$, and $\rho_{T,\mathbb{D}}$ is the uniform distance on $\mathbb{D}([0,T];\mathbb{D})$ given by:

$$\rho_{T,\mathbb{D}}(x,y) = \sup_{t\in[0,T]} d^0_{J_1}(x(t),y(t)). \tag{3.5.6}$$

We denote by $\|\cdot\|_{T,\mathbb{D}}$ the super-uniform norm on $\mathbb{D}([0,T];\mathbb{D})$ given by:

$$\|x\|_{T,\mathbb{D}} = \sup_{t\in[0,T]} \|x(t)\|.$$

For any $x,y \in \mathbb{D}([0,T];\mathbb{D})$, we have

$$d_{T,\mathbb{D}}(x,y) \le \rho_{T,\mathbb{D}}(x,y) \le \|x-y\|_{T,\mathbb{D}}. \tag{3.5.7}$$

The Skorohod distance on the space $\mathbb{D}([0,\infty);\mathbb{D})$ is given by: (see (2.2) of [34])

$$d_{\infty,\mathbb{D}}(x,y) = \int_0^\infty e^{-t}\Big(d_{t,\mathbb{D}}\big(r_t(x),r_t(y)\big)\wedge 1\Big)dt, \tag{3.5.8}$$

where $r_t(x)$ is the restriction to $[0,t]$ of the function $x \in \mathbb{D}([0,\infty);\mathbb{D})$.

By Theorem 2.6 of [34], $\mathbb{D}([0,\infty);\mathbb{D})$ equipped with distance $d_{\infty,\mathbb{D}}$ is a Polish space. Its Borel σ-field $\mathcal{D}_{\infty,\mathbb{D}}$ coincides (by Lemma 2.7 of [34]) with the σ-field generated by the projections $\{\pi^{\mathbb{D}}_t; t \ge 0\}$, where $\pi^{\mathbb{D}}_t : \mathbb{D}([0,\infty);\mathbb{D}) \to \mathbb{D}$ is given by $\pi^{\mathbb{D}}_t(x) = x(t)$.

Similarly to page 174 of [6], if $(P_n)_{n\ge 1}$ and P are probability measures on $\mathbb{D}([0,\infty);\mathbb{D})$ such that $P_n \stackrel{w}{\to} P$ then the marginal convergence (3.3.1) holds for all $t_1,\ldots,t_k \in T_P$, where the set T_p (defined as in Section 3.2 above) has a countable complement. In fact, $P_n \stackrel{w}{\to} P$ if and only if $P_n \circ r_t^{-1} \stackrel{w}{\to} P \circ r_t^{-1}$ for any $t \in T_p$ (see also Theorem 2.8 of [34]).

Chapter 4

The $\mathbb{D}$-valued α-stable Lévy motion

In this chapter, we give the construction of the α-stable Lévy motion $\{Z(t)\}_{t\geq 0}$ with values in $\mathbb{D}$, following the method described in Section 5.5 of [28] for Lévy process with values in $\mathbb{R}^d$.

For each $t \geq 0$, $Z(t)$ is a random element in $\mathbb{D}$ which we denote by $\{Z(t,s)\}_{s\in[0,1]}$, i.e $Z(t,s) = Z(t)(s)$. Intuitively, the process Z evolues in time and space: $Z(t,s)$ gives the value of this process at time $t \geq 0$ and location $s \in [0,1]$ in space.

We consider the function $T : \mathbb{D}_0 \to (0,\infty) \times \mathbb{S}_{\mathbb{D}}$ given by $T(x) = \left(\|x\|, \frac{x}{\|x\|}\right)$, where $\mathbb{D}_0 = \mathbb{D}\backslash\{0\}$ and $\mathbb{S}_{\mathbb{D}} = \{x \in \mathbb{D};\ \|x\| = 1\}$. Recall that $\|\cdot\|$ denotes the uniform norm on $\mathbb{D}$ and $\pi_{s_1,\ldots,s_n} : \mathbb{D}_0 \to \mathbb{R}^m$ is the projection given by $\pi_{s_1,\ldots,s_n}(x) = (x(s_1),\ldots,x(s_n))$ for fixed $s_1,\ldots,s_m \in [0,1]$. We let ν_α the measure on $(0,\infty]$ given by:

$$\nu_\alpha(dr) = \alpha r^{-\alpha-1} 1_{(0,\infty)}(r)dr$$

We introduce the following assumptions that speak about a probability measure Γ_1 on $\mathbb{S}_{\mathbb{D}}$:

Assumption A. For any $s \in [0,1]$, $\Gamma_1(\{z \in \mathbb{S}_{\mathbb{D}}; z(s) = 0\}) = 0$.

Assumption B. For any $s \in [0,1]$, $\Gamma_1(\{z \in \mathbb{S}_{\mathbb{D}}; s \in \mathrm{Disc}(z) = 0\}) = 0$,
where $\mathrm{Disc}(z)$ is the set of discontinuity of $z \in \mathbb{S}_{\mathbb{D}}$.

The results presented in this chapter are taken from the companion paper [2].

Definition 4.0.1. *Let ν be a measure on $(\mathbb{D},\mathcal{D})$ such that $\nu(\{0\}) = 0$ and*

$$\overline{\nu} := \nu \circ T^{-1} = c\nu_\alpha \times \Gamma_1 \qquad (4.0.1)$$

*for some $c > 0$, $\alpha \in (0,2), \alpha \neq 1$ and a probability measure Γ_1 on $\mathbb{S}_{\mathbb{D}}$ (which satisfies Assumption A). A collection $\{Z(t)\}_{t\geq 0}$ of random elements in $\mathbb{D}$, defined on a probability space $(\Omega,\mathcal{F},P)$ is a $\mathbb{D}$-**valued** α-**stable Lévy motion** (corresponding to ν) if*

(i) $Z(0) = 0$ *a.s.;*

(ii) $Z(t_2) - Z(t_1),\ldots,Z(t_K) - Z(t_{K-1})$ *are independent, for any* $0 \leq t_1 < \ldots < t_K$,

$K \geq 3$;
(iii) *$Z(t_2) - Z(t_1) \stackrel{d}{=} Z(t_2 - t_1)$ for any $0 \leq t_1 < t_2$, where $\stackrel{d}{=}$ means equality in distribution;*
(iv) *for any $t > 0$, $Z(t) = \{Z(t,s)\}_{s \in [0,1]}$ is an α-stable process (with sample paths in $\mathbb{D}$) such that for any $s_1, \ldots, s_m \in [0,1]$ and for any $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$,*

$$E\left(e^{iu_1 Z(t,s_1)+\ldots+iu_m Z(t,s_m)}\right) = \exp\left\{t \int_{\mathbb{R}^m} (e^{iu\cdot y} - 1)\mu_{s_1,\ldots,s_m}(dy)\right\} \quad \text{if} \quad \alpha < 1, \quad (4.0.2)$$

$$E\left(e^{iu_1 Z(t,s_1)+\ldots+iu_m Z(t,s_m)}\right) = \exp\left\{t \int_{\mathbb{R}^m} (e^{iu\cdot y} - 1 - iu\cdot y)\mu_{s_1,\ldots,s_m}(dy)\right\} \quad \text{if} \quad \alpha > 1 \quad (4.0.3)$$

where $y = (y_1, \ldots, y_m)$, $u \cdot y = \sum_{i=1}^m u_i y_i$, and $\mu_{s_1,\ldots,s_m} = \nu \circ \pi^{-1}_{s_1,\ldots,s_m}$.

From this definition, it follows that $Z(t,s)$ has an α-stable $S_\alpha(t^{1/\alpha}\sigma_s, \beta_s, 0)$-distribution, for some constants $\sigma_s > 0$ and $\beta_s \in [-1,1]$ depending on s (see in Proposition 4.2.1 below). Note that property (4.0.1) implies that $\int_{\mathbb{D}_0}(\|x\|^2 \wedge 1)\nu(dx) < \infty$, by a change of variables.

The goal of this chapter is to prove the following result.

Theorem 4.0.2. *a) For any measure ν on $(\mathbb{D}, \mathcal{D})$ such that $\nu(\{0\}) = 0$ and (4.0.1) holds, there exists a $\mathbb{D}$-valued α-stable Lévy motion $\{Z(t)\}_{t\geq 0}$ (corresponding to measure ν).*

b) There exists a collection $\{\widetilde{Z}(t)\}_{t\geq 0}$ of random elements in $\mathbb{D}$ such that $P(Z(t) = \widetilde{Z}(t)) = 1$ for any $t \geq 0$, and the map $t \mapsto \widetilde{Z}(t)$ is in $\mathbb{D}([0,\infty);\mathbb{D})$ with probability 1.

Remark 4.0.3. The authors of [10] considered α-stable Lévy processes $\{Z(t)\}_{t\geq 0}$ with values in a normed cone $\mathbb{K}$ with a sub-invariant norm. By definition, these processes have independent and stationary StαS increments, where StαS stands for "strictly α-stable". If $\alpha < 1$, a $\mathbb{D}$-valued α-stable Lévy motion (in the sense of Definition 4.0.1) is an α-stable Lévy process on the cone $\mathbb{K} = \mathbb{D}$, and therefore has the series representation given by Theorem 3.10 of [10]. (Note that the space $\mathbb{D}$ equipped with $d^0_{J_1}$ is a normed cone, as specified by Definition 2.6 of [10], and the sup-norm $\|\cdot\|$ is sub-invariant, as defined by relation (2.9) of [10], i.e. $d^0_{J_1}(x+h, x) \leq \|h\|$ for any $x, h \in \mathbb{D}$.)

This chapter is organized as follows. In Section 4.1, we introduce some compound Poisson random variables which constitute the basis of our construction, and we discuss some proprieties of the measure $\mu_{s_1,\ldots,s_m}$. In Section 4.2 and 4.3, we give the proof of 4.0.2 in the case $\alpha < 1$, respectively $\alpha > 1$.

4.1 The compound Poisson building blocks

In this section, we introduce the compound Poisson random variables which constitute the building blocks of the construction, and we derive some important properties of a Lévy measure $\mu_{s_1,\ldots,s_n}$ associated to the finite-dimensional distribution of the process $\{Z(t,s)\}_{s\in[0,1]}$ at spatial location $s_1,\ldots,s_n \in [0,1]$.

Let $N = \sum_{i\geq 1} \delta_{(T_i,R_i,W_i)}$ be Poisson random measure on $[0,\infty) \times \overline{\mathbb{D}}_0$ of intensity $\text{Leb} \times \overline{\nu}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\overline{\mathbb{D}}_0 = (0,\infty] \times \mathbb{S}_{\mathbb{D}}$ and $\overline{\nu}$ is given by (4.0.1). By an extension of Proposition 5.3 of [28] to point processes on Polish spaces, we can represent the points (T_i, R_i, W_i) as follows : $\{(T_i, R_i)\}$ are the points of a Poisson random measures on $[0,\infty) \times (0,\infty]$ of intensity $\text{Leb} \times \overline{\nu}_\alpha$, and $(W_i)_{i\geq 1}$ is an independent sequence of i.i.d. random elements in $\mathbb{S}_{\mathbb{D}}$. Note that $\overline{\mathbb{D}}_0$ is a Polish space endowed with the distance $d_{\overline{\mathbb{D}}_0}$ given by (5.2.1) in Chapter 5 below.

As in Section 5.5 of [28], let $(\varepsilon_j)_{j\geq 0}$ be a sequence of positive numbers such that $(\varepsilon_j)_j \downarrow 0$ and $\varepsilon_0 = 1$. We let $I_j = (\varepsilon_j, \varepsilon_{j-1}]$, $j \geq 1$ and $I_0 = (1,\infty]$. For each $j \geq 0$, $t \geq 0$ and $s \in [0,1]$, $[0,t] \times I_j \times \mathbb{S}_{\mathbb{D}}$ is a bounded set in $[0,\infty) \times \overline{\mathbb{D}}_0$, due to the form of the distance $d_{\overline{\mathbb{D}}_0}$.

For any $t > 0$ and $j \geq 0$, we define the random variable

$$Z_j(t,s) = \int_{[0,t]\times I_j \times \mathbb{S}_{\mathbb{D}}} rz(s) N(du,dr,dz) = \sum_{T_i \leq t} R_i W_i(s) 1_{\{R_i \in I_j\}}. \tag{4.1.1}$$

Note that for any $j \geq 0$ and $s \in [0,1]$, $Z_j(0,s) = 0$.

Lemma 4.1.1. *a) $Z_j(t,s)$ is well-defined and $\mathcal{F}$-measurable for any $j \geq 0, t \geq 0, s \in [0,1]$. b) For any $t \geq 0$ and $j \geq 0$, the process $Z_j(t) = \{Z_j(t,s)\}_{s\in[0,1]}$ has all sample paths in $\mathbb{D}$, with left limit at point $s \in (0,1]$ given by*

$$Z_j(t,s-) = \int_{[0,t]\times I_j \times \mathbb{S}_{\mathbb{D}}} rz(s-) N(du,dr,dz) = \sum_{T_i \leq t} R_i W_i(s-) 1_{\{R_i \in I_j\}}.$$

Proof: a) $Z_j(t,s)$ is well-defined since $[0,t] \times I_j \times \mathbb{S}_{\mathbb{D}}$ is a *bounded* set in $[0,\infty) \times \overline{\mathbb{D}}_0$ (due to definition 5.2.1 of the metric $d_{\overline{\mathbb{D}}_0}$ on $\overline{\mathbb{D}}_0$), and the sum in (4.1.1) contains finitely many terms. $Z_j(t,s)$ is $\mathcal{F}$-measurable since N is a point process and the map $\mu \mapsto \mu(\overline{\pi}_s) = \int_{(0,\infty)\times \mathbb{S}_{\mathbb{D}}} rz(s) \mu(dr,dz)$ is $\mathcal{M}_p([0,\infty) \times \overline{\mathbb{D}}_0)$-measurable, where $\overline{\pi}_s(r,z) = rz(s)$ (see Section 5.1 below for the definition of a point process).

b) This follows by the dominated convergence theorem, whose application is justified by the fact that $\int_{[0,t]\times I_j \times \mathbb{S}_{\mathbb{D}}} rN(du,dr,dz) < \infty$. $\square$

For any $s_1,\ldots,s_m \in [0,1]$, we consider the function $\overline{\pi}_{s_1,\ldots,s_m} : (0,\infty) \times \mathbb{S}_{\mathbb{D}} \to \mathbb{R}^m$ given by:

$$\overline{\pi}_{s_1,\ldots,s_m}(r,z) = (rz(s_1),\ldots,rz(s_m))$$

Note that $\overline{\pi}_{s_1,\dots,s_m} \circ T^{-1} = \pi_{s_1,\dots,s_m}$.

Lemma 4.1.2. *For any $j \geq 0$, $t \geq 0$ and $s \in [0,1]$, $Z_j(t,s)$ is compound Poisson random variable with the characteristic function given by : for any $u \in \mathbb{R}$*

$$E(\exp(iuZ_j(t,s))) = \exp\left\{ t \int_{I_j \times \mathbb{S}_{\mathbb{D}}} \big(\exp(iurz(s)) - 1\big)\overline{\nu}(dr,dz)\right\}. \tag{4.1.2}$$

In particular,

$$E\big(Z_j(t,s)\big) = t\varphi(s)\Big(\int_{I_j} r\nu_\alpha(dr)\Big) \text{ and } \mathrm{Var}\big(Z_j(t,s)\big) = t\psi(s)\Big(\int_{I_j} r^2\nu_\alpha(dr)\Big),$$

where $\varphi(s) = \int_{\mathbb{S}_{\mathbb{D}}} z(s)\Gamma_1(dz)$ and $\psi(s) = \int_{\mathbb{S}_{\mathbb{D}}} |z(s)|^2\,\Gamma_1(dz)$

Proof: Note that, for fixed $t > 0$ and $j \geq 0$, the process N restricted to the set $[0,t] \times I_j \times \mathbb{S}_{\mathbb{D}}$ has the following representation :

$$N|_{[0,t]\times I_j \times \mathbb{S}_{\mathbb{D}}} \stackrel{d}{=} \sum_{i=1}^{K} \delta_{(\tau_i, J_i, W_i)}$$

where $\stackrel{d}{=}$ denotes equality in distribution, K is a Poisson random variable of intensity $t\overline{\nu}(I_j \times \mathbb{S}_{\mathbb{D}})$, $(\tau_i)_{i\geq 1}$ are i.i.d. uniformly distributed on $[0,t]$, $\{(J_i,W_i)\}_{i\geq 1}$ are i.i.d. on $I_j \times \mathbb{S}_{\mathbb{D}}$ of law $\frac{1}{\overline{\nu}(I_j\times\mathbb{S}_{\mathbb{D}})}\overline{\nu}|_{I_j\times\mathbb{S}_{\mathbb{D}}}$ (i.e. $(J_i)_{i\geq 1}$ are i.i.d. on I_j of law $\frac{1}{\nu_\alpha(I_j)}\nu_\alpha|_{I_j}$, $(W_i)_{i\geq 1}$ are i.i.d. on $\mathbb{S}_{\mathbb{D}}$ of law Γ_1 and $(J_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$ are independent) and K, $(\tau_i)_{i\geq 1}$ and $\{(J_i,W_i)\}_{i\geq 1}$ are independent.

Therefore $Z_j(t,s) \stackrel{d}{=} \sum_{i=1}^{K} J_iW_i(s)$. Note that $J_iW_i(s) = \overline{\pi}_s(J_i,W_i)$. Hence $\{J_iW_i(s)\}_{i\geq 1}$ are i.i.d. with law $P\circ(J_i,W_i)^{-1}\circ\overline{\pi}_s^{-1} = \frac{1}{\overline{\nu}(I_j\times\mathbb{S}_{\mathbb{D}})}\overline{\nu}|_{I_j\times\mathbb{S}_{\mathbb{D}}}\circ\overline{\pi}_s^{-1}$ and $Z_j(t,s)$ has a compound Poisson distribution with characteristic function given by (4.1.2). In particular, it follows that $Z_j(t,s)$ has mean and variance given by :

$$E\big(Z_j(t,s)\big) = t\int_{I_j\times\mathbb{S}_{\mathbb{D}}} rz(s)\overline{\nu}(dr,dz) = t\varphi(s)\Big(\int_{I_j} r\nu_\alpha(dr)\Big)$$

$$\mathrm{Var}\big(Z_j(t,s)\big) = t\int_{I_j\times\mathbb{S}_{\mathbb{D}}} r^2z^2(s)\overline{\nu}(dr,dz) = t\psi(s)\Big(\int_{I_j} r^2\nu_\alpha(dr)\Big).$$

This concludes the proof. □

Since $\alpha < 2$,

$$\sum_{j\geq 1}\mathrm{Var}\big(Z_j(t,s)\big) = t\psi(s)\sum_{j\geq 1}\int_{I_j} r^2\nu_\alpha(dr) = t\psi(s)\int_{(0,1]} r^2\nu_\alpha(dr) < \infty.$$

Moreover $\{Z_j(t,s)\}_{j\geq 0}$ are independent since they are integrals with respect to N of functions with disjoint support. By Kolmogorov convergence criterion (Theorem 22.6 of [7])

$$\sum_{j\geq 1}\Big(Z_j(t,s)-E(Z_j(t,s))\Big) \text{ converges a.s..} \tag{4.1.3}$$

We denote by $\Omega_{t,s}$ the event that this series converges, which $P(\Omega_{t,s})=1$.

We now examine the distribution of the vector $(Z_j(t,s_1),\ldots,Z_j(t,s_m))$.

Lemma 4.1.3. *For any $j\geq 0, t\geq 0$ and $s_1,\ldots,s_m\in[0,1]$, the vector $(Z_j(t,s_1),\ldots,Z_j(t,s_m))$ has a compound Poisson distribution in $\mathbb{R}^m$ with characteristic function : for any $(u_1,\ldots,u_m)\in\mathbb{R}^m$*

$$\begin{aligned}
&E\Big(\exp\Big\{\sum_{k=1}^{n} iu_k Z_j(t,s_k)\Big\}\Big) = \\
&= \exp\Big\{t\int_{I_j\times\mathbb{S}_{\mathbb{D}}}\big(\exp\{iu_1 z(s_1)+\ldots+iu_m z(s_m)\}-1)\big)\overline{\nu}(dr,dz)\Big\}
\end{aligned}$$

Proof: Note that

$$\begin{aligned}
(Z_j(t,s_1),\ldots,Z_j(t,s_m)) &= \sum_{i=1}^{K} J_i(W_i(s_1),\ldots,W_i(s_m)) \\
&= \sum_{i=1}^{K}\overline{\pi}_{s_1,\ldots,s_m}(J_i,W_i).
\end{aligned}$$

The vectors $\{\overline{\pi}_{s_1,\ldots,s_m}(J_i,W_i)\}_{i\geq 1}$ are i.i.d. in $\mathbb{R}^m$ with law

$$P\circ(J_i,W_i)^{-1}\circ\overline{\pi}^{-1}_{s_1,\ldots,s_m}=\frac{1}{\overline{\nu}(I_j\times\mathbb{S}_{\mathbb{D}})}\overline{\nu}|_{I_j\times\mathbb{S}_{\mathbb{D}}}\circ\overline{\pi}^{-1}_{s_1,\ldots,s_m}.$$

By Lemma A.0.4 (Appendix A), it follows that $(Z_j(t,s_1),\ldots,Z_j(t,s_m))$ has a compound Poisson distribution with characteristic function :

$$\begin{aligned}
&E\big(\exp\{iu_1 Z_j(t,s_1)+\ldots+iu_m Z_j(t,s_m)\}\big) \\
&= \exp\Big\{t\int_{\mathbb{R}^m}\big(\exp(iu\cdot y)-1)\big)\overline{\nu}|_{I_j\times\mathbb{S}_{\mathbb{D}}}\circ\overline{\pi}^{-1}_{s_1,\ldots,s_m}(dy)\Big\} \\
&= \exp\Big\{t\int_{I_j\times\mathbb{S}_{\mathbb{D}}}\big(\exp\{iu_1 z(s_1)+\ldots+iu_m z(s_m)\}-1)\big)\overline{\nu}(dr,dz)\Big\}
\end{aligned}$$

This concludes the proof. □

The following scaling property of the measure $\overline{\nu}$ plays an important role in this thesis.

Lemma 4.1.4. *The measure $\overline{\nu}$ given by (4.0.1) has the following scaling property: for any $h > 0$ and for any set $H \in \mathcal{B}(\overline{\mathbb{D}}_0)$,*

$$\overline{\nu}(hH) = h^{-\alpha}\overline{\nu}(H),$$

where

$$hH = \{(hr, z);\ (r, z) \in H\}. \tag{4.1.4}$$

Proof. Let $h > 0$ and $H \in \mathcal{B}(\overline{\mathbb{D}}_0)$ be arbitrary. We denote by $H_z = \{r \in (0, \infty], (r, z) \in H\}$ the section of the set H at the point $z \in \mathbb{S}_{\mathbb{D}}$. Then

$$\begin{aligned} hH_z &= h\{r;\ (r,z) \in H\} = \{hr;\ (r,z) \in H\} = \{hr;\ (hr,z) \in hH\} \\ &= \left\{r';\ (r',z) \in hH\right\} = (hH)_z \end{aligned}$$

where $(hH)_z$ is the section of the set hH at z. By Fubini theorem and the scaling property of ν_α we have :

$$\begin{aligned} \overline{\nu}(hH) &= c\,(\nu_\alpha \times \Gamma_1)\,(hH) = c\int_{\mathbb{S}_{\mathbb{D}}} \nu_\alpha((hH)_z)\Gamma_1(dz) \\ &= h^{-\alpha}c\int_{\mathbb{S}_{\mathbb{D}}} \nu_\alpha(H_z)\Gamma_1(dz) = h^{-\alpha}\,(c\nu_\alpha \times \Gamma_1)\,(H) \\ &= h^{-\alpha}\overline{\nu}(H) \end{aligned}$$

This finishes the proof. □

For any $s_1, \ldots, s_m \in [0, 1]$, we consider the following measure on $\mathbb{R}^m$:

$$\mu_{s_1,\ldots,s_m} = \nu \circ \pi^{-1}_{s_1,\ldots,s_m} = \overline{\nu} \circ \overline{\pi}^{-1}_{s_1,\ldots,s_m}. \tag{4.1.5}$$

To conclude that the vector $(Z_j(t, s_1), \ldots, Z_j(t, s_m))$ has an α-stable distribution in $\mathbb{R}^m$, it suffices to show that $\mu_{s_1,\ldots,s_m}$ is a Lévy measure which satisfies an appropriate scaling property. Assumption A is needed to guarantee that $\mu_{s_1,\ldots,s_m}(\{0\}) = 0$.

Assumption A : For any $s \in [0, 1]$, $\Gamma_1(\{z \in \mathbb{S}_{\mathbb{D}};\ z(s) = 0\})$.

Lemma 4.1.5. *Suppose that Assumption A holds.*
a) For any $s_1, \ldots, s_m \in [0, 1]$, $\mu_{s_1,\ldots,s_m}$ is a Lévy measure on $\mathbb{R}^m$, i.e.

$$\mu_{s_1,\ldots,s_m}(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} \left(|y|^2 \wedge 1\right)\mu_{s_1,\ldots,s_m}(dy) < \infty.$$

b) For any $s_1, \ldots, s_m \in [0, 1]$, $\mu_{s_1,\ldots,s_m}$ satisfies the following scaling property: for any $h > 0$ and for any Borel set $A \subset \mathbb{R}^m_0$,

$$\mu_{s_1,\ldots,s_m}(hA) = h^{-\alpha}\mu_{s_1,\ldots,s_m}(A).$$

Proof. a) By the definition of $\mu_{s_1,\ldots,s_m}$ and Assumption A, we have

$$\begin{aligned}\mu_{s_1,\ldots,s_m}(\{0\}) &= \overline{\nu}\left(\{(r,z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ rz(s_1)=\ldots=rz(s_m)=0\}\right)\\ &= \nu_\alpha(0,\infty)\,\Gamma_1\left(\{(z\in\mathbb{S}_{\mathbb{D}};\ z(s_1)=\ldots=z(s_m)=0\}\right)=\infty\cdot 0=0.\end{aligned}$$

Here, we use the usual convention from measure theory that $\infty\cdot 0=0$ (see page 199 of [7]). The second property follows because

$$\begin{aligned}\int_{|y|\le 1}|y|^2\,\mu_{s_1,\ldots,s_m}(dy) &= \int_{(0,\infty)\times\mathbb{S}_{\mathbb{D}}}|\overline{\pi}_{s_1,\ldots,s_m}(r,z)|^2\,1_{\left\{|\overline{\pi}_{s_1,\ldots,s_m}(r,z)|\le 1\right\}}\overline{\nu}(dr,dz)\\ &= c\int_{\left\{r\sqrt{\sum_{i=1}^m|z(s_i)|^2}\le 1\right\}} r^2\sum_{i=1}^m|z(s_i)|^2\,\nu_\alpha(dr)\Gamma_1(dz)\\ &= c\int_{\mathbb{S}_{\mathbb{D}}}\Big(\int_0^{\left(\sum_{i=1}^m|z(s_i)|^2\right)^{-1/2}} r^2\nu_\alpha(dr)\Big)\sum_{i=1}^m|z(s_i)|^2\,\Gamma_1(dz)\\ &= c\frac{\alpha}{2-\alpha}\int_{\mathbb{S}_{\mathbb{D}}}\left(\sum_{i=1}^m|z(s_i)|^2\right)^{\alpha/2}\Gamma_1(dz)<\infty\end{aligned}$$

and

$$\begin{aligned}\int_{|y|>1}\mu_{s_1,\ldots,s_m}(dy) &= \int_{(0,\infty)\times\mathbb{S}_{\mathbb{D}}}1_{\left\{|\overline{\pi}_{s_1,\ldots,s_m}(r,z)|>1\right\}}\overline{\nu}(dr,dz)\\ &= c\int_{\left\{r\sqrt{\sum_{i=1}^m|z(s_i)|^2}>1\right\}}\nu_\alpha(dr)\Gamma_1(dz)\\ &= c\int_{\mathbb{S}_{\mathbb{D}}}\Big(\int_{\left(\sum_{i=1}^m|z(s_i)|^2\right)^{-1/2}}^\infty\nu_\alpha(dr)\Big)\Gamma_1(dz)\\ &= c\int_{\mathbb{S}_{\mathbb{D}}}\left(\sum_{i=1}^m|z(s_i)|^2\right)^{\alpha/2}\Gamma_1(dz)<\infty.\end{aligned}$$

b) For any $h>0$ and for any Borel $A\subset\mathbb{R}^m$ we have:

$$\mu_{s_1,\ldots,s_m}(hA)=\overline{\nu}(\overline{\pi}_{s_1,\ldots,s_m}^{-1}(hA))=\overline{\nu}(B)$$

where $B=\{(r,z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ (rz(s_1),\ldots,rz(s_m))\in hA\}$.

Let $H=\{(r,z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ (rz(s_1),\ldots,rz(s_m))\in A\}=\overline{\pi}_{s_1,\ldots,s_m}^{-1}(A)$. Note that the blow up of a set in $(0,\infty)\times\mathbb{S}_{\mathbb{D}}$ is defined by blowing up only the variable r and not the variable z (see relation (4.1.4)). Therefore

$$hH = h\{(r,z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ (rz(s_1),\ldots,rz(s_m))\in A\}$$

$$\begin{aligned} &= \{(hr,z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ (hrz(s_1),\ldots,hrz(s_m))\in hA\} \\ &= \left\{(r',z)\in(0,\infty)\times\mathbb{S}_{\mathbb{D}};\ (r'z(s_1),\ldots,r'z(s_m))\in hA\right\} = B. \end{aligned}$$

By the scaling property of $\overline{\nu}$ (given by Lemma 4.1.4), we have

$$\mu_{s_1,\ldots,s_m}(hA)=\overline{\nu}(B)=\overline{\nu}(hH)=h^{-\alpha}\overline{\nu}(H)=h^{-\alpha}\overline{\nu}(\pi^{-1}_{s_1,\ldots,s_m}(A))=h^{-\alpha}\mu_{s_1,\ldots,s_m}(A).$$

This finishes the proof. $\square$

In particular, the scaling property of the measure μ_s in $\mathbb{R}$ allows us to derive an explicit formula for this measure, as the next result shows.

Lemma 4.1.6. *For any $s\in[0,1]$, the measure μ_s is given by :*

$$\mu_s(dy)=\left(c_s^+\alpha y^{-\alpha-1}1_{(y>0)}+c_s^-\alpha(-y)^{-\alpha-1}1_{(y<0)}\right)dy$$

where $c_s^+=\mu_s\,(1,\infty)$ and $c_s^-=\mu_s\,(-\infty,-1)$.

Proof. For any $a>0$, since $(a,\infty)=a\,(1,\infty)$, by the scaling property of μ_s we have:

$$\mu_s\,(a,\infty)=a^{-\alpha}\mu_s\,(1,\infty)=c_s^+\int_a^\infty \alpha y^{-\alpha-1}dy.$$

Similarly, since $(-\infty,-a)=a\,(-\infty,-1)$ for any $a>0$,

$$\mu_s\,(-\infty,-a)=a^{-\alpha}\mu_s\,(-\infty,-1)=c_s^-\int_{-\infty}^{-a}\alpha(-y)^{-\alpha-1}dy.$$

The conclusion follows since $\mu_s(\{0\})=0$. $\square$

We denote by $\mathbb{D}_u([0,\infty);\mathbb{D})$ the set of functions $x:[0,\infty)\to\mathbb{D}$ which are right-continuous and have left limits with respect to the uniform norm $\|\cdot\|$ on $\mathbb{D}$. Clearly, $\mathbb{D}_u([0,\infty);\mathbb{D})$ is a subset of $\mathbb{D}([0,\infty);\mathbb{D})$.

Lemma 4.1.7. *For any $j\geq 0$, the process $\{Z_j(t)\}_{t\geq 0}$ has all sample paths in $\mathbb{D}_u([0,\infty);\mathbb{D})$, with left limit at $t>0$ given by $Z_j(t-)=\{Z_j(t-,s)\}_{s\in[0,1]}$, where*

$$Z_j(t-,s)=\int_{[0,t)\times I_j\times\mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz).$$

Proof: We first show that the map $t\mapsto Z_j(t)$ is right-continuous in $(\mathbb{D},\|\cdot\|)$. Let $t\geq 0$ be arbitrary and $(t_n)_{n\geq 1}$ such that $t_n\to t$ and $t_n\geq t$ for all $n\geq 1$. Then

$$\|Z_j(t_n)-Z_j(t)\|=\sup_{s\in[0,1]}\left|\int_{(t,t_n]\times I_j\times\mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz)\right|\leq\int_{(t,t_n]\times I_j\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz),$$

and the last integral converges to 0 as $n \to \infty$ by the dominated convergence theorem. Next, we show that the map $t \mapsto Z_j(t)$ has left limit $Z_j(t-)$ in $(\mathbb{D}, \|\cdot\|)$. Let $t > 0$ be arbitrary and $(t_n)_{n\geq 1}$ such that $t_n \to t$ and $t_n \leq t$ for all $n \geq 1$. Then

$$\|Z_j(t-)-Z_j(t_n)\| = \sup_{s\in[0,1]} \left| \int_{(t_n,t)\times I_j \times \mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz) \right| \leq \int_{(t_n,t)\times I_j\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz),$$

and the last integral converges to 0 as $n \to \infty$ by the dominated convergence theorem. $\square$

For any $\varepsilon > 0$, $t \geq 0$ and $s \in [0,1]$, we let

$$Z^{(\varepsilon)}(t,s) = \int_{[0,t]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz) = \sum_{T_i\leq t} R_i W_i(s) 1_{\{R_i\in(\varepsilon,\infty)\}}. \tag{4.1.6}$$

Using this notation, we have:

$$Z^{(\varepsilon_k)}(t,s) = \sum_{j=0}^{k} Z_j(t,s), \quad \text{for all} \quad k \geq 0. \tag{4.1.7}$$

Remark 4.1.8. Similarly to Lemma 4.1.1 and Lemma 4.1.7 for $j = 0$, the process $Z^{(\varepsilon)}(t) = \{Z^{(\varepsilon)}(t,s)\}_{s\in[0,1]}$ has all sample paths in $\mathbb{D}$ for any $t \geq 0$, and the process $Z^{(\varepsilon)} = \{Z^{(\varepsilon)}(t)\}_{t\geq 0}$ has all sample paths in $\mathbb{D}_u([0,\infty);\mathbb{D})$.

4.2 Construction in the case $\alpha < 1$

In this section, we give the construction of the α-stable Lévy motion with values in $\mathbb{D}$, in the case $\alpha < 1$, i.e. we prove Theorem 4.0.2.a). Throughout this section we assume that the probability measure Γ_1 satisfies Assumption A.

In this case, for any $t > 0$ and $s \in [0,1]$ fixed, the following series is convergent :

$$\sum_{j\geq 1} E(Z_j(t,s)) = t\sum_{j\geq 1}\int_{I_j\times\mathbb{S}_{\mathbb{D}}} rz(s)\overline{\nu}(dr,dz) = tc\varphi(s)\int_{(0,1]} r\nu_\alpha(dr).$$

Therefore on the event $\Omega_{t,s}$ we can split the series (4.1.3) into two convergent series : $\sum_{j\geq 1} Z_j(t,s)$ and $\sum_{j\geq 1} E(Z_j(t,s)$. We consider only the first term of these series, to which we add $Z_0(t,s)$. On the event $\Omega_{t,s}$, we define

$$\overline{Z}(t,s) = \sum_{j\geq 0} Z_j(t,s). \tag{4.2.1}$$

On the event $\Omega_{t,s}^c$, we let $\overline{Z}(t,s) = x_0$, for arbitrary $x_0 \in \mathbb{D}$.

We denote by $S_\alpha(\sigma,\beta,\mu)$ the α-stable distribution with characteristic function (2.2.2) (see Remark 2.2.6) and by C_α the constant given by (2.2.5)

Proposition 4.2.1. *For any $t > 0$, the process $\overline{Z}(t) = \{\overline{Z}(t,s)\}_{s\in[0,1]}$ given by (4.2.1) is α-stable with finite-dimensional distributions given by (4.0.2). In particular, for any $t > 0$ and $s \in [0,1]$, $\overline{Z}(t,s)$ has a $S_\alpha(t^{1/\alpha}\sigma_s, \beta_s, 0)$ distribution with parameters*

$$\sigma_s = C_\alpha^{-1}(c_s^+ + c_s^-) \quad \textit{and} \quad \beta_s = \frac{c_s^+ - c_s^-}{c_s^+ + c_s^-}, \tag{4.2.2}$$

where c_s^+ and c_s^- are given in Lemma 4.1.6. Moreover, $\overline{Z}(t,s_k) \xrightarrow{d} \overline{Z}(t,s)$ as $k \to \infty$, for any $s \in [0,1]$ and for any sequence $(s_k)_{k\geq 1}$ with $s_k \to s$ and $s_k \geq s$ for all $k \geq 1$.

Proof: By the independence of $\{Z_j(t,s)\}_{j\geq 0}$, the characteristic function of the random variable $\overline{Z}(t,s)$ is given by

$$\begin{aligned} E(\exp(iu\overline{Z}(t,s))) &= \prod_{j\geq 0} E\big(\exp(iuZ_j(t,s))\big) \\ &= \prod_{j\geq 0} \exp\left\{t\int_{I_j\times\mathbb{S}_\mathbb{D}} \big(\exp(iurz(s)) - 1\big)\overline{\nu}(dr,dz)\right\} \\ &= \exp\left\{t\int_{(0,\infty)\times\mathbb{S}_\mathbb{D}} \big(\exp(iurz(s)) - 1\big)\overline{\nu}(dr,dz)\right\} \\ &= \exp\left\{t\int_{\mathbb{R}} \big(\exp(iuy) - 1\big)\mu_s(dy)\right\}. \end{aligned}$$

By Corollary 2.2.8 a), it follows that $\overline{Z}(t,s)$ has a $S_\alpha(t^{1/\alpha}\sigma_s, \beta_s, 0)$, with parameters σ_s and β_s given by 4.2.2. We now examine the distribution of $(\overline{Z}(t,s_1),\ldots,\overline{Z}(t,s_m))$ for fixed $t > 0$ and $s_1,\ldots,s_m \in [0,1]$. By the independence of the vectors $(\overline{Z}_j(t,s_1),\ldots,\overline{Z}_j(t,s_m))_{j\geq 1}$, we infer that the characteristic function of the vector $(Z(t,s_1),\ldots,Z(t,s_m))$ is

$$\begin{aligned} & E(\exp(iu_1\overline{Z}(t,s_1) + \ldots + iu_m\overline{Z}(t,s_m))) \\ = & \prod_{j\geq 0} E(\exp\{iu_1Z_j(t,s_1) + \ldots + iu_mZ_j(t,s_m)\}) \\ = & \exp\left\{t\int_{(0,\infty)\times\mathbb{S}_\mathbb{D}} \big(\exp\{iu_1rz(s_1) + \ldots + iu_mrz(s_m)\} - 1\big)\overline{\nu}(dr,dz)\right\} \\ = & \exp\left\{t\int_{\mathbb{R}^m} \big(\exp(iu\cdot y) - 1\big)\mu_{s_1,\ldots,s_m}(dy)\right\}. \end{aligned}$$

Since $\mu_{s_1,\ldots,s_m}$ is a Lévy measure on $\mathbb{R}^m$ which satisfies the scaling property $\mu_{s_1,\ldots,s_m}(hH) = h^{-\alpha}\mu_{s_1,\ldots,s_m}(H)$, we conclude that the vector $(\overline{Z}(t,s_1),\ldots,\overline{Z}(t,s_m))$ has an α-stable distribution (see Remark 2.3.15 a) and Theorem 2.3.9).

The last statement follows from the fact that $E(e^{iu\overline{Z}(t,s_k)}) \to E(e^{iu\overline{Z}(t,s)})$ as $k \to \infty$. To see this, note that $\lim_{k\to\infty} z(s_k) = z(s)$ for any $z \in \mathbb{S}_{\mathbb{D}}$. By the dominated convergence theorem,

$$\int_{\overline{\mathbb{D}}_0} (e^{iurz(s_k)} - 1)\overline{\nu}(dr, dz) \to \int_{\overline{\mathbb{D}}_0} (e^{iurz(s)} - 1)\overline{\nu}(dr, dz), \quad \text{as} \quad k \to \infty.$$

The application of this theorem is justified using the inequalities $|e^{iurz(s)}-1| \leq |urz(s)|$ if $r \leq 1$ and $|e^{iurz(s)} - 1| \leq 2$ if $r > 1$. □

The next result shows that for any $t > 0$ fixed, the process $\overline{Z}(t) = \{\overline{Z}(t,s)\}_{s\in[0,1]}$ given by (4.2.1) has a càdlàg modification which can be obtained as an almost sure limit with respect to the uniform norm. Recall that $\{X(s)\}_{s\in[0,1]}$ is a *modification* of $\{Y(s)\}_{s\in[0,1]}$ if $P(X(s) = Y(s)) = 1$ for all $s \in [0,1]$.

Lemma 4.2.2. *If $\alpha < 1$, then for any $t \geq 0$, there exists a random element $Z(t) = \{Z(t,s)\}_{s\in[0,1]}$ in $\mathbb{D}$ such that $P(Z(t,s) = \overline{Z}(t,s)) = 1$ for all $s \in [0,1]$, and*

$$\lim_{k\to\infty} \|Z^{(\varepsilon_k)}(t) - Z(t)\| = 0 \quad a.s.$$

Proof: For $t = 0$, we define $Z(0) = 0$. We consider the case $t > 0$. By (4.1.1), $\|Z_j(t)\| \leq \sum_{i\geq 1} R_i 1_{\{R_i\in I_j\}} 1_{\{T_i\leq t\}} = \int_{[0,t]\times I_j\times\mathbb{S}_{\mathbb{D}}} rN(du, dr, dz)$. Hence,

$$E\sum_{j\geq 1} \|Z_j(t)\| \leq E\sum_{j\geq 1}\int_{[0,t]\times I_j\times\mathbb{S}_{\mathbb{D}}} rN(du, dr, dz) = t\int_{(0,1]\times\mathbb{S}_{\mathbb{D}}} r\overline{\nu}(dr, dz) < \infty,$$

which implies that $\sum_{j\geq 1}\|Z_j(t)\| < \infty$ a.s. We denote by Ω_t the event that this series converges, with $P(\Omega_t) = 1$. On the event Ω_t, the sequence $\{Z^{(\varepsilon_k)}(t) = \sum_{j=0}^{k} Z_j(t)\}_{k\geq 0}$ is Cauchy in $(\mathbb{D}, \|\cdot\|)$, and we denote its limit by $Z(t)$. On the event Ω_t^c, we let $Z(t) = x_0$. By Lemma 4.1.1.a), $Z(t,s)$ is $\mathcal{F}$-measurable for any $s \in [0,1]$. Hence, $Z(t)$ is a random element in $\mathbb{D}$. On the event $\Omega_{t,s}\cap\Omega_t$, $\overline{Z}(t,s) - Z^{(\varepsilon_k)}(t,s) = \sum_{j\geq k+1} Z_j(t,s)$, and hence

$$|Z^{(\varepsilon_k)}(t,s) - \overline{Z}(t,s)| \leq \sum_{j\geq k+1} |Z_j(t,s)| \leq \sum_{j\geq k+1} \|Z_j(t)\| \to 0.$$

On the other hand, on the event Ω_t, $Z^{(\varepsilon_k)}(t,s) \to Z(t,s)$ for any $s \in [0,1]$. By the uniqueness of the limit, $Z(t,s) = \overline{Z}(t,s))$ on the event $\Omega_{t,s}\cap\Omega_t$. □

Lemma 4.2.3. *Let (S,d) be a separable metric space. Let $X_n^{(1)}, \ldots, X_n^{(k)}$ and $X^{(1)}, \ldots, X^{(k)}$ be random elements in S defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $d(X_n^{(i)}, X^{(i)}) \to 0$ a.s. for any $i = 1, \ldots, k$. If $X_n^{(1)}, \ldots, X_n^{(k)}$ are independent for any $n \geq 1$, then $X^{(1)}, \ldots, X^{(k)}$ are independent.*

Proof: We assume for simplicity that $k = 2$, the general case being similar. To simplify the notation, we let $X_n = X_n^{(1)}$ and $Y_n = X_n^{(2)}$. Clearly, $d(X_n, X) \xrightarrow{P} 0$ and $d(Y_n, Y) \xrightarrow{P} 0$. Note that the space $S \times S$ equipped with the product metric is separable and (X_n, X) is a random element in $S \times S$ (see p.225 of [5]). By Corollary to Theorem 3.1 of [6], $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. By Theorem 3.2 of [5],

$$(P \circ X_n^{-1}) \times (P \circ Y_n^{-1}) \xrightarrow{w} (P \circ X^{-1}) \times (P \circ Y^{-1}) \quad \text{on} \quad S \times S. \tag{4.2.3}$$

On the other hand, $(X_n, Y_n) \to (X, Y)$ a.s. with respect to the product distance in $S \times S$. Hence, again by Corollary to Theorem 3.1 of [6], $(X_n, Y_n) \xrightarrow{d} (X, Y)$ in $S \times S$, i.e.

$$P \circ (X_n, Y_n)^{-1} \xrightarrow{w} P \circ (X, Y)^{-1} \quad \text{on} \quad S \times S. \tag{4.2.4}$$

Finally, $P \circ (X_n, Y_n)^{-1} = (P \circ X_n^{-1}) \times (P \circ Y_n^{-1})$ for any $n \geq 1$, since X_n and Y_n are independent for any $n \geq 1$. The fact that $P \circ (X, Y)^{-1} = (P \circ X^{-1}) \times (P \circ Y^{-1})$ follows from (4.2.3) and (4.2.4), by the uniqueness of the limit. $\square$

The next result proves Theorem 4.0.2 a) in the case $\alpha < 1$.

Theorem 4.2.4. *If $\alpha \in (0, 1)$, the process $\{Z(t)\}_{t \geq 0}$ defined in Lemma 4.2.2 is a $\mathbb{D}$-valued α-stable Lévy motion (corresponding to ν). This process is $(1/\alpha)$-self-similar, i.e.*

$$\{Z(ct)\}_{t \geq 0} \stackrel{d}{=} c^{1/\alpha} \{Z(t)\}_{t \geq 0} \quad \textit{for any} \quad c > 0, \tag{4.2.5}$$

where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions.

Proof: We first show that the process $\{Z(t)\}_{t \geq 0}$ satisfies properties *(i)*-*(iv)* given in Definition 4.0.1. Property *(i)* is clear. To verify property *(ii)*, we apply Lemma 4.2.3 to the space $S = \mathbb{D}$ equipped with $d_{J_1}^0$. By Lemma 4.2.2, for $i = 2, \ldots, K$, $X_k^{(i)} := Z^{(\varepsilon_k)}(t_i) - Z^{(\varepsilon_k)}(t_{i-1}) \to X^{(i)} := Z(t_i) - Z(t_{i-1})$ a.s. as $k \to \infty$, in $(\mathbb{D}, \| \cdot \|)$, and hence also in $(\mathbb{D}, J_1)$. The variables $X_k^{(2)}, \ldots, X_k^{(K)}$ are independent for any k, since $X_k^{(i)}$ is $\mathcal{F}_{t_{i-1}, t_i}^N$-measurable and the σ-fields $\mathcal{F}_{t_{i-1}, t_i}^N, i = 2, \ldots, K$ are independent. Here $\mathcal{F}_{s,t}^N$ is the σ-field generated by $N((a, b] \times B)$ for any $s < a < b \leq t$ and $B \in \mathcal{B}(\overline{\mathbb{D}}_0)$. It follows that $X^{(2)}, \ldots, X^{(K)}$ are independent.

For property *(iii)*, we have to show that vectors $X = (Z(t_2, s_1) - Z(t_1, s_1), \ldots Z(t_2, s_1) - Z(t_1, s_m))$ and $Y = (Z(t_2 - t_1, s_1), \ldots, Z(t_2 - t_1, s_m))$ have the same distribution, for any $s_1, \ldots, s_m \in [0, 1]$. By (4.2.1) and Lemma 4.2.2, on the event $\Omega_{t_1,s} \cap \Omega_{t_2,s} \cap \Omega_{t_1} \cap \Omega_{t_2}$,

$$Z(t_2, s) - Z(t_1, s) = \overline{Z}(t_2, s) - \overline{Z}(t_1, s) = \sum_{j \geq 0} \big(Z_j(t_2, s) - Z_j(t_1, s) \big).$$

As in the proof of Proposition 4.2.1, it follows that the characteristic function of X is

$$E(e^{iu\cdot X}) = \exp\left\{(t_2 - t_1)\int_{\mathbb{R}^m}(e^{iu\cdot y}-1)\mu_{s_1,\ldots,s_m}(dy)\right\}, \quad u \in \mathbb{R}^m,$$

which is the same as the characteristic function of Y. Hence $X \stackrel{d}{=} Y$. Finally, property *(iv)* was shown in Proposition 4.2.1 for $\overline{Z}(t)$, and remains valid for its modification $Z(t)$.

To prove relation (4.2.5), we have to show that $\{Z(ct)\}_{t\geq 0} \stackrel{d}{=} \{c^{1/\alpha}Z(t)\}_{t\geq 0}$ for any $c > 0$. Since both processes have stationary and independent increments, it is enough to show that $Z(ct) \stackrel{d}{=} c^{1/\alpha}Z(t)$ for any $t > 0$, i.e. vectors $U = (Z(ct, s_1), \ldots, Z(ct, s_m))$ and $V = c^{1/\alpha}(Z(t, s_1), \ldots, Z(t, s_m))$ have the same distribution, for any $s_1, \ldots, s_m \in [0, 1]$ and $t > 0$. Let $h_c(y) = c^{1/\alpha}y$ for $y \in \mathbb{R}^m$. By the scaling property of the measure $\mu_{s_1,\ldots,s_m}$ given in Lemma 4.1.5.b),

$$\mu_{s_1,\ldots,s_m}(h_c^{-1}(A)) = \mu_{s_1,\ldots,s_m}(c^{-1/\alpha}A) = c\mu_{s_1,\ldots,s_m}(A),$$

for any Borel set $A \subset \mathbb{R}^m$. Therefore, the characteristic function of V is

$$\begin{aligned} E(e^{iu\cdot V}) &= \exp\left\{t\int_{\mathbb{R}^m}(e^{iu\cdot y}-1)(\mu_{s_1,\ldots,s_m}\circ h_c^{-1})(dy)\right\} \\ &= \exp\left\{ct\int_{\mathbb{R}^m}(e^{iu\cdot y}-1)\mu_{s_1,\ldots,s_m}(dy)\right\} \end{aligned}$$

for any $u \in \mathbb{R}^m$, which is the same as the characteristic function of U. Hence $U \stackrel{d}{=} V$.
$\square$

The following result is an extension of Lemma 5.2 of [28] to the case of functions with values in an arbitrary metric space. Its proof is elementary and we include it for the sake of completeness.

The following result proves Theorem 4.0.2.b) in the case $\alpha < 1$.

It shows that the $\mathbb{D}$-valued process $\{Z(t)\}_{t\geq 0}$ has a modification $\{\widetilde{Z}(t)\}_{t\geq 0}$ whose sample paths are in $\mathbb{D}_u([0,\infty);\mathbb{D})$. This modification will be needed in the approximation result (Theorem 5.5.2 below).

Theorem 4.2.5. *If $\alpha < 1$ and $\{Z(t)\}_{t\geq 0}$ is the process defined in Lemma 4.2.2, then there exists a collection $\{\widetilde{Z}(t)\}_{t\geq 0}$ of random elements in $\mathbb{D}$, such that $P(Z(t) = \widetilde{Z}(t)) = 1$ for all $t \geq 0$, and for any $T > 0$,*

$$\sup_{t\leq T}\|Z^{(\varepsilon_k)}(t) - \widetilde{Z}(t)\| \to 0 \quad a.s. \quad as\ k \to \infty. \tag{4.2.6}$$

Moreover, the map $t \mapsto \widetilde{Z}(t)$ lies in $\mathbb{D}_u([0,\infty);\mathbb{D})$ a.s.

Proof: For any $T > 0$, we denote by $\mathbb{D}_u([0,T];\mathbb{D})$ the set of functions $x : [0,T] \to \mathbb{D}$ which are right-continuous and have left-limits with respect to the norm $\|\cdot\|$ on $\mathbb{D}$. Note that $\mathbb{D}_u([0,T];\mathbb{D})$ is a Banach space with respect to the super-uniform norm given by (3.3.3):

$$\|x\|_{T,\mathbb{D}} = \sup_{t\leq T} \|x(t)\|.$$

Using the same idea as in the proof of Theorem 5.4 of [28], we will show that there exists an event $\widetilde{\Omega}$ of probability 1, on which we can say that for any $T > 0$,

$$\{Z^{(\varepsilon_k)}(\cdot)\}_{k\geq 1} \quad \text{is a Cauchy sequence in } \mathbb{D}_u([0,T];\mathbb{D}), \tag{4.2.7}$$

where $\mathbb{D}_u([0,T];\mathbb{D})$ is equipped with the norm $\|\cdot\|_{T,\mathbb{D}}$. We denote by $\{\widetilde{Z}(t)\}_{t\in[0,T]}$ the limit of this sequence in $\mathbb{D}_u([0,T];\mathbb{D})$ (on the event $\widetilde{\Omega}$). Relation (4.2.6) then holds by definition. Since $T > 0$ is arbitrary, $\widetilde{Z}(\omega,t)$ is a well-defined element in $\mathbb{D}$ for any $t \geq 0$ and $\omega \in \widetilde{\Omega}$. For $\omega \notin \widetilde{\Omega}$, we let $\widetilde{Z}(\omega,t) = x_0$ for any $t \geq 0$, where $x_0 \in \mathbb{D}$ is arbitrary. For any $\omega \in \Omega$ and $t \geq 0$, $Z(\omega,t) \in \mathbb{D}$ and we denote $\widetilde{Z}(\omega,t,s) := \widetilde{Z}(\omega,t)(s)$ for any $s \in [0,1]$. Clearly, $\widetilde{Z}(t,s)$ is $\mathcal{F}$-measurable for any $s \in [0,1]$, being the a.s. limit of the sequence $\{Z^{(\varepsilon_k)}(t,s)\}_{k\geq 1}$ This proves that $\widetilde{Z}(t)$ is a random element in $\mathbb{D}$ for any $t \geq 0$.

By Lemma 4.2.3 with $S = \mathbb{D}$ equipped with the norm $\|\cdot\|$, the map $t \mapsto \widetilde{Z}(t)$ lies in $\mathbb{D}_u([0,\infty);\mathbb{D})$ (on the event $\widetilde{\Omega}$). From relation (4.2.6) and Lemma 4.2.2, we infer that $\|Z(t) - \widetilde{Z}(t)\| = 0$ a.s. for any $t \geq 0$

It remains to prove (4.2.7). For this, it suffices to prove that for any $\delta > 0$,

$$\lim_{K\to\infty} \lim_{L\to\infty} P(\max_{K<k\leq L} \|Z^{(\varepsilon_k)} - Z^{(\varepsilon_K)}\|_{T,\mathbb{D}} > \delta) = 0. \tag{4.2.8}$$

Let $\delta > 0$ be arbitrary. For any $K < k \leq L$, $t > 0$ and $s \in [0,1]$,

$$Z^{(\varepsilon_k)}(t,s) - Z^{(\varepsilon_K)}(t,s) = \int_{[0,t]\times(\varepsilon_k,e_K]\times\mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz) = \sum_{T_i\leq t} R_i W_i(s) 1_{\{\varepsilon_k<R_i\leq\varepsilon_K\}},$$

and hence

$$\|Z^{(\varepsilon_k)}(t) - Z^{(\varepsilon_K)}(t)\| \leq \sum_{T_i\leq t} R_i 1_{\{\varepsilon_k<R_i\leq\varepsilon_K\}} = \int_{[0,t]\times(\varepsilon_k,\varepsilon_K]\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz).$$

Taking the supremum over $t \in [0,T]$ followed by the maximum over k with $K < k \leq L$, we obtain:

$$\max_{K<k\leq L} \|Z^{(\varepsilon_k)} - Z^{(\varepsilon_K)}\|_{T,\mathbb{D}} \leq \int_{[0,T]\times(\varepsilon_L,\varepsilon_K]\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz).$$

By Markov's inequality,

$$P(\max_{K<k\leq L} \|Z^{(\varepsilon_k)} - Z^{(\varepsilon_K)}\|_{T,\mathbb{D}} > \delta) \leq \frac{1}{\delta} E\left(\int_{[0,T]\times(\varepsilon_L,\varepsilon_K]\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz)\right)$$
$$= \frac{T}{\delta}\int_{(\varepsilon_L,\varepsilon_K]\times\mathbb{S}_{\mathbb{D}}} r\overline{\nu}(dr,dz) = \frac{T}{\delta}\int_{(\varepsilon_L,\varepsilon_K]} r\nu_\alpha(dr) \to 0 \quad \text{as} \quad K,L\to\infty,$$

using the fact that $\int_{(\varepsilon_L,1]} r\nu_\alpha(dr) \to \int_0^1 r\nu_\alpha(dr) < \infty$, as $L\to\infty$. This proves (4.2.8). □

4.3 Construction in the case $\alpha > 1$

In this section, we prove Theorem 4.0.2.b). In this case, $E(Z_0(t,s)) = t\varphi(s)\int_1^\infty r\nu_\alpha(dr)$ is finite. Recall that $\Omega_{t,s}$ is the event where (4.1.3) holds. For any $t\geq 0$ and $s\in[0,1]$, on the event $\Omega_{t,s}$ we define

$$\overline{Z}(t,s) = \sum_{j\geq 0}\big(Z_j(t,s) - E(Z_j(t,s))\big). \tag{4.3.1}$$

Note that $\overline{Z}(0,s)=0$ for all $s\in[0,1]$. On the event $\Omega_{t,s}^c$, we let $\overline{Z}(t,s)=x_0$, for arbitrary $x_0\in\mathbb{D}$.

Proposition 4.3.1. *For any $t>0$, the process $\overline{Z}(t) = \{\overline{Z}(t,s)\}_{s\in[0,1]}$ given by (4.3.1) is α-stable with finite-dimensional distributions given by (4.0.3). In particular, for any $t>0$ and $s\in[0,1]$, $\overline{Z}(t,s)$ has a $S_\alpha(t^{1/\alpha}\sigma_s,\beta_s,0)$ distribution with parameters σ_s and β_s given by (4.2.2). Moreover, $\overline{Z}(t,s_k)\xrightarrow{d}\overline{Z}(t,s)$ as $k\to\infty$, for any $s\in[0,1]$ and for any sequence $(s_k)_{k\geq 1}$, with $s_k\to s$ and $s_k\geq s$ for all $k\geq 1$.*

Proof: Using the independence of $\{Z_j(t,s)\}_{j\geq 0}$, we infer that $\overline{Z}(t,s)$ has characteristic function:

$$\begin{aligned}
E(\exp(iu\overline{Z}(t,s))) &= \prod_{j\geq 0} E\Big(\exp(iuZ_j(t,s) - E(Z_j(t,s)))\Big)\\
&= \prod_{j\geq 0}\exp\left\{t\int_{I_j\times\mathbb{S}_{\mathbb{D}}}\big(\exp(iurz(s)) - 1 - iurz(s)\big)\overline{\nu}(dr,dz)\right\}\\
&= \exp\left\{t\int_{(0,\infty)\times\mathbb{S}_{\mathbb{D}}}\big(\exp(iurz(s)) - 1 - iurz(s)\big)\overline{\nu}(dr,dz)\right\}\\
&= \exp\left\{t\int_{\mathbb{R}}\big(\exp(iuy) - 1 - iuy\big)\mu_s(dy)\right\}
\end{aligned}$$

By Corollary 2.2.8.b) $\overline{Z}(t,s)$ has a $S_\alpha(t^{1/\alpha}\sigma_s,\beta_s,0)$ distribution with the same parameters σ_s and β_s as in the case $\alpha<1$. Similarly it can be seen that $(\overline{Z}(t,s_1),\ldots,\overline{Z}(t,s_m))$ has characteristic function given by (4.0.3). Since $\mu_{s_1,\ldots,s_m}$ is a Lévy measure on $\mathbb{R}^m$ which satisfies the scaling property given by Lemma 4.1.4, the vector $(Z(t,s_1),\ldots,Z(t,s_m))$ has an α-stable distribution (see Remark 2.3.15.b) and Theorem 2.3.9).

The last statement follows from the fact that $E(e^{iu\overline{Z}(t,s_k)})\to E(e^{iu\overline{Z}(t,s)})$, since

$$\int_{\mathbb{D}_0}(e^{iurz(s_k)}-1-iurz(s))\overline{\nu}(dr,dz)\to\int_{\mathbb{D}_0}(e^{iurz(s)}-1-iurz(s))\overline{\nu}(dr,dz).$$

The application of the dominated convergence theorem is justified using the inequalities $|e^{iurz(s)}-1-iurz(s)|\le\frac{1}{2}|urz(s)|^2$ if $r\le1$ and $|e^{iurz(s)}-1-iurz(s)|\le 2|urz(s)|$ if $r>1$. $\square$

For any $\varepsilon>0$, $t>0$ and $s\in[0,1]$, let

$$\begin{aligned}\overline{Z}^{(\varepsilon)}(t,s) &= Z^{(\varepsilon)}(t,s)-E(Z^{(\varepsilon)}(t,s))\\ &= \int_{[0,t]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} rz(s)N(du,dr,dz)-t\varphi(s)\int_\varepsilon^\infty r\nu_\alpha dr.\end{aligned}$$

By (4.1.7)

$$\overline{Z}^{(\varepsilon_k)}(t,s)=\sum_{j=0}^{k}(Z_j(t,s)-E(Z_j(t,s))) \tag{4.3.2}$$

Remark 4.3.2. For any probability measure Q on $(\mathbb{D},\mathcal{D})$, there exists a càdlàg process $\{Y(s)\}_{s\in[0,1]}$, defined on a probability space $(\Omega',\mathcal{F}',P')$, whose law under P' is Q. This is simply because we may take $(\Omega',\mathcal{F}',P')=(\mathbb{D},\mathcal{D},Q)$ and $Y(s)=\pi_s$ for all $s\in[0,1]$. This fact will be used in the proof of Lemma 4.3.4 below.

Lemma 4.3.4 below is the analogue of Lemma 4.2.2 for the case $\alpha>1$. The crucial elements of its proof are: *(i)* tightness of the sequence $\{\overline{Z}^{(\varepsilon_k)}(t)\}_{k\ge1}$ in $\mathbb{D}$, proved in [29]; and *(ii)* the improved version of Itô-Nisio theorem for random elements in $\mathbb{D}$, given in [4]. (The original version of Itô-Nisio theorem in $\mathbb{D}$ can be found in [19].) Recall that in the case $\alpha>1$, the process $\overline{Z}(t)=\{\overline{Z}(t,s)\}_{s\in[0,1]}$ is given by (4.3.1). We recall below the Itô-Nisio theorem from [4].

Theorem 4.3.3 (Theorem 2.1 of [4])**.** *Let $\{X_j\}$ be a sequence of independent random element in $\mathbb{D}([0,1];E)$ where $(E,|\cdot|_E)$ is a separable complete Banach space and let $S_n=\sum_{j=1}^n X_j$. Suppose there exist a random element Y in $\mathbb{D}([0,1];E)$ and a dense set T of $[0,1]$ such that $1\in T$ and for any $t_1,\cdots,t_k\in T$*

$$(S_n(t_1),\cdots,S_n(t_k))\xrightarrow{d}(Y(t_1),\cdots,Y(t_k))\quad \textit{as } n\to\infty$$

Then there exist a random element S in $\mathbb{D}([0,1];E)$ with the same distribution as Y such that:

- $S_n \to S$ *a.s. uniformly on $[0,1]$, provided X_n are symmetric.*
- *if X_n are not symmetric, then*
$$S_n + y_n \to S \quad a.s.\ uniformly\ on\ \ [0,1] \tag{4.3.3}$$
for some $y_n \in \mathbb{D}([0,1];E)$ such that $lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$.
- *Moreover, if the family $\{|S(t)|_E : t \in T\}$ is uniformly integrable and the functions $t \longmapsto E(X_n(t))$ belong to $\mathbb{D}([0,1];E)$, then one can take in (4.3.3) given by*
$$y_n(t) = E(S(t) - S_n(t)).$$

Lemma 4.3.4. *If $\alpha > 1$, then for any $t \geq 0$, there exists a random element $Z(t) = \{Z(t,s)\}_{s\in[0,1]}$ in $\mathbb{D}$ such that $P(Z(t,s) = \overline{Z}(t,s)) = 1$ for all $s \in [0,1]$, and*

$$\lim_{k\to\infty} \|\overline{Z}^{(\varepsilon_k)}(t) - Z(t)\| = 0 \quad a.s. \tag{4.3.4}$$

Proof: For $t = 0$, we define $Z(0,s) = 0$ for all $s \in [0,1]$. We will assume for simplicity that $t = 1$, the case of arbitrary $t > 0$ being similar. To simplify the notation, in this proof we denote $\overline{Z}^{(\varepsilon_k)} = \{\overline{Z}^{(\varepsilon_k)}(s) = \overline{Z}^{(\varepsilon_k)}(1,s)\}_{s\in[0,1]}$ and $\overline{Z} = \{\overline{Z}(s) = \overline{Z}(1,s)\}_{s\in[0,1]}$.

From the last part of the proof of Theorem 2.12 of [29], we know that $(\overline{Z}^{(\varepsilon_k)})_{k\geq1}$ is tight in $(\mathbb{D}, J_1)$. By Prohorov's theorem, $(\overline{Z}^{(\varepsilon_k)})_{k\geq1}$ is relatively compact in $(\mathbb{D}, J_1)$. Hence, there exists a subsequence $N' \subset \mathbb{Z}_+$ and a probability measure Q on $(\mathbb{D}, \mathcal{D})$ such that $P \circ (\overline{Z}^{(\varepsilon_k)})^{-1} \stackrel{w}{\to} Q$ as $k \to \infty, k \in N'$. By Remark 4.3.2, let Y be a random element in $\mathbb{D}$ with law Q, defined on a probability space $(\Omega', \mathcal{F}', P')$. Then, $\overline{Z}^{(\varepsilon_k)} \stackrel{d}{\to} Y$ in $(\mathbb{D}, J_1)$ as $k \to \infty, k \in N'$, which implies that

$$(\overline{Z}^{(\varepsilon_k)}(s_1), \ldots, \overline{Z}^{(\varepsilon_k)}(s_m)) \stackrel{d}{\to} (Y(s_1), \ldots, Y(s_m)), \tag{4.3.5}$$

as $k \to \infty, k \in N'$, for any $s_1, \ldots, s_m \in T$, where $T = \{s \in (0,1); P'(s \in \mathrm{Disc}(Y)) = 0\} \cup \{0,1\}$ is dense in $[0,1]$ (see p.124 of [5]). By (4.3.1) and (4.3.2),

$$\overline{Z}(s) = \lim_{k\to\infty} \overline{Z}^{(\varepsilon_k)}(s) \quad \text{a.s.} \quad \text{for any } s \in [0,1]. \tag{4.3.6}$$

By (4.3.5) and the uniqueness of the limit, it follows that for any $s_1, \ldots, s_m \in T$,

$$(\overline{Z}(s_1), \ldots, \overline{Z}(s_m)) \stackrel{d}{=} (Y(s_1), \ldots, Y(s_m)).$$

Consider now another subsequence $N'' \subset \mathbb{Z}_+$ such that $P \circ (\overline{Z}^{(\varepsilon_k)})^{-1} \xrightarrow{w} Q'$ as $k \to \infty, k \in N''$, for a probability measure Q' on $(\mathbb{D}, \mathcal{D})$. Let Y' be a random element in $\mathbb{D}$ with law Q', defined on a probability space $(\Omega'', \mathcal{F}'', P'')$. Let $T' = \{s \in (0,1); P''(s \in \text{Disc}(Y')) = 0\} \cup \{0,1\}$. The same argument as above shows that for any $s_1, \ldots, s_m \in T'$

$$(\overline{Z}(s_1), \ldots, \overline{Z}(s_m)) \stackrel{d}{=} (Y'(s_1), \ldots, Y'(s_m)).$$

Hence, $(Y(s_1), \ldots, Y(s_m)) \stackrel{d}{=} (Y'(s_1), \ldots, Y'(s_m))$ for any $s_1, \ldots, s_m \in T \cap T'$. Since $T \cap T'$ is dense in $[0,1]$ and contains 1, by Theorem 12.5 of [6], we conclude that $Q = Q'$. This shows that any subsequence of $\{P \circ (\overline{Z}^{(\varepsilon_k)})^{-1}\}_k$ which converges weakly, in fact converges weakly to Q. Therefore, $P \circ (\overline{Z}^{(\varepsilon_k)})^{-1} \xrightarrow{w} Q$ as $k \to \infty$, and relation (4.3.5) holds as $k \to \infty$ (not only along the subsequence N').

Note that $\overline{Z}^{(\varepsilon_k)}(s) = \sum_{j=0}^{k} \big(Z_j(1,s) - E(Z_j(1,s))\big)$ and $\{X_j = Z_j(1,\cdot) - E(Z_j(1,\cdot))\}_{j \geq 0}$ are random elements in $\mathbb{D}$ (by Lemma 4.1.1), which are independent and have mean zero. The existence of a càdlàg process $\{Z(s)\}_{s \in [0,1]}$ such that $\lim_{k\to\infty} \|\overline{Z}^{(\varepsilon_k)} - Z\| = 0$ a.s. will follow by Theorem 2.1.(iii) of [4]. Relation (2.1) of [4] holds, due to (4.3.5). We only have to prove that $\{|Y(s)|\}_{s\in[0,1]}$ is uniformly integrable, which is equivalent to $\{|\overline{Z}(s)|\}_{s\in[0,1]}$ being uniformly integrable. This will follow from the fact that:

$$\sup_{s\in[0,1]} E|\overline{Z}(s)|^p < \infty \quad \text{for any } 1 < p < \alpha. \tag{4.3.7}$$

To prove (4.3.7), recall from Proposition 4.2.1 that $\overline{Z}(s)$ has a $S_\alpha(\sigma_s, \beta_s, 0)$-distribution. By Property 1.2.17 of [30], $E|\overline{Z}(s)|^p = \sigma_s^p (c_{\alpha,\beta_s}(p))^p$, where

$$\begin{aligned}(c_{\alpha,\beta_s}(p))^p &= c_p \left(1 + \beta_s^2 \tan^2 \frac{\alpha\pi}{2}\right)^{p/2\alpha} \cos\left(\frac{p}{\alpha} \arctan\left(\beta_s \tan \frac{\alpha\pi}{2}\right)\right) \\ &\leq c_p \left(1 + \tan^2 \frac{\alpha\pi}{2}\right)^{p/2\alpha} \quad \text{for all} \quad s \in [0,1],\end{aligned}$$

and $c_p > 0$ is a constant depending only on p. (The form of the constant $c_{\alpha,\beta}(p)$ plays an important roles in the argument above. This constant was computed in [15].) Note that for any $s \in [0,1]$,

$$\begin{aligned}\sigma_s &= C_\alpha^{-1}(c_s^+ + c_s^-) = C_\alpha^{-1}\mu_s(\{y \in \mathbb{R}; |y| > 1\}) \\ &= C_\alpha^{-1}\overline{\nu}(\{(r,z) \in (0,\infty) \times \mathbb{S}_{\mathbb{D}}; r|z(s)| > 1\}) \\ &\leq C_\alpha^{-1}\overline{\nu}((1,\infty) \times \mathbb{S}_{\mathbb{D}}) = C_\alpha^{-1} c\nu_\alpha((1,\infty)) < \infty,\end{aligned}$$

where for the last equality we used definition (4.0.1) of $\overline{\nu}$. Relation (4.3.7) follows. □

The following result proves Theorem 4.0.2.a) in the case $\alpha > 1$.

Theorem 4.3.5. *If $\alpha \in (1,2)$, the process $\{Z(t)\}_{t\geq 0}$ defined in Lemma 4.3.4 is a $\mathbb{D}$-valued α-stable Lévy motion (corresponding to ν). This process is $(1/\alpha)$-self-similar, i.e.*

$$\{Z(ct)\}_{t\geq 0} \stackrel{d}{=} c^{1/\alpha}\{Z(t)\}_{t\geq 0} \quad \text{for any} \quad c>0, \tag{4.3.8}$$

where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions. Moreover, for any $t \geq 0$ and for any monotone sequence $(t_k)_{k\geq 0}$ with $t_k \downarrow t$,

$$\lim_{k\to\infty} \|Z(t_k) - Z(t)\| = 0 \quad a.s. \tag{4.3.9}$$

Proof: We first show that the process $\{Z(t)\}_{t\geq 0}$ satisfies properties *(i)*-*(iv)* given in Definition 4.0.1. Property *(i)* is clear. To verify property *(ii)*, we assume for simplicity that $k=3$. The case of arbitrary $k>3$ is similar. We apply Lemma 4.2.3 to the space $S=\mathbb{D}$ equipped with $d^0_{J_1}$. By Lemma 4.3.4, for $i=1,2$, $X_i^k = \overline{Z}^{(\varepsilon_k)}(t_{i+1}) - \overline{Z}^{(\varepsilon_k)}(t_i) \to X_i = Z(t_{i+1}) - Z(t_i)$ a.s. as $k\to\infty$ in $(\mathbb{D}, \|\cdot\|)$ and hence in $(\mathbb{D}, J_1)$. The variables X_1^k and X_2^k are independent for any k, since X_i^k is $\mathcal{F}^N_{t_i,t_{i+1}}$-measurable and the σ-fields $\mathcal{F}^N_{t_1,t_2}, \mathcal{F}^N_{t_2,t_3}$ are independent. Here $\mathcal{F}^N_{s,t}$ is the σ-field generated by $N((a,b]\times B)$ for any $s<a<b\leq t$ and $B\in\mathcal{B}(\overline{\mathbb{D}}_0)$. It follows that X_1 and X_2 are independent.

For property *(iii)*, we have to show that vectors $X=(Z(t_2,s_1)-Z(t_1,s_1), \ldots Z(t_2,s_1)-Z(t_1,s_m))$ and $Y=(Z(t_2-t_1,s_1),\ldots,Z(t_2-t_1,s_m))$ have the same distribution, for any $s_1,\ldots,s_m \in [0,1]$. We let $\Omega'_{t,s} = \left\{Z(t,s) = \overline{Z}(t,s)\right\}$, by (4.2.1) and Lemma 4.3.4, on the event $\Omega_{t_1,s}\cap\Omega_{t_2,s}\cap\Omega'_{t_1,s}\cap\Omega'_{t_2,s}$,

$$\begin{aligned} Z(t_2,s)-Z(t_1,s) &= \overline{Z}(t_2,s)-\overline{Z}(t_1,s) \\ &= \sum_{j\geq 0}\Big(Z_j(t_2,s) - E(Z_j(t_2,s) - Z_j(t_1,s) + E(Z_j(t_1,s)\Big) \end{aligned}$$

As in the proof of Proposition 4.3.1, it follows that the characteristic function of X is

$$E(e^{iu\cdot X}) = \exp\left\{(t_2-t_1)\int_{\mathbb{R}^m}(e^{iu\cdot y}-1-iu\cdot y)\mu_{s_1,\ldots,s_m}(dy)\right\}, \quad u\in\mathbb{R}^m,$$

which is the same as the characteristic function of Y. Hence $X \stackrel{d}{=} Y$. Finally, property *(iv)* was shown in Proposition 4.3.1 for $\overline{Z}(t)$, and remains valid for its modification $Z(t)$.

To prove relation (4.3.8), we have to show that $\{Z(ct)\}_{t\geq 0} \stackrel{d}{=} \{c^{1/\alpha}Z(t)\}_{t\geq 0}$ for any $c>0$. Since both processes have stationary and independent increments, it is enough to show that $Z(ct) \stackrel{d}{=} c^{1/\alpha}Z(t)$ for any $t>0$, i.e. vectors $U=(Z(ct,s_1),\ldots,Z(ct,s_m))$ and $V=c^{1/\alpha}(Z(t,s_1),\ldots,Z(t,s_m))$ have the same distribution, for any $s_1,\ldots,s_m\in$

$[0,1]$ and $t>0$. Let $h_c(y)=c^{1/\alpha}y$ for $y\in\mathbb{R}^m$. By the scaling property of the measure $\mu_{s_1,\ldots,s_m}$ given in Lemma 4.1.5.b),

$$\mu_{s_1,\ldots,s_m}(h_c^{-1}(A))=\mu_{s_1,\ldots,s_m}(c^{-1/\alpha}A)=c\mu_{s_1,\ldots,s_m}(A),$$

for any Borel set $A\subset\mathbb{R}^m$. Therefore, the characteristic function of V is

$$\begin{aligned} E(e^{iu\cdot V}) &= \exp\left\{t\int_{\mathbb{R}^m}(e^{iu\cdot y}-1-iu\cdot y)(\mu_{s_1,\ldots,s_m}\circ h_c^{-1})(dy)\right\}\\ &= \exp\left\{ct\int_{\mathbb{R}^m}(e^{iu\cdot y}-1-iu\cdot y)\mu_{s_1,\ldots,s_m}(dy)\right\}\end{aligned}$$

for any $u\in\mathbb{R}^m$, which is the same as the characteristic function of U. Hence $U\stackrel{d}{=}V$.

We now prove (4.3.9). For this, we apply again Theorem 2.1.(iii) of [4] with $E=\mathbb{R}$. For any $i\geq 1$, let $X_i=Z(t_{i-1})-Z(t_i)$. By property *(ii)* in Definition 4.0.1, $(X_i)_{i\geq 1}$ are independent random elements in $\mathbb{D}$ (with zero mean). Let $S_k=\sum_{i=1}^k X_i=Z(t_0)-Z(t_k)$ for all $k\geq 1$, and $Y=Z(t_0)-Z(t)$. We first show that for any $s_1,\ldots,s_m\in[0,1]$,

$$(S_k(s_1),\ldots,S_k(s_m))\stackrel{d}{\to}(Y(s_1),\ldots,Y(s_m))\quad\text{as } k\to\infty.$$

To see this, note that $(S_k(s_1),\ldots,S_k(s_m))\stackrel{d}{=}(Z(t_0-t_k,s_1),\ldots,Z(t_0-t_k,s_m))$ by property *(iii)* in Definition 4.0.1) (stationarity of the increments). It is now clear that we have the following convergence the characteristic functions: for any $u=(u_1,\ldots,u_m)\in\mathbb{R}^m$,

$$\begin{aligned} &E(e^{iu_1S_k(s_1)+\ldots+iu_mS_k(s_m)})=\exp\left\{(t_0-t_k)\int_{\mathbb{R}^m}(e^{iu\cdot y}-1-iu\cdot y)\mu_{s_1,\ldots,s_m}(dy)\right\},\\ &\qquad\to E(e^{iu_1Y(s_1)+\ldots+iu_mY(s_m)})=\exp\left\{(t_0-t)\int_{\mathbb{R}^m}(e^{iu\cdot y}-1-iu\cdot y)\mu_{s_1,\ldots,s_m}(dy)\right\},\end{aligned}$$

as $k\to\infty$. It remains to show that $\{|Y(s)|\}_{s\in[0,1]}$ is uniformly integrable, which is equivalent to saying that $\{|Z(t_0-t,s)|\}_{s\in[0,1]}$ is uniformly integrable, by the stationarity of the increments. By the self-similarity of $\{Z(t)\}_{t\geq 0}$, $Z(t_0-t,s)\stackrel{d}{=}(t_0-t)^{1/\alpha}Z(1,s)$ for all $s\in[0,1]$. Using (4.3.7) and the fact that $Z(1,s)=\overline{Z}(1,s)$ a.s. for any $s\in[0,1]$, it follows that for any $1<p<\alpha$,

$$\sup_{s\in[0,1]}E|Z(t_0-t,s)|^p=(t_0-t)^{p/\alpha}\sup_{s\in[0,1]}E|Z(1,s)|^p<\infty.$$

(Recall that in (4.3.7) we used the notation $\overline{Z}(s)=\overline{Z}(1,s)$.) Hence, $\{|Z(t_0-t,s)|\}_{s\in[0,1]}$ is uniformly integrable. By Theorem 2.1.(iii) of [4], it follows that $S_k\to Z(t_0)-Z(t)$ a.s. in $(\mathbb{D},\|\cdot\|)$, as $k\to\infty$, which is the same as $Z(t_k)\to Z(t)$ a.s. in $(\mathbb{D},\|\cdot\|)$, as $k\to\infty$. $\square$

The following preliminary result will be used in the proof of tightness of $(\overline{Z}^{(\varepsilon_k)})_{k\geq 1}$.

Lemma 4.3.6. *For any $\varepsilon > 0$ and $T > 0$,*

$$E\|Z^{(\varepsilon)}\|_{T,\mathbb{D}} \leq Tc\frac{\alpha}{\alpha-1}\varepsilon^{1-\alpha}.$$

Proof: By definition, for any $t \in [0,T]$ and $s \in [0,1]$, we have

$$|Z^{(\varepsilon)}(t,s)| \leq \int_{[0,t]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} r|z(s)|N(du,dr,dz) \leq \int_{[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} rN(du,dr,dz) =: Y.$$

Hence $\|Z^{(\varepsilon)}\|_{T,\mathbb{D}} \leq Y$ and $E\|Z^{(\varepsilon)}\|_{T,\mathbb{D}} \leq E(Y) = T\int_{(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} r\overline{\nu}(dr,dz) = Tc\frac{\alpha}{\alpha-1}\varepsilon^{1-\alpha}$. □

The next result plays a crucial role in the proof of Theorem 4.0.2.b) in the case $\alpha > 1$. Its proof uses some results related to sums of i.i.d. regularly varying random elements in $\mathbb{D}$, which are given in Section 5.6 below.

Theorem 4.3.7. *If Assumption B holds, then $(\overline{Z}^{(\varepsilon_k)})_{k\geq 1}$ is tight in $\mathbb{D}([0,\infty);\mathbb{D})$.*

Proof: It is enough to prove that $(\overline{Z}^{(\varepsilon_k)})_{k\geq 1}$ is tight in $\mathbb{D}([0,T];\mathbb{D})$ for any $T > 0$. Without loss of generality, we assume that $T = 1$. Let P_k be the law of $\overline{Z}^{(\varepsilon_k)}$. We verify that $(P_k)_{k\geq 1}$ satisfies conditions *(i)*-*(iii)* of Theorem 3.3.6. To prove this, we argue as in the last part of the proof of Theorem 2.12 of [29].

For condition *(i)*, it suffices to show that the following two relations hold:

$$\lim_{A\to\infty} P(\|\overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > A) = 0 \quad \text{for all } \varepsilon_0 > 0 \tag{4.3.10}$$

$$\lim_{\varepsilon_0\downarrow 0} \sup_{0<\varepsilon<\varepsilon_0} P(\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta) = 0 \quad \text{for all } \eta > 0. \tag{4.3.11}$$

To see this, let $\eta > 0$ and $\rho > 0$ be arbitrary. By (4.3.11) and the fact that $\varepsilon_k \downarrow 0$, there exist $\varepsilon_0^* \in (0,1)$ and k_0 such that $P(\|\overline{Z}^{(\varepsilon_k)} - \overline{Z}^{(\varepsilon_0^*)}\|_{\mathbb{D}} > \eta) < \rho/2$ for any $k \geq k_0$. By (4.3.10), there exists $A_0 > 0$ such that $P(\|\overline{Z}^{(\varepsilon_0^*)}\|_{\mathbb{D}} > A_0) < \rho/2$. Let $a_0 = \eta + A_0$. Then, for all $k \geq k_0$,

$$P(\|\overline{Z}^{(\varepsilon_k)}\|_{\mathbb{D}} > a_0) \leq P(\|\overline{Z}^{(\varepsilon_k)} - \overline{Z}^{(\varepsilon_0^*)}\|_{\mathbb{D}} > \eta) + P(\|\overline{Z}^{(\varepsilon_0^*)}\|_{\mathbb{D}} > A_0) < \rho.$$

This proves that condition *(i)* holds.

To prove (4.3.10), let $\varepsilon_0 > 0$ be arbitrary. For any $A > 2\|E(Z^{(\varepsilon_0)})\|_{\mathbb{D}}$,

$$P(\|\overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > A) \leq P(\|Z^{(\varepsilon_0)}\|_{\mathbb{D}} > A/2) \leq \frac{2}{A}\|E(Z^{(\varepsilon_0)})\|_{\mathbb{D}} \leq \frac{2}{A}Tc\frac{\alpha}{\alpha-1}\varepsilon_0^{1-\alpha},$$

using Markov inequality and Lemma 4.3.6. Relation (4.3.10) follows letting $A \to \infty$.

To prove (4.3.11), we use an indirect argument. Consider a sequence $(X_i)_{i\geq 1}$ of i.i.d. regularly varying elements in $\mathbb{D}$ (as given by Definition 5.2.1 with limiting measure $\overline{\nu}$ given by (4.0.1)). Let $S_n^{(\varepsilon)}$ be given by relation (5.4.1) below. Similarly to Theorem 5.6.4 below (which is based on the fact that the probability measure Γ_1 satisfies Assumptions B), it can be proved that for any $0<\varepsilon<\varepsilon_0$,

$$S_n^{(\varepsilon)} - S_n^{(\varepsilon_0)} - E(S_n^{(\varepsilon)} - S_n^{(\varepsilon_0)}) \xrightarrow{d} \overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)} \quad \text{in} \quad \mathbb{D}([0,1];\mathbb{D}), \tag{4.3.12}$$

where $\mathbb{D}([0,1];\mathbb{D})$ is equipped with distance $d_{\mathbb{D}}$. For any $t>0$ and $s\in[0,1]$, we define

$$S_n^{<\varepsilon}(t,s) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i(s) 1_{\{\|X_i\|\leq a_n\varepsilon\}}.$$

Then $S_n^{(\varepsilon)} = S_n - S_n^{<\varepsilon}$. Hence, $S_n^{(\varepsilon)} - S_n^{(\varepsilon_0)} = S_n^{<\varepsilon_0} - S_n^{<\varepsilon}$ and relation (4.3.12) becomes:

$$S_n^{<\varepsilon_0} - S_n^{<\varepsilon} - E(S_n^{<\varepsilon_0} - S_n^{<\varepsilon}) \xrightarrow{d} \overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)} \quad \text{in} \quad \mathbb{D}([0,1];\mathbb{D}).$$

Since $\|\cdot\|_{\mathbb{D}}$ is $d_{\mathbb{D}}$-continuous (see Lemma 3.2.2), by the continuous mapping theorem, we have: $\|S_n^{<\varepsilon_0} - S_n^{<\varepsilon} - E(S_n^{<\varepsilon_0} - S_n^{<\varepsilon})\|_{\mathbb{D}} \xrightarrow{d} \|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}}$ as $n\to\infty$. Let $\eta>0$ be arbitrary. By Portmanteau theorem,

$$\begin{aligned} P(\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta) &\leq \liminf_{n\to\infty} P(\|S_n^{<\varepsilon_0} - S_n^{<\varepsilon} - E(S_n^{<\varepsilon_0} - S_n^{<\varepsilon})\|_{\mathbb{D}} > \eta) \\ &\leq \limsup_{n\to\infty} P(\|S_n^{<\varepsilon_0} - E(S_n^{<\varepsilon_0})\|_{\mathbb{D}} > \eta/2) + P(\|S_n^{<\varepsilon} - E(S_n^{<\varepsilon})\|_{\mathbb{D}} > \eta/2). \end{aligned}$$

We take the supremum over all $\varepsilon\in(0,\varepsilon_0)$, followed by the limit as $\varepsilon_0\downarrow 0$. We obtain that $\lim_{\varepsilon_0\downarrow 0}\sup_{0<\varepsilon<\varepsilon_0} P(\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta)$ is less than

$$\lim_{\varepsilon_0\downarrow 0}\limsup_{n\to\infty} P(\|S_n^{<\varepsilon_0} - E(S_n^{<\varepsilon_0})\|_{\mathbb{D}} > \eta/2) + \lim_{\varepsilon_0\downarrow 0}\sup_{0<\varepsilon<\varepsilon_0}\limsup_{n\to\infty} P(\|S_n^{<\varepsilon} - E(S_n^{<\varepsilon})\|_{\mathbb{D}} > \eta/2).$$

Since $S_n^{<\varepsilon} = S_n - S_n^{(\varepsilon)}$, both these terms are zero, by relation (5.6.1) below (with $T=1$). This concludes the proof of (4.3.11).

We prove that $(P_k)_{k\geq 1}$ satisfies condition *(ii)* of Theorem 3.3.6. Let $\eta>0$ and $\rho>0$ be arbitrary. It suffices to show that there exist $\delta\in(0,1)$ and $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$,

$$\begin{cases} (a) & P(w''(\overline{Z}^{(\varepsilon)}(t,\delta) > \eta \text{ for some } t\in[0,1]) < \rho \\ (b) & P(|\overline{Z}^{(\varepsilon)}(t,\delta) - \overline{Z}^{(\varepsilon)}(t,0)| > \eta \text{ for some } t\in[0,1]) < \rho \\ (c) & P(|\overline{Z}^{(\varepsilon)}(t,1-) - \overline{Z}^{(\varepsilon)}(t,1-\delta)| > \eta \text{ for some } t\in[0,1]) < \rho. \end{cases} \tag{4.3.13}$$

By (4.3.11), there exists $\varepsilon_0>0$ such that

$$P(\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta/4) < \rho/2 \quad \text{for all} \quad \varepsilon\in(0,\varepsilon_0). \tag{4.3.14}$$

Since $\mathbb{D}([0,1];\mathbb{D})$ endowed with $d_{\mathbb{D}}^0$ is separable and complete (see Theorem 3.2.6), by Theorem 1.3 of [6], the single probability measure $P \circ (\overline{Z}^{(\varepsilon_0)})^{-1}$ is tight. Hence, by condition *(ii)* of Theorem 3.3.6, there exists $\delta \in (0,1)$ such that

$$P(w''(\overline{Z}^{(\varepsilon_0)}(t,\delta) > \eta/2 \text{ for some } t \in [0,1]) < \rho/2 \tag{4.3.15}$$

$$P(|\overline{Z}^{(\varepsilon_0)}(t,\delta) - \overline{Z}^{(\varepsilon_0)}(t,0)| > \eta/2 \text{ for some } t \in [0,1]) < \rho/2 \tag{4.3.16}$$

$$P(|\overline{Z}^{(\varepsilon_0)}(t,1-) - \overline{Z}^{(\varepsilon_0)}(t,1-\delta)| > \eta/2 \text{ for some } t \in [0,1]) < \rho/2. \tag{4.3.17}$$

Using the fact that

$$w''(x+y,\delta) \leq w''(x,\delta) + 2\|y\| \quad \text{for all} \quad x, y \in \mathbb{D},$$

we infer that $w''(\overline{Z}^{(\varepsilon)}(t),\delta) \leq w''(\overline{Z}^{(\varepsilon_0)}(t),\delta) + 2\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}}$, and hence $P(w''(\overline{Z}^{(\varepsilon)}(t),\delta) > \eta$ for some $t \in [0,1])$ is smaller than

$$P(w''(\overline{Z}^{(\varepsilon_0)}(t),\delta) > \eta/2 \text{ for some } t \in [0,1]) + P(\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta/4).$$

Part (a) of (4.3.13) follows from (4.3.14) and (4.3.15). Similarly, part (b) of (4.3.13) follows from (4.3.14) and (4.3.16), using the fact that

$$|\overline{Z}^{(\varepsilon)}(t,\delta) - \overline{Z}^{(\varepsilon)}(t,0)| \leq |\overline{Z}^{(\varepsilon_0)}(t,\delta) - \overline{Z}^{(\varepsilon_0)}(t,0)| + 2\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}},$$

whereas part (c) of (4.3.13) follows from (4.3.14) and (4.3.17), since

$$|\overline{Z}^{(\varepsilon)}(t,1-) - \overline{Z}^{(\varepsilon)}(t,1-\delta)| \leq |\overline{Z}^{(\varepsilon_0)}(t,1-) - \overline{Z}^{(\varepsilon_0)}(t,1-\delta)| + 2\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}}.$$

It remains to prove that $(P_k)_{k\geq 1}$ satisfies condition *(iii)* of Theorem 3.3.6. Let $\eta > 0$ and $\rho > 0$ be arbitrary. Note that $\overline{Z}^{(\varepsilon)}(0) = 0$. We will show that there exist $\delta \in (0,1)$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\begin{cases} (a) & P(w''_{\mathbb{D}}(\overline{Z}^{(\varepsilon)},\delta) > \eta) < \rho \\ (b) & P(\|\overline{Z}^{(\varepsilon)}(\delta)\| > \eta) < \rho \\ (c) & P\big(d_{J_1}^0\big(\overline{Z}^{(\varepsilon)}(1-), \overline{Z}^{(\varepsilon)}(1-\delta)\big) > 3\eta/2\big) < \rho. \end{cases} \tag{4.3.18}$$

Let ε_0 be such that (4.3.14) holds. Using again the fact that $P \circ (\overline{Z}^{(\varepsilon_0)})^{-1}$ is tight, but invoking this time condition *(iii)* of Theorem 3.3.6, we infer that there exists $\delta \in (0,1)$ such that

$$P(w''_{\mathbb{D}}(\overline{Z}^{(\varepsilon_0)},\delta) > \eta/2) < \rho/2 \tag{4.3.19}$$

$$P(\|\overline{Z}^{(\varepsilon_0)}(\delta)\| > \eta/2) < \rho/2 \tag{4.3.20}$$

$$P\big(d^0_{J_1}\big(\overline{Z}^{(\varepsilon_0)}(1-) - \overline{Z}^{(\varepsilon_0)}(1-\delta)\big) > \eta/2\big) < \rho/2. \tag{4.3.21}$$

By Lemma 3.2.8, $P(w''_{\mathbb{D}}(\overline{Z}^{(\varepsilon)}, \delta) > \eta) \leq P(w''_{\mathbb{D}}(\overline{Z}^{(\varepsilon_0)}, \delta) > \eta/2) + P(2\|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}} > \eta/2) < \rho$. Part (a) of (4.3.18) follows using (4.3.19) and (4.3.14). Part (b) of (4.3.18) follows using (4.3.20) and (4.3.14), since $\|\overline{Z}^{(\varepsilon)}(\delta)\| \leq \|\overline{Z}^{(\varepsilon_0)}(\delta)\| + \|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}}$. To see that part (c) of (4.3.18) holds, note that by the triangular inequality, $d^0_{J_1}\big(\overline{Z}^{(\varepsilon)}(1-), \overline{Z}^{(\varepsilon)}(1-\delta)\big)$ is smaller than

$$d^0_{J_1}\big(\overline{Z}^{(\varepsilon)}(1-), \overline{Z}^{(\varepsilon_0)}(1-)\big) + d^0_{J_1}\big(\overline{Z}^{(\varepsilon_0)}(1-), \overline{Z}^{(\varepsilon_0)}(1-\delta)\big) + d^0_{J_1}\big(\overline{Z}^{(\varepsilon_0)}(1-\delta), \overline{Z}^{(\varepsilon)}(1-\delta)\big).$$

We treat separately these three terms. For the second term, we use (4.3.21). For the last term, we use (4.3.14), since this term is bounded by $\|\overline{Z}^{(\varepsilon_0)}(1-\delta) - \overline{Z}^{(\varepsilon)}(1-\delta)\|$ which is smaller than $\|\overline{Z}^{(\varepsilon_0)} - \overline{Z}^{(\varepsilon)}\|_{\mathbb{D}}$. For the first term, we also use (4.3.21), since this term is bounded by $\|\overline{Z}^{(\varepsilon)}(1-) - \overline{Z}^{(\varepsilon_0)}(1-)\|$ which is smaller than $\|\overline{Z}^{(\varepsilon_0)} - \overline{Z}^{(\varepsilon)}\|_{\mathbb{D}}$. To see this, note that by Remark 4.1.8, $\overline{Z}^{(\varepsilon)}(1-) = \lim_{\delta\to 0} \overline{Z}^{(\varepsilon)}(1-\delta)$ in $(\mathbb{D}, \|\cdot\|)$ and $\overline{Z}^{(\varepsilon_0)}(1-) = \lim_{\delta\to 0} \overline{Z}^{(\varepsilon_0)}(1-\delta)$ in $(\mathbb{D}, \|\cdot\|)$, and hence

$$\|\overline{Z}^{(\varepsilon)}(1-) - \overline{Z}^{(\varepsilon_0)}(1-)\| = \lim_{\delta\to 0} \|\overline{Z}^{(\varepsilon)}(1-\delta) - \overline{Z}^{(\varepsilon_0)}(1-\delta)\| \leq \|\overline{Z}^{(\varepsilon)} - \overline{Z}^{(\varepsilon_0)}\|_{\mathbb{D}}.$$

□

The following result proves Theorem 4.0.2.b) in the case $\alpha > 1$.

Theorem 4.3.8. *If $\alpha \in (1, 2)$ and Assumption B holds, then there exists a collection $\{\widetilde{Z}(t)\}_{t\geq 0}$ of random elements in $\mathbb{D}$ such that $P(Z(t) = \widetilde{Z}(t)) = 1$ for all $t \geq 0$, the map $t \mapsto \widetilde{Z}(t)$ is in $\mathbb{D}([0,\infty);\mathbb{D})$, and*

$$\overline{Z}^{(\varepsilon_k)}(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad in \quad \mathbb{D}([0,\infty);\mathbb{D}) \tag{4.3.22}$$

as $k \to \infty$, $k \in N'$, for a subsequence $N' \subset \mathbb{Z}_+$, where $\mathbb{D}([0,\infty);\mathbb{D})$ is equipped with the Skorohod distance $d_{\infty,\mathbb{D}}$ given by (3.5.8).

Proof: *Step 1.* By Theorem 4.3.7, there exists a subsequence $N' \subset \mathbb{Z}_+$ such that

$$\overline{Z}^{(\varepsilon_k)}(\cdot) \xrightarrow{d} Y(\cdot) \quad \text{in} \quad \mathbb{D}([0,\infty);\mathbb{D}), \tag{4.3.23}$$

as $k \to \infty, k \in N'$, where Y is a random element in $\mathbb{D}([0,\infty);\mathbb{D})$, defined on a probability space $(\Omega', \mathcal{F}', P')$. We prove that for any $t_1, \ldots, t_n \geq 0$,

$$(Z(t_1), \ldots, Z(t_n)) \stackrel{d}{=} (Y(t_1), \ldots, Y(t_n)) \quad \text{in } \mathbb{D}^n. \tag{4.3.24}$$

To see this, note that (4.3.23) implies that $(\overline{Z}^{(\varepsilon_k)}(t_1), \ldots, \overline{Z}^{(\varepsilon_k)}(t_n)) \stackrel{d}{\to} (Y(t_1), \ldots, Y(t_n))$ in $(\mathbb{D}^n, J_1^n)$, for any $t_1, \ldots, t_n \in T_Y = T_{P' \circ Y^{-1}}$ (see page 138 of [6]). On the other hand, by (4.3.4), $(\overline{Z}^{(\varepsilon_k)}(t_1), \ldots, \overline{Z}^{(\varepsilon_k)}(t_n)) \stackrel{p}{\to} (Z(t_1), \ldots, Z(t_n))$ in $(\mathbb{D}^n, J_1^n)$ for any $t_1, \ldots, t_n \geq 0$. By the uniqueness of the limit, (4.3.24) holds for any $t_1, \ldots, t_n \in T_Y$. To see that (4.3.24) holds for arbitrary $t_1, \ldots, t_n \geq 0$, we proceed by approximation. Since T_Y is dense in $[0, \infty)$, for any $i = 1, \ldots, n$, there exists a monotone sequence $(t_i^k)_k \subset T_Y$ such that $t_i^k \downarrow t_i$ as $k \to \infty$. By (4.3.9), $(Z(t_1^k), \ldots, Z(t_n^k)) \stackrel{p}{\to} (Z(t_1), \ldots, Z(t_n))$ in $(\mathbb{D}^n, J_1^n)$ as $k \to \infty$. Since Y has all sample paths in $\mathbb{D}([0,\infty); \mathbb{D})$, $(Y(t_1^k), \ldots, Y(t_n^k)) \to (Y(t_1), \ldots, Y(t_n))$ in $(\mathbb{D}^n, J_1^n)$ as $k \to \infty$. Relation (4.3.24) follows again by the uniqueness of the limit.

Step 2. Relation (4.3.24) shows that processes $\{Z(t)\}_{t \geq 0}$ and $\{Y(t)\}_{t \geq 0}$ have the same finite-dimensional distributions. The process $\{Y(t)\}_{t \geq 0}$ has sample paths in $\mathbb{D}([0,\infty); \mathbb{D})$, which is a Borel space (being a Polish space). By Lemma 3.24 of [21], there exists a process $\{\widetilde{Z}(t)\}_{t \geq 0}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, whose sample paths are in $\mathbb{D}([0,\infty); \mathbb{D})$, such that $P(Z(t) = \widetilde{Z}(t)) = 1$ for all $t \geq 0$. In particular, $\{\widetilde{Z}(t)\}_{t \geq 0}$ has the same finite-dimensional distributions as $\{Z(t)\}_{t \geq 0}$, hence also as $\{Y(t)\}_{t \geq 0}$. Since finite-dimensional distributions uniquely determine the law, it follows that the random elements $\widetilde{Z}(\cdot) = \{\widetilde{Z}(t)\}_{t \geq 0}$ and $Y(\cdot) = \{Y(t)\}_{t \geq 0}$ have the same law in $\mathbb{D}([0,\infty); \mathbb{D})$. Relation (4.3.22) follows from (4.3.23). $\square$

Chapter 5

Stable FCLT in $\mathbb{D}$

In this chapter, we show that the α-stable process with values in $\mathbb{D}$ constructed in Chapter 4 can be obtained as the limit (in distribution) of the partial sum sequence associated to i.i.d. regularly varying element in $\mathbb{D}$, with suitable normalization and centering. This result can be viewed as an extension of the stable functional central limit theorem (FCLT) to the case of random elements in $\mathbb{D}$. Our proofs rely on arguments borrowed form [29] which we extend to include the time variable. Similarly to [29], we use the method based on point process convergence.

Recall that Assumptions A and B where given at the beginning of Chapter 4. The goal of this chapter is to prove the following result. Note that the concept of regularly varying element in $\mathbb{D}$ is given by Definition 5.2.1 below.

The results presented in this chapter are taken from the companion paper [2].

Theorem 5.0.1. *Let $X, (X_i)_{i\geq 1}$ be i.i.d. random elements in $\mathbb{D}$ such that $X \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$. Let α be the index of stability of X and Γ_1 be the spectral measure of X. Suppose that the probability measure Γ_1 given by (5.2.5) satisfies Assumptions A and B. For any $n \geq 1$, $t \geq 0$, let $S_n(t) = \{S_n(t,s)\}_{s\in[0,1]}$, where $S_n(t,s) = a_n^{-1}\sum_{i=1}^{[nt]} X_i(s)$ for $s \in [0,1]$. Let $\{\widetilde{Z}(t)\}_{t\geq 0}$ be the process constructed in Theorem 4.0.2, which may not be defined on the same probability space as the sequence $(X_i)_{i\geq 1}$.*

a) *If $\alpha < 1$, then*

$$S_n(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad in \quad \mathbb{D}([0,\infty);\mathbb{D}).$$

b) *If $\alpha > 1$, let $\overline{S}_n(t) = S_n(t) - E[S_n(t)]$, where $E[S_n(t)] = \{E[S_n(t,s)]\}_{s\in[0,1]}$. If*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\max_{k\leq[nT]} P\left(\left\|\sum_{i=1}^{k}\left(X_i 1_{\{\|X_i\|\leq a_n\varepsilon\}} - E[X_i 1_{\{\|X_i\|\leq a_n\varepsilon\}}]\right)\right\| > a_n\delta\right) = 0 \quad (5.0.1)$$

for any $\delta > 0$ and $T > 0$, then

$$\overline{S}_n(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad in \quad \mathbb{D}([0,\infty);\mathbb{D}).$$

Assumption B is the same as Condition (A-i) of [20], whereas (5.0.1) is a stronger form of Condition (A-ii) of [20], which is needed for the functional convergence. The proof of Theorem 5.0.1 uses the method of point process convergence, instead of the classic method of finite-dimensional convergence and tightness. In Section 5.1, we introduce some basic facts related to point processes on Polish spaces.

In Section 5.2, we recall the definition of regularly variation in $\mathbb{D}$ and we give some of its proprieties. In Section 5.3, we establish the continuity of the summation functional. In Section 5.4, we prove the convergence of the truncated sums. In Sections 5.5 and 5.6, we give the proof of Theorem 5.0.1 in the case $\alpha < 1$, respectively $\alpha > 1$.

The limit process $\widetilde{Z} = \{\widetilde{Z}(t)\}_{t\in[0,1]}$ has simple paths on $\mathbb{D}([0,1];\mathbb{D})$. One can also develop an analogue limit theorem in which the limit is a stable Lévy sheet with sample paths in $\mathbb{D}([0,1]^2)$ (see Theorem C.0.12, Appendix C). The advantage of Theorem 5.0.1 is that it yields a process which may not have independent increments in the space variable s.

5.1 Point process on Polish spaces

In this section, we review some basic concepts related to point processes on Polish spaces, following [8]. Similar concept are considered in [28, 27] for point processes on LCCB spaces (i.e. a locally compact space with countable base).

Let (E, d) be a Polish space (i.e. a complete separable metric space) and $\mathcal{E}$ its Borel σ-field. A measure μ on E is *boundedly finite* if $\mu(A) < \infty$ for any bounded set $A \in \mathcal{E}$. (Recall that a set A is bounded if it is contained in an open ball $B_r(x) = \{y \in E; d(x,y) < r\}$). We denote by $\widehat{M}_+(E)$ the set of all boundedly finite measures on E and by $\widehat{M}_p(E)$ its subset consisting of point (or counting) measures (i.e $\mathbb{Z}_+$-valued measures where $\mathbb{Z}_+ = \{0, 1, \ldots\}$). A measure $\mu \in \widehat{M}_p(E)$ can be represented as $\mu = \sum_{i\geq 1} \delta_{x_i}$ for some $(x_i)_{i\geq 1} \subset E$, where δ_x is the Dirac measure at x. In this case, $(x_i)_{i\geq 1}$ are called the atoms (or points) of μ. A measure $\mu = \sum_{i\geq 1} \delta_{x_i} \in \widehat{M}_p(E)$ is called *simple* if $\mu(\{x\}) \leq 1$ for all x, i.e. $(x_i)_{i\geq 1}$ are distinct.

The set $\widehat{M}_+(E)$ is equipped with the topology of $\widehat{w}$*-convergence*: $\mu_n \stackrel{\widehat{w}}{\to} \mu$ on E if for any bounded set $A \in \mathcal{E}$ with $\mu(\partial A) = 0$,

$$\mu_n(A) \to \mu(A). \tag{5.1.1}$$

This is equivalent to saying that for any $f \in \widehat{C}(E)$,

$$\mu_n(f) \to \mu(f), \tag{5.1.2}$$

where $\mu(f) = \int_E f d\mu$ and $\widehat{C}(E)$ is the set of continuous bounded functions $f : E \to \mathbb{R}$ which vanish outside a bounded set.

We denote by $\widehat{\mathcal{M}}_+(E)$ and $\widehat{\mathcal{M}}_p(E)$ the Borel σ-fields of $\widehat{M}_+(E)$ respectively $\widehat{M}_p(E)$. By Proposition 9.1.IV. of [8], $\widehat{M}_+(E)$ and $\widehat{M}_p(E)$ are Polish spaces, $\widehat{\mathcal{M}}_+(E)$ is the σ-field generated by the projections $\widehat{M}_+(E) \ni \mu \longmapsto \mu(A),\ A \in \mathcal{E}$, and $\widehat{\mathcal{M}}_p(E)$ is the σ-field generated by the functions $\widehat{M}_p(E) \ni \mu \longmapsto \mu(A),\ A \in \mathcal{E}$.

A *point process* on E is a function $N:\ \Omega \to \widehat{M}_p(E)$ which is $\mathcal{F}/\widehat{\mathcal{M}}_p(E)$-measurable, where $(\Omega, \mathcal{F}, P)$ is a probability space. Since $\widehat{\mathcal{M}}_p(E)$ is the σ-field generated by projections, this is equivalent to saying that $N(A):\ \Omega \to \mathbb{Z}_+$ is $\mathcal{F}$-measurable, for any $A \in \mathcal{E}$. We denote by $P \circ N^{-1}$ the law of N.

The *Laplace functional* of a point process N on E is defined by

$$L_N(f) = E(e^{-N(f)})$$

for any bounded $\mathcal{E}$-measurable function $f:\ E \to \mathbb{R}$ with bounded support.

We say that a sequence $(N_n)_n$ of point processes on E *converge in distribution* to the point process N on E (and we write $N_n \stackrel{d}{\to} N$ in $\widehat{M}_p(E)$) if $(P \circ N_n^{-1})_{n\geq 1}$ converges weakly to $P \circ N^{-1}$ as measures on $\widehat{M}_p(E)$. By Proposition 11.1.VIII of [8], this is equivalent to saying that

$$L_{N_n}(f) \to L_N(f) \text{ for any } f \in \widehat{C}(E).$$

Definition 5.1.1. *Let $\nu \in \widehat{M}_+(E)$ be arbitrary. A point process N on E is called a* **Poisson random measure** *on E of intensity ν if it satisfies the following two conditions:*

- *for any bounded $A \in \mathcal{E}$, $N(A)$ has a Poisson distribution with mean $\nu(A)$*
- *for any bounded disjoint sets $A_1, \ldots, A_k \in \mathcal{E}$, $N(A_1), \ldots, N(A_k)$ are independent.*

The Laplace functional of a Poisson random measure N of intensity ν on E is:

$$L_N(f) = \exp\left\{-\int_E (1 - \exp(-f(x)))\nu(dx)\right\} \tag{5.1.3}$$

for any bounded $\mathcal{E}$-measurable function $f:\ E \to [0, \infty)$ with bounded support.

The following result plays a crucial role in this thesis. It is an extension of Proposition 3.21 of [27] to point process on Polish spaces, with which it shares the same proof based on Laplace functionals. Recall that a *random element* in E is a function $X:\ \Omega \to E$ which is $\mathcal{F}/\mathcal{E}$-measurable where $(\Omega, \mathcal{F}, P)$ is a probability space.

Proposition 5.1.2. *Let E be a Polish space and $\nu \in \widehat{M}_+(E)$. For any $n \geq 1$, let $(X_{i,n})_{i\geq 1}$ be i.i.d random elements in E and $N_n = \sum \delta_{(i/n, X_{i,n})}$. Let N be a Poisson* *random measure on* $[0,\infty) \times E$ *of intensity* $Leb \times \nu$*, where Leb denotes the Lebesgue measure. Then* $N_n \xrightarrow{d} N$ *in* $\widehat{M}_p([0,\infty) \times E)$ *if and only if*

$$nP(X_{1,n} \in \cdot) \xrightarrow{\widehat{w}} \nu \text{ on } E.$$

We conclude this section with few words about finite measures. We denote by $\widehat{M}_f(E)$ the set of finite measures μ on $(E, \mathcal{E})$, i.e. measures μ satisfying the condition $\mu(E) < \infty$. This space is equipped with the usual topology of weak convergence: $\mu_n \xrightarrow{w} \mu$ if (5.1.1) holds for any set $A \in \mathcal{E}$ with $\mu(\partial A) = 0$. This is equivalent to saying that (5.1.2) holds for any $f \in C_b(E)$, where $C_b(E)$ is the class of continuous bounded functions $f: E \to \mathbb{R}$. Finally, we denote by $\widehat{M}_{p,f}(E)$ the set of finite point measures on E, equipped also with the topology of weak convergence.

5.2 Regular variation in $\mathbb{D}$

In this section, we introduce the notion of regular variation for random elements in $\mathbb{D}$ and discuss some of its properties.

In Section 2.5, we saw that the regular variation of a random element in $\mathbb{R}^d$ was defined by removing 0 from the space and adding the ∞-hyperplanes . More precisely, we considered the vague convergence of the sequence $\left\{\mu_n = nP\left(\frac{X}{a_n} \in \cdot\right)\right\}_{n\geq 1}$ of measures on the space $\overline{\mathbb{R}}_0^d = [-\infty,\infty]^d \setminus \{0\}$ to a non-null radon measure ν with the property $\nu(\overline{\mathbb{R}}_0^d \backslash \mathbb{R}_0^d) = 0$. Equivalently this condition can be expressed in polar coordinates as the vague convergence of the sequence $\left\{\overline{\mu}_n = nP\left(\frac{|X|}{a_n}, \frac{X}{|X|}\right) \in \cdot\right\}_{n\geq 1}$ of measures on the space $(0,\infty] \times \mathbb{S}_d$ to the product measure $c\nu_\alpha \times \Gamma_1$, for some $c > 0$, $\alpha > 0$ and a probability measure Γ_1 on the sphere $\mathbb{S}_d = \left\{x \in \mathbb{R}^d;\ |x| = 1\right\}$, where $\nu_\alpha(dr) = \alpha r^{-\alpha-1} 1_{(0,\infty)} dr$.

In the case of random elements in $\mathbb{D}$, there is no natural analogue of an ∞-hyperplane. However, in his Ph.D thesis [25], Lindskog introduced an ingenious method of defining regular variation for random elements in $\mathbb{D}$, based on the product space

$$\overline{\mathbb{D}}_0 = (0,\infty] \times \mathbb{S}_{\mathbb{D}}$$

where $\mathbb{S}_{\mathbb{D}} = \{x \in \mathbb{D};\ \|x\| = 1\}$. This method was based on the earlier article [12] (for the case of random elements with values in the space $\mathbb{C}\,[0,1]$ of continuous functions on $[0,1]$) and was developed further in [17]. We will use this method here.

Since the supremum norm $\|\cdot\|$ is J_1-continuous, the polar coordinate transformation $T:\ \mathbb{D}_0 \to (0,\infty] \times \mathbb{S}_{\mathbb{D}}$ given by

$$T(x) = \left(\|x\|, \frac{x}{\|x\|}\right)$$

is a homeomorphism on $\mathbb{D}_0 = \mathbb{D}\backslash\{0\}$. Unlike [17] we do not identify $\mathbb{D}_0$ with $T(\mathbb{D}_0) = (0,\infty)\times\mathbb{S}_{\mathbb{D}}$. Therefore, we will not say that $\mathbb{D}_0$ is a subset of $\overline{\mathbb{D}}_0$. Note that the space $\overline{\mathbb{D}}_0$ is a Polish space equipped with the the distance $d_{\overline{\mathbb{D}}_0}$ given by: for any (r,z), $(r^{'},z^{'}) \in \overline{\mathbb{D}}_0$

$$d_{\overline{\mathbb{D}}_0}\left((r,z),(r^{'},z^{'})\right) = \left|\frac{1}{r} - \frac{1}{r^{'}}\right| \vee d^0_{J_1}(z,z^{'}) \tag{5.2.1}$$

with the convention $1/\infty = 0$ where $d^0_{J_1}$ is given by (3.1.1).

Definition 5.2.1. *We say that a random element X in $\mathbb{D}$ is* **regularly varying** *if there exist a sequence $(a_n)_{n\geq 1}\subset\mathbb{R}_+$ with $a_n\uparrow\infty$ and a boundedly finite measure $\overline{\nu}$ on $\overline{\mathbb{D}}_0$ with $\overline{\nu}(\overline{\mathbb{D}}_0\backslash T(\mathbb{D}_0)) = 0$ such that*

$$nP\left(\left(\frac{\|X\|}{a_n},\frac{X}{\|X\|}\right)\in\cdot\right)\overset{\widehat{w}}{\to}\overline{\nu}\quad on\quad \overline{\mathbb{D}}_0. \tag{5.2.2}$$

In this case, we write $X\in RV\left(\{a_n\},\overline{\nu},\overline{\mathbb{D}}_0\right)$.

It can be proved that the measure $\overline{\nu}$ in the Definition 5.2.1 satisfies the following scaling property : for any $a>0$ and $A\in\mathcal{B}(\overline{\mathbb{D}}_0)$

$$\overline{\nu}(aA) = a^{-\alpha}\overline{\nu}(A) \tag{5.2.3}$$

for some $\alpha>0$ (see Remark 3 of [17]). We say that α is the **index** of X. Note that in the present thesis, we define for any $a>0$ and $A\in\mathcal{B}(\overline{\mathbb{D}}_0)$,

$$aA = \{(ar,z);(r,z)\in A\}. \tag{5.2.4}$$

Moreover, by Theorem 4 of [17], a random element X in $\mathbb{D}$ is regularly varying if and only if for any $r>0$

$$\frac{P\left(\|X\|>tr,\frac{X}{\|X\|}\in\cdot\right)}{P\left(\|X\|>t\right)}\overset{w}{\to} r^{-\alpha}\Gamma_1(\cdot)\quad\text{on}\quad\mathbb{S}_{\mathbb{D}} \tag{5.2.5}$$

as $t\to\infty$, for a probability measure Γ_1 on $\mathbb{S}_{\mathbb{D}}$ (called the **spectral measure** of X). We denote by $\mathcal{B}(\mathbb{S}_{\mathbb{D}})$ the Borel σ-field on $\mathbb{S}_{\mathbb{D}}$ (under the distance $d^0_{J_1}$) .

If $X\in\mathrm{RV}(\{a_n\},\overline{\nu},\overline{\mathbb{D}}_0)$, then $\|X\|$ is regularly varying of index $-\alpha$: for any $\varepsilon>0$,

$$nP(\|X\|>a_n\varepsilon)\to c\varepsilon^{-\alpha},\quad\text{as}\quad n\to\infty, \tag{5.2.6}$$

where the constant $c>0$ is given by Proposition 5.2.2 below. From this, we infer that if $\alpha>1$, $E\|X\|<\infty$, and hence $E|X(s)|<\infty$ for all $s\in[0,1]$. In this case, we define $E[X] = \{E[X(s)]\}_{s\in[0,1]}$.

The following result gives the relationship between the measures $\overline{\nu}$ and Γ_1.

Proposition 5.2.2. *If $X \in RV\left(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0\right)$ then,*

$$\overline{\nu} = c\nu_\alpha \times \Gamma_1 \tag{5.2.7}$$

with α and Γ_1 given by (5.2.5), $c = \overline{\nu}\left((1,\infty) \times \mathbb{S}_{\mathbb{D}}\right)$ and $\nu_\alpha(dr) = \alpha r^{-\alpha-1} 1_{(0,\infty)}(r)dr$.

Proof. Let $\mathcal{P}$ be the class of the sets $A_{r,S} = (r,\infty) \times S$ with $r > 0$ and $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$. If we denote $V_{r,S} = \left\{x \in \mathbb{D};\ \|x\| > r,\ \frac{x}{\|x\|} \in S\right\}$, then $T(V_{r,S}) = A_{r,S}$. By Remark 5 of [17]

$$\Gamma_1(S) = \frac{\overline{\nu}(A_{1,S})}{c} \text{ for any } S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}}), \tag{5.2.8}$$

where $c = \overline{\nu}(A_{1,\mathbb{S}_{\mathbb{D}}})$. (This fact is stated in [17] in terms of the subsets $V_{r,S}$ of $\mathbb{D}_0$. Since we do not identify $\mathbb{D}_0$ with $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$, we expressed this relation in terms of the subsets $A_{r,S}$ of $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$.) Note that for any $a > 0$

$$aV_{r,S} = \left\{ax \in \mathbb{D};\ \|ax\| > ar,\ \frac{ax}{\|ax\|} \in S\right\} = V_{ar,S}.$$

Similarly, for any $a > 0$, by relation (5.2.4),

$$\begin{aligned} aA_{r,S} &= a\left\{(s,z) \in (0,\infty) \times \mathbb{S}_{\mathbb{D}};\ s > r,\ z \in S\right\} \\ &= \left\{(as,z) \in (0,\infty) \times \mathbb{S}_{\mathbb{D}};\ as > ar,\ z \in S\right\} = A_{ar,S}. \end{aligned}$$

In particular, $A_{r,S} = rA_{1,S}$. Using (5.2.3) and (5.2.8), it follows that for any $r > 0$ and $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$

$$\overline{\nu}(A_{r,S}) = r^{-\alpha}\overline{\nu}(A_{1,S}) = cr^{-\alpha}\Gamma_1(S) = (c\nu_\alpha \times \Gamma_1)(A_{r,S}).$$

Hence, when restricted to $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$, the measures $\overline{\nu}$ and $c\nu_\alpha \times \Gamma_1$ coincide for sets in the class $\mathcal{P}$. It is easy to see that $\mathcal{P}$ is a π-system: $A_{r,S} \cap A_{r',S'} = A_{r \vee r', S \cap S'}$ for any $r > 0, r' > 0$ and $S, S' \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$. To show that $\overline{\nu} = c\nu_\alpha \times \Gamma_1$ on $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$, it suffices to show that

$$\sigma(\mathcal{P}) = \mathcal{B}((0,\infty) \times \mathbb{S}_{\mathbb{D}}), \tag{5.2.9}$$

where $\mathcal{B}((0,\infty) \times \mathbb{S}_{\mathbb{D}})$ is the Borel σ-field on $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$ (with respect to the distance $d_{\overline{\mathbb{D}}_0}$). The fact that $\overline{\nu} = c\nu_\alpha \times \Gamma_1$ on $\overline{\mathbb{D}}_0$ will follow since both measures are zero on $\overline{\mathbb{D}}_0 \backslash T(\mathbb{D}_0)$.

We prove (5.2.9). Since $(0,\infty)$ and $\mathbb{S}_{\mathbb{D}}$ are separable, $\mathcal{B}((0,\infty) \times \mathbb{S}_{\mathbb{D}}) = \mathcal{B}((0,\infty)) \times \mathcal{B}(\mathbb{S}_{\mathbb{D}})$ (see e.g. p.225 of [5]). Since the product of two σ-fields is the σ-field generated by the rectangles, relation (5.2.9) is equivalent to

$$\sigma(\mathcal{P}) = \sigma\left\{A \times S, A \in \mathcal{B}((0,\infty), S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})\right\}.$$

The $\subset$ inclusion is clear since $\mathcal{P} \subset \mathcal{B}((0,\infty) \times \mathbb{S}_{\mathbb{D}})$. To prove the other inclusion, it suffices to show that

$$A \times S \in \sigma(\mathcal{P}) \text{ for any } A \in \mathcal{B}((0,\infty)), S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}}). \tag{5.2.10}$$

For this, we use a $\pi-\lambda$-system argument. Let $\mathcal{J} = \{(r,\infty)\,; r > 0\}$ and

$$\mathcal{M} = \{A \in \mathcal{B}((0,\infty)); A \times S \in \sigma(\mathcal{P}) \text{ for all } S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}}\}\,.$$

Clearly, $\mathcal{J}$ is a π-system in $(0,\infty)$, and $\mathcal{J} \subset \mathcal{M}$ (since $(r,\infty) \times S \in \mathcal{P} \subset \sigma(\mathcal{P})$ for any $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$). We show that $\mathcal{M}$ is a λ-system. First $(0,\infty) \in \mathcal{M}$ since for any $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$, $(0,\infty) \times S = \cup_{n \geq 1} \left(\left(\frac{1}{n},\infty\right) \times S\right) \in \sigma(\mathcal{P})$. Second, if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$ since $A^c \times S = ((0,\infty) \times S) \setminus [(A \times S) \cup ((0,\infty) \times S^c)] \in \sigma(\mathcal{P})$. Third if $(A_n)_n \subset \mathcal{M}$ are disjoint then $\cup_n A_n \in \mathcal{M}$ since $(\cup_n A_n) \times S = \cup_n (A_n \times S)$. By the $\pi - \lambda$ theorem (Theorem 3.2 of [7]) it follows that $\sigma(\mathcal{J}) \subset \mathcal{M}$. Since $\sigma(\mathcal{J}) = \mathcal{B}((0,\infty))$, we conclude that $\mathcal{M} = \mathcal{B}((0,\infty))$. This concludes the proof (5.2.10). $\square$

Using Proposition 5.1.2, we obtain a useful characterization of regular variation on $\mathbb{D}$ using the point process convergence.

Proposition 5.2.3. *Let $(X_i)_{i \geq 1}$ be i.i.d. random elements in $\mathbb{D}$. For each $n \geq 1$, consider the following point process on $[0,\infty) \times \overline{\mathbb{D}}_0$:*

$$N_n = \sum_{i \geq 1} \delta_{\left(\frac{i}{n}, \frac{\|X_i\|}{a_n}, \frac{X_i}{\|X_i\|}\right)}$$

Let N be a Poisson random measure on $[0,\infty) \times \overline{\mathbb{D}}_0$ of intensity $Leb \times \overline{\nu}$ where $\overline{\nu}$ is given by (5.2.7). Then $X_1 \in RV\left(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0\right)$ if and only if

$$N_n \xrightarrow{d} N \text{ in } \widehat{M}_p([0,\infty) \times \overline{\mathbb{D}}_0).$$

5.3 Continuity of summation functional

In this section, we establish the continuity of the truncated summation functional defined on the set on the set of point measures on $(0,\infty) \times \overline{\mathbb{D}}_0$. This will constitute an important step in the proof of the main result. The proofs presented in this section are extension of those of [29] to point measures where atoms include also a time variable.

We endow the spaces $[0,\infty) \times \overline{\mathbb{D}}_0$ and $[0,T] \times \mathbb{D}$ with the product topologies, $\mathbb{D}$ being equipped with Skorohod's J_1-topology.

For fixed $T > 0$ and $\varepsilon > 0$, we define $\Psi : \widehat{M}_p([0,\infty) \times \overline{\mathbb{D}}_0) \to M_{p,f}([0,T] \times \mathbb{D})$ by:

$$\Psi(m) = m|_{[0,T] \times (\varepsilon,\infty) \times \mathbb{S}_{\mathbb{D}}} \circ \psi^{-1}$$

where $m|_{[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}}$ denotes the restriction of m to $[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}$, and the function $\psi:[0,\infty)\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}\to[0,T]\times\mathbb{D}$ is given by $\psi(t,r,z)=(t,rz)$. Note that $\Psi(m)$ is a finite measure since $[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}$ is a bounded set.

The application of the function Ψ has a double effect on a measure m: it removes the atoms (t_i,r_i,z_i) of m whose second coordinate r_i is less than ε or is ∞, and transforms the remaining atoms using the "inverse polar-coordinate" map $(r,z)\mapsto rz$, while leaving the first coordinate t_i of these atoms unchanged (provided that $t_i\leq T$). More precisely, if $m=\sum_{i\geq1}\delta_{(t_i,r_i,z_i)}\in\widehat{M}_p([0,\infty)\times\overline{\mathbb{D}}_0)$ then $\Psi(m)=\sum_{t_i\leq T}\delta_{(t_i,r_iz_i)}1_{\{r_i\in(\varepsilon,\infty)\}}$.

For any $m\in\widehat{M}_p([0,\infty)\times\overline{\mathbb{D}}_0)$ and for any measurable function $f:[0,T]\times\mathbb{D}\to[0,\infty)$,

$$\int_{[0,T]\times\mathbb{D}}f(t,x)\Psi(m)(dt,dx)=\int_{[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}}f(t,rz)m(dt,dr,dz). \tag{5.3.1}$$

Lemma 5.3.1. *The function Ψ is continuous on the set $\mathcal{A}$ of measures $m\in M_p([0,\infty)\times\overline{\mathbb{D}}_0)$ which satisfy the following two conditions:*

$$m([0,\infty)\times\{\varepsilon,\infty\}\times\mathbb{S}_{\mathbb{D}})=0\quad\textit{and}\quad m(\{0,T\}\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}})=0.$$

(The function $\Psi=\Psi_{\varepsilon,T}$ and the set $\mathcal{A}=\mathcal{A}_{\varepsilon,T}$ depend on ε and T. To simplify the writing, we drop the indices ε, T.)

Proof: Let $E=[0,\infty)\times\overline{\mathbb{D}}_0$, $E'=[0,\infty)\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}$ and $E''=[0,T]\times\mathbb{D}$. Since E' is a bounded set, $\widehat{M}_p(E')=M_{p,f}(E')$. Note that $\Psi=\Psi_2\circ\Psi_1$, where $\Psi_1:\widehat{M}_p(E)\to M_{p,f}(E')$ is the restriction $\Psi_1(m)=m|_{E'}$ and $\Psi_2:M_{p,f}(E')\to M_{p,f}(E'')$ is given by $\Psi_2(m)=m\circ\psi^{-1}$.

Similarly to Proposition 3.3 of [14], it can be shown that Ψ_1 is continuous on $\mathcal{A}$. The fact that Ψ_2 is continuous follows from the continuity of function ψ, exactly as in the proof of Proposition 5.6.(a) of [28]. $\Box$

Definition 5.3.2. *We consider the space $M^*_{p,f}([0,T]\times\mathbb{D})$ of all finite point measure $\mu=\sum_{i=1}^p\delta_{(t_i,x_i)}$ on $[0,T]\times\mathbb{D}$ which satisfy the following conditions :*

1. *The points $(t_1,x_1),\ldots,(t_p,x_p)$ are distinct.*

2. $\mathrm{Disc}(x_i)\cap\mathrm{Disc}(x_j)=\emptyset$ *for all $i=1,\ldots,p$ with $i\neq j$.*

3. *No vertical line contains two points of μ.*

The following result gives an alternative characterization of the set $M^*_{p,f}([0,T]\times\mathbb{D})$. Its proof is straightforward and we omit it.

Lemma 5.3.3. *The set $M^*_{p,f}([0,T]\times\mathbb{D})$ coincides with the set of all finite point measures μ on $[0,T]\times\mathbb{D}$ which satisfy the following conditions :*

(i) μ *is simple, i.e.* $\mu\Big(\{(t,x)\}\Big) \leq 1$ *for any* $(t,x) \in [0,T] \times \mathbb{D}$*;*

(ii) $\mu\Big(\{(t,x),(t^{'},x^{'})\}\Big) \leq 1$ *for any* $(t,x),(t^{'},x^{'}) \in [0,T] \times \mathbb{D}$ *such that* $x \neq x^{'}$ *and* $\mathrm{Disc}(x) \cap \mathrm{Disc}(x^{'}) \neq \emptyset$*;*

(iii) $\mu(\{t_0\} \times \mathbb{D}) \leq 1$ *for any* $t_0 \in [0,T]$*.*

Recall that $\mathbb{D}([0,T]\,;\mathbb{D})$ is the space of functions $x: \ [0,T] \to \mathbb{D}$ which are right-continuous and have left limits, which respect to J_1. (See Section 3.8).

Theorem 5.3.4. *The summation functional*

$$\Phi : M_{p,f}([0,T] \times \mathbb{D}) \to \mathbb{D}\left([0,T]\,;\mathbb{D}\right)$$

defined by

$$\Phi\left(\mu\right) = \left(\sum_{t_i \leq t} x_i\right)_{t \in [0,T]} \quad \text{if } \mu = \sum_{i=1}^{p} \delta_{(t_i,x_i)}$$

is continuous on $M^{*}_{p,f}([0,T] \times \mathbb{D})$*, where* $\mathbb{D}([0,T]\,;\mathbb{D})$ *is equipped with the Skorohod distance given by (3.5.1).*

Proof. We use a similar argument to page 221 of [28] combined with the argument contained in the proof of Lemma 2.9 of [29]. Let $(\mu_n)_n \subset M_{p,f}([0,T] \times \mathbb{D})$ and $\mu \in M^{*}_{p,f}([0,T] \times \mathbb{D})$ be such that $\mu_n \xrightarrow{w} \mu$. Say $\mu = \sum_{i=1}^{p} \delta_{(t_i,x_i)}$. Then

$$\mu_n(A) \to \mu(A), \tag{5.3.2}$$

for any set $A \in \mathcal{B}([0,T] \times \mathbb{D})$ with $\mu(\partial A) = 0$. In particular,

$$\mu_n([0,T] \times \mathbb{D}) \to \mu([0,T] \times \mathbb{D}) = p.$$

Since $\mu_n([0,T] \times \mathbb{D}) \in \mathbb{Z}_+$ for all $n \geq 1$, there exists an integer $N_0 \geq 1$ such that $\mu_n([0,T] \times \mathbb{D}) = p$, for any $n \geq N_0$. Since μ is simple, there exist $r > 0$ small enough such that $\mu(B_r(t_i,x_i)) = 1$ and for any $r^{'} \in (0,r)$, $\mu(\partial B_{r^{'}}(t_i,x_i)) = 0$.

By (5.3.2), it follows that for any $i = 1,\ldots,p$ and $r^{'} \in (0,r)$,

$$\mu_n(B_{r^{'}}(t_i,x_i)) \to \mu(B_{r^{'}}(t_i,x_i)) = 1.$$

Since $\mu_n(B_{r^{'}}(t_i,x_i)) \in \mathbb{Z}_+$ for all $n \geq 1$, then there exists an integer $N_i(r^{'}) \geq N_0$ such that

$$\mu_n(B_{r^{'}}(t_i,x_i)) = 1, \text{ for all } n \geq N_i(r^{'}).$$

In particular for $r^{'} = \frac{r}{2}$, if we let $N_i = N_i(r/2)$, then for all $n \geq N_i$

$$\mu_n(B_{r/2}(t_i,x_i)) = 1 \tag{5.3.3}$$

i.e. μ_n has exactly one atom in the ball $B_{r/2}(t_i, x_i)$. We denote this atom by $(t_i^{(n)}, x_i^{(n)})$. Note that this atom may be different than (t_i, x_i).

Let $N = \max_{1\le i\le p} N_i$. Then for any $n \ge N$, $\mu_n = \sum_{i=1}^{p} \delta_{(t_i^{(n)}, x_i^{(n)})}$ and hence

$$\Phi(\mu_n) = \left(\sum_{t_i^{(n)} \le t} x_i^{(n)} \right)_{t\in[0,T]}.$$

We now prove that for any $i = 1, \dots, p$

$$(t_i^{(n)}, x_i^{(n)}) \to (t_i, x_i) \text{ in } [0,T] \times \mathbb{D}, \tag{5.3.4}$$

i.e. $t_i^{(n)} \to t_i$ in $[0,T]$ and $x_i^{(n)} \to x_i$ in $(\mathbb{D}, J_1)$.

To prove (5.3.4), fix $i = 1, \dots, p$ and let $r' \in (0, r/2)$ be arbitrary. By (5.3.3) for any $n \ge N_i(r')$, μ_n has exactly one atom in the ball $B_{r'}(t_i', x_i')$. Since $B_{r'}(t_i', x_i') \subset B_{r/2}(t_i, x_i)$ and in the ball $B_{r/2}(t_i, x_i)$ we already know that μ_n has *only one atom*, namely $(t_i^{(n)}, x_i^{(n)})$, the atom of μ_n in $B_{r'}(t_i, x^{i)}$ must be $(t_i^{(n)}, x_i^{(n)})$. Hence for any $n \ge N_i(r')$, $(t_i^{(n)}, x_i^{(n)}) \in B_{r'}(t_i, x_i)$. This finishes the proof of (5.3.4).

Note that $t_1, t_2, \dots, t_p$ are distinct since μ cannot have two atoms with the same time coordinate, by property 3) in Definition 5.3.2 of $M_{p,f}^*([0,T] \times \mathbb{D})$. Suppose that $t_1 < t_2 < \dots < t_p$. Pick $\delta_0 > 0$ such that $t_{i+1} - t_i > 2\delta_0$ for any $i = 1, \dots, p-1$. Let $\delta \in (0, \delta_0)$ be arbitrary. Then the intervals $(t_i - \delta, t_i + \delta)$ $i = 1, \dots, p$ are non overlapping.

Since $x_i^{(n)} \xrightarrow{J_1} x_i$ for all $i = 1, \dots, p$ and $\mathrm{Disc}(x_i) \cap \mathrm{Disc}(x_j) = \emptyset$ for all $i \ne j$, by Theorem 4.1 of [34],

$$\sum_{i=1}^{k} x_i^{(n)} \xrightarrow{J_1} \sum_{i=1}^{k} x_i \text{ for all } k \le p. \tag{5.3.5}$$

It follows that there exists $n_1(\delta) \ge N$ such that for all $n \ge n_1(\delta)$

$$\left| t_i^{(n)} - t_i \right| < \delta \text{ for all } i = 1, \dots, p \tag{5.3.6}$$

$$d_{J_1}^0 \left(\sum_{i=1}^{k} x_i^{(n)}, \sum_{i=1}^{k} x_i \right) < \delta \text{ for all } k = 1, \dots, p. \tag{5.3.7}$$

For any $n \ge N$ we consider the function $\lambda_n : [0,T] \to [0,T]$ such that: $\lambda_n(0) = 0, \lambda_n(T) = T, \lambda_n(t_i^{(n)}) = t_i$ for any $i = 1, \dots, p$ and λ_n is defined by linear interpolation between $t_i^{(n)}$ and $t_{i+1}^{(n)}$ for $i = 0, \dots, p$ where we let $t_0^{(n)} = 0$ and $t_{p+1}^{(n)} = T$.

Clearly, for any $t \in [0,T]$

$$\Phi(\mu_n)(t) = \sum_{t_i^{(n)} \le t} x_i^{(n)} \text{ and } \Phi(\mu)(t) = \sum_{t_i \le t} x_i. \tag{5.3.8}$$

Recalling the definition of the uniform distance (3.5.2) on $\mathbb{D}([0,T];\mathbb{D})$, we have

$$\begin{aligned}
\rho_{T,\mathbb{D}}(\Phi(\mu),\Phi(\mu_n)\circ\lambda_n^{-1}) &= \sup_{t\in[0,T]} d^0_{J_1}(\Phi(\mu)(t),\Phi(\mu_n)(\lambda_n^{-1}(t))) \\
&= \sup_{t\in[0,T]} d^0_{J_1}\Bigg(\sum_{t_i\le t} x_i, \sum_{t_i^{(n)}\le\lambda_n^{-1}(t)} x_i^{(n)}\Bigg) \\
&= \sup_{t\in[0,T]} d^0_{J_1}\Bigg(\sum_{t_i\le t} x_i, \sum_{\lambda_n(t_i^{(n)})\le t} x_i^{(n)}\Bigg) \\
&= \sup_{t\in[0,T]} d^0_{J_1}\Bigg(\sum_{t_i\le t} x_i, \sum_{t_i\le t} x_i^{(n)}\Bigg) \\
&= \max_{1\le k\le p} d^0_{J_1}\Bigg(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} x_i^{(n)}\Bigg) \qquad (5.3.9)\\
&< \delta, \qquad (5.3.10)
\end{aligned}$$

where for the last inequality we used (5.3.7). Note that (5.3.9) is justified by the fact that

$$\sum_{t_i\le t} x_i = \sum_{i=1}^{k} x_i \text{ if } t_k \le t < t_{k+1} \text{ for } k=0,\dots,p$$

$$\sum_{t_i\le t} x_i^{(n)} = \sum_{i=1}^{k} x_i^{(n)} \text{ if } t_k \le t < t_{k+1} \text{ for } k=0,\dots,p$$

with $t_0 = 0$ and $t_{p+1} = T$. Hence

$$d^0_{J_1}\Bigg(\sum_{t_i\le t} x_i^{(n)}, \sum_{t_i\le t} x_i\Bigg) = \begin{cases} 0 & \text{if } 0\le t_1 < t \\ d^0_{J_1}(x_1, x_1^{(n)}) & \text{if } t_1 \le t < t_2 \\ d^0_{J_1}(x_1+x_2, x_1^{(n)}+x_2^{(n)}) & \text{if } t_2\le t<t_3 \\ \vdots & \end{cases} \qquad (5.3.11)$$

By relation (7.20) of [28], we know that for any $n \ge n_1(\delta)$,

$$\|\lambda_n - e\|_T \le 3\delta. \qquad (5.3.12)$$

Then recalling definition (3.5.1) of the Skorohod distance $d_{T,\mathbb{D}}$ on $\mathbb{D}([0,T];\mathbb{D})$, we infer from (5.3.10) and (5.3.12) that for all $n > n_1(\delta)$

$$d_{T,\mathbb{D}}(\Phi(\mu),\Phi(\mu_n)) \le \|\lambda_n - e\|_T \vee \rho_{T,\mathbb{D}}(\Phi(\mu),\Phi(\mu_n)\circ\lambda_n^{-1}) \le 3\delta.$$

Since $\delta \in (0, \delta_0)$ is arbitrary, we conclude that $\Phi(\mu_n) \to \Phi(\mu)$ in $\mathbb{D}([0,T];\mathbb{D})$ equipped with the Skhorohod distance $d_{T,\mathbb{D}}$. Hence, Φ is continuous at μ. $\square$

The following corollary is an immediate consequence of Lemma 5.3.1 and Theorem 5.3.4.

Corollary 5.3.5. *The function $Q = \Phi \circ \Psi$ given by*

$$\begin{aligned} Q : \widehat{M}_p([0,\infty) \times \overline{\mathbb{D}}_0) &\to \mathbb{D}([0,T];\mathbb{D}) \\ m = \sum_{i \geq 1} \delta_{(t_i, r_i, z_i)} &\longmapsto \left(\sum_{t_i \leq t} r_i z_i 1_{\{r_i \in (\varepsilon, \infty)\}} \right)_{t \leq T} \end{aligned}$$

is continuous on the set $\mathcal{U} = \mathcal{A} \cap \Psi^{-1}\left(M^*_{p,f}([0,T] \times \mathbb{D})\right)$ where $\mathbb{D}([0,T];\mathbb{D})$ is equipped with the Skorohod distance $d_{T,\mathbb{D}}$. (The function $Q = Q_{\varepsilon,T}$ and the set $\mathcal{U} = \mathcal{U}_{\varepsilon,T}$ depend on ε and T. To simplify the writing, we omit the indices ε, T.)

Proof. Let $m \in \mathcal{U}$ and $(m_n)_n \subset \widehat{M}_p([0,\infty) \times \overline{\mathbb{D}_0})$ be such that $m_n \stackrel{\hat{w}}{\to} m$. We want to prove $Q(m_n) \to Q(m)$ in the space $\mathbb{D}([0,T];\mathbb{D})$ equipped with the distance $d_{T,\mathbb{D}}$. By Lemma 5.3.1, since $m \in \mathcal{A}$

$$\Psi(m_n) \stackrel{w}{\to} \Psi(m) \text{ as measures on } M_{p,f}([0,T] \times \mathbb{D}).$$

Then by Lemma 5.3.4, since $\Psi(m) \in M^*_{p,f}([0,T] \times \mathbb{D})$

$$\Phi(\Psi(m_n)) \to \Phi(\Psi(m)) \text{ on } \mathbb{D}([0,T];\mathbb{D})$$

where $\mathbb{D}([0,T];\mathbb{D})$ is equipped with distance $d_{T,\mathbb{D}}$. $\square$

Remark 5.3.6. The set $\mathcal{U}$ consists of measures $m = \sum_{i \geq 1} \delta_{(t_i, r_i, z_i)}$ which satisfy the following conditions :

1. $m([0,\infty) \times \{\varepsilon, \infty\} \times S_{\mathbb{D}}) = 0$;
2. $m(\{0,T\} \times (\varepsilon, \infty] \times S_{\mathbb{D}}) = 0$;
3. $\Psi(m) = \sum_{i \geq 1} \delta_{(t_i, r_i z_i)}(\cdot) 1_{\{t_i \leq T, r_i \in (\varepsilon, \infty)\}}$ is simple;
4. $\text{Disc}(z_i) \cap \text{Disc}(z_j) = \emptyset$ for any $i \neq j$ such that $t_i \leq T$, $r_i \in (\varepsilon, \infty)$, $t_j \leq T$ and $r_j \in (\varepsilon, \infty)$;
5. No vertical line contains two atoms of $\Psi(m)$, i.e. $\Psi(m)(\{t_0\} \times \mathbb{D}) \leq 1$ for any $t_0 \in [0,T]$.

5.4 Convergence of truncated sums

In this section, we consider a sequence $(X_i)_{i\geq 1}$ of i.i.d. regularly varying random elements in $\mathbb{D}$, and we prove that for any $\varepsilon > 0$ the sequence $(S_n^{(\varepsilon)})_{n\geq 1}$ of truncated sums defined by:

$$S_n^{(\varepsilon)}(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i 1_{\{\|X_i\|>a_n\varepsilon\}}, \quad \text{for any } t \geq 0 \tag{5.4.1}$$

converges in distribution in the space $\mathbb{D}([0,\infty);\mathbb{D})$ to the process $Z^{(\varepsilon)}$ given by (4.1.6).

The following result together with Corollary 5.3.5 will allow us to apply the continuous mapping theorem. Recall that Assumption B was stated at the beginning of Chapter 4.

Theorem 5.4.1. *Let N be a Poisson random measure on $[0,\infty)\times\overline{\mathbb{D}}_0$ of intensity* Leb $\times\,\overline{\nu}$*, where $\overline{\nu}$ is given by (5.2.7). If Γ_1 satisfies Assumption B, then*

$$N \in \mathcal{U}_{\varepsilon,T} \ \textit{a.s.}$$

for any $\varepsilon > 0$ and $T > 0$, where $\mathcal{U}_{\varepsilon,T}$ is the set given in Corollary 5.3.5.

Proof: We have to show that with probability 1, N satisfies the two conditions listed in Lemma 5.3.1, and $\xi = \Psi_{\varepsilon,T}(N) \in M_{p,f}^*([0,T]\times\mathbb{D})$.

We begin with the conditions of Lemma 5.3.1. For any $n \geq 1$, $E[N([n-1,n)\times\{\varepsilon,\infty\}\times\mathbb{S}_{\mathbb{D}})] = c\nu_\alpha(\{\varepsilon,\infty\}) = 0$ and hence $N([n-1,n)\times\{\varepsilon,\infty\}\times\mathbb{S}_{\mathbb{D}}) = 0$ a.s. By additivity, $N([0,\infty)\times\{\varepsilon,\infty\}\times\mathbb{S}_{\mathbb{D}}) = 0$ a.s. Similarly, $N(\{0,T\}\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}) = 0$ a.s. since $E(N(\{0,T\}\times(\epsilon,\infty)\times\mathbb{S}_{\mathbb{D}})) = \text{Leb}(\{0,T\})\nu_\alpha(\epsilon,\infty) = 0$.

Next, we show that with probability 1, ξ satisfies conditions *(i)-(iii)* given in Lemma 5.5.3. First, we show that ξ is a Poisson random measure on $[0,T]\times\mathbb{D}$ of intensity Leb $\times\,\nu^{(\varepsilon)}$ where $\nu^{(\varepsilon)} = \overline{\nu}|_{(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}}\circ U^{-1}$ and $U : (\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}} \to \mathbb{D}$ is given by $U(r,z) = rz$. Note that ξ is a point process since N is a point process and $\Psi_{\varepsilon,T}$ is measurable. So, it suffices to show that the Laplace functional of ξ is given by (5.1.3). Let $g : [0,T]\times\mathbb{D} \to [0,\infty)$ be a bounded measurable function with bounded support. By (5.3.1),

$$\begin{aligned}
L_\xi(g) &= E\left[\exp\left(-\int_{[0,T]\times\mathbb{D}} g d\xi\right)\right] = E\left[\exp\left(-\int_{[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} g(t,rz)N(dt,dr,dz)\right)\right] \\
&= \exp\left\{-\int_{[0,T]\times(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} (1-e^{-g(t,rz)})dt\overline{\nu}(dr,dz)\right\} \\
&= \exp\left\{-\int_{[0,T]\times\mathbb{D}} (1-e^{-g(t,x)})\, dt\, \overline{\nu}^{(\varepsilon)}(dx)\right\}.
\end{aligned}$$

Since $\text{Leb} \times \nu^{(\varepsilon)}$ is diffuse, ξ is simple a.s., so ξ satisfies condition *(i)* with probability 1.

To show that ξ satisfies condition *(ii)* with probability 1, we represent its points as follows. Let $P_i = c^{1/\alpha}\gamma_i^{-1/\alpha}$ where $\gamma_i = \sum_{j=1}^{i} E_j$ and $(E_i)_{i\geq 1}$ are i.i.d. exponential random variables of mean 1. Let $(W_i)_{i\geq 1}$ be an independent sequence of i.i.d. random elements in $\mathbb{S}_{\mathbb{D}}$ of law Γ_1. By the extension of Proposition 5.3 of [28] to Polish spaces, $\sum_{i\geq 1} \delta_{(P_i,W_i)}$ is a Poisson random measure on $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$ of intensity $\overline{\nu}$, and so, $\sum_{i\geq 1} \delta_{(P_i,W_i)} 1_{\{P_i>\varepsilon\}}$ is a Poisson random measure on $(\varepsilon,\infty) \times \mathbb{S}_{\mathbb{D}}$ of intensity $\overline{\nu}|_{(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}}$. By the extension of Proposition 5.2 of [28] to Polish spaces, $\sum_{i\geq 1} \delta_{P_i W_i} 1_{\{P_i>\varepsilon\}}$ is a Poisson random measure on $\mathbb{D}$ of intensity $\nu^{(\varepsilon)}$. Finally, by the extension of Proposition 5.3 of [28], $\xi' = \sum_{i\geq 1} \delta_{(\tau_i,P_i W_i)} 1_{\{P_i>\varepsilon\}}$ is a Poisson random measure on $[0,T] \times \mathbb{D}$ of intensity $\text{Leb} \times \nu^{(\varepsilon)}$, where $(\tau_i)_{i\geq 1}$ are i.i.d. uniformly distributed on $[0,T]$, independent of $(E_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$. Hence $\xi \stackrel{d}{=} \xi'$.

Consider the event $A = \cap_{i\neq j} A_{i,j}$, where $A_{i,j} = \{\text{Disc}(W_i) \cap \text{Disc}(W_j) = \emptyset\}$. Let $F = \{(x,y) \in \mathbb{S}_{\mathbb{D}} \times \mathbb{S}_{\mathbb{D}}; \text{Disc}(x) \cap \text{Disc}(y) \neq \emptyset\}$. By Fubini's theorem and Assumption B,

$$P(A_{i,j}^c) = P((W_i,W_j) \in F) = (\Gamma_1 \times \Gamma_1)(F) = \int_{\mathbb{S}_{\mathbb{D}}} \Gamma_1(F_x)\Gamma_1(dx) = 0,$$

where $F_x = \{y \in \mathbb{S}_{\mathbb{D}}; (x,y) \in F\} = \cup_{s\in \text{Disc}(x)} \{y \in \mathbb{S}_{\mathbb{D}}; s \in \text{Disc}(y)\}$. Hence, $P(A) = 1$.

Let B be the event on which $\xi(\{(t,x),(t',x')\}) \leq 1$ for all $(t,x),(t',x') \in [0,T]\times\mathbb{D}$ with $x \neq x'$ and $\text{Disc}(x) \cap \text{Disc}(x') \neq \emptyset$, and B' the similar event with ξ replaced by ξ'. Since $\xi \stackrel{d}{=} \xi'$, $P(B) = P(B')$. We claim that $A \subset B'$. $\big($To see this, let $\omega \in (B')^c$. Then, there exist $(t,x),(t',x') \in [0,T] \times \mathbb{D}$ with $x \neq x'$ and $\text{Disc}(x) \cap \text{Disc}(x') \neq \emptyset$ such that $\xi'(\omega; \{(t,x),(t',x')\}) \geq 2$. This means that both (t,x) and (t',x') are atoms of $\xi'(\omega)$. But the atoms of $\xi'(\omega)$ are of the form $(\tau_i(\omega), P_i(\omega)W_i(\omega))$ with $P_i(\omega) > \varepsilon$. Hence, there exist $i \neq j$ with $P_i(\omega) > \varepsilon$ and $P_j(\omega) > \varepsilon$ such that $(t,x) = (\tau_i(\omega), P_i(\omega)W_i(\omega))$ and $(t',x') = (\tau_j(\omega), P_j(\omega)W_j(\omega))$. This proves that $\omega \in A_{ij}^c \subset A^c$.$\big)$ Hence, $P(B) = P(B') = P(A) = 1$. This proves that ξ satisfies condition (ii) with probability 1.

Finally, to show that ξ satisfies condition (iii) with probability 1, we let $C = \cap_{i\neq j} C_{i,j}$, where $C_{i,j} = \{\tau_i \neq \tau_j\}$. Note that $P(C) = 1$ since for all $i \neq j$

$$P(C_{i,j}^c) = P(\tau_i = \tau_j) = \frac{1}{T^2} \int_0^T \int_0^T 1_{\{x=y\}} dx dy = 0.$$

Let D be the event on which $\xi(\{t_0\} \times \mathbb{D}) \leq 1$ for all $t_0 \in [0,T]$, and D' the similar event with ξ replaced by ξ'. Since $\xi \stackrel{d}{=} \xi'$, $P(D) = P(D')$. We claim that $C \subset D'$. $\big($To see this, let $\omega \in (D')^c$. Then there exists $t_0 \in [0,T]$ such that $\xi'(\omega; \{t_0\} \times \mathbb{D}) \geq 2$. This means that $\xi'(\omega)$ has at least two atoms with time coordinate t_0. Using the form of the atoms of $\xi'(\omega)$, we infer that there exist $i \neq j$ such that $\tau_i(\omega) = \tau_j(\omega) = t_0$.

This proves that $\omega \in C_{i,j}^c \subset C^c$.) Hence, $P(D) = P(D') = P(C) = 1$. This proves that ξ satisfies condition (iii) with probability 1. $\square$

The next result gives the convergence of the truncated sums of i.i.d. regulary varying elements in $\mathbb{D}$.

Theorem 5.4.2. *Let $(X_i)_{i\geq 1}$ be i.i.d random elements in $\mathbb{D}$ such that $X_1 \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$. Let α be the index of X and Γ_1 be the spectral measure of X. Suppose that $\alpha < 2$, $\alpha \neq 1$ and Γ_1 satisfies Assumption B. Let $\left\{S_n^{(\varepsilon)}(t)\right\}_{t\geq 0}$ be given by (5.4.1) and $Z^{(\varepsilon)}(\cdot)$ given by (4.1.6). Then for any $\varepsilon > 0$ and $T > 0$,*

$$S_n^{(\varepsilon)}(\cdot) \xrightarrow{d} Z^{(\varepsilon)}(\cdot) \text{ in } \mathbb{D}([0,T]\,;\mathbb{D})$$

where the space $\mathbb{D}([0,T]\,;\mathbb{D})$ is equipped with the metric $d_{T,\mathbb{D}}$ given by (3.3.1). Moreover $P(s \in \mathrm{Disc}(Z^{(\varepsilon)}(t))$ for some $t > 0) = 0$ for any $s \in [0,1]$.

Proof: By Proposition 5.1.2, with $E = \overline{\mathbb{D}}_0$ and $X_{i,n} = (\|X_i\|/a_n, X_i/\|X_i\|)$,

$$N_n = \sum_{i\geq 1} \delta_{\left(\frac{i}{n}, \frac{\|X_i\|}{a_n}, \frac{X_i}{\|X_i\|}\right)} \xrightarrow{d} N,$$

where N is a Poisson random measure on $[0,\infty) \times \overline{\mathbb{D}}_0$ of intensity Leb $\times\, \overline{\nu}$.

We apply the continuous mapping theorem with the map Q of Corollary 5.3.5. Note that $Q(N_n) = S_n^{(\varepsilon)}$ and $Q(N) = Z^{(\varepsilon)}$. Using Theorem 5.4.1, we obtain that $S_n^{(\varepsilon)} \xrightarrow{d} Z^{(\varepsilon)}$ in $\mathbb{D}([0,T];\mathbb{D})$, equipped with distance $d_{T,\mathbb{D}}$.

To prove the last statement, we fix $s \in [0,1]$ and we let $\Omega_T = \cup_{t\in[0,T]}\{s \in \mathrm{Disc}(Z^{(\varepsilon)}(t))\}$. It is enough to prove that $P(\Omega_T) = 0$ for all $T > 0$. From (4.1.6), we see that if W_i is continuous at s for all $i \geq 1$, then $Z^{(\varepsilon)}(t)$ is continuous at s for all $t \in [0,T]$. Hence, $\Omega_T \subset \cup_{i\geq 1}\{s \in \mathrm{Disc}(W_i)\}$. The fact that $P(\Omega_T) = 0$ follows by Assumption B, since $P(s \in \mathrm{Disc}(W_i)) = \Gamma_1\left(\{z \in \mathbb{S}_{\mathbb{D}}; s \in \mathrm{Disc}(z)\}\right) = 0$. $\square$

5.5 Approximation in the case $\alpha < 1$

In this section, we give the proof of Theorem 5.0.1.a). The first result shows that a certain asymptotic negligibility condition holds automatically in the case $\alpha < 1$.

Lemma 5.5.1. *Let $(X_i)_{i\geq 1}$ be i.i.d. random elements in $\mathbb{D}$ such that $X_1 \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$, Let α be the index of X. Suppose that $\alpha \in (0,1)$. Let $\{S_n^{(\varepsilon)}, n \geq 1\}$ be given by (5.4.1) and $S_n(t) = a_n^{-1}\sum_{i=1}^{[nt]} X_i$ for all $t \geq 0, n \geq 1$. Then for any $\delta > 0$ and $T > 0$,*

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P(\|S_n - S_n^{(\varepsilon)}\|_{T,\mathbb{D}} > \delta) = 0,$$

and in particular, $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P(d_{T,\mathbb{D}}(S_n, S_n^{(\varepsilon)}) > \delta) = 0$.

Proof: Let $\delta > 0$ and $T > 0$ be arbitrary. Since $S_n(t) - S_n^{(\varepsilon)}(t) = a_n^{-1}\sum_{i=1}^{[nt]} X_i 1_{\{|X_i\| \le a_n\varepsilon\}}$,

$$\|S_n - S_n^{(\varepsilon)}\|_{T,\mathbb{D}} = \frac{1}{a_n}\max_{k\le[nT]}\left\|\sum_{i=1}^{k} X_i 1_{\{\|X_i\|\le a_n\varepsilon\}}\right\| \le \frac{1}{a_n}\sum_{i=1}^{[nT]}\|X_i\|1_{\{\|X_i\|\le a_n\varepsilon\}}.$$

By Markov's inequality,

$$P(\|S_n - S_n^{(\varepsilon)}\|_{T,\mathbb{D}} > \delta) \le \frac{1}{\delta a_n}[nT]\, E(\|X_1\|1_{\{\|X_1\|\le a_n\varepsilon\}}).$$

Since $\|X_1\|$ is regularly varying of index $\alpha < 1$, $E(\|X_1\|1_{\{\|X_1\|\le x\}}) \sim \frac{\alpha}{1-\alpha}xP(\|X_1\| > x)$ as $x \to \infty$, by Karamata's theorem (e.g. Theorem 2.1 of [28]). Here $f(x) \sim g(x)$ as $x \to \infty$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. Hence, by (5.2.6),

$$\frac{n}{a_n}E(\|X_1\|1_{\{\|X_i\|\le a_n\varepsilon\}}) \sim \frac{\alpha}{1-\alpha}\varepsilon n P(\|X_1\| > a_n\varepsilon) \sim \frac{\alpha}{1-\alpha}\, c\,\varepsilon^{1-\alpha} \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\limsup_{n\to\infty} P(\|S_n - S_n^{(\varepsilon)}\|_{T,\mathbb{D}} > \delta) \le \frac{T}{\delta}\cdot\frac{\alpha}{1-\alpha}\, c\,\varepsilon^{1-\alpha}.$$

The conclusion follows letting $\varepsilon \downarrow 0$, and using the fact that $\alpha < 1$. □

The next result gives the proof of Theorem 5.0.1.a).

Theorem 5.5.2. *Let $(X_i)_{i\ge1}$ be i.i.d. random elements in $\mathbb{D}$ such that $X_1 \in RV(\{a_n\}, \overline{\nu}, \mathbb{D})$. Let α be the index of X and Γ_1 be the spectral measure of X. Suppose that $\alpha < 1$ and Γ_1 satisfies Assumptions A and B. For any $n \ge 1$, $t \ge 0$, let $S_n(t) = \{S_n(t,s)\}_{s\in[0,1]}$, where $S_n(t,s) = \sum_{i=1}^{[nt]} X_i(s)$ for $s \in [0,1]$. Let $\{\widetilde{Z}(t)\}_{t\ge0}$ be the process constructed in Theorem 4.0.2, which may not be defined on the same probability space as the sequence $(X_i)_{i\ge1}$. Then*

$$S_n(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad \textit{in} \quad \mathbb{D}([0,\infty);\mathbb{D}).$$

Proof: By Corollary 2.8 of [34], it is enough to prove that

$$S_n(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad \text{in} \quad \mathbb{D}([0,T];\mathbb{D})$$

for any $T > 0$. This follows by Theorem A.0.2, Appendix A (Theorem 4.2 of [5]), whose hypotheses are verified due to Theorem 4.2.5, Theorem 5.4.2 and Lemma 5.5.1. □

5.6 Approximation in the case $\alpha > 1$

In this section, we prove Theorem 5.b).

The following result is the counterpart of Lemma 5.5.1 for the case $\alpha > 1$.

Lemma 5.6.1. *Let $(X_i)_{i\geq 1}$ be i.i.d.random elements in $\mathbb{D}$ such that $X_1 \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$. Suppose that $\alpha \in (1,2)$, where α is the index of X_1. Let $\{S_n^{(\varepsilon)}, n \geq 1\}$ be given by (5.4.1). For any $t \geq 0$ and $n \geq 1$, let $S_n(t) = \sum_{i=1}^{[nt]} X_i/a_n$,*

$$\overline{S}_n^{(\varepsilon)}(t) = S_n^{(\varepsilon)}(t) - E[S_n^{(\varepsilon)}(t)] \quad \textit{and} \quad \overline{S}_n(t) = S_n(t) - E[S_n(t)].$$

If (5.6.6) holds for any $\delta > 0$ and $T > 0$, then for any $\delta > 0$ and $T > 0$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} P(\|\overline{S}_n - \overline{S}_n^{(\varepsilon)}\|_{T,\mathbb{D}} > \delta) = 0, \tag{5.6.1}$$

and in particular, $\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P(d_{T,\mathbb{D}}(\overline{S}_n, \overline{S}_n^{(\varepsilon)}) > \delta) = 0$.

Proof: Let

$$Y_{i,n} = a_n^{-1}\big(X_i 1_{\{\|X_i\| \leq a_n\varepsilon\}} - E(X_i 1_{\{\|X_i\|\leq a_n\varepsilon\}})\big).$$

Since $\overline{S}_n(t) - \overline{S}_n^{(\varepsilon)}(t) = \sum_{i=1}^{[nt]} Y_{i,n}$, then

$$\|\overline{S}_n - \overline{S}_n^{(\varepsilon)}\|_{T,\mathbb{D}} = \sup_{t\in[0,T]} \|\overline{S}_n(t) - \overline{S}_n^{(\varepsilon)}(t)\| = \max_{k\leq[nT]} \left\| \sum_{i=1}^{k} Y_{i,n} \right\|.$$

By Lévy-Octaviani inequality, which is valid for independent random elements in a normed space (see Proposition 1.1.1 of [22]), for any $\delta > 0$,

$$P(\|\overline{S}_n - \overline{S}_n^{(\varepsilon)}\|_{T,\mathbb{D}} > \delta) \leq 3 \max_{k\leq[nT]} P\left(\left\| \sum_{i=1}^{k} Y_{i,n} \right\| > \delta/3 \right).$$

The conclusion follows by (5.6.6). $\Box$

To deal with the centering constants, we need to use the fact that addition is continuous in the space $\mathbb{D}([0,T];\mathbb{D})$ equipped with the distance $d_{T,\mathbb{D}}$. To deduce this, we cannot simply apply Theorem 4.1 of [34] with $(S,m) = (\mathbb{D}, d_{J_1}^0)$, since we do not know if the relation $d_{J_1}^0(x+y, x'+y') \leq d_{J_1}^0(x,x') + d_{J_1}^0(y,y')$ holds for any $x, x', y, y' \in \mathbb{D}$, as required on p.78 of [34]. Although the general question of continuity of the addition on $\mathbb{D}([0,T];\mathbb{D})$ remains open, we were able to find a weaker version of this result which is sufficient for our purposes. This is contained in the lemma below.

Lemma 5.6.2. *Let $(f_n)_{n\geq 1} \subset \mathbb{D}$ and $f \in \mathbb{D}$ be such that $f_n \overset{J_1}{\to} f$. Consider $(y_n)_{n\geq 1} \subset \mathbb{D}([0,T];\mathbb{D})$ and $y \in \mathbb{D}([0,T];\mathbb{D})$ defined as follows: for any $t \in [0,T]$,*

$$y_n(t) = \frac{[nt]}{n} f_n \quad \text{and} \quad y(t) = tf. \tag{5.6.2}$$

Then $\rho_{T,\mathbb{D}}(y_n, y) \to 0$. Moreover, if f is continuous, then for any sequence $(x_n)_{n\geq 1} \subset \mathbb{D}([0,T];\mathbb{D})$ and $x \in \mathbb{D}([0,T];\mathbb{D})$ such that $d_{T,\mathbb{D}}(x_n, x) \to 0$, we have:

$$d_{T,\mathbb{D}}(x_n + y_n, x + y) \to 0. \tag{5.6.3}$$

Proof: We first prove that $\rho_{T,\mathbb{D}}(y_n, y) \to 0$. Since $f_n \overset{J_1}{\to} f$, there exists a sequence $(\rho_n)_{n\geq 1} \subset \Lambda$ such that $\|\rho_n\|^0 \to 0$ and $\|f_n - f \circ \rho_n\| \to 0$. Let $z_n(t) = \frac{[nt]}{n} f$. Let $\varepsilon > 0$ be arbitrary. Then, there exists N_ε such that for all $n \geq N_e$, $\|\rho_n\|^0 < \varepsilon$ and $\|f_n - f \circ \rho_n\| < \varepsilon/T$. Hence, for any $t \in [0,T]$ and $n \geq N_\varepsilon$, $\|y_n(t) - z_n(t) \circ \rho_n\| \leq t\|f_n - f \circ \rho_n\| < \varepsilon$ and

$$d^0_{J_1}(y_n(t), z_n(t)) \leq \|\rho_n\|^0 \vee \|y_n(t) - z_n(t) \circ \rho_n\| < \varepsilon.$$

On the other hand, there exists N'_ε such that, for any $t \in [0,T]$ and $n \geq N'_\varepsilon$,

$$d^0_{J_1}(z_n(t), y(t)) \leq \|z_n(t) - y(t)\| = \left|\frac{[nt]}{n} - t\right| \cdot \|f\| \leq \frac{1}{n}\|f\| < \varepsilon.$$

This shows that $\rho_{T,\mathbb{D}}(y_n, y) = \sup_{t\in[0,T]} d^0_{J_1}(y_n(t), y(t)) < 2\varepsilon$ for any $n \geq N_\varepsilon \vee N'_\varepsilon$.

We now prove (5.6.3). For any $t \in [0,T]$, we denote $x(t) = \{x(t,s)\}_{s\in[0,1]}$, and we use a similar notation for $y(t), x_n(t)$ and $y_n(t)$. Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists $\delta_\varepsilon \in (0, \varepsilon)$ such that for any $s, s' \in [0,1]$ with $|s - s'| < \delta_\varepsilon$,

$$|f(s) - f(s')| < \varepsilon. \tag{5.6.4}$$

Because $d_{T,\mathbb{D}}(x_n, x) \to 0$, there exists a sequence $(\lambda_n)_{n\geq 1} \subset \Lambda_T$ such that $\|\lambda_n - e\|_T \to 0$ and $\rho_{T,\mathbb{D}}(x_n \circ \lambda_n, x) \to 0$. Pick $0 < \eta_\varepsilon < \varepsilon \wedge \ln(\delta_\varepsilon + 1)$ arbitrary. Then, there exists $N^{(1)}_\varepsilon$ such that for any $n \geq N^{(1)}_\varepsilon$, $\sup_{t\in[0,T]} |\lambda_n(t) - t| < \varepsilon$ and $\sup_{t\in[0,T]} d^0_{J_1}(x_n(\lambda_n(t)), x(t)) < \eta_\varepsilon$. Using definition (3.1.1) of $d^0_{J_1}$, it follows that for any $n \geq N^{(1)}_\varepsilon$ and for any $t \in [0,T]$, there exists $\mu^{(n)}_t \in \Lambda$ such that $\|\mu^{(n)}_t\|^0 < \eta_\varepsilon$ and

$$\sup_{s\in[0,1]} |x_n\big(\lambda_n(t), \mu^{(n)}_t(s)\big) - x(t,s)| < \eta_\varepsilon. \tag{5.6.5}$$

By inequality (3.1.3) and the choice of η_ε, $\sup_{s\in[0,1]} |\mu^{(n)}_t(s) - s| < e^{\eta_\varepsilon} - 1 < \delta_\varepsilon$.

Note that $\|f_n - f\| \to 0$, since $f_n \overset{J_1}{\to} f$ and f is continuous. Hence, there exists $N^{(2)}_\varepsilon$ such that $\sup_{s\in[0,1]} |f_n(s) - f(s)| < \varepsilon$ for any $n \geq N^{(2)}_\varepsilon$. By (5.6.4), for any $n \geq N^{(1)}_\varepsilon \vee N^{(2)}_\varepsilon$,

$$|f_n(\mu^{(n)}_t(s)) - f(s)| \leq |f_n(\mu^{(n)}_t(s)) - f(\mu^{(n)}_t(s))| + |f(\mu^{(n)}_t(s)) - f(s)| < 2\varepsilon.$$

Choose $N_\varepsilon^{(0)}$ such that $1/n < \varepsilon$ for any $n \geq N_\varepsilon^{(0)}$. Then, for any $n \geq N_\varepsilon^{(0)}$ and $t \in [0, T]$,

$$\left|\frac{[n\lambda_n(t)]}{n} - t\right| \leq \left|\frac{[n\lambda_n(t)]}{n} - \lambda_n(t)\right| + |\lambda_n(t) - t| \leq \frac{1}{n} + \varepsilon < 2\varepsilon.$$

Since $\|f_n - f\| \to 0$, it follows that $C := \sup_{n\geq 1} \|f_n\| < \infty$. Let $N_\varepsilon = N_e^{(0)} \vee N_e^{(1)} \vee N_e^{(2)}$. Using the definitions of y_n and y, it follows that for any $n \geq N_\varepsilon$, $t \in [0, T]$ and $s \in [0, 1]$,

$$\begin{aligned} |y_n\big(\lambda_n(t), \mu_t^{(n)}(s)\big) - y(t, s)| &\leq \left|\frac{[n\lambda_n(t)]}{n} - t\right| |f_n(\mu_t^{(n)}(s))| + t|f_n(\mu_t^{(n)}(s)) - f(s)| \\ &< 2\varepsilon(C + T). \end{aligned}$$

and hence, by (5.6.5),

$$|(x_n + y_n)\big(\lambda_n(t), \mu_t^{(n)}(s)\big) - (x + y)(t, s)| < \eta_\varepsilon + 2\varepsilon(C + T) < \varepsilon[1 + 2(C + T)].$$

To summarize, we have proved that for any $n \geq N_\varepsilon$, and $t \in [0, T]$, there exists $\mu_t^{(n)} \in \Lambda$ such that $\|\mu_t^{(n)}\|^0 < \eta_\varepsilon < \varepsilon$ and $\|(x_n+y_n)(\lambda_n(t))\circ\mu_t^{(n)} - (x+y)(t)\| < \varepsilon[1+2(C+T)]$. By definition (3.1.1) of $d_{J_1}^0$, this implies that for any $n \geq N_\varepsilon$ and $t \in [0, T]$,

$$d_{J_1}^0\big((x_n + y_n)(\lambda_n(t)), (x + y)(t)\big) < \varepsilon[1 + 2(C + T)].$$

Therefore, for any $n \geq N_\varepsilon$

$$\rho_{T,\mathbb{D}}((x_n + y_n) \circ \lambda_n, x + y) = \sup_{t\in[0,T]} d_{J_1}^0\big((x_n + y_n)(\lambda_n(t)), (x + y)(t)\big) < \varepsilon[1 + 2(C + T)].$$

Since $\|\lambda_n - \varepsilon\|_T < \varepsilon$, using definition (3.5.1) of $d_{T,\mathbb{D}}$, we conclude that $d_{T,\mathbb{D}}(x_n + y_n, x + y) < \varepsilon[1 + 2(C + T)]$ for any $n \geq N_\varepsilon$. $\square$

Remark 5.6.3. In the proof of Theorem 2.12 of [29], it was shown that, in a more general context than here, the function $s \mapsto E[Z^{(\varepsilon)}(1, s)]$ is continuous on $[0, 1]$. In our case, $E[Z^{(\varepsilon)}(1, s)] = c\varphi(s) \int_\varepsilon^\infty r\nu_\alpha(dr)$, and the continuity of φ can be proved directly as follows. By the dominated convergence theorem, φ is a càdlàg function. To show that φ is left-continuous, note that for any $s \in [0, 1]$,

$$\varphi(s) - \varphi(s-) = \int_{\mathbb{S}_\mathbb{D}} \big(z(s) - z(s-)\big)\Gamma_1(dz) = \int_{\{z\in\mathbb{S}_\mathbb{D}; z\{s\}>0\}} z\{s\}\Gamma_1(dz),$$

where $z\{s\} = z(s) - z(s-)$ is the jump of $z \in \mathbb{S}_\mathbb{D}$ at s. By Assumption B, the set in the last integral above has Γ_1-measure 0, and hence this integral is equal to 0.

Theorem 5.6.4. *Let $(X_i)_{i\geq 1}$ be i.i.d. random elements in $\mathbb{D}$ such that $X_1 \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$. Let α be the index of X and Γ_1 be the spectral measure of X. Suppose that $\alpha \in (1,2)$ and Γ_1 satisfies Assumption B. Let $\{S_n^{(\varepsilon)}, n \geq 1\}$ and $Z^{(\varepsilon)}$ be given by (5.4.1),* (4.1.6) *respectively. For any $t \geq 0$, let $\overline{S}_n^{(\varepsilon)}(t) = S_n^{(\varepsilon)}(t) - E[S_n^{(\varepsilon)}(t)]$ and $\overline{Z}^{(\varepsilon)}(t) = Z^{(\varepsilon)}(t) - E[Z^{(\varepsilon)}(t)]$.*

Then, for any $\varepsilon > 0$ and $T > 0$

$$\overline{S}_n^{(\varepsilon)}(\cdot) \xrightarrow{d} \overline{Z}^{(\varepsilon)}(\cdot) \quad in \quad \mathbb{D}([0,T];\mathbb{D}),$$

where $\mathbb{D}([0,T];\mathbb{D})$ is equipped with distance $d_{T,\mathbb{D}}$.

Proof: Let $X_n = S_n^{(\varepsilon)}$ and $X = Z^{(\varepsilon)}$. For any $t \geq 0$ and $s \in [0,1]$,

$$y_n(t,s) := -E[S_n^{(\varepsilon)}(t,s)] = -\frac{[nt]}{a_n} E[X_1(s) 1_{\{\|X_1\| > a_n\varepsilon\}}] = \frac{[nt]}{n} f_n(s),$$

with $f_n(s) = -\frac{n}{a_n} E[X_1(s) 1_{\{\|X_1\| > a_n\varepsilon\}}]$, and

$$y(t,s) := -E[Z^{(\varepsilon)}(t,s)] = -tc \int_{(\varepsilon,\infty)\times\mathbb{S}_{\mathbb{D}}} r z(s) \nu_\alpha(dr)\Gamma_1(dz) = t f(s),$$

with $f(s) = -c\frac{\alpha}{\alpha-1}\varepsilon^{1-\alpha}\varphi(s)$ and $\varphi(s) = \int_{\mathbb{S}_D} z(s)\Gamma_1(dz)$. This shows that the functions $(y_n)_{n\geq 1}$ and y are of the same form as in (5.6.2). By Remark 5.6.3, φ is continuous on $[0,1]$.

By Theorem 5.4.2, $X_n \xrightarrow{d} X$ in the space $\mathbb{D}([0,T];\mathbb{D})$ equipped with $d_{T,\mathbb{D}}$. Since this space is separable (by Theorem 3.2.6), by Skorohod's embedding theorem (Theorem 6.7 of [6]), there exist random elements $(X'_n)_{n\geq 1}$ and X' defined on a probability space $(\Omega', \mathcal{F}', P')$ such that $X'_n \stackrel{d}{=} X_n$ for all n, $X' \stackrel{d}{=} X'$ and $d_{T,\mathbb{D}}(X'_n, X') \to 0$ a.s. By Lemma 5.6.2, it follows that

$$d_{T,\mathbb{D}}(X'_n + y_n, X' + y) \to 0 \quad \text{a.s.}$$

This implies that $d_{T,\mathbb{D}}(X'_n + y_n, X' + y) \to 0$ in probability (and in distribution). By Corollary to Theorem 3.1 of [6] (and using again the fact that $\mathbb{D}([0,T];\mathbb{D})$ equipped with $d_{T,\mathbb{D}}$ is a separable space), we infer that $X'_n + y_n \xrightarrow{d} X' + y$ in $\mathbb{D}([0,T];\mathbb{D})$ equipped with $d_{T,\mathbb{D}}$. Since $(y_n)_{n\geq 1}$ and y are deterministic, $X_n + y_n \stackrel{d}{=} X'_n + y_n$ for any n, and $X + y \stackrel{d}{=} X + y$. It follows that $X_n + y_n \xrightarrow{d} X + y$ in $\mathbb{D}([0,T];\mathbb{D})$ equipped with $d_{T,\mathbb{D}}$. $\square$

The following result proves Theorem 5.0.2.(b)

Theorem 5.6.5. *Let $X, (X_i)_{i\geq 1}$ be i.i.d. regulary varying random elements in $\mathbb{D}$. Let $\alpha > 1$ be the index of stability of X and Γ_1 be the spectral measure of X. Suppose that* *the probability measure Γ_1 given by (5.2.5) satisfies Assumptions A and B. For any $n \geq 1$, $t \geq 0$, let $S_n(t) = \{S_n(t,s)\}_{s \in [0,1]}$, where $S_n(t,s) = \sum_{i=1}^{[nt]} X_i(s)$ for $s \in [0,1]$. Let $\{\widetilde{Z}(t)\}_{t \geq 0}$ be the process constructed in Theorem 4.0.2, which may not be defined on the same probability space as the sequence $(X_i)_{i \geq 1}$. Let $\overline{S}_n(t) = S_n(t) - E[S_n(t)]$, where $E[S_n(t)] = \{E[S_n(t,s)]\}_{s \in [0,1]}$. If*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \max_{k \leq [nT]} P\left(\left\| \sum_{i=1}^{k} \left(X_i 1_{\{\|X_i\| \leq a_n \varepsilon\}} - E[X_i 1_{\{\|X_i\| \leq a_n \varepsilon\}}] \right) \right\| > a_n \delta \right) = 0 \quad (5.6.6)$$

for any $\delta > 0$ and $T > 0$, then

$$\overline{S}_n(\cdot) \xrightarrow{d} \widetilde{Z}(\cdot) \quad in \quad \mathbb{D}([0,\infty);\mathbb{D}).$$

Proof: This follows by Theorem 4.2 of [5] whose hypotheses are verified due to Theorem 4.3.8, Lemma 5.6.1 and Theorem 5.6.4. $\square$

Chapter 6

Simulations

In this chapter, we simulate an approximation of the sample paths. the sample paths of a $\mathbb{D}$-valued α-stable Lévy motion using Theorem 5.0.1, by focusing on two examples of a regularly varying process X in $\mathbb{D}$.

6.1 The first example

The simplest example of a regularly varying process $X = \{X(s)\}_{s\in[0,1]}$ in $\mathbb{D}$ is the α-stable Lévy motion, which can be simulated using the stable central limit theorem. We recall briefly this result below. Let $\xi, (\xi_j)_{j\geq 1}$ be i.i.d. regularly varying random variables in $\mathbb{R}$, i.e.

$$P(|\xi| > x) = x^{-\alpha}L(x) \quad \text{and} \quad \lim_{x\to\infty} \frac{P(\xi > x)}{P(|\xi| > x)} = p, \tag{6.1.1}$$

for some $\alpha \in (0, 2)$, $p \in [0, 1]$ and a slowly varying function L. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers with $a_n \uparrow \infty$ such that $nP(|\xi| > a_n) \to 1$ as $n \to \infty$, i.e. $a_n^\alpha \sim nL(a_n)$ as $n \to \infty$. Condition (6.1.1) is equivalent to the vague convergence $nP(\xi/a_n \in \cdot) \stackrel{v}{\to} \nu_{\alpha,p}$ in $\overline{\mathbb{R}}_0$, where

$$\nu_{\alpha,p}(dz) = \big(p\alpha z^{-\alpha-1}1_{(0,\infty)}(z) + q\alpha(-z)^{-\alpha-1}1_{(-\infty,0)}(z)\big)dz \tag{6.1.2}$$

with $q = 1 - p$. In other words, for any $x > 0$,

$$\lim_{n\to\infty} nP\left(\frac{\xi}{a_n} > x\right) = px^{-\alpha} \quad \text{and} \quad \lim_{n\to\infty} nP\left(\frac{\xi}{a_n} < -x\right) = qx^{-\alpha}.$$

In this case, we write $\xi \in RV(\{a_n\}, \nu_{\alpha,p}, \overline{\mathbb{R}}_0)$. In particular, if

$$\lim_{x\to\infty} L(x) = C > 0, \tag{6.1.3}$$

then $a_n^\alpha \sim Cn$. We assume that $\alpha \neq 1$. Let $\mu = 0$ if $\alpha < 1$ and $\mu = E(\xi)$ if $\alpha > 1$. A classical result, which can be deduced for instance from Theorem 2.7 of [33], states that

$$\frac{1}{a_n}\sum_{j=1}^{[n\cdot]}(\xi_j - \mu) \xrightarrow{d} X(\cdot) \quad \text{in} \quad \mathbb{D} \tag{6.1.4}$$

where $X = \{X(s)\}_{s\in[0,1]}$ is an α-stable Lévy motion, with $X(1)$ having a $S_\alpha(\sigma_\alpha, \beta, 0)$-distribution. Here $\sigma_\alpha^\alpha = C_\alpha^{-1}$ with C_α given by (2.2.5), and $\beta = p - q$. By Property 1.2.15 of [30], $\lim_{x\to\infty} x^\alpha P(X(1) > x) = p$ and $\lim_{x\to\infty} x^\alpha P(X(1) < -x) = q$. If L satisfies (6.1.3), this implies that $X(1) \in RV(\{a_n\}, C\nu_{\alpha,p}, \overline{\mathbb{R}}_0)$, since

$$nP\left(\frac{X(1)}{a_n} > x\right) = (na_n^{-\alpha}) \cdot (a_n x)^\alpha P(X(1) > a_n x) \cdot x^{-\alpha} \to Cpx^{-\alpha}$$

as $n \to \infty$, and similarly, $nP\left(\frac{X(1)}{a_n} < -x\right) \to Cqx^{-\alpha}$. By Lemma 2.1 of [18], it follows that $X \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$ for a boundedly finite measure $\overline{\nu}$ on $\overline{\mathbb{D}}_0$. Note that the normalizing sequence $\{a_n\}_n$ for the regular variation of X in $\mathbb{D}$ is *the same* as for ξ, if L satisfies (6.1.3). In the simulations, we take $a_n = (Cn)^{1/\alpha}$, where C is given by (6.1.3).

In view of (6.1.4), for any $s \in [0,1]$, $X(s) \approx \frac{1}{a_m}\sum_{j=1}^{[ms]}(\xi_j - \mu)$, when m is large.

Next, we consider n i.i.d. copies of X. For this, let $(\xi_{ij})_{i,j\geq 1}$ be i.i.d. copies of ξ. When m is large, we have the following approximations for any $s \in [0,1]$:

$$X_i(s) \approx \frac{1}{a_m}\sum_{j=1}^{[ms]}(\xi_{ij} - \mu), \quad \text{for all} \quad i = 1, \ldots, n.$$

By Theorem 5.5.2, the following approximation gives a $\mathbb{D}$-valued α-stable Lévy motion Z:

$$Z(t,s) \approx \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i(s) \approx \frac{1}{a_n a_m}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ms]}(\xi_{ij} - \mu),$$

for any $t, s \in [0,1]$, when n and m are large. (By Theorem C.0.12 below, this approximation yields in fact an *α-stable Lévy sheet*, which is an example of a $\mathbb{D}$-valued α-stable Lévy motion, according to Theorem C.0.11 below.)

We consider 6 examples of regularly varying random variables ξ which satisfy (6.1.3):

(i) $\xi \sim$ Pareto(α), i.e. ξ has density $f(x) = \alpha x^{-\alpha-1}$ if $x > 1$; then $L(x) = 1$;

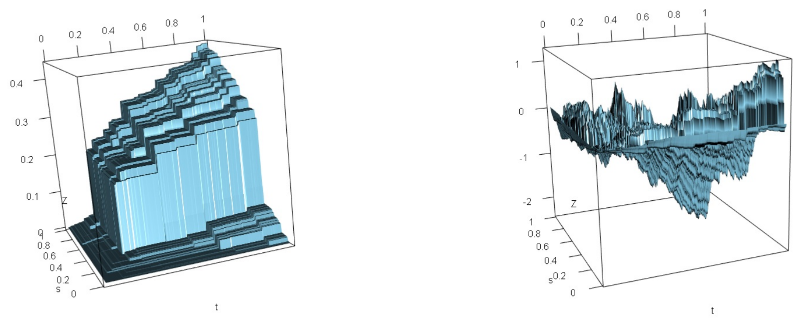
(ii) ξ has a two-sided Pareto distribution, i.e. ξ has density given by $f(x) = p\alpha x^{-\alpha-1}$ if $x > 1$ and $f(x) = q\alpha(-x)^{-\alpha-1}$ if $x < -1$, for $p \in (0,1)$ and $q = 1 - p$; then $L(x) = 1$;

(iii) $\xi \sim$ Fréchet(α), i.e. ξ has density $f(x) = \alpha x^{-\alpha-1}e^{-x^{-\alpha}}$ if $x > 0$; then $L(x) = x^{\alpha}(1 - e^{-x^{-\alpha}}) \to 1$ as $x \to \infty$;

(iv) $\xi \sim$ Burr(a, b) with $a, b > 0$, i.e. ξ has density $f(x) = abx^{b-1}(1 + x^b)^{-a-1}$ for $x > 0$; in this case $\alpha = ab$ and $L(x) = (1 + x^{-b})^a \to 1$ as $x \to \infty$;

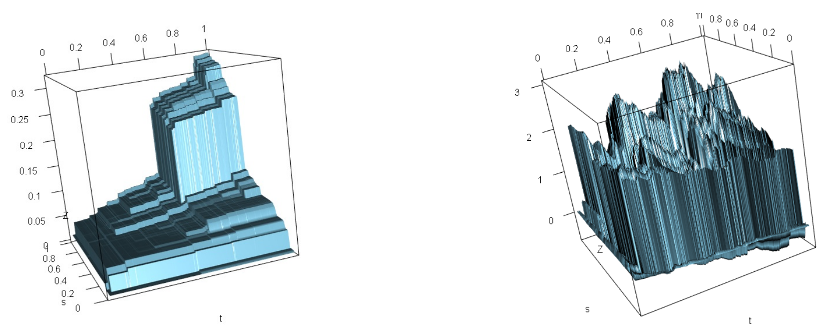
(v) $\xi \sim S_{\alpha}(\sigma, \beta, \mu)$; in this case $L(x) \to C := C_{\alpha}\sigma^{\alpha}$ as $x \to \infty$.

The following pictures are the 3-dimensional plots of $(t_k, s_l, Z(t_k, s_l))$ for $k = 1, \ldots, n$ and $l = 1, \ldots, m$, with $t_k = k/n$ and $s_l = l/m$, when $n = 400$ and $m = 250$.



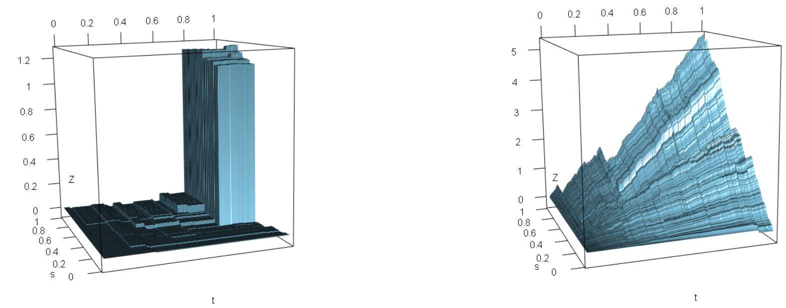
(a) $\alpha = 0.5$ (b) $\alpha = 1.5$

Figure 6.1: α-stable Lévy sheet based on Pareto distribution



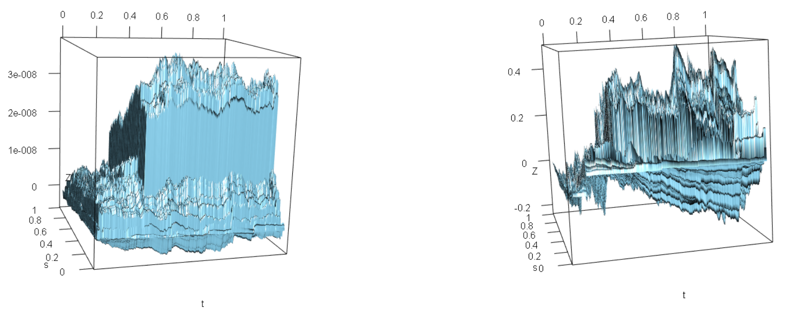
(a) $\alpha = 0.5$ (b) $\alpha = 1.5$

Figure 6.2: α-stable Lévy sheet based on Fréchet distribution



(a) $\alpha = 0.5$ (b) $\alpha = 1.5$

Figure 6.3: α-stable Lévy sheet based on stable distribution



(a) $\alpha = 0.5$ (b) $\alpha = 1.5$

Figure 6.4: α-stable Lévy sheet based on Burr distribution

6.2 The second example

In this example, $X = \{X(s)\}_{s\in[0,1]}$ is a regularly varying random element in $\mathbb{D}$ given by a series, as explained in Example 4.1 of [9]. Let $Y, (Y_j)_{j\geq 1}$ be i.i.d. random elements in the space $\mathbb{C} = \mathbb{C}([0,1])$ of continuous functions on $[0,1]$, such that

$$0 < C_{Y,\alpha} := E\Big(\sup_{s\in[0,1]} |Y(s)|^{\alpha}\Big) < \infty \tag{6.2.1}$$

for some $\alpha \in (0,2)$. Let $(\varepsilon_j)_{j\geq 1}$ be i.i.d. random variables which take values 1 and -1 with probability $1/2$, and $\Gamma_j = \sum_{i=1}^{j} E_i$ where $(E_i)_{i\geq 1}$ are i.i.d. exponential random variables of mean 1. Assume that $(Y_j)_{j\geq 1}, (\varepsilon_j)_{j\geq 1}$ and $(E_j)_{j\geq 1}$ are independent. By Theorem 1.4.2 of [30], for any $s \in [0,1]$, the series

$$X(s) = \sum_{j\geq 1} \varepsilon_j \Gamma_j^{-1/\alpha} Y_j(s) \quad \text{converges a.s.} \tag{6.2.2}$$

and has a $S_\alpha(\sigma_s, 0, 0)$-distribution, with $\sigma_s^{\alpha} = C_{\alpha}^{-1} E|Y(s)|^{\alpha}$ and C_α given by (2.2.5). Moreover, the process $X = \{X(s)\}_{s\in[0,1]}$ has sample paths in $\mathbb{C}$, and is regularly varying in $\mathbb{D}$. More precisely, $X \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$ with sequence $(a_n)_n$ chosen such that $a_n^\alpha \sim nC_{Y,\alpha}$, and limiting measure $\overline{\nu}$ specified by (4.3) of [9].

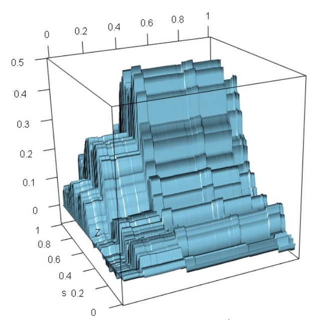
In the simulation below, we truncate the series in (6.2.2) by considering only the first K terms (for K large), and we take $Y = W$ where $W = \{W(s)\}_{s\in[0,1]}$ is the Brownian motion. (The fact that W satisfies condition (6.2.1) is proved in Appendix A.) We simulate K i.i.d. copies of W using Donsker theorem. Let $\xi, (\xi_{jk})_{j,k\geq 1}$ be i.i.d. random variables with mean 0 and variance 1. When m is large, $W_j(s) \approx \frac{1}{\sqrt{m}} \sum_{k=1}^{[ms]} \xi_{jk}$ for any $j = 1, \ldots, K$, and $X(s) \approx \sum_{j=1}^{K} \varepsilon_j \Gamma_j^{-1/\alpha} W_j(s) \approx \frac{1}{\sqrt{m}} \sum_{j=1}^{K} \sum_{k=1}^{[ms]} \varepsilon_j \Gamma_j^{-1/\alpha} \xi_{jk}$ for any $s \in [0, 1]$.

Next, we consider n i.i.d. copies of X. Let $(\varepsilon_{ij})_{i,j\geq 1}$ be i.i.d. copies of ε_1, $(E_{ij})_{i,j\geq 1}$ i.i.d. copies of E_1 and $(\xi_{ijk})_{i,j,k\geq 1}$ i.i.d. copies of ξ. Let $\Gamma_{ij} = \sum_{k=1}^{j} E_{ik}$. We take $a_n = (nC_{W,\alpha})^{1/\alpha}$ where $C_{W,\alpha}$ is computed by approximation. By Theorem 5.5.2,

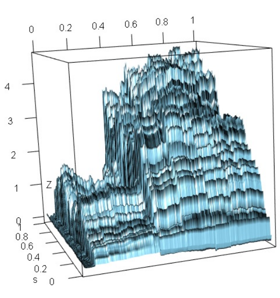
$$Z(t, s) \approx \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i(s) \approx \frac{1}{a_n\sqrt{m}} \sum_{i=1}^{[nt]} \sum_{j=1}^{K} \sum_{k=1}^{[ms]} \varepsilon_{ij} \Gamma_{ij}^{-1/\alpha} \xi_{ijk}$$

is an approximation of a $\mathbb{D}$-valued α-stable Lévy motion, when n, m and K are large.

The following pictures are the 3-dimensional plots of $(t_k, s_l, Z(t_k, s_l))$ for $k = 1, \ldots, n$ and $l = 1, \ldots, m$, with $t_k = k/n$ and $s_l = l/m$, when $n = 400$ and $m = 250$.



(a) $\alpha = 0.5$



(b) $\alpha = 1.5$

Figure 6.5: $\mathbb{D}$-valued α-stable Lévy motion based on a regularly process in $\mathbb{D}$ given by series (6.2.2) in which $(Y_j)_{j\geq 1}$ are i.i.d. Brownian motions

Chapter 7

Conclusion

The study of the asymptotic behaviour of the sum or maximum of regularly varying variables is a classical problem, with applications in finance and environmental studies. In the case of independent observations, under suitable normalization, these converge respectively to a stable distribution, and a Fréchet distribution. A method which is used for proving both results relies on the convergence of a sequence of point processes.

In chapter 3, we introduce the space $\mathbb{D}([0,1]\,;\mathbb{D})$ of functions defined on $[0,1]$ with values in the Skorohod space $\mathbb{D}$, which are right-continuous and have left limits with respect to the J_1 topology. This space is equipped with a Skorohod-type distance. Following the classical approach of [5, 6], we give several criteria for tightness of probability measures on this space, by characterizing the relatively compact subsets of this space. In particular, one of this criteria is used for proving the existence of a $\mathbb{D}$-valued α-stable Lévy motion. Finally, we give a criterion for weak convergence of random elements in $\mathbb{D}([0,1]\,;\mathbb{D})$, and a criterion for the existence of a process with sample paths in $\mathbb{D}([0,1]\,;\mathbb{D})$ based on its finite-dimensional distributions.

The first goal of this thesis was to give a construction of an α-stable process Z with values in the Skorohod space $\mathbb{D}$ of cadlag functions defined on the interval $[0,1]$, i.e. right continuous functions with left limits. The process Z shares many properties with the classical α-stable Lévy motion (such as self-similarity and stationarity and independence of increments), but takes values in the infinite-dimensional space $\mathbb{D}$. This task was a main objective of the thesis and has been completed in Chapter 4.

The second goal of this thesis was to shows that the process Z can also be obtained as the limit (in distribution) of the partial sum sequence associated with independent identically distributed regularly varying elements in $\mathbb{D}$, with suitable normalization and centering. This result is a functional counterpart of the stable CLT obtained recently by Roueff and Soulier [29] for random elements in $\mathbb{D}$, and can be viewed as an infinite-dimensional extension of the classical stable functional central limit theorem (FCLT).This goal has been achieved in Chapter 5. For the proofs of these results, I used techniques from the theory of point processes, weak convergence and tightness of probability measures on arbitrary metric spaces, combined with a recent version of $It\widehat{o}$-Nisio theorem for random elements in $\mathbb{D}$. First, the mathematical techniques needed to construct the approximations have been discussed, then the proofs showing the convergence of the approximations to the actual processes have been provided.

The research contained in this thesis can be extended in various directions. Some of these are:

1. Using the stable functional central limit theorem (FCLT) of Chapter 5 of the thesis in some real applications.

2. Develop an explicit construction for the α-stable Lévy process motion with values in $\mathbb{D}([0;1])$ and a stable functional central limit theorem (FCLT) for the case $\alpha = 1$.

3. Develop a max-stable limit theorem instead of working with the sum in the case of i.i.d. regularly varying random variables with values in $\mathbb{D}$.

Appendix A

Some fundamental results

Lemma A.0.1. *(Lemma 14.11 of [31]) For* $0 < \alpha < 1$

$$\int_0^\infty \Big(\exp(ir) - 1\Big) r^{-1-\alpha} dr = \Gamma(-\alpha)\exp(i\frac{\pi\alpha}{2}), \tag{A.0.1}$$

for $1 < \alpha < 2$

$$\int_0^\infty \Big(\exp(ir) - 1 - ir\Big) r^{-1-\alpha} dr = \Gamma(-\alpha)\exp(i\frac{\pi\alpha}{2}), \tag{A.0.2}$$

and for $z > 0$

$$\int_0^\infty \Big(\exp(izr) - 1 - izr1_{[0,1]}(r)\Big) r^{-2} dr = -\frac{\pi z}{2} - iz\ln(z) + iaz \tag{A.0.3}$$

where

$$a = \int_0^\infty \Big(sin(r) - r\mathbf{1}_{[0,1]}(r)\Big) r^{-2} dr.$$

Theorem A.0.2 (Theorem 4.2 of [5])**.** *Suppose that* $\Big\{X_n^{(k)}, X^{(k)}, Y_n, X; n \geq 1, k \geq 1\Big\}$ *are random elements of metric space* (S, d) *and are defined on the same probability space. Assume that for each* $k \geq 1$,

$$X_n^{(k)} \xrightarrow{d} X^{(k)} \text{ as } n \to \infty$$

and

$$X^{(k)} \xrightarrow{d} X \text{ as } k \to \infty$$

Suppose further that for all $\delta > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[d(X_n^{(k)}, Y_n) > \delta\right] = 0.$$

Then

$$Y_n \xrightarrow{d} X \text{ as } n \to \infty$$

Lemma A.0.3. *Let N be a Poisson random measure on E of intensity ν and f, g two positives measurable functions with disjoint supports, Then $N(f)$ and $N(g)$ are two independent random variables, where $N(f) = \int_E f dN$ and $N(g) = \int_E g dN$.*

Lemma A.0.4. *Let τ be a Poisson random variable of mean $\lambda > 0$. Let $(X_i)_{i\geq 1}$ iid random vectors in $\mathbb{R}^m$ with law F. Then $X = \sum_{i=1}^{\tau} X_i$ has a compound Poisson distribution with characteristic function*

$$E(\exp(iu \cdot X)) = \exp\left\{\lambda \int_{\mathbb{R}^n} \big(\exp(iu\cdot x) - 1\big) F(dx)\right\} \ , \ u \in \mathbb{R}^m.$$

In particular, if $m = 1$, $E(X) = \lambda \int_{\mathbb{R}} xF(dx)$ and $\mathrm{Var}(X) = \lambda \int_{\mathbb{R}} x^2 F(dx)$

Theorem A.0.5. *Let $W = \{W(s)\}_{s\in[0,1]}$ be the Brownian motion. Then,*

$$E\Big(\sup_{s\in[0,1]} |W(s)|^{\alpha}\Big) < \infty \quad for\ all\ \alpha \in (0, 2).$$

Proof: We give an indirect argument, based on Donsker's theorem. Let $X, (X_i)_{i\geq 1}$ be i.i.d. random variables with $E(X) = 0$ and $E(X^2) = 1$. By Donsker's theorem, $n^{-1/2} \sum_{i=1}^{[n\cdot]} X_i \xrightarrow{d} W(\cdot)$ in $\mathbb{C}([0, 1])$. Since the supremum is continuous on $\mathbb{C}([0, 1])$,

$$M_n := \frac{1}{\sqrt{n}} \sup_{s\in[0,1]} |\sum_{i=1}^{[ns]} X_i| \xrightarrow{d} M := \sup_{s\in[0,1]} |W(s)|.$$

By Skorokhod's representation theorem, there exist random variables $(M_n')_n$ and M' (defined on another probability space), such that $M_n \stackrel{d}{=} M_n'$ for all n, $M \stackrel{d}{=} M'$ and $M_n' \to M'$ a.s. as $n \to \infty$. Hence $M_n'^{\alpha} \to M'^{\alpha}$ a.s. We will prove below that $(M_n'^{\alpha})_{n\geq 1}$ is uniformly integrable. Then, by Theorem 16.14 of [7], we infer that $E(M'^{\alpha}) < \infty$. This concludes the proof, since $E(M^{\alpha}) = E(M'^{\alpha})$.

To show the uniform integrability of $(M_n'^{\alpha})_{n\geq 1}$, it is enough to prove that $\sup_n E(M_n'^{\alpha p}) < \infty$ for some $p > 1$, or equivalently

$$\sup_{n\geq 1} E(M_n^{\alpha p}) < \infty \quad \text{for some } p > 1. \tag{A.0.4}$$

To show (A.0.4), let $q = \alpha p$. Note that $E(M_n^q) = \int_0^{\infty} P(M_n^q > x)dx = \int_0^{\infty} P(M_n > y)qy^{q-1}dy$. We consider separately the integrals on $[0, 1]$ and $(1, \infty)$. The integral on $[0, 1]$ is bounded by 1, since $P(M_n > y) \leq 1$. For the integral on $(1, \infty)$, we use Kolmogorov's inequality:

$$P(M_n > y) = P(\max_{k\leq n} |S_k| > y\sqrt{n}) \leq \frac{1}{ny^2} E(S_n^2) = \frac{1}{y^2},$$

where $S_k = \sum_{i=1}^{k} X_i$. Hence, $\int_1^{\infty} P(M_n > y)qy^{q-1}dy \leq q \int_1^{\infty} y^{q-3}dy = q/(2 - q)$ if $q < 2$. To satisfy the condition $q < 2$, we choose $1 < p < 2/\alpha$. This proves (A.0.4). $\square$

Appendix B

Vague Convergence

In this section, we assume that E is a locally compact space with a countable basis (LCCB). We denote by $\mathcal{E}$ the class of its Borel sets.

Lemma B.0.6 (Therorem 5.3 of [28])**.** *For each n, suppose that $\{X_{n,j}; j \geq 1\}$ are i.i.d random elements in $(E, \mathcal{E})$. Let μ be a Radon measure on $(E, \mathcal{E})$. Define $N_n = \sum_{i=1}^{n} \delta_{X_{n,j}}$. Let N be a Poisson random measure on E of intensity μ. Then*

$$nP(X_{n,1} \in \cdot) \xrightarrow{v} \mu \text{ on } E \text{ if and only if } N_n \xrightarrow{d} N \text{ in } M_p(E). \tag{B.0.1}$$

Lemma B.0.7 (Therorem 15.7.6 of [20])**.** *Let $(\mu_n)_{n\geq 1}$ and μ be finite measures on E. The following statements are equivalent:*
(i) $\mu_n \xrightarrow{w} \mu$
(ii) $\mu_n \xrightarrow{v} \mu$ *and* $\mu_n(E) \to \mu(E)$
(iii) $\mu_n \xrightarrow{v} \mu$ *and* $\inf_{B\in\mathcal{B}} \limsup_{n\to\infty} \mu_n(B^c) = 0$

Lemma B.0.8 (Proposition 3.3. of [14])**.** *If*

$$\mu_n \xrightarrow{v} \mu,$$

then for any set $K \in \mathcal{E}$ such that $\mu(\partial K) = 0$, we have

$$\mu_n(K \cap \cdot) \xrightarrow{v} \mu(K \cap \cdot),$$

as measures in K (or in E)

Lemma B.0.9 (Therorem 3.2 of [28])**.** *Let μ and $(\mu_n)_{n\geq 1}$ be Radon measures on E. The following statements are equivalent:*

1. $\mu_n \xrightarrow{v} \mu$.

2. $\mu_n(B) \to \mu(B)$ *for all relatively compact B satisfying $\mu(\partial B) = 0$.*

3. For all $K \in K(\mathbb{E})$, we have

$$\limsup_{n\to\infty} \mu_n(K) \le \mu(K)$$

and for all $G \in$ that are relatively compact, we have

$$\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$$

where $\stackrel{v}{\to}$ denotes the vague convergence.

Lemma B.0.10 (Lemma 7.1 of [28])**.** *Suppose $m_n, n > 0$, are point measures on a LCCB space E and $m_n \stackrel{v}{\to} m_0$. Let K be a relatively compact set in E, such that $m_0(\partial K) = 0$. Then there exist an integer $n(K) \ge 1$ such that for all $n \ge n(K)$, there exist labeling of the points of m_n and m_0 in K for which*

$$m_n(\cdot \cap K) = \sum_{i=1}^{p} \delta_{x_i^{(n)}}(\cdot) \text{ and } m_0(\cdot \cap K) = \sum_{i=1}^{p} \delta_{x_i^{(0)}}(\cdot)$$

and

$$(x_i^{(n)}, 1 \le i \le p) \to (x_i^{(0)}, 1 \le i \le p) \text{ in } \mathbb{E}^p, \text{ as } n \to \infty.$$

Appendix C

The α-stable Lévy sheet

In this section, we show that the α-stable Lévy sheet can be viewed as an example of a $\mathbb{D}$-valued α-stable Lévy motion restricted to the time interval $[0,1]$.

First, we recall briefly the construction of the α-stable Lévy sheet, as described in Section 4.8 of [26]. Let $M = \sum_{i\geq 1}\delta_{(T_i,S_i,J_i)}$ be a Poisson random measure on $[0,\infty)\times[0,\infty)\times\overline{\mathbb{R}}_0$ of intensity Leb $\times$ Leb $\times\, \nu_{\alpha,p}$, where $\nu_{\alpha,p}$ is given by (6.1.2), for some $\alpha\in(0,2), \alpha\neq 1$ and $p\in[0,1]$, with $q=1-p$. Let $(\varepsilon_j)_{j\geq 0}$ be a sequence of real numbers such that $\varepsilon_j \downarrow 0$ and $\varepsilon_0 = 1$. Let $I_j = (\varepsilon_j, \varepsilon_{j-1}]$ for $j\geq 1$ and $I_0=(1,\infty)$. For any $t,s\in[0,1]$ and $j\geq 0$, let

$$L_j(t,s) = \int_{[0,t]\times[0,s]\times\Gamma_j} zM(du_1,du_2,dz) = \sum_{i\geq 1} J_i 1_{\{J_i\in\Gamma_j\}}1_{\{T_i\leq t, S_i\leq s\}}.$$

Note that $L_j(t,s)$ is a compound Poisson random variable with characteristic function

$$E[e^{iuL_j(t,s)}] = \exp\left\{ts\int_{\Gamma_j}(e^{iuz}-1)\nu_{\alpha,p}(dz)\right\}, \quad u\in\mathbb{R}.$$

By Kolmogorov's criterion, the series $\sum_{j\geq 1}\big(L_j(t,s)-E(L_j(t,s))\big)$ converges a.s., since $\mathrm{Var}\big(L_j(t,s)\big) = ts\int_{\Gamma_j} z^2\nu_{\alpha,p}(dz)$ for any $j\geq 1$ and $\int_{|z|\leq 1} z^2\nu_{\alpha,p}(dz)<\infty$.

We define $\overline{L}(t,s) = \sum_{j\geq 0} L_j(t,s)$ if $\alpha<1$ and $\overline{L}(t,s) = \sum_{j\geq 0}\big(L_j(t,s) - E(L_j(t,s))\big)$ if $\alpha>1$. It can be proved that there exists a process $\{L(t,s)\}_{(t,s)\in[0,1]^2}$ with sample paths in $\mathbb{D}([0,1]^2)$ such that $L(t,s)=\overline{L}(t,s)$ a.s. for any $t,s\in[0,1]$, and

$$\sup_{(t,s)\in[0,1]^2} |L^{(\varepsilon_k)}(t,s) - L(t,s)| \to 0 \quad \text{a.s.} \quad \text{if} \quad \alpha<1, \tag{C.0.1}$$

$$\sup_{(t,s)\in[0,1]^2} |\overline{L}^{(\varepsilon_k)}(t,s) - L(t,s)| \to 0 \quad \text{a.s.} \quad \text{if} \quad \alpha>1, \tag{C.0.2}$$

where $L^{(\varepsilon_k)}(t,s) = \sum_{j=0}^{k} L_j(t,s)$ and $\overline{L}^{(\varepsilon_k)}(t,s) = L^{(\varepsilon_k)}(t,s) - E(L^{(\varepsilon_k)}(t,s))$ (if $\alpha>1$). Here $\mathbb{D}([0,1]^2)$ is the space of functions $x:[0,1]^2\to\mathbb{R}$ which are continuous at any point (t,s) when this point is approached from the upper right quadrant, and have limits when the point is approached from the other three quadrants. Moreover,

$$E[e^{iuL(t,s)}] = \exp\left\{ts\int_{\mathbb{R}}(e^{iuz}-1)\nu_{\alpha,p}(dz)\right\} \quad \text{if} \quad \alpha<1,$$

$$E[e^{iuL(t,s)}] = \exp\left\{ts\int_{\mathbb{R}}(e^{iuz}-1-iuz)\nu_{\alpha,p}(dz)\right\} \quad \text{if} \quad \alpha>1.$$

Consequently, $L(t,s)$ has a $S_\alpha\big((ts)^{1/\alpha}C_\alpha^{-1},\beta,0\big)$-distribution with $\beta=p-q$ and C_α given by (2.2.5). The process $\{L(t,s)\}_{(t,s)\in[0,1]^2}$ is called an *α-stable Lévy sheet*. Note that both processes $\{L(t,s)\}_{t\in[0,1]}$ and $\{L(t,s)\}_{s\in[0,1]}$ are α-stable Lévy motions with paths in $\mathbb{D}$.

Theorem C.0.11. *Let $L(t)=\{L(t,s)\}_{s\in[0,1]}$ for any $t\in[0,1]$. The process $\{L(t)\}_{t\in[0,1]}$ is an $\mathbb{D}$-valued α-stable Lévy motion (according to Definition 5.2.1).*

Proof: We show that $\{L(t)\}_{t\in[0,1]}$ satisfies conditions *(i)-(iv)* of Definition 4.0.1. We assume that $\alpha<1$, the case $\alpha>1$ being similar. Clearly $L(0)=0$, so property *(i)* holds.

For property *(ii)*, note that by (C.0.1), $L^{(\varepsilon_k)}(t_i)\to L(t_i)$ a.s. in $(\mathbb{D},\|\cdot\|)$ as $k\to\infty$ for $i=1,\ldots,K$, and hence $L^{(\varepsilon_k)}(t_i)-L^{(\varepsilon_k)}(t_{i-1})\to L(t_i)-L(t_{i-1})$ a.s. in $(\mathbb{D},\|\cdot\|)$ as $k\to\infty$, for any $i=2,\ldots,K$. By Lemma 4.2.3, $L(t_i)-L(t_{i-1}), i=2,\ldots,K$ are independent, since $L^{(\varepsilon_k)}(t_i)-L^{(\varepsilon_k)}(t_{i-1}), i=2,\ldots,K$ are independent for any k.

To verify property *(iii)*, we observe that for any $t_1<t_2$ and $s\in[0,1]$,

$$L(t_2,s)-L(t_1,s)=\overline{L}(t_1,s)-\overline{L}(t_2,s)=\sum_{j\geq 0}\int_{(t_1,t_2]\times[0,s]\times\Gamma_j} zM(du_1,du_2,dz) \quad \text{a.s.}$$

From this, it can be proved that $L(t_2)-L(t_1)=\{L(t_2,s)-L(t_1,s)\}_{s\in[0,1]}$ is an α-stable Lévy motion with characteristic function

$$E[e^{iu(L(t_2,s)-L(t_1,s))}]=\exp\left\{(t_2-t_1)s\int_{\mathbb{R}}(e^{iuz}-1)\nu_{\alpha,p}(dz)\right\}, \quad \text{for all } u\in\mathbb{R}.$$

On the other hand, $L(t_2-t_1)=\{L(t_2-t_1,s)\}_{s\in[0,1]}$ is also an α-stable Lévy motion with the same characteristic function. Hence, $L(t_2)-L(t_1)\stackrel{d}{=}L(t_2-t_1)$.

To verify property *(iv)*, we assume first that $t=1$. The process $L(1)=\{L(1,s)\}_{s\in[0,1]}$ is an α-stable Lévy motion, so it is an α-stable process. It follows that for any $s_1,\ldots,s_m\in[0,1]$, $(L(1,s_1),\ldots,L(1,s_m))$ has an α-stable distribution in $\mathbb{R}^m$ with Lévy measure $\mu_{s_1,\ldots,s_m}$:

$$E\big(e^{iu_1L(1,s_1)+\ldots+iu_mL(1,s_m)}\big)=\exp\left\{\int_{\mathbb{R}^m}(e^{iu\cdot y}-1)\mu_{s_1\ldots,s_m}(dy)\right\}, \quad u=(u_1,\ldots,u_m)\in\mathbb{R}^m.$$

In particular, $(L(1,s_1),\dots,L(1,s_m))$ is regularly varying with limiting measure $\mu_{s_1,\dots,s_m}$.

On the other hand, by Lemma 2.1 of [18], $L(1)$ is regularly varying in $\mathbb{D}$ (in the sense of Definition 5.2.1), i.e. $L(1) \in RV(\{a_n\}, \overline{\nu}, \overline{\mathbb{D}}_0)$ for a boundedly finite measure $\overline{\nu}$ on $\overline{\mathbb{D}}_0$ with $\overline{\nu}(\overline{\mathbb{D}}_0 \backslash T(\mathbb{D}_0)) = 0$. Moreover, $\overline{\nu} = c\nu_\alpha \times \Gamma_1$ for some $c > 0$ and a probability measure Γ_1 on $\mathbb{S}_{\mathbb{D}}$. Let $\nu = \overline{\nu} \circ S^{-1}$, where $S : (0,\infty) \times \mathbb{S}_{\mathbb{D}} \to \mathbb{D}_0$ is the inverse of the map T, i.e. $S(r,z) = rz$. By Theorem 8 of [17], $(L(1,s_1),\dots,L(1,s_m))$ is regularly varying with limiting measure $\nu_{s_1,\dots,s_m} = \nu \circ \pi^{-1}_{s_1,\dots,s_m}$. By the unicity of the limit, $\mu_{s_1,\dots,s_m} = \nu_{s_1,\dots,s_m}$. Finally, property *(iv)* for general t follows using the scaling property of $\mu_{s_1,\dots,s_m}$ and the fact that $\{L(t,s)\}_{s\in[0,1]} \stackrel{d}{=} \{t^{1/\alpha}L(1,s)\}_{s\in[0,1]}$. $\square$

In relation with the simulation procedure described in Example 6.1, we include the following result, which can be proved using the same argument as in Section 48 of [26].

Theorem C.0.12. *Let $\xi, (\xi_{ij})_{i,j\geq 1}$ be i.i.d. regularly varying random variables, i.e.*

$$nP\left(\frac{\xi}{a_n} \in \cdot\right) \stackrel{v}{\to} \nu_{\alpha,p} \quad in \quad \overline{\mathbb{R}}_0,$$

for some $a_n \uparrow \infty$, $\alpha \in (0,2), \alpha \neq 1$ and $p \in [0,1]$, where $\nu_{\alpha,p}$ is given by (6.1.2). For any $t,s \in [0,1]$, let $T_{n,m}(t,s) = a_n^{-1}a_m^{-1}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ms]}(\xi_{ij}-\mu)$, where $\mu = 0$ if $\alpha < 1$ and $\mu = E(\xi)$ if $\alpha > 1$. Let $L = \{L(t,s)\}_{(t,s)\in[0,1]^2}$. Then

$$T_{n,m} \stackrel{d}{\to} \widetilde{L} \quad in \quad \mathbb{D}([0,1]^2), \quad as \quad n,m \to \infty.$$

where $\mathbb{D}([0,1]^2)$ is equipped with a Skorohod topology similar to the J_1-topology. Here, $\widetilde{L}$ is a modification of L with sample paths in $\mathbb{D}([0,1]^2)$.

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