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EXISTENCE OF TIME-OPTIMAL CONTROLS FOR SYSTEMS WITH DISCONTINUOUS RIGHT-HAND SIDE

by

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ABSTRACT

In this thesis we investigate the existence of a time optimal control for systems with Discontinuous Right-Hand Side, whose solutions are defined in the sense of Filippov.

We define a relaxed system corresponding to the original one and prove that the attainable set for the relaxed system is closed. This result is used further to prove the existence of a time optimal control for the relaxed system.

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CHAPTER I

INTRODUCTION

Up to the present time the existence of a time-optimal control for the non-linear case has been investigated taking into consideration systems with continuous right-hand side, ([2], part III). The answer to this problem has been positively given under a certain condition ([2], condition of convexity of R(t,x) in theorem 20.1 pg. 107).In this thesis we investigate the same problem for the case at which the right-hand side of the given dynamical system has a countable number of discontinuities.

In 1960 Filippov introduced a new concept of solution of a differential equation, [1], and he studied properties of usual interest of these solutions (existence, uniqueness, etc.). Under certain limitations he proved a number of theorems regarding the existence uniqueness and continuous dependence (on the initial value) of solutions with piecewise continuous right-hand side. As it is mentioned in the Abstract, Filippov's definition is adopted in this thesis and gives a convenient tool of insighting the situation in our problem.

As one can observe in the following paragraph, 1.1, the right-hand side of the given system is considered to be defined only on the interval $[0,T]$; this does not restrict our problem since (by assumption) a solution exists and it is of the following form

$$x(t) = \int_0^t f(\tau, x(\tau), u(\tau))\, d\tau \text{ for almost all } t \in [0,T].$$

Hence, if an optimal (minimum) time exists, it should lie in the interval $[0,T]$.

1.1 Description of the systems

Under consideration are differential systems of the following type:

$$S: \quad \begin{cases} \dot{x}(t) = f(t,x(t), u(t)) & \text{a.e in } I = [0,T] \\ x(0) = x_o, \quad u \in B \end{cases}$$

where for each $t \in [0,T]$, $x(t) \in R^n$, $u(t) \in R^r$; the function $f: I \otimes R^n \otimes R^r \rightarrow R^n$ satisfies the basic assumptions stated below; B is a class of functions (class of admissible controls), also defined later in this part of the thesis.

1.2 Basic Assumptions

Throughout this thesis the function f is assumed to satisfy the following properties:

P_1: The function $f(.,x,u)$ has, at most, a countable number of discontinuities in I, for each $(x,u) \in R^n \otimes R^r$ at which it is defined; for almost all $t \in I$ and $u \in R^r$, $f(t,.,u)$ has at most a countable set of discontinuities on R^n, and for almost all $t \in I$, and $x \in R^n$ $(f(t,x,.)$ is continuous in u on R^r.

P_2: f is bounded on every bounded set $Q = I \otimes D \otimes U \subset I \otimes R^n \otimes R^r$.

The investigation that follows is based on a recent definition of solution of differential equations with discontinuous right-hand side given by Filippov, [1] pg.202). This is briefly discussed below.

Let a vector differential equation (system) be given as

(Σ) $\dot{x}(t) = \varphi(t,x(t))$ a.e in I.

Definition 1

A vector function x(t) defined on an interval I is called a solution of (Σ) if it is absolutely continuous and satisfies the following inclusion:

$\dot{x}(t) \in K\{\varphi(t,x(t))\}$ a.e in I, where,

$$K\{\varphi(t,x)\} = \bigcap_{\delta > o} \; \bigcap_{M \in \Lambda_\delta(x)} \overline{co}\, \varphi(t, N_\delta(x)/M)$$

$\overline{co}A$: is the closed convex hull of A

$N_\delta(x)$: is a δ-neighborhood of x

$\Lambda_\delta(x) = \{M; M \subset N_\delta(x)$ with $\mu(M) = 0\}$ and

μ is Lebesque measure, and

N/M stands for set-theoretic difference .

1.3 The Class of Admissible Controls B

The class of admissible controls B is defined as follows:

$B = \{ u \in L_\infty(I, R^r);\ u(t) \in U \}$ where $U \subset R^r$ is compact and convex.

A solution of the system S will be understood in the sense of definition, i.e.,

$$S: \begin{cases} \dot{x}(t) \in K \{ f(t,x(t), u(t)) \} & \text{a.e in } I \\ x(0) = x_o \end{cases}$$

where $K \{f(t,x,u)\} = \bigcap_{\delta > o} \bigcap_{M \in \Lambda_\delta(x)} \overline{co}\, f(t, N_\delta(x)/M, u)$

For almost all $(t,x) \in I \otimes R^n$ we define the set $G(t,x)$ by

$G(t,x) = \{ y \in R^n;\ y \in K \{ f(t,x,u) \},\ u \in U \}$

Clearly $G(t,x)$ is a set valued function of (t,x). We define now the system:

$$S_o: \begin{cases} \dot{x}(t) \in G(t,x(t)) & \text{a.e in } I \\ x(0) = x_o \end{cases}$$

We consider also the system

$$S_r: \begin{cases} \dot{x}(t) \in \overline{co}\, G(t,x(t)) & \text{a.e in } I \\ x(0) = x_o \end{cases}$$

Definition 2

Let $AC(I,R^n)$ be the class of all absolutely continuous functions from I to R^n. We call the "solution set of system S_o" the following class X_o:

$X_o = \{x \in AC(I,R^n);\ \dot{x}(t) \in G(t,x(t))$ a.e in I, $x(0) = x_o\}$

Similarly, we define the solution set X_r of the system S_r and the one of the system S: $X = \{x \in AC(I,R^n;\ \dot{x}(t) = f(t,x(t),u(t))$, a.e in I, $x(0) = x_o\}$.

Definition 3

By the attainable set $A_o(t)$ of system S_o at the point t we mean the following set:

$A_o(t) = \{x(t);\ x \in X_o\} \subset R^n$(Similarly for $A_r(t)$, and $A(t)$).

1.4 Statement of the Problem

The purpose of this investigation is to prove certain properties of the sets X and X_r: equicontinuity and uniform boundedness (conditional compactness) and furthermore the existence of a time optimal control for systems of the above type S. We approach the problem proving lemma 4 which allows us to take under consideration the set X_o instead of its equivalent X and prove the above mentioned properties for the set X_o. Since the absence of convexity in the right hand side of the system S_o complicates the problem we direct our effort to solve a more relaxed problem and we investigate the properties of X_r with the further intention to obtain some general results for X_o (and therefore X) searching the relationship between X_r and X_o (we hope that X_o is dense in X_r). However, in the special case that G(t,x) is a convex-set-valued function, the existence of a time-optimal control for the system S_o (and consequently S) is proven in this thesis.

CHAPTER 2

EQUIVALENCE OF THE SYSTEMS S AND S_o.

2.1 Condition B

There is a constant $C > 0$ such that for any $x \in R^n$ the inequality

$$(B) \quad (x.y) \leq C(1 + \|x\|^2)$$

holds for all $y \in G(t,x) = \{ y \in R^n;\ y \in K\{f(t,x,u)\},\ u \in U\}$ uniformly on I, where $(x \cdot y)$ stands for inner product and $\| . \|$ is the usual norm in R^n.

Note that the boundedness does not depend on the continuity of f at x. We give here two examples of discontinuous functions that satisfy condition B.

2.2 Examples and Boundedness of X_o

Example 1

Consider the function $f : R \otimes [-1, 1] \longrightarrow R$ defined by

$$f(x,u) = \begin{cases} x^2 + u, & x < 1 \\ x + u + 1, & x > 1 \end{cases} \quad u \in [-1, 1]$$

We have that $G(x) = \{y \in R,\ y \in K\{f(x,u)\},\ u \in [-1,1]\}$ and we chose $C = 3$. Note that $K\{f(x,u)\} = f(x,u) = x^2+u$ for $x < 1$ and therefore substituting in (B) we are to verify that,

$$(1) \qquad x(x^2+u) \leq 3(1+|x|^2) \quad , \text{ for } x < 1$$

Indeed, for $u \geq o$ we elicit

$$(1a) \qquad (x^2+u)\,x < 3 \text{ for every } x \leq o \text{ (left side is always negative).}$$

For $u \leq o$ and $x \leq o$ it is true that

$$(1b) \qquad 0 \leq (x^2+u)\,x \leq (x^2-1)\,x \leq \sup_{x \in (-\infty, o]} (x^2-1)\,x < 3 .$$

Also,

$$(1c) \qquad \sup_{\substack{x \in (o,1) \\ u \in [-1,1]}} (x^2+u)\,x = 2 < 3$$

Thus from (1a), (1b), 1c) we imply (1) . -

On the other hand, note that for $x > 1$

$$K\{f(x,u)\} = x + u + 1$$ and therefore (B)

becomes $x(x+u+1) \leq 3(1+x^2)$ or $2x^2 - (u+1)x + 3 \geq 0$ which we are to verify. But this is true again for every $x \in R$ and $|u| \leq 1$ since $\Delta = (u+1)^2 - 24 < 0$. To complete the verification we note that

$$G(1) = \{ y \in R,\ y \in K\{f(1,u)\},\ u \in [-1,1] \}$$ and that

$K\{f(1,u)\} = [1+u, 2+u]$. It is obvious then that (B) becomes

$$1 . y < 6$$ which is true for every

$$y \in G(1) = \bigcup_{u \in [-1,1]} [1+u, 2+u] = [0,3]$$

<u>Example 2</u>

We consider the function $f : R^2 \longrightarrow R^2$ defined by:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} 4 + 2 \text{ sign } x_2 \\ 2 - 2 \text{ sign } x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 6 \\ -2 \end{pmatrix}, & x_2 > 0 \\ \begin{pmatrix} 2 \\ 6 \end{pmatrix}, & x_2 < 0 \end{cases}$$

where $x = (x_1, x_2)$. We can verify first that $K\{f(x_1, o)\} = \overline{AB} \subset R^2$, i.e a closed segment of the plane, which can be analytically expressed as

$$K\{f(x_1, o)\} = \{ (z,w) \in [2,6] \otimes [-2,6] ;\ 2z + w = 10 \} .$$

It is clear that for $x = (x_1, x_2)$ with $x_2 > o$ we have that

$$K\{f(x)\} = f(x) = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$, therefore (B), which

we are to verify, takes the following form

$$(1) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -2 \end{pmatrix} \leq 9(1 + x_1^2 + x_2^2)$$

where we chose $C = 9$, and the dot stands for inner product. A simple calculation transforms (1) into

$$(9x_1^2 - 6x_1 + 9) + (9x_2^2 + 2x_2) \geq 0$$

and this is true for every $x_1 \in R$ since $9x_2^2 + 2x_2 > 0 \quad (x_2 > 0)$ and $9x_1^2 - 6x_1 + 9 > 0$ (complex roots).

Similarly for $x_2 < 0$ we have that $K\{f(x)\} = f(x) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, and (B) becomes

$$(2) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} \leq 9(1 + x_1^2 + x_2^2), \text{ which gives}$$

$(9x_1^2 - 2x_1 + 9) + (9x_2^2 - 6x_2) \geq 0$. But this is again true for similar reasons for every $x_1 \in R$. To complete the verification it suffices to note that

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} z \\ w \end{pmatrix} = x_1 z \leq 9(1 + x_1^2)$$

for every $y = (z,w) \in K\{f(x_1, o)\} = \{(z, w) \in [2,6] \otimes [-2,6] ; 2z + w = 10\}$, and for every $x_1 \in R$.

Theorem 1

If the function $G(t,x)$ satisfies condition B then the solution set X_o of S_o is uniformly bounded.

Proof

Let $x \in X_o$, then if we set in (B) $y = \dot{x}(t)$ and $x = 2x(t)$ we obtain

$$\frac{d}{dt} \| x(t) \|^2 \leq 2C(1 + \|x\|^2) \text{ a.e in } I, \text{ and this implies}$$

$$\| x(t) \| \leq (\| x_o \|^2 + 1) \exp\{2CT\}$$

Q.E.D.

2.3 The Function K(u) and its Properties

By the definition of $K\{f(t,x,u)\}$ we imply that it is closed and by property P_2 bounded, thus it is compact. Consequently if we fix (t,x) we can define the following compact-set-valued function.

$k: \quad U \longrightarrow CP(R^n)$ defined by

$k(u) = K\{f(t,x,u)\}$, where $CP(R^n)$ is the class of all compact subsets of R^n, (provided that f is defined at t).

Definition 4

The following function

$$\rho_H : \quad CP(R^n) \otimes CP(R^n) \longrightarrow R \quad \text{defined by}$$

$$2\rho_H \quad (A,B) = \max_{x \in A} \; d(x,B) + \max_{y \in B} \; (y,A), \quad \text{where,}$$

$$d(x,B) = \min_{s \in B} \|x-s\| \;, \quad (\text{respectively for } d(y,A)\;)$$

is a metric which we call Hausdorff metric.

Since the set $K\{f(t,x.u)\}$ is defined in a manner that makes its use difficult , we wish to simplify it. The following lemma, along with remark 1, serves this purpose and enables us to express $K\;\{f(t,x,u)\}$ in a more convenient and simple way.

Lemma 1

Let φ be a function φ: $E \longrightarrow R^n$ where E is any subset of a Eucledian space. Suppose that φ is discontinuous on a subeet N^o of E with $\mu\,(N^o) = o$ (μ is Lebesgue measure). Then the following equality is true,

$$\bigcap_{\mu\,(N)=o} \overline{\varphi\,(E/N)} = \overline{\varphi(E/N^o)}$$

where the intersection is meant over all the subsets $N \subset E$ with measure zero and the bar denotes closure.

Proof

Since $\overline{\varphi(E/N^o)}$ is a member of the intersection we imply that

$$\bigcap_{\mu(N)=o} \overline{\varphi\,(E/N)} \subset \overline{\varphi(E/N^o)}$$

It suffices then to prove that $\overline{\varphi(E/N^o)} \subset \overline{\varphi\,(E/N)}$ for every N with $\mu(N) = 0$. Let $y \in \overline{\varphi(E/N^o)}$, then there is a sequence $\varphi(x_n)$ with $x_n \in E/N^o$ $(n=1,2,\ldots)$ such that $\lim_{n\to\infty} \varphi(x_n) = y$; i.e for every $\varepsilon > 0$ there is a $n_o(\varepsilon)$ such that

(1) $$\|\varphi(x_n) - y\| < \varepsilon/2 \text{ for } n \geq n_o(\varepsilon).$$

Since $x_n \in E/N^o$ $(n = 1,2,\ldots)$ we have, by assumption, that φ is continuous at x_n for every n ; therefore, for every $\varepsilon > o$ there is a $\delta(\varepsilon)$ such that

(2) $$|x_n - z| < \delta(\varepsilon) \text{ implies } |\varphi(x_n) - \varphi(z)| < \varepsilon/2$$

and this is true for every n. But for each x_n and every N with $\mu(N) = 0$ we have that $(E/N) \cap S_\delta(x_n) \neq \phi$ (S_δ is an open neighborhood of x_n and $\delta = \delta(\varepsilon)$). Indeed, if we assume the opposite , i.e $(E/N) \cap S_\delta(x_n) = \phi$, we arrive to the following inclusion $S_\delta(x_n) \subset N$; thus $\mu(N) \geq \delta > o$ and this is contradiction. Therefore, for each x_n and every N with $\mu(N) = o$ there is a $y_n \in E/N$ such that

(3) $$|\varphi(x_n) - \varphi(y_n)| < \varepsilon/2, \quad y = 1,2,\ldots..$$

Adding now (1) and (3) and using the triangle inequality we obtain

$$\|\varphi(y_n) - y\| < \varepsilon \qquad \text{for } n \geq n_o(\varepsilon)$$

Thus $y \in \overline{\varphi(E/N)}$ for any $N \subset E$ with $\mu(N) = 0$ and since $y \in \overline{\varphi(E/N^o)}$ is chosen arbitrary,we imply that $\overline{\varphi(E/N^o)} \subset \overline{\varphi(E/N)}$ for every such N. The equality that was to be proved follows now directly. Q.E.D.

Remark 1

Note that the following is also true.

$$\bigcap_{\mu(N)=o} \overline{co}\,\varphi(E/N) = \overline{co}\,\varphi(E/N^o)$$

Indeed, it is obvious first that

(1) $$\bigcap_{\mu(N)=o} \overline{co}\,\varphi(E/N) \subset \overline{co}\,\varphi(E/N^o)$$

But it is true that,

$$(2) \qquad \varphi(E/N^{o}) \subset \overline{\varphi(E/N^{o})} \subset \overline{\varphi(E/N)} \subset \overline{co}\,\varphi(E/N)$$

for every $N \subset E$ with $\mu(N) = 0$. The second inclusion is a consequence of lemma 1 and the third one comes from the definition of the closure of a set and the fact that $\overline{co}\,\varphi(E/N)$ is a closed set which contains the set $\varphi(E/N)$. Finally we have from (2) that

$$\overline{co}\,\varphi(E/N^{o}) \subset \overline{co}\,\varphi(E/N) \quad \text{for every} \quad N \text{ with } \mu(N) = 0$$

because, by definition, $\overline{co}\,\varphi(E/N^{o})$ is the smallest convex closed set that contains $\varphi(E/N^{o})$. The desired result follows now directly.

Corrolary 1

The set $K\{f(t,x,u)\}$ can be expressed as follows

$$(F) \qquad K\{f(t,x,u)\} = \bigcap_{n=1}^{\infty} \overline{co}\, f(t, N_{1/n}(x) \,/\, M^{o}, u)$$

where M^{o} is the set of all points of R^{n} at which the function f is discontinuous.

Proof

The proof is direct consequence of remark 1.

Q.E.D.

For our convenience we introduce the following notation

$$S_{n}(u,M) = f(t, N_{1/n}(x)/M, u) \quad \text{where } M \in \Lambda_{1/n}(x) .$$

Lemma 2

For arbitrary but fixed $n \in \Phi$ (natural numbers) the set valued function $\overline{co}\, S_{n}(u, M^{o})$ is a continuous function of $u \in U$ in respect to the Hausdorff metric.

Proof

Consider $u_{o} \in U$ to be an arbitrary but fixed point. By the definition of ρ_{H} we have that,

$$(1) \qquad 2\,\rho_H(\overline{co}\,S_n(u_o, M^o),\ \overline{co}\,S(u, M^o)) = \max_{s' \in \overline{co}\,S_n(u, M^o)} d(s', \overline{co}\,S_n(u_o, M^o)) +$$

$$+ \max_{s \in \overline{co}\,S_n(u_o, M^o)} d(s, \overline{co}\,S_n(u, M^o)) \le d(s'_o, \overline{co}\,S_n(u_o, M^o)) +$$

$$+\ d(s_o, \overline{co}\,S_n(u, M^o))$$

where $s_o \in \overline{co}\,S_n(u_o, M^o)$ and $s'_o \in \overline{co}\,S_n(u\ , M^o)$.

It is obvious, from (1), that for every $s \in \overline{co}\,S_n(u_o, M^o)$ and every $s' \in \overline{co}\,S_n(u, M^o)$ the following is true.

$$(2) \qquad 2\,\rho_H\,(\overline{co}\,S_n(u_o, M^o), \overline{co}\,S_n(u, M^o)) \le \| s'_o - s \| + \| s_o - s' \|$$

A theorem by Caratheodory ([5] Valentine, "Convex Sets" pg 14) allows us to express s_o as a (n+1) - convex combination of points of $S_n(u_o, M^o)$, where n, here, stands for the dimension of R^n, that means

$$(3) \qquad s_o = \sum_{i=o}^{n} \xi_i\, f(t, x_i, u_o), \quad \text{where } x_i \in N_{1/n}(x)/M^o,\ i=0, 1, \ldots, n$$

$$\sum_{i=o}^{n} \xi_i = 1 \quad \text{and} \quad \xi_i \in [0, 1],\ i = 0, 1, \ldots n . \quad \text{Then}$$

it is obvious that $s' = \sum_{i=o}^{n} \xi_i\, f(t, x_i, u) \in \overline{co}\,S_n(u\ , M^o\ )$

and

$$(4) \qquad \| s_o - s' \| = \| \sum_{i=o}^{n} \xi_i\,[f(t, x_i, u_o) - f(t, x_i, u)] \| \le$$

$$\le n\, \| f(t, x_i, u_o) - f(t, x_i, u) \|$$

Thus from (4) and the continuity of f in respect to u we have that, for every $\varepsilon > o$ there is a $\delta_1(\varepsilon)$ such that

$$(6) \qquad \| s_o - s' \| < \varepsilon \quad \text{for} \quad \| u_o - u \| < \delta_1(\varepsilon) \; .$$

Similarly we can prove that for every $\varepsilon > o$ there is a $\delta_2(\varepsilon)$ such that

$$(7) \qquad \| s'_o - s \| < \delta_2(\varepsilon) \; .$$

Finally combining (2), (6) and (7) we obtain that for every $\varepsilon > o$ there is a $\delta(\varepsilon) = \min \{ \delta_1(\varepsilon), \delta_2(\varepsilon) \} > o$ such that

$$\rho_H (\text{co}\, S_n(u, M^o), \text{co}\, S_n(u_o, M^o)) < \varepsilon \quad \text{for every} \quad u$$

that satisfies $\| u_o - u \| < \delta(\varepsilon)$; This completes the proof.

Q.E.D.

Lemma 3

The set valued function k defined as a function of u on U is continuous in respect to the Hausdorff metric.

Proof

From formula (F) (pg. 10) it is clear that for each $u \in U$ we have $\lim_{n \to \infty} \overline{\text{co}}\, S_n(u, M^o) = K \{ f(t,x,u) \} = k(u)$, or equivalently, for every $\varepsilon > o$ there is a $n_o(\varepsilon, u)$ such that

$$(1) \qquad \rho_H (k(u), \overline{\text{co}}\, S_n(u, M^o)) < \varepsilon/3 \quad \text{for} \quad n \geq n_o(\varepsilon, u)$$

and every $u \in U$. In particular we have that

$$(2) \qquad \rho_H (k(u_o), \overline{\text{co}}\, S_n(u_o, M^o)) < \varepsilon/3 \quad \text{for} \quad n \geq n_o(\varepsilon, u_o)$$

But lemma 2 certifies that for every $\varepsilon > o$ there is $\delta(\varepsilon) > o$ such that

(3) $\rho_H(\overline{co}\, S_n(u, M^o), \overline{co}\, S_n(u_o, M^o)) < \varepsilon/3$ for $\|u_o - u\| < \delta(\varepsilon)$

By addition of (1) and (3) and use of the triangle inequality we obtain

(4) $\rho_H(k(u), \overline{co}\, S_n(u_o, M^o)) < 2\varepsilon/3$

and this is true for every $n \geq n_o(\varepsilon, u)$ and every $u \in U$ such that $\|u_o - u\| < \delta(\varepsilon)$. To the end of the proof we add (2) and (4) and using again the triangle inequality we obtain

(5) $\rho_H(k(u_o), k(u)) < \varepsilon$ for every $u \in U$ such that

$\|u - u_o\| < \delta(\varepsilon)$. This completes the proof. -

Q.E.D.

2.4 Equivalence between S and S_o

Lemma 4

The systems S and S_o are equivalent i.e a function $\varphi(t)$ on I is a solution of S_o iff it is a solution of S.

Proof

The necessity is obvious: $\dot{\varphi}(t) \in K\{f(t, \varphi(t), u(t))\} \subset G(t, \varphi(t))$. For the sufficiency we prove first the following. Suppose that $r \in R^n$ is fixed, then we show that for fixed (t,x) the set

$V_o(r) = \{u \in U;\ r \in K\{f(t,x,u)\}\}$ is compact

It suffices to show that it is closed. Indeed let u_n be a sequence of points of $V_o(r)$ such that $u_n \longrightarrow u$. By lemma 3 we have that for every $\varepsilon > 0$ there is a $n_o(\varepsilon)$ such that

(1) $\rho_H(k(u), k(u_n)) < \varepsilon/2$ for every $n \geq n_o(\varepsilon)$ or

$$2\rho_H(k(u),k(u_n)) = \max_{s \in k(u)} d(s,\, k(u_n)) + \max_{z \in k(u_n)} d(z,k(u)) < \varepsilon\,, \tag{2}$$

for every $n \ge n_o(\varepsilon)$. Since $u_n \in V_o(r)$, $n = 1,2,\ldots$ it follows that $r \in k(u_n)$, $n = 1,2,\ldots$. Hence from (2) we imply that $d(r,k(u)) < \varepsilon$ for every $\varepsilon > 0$, and since $k(u)$ is closed, $r \in k(u)$; thus $u \in V_o(r)$.

Suppose now that φ is a solution of S_o, then

$$\dot{\varphi}(t) \in G(t,\varphi(t)) = \{y \in R^n;\ y \in K\{f(t,\varphi(t),u)\}, u \in U\} \text{ a.e in } I, \tag{3}$$

or equivalently we can say that there is a set I_o, with $\mu(I_o) = 0$ such that (3) is true for every $t \in I' = I/I_o$. This implies that for every $t \in I'$ there is at least one $u^* \in U$ depending on t that satisfies

$$\dot{\varphi}(t) \in K\{f(t,\varphi(t),\ u^*)\} \tag{4}$$

Suppose that,

$$V_o(\dot{\varphi}(t)) = \{u \in U;\ \dot{\varphi}(t) \in K\ \{f(t,\varphi(t),u)\}\}$$

We know already that $V_o(\dot{\varphi}(t))$ is compact for every $t \in I'$ and therefore its projection to any coordinate axis of R^r is a compact subset of R and it has a minimal element. Thus we can select from $V_o(\dot{\varphi}(t))$ the element u with the smallest first coordinate. If there is more than one we select the one with the smallest second coordinate and so on. Let then $u(t) = (u_1(t),\ u_2(t), \ldots, u_r(t))$ so chosen. We shall show by induction that the functions $u_1(t), \ldots, u_r(t)$ are measurable. Suppose that $u_1(t), \ldots, u_{s-1}(t)$ are measurable. (If $s = 1$ there is nothing to assume). We must show that $u_s(t)$ is measurable. By Lusin's theorem, for any $\varepsilon>0$ there is a closed set $E \subset I'$ such that $\mu(I'/E) < \varepsilon$ and the functions $\dot{\varphi}(t)$, $u_1(t), \ldots, u_{s-1}(t)$ are continuous on E. We shall show that for any number $\beta \in R$ the set

(5) $\{t \in E\ ;\ u_s(t) \le \beta\}$ is closed. Suppose it is not, then there is a sequence $t_n \in E$ such that

(6) $t_n \to t' \in E$ and $u_s(t_n) \leq u_s(t') - \varepsilon_1$, $\varepsilon_1 > 0$.

Since $u(t_n)$ is a sequence of points of U it is bounded and therefore there is a convergent subsequence of $u(t_n)$. (We take it to be the original one, i.e.

$$\lim_{n\to\infty} u(t_n) = u' = (u_1', u_2', \dots, u_r')) .$$

The continuity of the function $k(u) = K\{f(t,x,u)\}$ implies that

(7) $K\{f(t', \varphi(t'), u(t_n))\} \to K\{f(t', \varphi(t'), u')\}$, with respect to

the Hausdorff metric. By the assumption that φ is a solution, we have that

(8) $\dot{\varphi}(t_n) \in K\{f(t_n, \varphi(t_n), u(t_n))\}$

and since $\dot{\varphi}$ is continuous on E, for every $\varepsilon > o$ there is a $n_o(\varepsilon)$ such that

(9) $d(\dot{\varphi}(t_n), \dot{\varphi}(t')) < \varepsilon$, for every $n \geq n_o(\varepsilon)$.

Therefore from (8) and (9) we obtain

$$d(\dot{\varphi}(t'), K\{f(t_n, \varphi(t_n), u(t_n))\}) < \varepsilon, \text{ for every } n \geq n_o(\varepsilon) .$$

Hence due to (7) we find that $\dot{\varphi}(t') \in K\{f(t', \varphi(t'), u')\}$.

This implies that $u' \in V_o(\varphi(t'))$ and on the other hand we have from (6) that $u_s' \leq u_s(t')$; this leads to contradiction of the fact that $u_s(t')$ is the smallest. Thus $u_s(t)$ is measurable on E. Since now $E \subset I'$ and $\mu(I'/E) < \varepsilon$, we can let $\varepsilon \to 0$ and obtain that u_s is measurable on I' and furthermore on I.

Q.E.D.

CHAPTER 3

EXISTENCE OF A TIME-OPTIMAL CONTROL FOR THE RELAXED SYSTEM.

3.1 Properties of the Function G(t,x)

The purpose of this paragraph is to prove that the set $G(t,x)$ is compact for almost all $(t,x) \in I \otimes R^n$. This is an important result because it provides the possibility of considering continuity of the function G with respect to the Housdorff metric.

Lemma 5

For almost all $(t,x) \in I \otimes R^n$ the set $G(t,x)$ is a closed subset of R^n.

Proof

Let $C = \{ k(u), u \in U\} \subset CP(R^n)$; the continuity of k implies that C is a compact subset of $CP(R^n)$ with respect to the Hausdorff topology. We express $G(t,x)$ as follows,

(1) $G(t,x) = \bigcup_{Q \in C} \{ y \in R^n;\ y \in Q \}$ and we let p to be an accumulation point of $G(t,x)$. Then by the definition of an accumulation point we have

(2) $$\bigcup_{Q \in C} [\{ N_{1/n}(p)/ \{p\} \cap Q] \neq \phi$$

where $N_{1/n}(p)$ is an open $1/n$ - neighborhood of p. Therefore there is a sequence $Q_n \in C$ such that $Q_n \cap \{N_{1/n}(p)/ \{p\}\} \neq \phi$, and a subsequence Q_{n_j} of Q_n that converges in C(C is compact), i.e.

(3) $$\lim_{j \to \infty} Q_{n_j} = Q^*$$ (with respect to the Hausdorff metric) where $Q^* \in C$.

Clearly $Q_{n_j} \cap \{N_{1/n_j}(p)/ \{p\}\} \neq \phi$ for $j = 1,2,\ldots$ and therefore there is a sequence of points $x_j \in Q_{n_j}$, ($j = 1,2,\ldots$ respectively) such that

(4) $$\| p-x_j \| < 1/n_j$$

On the other hand we have from (3) that for every $\varepsilon > 0$ there is a $j_o(\varepsilon)$ such that

(5) $$\rho_H(Q_{n_j}, Q^*) < \varepsilon \text{ for every } j \geq j_o(\varepsilon) .$$

From (4) and (5) we obtain

$$d(p, Q^*) \leq d(x_j, Q^*) + \|p-x_j\| < \varepsilon + 1/n_j$$

for every $j \geq j_o(\varepsilon)$ and for every $\varepsilon > 0$. Thus if we let $\varepsilon \to 0$ and $j \to \infty$ we obtain that $d(p, Q^*) = 0$ and since $Q^* \in C$(i.e. Q^* is compact) we imply that $p \in Q^*$ and consequently, from (1), $p \in G(t,x)$.

Q.E.D.

Lemma 6

For almost all $(t,x) \in I \otimes R^n$ the set $G(t,x)$ is a bounded subset of R^n.

Proof

The proof follows from the condition B since $\frac{(x,y)}{1+\|x\|^2} \leq C$ for every $x \in R^n$ and $y \in G(t,x)$ uniformly on I.

Q.E.D.

Theorem 2

For almost all $(t,x) \in I \otimes R^n$ the set $G(t,x)$ is a compact subset of R^n.

Proof

The proof follows directly from lemmas 3 and 4.

Q.E.D.

Remark 2 Theorem 2 allows us to talk in terms of continuity (with respect to the Hausdorff metric) about the function $G(t,x)$ defined as follows:

$$G(t,x) = \begin{cases} f(t,x,U) & , (t,x) \in I \otimes (R^n/M^o) \\ K\{f(t,x,U)\} & , (t,x) \in I \otimes M^o \end{cases}$$

where $M^o \subset R^n$ is the set of all points of R^n at which the function f is discontinuous in x; clearly $\mu(M^o) = 0$. Due to the nature of the Hausdorff metric the function $G(t,x)$ is defined and continuous at every point (t,x) at which f is, and generally a discontinuity of f at t or x results a discontinuity of $G(t,x)$. To maintain this argument it suffices to point out that for $(t_o, x_o) \in I \otimes (R^n/M^o)$ and $(t,x) \in I \otimes (R^n/M^o)$ the following is true.

$$\rho_H(f(t_o,x_o,U), f(t,x,U)) \leq \| f(t_o,x_o,u) - f(t,x,u')\|$$

for some $u \in U$ and $u' \in U$.

3.2 An Approximation of G(t,x) by Continuous Functions

Definition 5

By a convex combination of two sets A and B we mean the following,

$$\lambda A + (1-\lambda) B = \{z \in R^n;\ z = \lambda x + (1-\lambda)y, x \in A, y \in B\}, \text{where } \lambda \in [0,1] .$$

Suppose that $Q = I \otimes D \subset I \otimes R^n$ is a bounded set and let (t_i, x_j), $i = 1,2;\ldots,\ j = 1,2,\ldots$ be the points of $I \otimes D$ at which the function f is discontinuous. We consider a sequence of disjoint open sets B^n_{ij} such that $(t_i, x_j) \in B^n_{ij}$ and $\mu(B^n_{ij}) < \varepsilon_{ij}/\alpha n^2$, where $\sum\limits_{i,j}^{\infty} \varepsilon_{ij} \leq \alpha < \infty$ and n is any natural number. For our convenience we define $y = (t,x) \in I \otimes R^n$. Since $\overline{B^n_{ij}}$ is compact and the functional $\mathcal{L}(y) = \|y\|$ is continuous and convex it attains its maximum and minimum in $\overline{B^n_{ij}}$. The norm $\|\cdot\|$ used here is meant as

$\|y\| = |t| + \|x\|$ where $|\cdot|$ and $\|\cdot\|$ are the usual norms in R respectively, in R^n. We define, in R^n.

$$\alpha_{ij} = \|z_{ij}\| = \max_{y \in \overline{B^n_{ij}}} \|y\| , \quad \beta_{ij} = \|w_{ij}\| = \min_{y \in \overline{B^n_{ij}}} \|y\|$$

and furthermore the following set-valued function :

$$F_n(y) = \begin{cases} G(y),\ y \in Q/B^n_{ij}\ ;\quad i = 1,2,\ldots;\quad j = 1,2,\ldots \\ \dfrac{\alpha_{ij} - \|y\|}{\alpha_{ij} - \beta_{ij}}\, G(w_{ij}) + \dfrac{\|y\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}}\, G(z_{ij}),\ y \in B^n_{ij},\ \begin{matrix} i = 1,2,\ldots \\ j = 1,2,\ldots \end{matrix} \end{cases}$$

Then $\mu\{y \in Q;\ G(y) \neq F_n(y)\} = \mu\{\bigcup\limits_{i=1}^{\infty} [\bigcup\limits_{j=1}^{\infty} B^n_{ij}]\} < 1/\alpha n^2 \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty} \varepsilon_{ij} \leq 1/n^2$.

Lemma 7

For any fixed $n \in \Phi$ the function $F_n(y)$ is continuous on Q.

Proof

Let y_o be an arbitrary but fixed point of Q.

If $y_o \in Q/B^n_{ij}$, $i = 1,2,\ldots;\ j = 1,2,\ldots$ then by remark 2, $F_n(y)$ is continuous at y_o. Suppose then that $y_o \in B^n_{ij}$ for some fixed $(i,j) \in \Phi \otimes \Phi$. It is known that

$$(1) \qquad 2\rho_H(F_n(y_o), F_n(y)) = \max_{s \in F_n(y_o)} d(s, F_n(y)) + \max_{s' \in F_n(y)} d(s', F_n(y_o))$$

$$= d(s_o, F_n(y)) + d(s'_o, F_n(y_o))$$

where $s_o \in F_n(y_o)$ and $s'_o \in F_n(y)$. But it is clear that s'_o depends uppon y and from the definition of F_n it follows that for every $y \in B^n_{ij}$ we have

$$(2) \qquad s'_o = \frac{\alpha_{ij} - \|y\|}{\alpha_{ij} - \beta_{ij}} g'_z + \frac{\|y\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}} g'_w$$

where $g'_z \in F_n(z_{ij})$ and $g'_w \in F_n(w_{ij})$. In particular we have that

$$(3) \qquad s_o = \frac{\alpha_{ij} - \|y_o\|}{\alpha_{ij} - \beta_{ij}} g_z + \frac{\|y_o\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}} g_w$$

where $g_z \in F_n(z_{ij})$ and $g_w \in F_n(w_{ij})$.

On the other hand, from (1) it ensues that

$$(4) \qquad 2\rho_H(F_n(y_o), F_n(y)) \leq \|s_o - s'\| + \|s'_o - s\|$$

and this is true for any $s \in F_n(y_o)$ and $s' \in F_n(y)$. Let us chose s and s' to be as follows,

$$(5) \qquad s = \frac{\alpha_{ij} - \|y_o\|}{\alpha_{ij} - \beta_{ij}} g'_z + \frac{\|y_o\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}} g'_w$$

$$(6) \qquad s' = \frac{\alpha_{ij} - \|y\|}{\alpha_{ij} - \beta_{ij}} g_z + \frac{\|y\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}} g_w$$

Then, again, from the definition of F_n we have that $s \in F_n(y_o)$ and $s' \in F_n(y)$ and it is matter of calculations to verify that

$$(7) \qquad \|s_o - s'\| = \frac{\|y\| - \|y_o\|}{\alpha_{ij} - \beta_{ij}} \|g_z - g_w\|$$

$$(8) \qquad \|s'_o - s\| = \frac{\|y\| - \|y_o\|}{\alpha_{ij} - \beta_{ij}} \|g'_z - g'_w\|$$

By addition of (7) and (8) and use of inequality (4) we obtain

$$(9) \qquad 2\rho_H(F_n(y_o), F_n(y)) \le \frac{\|y\| - \|y_o\|}{\alpha_{ij} - \beta_{ij}} (\|g_z - g_w\| + \|g'_z - g'_w\|)$$

If $g_z = g_w$ and $g'_z = g'_w$ the continuity of F_n at y_o follows trivially.

If $g_z \neq g_w$ or $g'_z = g'_w$ then for every $\varepsilon > o$ we chose $\delta(\varepsilon)$ to be

$$\delta(\varepsilon) = \frac{2\varepsilon(\alpha_{ij} - \beta_{ij})}{\|g_z - g_w\| + \|g'_z - g'_w\|} \quad \text{and we claim}$$

that $\qquad \rho_H(F_n(y_o), F_n(y)) < \varepsilon \qquad$ for every $y \in B^n_{ij}$

such that $\|y - y_o\| < \delta(\varepsilon)$. Indeed, this is verified by substitution of $\delta(\varepsilon)$ in (9) and therefore proves the stated lemma.

Q.E.D.

The same way we can define on Q the following function,

$$G_n(y) = \begin{cases} \overline{co}G(y), \ y \in Q/B^n_{ij}, \ j = 1,2,\dots; \ i = 1,2,\dots \\ \dfrac{\alpha_{ij} - \|y\|}{\alpha_{ij} - \beta_{ij}} \overline{co}\,G(w_{ij}) + \dfrac{\|y\| - \beta_{ij}}{\alpha_{ij} - \beta_{ij}} \overline{co}\,G(z_{ij}), \ y \in B^n_{ij} \quad \begin{matrix} i = 1,2,\dots \\ j = 1,2,\dots \end{matrix} \end{cases}$$

where α_{ij}, β_{ij}, z_{ij}, w_{ij} are defined as before. The G'_n s are also continuous and this is due to the following lemma.

<u>Lemma 8</u>

If $G(y)$ is a set valued function with values in $CP(R^n)$, then the function $\overline{co}G(y)$ is continuous at every point y at which G is.

Proof

Suppose that the function G is continuous at the point y_o, then, for every $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that

$$\|y_o - y\| < \delta(\varepsilon) \text{ implies } \rho_H (G/y_o), g(y)) < \varepsilon/2(n+1)$$

where n stands for the dimension of R^n. Hence, by the definition of ρ_H we have,

$$(1) \qquad 2\rho_H (G(y_o), G(y)) = \max_{s \in G(y_o)} d(s, G(y)) + \max_{s' \in G(y)} d(s', G(y_o)) < \varepsilon/n+1$$

Consider p to be any point of $G(y_o)$, then we claim that for every y with $\|y_o - y\| < \delta(\varepsilon)$ there is a point $p' \in G(y)$ such that $\|p - p'\| < \varepsilon/n+1$. Indeed, if we assume that this is not true then we imply that there is a $p^* \in G(y_o)$ such that $\|p^* - z\| \geq \varepsilon/n+1$ for every $z \in G(y)$, where y issome element of the δ-neighborhood of y_o, namely, $y \in N_\delta(y_o)$. But this implies,

$$d(p^*, G(y)) = \inf_{z \in G(y)} \|p^* - z\| \geq \varepsilon/n+1$$

which contradicts to inequality (1). Consider now the following equality

$$(2) \qquad 2\rho_H(\overline{co}\, G(y_o), \overline{co}\, G(y)) = \max_{x \in \overline{co}\, G(y_o)} d(x, \overline{co}\, G(y)) + \max_{w \in \overline{co} G(y)} d(w, \overline{co}\, G(y_o)) =$$

$$= d(x^*, \overline{co}\, G(y)) + d(w^*, \overline{co}\, G(y_o))$$

where $x^* \in \overline{co}\, G(y_o)$ and $w^* \in \overline{co}\, G(y)$.

It is known already that since x^* belongs to $\overline{co}\, G(y_o)$ it can be expressed as a $(n+1)$ - convex combination of points of $G(y_o)$, i.e.

$x^* = \sum_{i=o}^{n} \xi_i x_i$ where $x_i \in G(y_o)$, $\xi_i \in [0, 1]$ for $i = 0, 1, 2, \ldots, n$ and $\sum_{i=o}^{n} \xi_i = 1$. We chose an element $y \in N_\delta(x)$ and the points $x_i' \in G(y)$; $i = 0, 1, \ldots, n$ such that $\|x_i - x_i'\| < \varepsilon/n+1$. Clearly

$$x = \sum_{i=o}^{n} \xi_i x_i' \in \overline{co}\, G(y) \text{ and consequently}$$

(3) $$d(x^*, \text{co}\, G(y)) \leq \|x^* - x\| = \Big\| \sum_{i=o}^{n} \xi_i (x_i - x_i') \Big\| \leq (n+1) \sum_{i=o}^{n} \|x_i - x_i'\| < \varepsilon$$

Similarly, setting $w^* = \sum_{i=o}^{n} \xi_i w_i$, where $w_i \in G(y)$, $\xi_i' \in [0,1]$ for $i = 0, 1, 2, \ldots, n$ and $\sum_{i=o}^{n} \xi_i' = 1$, and introducing the point $w = \sum_{i=o}^{n} \xi_i' w_i'$ where $\|w_i - w_i'\| < \varepsilon/n+1$, we proceed with the same technique and we find,

(4) $$d(w^*, \overline{\text{co}}\, G(y_o)) < \varepsilon$$

Finally, by addition of (3) and (4) and comparison with (2) we obtain that, for every $\varepsilon > o$ there is a $\delta(\varepsilon)$ such that

$$\| y_o - y \| < \delta(\varepsilon) \text{ implies } \rho_H(\overline{\text{co}}\, G(y), \overline{\text{co}}\, G(y_o)) < \varepsilon$$

Q.E.D.

Lemma 9

For every bounded set $Q \subset I \otimes R^n$ the sequence $G_n(y)$ converges to $\overline{\text{co}}\, G(y)$ a.e in Q with respect to the Hausdorff metric.

Proof

We set

$$B_n = \bigcup_{i=1}^{\infty} \Big[\bigcup_{j=1}^{\infty} B_{ij}^n \Big],$$ then we have that (by definition of B_{ij}^n's(pg. 18)

(1) $$\sum_{n=1}^{\infty} \mu(B_n) < \sum_{n=1}^{\infty} 1/n^2 < \infty \qquad \text{and}$$

(2) $$B_n = \{ y \in Q \; ; \; G_n(y) \neq \overline{\text{co}}\, G(y) \}$$

But (1) implies that almost all $x \in Q$ lie in at most finitely many of the sets B_n, ([4], Rudin, "Real and Complex Analysis" pg. 30) and therefore from (2) it ensues that $G_n(y) = \overline{\text{co}}\, G(y)$ a.e in Q for all large enough $n \in \Phi$ (natural numbers).

Remark 3

One can see that the only points at which the convergence $G_n(y) \longrightarrow \overline{\text{co}}\, G(y)$ is not realized are the points $y_{ij} = (t_i, x_j)$ of discontinuity of f. Indeed, we know that $(t_i, x_j) \in B_{ij}^n$ for every n and

$\mu(B^n_{ij}) < \varepsilon_{ij}/a n^2$. If B^n_{ij} 's are chosen to be open balls then

$$\lim_{n\to\infty} B^n_{ij} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B^k_{ij} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B^k_{ij} = \bigcap_{n=1}^{\infty} B^n_{ij} = (t_i, x_j)$$

Therefore,

$$B_\infty = \lim_{n\to\infty} B_n = \lim_{n\to\infty} \bigcup_{i,j}^{\infty} B^n_{ij} = \{ (t_i, x_j);\ i = 1,2,\ldots;\ j = 1,2,\ldots \}$$

Also, it is obvious from (2) of the previous lemma that

$$B_\infty = \{ y \in Q;\ \overline{co}\, G(y) \neq \lim G_n(y) \}$$

3.3 Boundedness and equicontinuity of X_r

Lemma 10

The solution set X_r of the relaxed system S_r is uniformly bounded.

Proof

It is sufficient to show that S_r satisfies the property B as stated for S. This follows from the observation that

$$\overline{co}\, G(t,x) = \{ a y_1 + (1-a) y_2;\ a \in [0,1],\ y_1, y_2 \in G(t,x) \}$$

and the fact that for every $x \in R^n$ the following is true,

$$(x.[a\, y_1 + (1-a) y_2]) = a(x.y_1) + (1-a)(x.y_2) \leq aC(1+\|x\|^2) + (1-a)C(1+\|x\|^2)$$

$$= C(1+\|x\|^2), \text{ for all } y_1, y_2 \in G(t,x) \text{ and for all } a \in [0,1].$$

Lemma 11

The set $\mathcal{L} = \bigcup_{y \in Q} G(y)$ is bounded for every bounded set $Q \subset I \otimes R^n$. The same holds for the set $\mathcal{L}_r = \bigcup_{y \in Q} \overline{co}\, G(y)$

Proof

This follows from property P_2 and the fact that $K\{f(t,x,u)\} \subset \overline{co}\, f(Q \otimes U)$ for every $(t,x) = y \in Q$ which implies that $\mathcal{L} \subset \overline{co}\, f(Q \otimes U)$. The second statement can be easily proved by considering the points of $\overline{co}\, G(y)$ as $(n+1)$ - convex combinations of points of $G(y)$, where n, here, stands for the dimension of R^n.

Q.E.D.

3.4 Existence of a Time Optimal Control

Lemma 12

For every $t \in I$ the attainable set $A_r(t)$ of the relaxed system S_r is compact and the set-valued function $A_r(t)$ is continuous on I with respect to the Hausdorff metric.

Proof

We prove first that $A_r(t)$ is compact for every $t \in I$. Since $A_r(t) \subset R^n$, it suffices to prove that it is closed and bounded for every $t \in I$. Let us consider $t \in I$ to be arbitrary, then it is obvious that $A_r(t)$ is bounded ; ($x \in A_r(t)$ implies that $\|x\| = \| x_o + \int_o^t f(\xi, x(\xi), u) d\xi \| \leq \|x_o\| + \int_o^t M d\xi$, where M is the bound of f). Suppose now that p is an accumulation point of $A_r(t)$. Then, there is a sequence p_n of points of $A_r(t)$ such that $p_n \to p$ and since $A_r(t) = \{ x(t), x \in X_r \}$ we imply that there is a sequence of functions $x_n \in X_r$ such that $x_n(t) \to p$. In lemma 10 we proved that X_r is bounded and from lemma 11 it ensues that it is also equicontinuous. If follows, then, that there is a subsequence of x_n which converges uniformly to an absolutely continuous function x. For simplicity in the notation, we take it to be the same original sequence and we write $x_n \xrightarrow{u} x$. Clearly $x(t) = p$, and we shall prove that $x \in X_r$.

It is known that,

$$\frac{x(t)-x(\tau)}{t-\tau} = \lim_{n\to\infty} \frac{x_n(t)-x_n(\tau)}{t-\tau} = \lim_{n\to\infty} \frac{1}{t-\tau} \int_\tau^t \dot{x}_n(\xi)\, d\xi =$$

$$= \lim_{n\to\infty} \int_o^1 \dot{x}_n(\tau+(t-\tau)s)\, ds$$

Using the approximation which we have already introduced previously, we obtain that, for every $\varepsilon < 0$ there is a $k_o(\varepsilon)$ such that

$$(1) \qquad \rho_H(G_k(\tau, x_n(\tau)), \overline{co}\, G(\tau, x_n(\tau))) < \varepsilon/2$$

for every $k \geq k_o(\varepsilon)$, every fixed n, and almost all $\tau \in I$. (Lemma 9 and the remark that follows it justify this result). Since $\dot{x}_n(\tau) \in \overline{co}\, G(\tau, x_n(t))$ for almost all $\tau \in I$, it easy to imply from (1) that $\dot{x}_n(\tau) \in G_k^{\varepsilon/2}(\tau, x_n(\tau))$ a.e in I, for every $\varepsilon > 0$, $k \geq k_o(\varepsilon)$ and every particular n; note that

$G_k^{\varepsilon/2}(\tau, x_n(\tau))$ is not a Hausdorff neighborhood but it is defined as:

$$G_k^{\varepsilon/2}(\tau, x_n(\tau)) = \{ y \in R^n;\ d(y, G_k(\tau, x_n(\tau))) \le \varepsilon/2 \}$$

The continuity of G_k's on Q and the continuity of x_n's on I assert that for every $\varepsilon > 0$, there is a $n_o(\varepsilon) > 0$ such that,

$$(2) \qquad \rho_H(G_k^{\varepsilon/2}(\tau, x_n(\tau)),\ G_k^{\varepsilon/2}(t, x(t))) < \varepsilon/2 \ \text{ for } n \ge n_o(\varepsilon),\ |t-\tau| < \varepsilon \ ,$$

and every $k \in \Phi$. Therefore, since $\dot{x}_n(\tau) \in G_k^{\varepsilon/2}(\tau, x_n(\tau))$ a.e. in I for every $k \ge k_o(\varepsilon)$ and for every $n \in \Phi$, we imply from (2) that $\dot{x}_n(\tau) \in G_k^{\varepsilon}(t, x(t))$ for almost all $t \in I$, $k \ge k_o(\varepsilon)$, and $|t-\tau| < \delta(\varepsilon)$. Hence $\dot{x}_n(\tau + (t-\tau)s) \in G_k^{\varepsilon}(t, x(t))$ for almost all $s \in [0,1]$, and since $G_k^{\varepsilon}(t, x(t))$ is convex, the mean value theorem for vector-valued functions asserts that for every $n \in \Phi$

$$\frac{x_n(t) - x_n(\tau)}{t - \tau} = \int_o^1 \dot{x}_n(\tau + (t-\tau)s)\,ds \in G_k^{\varepsilon}(t, x(t))$$

for almost all $t \in I$ where $k \ge k_o(\varepsilon)$ and $|t-\tau| < \delta(\varepsilon)$. Consequently the fact that $G_k^{\varepsilon}(t, x(t))$ is closed, shows that $\frac{x(t) - x(\tau)}{t - \tau} \in G_k^{\varepsilon}(t, x(t))$ for $k \ge k_o(\varepsilon)$ and $|t-\tau| < \delta(\varepsilon)$. If we let now $\tau \to t$, $k \to \infty$ and $\varepsilon \to o$ we obtain $\dot{x}(t) \in \overline{co}\, G(t, x(t))$ provided that $\dot{x}(t)$ exists. But x(t) is absolutely continuous and therefore the above inclusion is meaningful almost everywhere in I. Thus, $x \in X_r$ and consequently $p = x(t) \in A_r(t)$, showing that $A_r(t)$ is closed. To complete the proof it remains to prove that $A_r(t)$ is continuous with respect to the Hausdorff topology. Indeed, suppose that $p_1 \in A_r(t)$ then $p_1 = \varphi(t_1)$ where $\varphi \in X_r$. Considering also the point $p_2 = \varphi(t_2) \in A_r(t_2)$, it is easy to see that $\|p_1 - p_2\| \le |\int_{t_1}^{t_2} M\, d\tau| \le M|t_1 - t_2|$, where M is the bound of the function f. This means that for every $p_1 \in A_r(t_1)$ there is a point $p_2 \in A_r(t_2)$ such that $\|p_1 - p_2\| \le M|t_1 - t_2|$. Hence, $\max\limits_{s \in A_r(t_1)} d(s, A_r(t_2)) \le M|t_1 - t_2|$. Similarly, we can see that $\max\limits_{s \in A_r(t_2)} d(s, A_r(t_2)) \le M|t_1 - t_2|$. It is trivial now to verify that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) = \varepsilon/M$ such that $\rho_H(A_r(t_1), A_r(t_2)) < \varepsilon$, for $|t_1 - t_2| < \delta(\varepsilon)$ showing that $A_r(t)$ is continuous on I.

Q.E.D.

As we have mentioned previously, the existence of solutions of the system S is assumed. An obvious consequence of the assumption is the existence of solutions of the system S. (lemma 4) and furthermore of the system S_r. Once this is established we can see that a solution of X_r must be of the form

$$\rho(t) = x_o + \int_o^t f(t,x(\tau),u)d\tau \quad \text{for almost all } t \in [0,T].$$

Consequently the question of the existence of a time-optimal control for the system S_r consists in finding a $t^* \in [0, T]$ such that

$$t^* = \inf \; [t \in [0,T]; \; z(t) \in A(t)\} \text{, where } z(t) \text{ is the}$$

trajectory of the target, which is assumed to be continuous. The answer is given by the following theorem.

Theorem 3

Let S be a dynamical system that satisfies properties P_1 and P_2. If S_r is the relaxed system corresponding to S, then there exists a time optimal control for S_r.

Proof

Let us suppose that $z(T) \in A_n(T)$ (i.e a solution exists). We define $t^* = \inf \{t \in [0,T]; \; z(t) \in A_n(t)\}$ and we shall show that $z(t^*) \in A_n(t^*)$. Suppose that t_n is a sequence of times that converges to t^*, namely, $t_n \longrightarrow t^*$, such that $z(t_n) \in A_n(t_n)$ for each n. Let $\varphi^n \in X_r$ with $\varphi^n(t_n) = z(t_n)$. Then,

$$||\varphi^n(t^*) - z(t^*)|| \leq ||\varphi^n(t^*) - \varphi^n_n(t_n)|| + ||z(t_n) - z(t^*)|| \leq$$

$$\leq ||z(t_n) - z(t^*)|| + \int_{t^*}^{t_n} M \, d\tau$$

where M is the bound of $||f(t,x,u)||$. Therefore $\lim_{n\to\infty} \varphi^n(t^*) = z(t^*)$ and since $\varphi^n(t^*) \in A_r(t^*)$ and $A_r(t^*)$ is (by lemma 12) closed, it follows that $z(t^*) \in A_r(t^*)$, which was to be proved.

Q.E.D.

CONCLUSION

As can be seen, theorem 3 partially solves the problem of existence of a time-optimal control, with Discontinuous Right-Hand Side. Strictly speaking the existence of a time-optimal control is a direct consequence of theorem 3 in the special case at which the right hand side of the system S_o is a convex set-valued function. This is due to the fact that, in this case, $G(t,x) = \overline{co}\, G(t,x)$ which implies $X_o \equiv X_r$ and furthermore, by lemma 4, $X_r \equiv X_o \equiv X$.

In order to solve the general problem of existence of a time-optimal control for a system with countably many discontinuities in the right hand side, a suggestion would be to prove that the solution set X is dense in X_r.

Another, more general, problem for the non-linear case would be, to prove the existence of a time-optimal control for systems whose right hand side is measurable. The approach followed in this thesis might prove successful if we use Plis's theorem [3] , to approximate the function $G(t,x)$.

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