Applications of impulsive differential equations to the control of malaria outbreaks and introduction to impulse extension equations: a general framework to study the validity of ordinary differential equation models with discontinuities in state

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Abstract

Impulsive differential equations are often used in mathematical modelling to simplify complicated hybrid models. We propose an inverse framework inspired by impulsive differential equations, called impulse extension equations, which can be used as a tool to determine when these impulsive models are accurate. The linear theory is the primary focus, for which theorems analoguous to ordinary and impulsive differential equations are derived. Results explicitly connecting the stability of impulsive differential equations to related impulse extension equations are proven in what we call time scale consistency theorems. Opportunities for future research in this direction are discussed.

Following the work of Smith? and Hove-Musekwa on malaria vector control by impulsive insecticide spraying, we propose a novel autonomous vector control scheme based on human disease incidence. Existence and stability of periodic orbits is established. We compare the implementation cost of the incidence-based control to a fixed-time spraying schedule. Hybrid control strategies are discussed.

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I dedicate this thesis to my parents. Thank you for loving me; thank you for believing in me.

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List of Common Symbols

E	identity matrix
$\prod_{k=r}^{m} A_k$	product of matrices $A_r \cdot A_{r\pm 1} \cdots A_{m\mp 1} \cdot A_m$
$\mathbb{R}_+^n$	non-negative orthant of $\mathbb{R}^n$
A°	interior of the set A
$\overline{A}$	closure of the set A
τ_k	sequence of impulses
a_k	duration sequence
$\mathcal{S}_j(a_k)$	$[\tau_j, \tau_j + a_j)$
$\mathcal{S}(a_k)$	$\bigcup_{j\in\mathbb{Z}} \mathcal{S}_j(a_k)$
$L(t;s)$	see equation (3.1.1)
$X(t;s)$	Cauchy matrix of $\dot{x} = A(t)x$
$U(t)$	matrix solution of the homogeneous equation (3.2.1)
[E]	see remark 3.1.5
[P]	see remark 3.1.5
[E$^+$]	see remark 3.6.5
$\mathcal{P}$	predictable set; see definition 3.6.4
M	monodromy matrix
$\sigma(A)$	spectrum of the matrix A
$\rho(A)$	spectral radius of the matrix A
$\vec{a}$	vectorial identification of $\{a\}$; see definition 3.5.1

$g = O(f)$	there exists $M > 0$ such that, asymptotically, $\lvert g\rvert \leq M\lvert f\rvert$
$\nearrow$ ($\searrow$)	increasing (resp. decreasing) and convergent to
$\gtrsim$	greater than but approximately equal to
S	susceptible human population
I	infected human population
R	temporarily immune human population
M	susceptible mosquito population
N	infected mosquito population
Θ	infection tracking component for incidence-based spraying
r	insecticide spraying efficacy
$\overline{\Theta}$	critical infection threshold
η	observability parameter for incidence-based spraying
Ψ	total mosquito population, $\Psi = M + N$
π	human birth rate
Λ	mosquito birth rate
μ_H	human death rate
μ	mosquito death rate
β_H	rate of infection of humans by mosquitoes
β_M	rate of infection of mosquitoes by humans
γ	death rate due to malaria
α	rate of recovery with temporary immunity
h	rate of recovery with no immunity
δ	rate of immunity loss

Chapter 1

Introduction

1.1 History of impulsive differential equations and the modelling framework

Impulsive differential equations — differential equations (DEs) with discrete jumps in state — have seen application in many fields of science. Historically, the first appearance of such systems was in the early 20th century in engineering, in the context of the Dirac delta distribution [10]. In its present form, most modern monographs on impulsive differential equations make no explicit mention of the delta distribution, as the impulse effect is considered synthetically as opposed to analytically (i.e. in the distributional sense). The theory in its present (synthetic) form was undoubtedly inspired by the work of Pavlidis [25, 26], who was interested in dynamical descriptions of pulse-frequency modulation [24]. He appears to have been the first to abstractly study systems with discontinuities without explicit reference to analytical constructions such as the Dirac delta function, instead describing the trajectories in terms of transition operators or jump maps; see Appendix A for the present treatment. Pavlidis' work now falls under the heading of "discontinuous flows" [3], which may be seen as flows (in the dynamical systems sense) that exhibit discontinuities in their trajectories. See the monographs [3, 5, 16, 23, 29] for some of the elementary theory of differential equations with impulse effects.

It is our view that mathematical modelling with impulsive differential equations is often done as a way of simplifying a more complicated model. Agarwaal and Leela [1], in their survey of the works of V. Lakshmikantham (one of the pioneers of impulsive system theory), describes the motivation behind such equations as follows.

"Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. This is due to short term perturbations whose duration is negligible in comparison with the duration of the process. It is natural, therefore, to assume that such perturbations act instantaneously, that is, in the form of impulses."

Another reference to the assumptions, from Samoilenko and Perestyuk [29]:

"To describe mathematically an evolution of a real process with a short-term perturbation, it is sometimes convenient to neglect the duration of the perturbation and to consider these perturbations to be 'instantaneous'."

Clearly, one of the assumption of modelling with impulsive differential equations is some kind of time-scale separation, in which the time-scale of the perturbation must be small compared to the time scale of the underlying dynamics. However, this begs the question of how much separation there should be.

On the other hand, in some situations, modelling perturbations by discontinuities could be argued to more accurately represent reality. Liu and Chen [18] consider a predator-prey model with Holling type II functional response and periodic impulsive introduction of predator. They opt for an impulsive addition of predator as opposed to continuous periodic addition for two reasons. First, continuous periodic forcing makes the analysis somewhat intractable. The other proposed reason is that discontinuous introduction of predator is more realistic than continuous introduction. We agree with this, at least for small quantities of predator, because, in real life, one cannot introduce a fractional amount of some organism into an environment. However, if the amount of predator to be introduced is very large, then a realistic model of predator introduction would necessitate a sequence of finely spaced impulses, the sum of which accounted for all new predators. At this point, it could be argued that a continuous perturbation of predator influx would be an adequate approximation, especially if it takes a non-negligible amount of time for the population to become well-mixed.

The above example illustrates that there may be a modelling choice to be made. In the above, introduction of a small amount of predator is more accurately modelled by a discontinuity. For higher level of predator introduction, a continuous time-dependent perturbation may provide a better approximation. However, certain analytical techniques may be easier to apply to impulsive systems than to continuous time-dependent systems. In particular, for autonomous impulsive differential equations, the local stability of periodic orbits can be reduced to calculating the eigenvalues associated with the trivial solution of a linear time-dependent system. Conversely, continuous hybrid systems with time-dependent perturbations (which, under certain circumstances, can be reduced to autonomous impulsive DEs under suitable time-scale separation assumptions; see Appendix D), stability is a very difficult problem.

There are therefore several reasons why impulsive differential equations may be favoured over continuous differential equations with continuous perturbations. It might not, however, be clear which method more accurately describes reality. As mentioned earlier, one assumption that must be satisfied whenever impulses are used to simplify perturbative behaviour is that these perturbations must occur on a short time scale. Our objective is to determine conditions under which impulsive systems and continuous systems that resemble them (in a way to be made precise later) both exhibit the same qualitative behaviour.

1.1.1 A survey of examples

To begin, we present a few examples from the literature in which impulsive differential equations are used in an applied context. We highlight some of the shortcomings of impulsive differential equations as models, and illustrate how accomplishing our objective, even in a restricted way, will strengthen the theory as a whole.

Example 1.1.1. One of the classical models of population-level infectious disease to incorporate impulse effects is the SIR model with pulse vaccination and permanent immunity, described in [13, 32]. With pulse vaccination, some proportion, $p \in (0, 1)$, of the susceptible population is vaccinated against a disease at sequential times. That is, if S denotes the fraction of susceptible individuals, then

$$\begin{aligned} \dot{S} &= m - (\beta I + m)S, && t \neq \tau_k, \\ \Delta S &= -pS, && t = \tau_k, \end{aligned}$$

describes how this fraction evolves, where I is the fraction of infected individuals, $1/m$ is the mean life expectancy and τ_k is a sequence of pulse vaccination times. Effectively, pulse vaccination "removes" susceptibles from the population. This is in contrast to continuous vaccination, which is modelled by an effective rescaling of the birth rate [32]:

$$\dot{S} = (1 - p)m - (\beta I + m)S.$$

Pulsed vaccination is an approximation, in the sense that, realistically, it is not possible to simultaneously vaccinate a nontrivial proportion of the susceptible population. Indeed, we have a very obvious problem if there are fewer people administering vaccines than there are people who need to be vaccinated. However, the impulsive pulse vaccination strategy can be naturally interpreted as a time-scale separation approximation of a system with fast, perturbative vaccination;

$$\dot{S} = m - (\beta I + m)S, \qquad t \notin [\tau_k, \tau_k + d),$$

$$\dot{S} = m - (\beta I + m)S - f_k(t, S(\tau_k)), \qquad t \in [\tau_k, \tau_k + d),$$

where d is the duration of the perturbation and $f_k(t, S)$ is the vaccination rate given input fraction S of unvaccinated susceptibles, which satisfies

$$\int_{\tau_k}^{\tau_k+d} f_k(t, S(\tau_k))dt = pS(\tau_k).$$

Example 1.1.2. Smith? and Schwartz [31] consider the problem of optimal vaccine strength and vaccination frequency in a model of HIV vaccination via cytotoxic T-lymphocyte (CTL) activation (note that, in this context, vaccination refers to activation of CTLs). For the purpose of their model, they assume that the effect of the vaccine is instantaneous, resulting in a system of impulsive differential equations

$$\begin{aligned} \frac{dT}{dt} &= \pi - dT - pCT, && t \neq \tau_k \\ \frac{dC}{dt} &= \alpha CT - \delta C, && t \neq \tau_k \\ \Delta C &= \tilde{C}, && t = \tau_k, \end{aligned}$$

where C denotes the number of activated CTLs, T the number of infected T-cells, and $\tilde{C}$ is the number of cells activated by vaccination. Other parameters are positive and related to the dynamics of the system in question. They determined that, with this model, the average number of infected T-cells can be brought down to an arbitrarily low level, given sufficiently high activation $\tilde{C} > 0$, or sufficiently small times between vaccination $\tau = \tau_k - \tau_{k-1}$. They then performed numerical simulations, which included replacing the impulse effect by a continuous, linear CTL activation; for example, describing the evolution of CTLs by

$$\begin{aligned} \frac{dC}{dt} &= \alpha CT - \delta C, && t \notin [\tau_k, \tau_k + d), \\ \frac{dC}{dt} &= \alpha CT - \delta C + \frac{\tilde{C}}{d}, && t \in [\tau_k, \tau_k + d), \end{aligned}$$

would suffice, where d is time required for the vaccination to have its intended effect. For the parameters used, both simulations predicted qualitatively the same result: an asymptotically stable periodic orbit. However, realistically, activation of CTLs may occur faster or slower at different times on the activation interval $[\tau_k, \tau_k + d)$, and a linear mechanism does not describe this. It would be useful to know that the stability of the periodic orbit doesn't crucially depend on the "shape" of the vaccination pulse.

Example 1.1.3. Harvesting of a single population exhibiting logistic growth can be modelled by the ordinary differential equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - Ex$$

where r is the intrinsic growth rate, K is the carrying capacity and E is the (continuous) harvesting effort. Zhang, et al. remark [38] that the assumption of continuous harvesting may not be realistic; fisherman do not fish for twenty-four hours consecutively, and they fish net by net. They conclude that it makes sense to model the harvesting process by short, temporary perturbations. They then formulate a model of logistic growth with harvesting described by impulsive differential equations.

However, this too comes at the cost of realism. Just as a continuous, constant harvesting E is unrealistic because fisherman do not fish constantly, impulsive harvesting is unrealistic because not all fisherman will fish at exactly the same time. In the rare situation in which the number of harvesting agents is small (e.g. a small farm), the impulse assumption still ignores the population dynamics during the harvesting process. With the logistic equation (which is very well understood), this may not be much of a problem. However, the same cannot be said in general, in particular if the underlying dynamics are chaotic, as can be seen in forced predator prey models [28, 33], the Duffing oscillator and the Lorentz attractor [34], among others. If there is competition for the population being harvested (for example, a predator-prey model with external harvesting by humans), one may not be able to ignore these short-term dynamics.

It therefore makes sense to consider what may happen if impulsive harvesting is replaced by fast, perturbative harvesting. If the dynamical properties are similar to the impulsive case, then whichever formulation provides the most analytically tractable results may be used. This essentially amounts to a reduction of bias in the qualitative results; how the harvesting is modelled does not influence the qualitative behaviour, so we may use whatever formulation we want, and the only difference is in the quantitative aspects of the results.

1.2 Modelling malaria vector control using impulsive differential equations

The World Health Organization [35] describes malaria as an infectious disease caused by plasmodium parasites that is spread to humans by the bites of female *Anopheles* mosquitoes. There are an estimated 219 million cases of malaria each year, with an estimated 660,000 deaths, with most of these occuring in the poorest countries. Interventions to abate the spread of the disease include treatment, insecticide-treated nets and indoor residual spraying (IRS).

IRS is the application of insecticides to walls and roofs of houses and animal shelters. Mosquitoes that land on these treated walls die, effetively reducing the lifespan of mosquitoes in the area, making it less likely that they can infect another human. IRS has been successfully used on many occasions to reduce the incidence of malaria [36]. In a 2008 article [30], Smith? and Hove-Musekwa propose a mathematical model of malaria that incorporates indoor residual spraying using impulsive differential equations. They determine the minimal period of insecticide spraying required to reduce the mosquito population below a desired level.

Smith? and Hove-Musekwa's results pertain to a strategy whereby insecticide is sprayed according to a predetermined schedule; it is not dependent on the state of the epidemic. We consider another strategy, which we refer to as incidence-based spraying, that is state-dependent. This strategy has the advantage of relying on readily obtained data (namely, incidence data), provided this data is being collected and is accessible. Moreover, this strategy may be more cost-effective than fixed spraying, so it is of interest to consider alternative strategies such as this.

Various analytical properties of the impulsive differential equations will be studied for both control strategies, including the existence and uniqueness of periodic orbits and their stability by linear analysis and bifurcation theory. Sampling methods will then be employed to compare the cost of each strategy. This is covered in Chapter 4,5 and the first half of Chapter 6.

Finally, in the latter half of Chapter 6, we briefly consider hybrid strategies, which combine fixed frequency spraying with some form of incidence-based spraying. These strategies are far more complex, and we only briefly describe a few types, without performing any analysis.

Chapter 2

Impulse Extension Equations: General Theory

Consider an impulsive differential equation with impulses at fixed times

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), && t \neq \tau_k \\ \Delta x &= I_k(x), && t = \tau_k. \end{aligned} \tag{2.0.1}$$

with phase space $\Omega \subset \mathbb{R}^n$. We will now construct a (functional) differential equation with continuous solutions that in some sense "approximates" the above impulsive differential equation. It will often be notationally convenient to identify an impulsive differential equation (2.0.1) with a triple (f, I_k, τ_k), where each symbol represents the function, sequence of functions or sequence appearing in (2.0.1). The only assumption that must be imposed at this point is monotonicity and unboundedness of the impulse times. That is, $\tau_k < \tau_{k+1}$ for all $k \in \mathbb{Z}$ and $\tau_k \to \infty$ as $k \to \infty$. We begin with some definitions.

Definition 2.0.1. *Consider an impulsive differential equation* (f, I_k, τ_k) *as in* (2.0.1).

- *A* duration sequence over τ_k *is sequence of positive real numbers* a_k *such that* $\tau_k + a_k < \tau_{k+1}$ *for all* $k \in \mathbb{Z}$. *We denote* $\mathcal{S}_j = \mathcal{S}_j(a_k) \equiv [\tau_j, \tau_j + a_j)$ *and*

$\mathcal{S} = \mathcal{S}(a_k) \equiv \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j$.

- *A sequence of functions* $\varphi_k : \mathcal{S}_k \times \Omega \to \mathbb{R}^n$ *is an* impulse extension for (f, I_k, τ_k) compatible with a_k *if for all* $k \in \mathbb{Z}$ *and* $x \in \Omega$*, the function* $\varphi_k(\cdot, x)$ *is locally integrable and*

$$\int_{\mathcal{S}_k(a_k)} \varphi_k(t,x)dt = I_k(x).$$

- *Given a duration sequence* a_k *and a compatible impulse extension* φ_k *for the impulsive differential equation* (f, I_k, τ_k)*, the* impulse extension equation associated to (f, I_k, τ_k) induced by (a_k, φ_k) *is the (functional) differential equation*

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= f(t,x) + \varphi_k(t, x(\tau_k)), && t \in \mathcal{S}_k. \end{aligned} \tag{2.0.2}$$

As described by Church and Smith? [9] for linear systems specifically, differential equations of this type can be seen as continuous versions of impulsive systems, where the "impulse" $I_k(x)$ is carried by the function $\varphi_k(t, x)$. This can be justified as follows. If the vector field is "turned off" artificially, so that we set $f(t, x) = 0$, then a solution for $t \in \mathcal{S}_k$ of the initial-value problem $x(\tau_k) = x_k$ is a solution of the differential equation

$$\frac{dx}{dt} = \varphi_k(t, x_k).$$

Consequently, a unique solution exists (see Carathéodory conditions; [15]) and is given by

$$x(t) = x_k + \int_{\tau_k}^{t} \varphi_k(s, x_k)ds.$$

For $t = a_k$, we obtain

$$x(a_k) = x_k + I_k(x_k) = x_k + \Delta x_k.$$

Therefore, the impulse extension φ_k "applies" the effect of the impulse over a finite, nonzero length of time a_k. However, in reality, the vector field is not "off", so the other dynamics might contribute. See Appendix B for further discussion.

2.1 Existence and uniqueness of solutions for admissible initial-value problems

One aspect of (2.0.2) that makes it difficult to study is that it cannot be considered as an ordinary differential equation, since the right-hand side in general depends on the solution at two different times. Indeed, a reference to a solution $x(t)$ satisfying $x(t_0) = x_0$ for $t_0 \in \mathcal{S}_k^\circ$ does not make sense, since the right-hand side of (2.0.2) requires knowledge of the solution at time τ_k; namely, it takes as one of its inputs the point $x(\tau_k)$. In a way, the dimension of an impulse extension equation is "almost" twice that of the original impulsive differential equation.

We therefore at the very least require a modified definition of "solution" for such an equation, and we will find that, in general, we must also restrict our attention to a limited class of initial-value problems if we wish to talk about existence and uniqueness of solutions in a satisfactory way. This can, however, be circumvented when the equations satisfy a linearity property, although the existence and uniqueness conditions are very strict and in general difficult to verify in all but the simplest of cases.

Alternatively, by doubling the dimension of the phase space, solutions of an impulse extension equation (2.0.2) can be seen as solutions of a particular impulsive differential equation. This approach will be elaborated upon in Section 2.3.

Definition 2.1.1. *Consider an impulse extension equation* (2.0.2)*. The point* $(t_0, x_0) \in \mathbb{R} \times \Omega$ *is*

- admissible for (2.0.2) *if* $(t_0, x_0) \in (\mathbb{R} \setminus \mathcal{S}^\circ) \times \Omega$*,*
- strongly admissible for (2.0.2) *if* $(t_0, x_0) \in (\mathbb{R} \setminus \overline{\mathcal{S}}) \times \Omega$*,*
- indeterminate for (2.0.2) *if* $(t_0, x_0) \in \mathcal{S}^\circ \times \Omega$*,*
- k-indeterminate for (2.0.2) *if* $(t_0, x_0) \in \mathcal{S}_k^\circ \times \Omega$*.*

We will find that admissible points can be used to define initial-value problems that have unique solutions under fairly "typical" conditions on the impulse extensions φ_k and the vector field f, while indeterminate points generate problems may have no solutions or infinitely many solutions, even under very strict conditions on the impulse extensions and vector field.

Definition 2.1.2. *A function $\phi : I \to \mathbb{R}^n$ defined on an interval $I \subset \mathbb{R}$ is a* solution *of* (2.0.2) *if it satisfies the following conditions:*

- *ϕ is continuous,*
- *if $I \cap \mathcal{S}_k$ is nonempty, then $\tau_k \in I$,*
- *$\phi(t)$ is differentiable almost everywhere in I,*
- *$\frac{d\phi}{dt}(t) = f(t, \phi(t))$ almost everywhere on $I \setminus \mathcal{S}$,*
- *$\frac{d\phi}{dt}(t) = f(t, \phi(t)) + \varphi_k(t, \phi(\tau_k))$ almost everywhere on $I \cap \mathcal{S}_k$.*

The function $\phi(t)$ is a solution of the initial-value problem (2.0.2)-(2.1.1)

$$x(t_0) = x_0 \tag{2.1.1}$$

for some $(t_0, x_0) \in \mathbb{R} \times \Omega$ if, in addition, $\phi(t_0) = x_0$.

The second condition guarantees that the derivative of a solution can always be given meaning when comparing to the right-hand side of (2.0.2). The other conditions are typical properties of solutions of differential equations (in the extended, Carathéodory sense; see [15] for details), together with the defining property of (2.0.2) describing the dynamics at times $t \in \mathcal{S}$.

We may sometimes make reference to "the initial-value problem $x(t_0) = x_0$". In this case, the differential equation in question should be clear from the context, and the quoted phrase should be interpreted as the initial-value problem in the sense of (2.0.2)-(2.1.1) with initial data (t_0, x_0).

Theorem 2.1.1. Consider the impulse extension equation (2.0.2). The following hold.

- If $f(t,x)$ is locally Lipschitz continuous in x and continuous in t, then there exists $\epsilon > 0$ such that for every *strongly admissible* point (t_0, x_0), the initial-value problem (2.0.2)-(2.1.1) has a unique solution defined on the interval $[t_0-\epsilon, t_0+\epsilon]$.

- If $f(t,x)$ is locally Lipschitz continuous in x, continuous in t and $\varphi_k(t,y)$ is continuous in t for all $k \in \mathbb{Z}$ and $y \in \Omega$, then there exists $\epsilon > 0$ and $\eta \geq 0$ such that for every *admissible* point (t_0, x_0), the initial-value problem (2.0.2)-(2.1.1) has a unique solution defined on the interval $[t_0 - \eta, t_0 + \epsilon]$, and if $t_0 \notin \{\tau_k + a_k\}_{k\in\mathbb{Z}}$, then $\eta > 0$.

Proof: It is obvious from the above definitions that if (t_0, x_0) is a strongly admissible point for an impulse extension equation, then locally solving the initial-value problem for a solution $x(t)$ satisfying $x(t_0) = x_0$ reduces to solving the ordinary differential equation

$$\frac{dx}{dt} = f(t,x).$$

The first result then follows by the Picard–Lindelöf theorem [15].

The second part of the theorem deals with initial times on the boundary of $\mathcal{S}$; that is, $t_0 \in \{\tau_k, \tau_k+a_k\}_{k\in\mathbb{Z}}$. The above analysis holds if we seek solutions $x(t)$ defined for time $t \geq t_0$, provided $t_0 = \tau_k + a_k$. Convsersely, if $t_0 = \tau_k$, then locally, a solution defined for $t \geq \tau_k$ can be obtained by solving the ordinary differential equation

$$\frac{dx}{dt} = f(t,x) + \varphi_k(t, x_0).$$

We must now extend these solutions locally for times $t < t_0$.

If $t_0 = \tau_k$ and $z(t)$ is the solution defined on $[t_0, t_0 + \epsilon]$ given above, we let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = f(t,y),$$

$$y(t_0) = x_0,$$

defined on the interval $[t_0 - \eta, t_0 + \eta]$ for some $\eta > 0$. Define

$$x(t) = \begin{cases} y(t), & \tau_k - \eta \le t < \tau_k, \\ z(t), & \tau_k \le t \le \tau_k + \epsilon \end{cases}$$

Then $x(t_0) = x_0$, $x(t)$ is differentiable on $(\tau_k - \eta, \tau_k + \epsilon)$ except possibly at $t = \tau_k$ and $x(t)$ satisfies the differential equations appearing in (2.0.2) on the appropriate intervals almost everywhere. Therefore, $x(t)$ is a solution. Moreover, it is unique. Indeed, for $t \le \tau_k$, there can be at most one solution that coincides with $x(t)$ where it is defined, and that is precisely $z(t)$. Similarly for $t \ge \tau_k$. Therefore, the solution is unique.

In the case where $t_0 = \tau_k + a_k$, it is in general not possible to extend to $t < t_0$ (see Theorem 2.3.1), so we take $\eta = 0$. Therefore, the result holds for $t_0 \in \partial\mathcal{S}$, and this concludes the proof. ∎

Forward continuation of solutions can be accomplished by similar means as with ordinary differential equations, and, as the proof is nearly identical, we omit it.

Theorem 2.1.2. Suppose $f(t, x)$ is locally Lipschitz continuous in x, continuous in t and $\varphi_k(t, x)$ is continuous in t for all $k \in \mathbb{Z}$ and $x \in \Omega$. Let (t_0, x_0) be admissible for (2.0.2). If $\phi : I \to \Omega$ is any solution of the initial-value problem (2.0.2)-(2.1.1) defined on an interval I containing t_0, then there is a unique forward continuation of ϕ to a maximal forward interval of existence I^+. Moreover, if $\phi^+ : I^+ \to \Omega$ is the solution of the initial-value problem $x(t_0) = x_0$ and I^+ is the maximal forward interval of existence, then $(t, \phi^+(t))$ approaches the boundary of $\mathbb{R} \times \Omega$ as $t \to \sup I^+$.

We do not discuss continuous/smooth dependence of solutions on initial conditions and parameters. This question will be investigated in subsequent research.

2.2 Failure of existence and uniqueness of solutions in a one-dimensional phase space

We now turn our attention to indeterminate initial-value problems; that is, the problem of finding a solution to the initial-value problem (2.0.2)-(2.1.1) with an indeterminate initial condition (t_0, x_0). The situation is easier to visualize and more results can be obtained if we consider a one-dimensional phase space, $\Omega \subset \mathbb{R}$. For the rest of this section, we assume that the conditions of Theorem 2.1.2 hold, so that admissible initial-value problems have unique solutions defined on a maximal forward interval of existence. We further assume that all solutions are continuable to $+\infty$.

Definition 2.2.1. *A point* $(\hat{t}, \hat{x}) \in \mathcal{S}_k^\circ \times \Omega$ splits on $[\underline{x}, \overline{x}]$ *if* $\underline{x} < \overline{x}$ *and the unique forward solutions* ϕ *and* ψ *of the admissible initial-value problems* $x(\tau_k) = \underline{x}$ *and* $x(\tau_k) = \overline{x}$ *respectively satisfy* $\phi(\hat{t}) = \hat{x} = \psi(\hat{t})$. *The number* $\hat{t}$ *is a* splitting time *and the interval* $[\underline{x}, \overline{x}]$ *is a* wall for $\hat{t}$.

In other words, a point splits if there are two distinct solutions to its associated initial-value problem; see Figure 2.1. If the phase space is compact, then some properties of the topological structure of the splitting times can be elucidated, given moderate continuity conditions on the impulse extensions.

Lemma 2.2.1. Suppose Ω is compact and $\varphi_k(t, x)$ is locally Lipschitz continuous in a neighbourhood of $\{\tau_k\} \times \Omega$. The following results hold.

- If $t_n \to \tau_k$ is a sequence of splitting times with walls M_n that are nested, then $\bigcap_{n\in\mathbb{N}} M_n$ is either empty or a singleton.
- If (t_n, x_n) splits on M_n and $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$, then $t_n \not\to \tau_k$.

Proof: Recall that a sequence of closed intervals is *nested* if $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}$. We proceed by contradiction. Suppose $\bigcap_{n\in\mathbb{N}} M_n = M$ is an interval and the

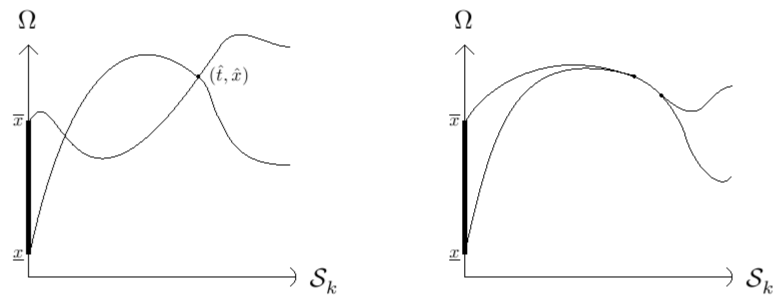

Figure 2.1: Left: The point $(\hat{t},\hat{x})$ splits on $[\underline{x},\overline{x}]$, indicated in bold on the Ω axis. Note that solution curves may intersect before reaching the point $(\hat{t},\hat{x})$. Right: Two solutions associated to different initial conditions may merge tangentially, stay together for a finite period of time and then split tangentially.

sequence M_n is nested. As Ω is bounded, by the Bolzano–Weierstrass theorem, we may assume that x_n converges by passing to a subsequence, $x_{n_k} \to x$. By compactness, $x \in \Omega$. Since M_n is closed for all n, M is closed, and the associated subsequence of closed intervals M_{n_k} must therefore also converge to M under intersection: $\bigcap M_{n_k} = M$. Therefore, by passing to subsequences, we may assume that (t_n, x_n) splits on M_n, that $(t_n, x_n) \to (\tau_k, x)$ and $\bigcap M_n = M$.

Since the M_n are closed and nested, defining $M_n^+ = \max M_n$ and $M_n^- = \min M_n$, we have $M_n^+ \to M^+$ and $M_n^- \to M^-$, where $M^+ = \max M$ and $M^- = \min M$. Without loss of generality, we may assume that $x \neq M^+$. Define the sequence

$$p_n^+ = \frac{x_n - M_n^+}{t_n - \tau_k}.$$

Since $x_n \nrightarrow M_n^+$ and $t_n \to \tau_k$, we conclude that p_n^+ is unbounded. However, we note that p_n^+ has the representation

$$p_n^+ = \frac{\varphi_k(t_n, M_n^+) - \varphi_k(\tau_k, M_n^+)}{t_n - \tau_k}.$$

By the Lipschitz condition on φ_k, there exists a neighbourhood N of (τ_k, M^+) on which

φ_k is Lipschitz continuous with constant K. Since (t_n, M_n^+) converges to (τ_k, M^+), for all n suficiently large, we have $(t_n, M_n^+) \in N$ and $(\tau_k, M_n^+) \in N$. Consequently, for all n sufficiently large,

$$|\varphi_k(t_n, M_n^+) - \varphi_k(\tau_k, M_n^+)| \leq K|t_n - \tau_k|,$$

from which we conclude that $|p_n^+| \leq K$, contradicting the above conclusion that p_n^+ is unbounded. Therefore, M is not an interval.

We now consider the other case. Suppose $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$. By contradiction, assume $t_n \to \tau_k$. By repeating the construction above, we may assume $(t_n, x_n) \to (\tau_k, x)$ by passing to a subsequence, and, indeed, since subsequences are order-preserving, the inclusions $M_n \subset M_{n+1}$ still hold.

Since Ω is compact and $M_n \subset \Omega$ is an increasing sequence of closed intervals, the union $\bigcup M_n = M \subset \Omega$ exists, and if one defines $M^+ = \sup M$ and $M^- = \inf M$, then $\overline{M} \subset [M^-, M^+] \subset \Omega$, $M_n^+ \to M^+$ and $M_n^- \to M^-$, where $M_n^{\pm}$ are defined as before. The same argument as above, involving the sequence p_n^+, provides a contradiction. We conclude that $t_n \nrightarrow \tau_k$. ∎

Although not incredibly interesting on its own, the above lemma suggests that nonuniqueness points may result if the phase space is not compact or if φ_k does not satisfy a Lipschitz condition. We now present two counterexamples that support this hypothesis.

Example 2.2.2. Let $\Omega = \mathbb{R}$ be the phase space, and let $\mathcal{S}_k = [2k, 2k+1)$. Let $q > 1$ be an integer, define $\varphi(x) = -x^q$ and consider the following impulse extension equation.

$$\begin{aligned} \frac{dx}{dt} &= 0, && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= \varphi(x(2k)), && t \in \mathcal{S}_k. \end{aligned}$$

Note that $\varphi(x)$ is smooth. The solution of the initial-value problem $x(2k) = x_k$ is given by, for $t \in \mathcal{S}_k$,

$$x(t; x_k) = x_k - (t - 2k)x_k^q.$$

We observe that $x(t; 0) = 0$ and, for $x_k \neq 0$,

$$x(2k + x_k^{1-q}; x_k) = 0.$$

For $x_k > 1$, we have $2k + x_k^{1-q} \in \mathcal{S}_k$. Moreover, the map $x_k \mapsto 2k + x_k^{1-q}$ is a surjection of $\{|x_k| > 1\}$ onto $\mathcal{S}_k^\circ$. It follows that, for all $t \in \mathcal{S}_k^\circ$, there exists x_k such that $x(t; 0) = x(t; x_k) = 0$. Consequently, every $t \in \mathcal{S}_k^\circ$ is a splitting time. We conclude that every $t \in \mathcal{S}^\circ$ is a splitting time. See Figure 2.2 for a plot that illustrates the presence of these splitting times.

Considering the sequence of impulses $\tau_k = 2k$, the function $\varphi(x)$ is seen to be compatible with the duration sequence $a_k \equiv a = 1$ and satisfies

$$\int_{\mathcal{S}_k} \varphi(x) dt = -x^q.$$

Therefore, the above impulse extension equation can be considered induced by $(\varphi, 1)$ and associated to the impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= 0, & t &\neq 2k, \\ \Delta x &= -x^q, & t &= 2k. \end{aligned}$$

The above impulsive differential equation has the forward global existence and uniqueness property; every initial-value problem $x(t_0) = x_0$ has a unique solution that is defined for all $t \geq t_0$, even though the induced impulse extension equation does not.

Example 2.2.3. Let $\Omega = [-1, 1]$ be the phase space, and let $\mathcal{S}_k = [2k, 2k+1)$. Define

$$\varphi(x) = \begin{cases} 0 & : \ x = 0, \\ -x|x|^{-1/2} & : \ x \neq 0. \end{cases}$$

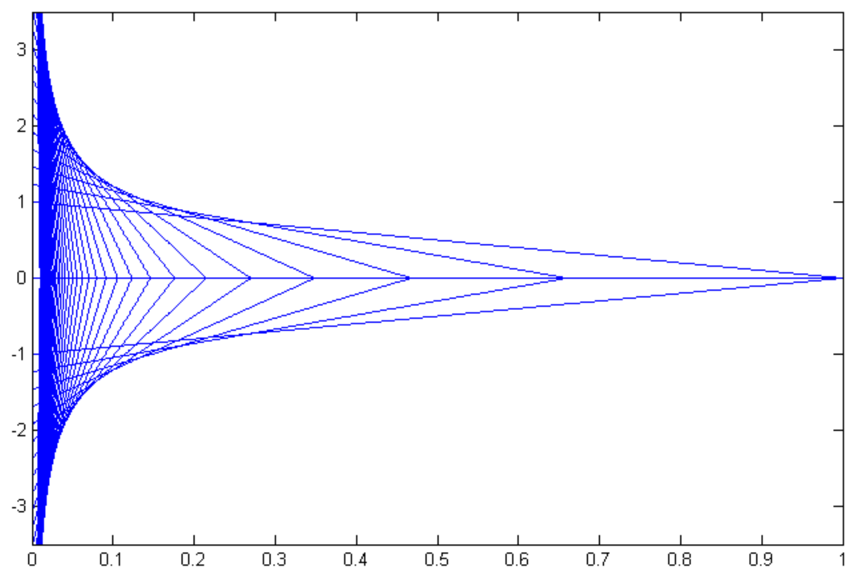

Figure 2.2: Solution curves of Example 2.2.2 with $q = 3$ plotted for $0 \leq t \leq x_0^{-2}$, with $x(0) = x_0 \in [-10, -1] \cup [1, 10]$ and 44 evenly spaced samples; the horizontal axis is t and the vertical axis x, with the viewing window restricted to $|x| \leq 3.5$ for clarity. The trivial solution $x(t) = 0$ is plotted for $0 \leq t \leq 1$ for reference. Notice that the lines become ever more steep as $|x_0|$ increases; this is not possible when Ω is compact.

As defined, $\varphi(x)$ is countinuous everywhere and differentiable everywhere except at $x = 0$. In particular, $\varphi(x)$ is not locally Lipschitz on Ω because $\varphi'(x) \to -\infty$ as $x \to 0$. Consider the following impulse extension equation.

$$\begin{aligned} \frac{dx}{dt} &= 0, && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= \varphi(x(2k)), && t \in \mathcal{S}_k. \end{aligned}$$

The solution of the initial-value problem $x(2k) = x_k$ is given by, for $t \in \mathcal{S}_k$,

$$x(t; x_k) = x_k - (t - 2k)x_k \cdot |x_k|^{-1/2}.$$

Notice that, for all $x_k \in \Omega$ and all $t \in \mathcal{S}_k$, we have $x(t; x_k) \in \Omega$; see Figure 2.3. We observe that $x(t; 0) = 0$ and, for $x_k \neq 0$,

$$x(2k + \sqrt{|x_k|}; x_k) = 0.$$

For $|x_k| < 1$, we have $2k + \sqrt{|x_k|} \in \mathcal{S}_k$. Moreover, the map $x_k \mapsto 2k + \sqrt{|x_k|}$ is a surjection of $\{|x_k| < 1\}$ onto $\mathcal{S}_k^\circ$. It follows that, for all $t \in \mathcal{S}_k^\circ$, there exists x_k such that $x(t; 0) = x(t; x_k) = 0$. Consequently, every $t \in \mathcal{S}_k^\circ$ is a splitting time; see Figure 2.3. We conclude that every $t \in \mathcal{S}^\circ$ is a splitting time.

Considering the sequence of impulses $\tau_k = 2k$, the function $\varphi(x)$ is seen to be compatible with the duration sequence $a_k \equiv a = 1$ and satisfies

$$\int_{\mathcal{S}_k} \varphi(x) dt = \varphi(x).$$

Therefore, the above impulse extension equation can be considered induced by $(\varphi, 1)$ and associated to the impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= 0, && t \neq 2k, \\ \Delta x &= \varphi(x), && t = 2k. \end{aligned}$$

The above impulsive differential equation has the global forward existence and uniqueness property; every initial-value problem $x(t_0^+) = x_0$ has a unique solution defined

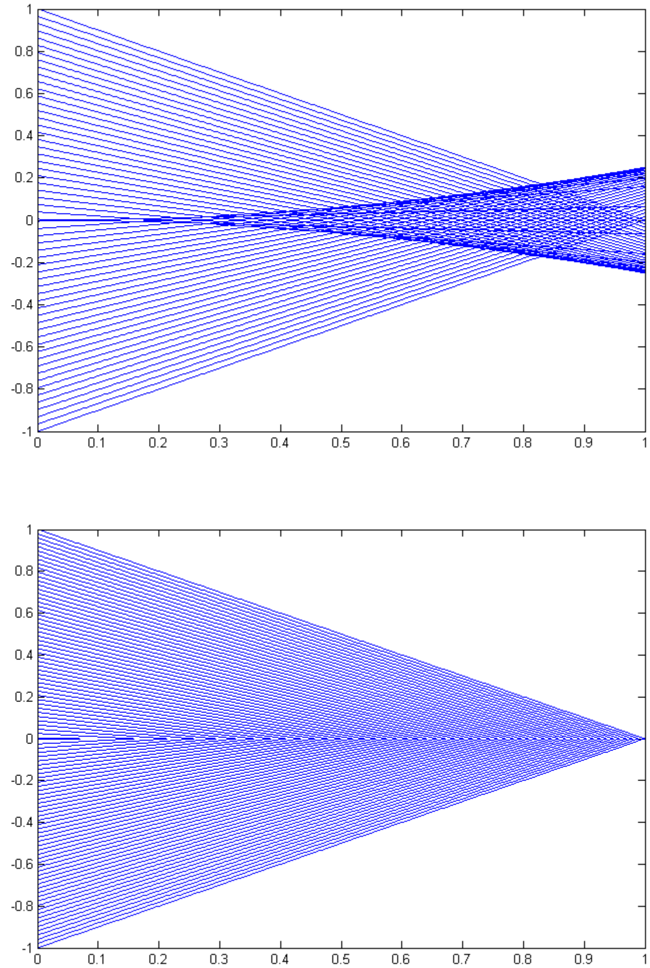

Figure 2.3: Top: Solution curves of Example 2.2.3 plotted for $0 \leq t \leq 1$, with t on the horizontal axis and x on the vertical. Notice that $x \in [-1, 1]$, so all solutions remain in the phase space. By analyzing the critical points of the solution $x - tx|x|^{-1/2}$ at $t = 1$, we find that all curves enter the interval $[-\frac{1}{4}, \frac{1}{4}]$ at $t = 1$. Bottom: Solution curves $x(t; x_0)$ plotted for $0 \leq t \leq \sqrt{|x_0|}$. The angle that the solution $x(t; x_0)$ makes with the line $x = 0$ is given by $\arctan(x_0 \cdot |x_0|^{-1/2})$, and this decreases arbitrarily fast as $x \to 0$. Figures are meant to illustrate the regions of phase space where solutions exist and/or are not unique, rather than visualize specific solutions..

for $t \geq t_0$, even though the induced impulse extension equation does not. If the phase space is restricted to $(-1, 1)$, then the solutions are defined uniquely for all $t \in \mathbb{R}$.

Moreover, one can see from Figure 2.3 (top) that the indeterminate initial-value problems $x(t_0) = \pm 1$ for $t_0 \in (0, 1]$ do not have solutions at all. In fact, there are many indeterminate problems that do not have solutions; they are all those points in the figure [2] contained in the whitespace. In this sense, the classical Peano existence theorem or an appropriate analogue does not hold; continuity of the relevant functions appearing in the right-hand side of (2.0.2) is insufficient to prove existence of solutions.

2.3 An analogue of the Peano existence theorem

For an indeterminate point (t_0, x_0), it has been demonstrated that there may be no solution $x(t)$ of the impulse extension equation that satisfies $x(t_0) = x_0$. This non-existence of solutions remains a difficulty even when the vector fields appearing in (2.0.2) are very regular. The following sufficient condition for existence of a solution for the indeterminate initial-value problem arises when one attempts to emulate the Peano existence proof from ordinary differential equations in the impulse extension equations case. We state it without proof, as the proof is nearly identical to the ordinary differential equations case (see [15] for a proof).

Theorem 2.3.1. *Suppose $f(t, x)$ is continuous and $\varphi_k(t, x)$ are continuous. Let $(t_0, x_0) \in \mathbb{R} \times \Omega$ be k-indeterminate for the impulse extension equation* (2.0.2). *For $\alpha, \beta > 0$, define the set*

$$U(\alpha, \beta) = (\tau_k, t_0 + \alpha) \times B_\beta(x_0)$$

where $B_\beta(x_0)$ is the open ball of radius β centered at x_0. Suppose there exist $\alpha, \beta > 0$

[2] Specifically, all those points not contained in the union of what appears to be a triangle and a hyperbolic cone.

such that $B_\beta(x_0) \subset \Omega$ *and*

$$\max\{\alpha, t_0 - \tau_k\} \cdot \left(\sup_{U(\alpha,\beta)} |f(t,x)| + \sup_{U(\alpha,\beta)} |\varphi_k(t,x)| \right) \leq \beta. \tag{2.3.1}$$

Then there exists a solution $x(t)$ *of* (2.0.2) *satisfying* $x(t_0) = x_0$.

The "usual" proof fails because the maximum term appearing on the left of the inequality in (2.3.1) cannot necessarily be made sufficiently small by an appropriate choice of α, and $U(\alpha,\beta)$ does not become arbitrarily small when we take α and β small.

Example 2.3.1. As an example of this failure in action, consider the following simple (linear) example.

$$\begin{aligned} \frac{dx}{dt} &= 0, && t \notin \mathcal{S} \\ \frac{dx}{dt} &= -x(3k), && t \in [3k, 3k+2). \end{aligned} \tag{2.3.2}$$

Here, $k \in \mathbb{Z}$. Let us attempt to solve the initial-value problem $x(1) = x_1$ for arbitrary $x_1 \in \mathbb{R}$. This essentially requires us to find some $x_0 = x_0(x_1) \in \mathbb{R}$ and a solution $x(t)$ that satisfies $x(0) = x_0$ and $x(1) = x_1$. It is easy to check that, for $x_0 \in \mathbb{R}$, the solution of the initial-value problem $x(0) = x_0$ is

$$x(t;x_0) = \begin{cases} (1-t)x_0, & t \in [0,2) \\ -x_0, & t \in [2,3) \\ -(4-t)x_0, & t \in [3,5) \\ x_0, & t \in [5,6), \end{cases}$$

extended periodically with period 6. However, we remark that $x(1;x_0) = 0$ for all $x_0 \in \mathbb{R}$. Consequently, there is no solution of the initial-value problem $x(1) = x_1$ for $x_1 \neq 0$. There are, however, infinitely many solutions to the initial-value problem $x(1) = 0$.

Consistent with this is the form that equation (2.3.1) takes. We find that, with $t_0 = 1$ and $x_1 \in \mathbb{R}$, we require

$$\max\{\alpha, 1\} \cdot (|x_1| + \beta) \leq \beta.$$

However, this obviously fails for all $x_1 \neq 0$. Conversely, when $x_1 = 0$, the inequality holds for all $\alpha \leq 1$, so Theorem 2.3.1 implies the existence of a solution $x(t)$ of the impulse extension equation satisfying $x(1) = 0$. The results of the theorem are therefore consistent with the above analysis.

In conclusion, uniqueness and even existence of solutions of indeterminate initial-value problems of general impulse extension equations is difficult to establish and may not hold. We have demonstrated through examples that a simple analogue of the Peano existence theorem, requiring only continuity of the vector fields appearing in (2.0.2), is not sufficient to guarantee existence of solutions. Theorem 2.3.1 generalizes this, although the statement is not nearly as strong as the classical result for ordinary differential equations.

2.4 Impulse extension equations as impulsive differential equations with a larger phase space

Recall the general impulse extension equation (2.0.2)

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= f(t, x) + \varphi_k(t, x(\tau_k)), && t \in \mathcal{S}_k, \end{aligned}$$

with $x \in \Omega \subset \mathbb{R}^n$ and $\mathcal{S}_k = [\tau_k, \tau_k + a_k)$. We will now outline a process by which we may transform the above into an impulsive differential with phase space $\Omega \times \Omega \subset \mathbb{R}^{2n}$. Define the function $\Phi : \mathbb{R} \times \Omega \to \Omega$ by

$$\Phi(t, y) = \sum_{k=-\infty}^{\infty} \mathbb{1}_{\mathcal{S}_k}(t) \cdot \varphi_k(t, y).$$

Consider now the impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) + \Phi(t,y), && t \neq \tau_k, \\ \frac{dy}{dt} &= 0, && t \neq \tau_k, \\ \Delta x &= 0, && t = \tau_k, \\ \Delta y &= x - y, && t = \tau_k, \end{aligned} \tag{2.4.1}$$

where $(x,y) \in \Omega \times \Omega$.

We will discuss the structure of equation (2.4.1). If $t \notin \mathcal{S}$, then $\Phi(t,y) = 0$, so the solutions are govered by the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), && t \notin \mathcal{S}, \\ \frac{dy}{dt} &= 0, && t \notin \mathcal{S}. \end{aligned}$$

If $t = \tau_k$, then any solution $(x(t), y(t))$ satisfies $x(\tau_k^+) = x(\tau_k)$ and $y(\tau_k^+) = x(\tau_k)$. Consequently, for $t \in \mathcal{S}_k^\circ$, since $\Phi(t,y) = \varphi_k(t,y)$, solutions satisfy the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) + \varphi_k(t, x(\tau_k)), && t \in \mathcal{S}_k, \\ \frac{dy}{dt} &= 0. && t \in \mathcal{S}_k. \end{aligned}$$

The above two differential equations are essentially (2.0.2) except with an extra piecewise-constant component, y. Moreover, it can be verified that, if $x(t)$ is a solution of the impulse extension equation (2.0.2) defined on $\mathcal{I} \subset \mathbb{R}$, then the function $(x(t), y(t))$ with

$$y(t) = \sum_{\tau_k \in \mathcal{I}} \mathbb{1}_{[\tau_k,\, \tau_{k+1})}(t) \cdot x(\tau_k)$$

is a solution of (2.4.1). [3]

[3] As a note of possible clarification, since $x(t)$ is a solution of (2.0.2), we cannot have $\alpha \equiv \inf \mathcal{I} \in \mathcal{S}^\circ$. Consequently, if necessary, we may define $y(t) = c$ where c is any constant, for $\alpha < \tilde{\tau}$ where $\tilde{\tau} = \min\{\tau_k \in \mathcal{I}\}$.

Therefore, all solutions of (2.0.2) can be seen as solutions of (2.4.1). However, the converse is not true. An intial value problem for (2.4.1) requires initial conditions in $\Omega \times \Omega$ as opposed to (2.0.2), which in some sense only requires an initial condition in Ω. Specifically, the following proposition is easily verified.

Proposition 2.4.1. Suppose $Z(t) = (x(t), y(t))$ is a solution of (2.4.1) defined on some $\mathcal{I} \subset \mathbb{R}$. Then $x(t)$ is a solution of (2.0.2) if and only if $\inf \mathcal{I} \notin \mathcal{S}^\circ$ and, for all $\tau_k \in \mathcal{I}$, there exists $v_k \in \Omega$ such that $Z(\tau_k^+) = (v_k, v_k)$.

We conclude this section by noting that, even in the linear case, (2.4.1) is very ill-conditioned. As will be discussed at length in the following sections, an impulse extension equation is *linear* if $f(t,x) = A(t)x + g(t)$ and the extension $\varphi_k(t,x)$ can be written as $\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$. In this case, with $\Phi^B(t) = \sum \mathbb{1}_{\mathcal{S}_k}(t) \cdot \varphi_k^B(t)$ and $\Phi^h(t) = \sum \mathbb{1}_{\mathcal{S}_k}(t) \cdot \varphi_k^h(t)$, (2.4.1) can be written in matrix notation as

$$\dot{X} = \begin{bmatrix} A(t) & \Phi^B(t) \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} f(t) + \Phi^h(t) \\ 0 \end{bmatrix}, \qquad t \neq \tau_k,$$

$$\Delta X = \begin{bmatrix} 0 & 0 \\ E & -E \end{bmatrix} X, \qquad t = \tau_k.$$

Most of the theory of linear stability of periodic orbits assumes the condition $\det(C_k + E) \neq 0$, where $\Delta X = C_k X$ when $t = \tau_k$. Clearly, this does not hold for the linear equation above. This requirement relates to the existence of fundamental matrix solutions; when this condition is not satisfied, certain classical properties (including uniqueness up to multiplication by an invertible, constant matrix) fail to hold. Consequently, Floquet theory cannot be applied to this equation.

For this reason, the above construction is useful more for illustrative purposes. It helps to explain why existence and uniqueness of solutions for (2.0.2) is in general, difficult. Namely, it provides a more reasonable explanation as to why the the phase space "should" be $\Omega \times \Omega \subset \mathbb{R}^{2n}$, even though the equation appears to be restricted to $\mathbb{R}^n$.

Chapter 3

Linear Impulse Extensions

As was briefly mentioned in Section 2.1, when an impulse extension equation satisfies a linearity property, existence and uniqueness conditions can be derived explicitly. We now elaborate on this.

Definition 3.0.1. *An impulse extension* $\varphi_k(t,x)$ *is* linear *if it is an affine function of* x*, that is, if* $\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$ *for some* $\varphi_k^B : \mathcal{S}_k \to \mathbb{R}^{n\times n}$ *and* $\varphi_k^h : \mathcal{S}_k \to \mathbb{R}^n$*.*

Example 3.0.2. Let us present some examples of linear impulse extensions. It will be assumed that the phase space is $\mathbb{R}^n$. For these examples, we will let $\Delta x = I_k(x) = B_k x + h_k$ be a linear jump condition to be described by an impulse extension, with impulse times τ_k.

- Perhaps the simplest impulse extension is the *constant extension*. Given a compatible duration sequence a_k, the constant extension is the pair $(\varphi_k^B, \varphi_k^h)$ defined by
$$\varphi_k^B(t) = \frac{1}{a_k}B_k, \qquad \varphi_k^h(t) = \frac{1}{a_k}h_k.$$
It is easy to verify that the function $\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$ constitutes an impulse extension for any impulsive differential equation with the affine jump $\Delta x = B_k x + h_k$. The extension is linear because of the above decomposition.

- Impulse extensions can be constructed from matrix exponentials. For each $k \in \mathbb{Z}$, let M_k and N_k be $n \times n$ matrices all of whose eigenvalues have positive real part. Given a square matrix H, define the function

$$C_k(t;H) = \frac{a_k}{(\tau_k + a_k - t)^2} H \exp\left(-H\left(\frac{t-\tau_k}{\tau_k + a_k - t}\right)\right).$$

Then we can construct a linear extension $(\varphi_k^B, \varphi_k^h)$ as follows:

$$\varphi_k^B(t) = C_k(t;M_k)B_k \qquad \varphi_k^h(t) = C_k(t;N_k)h_k.$$

We now show that this defines a valid impulse extension. Since the defining quality is an integral condition, by linearity of the integral, it suffices to show that $\int_{\mathcal{S}_k} C_k(t;H)dt = E$ whenever all eigenvalues of H have positive real part; then we will have

$$\int_{\mathcal{S}_k} \varphi_k^B(t)x + \varphi_k^h(t)dt = EB_kx + Eh_k = B_kx + h_k$$

as required. We calculate

$$\begin{aligned}
\int_{\mathcal{S}_k} C_k(t;H)dt &= H\int_{\tau_k}^{\tau_k+a_k} \frac{a_k}{(\tau_k+a_k-t)^2}\exp\left(-H\left(\frac{t-\tau_k}{\tau_k+a_k-t}\right)\right)dt \\
&= H\int_0^{a_k} \frac{a_k}{(a_k-u)^2}\exp\left(-H\left(\frac{u}{a_k-u}\right)\right)du \\
&= H\int_0^{\infty} \exp(-Hr)dr \\
&= HH^{-1} \\
&= E,
\end{aligned}$$

where the fourth equality follows by the eigenvalues of H having positive real part; the eigenvalues of $-H$ all have negative real part, and consequently, the integral converges to H^{-1}. From this calculation, we conclude that the pair $(\varphi_k^B, \varphi_k^h)$ defines a linear impulse extension. In fact, for each k, we have a choice; the function $\varphi_k^B(t) = B_kC_k(t;M_k)$ is another suitable candidate for the matrix-valued part φ_k^B.

With Definition 3.0.1 in mind, we define the general linear impulse extension equation. We study these equations abstractly for the moment, without any specific reference to impulsive differential equations.

Definition 3.0.2. *An impulse extension equation is* linear *if it is of the form*

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t)x(\tau_k) + \varphi_k^h(t), && t \in \mathcal{S}_k, \end{aligned} \tag{3.0.1}$$

where the impulse extension $\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$ is linear and the following conditions are met:

- *The functions $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $g : \mathbb{R} \to \mathbb{R}^n$ are integrable on every compact set.*
- *The functions $\varphi_k^B : \mathcal{S}_k \to \mathbb{R}^{n\times n}$ and $\varphi_k^h : \mathcal{S}_k \to \mathbb{R}^n$ are integrable for all $k \in \mathbb{Z}$.*

A remark should be made that the requirement in the usual impulsive differential equations monographs is to have $A(t)$ continuous except at impulse times, where it has discontinuities of the first kind [3, 5, 16]. This requirement is only stated for ease of presentation; with ordinary differential equations, integrability of $A(t)$ on all compact sets is sufficient to establish all of the basic results [15], including existence, uniqueness and continuability of solutions, fundamental matrices and Floquet's theorem for periodic equations. The proofs in the impulsive case are indeed identical to those for ordinary differential equations.

3.1 Existence and uniqueness of solutions

We begin with the admissible initial-value problems, and then work backwards to backward continuation and indeterminate initial-value problems. Theorem 2.1.1 is not applicable, since the functions appearing in (3.0.1) are in general not continuous, so we must use alternative techniques.

A brief remark; since our definition of a solution of an impulse extension equation is with respect to the concept of an extended solution, any statement about the derivative of a solution may hold almost everywhere, but not necessarily at every point in its domain of definition. In the following section, this fact should be clear from context, so we will refrain from writing "almost everywhere".

Lemma 3.1.1. Let (t_0, x_0) be admissible for (3.0.1) with $\tau_{k-1} + a_{k-1} < t_0 \leq \tau_k$. The initial-value problem $x(t_0) = x_0$ for (3.0.1) has a unique solution defined for all $t > \tau_{k-1} + a_{k-1}$.

Proof: There are two cases to consider: $t_0 \notin \mathcal{S}$ and $t_0 = \tau_k$ for some k. However, the proof of each is essentially the same, so we will only prove the second, slightly more technical case. Let $x(\tau_k) = x_0$. We first construct a solution that is valid for all $t \geq t_0$. Since we are only interested in solutions defined for $t \geq t_0$, we may fix

$$x'(t) = A(t)x(t) + g(t) + \varphi_k^B(t)x_0 + \varphi_k^h(t)$$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ and ϵ small, since the solution for $t < t_0$ will be disregarded. Locally then, a unique solution exists by standard results of ordinary differential equations [15]. Moreover, since the equation is linear, this solution is continuable until $t = \tau_k + a_k$. Let this solution, defined for $\tau_k \leq t < \tau_k + a_k$ be denoted $x(t; x_0)$. Since the solution is continuous, we define

$$x(\tau_k + a_k; x_0) \equiv \lim_{t \to \tau_k^-} x(t; x_0).$$

Now define the auxilary initial-value problem

$$y'(t) = A(t)y(t) + g(t),$$
$$y(\tau_k + a_k) = x(\tau_k + a_k; x_0).$$

Again, by standard results, this has a unique solution defined for, in particular, $\tau_k + a_k \leq t < \tau_{k+1}$. By continuity, we define $y(\tau_k)$ by a left limit, similarly to above.

Now extend $x(t;x_0)$ as follows:

$$x(t;x_0) = \begin{cases} x(t;x_0), & \tau_k \le t < \tau_k + a_k, \\ y(t), & \tau_k + a_k \le t \le \tau_{k+1}. \end{cases}$$

It is easy to verify that $x(t;x_0)$ as defined above is still a solution of the initial-value problem. To extend the solution for $t > \tau_{k+1}$, the procedure can be repeated inductively. Since, at each step, the resulting solution is unique, a unique solution exists and is defined for all $t \ge t_0$.

To continue backward in time, we consider the auxiliary initial-value problem

$$\begin{aligned} z'(t) &= A(t)z(t) + g(t), \\ z(\tau_k) &= x_0. \end{aligned}$$

By standard existence and uniqueness for linear differential equations, a unique solution exists and is continuable backwards in time until $t = \tau_{k-1} + a_{k-1}$. If $x(t;x_0)$ is the unique maximal forward solution, then its unique backward continuation for $t > \tau_{k-1} + a_{k-1}$ is given by

$$x(t;x_0) = \begin{cases} x(t;x_0), & t \ge \tau_k, \\ z(t), & \tau_{k-1} + a_{k-1} < t < \tau_k. \end{cases}$$

This proves the desired result. ∎

It is worth mentioning that the solution as obtained above, though unique, is a solution in the extended sense; see the Carathéodory existence theorem [15] for more details.

With impulsive differential equations, the condition required for uniqueness of solutions is $\det(E + B_k) \ne 0$, where the impulse condition is $\Delta x = B_k x + h_k$ at time $t = \tau_k$. The condition for (3.0.1) is similar.

Lemma 3.1.2. Let (t_0, x_0) be k-indeterminate for (3.0.1). The initial-value problem $x(t_0) = x_0$ has a solution if and only if

$$X^{-1}(t_0;\tau_k)x_0 - \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^h(s)\big]\,ds$$

is in the column space of $L(t_0;\tau_k)$, where, for $t_0 \in \overline{\mathcal{S}_k}$, $L(t_0,\tau_k)$ is defined by

$$L(t_0;\tau_k) \equiv E + \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\varphi_k^B(s)ds, \tag{3.1.1}$$

and $X(t;s)$ is the Cauchy matrix of the homogeneous ordinary differential equation $x'(t) = A(t)x$. If a solution exists, then it exists for all $t \geq \tau_{k-1} + a_{k-1}$. The solution is unique if and only if $\det L(t_0;\tau_k) \neq 0$.

Proof: Suppose a solution exists. Since $t_0 \in \mathcal{S}_k$, this solution must be defined at τ_k. In particular, there must exist some $x_{\tau_k} \in \mathbb{R}^n$ such that the solution $\phi(t)$ of the initial-value problem

$$\begin{aligned} &\phi'(t) = A(t)\phi(t) + g(t) + \varphi_k^B(t)x_{\tau_k} + \varphi_k^h(t), \\ &\phi(\tau_k) = x_{\tau_k} \end{aligned} \tag{3.1.2}$$

satisfies $\phi(t_0) = x_0$. Since the above ODE is linear, it has a solution defined, in particular, for all $t \in (\tau_k, \tau_k + a_k)$. By similar arguments to the proof of Lemma 3.1.1, this solution $\phi(t)$ can be extended backward in time until $t = \tau_{k-1} + a_{k-1}$ such that, for $\tau_{k-1} + a_{k-1} < t < \tau_k$, it is continuous and satisfies the differential equation

$$\phi'(t) = A(t)\phi(t) + g(t).$$

Conversely, if said x_{τ_k} should exist, then the solution $\phi(t)$ defined above is a solution of the IVP in question, defined for $\tau_{k-1} + a_{k-1} \leq t < \tau_k + a_k$. Extending by continuity to $t = \tau_k + a_k$, we can extend further to all $t > \tau_k + a_k$ by applying Lemma 4.2. We conclude that the existence criteria of the lemma is equivalent to showing that such a point x_{τ_k} exists.

Such a point x_{τ_k} exists if and only if there exists some x_{τ_k} such that the solution of the initial-value problem (3.1.2) satisfies $\phi(t_0) = x_0$. The solution of said initial-value problem is

$$\phi(t) = X(t;\tau_k)x_{\tau_k} + X(t;\tau_k)\int_{\tau_k}^{t} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^B(s)x_{\tau_k} + \varphi_k^h(s)\big]ds,$$

where $X(t;s)$ is the Cauchy matrix of $x'(t) = A(t)x(t)$. Rearranging the above and imposing the condition $\phi(t_0) = x_0$, we arrive at the condition

$$\begin{aligned} x_0 - X(t_0;\tau_k)\int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^h(s)\big]ds &= X(t_0;\tau_k)\left[E + \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\varphi_k^B(s)ds\right]x_{\tau_k} \\ &= X(t_0;\tau_k)L(t_0;\tau_k)x_{\tau_k}. \end{aligned}$$

Taking into account the invertibility of the Cauchy matrix, we arrive at

$$X^{-1}(t_0;\tau_k)x_0 - \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^h(s)\big]ds = L(t_0;\tau_k)x_{\tau_k}.$$

A point x_{τ_k} that satisfies the above relation exists if and only if the vector on the left-hand side is in the column space of $L(t_0;\tau_k)$. A unique point x_{τ_k} exists if and only if $L(t_0;\tau_k)$ is invertible, or, equivalently, if and only if $\det L(t_0;\tau_k) \neq 0$. This proves the lemma. ∎

By combining the above two lemmas, we can state a global existence and uniqueness result.

Theorem 3.1.3. The initial-value problem $x(t_0) = x_0$ associated to (3.0.1) has a unique solution defined for all $t \in \mathbb{R}$ if and only if one of the following is satisfied:

- $t_0 \in [\tau_k + a_k, \tau_{k+1})$ and $\det L(\tau_j + a_j;\tau_j) \neq 0$ for all $j \leq k$,
- $t_0 \in \mathcal{S}_k$, $\det L(t_0;\tau_k) \neq 0$ and $\det L(\tau_j + a_j;\tau_j) \neq 0$ for all $j < k$.

Proof: If $t_0 \in [\tau_k, \tau_k + a_k)$, then, by Lemma 3.1.2, a unique solution exists if and only if $\det L(t_0; \tau_k) \neq 0$. If this condition is satisfied, then a unique solution $x(t)$ exists for all $t > \tau_{k-1} + a_{k-1}$. By continuity, this solution can be extended to $t = t_1 := \tau_{k-1} + a_{k-1}$. To continue the solution earlier in time requires solving an initial-value problem of the form $x(t_1) = x_{t_1}$. For $\tau_{k-1} < t < t_1$, we have to solve

$$\phi'(t) = A(t)\phi(t) + g(t) + \varphi^B_{k-1}(t)x_{\tau_{k-1}} + \varphi^h_k(t),$$
$$\phi(\tau_{k-1}) = x_{\tau_{k-1}},$$

where the vector $x_{\tau_{k-1}}$ has yet to be determined, and such that $\lim_{t\to t_1^-} \phi(t)$ coincides with $x(t_1)$. By the proof of Lemma 3.1.2, existence of a unique vector $x_{\tau_{k-1}}$ with this property is equivalent to having $\det L(\tau_{k-1} + a_{k-1}; \tau_{k-1}) \neq 0$, and, when this is satisfied, the solution $x(t)$ exists for all $t > \tau_{k-2} + a_{k-2}$. By an inductive argument, we see that the solution exists for all $t \in \mathbb{R}$ if and only if the first condition $\det L(t_0; \tau_k)$ is satisfied, along with $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j < k$.

On the other hand, if $t_0 \in [\tau_k + a_k, \tau_{k+1})$, then Lemma 3.1.1 ensures that a unique solution exists for all $t \geq \tau_k + a_k$. Then, to uniquely continue backward in time beyond $t = \tau_k + a_k$ is, by the above argument, equivalent to having $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j \leq k$. This completes the proof. ∎

Since $L(\tau_k; \tau_k) = E$ for all $k \in \mathbb{Z}$, we have the following corollary.

Corollary 3.1.4. The initial-value problem $x(t_0) = x_0$ for (3.0.1) has a solution for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}$ if and only if $\det L(t; \tau_j) \neq 0$ for all $j \in \mathbb{Z}$ and all $t \in (\tau_j, \tau_j + a_j]$.

It should be emphasized that solutions of (3.0.1) are in fact absolutely continuous [15]. This regularity will be important later.

With linear impulsive differential equations, to have global existence and uniqueness of all initial-value problems, we require

$$\det(E + B_k) \neq 0$$

for all k, where the "homogeneous part" of the jump condition is given by $\Delta x = B_k x$. With impulse extension equations, we require

$$\det\left(E + \int_{\tau_k}^{t} X^{-1}(s;\tau_k)\varphi_k^B(s)ds\right)$$

for all k and all $t \in (\tau_k, \tau_k + a_k]$, where the "homogeneous part" of the impulse extension is given by $\varphi_k^B(t)$. Formally, in the determinant conditions above, the integrals in the extension case play the role of the matrix B_k in the impulsive case. In the limiting case where

$$\varphi_k^B(t) = \delta(t - \tau_k)B_k,$$

and $\delta(t)$ is the Dirac delta distribution, we have

$$\begin{aligned}\int_{\tau_k}^{t} X^{-1}(s;\tau_k)\varphi_k^B(s)ds &= \int_{\tau_k}^{t} \delta(s-\tau_k)X^{-1}(s;\tau_k)B_k ds \\ &= X^{-1}(\tau_k;\tau_k)B_k \\ &= EB_k \\ &= B_k\end{aligned}$$

for $t > \tau_k$. Therefore, in the limiting case of an impulsive differential equation, the existence and uniqueness criteria reduce to the impulsive existence and uniqueness conditions. In this sense, the conditions are consistent.

Periodic systems of type (3.0.1) will be of great interest.

Definition 3.1.1. *A linear impulse extension equation is* periodic with period T and cycle number c *if the following identities are satisfied.*

P3.1 *$A(t)$ and $g(t)$ are T periodic,* P3.2 $c \in \mathbb{N}$ *is the smallest integer for which* $\tau_{k+c} = \tau_k + T$, $a_{k+c} = a_k$, *and the shift property*

$$\varphi_{k+c}^{\alpha}(t+T) = \varphi_k^{\alpha}(t)$$

holds for all $t \in \mathcal{S}_k$ *and integers* k, *where* $\alpha \in \{B, h\}$ *and* $(\varphi_k^B, \varphi_k^h)$ *is the linear extension pair.*

Remark 3.1.5. We will refer to conditions P3.1 and P3.2 collectively as conditions [P]. From this point on, we will assume that every initial-value problem has a unique solution defined for all $t \in \mathbb{R}$. By Corollary 3.1.4, this is equivalent to condition [E]:

E: $\det L(t;\tau_j) \neq 0$ for all $j \in \mathbb{Z}$ and all $t \in (\tau_j, \tau_j + a_j]$, where $L(t, \tau_k)$ for $t \in \overline{\mathcal{S}_k}$ is as defined in (3.1.1).

We will later relax this requirement, as it is too stringent for most applications; see Section 3.6. It is important to note that condition [E] is simplified in the periodic case.

Theorem 3.1.6. Suppose condition [P] is satisfied. Then condition [E] holds if and only if $\det L(t, \tau_j) \neq 0$ for all $j = 0, \ldots, c-1$ and all $t \in (\tau_j, \tau_j + a_j]$, where c is the cycle number.

Proof: Necessity is obvious. To prove sufficiency, let $j \in \mathbb{Z}$ and write $j = kc + r$ where $0 \leq r \leq c-1$ is the remainder modulo c. Notice that, by Definition 3.1.1,

$$(\tau_j, \tau_j + a_j] = (\tau_r + kT, \tau_r + cT + a_r], \qquad \varphi_j^B(t) = \varphi_r^B(t - kT).$$

Consequently,

$$\begin{aligned} L(t;\tau_j) &= L(t;\tau_r + kT) \\ &= E + \int_{\tau_r + kT}^{t} X^{-1}(s;\tau_r + kT)\varphi_r^B(s - kT)ds \\ &= E + \int_{\tau_r}^{t-kT} X^{-1}(u + kT;\tau_r + kT)\varphi_r^B(u)du \end{aligned}$$

$$= E + \int_{\tau_r}^{t-kT} X^{-1}(u;\tau_r)\varphi_r^B(u)du$$
$$= L(t-kT;\tau_r).$$

However, since $t \in (\tau_r + kT, \tau_r + Kt + a_r]$, we have $t - kT \in (\tau_r, \tau_r + a_r]$. It follows that if $\det L(t;\tau_r) \neq 0$ for some $r \in \{0, \ldots, c-1\}$ and all $t \in (\tau_r, \tau_r + a_r]$, then $\det L(t;\tau_j) \neq 0$ for $t \in (\tau_j, \tau_j + a_j]$ whenever j can be written $j = kc + r$ for some $k \in \mathbb{Z}$. Since any integer can be decomposed in this way for some $r \in \{0, \ldots, c-1\}$, we have the desired result. ∎

3.2 Fundamental matrix solutions

To a linear impulse extension equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t)x(\tau_k) + \varphi_k^h(t), && t \in \mathcal{S}_k, \end{aligned}$$

we will associate the *homogeneous equation*

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, && t \notin \mathcal{S}, \\ \frac{dx}{dt} &= A(t)x + \varphi_k^B(t)x(\tau_k), && t \in \mathcal{S}_k. \end{aligned} \tag{3.2.1}$$

To understand the inhomogeneous equation (3.0.1), it is fruitful first to consider the simpler homogeneous equation (3.2.1). To begin, we demonstrate that under condition [E], the homogeneous equation admits fundamental matrix solutions, analoguously to ordinary and impulsive differential equations.

Theorem 3.2.1. Let condition [E] hold. Let $U(t)$ be an $n \times n$ matrix whose columns are solutions of the homogeneous equation (3.2.1), and such that $\det U(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$. The following results hold.

1. For any $x_0 \in \mathbb{R}^n$, $U(t)x_0$ is a solution of the homogeneous equation,

2. $U(t)$ is invertible for all $t \in \mathbb{R}$,

3. the unique solution of the initial-value problem $x(t_1) = x_0$ associated to (3.2.1) for $x_0 \in \mathbb{R}^n$ is given by $U(t)U^{-1}(t_1)x_0$,

4. denote $U(a;b) = U(a)U^{-1}(b)$. The identity $U(t;s) = U(t;r)U(r;s)$ is valid.

Proof: Part 1: Let $x_0 \in \mathbb{R}^n$. The function $x(t) = U(t)x_0$ clearly satisfies the differential equation for $t \notin \mathcal{S}$. For $t \in \mathcal{S}_k$, we have

$$\frac{d}{dt}x(t) = A(t)\sum_{j=1}^{n} A(t)x_j(t)x_0^j + \sum_{j=1}^{n} \varphi_k^B(t)x_j(\tau_k)x_0^j = A(t)x(t) + \varphi_k^B(t)x(\tau_k),$$

as required. Therefore, $U(t)x_0$ is a solution of the homogeneous equation (3.2.1).

Part 2: Suppose that $U(t^*)$ is not invertible for some $t^* \in \mathbb{R}$. Then there exists some $x^* \in \mathbb{R}^n \setminus 0$ such that $U(t^*)U^{-1}(t_0)x^* = 0$. For all $y \in \operatorname{span}\{x^*\}$, we have $U(t^*)U^{-1}(t_0)y = 0$. Since $U(t)z$ is a solution of the impulse extension equation for any $z \in \mathbb{R}^n$, it follows that there are infinitely many solutions of the initial-value problem $x(t^*) = 0$, contradicting the uniqueness of solutions guaranteed by the conditions [E]. We conclude that $U(t)$ is invertible.

Part 3: We now remark that if H is a nonsingular $n \times n$ matrix, then $y(t) = U(t)Hx_0$ is a solution for all $x_0 \in \mathbb{R}^n$. Indeed, for $t \notin \mathcal{S}$,

$$y'(t) = \frac{d}{dt}\sum_{j=1}^{n} x_j(t)[Hx_0]^j = \sum_{j=0}^{n} A(t)x_j(t)[Hx_0]^j = A(t)y(t),$$

and for $t \in \mathcal{S}_k$,

$$y'(t) = A(t)y(t) + \sum_{j=0}^{n} \varphi_k^B(t)x_j(\tau_k)[Hx_0]^j = A(t)y(t) + \varphi_k^B(t)y(\tau_k).$$

By condition 2, $U^{-1}(t_1)$ exists. Consequently, by the above calculations, $U(t)U^{-1}(t_1)x_0$ is a solution. Specifically, it is a solution of the initial-value problem $x(t_1) = x_0$, since $U(t_1)U^{-1}(t_1)x_0 = x_0$. By conditions [E], it is the only solution.

Part 4: The proof of this assertion follows by uniqueness of solutions and the above properties. ∎

Any matrix that satisfies the conditions of Theorem 3.2.1 is called a *fundamental matrix*. If $U(t)$ is a fundamental matrix and $U(t_0) = E$ for some $t_0 \in \mathbb{R}$, then $U(t)$ is called a *principal matrix solution at* t_0.

The superposition principle remains valid for linear impulse extension equations.

Lemma 3.2.2. Let $x_1(t)$ and $x_2(t)$ be two solutions of the homogeneous impulse extension equation (3.2.1) defined on the intervals I_1 and I_2, respectively. If $I = I_1 \cap I_2$ is an interval, the sum $x(t) \equiv x_1(t) + x_2(t)$ is also a solution when restricted to I.

Proof: For $t \in \mathcal{S}_k \cap I \neq \emptyset$, we have

$$\frac{dx}{dt} = A(t)x_1 + \varphi_k^B(t)x_1(\tau_k) + A(t)x_2 + \varphi_k^B(t)x_2(\tau_k) = A(t)x + \varphi_k^B(t)x(\tau_k)$$

almost everywhere. The analoguous result holds for $t \in I \setminus \mathcal{S}$. ∎

Lemma 3.2.3. Let $p(t)$ be a solution of the inhomogeneous impulse extension equation (3.0.1) and let $r(t)$ be a solution of its corresponding homogeneous equation (3.2.1). Let these functions be defined respectively on intervals I_p and I_r. If $I = I_p \cap I_r$ is an interval, then $x(t) \equiv r(t) + p(t)$ is a solution of the inhomogeneous equation (3.0.1).

Proof: For $t \in \mathcal{S}_k \cap I \neq \emptyset$, we find

$$\begin{aligned}\frac{dx}{dt} &= A(t)r + \varphi_k^B(t)r(\tau_k) + A(t)p + g(t) + \varphi_k^B(t)p(\tau_k) + \varphi_k^h(t)\\ &= A(t)(r+p) + g(t) + \varphi_k^B(t)(r(\tau_k) + p(\tau_k)) + \varphi_k^h(t)\\ &= A(t)x + g(t) + \varphi_k^B(t)x(\tau_k) + \varphi_k^h(t)\end{aligned}$$

almost everywhere. The analoguous result holds for $t \in I \setminus \mathcal{S}$. ∎

Similarly to linear ordinary and impulsive differential equations, the solutions of the linear inhomogeneous impulse extension equation (3.0.1) are completely determined by a single, particular solution.

Corollary 3.2.4. Let condition [E] hold and let $U(t)$ be a fundamental matrix solution. Let $p(t)$ be a solution of the linear inhomogeneous impulse extension equation (3.0.1). The solution of the initial-value problem $x(t_0) = x_0$ for the inhomogeneous equation (3.0.1) is precisely

$$x(t) = U(t)U^{-1}(t_0)[x_0 - p(t_0)] + p(t).$$

Proof: As can be verified by Theorem 3.2.1, $U(t)U^{-1}(t_0)[x_0 - p(t_0)]$ is a solution of the homogeneous equation. Since $p(t)$ is a solution of the inhomogeneous equation, the previous lemma implies that their sum, $x(t)$, is a solution of the inhomogeneous equation as well. Also,

$$x(t_0) = x_0 - p(t_0) + p(t_0) = x_0,$$

indicating that this solution satisfies the initial condition. Uniqueness is guaranteed because of condition [E]. ∎

3.3 Homogeneous periodic equations and Floquet theory

Floquet's theorem for ordinary differential equations (or its analogue for impulsive differential equations) is a statement about the structure of the fundamental matrix solution of a homogeneous equation. We note that $U(t)$ is a fundamental matrix if and only if it is invertible and satisfies the matrix differential equation

$$\begin{aligned} U'(t) &= A(t)U(t), && t \notin \mathcal{S} \\ U'(t) &= A(t)U(t) + \varphi_k^B(t)U(\tau_k), && t \in \mathcal{S}_k. \end{aligned} \tag{3.3.1}$$

This is the necessary ingredient we need to establish an analogue of Floquet's theorem.

Theorem 3.3.1. Let the conditions [E] and [P] hold. Then each fundamental matrix $U(t)$ of the homogeneous T-periodic impulse extension equation (3.2.1) can be respresented in the form

$$U(t) = \phi(t)e^{\Lambda t}, \tag{3.3.2}$$

where $\Lambda \in \mathbb{C}^{n\times n}$ is nonsingular and the matrix $\phi(\cdot)$ is absolutely continuous, differentiable almost everywhere, complex-valued, non-singular and T-periodic.

Proof: Let $U(t)$ be a fundamental matrix. We claim first that $U(t+T)$ is a fundamental matrix. We show this by demonstrating that $Y(t) = U(t+T)$ satisfies (3.3.1). For $t \notin \mathcal{S}$, we have

$$\frac{dY}{dt}(t) = A(t+T)U(t+T) = A(t)Y(t),$$

and, for $t \in \mathcal{S}_k$, if we remark that $t+T \in \mathcal{S}_{k+c}$ and $\tau_{k+c} = \tau_k + T$ (by conditions [P]), then

$$\frac{dY}{dt}(t) = A(t+T)U(t+T) + \varphi_{k+c}^B(t+T)U(\tau_{k+c}) = A(t)Y(t) + \varphi_k^B(t)Y(\tau_k).$$

So $Y(t)$ satisfies (3.3.1) and therefore $Y(t)$ is a fundamental matrix. Define $M = U(\tau_0)^{-1}U(\tau_0+T)$. By the proof of Theorem 3.2.1, $U(t)M$ is a fundamental matrix. Define $Z(t) = Y(t) - U(t)M$. For $t \notin \mathcal{S}$, we have

$$\frac{dZ}{dt}(t) = A(t)Y(t) - A(t)U(t)M = A(t)Z(t),$$

and for $t \in \mathcal{S}_k$, we have

$$\frac{dZ}{dt}(t) = A(t)Y(t)+\varphi_k^B(t)Y(\tau_k)-[A(t)U(t)M+\varphi_k^B(t)U(\tau_k)M] = A(t)Z(t)+\varphi_k^B(t)Z(\tau_k),$$

so $Z(t)$ is matrix solution of (3.3.1). We have

$$Z(\tau_0) = Y(\tau_0) - U(\tau_0)M = U(\tau_0+T) - U(\tau_0)U(\tau_0)^{-1}U(\tau_0+T) = 0,$$

so, in particular,

$$\frac{dZ}{dt}(\tau_0^+) = 0.$$

By continuity of the solution, we also have

$$\frac{dZ}{dt}(\tau_0^-) = 0.$$

It follows that $\frac{dZ}{dt} = 0$ almost everywhere, and since the entries of $Z(t)$ are products and sums of absolutely continuous functions [15], they too are absolutely continuous, from which we conclude [27] that $Z(t) \equiv 0$. Consequently, $U(t+T) = X(t)M$.

Define the following:

$$\begin{aligned} \Lambda &= \frac{1}{T}\ln M, \\ \phi(t) &= U(t)e^{-\Lambda t}. \end{aligned} \tag{3.3.3}$$

Note that Λ exists since M is non-singular [15], although it need not be unique (however, any logarithm will suffice). With these representations, formula (3.3.2) holds and we have

$$\phi(t+T) = U(t+T)e^{-\Lambda(t+T)} = U(t)Me^{-\Lambda T}e^{-\Lambda t} = U(t)e^{-\Lambda t} = \phi(t),$$

so $\phi(t)$ is T-periodic. It is also clearly nonsingular, absolutely continuous and differentiable almost everywhere since $U(t)$ is, as each of its columns are solutions and by definition have this level of regularity. ∎

We have obtained a representation of every fundamental matrix in terms of the product of a periodic matrix and a matrix exponential. These matrices can be complex-valued, however. If one wishes to have this representation in terms of real matrices, then the choice of

$$\Lambda = \frac{1}{2T}\ln M^2$$

and

$$\phi(t) = X(t)e^{-\Lambda t}$$

would satisfy (3.3.2) and these matrices would be real; however, $\phi(t)$ would be $2T$-periodic. See Montagnier, Paige and Spiteri [20] for a discussion on this and other results in real Floquet factorizations.

The eigenvalues of M in (3.3.3) are called the *floquet multipliers* of the periodic impulse extension equation. It is not difficult to verify that the floquet multipliers do not depend on the choice fundamental matrix used to calculate M; choosing another fundamental matrix $X_1(t)$ and using this to derive the matrix M_1, one finds that M and M_1 are similar. The proof of this fact is the same as with ordinary differential equations, and in fact holds under more general conditions. See later the proof of Corollary 3.6.7.

Corollary 3.3.2. The homogeneous periodic impulse extension equation (3.2.1) has a kT-periodic solution if and only if there exists a multiplier $\mu \in \sigma(M)$ such that $\mu^k = 1$, where M is the monodromy matrix.

Proof: We seek conditions under which $X(kT)x_0 = X(0)x_0$ for some x_0, for this is precisely the condition under which we have a kT-periodic solution. By Theorem 3.3.1, this is equivalent to having $e^{\Lambda kT}x_0 = x_0$, by cancellation of the T-periodic matrix $\phi(t)$. Since $\Lambda = \frac{1}{T}\ln M$, this is equivalent to $M^k x_0 = x_0$. The result follows. ∎

Just as it does with ordinary and impulsive differential equations, one finds that the time-dependent change of coordinates $x = \phi(t)y$ transforms solutions of the time-dependent periodic system (3.2.1) into solutions of an autonomous, linear ordinary differential equation.

Corollary 3.3.3. Let conditions [E] and [P] hold. There exists a continuous, invertible, time-dependent change of coordinates that converts the periodic system (3.2.1) into the linear differential equation with constant coefficients

$$y' = \Lambda y,$$

where $\Lambda = \frac{1}{T}\ln M$ is defined as in (3.3.3). In particular, the periodic impulse extension equation is:

- asymptotically stable if and only if $\rho(M) < 1$,
- stable if and only if $\rho(M) \leq 1$ and, for any eigenvalue μ of M with unit modulus, the geometric and algebraic multiplicities of μ coincide.

Proof: To any solution $x(t)$ of the periodic impulse extension equation (3.2.1), define the change of coordinates

$$x(t) \mapsto y(t) \equiv \phi^{-1}(t)x(t).$$

This map is clearly continuous since $\phi(t)$ is nonsingular and periodic, and hence has a bounded inverse. We find

$$\begin{aligned}
\frac{d}{dt}y(t) &= \frac{d}{dt}\left(\phi^{-1}(t)X(t)X^{-1}(0)x(0)\right) \\
&= \frac{d}{dt}\left(e^{\Lambda t}X^{-1}(0)x(0)\right) \\
&= \Lambda\left(e^{\Lambda t}X^{-1}(0)x(0)\right) \\
&= \Lambda\left(\phi^{-1}(t)X(t)X^{-1}(0)x(0)\right) \\
&= \Lambda\phi^{-1}(t)x(t)
\end{aligned}$$

$$= \Lambda y,$$

as required. The stability results then follow by standard theorems from ordinary differential equations [15, 34]. ∎

So, to determine stability of the periodic impulse extension equation, all that is needed is a monodromy matrix M and its eigenvalues. As can be verified by the variation of constants formula for ordinary differential equations (see the proof of Theorem 3.4.2 for a similar calculation), the solution of the initial-value problem $x(\tau_0) = x_0$ is given by, for $t \in [\tau_k, \tau_{k+1})$,

$$x(t; x_0) = Y(t)x_0 := X(t; \tau_k)L(\bar{t}; \tau_k)\left[\prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)L(\tau_r + a_r; \tau_r)\right] x_0, \quad (3.3.4)$$

where $\bar{t} = \min\{t, \tau_k + a_k\}$, $X(t; s)$ is the Cauchy matrix of $x'(t) = A(t)x$ and $L(t; s)$ is defined as in (3.1.1). Using this, we can construct a monodromy matrix, since the above expression implies that $Y(t)$ is a fundamental matrix. We have $Y(\tau_0) = I$, so the monodromy matrix is $M = Y^{-1}(\tau_0)Y(T + \tau_0) = Y(\tau_c)$. Therefore

$$M = \prod_{k=c-1}^{0} X(\tau_{k+1}; \tau_k)L(\tau_k + a_k; \tau_k), \quad (3.3.5)$$

where c is the cycle number.

We now provide a rather in-depth example motivated by a simple biological model.

Example 3.3.4. Consider the simple exponential bacterial growth model without carrying capacity

$$\frac{db}{dt} = \alpha b,$$

where $b(t)$ is the number of bacterial cells at time t and $\alpha > 0$ is the growth rate. If one wishes to control the growth of bacteria, then one approach could be to, through some chemical process, remove some proportion $r \in (0,1)$ of the bacteria every T units of time.

Modelling the above with an impulsive differential equation, we would have

$$\begin{aligned} \frac{db}{dt} &= \alpha b, && t \neq kT, \\ \Delta b &= -rb, && t = kT. \end{aligned} \tag{3.3.6}$$

The above impulsive differential equation is stable if and only if

$$\mu_0 = e^{\alpha T}(1-r) < 1,$$

(see [5]) which is to say that the growth of bacteria can be controlled, and the population brought down to zero as time approaches infinity if and only if $\mu_0 < 1$. By taking logarithms, we may arrive at the equivalent stability criteria

$$\alpha < \frac{|\log(1-r)|}{T}.$$

This has a reasonable biological interpretation: the control frequency is $1/T$, and so the above criteria reads

$$\text{growth rate} < |\log(1 - \text{control efficacy})| \times \text{control frequency}.$$

Conversely, if the effect of the chemical control is not instantaneous, but instead takes $\delta \in (0,T)$ units of time to complete, then with control dynamics occuring in intervals of the form $[kT, kT+\delta) = \mathcal{S}_k$, we obtain an impulse extension equation

$$\begin{aligned} \frac{db}{dt} &= \alpha b, && t \notin \mathcal{S} \\ \frac{db}{dt} &= \alpha b - \overline{r}(t-kT;\delta)b(kT), && t \in \mathcal{S}_k, \end{aligned} \tag{3.3.7}$$

where $\varphi_k(t;\delta) = -\overline{r}(t-kT;\delta)$ defines an impulse extension for each integer k and $\delta \in (0,T)$ such that

$$\int_{\mathcal{S}_k} \overline{r}(t-kT;\delta)dt = r.$$

Note that, for certain functional forms of $\overline{r}$, a solution (3.3.7) with positive initial condition may not remain positive. Glossing over this issue for the moment, Corollary 3.3.3 and equation (3.3.5) imply that the bacteria can be controlled as time approaches infinity if and only if

$$\mu_\delta = e^{\alpha T}\left(1 - \int_0^\delta \overline{r}(t;\delta)e^{-\alpha t}dt\right) < 1. \tag{3.3.8}$$

The choice of $\overline{r}(t;\delta)$ may have different effects on μ_δ, so we consider several cases. For the rest of this example, we will assume that condition [E] is satisfied; we will comment on this briefly at the end.

Constant control: $\overline{r}(t;\delta) = r/\delta$. The constant control is in some sense the baseline case. The floquet multiplier is

$$\mu_\delta^0 = e^{\alpha T}\left(1 - \frac{r(1-e^{-\alpha\delta})}{\alpha\delta}\right).$$

The above is an increasing function of δ; indeed,

$$\frac{d\mu_\delta^0}{d\delta} = \frac{re^{\alpha(T-\delta)}(e^{\alpha\delta} - (\alpha\delta+1))}{\alpha\delta^2} > 0.$$

Also, $\mu_\delta^0 \to \mu_0$ as $\delta \to 0^+$. In the limit as $\delta \to T^-$, we have

$$\mu_\delta^0 \to \frac{e^{\alpha T}(\alpha T - r) + r}{\alpha T} \equiv \mu_T^0. \tag{3.3.9}$$

We then have $\mu_T^0 < 1$ if and only if

$$e^{\alpha T}(\alpha T - r) < \alpha T - r.$$

Assuming positive parameter values, it is clear that the above condition is satisfied if and only if

$$\alpha T < r.$$

Similarly to before, this condition may be interpreted biologically as

growth rate < control efficacy × control frequency.

We make a few remarks. For $r \in (0,1)$, we always have $r < |\log(1-r)|$. Consequently, in the limit as $\delta \to T^-$, we always have $\mu_0 < \mu_T^0$. Since we observed above that $\mu_{\delta\to 0^+}^0 = \mu_0$ and μ_δ^0 is an increasing function of δ, we conclude that $\mu_0 < \mu_\delta^0$ for all $\delta \in (0,T)$. For example, if $\alpha = \log 2$ and $T = 1$, then

$$\mu_\delta^0 \to \frac{\log 4 - r}{\log 2},$$

which is less than 1 if and only if

$$r > \tilde{r} := \log 2 \approx 0.693.$$

On the other hand, for these parameter values, we have

$$\mu_0 = 2(1-r),$$

which is less than 1 if and only if $r > \frac{1}{2}$. Thus, for $r \in (0.5, \tilde{r})$, the impulsive differential equation predicts control of the bacteria, while the impulse extension equation might not, if δ is too large.

In this parameter range, the continuous control is too "slow" to control the bacterial growth, regardless if the initial population. This can be seen as follows. The characteristic exponent of the impulsive system is

$$\lambda_T = \alpha + \frac{\log(1-r)}{T}.$$

In comparison, the characteristic exponent of the continuous impulse extension equation is

$$\lambda_\delta = \alpha + \frac{1}{T}\log\left(1 - r\frac{1-e^{-\alpha\delta}}{\alpha\delta}\right).$$

Notice that $\alpha\delta$ is a dimensionless quantity and α is in units of time^{-1}, so that both of the characteristic exponents are dimensionless rates. These rates can be interpreted as periodically averaged growth or decay rates of the bacterial population relative to the periodic orbit. Increasing the duration of impulse effect corresponds to increasing the logarithmic term in λ_δ, which, if $\lambda_\delta = 0$, causes a bifurcation from exponential decay to exponential growth, relative to the periodic orbit.

For the special case where $r = 1 - e^{-\alpha T}$, the impulsive floquet multiplier is $\mu_0 = 1$. It follows that there is a nontrivial periodic orbit that is stable [5]. Specifically, all solutions are periodic and stable; we have that

$$x(t; x_0) = x_0 e^{\alpha |t|_T}$$

is the solution of the initial-value problem $x(0) = x_0$, where $|t|_T$ denotes t mod T (see Figure 3.1, top).

However, since $\mu_\delta^0 > \mu_0 = 1$ for all $\delta \in (0, T)$, the impulse extension equation has no periodic orbits, the trivial solution is unstable, and there are no bounded solutions (see Figure 3.1, bottom).

Sinusoidal "bump" control: $\overline{r}(t; \delta) = \frac{r\pi}{2\delta} \sin(\frac{\pi t}{\delta})$**.** If $r(t; \delta)$ is sinusoidal, we have

$$\mu_\delta = e^{\alpha T} \left(1 - \frac{\pi^2 r (e^{-\alpha\delta} + 1)}{2(\alpha^2 \delta^2 + \pi^2)} \right),$$

which is again an increasing function of δ that approaches μ_0 as $\delta \to 0^+$. With the choice of parameter values $\alpha = \log 2$ and $T = 1$ as before, as $\delta \to T^-$, we have

$$\mu_\delta \to 2 \left(1 + \frac{\pi^2 r}{2((\log 2)^2 + \pi^2)} \right),$$

which is greater than or equal to 1 for any positive r. Hence, for a sinusoidal "bump" control, if δ is too large, then the bacterial population cannot be controlled for any value of $r \in (0, 1)$.

It should be mentioned that, for arbitrary positive parameter values, it can be shown that we still have

$$\frac{d\mu_\delta}{d\delta} > 0$$

for all δ, so in particular, if $\mu_0 \geq 1$, then $1 \leq \mu_0 < \mu_\delta$ for all $\delta > 0$.

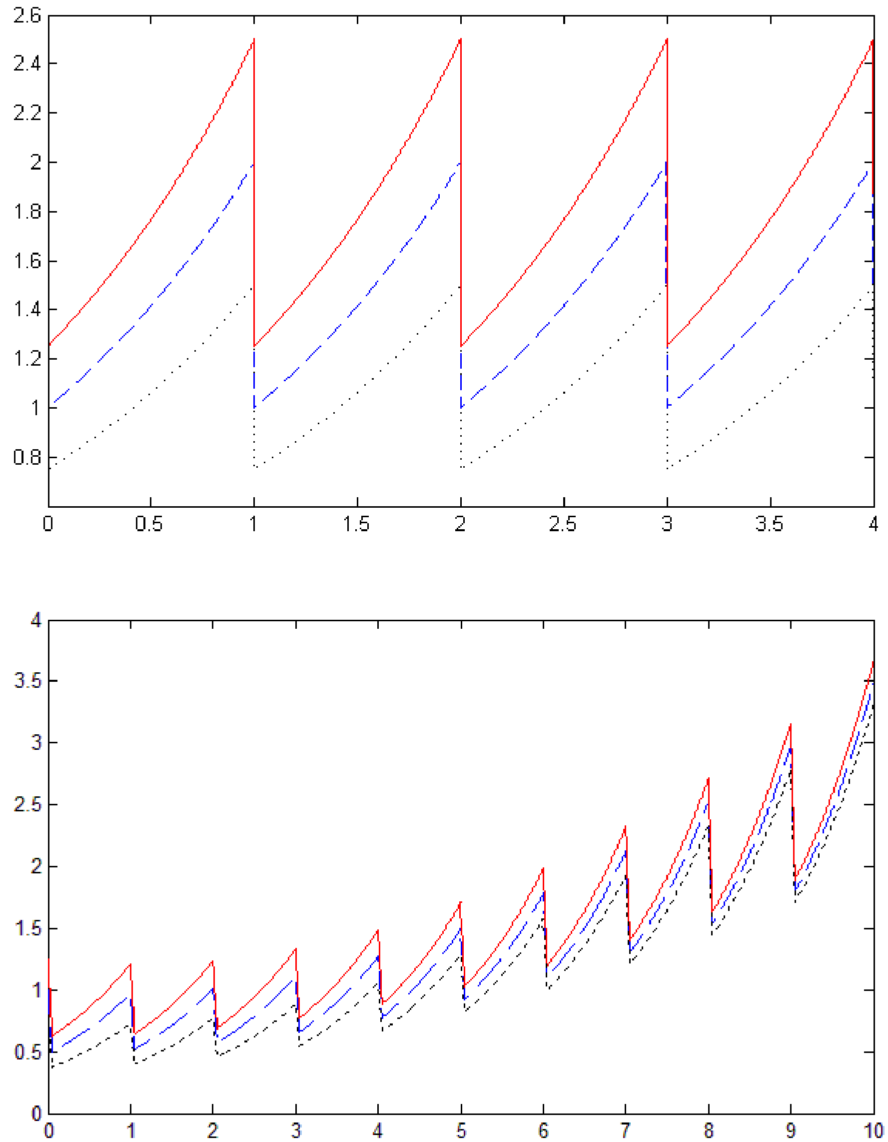

Figure 3.1: Top: Three periodic solution of the impulsive differential equation (3.3.6) with parameters $\alpha = \log(2)$, $T = 1$ and $r = 1 - e^{-\alpha T}$, plotted for $0 \leq t \leq 4$. Solid, dashed, and dotted lines are solutions with initial conditions 1.25, 1 and 0.75 respectively. Bottom: Three solutions of the impulse extension equation (3.3.7) with parameters $\alpha = \log(2)$, $T = 1$, $r = 1 - e^{-\alpha T}$ and $\delta = 0.05$, plotted for $0 \leq t \leq 10$. Solid, dashed, and dotted lines are solutions with initial conditions 1.25, 1 and 0.75 respectively. In contrast to the impulsive case, the solutions are not periodic and grow exponentially.

Sinusoidal perturbation of a constant control: $\overline{r}(t;\delta,h) = \frac{r}{\delta} + h\sin\left(\frac{2\pi}{\delta}t\right)$. For any $h \in \mathbb{R}$, we have

$$\int_{kT}^{kT+\delta} \overline{r}(t-kT;\delta,h)dt = r,$$

so this induces a valid impulse extension. Note that, if $|h| \geq \frac{r}{\delta}$, then it is not the case that $\overline{r} > 0$. However, it is not required that $\overline{r}$ be strictly positive.[4] The floquet multiplier is

$$\mu_\delta(h) = \mu_\delta^0 - \frac{2\pi e^{\alpha T} h\delta(1-e^{-\alpha\delta})}{\alpha^2\delta^2 + 4\pi^2}. \tag{3.3.10}$$

If h is fixed, we have $\mu_\delta(h) \to \mu_0$ as $\delta \to 0^+$, and it can be shown that

$$\left.\frac{d\mu_\delta}{d\delta}\right|_{\delta\to 0^+} = \frac{r\alpha e^{\alpha T}}{2} > 0.$$

From this, we conclude that $\mu_0 < \mu_\delta(h)$ provided $\delta < \overline{\delta}(h)$ for some $\overline{\delta}(h)$. However, for fixed δ, the function $\mu_\delta(h)$ is linearly decreasing in h, and, consequently, for all $\delta > 0$ and all $x \in \mathbb{R}$, there exists $h_x \in \mathbb{R}$ such that $\mu_\delta(h_x) = x$. Therefore, regardless the stability of the impulsive periodic orbit, given a length of time δ on which the control should occur, there exists an impulse extension whose corresponding periodic orbit can be "stabilized" or "destabilized" by choosing h wisely.

Let us consider a slightly different problem. Suppose a constant control is to be applied, which is subject to low-order perturbation uncertainty. We may then suppose that the control is described by $r(t;\delta,h)$ as given above. Suppose the maximum amplitude h^+ is known, so that $|h| \leq h^+$. We ask the following question: how small must δ be to guarantee that $|\mu_\delta(h)| < 1$ for all $h \in [-h^+, h^+]$?

An explicit δ could be found numerically should one exist; in general, such a δ will exist if and only if $|\mu_0| < 1$. This follows by the relations

$$\mu_\delta(h) \to \mu_0 > 0 \qquad \text{and} \qquad \frac{d\mu_\delta}{d\delta} > 0$$

[4] If $\overline{r}$ is not strictly nonnegative, this may be interpreted biologically as a control that removes too much bateria and compensates by putting some back at a later time, or vice versa (although the latter possibility is somewhat unrealistic).

in the limit as $\delta \to 0^+$, provided h is fixed. However, we may obtain an explicit lower bound for said δ as follows. Recall $0 < \mu_\delta^0 < \mu_T$. Then

$$\begin{aligned} |\mu_\delta(h)| &\leq |\mu_\delta^0| + \frac{2\pi e^{\alpha T} h^+ \delta(1 - e^{-\alpha\delta})}{(\alpha\delta)^2 + 4\pi^2} \\ &< \mu_T^0 + \frac{e^{\alpha T} h^+ \delta(1 - e^{-\alpha\delta})}{2\pi}, \end{aligned}$$

where μ_T^0 is as defined in (3.3.9). The following inequalities are valid.

$$|\mu_\delta(h)| < \mu_T^0 + \frac{e^{\alpha T}\delta h^+}{2\pi}, \tag{3.3.11}$$

$$|\mu_\delta(h)| < \mu_T^0 + \frac{T e^{\alpha T} h^+ (1 - e^{-\alpha\delta})}{2\pi}. \tag{3.3.12}$$

Suppose $\mu_T^0 < 1$; that is, $\alpha T < r$. By bounding (3.3.11) by 1, we will have $|\mu_\delta(h)| < 1$ if $\delta < \delta_1$, where

$$\delta_1 = \frac{2\pi e^{-\alpha T}(1 - \mu_T^0)}{h^+}.$$

Conversely, bounding (3.3.12) by 1, the bound $|\mu_\delta(h)| < 1$ is achieved if $\delta < \delta_2$, where

$$\delta_2 = \frac{1}{\alpha} \log\left(\frac{T e^{\alpha T} h^+}{T e^{\alpha T} h^+ - 2\pi(1 - \mu_T^0)}\right).$$

Therefore, if $\alpha T < r$ and $\delta < \max\{\delta_1, \delta_2\}$, then $|\mu_\delta(h)| < 1$ for $|h| \leq h^+$. When $\alpha T \geq r$, these bounds will not suffice and numerical methods are needed.

The previous results depended on h being fixed when δ was varied. Let us consider the case where we allow h to depend on δ; we write $h = h^\delta$. Suppose

$$h^\delta = -\frac{(\alpha^2\delta^2) + 4\pi^2}{e^{\alpha T} 2\pi\delta(1 - e^{-\alpha\delta})}.$$

The resulting function $r(t; \delta, h^\delta)$ still induces an impulse extension for all $\delta > 0$. However, by (3.3.10), we have

$$\mu_\delta(h^\delta) = \left(1 - \frac{r(1 - e^{-\alpha\delta})}{\alpha\delta} + \frac{1}{e^{\alpha T}}\right) \to \mu_0 + 1$$

as $\delta \to 0^+$. This is slightly disturbing; we have a sequence of impulse extensions $r(t; \delta, h^\delta)$ for which the associated solutions do not converge to the impulsive solutions (in any way!) as the duration sequence length converges to zero. However, notice that $|h^\delta| \to \infty$ as δ approaches zero. Let us consider the general case.

Mean-zero perturbation of a constant control: $\overline{r}(t;\delta) = \frac{r}{\delta} + \nu(t,\delta)$. Suppose $\nu(t,\delta)$ has mean zero; that is,

$$\int_0^\delta \nu(t,\delta)dt = 0$$

for all $\delta > 0$. Then $\int_{\mathcal{S}_k} r(t-kT;\delta)dt = r$, as required. The floquet multiplier is

$$\mu_\delta = \mu_\delta^0 - \int_0^\delta \nu(t,\delta)e^{-\alpha t}dt.$$

From this representation, it is clear that in the limit as $\delta \to 0^+$, we have

$$\mu_\delta \to \mu_0 \qquad \text{if and only if} \qquad \int_0^\delta \nu(t,\delta)e^{-\alpha t}dt \to 0.$$

Therefore, the behaviour seen in the previous example (where the impulse extension equation exhibited markedly different behaviour to the impulsive differential equation) is only possible if

$$\sup_{t\in[0,\delta)} |\nu(t,\delta)| \to \infty.$$

More constructively, suppose $\nu(t,\delta) = O\left(w(t,\delta)\frac{1}{e^{\alpha\delta}-1}\right)$ as $\delta \to 0$, where $w(t,\delta)$ is continuous and $w(0,0)$=0. Then, for δ small and some $M > 0$, we have

$$\begin{aligned}\left|\int_0^\delta e^{-\alpha t}\nu(t,\delta)dt\right| &\le M||w||_\infty \frac{1}{|e^{\alpha\delta}-1|}\int_0^\delta e^{-\alpha t}dt \\ &= M||w||_\infty \frac{1}{e^{\alpha\delta}-1}\frac{1-e^{-\alpha\delta}}{\alpha} \\ &= \frac{M}{\alpha}||w||_\infty e^{-\alpha\delta} \to 0,\end{aligned}$$

where the uniform norm $||\cdot||_\infty$ is taken over $[0,\delta)$ and the final limit is with respect to $\delta \to 0^+$. A nontrivial example for the asymptotic bound on ν, could be, say,

$$\nu(t,\delta) = O\left(\frac{\sqrt{\delta}}{e^{\alpha\delta}-1}\right).$$

The use of these asymptotic bounds on mean zero decompositions of impulse extensions will be elaborated upon in Section 3.4.

Condition [E] may in fact violated in some of these examples. For the constant control, we can determine exactly when this occurs. With the constant control, $\overline{r}(t;\delta) = r/\delta$, we find that $L(t^*;0) = 0$ if there exists $t^* \leq \delta$ such that

$$1 + \frac{r}{\alpha\delta}(e^{-\alpha t^*} - 1) = 0.$$

Since the function on the left of the equality is strictly decreasing in t^*, it suffices to determine if

$$q(\delta, r) := 1 + \frac{r}{\alpha\delta}(e^{-\alpha\delta} - 1)$$

is positive or negative; if it is negative or zero, such a t^* exists, but if it is positive, t^* does not exist. Notice that $q(\cdot, r)$ is an increasing function of δ, and that $\lim_{\delta\to 0} q(\delta, r) = 1 - r$. It follows that $q(\delta, r) > 0$ for all δ positive, provided $r \in (0, 1]$. From this, we conclude that for the constant control, positivity of solutions is maintained and Condition [E] holds.

For the other controls, condition [E] might be violated. Violation of condition [E] is in fact equivalent to the lack of invariance of $\mathbb{R}_+$, and it can be seen by numerical simulation that several of the examples presented in this section fail this condition (eg. sinusoidal perturbation of the constant control with $|h|$ large enough). However, this does not alter the stability results we have obtained in these examples. The Floquet theorem and related corollaries are valid under slightly weaker hypotheses. In particular, condition [E] can fail almost everywhere on $\overline{\mathcal{S}}$, and the only modifications necessary are that our definition of stability has to be weakened slightly. We will explain this in more detail in Section 3.6.

3.4 Periodic solutions of inhomogeneous equations

We now consider the fully inhomogeneous periodic equation (3.0.1). Before we begin, it is important to mention that the inhomogeneous equation (3.0.1) and its associated homogeneous equation (3.2.1) have the same stability. To be more precise, we have the following theorem.

Theorem 3.4.1. Let conditions [E] and [P] be satisfied. The stability of the linear inhomogeneous equation (3.0.1) is completely determined by the stability of its associated homogeneous equation (3.2.1). That is, the periodic equation (3.0.1) is

- asymptotically stable if and only if $\rho(M) < 1$,
- stable if and only if $\rho(M) \leq 1$ and for any eigenvalue μ of M with unit modulus, the geometric and algebraic multiplicities of μ coincide,

where M is the monodromy matrix of the homogeneous equation (3.2.1).

Proof: If $x(t)$ and $y(t)$ are solutions of (3.0.1) satisfying $x(t_0) = x_0$ and $y(t_0) = y_0$ respectively, Corollary 3.2.4 implies

$$x(t) = U(t)x_0 + p(t),$$
$$y(t) = U(t)y_0 + p(t),$$

where $U(t)$ is the principal fundamental matrix solution at t_0 and $p(t)$ is the solution of (3.0.1) satisfying $p(t_0) = 0$. Therefore,

$$x(t) - y(t) = U(t)(x_0 - y_0).$$

Therefore, stability of the linear equation (3.0.1) is completely determined by the stability of the homogeneous equation (3.2.1). ∎

Now that stability is understood, we are interested in conditions under which this inhomogeneous equation has a periodic solution. We begin with a proposition.

Proposition 3.4.2. Consider a linear impulse extension equation (3.0.1) satisfying condition [E] on existence and uniqueness of solutions. For each integer $k \geq 1$, there exists $v_k \in \mathbb{R}^n$ such that the solution $x(t; x_0)$ of the initial-value problem $x(\tau_0) = x_0$ can be written as

$$x(\tau_k; x_0) = \left[\prod_{j=k-1}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j)\right] x_0 + v_k = U(\tau_k; \tau_0)x_0 + v_k,$$

where $X(t, s)$ is the Cauchy matrix of the homogeneous ordinary differential equation $\dot{x} = A(t)x$, $L(t, s)$ is as defined in (3.1.1) and $U(t; \tau_0)$ is the principal fundamental matrix at τ_0 of (3.2.1). Specifically, the vectors v_k are independent of x_0 and are generated by the recurrence relation

$$\begin{aligned} v_0 &= 0, \\ v_{k+1} &= X(\tau_{k+1}, \tau_k) L(\tau_k + a_k, \tau_k) v_k + X(\tau_{k+1}, \tau_k) Q_k, \\ Q_k &= \int_{\tau_k}^{\tau_{k+1}} X^{-1}(s, \tau_k) g(s) ds + \int_{\mathcal{S}_k} X^{-1}(s, \tau_k) \varphi_k^h(s) ds. \end{aligned} \tag{3.4.1}$$

Proof: We proceed by induction on k. For $k = 1$, by the variaton of constant formula for ordinary differential equations [15, 34], we have

$$\begin{aligned} x(\tau_1; x_0) &= X(\tau_1, \tau_0)x_0 + X(\tau_1, \tau_0) \int_{\tau_0}^{\tau_1} X^{-1}(s, \tau_0) \left[g(s) + \mathbb{1}_{\mathcal{S}_0}(s)\varphi_0^h(s)\right] ds \ldots \\ &\quad + X(\tau_1, \tau_0) \int_{\tau_0}^{\tau_1} X^{-1}(s, \tau_0) \mathbb{1}_{\mathcal{S}_0}(s) \varphi_0^B(s) x_0 ds \\ &= X(\tau_1, \tau_0) \left[E + \int_{\mathcal{S}_0} X^{-1}(s, \tau_0) \varphi_0^B(s) ds\right] x_0 + X(\tau_1, \tau_0) Q_1 \\ &= X(\tau_1, \tau_0) L(\tau_0 + a_0, \tau_0) x_0 + v_1 \\ &= \prod_{j=1-1}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j) + v_1, \end{aligned}$$

where $v_1 = X(\tau_1, \tau_0)Q_1$ as required.

Suppose the result holds for some $k > 1$. Denote $x(\tau_k) = x(\tau_k; x_0)$ Then by the variation of constants formula, we have

$$x(\tau_{k+1}; x_0) = X(\tau_{k+1}, \tau_k) \left[x(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} X^{-1}(s, \tau_k) \left[g(s) + \mathbb{1}_{\mathcal{S}_k}(s) \varphi_k^h(s)\right] ds\right] \ldots$$

$$
\begin{aligned}
&+ X(\tau_{k+1}, \tau_k) \int_{\tau_k}^{\tau_{k+1}} X^{-1}(s, \tau_k) \mathbb{1}_{\mathcal{S}_k}(s) \varphi_k^B(s) x(\tau_k) ds \\
&= X(\tau_{k+1}, \tau_k) \left[E + \int_{\mathcal{S}_k} X^{-1}(s, \tau_k) \varphi_k^B(s) ds \right] x(\tau_k; x_0) + X(\tau_{k+1}, \tau_k) Q_k \\
&= X(\tau_{k+1}, \tau_k) L(\tau_k + a_k, \tau_k) \left(\left[\prod_{j=k-1}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j) \right] x_0 + v_k \right) \dots \\
&+ X(\tau_{k+1}, \tau_k) Q_k \\
&= \left[\prod_{j=k}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j) \right] x_0 + X(\tau_{k+1}, \tau_k) L(\tau_k + a_k, \tau_k) v_k \dots \\
&+ X(\tau_{k+1}, \tau_k) Q_k \\
&= \left[\prod_{j=k+1-1}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j) \right] x_0 + v_{k+1}.
\end{aligned}
$$

As required, v_{k+1} satisfies the required recurrence relation (3.4.1). By induction, the lemma is proven, where the representation in terms of fundamental matrices is furnished by equation (3.3.4). ∎

The above is essentially a specification of Corollary 3.2.4 to the times $t = \tau_k$. In terms of that notation, $p(t)$ is chosen to be the solution of the linear, periodic inhomogeneous equation (3.0.1) satisfying $p(\tau_0) = 0$.

Theorem 3.4.3. Consider a linear, periodic impulse extension equation (3.0.1) with period T and cycle number c. Let the conditions [E] and [P] be satisfied. This equation has a unique kT-periodic solution if and only if $\det(E - M^k) \neq 0$, where M is a monodromy matrix.

Proof: Without loss of generality, by appropriate shifting in the time domain, we may assume $\tau_0 = 0$. By the periodic structure of (3.0.1) guaranteed by the conditions [P], $x(t; x_0)$ is a kT-periodic solution satisfying $x(0) = x_0$ if and only if $x(kT; x_0) = x_0 = x(0; x_0)$. By the identity $\tau_{kc} = \tau_{0+kc} = \tau_0 + kT = kT$, this condition is equivalent to having $x(\tau_{kc}; x_0) = x_0$. By Proposition 3.4.2, Corollary 3.1.6 and equation (3.3.5), we have the representation

$$x_0 = x(\tau_{kc}; x_0) = \left[\prod_{j=c-1}^{0} X(\tau_{j+1}, \tau_j)L(\tau_j + a_j, \tau_j)\right]^k x_0 + v_{kc} = M^k x_0 + v_{kc}.$$

Equivalently,

$$(E - M^k)x_0 = v_{kc}. \tag{3.4.2}$$

The above equation has a unique solution x_0 if and only if $\det(E - M^k) \neq 0$. Consequently, this condition is necessary and sufficient for the existence of a unique periodic solution. Since all monodromy matrices are similar, this condition is independent of the choice of monodromy matrix. ∎

Theorem 3.4.3 provides an impulse extensions analogue of the "Non-Critical" case from impulsive differential equations (see Appendix A). The "Critical case", where $\det(E - M) = 0$, is more difficult. Establishing conditions under which a periodic solution exists in this case amounts to determining when (3.4.2) has a solution x_0.

Definition 3.4.1. *Let* $\tau_0 = 0$ *and let* $U(t)$ *be the principal matrix solution of* (3.2.1) *at* $t_0 = 0$. *The* adjoint equation *to the homogeneous periodic system* (3.2.1) *is the ordinary differential equation*

$$\begin{aligned} \frac{dy}{dt} &= -A^*(t)y, && t \notin \mathcal{S}, \\ \frac{dy}{dt} &= -A^*(t)y - \big(U(\tau_k)U^{-1}(t)\big)^* [\varphi_k^B]^*(t)y, && t \in \mathcal{S}. \end{aligned} \tag{3.4.3}$$

We now prove that the above differential equation truly is adjoint to (3.2.1). We will refer to these equations as *mutually adjoint*.

Proposition 3.4.4. Let condition [E] hold. The homogeneous equation (3.2.1) and the adjoint equation (3.4.3) satisfy the following properties.

1. For any two solutions $x(t)$, $y(t)$ of the mutually adjoint equations (3.2.1) and (3.4.3), the following identity is valid

$$\langle x(t), y(t)\rangle = \langle x(0), y(0)\rangle$$

for all $t \in \mathbb{R}$. That is, $\langle x(t), y(t)\rangle$ is constant.

2. Any fundamental matrices $U(t)$ and $Y(t)$ of the mutually adjoint equations (3.2.1) and (3.4.3) satisfy the identity

$$Y^*(t)U(t) = C$$

for some $C \in \mathbb{C}^{n\times n}$.

3. If the identity in part 2 is valid for a fundamental matrix $U(t)$ of (3.2.1) and C is a non-singular matrix, then $Y(t)$ is a fundamental matrix of (3.4.3).

Proof: To prove part 1, if $t \notin \mathcal{S}$, the proof is the same as for ordinary differential equations. We therefore prove only the (more difficult) case where $t \in \mathcal{S}_k$. Supressing the dependence on t (except where there may be ambiguity), we have

$$\begin{aligned}
\frac{d}{dt}\langle x, y\rangle &= \langle x', y\rangle + \langle x, y'\rangle \\
&= \langle Ax + \varphi_k^B x(\tau_k), y\rangle + \langle x, -A^*y - \left(U(\tau_k)U^{-1}(t)\right)^* [\varphi_k^B]^* y\rangle \\
&= \langle Ax, y\rangle + \langle \varphi_k^B x(\tau_k), y\rangle - \langle x, A^* y\rangle - \langle x, \left(U(\tau_k)U^{-1}(t)\right)^* [\varphi_k^B]^* y\rangle \\
&= \langle Ax, y\rangle + \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle Ax, y\rangle - \langle U(t)x(0), \left[\varphi_k^B U(\tau_k)U^{-1}(t)\right]^* y\rangle \\
&= \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle \varphi_k^B U(\tau_k)U^{-1}(t)U(t)x(0), y\rangle \\
&= \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle \varphi_k^B U(\tau_k)x(0), y\rangle = 0
\end{aligned}$$

almost everywhere on $\mathcal{S}_k$. This also holds on $\mathbb{R}\setminus\mathcal{S}$. Since $x(t)$ and $y(t)$ are absolutely continuous, their inner product is as well [27]. Therefore, $\langle x(t), y(t)\rangle$ is absolutely continuous with zero derivative almost everywhere. It follows that $\langle x(t), y(t)\rangle$ is constant for all $t \in \mathbb{R}$. Consequently, $\langle x(t), y(t)\rangle = \langle x(0), y(0)\rangle$. This proves part 1.

Part 2 is a direct consequence of part 1. Indeed,

$$Y^*U[i,j] = [Y^*]^i X_j = \sum_k (Y^*)^i(k)U_j(k) = \sum_k (Y_i)^*(k)U_j(k) = \langle Y_i, U_j\rangle,$$

which, by part 1, is constant.

To prove part 3, suppose $Y = (CU^{-1})^*$, so that it is defined by the identity $Y^*U = C$ for a nonsingular matrix C. It is easy to verify that, for $t \notin \mathcal{S}$, we have

$$\frac{dY}{dt} = -A^*Y.$$

Conversely, if $t \in \mathcal{S}_k$, then

$$\begin{aligned}
\frac{dY}{dt} &= \frac{d(U^{-1})^*}{dt} C^* \\
&= -\left(U^{-1}\left[AU + \varphi_k^B U(\tau_k)\right]U^{-1}\right)^* C^* \\
&= -\left(U^{-1}AUU^{-1}\right)^* C^* - \left(U^{-1}\varphi_k^B U(\tau_k)U^{-1}\right)^* C^* \\
&= -\left(U^{-1}A\right)^* C^* - \left(U^{-1}\varphi_k^B U(\tau_k)U^{-1}\right)^* C^* \\
&= -A^*(U^{-1})^*C^* - \left(U(\tau_k)U^{-1}\right)^* \left[\varphi_k^B\right]^* (U^{-1})^*C^* \\
&= -A^*Y - \left(U(\tau_k)U^{-1}\right)^* \left[\varphi_k^B\right]^* Y.
\end{aligned}$$

Therefore $Y(t)$ is a matrix solution of (3.4.3) and, since $\det Y \neq 0$, it is a fundamental matrix for (3.4.3). ∎

We establish now the existence criteria for periodic solutions in the critical case. The proof is somewhat technical, and at times the notation can be a bit cumbersome.

Theorem 3.4.5. Let conditions [E] and [P] be satisfied and let the homogeneous equation (3.2.1) have $m \le n$ linearly independent T-periodic solutions $p_1(t), \ldots, p_m(t)$. Then:

1. The adjoint equation (3.4.3) has m linearly indepenent T-periodic solutions $r_1(t), \dots, r_m(t)$.

2. Equation (3.0.1) has a nontrivial T-periodic solution if and only if, for $j = 1, \dots m$, the following condition is satisfied:

$$\sum_{k=0}^{c-1} \int_{\mathcal{S}_k} r_j^*(t)\mathcal{H}_k(t)[g(t) + \varphi_k^h(t)]dt + \int_{\tau_k + a_k}^{\tau_{k+1}} r_j^*(t)g(t)dt = 0, \tag{3.4.4}$$

   where $\mathcal{H}_k(t)$, defined by $\mathcal{H}_k(t) = X(t, \tau_k)L(t, \tau_k)L^{-1}(\tau_k + a_k, \tau_k)X^{-1}(t, \tau_k)$ is the *homogeneity matrix* [5]

3. If condition (3.4.4) is met, then each T-periodic solution of (3.0.1) has the form

$$x(t) = c_1 p_1(t) + \cdots + c_m p_m(t) + x_0(t)$$

   for a particular T-periodic solution $x_0(t)$ of (3.0.1).

Proof: 1. By the conditions of the theorem, $(E - M)x = 0$ has m linearly independent solutions x_i to which there correspond m linearly independent T-periodic solutions $p_i(t)$ of (3.2.1). By elementary linear algebra (see Fredholm alternative [11]), this is true if and only if $(E - M^*)y = 0$ has m linearly independent solutions y_i. Without loss of generality, by taking $\tau_0 = 0$ and and setting $U(t)$ to be the principal matrix solution at $t_0 = 0$ for (3.2.1), we can choose $M = U(T)$ as the monodromy matrix. Now let $Y(t)$ be the principal fundamental matrix of the adjoint equation (3.4.3) at $t_0 = 0$. By Proposition 3.4.4, we have $Y^*(t)U(t) = Y^*(0)U(0) = E$, from which it follows that $M^* = U^*(T) = Y^{-1}(T)$. Then, for each solution y_i of $(E - M^*)y = 0$, we have

$$M^* y_i = y_i \Longrightarrow Y^{-1}(T)y_i = y_i \Longrightarrow Y(T)y_i = y_i.$$

[5] We name $\mathcal{H}_k$ as such because, when the homogeneous part of the impulse extension is zero (ie. $\varphi_k = 0$), the homogeneity matrix reduces to the identity.

Since $Y(T)$ is a monodromy matrix for (3.4.3), we conclude that the adjoint equation has m linearly independent periodic solutions $r_i(t)$ satisfying $r_i(0) = y_i$.

2. We must determine the solvability of $(E - M)x_0 = v_c$. By elementary results from linear algebra, a solution exists if and only if $y_j^* v_c = 0$ for each y_j satisfying $r_j(0) = y_j$, as described in part 1. We must now describe the vector v_c in more detail. We claim that v_c can be written as a sum of c products. For brevity, let $X_k = X(\tau_k, \tau_{k-1})$ and $L_k = L(\tau_{k-1} + a_{k-1}, \tau_{k-1})$. For $k = 1 \ldots c$, denote

$$v_c^k = X_c L_c X_{c-1} L_{c-1} \cdots X_{k+1} L_{k+1} X_k \int_{\tau_{k-1}}^{\tau_k} X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds,$$

where dots denote sequential multiplication and, for succinctness, we abuse notation and write $\varphi_k^h = \mathbb{1}_{\mathcal{S}_k} \varphi_k^h$. Then, by the recurrence relation (3.4.1) of Proposition 3.4.2, we find

$$v_c = \sum_{k=1}^{c} v_c^k.$$

We will need to simplify the individual terms of this sum. Introduce the symbol $X(a|b)$, defined by

$$X(a|b) = X_a L_a \cdots X_{b+1} L_{b+1} X_b.$$

With this notation, the identities

$$X(a|b) = X_a L_a X(a{-}1|b), \qquad X(a|b) L_b X(b{-}1|d) = X(a|d) \tag{3.4.5}$$

hold when defined. By taking the constant matrices under the integral sign, with the above symbolic notation, we can write v_c^k as

$$\begin{aligned}
v_c^k &= \int_{\tau_{k-1}}^{\tau_k} X(c|k) \cdot X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X(c|k) \left[L_k X(k{-}1|1) L_1\right] \cdot \left[L_k X(k{-}1|1) L_1\right]^{-1} X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X_c L_c X(c{-}1|k) L_k X(k{-}1|1) L_1 \left[L_k X(k{-}1|1) L_1\right]^{-1} X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X(c|1) L_1 \cdot \left[X(s, \tau_{k-1}) L_k X(k{-}1|1) L_1\right]^{-1} [g(s) + \varphi_{k-1}^h(s)]ds
\end{aligned}$$

$$= \int_{\tau_{k-1}}^{\tau_k} U(T)[X(s,\tau_{k-1})L_k X(k-1|1)L_1]^{-1}[g(s)+\varphi_{k-1}^h(s)]ds$$
$$= \int_{\mathcal{S}_{k-1}} U(T)U^{-1}(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} U(T)U^{-1}(s)g(s)ds. \tag{3.4.6}$$

Most of the above calculations involve use of the identities (3.4.5). We make a few clarifications, however. By (3.3.4), if $t \in [\tau_{q-1}+a_{q-1}, \tau_q)$ for some q, then

$$U(t) = X(t,\tau_{q-1})L(\tau_{q-1}+a_{q-1})\cdots X(\tau_1,\tau_0)L(\tau_0+a_0,\tau_0) = X(t,\tau_{q-1})L_q X(q-1|1)L_1.$$

This implies $U(T) = X(\tau_c,\tau_{c-1})L_c X(c-1|1)L_1 = X(c|1)L_1$. Conversely, if $t \in \mathcal{S}_{k-1}$, then by (3.3.4), we know

$$\begin{aligned} U(t) &= X(t;\tau_{k-1})L(t;\tau_{k-1}) \prod_{r=k-1}^{1} X(\tau_r;\tau_{r-1})L(\tau_{r-1}+a_{r-1};\tau_{r-1}) \\ &= X(t,\tau_{k-1})L(t,\tau_{k-1})X_{k-1}L_{k-1}\cdots X_1 L_1 \\ &= X(t,\tau_{k-1})L(t,\tau_{k-1})X(k-1|1)L_1. \end{aligned}$$

From here it follows that

$$\begin{aligned} X(t,\tau_{k-1})L_k X(k-1|1)L_1 &= X(t,\tau_{k-1})L(\tau_{k-1}+a_{k-1},\tau_{k-1})\left[X(t,\tau_{k-1})L(t,\tau_{k-1})\right]^{-1}U(t) \\ &= \mathcal{H}_{k-1}^{-1}(t)U(t). \end{aligned}$$

Therefore

$$[X(t,\tau_{k-1})L_k X(k-1|1)L_1]^{-1} = U^{-1}(t)\mathcal{H}_{k-1}(t).$$

We have therefore established (3.4.6). Next, let $r_j(t)$ be one of the T-periodic solutions of (3.4.3). We now calculate $r_j^*(0)v_c^k$ for each $k = 1,\ldots,c$.

By periodicity, we have $r_j^*(0) = r_j^*(T)$. By Lemma 3.4.4 part 1, we have $r_j^*(T)U(T) = r_j^*(t)U(t)$ for all $t \in \mathbb{R}$, since each column of $U(t)$ is a solution of (3.2.1). This establishes the identity

$$r_j^*(T)U(T)U^{-1}(s) = r_j^*(s)U(s)U^{-1}(s) = r_j^*(s).$$

Therefore, multiplying (3.4.6) on the left by $r_j^*(0)$, we obtain

$$\begin{aligned} r_j^*(0)v_c^k &= \int_{\mathcal{S}_{k-1}} r_j^*(T)U(T)U^{-1}(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds \\ &\quad + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(T)U(T)U^{-1}(s)g(s)ds \\ &= \int_{\mathcal{S}_{k-1}} r_j^*(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(s)g(s)ds. \end{aligned}$$

Since $v_c = \sum_{k=1}^c v_c^k$, we arrive at

$$r_j^*(0)v_c = \sum_{k=1}^{c} \int_{\mathcal{S}_{k-1}} r_j^*(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(s)g(s)ds,$$

from which we obtain the left-hand side of (3.4.4) by shifting summation indices.

3. The proof of this assertion is the same as in the case of ordinary differential equations. As such, we omit it. ∎

Note that if the homogeneous equation has no linearly independent T-periodic solutions, then condition 2 is vacuously true and by Theorem 3.4.3, the inhomogeneous equation has a unique T-periodic solution.

When $\varphi_k^B = 0$ for all k, the homogeneity matrices $\mathcal{H}_k$ reduce to identity matrices, and condition (3.4.4) reduces to

$$\int_{\tau_0}^{\tau_c} r_j^*(s)g(s)ds + \sum_{k=0}^{c-1} \int_{\mathcal{S}_k} r_j^*(s)\varphi_k^h(s)ds = 0.$$

This formula was previously established by Church and Smith? in [9]. Condition (3.4.4) provides a generalization of this to the case where we do not necessarily have $\varphi_k^B = 0$.

We now prove a generalization of the Massera theorem for (3.0.1). The proof is the same as in the continuous or impulsive case, as it relies almost entirely on elementary results from linear algebra. However, we include it for completeness.

Theorem 3.4.6. Let conditions [E] and [P] hold and suppose (3.0.1) has a bounded solution for $t \geq \tau_0 = 0$. Then this equation has a nontrivial T-periodic solution.

Proof: Let $\hat{y}(t)$ be a bounded solution of (3.0.1). By Proposition 3.4.2, we have $\hat{y}(T) = M\hat{y}(0) + v$ for some $v \in \mathbb{R}^n$. A straightforward inductive argument shows that

$$\hat{y}(rT) = M^r\hat{y}(0) + \sum_{k=0}^{r-1} M^k v$$

for any integer $r \geq 1$. Now suppose (3.0.1) has no nontrivial periodic solutions. This is equivalent to the equation $(E - M)y = v$ having no solutions. By the Fredholm alternative theorem, this is true if and only if there is a solution z of $(E - M^*)z = 0$ for which $\langle z, v\rangle \neq 0$. Consequently, $M^*z = z$, from which it follows that

$$z = (M^*)^k z = (M^k)^* z$$

for all $k \in \mathbb{Z}$. We take the inner product of $\hat{y}(rT)$ with z:

$$\begin{aligned}
\langle z, \hat{y}(rT)\rangle &= \langle z, M^r\hat{y}(0)\rangle + \sum_{k=0}^{r-1}\langle z, M^k v\rangle \\
&= \langle (M^r)^* z, \hat{y}(0)\rangle + \sum_{k=0}^{r-1}\langle (M^k)^* z, v\rangle \\
&= \langle z, \hat{y}(0)\rangle + \sum_{k=0}^{r-1}\langle z, v\rangle \\
&= \langle z, \hat{y}(0)\rangle + r\langle z, v\rangle.
\end{aligned}$$

It then follows that $\langle z, \hat{y}(rT)\rangle \to \infty$ as $r \to \infty$, contradicting the boundedness of $\hat{y}(t)$. We conclude that (3.0.1) must have a nontrivial T-periodic solution. ∎

Theorem 3.4.6 is not necessarily very interesting on its own, but more so because of the following obvious corollary.

Corollary 3.4.7. Suppose condition [E] and [P] are satisfied and (3.0.1) has no nontrivial T-periodic solution. Then it has no bounded solution.

Therefore, if condition (3.4.4) does not hold, then the linear periodic impulse extension equation (3.0.1) has no bounded solution.

3.5 Time scale consistency theorems

This section contains what is in a certain sense the crux of this thesis. In Example 3.3.4, we saw that a linear, periodic impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), \qquad && t \neq \tau_k \\ \Delta x &= B_k x + h_k, && t = \tau_k, \end{aligned} \tag{3.5.1}$$

might exhibit very different behaviour to a related impulse extension equation, even as the duration sequence approaches zero. We now prove that, under certain regularity conditions, this difference of behaviour does not happen, provided the duration sequence is sufficiently small. By so doing, we will be lead to a fundamental theorem that relates the stability of an impulsive differential equation to its associated impulse extension equations. It will be assumed for the rest of this section that the matrices $E + B_k$ are invertible [6] for all $k \in \mathbb{Z}$. We begin with an obvious proposition, whose proof we omit.

Proposition 3.5.1. Every linear impulse extension $(\varphi_k^B, \varphi_k^h)$ defined on a sequence of impulse times τ_k and associated duration sequence a_k can be uniquely represented by the sum of the constant extension and a mean zero function on $\mathcal{S}_k$. That is, $\varphi_k^\alpha(t)$ can be written as

$$\varphi_k^\alpha(t) = \frac{1}{a_k}\alpha_k + \epsilon_k^\alpha(t)$$

for $\alpha \in \{B, h\}$, where $\epsilon_k^\alpha(t)$ has mean zero on $\mathcal{S}_k$.

[6] This assumption is necessary for most of the linear theory of impulsive differential equations.

By identifying a duration sequence with cycle number c with a vector in $\mathbb{R}^c$, we can define the convergence of sequences of duration sequences. In what follows, it will become convenient to not write the subscripts when referring to duration sequences, and to encase them in braces to distinguish sequences of points from sequences of functions. Subscripts will now be used to refer only to specific elements; that is $a_k = a(k)$ whenever

$$a : \mathbb{Z} \to \mathbb{R}$$

is a sequence.

Definition 3.5.1. *Let $\{a\}$ be a duration sequence over $\{\tau_k\}$ with cycle number c. The* vector identification *of $\{a\}$ is the vector $\vec{a} = (a_0, \ldots, a_{c-1}) \in \mathbb{R}^c$. If $\{a\}_n$ is a sequence of duration sequences over $\{\tau_k\}$ with cycle number c, we say $\{a\}_n$* converges to 0 *and write $\{a\}_n \to 0$ if the sequence of vector identifications $\vec{a}_n$ converges to zero in $\mathbb{R}^c$.*

Definition 3.5.2. *A sequence of functions $\varphi_k(t, x, r)$,*

$$\varphi_k(\cdot, \cdot, \cdot) : \mathbb{R} \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^n,$$

is a family of impulse extensions for the impulsive differential equations (2.0.1) *if for each index k and any duration sequence $\{a\}$ over $\{\tau_k\}$, the following conditions hold:*

- $\operatorname{supp}(\varphi_k(\cdot, x, a_k)) \subset \mathcal{S}_k(a)$ *for all $x \in \Omega$,*
- *the sequence of functions obtained by restricting for each k the function $\varphi_k(\cdot, \cdot, a_k)$ to $\mathcal{S}_k(a)$ defines an impulse extension for* (2.0.1)*.*

In the case of a linear impulse extension, the variable x appearing in the above definition can be ignored. It will be convenient to generalize Proposition 3.5.1 to the case of families of impulse extensions. The statement is obvious, and we do not prove it.

Lemma 3.5.2. Let $(\varphi_k^B(t,r), \varphi_k^h(t,r))$ be a linear family of impulse extensions for (3.5.1). Then for any duration sequence $\{a\}$,

$$\varphi_k^B(t,a_k) = \frac{1}{a_k}B_k + \epsilon_k^B(t,a_k),$$
$$\varphi_h^h(t,a_k) = \frac{1}{a_k}h_k + \epsilon_k^h(t,a_k)$$

in the restriction to $\mathcal{S}_k(a)$, where the functions $\epsilon_k^\alpha(t,a_k)$ for $\alpha \in \{B,h\}$ satisfy

$$\text{supp}(\epsilon_k^\alpha(\cdot,a_k)) \subset \mathcal{S}_k(a)$$

and have mean zero on $\mathcal{S}_k(a)$.

Proposition 3.5.3. If $\varphi_k^B(t,a_k) = \frac{1}{a_k}B_k$ is the family of constant homogeneous impulse extensions, then

$$L(\tau_k + a; \tau_k) = E + \int_{\tau_k}^{\tau_k+a} X^{-1}(s;\tau_k)\varphi_k^B(s,a)ds \to (E + B_k)$$

as $a \to 0$.

Proof: Write

$$L(\tau_k + a; \tau_k) = E + \left[\frac{1}{a}\int_{\tau_k}^{\tau_k+a} X^{-1}(s;\tau_k)ds\right] B_k.$$

Since $X(s,\tau_k)$ is continuous at τ_k, we have that, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for $|\tau_k - s| < \delta$, or, equivalently, for $a < \delta$,

$$\left|\left|\frac{1}{a}\int_{\tau_k}^{\tau_k+a} X^{-1}(s,\tau_k) - X^{-1}(\tau_k,\tau_k)ds\right|\right| < \frac{1}{a}\int_{\tau_k}^{\tau_k+a} \epsilon \cdot ds = \epsilon.$$

Since $X(\tau_k,\tau_k) = E$, we conclude that

$$\frac{1}{a}\int_{\tau_k}^{\tau_k+a} X^{-1}(s,\tau_k)ds \to E.$$

The result follows. ∎

Lemma 3.5.4. Let $t_0 \in \mathbb{R}$ and let $\varphi_k^B(t, a_k) = \frac{1}{a_k}B_k$ be the family of constant homogeneous impulse extensions. Denote by $U_{\{a\}}(t; t_0)$ the fundamental matrix of the homogeneous impulse extension equation (3.2.1) with duration sequence $\{a\}$, normalized at t_0. Then $U_{\{a\}}(t; t_0)$ converges pointwise to the fundamental matrix of the homogeneous impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \qquad && t \neq \tau_k \\ \Delta x &= B_k x, && t = \tau_k \end{aligned}$$

normalized at t_0 as $\{a\} \to 0$.

Proof: Without loss of generality, let $t_0 = \tau_0$. For $t \in (\tau_k, \tau_{k+1}]$, we have

$$U_{\{a\}}(t) = X(t; \tau_k) L_{\{a\}}(\bar{t}(t, a); \tau_k) \prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r) L_{\{a\}}(\tau_r + a_r; \tau_r)$$

where $\bar{t}(t, a) = \min\{t, \tau_k + a_k\}$ and

$$L_{\{a\}}(t; \tau_k) = E + \int_{\tau_k}^{t} X^{-1}(s; \tau_k) \varphi_k^B(s, a_k) ds.$$

As $\{a\} \to 0$, we have $\bar{t}(t, a) \approx \tau_k + a_k \to \tau_k$ pointwise for $t \in (\tau_k, \tau_{k+1})$. Consequently, by the previous proposition,

$$L_{\{a\}}(\bar{t}(t, a); \tau_k) \to (E + B_k)$$

pointwise on this interval. Also, each $L_{\{a\}}(\tau_r + a_r; \tau_r)$ in the product term converges to $E + B_r$ under the same limit. It follows that

$$U_{\{a\}}(t) \to X(t; \tau_k)(E + B_k) \prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)(E + B_r)$$

pointwise for all $t \in (\tau_k, \tau_{k+1}]$. The above is indeed a representation of the fundamental matrix at $t \in (\tau_k, \tau_{k+1}]$ normalized at t_0, and this concludes the proof. ∎

With the above lemma at our disposal, we are ready to state and prove what we call our first time-scale consistency theorem.

Theorem 3.5.5 (Time-Scale Consistency Theorem I)**.** Consider a linear T-periodic impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), & t &\neq \tau_k, \\ \Delta x &= B_k x + h_k, & t &= \tau_k. \end{aligned} \tag{I}$$

with cycle number c. Let $\varphi_k = (\varphi_k^B(t,r), \varphi_k^h(t,r))$ be family of linear impulse extensions for the impulsive differential equation (I), and, for each duration sequence $\{a\}$, consider the induced impulse extension equation:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), & t &\notin \mathcal{S}(a), \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t,a_k)x(\tau_k) + \varphi_k^h(t,a_k), & t &\in \mathcal{S}_k(a). \end{aligned} \tag{C}$$

Let

$$\varphi_k^B(t,a_k) = \frac{1}{a_k}B_k + \epsilon_k^B(t,a_k)$$

be the factorization furnished by Lemma 3.5.2. Suppose conditions [E] and [P] hold for each duration sequence. There exists a number $\sigma_A > 0$ depending only on the matrix $A(t)$ with the following property.

Suppose the following conditions hold:

W1 there exist scalar functions $w^k(t,r)$ for $k = 0,\dots,c-1$, such that w^k is continuous at $(\tau_k, 0)$ and $w^k(\tau_k, 0) = 0$,

W2 for any duration sequence $\{a\}$, the representation $\varphi_k^B(t,a_k) = \frac{1}{a_k}B_k + \epsilon_k^B(t,a_k)$ provided by Lemma 3.5.2 satisfies

$$\epsilon_k^B(t,r) = O\left(w^k(t,r)\frac{1}{e^{\sigma_A r} - 1}\right) \tag{3.5.2}$$

on $[\tau_k, \tau_k + r)$ as $r \to 0^+$ for $k = 0,\dots,c-1$.

The following results hold:

1. If $\{a\}_n \to 0$ is a convergent sequence of duration sequences and $U_{\{a\}_n}(t)$ is the corresponding sequence of fundamental matrices for the impulse extension equation (C) mutually normalized at some $t_0 \in \mathbb{R}$, then

$$U_{\{a\}_n}(t) \to U_0(t)$$

pointwise, where $U_0(t)$ is the fundamental matrix of the impulsive differential equation (I) normalized at t_0.

2. If $\{a\}_n \to 0$ is a convergent sequence of duration sequences, $\sigma(\mathrm{I})$ is the spectrum of the monodromy matrix for the impulsive equation (I) and $\sigma(\mathrm{C}_n)$ is the spectrum of the monodromy matrix of the impulse extension equation (C) with duration sequence $\{a\}_n$, then

$$\sigma(\mathrm{C}_n) \to \sigma(I).$$

3. There exists $\delta > 0$ such that, for any duration sequence $\{a\}$ whose vector identification $\vec{a}$ satisfies $|\vec{a}| < \delta$, the impulse extension equation (C) is asymptotically stable (resp. strongly unstable[7]) if and only if the impulsive differential equation (I) is asymptotically stable (resp. strongly unstable).

Proof: Let $X(s, \tau_0)$ be the fundamental matrix solution of the ordinary differential equation

$$\dot{x} = A(t)x$$

normalized at τ_0. Then $X(s, \tau_0) = \phi(s)e^{\Lambda s}$ for a nonsingular, continuous, differentiable (almost everywhere), τ-periodic matrix $\phi(s)$ and a nonsingular Λ. Define the quantities

$$K_\phi = ||\phi^{-1}(s)|| \max_{k=0,\ldots,c-1} ||X^{-1}(\tau_0, \tau_k)||, \qquad \sigma_A = ||\Lambda||.$$

[7]We say a linear periodic system is strongly unstable if it is unstable and has no floquet multipliers on the unit circle.

Then, for $k = 0, \dots, c-1$, the inequality

$$||X^{-1}(s, \tau_k)|| \leq K_\phi e^{\sigma_A s} \tag{3.5.3}$$

is valid, where $||\cdot||$ denotes the euclidean norm.

Part 1. Let $t \in (\tau_k, \tau_{k+1}]$ for $k \in \{0, \dots, c-1\}$. By the proof of Theorem 3.1.6, we need only verify that the theorem holds for these finite indices k. We have, with the notation of Lemma 3.5.4,

$$\begin{aligned} L_{\{a\}}(\bar{t}(t,a);\tau_k) &= E + \int_{\tau_k}^{\bar{t}(t,a)} X^{-1}(s,\tau_k)\left[\frac{1}{a_k}B_k + \epsilon_k^B(s,a_k)\right] ds \\ &= E + \int_{\tau_k}^{\bar{t}(t,a)} X^{-1}(s,\tau_k)\frac{1}{a_k}B_k ds + \int_{\tau_k}^{\bar{t}(t,a)} X^{-1}(s,\tau_k)\epsilon_k^B(s,a_k) ds. \end{aligned}$$

It is known by the proof of Lemma 3.5.4 that the first of these integrals converges to B_k pointwise as $\{a\} \to 0$. We now show that the second one converges to zero. Since $\bar{t}(t,a) = \tau_k + a_k$ for $\{a\}$ suffciently small, we may assume $\bar{t}(t,a) = \tau_k + a_k$. For a_k small, by the asymptotic properties of $\epsilon_k^B(t, a_k)$, there exists $M > 0$ such that

$$\begin{aligned} \left|\left|\int_{\tau_k}^{\tau_k+a_k} X^{-1}(s,\tau_k)\epsilon_k^B(s,a_k)ds\right|\right| &\leq \int_{\tau_k}^{\tau_k+a_k} \left|\left|X^{-1}(s,\tau_k)\right|\right| \cdot \left|\left|\epsilon_k^B(s,a_k)\right|\right| ds \\ &\leq M||w^k(\cdot,a_k)||_\infty \frac{1}{e^{\sigma_A a_k}-1}\int_{\tau_k}^{\tau_k+a_k} ||X^{-1}(s,\tau_k)||ds, \end{aligned}$$

where the uniform norm is taken over $[\tau_k, \tau_k + a_k)$. However, by inequality (3.5.3), the integrand can be bounded further. We find

$$\begin{aligned} \left|\left|\int_{\tau_k}^{\tau_k+a_k} X^{-1}(s,\tau_k)\epsilon_k^B(s,a_k)ds\right|\right| &\leq MK_\phi||w^k(\cdot,a_k)||_\infty \frac{1}{e^{\sigma_A a_k}-1}\int_{\tau_k}^{\tau_k+a_k} e^{\sigma_A s}ds \\ &= MK_\phi||w^k(\cdot,a_k)||_\infty \frac{1}{e^{\sigma_A a_k}-1}\cdot\frac{e^{\sigma_A(\tau_k+a_k)}-e^{\sigma_A\tau_k}}{\sigma_A} \\ &= MK_\phi||w^k(\cdot,a_k)||_\infty \frac{e^{\sigma_A\tau_k}}{\sigma_A} \to 0 \end{aligned}$$

as $a_k \to 0$, due to the continuity of $w^k(t,r)$ and the property $w(\tau_k, 0) = 0$. We therefore conclude that

$$L_{\{a\}}(\bar{t}(t,a);\tau_k) \to E + B_k$$

pointwise as $\{a\} \to 0$. It follows that the associated fundamental matrices converge pointwise, so that Lemma 3.5.4 holds under the more general hypotheses of the present theorem. This is the precisely the statement of part 1.

Part 2. This follows directly from part 1. Since we can choose $M_{\{a\}_n} = U_{\{a\}_n}(\tau_c)$ as the monodromy matrix (with the U's mutually normalized at τ_0), we know that

$$M_{\{a\}_n} \to M_0,$$

where $M_0 = U_0(\tau_c)$ and U_0 is the fundamental matrix of the impulsive differential equation (normalized at τ_0). Since the roots of a monic polynomial are continuous with respect to the coefficients of the polynomial [21], the eigenvalues of $M_{\{a\}_n}$ converge to those of M_0.

Part 3. Let a duration sequence $\{a\}$ have vectorial identification $\vec{a} = (a_0, \ldots, a_{c-1})$. The monodromy matrix is

$$M_{\{a\}} = \prod_{k=c-1}^{0} X(\tau_{k+1}; \tau_k) L_{\{a\}}(\tau_k + a_k; \tau_k).$$

Parts 1 and 2 imply that the function $M_{\{\cdot\}} : \mathbb{R}_+^c \cup \{0\} \to \mathbb{R}^{n \times n}$ defined by

$$\{a\} \mapsto \begin{cases} M_{\{a\}}, & \vec{a} \in \mathbb{R}_c^+, \\ M_0, & \vec{a} = 0, \end{cases}$$

is continuous at zero. Since the roots of a monic polynomial depend continuously on its coefficients [21], the eigenvalues of $M_{\{a\}}$ depend continuously on the input $\{a\}$.

Let ρM denote the spectral radius of the matrix M. Then $\rho : M \mapsto \rho M$ is continuous [8]. It follows that $\rho M_{\{\cdot\}}$ is continuous at zero. Let $\epsilon = |1 - \rho M_0|$. Then there exists $\delta_1 > 0$ such that for all duration sequence $\{a\}$ with $|\vec{a}| < \delta_1$, we have

$$|\rho M_{\{a\}} - \rho M_0| < |1 - \rho M_0|.$$

If this is satisfied, then $\rho M_0 < 1$ implies $\rho M_{\{a\}} < 1$ and $\rho M_0 > 1$ implies $\rho M_{\{a\}} > 1$.

Now, let $\gamma M = \min\{|1 - |\lambda|| : \lambda \in \sigma(M)\}$. Then γ is continuous [9]. Recall that,

[8] This can be proven by appealing to the correspondence of roots of monic polynomials with their coefficients as in Naulin and Pabst [21].

[9] This can be proven similarly to the continuity of the spectral radius map.

by assumption, we have $\gamma M_0 > 0$. It follows that there exists $\delta_2 > 0$ such that, for all duration sequences $\{a\}$ with $|\vec{a}| < \delta_2$, we have $\gamma M_{\{a\}} > 0$.

All aspects of part 3 are satisfied with $\delta = \min\{\delta_1, \delta_2\}$ except for the sufficiency. That is, it remains to prove that $\rho M_{\{a\}} < 1$ implies $\rho M_0 < 1$ and vice versa (with strict inequalities). Suppose to the contrary that $\rho M_0 \geq 1$. By hypothesis, we cannot have $\rho M_0 = 1$, so we must have $\rho M_0 > 1$. But then we must have $\rho M_{\{a\}} > 1$ as well, which is a contradiction. The opposite (strict) inequality is shown by a symmetric argument. ∎

The above theorem describes the case where the functions $\epsilon_k(t,r)$ are asymptotically bounded. Indeed, take $w^k(t,r) = e^{\sigma_A r} - 1$. This function is clearly continuous, and $w^k(t,0) = 0$ for all t. The condition of the theorem reduces to $\epsilon_k^B(t,r) = O(1)$ as $r \to 0$. However, the theorem is more general; taking $w^k(t,r) = \sqrt{r}$ produces an upper bound that approaches infinity as $r \to 0$.

In summary, there are many families of homogeneous impulse extensions that exhibit singular behaviour as the duration sequence becomes arbitrarily small, but those that are well-behaved are asymptotically dominated (up to multiplication by a continuous function with certain continuity properties) by a particular singular exponential function. In other words, as long as the impulse extension functions do not "blow up" too badly for small duration sequences, the floquet multipliers of the continuous system will eventually coincide with those of the impulsive system. In particular, since the operation of extracting roots from monic polynomials is continuous with respect to their coefficients, we are able to say that, for sufficiently small duration sequences, the stability of an impulsive differential equation completely describes the stability of its well-behaved impulse extensions equations.

We have therefore answered one of the questions set out in the introduction. Under certain asymptotic criteria (see (3.5.2)), linear periodic impulsive differential equations with impulses at fixed times will exhibit the same qualitative behaviour as a class of approximating continuous systems (impulse extension equations) as long as the perturbation times are short enough, and the impulsive system is either asymptotically stable or strongly unstable. If, however, the impulsive system has a floquet multiplier on the unit circle (that is, part 3 of Theorem 3.5.5 is not satisfied), then it is impossible to predict how a similar continuous system (i.e., an induced impulse extension equation) will behave, since its floquet multipliers will converge, but for all nonzero-length perturbations, the results are uninformative (since the multipliers could be inside or ourside the unit disc). We now present such a counterexample.

Example 3.5.6. Somewhat trivial impulsive differential equations such as

$$\begin{aligned} \frac{dr}{dt} &= r\sin(t), \qquad && t \neq 2k\pi \\ \Delta r &= 0, && t = 2k\pi. \end{aligned} \tag{3.5.4}$$

are easy to work with computationally and provide a wealth of counterexamples. The floquet multiplier of this system is $\mu_0 = 1$, and the fundamental matrix solution at $t_0 = 0$ is $X(t) = \exp(1 - \cos(t))$. Therefore, part 3 of the time scale consistency theorem does not apply. Let us consider for any $a > 0$, the family of impulse extensions

$$\varphi(t, a) = a^5 \sin\left(\frac{2\pi t}{a}\right) \sin\left(\frac{1}{a}\right)$$

with $\varphi(t, 0) \equiv 0$ for all t. Note that

$$c(a) \equiv \int_0^a e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt > 0$$

for $a < \pi$ We argue this as follows. For $0 \leq t < \pi$, the function $e^{\cos(t)}$ is positive and decreasing. Consequently, $e^{\cos(t)} > e^{\cos(a/2)}$ for $t < a/2$ and $e^{\cos(t)} < e^{\cos(a/2)}$ for $t > a/2$. Then

$$c(a) = \int_0^{a/2} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt$$

$$> \int_0^{a/2} \min_{[0,a/2]} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} \max_{[a/2,a]} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt$$
$$= \int_0^{a/2} e^{\cos(a/2)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} e^{\cos(a/2)} \sin\left(\frac{2\pi t}{a}\right) dt = 0.$$

Therefore $c(a) > 0$ for $t < \pi$. By (3.3.5), the floquet multiplier of the induced impulse extension equation with duration sequence a is

$$\mu_a = X(2\pi)\left[1 + \int_0^a e^{\cos(t)-1} a^5 \sin\left(\frac{2\pi t}{a}\right) \sin\left(\frac{1}{a}\right) dt\right]$$
$$= 1 + \frac{1}{e} a^5 \sin\left(\frac{1}{a}\right) c(a).$$

The function $a^5 \sin\left(\frac{1}{a}\right)$ has roots at $(2\pi n)^{-1}$ for all integers n, with derivative oscillating in sign from positive to negative. Consequently, $a^5 \sin\left(\frac{1}{a}\right)$ assumes both positive and negative values on any interval $(0, \epsilon)$. We conclude that μ_a oscillates between greater than and less than 1 on any interval $(0, \epsilon)$ for $\epsilon < \pi$; see Figure 3.2.

This particular family of impulse extensions was chosen to illustrate that this phenomenon can occur even when the impulse extensions are very regular. In particular, the choice $\varphi(t, a) = a^n \sin\left(\frac{2\pi t}{a}\right) \sin\left(\frac{1}{a}\right)$ for $n \geq 3$ an odd integer is in fact $\frac{n-1}{2}$ times continuously differentiable, and the same oscillatory behaviour of the floquet multipliers will ensue.

The first time scale consistency theorem (Theorem 3.5.5) is a statement about the convergence of floquet multipliers; it says nothing about the convergence of solutions. For this purpose, we have the following.

Theorem 3.5.7 (Time Scale Consistency Theorem II)**.** Consider the situation as in Theorem 3.5.5. Let

$$\varphi_k^B(t, a_k) = \frac{1}{a_k} B_k + \epsilon_k^B(t, a_k)$$
$$\varphi_k^h(t, a_k) = \frac{1}{a_k} h_k + \epsilon_k^h(t, a_k)$$

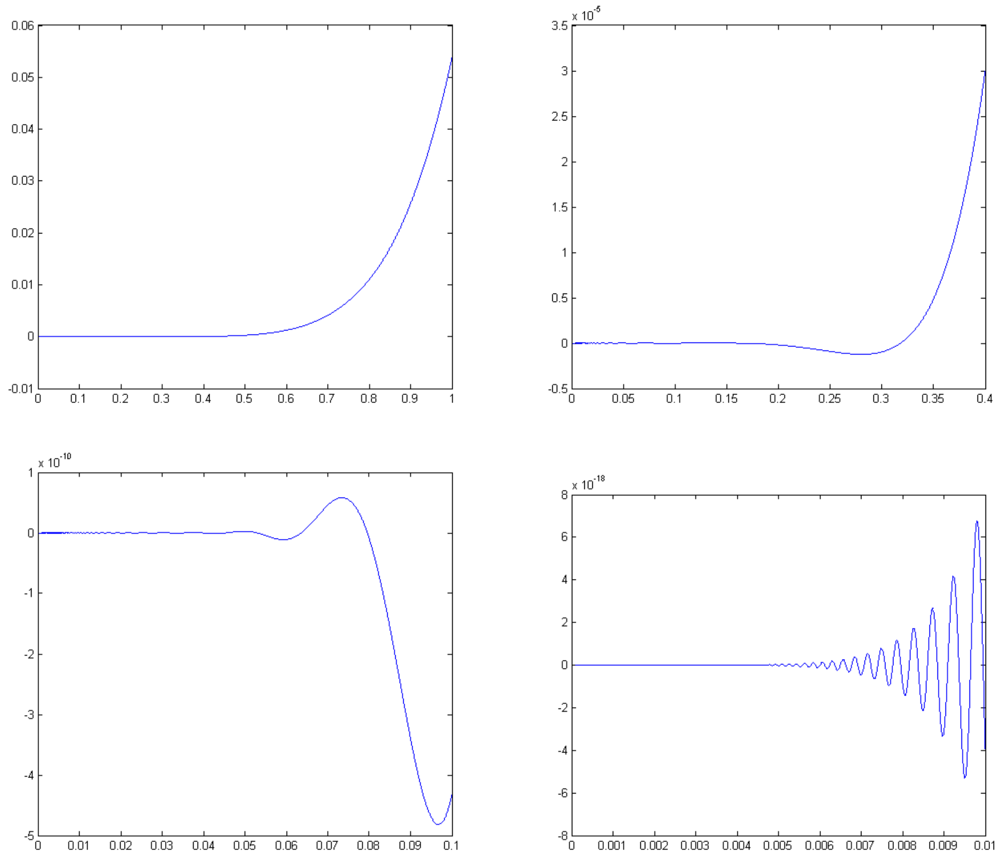

Figure 3.2: Plots of $\mu_a - 1$ on four different scales, with 4000 sample points. Notice that oscillation is more easily seen on the smaller scales. This is to be expected, as the amplitude is a fifth-order polynomial in a.

be the factorization furnished by Lemma 3.5.2. Suppose $\epsilon_k^B(t,r)$ and $\epsilon_k^h(t,r)$ satisfy the asymptotic bounds

$$\epsilon_k^\alpha = O\left(w_\alpha^k(t,r)\frac{1}{e^{\sigma_A r}-1}\right)$$

on $[\tau_k, \tau_k + r)$ as $r \to 0^+$ for $k = 0, \ldots, c-1$ and $\alpha \in \{B, h\}$, where the functions w_k^α satisfy the properties *W1–2* of Theorem 3.5.5. If $\{a\}$ is a duration sequence, let $x(t, x_0, a)$ denote the solution of (C) with duration sequence $\{a\}$ satisfying $x(t_0, x_0, a) = x_0$, and let $x(t, x_0, 0)$ denote the solution of the the impulsive differential equation (I) satisfying $x(t_0, x_0, 0) = x_0$. Then for any sequence of duration sequences $\{a\}_n \to 0$, we have

$$x(t, x_0, a_n) \to x(t, x_0, 0)$$

pointwise for all $t \in \mathbb{R}$.

We omit the proof of this theorem, since the argument is similar in form to the proofs of Theorem 3.5.5 and Lemma 3.5.4, if one takes into account the variation of constants formula for impulsive differential equations (see 2.18 of [5]). Corollary 3.2.4 guarantees that such a variation of constants formula exists for the impulse extension equation, and in particular, that it suffices to prove the pointwise convergence of a single, particular solution of the inhomogeneous equation (since Theorem 3.5.5 guarantees convergence of the homogeneous part).

3.6 The predictable set

As was suggested in Example 3.3.4, condition [E] to fails in general. It is not difficult to see why this failure occurs. Consider the following scalar equation with cycle number 1.

$$\begin{aligned} \frac{dx}{dt} &= ax, && t \notin \mathcal{S}(\delta) \\ \frac{dx}{dt} &= ax + \frac{1}{\delta}bx(\tau_k), && t \in \mathcal{S}_k(\delta). \end{aligned} \tag{3.6.1}$$

Solutions will "cross" at time $t^* \in (0, \delta)$ and then periodically if

$$\delta > t^* = \frac{-1}{a}\log(1 + a\delta/b) > 0.$$

This crossing can only occur if a and b have opposite signs. However, this is problematic from a modelling (specifically, control) perspective because having such models typically involve precisely the aforementioned sign pattern with a and b (see the models appearing in the introduction, for example). Moreover, the above problem will also be apparent in more general situations. If the coefficient a is replaced by a periodic function $a = a(t)$, the same crossing problem will arise if $a(0)$ and b have opposite signs, provided a is continuous at 0. This problem persists in higher dimensions as well. As has been previously illustrated, since an impulse extension can always be factored as the sum of a constant extension (as above) and a mean-zero function, this crossing behaviour is always possible.

This problem illustrates the strength of impulsive differential equations, for this issue is (almost) absent in the impulsive case. However, if we wish to make meaningful comparisons between discontinuous impulsive systems and their continuous impulse extension counterparts, then we will need to weaken our hypotheses. Specifically, the existence of fundamental matrix solutions is too strong a requirement to work with. The problem is the merging of solutions, which is quantified (linearly) by the presence of times where the fundamental matrix is noninvertible. This condition will now be relaxed.

Definition 3.6.1. *A matrix function $U : I \subset \mathbb{R} \to \mathbb{R}^{n\times n}$ is a* matrix solution *of the homogeneous impulse extension equation* (3.2.1) *if the following conditions are satisfied.*

1. *for all $x_0 \in \mathbb{R}^n$, $x(t) = U(t)x_0$ is a solution of* (3.2.1),
2. *there exists $t^* \in I$ such that $U(t^*)$ is invertible.*

A matrix solution $U(t)$ *is* maximal *if there is no matrix solution* $V : I^+ \to \mathbb{R}^{n\times n}$ *with* $I^+ \supsetneq I$ *such that* $U(t) = V(t)$ *on* I.

Under this definition, all fundamental matrix solutions are matrix solutions, but not vice-versa.

One property of fundamental matrix solutions used in the classical proof of Floquet's theorem is, that, for any two fundamental matrix solutions, $X(t)$ and $Y(t)$, there is a unique invertible matrix C, such that $Y(t) = X(t)C$ for all $t \in \mathbb{R}$. In the classical case, fundamental matrices are invertible everywhere, so one can simply choose $C = X^{-1}(0)Y(0)$. Since we do not require matrix solutions to be invertible, however, this may not be possible. Because this property is so crucial, we state a definition.

Definition 3.6.2. *Let* $\mathcal{U}$ *denote the set of all maximal matrix solutions for the homogeneous equation* (3.2.1)*.* $\mathcal{U}$ *satisfies the* uniqueness property *if, for each pair* $X(t), Y(t) \in \mathcal{U}$*, there exists a unique invertible matrix* C *such that* $Y(t) = X(t)C$ *whenever* t *is in the domain of both* X *and* Y.

Definition 3.6.3. *A solution* $x : I \to \mathbb{R}^n$ *of the linear impulse extension equation* (3.0.1) *is* unique on I *if, for any other solution* $y : I' \to \mathbb{R}^n$ *with* $I' \subseteq I$*, we have* $y|_I = x$*. A solution is* locally unique at $t_0 \in \mathbb{R}$ *if there exists a nonempty open interval* I *containing* t_0 *on which said solution is unique.*

The uniqueness property is satisfied under fairly mild conditions. The following notion will be central to both this, and later, stability results.

Definition 3.6.4. *The* predictable set*,* $\mathcal{P} \subset \mathbb{R}$*, of a linear impulse extension equation* (3.0.1)*, is the set of all* $t_0 \in \mathbb{R}$ *such that, for any* $x_0 \in \mathbb{R}^n$*, there exists a solution* $x(t)$ *of* (3.0.1) *that satisfies* $x(t_0) = x_0$ *and is locally unique at* t_0.

We have the following obvious proposition, which follows directly from Lemma 3.1.2.

Proposition 3.6.1. Define the map $p : \mathcal{S}^+ \to \mathbb{R}$ by

$$p(t) = \det L(t; \max\{\tau_k : \tau_k \leq t\}),$$

where $\mathcal{S}_k^+ = (\tau_k, \tau_k + a_k]$ and $\mathcal{S}^+ = \bigcup_k \mathcal{S}_k^+$. Then

$$\mathcal{P} = \mathbb{R} \setminus p^{-1}(0).$$

The utility of the predictable set is that it it precisely the set on which any matrix solution can hope to be invertible.

Lemma 3.6.2. Let $U : (\alpha, \beta) \to \mathbb{R}^{n\times n}$ be a matrix solution of the homogeneous impulse extension equation (3.2.1). If $U(t_0)$ is invertible, then t_0 is *predictable*; that is, $t_0 \in \mathcal{P}$. Also, if this is the case, then the function $U(t)U^{-1}(t_0)x_0$ is, locally, the unique solution of the initial value problem $x(t_0) = x_0$.

Proof: Suppose $U(t_0)$ is invertible. Then, for all $x_0 \in \mathbb{R}^n$, the function

$$x(t; x_0) = U(t)U^{-1}(t_0)x_0$$

is a solution of the homogeneous equation satisfying $x(t_0; x_0) = x_0$. If (t_0, x_0) is k-indeterminate, it follows by Lemma 3.1.2 that $L(t_0; \tau_k)$ must have full rank because the indeterminate initial value problem $x(t_0) = x_0$ has a solution for any $x_0 \in \mathbb{R}^n$. Therefore, $\det L(t_0; \tau_k) \neq 0$, so that $t_0 \in \mathcal{P}$. Conversely, if $t_0 \notin \mathcal{S}^+$, then $t_0 \in \mathcal{P}$; see Proposition 3.6.1. We have that $U(t)U^{-1}(t_0)x_0$ is, locally, the unique solution of the IVP $x(t_0) = x_0$ because of Lemma 3.1.2. ∎

The converse to this statement is not, in general, true. It is possible for t_0 to be predictable and for a matrix solution $U(\cdot)$ to be non-invertible at t_0, as Example 3.6.3 demonstrates.

Example 3.6.3. Consider the linear impulse extension equation

$$\begin{aligned} \frac{dr}{dx} &= 0, & t &\notin [2k, 2k+1) \\ \frac{dx}{dt} &= -x(2k), & t &\in [2k, 2k+1). \end{aligned} \tag{3.6.2}$$

The function $U : [0, \infty) \to \mathbb{R}$, defined by

$$U(t) = \begin{cases} 1 - t, & t \leq 1, \\ 0, & t > 1, \end{cases}$$

is a matrix solution of (3.6.2). The times $2k$ for $k \in \mathbb{N}$ are predictable, yet $U(t) = 0$ for $t \geq 1$, implying that, for any $x_0 \in \mathbb{R}$, the solution of the initial value problem $x(0) = x_0$ permanently merges with the trivial solution in finite time.

If each "endpoint" of impulse effect, $\tau_k + a_k$, is predictable, then the predictable set is precisely where any maximal matrix solution will be invertible. As an added benefit, the uniqueness property holds and any maximal matrix solution is defined on the entire real line.

Theorem 3.6.4 (Predictable Endpoints)**.** Suppose $\tau_k + a_k \in \mathcal{P}$ for all $k \in \mathbb{Z}$. Then every maximal matrix solution of the homogeneous equation (3.2.1) has domain equal to $\mathbb{R}$, the uniqueness property is satisfied and any matrix solution is invertible at $t \in \mathbb{R}$ if and only if $t \in \mathcal{P}$.

Proof: Let $U(t)$ be a maximal matrix solution with domain I. We claim that $I = \mathbb{R}$. Indeed, $U(t^*)$ is nonsingular at some t^*, so for all $x_* \in \mathbb{R}^n$, there is unique solution of the initial value problem $x(t^*) = x_*$; see Lemma 3.6.2. However, by Lemma 3.1.2 and Theorem 3.1.3, this solution must be defined on $\mathbb{R}$ if it is to be maximal. It follows that $U(t)$ has domain $\mathbb{R}$ since each of its columns does.

We now show that $U(t)$ is invertible on $\mathcal{P}$. Suppose $t_0 \in \mathcal{P} \cap \mathcal{S}_k^+$. Under the conditions of the theorem, we have $\det L(t_0; \tau_k) \neq 0$ and $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j \in \mathbb{Z}$. By Theorem 3.1.3, there is a unique solution of the initial value problem

$x(t_0) = 0$. Suppose the conclusion is false, so that $U(t_0)$ is not invertible. Then there are two distinct $x, y \in \mathbb{R}^n$ for which $U(t_0)x = 0 = U(t_0)y$. By definition of the matrix solution, both $U(t)x$ and $U(t)y$ are solutions of the initial value problem $x(t_0) = 0$, and since $U(t^*)$ is invertible for some t^*, these solutions are distinct because $U(t^*)x \neq U(t^*)y$, which is a contradiction. We conclude that $U(t_0)$ is invertible. An identical argument serves to prove the result for $t_0 \in \mathcal{P} \setminus \mathcal{S}^+$. Therefore, $U(t)$ is invertible on $\mathcal{P}$. The converse has previously been established in Lemma 3.6.2.

Now, let $X(t)$ and $Y(t)$ be two maximal matrix solutions. Define $C = X^{-1}(\tau_0)Y(\tau_0)$, which exists and is invertible by the above result. We will show that $Y(t) = X(t)C$. Define the function $Z(t) = Y(t) - X(t)C$. It is easily verified that $Z(t)$ is a solution of the matrix impulse extension equation (3.3.1), and, by construction, that $Z(\tau_0) = 0$. Under the conditions of the theorem, the unique solution of the initial value problem $x(\tau_0) = 0$ of the homogeneous equation is precisely the trivial solution, $x(t) = 0$; see Theorem 3.1.3. Since each column of $Z(t)$ is a solution of the homogeneous equation and $z(\tau_0) = 0$, we conclude that $Z(t) = 0$, and consequently, that $Y(t) = X(t)C$. Uniqueness is obvious. ∎

Effectively, requiring the impulse endpoints to be predictable forces the solution to be uniquely continuable backwards to each corresponding impulse start time. Correspondingly, it is never possible for two distinct solutions that begin at an impulse time to leave an impulse set $\mathcal{S}_k$ in a merged state. This prevents the somewhat pathological behaviour of two solutions permanently merging together in finite time, which can happen if this hypothesis is not satisfied; see Example 3.6.3.

Remark 3.6.5. Condition [E] is precisely the situation in which the predictable set is the entire real line; $\mathcal{P} = \mathbb{R}$. This requirement will now be weakened significantly, since Theorem 3.6.4 has demonstrated that a predictable set containing all impulse start points, τ_k, and endpoints, $\tau_k + a_k$, is all that is necessary to maintain the uniqueness property. We define the condition [E^+]:

E^+: $\det L(\tau_k + a_k; \tau_j) \neq 0$ for all $k \in \mathbb{Z}$; that is, $\{\tau_k + a_k\}_{k\in\mathbb{Z}} \subset \mathcal{P}$.

We are now ready to state and prove a weakened form of Floquet's theorem.

Theorem 3.6.6. Let conditions [E^+] and [P] hold. Every maximal matrix solution $U(t)$ of the homogeneous T-periodic impulse extension equation (3.2.1) can be represented in the form

$$U(t) = \phi(t)e^{\Lambda t}, \tag{3.6.3}$$

where $\Lambda \in \mathbb{C}^{n\times n}$ is nonsingular and the matrix function $\phi(t)$ is absolutely continuous, differentiable almost everywhere, complex-valued, T-periodic and invertible on the predictable set $\mathcal{P}$.

Proof: The proof is almost identical to the fundamental matrix case. We outline the steps, refering the reader to the proof of Theorem 3.3.1 when the argument is nearly identical.

Step 1: If $U(t)$ is a matrix solution, then so is $U(t+T)$. See the proof of Theorem 3.3.1.

Step 2: There is a unique matrix M with the property that $U(t+T) = U(t)M$. Also, M can be chosen to be $M = U^{-1}(\tau_0)U(\tau_0 + T)$. This is guaranteed by Theorem 3.6.4.

Step 3: Define $\Lambda = \frac{1}{T}\ln M$ and $\phi(t) = U(t)e^{-\Lambda t}$. Then formula (3.6.3) is valid, and Λ and $\phi(t)$ satisfy the properties of the theorem. Since M is nonsingular, so is Λ. $\phi(t)$ is absolutely continuous and almost everywhere differentiable because $U(t)$ is. Periodicity of $\phi(t)$ can be shown in the same way as in the proof of Theorem 3.3.1. By Theorem 3.6.4, $U(t)$ is invertible on $\mathcal{P}$, from which we conclude that $\phi(t)$ is as well. ∎

Therefore, matrix solutions of the homogeneous periodic impulse extension equation (3.2.1) have Floquet representations under the weakened assumption [E^+]. Recall that the matrix M appearing in the theorem is called the *monodromy matrix*.

Corollary 3.6.7. Let conditions [P] and [E^+] hold. If $U_1(t)$ and $U_2(t)$ are two maximal matrix solutions of (3.2.1), then their monodromy matrices M_1 and M_2 are similar.

Proof: By Theorem 3.6.4, we have $U_1(t) = U_2(t)C$ for some invertible matrix C. Then

$$\begin{aligned} M_1 &= U_1(\tau_0)^{-1}U_1(\tau_0+T) \\ &= (U_2(\tau_0)C)^{-1}U_2(\tau_0+T)C \\ &= C^{-1}U_2(\tau_0)^{-1}U_2(\tau_0+T)C \\ &= C^{-1}M_2C. \end{aligned}$$

By definition, M_1 and M_2 are similar. ∎

It can be shown that many of the results obtained in Sections 3.3–3.5 remain valid in this more general framework. In the following subsections, we will state (and occasionally prove) the generalizations of the most poignant of those theorems. Many of the proofs are nearly identical to the analoguous ones appearing in Sections 3.3–3.5, and whenever the proofs do not require a serious change, they will be omitted. Finally, we will show that making the assumption [E^+] is "reasonable". To begin, we will state a modified definition of stability that is more applicable to our purposes.

Definition 3.6.5. *Let $x(t)$ be a solution of the linear impulse extension equation* (3.0.1) *and let I be a subset of $\mathbb{R}$. The solution $x(t)$ is*

- stable on I *if, for all $t_0 \in I$ and all $\epsilon > 0$, there exists $\delta > 0$ such that for* *any solution $y(t)$ satisfying $|x(t_0) - y(t_0)| < \delta$, we have $|x(t) - y(t)| < \epsilon$ for all $t \geq t_0$,*

- uniformly stable on I *if it is stable on I and δ can be chosen independently of t_0,*
- attracting on I *if, for all $t_0 \in I$, there exists $\delta > 0$ such that, for any solution $y(t)$ satisfying $|x(t_0) - y(t_0)| < \delta$, we have $|x(t) - y(t)| \to 0$ as $t \to \infty$,*
- uniformly attracting on I *if it is attracting on I and δ can be chosen independently of t_0,*
- asymptotically stable on I *if it is stable and attracting on I,*
- uniformly asymptotically stable on I *if it is uniformly stable and uniformly attracting on I.*

We say a linear impulse extension equation has one of the above stability properties if all of its solutions do and the various δ's appearing above do not depend on the choice of solution.

3.6.1 Results from Section 3.3

Corollary 3.6.8 (See Corollary 3.3.2)**.** Let conditions [P] and [E^+] hold. The homogeneous periodic impulse extension equation (3.2.1) has a kT-periodic solution if and only if there exists a multiplier $\mu \in \sigma(M)$ such that $\mu^k = 1$, where M is the monodromy matrix.

Corollary 3.6.9 (See Corollary 3.3.3)**.** Let condition [P] and [E^+] hold. Let $N \subseteq \mathcal{P}$ be a subset of the predictable set. The linear homogeneous impulse extension equation is

- asymptotically stable on N and uniformly attracting on $\mathbb{R}$ if and only if $\rho(M) < 1$,
- stable on N if and only if $\rho(M) \leq 1$ and, for any eigenvalue μ of M with unit modulus, the geometric and algebraic multiplicities of μ coincide.

Proof: We will prove the uniform attractivity first. Let $x(t)$ and $y(t)$ be two solutions of (3.2.1). Since $U(\tau_0)$ is nonsingular, we must have $x(t) = U(t)U^{-1}(\tau_0)x(\tau_0)$ and $y(t) = U(t)U^{-1}(\tau_0)y(\tau_0)$ by Lemma 3.6.2. Then

$$x(t) - y(t) = U(t)U^{-1}(\tau_0) \cdot [(x(\tau_0) - y(\tau_0)].$$

It follows that $x(t)$ is uniformly attracting if and only if $U(t) \to 0$ as $t \to \infty$. The rest of the proof is identical to that of Corollary 3.3.3.

We now prove stability results. Let $N \subseteq \mathcal{P}$. Note that, by the previous calculation, it suffices to check the stability of the trivial solution $x = 0$. By Lemma 3.6.2 and Theorem 3.6.6, the solution of the initial-value problem $x(t_0; t_0, x_0) = x_0$ for $t_0 \in N$ can be written

$$x(t; t_0, x_0) = \phi(t)e^{\Lambda t}K_{t_0}x_0,$$

where $K_{t_0} = e^{-\Lambda t_0}\phi^{-1}(t_0)$. Since $\phi(t)$ is periodic, all solutions remain bounded for all $t \geq t_0$ if and only if $||e^{\Lambda t}|| \leq K$ for some constant $K > 0$, which is true if and only if all eigenvalues of Λ have negative real parts, and any eigenvalue λ with zero real part has geometric multiplicity equal to its algebraic multiplicity [15]. Since $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic and $M = \exp(\Lambda T)$, the spectral mapping theorem (see [22], Theorem 2.1.10) implies that any eigenvalue m_j of M is of the form $m_j = e^{\lambda_j T}$ for an eigenvalue λ_j of Λ, with multiplicities preserved. These will all have modulus less than or equal to one if and only if the eigenvalues of Λ all have real part less than or equal to zero, thus establishing the required equivalence of $\rho(M) \leq 1$ with $\Re\sigma(\Lambda) \leq 0$. If this condition is satisfied, then if $\epsilon > 0$ is given and

$$\delta = \frac{\epsilon}{K K_{t_0} \max||\phi(\cdot)||},$$

then $||x_0|| < \delta$ implies $||x(t; t_0, x_0)|| < \epsilon$ for all $t \geq t_0$, so we have stability. Note that δ exists since $\phi(t)$ is bounded and nonzero. Conversely, if any multiplier has modulus greater than 1, then Λ has an eigenvalue with positive real part and at least one column of $e^{\Lambda t}$ grows without bound, so we have instability.

On the other hand, we have $||e^{\Lambda t}|| \leq K e^{-\alpha t}$ for some constants $K > 0$ and $\alpha > 0$ if and only if all eigenvalues of Λ have negative real part, or, equivalently, if and only if all multipliers of M have modulus less than 1. When this is satisfied, we have the stability result of the above, together with the property that, for some $K' = K \cdot K_{t_0}$ indepdent of x_0,

$$||x(t; t_0, x_0)|| \leq K' e^{-\alpha t} ||x_0|| \to 0$$

as $t \to \infty$. That is, the periodic impulse extension equation is asymptotically stable.

Since these results hold for any $t_0 \in N$, the stability results hold. ∎

Corollary 3.6.10. Formula (3.3.5) for a monodromy matrix,

$$M = \prod_{k=c-1}^{0} X(\tau_{k+1}; \tau_k) L(\tau_k + a_k; \tau_k),$$

remains valid provided conditions [P] and [E$^+$] are satisfied.

3.6.2 Results from Section 3.4

Theorem 3.6.11 (See Theorem 3.4.1)**.** Let conditions [P] and [E$^+$] be satisfied. The stability/attractivity of the linear inhomogeneous equation (3.0.1) is completely determined by the stability/attractivity of its associated homogeneous equation (3.2.1).

That is, the inhomogeneous equation is stable (uniformly stable), attracting (uniformly attracting) or asymptotically stable (uniformly asymptotically stable) on a predictable subset $I \subset \mathcal{P}$ if and only if the homogeneous equation is.

Theorem 3.6.12 (See Theorem 3.4.3)**.** Consider a linear, periodic impulse extension equation (3.0.1) with period T and cycle number c. Let the conditions [P] and [E$^+$] be satisfied. This equation has a unique kT-periodic solution if and only if $\det(E-M^k) \neq 0$, where M is a monodromy matrix.

Contrary to the "strong" existence and uniqueness case where condition [E] is satisfied, the homogeneous adjoint equation (3.2.1) does not have very desirable properties if only the weaker condition [E$^+$] is satisfied. In particular, the adjoint equation is only defined on the predictable set, since its definition involves the inverse of a matrix solution. For this reason, a result analoguous to Theorem 3.4.5 may not exist.

That being said, Massera's theorem remains valid.

Theorem 3.6.13 (See Theorem 3.4.6)**.** Let conditions [P] and [E$^+$] be satisfied and suppose the inhomogeneous T-periodic equation (3.0.1) has a bounded solution for $t \geq \tau_0 = 0$. Then this equation has a nontrivial T-periodic solution.

3.6.3 Results from Section 3.5

As in Section 3.5, we assume the matrices $E+B_k$ are invertible for all $k \in \mathbb{Z}$, where the matrices B_k are those appearing in the linear impulsive differential equation (3.5.1).

Theorem 3.6.14 (See Theorem 3.5.5)**.** Consider a linear T-periodic impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \neq \tau_k, \\ \Delta x &= B_k x + h_k, && t = \tau_k, \end{aligned} \tag{I}$$

with cycle number c. Let $\varphi_k = (\varphi_k^B(t,r), \varphi_k^h(t,r))$ be a family of linear impulse extensions for the impulsive differential equation (I), and, for each duration sequence $\{a\}$, consider the induced impulse extension equation:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \notin \mathcal{S}(a), \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t,a_k)x(\tau_k) + \varphi_k^h(t,a_k), && t \in \mathcal{S}_k(a). \end{aligned} \tag{C}$$

Let

$$\varphi_k^B(t,a_k) = \frac{1}{a_k} B_k + \epsilon_k^B(t,a_k)$$

be the factorization furnished by Lemma 3.5.2. Suppose conditions [P] and [E$^+$] hold for each duration sequence. There exists a number $\sigma_A > 0$ depending only on the matrix $A(t)$ with the following property:

Suppose that the conditions *W1–2* of Theorem 3.5.5 are satisfied. The following results hold:

1. If $\{a\}_n \to 0$ is a convergent sequence of duration sequences and $U_{\{a\}_n}(t)$ is the corresponding sequence of matrix solutions for the impulse extension equation (C) mutually normalized at some $t_0 \in \mathbb{R}$, then

   $$U_{\{a\}_n}(t) \to U_0(t)$$

   pointwise, where $U_0(t)$ is the fundamental matrix of the impulsive differential equation (I) normalized at t_0.

2. If $\{a\}_n \to 0$ is a convergent sequence of duration sequences, $\sigma(\mathrm{I})$ is the spectrum of the monodromy matrix for the impulsive equation (I) and $\sigma(\mathrm{C}_n)$ is the spectrum of the monodromy matrix of the impulse extension equation (C) with duration sequence $\{a\}_n$, then

   $$\sigma(\mathrm{C}_n) \to \sigma(I).$$

3. Let $I \subset [\tau_0, \tau_0 + T]$ be closed and disjoint from $\{\tau_k\}_{k\in\mathbb{Z}}$. There exists $\delta > 0$ such that, for any duration sequence $\{a\}$ whose vector identification $\vec{a}$ satisfies $|\vec{a}| < \delta$, the impulse extension equation (C) is asymptotically stable on

$$\bigcup_{k\in\mathbb{Z}} I + kT$$

if and only if the impulsive differential equation (I) is asymptotically stable, where $I + y = \{x + y : x \in I\}$ is the translation of the set I by y.

4. There exists $\delta > 0$ such that, for any duration sequence $\{a\}$ whose vector identification $\vec{a}$ satisfies $|\vec{a}| < \delta$, the impulse extension equation (C) is uniformly attracting on $\mathbb{R}$ if and only if the impulsive differential equation (I) is asymptotically stable.

Proof: The proofs of parts 1–2 are identical to those of Theorem 3.5.5. We prove only part 3 and provide an outline of part 4.

Part 3. Denote $I^- = \min I$ and let

$$k(t) = \{k : \tau_k \le t < \tau_{k+1}\}.$$

If $\{a\}$ is a duration sequence, let us consider the function $L_{\{a\}}(t; \tau(t))$ defined on $I \cap \overline{\mathcal{S}}$ analoguously to the proof of Theorem 3.5.5. Then

$$\det U(t) = \det X(t; \tau_{k(t)}) \det L_{\{a\}}(h(t); \tau_{k(t)}) \prod_{I^- \le \tau_j < \tau_{k(t)}} \det X(\tau_{j+1}; \tau_j) \det L_{\{a\}}(\tau_j + a_j; \tau_j),$$

where $h(t) = \min\{t, \tau_{k(t)} + a_{k(t)}\}$. Since $X(t; s)$ is invertible for all t, s, the only terms that could be zero are the determinants of the matrices $L_{\{a\}}(\tau_j + a_j; \tau_j)$ and $L_{\{a\}}(h(t); \tau_{k(t)})$.

By Proposition 3.5.3, $L_{\{a\}}(\tau_j + a_j; \tau_j) \to (E + B_j)$ as $\{a\} \to 0$. Consequently, each of these terms eventually has nonzero determinant. In particular, since I is bounded, there exists $\delta_1 > 0$ such that if $|\vec{a}| < \delta$, then $L_{\{a\}}(\tau_j + a_j; \tau_j)$ is invertible for all $\tau_j \in I$, since there are at most finitely many indices j for which this inclusion occurs.

If $t \notin I \cap \mathcal{S}$, then $L_{\{a\}}(h(t);\tau_{k(t)}) = L_{\{a\}}(\tau_{k(t)} + a_{k(t)};\tau_{k(t)})$, which is invertible by the above argument if $|\vec{a}| < \delta_1$.

Let $t \in I \cap \mathcal{S}_k$. Then

$$L_{\{a\}}(h(t);\tau_{k(t)}) = E + \int_{\tau_k}^{h(t)} X^{-1}(s;\tau_k)\varphi_k^B(s)ds.$$

Since I closed and bounded with $\tau_k \notin I$, there exists $q_k \in I$ with

$$q_k = \inf I \cap \mathcal{S}_k > \tau_k.$$

If $a_k < q_k - \tau_k$, then $L_{\{a\}}(h(t);\tau_{k(t)}) = L_{\{a\}}(\tau_k + a_k;\tau_k)$. This construction can be repeated on $I \cap \mathcal{S}_j$ whenever this intersection is nonempty, and there will be only finitely many indices j where this holds since I is bounded. Therefore $\delta_2 \equiv \min\{q_k - \tau_k\}$ exists. δ_2 has the property that if $|\vec{a}| < \delta_2$ (where $|\cdot|$ denotes the uniform norm), then $L_{\{a\}}(h(t);\tau_{k(t)}) = L_{\{a\}}(\tau_{k(t)} + a_{k(t)};\tau_{k(t)})$.

Define $\delta \equiv \min\{\delta_1, \delta_2\}$. If $\{a\}$ is such that $|\vec{a}| < \delta$, then $L_{\{a\}}(\tau_j + a_j;\tau_j)$ is invertible for all $\tau_j \in I$ and for all $t \in I$, $L_{\{a\}}(h(t);\tau_{k(t)}) = L(\tau_k + a_k;\tau_k)$ for some $\tau_k \in I$. By the proof of Theorem 3.1.6, this holds[10] for all $k \in \mathbb{Z}$ and all $t \in I + kT$. Consequently, $\det U(t) \neq 0$ for all $t \in \bigcup_{k\in\mathbb{Z}} I + kT$. The rest of the proof follows a similar argument to that of part 3 of Theorem 3.5.5, and is omitted.

Part 4. This follows from part 1, the proof of part 3 of Theorem 3.5.5 and Theorem 3.6.11. ∎

Theorem 3.6.15 (See Theorem 3.5.7)**.** Theorem 3.5.7 remains valid if the condition [E] is replaced with the weaker condition [E$^+$].

[10] Because if $t \in \mathcal{S}_j$, then $L_{\{a\}}(t + kT;\tau_j + kT) = L_{\{a\}}(t;\tau_j)$ for all $k \in \mathbb{Z}$.

3.6.4 Condition [E$^+$] is asymptotically satisfied in the periodic case

We now prove a theorem that confirms that the results of Section 3.6 can always be applied, provided the duration of impulse effect is short.

Theorem 3.6.16. Consider a linear, T-periodic[11] impulse extension equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \notin \mathcal{S}(a), \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t, a_k)x(\tau_k) + \varphi_k^h(t, a_k), && t \in \mathcal{S}_k(a), \end{aligned} \tag{C}$$

with cycle number c induced by a family of impulse extensions $\varphi_k = (\varphi_k^B(t, r), \varphi_k^h(t, r))$ with duration sequence $\{a\}$. Let

$$\varphi_k^B(t, a_k) = \frac{1}{a_k} B_k + \epsilon_k^B(t, a_k)$$

be the factorization furnished by Lemma 3.5.2. Suppose conditions *W1–2* of Theorem 3.5.5 are satisfied. There exists $\delta > 0$ such that, if the duration sequence $\{a\}$ satisfies $|\vec{a}| < \delta$, then condition [E$^+$] is satisfied.

Proof: Under the conditions of the theorem, we know[12]

$$L_{\{a\}}(\tau_k + a_k; \tau_k) \to E + B_k$$

as $\{a\} \to 0$ for each k. Since $E + B_k$ has nonzero determinant for all k and the set $\{0, \ldots, c-1\}$ is finite, it follows that there exists $\delta > 0$ such that, for $|\vec{a}| < \delta$, $\det L_{\{a\}}(\tau_k + a_k; \tau_k) \neq 0$ for $k \in \{0, \ldots, c-1\}$. By Theorem 3.1.6, $\det L_{\{a\}}(\tau_k + a_k; \tau_k) \neq 0$ for all $k \in \mathbb{Z}$, so that condition [E$^+$] is satisfied. ∎

Therefore, assuming certain asymptotic requirements on the family of impulse extensions, it is always possible to (asymptotically) apply the techniques of Section

[11] That is, condition [P] holds.

[12] See the proof of part 1 of Theorem 3.5.5.

3.6 to study the stability, attractivity and existence of periodic solutions of linear, periodic impulse extension equations.

3.7 Summary and further research questions

We will end our presentation of impulse extension theory with a summary of key results. Following this, we will present some questions to be answered in future research.

We have presented impulse extension equations as a framework in which to study the time scale assumptions inherent in mathematical modelling with impulsive differential equations. It was shown that, for nonlinear equations, existence and uniqueness of solutions is a difficult problem unless one restricts attention to only a small class of initial-value problems; see Theorem 2.1.2. One nonlinear existence theorem was supplied (Theorem 2.3.1). For linear equations, these issues can be overcome; Theorem 3.1.3 and Corollary 3.1.4 provide necessary and sufficient conditions for existence and uniqueness of solutions of initial-value problems. Following this, we proved the existence of fundamental matrix solutions for homogeneous equations; see Theorem 3.2.1.

We then developed the theory of linear periodic impulse extension equations. This began with homogeneous equations. Included was a generalization of Floquet's Theorem 3.3.1 with applications to stability (Theorem 3.3.3) and existence of periodic solutions. In particular, the monodromy matrix plays a central role, as it does with ordinary and impulsive differential equations.

It was shown that a T-periodic inhomogeneous equation has a unique T-periodic solution if and only if the homogeneous equation no floquet multiplier equal to one (Theorem 3.4.3). Conversely, when there is a multiplier equal to one, we proved an analogue of Condition A.3.9 from impulsive differential equations that establishes necessary and sufficient conditions for existence of periodic solutions; see Theorem 3.4.5.

This required a derivation of the homogeneous adjoint equation (3.4.3). Following this, we demonstrated that stability of the inhomogeneous equation was equivalent to that of the homogeneous equation.

We concluded with what we refer to as time-scale consistency theorems. These theorems provide criteria under which certain properties of impulsive differential equations will carry over to induced impulse extension equations, provided the duration sequences are small. The first theorem (Theorem 3.5.5) says that an impulsive differential equation and an associated impulse extension equation will have the same stability properties provided the homogeneous part of the impulse extension satisfies an asymptotic bound and the duration sequence is small enough. The second theorem (Theorem 3.5.7) states that the trajectories of the impulse extension equation will converge pointwise to those of the impulsive differential equation as the duration sequence becomes arbitrarily small, provided a slightly stronger asymptotic bound is satisfied.

In Section 3.6, the aforementioned results for linear systems were generalized in such a way that stability and attractivity of solutions of periodic equations can still be determined by Floquet theory even if the system in question does not satisfy the strict requirement of existence and uniqueness of solutions outlined in Corollary 3.1.4. In particular, it was shown that this analysis can always be done if the duration sequence is small enough and the impulse extensions satisfy an asymptotic bound.

We now pose several questions related to the theory presented so far. They deal mostly with nonlinear equations.

Q1. Can we study the linearization at a periodic orbit of a nonlinear impulse extension equation?

Q2. Can anything constructive be said about existence and uniqueness of solutions of nonlinear impulse extension equations?

Q3. Does the first time scale consistency theorem apply to nonlinear impulse extension equations?

Q4. Is every periodic solution of an impulsive differential equation the pointwise limit of a periodic solution of an associated impulse extension equation, as the duration sequence approaches zero?

We now suggest possible solutions to these problems.

A1. Most likely, yes. In fact, Section 2.3 provides a hint at a possible form of such a linearization. Since every impulse extension equation can be seen as an impulsive differential equation by doubling the dimension of the phase space, and periodic orbits are preserved by this transformation, this equation can be linearized at the periodic orbit. Transforming back to an impulse extension equation, the result may very well be "the linearization" about the periodic orbit. See the schematic diagram (Figure 3.3).

Naturally, there are other questions to consider such as continuous/smooth dependence of solutions on initial conditions and parameters. These questions can certainly be handled for admissible initial-value problems. For indeterminate initial-value problems, the associated proofs will likely be more technical or the theorems limited in applicability. Further research is necessary.

A2. There may be local existence (and under strict assumptions, uniqueness) of solutions near periodic orbits. We would hypothesize by analogy with ordinary and impulsive differential equations that if the linearization at a periodic orbit (or trajectory) satisfies the conditions of existence and uniqueness of solutions, then this may also be true locally for the nonlinear equation, for initial conditions near the periodic orbit (or trajectory). In the terminology of Section 3.6, it may be fruitful to examine the structure of the "local" predictable sets of periodic orbits (or trajectories) by linearization.

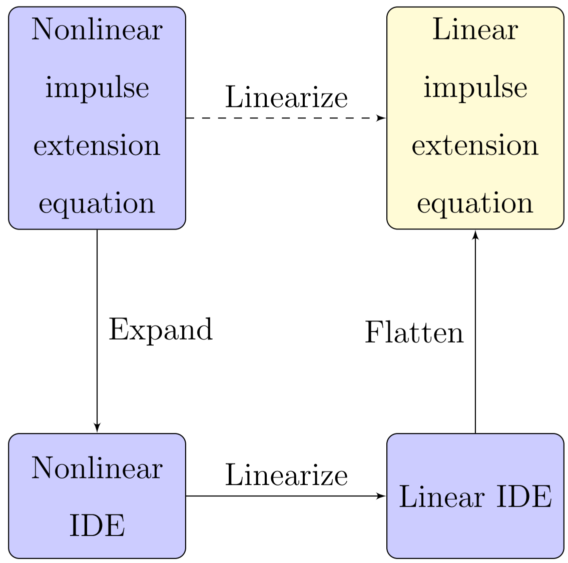

Figure 3.3: Diagram of possible linearization of a nonlinear impulse extension equation (impulse extension equation) at a periodic orbit. First, one could expand then n-dimensional nonlinear impulse extension equation into the impulsive differential equation (IDE) (2.4.1) with dimension $2n$. Next, one could linearize this IDE at the periodic orbit. Following this, the resulting linear IDE could be flattened back into an impulse extension equation of dimension n. We therefore have the following question: does the dashed arrow exist, and, if so, does the diagram commute?

A3. If linearization is possible, then the first time scale consistency theorem would be directly applicable to *each* periodic orbit of the nonlinear impulse extension equation. However, drawing useful conclusions from this in a mathematical modelling context would rely heavily on an affirmative answer to Q4, since a priori, one may not even know that a given associated impulse extension equation *has* a periodic orbit.

The ideal situation would be that each periodic orbit of the impulsive system correspond to (at least) one associated periodic orbit for a well-behaved family of impulse extensions equations (for small duration sequences), and that these periodic orbits converge to the impulsive periodic orbits as the duration sequence approaches zero. In this case, the first time scale consistency theorem would relate the stability of the impulsive periodic orbits to those periodic orbits of the impulse extension equations.

A4. This question of how to perturb a discontinous impulsive periodic orbit into a continuous periodic orbit in an associated impulse extension equation is probably the most difficult question to answer of the four. It is known that this is not always possible for linear systems [9]. We hypothesize that such a perturbation is possible and well-behaved with respect to stability if no floquet multiplier of the associated impulsive differential equation is equal to one (recall that this is a requirement in the linear case) and the impulse extensions and vector fields are sufficiently regular.

Chapter 4

Malaria Vector Control with Impulsive Differential Equations: The Full Model

We will be interested in two different, but related systems of impulsive differential equations. The first is the model of Smith? and Hove-Muskewa [30], and the second is an autonomous equation with impulses governed by an infection tracking strategy.

The model by Smith? and Hove-Muskewa consists of the following system of differential equations:

$$\begin{aligned} \dot{S} &= \pi - \beta_H SN + hI + \delta R - \mu_H S, \\ \dot{I} &= \beta_H SN - hI - \alpha I - (\mu_H + \gamma) I, \\ \dot{R} &= \alpha I - \delta R - \mu_H R, \\ \dot{M} &= \Lambda - \mu M - \beta_M M I, \\ \dot{N} &= \beta_M M I - \mu N, \end{aligned} \tag{4.0.1}$$

for $t \neq t_k$, with impulsive conditions given by

$$\begin{aligned} \Delta M &= -rM^-, \\ \Delta N &= -rN^-, \end{aligned} \tag{4.0.2}$$

when $t = t_k$.

Here, S, I and R represent, respectively, the number of susceptible, infected and temporarily immune humans, while M and N are, respectively, the number of susceptible and infected mosquitoes. The birth rate of humans is π, while that of mosquitoes is Λ. Both populations are assumed to experience a constant background death rate; μ_H is the death rate of humans and μ the death rate of mosquitoes. The rate of infection of a susceptible individual is β_H, and the rate of infection of a mosquito is β_M. Infected humans experience recovery at rate h (acquiring no immunity) or gain temporary immunity at rate α. Those individuals who have gained immunity lose it and become susceptible at rate δ. The death rate of infected humans due to malaria is γ. See [30] for further details.

The authors determined that spraying at fixed periodic times (so that $t_{k+1} - t_k = \tau$ for some constant τ) was always more beneficial (in the sense of reducing the number of mosquitoes) than spraying at times that did not satisfy such a periodicity condition. We will therefore assume without loss of generality that we are in the beneficial case, so that spraying occurs at times $t_k = k\tau$.

We consider also the following autonomous impulsive differential equation:

$$\begin{aligned} \dot{S} &= \pi - \beta_H SN + hI + \delta R - \mu_H S, \\ \dot{I} &= \beta_H SN - hI - \alpha I - (\mu_h + \gamma)I, \\ \dot{R} &= \alpha I - \delta R - \mu_H R, \\ \dot{M} &= \Lambda - \mu M - \beta_M MI, \\ \dot{N} &= \beta_M MI - \mu N, \\ \dot{\Theta} &= \eta \beta_H SN \end{aligned} \tag{4.0.3}$$

for $\Theta \neq \overline{\Theta}$, with impulse conditions given by

$$\begin{aligned} \Delta M &= -\, rM^-, \\ \Delta N &= -\, rN^-, \\ \Delta\Theta &= -\, \overline{\Theta} \end{aligned} \tag{4.0.4}$$

for $\Theta = \overline{\Theta}$. Here, the variable Θ tracks the total number of infections since the previous spraying event, scaled by an "observability" parameter, η, which is the probability that, on infection, a newly infected individual reports their infectious status. That is, $\Theta(t)$ is the number of new observed infections between the previous spraying event and time $t \geq 0$. The impulse effect is then triggered when a critical number $\overline{\Theta}$ of new infections is reached, at which point Θ is reset to zero.

Existence of periodic solutions for this set of differential equations is fairly easy to establish because of the condition by which the impulse occurs.

4.1 Equivalence and stability of periodic solutions of the fixed-time and autonomous models

It is fairly obvious that periodic solutions of the autonomous model give rise to periodic solutions of the model with fixed-time spraying events, provided the correct spraying interval is chosen. It turns out that the converse of this statement holds as well. Additionally, certain uniqueness results are transferable. This equivalence of periodic orbits is applicable to many impulsive vector-control models, so we state and prove the following lemma in full generality. In the following, we assume that all differential equations in question admit unique, globally defined solutions.

Lemma 4.1.1. Consider the following impulsive differential equation undergoing impulse effects at fixed times $t_k = k\tau$ with $k \in \mathbb{Z}$, $\tau \in \mathbb{R}^+$ and phase space $\Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n_+$, the nonnegative orthant:

$$\frac{dx}{dt} = g(x), \qquad \Delta x|_{t=k\tau} = a(x^-). \tag{4.1.1}$$

Suppose (4.1.1) has a unique, non-trivial τ-periodic solution with one impulse per cycle. Denote said periodic solution by $\tilde{x}_\tau(t)$. Then, for any $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $u > 0$ on $\tilde{x}_\tau(t)$, the *autonomous tracking system*

$$\begin{aligned} \frac{dx}{dt} &= g(x), \qquad \Delta x|_{\theta=\overline{\theta}} = a(x^-), \\ \frac{d\theta}{dt} &= u(x), \qquad \Delta \theta|_{\theta=\overline{\theta}} = -\overline{\theta} \end{aligned} \tag{4.1.2}$$

has a unique τ-periodic solution up to phase shift, with one impulse per cycle, given by $(\tilde{x}_\tau(t), \tilde{\theta}_\tau(t))$, whenever

$$\overline{\theta} = \int_0^\tau u(\tilde{x}_\tau(t))dt. \tag{4.1.3}$$

Conversely, if (4.1.2) has a non-trivial τ_0-periodic solution with one impulse per cycle for some $\overline{\theta} \in \mathbb{R}_+$, then (4.1.1) has a τ_0-periodic solution when $\tau = \tau_0$. Moreover, $\overline{\theta}$ satisfies equation (4.1.3).

Proof: Let (4.1.1) have a unique τ-periodic solution, $\tilde{x}_\tau(t)$, and let (4.1.3) hold. Consider then the point $(\tilde{x}_\tau(0^+), 0) \equiv z_0^+ \in \Omega \times \mathbb{R}_+ \equiv X$. Then the left limit of the forward flow $\phi : X \times \mathbb{R}_+ \to X$ of (4.1.2) at time τ is

$$\begin{aligned} \phi(z_0^+, \tau^-) = \lim_{t \to \tau^-} \phi(z_0^+, t) &= \left(\tilde{x}_\tau(\tau^-), \int_0^\tau u(\tilde{x}_\tau(t))dt \right) \\ &= \left(\tilde{x}_\tau(\tau^-), \overline{\theta} \right). \end{aligned}$$

Hence, an impulse occurs along the forward orbit from z_0^+ at time $t = \tau$. By periodicity of $\tilde{x}_\tau(t)$ and the impulse condition, we have

$$\phi(z_0^+, \tau^+) = \lim_{t \to \tau^+} \phi(z_0^+, t) = (\tilde{x}_\tau(0), 0) = z_0^+.$$

Note also that $\overline{\theta} \neq 0$, since u is strictly positive along $\tilde{x}_\tau$. Hence, there is always a finite amount of time between impulses. We may therefore conclude that equation

(4.1.2) has a τ-periodic solution. Moreover, it is unique up to phase shift. If (4.1.2) had another τ-periodic solution, say $(x_1(t), \theta_1(t))$, which had an impulse effect at $t = T$, then $x_1(t+T)$ would be a τ-periodic solution of (4.1.1). By uniqueness of periodic solutions of the fixed-time equation, we would then have $x_1(t+T) = \tilde{x}_\tau(t)$. By the impulse condition, we have $\tilde{\theta}(0^+) = 0 = \theta_1(T^+)$. Finally, for $t \in (0, \tau)$,

$$\begin{aligned}
\tilde{\theta}_\tau(t) &= \tilde{\theta}_\tau(0^+) + \int_0^t u(\tilde{x}_\tau(s))ds \\
&= \theta_1(T^+) + \int_0^t u(x_1(T+s))ds \\
&= \theta_1(T^+) + \int_T^{T+t} u(x_1(s))ds \\
&= \theta_1(T+t).
\end{aligned}$$

Therefore, (x_1, θ_1) is a phase shift of $(\tilde{x}_\tau, \tilde{\theta})$. There can thus be only one τ-periodic solution of the autonomous equation (4.1.2) up to phase shift equivalence. The converse statement is obvious. ∎

For our model, we take

$$u(S, I, R, M, N) = \eta\beta_H SN.$$

This function satisfies the conditions of Lemma 4.1.1. Therefore, to establish the existence of a unique (up to phase shift) τ-periodic solution of the autonomous spraying model, it suffices to prove the existence of a unique τ-periodic solution of the fixed-time spraying model.

The next lemma states that stability of periodic orbits in the system with impulses at fixed times is equivalent to the stability of the corresponding periodic orbit in the autonomous tracking system.

Lemma 4.1.2. Consider the situation as in Lemma 4.1.1. Let $\varphi(t)$ be a τ-periodic solution of the system with fixed impulses (4.1.1) and let $\tilde{\varphi}(t)$ denote the corresponding periodic solution of the autonomous tracking system in (4.1.2). Suppose the conditions of Theorem A.2.6 are satisfied for both systems. Then $\varphi(t)$ is exponentially stable if and only if $\tilde{\varphi}(t)$ is orbitally asymptotically stable with asymptotic phase.

Proof: Without loss of generality, we have $\tilde{\varphi} = (\varphi, \Theta)$ for a τ-periodic function Θ satisfying $\Theta(0) = 0$. The variational equation associated to the periodic orbit $\varphi(t)$ is

$$\begin{aligned} \dot{z} &= \frac{dg}{dx}(\varphi(t))z, && t \neq k\tau, \\ \Delta z &= \frac{da}{dx}z, && t = k\tau. \end{aligned} \tag{4.1.4}$$

Conversely, the variational equation at $\tilde{\varphi}(t)$ is

$$\begin{aligned} \dot{w} &= \left[\begin{array}{c|c} \frac{dg}{dx}(\varphi(t)) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \nabla u(\varphi(t)) & 0 \end{array}\right] w, && t \neq k\tau, \\ \Delta w &= \left[\begin{array}{c|c} \frac{da}{dx} & \xi \\ \hline 0 \quad \cdots \quad 0 & -1 \end{array}\right] w, && t = k\tau, \end{aligned} \tag{4.1.5}$$

where

$$\xi = g(\varphi(\tau^+)) - g(\varphi(\tau)) - \frac{da}{dx}g(\varphi(\tau)),$$

and $w = (w_{1:n}, w_{n+1})$. By Theorem A.4.2, the above equation (4.1.5) has a nontrivial τ-periodic solution, namely $\tilde{\varphi}'(t)$. Without loss of generality, we may assume $\tilde{\varphi}'(0) = e^{n+1}$. If $\varphi(t)$ is exponentially stable, then there exist n linearly independent solutions $v_1, \ldots, v_n$ of (4.1.4) satisfying $v_j(0) = e^j$ for $j = 1, \ldots, n$. It is then easy to check that $(v_j, 0)$ are linearly independent solutions of (4.1.5). Therefore a monodromy matrix for the linearized tracking system is

$$M = \begin{bmatrix} v_1(\tau) & \cdots & v_n(\tau) & \varphi'(\tau) \\ 0 & \cdots & 0 & \Theta'(\tau) \end{bmatrix}.$$

The matrix has a block structure; the floquet multipliers are precisely $\Theta'(\tau) = \Theta'(0) = 1$ and the multipliers of $\varphi(t)$, by hypothesis, within the unit disc, since this solution is exponentially stable. It follows by Theorem A.4.2 that $\tilde{\varphi}(t)$ is orbitally asymptotically stable and has the property of asymptotic phase. The converse follows by similar reasoning. ∎

In a sense, the previous lemmas prove that the system with impulses at fixed times (4.1.1) and the autonomous tracking system (4.1.2) are asymptotically equivalent; any periodic orbit appearing in one system appears in the other and has the same stability properties.

4.2 Uniqueness of solutions of autonomous impulsive differential equations under certain domain restrictions

Existence and uniqueness of solutions of the autonomous equation (4.0.3)–(4.0.4) will now be discussed. Because of the structure of the impulse effect, we have uniqueness of solutions in backward time, which is a rather desirable property in dynamical systems. Even with simple discontinuous flows, this is not always satisfied; see Appendix A for an example. First, some background.

Akalin and Akhmet [2] proved an existence and uniqueness results for a class of discontinuous flows where problems of uniqueness in backward continuation of solutions do not exist. Overcoming this hurdle requires imposing what they call a "time symmetry" condition to the impulse effect. In a sense, the time symmetry condition makes it so that, when considering backward continuation of solutions, one can consider the differential equation under time reversal, and time symmetry causes this to also be an autonomous impulsive differential equation. However, such a condition may not be realistic to impose in a model, and as such we do not elaborate on this.

The problem of uniqueness of backward continuation can be solved if the phase space is adequately restricted. Such a restriction may be a more natural condition to impose in certain models. In particular, the following is applicable to our problem (4.0.3)–(4.0.4).

Theorem 4.2.1. Consider the autonomous impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(x), && \phi(x) \neq 0, \\ \Delta x &= I(x), && \phi(x) = 0. \end{aligned} \tag{4.2.1}$$

Let $\Omega \subset \mathbb{R}^n$ have a piecewise C^1 orientable boundary $\partial\Omega$. Denote $\Gamma = \phi^{-1}(0)$ and $J(x) = x + I(x)$. Suppose the following conditions are satisfied:

- $J(\Gamma) \subset \partial\Omega$, the exterior unit normal $\vec{n}(x)$ is continuous on $J(\Gamma)$ and $f(x)\cdot\vec{n}(x) < 0$ for all $x \in J(\Gamma)$,
- $\Gamma \cap J(\Gamma) = \emptyset$,
- ϕ is continuously differentiable on some ϵ-neighbourhood (Γ_ϵ) of Γ and $\nabla\phi(x) \neq 0$ for all $x \in \Gamma$,
- J is continuously differentiable on Γ_ϵ and $\det(D_x J) \neq 0$ for all $x \in \Gamma_\epsilon$,
- f is locally Lipschitz continuous in Ω.

Then, for all $x_0 \in \Omega$, there is at most one solution of the intial-value problem $x(0) = x_0$ for (4.2.1) contained in Ω defined on any interval of the form (α, β) for $\alpha < 0 < \beta$.

Proof: On the contrary, suppose that on some interval (α, β), there are defined two solutions $y(t)$ and $z(t)$ contained in Ω that are not equal. Since local existence and uniqueness is ensured by Theorem A.2.5 [13], there must exist a subinterval (a, b) on which these solutions agree. Suppose this interval is maximal. We will show that $a = \alpha$ and $b = \beta$, contradicting the existence of two different solutions.

Since solutions are continuous from the left, we have $r \equiv y(b) = y(b^-) = z(b^-) = z(b)$, so in fact the solutions must agree on the half-open interval $(a, b]$. However, by local existence and uniqueness, both solutions can be uniquely extended forward to some $c > b$ by solving the intial-value problem $x(b) = r$. Therefore the interval on which solutions agree can be enlarged to $(a, c) \supset (a, b)$. Since (a, b) was assumed to be maximal, we must have $b = \beta$.

Consider the one-sided limits

$$l \equiv \lim_{t \to a^+} y(t) = \lim_{t \to a^+} z(t).$$

By continuity, these (sided) limits coincide. There are three subcases to consider.

a) Suppose $l \in \Gamma$. If $\alpha < a$, then $y(a)$ and $z(a)$ must both exist. If $y(a) \in \Gamma$, then $Jy(a) = y(a^+) = l \in \Gamma$, contradicting the assumption that Γ and $J(\Gamma)$ are disjoint. Similarly, we cannot have $z(a) \in \Gamma$. We may therefore assume that $l \notin \Gamma$.

b) Suppose $l \in \Omega \setminus J(\Gamma)$. This case is then similar to part I, since there exists a unique solution to the intial-value problem $x(a) = l$ defined on the interval (v, w) for some $v < a$, and, by uniqueness, $y(t)$ and $z(t)$ must coincide in some subinterval of (v, w) containing a. As (a, b) was assumed to be maximal, we must have $a = \alpha$.

[13] Technically, this only holds if $x_0 \notin J(\Gamma)$. If $x_0 \in J(\Gamma)$, however, the condition on the normal derivative given in the theorem ensures that any solution of $x(0) = x_0$ cannot be defined for negative time, since it will immediately leave Ω. Moreover, it cannot have reached this point by undergoing an impulse because the formalism is that solutions of impulsive differential equations are continuous from the *left*. The theorem is therefore vacuously satisfied for these particular boundary points, because there are *no* solutions of the IVP $x(0) = x_0 \in J(\Gamma)$ defined on an open interval containing 0.

c) The last possible outcome is $l \in J(\Gamma)$. There are now two subcases to consider.

1. Suppose $y(a) = l$ or $z(a) = l$. Without loss of generality, $y(a) = l$. Since $l \in J(\Gamma)$, we have $f(l) \cdot \vec{n}(l) < 0$, which implies $y'(a) \cdot \vec{n}(y(a)) < 0$. Since l is on the boundary of Ω, there must exist a sequence $t_n \nearrow a$ with $y(t_n) \to y(a) = l$ and $y(t_n) \notin \Omega$. Therefore, $y(t)$ is not contained in Ω, which is a contradiction.

2. We must have $y(a), z(a) \in \Omega$, and since solutions are continuous from the left, we conclude that $y(a), z(a) \in \Gamma$. By the hypotheses of Theorem 2.4 (which are assumed to hold), J is invertible [2], so $Jy(a) = y(a^+) = l = z(a^+) = Jz(a)$ implies $y(a) = z(a) \equiv l^-$. The result then follows by a similar argument to part b), where, by solving the intial-value problem $x(a) = l^-$, the interval (a, b) can be extended to the left. By maximality, we must have $a = \alpha$.

We therefore have a contradiction; both solutions are in fact equal. ∎

With the above result in mind, the only possible issue one might encounter with existence of solutions occurs if one of the following two problems is encountered: the solution leaves the phase space Ω, or it hits the manifold Γ (or $J(\Gamma)$, in backward time) an infinite number of times over a finite time interval; that is, the discontinuous flow has a limit point on the manifold. The first of these problems is uncommon in applications, since the differential equations are usually formulated in such a way that their solutions are bounded (at least in forward time), and thus the dynamics can be restricted to a compact set. To deal with the latter, we have the following result.

Theorem 4.2.2. Assume the following.

1. Every solution of the intial-value problem $\dot{y} = f(t, x)$, $y(0) = y_0 \in \Omega$ is continuable either to $+\infty$ or Γ,

2. for all $x \in J(\Gamma)$, there exists $\epsilon_x > 0$ such that $\overline{B_{\epsilon_x}(x)} \cap \Gamma = \emptyset$, and

$$\inf_{x \in J(\Gamma)} \left\{ \frac{\epsilon_x}{\sup_{B_{\epsilon_x}(x)\cap\Omega} ||f(x)||} \right\} > 0.$$

Then every solution of the impulsive differential equation (4.2.1) is continuable to $+\infty$. Conversely, if

1′. every solution of the intial-value problem $\dot{y} = f(t,x)$, $y(0) = y_0 \in \Omega$ is continuable either to $-\infty$ or $J(\Gamma)$,

2′. for all $x \in \Gamma$, there exists $\epsilon_x > 0$ such that $\overline{B_{\epsilon_x}(x)} \cap J(\Gamma) = \emptyset$, and

$$\inf_{x \in \Gamma} \left\{ \frac{\epsilon_x}{\sup_{B_{\epsilon_x}(x)\cap\Omega} ||f(x)||} \right\} > 0,$$

then every solution of the impulsive differential equation (4.2.1) is continuable to $-\infty$.

The proof of the above theorem follows almost verbatim the proof of the analoguous result for the previously mentioned discontinuous flows of Akalin and Akhmet. See [2, 3] for details. These authors provide other criteria which may be used to guarantee continuability on $\mathbb{R}$; however, this one will be the most use to us, mainly because of the following obvious corollary.

Corollary 4.2.3. Suppose Ω is bounded and there exists $L > 0$ such that $d(x, J(\Gamma)) \geq L$ for all $x \in \Gamma$. If every solution of the intial-value problem $\dot{y} = f(t,y)$, $y(0) = y_0 \in \Omega$ is continuable either to $+\infty$ (resp. $-\infty$) or Γ (resp. $J(\Gamma)$), then the solution of the impulsive differential equation (4.2.1) is continuable to $+\infty$ (resp. $-\infty$).

4.2.1 Existence and uniqueness for the malaria models

We now establish existence and uniqueness of solutions of the autonomous equation (4.0.3)–(4.0.4).

Theorem 4.2.4. Every intial-value problem for the autonomous malaria control model (4.0.3)–(4.0.4) has a unique (nonlocal) solution continuable to $+\infty$.

Proof: We take the domain to be $\Omega = \{x \in \mathbb{R}^6_+ : 0 \le x^6 \le \overline{\Theta}\}$. Then the impulse map is

$$J(S, I, R, M, N, \Theta) = (S, I, R, (1-r)M, (1-r)N, \Theta - \overline{\Theta}),$$

and the surface Γ is defined by the zero set of

$$\phi(S, I, R, M, N, \Theta) = \Theta - \overline{\Theta}.$$

This function is smooth, and

$$\Gamma = \{x \in \Omega : x^6 = \overline{\Theta}\}.$$

Consequently,

$$J(\Gamma) = \{x \in \mathbb{R}^6_+ : x^6 = 0\} \subset \partial\Omega.$$

Moreover, Γ and $J(\Gamma)$ are disjoint, J and ϕ are C^1 and $\det D_x J = (1-r)^2 \neq 0$. The vector field is C^1 and is therefore locally Lipschitz continuous. Finally, the exterior normal along $J(\Gamma)$ is $\vec{n}(x) = (0, 0, 0, 0, 0, -1)$, so that $f(x) \cdot \vec{n}(x) = \eta\beta_H SN < 0$. We conclude that every intial-value problem has (locally) a unique solution (by Theorem A.2.5) and, nonlocally, there is at most one solution passing through any point in the extended phase space $\Omega \times \mathbb{R}$ (by Theorem 4.2.1).

It can be verified that the region

$$\Omega^*_\epsilon = \left\{0 \le S + I + R \le \frac{\pi}{\mu_H} + \epsilon,\ \ 0 \le M + N \le \frac{\Lambda}{\mu} + \epsilon\right\} \cap \Omega$$

is compact and attracting for all $\epsilon \ge 0$ and is reached in finite time by any positive solution, provided $\epsilon > 0$. We have

$$d(x, J(\Gamma)) \ge \overline{\Theta} > 0$$

for all $x \in \Gamma$, so by Corollary 4.2.3, every solution of the autonomous malaria control model (4.0.3)–(4.0.4) beginning in $\overline{\Omega}^*_\epsilon$ is continuable to $+\infty$. Since every initial condition is contained in Ω^*_ϵ for some $\epsilon > 0$, we conclude that all solutions exist, are unique and are continuable to $+\infty$. ∎

It can also be verified that the system with impulses at fixed times has unique solutions defined for all time. We state the following without proof; see Appendix A for the relevant theorems.

Theorem 4.2.5. Every intial-value problem of the fixed-time malaria control model (4.0.1)–(4.0.2) has a unique solution defined for all time.

Finally, the nonnegative orthant is positively invariant.

Lemma 4.2.6. For both malaria control models, (4.0.1)–(4.0.2) and (4.0.3)–(4.0.4), the nonnegative orthant is positively invariant.

Proof: The impulse effect maps the nonnegative orthant to itself in both models, so it suffices to prove the result for the underlying ordinary differential equation(s). In the model with autonomous impulses, the Θ component is nondecreasing and positive except at impulse times, provided S and N are nonnegative. It therefore suffices to prove that any nonnegative solution of the (S, I, R, M, N) system remains nonnegative.

Suppose a solution $x = (S, I, R, M, N)$ is positive at time $t = 0$, but at least one component is negative at some future time $t^* > 0$. Then there exists a time T with $0 < T \leq t^*$ such that at least one component of $x(T)$ is zero, any nonzero components are positive and there is at least one component that is negative on an interval $(T, T + \epsilon)$ for some $\epsilon > 0$. There are five cases to consider.

- $S(T) = 0$: Then $S' = \pi + hI + \delta R \geq \pi > 0$. It follows that $S(t) > 0$ for $t \gtrsim T$.
- $M(T) = 0$: Then $M' = \Lambda > 0$. If follows that $M(t) > 0$ for $t \gtrsim T$.

- $N(T) = 0$: Then $N' = \beta_M MI$. It follows that if $M, I \geq 0$ in an interval of the form $(T, T + \epsilon)$, then $N(t) \geq 0$ for $t \gtrsim T$. Since M is guaranteed to be strictly positive or zero, it suffices to consider what happens if $I(T) = 0$. This is the disease-free state, so we have $N = 0$ for all $t \geq T$. We conclude that $N(t) \geq 0$ for $t \gtrsim T$.

- $I(T) = 0$: Then $I' = \beta_H SN \geq 0$. It follows that if $S, N \geq 0$ in an interval of the form $(T, T + \epsilon)$, then $I(t) \geq 0$ for $t \gtrsim T$. Since $S(t)$ is strictly positive or zero, it suffices to consider the special case where $N(T) = 0$. From the previous case, we know that $N(t)$ must be either zero or strictly positive for $t \gtrsim T$. We conclude that $I(t) > 0$ for $t \gtrsim T$.

- $R(T) = 0$: Then $R' = \alpha I$. From the previous case, we know that $I(t) \geq 0$ for $t \gtrsim T$. We conclude that $R(t) \geq 0$ for $t \gtrsim T$.

Therefore, the solution must be nonnegative for $t \gtrsim T$. This is a contradiction. ∎

4.3 Stability of vector-free and disease-free periodic orbits

We can study the stability of vector-free periodic orbits if we have a non-constant mosquito growth term, as opposed to a linear one. Suppose that birth rate of mosquitoes is not constant, but is rather a (possibly nonlinear) function M and N, denoted $\Lambda(M, N)$, for which $\Lambda(0, 0) = 0$ (i.e. there is no state-independent migration of mosquitoes into the region of interest). Then the situation can be described by the system of ordinary differential equations

$$\begin{aligned}\dot{S} &= \pi - \beta_H SN + hI + \delta R - \mu_H S,\\ \dot{I} &= \beta_H SN - hI - \alpha I - (\mu_h + \gamma)I,\\ \dot{R} &= \alpha I - \delta R - \mu_H R, \\ \dot{M} &= \Lambda(M,N) - \mu M - \beta_M MI,\\ \dot{N} &= \beta_M MI - \mu N,\end{aligned} \tag{4.3.1}$$

together with the appropriate impulse conditions (fixed-time or autonomous, where in the latter case we adjoin the $\dot{\Theta}$ compartment). We will now study the stability of the vector-free state of the system with impulses at fixed times, if they exist. The reason for not studying stability in the autonomous case will be explained later.

This model encompasses a variety of situations. For example, if vectors undergo logistic growth, one could posit

$$\Lambda(M,N) - \mu M = (g_M M + g_N N) \cdot \left(1 - \frac{M+N}{P}\right),$$

where g_M and g_N are the intrinsic growth rates resulting from reproduction of uninfected and infected vectors respectively, and P is the carrying capacity.

4.3.1 Stability of the vector-free periodic orbit

When $M = N = 0$, we have $\dot{M} = \dot{N} = 0$, and the impulse effects (4.0.2) and (4.0.4) have no effect on the system. Moreover, in this case, we have $I(t) \to 0$ and $R(t) \to 0$ for all initial conditions $I(0)$ and $R(0)$, and $\dot{\Theta} = \eta\beta_H SN = 0$ (if we are considering the autonomous model). It then follows that if $M = N = 0$, we have $(S, I, R, M, N) \to (\pi/\mu_H, 0, 0, 0, 0)$ as $t \to \infty$, and this is the unique vector-free orbit for the equation with impulses at fixed times. Since there is no human-human infection, the vector-free orbit is also one of the possible disease-free states.

Denote $\partial_M \Lambda(0,0) = \Lambda_M$ and $\partial_N \Lambda(0,0) = \Lambda_N$. To determine stability, we consider the variational equation

$$\dot{u} = \begin{bmatrix} -\mu_H & h & \delta & 0 & -\beta_H \frac{\pi}{\mu_H} \\ 0 & -(h+\alpha+\gamma+\mu_H) & 0 & 0 & \beta_H \frac{\pi}{\mu_H} \\ 0 & \alpha & -(\delta+\mu_H) & 0 & 0 \\ 0 & 0 & 0 & \Lambda_M - \mu & \Lambda_N \\ 0 & 0 & 0 & 0 & -\mu \end{bmatrix} u = Zu$$

$$\Delta u = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 & -r \end{bmatrix} u^- = Ru^-,$$

with impulses at $t = t_k = k\tau$. This system clearly has no nontrivial periodic solutions. It can be shown that a fundamental matrix $X(t)$ of this linear non-impulsive system satisfies

$$X(\tau) = \begin{bmatrix} e^{-\mu_H \tau} & \cdot & \cdot & 0 & \cdot \\ 0 & e^{-(\mu_H+h+\alpha+\gamma)\tau} & 0 & 0 & \cdot \\ 0 & \cdot & e^{-(\mu_H+\delta)\tau} & 0 & \cdot \\ 0 & 0 & 0 & e^{(\Lambda_M-\mu)\tau} & \cdot \\ 0 & 0 & 0 & 0 & e^{-\mu\tau} \end{bmatrix},$$

and $X(0) = E$, where dots are used as placeholders for other elements that, as will be seen, are not of great importance. A monodromy matrix is given by

$$M = X(\tau)(E+R) = X(\tau)\mathrm{diag}(1,1,1,1-r,1-r).$$

This matrix has the same pattern of zero and nonzero entries as $X(\tau)$, and therefore the eigenvalues are on the main diagonal. Thus, the largest eigenvalue of M is precisely

$$R_0^V := e^{(\Lambda_M-\mu)\tau}(1-r).$$

This eigenvalue is always positive; if it is less than one, then the vector-free orbit is exponentially stable [5]. We have therefore proven the following theorem.

Theorem 4.3.1. The vector-free orbit is exponentially stable if

$$R_0^V := e^{(\partial_M \Lambda(0,0)-\mu)\tau}(1-r) < 1.$$

It follows that, provided τ is sufficiently small and r sufficiently large, it is always possible to, at least in theory, eradicate the vector and, subsequently, eradicate the disease. This result assumes that there is no immigration of mosquitoes into the region of interest, which is not reasonable in practice. Moreover, eradicating the vector is not ecologically desirable.

We remark that, in the non-impulsive system, local asymptotic stability of the vector-free state is equivalent to having $\lambda_{max} = \partial_M \Lambda(0,0) - \mu < 0$. This coefficient appears in Theorem 4.3.1 in that stability of the vector-free orbit is guaranteed if $e^{\lambda_{max}\tau}(1-r) < 1$. Consequently, if the vector-free state is asymptotically stable in the impulse-free system, then it is asymptotically stable in the impulsive system, as is expected.

4.3.2 Conditional stability of the disease-free periodic orbit for low infection rates

Suppose that (4.3.1) has a disease-free periodic orbit; that is, it has a periodic solution of the form

$$(S, I, R, M, N) = (\pi/\mu_H, 0, 0, M(t), 0)$$

for a τ-periodic function $M(t)$.

Theorem 4.3.2. If β_H or β_M is sufficiently small and

$$R_0^D \equiv (1-r)\exp\left(-\mu\tau + \int_0^\tau \frac{\partial \Lambda}{\partial M}(M(t), 0)dt\right)$$

is less than one, then the disease-free orbit is exponentially stable.

Proof: In general, the variational equation about the disease-free periodic solution is

$$\dot{u} = \begin{bmatrix} -\mu_H & h & \delta & 0 & -\beta_H \frac{\pi}{\mu_H} \\ 0 & -(h+\alpha+\gamma+\mu_H) & 0 & 0 & \beta_H \frac{\pi}{\mu_H} \\ 0 & \alpha & -(\delta+\mu_H) & 0 & 0 \\ 0 & -\beta_M M(t) & 0 & \Lambda_M(M(t),0)-\mu & \Lambda_N(M(t),0) \\ 0 & \beta_M M(t) & 0 & 0 & -\mu \end{bmatrix} u$$

$$\Delta u = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 & -r \end{bmatrix} u^-, \tag{4.3.2}$$

where $M(t)$ is the mosquito component of the periodic solution. If $\beta_M = 0$, then the variational equation is essentially identical to the one associated to the vector-free periodic orbit of Section 2.1, except for the time dependence $\Lambda_M = \Lambda_M(M(t), 0)$ and $\Lambda_N = \Lambda_N(M(t), 0)$. A monodromy matrix has the form

$$\begin{bmatrix} e^{-\mu_H \tau} & \cdot & \cdot & 0 & \cdot \\ 0 & e^{-(\mu_H+h+\alpha+\gamma)\tau} & 0 & 0 & \cdot \\ 0 & \cdot & e^{-(\mu_H+\delta)\tau} & 0 & \cdot \\ 0 & 0 & 0 & R_0^D & \cdot \\ 0 & 0 & 0 & 0 & (1-r)e^{-\mu\tau} \end{bmatrix},$$

where dots are used as placeholders for other elements which do not factor into the calculation of the eigenvalues. The eigenvalues consist of the diagonal entries, and it follows that, if $\beta_M = 0$ and $R_0^D < 1$, the disease-free orbit is exponentially stable.

When $\beta_H = 0$, it can be shown that a monodromy matrix for this linear system takes the same form as in the case $\beta_M = 0$, and we therefore have stability when

$R_0^D < 1$.

Stability for β_H or β_M small follows from continuity of solutions of the linear variational equation with respect to parameters. This technique will be used in the proof of Theorem 5.2.2. The proof is similar here, so we omit it. ∎

4.3.3 No vector-free or disease-free periodic orbit in the autonomous system can be asymptotically stable

In the system with autonomous impulses, every point of the form

$$(S, I, R, M, N, \Theta) = (\pi/\mu_H, 0, 0, 0, 0, \Theta) := V_\Theta$$

is a vector-free state for all $\Theta \neq \overline{\Theta}$. In particular, no vector-free state in this system can be asymptotically stable, as there are infinitely-many other suitable points in any neighbourhood of V_Θ for any Θ. For this reason, any criteria that establishes asymptotic stability only is not useful to us. We therefore do not consider stability of such a periodic orbit. The situation is similar for disease-free periodic orbits, where every solution of the form

$$(S, I, R, M, N, \Theta) = (\pi/\mu_H, 0, 0, M(t), 0, \Theta)$$

is a disease-free orbit for some periodic solution $M(t)$ and any $\Theta \neq \overline{\Theta}$.

4.4 Existence of a bifurcation curve for the disease-free periodic orbit in the system with impulses at fixed times

For simplicity, we will assume that $\Lambda(M,N) = \Lambda$ is constant. By [30], the system without impulses (4.0.1) has a positive endemic equilibrium if and only if

$$R_0 \equiv \frac{\beta_H \beta_M \Lambda \pi}{\mu^2 \mu_H(\mu_H + \alpha + \gamma + h)} \geq 1,$$

where a transcritical bifurcation occurs when $R_0 = 1$. When $R_0 < 1$, the disease-free equilibrium is stable, and, when it is greater than one, the equilibrium is unstable, and the endemic equilibrium becomes stable. Define

$$\beta_M^* = \frac{\mu^2 \mu_H(\mu_H + \alpha + \gamma + h)}{\beta_H \Lambda \pi}. \tag{4.4.1}$$

Taking all parameters except for β_M constant, we have a transcritical bifurcation when $\beta_M = \beta_M^*$.

The impulsive equation (4.0.1)–(4.0.2) has a disease-free periodic orbit given by

$$\left(\frac{\pi}{\mu_H} 0, 0, M(t;r), 0\right), \qquad M(t;r) = \frac{\Lambda}{\mu} \cdot \left(1 + \frac{re^{-\mu t}}{(1-r)e^{-\mu\tau} - 1}\right), \tag{4.4.2}$$

for $t = t \bmod \tau$. When $r = 0$, we are in the system without impulses, and there is a transcritical bifurcation at $\beta_M = \beta_M^*$. We will now establish conditions under which, for $|r|$ small, there exists a function $\beta_M^*(r)$ satisfying $\beta_M^*(0) = \beta_M^*$, and such that the disease-free orbit of the impulsive system (4.0.1)–(4.0.2) for $r \neq 0$ small has a floquet multiplier $\mu = 1$ when $\beta_M = \beta_M^*(r)$. This is a necessary ingredient to have a bifurcation of periodic solutions.

Lemma 4.4.1. Consider the linear homogeneous time-dependent impulsive system

$$\dot{u} = \begin{bmatrix} -(\alpha + h + \gamma + \mu_H) & \beta_H \frac{\pi}{\mu_H} \\ \beta_M M(t;r) & -\mu \end{bmatrix} u \tag{4.4.3}$$

$$\Delta u = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} u^- \tag{4.4.4}$$

where

$$M(t;r) = \frac{\Lambda}{\mu} \cdot \left(1 + \frac{re^{-\mu t}}{(1-r)e^{-\mu\tau} - 1}\right),$$

and impulse effects occur at times $t_k = k\tau$. Let $\mu_i(r, \beta_M)$ denote the Floquet multipliers of this linear system, $i \in \{1, 2\}$. Then

$$\frac{\partial \mu_1}{\partial \beta_M}(0, \beta_M^*) \neq 0 \quad \text{if and only if} \quad \frac{\partial \mu_2}{\partial \beta_M}(0, \beta_M^*) \neq 0.$$

If this condition is satisfied, then there exists a C^1 function $\beta_M^* : \mathcal{O} \subset \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

1. $\beta_M^*(0) = \beta_M^*$ as defined in (4.4.1),
2. if $\beta_M = \beta_M^*(r)$, then the variational equation about the disease-free orbit has exactly one floquet multiplier of unit modulus, given by $\mu = 1$.

Proof: Note that $M(t;r)$ is the mosquito component of the disease-free periodic solution, when restricted to $(0, \tau)$. By considering the variational equation (4.3.2), we see that a monodromy matrix has the form

$$\begin{bmatrix} e^{-\mu_H\tau} & a_1(x_1) & e^{-\mu_H\tau}(e^{\delta\tau} - 1) & 0 & a_1(x_2) \\ 0 & x_1^1(r, \beta_M) & 0 & 0 & x_2^1(r, \beta_M) \\ 0 & a_2(x_1) & e^{-(\delta+\mu_H)\tau} & 0 & a_2(x_2) \\ 0 & a_3(x_1) & 0 & (1-r)e^{-\mu\tau} & a_3(x_3) \\ 0 & x_1^2(r, \beta_M) & 0 & 0 & x_2^2(r, \beta_M) \end{bmatrix},$$

where the various $a_i(x_j)$ are constants that will not factor into the calculation of the eigenvalues. The eigenvalues consist of the first, third and fourth diagonal entries (all of which are within the complex unit disc), along with the eigenvalues of

$$\begin{bmatrix} x_1^1(r, \beta_M) & x_2^1(r, \beta_M) \\ x_1^2(r, \beta_M) & x_2^2(r, \beta_M) \end{bmatrix}.$$

The vectors $x_i = (x_i^1, x_i^2)$, $i \in \{1,2\}$ are two linearly indpendent solutions of the linear system given in the theorem, evaluated at time $t = \tau$, and satisfying the initial condition $x_i(0) = e^i$. Therefore the μ_1 and μ_2 described in the theorem coincide with the eigenvalues of the matrix above, and, when $r = 0$, it follows [30] that

$$\mu_1(0, \beta_M^*) = e^{-(h+\alpha+\gamma+\mu_H)\tau}, \qquad \mu_2(0, \beta_M^*) = 1.$$

Moreover, by the analogue of Liouville's formula for impulsive differential equations [5], we know that the product of these two floquet multipliers satisfies

$$\mu_1(r, \beta_M)\mu_2(r, \beta_M) = (1-r)e^{-(h+\alpha+\gamma+\mu_H+\mu)\tau}.$$

By applying the product rule, we find

$$0 = \frac{\partial \mu_1}{\partial \beta_M}(0, \beta_M^*) + e^{-(h+\alpha+\gamma+\mu_H+\mu)\tau}\frac{\partial \mu_2}{\partial \beta_M}(0, \beta_M^*),$$

from which it follows that $\partial_{\beta_M}\mu_1(0, \beta_M^*) \neq 0$ if and only if $\partial_{\beta_M}\mu_2(0, \beta_M^*) \neq 0$. If this should be the case, then, by the implicit function theorem, there exists a unique C^1 function $\beta_M^* : r \mapsto \beta_M^*(r)$ defined in some neighbourhood of $r = 0$ such that $\mu_2(r, \beta_M^*(r)) = 1$. Since $\mu_2(r, \beta_M)$ is a Floquet multiplier of the disease-free orbit, the lemma is therefore proven. ∎

A remark: throughout Lemma 4.4.1, all parameters except for r and β_M are assumed to be constant, so, in reality, the bifurcation curve $\beta_M^* : \Omega \to \mathbb{R}$ will depend on the choice of the other parameters. Calculation of the bifurcation curve is not difficult to implement numerically; see Figure 4.1 for a plot of this curve.

We briefly mention that determination of the existence of this bifurcation curve can be accomplished analytically with the aid of special functions. Solutions $u(t) = (u_1(t), u_2(t))$ of the continuous part (4.4.3) of the linear impulsive system given in the previous lemma are solutions of the following system of second-order linear differential equations:

$$\begin{aligned}\ddot{u}_1 + (\alpha + h + \gamma + \mu_H + \mu)\dot{u}_1 + \left(\frac{\beta_H \beta_M \pi}{\mu_H} M(t;r) - \mu(\alpha + h + \gamma + \mu_H)\right) u_1 = 0,\\ \dot{u}_2 + \mu u_2 - \beta_M M(t;r) u_1 = 0,\\ \dot{u}_1(0) + (\alpha + h + \gamma + \mu_H) u_1(0) - \beta_H \frac{\pi}{\mu_H} u_2(0) = 0.\end{aligned} \tag{4.4.5}$$

The above system can be solved using Bessel functions. To begin, the first equation can be written

$$\ddot{u}_1 + a\dot{u}_1 + \left(b + ce^{-\mu t}\right) u_1 = 0,$$

with constants $a, b, c \in \mathbb{R}$. Under the change of coordinates $t = -\log x$, this is equivalent to

$$x^2 u_1'' + x(1-a)u_1' + (b + cx^\mu)u_1 = 0.$$

By Bowman [6], the general solution of this transformed Bessel-type equation is

$$u_1(x) = C_1 x^{\frac{a}{2}} J_{-\frac{\sqrt{a^2-4b}}{\mu}}\left(\frac{2\sqrt{cx^\mu}}{\mu}\right) + C_2 x^{\frac{a}{2}} J_{\frac{\sqrt{a^2-4b}}{\mu}}\left(\frac{2\sqrt{cx^\mu}}{\mu}\right),$$

where $J_n(x)$ denotes the modified Bessel function of the first kind and C_1 and C_2 are arbitrary constants. With $t = -\log x$, we can return to the original coordinates. Following this, the second equation of (4.4.5) can be solved by an integrating factor.

This basis of solutions can then be used to construct solutions of (4.4.5). The impulse condition can then be applied, the linear system appearing in Lemma 4.4.1 can be solved, and, finally, the Floquet multipliers can be determined analytically. We do not continue with this computation, however, since the resulting expressions are difficult to work with by hand.

4.4.1 Discussion: Existence of a transcritical bifurcation

We do not explicitly determine conditions under which a transcritical bifurcation occurs as β_M passes through the bifurcation curve given in the lemma. The obvious approach is to define a return map Ψ in the extended phase space; $\tilde{\Omega} = \mathbb{R}^5 \times \mathbb{R}^+$, with

$$\Psi : (x, \beta_M) \mapsto \Phi(\tau, Jx, \beta_M),$$

where $\Phi : \mathbb{R}^+ \times \tilde{\Omega} \to \tilde{\Omega}$ is the flow of the continuous vector field (4.0.1) in the extended phase space, obtained by adjoining the extra equation $\dot{\beta}_M = 0$, and $J = \text{diag}(1, 1, 1, 1-r, 1-r)$. The map Ψ has a non-hyperbolic fixed point $(\hat{x}, \beta_M)$ associated to the disease-free periodic orbit, for all β_M, where

$$\hat{x} = \left(\frac{\pi}{\mu_H}, 0, 0, M(\tau), 0 \right),$$

due to the presence of the $\dot{\beta}_M = 0$ equation. When $\beta_M = \beta_M^*(r)$, there are two eigenvalues equal to one because of Lemma 4.4.1. The approach would then be to perform a center manifold reduction at the fixed point $(\hat{x}, \beta_M^*(r))$. The calculations associated with this reduction are not incredibly interesting. Moreover, the resulting truncated center manifold map

$$v \mapsto v + \frac{1}{2} A v^2 + B v \eta,$$

in a simpler coordinate system, is difficult or impossible to compute exactly.

Although some model-specific simplifications can be made, calculating the coefficients in the above map requires solving at least eight five-dimensional non-autonomous ordinary differential equations. This is one reason we do not delve further, the other being that we are guaranteed only local existence of bifurcating periodic solutions from this analysis. That is, β_M must be close to $\beta_M^*(r)$. However, in practice, $\beta_M^*(r)$ is several orders of magnitude smaller than more realistic estimates of the transmission rate β_M; see, for example, Figure 4.1. The parameters in Table 4.1 are the same as those used for the numerical simulations in [30] and, excluding the transmission rate β_M, are used to produce the aforementioned figure. A baseline estimate for the human-to-mosquito transmission rate is $\beta_M = 0.05$; however, from the figure, we see that a bifurcation may occur when $\beta_M = \beta_M^*(r) \approx 10^{-6} \ll 5 \cdot 10^{-2}$. It is something

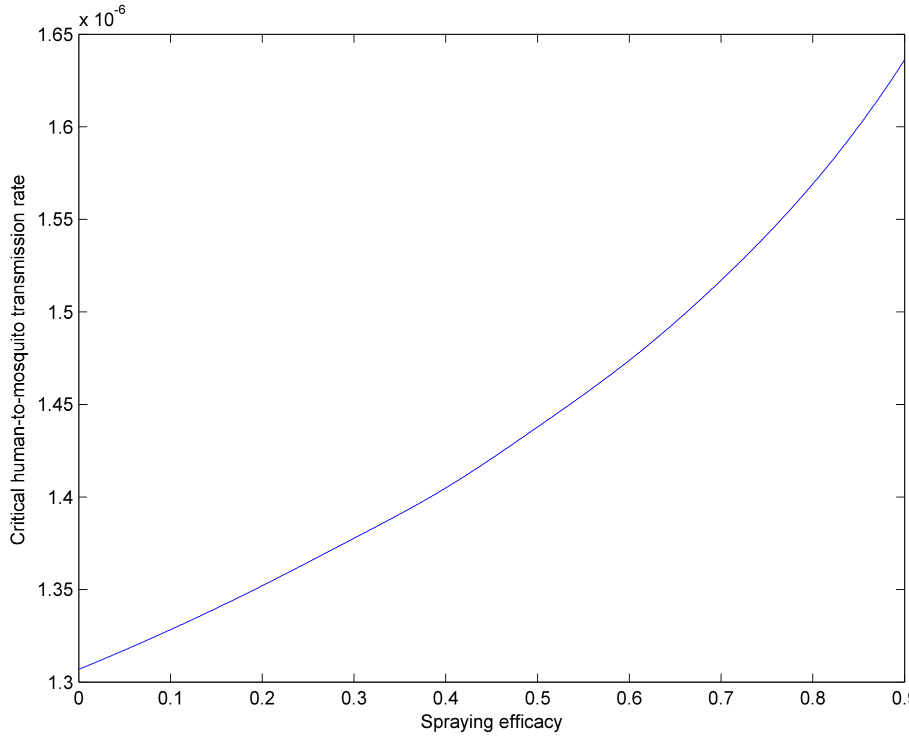

Figure 4.1: Plot of the curve $r \mapsto \beta_M^*(r)$ for $r \in [0, 0.9]$ and parameters given in Table 4.1. Notice that β_M^* is on the order of 10^{-6}, and high spraying efficacy (80%) does not confer more than a 30% increase in tolerable mosquito transmission rate, as compared to no spraying at all ($r = 0$).

of a stretch to suggest that four orders of magnitude is negligible, so it is not clear that further bifurcation analysis would be very useful.

Additionally, there are numerical issues involved in detecting the bifurcating periodic solutions for values of β_M near $\beta_M^*(r)$. If $\beta_M \approx \beta_M^*(r)$ and there is a stable endemic periodic orbit, then all of its Floquet multipliers must lie within the unit circle; however, one of these eigenvalues, μ^*, will be very close to the unit circle. Indeed, when $\beta_M = \beta_M^*(r)$, this eigenvalue is equal to one. Locally, near this periodic orbit, Floquet's Theorem for impulsive differential equations [5] implies that the linearized

Parameter	Value
π	100/365 humans/days
Λ	1000/365 mosquitoes/days
β_H	0.5 (mosquitoes·days)$^{-1}$
β_M	0.05 (humans·days)$^{-1}$
μ_H	1/(30·365) (days)$^{-1}$
γ	1/20 humans/days
h	1/9 humans/days
α	1/8 humans/days
δ	1/30 humans/days
μ	1/7.3 humans/days
τ	(3/12)·365 days

Table 4.1: Parameters used to produce the the bifurcation curve in Figure 4.1.

equation is reducible to

$$y' = Ly$$

by an invertible time-dependent change of coordinates, where all eigenvalues of L have negative real part. However, in these coordinates, μ^* corresponds to λ^*, which has negative real part, but is approximately zero. Consequently, the stiffness ratio [8] in the new coordinate system is large. Since solving the linearized equation and solving the reduced system $\dot{y} = Ly$ are equivalent, the linearized equation at the endemic orbit is stiff. Due to this stiffness, it may be difficult to numerically observe an endemic periodic orbit near the critical transmission rate $\beta_M^*(r)$.

4.5 Conditional existence of periodic orbits in the system with autonomous impulses for a range of spraying thresholds

Suppose that the system with impulses at fixed times (4.0.1)-(4.0.2) has an endemic periodic orbit with period τ. The discussion following Lemma 4.1.1 then guarantees that, for some critical spraying threshold $\overline{\Theta}$, the system with autonomous impulses (4.0.3)-(4.0.4) has a periodic orbit. We now show that there is in some sense a correspondence between small periods τ and small spraying thresholds $\overline{\Theta}$. This correspondence will later be strengthened to a global result when we consider certain model simplifications.

Theorem 4.5.1. Suppose that there exists $T^+ > 0$ such that, for $\tau \in (0, T^+)$, the system with impulses at fixed times (4.0.1)-(4.0.2) has a unique nonhyperbolic endemic periodic solution with period τ. Then there exists $\Theta^+ > 0$ such that, for all $\overline{\Theta} \in (0, \Theta^+)$, the system with autonomous impulses (4.0.3)-(4.0.4) has a periodic solution.

Proof: Define the map $P : (0, T^+) \to \mathbb{R}_+$,

$$P(\tau) = \int_0^\tau \tilde{S}_\tau(t)\tilde{N}_\tau(t)dy,$$

where for each τ, $\tilde{S}_\tau(t)$ and $\tilde{N}_\tau(t)$ are the S and N components of the τ-periodic solution of the system with impulses at fixed times.

Note that the total human population $H = S + I + R$ and the total mostuito population $\Psi = M + N$ satisfy the inequalities

$$\dot{H} \leq \pi - \mu_H H, \qquad \dot{\Psi} = \Lambda - \mu\Psi$$

for positive initial conditions. Since solutions on the periodic orbit are positive, it follows by Gronwall's inequality that the periodic solution must satisfy

$$0 \leq S_\tau + I_\tau + R_\tau \leq \frac{\Lambda}{\mu}$$

and

$$0 \leq M_\tau + N_\tau \leq \frac{\pi}{\mu_H}.$$

Each quantity being summed is positive (or zero). Thus,

$$0 \leq S_\tau \leq \frac{\pi}{\mu_H}, \qquad 0 \leq N_\tau \leq \frac{\Lambda}{\mu}.$$

From this, we conclude

$$\lim_{\tau \to 0^+} P(\tau) = 0.$$

We now show that $P(\tau)$ is continuous. If $\tau > 0$ and $a_n \nearrow \tau$, then

$$\begin{aligned}
|P(a_n) - P(\tau)| &\leq \left| \int_0^{a_n} \left[\tilde{S}_{a_n}(t)\tilde{N}_{a_n}(t) - \tilde{S}_\tau(t)\tilde{N}_\tau(t) \right] dt - \int_{a_n}^{\tau} \tilde{S}_{a_n}(t)\tilde{N}_{a_n}(t) dt \right| \\
&\leq \int_0^\tau \left| \tilde{S}_{a_n}(t)\tilde{N}_{a_n}(t) - \tilde{S}_\tau(t)\tilde{N}_\tau(t) \right| dt + \int_{a_n}^{\tau} \tilde{S}_{a_n}(t)\tilde{N}_{a_n}(t) dt \\
&\leq \int_0^\tau \left| \tilde{S}_{a_n}(t)\tilde{N}_{a_n}(t) - \tilde{S}_\tau(t)\tilde{N}_\tau(t) \right| dt + \frac{\pi\Lambda}{\mu_H \mu}(\tau - a_n),
\end{aligned}$$

where the last inequality follows by boundedness of the periodic solutions. It can be shown that the system with impulses at fixed times exhibits continuous dependence on initial conditions and parameters. By hyperbolicity of the periodic solutions, the terminal points of the periodic orbits depend continuously on the parameter τ. For details, see section A.2.2 and Theorem A.4.1. Consequently, $\tilde{S}_{a_n} \to \tilde{S}_\tau$ and $\tilde{N}_{a_n} \to \tilde{N}_\tau$ pointwise almost everywhere. In addition, by boundedness of the periodic solutions, the integrand in the above expression is bounded by $2\pi\Lambda/\mu_H\mu$. Hence, by Lebesgue's dominated convergence theorem, the above integral converges to zero. It then follows that $|P(\tau) - P(a_n)| \to 0$. The same result holds for $a_n \searrow \tau$. We may therefore conclude that P is continuous.

Since P is continous and strictly positive in an open right-interval of $\tau = 0$ and $P(0^+) = 0$, and since the same holds for $\eta\beta_H P$ it follows that, for some $\Theta^+ > 0$, there exists $\tau(\Theta)$ such that $\eta\beta_H P(\tau(\Theta)) = \Theta$ for all $\Theta \in (0, \Theta^+)$. The result follows by Lemma 4.1.1. ∎

Unfortunately, we cannot say more at this stage. We will later study a simplification of this model in which further analysis is possible.

Chapter 5

Malaria Vector Control with Impulsive Differential Equations: The Simplified Model

The model from the previous chapter can be simplified — and much more information obtained about the nature of its solutions — if we make the assumption that all mosquitoes are infectious. That is, we assume $\Psi = M + N \approx N$. Under this assumption, the disease-free equilibria or periodic orbits cannot exist, but we can say much more about the endemic orbits. The impulsive differential equations are

$$\begin{aligned}
\dot{S} &= \pi - \beta_H S\Psi + hI + \delta R - \mu_H S, && t \neq k\tau, \\
\dot{I} &= \beta_H S\Psi - hI - \alpha I - (\mu_H + \gamma)I, && t \neq k\tau, \\
\dot{R} &= \alpha I - \delta R - \mu_H R, && t \neq k\tau, \\
\dot{\Psi} &= \Lambda - \mu\Psi, && t \neq k\tau, \\
\Delta\Psi &= -r\Psi^-, && t = k\tau,
\end{aligned} \tag{5.0.1}$$

for spraying at fixed times, and

$$
\begin{aligned}
\dot{S} &= \pi - \beta_H S\Psi + hI + \delta R - \mu_H S, && \Theta \neq \overline{\Theta}, \\
\dot{I} &= \beta_H S\Psi - hI - \alpha I - (\mu_h + \gamma)I, && \Theta \neq \overline{\Theta}, \\
\dot{R} &= \alpha I - \delta R - \mu_H R, && \Theta \neq \overline{\Theta}, \\
\dot{\Psi} &= \Lambda - \mu\Psi, && \Theta \neq \overline{\Theta}, \\
\dot{\Theta} &= \eta\beta_H S\Psi, && \Theta \neq \overline{\Theta}, \\
\Delta\Psi &= -r\Psi^-, && \Theta = \overline{\Theta}, \\
\Delta\Theta &= -\overline{\Theta}, && \Theta = \overline{\Theta}
\end{aligned} \tag{5.0.2}
$$

for incidence-based spraying. The key to this simplification is that the mosquito equations, when written as a sum, are decoupled;

$$\dot{M} + \dot{N} = \Lambda - \mu(M + N).$$

The results on existence and uniqueness of solutions are easily seen to hold for equations (5.0.1) and (5.0.2) as well.

5.1 The system without spraying

We briefly comment on stability of the malaria model in the absence of spraying.

Theorem 5.1.1. System (5.0.1) satisfies, in the absence of impulse effect, the following conditions:

- there is a single biologically meaningful fixed point that exists for all positive parameter values and is locally asymptotically stable,
- if $\gamma = 0$, the fixed point is a global attractor.

Proof: We first note that no solution of (5.0.1) in the absence of impulses (or otherwise) can be unbounded. This claim can be established as follows. We note that

$H = S + I + R$ satisfies the inequality

$$-\mu_H H \leq \frac{dH}{dt} \leq \pi - \mu_H H.$$

It then follows that $H(t)$ satisfies the inequality

$$H(0)e^{-\mu_H t} \leq H(t) \leq H(0)e^{-\mu_H t} + \frac{\pi}{\mu_H}\left(1 - e^{-\mu_H t}\right),$$

so the human population is bounded. Moreover, it is clear that the mosquito equation has a unique globally asymptotically stable equilibrium $\Psi^* = \frac{\Lambda}{\mu}$, so the mosquito population is bounded as well. Hence solutions are bounded for all positive time.

For completeness, we remark that the unique equilibrium point of this system is given by

$$\begin{aligned}
\Psi^* &= \frac{\Lambda}{\mu}, \\
R^* &= \frac{\pi}{\mu_H + \beta_H\Psi^*} \cdot \left(\frac{\alpha^2\beta_H\Psi^*(\mu_H + \beta_H\Psi^*)}{\alpha(\mu_H(\delta + \mu_H)(h + \alpha + \mu_H + \gamma) + \beta_H\Psi^*((\delta + \mu_H)(\mu_H + \gamma) + \mu_H)}\right), \\
I^* &= \frac{R^*(\delta + \mu_H)}{\beta_H\Psi^*}, \\
S^* &= \frac{I^*(h + \alpha + \mu_H + \gamma)}{\beta_H\Psi^*},
\end{aligned} \tag{5.1.1}$$

which exists for all positive parameter values. It can be shown that the characteristic polynomial of the linearization at the equilibrium point is given by

$$(-\mu_H - \lambda)(-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3) = 0,$$

where the coefficients a_j are

$$\begin{aligned}
a_1 &= h + \alpha + \gamma + \delta + \beta_H\Psi^* + \mu_H S^* + 2\mu_H, \\
a_2 &= (\beta_H\Psi^* + \delta + 2\mu_H)(h + \alpha + \gamma + \mu_H) + (\delta + \mu_H)(\beta_H\Psi^* + \mu_H) + \beta_H\Psi^* h, \\
a_3 &= \mu_H(\beta_H\Psi^*(\alpha + \gamma + \mu_H) + \mu_H(h + \alpha + \gamma + \mu_H)) + \delta((\beta_H\Psi^* + \mu_H)(h + \alpha + \gamma + \mu_H)).
\end{aligned}$$

All of these coefficients are positive, and it is simple to verify that $a_1a_2 > a_3$. By the Routh–Hurwitz criterion, all roots of the cubic have negative real part. As a result, all eigenvalues have negative real part, and the equilibrium is locally asymptotically stable.

To prove the second part, we notice that this system can be transformed into an asymptotically autonomous planar system. If $\gamma = 0$, then $\dot{H} = \pi - \mu_H H$, so we can write

$$\begin{aligned}
\dot{I} &= \beta_H (H(t) - I - R)\Psi(t) - (h + \alpha + \mu_H)I, \\
\dot{R} &= \alpha I - (\delta + \mu_H)R.
\end{aligned}$$

Since $H(t) - H^*$ and $\Psi(t) - \Psi^*$ are decaying exponentials, the functions

$$(H(t) - H^*)(\Psi(t) - \Psi^*), \qquad I(\Psi(t) - \Psi^*), \qquad R(\Psi(t) - \Psi^*)$$

are locally uniformly convergent as $t \to \infty$ on all compact sets, where $H^* = \pi/\mu_H$ and $\Psi^* = \Psi/\mu$. This system is therefore asymptotically autonomous, all of its (positive) solutions are bounded, and it has as its limiting system

$$\begin{aligned}
\dot{I} &= \beta_H \left(\frac{\pi}{\mu_H} - I - R \right) \frac{\Lambda}{\mu} - (h + \alpha + \mu_H)I, \\
\dot{R} &= \alpha I - (\delta + \mu_H)R.
\end{aligned}$$

This system is linear, and its equilibrium point is asymptotically stable. Consequently, all orbits of the limiting system tend to the single equilibrum point. From a result of Markus, cited by Castillo-Chavez and Thieme [7], the omega limit set of any bounded solution of the asymptotically autonomous system must either be:

- an equilibrium of the limiting system,
- the union of periodic orbits of the limiting system and possibly centers of the limiting system and its surrounding periodic orbits,

- equilibria of the limiting system cyclically chained to each other by orbits of the limiting system.

Since the limiting system is linear, it has no periodic orbits, so the second option cannot happen. Since all orbits converge to the unique equilibrium point (since the linear system is asymptotically stable), there can be no cyclical chain of the equilibrium point to itself by its orbits. Consequently, the third option cannot happen. Therefore every bounded solution of the asymptotically autonomous system converges to the endemic equilibrium. Since all positive solutions are bounded, they all converge to this equilibrium. ∎

5.2 Existence and stability of periodic orbits

In contrast to the full model, the model with spraying at fixed times always has a periodic orbit, and we can provide conditions by which it is positive, unique and stable.

Lemma 5.2.1. The system with impulses at fixed times (5.0.1) has a periodic solution.

Proof: The mosquito compartment has a unique periodic solution

$$\tilde{\Psi}_\tau(t) = \frac{\Lambda}{\mu} \cdot \left(1 + \frac{re^{-\mu t}}{(1-r)e^{-\mu\tau} - 1}\right). \tag{5.2.1}$$

Therefore, to search for periodic solutions of (5.0.1), we may take $\Psi = \tilde{\Psi}_\tau$. We must prove the existence of positive periodic solutions of

$$\begin{aligned} \dot{S} &= \pi - \beta_H S\tilde{\Psi}_\tau(t) + hI + \delta R - \mu_H S, \\ \dot{I} &= \beta_H S\tilde{\Psi}_\tau(t) - hI - \alpha I - \gamma I - \mu_h I, \\ \dot{R} &= \alpha I - \delta R - \mu_H R. \end{aligned} \tag{5.2.2}$$

However, this ordinary differential equation is linear, R^3_+ is positively invariant, and the sum $H = S + I + R$ satisfies

$$\dot{H} \leq \pi - \mu_H H$$

for positive initial conditions, implying that this linear system has a bounded solution. By the Massera theorem [17], it has a periodic solution. ∎

Theorem 5.2.2. There exist $\epsilon_1, \ldots, \epsilon_4 > 0$ such that, for

$$\gamma < \epsilon_1, \qquad \delta < \epsilon_2, \qquad h < \epsilon_3, \qquad \alpha < \epsilon_4$$

the system with impulses at fixed times (5.0.1) has a unique, nonnegative periodic solution that is asymptotically stable.

Proof: This will be proven in several parts.

Part 1: We start with the case where $\gamma = h = \alpha = 0$. In this case, (5.2.2) reduces to

$$\begin{aligned}
\dot{S} &= \pi - \beta_H S \tilde{\Psi}_\tau(t) + \delta R - \mu_H S, \\
\dot{I} &= \beta_H S \tilde{\Psi}_\tau(t) - \mu_H I, \\
\dot{R} &= -\delta R - \mu_H R.
\end{aligned}$$

The homogeneous equation associated to the above linear system has no periodic solutions[14] and, consequently, the inhomogeneous equation has a unique periodic solution. It follows that (5.0.1) has a unique periodic solution.

Part 2: Suppose $\gamma = 0$. We wish to establish conditions under which the periodic solution

$$(\tilde{S}(t), \tilde{I}(t), \tilde{R}(t), \tilde{\Psi}(t))$$

[14] Without the inhomogeneous birth rate, π, any periodic solution must satisfy $R = 0$. Consequently, $S = 0$, and following this, $I = 0$, so there is no non-trivial periodic solution.

is nonnegative for all time. Since the positive orthant is positively invariant, it suffices to show that the initial conditions on this periodic orbit are positive. We clearly have $\tilde{\Psi}(t) > 0$, and, by using the identity $\tilde{S} + \tilde{I} + \tilde{R} = \frac{\pi}{\mu_H}$, we find

$$\tilde{S}(t) = \exp\left(-\int_0^t a(s)ds\right)\left[\tilde{S}(0) + \int_0^t \left(\pi\left(1 + \frac{h}{\mu_H}\right) + \tilde{R}(s)(\delta - h)\right)\exp\left(\int_0^s a(w)dw\right)ds\right]$$

where

$$a(s) = \beta_H \tilde{\Psi}(s) + h + \mu_H,$$

and $\tilde{S}(0)$ can be shown to be positive provided

$$\tilde{R}(t)(\delta - h) \geq 0.$$

By taking $\delta = h$, this condition is satisfied. Therefore if $\delta = h$, then $\tilde{S}(t) > 0$. Conversely, if $\tilde{S}(t) > 0$, then it is simple to verify by the variation of constants formula that $\tilde{I}(t) \geq 0$ and, following this, that $\tilde{R}(t) \geq 0$, where this result does not depend on the parameters δ, γ and h.

Part 3: Define the multiparameter $\lambda := (\gamma, \delta, h, \alpha)$ and let $\varphi(t)$ denote the periodic solution when $\lambda = 0$. By parts 1 and 2, we have $\tilde{R} = 0$ and $\tilde{S}(t) > 0$ on this solution.

- Let $f(t, x, \lambda)$ denote the vector field. We trivially have $f(t+T, x, \lambda) = f(t, x, \lambda)$; the impulse transition matrix I_k is constant, so $I_{k+1} = I_k$; and the sequence of impulses satisfies $\tau_{k+1} = \tau_k + \tau$, where τ is the period. Since the sequence of impulses $\tau_k = k\tau$ does not depend on x, we trivially have the relation

$$1 - \frac{\partial \tau_k}{\partial x}(\varphi(\tau_k), \lambda) f(\tau_k, \varphi(\tau_k), \lambda) = 1 \neq 0.$$

- In part 1, it was shown that solutions are bounded. It follows that we can choose some $\kappa > 0$ so that, for any $x_0 \in B_\kappa(\varphi_0)$, the solution $x(t; x_0, \lambda)$ exists for $t \in [0, \tau]$.

- We now establish smooth dependence on initial conditions and parameters sufficiently close to the periodic solution $\varphi(t)$. In a sufficiently small neighbourhood of $(\tau, \varphi_0, 0) \in \mathbb{R} \times \mathbb{R}^4 \times L$, where $(\gamma, \delta, h, \alpha) \in L$, the impulse effect occurs when the solution $x(t; x_0, \lambda)$ satisfies $\phi(t, x) = t - \tau = 0$. ϕ is C^1 and partitions $\mathbb{R} \times \mathbb{R}^3$ into disjoint domains D_1 and D_2 such that

$$\mathbb{R} \times \mathbb{R}^4 = D_1 \cup D_2 \cup \{(\tau, x) : x \in \mathbb{R}^4\}.$$

  Any solution defined in the aforementioned neighbourhood has moments of impulse effect at time $t = \tau$, and we have $(t, x(t)) \in D_1$ when $t_0 \leq t < \tau$ and $(t, x(t)) \in D_2$ when $\tau < t$. We have $\partial_x \phi(t, x) = 0$, so

$$\partial_t \phi(\tau, \varphi_0, 0) + \partial_x \phi(\tau, \varphi_0, 0) = 1 \neq 0.$$

  Since the impulses occur at fixed times, each solution meets the hypersurface $\{(\tau, x) : x \in \mathbb{R}^4\}$ at most once for any $\lambda \in L$ and any initial condition.

  Finally, $f(t, x, \lambda)$ is C^1, as is I, being constant. By Theorem A.2.6, we have smooth dependence on initial conditions and parameters in a neighbourhood of $(\tau, \varphi_0, 0)$.

By the above calculations, we may perform a linearization at the periodic orbit. When $\lambda = 0$, the linearized equation is

$$\begin{aligned}
\dot{u} &= \begin{bmatrix} -\beta_H \tilde{\Psi}(t) - \mu_H & 0 & 0 & -\beta_H \varphi_S(t) \\ \beta_H \tilde{\Psi}(t) & -\mu_H - \alpha & 0 & \beta_H \varphi_S(t) \\ 0 & \alpha & -\mu_H & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} u, \quad && t \neq k\tau \\
\Delta u &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \end{bmatrix} u^-, && t = k\tau.
\end{aligned} \tag{5.2.3}$$

This linear system has no nontrivial periodic solutions, and the Floquet multipliers are found to be exponentials of the diagonal entries multiplied by τ (with one multiplier scaled by a factor of $1-r$), and, since these diagonal entries are negative, all multipliers are within the unit circle. Consequently, by Theorem A.4.2, the periodic orbit is exponentially stable in the nonlinear system. Moreover, for $|\lambda| < \epsilon$ sufficiently small, there is a unique exponentially stable periodic solution [15] φ_λ that converges to φ as $\lambda \to 0$ in the B topology [16]. In other words, the unique periodic solution persists for $\lambda \approx 0$ and is close to φ.

Let $\tilde{S}_\lambda(t)$ denote the S component of the periodic solution at parameter λ. Since $\tilde{S}_0(t) > 0$, the above implies that $\tilde{S}_\lambda(t) > 0$ for $\lambda \approx 0$. By part 2, $\tilde{S}_\lambda(t) > 0$ is a sufficient condition for nonnegativity of the periodic solution. We have therefore proven the theorem. ∎

Unfortunately, we cannot say more about this periodic solution. In general, determining its stability away from $\lambda = 0$ or proving that it is the only periodic solution for arbitrary parameters requires solving a two-dimensional linear, non-autonomous system that is not in triangular form. If this could be done, then the perturbation result above would be unnecessary. Specifically, we have the following corollary.

Corollary 5.2.3. Suppose the linear system

$$\dot{u} = \begin{bmatrix} -\beta_H \tilde{\Psi}_\tau(t) - \mu_H & h & \delta \\ \beta_H \tilde{\Psi}_\tau(t) & -h - \alpha - \mu_H - \gamma & 0 \\ 0 & \alpha & -\delta - \mu_H \end{bmatrix} u \tag{5.2.4}$$

has no floquet multipliers on the unit circle. Then (5.0.1) has a unique τ-periodic solution.

[15] Theorem A.4.2 is stated in terms of a one-dimensional parameter. However, the proof can be easily modified to include the multiparameter case, being a typical application of the implicit function theorem. See [5] for details.

[16] See Remark A.3.3 of Appendix A for a heeuristic definition of this topology and a reference.

Proof: The proof is essentially the same as the proof of Lemma 5.2.1. There, we proved by the Massera theorem that a periodic solution exists. Equation (5.2.4) is the homogeneous equation associated to (5.2.2), and, if this has no periodic solutions (equivalently, no mutlipliers on the unit circle), then the inhomogeneous equation has a unique periodic solution [15]. ∎

As to whether the periodic solution remains positive for all positive parameter values, we remark that the positive orthant is invariant; see Lemma 4.2.6[17]. Therefore, if the periodic solution depends continuously on the parameter $\lambda = (\gamma, \delta, h, \alpha) > 0$ and exists for all $\lambda > 0$, then it must be positive for all $\lambda > 0$. This is because it has been established that the periodic solution is positive when $\lambda = 0$, and by invariance of the positive orthant, the periodic orbit cannot continuously "exit" the nonnegative orthant.

We now consider the autonomous equation.

Lemma 5.2.4. Suppose, for all $\tau > 0$, the system with spraying at fixed times (5.0.1) has a unique, positive periodic solution. Then the system with autonomous spraying (5.0.2) has a periodic solution for every $\overline{\Theta} > 0$. If $\overline{\Theta}$ is sufficiently small and periodic solutions are hyperbolic for τ small, there is a unique periodic solution.

Proof: Define the map $P : \mathbb{R}_+ \setminus 0 \to \mathbb{R}_+$,

$$P(\tau) = \int_0^\tau \tilde{S}_\tau(t)\tilde{\Psi}_\tau(t)dy,$$

where $\tilde{S}_\tau(t)$ and $\tilde{\Psi}_\tau(t)$ are the S and Ψ components of the unique τ-periodic solution of the fixed-impulse equation. It has been shown previously (Theorem 4.5.1) that P is continuous and $P(0^+) = 0$. Note that $\dot{S} \geq \pi - \beta_H S\Lambda/\mu - \mu_H S$ for all $\tau > 0$, so, in particular, there exists a neighbourhood of $S = 0$ where $\dot{S} > 0$. Hence we must have

[17] Lemma 4.2.6 is valid for the simplified models.

$S_{\min} := \inf_{\tau>0} ||\tilde{S}_\tau||_\infty > 0$. Consequently,

$$0 < S_{\min} \int_0^\tau \tilde{\Psi}_\tau(t)dt \leq P(\tau). \tag{5.2.5}$$

The lower bound can easily be seen to approach infinity with $\tau \to \infty$, and so P is unbounded. P is therefore surjective, and, since the same holds for $\eta\beta_H P$, there exists $\overline{\tau}$ such that $\eta\beta_H P(\overline{\tau}) = \overline{\Theta}$ for all $\overline{\Theta} > 0$. Consequently, for all $\overline{\Theta} > 0$, the system with autonomous spraying (5.0.2) has a periodic solution.

To show uniqueness for small values of $\overline{\Theta}$, we require several auxillary results. We claim P is differentiable. To show this, consider the differential equation

$$\begin{aligned}
\dot{S} &= \pi - \beta_H S\overline{\Psi}_\tau(t) + hI + \delta R - \mu_H S \\
\dot{I} &= \beta_H S\overline{\Psi}_\tau(t) - \alpha I - hI - \mu_H I - \gamma I \\
\dot{R} &= \alpha I - \delta R - \mu_H R,
\end{aligned} \tag{5.2.6}$$

where $\overline{\Psi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, a piecewise linear extension of Ψ, is defined by

$$\overline{\Psi}(t,\tau) = \begin{cases} \overline{\Psi_\tau^+}(t), & \tau > 0, \\ \overline{\Psi_\tau^-}(t), & \tau \leq 0, \end{cases}$$

$$\overline{\Psi_\tau^+}(t) = \begin{cases} \frac{d}{dt}\tilde{\Psi}_\tau(0) \cdot t + \tilde{\Psi}_\tau(0), & t < 0 \\ \tilde{\Psi}_\tau(t), & 0 \leq t \leq \tau, \\ \frac{d}{dt}\tilde{\Psi}_\tau(\tau) \cdot (t-\tau) + \tilde{\Psi}_\tau(\tau), & t \geq \tau, \end{cases}$$

$$\overline{\Psi_\tau^-}(t) = \begin{cases} \frac{d}{dt}\tilde{\Psi}_\tau(\tau) \cdot (t-\tau) + \overline{\Psi}_\tau(\tau), & t < \tau \\ \tilde{\Psi}_\tau(t), & \tau \leq t \leq 0, \\ \frac{d}{dt}\tilde{\Psi}_\tau(0) \cdot (t) + \tilde{\Psi}_\tau(0), & t \geq 0, \end{cases}$$

and

$$\Omega = \left(\frac{\log(1-r)}{\mu}, \infty\right),$$

and, in the definition of $\overline{\Psi_\tau^-}(t)$, we formally allow $\tilde{\Psi}_\tau(t)$ to be defined for nonpositive arguments by evaluating it for said arguments as in (5.2.1). Note that $\overline{\Psi}$ is C^1.

For every $\tau > 0$, the human components of the periodic solution of the impulsive differential equation (5.0.1), denoted $\tilde{S}_\tau(t)$, $\tilde{I}_{\tau(}t)$ and $\tilde{R}_\tau(t)$, coincide with the forward solution of (5.2.6) from $S(0) = \tilde{S}_\tau(0)$, $I(0) = \tilde{S}_\tau(0)$, $R(0) = \tilde{R}_\tau(0)$ for $t < \tau$. By standard results of smooth dependence on initial conditions and parameters, the solution map $x(t; x_0, \tau)$ associated to (5.2.6) is a continuously differentiable function of τ. By the hyperbolicity hypothesis of the theorem, $\tilde{S}_\tau(0)$, $\tilde{I}_\tau(0)$ and $\tilde{R}_\tau(0)$ are differentiable with respect to τ for τ small [18]. The differential equation is C^1, so we have smooth dependence on initial conditions and parameters. It then follows that the composition

$$x(t; \tilde{x}_0(\tau), \tau),$$

where $\tilde{x}_0(\tau) \equiv (\tilde{S}_\tau(0), \tilde{I}_\tau(0), \tilde{R}_\tau(0))$, is C^1 in τ small for each fixed t. By the Leibniz integral formula [14], the map $\overline{P} : \mathbb{R}_+ \setminus 0 \to \mathbb{R}_+$ defined by

$$\overline{P}(\tau) = \int_0^\tau x^1(t; \tilde{x}_0(\tau), \tau)\overline{\Psi}_\tau(t)dt$$

is continuously differentiable for small τ, where the superscript x^1 denotes the first component. Since $x^1(t; \tilde{x}_0(\tau), \tau) = \tilde{S}_\tau(t)$ and $\overline{\Psi}_\tau(t) = \tilde{\Psi}_\tau(t)$ for $t < \tau$, we have $\overline{P} = P$. We therefore conclude that P is C^1 for τ small.

Next we claim that $P'(0^+)$ exists and is zero. Again, first considering the map $\overline{P}$, by the Leibniz integral formula [14], the limit $\overline{P'}(0^+)$ is given by

$$\lim_{y \to 0^+} \int_0^y \left[\frac{d}{d\tau}\overline{\Psi}(t; y)\right] x^1(t; \tilde{x}_0(y), y) + \overline{\Psi}(t; h) \left[\frac{d}{d\tau}x^1(t; \tilde{x}_0(y), y)\right] dt + \tilde{x}^1(t; \tilde{x}_0(y), y)\overline{\Psi}(y; y),$$

provided the above exists, where we write $\overline{\Psi}_\tau(t) = \overline{\Psi}(t; \tau)$ to ensure no ambiguity in the above expression. The rightmost term converges to zero. By smooth dependence on parameters, $\frac{d}{d\tau}x^1(t; \tilde{x}_0(y), y)$ remains bounded in a neighbourhood of $y = 0$, say

[18] See Corollary 5.2.3. This condition is equivalent to hyperbolicity, and, if this holds, the unique periodic point is $\tilde{x}_0(\tau) = (E - X_\tau(\tau; 0))^{-1}v(\tau)$ where $X_\tau(\cdot, 0)$ is fundamental matrix normalized at 0 with parameter τ and $v(\tau)$ is a C^1 function of τ. Since matrix inversion is smooth where defined, $\tilde{x}_0(\tau)$ is C^1.

$\frac{d}{d\tau}x^1(t;\tilde{x}_0(y),y) \leq K_S$ in this neighbourhood. Thus, for y sufficiently small,

$$\left|\int_0^y \overline{\Psi}(t;y)\left[\frac{d}{d\tau}x^1(t;\tilde{x}_0(y),y)\right]dt\right| \leq y\frac{\Lambda}{\mu}K_S \to 0.$$

Hence

$$\begin{aligned}\overline{P'}(0^+) &= \lim_{y\to 0^+}\int_0^y \left[\frac{d}{d\tau}\overline{\Psi}(t;y)\right]x^1(t;\tilde{x}_0(y),y)dt \\ &= \lim_{y\to 0^+}\int_0^y \left[\frac{d}{d\tau}\overline{\Psi}(t;y)\right]\tilde{S}_y(t)dt,\end{aligned}$$

if the above exists. We have

$$\frac{d}{d\tau}\overline{\Psi}(t;y) = \frac{\mu(1-r)re^{-\mu(t+y)}}{(1-(1-r)e^{-\mu y})^2} > 0$$

where defined and $\frac{d}{d\tau}\overline{\Psi}(t;y) \leq \frac{d}{d\tau}\overline{\Psi}(0;y)$. Hence

$$\left|\int_0^y \left[\frac{d}{d\tau}\overline{\Psi}(t;y)\right]\tilde{S}_y(t)dt\right| \leq \frac{\mu(1-r)re^{-\mu y}}{(1-(1-r)e^{-\mu y})^2}\cdot\frac{\pi}{\mu_H}y \to 0,$$

so $P'(0^+) = \overline{P'}(0^+)$ exists and $P'(0^+) = 0$. Define $A = \eta\beta_H P$. Then A is C^1 and $A'(0^+) = 0$.

If $A' > 0$ for all $\tau > 0$, then the result holds for all $\overline{\Theta} > 0$ since A is invertible. Suppose then $A'(z) = 0$ for some $z > 0$. Since $A'(0^+) = 0$ and $A(0^+) = 0$, we cannot have $A' \leq 0$ on any interval of the form $(0,a)$, since then A would be nonpositive. Hence, said z can be chosen so that $A'(\tau) > 0$ for $0 < \tau < z$. Define

$$A_\downarrow(\tau) = \eta\beta_H S_{\min}\int_0^\tau \tilde{\Psi}_\tau(t).$$

Then by (5.2.5), $A_\downarrow(\tau) \leq A(\tau)$, and it can be shown that $A_\downarrow$ is strictly increasing.

Now, define $A^* = A_\downarrow(z)$. Since $A_\downarrow(\tau) \leq A(\tau)$, the inequality $A^* < A(\tau)$ holds for $\tau > z$. Consequently, the inequality $A(\tau) < A^*$ has no solutions for $\tau > z$. However, since $A(\tau)$ is increasing on $(0,\tau]$, there is a unique solution of the equation $A(\tau) = \overline{\Theta}$ for all $\overline{\Theta} \in (0, A^*]$, and this solution satisfies $0 < \tau < z$. See Figure 5.1 for a diagram. It follows from the existence proof that for $\overline{\Theta} \leq A^*$, that the autonomous impulsive

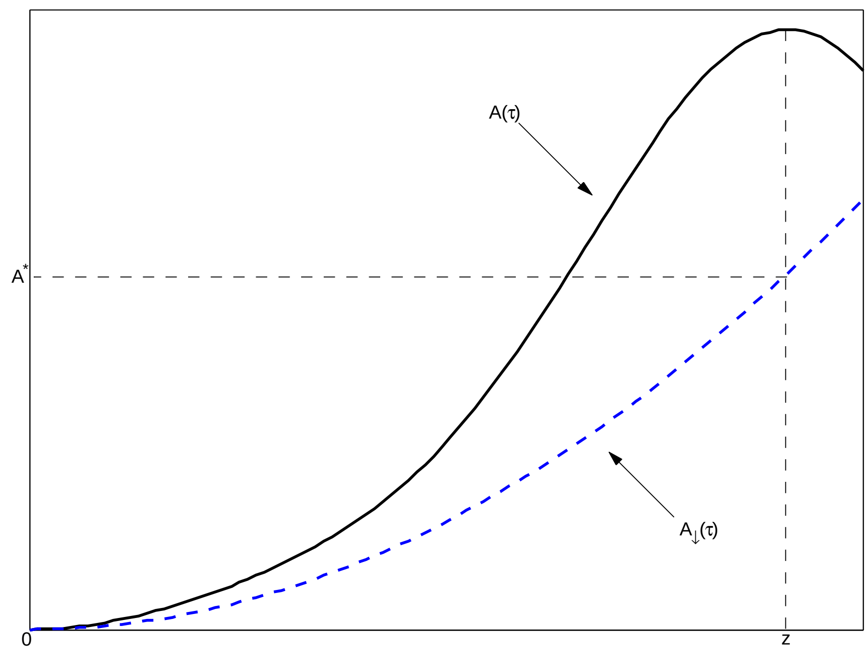

Figure 5.1: In the region $(0, A^*)$ on the vertical axis, the function A is invertible. Since this function dominates the strictly increasing function $A_\downarrow$, there can be no other solution to the equation $A(\tau) = \overline{\Theta}$, for $\overline{\Theta} \in (0, A^*)$, except the unique solution in the interval $(0, z)$. Curves appearing in the figure are for illustrative purposes only.

equation has a unique periodic solution. ∎

This lemma establishes an equivalence between the parameters τ and $\overline{\Theta}$. When these are small, the equivalence is guaranteed to be bijective, provided the periodic orbit is hyperbolic in this range. However, this result relies on the periodic solutions being positive. Lemma 4.2.6, Lemma 5.2.1, Theorem 5.2.2 together are suggestive of the hypothesis that the periodic solution is positive for all positive parameter values, although it does not appear that this can be proven analytically, at least without using more sophisticated techniques.

We conclude this section with a theorem that ties all of these ideas together. In particular, we now state stability results for the autonomous model. This proof of this theorem follows directly the results of Lemma 4.1.2, Theorem 5.2.2 and Lemma 5.2.4.

Theorem 5.2.5. There exist $\epsilon_1, \ldots, \epsilon_4 > 0$ and $\Theta^+ > 0$ such that, for

$$\gamma < \epsilon_1, \qquad \delta < \epsilon_2, \qquad h < \epsilon_3, \qquad \alpha < \epsilon_4, \qquad \overline{\Theta} < \Theta^+,$$

the following are true.

1. The system with spraying at fixed times (4.0.1) has a unique, nonnegative periodic solution that is asymptotically stable,
2. The system with incidence-based spraying (4.0.3) has a unique, nonnegative periodic solution that is orbitally asymptotically stable and enjoys the property of asymptotic phase.

As discussed previously, there are several reasons why we would expect these results to hold for larger parameter values (especially the nonnegativity of the periodic orbit). We will content ourselves, however, with these local results.

Chapter 6

Numerical Results, Hybrid Controls for the Malaria Model and Future Research

In this chapter, we numerically determine the implementation cost of the incidence-based vector control scheme and compare it to that of a more traditional fixed-time spraying schedule. Following this, we discuss alternatives to incidence-based spraying, including hybrid strategies.

6.1 Incidence rate

By Lemma 4.1.2, spraying regularly at fixed times is asymptotically equivalent to spraying when the post-spraying incidence reaches a critical threshold; the resulting periodic orbits, should they exist, have the same stability. Therefore, if the periodic orbit is (nearly) attained within relatively few spraying events, then the difference in cost between the two strategies is determined by the numer of spraying events at the beginning of their implementation. For spraying at fixed times, the formula is obvious:

$$\text{times sprayed in first } x \text{ days} = \left\lceil \frac{x}{\tau} \right\rceil$$

where τ is the spraying period, and the first spraying event is assumed to take place on day 0. For autonomous spraying, however, it is not as simple. For a given critical spraying theshold $\overline{\Theta}$, the period τ is the solution of the equation

$$\begin{aligned} &\eta\beta_H \int_0^\tau \phi^1(t,x)\phi^5(t,x)dt = \phi^6(\tau,x) = \overline{\Theta}, \\ &\phi(0,x) = J\phi(\tau,x), \end{aligned} \tag{6.1.1}$$

where $\phi(t,x)$ is the flow of the contnuous part of (4.0.3), and J is the jump operator;

$$J(S,I,R,M,N,\Theta) = (S,I,R,(1-r)M,(1-r)N,\Theta-\overline{\Theta}).$$

In the simplified case where we take $M+N \approx N$ (see Chapter 5), it is known that there is a bijection between periods τ and thresholds $\overline{\Theta}$ when these are small and certain parameters are also small. In the general case, it is not obvious that the correspondence between period and infecton threshold is as strict. Let

$$T : \tau \mapsto \overline{\Theta}$$

denote the function that associates to a spraying frequency τ the number of infections $\overline{\Theta}$ observed in one cycle of the associated periodic orbit. The pair $(\tau,\overline{\Theta})$ will then satisfy equation (6.1.1).

We obtain a numerical estimate of a possible branch of T by simulating impulsive differential equation (4.0.1) for the parameter values appearing in Table 4.1. The differential equation is run with period τ until a periodic orbit is (approximately) found. The number of infections $\mathcal{I}$ that occur along this periodic orbit is then extracted. We should have $T(\tau) \approx \eta\mathcal{I} = \overline{\Theta}$. Given an arbitrary initial condition, the found periodic orbit must be asymptotically stable, and, in view of Lemma 4.1.2, it is stable in both the system with fixed-time spraying and incidence-based spraying.

To obtain a coarse estimate for η, we cite 2010 data from the World Health Organization [37], and make the choice to use the data from Zimbabwe. With 1,720,767 estimated cases and 249,379 confirmed reported cases in 2010, this data produces a value of η of approximately 0.145. Recall that η is defined as the probability that an individual infected with malaria reports their infection (seeks medical help, etc.). See Figure 6.1 for a plot of the resulting branch.

It will be useful in the following discussion to recall the definition of incidence rate. This is given by the average incidence per time. If the actual population level incidence (which is in general not known) over a period of time $[0, \tau]$ is $\mathcal{I}$, then the incidence rate is $\frac{\mathcal{I}}{\tau}$. See Figure 6.1 for plots of the incidence rate with respect to spraying period and confirmed infection threshold.

Conversely, if a periodic orbit is attained and one observes $\overline{\Theta}$ infections during one period, then the *expected* incidence rate along the periodic orbit is given by

$$\frac{\overline{\Theta}}{\eta\tau}, \tag{6.1.2}$$

where η appears in the denominator due to the identity

$$\overline{\Theta} = \eta \int_0^\tau \beta_H \tilde{S}(t)\tilde{N}(t)dt.$$

This interpretation will be useful later.

6.2 Implementation cost

To estimate the short-term cost of implementing the incidence-based spraying strategy, we use latin hypercube sampling to sample initial conditions near the endemic equilibrium $(S^*, I^*, R^*, M^*, N^*)$ and count the number of spraying events in the first two years. We consider three asymptotic spraying periods: bi-monthly, quarterly and semi-annually.

The results are summarized in Table 6.1. The parameter $\eta = 0.145$ and $r = 0.85$ were used for all calculations, and the statistics related to the incidence-based

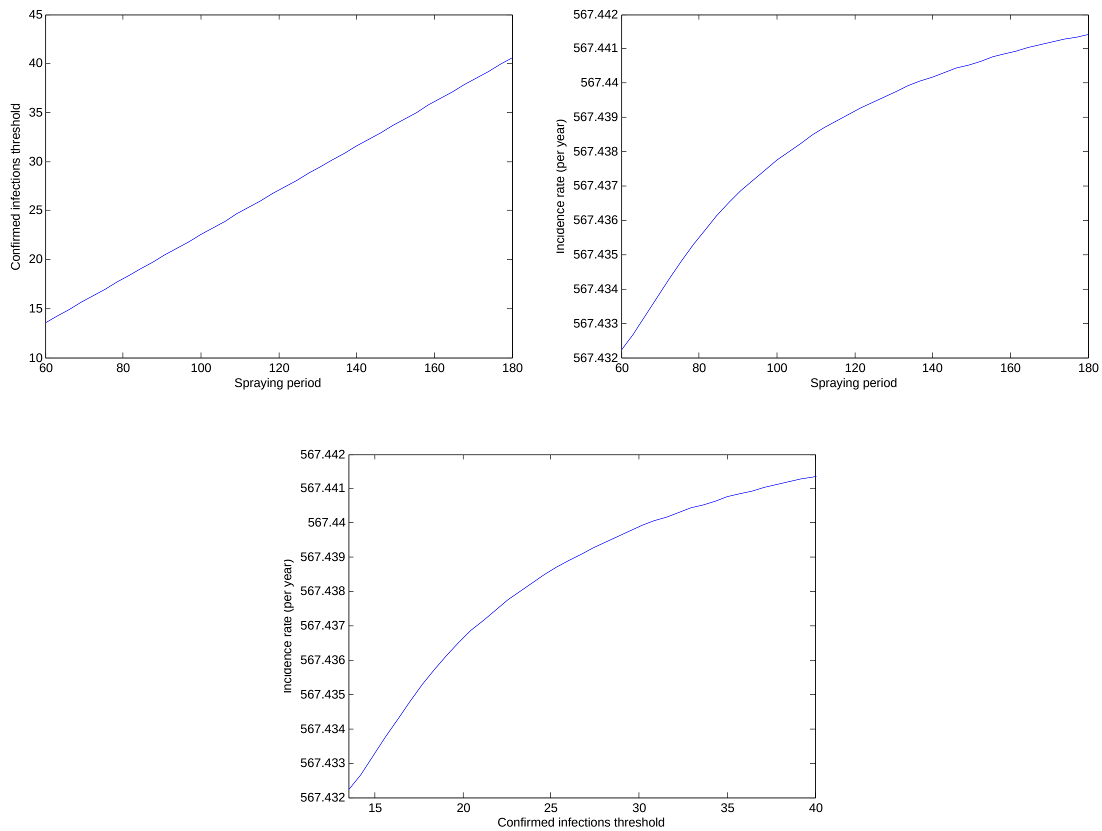

Figure 6.1: Top left: Plot of $T(\tau)$ for the range $\tau \in [60, 180]$, with units in days. Top right: Plot of incidence rate as a function of τ, with incidence rate scaled over years. Bottom: Plot of incidence rate as a function of confirmed infection threshold. Notice that although $T(\tau)$ seems approximately linear, the incidence rate plots suggest otherwise. All plots using parameters from Table 4.1 as well as $r = 0.85$ and $\eta = 0.145$.

Period	C_1^F	$\mu(C_1^I)$	C_2^F	$\mu(C_2^I)$
Bi-monthly	6	5.380	12	11.29
Quarterly	4	3.630	8	7.775
Semi-annual	2	1.495	4	3.550

Table 6.1: Cost (spraying events) during the first year and first two years of implementation of the fixed-time and incidence-based spraying strategies. C_j^F denotes the j-year cost for fixed-time spraying, while $\mu(C_j^I)$ is the sample mean of the j-year cost for incidence-based spraying given a 200-sample latin hypercube of initial conditions x_0 satisfying bounds of $x_0 \in (1 \pm 0.5)x^*$ for endemic equilibrium x^*.

spraying strategy were obtained by latin hypercube sampling with 200 samples for initial conditions near endemic equilibrium.

As can be seen from the table data, it appears as though the incidence-based spraying strategy is more cost-effective than spraying at fixed times. In the short term, there are fewer spraying events for this strategy than for fixed-time spraying, even though they are asymptotically equivalent. The histogram data (Figure 6.2) is even more promising; no sample ever had a greater cost than the fixed-time spraying strategy.

6.3 Hybrid controls

Since it has been suggested (Section 6.2) that incidence-based spraying is more cost-effective than spraying at fixed times, it makes sense to consider strategies that include a combination of both. The following examples come to mind.

- Let impulse effects occur at moments of $t = \tau_k(S, N)$, where

$$\tau_k(S, N) = k\tau G_Q(\eta\beta_H SN)$$

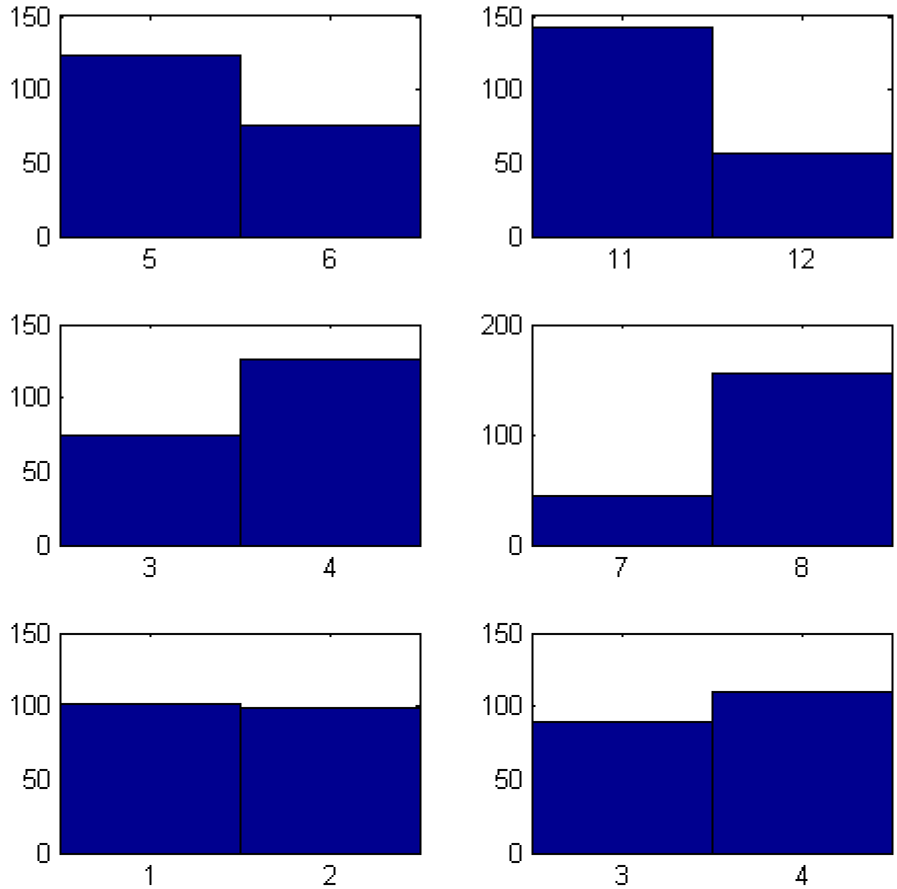

Figure 6.2: Histograms associated to the data used to derive the statistical results in Table 6.1. Top to bottom, bi-monthly, quarterly and semi-annual spraying. Left to right: first year and first two years. Horizontal axes correspond to the number of spraying events, with the vertical axis representing the number of samples (initial conditions) having this cost.

where $Q > 0$ represents a maximum desired instantaneous incidence rate and $G_Q : \mathbb{R} \to \mathbb{R}_+$ is a (continuous or C^1) decreasing function satisfying

$$G_Q(Q) = 1, \qquad \lim_{x\to\infty} G_Q(x) = 0.$$

Under this strategy, if the observed instantaneous incidence rate is less than Q, then spraying will occur less frequently than fixed-time spraying, while, in the opposite case, spraying occurs more frequently.

This control has the advantage of having good regularity properties and being in terms of observable data. Since $\eta\beta_H SN$ is the rate of observed new infections, it can be approximated in the field by averaging the number of reported infections in a range of days prior to the day of interest.

- Let impulse effects occur when $t = \tau_K(\Theta)$, where

$$\tau_k(\Theta) = k\tau G_{\overline{\Theta}}(\Theta)$$

where $G_{\overline{\Theta}} : \mathbb{R} \to \mathbb{R}_+$ is a (continuous or C^1) decreasing function satisfying

$$G_{\overline{\Theta}}(\overline{\Theta}) = 1, \qquad \lim_{x\to\infty} G_{\overline{\Theta}}(x) = 0.$$

How the jump in Θ is defined at impulse points will affect how the control is interpreted, as well as the analytical tractability.

Case 1: $\Theta \mapsto 0$.

Notice that $\Theta \mapsto 0$ is in some sense equivalent to the map $\Theta \mapsto \Theta - \overline{\Theta}$ if incidence-based spraying (4.0.3) is employed, since, with that strategy, impulse effects are only triggered when $\Theta = \overline{\Theta}$. Suppose $\tau_k < t < \tau_{k+1}$. Then there are two cases to consider.

– If the state $\Theta \geq \overline{\Theta}$ is not attained — that is, if

$$0 = \Theta(\tau_k^+) < \Theta(t) < \overline{\Theta}$$

for $\tau_k < t < \tau_{k+1}$ — then $\tau_k(\Theta(t)) > (k+1)\tau$. This means that the next spraying event will occur later than it would with spraying at fixed times. However, since $0 = \Theta(\tau_k^+) < \Theta(t) < \overline{\Theta}$, spraying occurs sooner than it would under incidence-based spraying.

– If the state $\Theta \geq \overline{\Theta}$ is attained — that is, $\Theta(t) \geq \overline{\Theta}$ on an interval $(a, \tau_{k+1}) \subset (\tau_k, \tau_{k+1})$ — then $\tau_{k+1}(\Theta(t)) < (k+1)\tau$ in this range. Consequently, the next spraying event occurs earlier than it would under fixed-time spraying, but later than it would with incidence-based spraying.

We remark also that this strategy is well-founded, in that

$$\tau_k(\Theta(\tau_k)) = k\tau G_{\overline{\Theta}}(\Theta(\tau_k)) < k\tau G_{\overline{\Theta}}(0) < \tau_{k+1} G_{\overline{\Theta}}(0) = \tau_{k+1} G(\Theta(\tau_k^+)),$$

$$\tau_k(\Theta) < \tau_{k+1}(\Theta) \qquad \text{for all } \Theta.$$

In other words, impulse times occur sequentially.

This strategy is one way in which an “average” of the fixed-time and incidence-based spraying strategies can be enacted. However, analysis of this system will be complicated due to the presence of the reset map $\Theta \mapsto 0$; this map will cause nonuniqueness of solutions of the linearized equation, which means that typical linearized stability will not work.

Case 2: $\Theta \mapsto \Theta - \overline{\Theta}$**.**

The interpretation is similar to before; however, it becomes more difficult to compare this system to the incidence-based spraying strategy, since it will now be possible for $\overline{\Theta}$ to take on negative values, and we do not necessarily have $\Theta(\tau_k^+) = 0$ as we did in the previous case.

In a way, since Θ is not reset to zero at each spraying event, the strategy has "memory" of whether or not the state $\Theta \geq \overline{\Theta}$ was achieved. Suppose $\Theta(\tau_k) > \overline{\Theta}$. Then we have

$$\Theta(\tau_k^+) = \Theta(\tau_k) - \overline{\Theta} > 0.$$

Consequently, the initial Θ state on (τ_k, τ_{k+1}) is larger than zero, and the $(k+1)$-st impulse will occur earlier than it would had this state not been reached. Colloquially, the strategy plays "catch up". The opposite situation occurs if the state $\Theta \geq \overline{\Theta}$ is not reached.

This strategy has the advantage of being more analytically tractable (since linearization will not have the same problems as in Case 1), but with a corresponding disadvantage in that it is more difficult to interpret. However, by defininition of $G_{\overline{\Theta}}$, the strategy is well-founded in the same way as in Case 1. The inequalities

$$\tau_k(\Theta(\tau_k)) < \tau_{k+1}(\Theta(\tau_k^+)),$$
$$\tau_k(\Theta) < \tau_{k+1}(\Theta)$$

still hold.

In both cases, if $\Theta^*(t)$ is the Θ component of a τ^*-periodic solution $x^*(t)$ of (4.0.3) satisfying $\Theta^*(\tau^*) = \overline{\Theta}$, then the same periodic solution is a τ^* periodic solution for the above strategies when $\tau = \tau^*$. Indeed,

$$\tau_k(\Theta^*(k\tau^*)) = k\tau^* G_{\overline{\Theta}}(\overline{\Theta}) = k\tau^*,$$

$$\Theta^*((k\tau^*)^+) = \begin{cases} 0, & \text{(Case 1)}, \\ \Theta^*(k\tau^*) - \overline{\Theta}, & \text{(Case 2)}, \end{cases}$$
$$= 0,$$

Since the impulse effects are the same for the other components at each impulse time and the continuous dynamics are identical, we conclude that $x^*(t)$ is a periodic solution of the hybrid control strategy when $\tau = \tau^*$.

We have demonstrated that this strategy is in many ways similar to both incidence-based spraying and fixed-time spraying. Periodic orbits are transferable from one system to the next, and "next" spraying times occur at points that are in some sense related to the fixed-time and incidence-based spraying times.

6.4 Summary of results and future research

Two models of malaria vector control using impulsive insecticide spraying have been considered; the first based on spraying at fixed times, and the second in which spraying occurs when a certain incidence threshold is reached. There is a fundamental connection between these two models: periodic orbits and their stability are transferable provided parameters are chosen wisely; this principle is proven in an abstract setting (see Lemmas 4.1.1 and 4.1.2). The incidence-based model, though autonomous, also displays uniqueness of solutions in forward and backward time; a property that is not always satisfied for autonomous impulsive differential equations; see Section 4.2.

Stability of vector-free and disease-free periodic orbits was examined analytically in Section 4.3. Unfortunately, most of the the results for stability of the disease-free orbit are local, being true only for low infection rates. This, however, is not of much concern, since we are not greatly interested in achieving eradication. Following this, we considered the possibility of bifurcations of disease-free periodic orbits with an endemic periodic orbit in Section 4.4. The difficulties in proving analytically that a bifurcation exists (e.g., transcritical with respect to β_M) were discussed, as were numerical difficulties in observing endemic periodic orbits near the bifurcation point.

Existence of periodic orbits was then considered in the system with autonomous impulses, assuming there exist periodic orbits in the system with impulses at fixed times. It was shown in Section 4.5 that, if there exist periodic orbits for a continuous range of small periods τ, then the incidence-based spraying model has a periodic solution for a continuous range of small confirmed infection thresholds $\overline{\Theta}$.

A simplified version of the model was considered in Chapter 5. The system without spraying is analyzed in Section 5.1; it is shown that, in the absence of disease-related death, there is a single, globally attracting fixed point. If there is disease-related death, then all that can be said is that the fixed point is locally asymptotically stable.

It was shown in Section 5.2 that, for γ, δ, h and $\overline{\Theta}$ small, both the system with fixed spraying and incidence-based spraying have unique, nonnegative periodic solutions that are asymptotically stable and orbitally asymptotically stable with asymptotic phase, respectively (see Theorem 5.2.5). For larger parameter values, conditions that can be checked numerically are supplied.

Investigation of the effect of hybrid controls such as those described in Section 6.3 is one research direction to pursue. The incidence-based spraying strategy is asymptotically equivalent to and was shown by numerical methods in Section 6.2 to be possibly cheaper to implement than a fixed spraying strategy. Other hybrid (or aperiodic) controls may be even cheaper to implement and have similar asymptotic properties.

The reporting coefficient η is a difficult parameter to estimate and likely varies from country to country. It may therefore be fruitful to consider the effect of uncertainty on this parameter, possibly in the presence of temporal variability. The faith of a people in its government and modern medicine, the accuracy of passive surveillance data and other factors relating to the reporting coefficient may be temporally varying, even random. Assuming η can be interpreted as a stochastic process η_t with mean μ_η, how does the asymptotic incidence rate (assuming it exists) in the population compare to the predicted incidence rate along the "ideal" periodic orbit? This would be $\frac{\overline{\Theta}}{\mu_\eta \tau}$, where we obtain this formula by replacing η in the incidence rate (6.1.2) by the "ideal" temporally invariant mean μ_η. The answer to this question is important, since it is desirable to know by how much the result of implementing a health intervention could differ from the baseline estimate.

One limitation of the model is that it does not take into account spatial heterogeneity. Spatial heterogeneity and its effects on malaria vector control by indoor residual spraying has previously been investigated by Al-arydah and Smith? [4]. There is room to perform a similar analysis with alternative spraying strategies, including the incidence-based stategy or any of the hybrid strategies proposed in the previous section. In a spatially heterogeneous context, it may furthermore be profitable to consider different forms of non-constant recruitment of mosquitoes and humans or alternative transmission dynamics (infection latency periods, mosquito life cycle, etc). This further complexity in the model may make analysis complicated, but could increase the applicability in real-world situations.

Appendix A

Background material on impulsive differential equations

We collect herein some of the elementary results of impulsive differential equations that will be frequently referenced in the following chapters. We begin with a description of impulsive differential equations, and define two subclasses that will be especially relevant to this thesis. Following this, we state basic existence and uniqueness results, as well as theorems on continuous/smooth dependence on parameters. Properties of periodic linear systems are stated, and we end with stability criteria for periodic orbits.

A.1 Description and classification of systems with impulse effect

The following description appears in Samoilenko and Perestyuk [29]. Let Ω be an open subset of $\mathbb{R}^n$, which will be the *phase space* of the evolutionary process. At each point in time $t \in \mathbb{R}$, suppose the state of the system can be described using n parameters, so that a point (t, x) in the set $\mathbb{R} \times \Omega$ can be identified with the state x

of the system at said time t.

Definition A.1.1. *An* (ordinary) impulsive differential equation *consists of an ordinary differnential equation*

$$\frac{dx}{dt} = f(t, x), \tag{A.1.1}$$

where $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, *and a family of maps*

$$A_t : M_t \to N_t,$$

indexed by $t \in \mathbb{R}$ *and* $M_t, N_t \subset \Omega$.

The trajectories of an impulsive differential are defined as follows: an initial point (t_0, x_0) moves along the curve $(t, x(t))$ given by the solution of (A.1.1) with initial condition $x(t_0) = x_0$. If there is a time $\tau_1 > t_0$ such that $x(\tau_1) \in M_{\tau_1}$, then the operator A_{τ_1} "instantaneously" transfers the point $x(\tau_1)$ to the position $x(\tau_1^+) := A_{\tau_1} x(\tau_1)$. The process then repeats itself by "resetting" the initial point to $(\tau_1, x(\tau_1^+))$. The solution of the impulsive differential equation is then the left-continuous function $x(t)$ which (in general) has discontinuities at times τ_k, corresponding to points at which

$$\lim_{t \to \tau_k^-} x(t) = x(\tau_k)$$

belongs to the set M_{τ_k}. These discontinuities are called *impulses*, and the times at which they occur are called *moments of impulsive effect*.

The possibility of different structures of the maps A_t allows a great flexibility in the dynamics that one may observe. For this thesis, we will be interested in two particular classes of impulsive differential equation.

Definition A.1.2 (Impulsive differential equation with impulses at fixed times)**.** *An impulsive differential equation has* impulses at fixed times *if the sets* M_t *are given by*

$$M_t = \begin{cases} \emptyset & : \quad t \neq \tau_k, \\ \Omega & : \quad t = \tau_k, \end{cases}$$

where $\{\tau_k : k \in K \subset \mathbb{Z}\}$ *is a strictly increasing sequence of real numbers;* $\tau_k < \tau_{k+1}$. *In this case, the equations are written as follows:*

$$\frac{dx}{dt} = f(t,x), \qquad t \neq \tau_k$$
$$\Delta x = I_k(x), \qquad t = \tau_k.$$

where $A_{\tau_k} = id_\Omega + I_k$ *for some* $I_k : \Omega \to \Omega$.

That is, an impulsive differential equation with impulses at fixed times is one where the discontinuities occur strictly at predetermined times τ_k indexed by a subset of the integers. Most of this thesis will pertain to *periodic* equations, which we briefly introduce here.

Definition A.1.3. *An impulsive differential equation with impulses at fixed times is* periodic *if there exist minimal* $q \in \mathbb{N}_+$ *and* $T \in \mathbb{R}_+$ *such that for all* $k \in \mathbb{Z}$,

- $f(t+T,x) = f(t,x)$ *for all* $t \in \mathbb{R}$,
- $\tau_{k+q} = \tau_k + T$,
- $I_{k+q} = I_k$.

The number T *is called the* period *and* q *is called the* cycle number *of the periodic impulsive differential equation.*

The term "cycle number" is not standard terminology. The first conditions states that the vector field is periodic in time with period T. The second condition ensures that the impulses occur periodically with period T and there are q impulses per period. The third condition ensures that the law governing the impulses is itself periodic.

Definition A.1.4 (Autonomous impulsive equations)**.** *An impulsive differential equation is* autonomous *if the vector field* $f(\cdot)$ *is autonomous and the family of operators* $A_t : M_t \to N_t$ *satisfy the following conditions:*

- *$M_t = \sigma$ for all $t \in \mathbb{R}$, where σ is an $(n-1)$-dimensional submanifold of Ω,*
- *there exists $I : \Omega \to \Omega$ such that $A_t = id_\Omega + I$ for all $t \in \mathbb{R}$.*

In this case, the equations are written as follows:

$$\begin{aligned} \frac{dx}{dt} &= f(x), \qquad && \phi(x) \neq 0, \\ \Delta x &= I(x), && \phi(x) = 0. \end{aligned}$$

An impulsive differential equation is therefore autonomous if the vector field is autonomous, and the times at which impulses occur and the nature of the impulses depends only on the state $x \in \Omega$. Autonomous impulsive differential equations are sometimes called *discontinuous dynamical systems* [29] or *discontinuous flows* [3].

Of relevance to us will be the case where the submanifold σ can be represented by a level set of a smooth function $\phi : \Omega \to \mathbb{R}$; that is, the condition $x \notin \sigma$ can be replaced with $\phi(x) \neq 0$, and vice versa for inclusion. Both types of impulsive differential equation can be connected in the following unifying way: an impulse effect occurs when some spatio-temporal relation is satisfied; that is, when $\phi(t, x) = 0$ for some function $\phi : \mathbb{R} \times \Omega \to \mathbb{R}$. For impulses at fixed times, we have $\phi(t, x) = 0$ if and only if $t = \tau_k$ for some k, and, with autonomous impulses, $\phi(t, x) = \phi(x)$ depends only on state. These types of equations are discussed at length in the literature cited in the introduction.

A.2 Fundamental properties of solutions

We will now establish properties of the solutions of the classes of impulsive differential equations introduced above. We state fundamental existence and uniqueness results (including some new results) and then move on to dependence on initial conditions and parameters. We begin with a definition of the solution of an impulsive differential equation.

Definition A.2.1. *Consider the impulsive differential equation*

$$\frac{dx}{dt} = f(t,x), \qquad x \notin M_t,$$

with $(t,x) \in \mathbb{R} \times \Omega$*,* $\Omega \subset \mathbb{R}^n$ *and discontinuity maps*

$$A_t : M_t \to N_t.$$

A function $\varphi : \langle \alpha, \beta \rangle \to \mathbb{R}^n$ *is a* solution *if the following conditions hold:*

- $(t, \varphi(t)) \in \mathbb{R} \times \Omega$ *for all* $t \in \langle \alpha, \beta \rangle$*,*
- *if* $\varphi(\tau) \in M_\tau$ *and* $\tau \neq \beta$*, then*

  $$\varphi(\tau^+) := \lim_{t \to \tau^+} \varphi(t)$$

  exists, $\varphi(\tau^+) = A_\tau \varphi(\tau)$ *and there exists* $\delta > 0$ *such that for* $\tau \leq s < t + \delta$*,* $\varphi(s) \notin M_s$*,*
- *for all* $t \in (\alpha, \beta)$ *with* $\varphi(t) \notin M_t$*, the function* $\varphi(t)$ *is differentiable and satisfies* $\frac{d\varphi}{dt}(t) = f(t, \varphi(t))$*,*
- *the function* $\varphi(t)$ *is continuous from the left in* $\langle \alpha, \beta \rangle$*.*

This definition is not standard, but, in the case of impulsive differential equations with impulses at fixed times, it is consistent with the usual definition; see Bainov and Simeonov [5]. Note that the third condition implies that the operators A_t have no fixed points.

A.2.1 Existence and uniqueness of solutions

For systems with impulses at fixed times, mild conditions on the impulse effect and sufficient continuity of the vector field are enough to guarantee local existence and uniqueness forward in time. The following results can be found in [5].

Theorem A.2.1 (Forward existence and uniqueness for systems with impulses at fixed times)**.** Consider the impulsive differential equation

$$\frac{dx}{dt} = f(t,x), \qquad t \neq \tau_k$$
$$\Delta x = I_k(x), \qquad t = \tau_k.$$

Suppose the following conditions are valid:

(a) The function $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous in the sets $(\tau_k, \tau_{k+1}] \times \Omega$ for $k \in \mathbb{Z}$, and, for each $k \in \mathbb{Z}$ and $x \in \Omega$, the limit of $f(t,y)$ as $(t,y) \to (\tau_k^+, x)$ exists and is finite,

(b) $f(t,\cdot)$ is locally Lipschitz continuous for fixed t,

(c) $x + I_k(x) \in \Omega$ for all $x \in \Omega$ and $k \in \mathbb{Z}$.

Then the initial-value problem

$$x(t_0^+) = x_0$$

has a unique solution for all $(t_0, x_0) \in \mathbb{R} \times \Omega$ defined in an interval of the form (t_0, ω), and is not continuable to the right of ω.

By time reversal, one finds analoguous conditions for existence of solutions defined on intervals that are not continuable to the left.

Corollary A.2.2 (Backward existence and uniqueness)**.** As in Theorem 1, the initial-value problem $x(t_0^+) = x_0$ has a unique solution for all $(t_0, x_0) \in \mathbb{R} \times \Omega$ defined on an interval of the form (γ, t_0) and is not continuable to the left of γ, if the following conditions hold:

(a) The function $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous in the sets $[\tau_k, \tau_{k+1}) \times \Omega$ for $k \in \mathbb{Z}$, and, for each $k \in \mathbb{Z}$ and $x \in \Omega$, the limit of $f(t,y)$ as $(t,y) \to (\tau_k^-, x)$ exists and is finite,

(b) $f(t, \cdot)$ is locally Lipschitz continuous for fixed t,

(c) The equation $x + I_k(x) = \eta$ has a unique solution $x \in \Omega$ for all $\eta \in \Omega$.

Consequently, to establish existence and uniqueness of solutions defined on maximal intervals that are not continuable on either side, it suffices that the vector field f be continuous, locally Lipschitz with respect to x, and for the map $J_k : \Omega \to \Omega$ defined by $J_k(x) = x + I_k(x)$ to exist and be invertible for all $k \in \mathbb{Z}$. This last conditon makes it impossible for two distinct solutions to "merge" together by a discontinuity.

Theorem A.2.3 (Global existence)**.** Let the conditions of Theorem 1 or Corollary 1 hold, and suppose $J^{\pm}(t_0, x_0)$ is the left ($-$) or right ($+$) maximal interval of existence of the initial-value problem $x(t_0^+) = x_0$, and let $\varphi(t)$ be the solution associated to this interval. If there exists a compact set $Q \subset \Omega$ such that $\varphi(t) \in Q$ for $t \in J^{\pm}(t_0, x_0)$, then $J^{\pm}(t_0, x_0)$ is unbounded; specifically,

$$J^{\pm} \in \{(-\infty, t_0), (t_0, \infty)\}.$$

The converse of this result then states that a solution that is not continuable must approach the boundary of Ω at one of the endpoints of its interval of existence. In the case where Ω is unbounded or equal to $\mathbb{R}^n$, this means that there may be finite-time "blow-up" of solutions.

Existence and uniqueness of solutions for the autonomous case is slightly more delicate. Consider the following simple example.

Example A.2.4. Let $\Omega = \mathbb{R}$ be the phase space.

$$\begin{aligned} \frac{dx}{dt} &= 1, & x &\neq 1, \\ \Delta x &= -x, & x &= 1. \end{aligned}$$

We now define two function x_1 and x_2 on the interval $(-1, \frac{1}{2})$:

$$x_1(t) = t + \frac{1}{2},$$

$$x_2(t) = \begin{cases} t + \frac{3}{2} & : \quad -1 < t \leq -\frac{1}{2}, \\ t + \frac{1}{2} & : \quad -\frac{1}{2} < t < \frac{1}{2}. \end{cases}$$

Notice that $x_1(0) = x_2(0) = \frac{1}{2}$ and that both functions are solutions of the impulsive differential equation. $x_1(t)$ has no discontinuities, $x_1 \neq 1$ on its interval of existence, and $\frac{dx_1}{dt} = 1$. As for $x_2(t)$, it has a discontinuity $\tau = -\frac{1}{2}$, where $x_2(\tau) = 1$. However,

$$x_2(\tau^+) = 0 = 1 - 1 = x_2(\tau) + \Delta x_2(\tau),$$

as required by Definition 5. Therefore, $x_2(t)$ is a solution. We therefore conclude that the initial-value problem $x(0) = \frac{1}{2}$ does not have a unique solution defined on the interval $(-1, 1/2)$, as both $x_1(t)$ and $x_2(t)$ satisfy the discontinuous flow and initial condition and are defined on said interval; see Figure A.1. This is especially surprising considering the equation is linear and the jump map is a bijection.

Even more surprising is the fact that the initial value problem $x(t_0) = 0$ does not have a unique solution for any t_0, no matter how small the domain of existence is. See Figure A.1; since this discontinuous dynamical system is autonomous, the figure can be freely "translated" so that the point where the two proposed solutions merge occurs at any $t_0 \in \mathbb{R}$.

As the above example suggests, the initial-value problem $x(0) = x_0$ does not in general possess a solution that is unique for $t < 0$ very far back in time, and if x_0 is in the range of the jump operator, there may not be a unique solution at all, even for small times $t \approx 0$. The following theorem and proof is adapted from a result in [2].

Theorem A.2.5 (Local existence and uniqueness of solutions for discontinuous flows)**.** Consider the discontinuous flow

$$\begin{aligned} \frac{dx}{dt} &= f(x), & \phi(x) &\neq 0, \\ \Delta x &= I(x), & \phi(x) &= 0. \end{aligned} \tag{A.2.1}$$

Denote $\Gamma = \phi^{-1}(0)$ and $J(x) = x + I(x)$. Suppose the following conditions are satisfied:

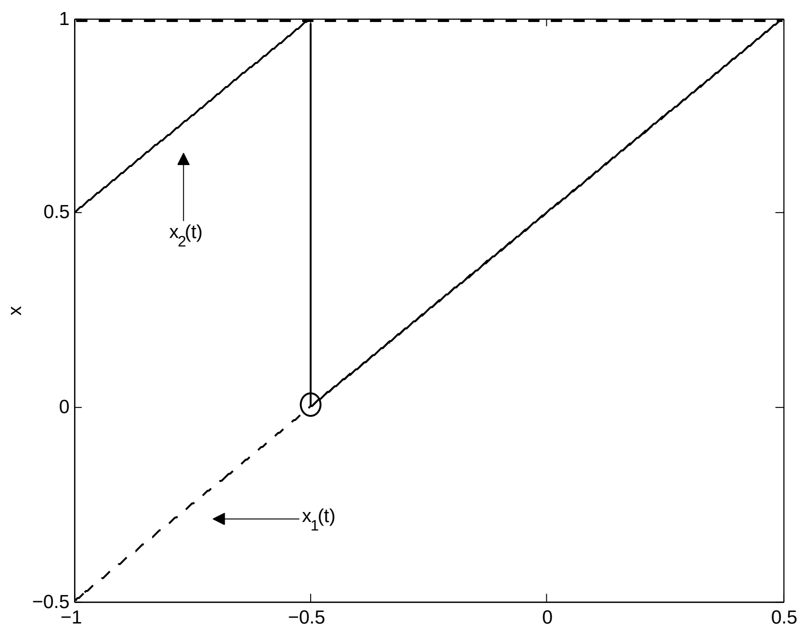

Figure A.1: A plot of the two solutions $x_1(t)$ and $x_2(t)$ given in Example A.2.4. Each solution is indicated by an arrow, the discontinuity surface is indicated by a dashed line at $x = 1$ and the point $(t, x) = (-0.5, 0)$ is circled. This is the point at which the two solutions “merge”.

- $\phi : \Omega \to \mathbb{R}$ is continuously differentiable on some ϵ-neighbourhood (Γ_ϵ) of Γ and $\nabla\phi(x) \neq 0$ for all $x \in \Gamma$,
- J is continuously differentiable on Γ_ϵ and $\det(D_x J) \neq 0$ for all $x \in \Gamma_\epsilon$,
- f is locally Lipschitz continuous in Ω,
- $\Gamma \cap J(\Gamma) = \emptyset$.

Then, for every $x_0 \in \Omega$, the initial-value problem $x(0) = x_0$ for (A.2.1) has a unique solution defined on an interval of the form $[0, \beta)$ for $\beta > 0$. If $x_0 \in \Omega \setminus J(\Gamma)$, the initial-value problem $x(0) = x_0$ for (A.2.1) has a unique solution defined on an interval of the form (α, β) for $\alpha < 0 < \beta$.

Proof: The first two conditions guarantee that Γ and $J(\Gamma)$ are C^1, $(n-1)$-dimensional submanifolds of Ω without boundary. There are now two cases to consider.

I: If $x_0 \in \Gamma$, set $\tilde{x}_0 = J(x_0)$. Since Γ and $J(\Gamma)$ are disjoint, we have $\tilde{x}_0 \notin \Gamma$. Since $J(\Gamma)$ is a smooth $(n-1)-$manifold, by part I, there exists a unique solution of the initial-value problem $x(0) = \tilde{x}_0$, defined on an interval (a, β). Denote this solution $y(t)$.

Let $z(t)$ denote the solution of the initial-value problem

$$\dot{x} = f(t, x), \qquad x(0) = x_0,$$

on an interval of the form (α, b) for $\alpha < 0 < b$. Now define

$$x(t) = \begin{cases} z(t) & : \quad \alpha < t \leq 0, \\ y(t) & : \quad 0 < t < \beta. \end{cases}$$

Then $x(t)$ has a discontinuity at $t = 0$, where $x(0) = x_0 \in \Gamma$. By continuity of $y(t)$, we have $x(0^+) = y(0) = \tilde{x}_0 = J(x_0)$, as required. The other defining qualities of a solution are easily verified. Uniqueness is obvious.

II: If $x_0 \notin \Gamma$, then since Γ is a smooth $(n-1)$-dimensional submanifold, there exists $\epsilon > 0$ such that $B_\epsilon(x_0) \cap \Gamma = \emptyset$, where $B_\epsilon(x_0)$ is the ball centered at x_0 with radius ϵ. Since f is locally Lipschitz continuous, standard existence and uniqueness theorems [15] guarantee that the initial-value problem $x(0) = x_0$ has a unique solution $y(t)$ defined on an interval of the form (a, b) for $a < 0 < b$, and such that $y(t) \in B_\epsilon(x_0)$. Since the aforementioned ball and the manifold Γ are disjoint, the solution never enters Γ, so there are no discontinuities.

Since this solution is the unique solution of the initial-value problem

$$\dot{x} = f(t, x), \qquad x(0) = x_0,$$

the restriction of this solution to $t \geq 0$ is automatically the unique solution of the impulsive differential equation defined on this interval. Uniqueness of it as a solution of the impulsive differential equation for $t < 0$ will follow if there are no other solutions that exhibit discontinuities on (a, b) and agree with $y(t)$ at $t = 0$. Assume now that $x_0 \notin \Gamma \cup J(\Gamma)$. Suppose $z(t)$ is another solution. Then $z(0) = x_0 \notin J(\Gamma) \cup J(\Gamma)$. This implies that $z(t)$ is continuous at $t = 0$, by the definition of solution. Therefore, $x(t)$ and $z(t)$ must agree on some subinterval $(\alpha, \beta) \subset (a, b)$ since both must solve the above initial-value problem. Uniqueness then follows by choosing the smaller interval (α, β).

∎

A.2.2 Dependence on initial conditions and parameters

Consider the impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, \lambda), \qquad && \phi(t, x, \lambda) \neq 0, \\ \Delta x &= I(t, x, \lambda), \qquad && \phi(t, x, \lambda) = 0, \end{aligned} \tag{A.2.2}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\lambda \in \Lambda \subset \mathbb{R}^m$ is a parameter. Let $x(t; \tau, y, \lambda)$ denote the solution of (A.2.2) satisfying the initial condition $x(\tau^+; \tau, y, \lambda) = y$. Let $(t_0, x_0, \lambda_0) \in \mathbb{R} \times \mathbb{R}^n \times \Lambda$ be such that $\phi(t_0, x_0, \lambda_0) \neq 0$, and let $\varphi(t) = x(t; t_0, x_0, \lambda_0)$ be a solution of (A.2.2) defined on an interval $[t_0, t_1]$, for which

$$\varphi(t_0) = x_0, \qquad \varphi(t_1, \varphi(t_1), \lambda_0) \neq 0.$$

Introduce the following conditions.

CA.1 The function $\phi : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}$ is C^1 and there exists a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$, the equation $\phi(t, x, \lambda) = 0$ defines a smooth hypersurface $S(\Lambda)$ that partitions the space $\mathbb{R} \times \mathbb{R}^n$ into a finite number of disjoint domains $D_1(\lambda), \ldots, D_{N+1}(\lambda)$;

$$\left[\bigcup_{k=1}^{N+1} D_k(\lambda)\right] \cup S(\lambda) = \mathbb{R} \times \mathbb{R}^n.$$

CA.2 The solution $\varphi(t)$ has moments of impulse effect at times $\tau_k = \tau_k(t_0, x_0, \lambda_0)$ $(k = 1, \ldots, N)$ and the relations

$$(t, \varphi(t)) \in D_k(\lambda_0), \qquad t \in \Delta_k$$

hold, where $\Delta_1 = [t_0, \tau_1)$, $\Delta_{N+1} = (\tau_N, t_1]$, $\Delta_k = (\tau_{k-1}, \tau_k)$.

CA.3 $\frac{\partial \phi}{\partial t}(\tau_k, \varphi(\tau_k), \lambda_0) + \frac{\partial \phi}{\partial x}(\tau_k, \varphi(\tau_k), \lambda_0) \cdot f(\tau_k, \varphi(\tau_k), \lambda_0) \neq 0$ for $k = 1, \ldots, N+1$.

Under the above three conditions, there exists $\delta > 0$ such that, for $|\lambda - \lambda_0| < \delta$, the sets

$$S_k(\lambda) = \{(t, x) : \phi(t, x, \lambda) = 0, |t - \tau_k| < \delta, |x - \varphi(\tau_k)| < \delta\}$$

are open, smooth n-dimensional manifolds, for $k = 1, \ldots, N$.

CA.4 There exists a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $|y - x_0| < \delta$, the solution $x(t) = x(t; t_0, y, \lambda)$ is defined for $t \in [t_0, t_1]$ and the curve $(t, x(t))$, $t \in [t_0, t_1]$, meet successively each $S_k(\lambda)$ $(k = 1, \ldots, N)$ just once.

CA.5 $f(t,x,\cdot)$ is continuous in Λ for each $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and the restriction of $f(\cdot,\cdot,\lambda)$ to $D_k(\lambda)$ is continuous for all $\lambda \in \Lambda$.

CA.6 The function I is continuous.

CA.7 $f(t,x,\cdot)$ is continuously differentiable in Λ for each $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and the restriction of $f(\cdot,\cdot,\lambda)$ to $D_k(\lambda)$ is continuously differentiable for all $\lambda \in \Lambda$

CA.8 The function I is continuously differentiable.

Theorem A.2.6. Suppose conditions CA.1-CA.6 hold. Then the solution $x(t;\tau,y,\lambda)$ of the initial-value problem associated to (A.2.2) is a continuous function in a neighbourhood of (t_1,t_0,x_0,λ_0). Moreover, the moments of impulse effect $\tau_k(\tau,y,\lambda)$ are continuous functions in a neighbourhood of (t_0,x_0,λ_0). If, in addition, conditions CA.7-CA.8 hold, then these functions are continuously differentiable in neighbourhoods of said points.

The above theorem is due to Bainov and Simeonov [5].

A.3 Linear, periodic systems with impulses at fixed times

Much of the theory of periodic solutions of nonlinear impulsive differential equations requires results concerning periodic linear systems. We collect here the necessary results on this topic, all of which are due to Bainov and Simeonov [5]. We begin with a definition.

Definition A.3.1. *Let $\{\tau_k\}$ be an increasing sequence indexed by $\mathbb{Z}$ with $\lim_{k\to\pm\infty}\tau_k = \pm\infty$, and $m,n,r \in \mathbb{N}$ with $D \subset \mathbb{R}$ and $F \subset \mathbb{R}^{n\times m}$ (or $F \subset \mathbb{C}^{n\times m}$). Denote $PC(D,F)$ the set of functions $\Psi : D \to F$ which are continuous on $D\backslash\{\tau_k\}$, are continuous from the left and have discontinuities of the first kind at points τ_k. The set $PC^r(D,F)$ is* *then the set of functions that are differentiable r times and whose rth derivative is in the set $PC(D,F)$.*

We will consider the *linear periodic IDE with impulses at fixed times*,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), \qquad && t \neq \tau_k, \\ \Delta x &= B_k x + h_k, && t = \tau_k. \end{aligned} \tag{A.3.1}$$

For the remainder of this chapter, the following conditions (H) will be assumed:

H1. $A(\cdot) \in PC(\mathbb{R}, \mathbb{C}^{n\times n})$ and there exists a minimal $T > 0$ such that $A(t+T) = A(t)$ for $t \in \mathbb{R}$.

H2. $B_k \in \mathbb{C}^{n\times n}$, $\det(E + B_k) \neq 0$ and $\tau_k < \tau_{k+1}$ for $k \in \mathbb{Z}$.

H3. There exists a minimal $q \in \mathbb{N}$ (the cycle number) such that

$$B_{k+q} = B_k, \qquad \tau_{k+q} = \tau_k + T \qquad (k \in \mathbb{Z}).$$

H4. $g(\cdot) \in PC(\mathbb{R}, \mathbb{C}^n)$ and $g(t+T) = g(t)$ for all $t \in \mathbb{R}$.

H5. $h_k \in \mathbb{C}^n$ and $h_{k+q} = h_k$ for all $k \in \mathbb{Z}$.

As a consequence of the linear structure of the equations, global existence and uniqueness can be readily established.

Theorem A.3.1. Let conditions (H) be satisfied, except possibly the condition $\det(E + B_k) \neq 0$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{C}^n$, there exists a unique solution $x(t)$ of equation (A.3.1) with $x(t_0^+) = x_0$ and this solution is defined for $t > t_0$. If, moreover, $\det(E + B_k) \neq 0$ $(k \in \mathbb{Z})$, then the solution is defined for all $t \in \mathbb{R}$.

A.3.1 Homogeneous equations

The homogeneous case must be treated first. This is the case where the function $g(t) = 0$ and $h_k = 0$ for all $k \in \mathbb{Z}$:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \qquad && t \neq \tau_k, \\ \Delta x &= B_k x, && t = \tau_k. \end{aligned} \tag{A.3.2}$$

The solutions of these equations satisfying the initial condition $x(t_0^+) = x_0$ can be written

$$x(t; t_0, x_0) = W(t, t_0^+)x_0,$$

where

$$W(t,s) = \begin{cases} U_k(t,s) & \text{for } t, s \in (\tau_k, \tau_{k+1}] \\ U_{k+1}(t,\tau_k^+)(E+B_k)U_k(\tau_k,s) & \text{for } \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1} \\ U_k(t,\tau_k)(E+B_k)^{-1}U_{k+1}(\tau_k^+,s) & \text{for } \tau_{k-1} < t \leq \tau_k < s \leq \tau_{k+1} \\ U_{k+1}(t,\tau_k^+)\prod_{j=k}^{i+1}(E+B_j)U_j(\tau_j,\tau_{j-1}^+)(E+B_i)U_i(\tau_i,s) & \text{for } \tau_{i-1} < s \leq \tau_i < \tau_k < t \leq \tau_{k+1} \\ U_i(t,\tau_i)\prod_{j=i}^{k+1}(E+B_j)^{-1}U_{j+1}(\tau_j^+,\tau_{j+1})(E+B_k)^{-1}U_{k+1}(\tau_k^+,s) & \text{for } \tau_{i-1} < t \leq \tau_i < \tau_k < s \leq \tau_{k+1} \end{cases}$$

and $U_k(t,s)$ is the Cauchy matrix for the linear homogeneous equation $\dot{x} = A(t)x$ with $t \in (\tau_{k-1}, \tau_k]$. We also have the following analogue of Liouville's formula for linear equations:

$$\det W(t,t_0) = \begin{cases} \prod\limits_{t_0<\tau_k<t} \det(E+B_k)\exp\Big(\int_{t_0}^t TrA(s)\,ds\Big) & \text{for } t > t_0 \\ \prod\limits_{t\leq\tau_k\leq t_0} \det(E+B_k)^{-1}\exp\Big(\int_{t_0}^t TrA(s)\,ds\Big) & \text{for } t \leq t_0\,. \end{cases}, \tag{A.3.3}$$

Definition A.3.2. *Let $x_1(t), ..., x_n(t)$ be solutions to* (A.3.2) *defined on the interval* $(0,\infty)$*. Let $X(t) = \{x_1(t), ..., x_n(t)\}$ be a matrix-valued function whose columns are these solutions. Then $x_1(t), ..., x_n(t)$ are linearly independent if and only if* $\det X(t_0^+) \neq 0$*. In this case, we say that $X(t)$ is a* fundamental matrix *of solutions of* (A.3.2).

A generalization of Floquet's theorem allows a convenient characterization of these fundamental matrices in the periodic case.

Theorem A.3.2. Each fundamental matrix of (A.3.2) can be represented in the form

$$X(t) = \varphi(t)e^{\Lambda t} \qquad (t \in \mathbb{R})$$

for a non-singular, T-periodic matrix $\varphi(\cdot) \in PC^1(\mathbb{R}, \mathbb{C}^{n\times n})$ and a constant matrix $\Lambda \in \mathbb{C}^{n\times n}$.

Remark A.3.1. To the fundamental matrix $X(t)$, there corresponds a unique matrix M such that $X(t+T) = X(t)M$ for all $t \in \mathbb{R}$. Here, M is called the *monodromy matrix* of equation (A.3.2). The eigenvalues $\mu_1, \ldots, \mu_n$ of M are called *Floquet multipliers* of (A.3.2). The eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ are called the *characteristic exponents* of (A.3.2). It follows that, to calculate the floquet multipliers, it suffices to choose a fundamental matrix $X(t)$ and calculate the eigenvalues of the matrix

$$M = X(t_0 + T)X^{-1}(t_0).$$

If $X(t)$ is normalized so that $X(0) = E$, then we may choose $M = X(T)$ as the monodromy matrix.

Theorem A.3.3. $\mu \in \mathbb{C}$ is a Floquet multiplier of (A.3.2) if and only if there exists a non-trivial solution $\varphi(t)$ such that $\varphi(t+T) = \mu\varphi(t)$ for all $t \in \mathbb{R}$.

Theorem A.3.4. Equation (A.3.2) has a non-trivial kT-periodic solution if and only if the kth power of at least one of its multipliers equals 1.

Now we consider stability of the linear T-periodic impulsive equation. Since impulses occur at fixed times, notions of stability are identical to that of ordinary differential equations. The multipliers of equation (A.3.2) completely characterize its stability; this is seen from the following theorem and the relation

$$\frac{1}{T}\ln|\mu_j| = Re(\lambda_j) \quad (j = 1, \ldots, n)$$

between the multipliers μ_j of the monodromy matrix M, and the real parts of the eigenvalues λ_j of the matrix Λ.

Theorem A.3.5. Suppose conditions H4.1-H4.3 hold. Then (A.3.2) is

1. stable if and only if all multipliers μ_j satisfy $|\mu_j| \leq 1$; for those multipliers for which $|\mu_j| = 1$, the corresponding characteristic exponent (which has zero real part) is a simple zero of the characteristic polynomial of Λ,

2. asymptotically stable if and only if all multipliers satisfy $|\mu_j| < 1$, and

3. unstable if $|\mu_j| > 1$ for some j.

A.3.2 Inhomogeneous equations

If $g(t)$ is not identically zero or $h_k \neq 0$ for some k, then (A.3.1) is said to be inhomogeneous. We shall investigate the question of existence of a T-periodic solution of this equation.

The variation of constants formula (see [5]) implies that the solution of (A.3.1) takes the form

$$x(t) = X(t)x(0) + \int_0^t X(t)X^{-1}(s)g(s)ds + \sum_{0\leq\tau_k<t} X(t)X^{-1}(\tau_k^+)h_k, \tag{A.3.4}$$

where $X(t) = W(t, 0)$ is the normalized (at $t_0 = 0$) fundamental matrix of equation (A.3.2). That is, $X(0) = E$.

The solution $x(t)$ will be T-periodic if $x(T) = x(0)$, or if

$$(E - X(T))x(0) = \int_0^T X(T)X^{-1}(s)g(s)ds + \sum_{0\leq\tau_k<T} X(T)X^{-1}(\tau_k^+)h_k. \tag{A.3.5}$$

Non-critical case: $\det(E - X(t)) \neq 0$

By Remark A.3.1, $M = X(T)$ is the monodromy matrix of the homogeneous equation (A.3.2) since $t_0 = 0$. This condition means that all multipliers of equation (A.3.2) are distinct from 1. Consequently, $E - X(T)$ is invertible, so equation (A.3.5) has a unique solution

$$x(0) = [E - X(t)]^{-1}\Big[\int_0^T X(T)X^{-1}(s)g(s)ds + \sum_{0\le\tau_k<T} X(T)X^{-1}(\tau_k^+)h_k\Big]. \quad \text{(A.3.6)}$$

Hence equation (A.3.1) has a unique T-periodic solution

$$\begin{aligned}\bar{x}(t) = & X(t)[E - X(T)]^{-1}\Big[\int_0^T X(T)X^{-1}(s)g(s)ds + \sum_{0\le\tau_k<T} X(T)X^{-1}(\tau_k^+)h_k\Big] \\ & + \int_0^t X(t)X^{-1}(s)g(s)ds + \sum_{0\le\tau_k<t} X(t)X^{-1}(\tau_k^+)h_k.\end{aligned} \quad \text{(A.3.7)}$$

Theorem A.3.6. Let the homogeneous equation (A.3.2) have no non-trivial T-periodic solutions. Then the non-homogeneous equation (A.3.1) has a unique T-periodic solution $\bar{x}(t)$.

Remark A.3.2. If all multipliers μ_j of equation (A.3.2) are such that $|\mu_j| < 1$, $j = 1, \ldots, n$, then the T-periodic solution $\bar{x}(t)$ of (A.3.1) is exponentially stable. If there is a multiplier with $|\mu_j| > 1$, then the periodic solution is unstable.

Critical case: $\det(E - X(T)) = 0$

To the homogeneous equation (A.3.2) we associate the *adjoint equation*

$$\begin{aligned}\frac{dy}{dt} &= -A^*(t)y \qquad && t \ne \tau_k \\ \Delta y &= -(E + B_k^*)^{-1}B_k^*y \qquad && t = \tau_k,\end{aligned} \quad \text{(A.3.8)}$$

where $A^*(t)$ is the conjugate transpose of $A(t)$.

Conditions for the existence of periodic solution of the inhomogeneous equation (A.3.1) in the critical case are established by the following theorem.

Theorem A.3.7. Let the homogeneous equation (A.3.2) have m linearly independent T-periodic solutions $\varphi_1(t), \ldots, \varphi_m(t)$ $(1 \le m \le n)$. Then:

1. The adjoint equation (A.3.8) also has m linearly independent T-periodic solutions $\psi_1(t), \dots, \psi_m(t)$.

2. Equation (A.3.1) has a T-periodic solution if and only if the following conditions are met
$$\int_0^T \psi_j^*(t) g(t) dt + \sum_{0 \le \tau_k < T} \psi_j^*(\tau_k^+) h_k = 0 \qquad (j = 1, \dots, m). \tag{A.3.9}$$

3. If condition (A.3.9) is met, then each T-periodic solution of equation (A.3.1) has the form
$$x(t) = c_1 \varphi_1(t) + \cdots + c_m \varphi_m(t) + x_p(t),$$
where $x_p(t)$ is a particular T-periodic solution of (A.3.1).

4. If condition (A.3.9) is met, then equation (A.3.1) has a unique T-periodic solution $\bar{x}(t)$ that satisfies the condition
$$(\varphi_i(0) | \bar{x}(0)) = 0 \qquad (i = 1, \dots, m).$$

A.4 Periodic solutions of non-linear equations

This section will be devoted to the presentation of notions of stability for non-linear impulsive differential equations and sufficient conditions that may be used to establish stability of periodic orbits (if these periodic orbits can be found). Perturbation of periodic solutions with respect to parameters is also covered. The proof of many of the following theorems may be found in [5] and others in the literature cited therein.

Nonlinear equations with impulses at fixed times

In the following, we consider the nonlinear impulsive differential equation
$$\begin{aligned} \frac{dx}{dt} &= f(t, x, \epsilon), \qquad && t \neq \tau_k(\epsilon) \\ \Delta x &= I_k(x, \epsilon), && t = \tau_k(\epsilon), \end{aligned} \tag{A.4.1}$$

where $\tau_k < \tau_{k+1}$ and $\epsilon \in (-\bar{\epsilon}, \bar{\epsilon})$ is a small parameter.

Suppose that, for $\epsilon = 0$, equation (A.4.1) has a T-periodic solution $\varphi(t)$. Associate with $\varphi(t)$ the variational equation

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}(t, \varphi(t), 0)z, && t \neq \tau_k(0), \\ \Delta z &= \frac{\partial I_k}{\partial x}(\varphi(\tau_k(0), 0)z, && t = \tau_k(0). \end{aligned} \tag{A.4.2}$$

The following theorem is valid.

Theorem A.4.1. Let the following conditions hold:

1: Equation (A.4.1) is T-periodic with cycle number q for all $\epsilon \in (-\bar{\epsilon}, \bar{\epsilon})$.

2: For $\epsilon = 0$, equation (A.4.1) has a T-periodic solution $\varphi(t)$.

3: The function $\tau_k(\epsilon)$ is differentiable in a neighbourhood of $\epsilon = 0$, and $\tau_0(0) < 0 < \tau_1(0)$.

4: There exists $\delta > 0$ such that, for any $\epsilon \in (-\delta, \delta)$ and $x_0 \in \mathbb{R}^n$ with $|x_0 - \varphi(0)| < \delta$, the solution $x(t; x_0, \epsilon)$ of the initial-value problem $x(0; x_0, \epsilon) = x_0$ associated to (A.4.1) is defined for $t \in [0, T]$, and, for the function $x(t; x_0, \epsilon)$, Theorem A.2.6 describing continuity on (t, x_0, ϵ) and differentiability on x_0 is valid in some neighbourhood of $(T, \varphi(0), 0)$.

5: The variational equation (A.4.2) has no non-trivial T-periodic solutions.

Then there exists $\epsilon_0 \in (0, \bar{\epsilon})$ such that, for $|\epsilon| < \epsilon_0$, equation (A.4.1) has a unique T-periodic solution $\varphi_\epsilon(t)$ such that, for $t \in [0, T]$,

$$\varphi_\epsilon \xrightarrow{B} \varphi \qquad \text{as} \qquad \epsilon \to 0.$$

Remark A.3.3 Note that, if the variational equation (A.4.2) is exponentially stable, then the T-periodic solution of (A.4.1) is also exponentially stable. By extension, if all of its multipliers satisfy $|\mu_j| < 1$, then $\varphi(t)$ is exponentially stable, and, since the linearized equation satisfies the conditions of continuity with respect to parameters, the multipliers $\mu_j(\epsilon)$ of the periodic solution φ_ϵ are continuous in ϵ near $\epsilon = 0$. The perturbation solution $\varphi_\epsilon(t)$ is then exponentially stable in some neighbourhood of $\epsilon = 0$.

The notation $\xrightarrow{B}$ denote B-convergence. See [5] for a precise definition. It is essentially a form of pointwise convergence that allows some flexibility when dealing with the fact that two solutions may not have impulses at the same time. Qualitatively, this convergence means that the impulse times τ_k must converge to those of φ, the domains on which the solutions are defined must, in the limit, agree up to a set of measure zero, and we must have pointwise convergence of the solutions at times t where their domains intersect and whenever t is not too close to an impulse time of φ.

Nonlinear autonomous equations

Consider the autonomous impulsive equation

$$\begin{aligned} \frac{dx}{dt} &= f(x, \epsilon), && x \notin \sigma(\epsilon), \\ \Delta x &= I_k(x, \epsilon), && x \in \sigma(\epsilon), \end{aligned} \tag{A.4.3}$$

where $\epsilon \in J = (-\bar{\epsilon}, \bar{\epsilon})$ is a small parameter and, for each $\epsilon \in J$, the set $\sigma(\epsilon)$ is a hypersurface in $\mathbb{R}^n$.

Suppose that $\sigma(\epsilon)$ consists of q non-intersecting smooth hypersurfaces $\sigma_k(\epsilon)$ which are given by the equation $\phi_k(x, \epsilon) = 0$ $(k = 1, \ldots, q)$. For $\epsilon = 0$, let equation (A.4.3) have a T_0-periodic solution $x = \varphi(t)$ with moments of an impulse effect τ_k and

$$\begin{aligned} \tau_{k+q} &= \tau_k + T_0 && (k \in \mathbb{Z}), \\ \phi_k(\varphi(\tau_k)) &= 0 && (k = 1, \ldots, q). \end{aligned}$$

Associate with the solution $\varphi(t)$ the variational equation

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}(\varphi(t),0)z \qquad && t \neq \tau_k \\ \Delta z &= N_k z && t = \tau_k, \end{aligned} \tag{A.4.4}$$

where

$$N_k = \frac{\partial I_k}{\partial x} + \left[f^+ - f - \frac{\partial I_k}{\partial x} f \right] \frac{\dfrac{\partial \phi}{\partial x}}{\dfrac{\partial \phi}{\partial x} f}$$

and

$$\begin{aligned} f &= f(\varphi(\tau_k),0), \qquad & f^+ &= f(\varphi(\tau_k^+),0), \\ \frac{\partial I_k}{\partial x} &= \frac{\partial I_k}{\partial x}(\varphi(\tau_k),0), & \frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial x}(\varphi(\tau_k),0). \end{aligned}$$

Theorem A.4.2. Let the following conditions hold.

1. For $\epsilon = 0$, equation (A.4.3) has a T_0-periodic solution $x = \varphi(t)$ with moments of an impulse effect τ_k satisfying $\tau_{k+q} = \tau_k + T_0$ $(k \in \mathbb{Z})$ and $\varphi'(t) \not\equiv 0$ $(t \in \mathbb{R})$.

2. For each $k = 1, \ldots, q$, the function $\phi_k(x, \epsilon)$ is differentiable in some neighbourhood of the point $(\varphi(\tau_k), 0)$ and

$$\begin{aligned} \phi_k(\varphi(\tau_k),0) &= 0, \\ \frac{\partial \phi_k}{\partial x}(\varphi(\tau_k),0) f(\varphi(\tau_k),0) &\neq 0. \end{aligned} \tag{A.4.5}$$

3. There exists $\delta > 0$ such that, for each $\epsilon \in (-\delta, \delta)$ and $x_0 \in \mathbb{R}^n$, $|x_0 - \varphi(0)| < \delta$, the solution $x(t; x_0, \epsilon)$ of equation (A.4.3) is defined for $t \in [0, T_0 + \delta]$ and, for the function $x(t; x_0, \epsilon)$, Theorem A.2.6 describing continuity on (t, x_0, ϵ) and differentiability on x_0 is valid in some neighbourhood of the point $(T_0, \varphi(0), 0)$.

4. The variational equation (A.4.4) has no nontrivial T_0-periodic solution other than $\varphi'(t)$.

Then there exists $\epsilon_0 \in (0, \bar{\epsilon})$ such that, for $|\epsilon| \leq \epsilon_0$, equation (A.4.3) has a unique periodic solution $x_\epsilon(t)$ with period $T(\epsilon)$ and moments of the impulse effect $t_k(\epsilon)$ such that for $t \in [t, T_0]$

$$x_\epsilon(t) \xrightarrow{B} \varphi(t), \quad T(\epsilon) \to T_0 \quad \text{and} \quad t_k(\epsilon) \to \tau_k \quad \text{as} \quad \epsilon \to 0. \tag{A.4.6}$$

Let $x = \gamma(t)$, $t \in \mathbb{R}_+$ be a solution of (A.4.3), with instants of impulsive effect τ_k, such that

$$\tau_k < \tau_{k+1}; \qquad \lim_{k\to\infty} \tau_k = \infty$$

and let $L_+ = \{x \in \mathbb{R}^2 : x = \gamma(t), t \in \mathbb{R}_+\}$. Denote by $J^+(t_0, x_0)$ the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, z_0)$ of (A.4.3) is defined.

Let $B_\eta(\gamma(\tau_1))$ be the ball of radius η centered at $\gamma(\tau_1)$.

Definition A.4.1. *The solution $x = \gamma(t)$ of (A.4.3) is called*

1. orbitally stable *if, for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$ such that, for all $x_0 \in \mathbb{R}^n$, $d(x_0, L^+) < \delta$ and $x_0 \notin \bar{B}_\eta(\gamma(\tau_k)) \cup \bar{B}_\eta(\gamma(\tau_k^+))$ implies $d(x(t), L^+) < \epsilon$ for $t \in J^+(t_0, x_0)$ and $|t_0 - t_k| > \eta$, where $x(t) = x(t; t_0, x_0)$ is any solution of (A.4.3) for which $x(t_0^+; t_0, x_0) = x_0$.*

2. orbitally attractive *if, for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$ and $T > 0$ such that $t_0 + T \in J^+(t_0, x_0)$, $d(x_0, L^+) < \delta$ and $x_0 \notin \bar{B}_\eta(\gamma(\tau_k)) \cup \bar{B}_\eta(\gamma(\tau_k^+))$ implies $d(x(t), L^+) < \epsilon$ for $t \geq t_0 + T$, $t \in J^+(t_0, x_0)$ and $|t_0 - \tau_k| > \eta$, where $x(t) = x(t; t_0, x_0)$ is any solution of (A.4.3) for which $x(t_0^+; t_0, x_0) = x_0$.*

3. orbitally asymptotically stable *if it is orbitally stable and orbitally attractive.*

Definition A.4.2. *The solution $x = \gamma(t)$ of (A.4.3) has the property of* asymptotic phase *if for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$, $c > 0$ and $T > |c|$ such that $t_0 + T \in J^+(t_0, x_0)$ and $|x_0 - \gamma(t_0)| < \delta$ implies $|x(t + c) - \gamma(t)| < \epsilon$ for $t \geq t_0 + T$, $t \in J^+(t_0, x_0)$ and $|t_0 - \tau_k| > \eta$, where $x(t + c) = x(t + c; t_0, x_0)$ is any solution of (A.4.3) for which $x(t_0^+; t_0, x_0) = x_0$.*

Theorem A.4.3. For $\epsilon = 0$, let the conditions of Theorem 2.17 hold and let the multipliers μ_j of equation (A.4.4) satisfy the conditions

$$\mu_1 = 1, \qquad |\mu_j| < 1 \qquad (j = 2, \ldots, n).$$

Then the T_0-periodic solution $x = \varphi(t)$ of (A.4.3) with $\epsilon = 0$ is orbitally asymptotically stable and enjoys the property of asymptotic phase.

Remark A.3.4. Similarly to Remark A.3.3, the above property holds in some neighbourhood of $\epsilon = 0$.

Appendix B

Continuous hybrid systems and impulsive differential equations

We elaborate on the proposition appearing in the introduction that autonomous impulsive differential equations arise from hybrid systems under time-scale separation assumptions. The discussion is very informal, and the conclusions should be interpreted as being non-rigorous.

Suppose the system to be modelled can be decomposed into a "background" process and a "switching" process, where this switching occurs when certain spatio-temporal conditions are satisfied. One model of such a system could consist of a hybrid system of differential equations

$$\begin{aligned} \dot{x} &= \mathrm{bg}(t,x) + f_k^{\epsilon}(t,x), && \phi_k(t,x) \neq 0, \\ f_k^{\epsilon} &\to f_{k+1}^{(t,x)}, && \phi_k(t,x) = 0, \end{aligned}$$

where

$$\mathrm{bg} : \mathbb{R} \times \Omega \to \Omega$$

is the background vector field,

$$f_k^{\epsilon} : \mathbb{R} \times \Omega \to \Omega$$

is a sequence of vector fields indexed by $k \in \mathbb{N}$ and parameterized by $\epsilon \in \mathbb{R} \times \Omega$, and

$$\phi_k : \mathbb{R} \times \Omega \to \mathbb{R}$$

is a sequence of "switching laws". The trajectories of such a system are then defined as follows. For the initial-value problem $x(0) = x_0$, if $\phi_0(0, x_0) \neq 0$, then the system evolves according to

$$\dot{x} = \mathrm{bg}(t, x) + f_0^{(0,x_0)}(t, x).$$

Let $x(t)$ denote the solution. If there is ever a time $\tau_0 \geq 0$ such that $\phi_0(\tau_0, x(\tau_0)) = 0$, then the vector field "switches" to

$$\dot{x} = \mathrm{bg}(t, x) + f_1^{(\tau_0, x\tau_0)}(t, x),$$

where $x\tau_0 \equiv x(\tau_0)$. The system then evolves according to the above differential equation. This switching of vector fields repeats; if ever the condition $\phi_k(\tau, x(\tau)) = 0$ is satisfied, then the switching of indexed vector fields

$$f_k^\epsilon \to f_{k+1}^{(\tau, x\tau)}$$

takes effect.

Impulsive differential equations may arise in an effort to simplify the description of the switching process. Assume the switching vector fields f_k^ϵ are functions of t alone; $f_k^\epsilon(t, x) = f_k^\epsilon(t, y) = f_k^\epsilon(t)$ for all $x, y \in \Omega$. These vector fields may then be considered as time-dependent "perturbations". Let $\gamma(t; t_0, \gamma_0)$ denote the solution of the initial-value problem $\gamma(t_0) = \gamma_0$ of the background process $\dot{\gamma} = \mathrm{bg}(t, \gamma)$. We propose the following simplification criteria.

Definition B.0.1. **(r-small impulsive simplification criteria):** *The switching system satisfies the* r-small impulsive simplification criteria *if, for all* $(\tau, z) \in \mathbb{R}^+ \times \Omega$ *and all* $k \in \mathbb{N}$ *for which* $\phi_k(\tau, z) \neq 0$*, there exists* $r > 0$ *and* $I_k(\tau, x)$ *such that*

- $\mathrm{supp}\left(f_k^{(\tau,z)}\right) \subset [\tau, \tau + r) := R_k(\tau, z)$,

- $\int f_k^{(\tau,z)} = I_k(\tau, x)$ *and* $\phi_k(\tau, z + I_k(\tau, z)) \neq 0$,
- $\inf\{t > \tau : \phi_k(t, \gamma(t)) = 0$ where $\gamma(t) = \gamma(t; \tau, z + I_k(\tau, z))\} \geq \tau + r$.

We now summarize and explain heuristically the significance of these conditions.

First condition: perturbation effects last a finite amount of time. The first condition requires that the perturbative vector field $f_k^{(\tau,z)}(t)$ is eventually zero.

Second condition: assuming no background dynamics, the effect of a perturbation is finite, and instantaneously perturbing the system by the increment associated to said perturbation does not instantaneously cause another switching of vector fields. The second condition states that the effect of the perturbation f_k^{ϵ}, ignoring the background dynamics, is a function of the parameter ϵ alone and is given by the vector $I_k(\epsilon)$. This comes about as follows: the solution of the initial-value problem

$$\begin{aligned} \dot{x} &= f_k^{(\tau,z)}(t), \\ x(\tau) &= z, \end{aligned}$$

for $t \geq \tau$ is precisely

$$z + \int_\tau^t f_k^{(\tau,z)}(s)ds.$$

Therefore, $I_k(z, \tau)$ is the displacement of z after the effect of the perturbation $f_k^{(\tau,z)}$ has completed, neglecting background dynamics. The second part is obvious; there is no switching time at τ if z is replaced by $z + I_k(\tau, z)$.

Third condition: if the effect of a perturbation is made artificially instantaneous, then the next switching time cannot occur in some half-closed neighbourhood of the support of said perturbation. The third condition may be interpreted as follows. Suppose $\phi_{k-1}(\tau, z) = 0$ and $\phi_k(\tau, z) \neq 0$. If the perturbative effect $f_k^{(\tau,z)}$ is assumed to occur instantaneously (that is, we synthetically "stop" the background dynamics, allow the perturbation to be applied and then "restart" the background dynamics, but set the clock back to τ), then $z + I_k(\tau, z)$ is the state of the system after the perturbation has occured. The condition then states that the next switching time must not occur in the interval $R_k(\tau, z)$ containing the support of $f_k^{(\tau,z)}$.

The above definition is very informal, but is seen to be similar in spirit to the typical assumptions made in modelling with impulsive differential equations. See the introduction for supporting information. The definition of r-small impulsive simplification criteria given above is tied to these assumptions as follows. Suppose (τ, z) is a switching time; $\phi_{k-1}(\tau, z) = 0$. If $\phi_k(\tau, z) \neq 0$, then, by passing to the equivalent integral representation of solutions, we have

$$x(t) = z + \int_\tau^t f_k^{(\tau,z)}(s)ds + \int_\tau^t \mathrm{bg}(s, x(s))ds.$$

When $t = \tau + r$, the second condition implies

$$x(\tau + r) = z + I_k(\tau, z) + \int_\tau^{\tau+r} \mathrm{bg}(s, x(s))ds.$$

Now, define

$$e(r) = \int_\tau^{\tau+r} \mathrm{bg}(s, x(s))ds.$$

If τ_{k+1} is the next switching time, then the first condition implies $x(t) = x(\tau + r) + \int_{\tau+r}^t \mathrm{bg}(s, x(s))ds$ for $\tau + r < t \leq \tau_{k+1}$. Thus

$$x(t) = z + I_k(\tau, z) + e(r) + \int_{\tau+r}^t \mathrm{bg}(s, x(s))ds$$

for $\tau + r < t \leq \tau_{k+1}$.

The hope is now the following: if r is small (that is, perturbations occur on a fast time scale), then $e(r) \approx 0$. Assuming this idea can be expressed rigorously, then, for r small,

$$x(t) \approx z + I_k(\tau, z) + \int_\tau^t \mathrm{bg}(s, x(s))ds$$

for $t > \tau$. The right-hand side of the above expression approaches $z + I_k(\tau, z)$ as $t \to \tau^+$. By the second part of condition 2, $\phi_k(\tau, z + I_k(\tau, z)) \neq 0$, so there is no obvious problem with the solution here.

The object on the right is the solution of the impulsive differential equation with variable structure

$$\begin{aligned} \dot{x} &= \mathrm{bg}(t,x), && \phi_k(t,x) \neq 0, \\ \Delta x &= I_k(t,x), && \phi_k(t,x) = 0. \end{aligned}$$

with initial condition $x(\tau) = z$. Equations similar to this type (and in a sense more general) were considered in [12]. One obtains an impulsive differential equation of the type studied in [3, 5, 16, 29] by assuming that $I_k(t,x)$ does not depend on t. Indeed, then, the three typical classes of impulsive differential equation are obtained by specifying how the switching functions $\phi_k(t,x)$ behave. If impulses occur at fixed times τ_k, then $\phi_k(t,x) = t - \tau_k$. If impulses are unfixed with functions $\tau_k(x)$, then $\phi_k(t,x) = t - \tau_k(x)$. If impulses are autonomous, then $\phi(t,x) = \phi(x)$ for some smooth ϕ.

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