NEW FINITE-STATE APPROACHES ON RELIABILITY MODELING

by

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Submitted to the Department of Electrical Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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June, 1973.

ABSTRACT

In this thesis, finite-state models for components and systems are presented. For a component, its reliability behavior is described by a Markov model with two or more states, rather than just two states, " good" and " bad", as often used in the literature. Thus a new reliability function is defined. A general birth and death process is also defined by the same Markov model with some modifications.

Reliability Prediction Techniques for a system of dependent failure and repair rates are given. Signal-flow graph approach is used to find reliability expressions.

ACKNOWLEDGEMENTS

The author is most grateful to his advisor, Professor C.L. Sheng, for his encouragement, understanding and guidance throughout the course of the research.

He is indebted to Dr. W.K. Chung for stimulating discussions and critical comments on various aspects of this work.

He wishes to express thanks to Prof. C. Lemyre, the Chairman of the Department of Electrical Engineering, University of Ottawa, for his encouragement and interest, and to Prof. W. Steenaart and Prof. J. V. Marsh for their useful suggestions.

Thanks are due to the members of the staff and the graduate students of the Electrical Engineering Department, for their friendship throughout the years of the research.

He also wishes to acknowledge the financial assistance of the National Research Council of Canada, and the Department of Electrical Engineering, University of Ottawa.

Finally he wishes to thank Madame L. LeBlanc for her skillful typing and patience .

Notations

n = either the number of operation states or the number of components in a r-out-of-n system.

s_i = either the operative state, $i = 1, \ldots n$, or the collapsed state in which i components have failed, $i = 0, 1, \ldots, n-r, n-r+1$.

s_f = the failure state.

λ_i = the transition rate from s_i to s_{i+1}, $i = 1, \ldots, n-1$, $\lambda_n = 0$.

λ_{n+i} = the transition rate from s_i to s_f, $i = 1, \ldots, n$.

λ_t = the set of parameters, $\lambda_t = (\lambda_1, \ldots, \lambda_{n-1})$

λ_f = the set of parameters, $\lambda_f = (\lambda_{n+1}, \ldots, \lambda_{2n})$

λ = the set of parameters, $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots, \lambda_{2n})$.

$P_i(t)$ = the probability that a system is in s_i at time t, $i = 1, \ldots, n$.

$P_f(t)$ = the probability that a system is in s_f at time t.

$R(t)$ = the reliability function; $R(t) = \sum_{i=1}^{n} P_i(t)$.

$h(t)$ = the hazard rate.

$h_e(t)$ = the expected hazard rate.

$h_e(t)$ = time-interval (piece-wise constant) hazard rate.

$\hat{h}$ = the midrange of h_j's ; $\hat{h} = \frac{1}{2}(h_{max} + h_{min})$.

I_i = the pseudo-interval, $i = 1, \ldots, n$.

a, b = the two adjustable parameters.

ω_i = the (initial value) transition rate from s_i to s_{i+1}, $i = 1, \ldots, n-1$, $\omega_n = 0$; $\lambda_i = a\,\omega_i$.

μ_{n+i} = the sample average of hazard rates (of the pseudo-interval I_i), $i = 1, \ldots, n-1$; $\lambda_{n+i} = b\,\omega_{n+i}$.

$P_{ik}(t)$ = the probability that a system enters s_k before time t, given that the system is initially in s_i.

$\hat{P}_{ik}(s)$ = the Laplace transform of $P_{ik}(t)$.

$R_i(t)$ = the (conditional) reliability of a system at time t, given that the system is initially in s_i.

μ_i the probability rates of repair from state s_i to state s_{i-1}, $i = 2, 3, \ldots, n$.

a_{t+1} the death rate of people at age $(t, t+1)$.

β_{t+1} the birth rate of people at age $(t, t+1)$.

r = the minimum number of operating components for a r-out-of-n system operating.

x_i = a notation representing success of component i

$\overline{x}_i$ = a notation representing failure of component i

s_i^j = the i-th state of j failure set of states (Table 5.1), $i = 1, 2, \ldots, n^j$ where $n^j = C_j^n$, $j = 0, 1, \ldots, n$.

s^j = the j-th failure set of states $s^j = (s_1^j, s_2^j, \ldots, s_{n^j}^j)$

$P_{s_i^j}$ = the probability of being in state s_i^j.

λ_i^j = the failure rate of component i when $j (n-r \geq j \geq 0)$ components have failed.

μ_i^j = the repair rate of failed component i when $j (n-r+1 \geq j \geq 1)$ components have failed.

λ^j = the probability rate of transition from state s_j to state s_{j+1}, i.e., the failure rate that any one of the remaining n-j components will fail when j components have failed, $j = 0, 1, \ldots, n-r$.

μ^j = the probability rate of transition from state s_j to state s_{j-1}, i.e., the repair rate that any one of the j failed components is repaired to operating condition given that j components have failed, $j = 1, 2, \ldots, n-r+1$

ω^j = $\sum_{k=j}^{n-1} \lambda^j_{i_k}$ intermediate parameters (page 70), $j = 0, 1, \ldots, n-r$.

i_m = the m+1-th failed component ($n-r \geq m \geq 0$).

$p^j_{i_0 i_1 \ldots i_{j-1} i_j}$ = $\lambda^j_{i_j} / (\omega^j)$; the probability that the j+1-th failed component is component i_j given that components $i_0, i_1, \ldots, i_{j-1}$ have failed, $j = 1, 2, \ldots, n-r$.

$P_i(t)$ = the probability that the system is in state s_i at time t given that the system is in state s_0 at time 0, $i=0, 1, 2, \ldots, n-r+1$.

$P_f(t)$ = the probability that the system is in the failure state s_f at time t, where $s_f = s_{n-r+1}$.

$\hat{P}_i(s)$ = the Laplace transform of $P_i(t)$.

$\hat{P}_f(s)$ = the Laplace transform of $P_f(t)$

P_i = the steady-state probability that the system is in state s_i.

$R(t)$ = the reliability function

$\hat{R}(s)$ = the Laplace transform of $R(t)$

$A(t)$ = the availability function

$\hat{A}(s)$ = the Laplace transform of $A(t)$

A_v = the steady-state availability

T_m = mean-time-to-first-system-failure.

Δ = the determinant of a signal flow graph.

Δ_{sf} = the co-factor of the path s-f.

T_{sf} = the path transmission of the path s-f.

L_i = the loop transmission.

C^n_j = combinations of j out of n.

P^n_j = permutations of j out of n.

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CHAPTER I

INTRODUCTION

Reliability of an item [1] is the probability of the item performing its purpose adequately for the period of time intended under the operating conditions encountered. Reliability theory has originated from practical technical problems in connection with the appearance of complex automatic systems containing a large number of components. Mathematical reliability theory contains a body of ideas, mathematical models, and methods directed toward solving problems in predicting, estimating, or optimizing the life distribution of components or systems. Finite-state Markov models are often used in the reliability analysis of systems.

Life test data or simply failure data are obtained from two sources, namely the failure times of various items in a population placed on a life test under normal or accelerated environmental conditions, and the repair reports listing operating hours of replaced items in equipment already in field use. It might appear that if life test data are available, there is no need for a mathematical model. However, in obtaining conclusions from life test data on the reliability behavior of other similar items, it is useful to describe the data with a mathematical model. So far, many mathematical models have been constructed [2,3]. The choice of a reliability model must encompass more than just the problem of fitting a curve with a formula. If possible, one should choose a model in which parameters have physical meaning.

In Chapter II the existing reliability functions are reviewed. Of these existing functions the Weibull functions are the most popular parametric family of failure models. It is known that the Weibull distribution has monotonic hazard rate,

$$h(t) = kt^m \quad , \quad m > -1 \qquad (1.1)$$

k is scale parameter

m is shape parameter

For most items the failure mechanism in time is best represented by a hazard rate curve in the form of bathtub. The monotonic character of the hazard rate of Weibull functions is in contrast with the bathtub curve, as Gorski [4] points out that practical considerations are not conducive to the acceptance of the Weibull functions as an analytic tool. If the Weibull function, (1.1), is going to be fitted to the wear-out period of a bathtub curve, then m must be greater than 0, whereas h(t) becomes zero at t = 0. Conversely if it is to be fitted to the burn-in period, then $m < 0$, and h(t) becomes initially very large and goes to zero as time increases indefinitely. Thus if a mathematical description of the bathtub curve is desired, it is justifiable to develop a new function.

In Chapter III, a new reliability function (with 2n-1 parameters) is defined, based on a special Markov model, in which parameters represent transition rates. To eliminate the difficulty of finding estimates of parameters, the concept of pseudo-intervals which correspond with states of the special Markov model is introduced.

In Chapter IV, the special Markov model is extended to a general finite-state birth and death model for a general birth and death process. A birth and death process is a stationary Markov process whose path functions X(t) are nonnegative integer values and whose transition probability functions for any two non-consecutive states are zero. The process has been defined by Feller and others [6, 7, 8]. In a reliability study, the repair and failure rates are equivalent to the birth and death rates and the state number i is equivalent to a value i of the path function X(t). Then for a finite-state (n+1 states) Markov model, i.e., $2 < X(t) \leq n+1$, the extended Markov model is a general model [9] for a birth and death process. The model is general in the sense that it covers the "conventional" birth and death model as a special case.

The matrix method of solving a system of simultaneous equations can be transformed into a topological method by the application of signal flow graphs. The method can be used for any linear discrete physical system that may be represented by a set of simultaneous equations. Topological methods of solving simulatneous equations were developed by Mason [10, 11] and Tustin and Percival [12, 13, 14]. These techniques have been applied to reliability studies [15, 16, 17, 18]. The linear signal-flow graph was extended to the stochastic transition signal-flow graph by Huggins [19]. Recently, Chan and Chung [20] applied a stochastic transition signal-flow graph to queuing systems. A summary of linear signal-flow approach is given in Chapter II. Since, in this thesis, only the linear signal-flow graph is applied to our problems, we will not review the stochastic transition signal-flow graph.

Reliability theory has been concerned to a great extent with the formulation of mathematical models to describe complex system reliability. Usually components in the system are assumed to have failure rates independent of their environments, and repair rates do not change with states of the system, so that a simple Markov model can be postulated. From experience and engineering judgement it was found that this kind of independence assumption often leads to a poor approximation. Shooman [21] presented a technique for the formulation of system models including dependence between components. The technique uses Markov chains for constructing reliability models for dependent system with n elements. After identifying 2^n system states, one ends up with determining 2^{2n} transition probabilities and finally solving 2^n first order differential equations. It is observed that the number of system states increases greatly as the number of components increases, which is not realistic even if using a computer to solve for the system reliability. Often reliability considerations of physical systems or subsystems suggest a r-out-of-n model, in which at least r out of n components must be functioning for the system to be considered operative. In Chapter V, reliability prediction techniques for a system with dependent failure and repair rates are proposed. In our model the number of system states is n-r+2, corresponding to sets of states of 0-failures, 1 failure, 2 failures, ..., n-r+1 failures, rather than 2^n, which greatly reduces complexity of the previous approach.

CHAPTER II

SUMMARIES OF RELIABILITY FUNCTIONS AND SIGNAL FLOW GRAPH AND SURVEY OF APPROACHES ON COMPUTING SYSTEM RELIABILITY

Section 2.1 reviews reliability functions which are used most frequently in reliability theory, and Section 2.2 summarizes the signal-flow graph and its applications to reliability studies. Section 2.3 surveys methods of computing the reliability of series, parallel, series-parallel systems and Techniques of approximating complex system reliability.

2.1 Existing Reliability Functions

This section describes distributions which are used most frequently in reliability.

Several functions are equally suitable for describing a failure distribution, defined as follows :

X :	A nonnegative real random variable that denotes the time to failure of an item.
$F(x)$:	The cumulative distribution function (CDF).
$R(x)$:	The reliability function, $R(x) + F(x) = 1$.
$f(x)$:	The probability density function (pdf) when it exists.
$h(x)$:	The hazard rate (or the conditional failure rate).
$\Phi(x,t)$:	$Pr \{ X \leq x \mid X \geq t \}$, the conditional distribution function, $\Phi(x,t) = 0 \ \forall \ t > x$.
$\phi(x,t)$:	$\partial \Phi(x,t) / \partial x$

The relationships among these reliability expressions have been evaluated [22] and are listed on Table 2.1. Since $\Phi(x,t)$ and $\phi(x,t)$ are little used in reliability theory, they are deleted from the table.

Table 2.1

Relationships Among the Reliability Expressions

Required \ Given	F(x)	R(x)	h(x)	f(x)
F(x)	-	$1 - R(x)$	$1 - \exp[-\int_0^x h(\sigma)d\sigma]$	$\int_0^x f(u)\,du$
R(x)	$1 - F(x)$	-	$\exp[-\int_0^x h(\sigma)d\sigma]$	$\int_x^\infty f(u)\,du$
h(x)	$\frac{F'(x)}{1-F(x)}$	$-\frac{R'(x)}{R(x)}$	-	$\frac{f(x)}{\int_x^\infty f(u)\,du}$
f(x)	$F'(x)$	$-R'(x)$	$h(x)\exp[-\int_0^x h(\sigma)d\sigma]$	-

Typical continuous failure distributions are:

(i) The exponential

$$f(x) = \lambda e^{-\lambda x} \quad ; \quad h(x) = \lambda \quad , \quad \lambda > 0 \tag{2.1}$$

The exponential function was first introduced for reliability analysis by Epstein and Sobel in 1951. They published an important paper "life testing" [23] in 1953. In this paper they found out that, in some cases, the exponential function is a good distribution function.

The constant hazard rate λ implies that previous use of an item does not affect its future life length. In other words, if an item has not failed up to a time t, then the probability distribution of its future life length T-t is the same as if the item was just placed in use at time t. The exponential function is the most simple distribution, but its applicability is limited because of the above-mentioned property.

There is, however, another situation in which the exponential distribution plays a prominent role. Consider a system consisting of many components, each subject to an individual pattern of malfunction and replacement and all parts making up the failure pattern of the system as a whole. Under some reasonable general conditions [24-25], the distribution of the time between system failure tends to the exponential as the complexity and time of operation increase.

(ii) The Weibull

$$f(x) = kx^m e^{-kx^{m+1}/(m+1)} \; ; \quad h(x) = kx^m, \quad m > -1 \qquad (2.2)$$

k scale parameter

m shape parameter

The Weibull function is called the Type III asymptotic distribution of extreme values by Gumbel [26], and it has been used to describe fatigue failure, vacuum tube failure [27], and ball-bearing failure [28] etc. It is the most popular parametric family of failure distribution. By appropriate choice of the two parameters k and m, a wide range of hazard curves can be approximated [29,30,31]. For $0 > m > -1$, the Weibull function fits a decreasing hazard case. For $m = 0$ it becomes an exponential function and for $m > 0$ it fits an increasing hazard case.

Physics of failure has told us that for most of items the failure mechanism in time is best represented by the bathtub curve [3]. Yet the intrinsic characteristics of the Weibull function contradict this theory. Practical considerations are not conducive for accepting the Weibull function as an analytic tool. If the Weibull function is going to be fitted to the wearout period of a bathtub curve, m must be greater than 0, whereas h becomes zero at $t = 0$. Conversely if it is to be fitted to the burn in period, then $m < 0$, and h becmes initially very large and goes to zero as time increases indefinitely, these are in direct contradiction to the behavior of the bathtub curve. Thus if a mathematical description of the bathtub curve (for practical usage, the period before wearout) is desired it is justified to develop such a new function (see Chapter 3).

(iii) The normal :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[- (x-u)^2 / 2\sigma^2 \right] \qquad (2.3)$$

or the truncated normal

$$\hat{f}(x) = \frac{1}{a\,\sigma\sqrt{2\pi}} \exp\left[- (x-u)^2 / 2\sigma^2 \right], \qquad (2.3a)$$

where $$a = 1 - \int_{-\infty}^{0} f(t)\,dt \qquad (2.3b)$$

where u and σ are the population mean and standard deviation of time to failure respectively. Since x is non-negative, the normal function is an acceptable description of failure only if $\int_{-\infty}^{0} f(t)\,dt$ is a negligible quantity, say $u > 3\sigma$. Thus if $u < 3\sigma$, a truncated normal distribution could be used. Let $f(x)$ be the normal probability density function and $\hat{f}(x)$ be the truncated distribution. Then the truncated function is defined as :

$$\hat{f}(x) = \frac{f(x)}{1 - \int_{-\infty}^{0} f(t)\,dt} \qquad (2.3d)$$

An item which follows a normal theory of failure displays a hazard rate which has a very small value in the early ages but increases at an accelerated rate throughout its life. The truncated normal distribution is observed in the case of gradual failures of electric and mechanical components and is widely used in the analysis of the reliability of complex systems when the deviations of the parameters of components beyond admissible limits are taken into account.

(iv) The gamma :

$$f(x) = \frac{1}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} x^{\alpha} \exp\left[- x/\beta \right], \ \alpha > -1, \ \beta > 0 \qquad (2.4)$$

α shape parameter

β scale parameter

The gamma function can be used to fit many sets of experimental data by proper choice of the two parameters α and β, but the Weibull function is superior for this task in reliability work [3, Chapter 4]. The gamma is a governing distribution for a standby system. Let $t_1, t_2, \ldots, t_n$ be the failure times of the on-line component and the n-1 cold standby components, the failure time of the system $t_s = t_1 + t_2 + \ldots + t_n$. It can be readily shown that if $t_1, t_2, \ldots, t_n$ are independent random variables and have an exponential distribution with the same β, then t_s is gamma distributed with the same β and $\alpha = n-1$.

(v) The modified extreme value distribution :

$$f(x) = \frac{1}{\lambda} \exp\left[- \frac{e^x - 1}{\lambda} + t\right] ; \quad h(t) = \frac{e^x}{\lambda}, \quad \lambda > 0 \tag{2.5}$$

which is a modification of the Type 1 asymptotic distribution of extreme values given by

$$F(x) = 1 - \exp\left[- e^{\lambda(x-u)}\right] \tag{2.5a}$$

where $-\infty < u < \infty$, $\lambda > 0$, and $-\infty < x < \infty$

(vi) The Rayleigh :

$$f(x) = k x e^{-kx^2/2} ; \quad h(x) = kx \tag{2.6}$$

It is observed that the hazard rate increases linearly with time. It can be readily shown that for all x such that $kx/2 < \lambda$, the probability of failure-free operation of an item with the Rayleigh law of distribution decreases with time much less rapidly than in the case of the exponential one, and that for all x such that $kx/2 > \lambda$, the probability of failure-free operation of the Rayleigh distributed item decreases with time much more rapidly than in the case of the exponentially distributed one. Apparently, the Rayleigh law can be used together with other laws of distribution when predicting the reliability of a system which has components with distinct aging effect. Cases were observed that some types of electrovacuum devices did obey a law of distribution which was close to the Rayleigh law.

Since in this thesis, attention is on continuous failure distributions, we will not present details of discrete failure distributions.

2.2. Fundamentals of Signal-Flow Graph and Its Applications to Reliability Studies.

In the 19th century, Kirchhoff noted that a set of linear equations could be represented by a flow-graph and that the determinant and co-factor of the set of equations could be written down directly by inspection of the flow-graph [32]. These topological methods were little explored until two decades ago. Mason , Tustin and Percival developed these methods and extended their applicability to modern control systems.

From a mathematical viewpoint, the flow-graph approach is a transformation from the matrix method of solving a system of simultaneous equations to a topological method. From the viewpoint of circuit theory, it may be convenient to analyze a linear system from its topological configuration. Evidently the signal-flow graph method need not be restricted in its application to electric circuits. It may be used for any linear physical system that may be represented by a set of simultaneous equations. Htun [15] has applied the method to reliability studies, Chung [16] has also appliedit to Probabilistic sequential circuits. Misra [17, 18] applied the method to reliability analysis of redundant networks. Chan and Chung [20] applied the stochastic transition flow graph to Queuing systems.

The work of Mason et al can be briefly summarized by a general expression of graph gain.

The general expression for graph gain may be written as

$$G = \frac{\sum_k G_k \Delta_k}{\Delta} \tag{2.7}$$

where

G_k = gain of the kth forward path (2.7a)

$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \dots$ (2.7b)

P_{mr} = gain product of the mth possible combination of r nontouching loops. (2.7c)

Δ_k = the value of Δ for that part of the graph not touching the kth forward path (2.7d)

By analogy with the case of solving a set of equations by determinants, we call Δ the determinant of the graph, and call Δ_k the cofactor of the forward path k. Δ_k is obtained from the graph determinant by striking out all terms containing transmittance products of loops which touch kth forward path.

Now a review of Htun's work on application of signal-flow graph to reliability follows.

In general, the set of differential equations which describe the relationships of the state probabilities of the system is of the form

$$\frac{d}{dt} P(t) = P(t) \cdot C \qquad (2.8)$$

where $P(t)$ is a row vector $P(t) = (P_1(t), P_2(t), \ldots, P_n(t), P_{n+1}(t))$,

where $P_i(t)$, $i = 1, \ldots n$, are the probabilities of being in system state i, and $P_{n+1}(t)$ is the probability of being in the system failure state n+1 (or the sink node in graph terminology).

The reliability (or availability) of a system is

$$R(t) = \sum_{i=1}^{n} P_i(t) \qquad (2.9)$$

Hence to obtain the reliability (or availability) of a system, a set of first-order differential equations has to be solved.

As mentioned earlier, if a system can be modeled by a linear flow graph, the solution may be obtained by inspection. Since the flow graph method is applicable only in solving a set of linear algebraic equations, to apply the flow graph method, we must first transform a set of differential equations to a set of linear algebraic equations. This can be achieved by Laplace transformations.

Denote $\hat{P}_i(s) = \int_0^\infty P_i(t)\, e^{-st}\, dt$ (2.10)

Then the Laplace transform of (2.8) is

$$s\hat{P}(s) - P(o) = \hat{P}(s) \cdot C$$

$$\hat{P}(s) = \hat{P}(s) \cdot \left(\frac{1}{s}\right) \cdot C + \frac{1}{s} P(o) \qquad (2.11)$$

(2.11) is a set of linear algebraic equations.

Denote $\hat{R}(s)$ = Laplace transform of $R(t)$

$$\hat{R}(s) = \int_0^\infty R(t)\, e^{-st}\, dt \qquad (2.12)$$

Then the Laplace transform of (2.9) is

$$\hat{R}(s) = \sum_{i=1}^{n} \hat{P}_i(s) = \frac{1}{s} - \hat{P}_{n+1}(s) \qquad (2.13)$$

Since from (2.11), a flow-graph of the system can be constructed (see Reference 15), $\hat{P}_i(s)$ can be evaluated by the flow-graph method, and then the reliability (or availability) in Laplace-transform form can be obtained from (2.13). The inverse transform of (2.13) can be done by using a computer.

The main advantage of using the flow-graph technique in reliability is that the mean and variance time to first system failure can be easily obtained from $\hat{R}(s)$, i.e., no inverse transformation is needed to obtain the mean and variance time to the first system failure.

The mean time to first system failure, T_m, is defined as

$$T_m = \bar{t} = \int_0^\infty t\, f(t)dt = \int_0^\infty R(t)\, dt$$

$$= \left| \hat{R}(s) \right|_{s=0} \qquad (2.14)$$

In Chapter 5 it will be shown that a closed form solution of T_m for a r-out-of-n system can be obtained.

The variance time to first system failure, T_v, is

$$T_v = \overline{t^2} - \bar{t}^2$$

where

$$\overline{t^2} = \int_0^\infty t^2 \ f(t)\, dt$$

Denote $\hat{f}(s)$ = The Laplace transform of $f(t)$

$$\hat{f}(s) = \int_0^\infty f(t)\ e^{-st}\ dt$$

Then $$\overline{t^2} = \left[\frac{d^2}{ds^2}\ \hat{f}(s)\right]_{s=0}$$

Since $$f(t) = \frac{dF(t)}{dt} = \frac{d(1-R(t))}{dt} = -\frac{d}{dt}\, R(t)$$

then

$$\hat{f}(s) = -\left[\ s\,\hat{R}(s) - 1\ \right]$$

where the initial condition $R(0) = 1$.

Hence

$$\overline{t^2} = -2\ \left[\frac{d}{ds}\ \hat{R}(s)\right]_{s=0}$$

Thus

$$T_v = -2\left[\frac{d}{ds}\ \hat{R}(s)\right]_{s=0} - \left\{\left[\hat{R}(s)\right]_{s=0}\right\}^2 \qquad (2.15)$$

2.3 Survey of Approaches on Evaluating System Reliability.

The simplest structure in reliability analysis is the series configuration. In the series case the reliable operation of the system depends on the proper operation of all system components. A system is represented by a parallel model if the system configuration is such that if one or more system components function properly then the system is in operative condition. A system is represented by a r-out-of-n model if at least r out of n components must be functioning for the system to be considered operative. Thus a 1-out-of-n system is a parallel system. Chan [33] has presented a generalized reliability function for systems of parallel components which includes the above-mentioned parallel and r-out-of-n systems. Obviously this function can be extended to be a generalized reliability function for series parallel systems by imposing logic "AND" to equation (4) of his paper. Chung [34] has given a generalized reliability function for systems of arbitrary configuration. He presents a modified path-enumeration approach, which can be applied to a system with components connected in any arbitrary configuration with the requirement that at least one special set of elementary paths must operate successfully.

Drenick [24] has shown that even if component hazards are not exponential, under some reasonably suitable conditions an exponential function can be used to approximate the reliability of a series system as the number of components becomes large. Messinger and Shooman [35] extended this limit theorem showing that under some special conditions, the reliability of any system approached a Weibull function. Evans [25] has simplified Drenick's limit theorem by using the time behavior instead of by the number of components.

Some algorithms have been proposed to approximate system reliability [36,37,38]. Basically they are derived from minimal tie sets or minimal cut sets. A tie set of a graph is a group of branches which forms a connection between input and output when traversed in the arrow direction [3]. A tie set is said to be minimal if no node is traversed more than once in tracing out a tie set. A cut set of a graph is a set of branches which interrupts all connections between input and output when removed from the graph. A cut set is minimal if the elimination of any one branch would no longer make it a cut. A set of minimal cuts is a set of cuts containing minimum number of minimal cut sets. Algorithms of finding minimal cuts have been presented in [37,38]. A tie set is good if all its components (branches) are operative, and is bad if at least one component has failed. A cut set is good if at least one of its components (branches) is operative, and is bad if all its components are bad.

Let T_i, $i = 1, \ldots, d$, be d minimal tie sets of a system. Let C_j, $j = 1, \ldots, m$, be m minimal cut sets of a system. The system reliability is thus given by

$$R = P_r \{\text{at least one tie set is good}\}$$

$$= P_r \{T_1 + T_2 + \ldots + T_d\} \tag{2.16}$$

or alternately in terms of the probability that all the tie sets are bad as :

$$R = 1 - P_r \{\text{all tie sets are bad}\}$$

$$= 1 - P_r \{\overline{T}_1 \cdot \overline{T}_2 \cdots \overline{T}_d\} \tag{2.17}$$

If expressed in terms of cut sets

$$R = P_r \{\text{all cut sets are good}\}$$

$$= P_r \{C_1 \cdot C_2 \ldots C_m\} \tag{2.18}$$

or

$$R = 1 - P_r \{ \text{at least one cut set is bad} \}$$
$$= 1 - P_r \{ \bar{C}_1 + \bar{C}_2 + \dots + \bar{C}_m \} \quad (2.19)$$

Although the formulation of a system in terms of equation (2.16) - (2.19) is simple, in a many-components system there will be many minimal tie sets and cut sets and hence the expansion of either equation will become formidable work. Bounds and approximations are obtained by taking the first few terms of the expansion of equations of (2.16), (2.19). Some results are summarized as shown in Table 2.1

Table 2.2 Bounds and Approximations of the System Reliability

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Shannon Approximation

$$R_U = 1 - \sum_{\{i \mid \xi_i = \xi\}} P_r \{ \bar{C}_i \}$$

$$R_L = \sum_{\{i \mid \eta_i = \eta\}} P_r \{ T_i \}$$

Expansion Bounds

$$R_{U1} = \Sigma P_r \{ T_i \}$$

$$R_{L1} = \Sigma P_r \{ T_i \} - \sum_{i<j} P_r \{ T_i T_j \}$$

$$R_{U2} = \Sigma P_r \{ T_i \} - \sum_{i<j} P_r \{ T_i T_j \} + \sum_{i<j<k} P_r \{ T_i T_j T_k \}$$

and so on

or

$$R_{L1} = 1 - \Sigma P_r \{ \bar{C}_i \}$$

$$R_{U2} = 1 - \Sigma P_r \{ \bar{C}_i \} + \sum_{i<j} P_r \{ \bar{C}_i \bar{C}_j \}$$

and so on

---

where in Shannon approximation [39], ξ_i is the number of components in the minimal tie set T_i and η_i is the number of components in the minimal cut set C_i,

and
$$\xi = \min \{ \xi_1, \xi_2, \ldots, \xi_d \}$$
$$\eta = \min \{ \eta_1, \eta_2, \ldots, \eta_m \} .$$

The bounds based on the tie sets are best in the low reliability region and those based on the cut sets are best in high reliability region. Recently a method of computing complex system reliability was proposed [40]. The method is composed of three phases. Phase 1 involves the reduction of all series, parallel, and series-parallel components to an irreducible non series-parallel system. In phase 2 an algorithmic recursive approach is proposed to enumerate all possible paths from the source to the sink of the graph. Phase 3 computes the value of the system reliability.

For repairable systems, analytic techniques and network approaches have been presented [41,42,43] to evaluate system reliability. Tillman et al [44,45,46,47] has been concerned with optimization of system reliability under multiple constraints.

On the formulation of mathematical models to describe complex reliability configurations, components in the system are assumed to fail independently so that a simple model can be postulated. Experience and engineering judgement suggest that there are many physical situations in which independence assumption often leads to a poor result. Shooman [21] has proposed a technique for the formulation of system models which consider dependence between components. The technique will be discussed in Chapter 5.

CHAPTER III
A NEW RELIABILITY FUNCTION

3.1 Introduction

In analyzing life test data on the reliability behavior of similar items, it is useful to describe the data with a mathematical model. So far, many mathematical models have been constructed, but the choice of reliability models must encompass more than just the problem of fitting a curve with a formula. If possible, one should choose a model which has physical significance.

The Weibull function is a popular parametric family of failure models. It has a monotone hazard rate,

$$h(t) = k\,t^{m}, \quad m > -1 ; \tag{3.1}$$

Since the Weibull hazard rate is monotonic, it can not fit the well-known hazard rate curve of Fig. 3.1, the bathtub curve.

Krohn [48] suggested selecting an appropriate hazard function for each of the three periods of decresing, constant, and incresing hazard, and calling these $h_1(t)$, $h_2(t)$, and $h_3(t)$ respectively. The population is presumed to be divided into 3 parts, each part with a probability p_i, $i = 1, 2, 3$. The hazard function is then developed :

$$h(t) = p_1 h_1(t) + p_2 h_2(t) + p_3 h_3(t) \tag{3.2}$$

As an example, utilizing a Weibull function for each $h_i(t)$, (3.2) can be written as $h(t) = p_1 k_1 t^{m_1} + p_2 k_2 + p_3 k_3 t^{m_3}$ (3.3)

where $m_1 < 0$, $m_2 = 0$, $m_3 > 0$. This $h(t)$ has 7 adjustable parameters.

In this Chapter, based on a special Markov model we define a Markov distribution (Sec. 3.2.1). There are $2n - 1$ parameters, $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots, \lambda_{2n})$, where a state s_i is sufficiently defined by its transition rates λ_i, λ_{n+i}. The maximum likelihood estimates of λ are difficult to find (see equation (3.7)). By the introducton of pseudo-intervals I_i, $i = 1, \ldots, n$, (Sec. 3.2.2), the parameter become $\lambda = (a\omega_1, \ldots, a\omega_{n-1}, b\omega_{n+1}, \ldots, b\omega_{2n})$, where ω_i, ω_{n+i} are uniquely determined for a given pseudo-interval I_i, and a and b are two adjustable parameters for the best fit to the given failure data. The technique of introducing pseudo-intervals eliminates the difficulty of finding maximum likelihood estimates of parameter λ and leaves only the parameters a, b to be determined for the best fit. Furthermore, since the parameters ω_i, ω_{n+i} are uniquely determined from I_i, the states s_i are corresponding to the pseudo-intervals I_i. Thus the state model dynamically characterizes the behavior of a system. In Section 3.3, three examples illustrate the applicability of the model. Numerical results demonstrate that the number of states or the number of pseudo-intervals, n, plays a role for obtaining the best fit.

3.2 A Special Markov Model

3.2.1 Formation of a Markov Distribution

Assume that we have a system in which the reliability behavior is defined by n + 1 states $s_1, s_2, \ldots, s_n, s_f$, where $s_1, s_2, \ldots, s_n$ are operative states and s_f is a failure state. The signal flow graph for the model is shown in Fig. 3.2, where

λ_i, $i = 1, \ldots, n - 1$, is the transition rate from s_i to s_{i+1}

λ_n is always 0

λ_{n+i}, $i = 1, \ldots, n$, is the transition rate from s_i to s_f.

The transitions satisfy the following conditions:

(1) At t = 0, a system is in s_1.

(2) If at the instant t the system enters state s_k, then the probability that it will enter state s_{k+1} in the infinitesimal interval (t, t + Δt) is

$$\lambda_k \Delta t + Q(\Delta t)$$

where Q (Δt) is the probability of more than one transition in the interval and is in the order of 0 (Δt) such that

$$\lim_{\Delta t \to 0} \frac{0(\Delta t)}{\Delta t} = 0.$$

The probability that it will enter state s_f in that interval, i.e., fail, is

$$\lambda_{n+k} \cdot \Delta t + Q(\Delta t)$$

and the probability that it remains in s_k in that interval is

$$1 - (\lambda_k + \lambda_{n+k}) \cdot \Delta t + Q(\Delta t)$$

(3) After entering state s_n the system will operate according to an exponential law with a parameter λ_{2n}.

The above 3 conditions completely determine the system behavior. Let $P_i(t)$ be the probability that the system is in s_i at time t, $i = 1, \ldots, n$, ans $P_f(t)$ be the probability that the system is in s_f at time t.

The differential equations for the system are

$$\begin{aligned} P_1'(t) &= -(\lambda_1 + \lambda_{n+1}) P_1(t) \\ P_k'(t) &= \lambda_{k-1} P_{k-1}(t) - (\lambda_k + \lambda_{n+k}) P_k(t) \qquad (k=2, \ldots, n) \\ P_1(0) &= 1, \; P_k(0) = 0, \; k=2, \ldots, n. \end{aligned} \tag{3.4}$$

The reliability of the system is

$$R(t) = \sum_{i=1}^{n} P_i(t)$$

and the probability of failure before time t is

$$P_f(t) = 1 - R(t) \tag{3.5}$$

The equations (3.4) are readily solved by Laplace transforms. Assume that $\lambda_i + \lambda_{n+i} \neq \lambda_j + \lambda_{n+j}$, $i, j = 1, \ldots, n$, $i \neq j$, we obtain

$$\begin{aligned} P_1(t) &= e^{-(\lambda_1 + \lambda_{n+1})t} \\ P_k(t) &= \left(\prod_{j=1}^{k-1} \lambda_j\right) \sum_{j=1}^{k} \left[\prod_{\substack{i=1 \\ i \neq j}}^{k} (\lambda_i - \lambda_j + \lambda_{n+i} - \lambda_{n+j})\right]^{-1} e^{-(\lambda_j + \lambda_{n+j})t}, \\ & \qquad (k = 2, \ldots, n). \end{aligned} \tag{3.6}$$

and

$$R(t) = \sum_{k=1}^{n} P_k(t) = \sum_{i=1}^{n} C_i \, e^{-(\lambda_i + \lambda_{n+i})t} \tag{3.7}$$

where

$$C_i = \sum_{k=i}^{n} \left(\prod_{j=1}^{k-1} \lambda_j\right) \left[\prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_j - \lambda_i + \lambda_{n+j} - \lambda_{n+i})\right]^{-1}, \tag{3.8}$$

$$\left(\prod_{j=1}^{0} \lambda_j\right) \triangleq 1$$

Hence $P_f(t) = 1 - R(t)$ is a function of 2n - 1 parameters, λ_k, k=1, ..., 2n.

In part 3.2.2, the reliability is made to be a function of only two parameters a and b.

3.2.2 Introduction of Pseudo-intervals to define the operation states of an item and the corresponding reduction of the number of parameters.

Step 1: Assume that the failure information consists of the following information. In the time interval $(t_{j-1}, t_j]$, n_j items have failed out of the N_j items that were on test at the beginning of that interval. Define time-interval hazard rate of failure data as

$$h_j(t) = \frac{n_j}{N_j(t_j - t_{j-1})} \quad t \in (t_{j-1}, t_j], \; j = 1, \ldots, Np \qquad (3.9)$$

where N_p is the number of original time intervals.

Step 2: Plot $h_j(t)$ as shown in Fig. 3.3. By observing the rate of change of time-interval hazard rate as shown in the histogram, it is feasible to determine boundaries of pseudo-intervals. For clarity, we use algorithmic steps to determine boundaries. There are three cases, a, b, c; they are used to pick an initial trial value (lower bound) of n.

Case a: The test of hypothesis on the exponentiality of the distribution of failure-free operation time [49], viz, n=1. If accepted, this is Fig. 3.a; go to step 3. Otherwise go to case b.

Case b: If the time-interval hazard rate ($h_j(t)$) exhibit a bathtub shape, let n=3, and go to step 3, after determining boundaries, otherwise go to case c. Given n=3, we divide the whole interval into 3 pseudo-intervals. The exact boundaries can be set by choosing three straight lines [50] respectively for decreasing, constant and increasing hazard rate. Let t', t'' be time coordinates of intersections of three consecutive lines. The three pseudo-intervals are $I_1 = (0, t'_1]$, $I_2 = (t'_1, t'_2]$, $I_3 = (t'_2, t_N]$, where $t'_1 = t_j$, $t'_2 = t_r$, $t_j(t_r)$ is the nearest original-time-interval boundary to t' (t''), as shown in Fig. 3.3b. The procedure of increasing the number of pseudo-intervals is the same as in case c.

Case c: (neither case a nor case b) The $h_j(t)$ is decreasing monotonically or increasing monotonically, let n=2, and go to step 3 after determining the center boundary (t') as follows.

Find the midrange of $h_j(t)$

$$\hat{h} = \frac{1}{2}(h_{max} + h_{min}).$$

Choose t'_1 as the far endpoint of the h_j which is nearest to that midrange, as shown in Fig. 3.3c.

In increasing n, we can further divide the pseudo-interval $I_i = (t'_{i-1}, t'_i]$ into two pseudo-intervals by the same procedure, where

$$|h(t'_i) - h(t'_{i-1})| \geq |h(t'_j) - h(t'_{j-1})|$$
$$i, j = 1, 2, \ldots, n, j \neq i .$$

In concluding step 2, since the hazard rates for most of components or systems belong to these three cases, we would not consider those cases other than cases a, b, and c. We continue to proceed step 2 by increasing the number of pseudo-intervals until an optimal n for best fit to failure data is obtained. During step 2, we have decided to use a constant hazard (case a) or a bathtub shape hazard (case b) or a strictly monotonic hazard (case c). The lower bound of n is 1 for case a, 3 for case b, 2 for case c, and the upper bound is N_p, the number of original time intervals. In step 2 we start from the lower bound and work up to an optimum n.

Step 3: Given n and the corresponding pseudo-intervals,

$$I_i = (t'_{i-1}, t'_i] = [(t_{p-1}, t_p], \ldots, (t_{p+q-1}, t_{p+q}]], \quad i = 1, \ldots, n.$$

The sample average of hazard rates, ω_{n+1}, is defined as

$$\omega_{n+i} \equiv \frac{h(t_p)\cdot(t_p - t_{p-1}) + \ldots + h(t_{p+q})\cdot(t_{p+q} - t_{p+q-1})}{t'_i - t'_{i-1}} \qquad (3.10)$$
$$i = 1, \ldots, n.$$

Next we form the set λ_{n+i}, $i = 1, \ldots, n$,
$\lambda_f = (\lambda_{n+1}, \ldots, \lambda_{2n}) = (b\omega_{n+1}, \ldots, b\omega_{2n})$, where b is an adjustable parameter to be determined for the best fit to life test data. The transition rate from s_i to s_{i+1} is inversely proportional to the length of the interval I_i. Thus we have a signal-flow graph as shown in Fig.3.4 with the set of probability rates of transition $\lambda_t = (\lambda_1, \ldots, \lambda_{n-1}) = (a\omega_1, \ldots, a\omega_{n-1})$, where a is the second parameter to be determined for best fit and where

$$\omega_i \equiv ((t'_i - t'_{i-1})^{-1}, \; i = 1, \ldots, n-1.$$

To conclude Part 3.2.2, it should be pointed out that the maximum likelihood estimate of $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots, \lambda_{2n})$ is difficult to find. Yet the formation of pseudo-intervals eliminates analytic and numerical difficulties in finding the maximum likelihood estimates of parameters $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots, \lambda_{2n})$ After determining the boundaries of the pseudo-intervals, the parameters ω_i, $i = 1, \ldots, 2n$, can be obtained and then only two parameters a, b need to be determined for the best fit. Hence the formation of n pseudo-intervals corresponds to the assignment of n operation states, and reduces the number of parameters from 2n -1 to 2. The step from $h_j(t)$ to the parameters ω is to smooth the piece-wise constant hazard data by a continuous function. The step from the parameters ω to the parameters λ, i.e., finding an optimal (a, b), and the increasing of n is to fit a curve.

The Markov hazard rate is

$$h(t) = -\frac{1}{R(t)} \cdot \frac{dR(t)}{dR} = \frac{\sum_{i=1}^{n} C_i(a, b) \cdot (\lambda_i + \lambda_{n+i}) \cdot e^{-(\lambda_i + \lambda_{n+i})t}}{\sum_{i=1}^{n} C_i(a, b) \cdot e^{-(\lambda_i + \lambda_{n+i})t}} \qquad (3.11)$$

by substitution of (3.7).

The expected hazard rate is

$$h_e(t) = \text{the expectation of hazard rate at time } t$$

$$= \sum_{i=1}^{n} \phi_i(t) \cdot \lambda_{n+i}$$

$$= \sum_{i=1}^{n} P_i(t) \cdot \lambda_{n+i} \Big/ \sum_{i=1}^{n} P_i(t) \qquad (3.12)$$

where $\phi_i(t)$ is the probability that the system is in s_i at time t given that it has survived up to time t, and $\sum_{i=1}^{n} \phi_i(t) = 1$. In Section 3.3, numerical results show that the expected hazard rate is very close to the Markov hazard rate.

3.2.3 Solving for parameters a, b.

Now the problem becomes that of evaluating optimal values a, b. It is a nonlinear simultaneous equation with order 2. The optimal values of a and b are subjected to computer evaluation as follow:

Let

$$g(t_i) = \frac{\text{number of the remaining operative items at time } t_i}{\text{total numbers of items tested}}, \qquad i = 1, 2, \ldots, N_p$$

where N_p is the number of data points, i.e., $g(t_i)$ is the sample fraction of failure-free items at time t_i and set

$$E(a, b) = \frac{1}{N_p} \sum_{i=1}^{N_p} \left(R(t_i) - g(t_i) \right)^2 \qquad (3.13)$$

be the mean square errors of approximating $g(t_i)$ by $R(t_i)$, then our problem is to find a, b such that E (a, b) is a minimum. For minimum E (a, b) we set $\partial E(a, b) / \partial a = 0$, $\partial E(a, b) / \partial b = 0$, and solve for the optimum a and b.

A computer program (given in the Appendix in FORTRAN IV) has been set up to find the optimal set (a, b) by half-interval iterative search criterion. The Markov model, with finite state n, has discrete state representation for a continuous function. In order to retain the correspondence between the defined pseudo-intervals and the failure test data (part 3.2.2 of this section), the optimal values of a, b are expected to be in the neighborhood of 1.0, say a, b $\in$ (0.5, 2.0). Hence the optimal values of a, b are computed by setting its initial values a, b = 1.0 and by half-interval iterative search criterion to find its local optimum values.

After the optimal a, b are evaluated, the hazard function can be from (3.11), and Mean Time to Failure is

$$MTTF = \int_0^\infty R(t)\,dt = \sum_{i=1}^{n} C_i(a, b) / (a\omega_i + b\omega_{n+i}) \qquad (3.14)$$

by substituting (3.7).

3.3 Examples

Example 1: A constant hazard case.

10 units are put on test at time 0. The failure data is listed on Table 3.1.

Table 3.1. Failure data for ten hypothetical electronic components [3]

Failure number	Operating time, hr	Time intervals	Hazard rate per hr. $h_j(t)$ ($* 10^{-2}$)
1	8	0 -- 8	1.25
2	20	8 -- 20	0.93
3	34	20 -- 34	0.96
4	46	34 -- 46	1.19
5	63	46 -- 63	0.98
6	86	63 -- 86	0.87
7	111	86 --111	1.00
8	141	111 --141	1.11
9	186	141 --186	1.11
10	266	186 --266	1.25

By the test of hypothesis on the exponentiality of the distribution of failure-free operation time, we find that this set of data fits an exponential reliability function $R(t) = e^{-\lambda t}$. The unbiased estimate for parameter λ [51]

$$\lambda = \frac{\text{total failures}}{\text{total time}} = \frac{10}{961} = 0.0104.$$

For the Markov distribution approach, let n = 1, then the Markov distribution becomes an exponential distribution

$$R(t) = e^{-b\omega_2 t},$$

as shown in Fig. 3.5.

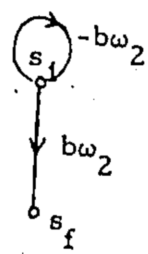


Fig. 3.5. A signal-flow graph with n = 1.

where ω_2, the sample average hazard rate, is

$$\omega_2 = \frac{\sum_{i=1}^{N_p} h_i(t)}{N_p} = 0.0107.$$

Let the initial value of b = 1.0. By computer evaluation we obtain the optimal b, b = 1.05, with $E^{1/2} = 0.0195$. The constant hazard rate is

$$h(t) = h_e(t) = \lambda = b\omega_2 = 0.011.$$

Example 2: A human-mortality case.

Given the data, column 1 and 2 of Table 3.2, with $N_p = 14$, the first step is to evaluate $h_j(t)$ by (3.9). The computed values are on the fourth column of Table 3.2, and Fig. 3.6. Since $h_j(t)$ has a bathtub shape, choose 3

Table 3. 2 Commissioner 1941 standard ordinary mortality table [3]

x	l_x	Time interval	$h_j(t)$ (per 100 yr)	Markov hazard (per 100 yr)	Expected hazard (per 100 yr)
0	1,023,102	0 - 1	2.2580	2.1045	2.1042
1	1,000,000	1 - 2	0.5770	1.1147	1.0828
2	994,230	2 - 3	0.4039	0.6684	0.6504
3	990,114	3 - 4	0.3380	0.4581	0.4500
4	986,767	4 - 5	0.2990	0.3523	0.3489
5	983,817	5 - 10	0.2442	0.2951	0.2937
10	971,804	10 - 15	0.1962	0.2232	0.2232
15	962,270	15 - 20	0.2238	0.2353	0.2353
20	951,483	20 - 25	0.2582	0.2733	0.2733
25	939,197	25 - 30	0.3106	0.3312	0.3312
30	924,609	30 - 35	0.3905	0.4037	0.4037
35	906,554	35 - 40	0.5121	0.4838	0.4838
40	883,342	40 - 45	0.6971	0.5639	0.5639
45	852,554	45 - 50	0.9772	0.7022	0.6379
50	810,900				

x = age in years ; l_x = number of living at age x pseudo-intervals, i.e., n = 3, with $I_1 = (0, 4]$, $I_2 = (4, 30]$, $I_3 = (30, 50]$. We have $\omega_1 = 0.25$, $\omega_2 = 0.038$, $\omega_3 = 0$. Evaluate sample average of hazard rate, $h_j(t)$, then $\omega_4 = 0.0089$, $\omega_5 = 0.0025$, $\omega_6 = 0.0080$. Let initial values of a, b = 1.0, the optimal set of (a, b) is (a = 0.78, b = 0.78), with rms error $E^{1/2} = 0.0119$.

To find an optimal value of n, we increase the number of states, i.e., the number of pseudo-intervals. Select the pseudo-intervals for each n as shown on Table 3.3.

Table 3.3 Pseudo-intervals for n = 6,9,11,13

n	I_1 I_2 I_3 I_4 I_5 I_6 I_7 I_8 I_9 I_{10} I_{11} I_{12} I_{13}
6	0 -- 1 -- 2 -- 4 -- 30 --45 -- 50
9	0 -- 1 -- 2 -- 3 -- 4 -- 30 --35 --40 --45 --50
11	0 -- 1 -- 2 -- 3 -- 4 -- 5 -- 25 -- 30 -- 35 --40 --45 --50
13	0 -- 1 -- 2 -- 3 -- 4 -- 5 -- 10 -- 20 --25 -- 30 --35 --40 --45 --50

For each n, find the optimal a, b, and the error. The results are listed on Table 3.4.

Table 3.4 Values of a, b, $E^{1/2}$ for n = 3, 6, 11, 13

n	a	b	$E^{1/2}$
3	0.78	0.78	0.0119
6	0.80	0.80	0.0079
9	0.82	0.83	0.0071
11	0.87	0.87	0.0061
13	0.92	0.93	0.0048

The reliability curves for n = 3, and n = 13 are graphed in Fig. 3.6. The computed values of the Markov hazard rate and the expected hazard rate for n = 13 are listed on the column 5 and 6 of Table 3.2. If we let n = 14, i.e., the number of states = the number of (failure data) time-intervals, the result is dominated by rounding errors.

Example 3: A decreasing hazard case.

1,000 B-52 aircraft are put on flight test at time 0. The failure data is listed on Table 3.5.

Table 3.5 Failure rate for B-52 aircraft [3]

Time till failure, hr	Numbers of failure in each interval	Hazard rate $h_i(t)$ (per 100 hr)	Markov hazard (per 100 hr)	Expected hazard (per 100 hr)
0 -- 2	222	11.10	12.10	12.10
2 -- 4	45	2.89	6.53	4.84
4 -- 6	32	2.18	3.85	3.07
6 -- 8	27	1.92	2.60	2.30
8 -- 10	21	1.56	1.97	1.86
10 -- 12	15	1.13	1.60	1.57
12 -- 14	17	1.33	1.36	1.35
14 -- 16	7	0.56	1.17	1.17
16 -- 18	14	1.14	1.02	1.02
18 -- 20	9	0.75	0.90	0.90
20 -- 22	8	0.68	0.80	0.80
22 -- 24	3/420	0.26	0.71	0.71

Observe the time-interval hazard rate, Table 3.5 and Fig. 3.7, choose $n = 6$ in case c. The corresponding pseudo-intervals are

$$\begin{array}{ccccccccccccc} & I_1 & & I_2 & & I_3 & & I_4 & & I_5 & & I_6 & \\ 0 & -- & 2 & -- & 4 & -- & 8 & -- & 14 & -- & 22 & -- & 24 \end{array}$$

We have $\omega_1 = 0.5$, $\omega_2 = 0.5$, $\omega_3 = 0.25$, $\omega_4 = 0.167$, $\omega_5 = 0.125$, $\omega_6 = 0$. Evaluating the sample average of hazard rate in each pseudo-interval, we have $\omega_7 = 0.110$, $\omega_8 = 0.0289$, $\omega_9 = 0.0205$, $\omega_{10} = 0.0134$, $\omega_{11} = 0.0078$, $\omega_{12} = 0.0026$. The optimal set of (a, b) is ($a = 0.84$, $b = 1.09$), with rms error = 0.0184. The associated values for comparison are in Fig. 3.7. From Fig. 3.7, it is clear that the deviation of curve ii from data i is in the interval (0, 8]. Hence when increasing n by 1 the pseudo-intervals are chosen as

$$\begin{array}{ccccccccccccccc} & I_1 & & I_2 & & I_3 & & I_4 & & I_5 & & I_6 & & I_7 & \\ 0 & -- & 2 & -- & 4 & -- & 6 & -- & 8 & -- & 14 & -- & 22 & -- & 24 \end{array}$$

The optimal set of (a, b) is ($a = 0.80$, $b = 1.05$) with rms error = 0.0200, which implies that by increasing n, $E^{1/2}$ does not monotonically decrease. Hence $n = 6$ is an optimum. It can also be observed that for a decreasing hazard case , whereas from data

$$h(t \in (10, 12]) < h(t \in (12, 14])$$

$$h(t \in (14, 16]) < h(t \in (16, 18])$$

which imply an optimal n is not near to 12, the original number of time-intervals.

The optimal n may not be N_p, the original number of time-intervals. In example 2 the optimal $n = 13$ ($N_p = 14$) and in example 3 the optimal $n = 6$ ($N_p = 12$).

3.4 Concluding Remarks

Numerical results (Tables 3.2 and 3.5) show that the expected hazard rate (3.12) is approximately equal to the Markov hazard rate (3.11). To prove the (approximation) identity of the Markov hazard rate and the expected hazard rate from analytical viewpoint remains as an open problem.

Compare the hazard rate (3.2) of Krohn [48] with the expected hazard rate. We conclude that the two approaches basically come from the same concept of using multiples of a hazard function as a mathematical model for fitting to the general failure curve. The difference is that in the Krohn's approach causes of failure, $p_1, \ldots, p_n$, are constant and hazard functions, $h_1(t), \ldots, h_n(t)$, are functions of time, whereas in (2.12) causes of failure are functions of time and hazard functions are constant. The latter approach is better than the former in that the hazard function of our method (3.11) or (3.12)) is derived in such a special Markov model that as far as the estimate of parameters is concerned the number of parameters of our method is reduced to 2 (a and b), whereas the number of parameters of the former method (3.2) increases as n increases.

From (3.7) we have

$$\frac{\partial R(t)}{\partial t} = -\sum_{i=1}^{n} C_i \cdot (\lambda_i + \lambda_{n+i}) \cdot e^{-(\lambda_i + \lambda_{n+i})t}$$

One nature of an negative exponential function is the smoothness especially in the tail end. Hence it is obvious that for any positive number ε there exist a time t' such that for $t > t'$, $|\partial R(t)/\partial t| < \varepsilon$. This is shown in Fig. 3.6 and 3.7 the smoothness of the Markov reliability function and the Markov hazard rate. Thus we may observe that for a life test data with relatively large $|\partial R(t)/\partial t|$ the Markov distribution can not catch up for the best fit. However we usually neglect the wear-out period of an item which has very large $|\partial R(t)/\partial t|$ in the tail end. Example 2 shows that the Markov function is a very good approximation for the human-mortality case (with age 0-50). Examples 1 and 3 also illustrate that the Markov function is suitable for the analysis of the decreasing and constant hazard cases. Note that an exponential distribution is a special case of the Markov distribution with n = 1.

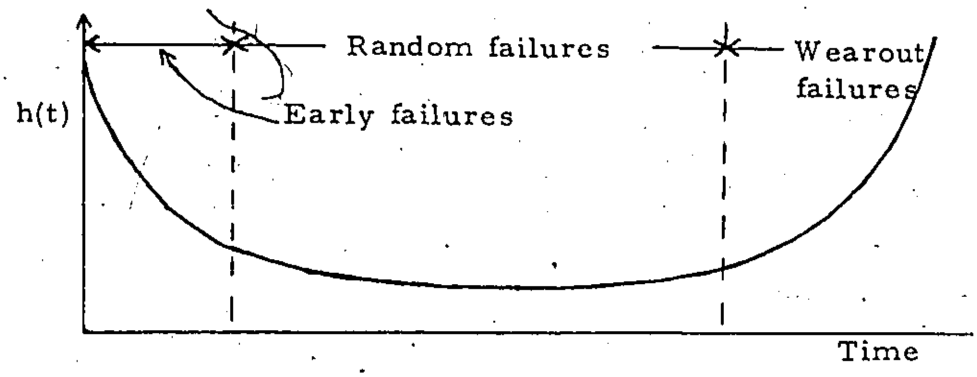


Fig. 3.1 Bathtub-shaped hazard-rate curve.

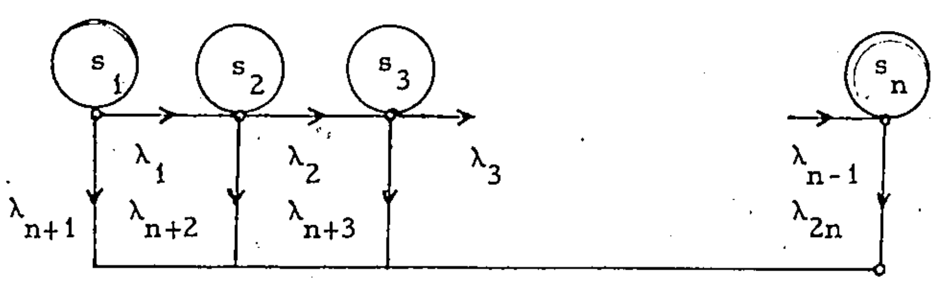


Fig. 3.2 A signal flow graph

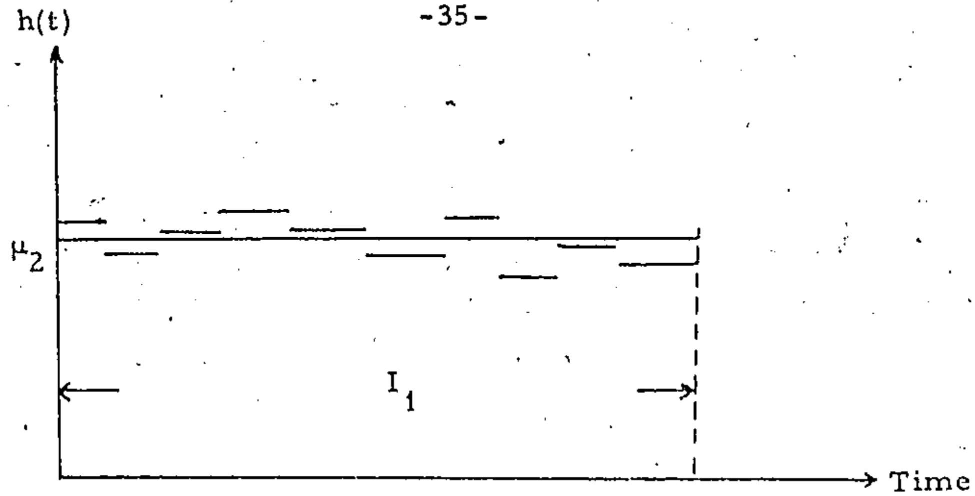


Fig. 3.3a Constant hazard case n = 1

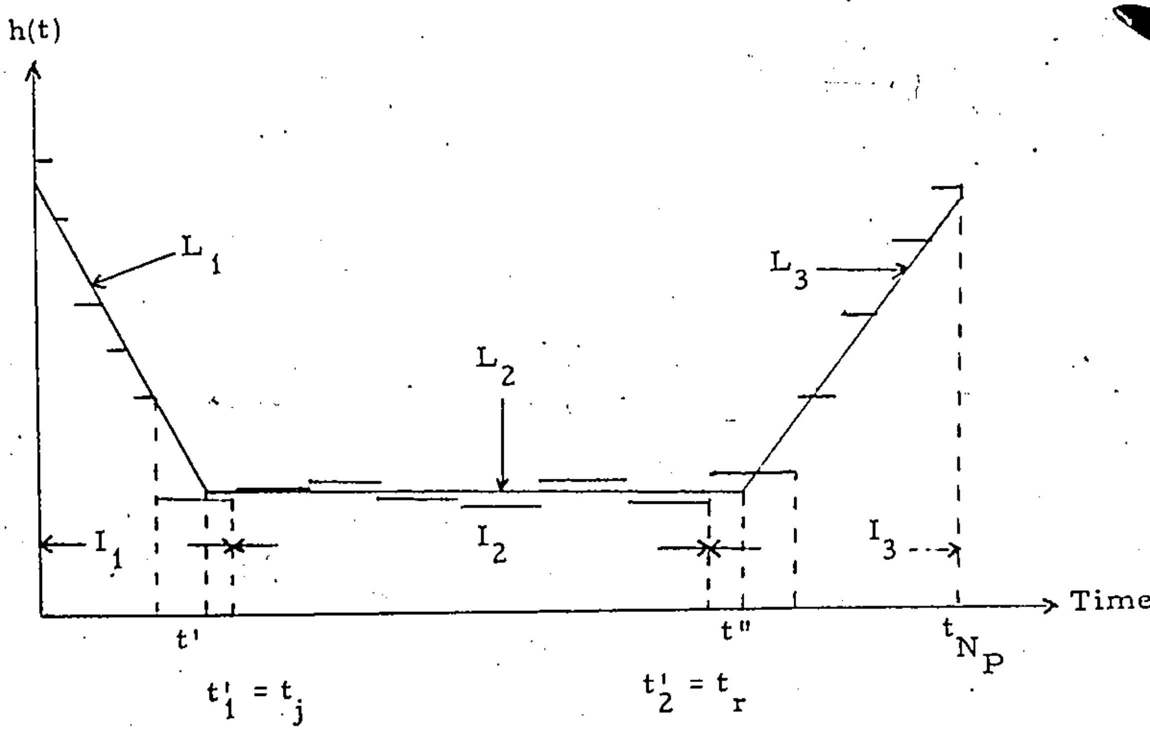


Fig. 3.3b Determining initial boundaries for a bathtub shape hazard case.

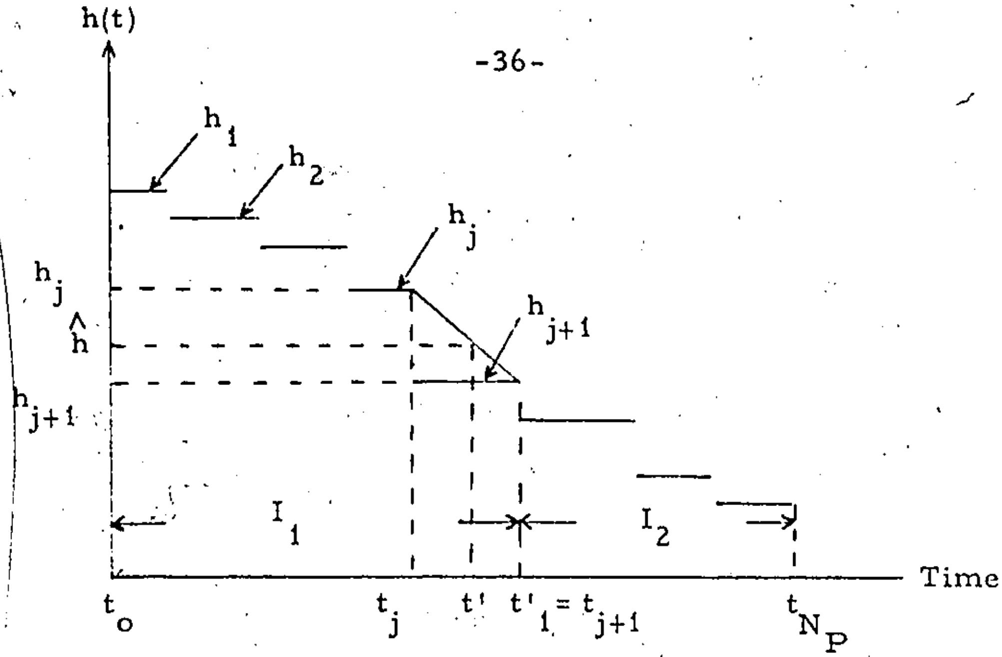


Fig. 3.3c Determining boundaries for montonic hazard case.

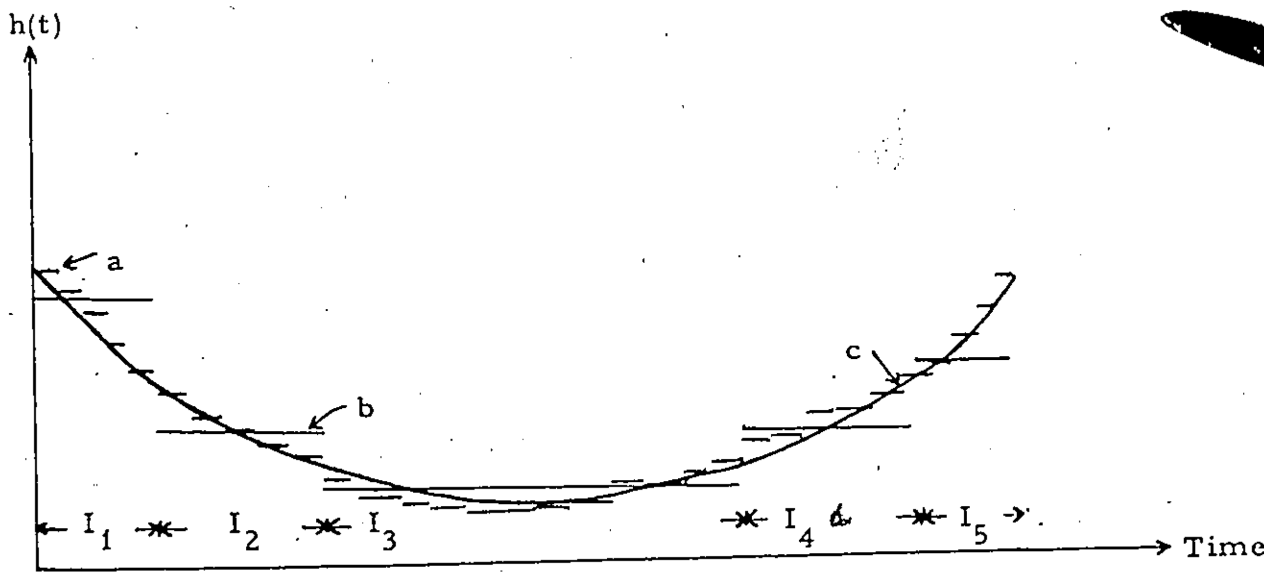


Fig. 3.3.d Construction of Pseudo-intervals by observing time-interval hazard rate.

a : Time-interval hazard rate of life test data

b : Pseudo-intervals hazard rate

c : Markov hazard curve.

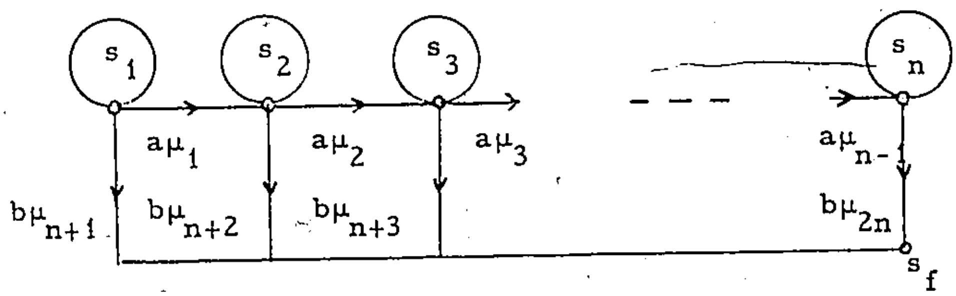


Fig. 3.4 A signal flow graph with Pseudo-intervals introduced.

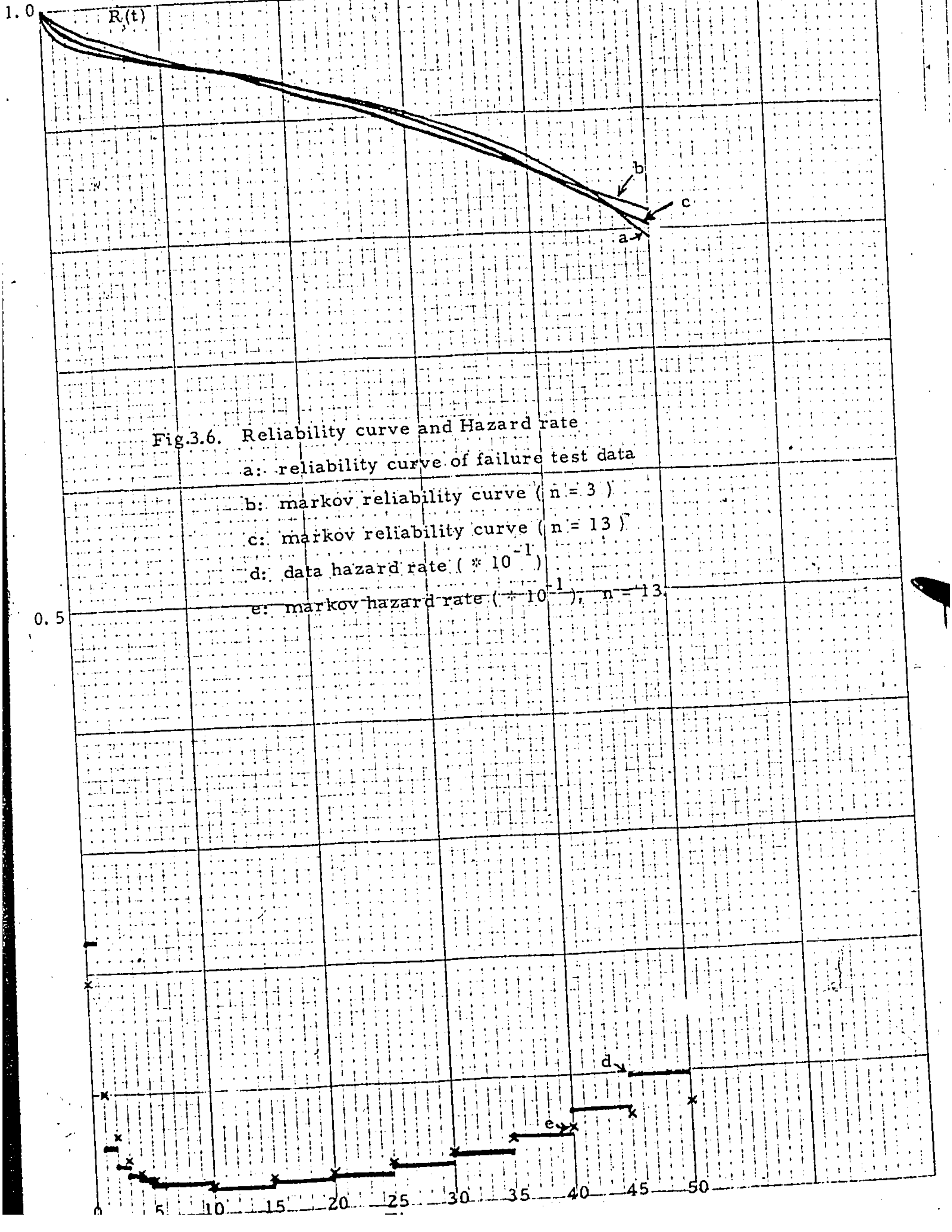


Fig.3.6. Reliability curve and Hazard rate

a: reliability curve of failure test data

b: markov reliability curve (n = 3)

c: markov reliability curve (n = 13)

d: data hazard rate ($* 10^{-1}$)

e: markov hazard rate ($* 10^{-1}$), n = 13

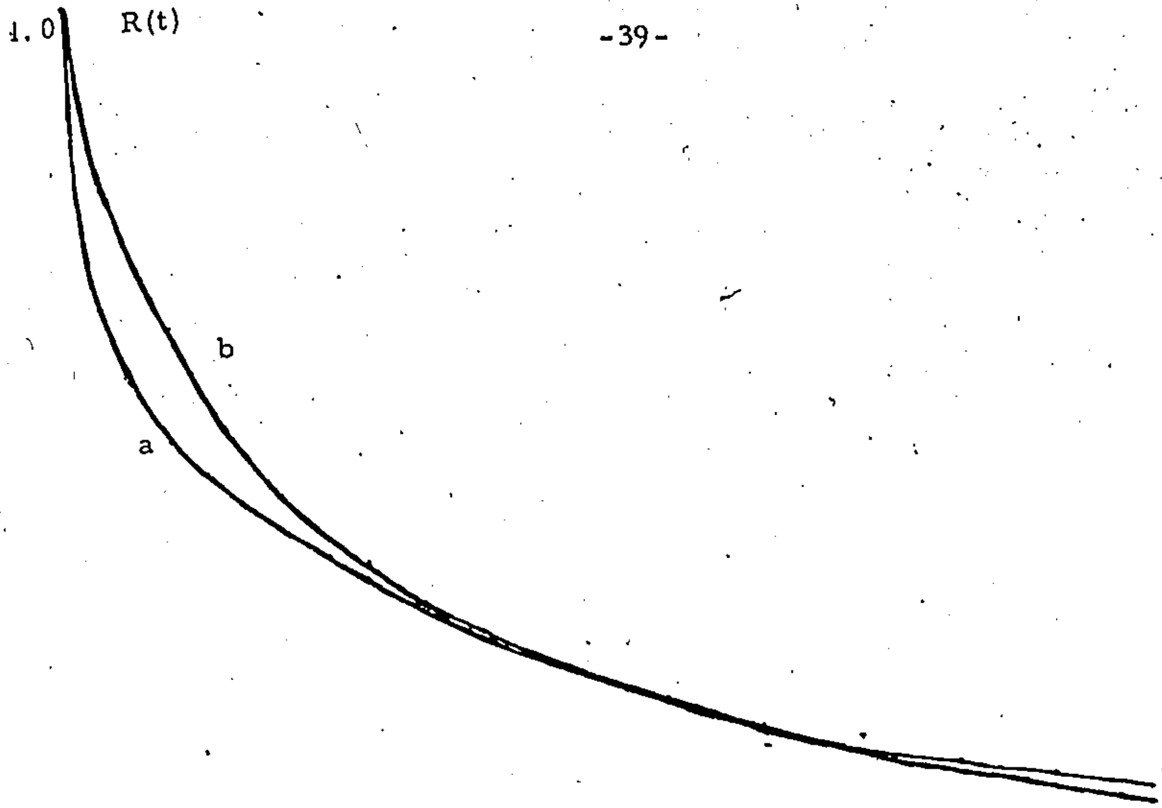


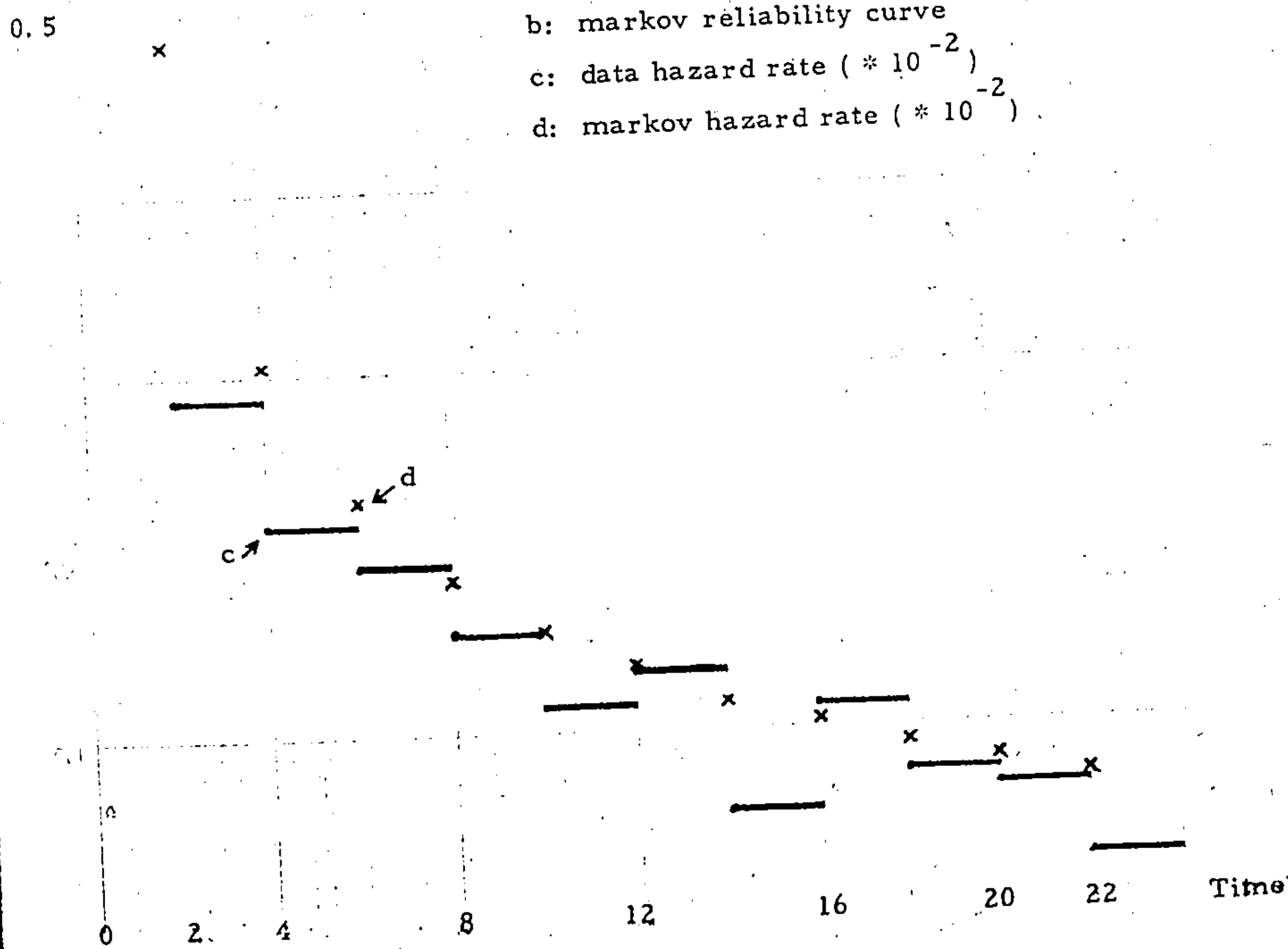
Fig.3.7. Reliability curve and Hazard rate

a: reliability curve of failure test data

b: markov reliability curve

c: data hazard rate (* 10^{-2})

d: markov hazard rate (* 10^{-2})



CHAPTER IV

A GENERAL BIRTH AND DEATH PROCESS

4.1 Introduction

In this Chapter, we extend the special Markov model (Part A, Section 3.2) to a general finite-state model for a birth and death process. The model is general in the sense that for bounded random variables, it covers the "conventional" birth and death model as a special case.

Denote X(t) as the path function of a random variable x [8]. A birth and death process is a stationary Markov process with X(t) having positive integer values and with a transition probability function

$$C_{ij}(t) = P_r \{ X(t+s) = j \mid X(s) = i \}$$

The function $C_{ij}(t)$ satisfies the conditions [for $\Delta t \rightarrow 0$]

$$C_{i,i+1}(\Delta t) = \lambda_i \Delta t + 0(\Delta t) \tag{4.1}$$

$$C_{i,i}(\Delta t) = 1 - (\lambda_i + \mu_i) \Delta t + 0(\Delta t)$$

$$C_{i,i-1}(\Delta t) = \mu_i \Delta t + 0(\Delta t)$$

where the constants λ_i, μ_i are the probability rates of transition from state i to state i + 1, or from state i to state i-1, with $i = 1, 2, \ldots, \infty$. The birth and death process has been discussed in detail in [6, 7, 8].

Note that in reliability studies the repair and failure rates are equivalent to the birth and death rates, and the state number i is equivalent to a value x = i of the path function X(t). The system model (as shown in Fig. 3.2), a general pure death model, is extended to include regeneration. Incorporating the probability rates μ_i in the state model of Fig. 3.2, with a finite number of states, we have a general finite state birth and death model with transition probability function satisfying the conditions

$$C_{i,i+1}(\Delta t) = \lambda_i \Delta t + 0(\Delta t)$$

$$C_{i,i}(\Delta t) = 1 - (\lambda_i + \mu_i + \lambda_{n+i}) \Delta t + 0(\Delta t), \quad \lambda_n = 0, \ \mu_1 = 0$$

$$C_{i,i-1}(\Delta t) = \mu_i \Delta t + 0(\Delta t) \qquad i \neq 1$$

$$C_{i,n+1}(\Delta t) = \lambda_{n+i} \Delta t + 0(\Delta t) \tag{4.2}$$

with i = 1, 2, ..., n

In Section 4.2, the derivation is described of a pure death process to a birth and death process. In Section 4.3, three examples illustrate the applicability of the proposed models. Note that for a finite number of states, the "conventional birth and death process is a special case of the general one proposed here with $\lambda_{n+i} = 0$.

4.2 Models

4.2.1 A General Pure Death Process

Assume that we have a system or a component (in term of probability, the concerned probability for a single component will be the same as the percentage for a large population of the same components), in which the behavior of operation condition could be defined by n+1 states, $s_1, s_2, \ldots, s_n, s_f$, where $s_1, s_2, \ldots, s_n$ are operation states and s_f is a failure state. We have a signal flow graph as shown in Fig. 4.1, where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the probability rates of transition between consecutive states, i.e., λ_1 is the probability rate of transition from s_1 to s_2, etc., and $\lambda_{n+1}, \ldots \lambda_{2n}$ are the probability rates of failure from states $s_1, s_2, \ldots, s_n$ respectively to state s_f.

Note that in the previous state space approach for system reliability and availability [3, 15], the behavior of a single component is expressed by only two states "good" or "bad", whereas in our model the behavior of a single component can be expressed by more than two states (inclusive). We may have some intermediate states from "good" to "bad".

The model, Fig. 4.1, is the same as the one shown in Fig. 3.2. Its expressions are the same as those in Section 3.2.1.

For the general initial conditions $P_i(0)$, $i = 1, 2, \ldots n$, the reliability function $R(t)$ can be generalized as follows.

Suppose that inital conditions satisfy

$$\sum_{i=1}^{n} P_i(0) = 1$$

and

$$1 \geq P_i(0) \geq 0$$

Let us denote symbols as below.

$P_{ik}(t)$: The probability that an item enters state s_k at time t, given that the item is initially in state s_i.

$\hat{P}_{ik}(s)$: The Laplace Transform of $P_{ik}(t)$.

$R_i(t)$, The reliability of an item at time t, given that the item is initially in state s_i.

Thus if an item starts at state s_i at time 0, then

$$\hat{P}_{ik}(s) = \frac{\lambda_{i-1}\lambda_i, \ldots \lambda_{k-1}}{(s+\lambda_i + \lambda_{n+i})\ldots(s+\lambda_k + \lambda_{n+k})} \qquad (4.1)$$

or in the time domain

$$P_{ik}(t) = \sum_{j=i}^{k} (s + \lambda_j + \lambda_{n+j})\hat{P}_{ik}(s)\Big|_{s = -(\lambda_j + \lambda_{n+j})} e^{-(\lambda_j+\lambda_{n+j})t} \qquad (4.2)$$

Hence $R_i(t) = \sum_{k=i}^{n} P_{ik}(t)$

and $R(t) = \sum_{i=1}^{n} P_i(0) \cdot R_i(t)$

Substituting equation (4.1) and (4.2), we obtain

$$R(t) = \sum_{i=1}^{n} P_i(0) \sum_{k=i}^{n}{}' \sum_{j=i}^{k} (s+\lambda_j+\lambda_{n+j}) \cdot \hat{P}_{ik}(s) \Big|_{s=-(\lambda_j+\lambda_{n+j})} e^{-(\lambda_j+\lambda_{n+j})t} \tag{4.3}$$

$$= \sum_{r=1}^{n} C_r e^{-(\lambda_r+\lambda_{n+r})t} \tag{4.4}$$

$$\text{where } C_r = \sum_{m=1}^{r} P_m(0) \cdot \hat{P}_{mr}(s) \cdot (s+\lambda_r+\lambda_{n+r}) \Big|_{s=-(\lambda_r+\lambda_{n+r})} \tag{4.5}$$

$$= \sum_{m=1}^{r} P_m(0) \cdot \left(\prod_{j=m-1}^{r-1} \lambda_j \right) \left[\prod_{j=m}^{r-1} (\lambda_j - \lambda_r + \lambda_{n+j} - \lambda_{n+r}) \right]^{-1}$$

4.2.2 A General Birth And Death Process

Modify the condition (2) of Section 3.2.1 as shown in Fig. 4.2, where μ_i is the probability rate of transition (repair) from the state s_i to state s_{i-1}, $i = 1,2,\ldots,n$. Then condition (2) becomes (2)'
If at the instant t the item enters state s_k, then the probability that it will enter state s_{k+1} in the infinitesimal interval $(t, t+\Delta t)$ is

$$\lambda_k \Delta t + 0(\Delta t),$$

the probability that it will go back to state s_{k-1} in the infinitesimal interval $(t, t+\Delta t)$ is

$$\mu_k \Delta t + 0(\Delta t)$$

and the probability that it will enter state s_{n+1} in that interval is

$$\lambda_{n+k} \Delta t + 0(\Delta t)$$

and the probability that it remains in state s_k in that interval is

$$1 - (\lambda_k + \lambda_{n+k} + \mu_k) \Delta t + 0(\Delta t) .$$

Thus conditions (1), (2)', (3) completely determine a finite flow of the operation behavior.

Similarly we can obtain the system of differential equations.

$$P_1'(t) = -(\lambda_1 + \lambda_{n+1}) P_1(t) + \mu_2 P_2(t)$$

$$P_k'(t) = \lambda_{k-1} P_{k-1}(t) - (\lambda_k + \lambda_{n+k} + \mu_k) P_k(t) + \mu_{k+1} P_{k+1}(t)$$

$$k = 2, 3, \ldots, n-1 \tag{4.6}$$

$$P_n'(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_{2n} + \mu_n) P_n(t)$$

with initial condition $P_1(0) = 1$, $P_k(0) = 0$, $k = 2, \ldots, n$.

or in matrix form

$$P'(t) = A\, P(t) \tag{4.7}$$

with initial condition $P(0) = I$, the identity matrix

where A is a nxn matrix

$$A = \begin{bmatrix} -(\lambda_1 + \lambda_{n+1}) & \mu_2 & & & \\ \lambda_1 & -(\lambda_2 + \lambda_{n+2} + \mu_2) & \mu_3 & & \\ & & \ddots & & \\ & & \lambda_{n-2} & -(\lambda_{n-1} + \lambda_{2n-1} + \mu_{n-1}) & \mu_n \\ & & & \lambda_{n-1} & -(\lambda_{2n} + \lambda_n) \end{bmatrix} = (a_{ij})$$

and $a_{ij} = \mu_{i+1}$, if $j = i+1$; $-(\lambda_i + \lambda_{n+i} + \mu_i)$ if $j = i$; λ_{i-1} if $j = i-1$; zero if $|j - i| > 1$.

The existence, uniqueness and the analytic properties of P(t) have been discussed in detail in [6]. The solution of the system (4.7) is

$$P(t) = e^{At} P(0) \tag{4.8}$$

The practical computation of (4.8) has been discussed in detail in [52].

4.3 Applications

For the proposed models, let us consider its application to some special cases, Example (i) is a case of the general death model, and example (ii) is a case of modified general death model with feedback, and example (iii) is a case of the general birth and death model.

(i) A system consists of two components in series. The failure law of one of the components follows an exponential distribution with a parameter σ, while that of the other follows an Erlang distribution of n type with a parameter λ. The sequential flow graph is the same as the one shown in Fig. 4.1, with $\lambda_i = \lambda$, $\lambda_{n+i} = \sigma$, $i = 1,\ldots n-1$, $\lambda_{2n} = \lambda + \sigma$.

We have

$$R(t) = \sum_{i=1}^{n} P_i(0) \cdot R_i(t)$$

where

$$R_i(t) = \sum_{k=i}^{n} P_{ik}(t)$$

and where $P_{ik}(t) = \mathcal{L}^{-1}\left[\dfrac{\lambda^{k-i}}{(s+\lambda+\sigma)^{k-i+1}}\right]$

$$= \frac{(\lambda t)^{k-i}}{(k-i)!} e^{-(\lambda+\sigma)t}$$

Thus we obtain

$$R(t) = \sum_{i=1}^{n} P_i(0) \cdot \sum_{k=i}^{n} \frac{(\lambda t)^{k-i}}{(k-i)!} \quad e^{-(\lambda+\sigma)t}$$

(ii) A Human Population Process.

Denote t as the age in years. We have a discrete time sequential flow graph as shown in Fig. 4.3, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the probability rates of failure from each state $s_1, s_2, \ldots, s_n$ to state s_f.

The resulting flow of transition satisfies the following conditions.

(1) As soon as a baby is born, he is in state s_1 (Hereafter we denote a baby as an item).

(2) T years after an item is born, the item is in state s_{t+1} and if the item survives in the interval $(t, t+1)$ then it will enter state s_{t+2} with probability one. During the interval $(t, t+1)$, α_{t+1} of the population in s_{t+1} will die, and the population (people) in state s_{t+1} will give birth to β_{t+1} of its population. Transitions are time sequential.

(3) After the instant of entering state s_n the item will operate according to an exponential law.

Thus we have the system of difference equations.

$$P_1(t+1) = f(t) = \sum_{i=2}^{n} \beta_i \, P_i(t) \qquad (4.9)$$

$$P_k(t+1) = P_{k-1}(t) \cdot (1-\alpha_{k-1}) \quad (k = 2, 3, \ldots, n)$$

Equation (4.9) is a set of recursive difference equations. Given the initial conditions $P_i(t_0)$, $i = 1, \ldots, n$, the solution for the system can be computed by successive application of (4.9)

$$[P_i(t_0)] \rightarrow [P_i(t_0 + 1)] \rightarrow \ldots \rightarrow [P_i(t)]$$

For the analytic solution, substitute $P_2(t+1), \ldots, P_n(t+1)$ of the second equations of (4.9) into the first equation of (4.9). We obtain

$$P_1(t+1) = \beta_2(1-\alpha_1)P_1(t-1) + \beta_3(1-\alpha_1)(1-\alpha_2)P_1(t-2) + \ldots + \beta_n(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1})P_1(t-n) \qquad (4.10)$$

Equation (4.10) is a linear difference equation of constant coefficients with degree n+1, the solution has been discussed in detail in [53] and $P_i(t+1)$ can be evaluated by

$$P_i(t+1) = \prod_{j=1}^{i-1}(1-\alpha_j)\cdot P_1(t+1-i) \quad (i = 2, 3, \ldots n) \qquad (4.11)$$

For large n, the solution of (4.10) can be obtained by computer computation.

Note that in this human population process, n may be as large as 80 or 100. To reduce the number of states, we regraph Fig. 4.3 as shown in Fig. 4.4a, where $s_{1\,1}, s_{1\,2}, \ldots, s_{1n_1}, s_{2\,1}, s_{2\,2}, \ldots, s_{2\,n_2}, \ldots, s_{k1}, s_{k\,2}, \ldots, s_{k\,n_k}$, are operation states, where $n_1+n_2+\ldots n_k = n$ and where β_{ij} and $P_{ij}(t)$ are respectively the corresponding rate of giving birth and the corresponding probability of being in state $s_{i\,j_i}$ at time t, where $i = 1, 2, \ldots, k$, $j_i = 1, 2, \ldots, n_i$.

We can merged the original state model of Fig. 4.4a into a modified state model as shown in Fig. 4.4b, where state $s_{i1}, s_{i2}, \ldots, s_{i\,n_i}$ are merged into a state s_i^*, $i = 1, \ldots, k$. The relationships of parameters and functions between the two state models, Fig. 4.4a and Fig. 4.4b, are as follows :

$$P_i^*(t) = \sum_{j=1}^{n_i} P_{ij}(t) \qquad (4.12)$$

$$f^*(t) = f(\sigma_i, P_i^*(t), i = 1, \ldots, k)$$

$$= \sum_{i=1}^{k} \sigma_i P_i^*(t) \qquad (4.12a)$$

where

$$\sigma_i = \text{Exp } \{\beta_{ij}\} \text{ with respect to } j$$

$$= \sum_{j=1}^{n_i} \phi_{ij} \beta_{ij}$$

where

ϕ_{ij} = the probability that an item is in state s_{ij} given that the item is in one of the states $s_{i1}, s_{i2}, \ldots, s_{in_i}$

$$= \frac{P_{ij}}{\sum_{j=1}^{n_i} P_{ij}} \qquad (4.12b)$$

From the second equation of (4.9),

$$P_{i2}(t) = (1 - a_{i1}) P_{i1}(t-1)$$

$$P_{i3}(t) = (1-a_{i1})(1-a_{i2}) P_{i1}(t-2)$$

$$\ldots$$

$$P_{in_i}(t) = (1-a_{i1})(1-a_{i2}) \ldots (1-a_{i\,n_i-1}) P_{i1}(t-n_i+1)$$

Assume that the population in state s_{i1} does not change much in the time interval $(t, t+n_i)$, $P_{i1}(t-1) = P_{i1}(t-2) = \ldots = P_{i1}(t-n_i+1) = P_{i1}$ then

$$P_{i2} = (1-a_{i1}) P_{i1}$$

$$P_{i3} = (1-a_{i1})(1-a_{i2}) P_{i1} \qquad (4.13)$$

$$\ldots$$

$$P_{in_i} = (1-a_{i1})(1-a_{i2}) \ldots (1-a_{i\,n_i-1}) P_{i1}$$

Hence ϕ_{ij} is evaluated as

$$\phi_{ij} = \frac{\prod_{m=1}^{j-1} (1-a_{im})}{1 + \sum_{j=1}^{n_i - 1} \prod_{m=1}^{j} (1 - a_{im})} \qquad j = 2, \ldots, n_i \qquad (4.13a)$$

The Numerator = 1, for j=1

The set of difference equations for the system of Fig. 4.4b is

$$P_1^*(t+1) = \sum_{i=1}^{k} \sigma_i P_i^*(t) + [1 - (\lambda_1 + \mu_1)] P_1^*(t)$$

$$P_i^*(t+1) = \lambda_{i-1} P_{i-1}^*(t) + [1 - (\lambda_i + \mu_i)] P_i^*(t) \qquad (i = 2, \ldots, k) \qquad (4.14)$$

where

$$\mu_i = \text{Exp}\{a_{ij}\} \qquad \text{with respect to } j$$

$$= \sum_{j=1}^{n_i} \phi_{ij} a_{ij} \qquad (4.14a)$$

and λ_i is obtained from the equivalent of the population of the flow from state s_{in_i} to state $s_{i+1\ 1}$ (Fig 4.4a) and that from state s_i^* to state s_{i+1}^* (Fig. 4.4b), i.e.,

$$(1-a_{in_i}) P_{in_i}(t) = \lambda_i P_i^*(t) \qquad (4.14b)$$

which implies

$$\lambda_i = \frac{P_{in_i}(t) \cdot (1-a_{in_i})}{P_i^*(t)},$$

$$= \frac{\prod_{j=1}^{n_i} (1 - a_{ij})}{1 + \sum_{j=1}^{n_i} \prod_{m=1}^{j} (1-a_{im})} \qquad (4.14c)$$

after substitution of (4.12) and (4.13).

(iii) A system consists of two components A_1 and A_2 in series. Component A_1 is expensive and non-replaceable, and therefore is designed with a very high reliability. Its failure law follows an exponential distribution with a parameter σ. Component A_2 is repairable (with constant repair rate μ) with n-1 redundant items, and the failure law of each item follows an exponential distribution with a parameter λ. Assume that the sensing of failed repairable items is perfect, and that there are sufficient repair men to repair any failed items. Then the sequential flow graph of the system is the same as that shown in Fig. 4.2, with $\lambda_i = (n-i+1)\lambda$, $\lambda_{n+1} = \sigma$, $i = 1, 2, \ldots, n-1$, and $\lambda_{2n} = \lambda + \sigma$, $\mu_i = (i-1)\mu$, $i = 1, 2, \ldots, n$.

4.4 Concluding remarks

For the system shown in Fig. 4.2, if $\lambda_{n+i} = 0$, $i = 1, \ldots, n$, then the general birth and death process becomes the "conventional" birth and death process. Since in the technical sense, the path function X(t) is often bounded, thus the proposed general birth and death model makes sense and does apply to real problems.

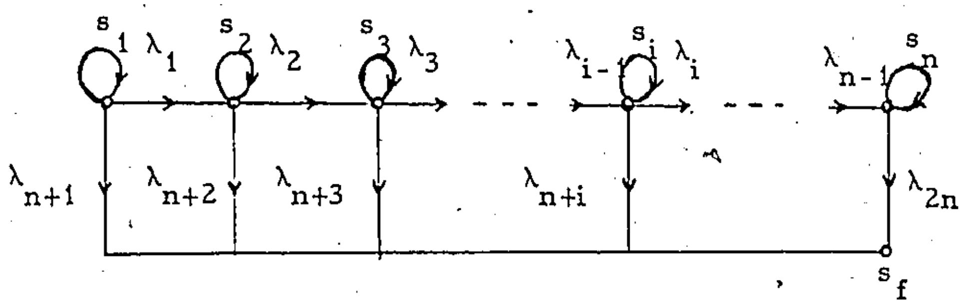


Fig. 4.1 A signal flow graph of a general death model.

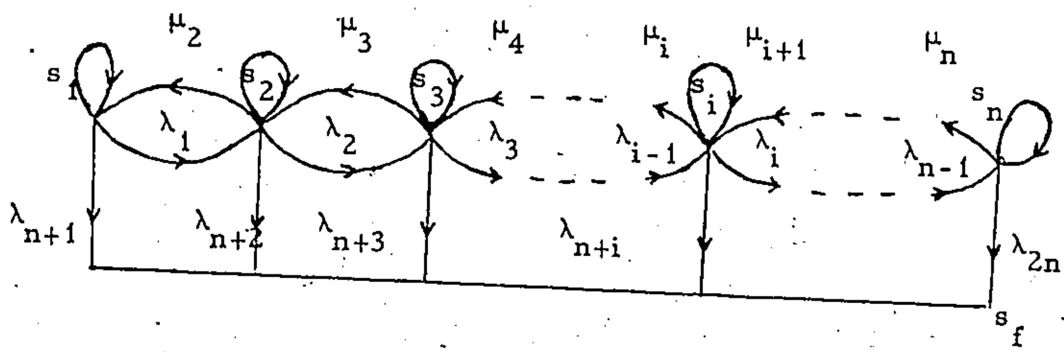


Fig. 4.2 A signal flow graph of a general birth and death model.

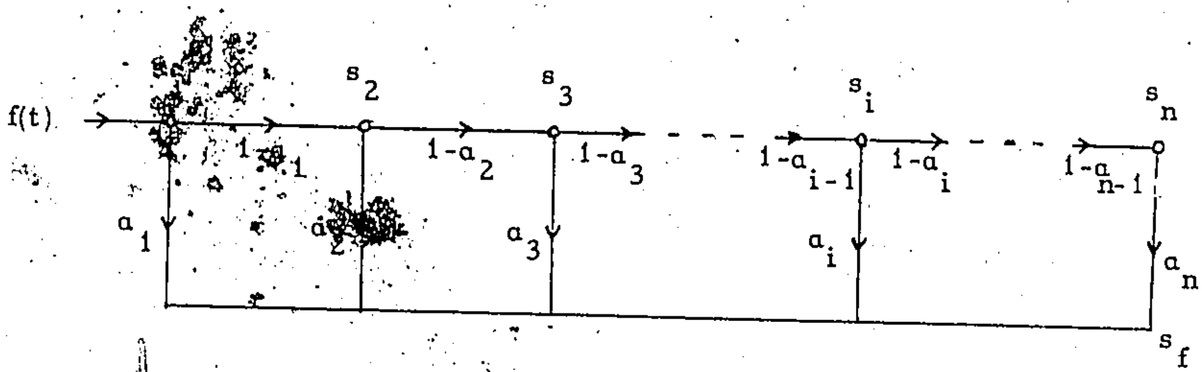


Fig. 4.3 A human population model.

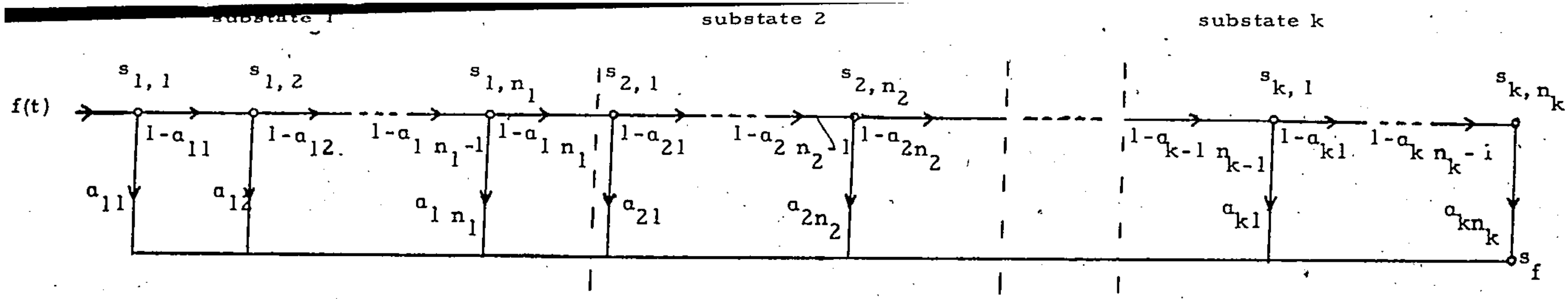


Fig. 4.4a A regraph of Fig. 4.3

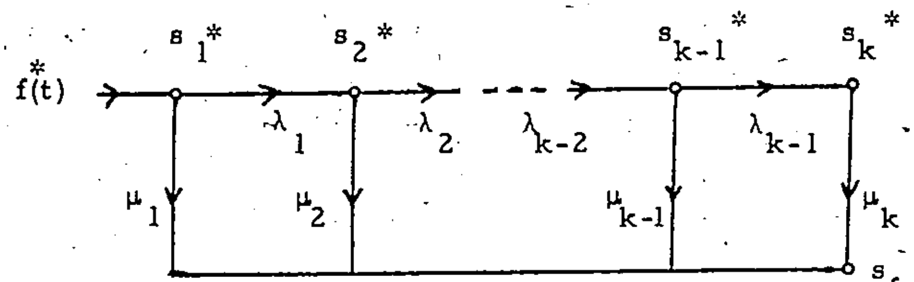


Fig. 4.4b A merging form of Fig. 4.4a

CHAPTER V

RELIABILITY PREDICTION TECHNIQUES FOR SYSTEMS WITH DEPENDENT FAILURE AND REPAIR RATES

5.1 Introduction

Usually in a reliability study we assume that components (or systems) in a system are functioning independently of each other and the repair rates do not change with system states, so that a simple Markov model can be formulated [3, 15]. However, in many physical situations (such as in the case of circuit designs) this kind of independence assumption often leads to poor results in reliability computation. Factors such as heat dissipation, vibration, etc., often cause dependence among components of a system. The repair rates of components in a system might be changed due to the system design and the maintenance policy. In this chapter a Markov model for systems of components with dependent failure and repair rates are formulated to obtain a better result in reliability computation.

In general a Markov model for an m-component system has 2^m states, but as far as the reliability and the availability of a system are concerned, the number of system states, corresponding to the number of failed componets, can be reduced to m+1 as proposed by Shooman [3, p 237]. Thus a r-out-of-n system can be described by n-r+2 system states. Evidently n-r+2 is a minimum number of states. It is assumed that failure rates of a component in different system states are not the same, but all are exponentially distributed. The system is expressed by a flow graph, in which only transition between consecutive states are possible so that it is straightforward to evaluate mean-time-to-first-system-failure and steady-state availability by signal flow graph techniques [10, 11, 15].

5.2. A r-out-of-n system

5.2.1 On defining system states and the corresponding transition rates.

Consider a system consisting of n components. The system is considered successful, if at least r-out-of-n components are in operating condition. Assume that n components are in perfect condition at $t = 0$, and one repair man is available for servicing "first-come-first-served". Further assume that the failure and repair time distributions of all components in all states are exponentially distributed. The failure rate of component i, $i = 1, 2, \ldots, n$ is λ_i^0 when all components are in operating condition, while the failure rate of component i is λ_i^1 when one component ha failed, and so on, i.e., λ_i^j is the failure rate of component i when j (excluding component i) out of n components have failed, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots n-r$. Note that the estimated value λ_i^j can be obtained from accelerated life testing based on predicted working conditions. Similarly, μ_i^j is the repair rate of failed component i when j (including component i) out of n components have failed, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n-r+1$.

We will define system states after reviewing the previous approach [21].

In the previous approach [21], $\hat{n}$ states are required for describing the system states to obtain reliability and availability formula, where

$$\hat{n} = n^0 + n^1 + n^2 + \ldots + n^{n-1} + n^n = 2^n \qquad (5.1)$$

and

$$n^i = C_i^n \qquad (5.2)$$

Let x_i represent success of component i

$\bar{x}_i$ represent failure of component i

Then system states are defined as shown on Table 5.1.

Table 5.1 System state notations

Zero failures	$s_1^0 = x_1 x_2 x_3 \ldots x_{n-1} x_n$
One failure	$s_1^1 = \bar{x}_1 x_2 x_3 \ldots x_{n-1} x_n$
	$s_2^1 = x_1 \bar{x}_2 x_3 \ldots x_{n-1} x_n$
	$s_3^1 = x_1 x_2 \bar{x}_3 \ldots x_{n-1} x_n$
$(n^1 = C_1^n)$	. . . .
	$s_{n^1-1}^1 = x_1 x_2 x_3 \ldots \bar{x}_{n-1} x_n$
	$s_{n^1}^1 = x_1 x_2 x_3 \ldots x_{n-1} \bar{x}_n$
Two failures	$s_1^2 = \bar{x}_1 \bar{x}_2 x_3 \ldots x_{n-1} x_n$
	$s_2^2 = \bar{x}_1 x_2 \bar{x}_3 \ldots x_{n-1} x_n$
	$s_3^2 = \bar{x}_1 x_2 x_3 \bar{x}_4 \ldots x_{n-1} x_n$
$(n^2 = C_2^n)$	. . .
	$s_{n^2-1}^2 = x_1 x_2 x_3 \ldots \bar{x}_{n-2} x_{n-1} \bar{x}_n$
	$s_{n^2}^2 = x_1 x_2 x_3 \ldots x_{n-2} \bar{x}_{n-1} \bar{x}_n$
	. . .
	. . .
	. . .
n failures	$s_1^n = \bar{x}_1 \bar{x}_2 \bar{x}_3 \ldots \bar{x}_{n-1} \bar{x}_n$

Total number of states $= C_0^n + C_1^n + \ldots + C_n^n = 2^n$.

Since if more than n-r+1 components have failed, the system is bad.

Thus the sufficient number of states for a r-out-of-n system

$= C_o^n + C_1^n + \ldots + C_{n-r}^n + 1$, where 1 is due to the merged failure state.

The corresponding transition matrix is shown on table 5.2.

Table 5.2 State Transition Matrix For A System with 2^n states

		0 Failures	1 Failures				2 Failures					n Failures
		s_1^0	s_1^1	s_2^1	...	$s_{n^1}^1$	s_1^2	s_2^2	...	$s_{n^2}^2$	...	s^n
0 Failures	s_1^0	$-\sum_1^n \lambda_i^0$	λ_1^0	λ_2^0	...	λ_n^0	0	0	...	0		0
	s_1^1	0	$\sum_{i\neq 1}^{n^1} \lambda_i^1$	0	...	0	λ_2^1	λ_3^1	...	$\lambda_{n^1}^1$, 0 ... 0		0
1	s_2^1	0	0	$\sum_{i\neq 2}^{n^1} \lambda_i^1$	...	0	λ_1^1	0	λ_3^1 ...	0		0
Failure	⋮	⋮	⋮			⋮	⋮					⋮
	$s_{n^1}^1$	0	0	...		$\sum_{i\neq n^1} \lambda_i^1$	0	0	... λ_{n^1-1}	0		0
	s_1^2	0	0	0		0	$-\sum_{i\neq 1,2}^{n^2} \lambda_i^2$	0 ...		0		0
2	s_2^2	0	0	0	...	0	0	$\sum_{i\neq 1,3}^{n^2} \lambda_i^2$	0 ... 0			⋮
Failures	⋮	⋮	⋮	⋮		⋮	⋮					0
	$s_{n^2}^2$	0	0	0		0	0	0	$\sum_{i\neq n^2-1, n^2}^{n^2} \lambda_i^2$			0
⋮												⋮ 0
n Failures	s_1^n	0	0	...		0	0	...		0 0 ... 0		0

It is observed that in obtaining the reliability function, one ends up with solving a set of 2^n first-order differential equations. It is not practical to process a $2^n \times 2^n$ matrix (for large n) even by a computer. Without obtaining solutions, the model is unrealistic. Thus if it is necessary to solve the system, some realistic method should be developed. In this chapter, an approximation method is developed to solve the system.

For clarity, firstly consider a simple 1-out-of-3 nonrepairable system. Its transition matrix is shown on Table 5.3, and its transition graph is shown in Fig. 5.1a. $\dot{p}(t) = p(t)\,A$

Table 5.3 State Transition Matrix For a 1-out-of-3 System

A =

	0 Failures	1 Failure			2 Failures			3 Failures
	s_1^0	s_1^1	s_2^1	s_3^1	s_1^2	s_2^2	s_3^2	s^3
s_1^0	$-(\lambda_1^0+\lambda_2^0+\lambda_3^0)$	λ_1^0	λ_2^0	λ_3^0	0	0	0	0
s_1^1	0	$-(\lambda_2^1+\lambda_3^1)$	0	0	λ_2^1	λ_3^1	0	0
s_2^1	0	0	$-(\lambda_3^1+\lambda_1^1)$	0	λ_1^1	0	λ_3^1	0
s_3^1	0	0	0	$-(\lambda_1^1+\lambda_2^1)$	0	λ_1^1	λ_2^1	0
s_1^2	0	0	0	0	$-\lambda_3^2$	0	0	λ_3^2
s_2^2	0	0	0	0	0	$-\lambda_2^2$	0	λ_2^2
s_3^2	0	0	0	0	0	0	$-\lambda_1^2$	λ_1^2
s_1^3	0	0	0	0	0	0	0	0

It can be easily shown that if components are identical,

i.e.,

$$\lambda_1^j = \lambda_2^j = \lambda_3^j = \lambda_j , \qquad j = 0, 1, 2 \tag{5.3}$$

then the probabilities in the general and the collapsed model (Fig. 5.1b) are related by

$$\begin{aligned} P_{s_0} &= P_{s_1^0} \\ P_{s_1} &= P_{s_1^1} + P_{s_2^1} + P_{s_3^1} , \\ P_{s_2} &= P_{s_1^2} + P_{s_2^2} + P_{s_3^2} \\ P_{s_3} &= P_{s_1^3} \end{aligned} \tag{5.4}$$

and hazards (transition rates) satisfy

$$\begin{aligned} \lambda^0 &= 3\lambda_0 \\ \lambda^1 &= 2\lambda_1 \\ \lambda^2 &= \lambda_2 \end{aligned} \tag{5.5}$$

It can also be shown that sufficient conditions for (5.4) and (5.5) are weaker than that of (5.3), and are

$$\lambda_1^j = \lambda_2^j = \lambda_3^j , \qquad j = 1, 2 \tag{5.6}$$

It can be extended to a general r-out-of-n system. A general Markov model, where system states are defined on Table 5.1 can be reduced to a general collapsed model as shown in Fig. 5.2,

if $\lambda_i^j = \lambda_k^j = \lambda_j$, $i,k=1,\ldots, n;\ j = 1,2,\ldots, n-r$.

Then the probabilities in the general and the collapsed models are related by

$$P_{s_0} = P_{s_1^0}$$

$$P_{s_j} = \sum_{i=1}^{n^i} P_{s_i^j} \;, \quad \text{where } n^i = C_i^n, \quad j = 1, \ldots, n-r \qquad (5.8)$$

$$P_{s_{n-r+1}} = P_{s_1^{n-r+1}}, \quad \text{where } s^{n-r+1} \text{ is the merged failure state,}$$

and hazards satisfy

$$\lambda^j = (n-j)\,\lambda_j \quad, \; j = 0, 1, \ldots, n-r \qquad (5.9)$$

For the general case $\lambda_i^j \neq \lambda_k^j \quad i, k = 1, \ldots, n, \; i \neq k; \; j=0, \ldots, n-r,$ (5.10)

we propose an equivalent collapsed model whose transition graph is the same as the one shown in Fig. 5.2, with approximated parameters λ^j as defined in equation (5, 14). Since dependences of failure rates have been incorporated in λ_i^j, $i = 1, \ldots, n$, $j = 0, 1, \ldots, n-r+1$, if j out of n components have failed then the occurence of failure of one of the n-j remaining components is independent of the octurence of any one of the other remaining components. A failure of one of the n-j remaining components willcause the system to change from a set of states $s^j = (s_1^j, s_2^j, \ldots, s_{n^j}^j)$ to a set of states $s^{j+1} = (s_1^{j+1}, s_2^{j+1}, \ldots, s_{n^{j+1}}^{j+1})$. Thus as far as the transition from s^j to s^{j+1} is concerned the n-j remaining components are in series connection in terms of reliability. Assign a number to each component, where numbers are in the order of 0, 1, ..., n-1. Let i_m, $m = 0, 1, 2, \ldots, n-r$ be the number of m+1-th failed component, $i_m \in (1, 2, \ldots, n)$. If components $i_0, i_1, \ldots, i_{j-1}$

have failed, then the transition from the j-th failure set of states $s^j = (s^j_1, s^j_2, \ldots, s^j_{n^j})$ to the j+1-th failure set of states $s^{j+1} = (s^{j+1}_1, s^{j+1}_2, \ldots, s^{j+1}_{n^{j+1}})$ will follow an exponential law with a

$$\text{parameter } \omega^j = \sum_{k=j}^{n-1} \lambda^j_{i_k} . \tag{5.11}$$

The parameter λ^j is obtained by taking the expectation of ω^j

$$\lambda^j = \text{Exp} \{ \omega^j \} \text{ with respect to all permutations obtained by taking } j \text{ out of } n \text{ in order, } P^n_j \tag{5.12}$$

where λ^j = the probability rate of transition from state s_j to state s_{j+1}, i.e., the failure rate that any one of the remaining n-j components will fail when j components have failed, $j = 0, 1, \ldots, n-r$.

Denote symbols as follows :

$i_0, i_1, \ldots, i_m$: represent component $i_0, i_1, \ldots, i_m$ have failed in the order of component i_0, component i_1, and so on.

$P^j_{i_0, i_1, \ldots, i_{j-1}, i_j}$: $\lambda^j_{i_j} / \omega^j$; the probability that the j+1-th failed component is component i_j given that components $i_0, i_1, \ldots, i_{j-1}$ have failed $j=0, 1, \ldots, n-r$ (5.13)

From (5.12) it can be shown that

λ^0 = the failure rate of any one of n components given that all n components are operative

$$= \sum_{i_m=1}^{n} \lambda^0_{i_m}$$

λ^1 = Exp { the failure rate of any one of the n-1 remaining components, given that one component has failed }

$$= \sum_{i_0=1}^{n} P^0_{i_0} \cdot \omega^1 = \sum_{i_0=1}^{n} P^0_{i_0} \left(\sum_{\substack{k=1 \\ k \neq i_0}}^{n} \lambda_k^1 \right)$$

$$\lambda^j = \sum_{i_0=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{n} \cdots \sum_{\substack{i_{j-1}=1 \\ i_{j-1} \neq i_{j-2} \neq \cdots \neq i_1 \neq i_0}} P^0_{i_0} P^1_{i_0 i_1} \cdots P^{j-1}_{i_0 i_1 \cdots i_{j-1}} \omega^j \qquad (5.14)$$

Denote

μ^j = the probability rate of transition from state s_j to state s_{j-1}, i.e., the repair rate that any one of the j failed components is repaired to operating condition given that j components have failed, $j = 1, 2, \ldots n-r+1$.

Similarly, μ^j can be obtained by taking the expectation of the j-th failure set of states $s^j = (s^j_1, s^j_2, \ldots, s^j_{n^j})$ to the (j-1)-th failure set of states $s^{j-1} = (s^{j-1}_1, s^{j-1}_2, \ldots, s^{j-1}_{n^{j-1}})$. It can be shown that for a first-come first-served one repairman system, the repair rates are

$$\mu^1 = \sum_{i_0=1}^{n} P^0_{i_0} \cdot \mu^1_{i_0}$$

$$\mu^2 = \sum_{i_0=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{n} P^0_{i_0} P^1_{i_0 i_1} \mu^2_{i_0} = \sum_{i_0=1}^{n} P^0_{i_0} \mu^2_{i_0}$$

$$\mu^j = \sum_{i_0=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{n} \cdots \sum_{\substack{i_{j-1}=1 \\ i_{j-1} \neq i_{j-2} \neq \cdots \neq i_1 \neq i_0}}^{n} P^0_{i_0} P^1_{i_0 i_1} \cdots P^{j-1}_{i_0 i_1 \cdots i_{j-1}} \mu^j_{i_0} \tag{5.15}$$

$$= \sum_{i_0=1}^{n} P^0_{i_0} \mu^j_{i_0} \qquad j = 1, 2, \ldots, n-r+1,$$

For a first-come first-served k repairman system, assume that any failed component is repaired by only one of k repairmen then the repair rates are

$$\mu^j = = \sum_{i_0=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{n} \cdots \sum_{\substack{i_{j-1}=1 \\ i_{j-1} \neq i_{j-2} \neq \cdots \neq i_1 \neq i_0}}^{n} P^0_{i_0} P^1_{i_0 i_1} \cdots P^{j-1}_{i_0 i_1 \cdots i_{j-1}} \cdot y^j \tag{5.15a}$$

where

$$y^j = \sum_{m=0}^{j-1} \mu^j_{i_m} \qquad \text{if} \qquad j \leq k$$

$$y^j = \sum_{m=0}^{k-1} \mu^j_{i_m} \qquad \text{if} \qquad j > k \tag{5.15b}$$

Let us denote

$P_i(t)$: Probability that the system is in state i at time t, $i = 0, 1, 2, \ldots, n-r+1$.

$$\sum_i P_i(t) = 1$$

Thus we can obtain the set of difference equations from the sequential flow graph. Assume that the probability of more than one transition in an infinitesimally small interval $(t, t+\Delta t)$ is $0(\Delta t)$ such that

$$\lim_{\Delta t \to 0} \frac{0(\Delta t)}{\Delta t} = 0$$

We obtain the set of differential equations for the reliability function

$$P'_0(t) = -\lambda^0 P_0(t) + \mu^1 P_1(t)$$

$$P'_i(t) = \lambda^{i-1} P_{i-1}(t) - (\lambda^i + \mu^i) P_i(t) + u^{i+1} P_{i+1}(t), \quad i=1,\ldots,n-r-1$$

$$P'_{n-r}(t) = \lambda^{n-r-1} P_{n-r-1}(t) - (\lambda^{n-r} + \mu^{n-r}) P_{n-r}(t) \qquad (5.16)$$

$$P'_f(t) = \lambda^{n-r} P_{n-r}(t)$$

or in matrix form

$$\dot{P}(t) = P(t) \cdot A \qquad (5.17)$$

where P(t) is a row matrix

$$P(t) = [P_0(t) \; P_1(t) \ldots P_{n-r}(t) \; P_f(t)]$$

and

A is a $(n-r+2)$ by $(n-r+2)$ matrix

$$A = \begin{bmatrix} -\lambda^0 & \lambda^0 & & & & & \\ \mu^1 & -(\lambda^1+\mu^1) & \lambda^1 & & & & \\ & \mu^2 & -(\lambda^2+\mu^2) & \lambda^2 & & & \\ & & \mu^3 & \ddots & \lambda^{n-r-2} & & \\ & & & \mu^{n-r-1} & -(\lambda^{n-r+1}+\mu^{n-r+1}) & \lambda^{n-r+1} & \\ & & & & \mu^{n-r} & -(\lambda^{n-r}+\mu^{n-r}) & \lambda^{n-r} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

The reliability of the system is

$$R(t) = \sum_{i=0}^{n-r} P_i(t) \tag{5.18}$$

$$= 1 - P_f(t)$$

Similarly the set of differential equations for the Availability function is

$$P'_0(t) = -\lambda^0 P_0(t) + \mu^1 P_1(t)$$

$$P'_i(t) = \lambda^{i-1} P_{i-1}(t) - (\lambda^i + \mu^i) P_i(t) + \mu^{i+1} P_{i+1}(t), \quad i = 1, 2, \ldots, n-r. \tag{5.19}$$

$$P'_f(t) = \lambda^{n-r} P_{n-r}(t) - \mu^{n-r+1} P_f(t)$$

or in matrix form

$$\dot{P}(t) = P(t) \cdot B \tag{5.20}$$

where B is a $(n-r+2)$ by $(n-r+2)$ matrix

$$B = \begin{bmatrix} -\lambda^0 & \lambda^0 & & & & & \\ \mu^1 & -(\lambda^1+\mu^1) & \lambda^1 & & & & \\ & \mu^2 & -(\lambda^2+\mu^2) & \lambda^2 & & & \\ & & & \cdot & & \lambda^{n-r-1} & \\ & & & & \cdot & & \\ & & & & \mu^{n-r} & -(\lambda^{n-r}+\mu^{n-r}) & \lambda^{n-r} \\ & & & & & \mu^{n-r+1} & -\mu^{n-r+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

The availability of the system is

$$A(t) = \sum_{i=1}^{n-r} P_i(t) \tag{5.21}$$

$$= 1 - P_f(t)$$

The reliability and availability functions can also be evaluated by a linear signal-flow graph [9, 10] or by a stochastic transition signal-flow graph [17]. However closed form solutions for the reliability function and the availability function are still not available; But, whereas we can find a closed-form solution for steady-state availability. The value of mean time to first system failure can also be computed as stated in Section 5.2.2 and 5.2.3.

5.2.2 Mean-time-to-first-system-failure

Since the reliability measure is also expressed by the mean time to first system failure,

$$T_m = \int_0^\infty R(t)\, dt = \left| \hat{R}(s) \right|_{s=0} \tag{5.22}$$

we evaluate T_m for the r-out-of-n system.

Define the notation

$$P_i = P_i(\infty) = \lim_{s \to o} s\hat{P}_i(s) \tag{5.23}$$

Since $R(t) = \sum_{i=o}^{n-r} P_i(t)$ (5.24)

Then

$$\left| \hat{R}(s) \right|_{s=o} = \left| \sum \hat{P}_i(s) \right|_{s=o} \tag{5.25}$$

Hence

$$T_m = \sum_{i=o}^{n-r} P_i \tag{5.26}$$

The flow graph for determining P_i is shown in Fig. 5.4

There are n-r loops

$$L_1 = \frac{\lambda^o}{\lambda^1+\mu^1} \cdot \frac{\mu^1}{\lambda^o} = \frac{\mu^1}{\lambda^1+\mu^1}$$

$$L_2 = \frac{\lambda^1 \mu^2}{(\lambda^1+\mu^1)(\lambda^2+\mu^2)}$$

$$L_{n-r} = \frac{\lambda^{n-r-1}\mu^{n-r}}{(\lambda^{n-r-1}+\mu^{n-r-1})(\lambda^{n-r}+\mu^{n-r})}$$

or $$L_i = \frac{\lambda^{i-1}\mu^i}{(\lambda^{i-1}+\mu^{i-1})(\lambda^i+\mu^i)} \qquad i = 1,\ldots, n-r \tag{5.27}$$

$$\mu^o = 0$$

The graph determinant is

$$\Delta = 1 - \sum_{i=1}^{n-r} L_i + \sum_{i=1}^{n-r-2} \sum_{j=3}^{n-r} L_i L_j - \sum_{i=1}^{n-r-4} \sum_{\substack{j=3 \\ j\neq i \\ j\neq i+1}}^{n-r-2} \sum_{k=5}^{n-r} L_i L_j L_k + \dots + R(\Delta) \tag{5.28}$$

where $R(\Delta) = (-1)^{n-r+1/2} L_1 L_3 \dots L_{n-r}$ if n-r is odd

$= (-1)^{n-r/2} L_1 L_3 \dots L_{n-r-1}$

$+ (-1)^{n-r/2} L_2 L_4 \dots L_{n-r}$ if n-r is even

P_i can be shown as

$$P_0 = \frac{1}{\lambda_0} / \Delta$$

$$P_1 = [\frac{1}{\lambda_0} \cdot \frac{\lambda^0}{\lambda^1 + \mu^1}] / \Delta$$

$$P_i = \frac{\prod_{j=o}^{i-1} \lambda^j}{\prod_{j=o}^{i} (\lambda^j + \mu^j)} / \Delta \qquad i = 1, \dots, n-r \quad \mu^o = 0 \tag{5.29}$$

Therefore $T_m = P_0 + P_1 + \dots + P_{n-r}$

$$= [\frac{1}{\lambda^0} + \frac{1}{\lambda^0} \cdot \frac{\lambda^0}{\lambda^1 + \mu^1} + \dots + \frac{1}{\lambda^0} \frac{\lambda^0}{(\lambda^1 + \mu^1)} \dots \frac{\lambda^{n-r-1}}{(\lambda^{n-r} + \mu^{n-r})}] / \Delta$$

$$= [1 + \sum_{i=1}^{n-r} (\frac{\prod_{j=o}^{i-1} \lambda^j}{\prod_{j=o}^{i} (\lambda^j + \mu^j)})] / \lambda^0 \Delta \tag{5.30}$$

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5.2.3 Steady-state availability

The steady-state flow graph of the system is the same as the one shown in Fig. 5.3 . The method of solving the problem is based on

{ steady-state probability flowing out of state s_i}

= {steady-state probability flowing into state s_i} ,

$i = 0, 1, 2, \ldots, n-r+1$

It can be readily shown that

$$P_i = \frac{\prod_{j=o}^{i-1} \lambda^j}{\prod_{j=1}^{i} \mu^j} P_0 \qquad i = 1, \ldots, n-r+1 \qquad (5.31)$$

Hence

$$P_f = P_{n-r+1} = \frac{\prod_{j=o}^{n-r} \lambda^j}{\prod_{j=1}^{n-r+1} \mu^j} P_0 \qquad (5.32)$$

Since $\sum_{i=o}^{n-r+1} P_i = 1$

then
$$P_0 = [1 + \sum_{i=1}^{n-r+1} (\frac{\prod_{j=o}^{i-1} \lambda^j}{\prod_{j=o}^{i} \mu^j})]^{-1} \qquad (5.33)$$

The steady-state availability is

$$A_v = 1 - P_f = 1 - (\frac{\prod_{j=o}^{n-r} \lambda^j}{\prod_{j=1}^{n-r+1} \mu^j}) [1 + \sum_{i=1}^{n-r+1} (\frac{\prod_{j=o}^{i-1} \lambda^j}{\prod_{j=o}^{i} \mu^j})]^{-1} \qquad (5.34)$$

5.3 An example

Consider a 3-out-of-5 system; the flow graph is shown in Fig. 5.5.

5.3.1. Reliability and Availability models

Denote

$\hat{P}_i(s)$: The Laplace transform of $P_i(t)$

$\hat{P}_f(s)$: The Laplace transform of $P_f(t)$

$\hat{R}(s)$: The Laplace transform of $R(t)$

The reliability flow graph for determining $\hat{P}_f(s)$ is shown in Fig. 5.6

Then

$$\hat{R}(s) = \hat{P}_0(s) + \hat{P}_1(s) + \hat{P}_2(s) = \frac{1}{s} - \hat{P}_f(s) \qquad (5.35)$$

$\hat{P}_f(s)$ is the Source-to-Sink Transmission,

$$\hat{P}_f(s) = \frac{T_{sf} \cdot \Delta_{sf}}{\Delta} \qquad (5.36)$$

where Δ is the determinant of the flow graph, T_{sf} is the path transmission from source to sink (s-f-path), and Δ_{sf} is the co-factor of the s-f-path.

There are 5 loops in Fig. 5.6,

$$L_0 = -\lambda^0 / s$$

$$L_1 = -\frac{(\lambda^1 + \mu^1)}{s}$$

$$L_2 = -\frac{(\lambda^2 + \mu^2)}{s}$$

$$L_{01} = \frac{\lambda^0 \quad \mu^1}{s^2}$$

$$L_{12} = \frac{\lambda^1 \cdot \mu^2}{s^2}$$

Hence

$$\Delta = 1 + \left[\frac{\lambda^0}{s} + \frac{(\lambda^1+\mu^1)}{s} + \frac{(\lambda^2+\mu^2)}{s} - \frac{\lambda^0\mu^1}{s^2} - \frac{\lambda^1\mu^2}{s^2} \right]$$

$$+ \left[\frac{\lambda^0(\lambda^1+\mu^1)}{s^2} + \frac{\lambda^0(\lambda^2+\mu^2)}{s^2} + \frac{(\lambda^1+\mu^1)(\lambda^2+\mu^2)}{s^2} - \frac{\lambda^0\lambda^1\mu^2}{s^3} - \frac{\lambda^0\mu^1(\lambda^2+\mu^2)}{s^3} \right]$$

$$+ \frac{\lambda^0(\lambda^1+\mu^1)(\lambda^2+\mu^2)}{s^3}$$

$$= \frac{s^3+(\lambda^0+\lambda^1+\lambda^2+\mu^1+\mu^2)s^2+(\lambda^0\lambda^1+\lambda^0\lambda^2+\lambda^0\mu^2+\lambda^1\lambda^2+\lambda^2\mu^1+\mu^1\mu^2)s+\lambda^0\lambda^1\lambda^2}{s^3}$$

$$\Delta_{sf} = 1$$

$$T_{sf} = \frac{\lambda^0\lambda^1\lambda^2}{s^4}$$

Thus $$\hat{P}_f(s) = \frac{\lambda^0\lambda^1\lambda^2}{s^4 \cdot \Delta} \qquad (5.37)$$

and from equation (5.35)

$$\hat{R}(s) = \frac{s^2 + C_1 s + C_2}{s^3 + C_1 s^2 + C_2 s + C_3} \qquad (5.38)$$

where $C_1 = \lambda^0 + \lambda^1 + \lambda^2 + \mu^1 + \mu^2$

$C_2 = \lambda^0\lambda^1 + \lambda^0\lambda^2 + \lambda^0\mu^2 + \lambda^1\lambda^2 + \lambda^2\mu^1 + \mu^1\mu^2$

$C_3 = \lambda^0\lambda^1\lambda^2$

Similarly, the availability flow graph for determining $\hat{P}_f(s)$ is shown in Fig. 5.7.

Then

$$\hat{A}(s) = \hat{P}_0(s) + \hat{P}_1(s) + \hat{P}_2(s) = \frac{1}{s} - \hat{P}_f(s)$$

and $\hat{P}_f(s)$ is given by equation (5.36).

There are 7 loops in Fig. 5.7,

$$L_0 = -\frac{\lambda^0}{s}$$

$$L_1 = -\frac{(\lambda^1+\mu^1)}{s}$$

$$L_2 = -\frac{(\lambda^2+\mu^2)}{s}$$

$$L_3 = -\frac{\mu^3}{s}$$

$$L_{01} = \frac{\lambda^0\mu^1}{s^2}$$

$$L_{12} = \frac{\lambda^1\mu^2}{s^2}$$

$$L_{23} = \frac{\lambda^2\mu^3}{s^2}$$

Hence

$$\begin{aligned}\Delta = \; & 1 - [L_0 + L_1 + L_2 + L_3 + L_{01} + L_{12} + L_{23}] \\ & + [(L_0L_1 + L_0L_2 + L_0L_3 + L_1L_2 + L_1L_3 + L_2L_3) \\ & \quad + L_{01}(L_2+L_3) + L_{12}(L_0+L_3) + L_{23}(L_0+L_1)] \\ & - [(L_0L_1L_2 + L_0L_1L_3 + L_0L_2L_3 + L_1L_2L_3) \\ & \quad + L_0L_1L_{23} + L_0L_3L_{12} + L_2L_3L_{01})] \\ & + L_0L_1L_2L_3\end{aligned}$$

$$= 1 + \frac{1}{s}(\lambda^0+\lambda^1+\lambda^2+\mu^1+\mu^2+\mu^3) + \frac{1}{s^2}(-\lambda^0(\lambda^1+\lambda^2+\mu^2+\mu^3)+\lambda^1(\lambda^2+\mu^3) + \lambda^2\mu^1+\mu^1\mu^2+\mu^1\mu^3+\mu^2\mu^3)$$

$$+ \frac{1}{s^3}(\lambda^0\lambda^1\lambda^2+\lambda^0\lambda^1\mu^3 + \lambda^0\mu^2\mu^3 + \mu^1\mu^2\mu^3) - \frac{\lambda^0\lambda^2\mu^1\mu^3}{s^4}$$

$$\Delta_{sf} = 1$$

$$T_{sf} = \frac{\lambda^0\lambda^1\lambda^2}{s^4}$$

Thus $\hat{P}_f(s) = \dfrac{\lambda^0\lambda^1\lambda^2}{s^4 \cdot \Delta}$

and

$$\hat{A}(s) = \frac{s^4 + d_1 s^3 + d_2 s^2 + d_3^1 s + d_4}{s(s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4)} \qquad (5.39)$$

where $d_1 = \lambda^0 + \lambda^1 + \lambda^2 + \mu^1 + \mu^2 + \mu^3$

$$d_2 = \lambda^0(\lambda^1+\lambda^2+\mu^2+\mu^3) + \lambda^1(\lambda^2+\mu^3) + \lambda^2\mu^1+\mu^1\mu^2 + \mu^1\mu^3 + \mu^2\mu^3$$

$$d_3 = \lambda^0\lambda^1\lambda^2 + \lambda^0\lambda^1\lambda^3 + \lambda^0\mu^2\mu^3 + \mu^1\mu^2\mu^3$$

$$d_3^1 = \lambda^0\lambda^1\lambda^3 + \lambda^0\mu^2\mu^3 + \mu^1\mu^2\mu^3$$

$$d_4 = -\lambda^0\lambda^2\mu^1\mu^3$$

For quantitative values of λ's, μ's, roots of the denominator in equations (5.38), (5.39) can be found and R(t) can be evaluated by using a computer.

5.3.2 Mean-Time-To-First-System-Failure Model

The flow graph for determining P_i is shown in Fig. 5.8. The graph determinant of Fig. 5.8 is

$$\Delta = 1 - \frac{\lambda^0}{\lambda^1 \mu^1} \cdot \frac{\mu^1}{\lambda^0} - \frac{\lambda^1}{\lambda^2+\mu^2} \cdot \frac{\mu^2}{\lambda^1+\mu^1}$$

$$= \frac{\lambda^1 \lambda^2}{(\lambda^1+\mu^1)(\lambda^2+\mu^2)}$$

Therefore

$$T_m = P_0 + P_1 + P_2$$

$$= \left[\frac{1}{\lambda^0} + \frac{1}{\lambda^0} \cdot \frac{\lambda^0}{(\lambda^1+\mu^1)} + \frac{1}{\lambda^0} \cdot \frac{\lambda^0}{(\lambda^1+\mu^1)} \cdot \frac{\lambda^1}{(\lambda^2+\mu^2)}\right] / \Delta$$

$$= \frac{\lambda^0 \lambda^1 + \lambda^0(\lambda^2+\mu^2) + (\lambda^1+\mu^1)(\lambda^2+\mu^2)}{\lambda^0 \lambda^1 \lambda^2}$$

5.3.3 Steady-state Availability

From equation (5.31) we have

$$P_1 = \frac{\lambda^0}{\mu^1} P_0$$

$$P_2 = \frac{\lambda^0 \lambda^1}{\mu^1 \mu^2} P_0$$

$$P_3 = \frac{\lambda^0 \lambda^1 \lambda^2}{\mu^1 \mu^2 \mu^3} P_0$$

Using equation (5.33)

$$P_0 = [1 + \frac{\lambda^0}{\mu^1} + \frac{\lambda^0\lambda^1}{\mu^1\mu^2} + \frac{\lambda^0\lambda^1\lambda^2}{\mu^1\mu^2\mu^3}]^{-1}$$

Thus the steady-state availability is

$$A_v = P_0 + P_1 + P_2$$

$$= \frac{(1 + \frac{\lambda^0}{\mu^1} + \frac{\lambda^0\lambda^1}{\mu^1\mu^2})}{(1 + \frac{\lambda^0}{\mu^1} \quad \frac{\lambda^0\lambda^1}{\mu^1\mu^2} \quad \frac{\lambda^0\lambda^1\lambda^2}{\mu^1\mu^2\mu^3})}$$

5.4. Concluding Remarks

Note that in equations (5.14), (5.15), λ^i's and μ^i's are expressed in summation form so that the quantitative values of λ^i's and μ^i's can be found by using a computer.

The advantage of defining system states corresponding to numbers of failed component is that only transition between consecutive states is possible so that evaluations of steady-state availability and mean time to first system failure are straightforward.

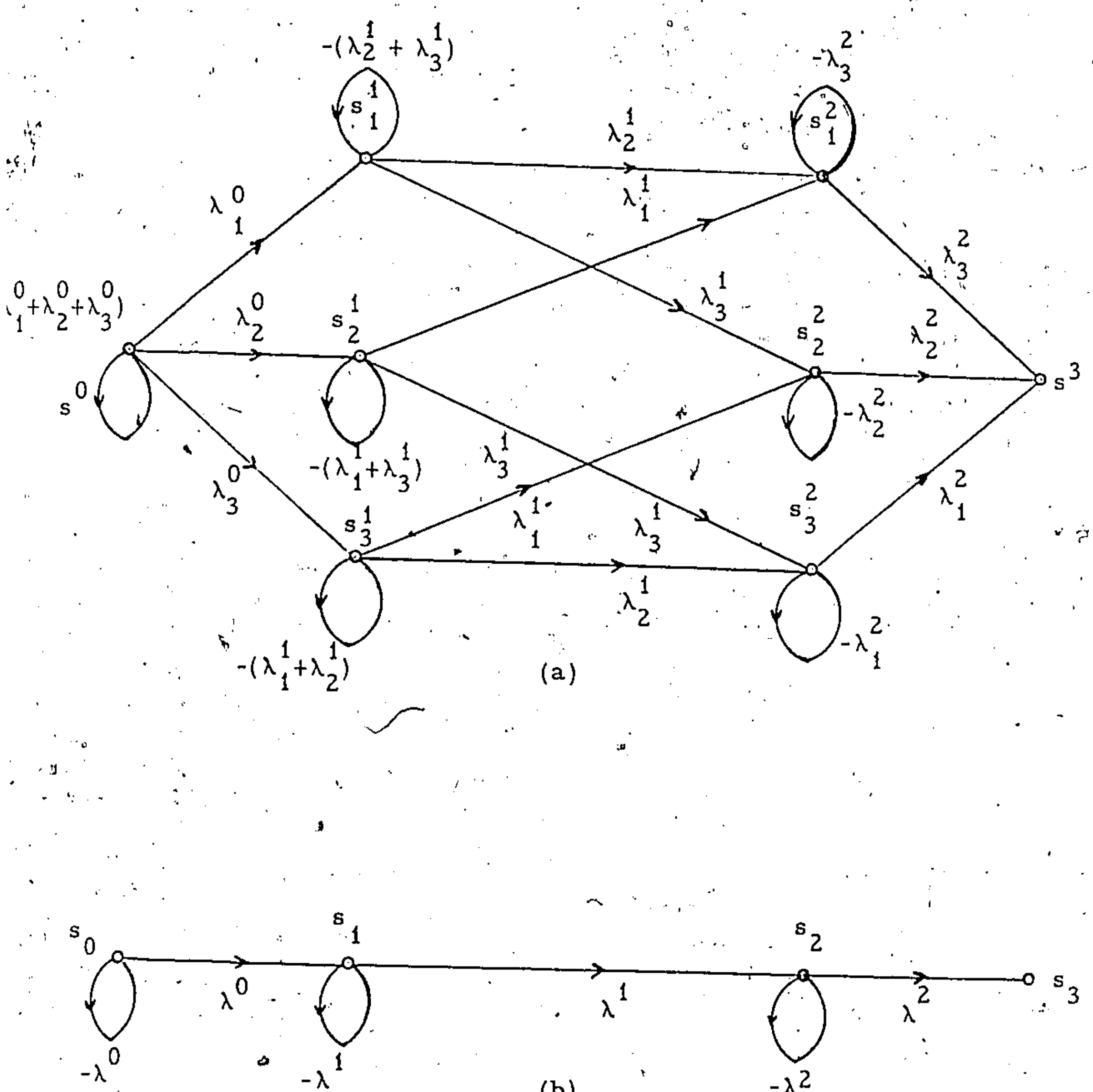

Fig. 5.1 Markov graph for a 1-out-of-3 system

(a) general model (b) collapsed model

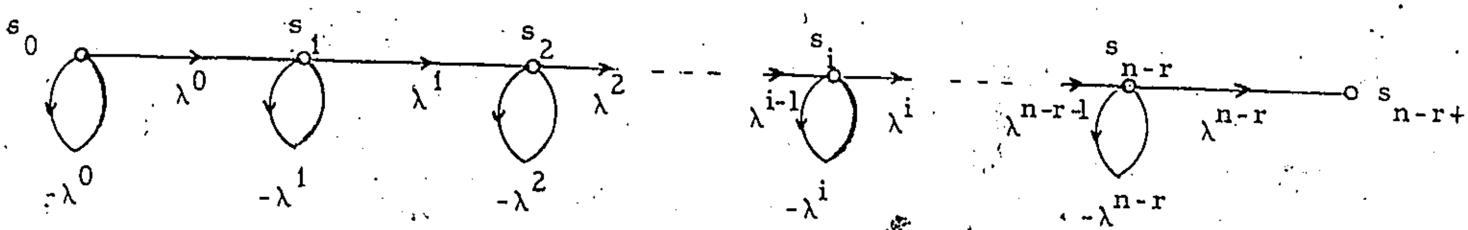

Fig. 5.2 A general collapsed model

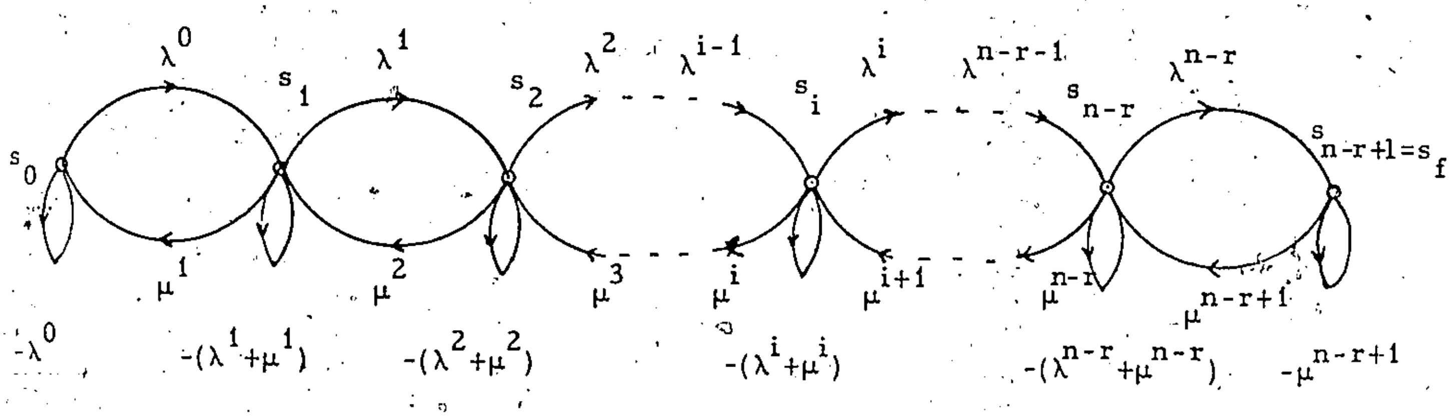

Fig. 5.3 A flow graph for a r-out-of-n repairable system.

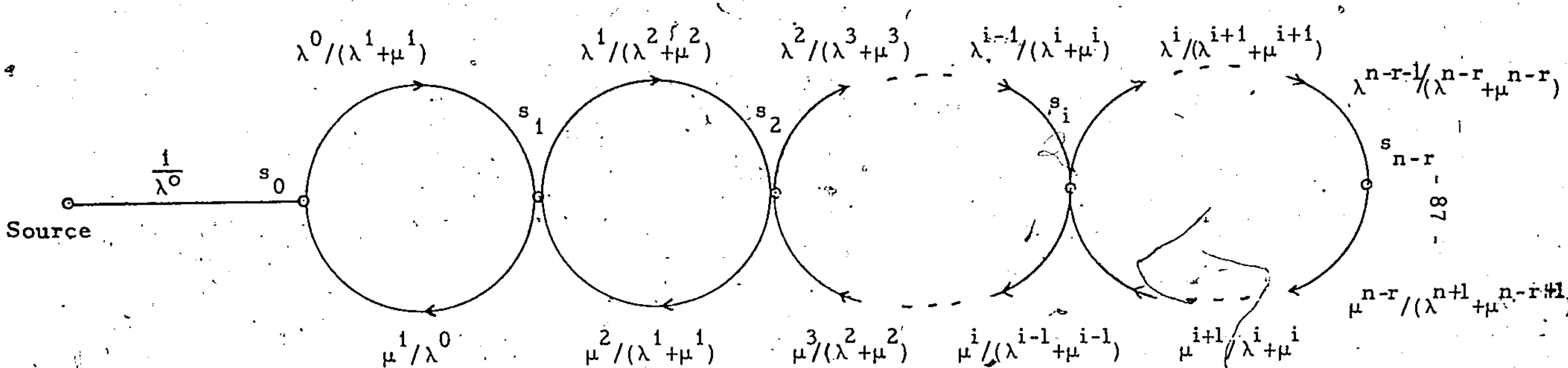

Fig. 5.4 The flow graph for determining P_i .

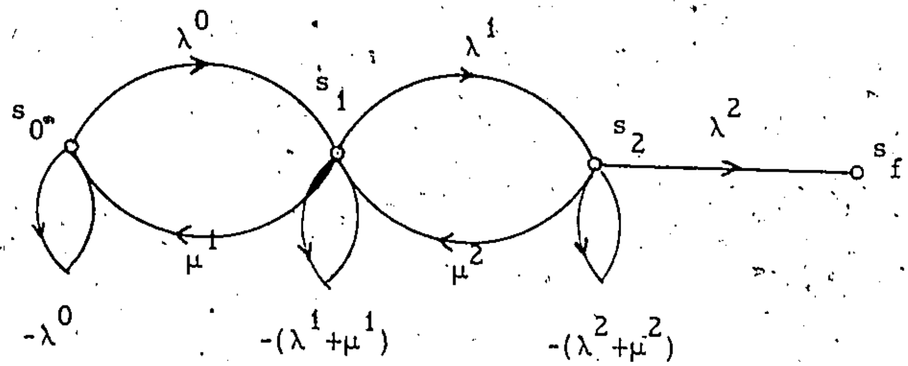

Fig. 5.5 The reliability flow graph of a 3-out-of-5 system.

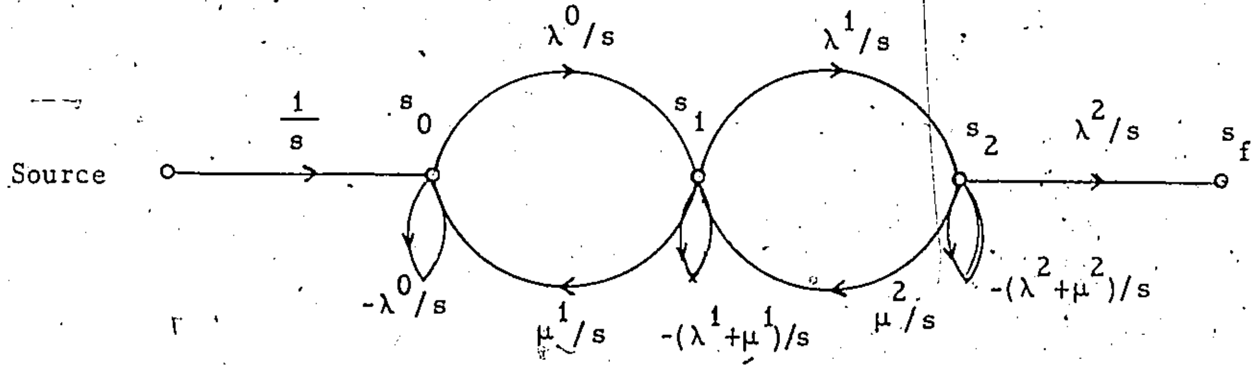

Fig. 5.6 The reliability flow graph for determining $P_f(s)$

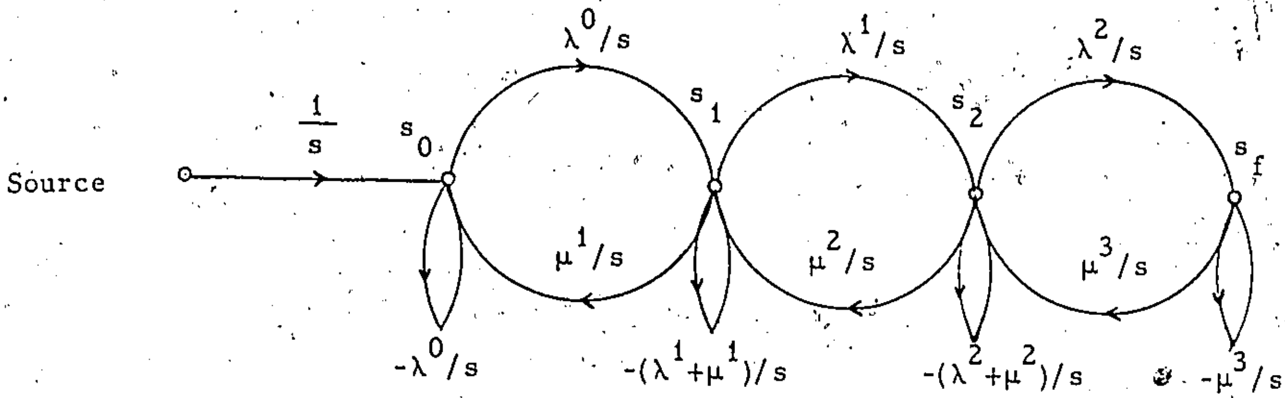

Fig. 5.7 The availability flow graph for determining $P_f(s)$.

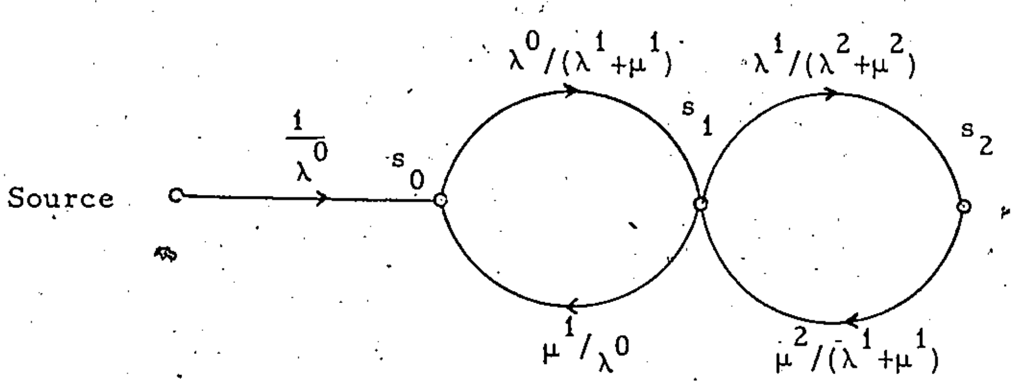

Fig. 5.8 The flow graph for determining P_i .

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

In this thesis, new finite-state Markov models are presented. A simple Markov model is presented to characterize dynamically the reliability behavior of a component. Furthermore, the model generalizes the birth and death process for a bounded path function. Reliability prediction techniques are presented for systems with dependent failure and repair rates.

The following problems may be worthy of further investigation.

1. Development of models and prediction techniques for the evaluation of the system reliability of practical systems in which dominant dependent factors are taken into account.

2. Designation of an optimal policy in terms of the cost of Burn-in and the cost for the degree of the improvement of manufacturing techniques.

REFERENCES

1. Naresky, Joseph J., "Reliability Definitions", IEEE Trans. on Rel., Vol. R-19, Nov. 1970, pp. 198-200.
2. Barlow and Proschan, "Mathematical Theory of Reliability", John Wiley and Sons, Inc., 1965.
3. Shooman, Martin L., "Probabilistic Reliability : An Engineering Approach", McGraw-Hill, 1968.
4. Gorski, Andrew C., "Beware of the Weibull Euphoria", IEEE Trans. Rel., Vol. R-17, Dec. 1968, pp. 202-203, Letters to the Editor.
5. Yeh, Y.C., and Sheng, C.L., "A New Reliability Function", TR 72-10, Dept. of Elect. Engg., Univ. of Ottawa. (Submitted to IEEE Trans. on Reliability).
6. Karlin, S., and McGregor, J.C., "The Differential Equations of Birth and Death Processes and Stieltjes Moment Problem, TAMS 85, 1957, pp. 489-546.
7. Kendall, D.G., "On the Generalized Birth and Death Processes", AMS 19, 1948, pp 1-15.
8. Feller, W., "An Introduction to Probability Theory and Its Applications", John Wiley and Sons, Vol. 1, 1968.
9. Yeh, Y.C., and Sheng, C.L., "A General Birth and Death Process", TR 73-3, Dept. of Elect. Engg., Univ. of Ottawa. (Submitted to Journal of the Institute of Management Sciences).
10. Mason, S.J., "Feedback Theory: Some Properties of Signal Flow-Graphs", Proc. IRE, 1953, 41, pp. 1144-1156.

11. Mason, S.J., "Feedback Theory: Further Properties of Signal Flow-Graphs", Proc. IRE, 1956, 44, pp. 920-926.

12. Tustin, Arnold, "The Mechanism of Economic Systems", Cambridge: Harvard University Press, 1953.

13. Tustin, Arnold, "Direct Current Machines For Control Systems", New York: MacMillan Co., 1952.

14. Percival, W.S., "The Graph of Active Networks", Proc. IRE. Vol. 102, Part C, 1955. pp. 270-277.

15. Htun, L. Tin, "Reliability Prediction Techniques For Complex Systems", IEEE Trans. on Rel., Vol. R-15, No. 2, Aug. 1966, pp 58-69.

16. Chung, Who-Kee, "Topics In Reliability of Sequential Circuits," Ph.D. Thesis, 1970, Univ. of Ottawa, Ottawa, Canada.

17. Misra, K.B. and Rao, T.S.M., "Reliability Analysis of Redundant Networks Using Flow Graph", IEEE Trans. Rel., Vol. R-19, Feb. 1970, pp 19-24.

18. Misra, K.B., "An Algorithm For Reliability Evaluation of Redundant Network," IEEE Trans. Rel., Vol. R-19, Nov. 1970, pp 146-151.

19. Huggins, W.H., "Signal-Flow Graph and Random Signal", Proc. IRE, 1957, (1), pp 74-86.

20. Chan, W.C. and Chung, W.K., "Transient In A Single Server Queuing Systems", to appear in Proc. of IEE.

21. Shooman, Martin L., "Reliability Computations for Systems with dependent Failure," IEEE Symposium on Reliability and Maintainability. 1966, pp 712-718.

22. Loo, S.G., "Evaluation of Conditional Failure Density From Hazard Rate", IEEE Trans. on Rel., May 1971, pp 87-88.

23. Epstein, B. and Sobel, M., "Life Testing", J. Amer. Statist. Assoc. Vol. 48, No. 263, 1953, pp 486-502.

24. Drenick, R.F., "The Failure Law of Complex Equipment", J. Soc. Indust. Appl. Math., Vol. 8, No. 4, 1960, pp 125-149.

25. Evans, Ralph A., "A Generalized Limited Theorem For Reliability", IEEE Trans. on Rel., Vol. R-18, No. 2, May 1969, pp 45-46.

26. Gumbel, E.J., "Statistics of Extremes", Columbia Univ. Press, New York, 1958, Chapter 5.

27. Kao, J.H.K., "A Computer Method For Estimating Weibull Parameters In Reliability Studies", IRE Trans. on Rel. and Quality Control, PGROC-13, 1958, pp 15-22.

28. Lieblein, J. and Zelen, M., "Statistical Investigation of the Fatigue Life of Deep-Groove Ball Bearings", J. Res., Nat. Bureau Stand., Vol. 57, 1956, pp 273-316.

29. Proschan, Frank, "Theoretical Explanation of Observed Decreasing Failure Rate", Technometrics, 5, 1963, pp 375-383.

30. Menon, M.V., "Estimation of The Shape and Scale Parameters of The Weibull Distribution", Technometrics, 5, 1963, pp. 175-182.

31. Kao, J.H.K., "A Graphical Estimation of Mixed Weibull Parameters in Life-Testing of Electron-Tubes, Technometrics, 1, 1959, pp 389-407.

32. Ku, Y.H., "Resume of Maxwell's and Kirchhoff's Rules For Network Analysis", J. of The Franklin Institute, Vol. 253, No. 1, January 1952, pp 211-224.

33. Chan, W.C., "A Generalized Reliability Function For Systems of Parallel Components," IEEE Trans. Rel. Vol. R-17, Dec. 1968, pp 199-201.

34. Chung, W.K., "Generalized Reliability Function of Arbitrary Configuration," IEEE Trans. Rel., Vol. R-20, May 1971, pp 85-87.

35. Messinger, M. and Shooman, M., " Exponential and Weibull Approximation for Chain Structure, " Proc. IEEE Ann. Symp. on Reliability, 1968, pp 494-508.

36. Messinger, M. and Shooman, M., "Reliability Approximations for Complex Structures", Proc. IEEE Ann. Symp. on Reliability, 1967, pp 292-301.

37. Jensen, P.A. and Bellmore, M., "An Algorithm to Determine the Reliability of A Complex System," IEEE. Trans. Rel., Vol. R-18, Nov. 1969, pp. 169-174.

38. Nelson, A.C. and Batts, J.R. and Beadles, R.L., "A Computer Program For Approximating System Reliability," IEEE. Trans. Rel., Vol. R-21, Nov. 1972, pp 61-65.

39. Moore, E.F. and Shannon, C.E., "Reliable Circuits Using Less Reliable Relays", J. Franklin Institute, 1956, pp 191-209, 281-297.

40. Kim, Y.H. and Case, K.E. and Ghare, P.M., "A Method For Computing Complex System Reliability", IEEE. Trans. Rel., Vol. R-21, pp 215-219, Nov. 1972.

41. Ascher, H.E., "Evaluation of Repairable System Reliability Using the Bad as Old Concept", IEEE Trans. Rel. Vol. R-17, June 1968, pp 103-110.

42. Christiaanse, W.R., "A Technique For The Analysis of Repairable Redundant Systems," IEEE. Trans. Rel., Vol. R-19, May 1970, pp 53-60.

43. Buzacott, J.A., "Network Approaches to Finding the Reliability of Repairable System", IEEE. Trans. Rel., Vol. R-19, Nov. 1970, pp 140-146.

44. Fan, L.T. and Wang, C.S. and Tillman, F.A. and Hwang, C.L., "Optimization of System Reliability", IEEE. Trans. Rel., Vol. R-16, Sep. 1967, pp 81-86.

45. Tillman, F.A., "Optimization by Integer Programming of Constrained Reliability Problems With Several Modes of Failure", IEEE. Rel., Vol. R-18, May 1969, pp 47-53.

46. Tillman, F.A. and Hwang, C.L. and Fan, L.T. and Balbale, S.A., "System Reliability Subject to Multiple Nonlinear Constraint", IEEE. Trans. Rel., Vol. R-17, Sep. 1969, pp 153-157.

47. Tillman, F.A. and Hwang, C.L. and Fan, L.T. and Lai, K.C., "Optimal Reliability of A Complex System", IEEE. Trans. Rel., Vol. R-19, Aug. 1970, pp 95-100.

48. Krohn, C.A., "Hazard Versus Renewal Rate of Electronic Items", IEEE. Trans. on Rel., Vol. R-18, No. 2, May 1969, pp 64-73.

49. Epstein, B., "Tests For The Validity of The Assumption That The Underlying Distribution of Life Is Exponential (Part I and II)", Technometrics, 2, 1960, pp 83-101, 167-183.

50. Stone, H., "Approximation of Curve by Line Segment," Math. Comput. 15 (1961), pp 40-47.

51. Gnedenko and Belyayev and Solovyev, "Mathematical Methods of Reliability Theory," Academic Press, 1969, pp 143-205.

52. Zadeh and Desoer, "Linear System Theory", McGraw-Hill, Appendix D.

53. Brand, L., "Differential and Difference Equation", John Wiley and Sons, Chapter 9.

APPENDIX

In this appendix the optimal values of a and b for a given n are evaluated.

Let

$$g(t_i) = \frac{\text{numbers of the remaining operative items at time } t_i}{\text{total numbers of items tested}} \qquad i = 1, 2, \ldots, N_p$$

i.e., $g(t_i)$ is the probability of failure-free operation at time t_i and set

$$E(a,b) = \frac{1}{N_p} \sum_{i=1}^{N_p} (R(t_i) - g(t_i))^2$$

be the mean square errors of approximating $g(t$ by $R(t_i)$, then our problem is to find a, b such that E (a, b) is a minimum.

For minimum E(a, b) we set $\delta E(a,b) / \delta a = 0$, $\delta E(a,b) / \delta b = 0$.

Solve

$$\frac{\delta E(a,b)}{\delta a} = \frac{2}{N_p} \sum_{j=1}^{N_p} (R(t_j) - g(t_j)) \frac{\delta R(t_j)}{\delta a}, \qquad (A.1)$$

$$\frac{\delta E(a,b)}{\delta b} = \frac{2}{N_p} \sum_{j=1}^{N_p} (R(t_j) - g(t_j)) \frac{\delta R(t_j)}{\delta b} \qquad (A.1')$$

where
$$\frac{\delta R(t_j)}{\delta a} = \sum_{i=1}^{n} \left(\frac{\delta C_i(a,b)}{\delta a} - C_i(a,b)\,\mu_i\, t_j \right) * e^{-(a\mu_i + b\mu_{n+i})\, t_j} \qquad (A.2)$$

$$\frac{\delta R(t_j)}{\delta b} = \sum_{i=1}^{n} \left(\frac{\delta C_i(a,b)}{\delta b} - C_i(a,b)\,\mu_{n+i}\, t_j \right) * e^{-(a\mu_i + b\mu_{n+i})\, t_j} \qquad (A.2')$$

after simple evaluation from (3.12),

and where
$$C_i(a,b) = \left(\prod_{m=1}^{i-1} \frac{a\mu_m}{s + a\mu_m + b\mu_{n+m}} \bigg|_{s=-(a\mu_i + b\mu_{n+i})} \right) \left(1 + \sum_{k=i}^{n-1} \prod_{m=i}^{k} \frac{a\mu_m}{s + a\mu_{m+1} + b\mu_{n+m+1}} \bigg|_{s=-(a\mu_i + b\mu_{n+i})} \right)$$

$$= U_i(a,b) * V_i(a,b) \qquad (A.3)$$

$$\frac{\delta C_i(a,b)}{\delta a} = \left(\frac{(i-1)}{a} - X_{i,1}(a,b) \right) * U_i(a,b) * V_i(a,b)$$

$$+ U_i(a,b) * \left(\sum_{k=i}^{n-1} W_{i,k}(a,b) \left(\frac{k-i+1}{a} - Y_{i,k,1}(a,b) \right) \right) \qquad (A.4)$$

$$\frac{\delta C_i(a,b)}{\delta b} = - \left(U_i(a,b)\, V_i(a,b)\, X_{i,2}(a,b) + U_i(a,b) \left(\sum_{k=i}^{n-1} W_{i,k}(a,b)\, Y_{i,k,2}(a,b) \right) \right) \qquad (A.5)$$

where
$$U_i(a,b) = \prod_{m=1}^{i-1} \frac{a\mu_m}{(s + a\mu_m + b\mu_{n+m})} \bigg|_{s=-(a\mu_i + b\mu_{n+i})} \qquad (A.6)$$

$$V_i(a,b) = \begin{cases} 1 + \displaystyle\sum_{k=i}^{n-1} \prod_{m=i}^{k} \frac{a\mu_m}{s + a\mu_{m+1} + b\mu_{n+m+1}} \bigg|_{s=-(a\mu_i + b\mu_{n+i})} & i=1,2,\ldots,n-1 \\ 1 & i=n \end{cases} \qquad (A.7)$$

$$W_{i,k}(a,b) = \prod_{m=i}^{k} \left. \frac{a\mu_m}{s + a\mu_{m+1} + b\mu_{n+m+1}} \right|_{s=-(a\mu_i + b\mu_{n+i})} \quad (A.8)$$

$$X_{i,1}(a,b) = \sum_{m=1}^{i-1} \left. \frac{a(\mu_m - \mu_i)}{s + a\mu_m + b\mu_{n+m}} \right|_{s=-(a\mu_i + b\mu_{n+i})} \quad (A.9)$$

$$X_{i,2}(a,b) = \sum_{m=1}^{i-1} \left. \frac{a(\mu_{n+m} - \mu_{n+i})}{s + a\mu_m + b\mu_{n+m}} \right|_{s=-(a\mu_i + b\mu_{n+i})} \quad (A.10)$$

$$Y_{i,k,1}(a,b) = \sum_{m=i}^{k} \left. \frac{a(\mu_{m+1} - \mu_i)}{s + a\mu_{m+1} + b\mu_{n+m+1}} \right|_{s=-(a\mu_i + b\mu_{n+i})} \quad (A.11)$$

$$Y_{i,k,2}(a,b) = \sum_{m=i}^{k} \left. \frac{a(\mu_{n+m+1} - \mu_{n+i})}{s + a\mu_{m+1} + b\mu_{n+m+1}} \right|_{s=-(a\mu_i + b\mu_{n+i})} \quad (A.12)$$

after some evaluation.

In the subsequent pages a FORTRAN program is given, comprising one main program and six subroutines, for computing the optimal values of a and b for a given n, and a FORTRAN program of computing the Markov hazard rate.

```
              C     THIS PROGRAM IS TO FIND THE OPTIMAL VALUES OF A AND B FOR A GIVEN N
0001                IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
0002                DIMENSION   D(30),C(20),ALPH(40),        T(30),G(30),E(30),R(30),
                   1 AINI(40), ALINI(20), AL(20),CD(20)
0003                COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
0004                COMMON/YCH/D,R,C,G,E,EP,T,LL,IIT,NP
              C     READ INPUT FAILURE DATA
0005                NP=15
0006                READ (1,90)(G(I),I=1, NP)
0007             90 FORMAT(13F5.4)
0008                READ(1,100)(T(I),I=1, NP)
0009            100 FORMAT(13I5)
              C     PRINT THE FAILURE DATA
0010                DO 99 I=1, NP
0011                WRITE(3,98) T(I), G(I)
0012             98 FORMAT(20X,I6, 10X, F10.8)
0013             99 CONTINUE
              C     ASSUME THE NUMBER OF STATES
0014                N=13
0015                NN=2*N
              C     READ THE ASSUMING INITIAL VALUES OF PARAMETERS
0016                READ(1, 101) (AINI(I), I=1, NN)
0017            101 FORMAT(8F10.6)
0018                DO 102 I=1, N
0019                ALINI(I)=0.
0020                ALINI(I)= AINI(I) + AINI(N+I)
0021            102 CONTINUE
              C     BEGIN TO FIND OPTIMAL A AND B BY HALF INTERVAL ITERATION SEARCH
              C     ADJUSTING INCREMENTS OF A AND B
0022                EP=0.02
              C     SET THE MINIMUM INCREMENT OF A
0023                EPS1=0.02
              C     SET THE MINIMUM INCREMENT OF B
0024                EPS2=0.02
              C     ASSUME THE INITIAL BALUE OF A AND B
0025                A=1.
0026                B=1.
              C     SET THE INITIAL VALUES OF INCREMENT DIVIDERS OF A AND B
              C     WHICH DECRESE INCREMENTS
0027                IA=2
0028                IB=1
              C     CALL SUBROUTINE ROOT TO FIND THE FIRST OUTPUT
0029                CALL ROOT(A,B)
```

```
                C       SET ITERATCN COUNTER
0030                    ITER=1
                C       SET INITIAL COUNTER TO DECIDE IF IT IS THE FIRST ITERATION OF A (B)
0031               49   IIT=1
0032                    IB=IB+1
0033                    IF(N.EQ.1) GO TO 500
0034                    EPS1=EPS1/2.
0035                    DELTA=A/(2*IA)
                C       COMPUTE C(I)= DERIVATIVE OF R(T(I)) W.R.T. A
                C       FIRST COMPUTE CD(M)= DERIVATIVE OF C(M) W.R.T. A
0036               50   NN=N-1
                C       THIS DO LOOP COMPUTES THE FIRST PART OF EQUATION(A.4)
0037                    DO 111 I=1, N
0038                    CD(I)=0.
0039                    S=-AL(I)
0040                    CALL PPROD(I)
0041                    CALL SSUMPR(I)
0042                    CALL SBS(I,1)
0043                    CD(I)= PROD*SUMPR*(DFLOAT(I-1)/A- SBSU)
0044              111   CONTINUE
                C       THIS DO LOOP COMPUTES THE SECOND PART OF EQUATION (A.4)
0045                    DO 150 I=1,NN
0046                    S=-AL(I)
0047                    CALL PPROD(I)
0048                    DO 150 K=I,NN
0049                    CALL SBP(I,K)
0050                    CALL SBSB(I,K,1)
0051                    CD(I)= CD(I)+ PROD*SBPR*(DFLOAT(K-I+1)/A-SBSBS)
0052              150   CONTINUE
                C       COMPUTE D(I)
0053                    DO 110 I=1, NP
0054                    D(I)=0.
0055                    DO 110 J=1, N
0056                    D(I)= D(I)+ (CD(J)-C(J)*AINI(J)*T(I))*DEXP(-AL(J)*T(I))
0057              110   CONTINUE
                C       COMPUTE THE DERIVATIVE OF M. S. ERROR W.R.T. A
0058                    SIGM=0.
0059                    DO 200 I=1, NP
0060              200   SIGM=SIGM+E(I)*D(I)
0061                    WRITE(3,401) SIGM
0062              401   FORMAT(//(20X, F16.8)//)
0063                    IF( IIT.EQ.1) GO TO 420
0064                    IF(EMP*SIGM) 410, 900, 412
```

```
0065        410 DELTA=DELTA/2.0
0066        412 IF(DELTA.LE.A/500.) GO TO 500
0067            IF(DABS(SIGM).LE.EPS1) GO TO 500
0068        420 IF(SIGM) 425, 900, 427
0069        425 A=A+DELTA
0070            GO TO 440
0071        427 A=A-DELTA
0072        440 ITER=ITER+1
0073            CALL ROOT(A,B)
0074            IF(ITER.GT.60) GO TO 900
0075            IF(LL.EQ.1) GO TO 500
0076            IIT=IIT+1
0077            EMP=SIGM
0078            GO TO 50
0079        500 IA=IA+1
          C     FROM THIS STEP WE PROCEED THE PARAMETER B
0080            EPS2=EPS2/2.
0081            IIT=1
0082            DELTB=B/(2*IB)
          C     COMPUTE D(I)= DERIVATIVE OF R(T(I)) W.R.T. B
          C     FIRST COMPUTE CD(M)= DERIVATIVE OF C(M) W.R.T. B
0083        550 NN=N-1
          C     THIS DO LOOP COMPUTES THE FIRST PART OF EQUATION(A.5)
0084            DO 601 I=1, N
0085            CD(I)=0.
0086            S=-AL(I)
0087            CALL PPROD(I)
0088            CALL SSUMPR(I)
0089            CALL SBS(I,2)
0090            CD(I)= -PROD*SUMPR*SBSU
0091        601 CONTINUE
          C     THIS DO LOOP COMPUTES THE SECOND PART OF EQUATION (A.5)
0092            DO 650 I=1,NN
0093            S=-AL(I)
0094            CALL PPROD(I)
0095            DO 650 K=I,NN
0096            CALL SBP(I,K)
0097            CALL SBSB(I,K,2)
0098            CD(I)= CD(I)-PROD*SBPR*SBSBS
0099        650 CONTINUE
          C     COMPUTE D(I)
0100            DO 660 I=1, NP
0101            D(I)=0.
```

```
0102              DO 660 J=1, N
0103              D(I)= D(I)+ (CD(J)-C(J)*AINI(N+J)*T(I))*DEXP(-AL(J)*T(I))
0104          660 CONTINUE
            C     COMPUTE THE DERIVATIVE OF M. S. ERROR W.R.T. B
0105              SIGM=0.
0106              DO 700 I=1, NP
0107          700 SIGM=SIGM+E(I)*D(I)
0108              WRITE(3,701) SIGM
0109          701 FORMAT(//(20X, F16.8)//)
0110              IF( IIT.EQ.1) GO TO 720
0111              IF(EMP*SIGM) 710, 900, 712
0112          710 DELTB=DELTB/2.0
0113          712 IF(DELTB.LE.B/500.)  GO TO 49
0114              IF(DABS(SIGM).LE.EPS2) GO TO 49
0115          720 IF(SIGM) 725, 900, 727
0116          725 B=B+DELTB
0117              GO TO 740
0118          727 B=B-DELTB
0119          740 ITER=ITER+1
0120              CALL ROOT(A,B)
0121              IF(ITER.GT.60) GO TO 900
0122              IF(LL.EQ.1) GO TO 49
0123              IIT=IIT+1
0124              EMP=SIGM
0125              GO TO 550
0126          900 STOP
0127              END
```

```
FORTRAN IV G LEVEL  21                    MAIN                   DATE = 73232        21/19/32

              C       THIS SUBROUTINE IS TO PRINT OUT THE RESULT OF FITTING DATA CURVE
              C       BY THE MARKOV FUNCTION
              C       THE OUTPUT IS TIME(I), DATA RELIABILITY(I), MARKOV RELIABILITY(I),
              C       M. S. ERROR(I), FOR EACH SET OF A AND B
 0001                 SUBROUTINE ROOT(A,B)
 0002                 IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
 0003                 DIMENSION   D(30),C(20),ALPH(40),      T(30),G(30),E(30),R(30),
                     1 AINI(40), ALINI(20), AL(20),CD(20)
 0004                 COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
 0005                 COMMON/YCH/D,R,C,G,E,EP,T,LL,IIT,NP
              C       FORMING ANOTHER SET OF PARAMETERS
 0006                 LL=0
 0007                 NN=2*N
 0008                 DO 100 I=1, N
 0009                 ALPH(I)= A*AINI(I)
 0010                 ALPH(N+I)= B*AINI(N+I)
 0011             100 CONTINUE
 0012                 DO 102 I=1, N
 0013                 AL(I) = 0.
 0014             102 AL(I) = ALPH(I) + ALPH(N+I)
 0015                 WRITE(3,97) (ALPH(I),I=1, NN),(AL(I), I=1, N)
 0016              97 FORMAT(//8(F10.6,4X)//)
              C       COMPUTE THE COEFFICIENTS OF C(M) BY DO LOOP
 0017                 DO 103 I=1, N
 0018                 S=-AL(I)
 0019                 CALL PPROD(I)
 0020                 CALL SSUMPR(I)
 0021             103 C(I)= PROD*SUMPR
              C       COMPUTE THE COEFFICIENTS OF R(I) BY DO LOOP
 0022                 DO 105 I=1, NP
 0023                 R(I)=0.
 0024                 DO 105 K=1, N
 0025             105 R(I)= R(I) + C(K)*DEXP(-AL(K)*DFLOAT(T(I)))
              C       COMPUTE THE APPROXIMATION  ERROR FOR EACH DISCRITE DATA POINT
 0026                 DO 106 I=1, NP
 0027                 E(I)=0.
 0028             106 E(I) = R(I) - G(I)
              C       COMPUTE THE MEAN SQUARE ROOT OF ERROR
 0029                 PEMP=0.
 0030                 DO 107 I=1, NP
 0031             107 PEMP = PEMP + E(I)**2
 0032                 DEV=DSQRT(PEMP/DFLOAT(NP))
```

```
FORTRAN IV G LEVEL  21                  ROOT              DATE = 73232        21/19/32

              C        PRONT THE OUTPUT
              C        FIRST PRINT MEAN SQUARE ROOT OF ERROR AND PARAMETER AL
0033                   JN=2*N -1
0034                   WRITE (3, 108) DEV
0035              108  FORMAT(1H1,20X, 'MEAN SQUARE ROOT OF ERROR = ', F10.8//)
0036                   WRITE(3, 150) A,B
0037              150  FORMAT(20X, 2(F10.6, 10X)///)
0038                   WRITE(3, 200)
0039              200  FORMAT(20X, 'T(I)', 11X, 'G(I)', 12X, 'R(I)', 12X, 'E(I)')
              C        PRINT COMPARATIVE DATAS
0040                   DO 202 I=1, NP
0041                   WRITE(3,201) T(I), C(I), R(I), E(I)
0042              201  FORMAT(20X, I6, 6X, 3(F10.6, 6X))
0043              202  CONTINUE
              C        CRITERION TO END THE EVALUATION
0044                   IF(DEV.GE.EP)   GO TO 365
              C        FOR DEV GT. 0.01 WE TRY TO FIND IMPROVING PARAMETER IN MAIN PROGRAM
0045                   EP=EP/2.
0046              360  Lb=1
0047              365  RETURN
0048                   END
```

```
C     THIS SUBROUTINE COMPUTES U(I) ( EQUATION (A.6))
      SUBROUTINE PPROD(M)
      IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
      DIMENSION ALPH(40), AL(20), AINI(40), ALINI(20)
      COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
      PROD=1.
      IF(M.EQ.1) GO TO 1001
      NR= M-1
      DO 1000 K=1, NR
 1000 PROD= PROD*ALPH(K)/(S+AL(K))
 1001 RETURN
      END
```

```
C     THIS SUBROUTINE COMPUTES V(I) ( EQUATION(A.7))
      SUBROUTINE SSUMPR(M)
      IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
      DIMENSION ALPH(40), AL(20), AINI(40), ALINI(20)
      COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
      SUMPR=0.
      IF(M.EQ.N) GO TO 2001
      DEMP=1.
      NK=N-1
      DO 2000 K=M, NK
      DEMP= DEMP*ALPH(K)/(S+AL(K+1))
 2000 SUMPR = SUMPR + DEMP
 2001 SUMPR = SUMPR + 1.
      RETURN
      END
```

```
FORTRAN IV G LEVEL  21                MAIN                DATE = 73232          21/19/32

C     THIS SUBROUTINE COMPUTES W(I,K) ( EQUATION (A.8))
      SUBROUTINE SBP(M,K)
      IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
      DIMENSION ALPH(40), AL(20), AINI(40), ALINI(20)
      COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SPSBS,ALPH,AL,AINI,ALINI,N
      SBPR=1.
      DO 3000 I=M,K
 3000 SBPR=SBPR*ALPH(I)/(S+AL(I+1))
      RETURN
      END
```

```
C     THIS SUBROUTINE COMPUTES X(I, 1$2) ( EQUATIONS (A.9-10))
      SUBROUTINE SBS(M,II)
      IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
      DIMENSION ALPH(40), AL(20), AINI(40), ALINI(20)
      COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
      SBSU=0.
      IF(M.EQ.1) GO TO 4200
      NR=M-1
      IF(II.EQ.2) GO TO 4100
      DO 4000 I=1,NR
 4000 SBSU=SBSU+(AINI(I)-AINI(M))/(S+AL(I))
      GO TO 4200
 4100 DO 4110 I=1, NR
 4110 SBSU=SBSU+(AINI(N+I)-AINI(N+M))/(S+AL(I))
 4200 RETURN
      END
```

```
C       THIS SUBROUTINE COMPUTES Y(I,K,1$2) ( EQUATIONS (A.11-12))
        SUBROUTINE SBSB(M,K,JJ)
        IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
        DIMENSION ALPH(40), AL(20), AINI(40), ALINI(20)
        COMMON/YAP/PROD,SUMPR,APROD,S,SBPR,SBSU,SBSBS,ALPH,AL,AINI,ALINI,N
        SBSBS=0.
        IF(JJ.EQ.2) GO TO 5100
        DO 5000 I=M,K
   5000 SBSBS=SBSBS+(AINI(I+1)-AINI(M))/(S+AL(I+1))
        GO TO 5200
   5100 DO 5110 I=M,K
   5110 SBSBS=SBSBS+(AINI(N+I+1)-AINI(N+M))/(S+AL(I+1))
   5200 RETURN
        END
```

```
FORTRAN IV G LEVEL  21            MAIN            DATE = 73232         20/47/22

C     THIS PROGRAM IS TO FIND THE MARKOV HAZARD RATE
      IMPLICIT INTEGER*4(I-N,T), REAL*8(A-H,O-S,U-Z)
      DIMENSION D(20), C(20), ALPH(40), AL(20), T(30), R(30), H(30)
      COMMON /YEH/ PROD, SUMPR,S, ALPH,AL,N
C     READ INPUT DATA
      NP=15
      N=13
      NN= 2*N
      READ (1,10) ( T(I), I=1, NP)
   10 FORMAT (13I5)
      READ (1,20) (ALPH(I), I=1,NN)
   20 FORMAT(8F10.6)
      DO 30 I=1, N
      AL(I)=0.
      AL(I)= ALPH(I) + ALPH(N+I)
   30 CONTINUE
      WRITE(3,40) (ALPH(I), I=1,NN), (AL(I), I=1, N)
   40 FORMAT(//8(F10.6, 4X)//)
C     COMPUTE THE COEFFICIENTS C(M) BY DO LOOP
      DO 50 I=1,N
      C(I) =0.
      S=-AL(I)
      CALL PPROD(I)
      CALL SSUMPR(I)
   50 C(I) = PROD*SUMPR
C     COMPUTE THE VALUES OF R(I) AND D(I)=-DR(T)/DT BY DO LOOP
      DO 105 I=1, NP
      R(I)=0.
      D(I)=0.
      DO 105 K=1, N
      R(I)= R(I) + C(K)*DEXP(-AL(K)*DFLOAT(T(I)))
      D(I)= D(I) +C(K)*AL(K)*DEXP(-AL(K)*DFLOAT(T(I)))
  105 CONTINUE
C     COMPUTE HAZARD VALUES H(I) = D(I) / R(I)
      DO 110 I=1,NP
  110 H(I) = D(I)/R(I)
C     PRINT OUTPUT
      DO 202 I=1, NP
      WRITE(3,201) T(I), H(I)
  201 FORMAT(20X, I6, 6X, F16.8)
  202 CONTINUE
      STOP
      END
```

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