STUDY OF CERTAIN ASPECTS OF A CLASS OF
NONLINEAR MULTIPLICATIVE SYSTEMS

by

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ABSTRACT

In this thesis we present a study of certain aspects of a class of nonlinear multiplicative systems. This study is carried out in three directions:

i) linear representation of nonlinear multiplicative systems

ii) improvement in the performance of systems by introducing intentional multiplicative nonlinearities

iii) stability study of systems in ii)

In Chapter II, it is shown that an open-loop nonlinear multiplicative system, if excited by a step function input, can be represented exactly by a linear system multiplied by a "gain factor". The method presented here is straightforward, and the representation for the nth order nonlinear multiplicative systems can be written after two complex convolution operations.

In Chapter III, it is shown that the performance of linear systems up to third order, and some second order nonlinear differential systems, can be improved by introducing intentional multiplicative nonlinearities in the for-

ward path of the feedback control system. The resulting control systems are more sensitive to parameter variations than direct feedback control systems. The transient response is extremely insensitive to parameter variations due to high controller gain and presence of multipliers. However, the steady state output of the system changes with parameter variations, but can be corrected by changing the gains of the linear systems $H_2(s)$ and $H_3(s)$ as discussed in Chapter III. The improvement in the response of the given system, the relation between the number of multipliers and the order of the given linear system, and the relation of the poles of the given linear system to the poles of $H_2(s)$ and $H_3(s)$, are consistent in every case considered.

In Chapter IV, the stability of the systems discussed in Chapter III is studied and shows that, according to Zubov's formulation, a closed form Liapunov function is not possible. However, these systems always have an equilibrium point, which is asymptotically stable and a Liapunov function of the quadratic form can be constructed to give a conservative estimate of the region of asymptotic stability around the equilibrium point.

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CHAPTER I

INTRODUCTION

In studying practical systems, we are aware that every system is nonlinear except in some region around the operating point. This region depends upon the type of the nonlinearity in the system. In case the nonlinearity is not sharp, (saturation or slow variation) linear models of control systems obtained by the perturbation method of Poincaré provide sufficient information about the performance of an actual system around the operating point. However, for large control signals, the behavior of the system becomes completely different and linear control system theory does not apply. Nevertheless, the perturbation technique has been successfully employed in cases, where one has confidence about the operating region. In cases, where little is known about the physical behavior of the system, this method is the only available approach. In order to eliminate the element of uncertainty in the performance of the control system because of idealization, we always desire the true nonlinear representation of the system. This is possible only if complete information about the system behavior is available. This representation makes study of the control system more difficult, because of nonlinear equations, (differential, integral, integro-differential) which are usually difficult to solve. However, if we overcome this difficulty, which is possible in a number of cases by computational methods, and in a few cases by analytic methods, the exact information obtained about the performance of the control system will eliminate the element of uncertainty due to idealization. Since, all physical systems are nonlinear, we also observe some beneficial aspects of the nonlinearity in the systems. This can be appreciated from the following examples.

i) Development of digital computers and large communication systems, which have on-off characteristics.

ii) Development of magnetic amplifiers, which depend on hysteresis loop.

iii) Saturation, without which electric machines would run to excessive speeds and destroy themselves.

These are a few of the numerous examples of inherent or intentional nonlinearities for achieving some desired performance.

In feedback control systems, (linear or nonlinear), where desired performance cannot be achieved by linear controllers, it is desirable to introduce nonlinear controllers. This technique has proved to be extremely successful especially when, in addition to linear elements, relays or multipliers are used as controller elements. These controller elements could be located either in a feedforward path, feedback path or both, of the control system.

Little attention has been given to systems which involve multiplicative nonlinearities in the forward path of the feedback control system, which will henceforth be called nonlinear multiplicative control systems. These systems are studied here for step function inputs, which are not too restrictive, since a fairly large number of systems operate on d.c. inputs and so are slow systems, (e.g., chemical processes). This study covers three aspects of these systems.

i) Representation of nonlinear multiplicative system by a linear system multiplied by a "gain factor". In this case, the system is assumed to have zero initial state.

ii) Improvement in the performance of the systems by introducing intentional multiplicative nonlinearities.

iii) Stability study of the nonlinear multiplicative control systems of (ii).

1.2 Definitions

In this section we define certain terms, which will be used frequently throughout this study.

i) System

A system s is a partially interconnected set of abstract objects $a_1, a_2, a_3, \ldots,$ termed the components of s. The components of s may be oriented or nonoriented, they may be finite or infinite in number; and each of them may be associated with a finite or infinite number of terminal variables.

ii) Linear System

Assume the responses of a system to two different inputs $x_1(t)$ and $x_2(t)$ to be $y_1(t)$ and $y_2(t)$, respectively and c_1 and c_2 to denote two constants.

A system s is linear if the response to $x(t) = c_1x_1(t) + c_2x_2(t)$ is $y(t) = c_1y_1(t) + c_2y_2(t)$ for all values of x_1, x_2, c_1 and c_2. This is known as superposition principle.

iii) Nonlinear System

If the superposition principle does not hold for a system s, then it is called a nonlinear system.

iv) Nonlinear Multiplicative System

A system s, which has linear systems and multipliers connected as shown inside the dotted lines in Fig. 1, is defined as a nonlinear multiplicative system. If there are n multipliers in the system referred to above then we call it an nth order nonlinear multiplicative system.

v) Control System

A system which employs a controller to maintain some variables at desired levels is called a control system.

vi) Optimum Control System

An optimal control system is defined as one, which drives the system towards the desired state along an optimum path with respect to some performance criterion.

vii) Performance Criterion

A system is required to perform some function in an environment. The measure of how well this function is performed, is called performance criterion.

1.3 The Class of Nonlinear Multiplicative Systems Under Study

The class of nonlinear multiplicative systems to be studied is shown in Fig. 1. The mathematical equation for this system can be written as follows:

$$y_m(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} H_{n+1}(\tau_1,\tau_2,\ldots\ldots,\tau_{n+1})y(t-\tau_1)y(t-\tau_2)\ldots. \\ \ldots. y(t-\tau_{n+1})d\tau_1 d\tau_2 \ldots d\tau_{n+1} \quad (1)$$

where

$$H_{n+1}(\tau_1,\tau_2,\ldots\ldots,\tau_{n+1}) \equiv h_1(\tau_1)h_2(\tau_2)\ldots.h_n(\tau_n)h_{n+1}(\tau_{n+1})$$

$$y(t) = Ke(t)$$

$$e(t) = r(t) - y_m(t)$$

Equation (1) shows that the operation of the overall system can be expressed compactly by $(n+1)$ tuple convolution of input $y(t)$ and a $(n+1)$ dimensional kernel $H_{n+1}(\tau_1,\tau_2,\ldots\ldots,\tau_{n+1})$. The expression of equation (1) is known as a "regular homogeneous functional" of $(n+1)$th degree. Func-

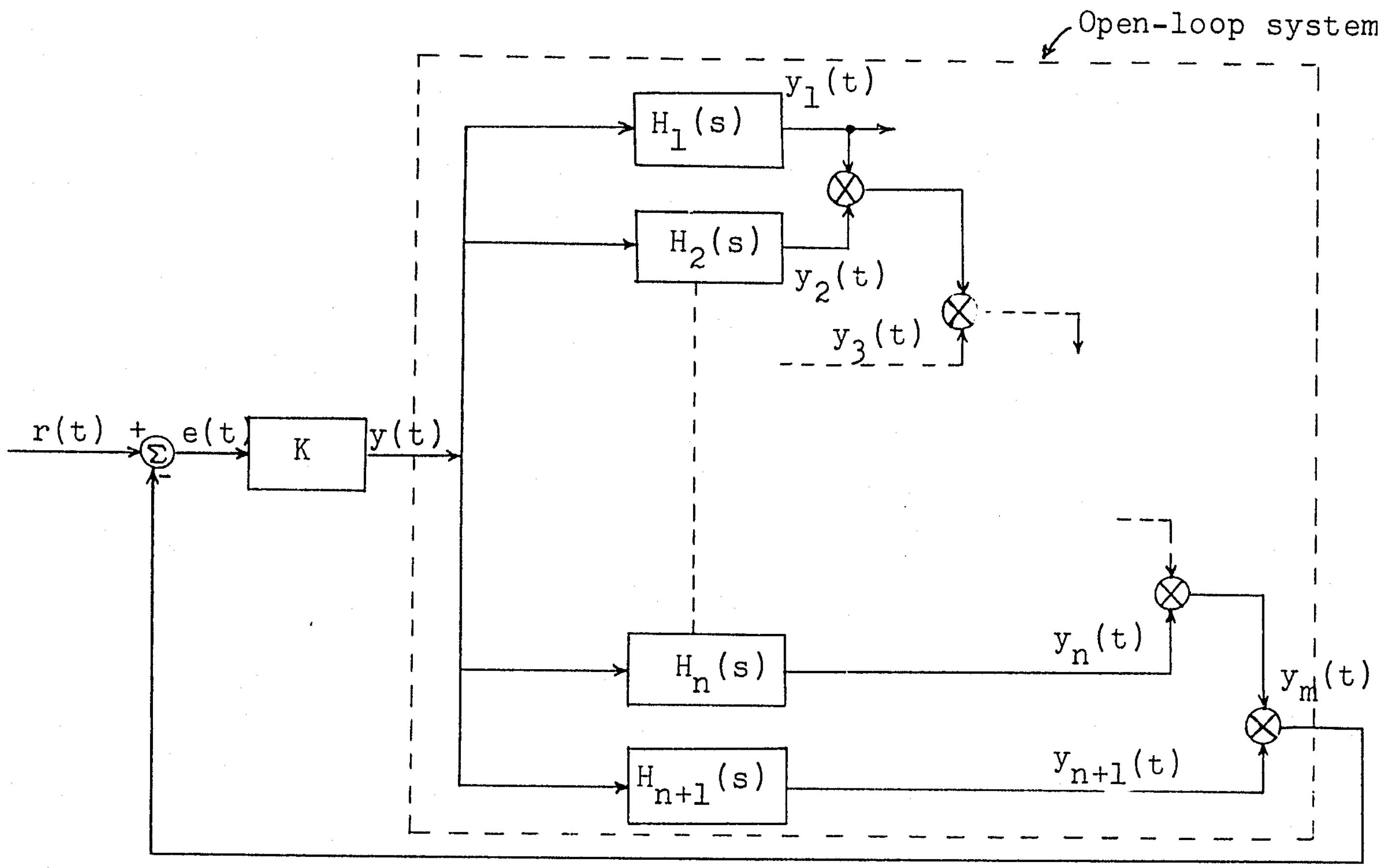


FIG. 1. A CLASS OF NONLINEAR MULTIPLICATIVE SYSTEMS.

$r(t)$ - Step Function Input
$e(t)$ - Error
K - Gain of the Proportional Controller
$y(t)$ - Output of the Controller
$H_i(s)$ - Transfer Function of the ith Linear System
$y_i(t)$ - Output of $H_i(s)$
$y_m(t)$ - Output of the Nonlinear Multiplicative System.

tionals of this type were introduced and extensively studied by Volterra[1]. The application of these functionals to non-linear system representation and analysis was by Wiener [2], Barret [3] and Smet[4].

Instead of the functional approach, which is of course general and well known, we will study some particular aspects of this class from the point of view of the Laplace transform, analog computations and differential equations. In this study, $H_i(s)$ $(i = 2, 3, \ldots\ldots, n + 1)$ is a first order linear system defined as $H_i(s) = \frac{g_i \cdot a_i}{s + a_i}$ and $H_1(s)$ is a first, second or third order linear system. In Chapters III and IV, $H_1(s)$ is replaced in two cases by a second order nonlinear differ-ential system.

It should be mentioned here that optimum output responses in Chapter III could be slightly improved with respect to the in-tegral squared error (ISE) criterion. This was not attempted for the following reasons.

i) at the controller gain $k > 14.0$, the amplifiers simulating the controller become saturated.

ii) the improvement in the ISE criterion is quite insignificant after $k > 10$ in almost all the cases considered.

CHAPTER II

REPRESENTATION OF A CLASS OF NONLINEAR MULTIPLICATIVE SYSTEMS

In equation (1), if the kernel is separable and $y(t)$ is a step function input, then equation (1) can be represented by the open loop system, shown inside the dotted lines in Fig. 1. This system can be represented exactly by a linear system multiplied by a "gain factor". If the step function input has a magnitude M and g_i's $(g_i = \frac{k_i}{\alpha_i})$ are the gains of the linear systems, then for n multipliers the "gain factor" is defined as $N(M, g_i, n) = M^n \cdot \prod_{i=1}^{n+1} g_i$. The equivalent linear system representation, as defined above, can be obtained by real multiplication, which is commonly called complex convolution. We state the following theorems[5] without proof.

2.1.1 Theorem 1. (Complex Convolution)

If $f_1(t)$ and $f_2(t)$ are Laplace-transformable functions having Laplace transforms $F_1(s)$ and $F_2(s)$ respectively, then

$$L[f_1(t)f_2(t)] = \frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_1(s-\omega)\, F_2(\omega)d\omega \tag{2}$$

$$\max(\sigma_{a1}, \sigma_{a2}, \sigma_{a1}+\sigma_{a2}) < \sigma ,$$

$$\sigma_{a2} < c_2 < \sigma - \sigma_{a1} ,$$

in which c_2 is a real constant, $\sigma = Re[s]$, and σ_{a1} and σ_{a2} are the abscissae of absolute convergence of the functions $f_1(t)$ and $f_2(t)$ respectively.

2.1.2 Theorem 1.a

If $f_1(t)$ and $f_2(t)$ are Laplace transformable functions having transforms $F_1(s)$ and $F_2(s)$ respectively, and if $F_1(s) \triangleq \frac{A_1(s)}{B_1(s)}$ is a rational algebraic fraction, having q first order poles and no others, then

$$L[f_1(t)f_2(t)] = \sum_{k=1}^{q} \frac{A_1(s_k)}{B_1'(s_k)} F_2(s-s_k) \tag{3}$$

2.1.3 Theorem 1.b

Let $f_1(t)$ and $f_2(t)$ be Laplace transformable functions, having Laplace-transforms $F_1(s)$, $F_2(s)$, respectively, and let $F_1(s)$ be a rational algebraic fraction having n distinct poles $s_1, s_2, \dots, s_n$ with

s_1 of multiplicity m_1

s_2 of multiplicity m_2

$\vdots$

s_n of multiplicity m_n

subject to the restriction $m_1 + m_2 + \dots + m_n = q$, then

$$L[f_1(t)f_2(t)] = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{(-1)^{m_k-j} K_{kj}}{(m_k - j)!} \left[\frac{d^{m_k-j}}{ds^{m_k-j}} F_2(s)\right]_{s=s-s_k} \tag{4}$$

in which

$$K_{kj} \triangleq \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} (s - s_k)^{m_k} F_1(s)\right]_{s=s_k} \tag{5}$$

The above theorems provide us with sufficient mathematical tool for an equivalent linear system representation of the open-loop system of Fig. 1, enclosed by dotted lines. The above theorems are stated for one multiplication. In a situation such as this, where n multiplications are involved, one can successively convolve each pair and obtain the final result after n complex convolution operations. At first, this procedure may appear laborious and time consuming. In fact, it is not, and after the first two complex convolution operations, the trend and form of the linear representation is such that the linear representation for nth order nonlinear multiplicative system can be written by inspection.

In this study, we will take systems $H_1(s)$ of different order and show that the form of the linear representation remains essentially the same in all cases for this class of systems. First, we work out in detail the case when $H_1(s) = \frac{\alpha_1}{s+\alpha_1}$ and show how the linear representation for the nth order nonlinear multiplicative system can be obtained with only the first two complex convolution operations. For this, we will use Theorem 1.a.

2.2 Representation of Nonlinear Multiplicative Systems

2.2.1 $H_1(s)$ is First Order Linear System

Let

$$H_1(s) = \frac{\alpha_1}{s+\alpha_1} \tag{6}$$

Then open loop system of Fig. 1 enclosed by the dotted lines can be broken into sections, each having one multiplier. The first section is shown in Fig. 2.

Define:

$$F_1(s) = \frac{\alpha_1}{s(s+\alpha_1)} \qquad (7)$$

$$F_2(s) = \frac{\alpha_2}{s(s+\alpha_2)} \qquad (8)$$

and $$F_{1,2}(s) = F_1(s)*F_2(s) \qquad (9)$$

where $*$ denotes convolution in the complex domain. Using Theorem 1.a.

$$F_{1,2}(s) = \sum_{k=1}^{2} \frac{1}{[s_k(s_k+\alpha_1)]'} \cdot \frac{1}{(s-s_k)(s-s_k+\alpha_2)} \qquad (10)$$

where s_k are the poles of $F_1(s)$

$'$ denotes differentiation with respect to s

k denotes the number of poles of $F_1(s)$

The poles of $F_1(s)$ lie at $s_1 = 0$, $s_2 = -\alpha_1$

Therefore,

$$F_{1,2}(s) = \alpha_2 \left[\frac{1}{s(s+\alpha_2)} - \frac{1}{(s+\alpha_1)(s+\alpha_1+\alpha_2)}\right] \qquad (11)$$

or $$F_{1,2}(s) = \frac{1}{s} - \frac{1}{s+\alpha_1} - \frac{1}{s+\alpha_2} + \frac{1}{s+\alpha_1+\alpha_2} \qquad (12)$$

or $$F_{1,2}(s) = \alpha_1\alpha_2 \left[\frac{(2s+\alpha_1+\alpha_2)}{s(s+\alpha_1)(s+\alpha_2)(s+\alpha_1+\alpha_2)}\right] \qquad (13)$$

After evaluating $F_{1,2}(s)$, we consider the next section as shown in Fig. 3.

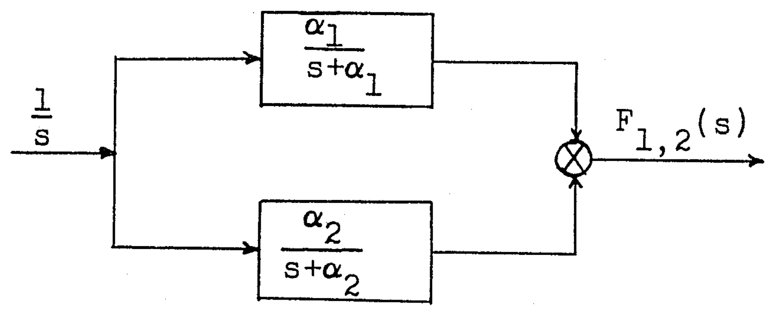


FIG. 2. FIRST SECTION OF THE OPEN-LOOP SYSTEM OF FIG. 1.

$$F_1(s) = \frac{\alpha_1}{s(s+\alpha_1)}$$

$$F_2(s) = \frac{\alpha_2}{s(s+\alpha_2)}, \quad F_{1,2}(s) = F_1(s) * F_2(s)$$

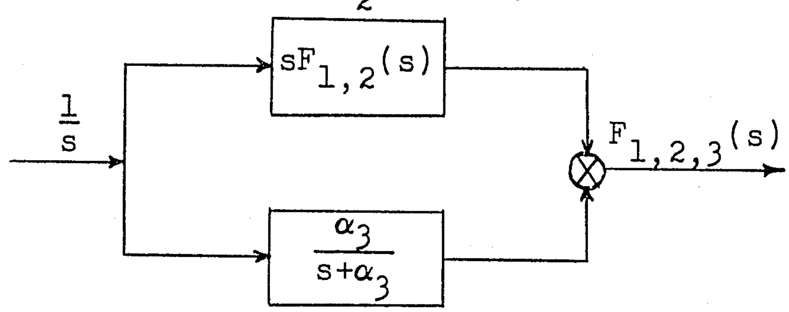


FIG. 3. SECOND SECTION OF THE OPEN-LOOP SYSTEM OF FIG. 1.

Again, applying Theorem 1.a for poles of $F_{1,2}(s)$ $s_1 = 0$, $s_2 = -\alpha_1$, $s_3 = -\alpha_2$, $s_4 = -(\alpha_1+\alpha_2)$ we obtain $F_{1,2,3}(s)$ as below.

$$F_{1,2,3}(s) = \alpha_3\left[\frac{1}{s(s+\alpha_3)} - \frac{1}{(s+\alpha_1)(s+\alpha_1+\alpha_3)} - \frac{1}{(s+\alpha_2)(s+\alpha_2+\alpha_3)} + \frac{1}{(s+\alpha_1+\alpha_2)(s+\alpha_1+\alpha_2+\alpha_3)}\right] \tag{14}$$

or

$$F_{1,2,3}(s) = \frac{1}{s} - \frac{1}{s+\alpha_1} - \frac{1}{s+\alpha_2} - \frac{1}{s+\alpha_3} + \frac{1}{s+\alpha_1+\alpha_2} + \frac{1}{s+\alpha_1+\alpha_3} + \frac{1}{s+\alpha_2+\alpha_3} - \frac{1}{s+\alpha_1+\alpha_2+\alpha_3} \tag{15}$$

Equation (12) and equation (15) are similar in form. Therefore, we do not have to repeat the complex convolution operation n times; we can write the linear representation for the entire system by inspection. Hence, if we have n multipliers, $F_{1,2,\ldots,n+1}(s)$ can readily be written as follows:

$$F_{1,2,\ldots,n+1}(s) = \frac{1}{s} - \sum_{i=1}^{n+1} \frac{1}{s+\alpha_i} + \frac{1}{2} \sum_{\substack{i=1 \\ i\neq j}}^{n+1} \sum_{j=1}^{n+1} \frac{1}{s+\alpha_i+\alpha_j} + \cdots + (-1)^n \frac{1}{n} \sum_{\substack{i=1 \\ i\neq j\neq \cdots \neq \ell}}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{\ell=1}^{n+1} \frac{1}{s+\alpha_i+\alpha_j+\cdots+\alpha_\ell} + (-1)^{n+1} \frac{1}{n+1} \sum_{\substack{i=1 \\ i\neq j\neq \cdots\neq p}}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{p=1}^{n+1} \frac{1}{s+\alpha_i+\alpha_j+\cdots+\alpha_p} \tag{16}$$

where

$F_{1,2,\ldots\ldots,n+1}(s)$ is the output response of the nonlinear multiplicative system for a unit step function input, and all g_i's are unity. (See appendix B for the proof of Eq. (16)).

If the magnitude of the step function input is M and the gains of linear system $H_i(s)$, $i = 1, 2, \ldots\ldots, n+1$ are g_i, then $F_{1,2,\ldots\ldots,n+1}(s)$ is to be multiplied by a "gain factor" $M^n \cdot \prod_{i=1}^{n+1} g_i$, so that

$$Y_m(s) = F_{1,2,\ldots\ldots,n+1}(s) \cdot M^n \cdot \prod_{i=1}^{n+1} g_i \tag{17}$$

where

$Y_m(s)$ is the output of the nonlinear multiplicative system. The transfer characteristic of the equivalent linear system is :

$$G(s) = \frac{\text{output}}{\text{input}} \tag{18}$$

so that

$$G(s) = s\Big[\frac{1}{s} - \sum_{i=1}^{n+1} \frac{1}{s+a_i} + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ i \neq j}}^{n+1} \frac{1}{s+a_i+a_j} + \ldots$$

$$+ (-1)^n \frac{1}{n} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{\substack{\ell=1 \\ i \neq j \neq \ldots \neq \ell}}^{n+1} \frac{1}{s+a_i+a_j+\ldots+a_\ell} +$$

$$(-1)^{n+1} \frac{1}{n+1} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{\substack{p=1 \\ i \neq j \neq \ldots \neq p}}^{n+1} \frac{1}{s+a_i+a_j+\ldots+a_p}\Big]$$

$$M^n \cdot \prod_{i=1}^{n+1} g_i \tag{19}$$

Equation (19) represents a series of decaying exponentials and thus is convergent. It also shows the simplicity with which a nonlinear multiplicative system of this class can be represented. Fig. 4 shows the linear representation of the open-loop non-linear multiplicative system of Fig. 1 for the above specified $H_i(s)(i=1,2,....,n+1)$.

Next we show that even if $H_1(s)$ is of higher order, the form of the linear representation remains essentially the same. We demonstrate this property with $H_1(s)$ a general second order linear system.

2.2.2 $H_1(s)$ a Second Order Linear System with Real Distinct Poles

Let

$$H_1(s) = \frac{\alpha_1\alpha_2}{(s+\alpha_1)(s+\alpha_2)} \tag{20}$$

Applying Theorem 1.a $F_{1,2}(s)$ is

$$F_{1,2}(s) = \frac{1}{s}\left[\frac{\alpha_2}{\alpha_2-\alpha_1}\,\frac{1}{s+\alpha_1} + \frac{\alpha_1}{\alpha_1-\alpha_2}\,\frac{1}{s+\alpha_2} + \frac{1}{s+\alpha_3}\right]$$

$$+ \frac{\alpha_2}{\alpha_2-\alpha_1}\cdot\frac{1}{s+\alpha_1+\alpha_3} + \frac{\alpha_1}{\alpha_1-\alpha_2}\,\frac{1}{s+\alpha_2+\alpha_3} \tag{21}$$

Let

$$\frac{\alpha_2}{\alpha_2-\alpha_1} = A\,, \qquad \frac{\alpha_1}{\alpha_1-\alpha_2} = B$$

Then equation (21) can be written as follows:

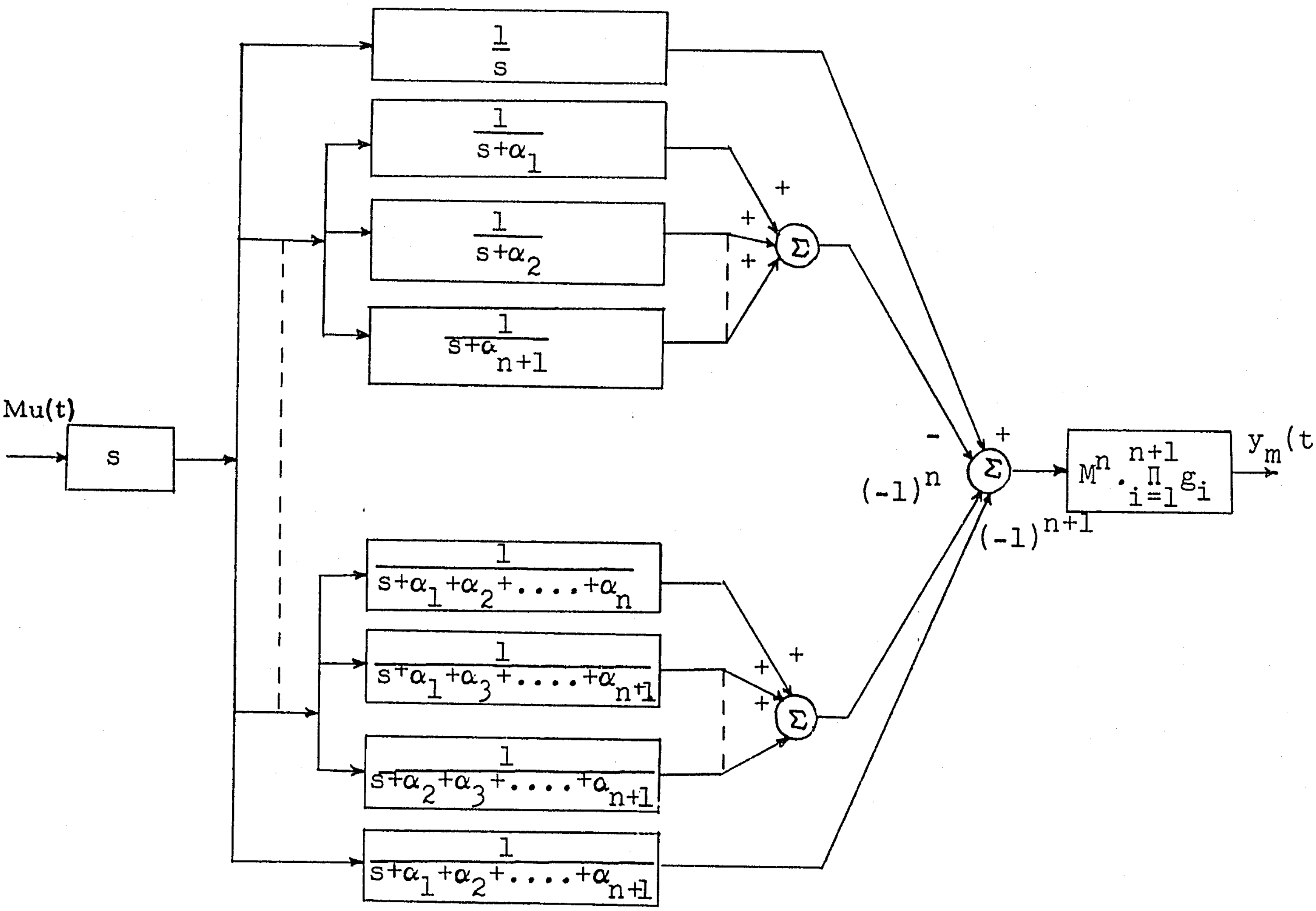


FIG. 4. LINEAR REPRESENTATION OF THE OPEN-LOOP SYSTEM OF FIG. 1.

$u(t)$ - Unit step function input

Note: Number of Blocks in each of the first Summers is given by $C_r^{n+1} = \frac{(n+1)!}{(n+1-r)!\,r!}$, where $(n+1)$ is the total number of variables, α_i, and r is the number of variables involved in any particular summation.

$$F_{1,2}(s) = \frac{1}{s} - \left[\frac{A}{s+\alpha_1} + \frac{B}{s+\alpha_2} + \frac{1}{s+\alpha_3}\right] + \frac{A}{s+\alpha_1+\alpha_3} + \frac{B}{s+\alpha_2+\alpha_3} \qquad (22)$$

Then, again, by Theorem 1.a.

$$F_{1,2,3}(s) = \frac{1}{s} - \left[\frac{A}{s+\alpha_1} + \frac{B}{s+\alpha_2} + \frac{1}{s+\alpha_3} + \frac{1}{s+\alpha_4}\right] + A\left[\frac{1}{s+\alpha_1+\alpha_3} + \frac{1}{s+\alpha_1+\alpha_4}\right]$$

$$+ B\left[\frac{1}{s+\alpha_2+\alpha_3} + \frac{1}{s+\alpha_2+\alpha_4}\right] + \frac{1}{s+\alpha_3+\alpha_4} - \left[\frac{A}{s+\alpha_1+\alpha_3+\alpha_4} + \frac{B}{s+\alpha_2+\alpha_3+\alpha_4}\right] \qquad (23)$$

From equation (22) and equation (23), we can write $F_{1,2,\ldots\ldots,n+1}(s)$ as follows:

$$F_{1,2,\ldots\ldots,n+1}(s) = \frac{1}{s} - \left[\frac{A}{s+\alpha_1} + \frac{B}{s+\alpha_2} + \sum_{i=3}^{n+2} \frac{1}{s+\alpha_i}\right]$$

$$+ \sum_{i=3}^{n+2} \frac{A}{s+\alpha_1+\alpha_i} + \sum_{i=3}^{n+2} \frac{B}{s+\alpha_2+\alpha_i} + \frac{1}{2} \sum_{\substack{i=3 \\ i\neq j}}^{n+2} \sum_{j=3}^{n+2} \frac{1}{s+\alpha_i+\alpha_j} +$$

$$\ldots\ldots + (-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \ldots\ldots\neq p}}^{n+2} \sum_{i=3}^{n+2} \ldots\ldots \sum_{p=3}^{n+2} \frac{A}{s+\alpha_1+\alpha_i+\alpha_j+\ldots+\alpha_p} +$$

$$(-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \ldots\ldots\neq p}}^{n+2} \sum_{j=3}^{n+2} \ldots\ldots \sum_{p=3}^{n+2} \frac{B}{s+\alpha_2+\alpha_i+\alpha_j+\ldots+\alpha_p} +$$

$$(-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \ldots\ldots\neq p}}^{n+2} \sum_{j=3}^{n+2} \ldots\ldots \sum_{p=3}^{n+2} \frac{1}{s+\alpha_i+\alpha_j+\ldots+\alpha_p} \qquad (24)$$

Equation (24) is similar to equation (16) except that we have more summation terms, which appear because of an additional factor in $H_1(s)$. An organized form of equation (24) is:

$$F_{1,2,\ldots,n+1}(s) = \frac{1}{s} + A\Big[-\frac{1}{s+\alpha_1} + \sum_{i=3}^{n+2}\frac{1}{s+\alpha_1+\alpha_i} - \frac{1}{2}\sum_{\substack{i=3\\ i\neq j}}^{n+2}\sum_{j=3}^{n+2}\frac{1}{s+\alpha_1+\alpha_i+\alpha_j} +$$

$$\ldots\ldots + (-1)^{n+1}\frac{1}{n}\sum_{i=3}^{n+2}\sum_{j=3}^{n+2}\cdots\sum_{p=3}^{n+2}{}_{i\neq j\neq\ldots\neq p}\frac{1}{s+\alpha_1+\alpha_i+\alpha_j+\ldots\ldots+\alpha_p}\Big]$$

$$B\Big[-\frac{1}{s+\alpha_2} + \sum_{i=3}^{n+2}\frac{1}{s+\alpha_2+\alpha_i} - \frac{1}{2}\sum_{\substack{i=3\\ i\neq j}}^{n+2}\sum_{j=3}^{n+2}\frac{1}{s+\alpha_2+\alpha_i+\alpha_j} + \ldots\ldots$$

$$\ldots\ldots + (-1)^{n+1}\frac{1}{n}\sum_{i=3}^{n+2}\sum_{j=3}^{n+2}\cdots\sum_{p=3}^{n+2}{}_{i\neq j\neq\ldots\neq p}\frac{1}{s+\alpha_2+\alpha_i+\alpha_j+\ldots\ldots+\alpha_p}\Big]$$

$$+\Big[-\sum_{i=3}^{n+2}\frac{1}{s+\alpha_i} + \frac{1}{2}\sum_{\substack{i=3\\ i\neq j}}^{n+2}\sum_{j=3}^{n+2}\frac{1}{s+\alpha_i+\alpha_j} + \ldots\ldots$$

$$\ldots\ldots + (-1)^{n+1}\frac{1}{n}\sum_{i=3}^{n+2}\sum_{j=3}^{n+2}\cdots\sum_{p=3}^{n+2}{}_{i\neq j\neq\ldots\neq p}\frac{1}{s+\alpha_i+\alpha_j+\ldots\ldots+\alpha_p}\Big] \tag{25}$$

If the magnitude of the input is M and g_i is the gain of the linear system $H_i(s)$ $(i = 1,2,\ldots\ldots,n+1)$, then equation (25) is to be multiplied by a "gain factor" $M^n \cdot \prod_{i=1}^{n+1} g_i$, so that $G(s) = \frac{\text{output}}{\text{input}}$.

$$G(s) = s[\frac{1}{s} + A \{-\frac{1}{s+\alpha_1} + \sum_{i=3}^{n+2} \frac{1}{s+\alpha_1+\alpha_i} - \frac{1}{2} \sum_{\substack{i=3 \\ i\neq j}}^{n+2} \sum_{j=3}^{n+2} \frac{1}{s+\alpha_1+\alpha_i+\alpha_j} + \cdots$$

$$\cdots + (-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \cdots \neq p}}^{n+2} \sum_{j=3}^{n+2} \cdots \sum_{p=3}^{n+2} \frac{1}{s+\alpha_1+\alpha_i+\alpha_j+\cdots+\alpha_p} \}$$

$$+ B\{-\frac{1}{s+\alpha_2} + \sum_{i=3}^{n+2} \frac{1}{s+\alpha_2+\alpha_i} - \frac{1}{2} \sum_{\substack{i=3 \\ i\neq j}}^{n+2} \sum_{j=3}^{n+2} \frac{1}{s+\alpha_2+\alpha_i+\alpha_j} + \cdots$$

$$\cdots + (-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \cdots \neq p}}^{n+2} \sum_{j=3}^{n+2} \cdots \sum_{p=3}^{n+2} \frac{1}{s+\alpha_2+\alpha_i+\alpha_j+\cdots+\alpha_p} \}$$

$$+ \{-\sum_{i=3}^{n+2} \frac{1}{s+\alpha_i} + \frac{1}{2} \sum_{\substack{i=3 \\ i\neq j}}^{n+2} \sum_{j=3}^{n+2} \frac{1}{s+\alpha_i+\alpha_j} + \cdots\cdots$$

$$\cdots + (-1)^{n+1} \frac{1}{n} \sum_{\substack{i=3 \\ i\neq j\neq \cdots \neq p}}^{n+2} \sum_{j=3}^{n+2} \cdots \sum_{p=3}^{n+2} \frac{1}{s+\alpha_i+\alpha_j+\cdots+\alpha_p} \}] M^n \prod_{i=1}^{n+1} g_i \quad (26)$$

The realization of equation (26) is similar to the one shown in Fig. 4.

2.2.3 $H_1(s)$ a Second Order Linear System with Real Repeated Poles

Let

$$H_1(s) = \frac{\alpha_1^2}{(s+\alpha_1)^2} \quad (27)$$

Applying Theorem 1.a.

$$F_{1,2}(s) = \frac{1}{s} - \frac{1}{s+\alpha_1} - \frac{\alpha_1}{(s+\alpha_1)^2} - \frac{1}{s+\alpha_2} + \frac{1}{s+\alpha_1+\alpha_2} + \frac{\alpha_1}{(s+\alpha_1+\alpha_2)^2} \qquad (28)$$

Similarly,

$$F_{1,2,3}(s) = \frac{1}{s} - \frac{1}{s+\alpha_1} - \frac{\alpha_1}{(s+\alpha_1)^2} - \frac{1}{s+\alpha_2} - \frac{1}{s+\alpha_3} + \frac{1}{s+\alpha_1+\alpha_2}$$

$$+ \frac{1}{s+\alpha_1+\alpha_3} + \frac{1}{s+\alpha_2+\alpha_3} + \frac{\alpha_1}{(s+\alpha_1+\alpha_2)^2} + \frac{\alpha_1}{(s+\alpha_1+\alpha_3)^2}$$

$$- \frac{1}{s+\alpha_1+\alpha_2+\alpha_3} - \frac{\alpha_1}{(s+\alpha_1+\alpha_2+\alpha_3)^2} \qquad (29)$$

From equation (28) and equation (29) we can immediately write $F_{1,2,\ldots\ldots,n+1}(s)$ as follows:

$$F_{1,2,\ldots\ldots,n+1}(s) = \frac{1}{s} + \Big[-\sum_{i=1}^{n+1} \frac{1}{s+\alpha_i} + \frac{1}{2}\sum_{\substack{i=1\\ i\neq j}}^{n+1}\sum_{j=1}^{n+1} \frac{1}{s+\alpha_i+\alpha_j} + \ldots\ldots$$

$$+ (-1)^{n+1}\frac{1}{n+1}\sum_{\substack{i=1\\ i\neq j\neq\ldots\neq p}}^{n+1}\sum_{j=1}^{n+1}\ldots\sum_{p=1}^{n+1} \frac{1}{s+\alpha_i+\alpha_j+\ldots+\alpha_p}\Big] +$$

$$\alpha_1\Big[\frac{1}{(s+\alpha_1)^2} + \sum_{i=2}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i)^2} - \frac{1}{2}\sum_{\substack{i=2\\ i\neq j}}^{n+1}\sum_{j=2}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i+\alpha_j)^2} +$$

$$\ldots\ldots + (-1)^{n+1}\frac{1}{n}\sum_{\substack{i=2\\ i\neq j\neq\ldots\neq p}}^{n+1}\sum_{j=2}^{n+1}\ldots\sum_{p=2}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i+\alpha_j+\ldots+\alpha_p)^2}\Big] \qquad (30)$$

Taking into account the magnitude of the input and the gains of the linear systems, the transfer characteristic of the equivalent linear representation of the open loop system of Fig. 1 is:

$$G(s) = s\Big[\frac{1}{s} + \Big\{-\sum_{i=1}^{n+1}\frac{1}{s+\alpha_i} + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{\substack{j=1\\ i\neq j}}^{n+1}\frac{1}{s+\alpha_i+\alpha_j} + \cdots\cdots\cdots\cdots$$

$$\cdots + (-1)^{n+1}\frac{1}{n+1}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}\cdots\sum_{\substack{p=1\\ i\neq j\neq\cdots\neq p}}^{n+1}\frac{1}{s+\alpha_i+\alpha_j+\cdots+\alpha_p}\Big\} +$$

$$\alpha_1\Big\{-\frac{1}{(s+\alpha_1)^2} + \sum_{i=2}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i)^2} - \frac{1}{2}\sum_{i=2}^{n+1}\sum_{\substack{j=2\\ i\neq j}}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i+\alpha_j)^2} +$$

$$\cdots\cdots + (-1)^{n+1}\frac{1}{n}\sum_{i=2}^{n+1}\sum_{j=2}^{n+1}\cdots\sum_{\substack{p=2\\ i\neq j\neq\cdots\neq p}}^{n+1}\frac{1}{(s+\alpha_1+\alpha_i+\alpha_j+\cdots+\alpha_p)^2}\Big\}\Big]$$

$$M^n \cdot \prod_{i=1}^{n+1} g_i \tag{31}$$

The realization of equation (31) is similar to the one shown in Fig. 4.

2.2.4 <u>$H_1(s)$ a Second Order Linear System with Complex Poles</u>

Let

$$H_1(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \tag{32}$$

where ξ is the damping factor

ω_n is the natural frequency of oscillation.

Applying Theorem 1.a.

$$F_{1,2}(s) = \frac{1}{s} - \frac{1}{A}(\omega_n A + \phi)\frac{1}{(s+\xi\omega_n)^2 + (\omega_n A + \phi)^2}$$

$$- \frac{1}{s+\alpha_1} + \frac{1}{A}(\omega_n A + \phi)\frac{1}{(s+\xi\omega_n+\alpha_1)^2 + (\omega_n A + \phi)^2} \qquad (33)$$

where

$$\phi = \cos^{-1}\xi$$

$$A = \sqrt{1 - \xi^2}$$

Similarly,

$$F_{1,2,3}(s) = \frac{1}{s} - \left[\frac{1}{s+\alpha_1} + \frac{1}{s+\alpha_2} + \frac{1}{A}\frac{(\omega_n A + \phi)}{(s+\xi\omega_n)^2 + (\omega_n A + \phi)^2}\right]$$

$$+ \frac{1}{s+\alpha_1+\alpha_2} + \frac{1}{A}\frac{(\omega_n A + \phi)}{(s+\xi\omega_n+\alpha_1)^2 + (\omega_n A + \phi)^2}$$

$$+ \frac{1}{A}\frac{(\omega_n A + \phi)}{(s+\xi\omega_n+\alpha_2)^2 + (\omega_n A + \phi)^2} - \frac{1}{A}\frac{(\omega_n A + \phi)}{(s+\xi\omega_n+\alpha_1+\alpha_2)^2+(\omega_n A+\phi)^2}$$

$$(34)$$

Let $(\omega_n A + \phi) = B$, then $F_{1,2,\ldots\ldots,n+1}(s)$ for the nth order nonlinear multiplicative system is as follows:

$$F_{1,2,\ldots\ldots,n+1}(s) = \frac{1}{s} + \left[-\sum_{i=1}^{n}\frac{1}{s+\alpha_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{1}{s+\alpha_i+\alpha_j} + \ldots\ldots\right.$$

$$\left.\ldots\ldots\ldots\ldots\ldots + (-1)^{n+1}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\ldots\sum_{\substack{p=1\\ i\neq j\neq\ldots\neq p}}^{n}\frac{1}{s+\alpha_i+\alpha_j+\ldots\ldots+\alpha_p}\right]+$$

$$\frac{B}{A}\Big[-\frac{1}{(s+\xi\omega_n)^2+B^2}+\sum_{i=1}^{n}\frac{1}{(s+\xi\omega_n+\alpha_i)^2+B^2}+\cdots\cdots\cdots\cdots$$

$$\cdots + (-1)^{n+1}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\cdots\sum_{\substack{p=1\\ i\neq j\neq\cdots\neq p}}^{n}\frac{1}{(s+\xi\omega_n+\alpha_i+\alpha_j+\cdots+\alpha_p)^2+B^2}\Big] \quad (35)$$

and

$$G(s) = s\Big[\frac{1}{s}+\Big\{-\sum_{i=1}^{n}\frac{1}{s+\alpha_i}+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{1}{s+\alpha_i+\alpha_j}+\cdots\cdots\cdots\cdots$$

$$\cdots + (-1)^{n+1}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\cdots\sum_{\substack{p=1\\ i\neq j\neq\cdots\neq p}}^{n}\frac{1}{s+\alpha_i+\alpha_j+\cdots+\alpha_p}\Big\} +$$

$$\frac{B}{A}\Big\{-\frac{1}{(s+\xi\omega_n)^2+B^2}+\sum_{i=1}^{n}\frac{1}{(s+\xi\omega_n+\alpha_i)^2+B^2}+\cdots\cdots\cdots\cdots$$

$$\cdots + (-1)^{n+1}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\cdots\sum_{\substack{p=1\\ i\neq j\neq\cdots\neq p}}^{n}\frac{1}{(s+\xi\omega_n+\alpha_i+\alpha_j+\cdots+\alpha_p)^2+B^2}\Big\}\Big]$$

$$M^n \prod_{i=1}^{n+1} g_i \quad (36)$$

The realization of equation (36) is similar to the one shown in Fig. 4.

If $\xi = 0$, then equation (36) becomes the equivalent representation of the nth order nonlinear multiplicative system where $H_1(s)$ has pure imaginary poles.

2.3 Comments

1. We have shown that the above class of nonlinear multiplicative systems can be represented by linear system multiplied by a "gain factor". It is also shown that the form of this representation does not change for system $H_1(s)$ of different order. Although we have not shown, but it is obvious from this study that if $H_1(s)$ is higher than second order, the form of the representation remains the same.

2. If a nonlinear system, excited by a step function input, can be represented by the functionals of equation (1) (separable kernel), then each functional can further be represented by linear system multiplied by a "gain factor".

3. The closed-loop response of a nonlinear multiplicative system differs from its open-loop equivalent linear representation during transient, but the response of the former is far superior to that of the latter as regards rise time and overshoot. This is an interesting property of these systems, since apparently, due to the presence of multipliers, the overshoot in the transient response is suppressed. This will be the subject of the next chapter, where we will show that the introduction of multipliers in

the forward path of feedback control systems greatly improves their performance.

Here, we are content to demonstrate the difference in the quality of closed-loop responses of the first order nonlinear multiplicative system and its open-loop equivalent linear representation. To demonstrate, we take $H_1(s) = H_2(s) = \frac{1}{s+1}$ and unit step function as the input. The equivalent linear representation of the open-loop nonlinear multiplicative system is $G(s) = \frac{2}{(s+1)(s+2)}$. The block diagrams of closed-loop systems for the above systems are shown in Fig. 5 and Fig. 6. The optimum closed-loop responses of these systems are shown in Fig. 7 and Fig. 8. The response of the nonlinear multiplicative control system of Fig. 5 (shown in Fig. 7), has no overshoot and also the rise time and settling time are negligible. This is the type of response which an engineer will strive to achieve for the optimum performance of a system. On the other hand, the optimum closed-loop response of the linear system of Fig. 6 (shown in Fig. 8) is comparatively sluggish and has some overshoot.

4. If an open-loop system transfer function is given and has a form similar to the transfer functions obtained in the preceding sections, then it can be realized by a nonlinear multiplicative system. In this way, one can achieve a better closed-loop response than is possible with the linear system.

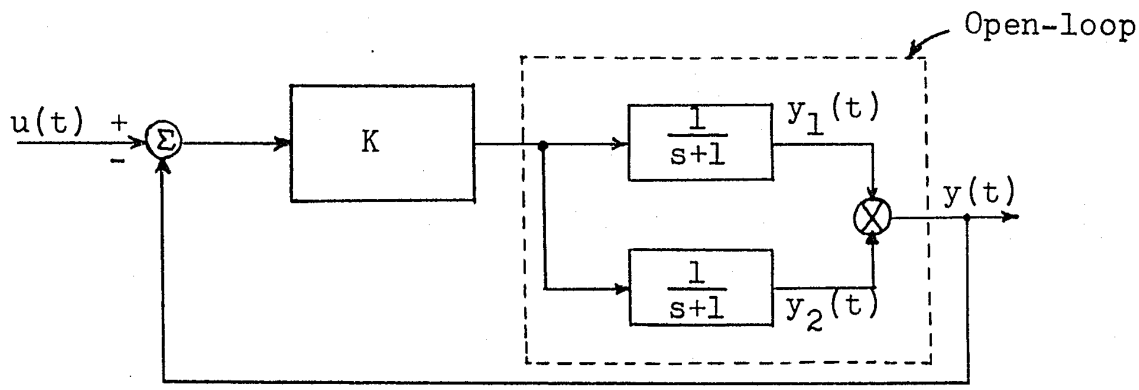


FIG. 5. FIRST ORDER NONLINEAR MULTIPLICATIVE FEEDBACK SYSTEM

K - gain of proportional controller
u(t) - unit step function input

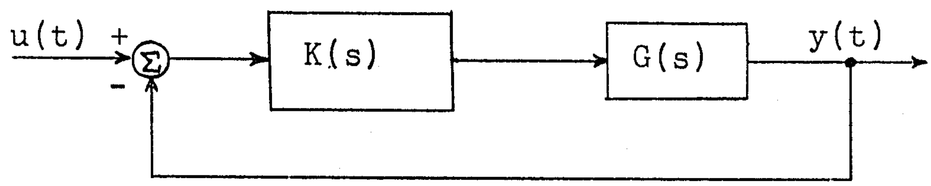


FIG. 6. FEEDBACK SYSTEM WHOSE SYSTEM TRANSFER FUNCTION $G(s) = \frac{2}{(s+1)(s+2)}$ IS EQUIVALENT TO THE OPEN-LOOP FIRST ORDER NONLINEAR MULTIPLICATIVE SYSTEM SHOWN INSIDE THE DOTTED LINES IN FIG. 5.

K(s) - Proportional plus integral controller

K = 14.0

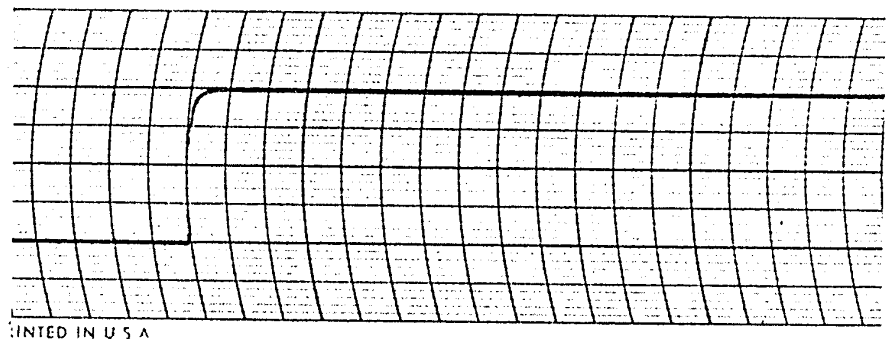


Fig. 7. Optimum Closed Loop Response of the First Order Nonlinear Multiplicative System with $H_1(s) = \frac{1}{s+1}$, $H_2(s) = \frac{1}{s+1}$.

K = 5.2

$T = \frac{1}{6.7}$ sec

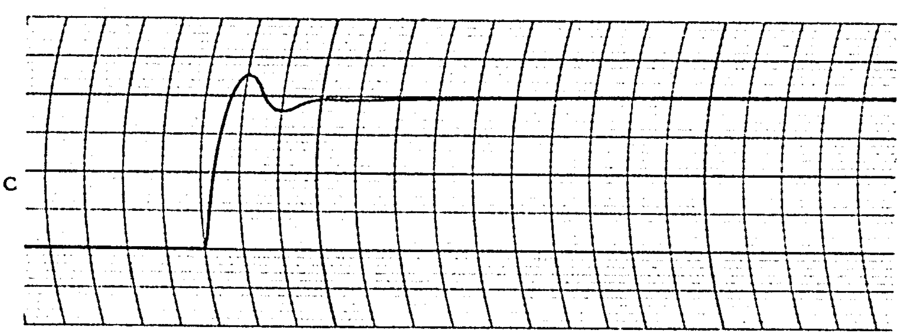


Fig. 8. Optimum Closed Loop Response of the Open Loop Equivalent Representation ($G(s) = \frac{2}{(s+1)(s+2)}$) of the First Order Nonlinear Multiplicative System with $H_1(s) = \frac{1}{s+1}$, $H_2(s) = \frac{1}{s+1}$.

T - Time constant of integration of the reset mode of the controller.

N.B. The scale throughout this study.

Horizontal scale 5 mm. / sec.

Vertical scale 20 mm. / volt.

CHAPTER III

INTRODUCTION OF INTENTIONAL MULTIPLICATIVE NONLINEARITIES TO IMPROVE THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

The optimum performance of a control system has always been a subject of great importance. Early researchers endeavored to achieve a compromise in performance criteria, such as overshoot, rise time, settling time, etc. of the transient response of a control system for the optimum performance. The tolerances on these criteria were specified and controllers designed to achieve them. Later, analytic criteria were introduced. Two of the earliest and most widely used performance criteria are:

i) Integral of the mean squared error criterion due to Wiener[6], and

ii) Integral squared error criterion due to Hall[7].

Later, several other performance criteria were introduced, an excellent account of which is given in a paper by Schultz and Rideout[8]. Since the choice of the performance criterion is a matter of individual ingenuity and since optimum performance is defined by the performance criterion, different persons may define a different "optimum" for the same system. In this study, since we will be dealing with regulator type systems, we define the optimum performance with respect to the "integral squared error criterion". The parameters of the controller for the optimum control system are obtained from the condition for an extremum of the performance criterion. This extremum is obtained either analytically or by analog or digital computations. In case analytic expressions become involved, analog or digital computation is adopted.

The systems considered in this chapter are time invariant, and analog computations will be sufficient to define the optimum controller parameters for the minimum of the integral squared error. In Chapter I, we pointed out that one has to try different combinations of linear and nonlinear elements for the controller to achieve the desired performance. Also, in Section 2.3 of Chapter II we remarked that the presence of a multiplier in a control system suppresses the overshoot and reduces the rise time of the transient response. In view of the above considerations, we propose the configuration of Fig. 1. for the improvement of the performance of such systems. In this configuration $H_1(s)$ is some existing system, whose performance we desire to improve. The remaining network is the, so called, "controller". We want to make $y_1(t)$ optimum with respect to the integral squared error criterion. At first, the proposed control system diagram appears to be quite involved. In fact, it is not, as we will see in the subsequent sections.

3.1 Review of the Previous Work

At this stage, it will be appropriate to discuss some of the previous work in nonlinear control of feedback systems. The concept of nonlinear feedback to improve the transient response of a servomechanism was first introduced by Lewis[9], whose system in block diagram form is shown in Fig. 9. The proposed system configuration shows that the system has nonlinear damping, which forces the output to follow the instantaneous values of the input.

Ku[10,11] studied the control of linear feedback systems by introducing the nonlinearities in either the feedforward, or feedback or both paths. Ku and Chen[12] studied the stability of a third order linear system with and without nonlinear control. They found some improvement in stability in comparison to the system without nonlinearity.

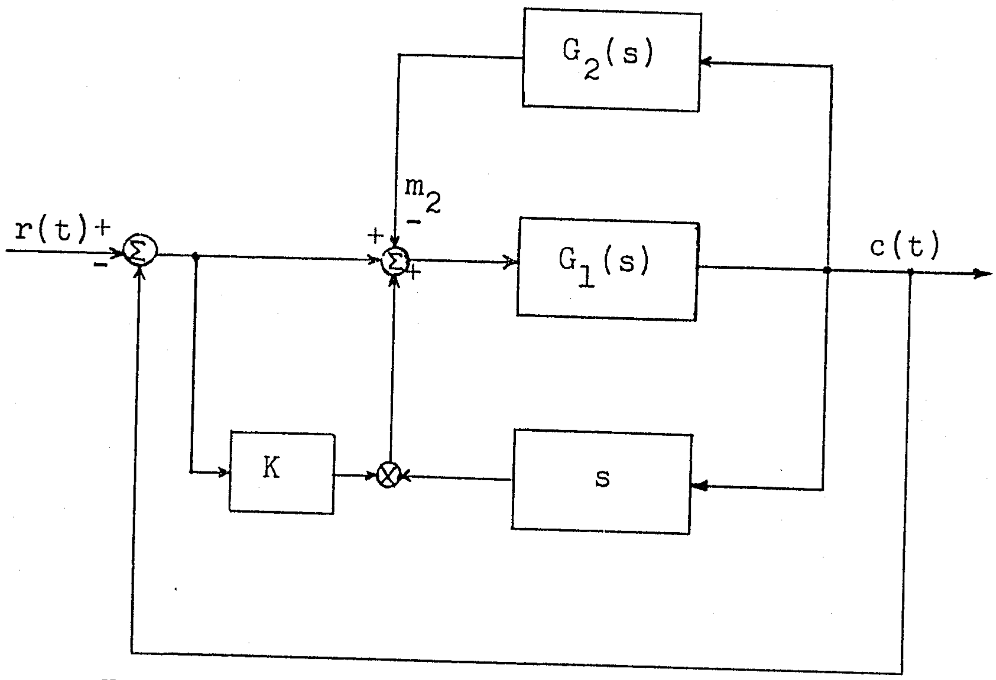


FIG. 9. NONLINEAR FEEDBACK CONTROL SYSTEM.

$G_1(s)$ - System Transfer Function
$G_2(s)$ - Linear System Intentionally Introduced
K - Controller Gain
$r(t)$ - Step Input
s - Differentiator

The work most relevent, is by Lewis[9]. The case considered, however, is very specific ($G_1(s) = \frac{1}{s(\tau s+1)}$). There is no discussion of the configuration of the control system, for a different system transfer function. By "different" we mean that the system transfer function is of higher order or has real multiple or complex poles. Also, the consistency of the system configuration in such situations, is not discussed.

In practice, many physical systems can be approximated by a second or third order linear system. The maximum improvement in the performance of a second order linear system employing the proposed scheme, is achieved with a proportional controller of gain K and one multiplier preceded by a first order linear system. In the case of a third order linear system, the maximum improvement in the performance is achieved with a proportional controller of gain K and two multipliers each preceded by a first order linear system. This scheme is simple and easy to instrument. It can also be employed to improve the performance of certain second order nonlinear differential systems. In all cases considered, the optimum output responses $y_1(t)$, of the existing system with a linear controller and with the proposed controller are compared.

We then proceed to show the improvement in the performance of the systems, mentioned above. The comparison is based on the minimum integral squared error criterion. In order to maintain the steady state output of the system equal to the reference input, the linear controller consists of proportional plus integral modes, while in our proposed scheme, for a first order nonlinear multiplicative system, this is achieved by a proportional controller of gain K and $\frac{K-1}{MK}$ as the gain of $H_2(s)$. In the case of second order nonlinear multiplicative systems, this is achieved by a proportional controller of gain K and the gains of $H_2(s)$ and $H_3(s)$ equal to $\frac{1}{M}\left(\frac{K-1}{K}\right)^{\frac{1}{2}}$, where M is the magnitude of the reference input. The block diagrams for the feedback control systems ($H_1(s)$ is a second or third order linear system) with linear controller and with the proposed scheme are shown in Fig. 10.

3.2 Improvement in the Performance of Second Order Linear Systems by the Proposed Scheme

In Fig. 10 let

$$H_1(s) = \frac{\gamma}{s^2 + \xi s + \gamma} \tag{37}$$

be a general second order linear system.

We will compare the optimum output responses, $y_1(t)$, in the cases indicated in Fig. 10, and show that by employing this scheme, the degree of improvement of the response

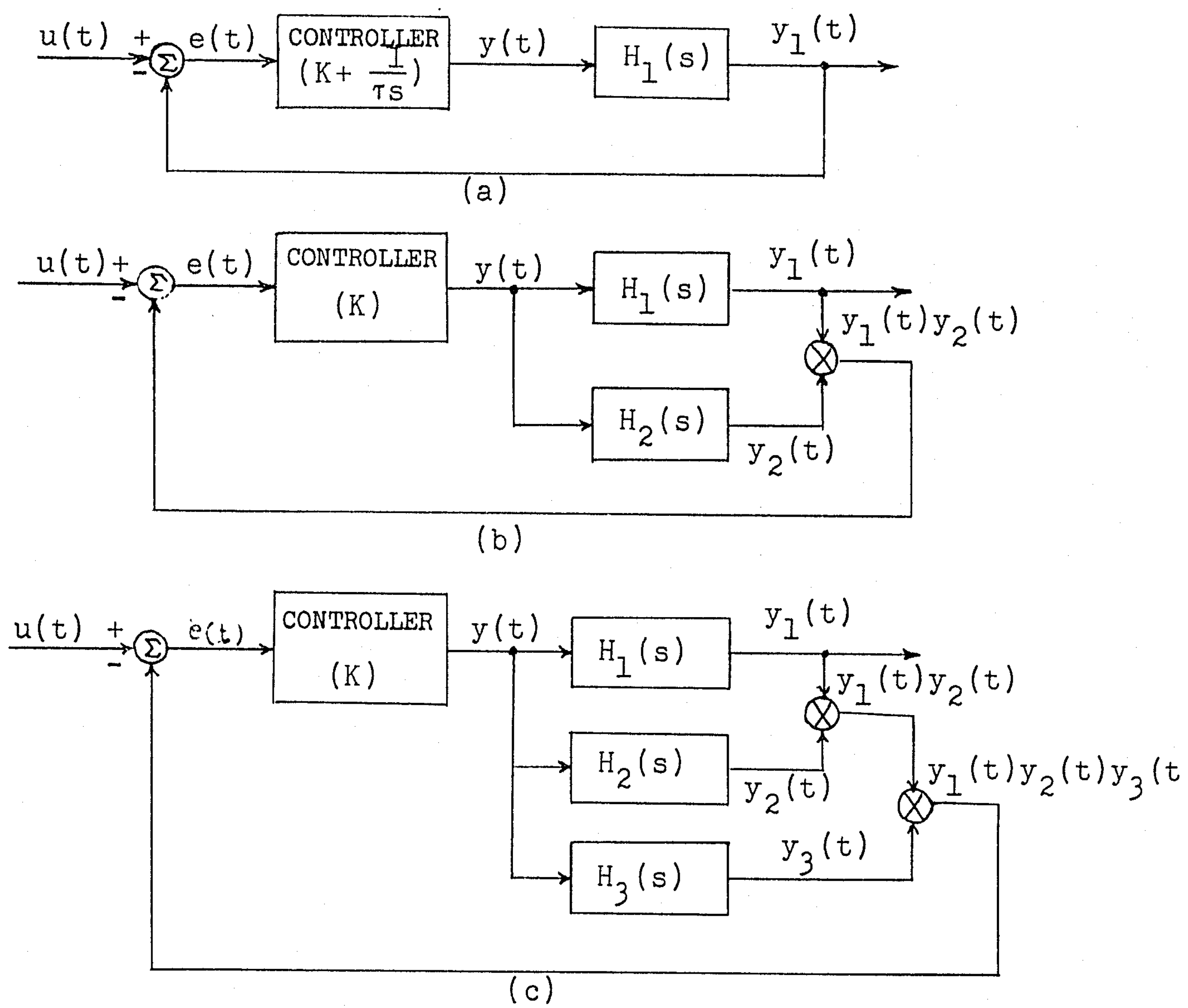


FIG. 10. BLOCK DIAGRAMS OF FEEDBACK CONTROL SYSTEMS.

a) with Linear Control
b) with Proposed Scheme using one Multiplier
c) with Proposed Scheme using two Multipliers

$H_1(s)$ - is some existing system whose performance we wish to improve.

$H_2(s), H_3(s)$ - are first order linear systems, which we introduce to improve the performance of $H_1(s)$.

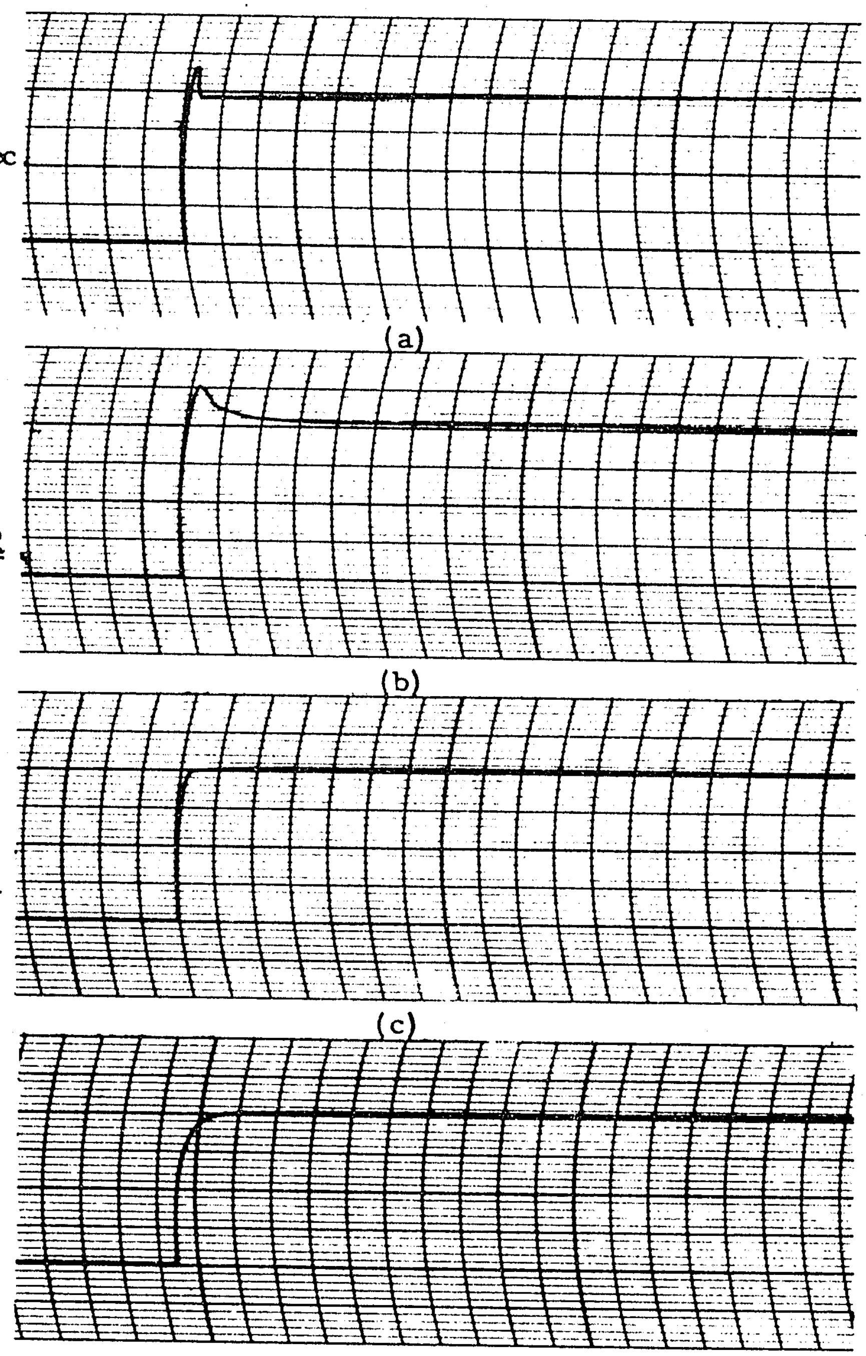
$H_1(s)$, $H_2(s)$ and $H_3(s)$ have poles in the left half plane including the origin.

$y_1(t)$ is remarkably high. A second order linear system can have different types of poles. In order to demonstrate the general applicability of our scheme, we will consider all types of second order systems.

3.2.1 Second Order Linear System with Real Distinct Poles

In equation (37), if $\xi = \alpha+\beta$ and $\gamma = \alpha\beta$, where $\xi, \gamma, \alpha, \beta$ are real nonnegative constants, then $-\alpha$ and $-\beta$ are the poles of $H_1(s)$. We will study the optimum closed-loop responses of the linear control and the control with the proposed scheme. For this study two examples are chosen i) $\alpha = 2$, $\beta = 5$, ($H_1(s) = \frac{10}{(s+2)(s+5)}$) and ii) $\alpha = 0.2$, $\beta = 0.5$,($H_1(s) = \frac{0.1}{(s+0.2)(s+0.5)}$; and optimum responses, $y_1(t)$, are recorded for various $H_2(s)$. The responses for examples (i) and (ii) are shown in Fig. 11 and Fig. 12 respectively. These figures show the degree of improvement as compared to the response of the system with linear control. The following conclusions are the results of extensive analog simulation study.

i) Only one multiplier is required to achieve the maximum improvement in the performance of the control system with the proposed scheme. It is found that, if we use two multipliers, the output response, $y_1(t)$, deteriorates. We demonstrate



K = 12.0

$T = \frac{1}{1.4}$ sec

ISE = 97

(a)

K = 12.0

$H_2(s) = \frac{0.92}{s+1}$

ISE = 120

(b)

K = 12.6

$H_2(s) = \frac{1.84}{s+2}$

ISE = 65

(c)

K = 11.0

$H_2(s) = \frac{4.55}{s+5}$

ISE = 98

(d)

Fig. 11. Closed Loop Optimum Responses of a Second Order Control System with Real Distinct Poles a) Linear Control, b, c, d) with Proposed Scheme. Using one Multiplier.

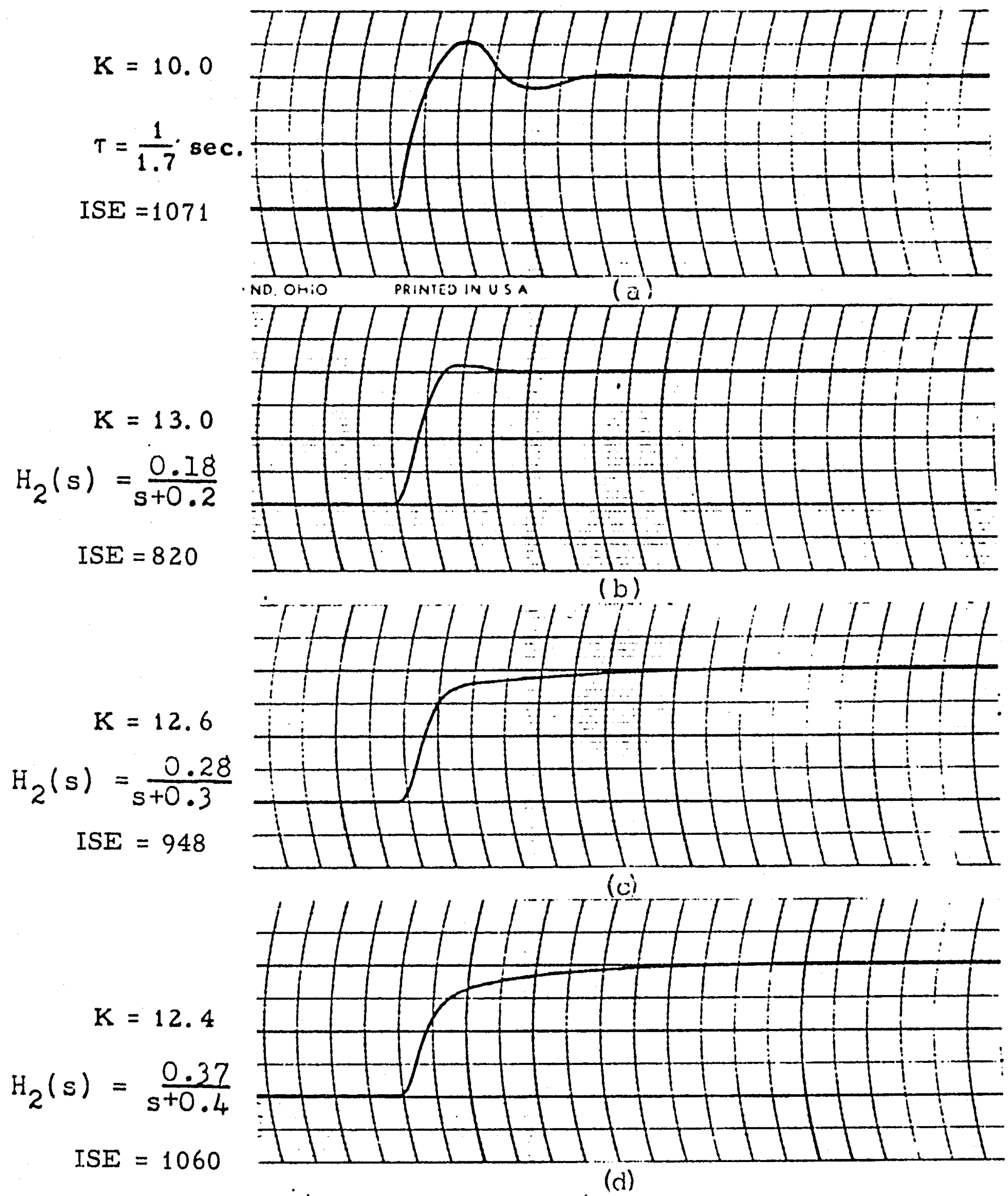


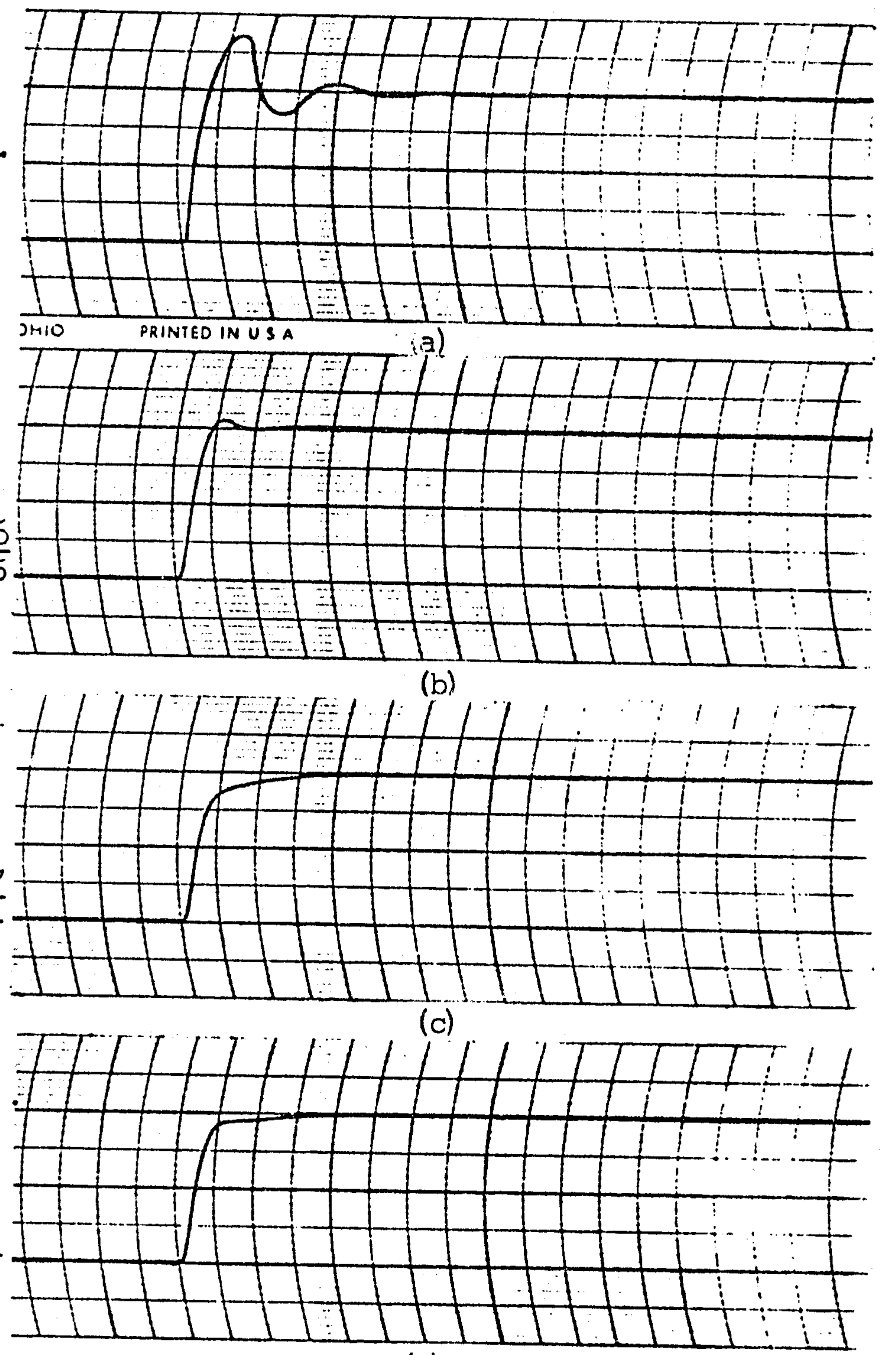
Fig.12 Optimum Closed Loop Responses of a Second Order System with Real Distinct Poles. a) with Linear Control, b, c, d) with Proposed Scheme Using one Multiplier.

this in 3.2.2 for a second order linear system with real repeated poles. Nevertheless, it is also true in this case.

ii) If $-\alpha$, $-\beta$ are the poles of the second order linear system such that $\alpha < \beta$ and $-\alpha_2$ (α_2 is a nonnegative constant) is the pole of $H_2(s)$, then the inequally $\alpha \leqslant \alpha_2 \leqslant \beta$ must be be satisfied for the maximum improvement in the performance of the system with the proposed scheme.

3.2.2 Second Order Linear System with Real Repeated Poles

In equation (37), if $\xi = 2\alpha$ and $\gamma = \alpha^2$, where ξ, γ, α are real nonnegative constants, then $H_1(s) = \frac{\alpha^2}{(s+\alpha)^2}$. For this study, we consider three examples: $\alpha = 0.5$, 1.0, 5.0. The optimum closed-loop responses for the above systems are shown in Figs. 13, 14, 15 and 16. In Fig. 15, ($H_1(s) = \frac{1}{(s+1)^2}$), we show that by introducing two multipliers, we make the control system sluggish. This implies that for a second order linear system, we need only one multiplier to achieve the optimum performance. Also, from these graphs, we find that the best response is obtained when $\alpha_2 = \alpha$. This supports our conclusions in Section. 3.2.1, where we stated that $\alpha \leqslant \alpha_2 \leqslant \beta$.



K = 12.0

$\tau = \frac{1}{2.2}$ sec.

ISE = 781

(a)

K = 13.0

$H_2(s) = \frac{0.46}{s+0.5}$

ISE = 580

(b)

K = 13.0

$H_2(s) = \frac{0.37}{s+0.4}$

ISE = 665

(c)

K = 13.0

$H_2(s) = \frac{0.55}{s+0.6}$

ISE = 630

(d)

Fig. 13. Optimum Closed Loop Responses of Second Order System with Real Repeated Poles a) with Linear Control, b, c, d) with Proposed Scheme Using one Multiplier.

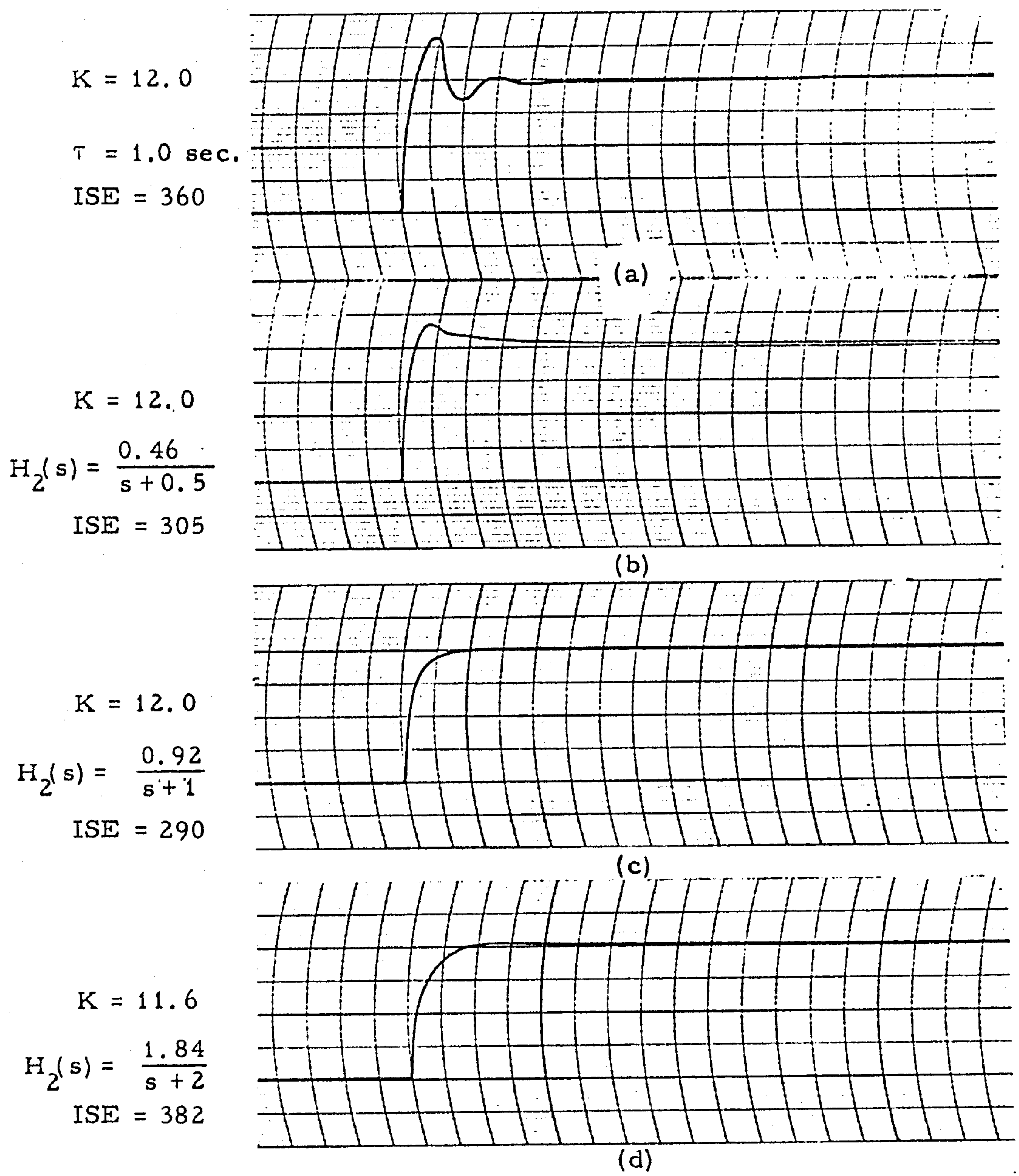
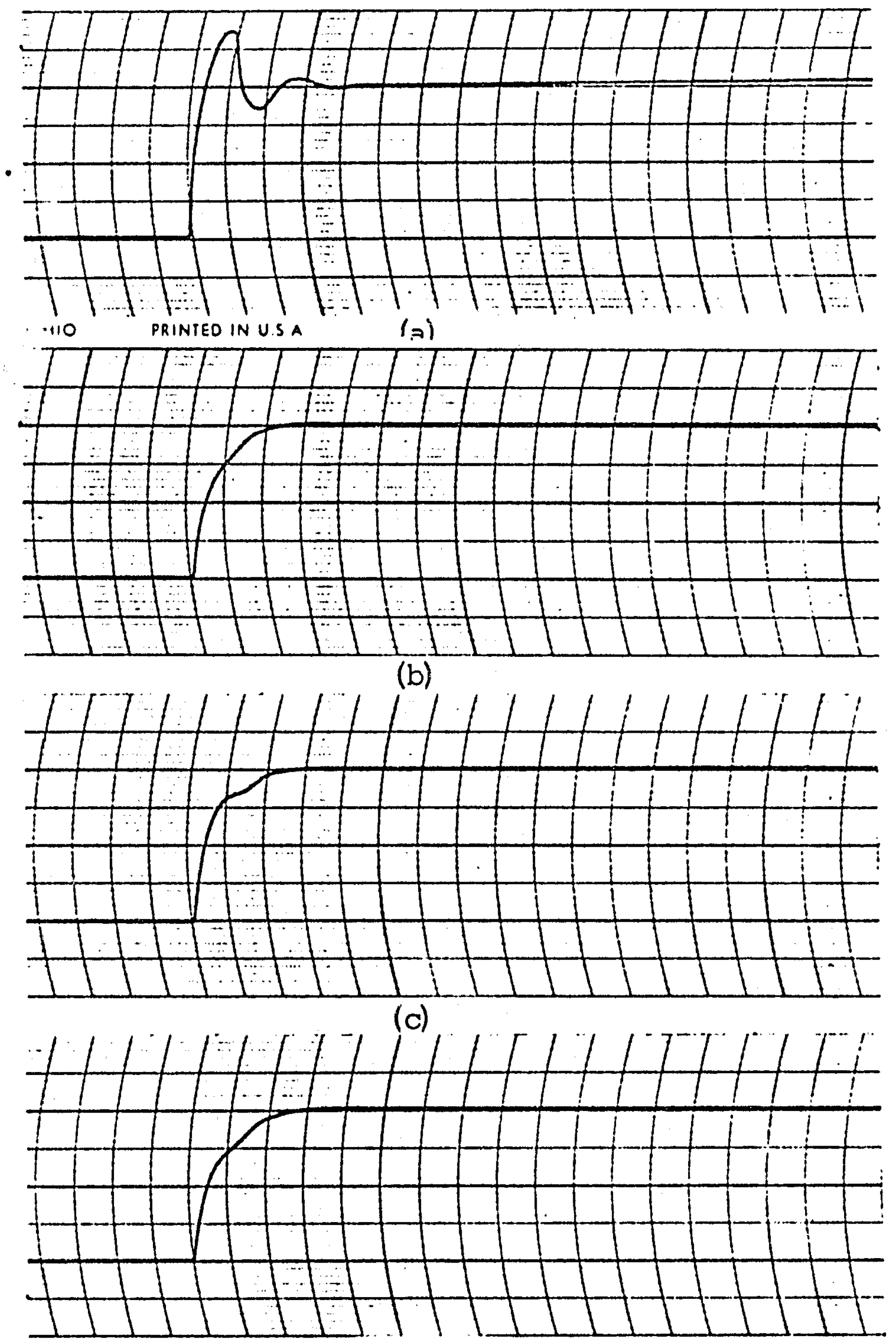


Fig. 14. Optimum Closed Loop Responses of a Second Order System with Real Repeated Poles. a) with Linear Control, b, c, d) with Proposed Scheme Using one Multiplier.



$K = 12.0$

$\tau = 1.0$ sec.

ISE = 360

$K = 13.0$

$H_2(s) = \frac{0.96}{s+1}$

$H_3(s) = \frac{0.96}{s+1}$

ISE = 422

$K = 13.3$

$H_2(s) = \frac{0.96}{s+1}$

$H_3(s) = \frac{1.92}{s+2}$

ISE = 492

$K = 13.3$

$H_2(s) = \frac{1.92}{s+2}$

$H_3(s) = \frac{1.92}{s+2}$

ISE = 561

Fig. 15. Optimum Closed Loop Responses of a Second Order System with Real Repeated Poles. a) with Linear Control, b, c, d) with Proposed Scheme Using two Multipliers.

K = 12.0

$T = \frac{1}{2.2}$ sec.

ISE = 87

K = 13.3

$H_2(s) = \frac{2.78}{s+3}$

ISE = 53

K = 13.0

$H_2(s) = \frac{4.6}{s+5}$

ISE = 51

K = 13.0

$H_2(s) = \frac{6.45}{s+7}$

ISE = 68

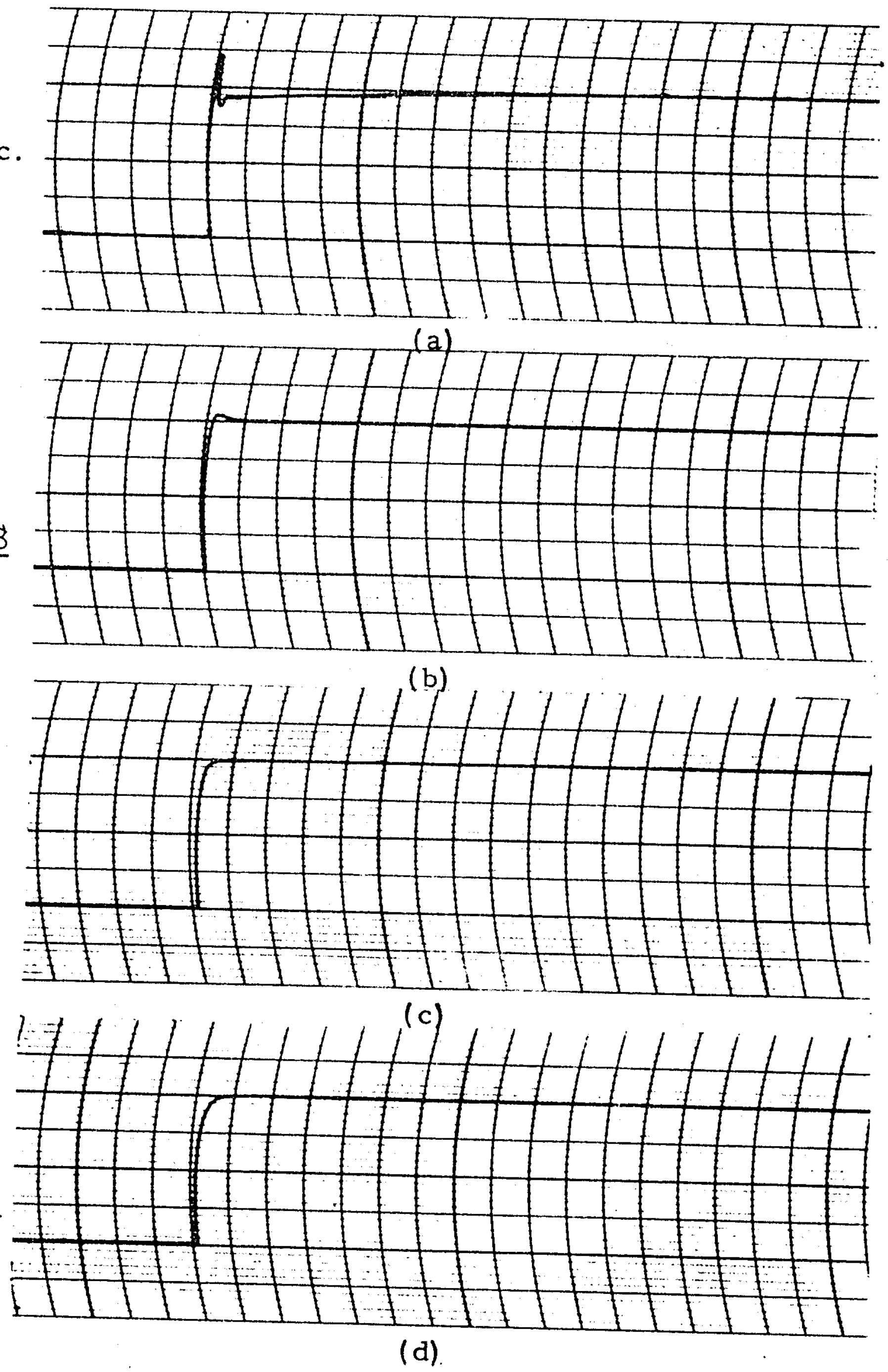


Fig. 16 Optimum Closed Loop Responses of a Second Order System with Real Repeated Poles. a) with Linear Control, b, c, d) with Proposed Scheme Using one Multiplier.

3.2.3 Second Order Linear Systems with Complex Poles

From equation (37), the poles of $H_1(s)$ are $\lambda_1, \lambda_2 = \dfrac{-\xi \pm (\xi^2 - 4\gamma)^{\frac{1}{2}}}{2}$. We consider here the case when $\xi = 1$ and $\gamma = 1$, so that $H_1(s)$ has complex poles $\gamma_1, \gamma_2 = \frac{1}{2}[-1 \pm j\sqrt{3}]$. The optimum output responses for the linear control, and with the proposed scheme with different $H_2(s)$ are given in Fig. 17. The improvement in the performance is apparent. Fig. 17.c gives the best response when $\alpha_2 = 1 = \frac{1}{2}[(-1)^2 + (\sqrt{3})^2]^{\frac{1}{2}}$. This suggests that if the complex poles of a second order system are at $-a \pm jb$ where a and b are positive constants, then $\alpha_2 = (a^2+b^2)^{\frac{1}{2}}$. This was checked for several cases, which are not included here.

3.2.4 Second Order Linear System with Pure Imaginary Poles

In equation (37), if $\xi = 0$ then $H_1(s)$ has pure imaginary poles. Here, we demonstrate the case with $\gamma = 1$ and show that by employing the proposed scheme, such an oscillatory system can be stabilized so that it attains a steady state. We also observe from the comparison of Fig. 18, b,c, and d, that the overshoot increases when we use two multipliers.

We conclude from the above study that the closed-loop response of a general second order linear system can be markedly improved by our proposed scheme. This scheme can readily be applied to practical problems, which involve second

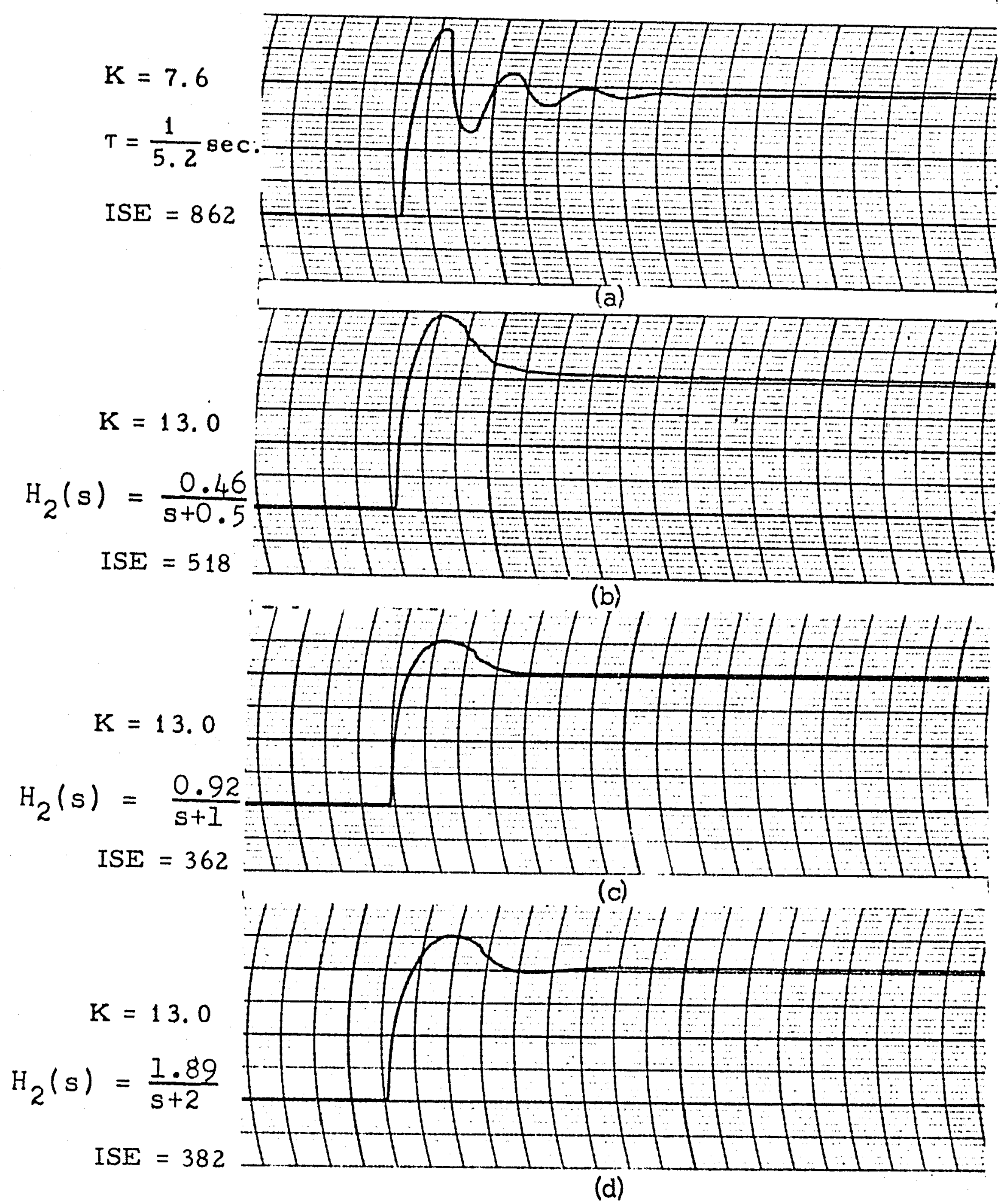


Fig. 17. Optimum Closed Loop Responses of a Second Order System with Complex Poles. a) with Linear Control, b, c, d) with Proposed Scheme Using one Multiplier.

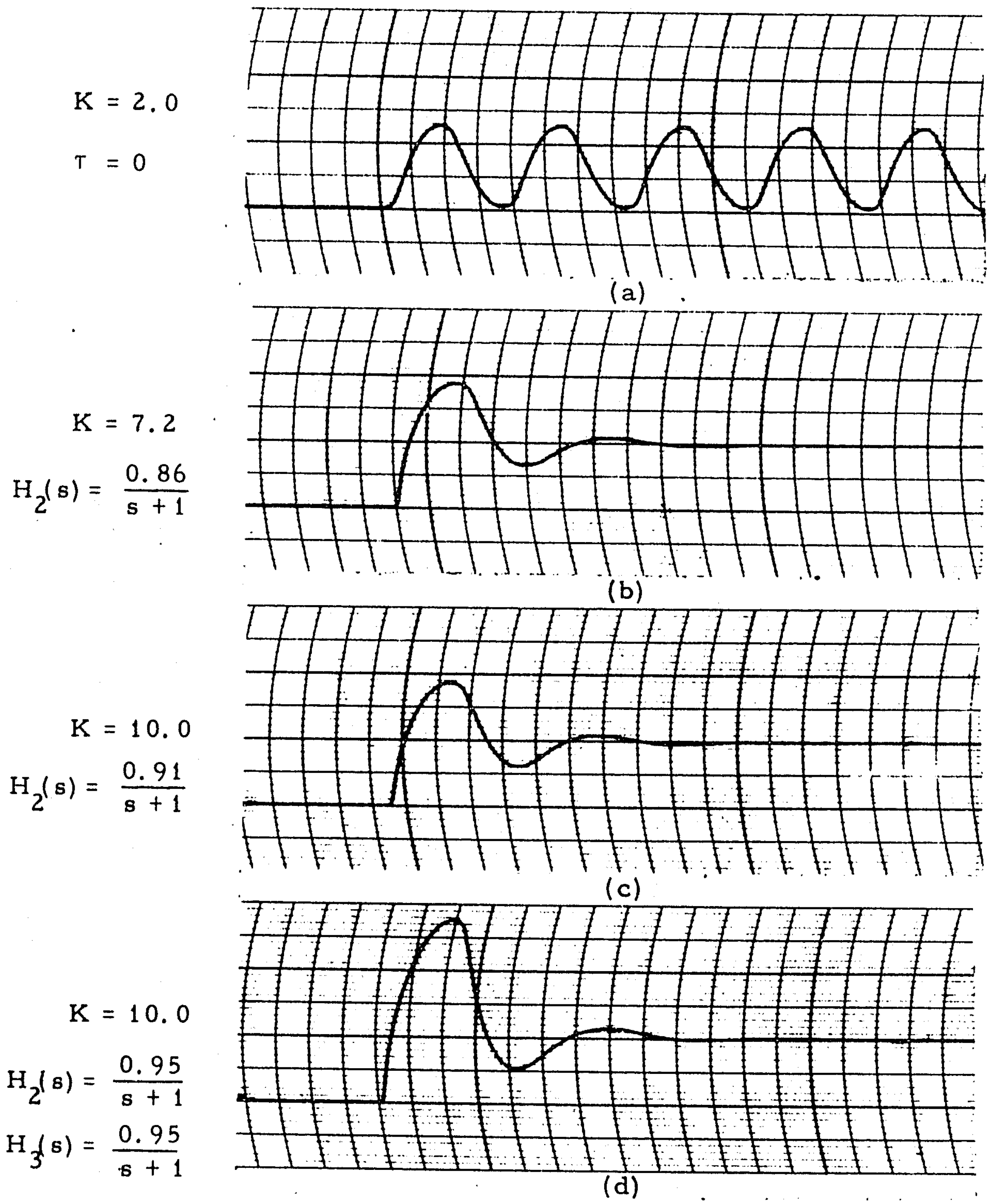


Fig. 18. Responses of Second Order Control System with Imaginary Poles. a) with Linear Control, b, c) with Proposed Scheme Using one Multiplier, d) with Proposed Scheme Using two Multipliers.

order linear systems. Also, this scheme can be used to stabilize oscillatory systems.

3.3 Improvement of the Performance of Third Order Linear Systems by the Proposed Scheme

In Fig. 10, let

$$H_1(s) = \frac{c}{s^3+as^2+bs+c} \tag{38}$$

be a general third order system, where a,b,c are nonnegative constants. We will compare the optimum responses, $y_1(t)$, in the cases indicated in Fig. 10, and show that by employing this scheme, the degree of improvement of the response $y_1(t)$ is remarkably high. We demonstrate the improvement in the performance of third order linear systems with:

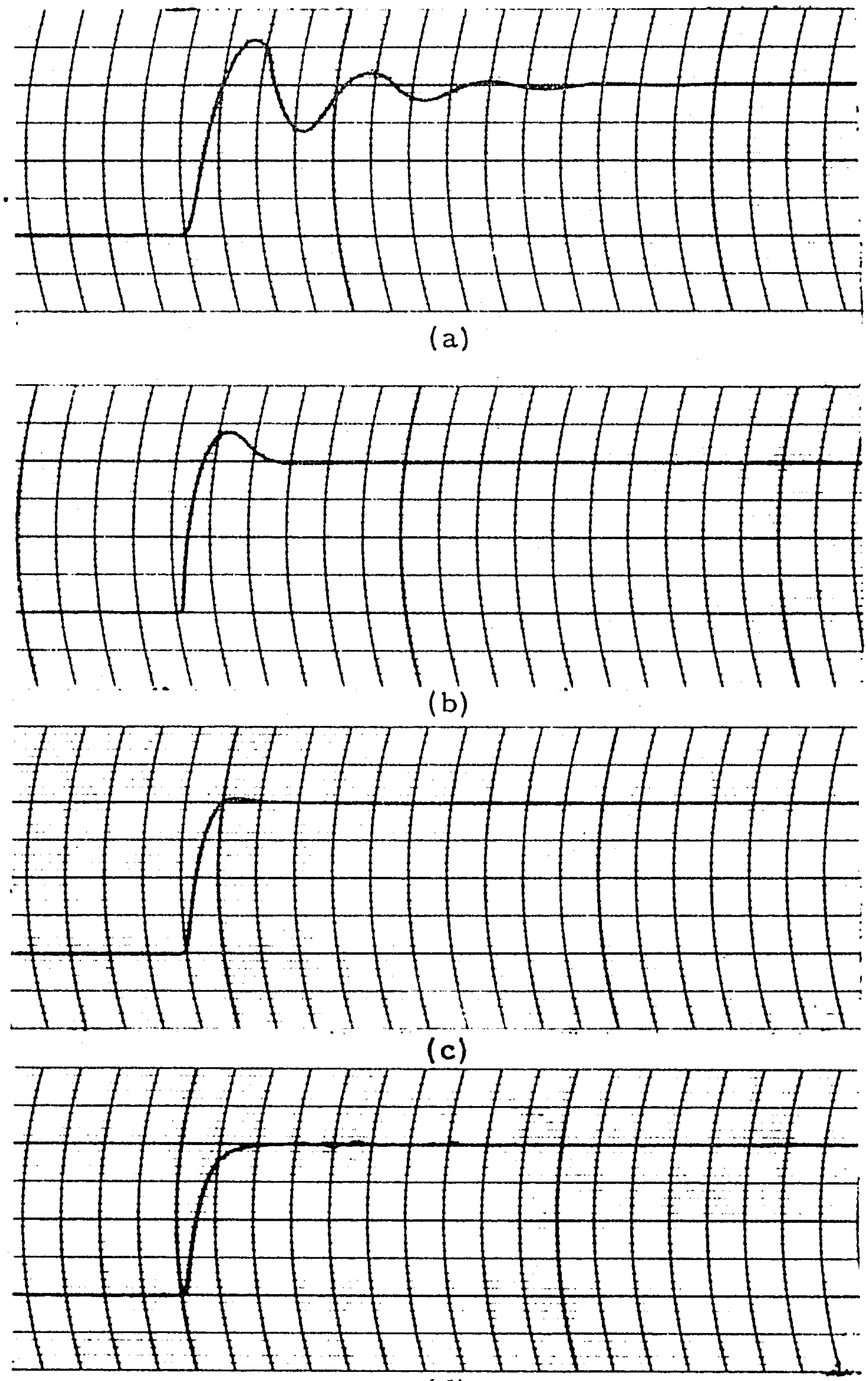
i) real distinct poles,

ii) real repeated poles,

iii) one real and a pair of complex poles.

3.3.1 Third Order Linear Systems with Real Distinct Poles

In equation (38), if $a = 6$, $b = 11$ and $c = 6$, then $H_1(s) = \frac{6}{(s+1)(s+2)(s+3)}$. The optimum output responses, $y_1(t)$, for this study are shown in Fig. 19. With a proportional controller, this system becomes unstable for



(a)

K = 3.55

$\tau = \frac{1}{0.78}$ sec.

ISE = 841

(b)

K = 13.3

$H_2(s) = \frac{0.92}{s+1}$

ISE = 294

(c)

K = 14.3

$H_2(s) = \frac{0.96}{s+1}$

$H_3(s) = \frac{1.93}{s+2}$

ISE = 262

(d)

K = 14.3

$H_2(s) = \frac{1.93}{s+2}$

$H_3(s) = \frac{1.93}{s+2}$

ISE = 292

Fig. 19. Optimum Closed Loop Responses of a Third Order System with Real Distinct Poles. a) with Linear Control b) with Proposed Scheme Using one Multiplier, c, d) with Proposed Scheme Using two Multipliers.

$k > 10$. In Fig. 19, b,c,d, we find that these systems can tolerate more gain than is permissible with linear control. This implies that the proposed scheme can be used to stabilize initially unstable systems. A comparison of Fig. 19.b with c and d shows that for maximum improvement in performance two multipliers are required. Also, the maximum improvement is obtained when $\alpha \leqslant \alpha_2 \leqslant \gamma$ and $\alpha \leqslant \alpha_3 \leqslant \gamma$, where $-\alpha$, $-\beta$, $-\gamma$ are the poles of $H_1(s)$ such that $\alpha < \beta < \gamma$ and $-\alpha_2, -\alpha_3$ are the poles of $H_2(s)$ and $H_3(s)$, respectively. $\alpha, \beta, \gamma, \alpha_2$ and α_3 are real nonnegative constants. These conclusions are based on extensive analog simulation not reported here, and are the same as those arrived at for the second order linear systems.

3.3.2 Third Order Linear Systems with Real Repeated Poles

In equation (38), if $a = 3$, $b = 3$ and $c = 1$, then $H_1(s) = \dfrac{1}{(s+1)^3}$. The optimum output responses, $y_1(t)$, are shown in Fig. 20. With a proportional controller, this system becomes unstable for $k > 8$. In Fig. 20, b,c,d, we find that these systems can tolerate more gain that is permissible with linear control. Also, we find that the maximum improvement in the performance is schieved with two multipliers and $\alpha_2 = \alpha_3 = \alpha$, where $-\alpha$ is the multiple of pole of $H_1(s)$,

K = 2.0

$\tau = \frac{1}{0.45}$ sec.

ISE = 1625

K = 13.3

$H_2(s) = \frac{1.85}{s+2}$

ISE = 792

K = 9.8

$H_2(s) = \frac{0.95}{s+1}$

$H_3(s) = \frac{0.95}{s+1}$

ISE = 545

K = 9.2

$H_2(s) = \frac{0.94}{s+1}$

$H_3(s) = \frac{1.88}{s+2}$

ISE = 583

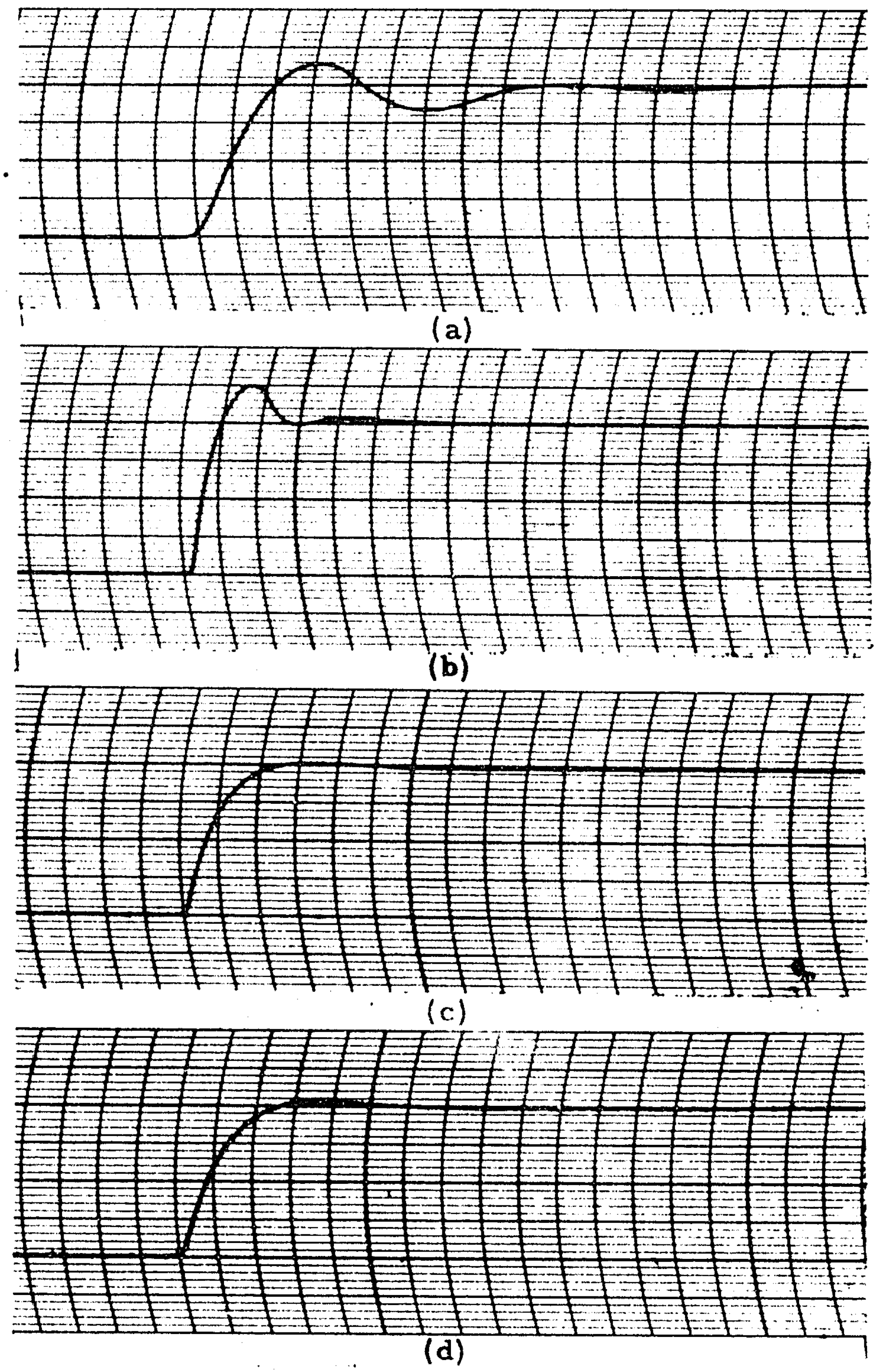


Fig. 20. Optimum Closed Loop Responses of a Third Order System with Real Repeated Poles. a) with Linear Control b) with Proposed Scheme Using one Multiplier. c, d) with Proposed Scheme Using two Multipliers.

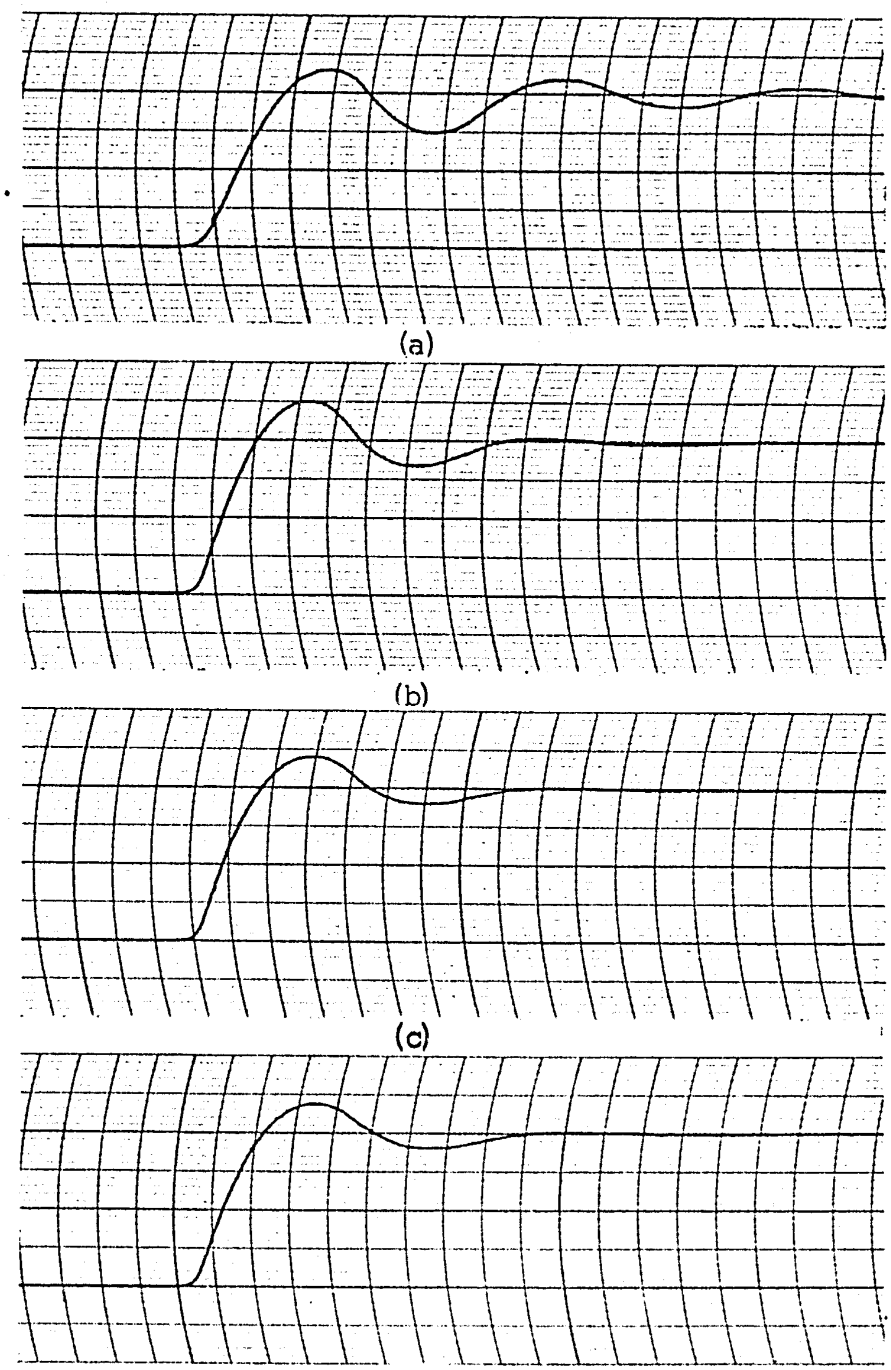
and $-\alpha_2, \alpha_3$ are the poles of $H_2(s)$ and $H_3(s)$ respectively. $\alpha, \alpha_2, \alpha_3$ are real nonnegative constants. These conclusions are again based on extensive analog simulation not reported here.

3.3.3 Third Order Linear Systems with One Real Pole and a Pair of Complex Poles

In equation (38), if $a = 2$, $b = 2$ and $c = 1$, then $H_1(s) = \frac{1}{(s^2+s+1)(s+1)}$, with poles at $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ and -1. The optimum output responses are shown in Fig. 21. With a proportional controller, this system becomes unstable for $k > 3$. In Fig. 21,b,c,d, we find that these systems can tolerate more gain than is permissible with linear control. This again demonstrates the stabilizing property of the proposed scheme. Also, we find that the maximum improvement in performance is obtained with two multipliers and $\gamma \leqslant \alpha_2 \leqslant \alpha_3 \leqslant (a^2+b^2)^{\frac{1}{2}}$, if $\gamma < (a^2+b^2)^{\frac{1}{2}}$, where $-\gamma$ is the real pole of $H_1(s)$ and $-a \pm jb$ are the complex poles of $H_1(s)$, $-\alpha_2, -\alpha_3$ are the poles of $H_2(s)$ and $H_3(s)$. These conclusions are again based on extensive analog simulation.

3.4 Improvement in the Performance of Certain Second Order Nonlinear Differential Systems

In this section, we consider second order systems with nonlinear velocity terms, and nonlinear position terms.



K = 1.0

$\tau = \frac{1}{0.438}$ sec.

ISE = 2054

(a)

K = 3.2

$H_2(s) = \frac{1.38}{s+2}$

ISE = 1142

(b)

K = 3.3

$H_2(s) = \frac{0.69}{s+1}$

$H_3(s) = \frac{0.69}{s+1}$

ISE = 1056

(c)

K = 3.3

$H_2(s) = \frac{1.38}{s+2}$

$H_3(s) = \frac{1.38}{s+2}$

ISE = 1143

(d)

Fig. 21. Optimum Closed Loop Responses of a Third Order System with one Real and a Pair of Complex Poles. a) with Linear Control, b) with Proposed Scheme Using one Multiplier, c, d) with Proposed Scheme Using two Multipliers.

The above properties are very common in physical systems. In this study, we initially investigated the use of both one multiplier and two multipliers. It was found that there was no further improvement in the performance of the nonlinear system with two multipliers, and therefore, we will not consider it in our subsequent sections.

The block diagrams for the feedback control systems with a linear controller and with proposed scheme are shown in Fig. 22.

We will compare the optimum output responses $y_1(t)$ in the two cases indicated in Fig. 22.

3.4.1 Second Order Nonlinear System with Nonlinear Velocity Term

In Fig. 22, let N be a nonlinear system which can be described by the following differential equation:

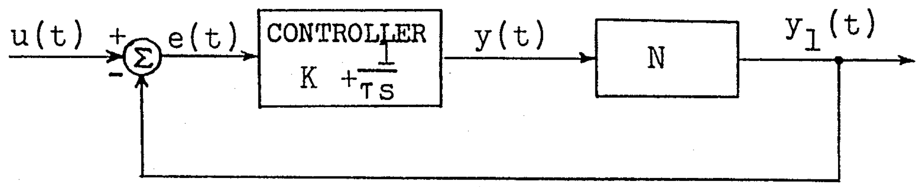
$$\ddot{y} + y\dot{y} + y = x \tag{39}$$

where

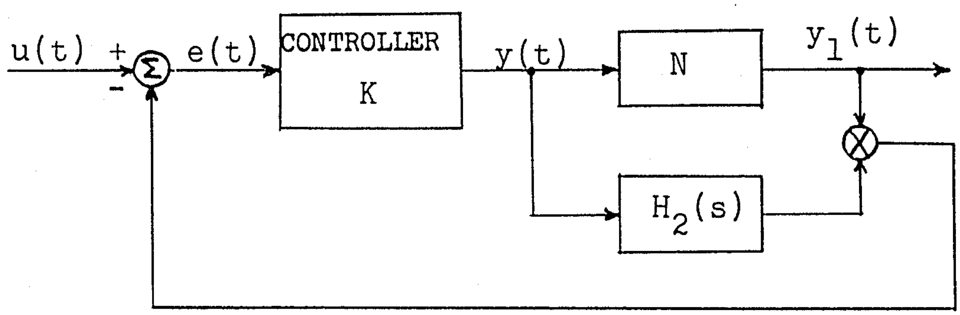
x is input to N

y is output of N

Equation (39) is a special case of Lienard's equation. The optimum output responses, $y_1(t)$ with linear con-



(a)



(b)

FIG. 22. BLOCK DIAGRAM OF FEEDBACK CONTROL SYSTEM.

(a) with Linear Control
(b) with Proposed Scheme using one Multiplier

N is some second order nonlinear differential system whose performance we desire to improve.

$H_2(s)$ is a first order linear system which we introduce to improve the performance of N.

trol and with the proposed scheme with one multiplier, are given in Fig. 23. The optimum output response of this system with linear control has a large overshoot and very long settling time, as shown in Fig. 23.a. Fig. 23.b and c show the degree of improvement in the output response, $y_1(t)$, with the proposed scheme.

3.4.2 Second Order Nonlinear System with a Nonlinear Position Term

In Fig. 22, let N be a nonlinear system which can be described by the following differential equation.

$$\ddot{y} + \dot{y} + y^2 = x \tag{40}$$

where

x is the input to N

y is the output of N

The optimum output responses $y_1(t)$ with linear control and with the proposed scheme using one multiplier are shown in Fig. 24. The optimum output response, $y_1(t)$, of this system with linear control shown in Fig. 24.a has a large overshoot and a long settling time. Fig. 24.b,c, show the improvement in the response, $y_1(t)$ with the proposed scheme.

3.5 Comments

1. The proposed scheme improves the performance of the system in every case studied. It is simple, easy to instrument, and hence practical.

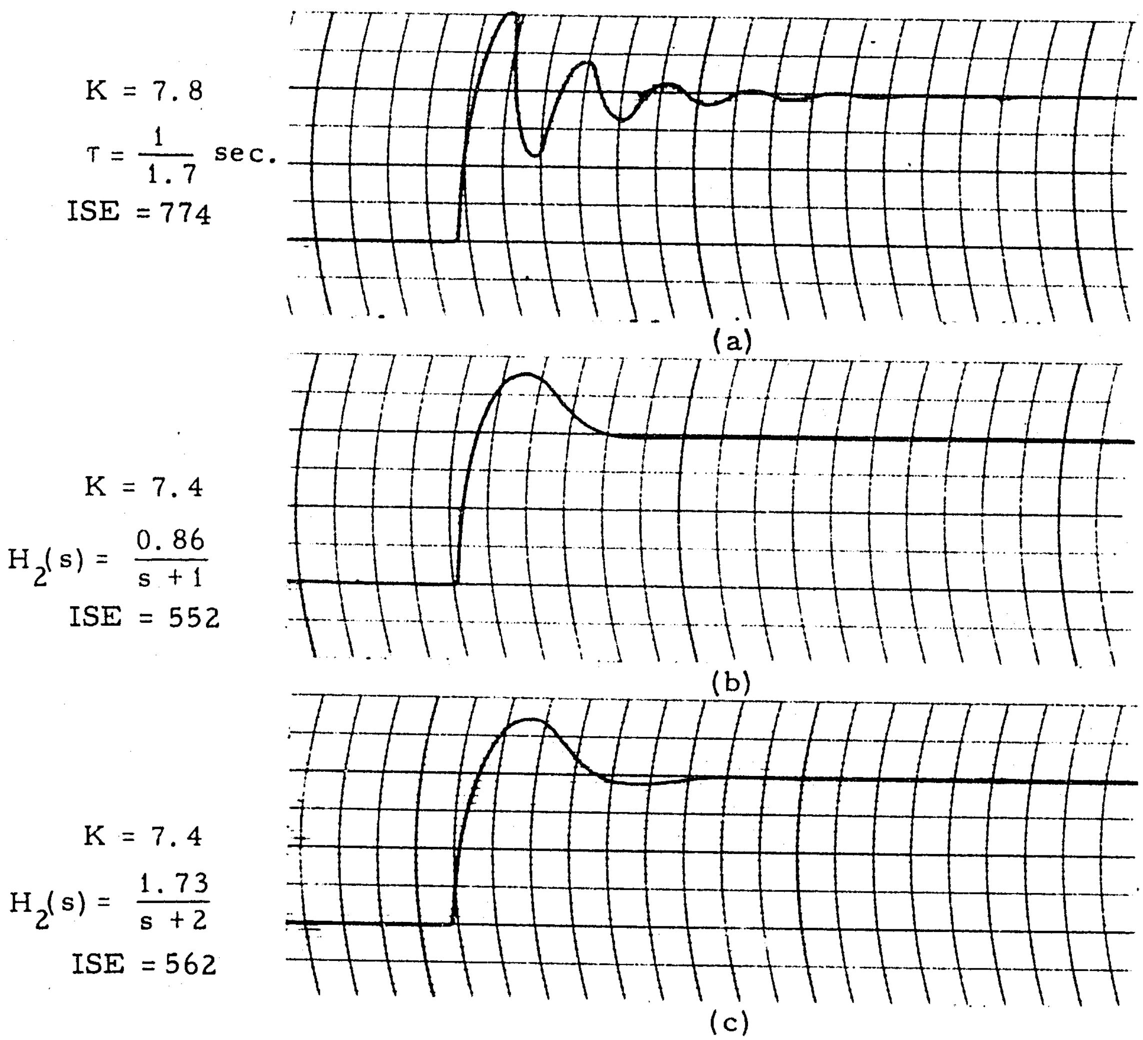


Fig. 23. Optimum Closed Loop Responses of the Nonlinear System with Nonlinear Velocity Term.
a) with Linear Control
b, c) with Proposed Scheme Using one Multiplier.

K = 10.0

$\tau = \frac{1}{2.4}$ sec.

ISE = 692

(a)

K = 10.0

$H_2(s) = \frac{0.90}{s+1}$

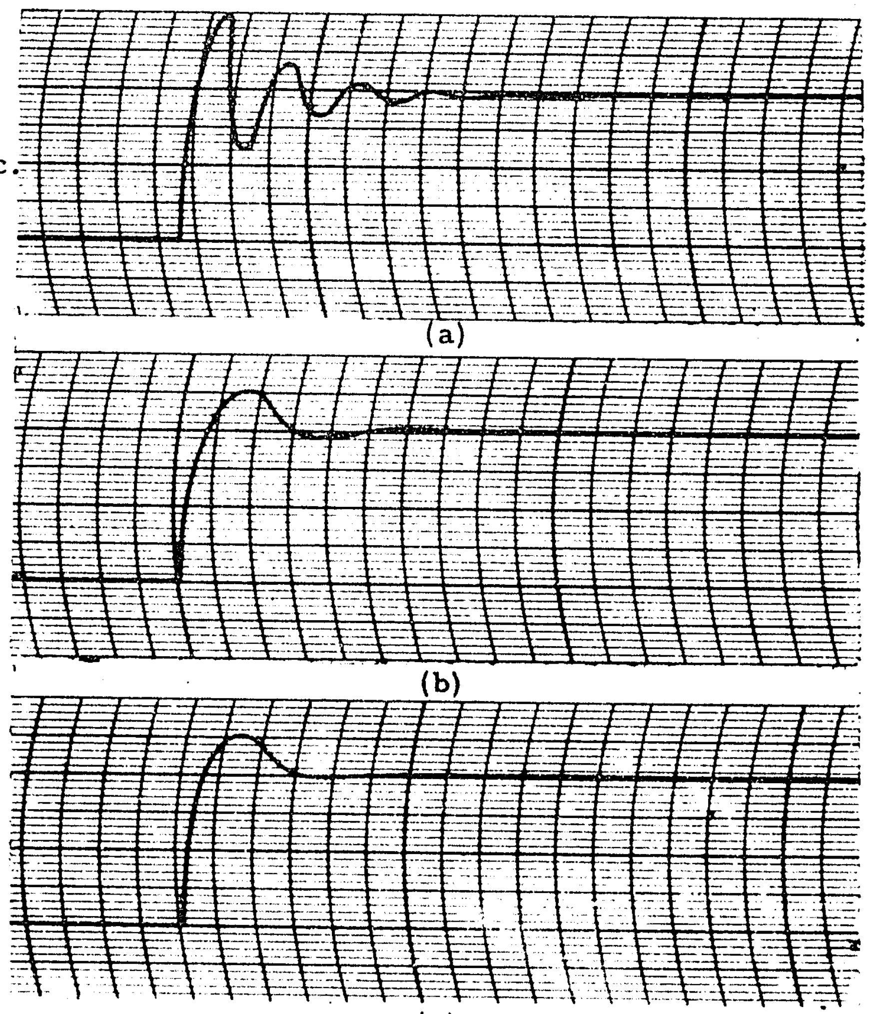
ISE = 360

(b)

K = 10.0

$H_2(s) = \frac{1.8}{s+2}$

ISE = 402



(c)

Fig. 24. Optimum Closed Loop Responses of a Second Order Nonlinear System with Nonlinear Position Term.
a) with Linear Control,
b, c) with Proposed Scheme Using one Multiplier.

2. In the case of second order linear systems and systems of Section 3.4 , we require only one multiplier for the maximum improvement of the performance of the given system. Whereas in the case of third order linear systems, we require two multipliers for the maximum improvement in performance.

3. If $-\alpha$ and $-\beta$ are the poles of a second order system and $-\alpha_2$ is the pole of $H_2(s)$, then $\alpha \leq \alpha_2 \leq \beta$ for maximum improvement of the performance. In the case of the real repeated poles $(\alpha = \beta)$, $\alpha_2 = \alpha$, if $\alpha = 0$ then $\alpha_2 = 0$. In the case of complex poles, if the poles are $\lambda_1, \lambda_2 = -a \pm jb$, then $\alpha_2 = (a^2+b^2)^{\frac{1}{2}}$. For the third order linear systems, if $-\alpha, -\beta$ and $-\gamma$ are the poles of $H_1(s)$ and $-\alpha_2$ and $-\alpha_3$ are the poles of $H_2(s)$ and $H_3(s)$, respectively, then for the maximum improvement in performance $\alpha \leq \alpha_2 \leq \gamma$ and $\alpha \leq \alpha_3 \leq \gamma$. If $\alpha = \beta = \gamma$, then $\alpha_2 = \alpha_3 = \alpha$, where $\alpha, \beta, \gamma, \alpha_2$ and α_3 are real nonnegative constants and a and b are real positive constants.

4. We have shown all the responses for a unit step function input. However, if the input magnitude

is M, the improvement in the performance of the existing system is just as good as in the case of the unit step function input.

5. The nonlinear systems differ so much in their characteristics that we cannot talk about the improvement in performance unless we study each case individually. However, from the two cases studied, it appears that this scheme can successfully be used for other second order nonlinear differential systems.

6. In the proposed scheme, since $\prod_{i=1}^{n} y_i \, (n=1,2,3,\ldots)$ rather than the system output $y_1(t)$, is compared with the reference input, the proposed control system is more sensitive to parameter variations than the control system where system output, $y_1(t)$, is directly compared with the reference input. For example, in the case of first order nonlinear multiplicative control systems, if the effective gain of $H_1(s)$ changes by x percent, then, for the range of controller gain K in this chapter, the steady state output of $H_1(s)$, changes by $x/2$ percent. Similarly, for a second order nonlinear control system, the

change in the steady state output of $H_1(s)$ is $2/3x$ percent. (See Appendix.) Since the system parameters may change over a period of time, and therefore changes occur in the effective gain of $H_1(s)$, the steady state output of $H_1(s)$ will not remain equal to the reference input. However, we can overcome this difficulty by comparing the steady state output of $H_1(s)$ (i.e., $y_1(t)$) with the reference input, passing the error through a low pass filter and using this signal to change the gain of $H_i(s)$ $(i = 2,3,\ldots.)$; so that the steady state output of $H_1(s)$ is maintained equal to the reference input.

CHAPTER IV

STABILITY ANALYSIS OF NONLINEAR MULTIPLICATIVE CONTROL SYSTEMS

4.1 Formulation of the Problem

In this chapter, we study the stability of the control systems discussed in Chapter III. More specifically, we will confine our study to first order nonlinear multiplicative control systems. Where possible, we will present the results in general terms.

In the following sections we present some definitions, theorems, and concepts necessary for the stability study.

4.2 Definitions

4.2.1 Stability in the Sense of Liapunov

Definition:

Let $\phi_i(t)$ be a solution of the set of equations,

$$\dot{x}_i = f_i\ (x_i, t) \tag{41}$$

We define a new set of variables; $q_i = x_i - \emptyset_i$. If these are substituted in (41), a new set of equations results, which has equilibrium position at $q = 0$. The equilibrium position is called stable in the sense of Liapunov if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $||q(t)|| < \epsilon$ whenever $||q(t_o)|| < \delta$ for all $t > t_o$. If q approaches zero as $t \to \infty$ the equilibrium position is called "asymptotically stable in the sense of Liapunov".

4.2.2 Definition:

The equilibrium of differential equation (41) is called "exponentially stable" if there exist two positive constants α and β which are independent of the initial values, such that for sufficiently small initial values the inequality,

$$|p(t,x_o,t_o)| < \beta|x_o| \exp(-\alpha(t-t_o)) \tag{42}$$

is satisfied. Where, $p(t,x_o,t_o)$ is a particular solution of equation (41).

4.2.3 Definition:

The equilibrium of the differential equation (41) is called "exponentially unstable" if two positive constants α and β exist and if there are initial values (x_o,t_o) in every domain $R_{h,\tau}$ with arbitrary small h and arbitrary large τ such that

$$|p(t,x_o,t_o)| > \beta |x_o| \exp\,(\alpha(t-t_o)) \qquad (43)$$

If this relation is valid for all initial points x_o with sufficiently small absolute values then the equilibrium is called "completely exponentially unstable".

Exponential stability, or exponential instability is known as the "significant behavior" of motions.

4.2.4 Definition of Liapunov Function

Let $v(x)$ be a function with the following properties:

i) $v(x)$ is continuous together with its first partial derivatives in a certain open region Ω about the origin.

ii) $v(0) = 0$

iii) Outside the origin (and always in Ω) $v(x)$ is positive.

iv) $\dot{v}(x) < 0$ in Ω ($\dot{v}(x)$ is the total time derivative of $v(x)$.

Then $v(x)$ is called a "Liapunov function".

4.2.5 Definition

A function $v(x,t)$ with $v(0,t) = 0$ is called "positive (negative) definite" if a continuous real function $\emptyset(r)$, (defined in the closed interval $o \leqslant r \leqslant h$ which vanishes at $r = 0$ and increases strictly monotonically with r) exists such that the relation

$$v(x,t) \geqslant \emptyset(|x|) \qquad (\leqslant -\emptyset(|x|)) \tag{44}$$

is satisfied in a half cylindrical neighborhood R_{h,t_o}.

4.2.6 Definition

A function $v(x,t)$ is called "radially unbounded" if inequality (44) is valid for arbitrarily large h where $\emptyset(r)$ increases unboundedly with r.

4.2.7 Definition

A function $v(x,t)$ is called "decrescent", if a continuous real function $\emptyset(r)$, with the above properties, exists such that in a half cylindrical neighborhood R_{h,t_o}

$$|v(x,t)| \leqslant \emptyset(|x|) \tag{45}$$

4.3 Variational Equations and Study of Singular Points

In the stability study of nonlinear systems, when it is not possible to find a closed form Liapunov function, Zubov's method[13] can be used to approximate v by numerical computations. This method may require a large number of computations for the exact determination of the stability region, especially when the system is higher than second order.

A study of the singular points provides information about the behavior of the system in the neighborhood of the singular point. Although this approach does not provide complete information regarding the extent of the region of asymptotic stability, it does give a fair idea of the behavior of the solutions in a certain neighborhood of the singular point.

Consider an nth order, time invariant, dynamical system described by the following equation

$$\dot{x} = f(x) \tag{46}$$

The singular points of equation (46) are given by $\dot{x} = 0$. Hence they are the solutions of

$$f(x) = 0 \tag{47}$$

Let $x = x_{ie}$ be the ith solution of equation (47), then equation (47) becomes

$$f(x_{ie}) = 0 \qquad i = 1,2,\ldots. \tag{48}$$

If each of the components of $f(x)$ can be expanded in a Taylor series about the ith singular point, the result of considering only the linear system is

$$\frac{d}{dt}(x - x_{ie}) = \dot{x} = J(x_{ie})(x - x_{ie}) \tag{49}$$

where $J(x_{ie})$ is the Jacobian matrix evaluated at $x = x_{ie}$, that is,

$$J(x_{ie}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots\cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x = x_{ie}} \tag{50}$$

In terms of $u = x - x_{ie}$, a variable measured from the singular point, equation (51) are the variational equations

$$\dot{u} = J(x_{ie})u \tag{51}$$

The variational equations are linear homogeneous differential equations, which for the autonomous case under consideration have constant coefficients. The stability of equation (51) can thus be determined from the roots of

$$|\lambda I - J\ (x_{ie})| = 0 \tag{52}$$

The nature of the roots determines the nature of the singular points. We will not discuss it here but refer to the book by Lefschetz[14].

4.4 Some Theorems on Stability

Let us assume that equation (46) can be written in the following form:

$$\dot{x} = Ax + g(x) \tag{53}$$

where

A is an nxn matrix with constant coefficients and $g(x)$ is the nonlinear part of equation (53)

Let

$$\dot{x} = Ax \tag{54}$$

be an autonomous linear differential equation. Suppose that functions $g_i(x)$ have, in the neighborhood of the origin, a power series expansion in the variables $x_1, x_2, \dots, x_n$ beginning with terms of at least second order. Then equation (54) is the first approximation of equation (53).

We state the following theorems, which will be used in the stability analysis of the nonlinear multiplicative control systems of Chapter III.

4.4.1 Theorem

The equilibrium of equation (54) is asymptotically stable if all the eigen-values of the matrix A have negative real parts.

4.4.2 <u>Theorem</u> (Liapunov)[15]

The equilibrium is asymptotically stable if there exists a positive definite decrescent function $v(x,t)$, so that its total time derivative $\dot{v}(x,t)$, for $\dot{x} = f(x,t)$ is negative definite.

4.4.3 <u>Theorem on Stability in the First Approximation</u>

If the stability behavior of the differential equation of the first approximation is "significant", then the equilibrium of the complete differential equation has the same behavior as the equilibrium of the reduced equation.

4.4.4 <u>Theorem</u> (Liapunov)[15]

The equilibrium state $x_e = 0$ of a continuous-time, free, linear, stationary dynamic system of equation (54) is asymptotically stable (a) if, and (b) only if, given any symmetric, positive - definite matrix Q there exists a symmetric, positive-definite matrix P which is the unique solution of the set of $\frac{n(n+1)}{2}$ linear equations

$$A^T P + PA = -Q \tag{55}$$

and $x^T Bx$ is a Liapunov function for equation (54).

4.5 Bounds for the Initial Values

When a closed form Liapunov function cannot be found for the complete differential equation of the system, one can have a conservative estimate of the asymptotic stability region by constructing a Liapunov function for the linear approximation of the differential equation.

Suppose that the linear part is asymptotically stable. We consider a Liapunov function $v(x)$ for the linear part and its derivative $\dot{v}(x)$ for the total equation. Let the function $v(x)$ be radially unbounded, which is the case in general for every definite form. If the nonlinearity is bounded and if $\dot{v}(x)$ is negative-definite throughout the entire state space, then according to a theorem due to Barbasin and Krasovskii[16], the equilibrium is asymptotically stable in the whole. However, if $\dot{v}(x)$ vanishes for certain points $x \neq 0$, we cannot draw this conclusion; but as long as one of the hypersurfaces $v(x) =$ constant lies completely in the interior of the domain determined by $\dot{v}(x) = 0$, then it certainly belongs to the domain of attraction of the origin. Hence it is necessary to determine these surfaces $v(x) =$ constant, which contact the surface $\dot{v}(x) = 0$ from the inside and to select from all these surfaces that one which corresponds to the smallest value v_o of the constant. Then

$$v(x) < v_o \tag{56}$$

is the subdomain of the domain of attraction.

4.6 Stability Study of First Order Nonlinear Multiplicative Control Systems

The control system to be analyzed for stability is shown in Fig. 25.

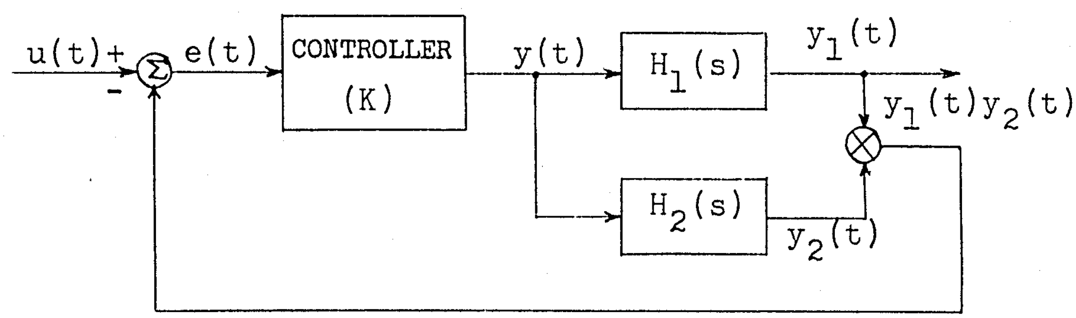


FIG. 25. BLOCK DIAGRAM FOR THE FIRST ORDER NONLINEAR MULTIPLICATIVE CONTROL SYSTEM

In this study, the controller has a proportional gain K, and $H_2(s)$ is always a first order linear system. The following cases are considered: $H_1(s)$ is

i) a first order linear system

ii) a second order linear system

iii) a third order linear system

4.6.1 $H_1(s)$ is a First Order Linear System

In Fig. 25, let $H_1(s) = \frac{k_1}{s+\alpha_1}$, $H_2(s) = \frac{k_2}{s+\alpha_2}$, then the differential equations for the system are

$$\begin{aligned} \dot{y}_1 + \alpha_1 y_1 &= k_1 K(M - y_1 y_2) \\ \dot{y}_2 + \alpha_2 y_2 &= k_2 K(M - y_1 y_2) \end{aligned} \tag{57}$$

The system assumes an equilibrium position when $\dot{y}_1 = \dot{y}_2 = 0$ and following conditions result.

$$\frac{y_1}{\frac{k_1}{\alpha_1}} = \frac{y_2}{\frac{k_2}{\alpha_2}} = r \tag{58}$$

where r is a constant.

From equilibrium condition of equation (57), $(\dot{y}_1 = \dot{y}_2 = 0)$ and conditions (58), following quadratic equation results.

$$k_1 k_2 K r^2 + \alpha_1 \alpha_2 r - K{\cdot}M\alpha_1\alpha_2 = 0 \tag{59}$$

The roots of equation (59) are

$$r_1, r_2 = \frac{-\alpha_1\alpha_2 \pm (\alpha_1{}^2\alpha_2{}^2 + 4k_1k_2K^2M\alpha_1\alpha_2)^{\frac{1}{2}}}{2k_1k_2K} \tag{60}$$

Equation (60) indicates that there are two equilibrium points r_1 and r_2 for the system of equation (57). Since Liapunov's theory is based on the trivial solution (input free equations), we define new variables (x_1, x_2) to shift the origin to the equilibrium point r_1 so that

$$x_1 = y_1 - \frac{r_1k_1}{\alpha_1}$$
$$x_2 = y_2 - \frac{r_1k_2}{\alpha_2} \qquad (61)$$

or

$$y_1 = x_1 + \frac{r_1k_1}{\alpha_1}$$
$$y_2 = x_2 + \frac{r_1k_2}{\alpha_2} \qquad (62)$$

Writing equation (57) in terms of new variables (x_1,x_2)

$$\dot{x}_1 = -(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2})x_1 - \frac{r_1k_1^2K}{\alpha_1}x_2 - k_1Kx_1x_2$$
$$\dot{x}_2 = -\frac{r_1k_2^2K}{\alpha_2}x_1 - (\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1})x_2 - k_2Kx_1x_2 \qquad (63)$$

It may be mentioned that equation (63) is a general case of the Volterra equations for the survival of two species, which conflict with each other[17]. Thus our study is relevant to the Volterra's problem.

The singular points of equation (63) are the origin O and $P(-\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_2K\alpha_1}, -\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_1K\alpha_2})$. First we study the origin. The Jacobian matrix for equation (63) evaluated at the origin is

$$J = \begin{bmatrix} -(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2}) & -\frac{r_1k_1^2K}{\alpha_1} \\ -\frac{r_1k_2^2K}{\alpha_2} & -(\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1}) \end{bmatrix} \text{ evaluated at } 0 \tag{64}$$

The stability of the origin can thus be determined from the roots of

$$|\lambda I - J| = 0 \tag{65}$$

which gives

$$\lambda^2 + (\alpha_1 + \alpha_2 + r_1k_1k_2K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}))\lambda + \alpha_1\alpha_2 + 2r_1k_1k_2K = 0 \tag{66}$$

In equation (66), since all the parameters are positive

$$\alpha_1 + \alpha_2 + r_1k_1k_2K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) > 0$$

$$\alpha_1\alpha_2 + 2r_1k_1k_2K > 0 \tag{67}$$

The above inequalities are necessary and sufficient for the negativity of the real parts of the roots of equation (65) (Routh Hurwitz Method[14]). According to Theorem 4.4.1, the system of the first approximation of equation (63) is asymptotically stable at the origin. Since all the eigen-values have negative real parts, the reduced equation has significant behavior and hence according to Theorem 4.4.3, the behavior of the complete equation (63) is the same at the origin as that of reduced equation (first approximation of equation (63)). This also indicates a stable region around the origin for which a quadratic Liapunov function can be constructed.

Next we consider the singular point P. The Jacobian matrix for equation (63) at P is

$$J = \begin{bmatrix} \dfrac{r_1 k_1 k_2 K}{\alpha_2} & k_1 K(\dfrac{\alpha_2}{k_2 K} + \dfrac{r_1 k_1}{\alpha_1}) \\ k_2 K(\dfrac{\alpha_1}{k_1 K} + \dfrac{r_1 k_2}{\alpha_2}) & \dfrac{r_1 k_1 k_2 K}{\alpha_1} \end{bmatrix}_{\substack{\text{evaluated} \\ \text{at } P}} \tag{68}$$

for which the resulting characteristic equation is

$$\lambda^2 - r_1 k_1 k_2 K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})\ \lambda - (\alpha_1 \alpha_2 + 2 r_1 k_1 k_2 K) = 0 \tag{69}$$

In equation (69)

$$\begin{aligned} -r_1 k_1 k_2 K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) &< 0 \\ -(\alpha_1 \alpha_2 K + 2 r_1 k_1 k_2 K) &< 0 \end{aligned} \tag{70}$$

The eigen-values of equation (69) are

$$\lambda_1, \lambda_2 = \frac{1}{2}\ [r_1 k_1 k_2 K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) \pm \{(r_1 k_1 k_2 K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}))^2 + 4(\alpha_1 \alpha_2 K + 2 r_1 k_1 k_2 K)\}^{\frac{1}{2}}] \tag{71}$$

Since all the parameters of the system of equation (63) are positive

$$\{(r_1 k_1 k_2 K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}))^2 + 4(\alpha_1 \alpha_2 K + 2 r_1 k_1 k_2 K)\}^{\frac{1}{2}} > 0 \tag{72}$$

Also,

$$\{(r_1k_1k_2K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}))^2 + 4(\alpha_1\alpha_2K + 2r_1k_1k_2K)\}^{\frac{1}{2}} > r_1k_1k_2K(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) \tag{73}$$

From inequalities (72) and (73), λ_1, λ_2 are real with opposite sign, so that P is a saddle point.

Since P is an unstable singular point, it is interesting to study its dependence on the system parameters. We consider here the following two cases:

i) $\dfrac{k_1}{\alpha_1} = 1, \quad \dfrac{k_2}{\alpha_2} = 1,$

For this case the unstable singular point P becomes

$$(x_1, x_2)_p = (-\frac{(1 + 4K^2M)^{\frac{1}{2}}}{K}, -\frac{(1 + 4K^2M)^{\frac{1}{2}}}{K}) \tag{74}$$

From equation (74) we find that, since K and M are positive, P never reaches the origin; and since P never reaches the origin, there always exists a certain stable region around the origin.

In Fig. 26 and Fig. 27, we show the trajectories of the solutions for this system in the space of initial conditions for $\alpha_1 = \alpha_2 = M = K = 1$ by analog simulations and isocline methods, respectively. From these figures, we can make a conservative estimate of the region of asymptotic stability, since any initial condition $(x_{10}, x_{20}) < OP$ reaches the origin asymptotically.

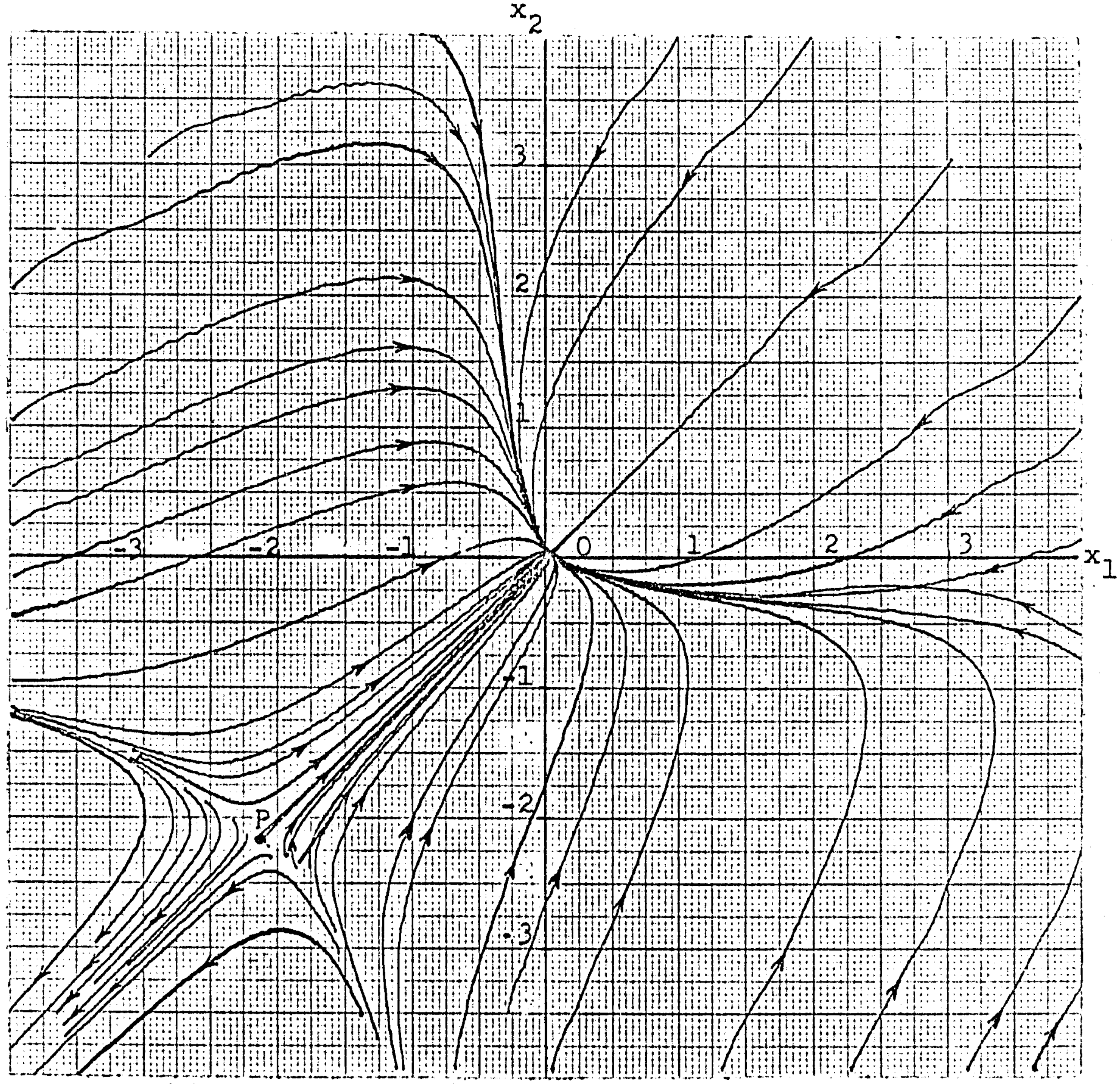


FIG. 26. STATE PLANE BY ANALOG COMPUTATIONS OF FIRST ORDER NONLINEAR MULTIPLICATIVE CONTROL SYSTEM FOR $\frac{k_1}{\alpha_1} = 1$, $\frac{k_2}{\alpha_2} = 1$, $M = 1$, $K = 1$, $\alpha_1 = \alpha_2 = 1$.

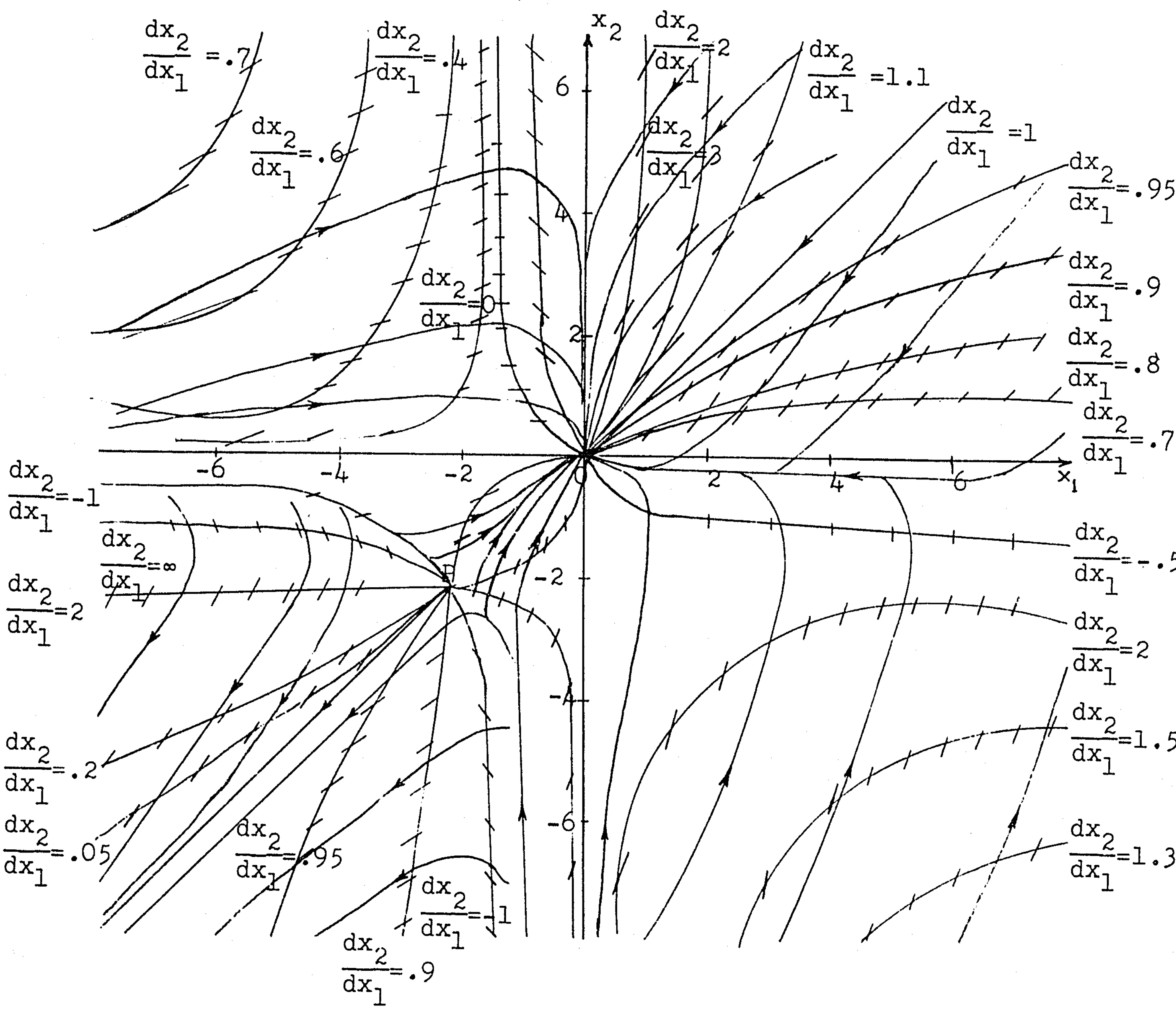


FIG. 27. STATE PLANE BY ISOCLINE METHOD FOR THE 1st ORDER NONLINEAR MULTIPLICATIVE CONTROL SYSTEM FOR $\frac{k_1}{\alpha_1} = 1$, $\frac{k_2}{\alpha_2} = 1$, $M = 1$, $K = 1$, $\alpha_1 = \alpha_2 = 1$.

ii) $\dfrac{k_1}{\alpha_1} = 1, \; \dfrac{k_2}{\alpha_2} = \dfrac{K-1}{KM}$,

This condition for the parameters was derived from the study of Chapter III. This is important, since from this study we can draw certain conclusions about the extent of the region of asymptotic stability of the trivial solution of the systems proposed in Chapter III. The singular point P for the above conditions for the parameters is independent of α_1 and α_2 and is

$$(x_1,x_2)_p = \left(-\frac{M(1+4(K-1)K)^{\frac{1}{2}}}{K-1}, \; -\frac{(1+4(K-1)K)^{\frac{1}{2}}}{K-1}\right) \quad (75)$$

as $K \to \infty$

$$(x_1,x_2)_p \simeq (-2M, -2) \quad (76)$$

Fig. 28 shows the dependence of x_{1p} and x_{2p} on K and M. For the above system a conservative estimate of the region of asymptotic stability is an ellipse. From equation (75), we note that x_{2p} is independent of the input. In the systems proposed in Chapter III, we intentionally introduced $H_2(s)$ and consequently x_2 and therefore, we have complete control over the initial conditions of x_2. This leaves us with the dependence of the stability region on x_1, which is a function of the magnitude of the input, and in the limiting case $(K \to \infty)$ is equal to $-2M$.

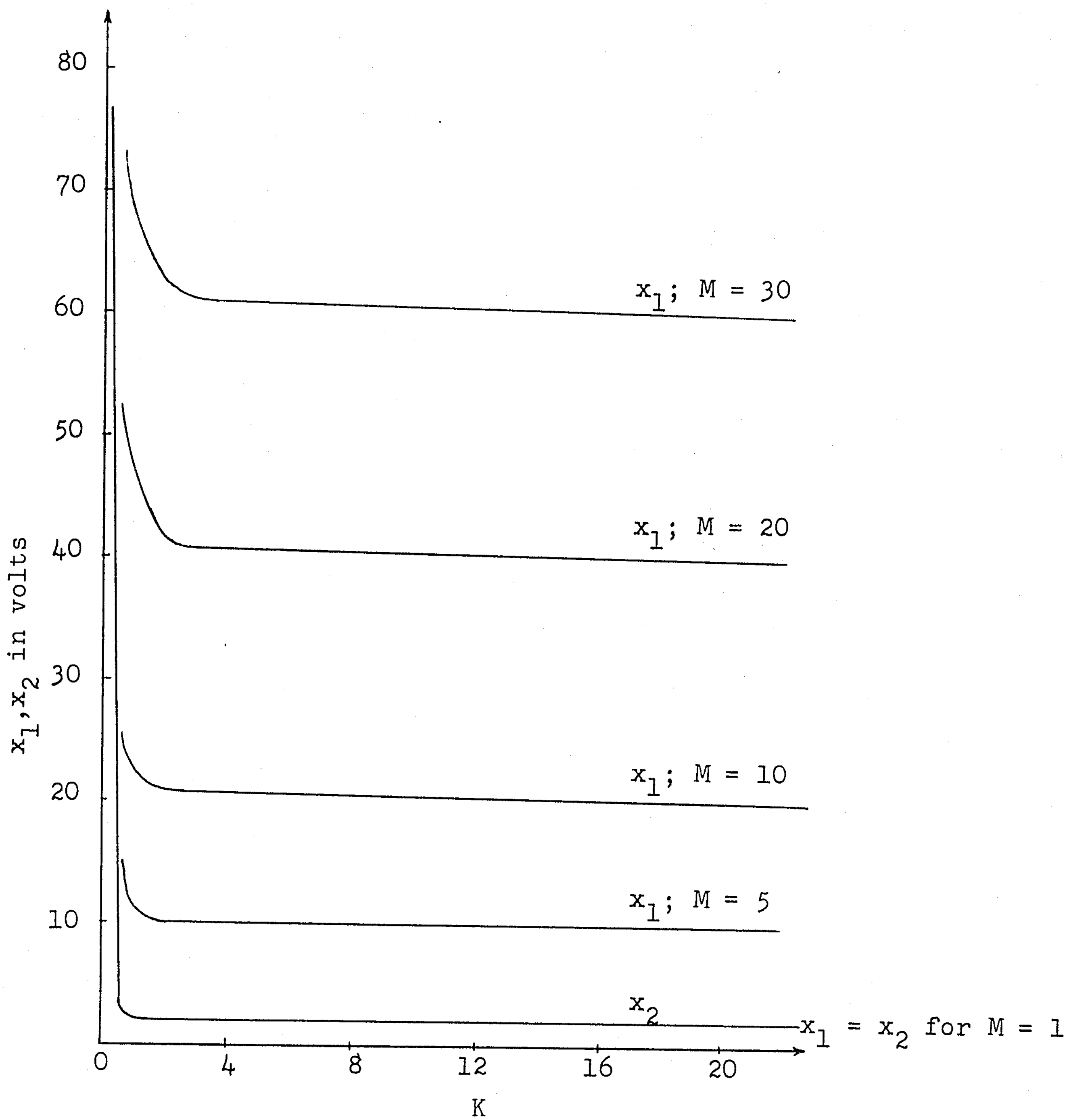


FIG. 28. LOCATION OF UNSTABLE SINGULAR POINT P FOR VARIOUS VALUES OF INPUT M AND CONTROLLER GAIN K.

x_1, x_2 to be taken negative

$$\frac{k_1}{\alpha_1} = 1, \quad \frac{k_2}{\alpha_2} = \frac{K-1}{KM}, \quad \alpha_1 = \alpha_2 = 1$$

4.6.2 An Estimate for the Bounds of Initial Value for Equation (63)

Using Zubov's method for obtaining the Liapunov function, the following partial differential equation results.

$$\frac{\partial v(x_1,x_2)}{\partial x_1}\left[\left(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2}\right)x_1 + \frac{r_1k_1^2K}{\alpha_1}x_2 + k_1K\ x_1x_2\right]$$

$$+ \frac{\partial v(x_1,x_2)}{\partial x_2}\left[\frac{r_1k_2^2K}{\alpha_2}\ x_1 + \left(\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1}\right)\ x_2 + k_2K\ x_1x_2\right]$$

$$= \phi(x_1,x_2)[1 - v\ (x_1,x_2)] \qquad (77)$$

where

$v(x_1,x_2)$ is a Liapunov function

$\phi(x_1,x_2)$ is some positive definite function.

We find that equation (77) is not integrable in the closed form.

Equation (77) was also considered for possible numerical solution with $v(x_1,x_2)$ and $\phi(x_1,x_2)$ of the following form.

$$v(x_1,x_2) = \sum_{j=2}^{JM}\sum_{k=1}^{j+1} d_{jk}\ x_1^{(j-k+1)}\cdot x_2^{(k-1)} \qquad (78)$$

and

$$\phi(x_1,x_2) = \gamma x_1^2 + \xi x_2^2 \qquad (79)$$

where

d_{jk} are the coefficients of the terms in the power series of $v(x_1,x_2)$.

JM is the highest power in the series

γ, ξ are two constants.

It was found that in order to satisfy equation (77), a high order power series for the approximation of the Liapunov function is required. Since up to JM = 3, the convergence was very slow, this approach for the determination of the stability region was abandoned. The computation time and labor do not justify it.

However, as we outlined in Section 4.5, an estimate of the stability region can be obtained from the linear part of the system differential equation. This we show for the following two cases.

i) Let the conditions for system parameters be as follows:

$$\alpha_1 = \alpha_2 = \frac{k_1}{\alpha_1} = \frac{k_2}{\alpha_2} = M = K = 1 \tag{80}$$

for conditions (80), equation (63) is

$$\begin{aligned} \dot{x}_1 &= -1.6x_1 - 0.6x_2 - x_1x_2 \\ \dot{x}_2 &= -0.6x_1 - 1.6x_2 - x_1x_2 \end{aligned} \tag{81}$$

The equations for the first approximation of equation (81) are

$$\begin{aligned} \dot{x}_1 &= - 1.6\, x_1 - 0.6x_2 \\ \dot{x}_2 &= - 0.6x_1 - 1.6x_2 \end{aligned} \tag{82}$$

The equilibrium of equation (82) is asymptotically stable, therefore according to Theorem 4.4.4, there exists a negative definite function $\dot{v}$, such that

$$\dot{v} = -x_1^2 - x_2^2 \tag{83}$$

for which the positive-definite quadratic form v is given by

$$v = 0.362\, x_1^2 - 0.272\, x_1 x_2 + 0.362\, x_2^2 \tag{84}$$

Then $\dot{v}$ for the complete equation (81) is

$$\dot{v}_{(81)} = -x_1^2 (0.954 + 0.453\, x_2) - x_2^2 (0.954 + 0.453\, x_1) \tag{85}$$

Equation (85) is then plotted for $\dot{v}_{(81)} = 0$, (Fig. 29), which gives the boundary of the stability region. The smallest value of v is at $(-2.2, -2.2)$,

$$v = 0.362\, x_1^2 - 0.272\, x_1 x_2 + 0.362\, x_2^2 = 2.27 \tag{86}$$

Then

$$v < 2.27 \tag{87}$$

is the subdomain of the domain of attraction and is plotted in Fig. 29.

ii) Let the conditions for system parameters be as follows:

$$\alpha_1 = \alpha_2 = \frac{k_1}{\alpha_1} = 1,\ K = 10,\ M = 10,\ \frac{k_2}{\alpha_2} = \frac{K-1}{KM} = 0.09 \tag{88}$$

For the above conditions for system parameters, equation (63) is as follows:

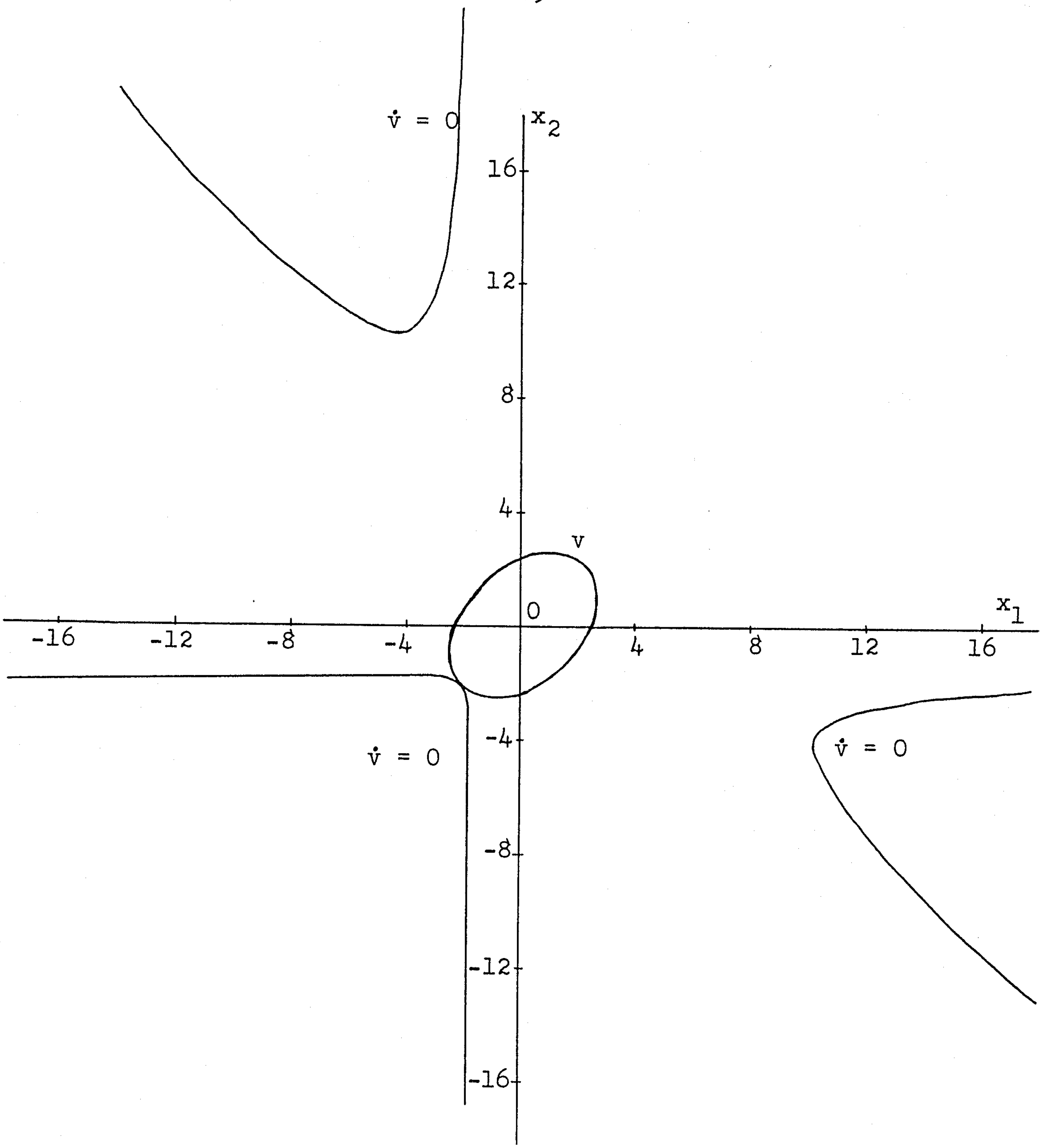


FIG. 29. AN ESTIMATE OF THE DOMAIN OF ATTRACTION BY CONSTRUCTING A LIAPUNOV FUNCTION v FOR THE FIRST APPROXIMATION OF EQUATION (81) AND $\dot{v}$ FOR EQUATION (81) FOR THE FIRST ORDER NON-LINEAR MULTIPLICATIVE CONTROL SYSTEM WITH $\frac{k_1}{\alpha_1} = \frac{k_2}{\alpha_2} = M = 1$ $K = \alpha_1 = \alpha_2 = 1$.

$$\dot{x}_1 = -10.5\,x_1 - 95\,x_2 - 10\,x_1x_2$$
$$\dot{x}_2 = -0.95\,x_1 - 10.5\,x_2 - 0.9\,x_1x_2 \qquad (89)$$

The first approximation for equation (89) is

$$\dot{x}_1 = -10.5\,x_1 - 95\,x_2$$
$$\dot{x}_1 = -0.95\,x_1 - 10.5\,x_2 \qquad (90)$$

The equilibrium of equation (90) is asymptotically stable, therefore according to Theorem 4.4.4, there exists a negative definite function $\dot{v}$ such that

$$\dot{v} = -x_1^2 - x_2^2 \qquad (91)$$

for which the positive-definite quadratic form v is given by

$$v = 0.165\,x_1^2 - 2.66\,x_1x_2 + 12.1\,x_2^2 \qquad (92)$$

The total time derivative $\dot{v}$ for the complete differential equation, equation (89) is

$$\dot{v}_{(89)} = -x_1^2(0.98 + 0.66\,x_2) - x_2^2(3 - 2.4\,x_1) + 1.4\,x_1x_2 \qquad (93)$$

The plot for $\dot{v}_{(89)} = 0$ is given in Fig. 30. The smallest value of v is at (1.5, 1.5) and is

$$v = 0.165\,x_1^2 - 2.66\,x_1x_2 + 12.1\,x_2^2 = 21.6 \qquad (94)$$

Then

$$v < 21.6 \qquad (95)$$

is the subdomain of the domain of attraction and is plotted in Fig. 30.

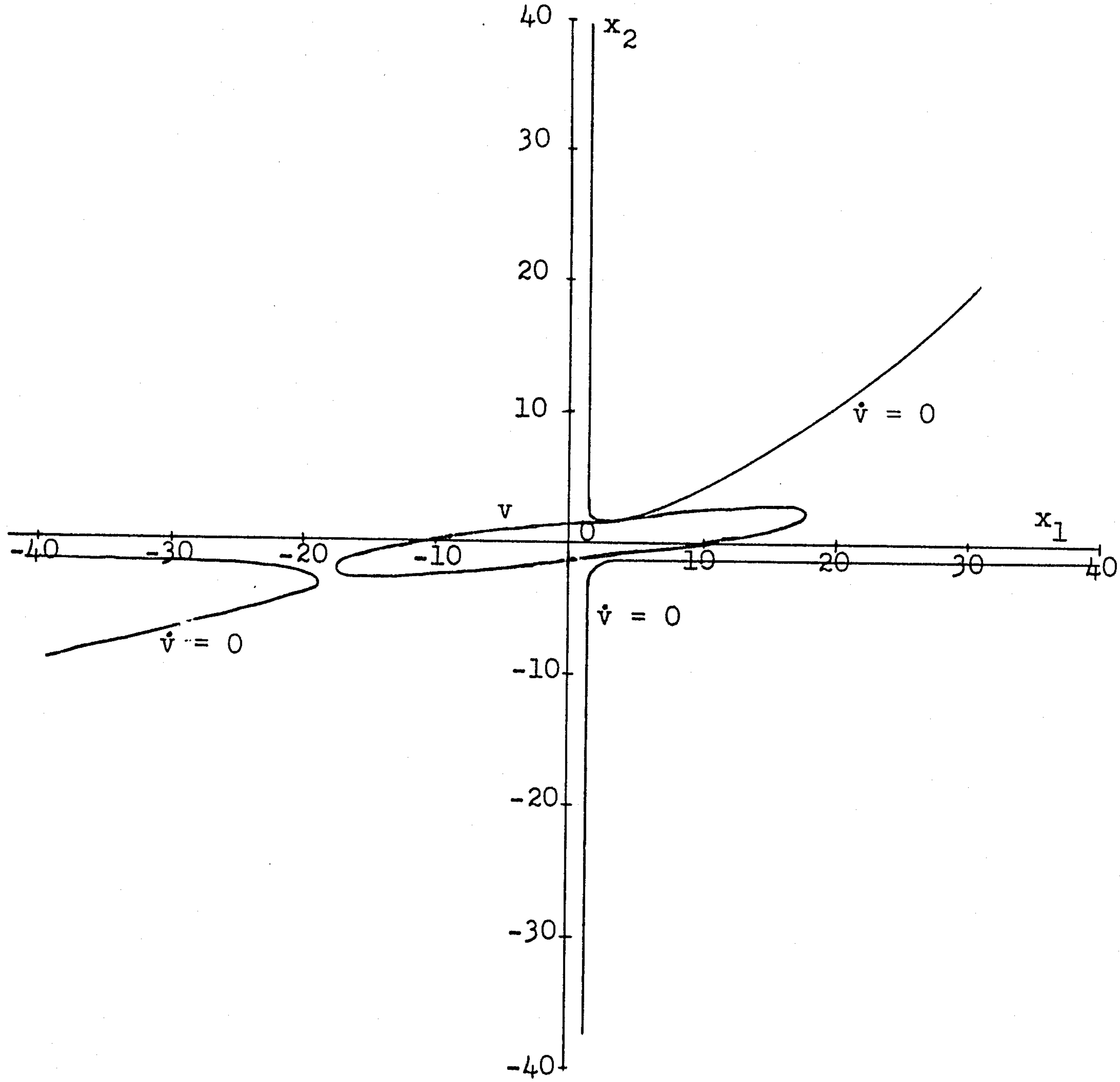


FIG. 30. AN ESTIMATE OF THE DOMAIN OF ATTRACTION BY CONSTRUCTING THE LIAPUNOV FUNCTION FOR THE FIRST APPROXIMATION OF EQUATION (89) AND $\dot{v}$ FOR EQUATION (89) FOR THE FIRST ORDER NON-LINEAR MULTIPLICATIVE CONTROL SYSTEM WITH $\alpha_1 = \alpha_2 = \frac{k_1}{\alpha_1} = 1$, $\frac{k_2}{\alpha_2} = 0.09$, K =10, M = 10.

The study of the above two cases indicates that for these systems, there always exists a certain region of asymptotic stability in the neighborhood of the origin, and a Liapunov function can be constructed. This method can also be used for higher order systems, although calculation becomes more involved.

4.6.3 $H_1(s)$ is a Second Order Linear System

In Fig. 25, let $H_1(s) = \dfrac{k_1}{s^2+\beta s+\alpha_1}$, $H_2(s) = \dfrac{k_2}{s+\alpha_2}$, then the system differential equations are

$$\ddot{y}_1 + \beta\dot{y}_1 + \alpha_1 y_1 = k_1 K(M-y_1y_2)$$

$$\dot{y}_2 + \alpha_2 y_2 = k_2 K(M-y_1y_2) \tag{96}$$

Writing equation (96) as a set of first order differential equations, let

$$y_1 = z_1, \quad \dot{y}_1 = z_2, \quad y_2 = z_3 \tag{97}$$

so that

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -\alpha_1 z_1 - \beta z_2 + k_1 K(M-z_1z_3)$$

$$\dot{z}_3 = -\alpha_2 z_3 + k_2 K(M-z_1z_3) \tag{98}$$

The system of equation (98) attains an equilibrium position when

$$\dot{z}_1 = \dot{z}_2 = \dot{z}_3 = 0 \tag{99}$$

From equation (98) and equation (99) the following set of equations results.

$$\alpha_1 z_1 = k_1 K(M-z_1 z_3)$$

$$\alpha_2 z_3 = k_2 K(M-z_1 z_3) \tag{100}$$

The following relations result from equation (100)

$$\frac{z_1}{\frac{k_1}{\alpha_1}} = \frac{z_3}{\frac{k_2}{\alpha_2}} = r \tag{101}$$

where r is a constant.

From equation (100) and relation (101), we get the following quadratic equation.

$$k_1 k_2 K r^2 + \alpha_1 \alpha_2 r - KM\alpha_1 \alpha_2 = 0 \tag{102}$$

The roots of equation (102) are

$$r_1, r_2 = \frac{-\alpha_1 \alpha_2 \pm (\alpha_1^2 \alpha_2^2 + 4k_1 k_2 K^2 M\alpha_1 \alpha_2)^{\frac{1}{2}}}{2k_1 k_2 K} \tag{103}$$

where r_1, r_2 are two equilibrium positions for the system of equation (98). We define another set of variables which shifts the origin to the equilibrium position r_1, so that

$$x_1 = z_1 - \frac{r_1 k_1}{\alpha_1}$$

$$x_2 = z_2 \tag{104}$$

$$x_3 = z_3 - \frac{r_1 k_2}{\alpha_2}$$

Writing equation (98) in terms of the new variables

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\left(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2}\right)x_1 - \beta x_2 - \frac{r_1k_1^2K}{\alpha_1}x_3 - k_1Kx_1x_3$$

$$\dot{x}_3 = -\frac{r_1k_2^2K}{\alpha_2}x_1 - \left(\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1}\right)x_3 - k_2Kx_1x_3 \qquad (105)$$

The singular points of equation (105) are the origin and

$$P\left(-\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_2K\alpha_1}, 0, -\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_1K\alpha_2}\right)$$

First we study the origin. The Jacobian matrix evaluated at the origin for equation (105) is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -\left(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2}\right) & -\beta & -\frac{r_1k_1^2K}{\alpha_1} \\ -\frac{r_1k_2^2K}{\alpha_2} & 0 & -\left(\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1}\right) \end{bmatrix}_{\text{evaluated at } 0} \qquad (106)$$

The stability of the origin, then, can be determined from the roots of the characteristic equation

$$|\lambda I - J| = 0 \qquad (107)$$

which results in the following equation

$$\lambda^3 + (\beta + \alpha_2 + \frac{r_1k_1k_2K}{\alpha_1})\lambda^2 + (\beta\alpha_2 + \frac{\beta r_1k_1k_2K}{\alpha_1}$$

$$+ \alpha_1 + \frac{r_1k_1k_2K}{\alpha_2})\lambda + \alpha_1\alpha_2 + 2r_1k_1k_2K = 0 \qquad (108)$$

Since all the system parameters are positive in equation (108)

$$\beta + \alpha_2 + \frac{r_1k_1k_2K}{\alpha_1} > 0$$

$$\beta\alpha_2 + \frac{\beta r_1k_1k_2K}{\alpha_1} + \alpha_1 + \frac{r_1k_1k_2K}{\alpha_2} > 0$$

$$\alpha_1\alpha_2 + 2r_1k_1k_2K > 0 \qquad (109)$$

$$(\beta + \alpha_2 + \frac{r_1k_1k_2K}{\alpha_1})(\beta\alpha_2 + \frac{\beta r_1k_1k_2K}{\alpha_1} + \alpha_1 + \frac{r_1k_1k_2K}{\alpha_2})$$

$$- (\alpha_1\alpha_2 + 2r_1k_1k_2K) > 0$$

Inequalities (109) are necessary and sufficient conditions for the presence of negative real parts in the roots of equation (108). Therefore, according to Theorem 4.4.1, the trivial solution for the first approximation of equation (104) is asymptotically stable and since the reduced equation has "significant behavior", according to Theorem 4.4.3, the complete equation (104) has the same stability behavior.

Next, consider the singular point P. The Jacobian matrix for equation (104) evaluated at P is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{r_1 k_1 k_2 K}{\alpha_2} & -\beta & k_1 K\left(\dfrac{\alpha_2}{k_2 K} + \dfrac{r_1 k_1}{\alpha_1}\right) \\ k_2 K\left(\dfrac{\alpha_1}{k_1 K} + \dfrac{r_1 k_2}{\alpha_2}\right) & 0 & \dfrac{r_1 k_1 k_2 K}{\alpha_1} \end{bmatrix} \quad \text{evaluated at } P \tag{110}$$

for which the resulting characteristic equation is

$$\lambda^3 + \left(\beta - \frac{r_1 k_1 k_2 K}{\alpha_1}\right) \lambda^2 - r_1 k_1 k_2 K \left(\frac{\beta}{\alpha_1} + \frac{1}{\alpha_2}\right)\lambda - (\alpha_1 \alpha_2 + 2 r_1 k_1 k_2 K) = 0 \tag{111}$$

We note that for a large λ the left hand side of equation (111) is positive, and, at $\lambda = 0$ the left hand side of equation (111) has a negative value. This means that it crosses the positive real axis at least once, and implies that equation (111) has at least one root with a positive real part. Therefore, P is an unstable singular point.

Nevertheless, as we have shown earlier, there always exists a small region of asymptotic stability around the origin. This can be studied further and a quantitative estimate of the region of asymptotic stability obtained according to the method outlined in Section 4.6.2.

4.6.4 $H_1(s)$ is a Third Order Linear System

In Fig. 25, let $H_1(s) = \dfrac{k_1}{s^3+\gamma s^2+\beta s+\alpha_1}$, $H_2(s) = \dfrac{k_2}{s+\alpha_2}$, then the system differential equations are

$$\begin{aligned} \dddot{y}_1 + \gamma\ddot{y}_1 + \beta\dot{y}_1 + \alpha_1 y_1 &= k_1 K(M-y_1y_2) \\ \dot{y}_2 + \alpha_2 y_2 &= k_2 K(M-y_1y_2) \end{aligned} \tag{112}$$

The transformed equations for the origin shift, with the new set of variables (x_1, x_2, x_3, x_4), are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -(\alpha_1 + \frac{r_1k_1k_2K}{\alpha_2})x_1 - \beta x_2 - \gamma x_3 - \frac{r_1k_1^2K}{\alpha_1}x_4 - k_1Kx_1x_4 \\ \dot{x}_4 &= -\frac{r_1k_2^2K}{\alpha_2}x_1 - (\alpha_2 + \frac{r_1k_1k_2K}{\alpha_1})x_4 - k_2Kx_1x_4 \end{aligned} \tag{113}$$

The singular points of equation (113) are the origin and

$$P\left(-\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_2K\alpha_1}, 0, 0, -\frac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_1K\alpha_2}\right)$$

First, we study the origin. The Jacobian matrix for equation (113) evaluated at the origin is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -(\alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2}) & -\beta & -\gamma & -\frac{r_1 k_1^2 K}{\alpha_1} \\ -\frac{r_1 k_2^2 K}{\alpha_2} & 0 & 0 & -(\alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1}) \end{bmatrix} \quad \text{evaluated at } 0 \qquad (114)$$

and the characteristic equation resulting from this Jacobian matrix is

$$\lambda^4 + (\gamma + \alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1})\lambda^3 + (\gamma\alpha_2 + \frac{\gamma r_1 k_1 k_2 K}{\alpha_1} + \beta)\lambda^2 + (\beta\alpha_2 + \frac{\beta r_1 k_1 k_2 K}{\alpha_1} + \alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2})\lambda + \alpha_1\alpha_2 + 2r_1 k_1 k_2 K = 0 \qquad (115)$$

In order that the trivial solution be asymptotically stable, the following necessary and sufficient conditions must be satisfied.

$$\gamma + \alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1} > 0$$

$$\gamma\alpha_2 + \frac{\gamma r_1 k_1 k_2 K}{\alpha_1} + \beta > 0 \qquad (116)$$

$$\beta\alpha_2 + \frac{\beta r_1 k_1 k_2 K}{\alpha_1} + \alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2} > 0$$

$$\alpha_1\alpha_2 + 2r_1 k_1 k_2 K > 0$$

and

$$\frac{r_1k_1k_2K}{\alpha_1}(\gamma^2 + 2\gamma\alpha_2 + \frac{\gamma r_1k_1k_2K}{\alpha_1}) + \gamma\ (\alpha_2^2 + \beta + \gamma\alpha_2\ )\ \rangle\ \alpha_1 + \frac{r_1k_1k_2K}{\alpha_2}$$

$$\frac{r_1^3k_1^3k_2^3K^3}{\alpha_1^3\alpha_2}\{\gamma\beta + \gamma\alpha_1 - 2\alpha_1\alpha_2\} +$$

$$\frac{r_1^2k_1^2k_2^2K^2}{\alpha_1^2\alpha_2^2}\{(3\gamma\beta - 5\alpha_1)\alpha_2^3 + (\gamma\alpha_1 + \gamma^2\beta - 2\gamma\alpha)\alpha_2^2$$

$$+ (\gamma^2 - \beta)\alpha_1\alpha_2 - \alpha_1^2\} + \qquad (117)$$

$$\frac{r_1k_1k_2K}{\alpha_1\alpha_2}\{(2\gamma\beta - 3\alpha_1)\alpha_2^3 + (\alpha_1\beta\gamma + 2\gamma^2\beta - \alpha_1^2 - 3\alpha_1\gamma - \beta^2)\alpha_2^2$$

$$+ (\beta - 3)\alpha_1\alpha_2\beta + (\beta - 2)\alpha_1^2 + (\beta\alpha_2 + \alpha_1)\gamma\beta\} +$$

$$(\beta\gamma - \alpha_1)\alpha_2^3 + (\beta\gamma - \alpha_1)\gamma\alpha_2^2 - \beta\alpha_1\alpha_2 + (\alpha_2 + \alpha_1)\beta\gamma - \alpha_1^2\ \rangle\ 0$$

Since all the system parameters are positive, inequalities (116), which are necessary conditions for the presence of negative real parts in the roots of equation (115), are in general readily satisfied. Inequality (117) may not be generally true, but it is satisfied for all the cases discussed in Chapter III. Therefore, according to Theorem 4.4.1 and Theorem 4.4.3, these systems are stable at the origin.

Next, we consider the singular point P. The Jacobian matrix for equation (113) evaluated at P is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{r_1k_1k_2K}{\alpha_2} & -\gamma & -\beta & k_1K(\dfrac{\alpha_2}{k_2K} + \dfrac{r_1k_1}{\alpha_1}) \\ k_2K(\dfrac{\alpha_1}{k_1K} + \dfrac{r_1k_2}{\alpha_2}) & 0 & 0 & \dfrac{r_1k_1k_2K}{\alpha_1} \end{bmatrix}_{\text{evaluated at } P} \qquad (118)$$

and the characteristic equation resulting from this Jacobian matrix is

$$\lambda^4 + (\beta - \frac{r_1k_1k_2K}{\alpha_1})\ \lambda^3 + (\gamma - \frac{\beta r_1k_1k_2K}{\alpha_1})\ \lambda^2$$

$$- r_1k_1k_2K\ (\frac{\gamma}{\alpha_1} + \frac{1}{\alpha_2})\ \lambda - (\alpha_1\alpha_2 + 2r_1k_1k_2K) = 0 \qquad (119)$$

We note that the left hand side of equation (119) changes sign from positive to negative when λ is very large and when λ is zero. This means that it crosses the positive real λ axis at least once. Therefore, equation (119) has at least one root which has a positive real part and P is an unstable singular point.

Since the systems of Chapter III are asymptotically stable at the origin, further study can be carried out to determine the extent of asymptotic stability as outlined in Section 4.6.2.

4.6.5 Stability Study of the Systems of Section 3.4.1

The stability study of the system in Section 3.4.1 is quite similar to that carried out in Section 4.6.3 Let the nonlinear system be described by the following differential equation

$$\ddot{y}_1 + \beta\dot{y}_1 y_1 + \alpha_1 y_1 = k_1 x \qquad (120)$$

where

x is the input to the nonlinear system

y_1 is the output of the nonlinear system

k_1 is a constant

Referring to Fig. 22, let $H_2(s) = \frac{k_2}{s+\alpha_2}$, then the system differential equations are

$$\ddot{y}_1 + \beta\dot{y}_1 y_1 + \alpha_1 y_1 = k_1 K(M-y_1 y_2)$$

$$\dot{y}_2 + \alpha_2 y_2 = k_2 K(M-y_1 y_2) \qquad (121)$$

As in Section 4.6.3, the input-free differential equations of the above system are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -(\alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2})x_1 - \beta r_1 x_2 - \frac{r_1 k_1^2 K}{\alpha_1}x_3 - \beta x_1 x_2 - k_1 K x_1 x_3$$

$$\dot{x}_3 = -\frac{r_1 k_2^2 K}{\alpha_2}x_1 - (\alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1})x_3 - k_2 K x_1 x_3 \qquad (122)$$

The singular points for equation (122) are the origin and $P(-\dfrac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_2K\alpha_1}, 0, -\dfrac{\alpha_1\alpha_2 + 2r_1k_1k_2K}{k_1K\alpha_2})$

First we study the origin. The Jacobian matrix for equation (122) evaluated at the origin is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -(\alpha_1 + \dfrac{r_1k_1k_2K}{\alpha_2}) & -\beta r_1 & -\dfrac{r_1k_1^2K}{\alpha_1} \\ -\dfrac{r_1k_2^2K}{\alpha_2} & 0 & -(\alpha_2 + \dfrac{r_1k_1k_2K}{\alpha_1}) \end{bmatrix}_{\substack{\text{evaluated} \\ \text{at } 0}} \tag{123}$$

The stability of the origin, then can be determined from the roots of the characteristic equation

$$|\lambda I - J| = 0 \tag{124}$$

which results in the following equation

$$\lambda^3 + (\beta r_1 + \alpha_2 + \frac{r_1k_1k_2K}{\alpha_1})\lambda^2 + (\beta r_1\alpha_2 + \frac{\beta r_1^2k_1k_2K}{\alpha_1} + \alpha_1 + \frac{r_1k_1k_2K}{\alpha_2})\lambda + \alpha_1\alpha_2 + 2r_1k_1k_2K = 0 \tag{125}$$

Since in equation (125) all the system parameters are positive, the following necessary and sufficient conditions for the presence of negative real parts in its roots are readily satisfied.

$$\beta r_1 + \alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1} > 0$$

$$\beta r_1 \alpha_2 + \frac{\beta r_1^2 k_1 k_2 K}{\alpha_1} + \alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2} > 0$$

$$\alpha_1 \alpha_2 + 2 r_1 k_1 k_2 K > 0 \tag{126}$$

$$\left(\beta r_1 + \alpha_2 + \frac{r_1 k_1 k_2 K}{\alpha_1}\right)\left(\beta r_1 \alpha_2 + \frac{\beta r_1^2 k_1 k_2 K}{\alpha_1} + \alpha_1 + \frac{r_1 k_1 k_2 K}{\alpha_2}\right) - (\alpha_1 \alpha_2 + 2 r_1 k_1 k_2 K) > 0$$

Therefore, due to Theorem 4.4.1, the trivial solution of the system of equation (122) is asymptotically stable. Since the reduced equation has "significant behavior", according to Theorem 4.4.3, the complete equation of equation (122) has the same behavior as that of reduced equation.

Next, we study the singular point P. The Jacobian matrix for equation (122) evaluated at P is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{r_1 k_1 k_2 K}{\alpha_2} & -\beta\left(r_1 - \dfrac{\alpha_1 \alpha_2 + 2 r_1 k_1 k_2 K}{k_2 K \alpha_1}\right) & k_1 K\left(\dfrac{\alpha_2}{k_2 K} + \dfrac{r_1 k_1}{\alpha_1}\right) \\ k_2 K\left(\dfrac{\alpha_1}{k_1 K} + \dfrac{r_1 k_2}{\alpha_2}\right) & 0 & \dfrac{r_1 k_1 k_2 K}{\alpha_1} \end{bmatrix} \tag{127}$$

evaluated at P

The characteristic equation for this Jacobian matrix is

$$\lambda^3 + (\beta r_1 - \frac{\beta(\alpha_1\alpha_2 + 2r_1k_1k_2K)}{k_2K\alpha_1} + \frac{r_1k_1k_2K}{\alpha_1})\lambda^2$$

$$+ (\frac{\beta r_1k_1}{\alpha_1}(\alpha_1\alpha_2 + 2r_1k_1k_2K) - \frac{\beta r_1{}^2k_1k_2K}{\alpha_1} - \frac{r_1k_1k_2K}{\alpha_2})\lambda$$

$$-(\alpha_1\alpha_2 + 2r_1k_1k_2K) = 0 \tag{128}$$

When λ is large the left hand side of equation (128) is positive and when λ is zero it is negative; it therefore crosses the positive real λ axis at least once. This indicates that equation (128) has at least one root with a positive real part and that P is an unstable singular point.

Again, the domain of attraction can be studied for this system in a way similar to Section 4.6.2.

4.6.6 Stability Study of a System of Section 3.4.2

The stability study of the systems of Section 3.4.2 with a nonlinear position term is similar to that of Section 4.6.5. Here, we discuss a particular case of Section 3.4.2 with the following differential equation which describes the nonlinear system.

$$\ddot{y}_1 + \dot{y}_1 + y_1{}^2 = x \tag{129}$$

where

x is input to the nonlinear system

y_1 is the output of the nonlinear system

The optimum response of this system for $H_2(s) = \frac{1.8}{s+2}$, $K = 10$ and $M = 1$, is shown in Fig. 24.b. The differential equations for the system of Fig. 22 with the above parameters are

$$\ddot{y}_1 + \dot{y}_1 + y_1^2 = 10\,(1-y_1y_2)$$

$$\dot{y}_2 + 2y_2 = 18\,(1-y_1y_2) \tag{130}$$

To write equation (130) as a set of first order differential equations, we define

$$y_1 = z_1, \quad \dot{y}_1 = z_2, \quad y_2 = z_3 \tag{131}$$

so that

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -z_2 - z_1^2 + 10\,(1-z_1z_3)$$

$$\dot{z}_3 = -2z_3 + 18\,(1-z_1z_3) \tag{132}$$

In order to shift the origin to the equilibrium position, we define a new set of variables (x_1, x_2, x_3). Writing equation (132) in terms of new variables we get

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -11x_1 - x_2 - 10x_3 - x_1^2 - 10x_1x_3$$

$$\dot{x}_3 = -16.2x_1 - 20x_3 - 18x_1x_3 \tag{133}$$

The only singular point of equation (133) is the origin, since the other two roots are complex. The Jacobian matrix for equation (133) evaluated at the origin is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -11 & -1 & -10 \\ -16.2 & 0 & -20 \end{bmatrix} \quad \text{evaluated at } 0 \tag{134}$$

The characteristic equation for the above Jacobian matrix is

$$\lambda^3 + 21\lambda^2 + 31\lambda + 58 = 0 \tag{135}$$

Equation (135) satisfies the necessary and sufficient conditions for the negativity of the real parts of its roots, and therefore the trivial solution for equation (133) is asymptotically stable. Since the reduced equation of equation (133) has "significant behavior", according to Theorem 4.4.3, the complete equation (133) has the same behavior. This assures stability in the small and that a Liapunov function can be constructed around the origin as in Section 4.6.2.

4.7 Stability Study of Second Order Nonlinear Multiplicative Control Systems of Section 3.3

The block diagram for the feedback system is given in Fig. 10.c. Let,

$$H_1(s) = \frac{k_1}{s^3+\lambda s^2 + \beta s+\alpha_1}, \quad H_2(s) = \frac{k_2}{s+\alpha_2}, \quad H_3(s) = \frac{k_3}{s+\alpha_3},$$

The differential equations for the above system are

$$\dddot{y}_1 + \gamma\ddot{y}_1 + \beta\dot{y}_1 + \alpha_1 y_1 = k_1 K(M-y_1y_2y_3)$$
$$\dot{y}_2 + \alpha_2 y_2 = k_2 K(M-y_1y_2y_3)$$
$$\dot{y}_3 + \alpha_3 y_3 = k_3 K(M-y_1y_2y_3) \tag{136}$$

To write equation (136) as a set of first order differential equations we define

$$y_1 = z_1,\ \dot{y}_1 = z_2,\ \ddot{y}_1 = z_3,\ y_2 = z_4,\ y_3 = z_5$$

so that

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = -\alpha_1 z_1 - \beta z_2 - \gamma z_3 + k_1 K(M-z_1z_4z_5)$$
$$\dot{z}_4 = -\alpha_2 z_4 + k_2 K(M-z_1z_4z_5)$$
$$\dot{z}_5 = -\alpha_3 z_5 + k_3 K(M-z_1z_4z_5) \tag{137}$$

When the system assumes an equilibrium state.

$$\dot{z}_1 = \dot{z}_2 = \dot{z}_3 = \dot{z}_4 = \dot{z}_5 = z_2 = z_3 = 0 \tag{138}$$

which results in the following relations:

$$\frac{\alpha_1 z_1}{k_1} = \frac{\alpha_2 z_4}{k_2} = \frac{\alpha_3 z_5}{k_3} = r \tag{139}$$

where r is a constant.

From equation (137), equation (138) and relation (139)

$$k_1k_2k_3Kr^3 + \alpha_1\alpha_2\alpha_3 r - \alpha_1\alpha_2\alpha_3 KM = 0 \tag{140}$$

Equation (140) has one real root and two complex roots.

Therefore, the system of equation (137) has only one equilibrium point, which is given as follows:

$$r_1 = \left[\frac{1}{2}\,\frac{M\alpha_1\alpha_2\alpha_3}{k_1k_2k_3K} + \left(\frac{1}{9}\,\frac{\alpha_1{}^3\alpha_2{}^3\alpha_3{}^3}{k_1{}^3k_2{}^3k_3{}^3K^3} + \frac{1}{4}\,\frac{M^2\alpha_1{}^2\alpha_2{}^2\alpha_3{}^2}{k_1{}^2k_2{}^2k_3{}^2K^2}\right)^{1/2}\right]^{1/3} +$$
$$\left[\frac{1}{2}\,\frac{M\alpha_1\alpha_2\alpha_3}{k_1k_2k_3K} - \left(\frac{1}{9}\,\frac{\alpha_1{}^3\alpha_2{}^3\alpha_3{}^3}{k_1{}^3k_2{}^3k_3{}^3K^3} + \frac{1}{4}\,\frac{M^2\alpha_1{}^2\alpha_2{}^2\alpha_3{}^2}{k_1{}^2k_2{}^2k_3{}^2K^2}\right)^{1/2}\right]^{1/3} \tag{141}$$

Define a new set of variables (x_1,x_2,x_3,x_4,x_5) which shifts the origin to the equilibrium point, such that

$$\begin{aligned} x_1 &= z_1 - \frac{r_1k_1}{\alpha_1} \\ x_2 &= z_2 \\ x_3 &= z_3 \\ x_4 &= z_4 - \frac{r_1k_2}{\alpha_2} \\ x_5 &= z_5 - \frac{r_1k_3}{\alpha_3} \end{aligned} \tag{142}$$

Then equation (137) in terms of the new variables is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\left(\alpha_1 + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_2\alpha_3}\right)x_1 - \beta x_2 - \gamma x_3 - \frac{k_1{}^2k_3Kr_1{}^2}{\alpha_1\alpha_3}\,x_4 \\ &\quad - \frac{k_1{}^2k_2Kr_1{}^2}{\alpha_1\alpha_2}\,x_5 - \frac{k_1k_3Kr_1}{\alpha_3}\,x_1x_4 - \frac{k_1k_2Kr_1}{\alpha_2}\,x_1x_5 \\ &\quad - \frac{k_1{}^2Kr_1}{\alpha_1}\,x_4x_5 - k_1Kx_1x_4x_5 \end{aligned}$$

$$\dot{x}_4 = -\frac{k_2{}^2k_3Kr_1{}^2}{\alpha_2\alpha_3}x_1 - (\alpha_2 + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_1\alpha_3})x_4 - \frac{k_1k_2{}^2Kr_1{}^2}{\alpha_1\alpha_2}x_5$$
$$- \frac{k_2k_3Kr_1}{\alpha_3}x_1x_4 - \frac{k_2{}^2Kr_1}{\alpha_2}x_1x_5 - \frac{k_1k_2Kr_1}{\alpha_1}x_4x_5 - k_2Kx_1x_4x_5$$

$$\dot{x}_5 = -\frac{k_2k_3{}^2Kr_1{}^2}{\alpha_2\alpha_3}x_1 - \frac{k_1k_3{}^2Kr_1{}^2}{\alpha_1\alpha_3}x_4 - (\alpha_3 + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_1\alpha_2})x_5$$
$$- \frac{k_3{}^2Kr_1}{\alpha_3}x_1x_4 - \frac{k_2k_3Kr_1}{\alpha_2}x_1x_5 - \frac{k_1k_3Kr_1}{\alpha_1}x_4x_5 - k_3Kx_1x_4x_5 \tag{143}$$

For equation (143) the origin is the only equilibrium point, since other two roots are complex. The Jacobian Matrix for equation (143) evaluated at the origin is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -(\alpha_1+\frac{k_1k_2k_3Kr_1}{\alpha_2\alpha_3}) & -\beta & -\gamma & -\frac{k_1{}^2k_3Kr_1{}^2}{\alpha_1\alpha_3} & -\frac{k_1{}^2k_2Kr_1{}^2}{\alpha_1\alpha_2} \\ -\frac{k_2{}^2k_3Kr_1{}^2}{\alpha_2\alpha_3} & 0 & 0 & -(\alpha_2+\frac{k_1k_2k_3Kr_1{}^2}{\alpha_1\alpha_3}) & -\frac{k_1k_2{}^2Kr_1{}^2}{\alpha_1\alpha_2} \\ -\frac{k_2k_3{}^2Kr_1{}^2}{\alpha_2\alpha_3} & 0 & 0 & -\frac{k_1k_3{}^2Kr_1{}^2}{\alpha_1\alpha_3} & -(\alpha_3+\frac{k_1k_2k_3Kr_1{}^2}{\alpha_1\alpha_2}) \end{bmatrix} \tag{144}$$

evaluated at 0

and the characteristic equation for the Jacobian matrix of (144) is

$$\lambda^5 + (\alpha_2 + \alpha_3 + \gamma + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_1} (\frac{1}{\alpha_2} + \frac{1}{\alpha_3}))\ \lambda^4 +$$

$$(\beta + \alpha_2\alpha_3 + \gamma\ (\alpha_2 + \alpha_3) + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_1} (2 + \gamma\ (\frac{1}{\alpha_2} + \frac{1}{\alpha_3})))\ \lambda^3 +$$

$$(\alpha_1 + \beta(\alpha_2 + \alpha_3) + \alpha_2\alpha_3\gamma + \frac{k_1k_2k_3Kr_1{}^2}{\alpha_1} (2\gamma+\beta\ (\frac{1}{\alpha_2} + \frac{1}{\alpha_3})+\frac{\alpha_1}{\alpha_2\alpha_3}))\ \lambda^2$$

$$+(\alpha_2\alpha_3\beta + \alpha_1(\alpha_2 + \alpha_3) + 2k_1k_2k_3Kr_1{}^2(\frac{\beta}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}))\ \lambda +$$

$$\alpha_1\alpha_2\alpha_3 + 3k_1k_2k_3Kr_1{}^2 = 0 \qquad (145)$$

Since all the system parameters are positive, the necessary condition for the presence of negative real parts in the roots of equation (145) is satisfied. The sufficiency condition for the presence of negative real parts in the roots of equation (145) depends upon system parameters, since in the equations for the sufficiency condition some terms are negative. Therefore this sufficiency condition can be determined if the system parameters are known. However, in all cases presented in Chapter III this condition is satisfied and the origin is asymptotically stable. This assures the stability of the system in the small.

Here we consider the following example

In Fig. 10.c, let

$$H_1(s) = \frac{6}{s^3+6s^2+11s+6}$$

$$H_2(s) = H_3(s) = \frac{1.93}{s+2}$$

$$K = 14.3, \quad M = 1.0 \tag{146}$$

The optimum response of this system is given in Fig. 19.d.

From equation (141)

$$r_1 = 1.0 \tag{147}$$

The system equations after shifting origin to the equilibrium point are

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -85.8x_1 - 11x_2 - 6x_3 - 82x_4 - 82x_5 - 82x_1x_4 \\
&\quad -82x_1x_5 - 85.8x_4x_5 - 85.8x_1x_4x_5 \\
\dot{x}_4 &= -26x_1 - 28.7x_4 - 26.7x_5 - 26.7x_1x_4 - 26.7x_1x_5 \\
&\quad -27.8x_4x_5 - 27.8x_1x_4x_5 \\
\dot{x}_5 &= -26x_1 - 26.7x_4 - 28.7x_5 - 26.7x_1x_4 \\
&\quad -26.7x_1x_5 - 27.8x_4x_5 - 27.8x_1x_4x_5
\end{aligned} \tag{148}$$

From equation (148), we find that the origin is the only equilibrium point. The characteristic equation for the origin as given by equation (145) for the above parameters, is

$$\lambda^5 + 63\lambda^4 + 462\lambda^3 + 1374\lambda^2 + 1780\lambda + 980 = 0 \tag{149}$$

Equation (149) satisfies both the necessary and sufficient conditions for the presence of negative real parts in its roots. Therefore, the origin is asymptotically stable. Since the reduced equation of equation (148) has "significant behavior", according to Theorem 4.4.3, the complete equation (148) has the same behavior at the equilibrium point. This assures stability in the small for the above system.

4.8 Comments

1. The stability study of the above systems shows that according to Zubov's formulation for the determination of a Liapunov function, a closed form Liapunov function is not possible. However, an approximate solution for the partial differential equation in Zubov's formulation can be obtained numerically; this requires a large number of computations.

2. In the first order nonlinear multiplicative control systems, there exist two equilibrium points. One equilibrium point is asymptotically stable, the other is asymptotically unstable. In second order nonlinear multiplicative control systems, there exists only one equilibrium point, which

is asymptotically stable. Since these systems always have a stable equilibrium point, there always exists a certain region of asymptotic stability, for which a Liapunov function can be constructed.

3. This analysis can be used to study the Volterra equations for the survival of species which conflict with each other and also certain chemical systems.

CHAPTER V

CONCLUSIONS AND SUGGESTIONS FOR FUTURE STUDY

In Chapter II of this thesis, it is shown that an open-loop nonlinear multiplicative system, if excited by a step function input, can be represented exactly by a linear system multiplied by a "gain factor". The method presented here is straightforward and, for the class of systems considered in this thesis, the representation for the nth order nonlinear multiplicative systems can be written after two complex convolution operations.

In Chapter III, it is shown that the performance of linear systems up to third order, and some second order nonlinear differential systems, can be improved by introducing intentional multiplicative nonlinearities in the forward path of the feedback control system. In this method, the output of the system under control is not compared directly with the reference input. This makes the control system more sensitive to parameter variations of the system than direct control. The transient response is extremely insensitive to the parameter variations, since the controller gain is fairly high and presence of multipliers suppresses the overshoot. However, as discussed in Chapter III, the steady state output of the system changes with variation in system parameters, but can be corrected by changing the gains of the linear systems $H_2(s)$ and $H_3(s)$ as proposed in Chapter III. The improvement in the output response of the given system, the relation between the number of multipliers and the order of the given linear system, and the relation of the poles of the given linear system to the poles of $H_2(s)$ and $H_3(s)$, are consistent in every case considered. Since the analytic treatment of the problem is extremely difficult, the conclusions of Chapter III are based on extensive analog simulation. It would be desirable to establish a sound mathematical basis for the results presented in Chapter III. It would also be interesting to investigate the applicability of this scheme to higher order linear systems and multivariable linear systems.

In Chapter IV, the stability of the systems discussed in Chapter III is studied and shows that, according to Zubov's formulation, a closed form Liapunov function is not possible. However, these systems always have an equilibrium point which is asymptotically stable, and a Liapunov function of the quadratic form can be constructed to give a conservative estimate of the region of asymptotic stability around the equilibrium point. This stability analysis can be used for the study of ecological system (Volterra equations for the survival of species) and certain chemical systems. Further work should be done to determine more closely the boundry of the stability regions of these systems by constructing more general Liapunov functions than quadratic forms.

If in the control systems of Chapter III, the proportional controller is replaced by a proportional plus integral controller, then the classical theory of Liapunov, which is based on the equilibrium of a point, cannot be applied. Since in this case, for first order nonlinear multiplicative control systems, one is concerned with the study of the stability of hyperbolas defined by $M = x_1 x_2$, where M is the magnitude of the step function input and x_1, x_2 are state variables. This system has little practical value for a control engineer, since there will be an infinite number of equilibrium points. Nevertheless, it is an interesting problem and needs further investigation.

APPENDIX

SENSITIVITY ANALYSIS OF NONLINEAR MULTIPLICATIVE CONTROL SYSTEMS

1. First Order Nonlinear Multiplicative Control Systems

The equations for the equilibrium of first order nonlinear multiplicative control systems, introduced in Chapter III, are as follow:

$$\begin{aligned} \alpha_1 y_1 &= k_1 K(M-y_1 y_2) \\ \alpha_2 y_2 &= k_2 K(M-y_1 y_2) \end{aligned} \tag{A-1}$$

The symbols are as previously defined. From equation (A-1)

$$\frac{y_1}{\dfrac{k_1}{\alpha_1}} = \frac{y_2}{\dfrac{k_2}{\alpha_2}} \tag{A-2}$$

Also from Chapter III, $\dfrac{k_2}{\alpha_2} = \dfrac{K-1}{KM}$.

Define $\dfrac{k_1}{\alpha_1} = A$ (A-3)

We wish to determine the percentage change in the steady state output, y_1, of the given system for a percentage change in A.

From equation (A-1), equation (A-2) and equation (A-3)

$$y_{1,2} = \frac{M\{-1 \pm \sqrt{1 + 4AK(K-1)}\}}{2(K-1)} \tag{A-4}$$

In equation (A-4), the first root is of interest. Then

$$\frac{dy_1}{dA} = \frac{KM}{\sqrt{1 + 4AK(K-1)}} \tag{A-5}$$

If $\frac{k_1}{\alpha_1} = A = 1$, then

$$\frac{dy_1}{dA} = \frac{KM}{2K-1} \tag{A-6}$$

From equation (A-6), we find that percentage change in the steady state output y_1 is approximately half the percentage change in A.

2. Second Order Nonlinear Multiplicative Control Systems

The equations for the equilibrium of second order nonlinear multiplicative control systems, introduced in Chapter III, are as follow:

$$\begin{aligned} \alpha_1 y_1 &= k_1 K(M - y_1 y_2 y_3) \\ \alpha_2 y_2 &= k_2 K(M - y_1 y_2 y_3) \\ \alpha_3 y_3 &= k_3 K(M - y_1 y_2 y_3) \end{aligned} \tag{A-7}$$

From equation (A-7)

$$\frac{\frac{y_1}{k_1}}{\alpha_1} = \frac{\frac{y_2}{k_2}}{\alpha_2} = \frac{\frac{y_3}{k_3}}{\alpha_3} \tag{A-8}$$

Also from Chapter III,

$$\frac{k_2}{\alpha_2} = \frac{k_3}{\alpha_3} = \frac{1}{M} \left(\frac{K-1}{K}\right)^{\frac{1}{2}} \tag{A-9}$$

Define

$$\frac{k_1}{\alpha_1} = A \tag{A-10}$$

From equations (A-7),(A-8),(A-9) and (A-10),

$$y_1^{\,3} + \frac{AM^2}{K-1} y_1 - \frac{A^2M^3K}{K-1} = 0 \tag{A-11}$$

Equation (A-11) has one real and two complex roots. The real root is of interest and is given by

$$y_1 = M\left[\left(\frac{1}{2} \frac{A^2K}{K-1} + \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2}\right)^{1/2}\right)^{1/3} + \right.$$

$$\left. \left(\frac{1}{2} \frac{A^2K}{K-1} - \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2}\right)^{1/2}\right)^{1/3} \right] \tag{A-12}$$

We wish to determine the percentage change in the steady state output, y_1, of the given system for percentage change in A. Thus

$$\frac{dy_1}{dA} = M \Big[\frac{1}{3} \left(\frac{1}{2} \frac{A^2K}{K-1} + \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2} \right)^{1/2} \right)^{-2/3}$$

$$\left(\frac{AK}{K-1} + \frac{1}{2} \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2} \right)^{1/2} \left(\frac{1}{3} \frac{A^2}{(K-1)^2} + \frac{A^3K}{(K-1)^2} \right) \right) +$$

$$\frac{1}{3} \left(\frac{1}{2} \frac{A^2K}{K-1} - \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2} \right)^{1/2} \right)^{-2/3}$$

$$\left(\frac{AK}{K-1} - \frac{1}{2} \left(\frac{1}{9} \frac{A^3}{(K-1)^3} + \frac{1}{4} \frac{A^4K^2}{(K-1)^2} \right) \left(\frac{1}{3} \frac{A^2}{(K-1)^3} + \frac{A^3K^2}{(K-1)^2} \right) \right) \Big] \tag{A-13}$$

In equation (A-13) for $A = 1$ and K in the range of values as in Chapter III, the percentage change in the steady state output y_1, is approximately two-thirds of the percentage change in A.

APPENDIX - B

PROOF OF EQUATION (16)

From Eq. (12) and Eq. (15), we conjecture the following form of the equation for n multipliers :

$$F_{1,2,\ldots\ldots,n+1}(s) = \frac{1}{s} - \sum_{i=1}^{n+1} \frac{1}{s+a_i} + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ i \neq j}}^{n+1} \frac{1}{s+a_i+a_j} + \ldots\ldots$$

$$+ (-1)^n \frac{1}{n} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{\substack{\ell=1 \\ i\neq j\neq \ldots\ldots \neq \ell}}^{n+1} \frac{1}{s+a_i+a_j+\ldots a_\ell} +$$

$$(-1)^n \frac{1}{n+1} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \cdots \sum_{\substack{p=1 \\ i\neq j\neq \ldots\ldots \neq p}}^{n+1} \frac{1}{s+a_i+a_j+\ldots+a_p} \tag{B-1}$$

To prove that this form is also true for n + 1 multipliers let us convolve Eq. (B-1) by $F_{n+2}(s) = \frac{a_{n+2}}{s+a_{n+2}}$ which results in the following

$$F_{1,2,\ldots\ldots n+1}(s) * F_{n+2}(s) = \frac{1}{s} - \frac{1}{s+a_{n+2}} - \sum_{i=1}^{n+1} \frac{1}{s+a_i} +$$

$$+ \sum_{i=1}^{n+1} \frac{1}{s+a_i+a_{n+2}} + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ i\neq j}}^{n+1} \frac{1}{s+a_i+a_j}$$

$$- \frac{1}{2} \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ i\neq j}}^{n+1} \frac{1}{s+a_i+a_j+a_{n+2}} + \ldots\ldots$$

$$+(-1)^{n}\ \frac{1}{n}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\ldots\neq \ell}}^{n+1}\ \sum_{j=1}^{n+1}\ \cdots\ \sum_{\ell=1}^{n+1}\ \frac{1}{s+a_i+a_j+\ldots a_\ell}\ +$$

$$(-1)^{n}\ \frac{1}{n}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\ldots\neq \ell}}^{n+1}\ \sum_{j=1}^{n+1}\ \cdots\ \sum_{\ell=1}^{n+1}\ \frac{1}{s+a_i+a_j+\ldots+a_\ell+a_{n+2}}$$

$$+\ (-1)^{n+1}\ \frac{1}{n+1}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\ldots\neq p}}^{n+1}\ \sum_{j=1}^{n+1}\ \cdots\ \sum_{p=1}^{n+1}\ \frac{1}{s+a_i+a_j+\ldots+a_p}$$

$$+(-1)^{n+1}\ \frac{1}{n+1}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\ldots\neq p}}^{n+1}\ \sum_{j=1}^{n+1}\ \cdots\ \sum_{p=1}^{n+1}\ \frac{1}{s+a_i+a_j+\ldots+a_p+a_{n+2}} \qquad \text{(B-2)}$$

or

$$F_{1,2,\ldots\ldots n+2}(s) = \frac{1}{s} - \sum_{i=1}^{n+2}\ \frac{1}{s+a_i} + \frac{1}{2}\ \sum_{\substack{i=1\\ i\neq j}}^{n+2}\ \sum_{j=1}^{n+2}\ \frac{1}{s+a_i+a_j}\ +$$

$$\ldots\ldots+(-1)^{n+1}\ \frac{1}{n+1}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\neq \ell}}^{n+2}\ \sum_{j=1}^{n+2}\ \cdots\ \sum_{\ell=1}^{n+2}\ \frac{1}{s+a_i+a_j+\ldots+a_\ell}$$

$$+(-1)^{n+2}\ \frac{1}{n+2}\ \sum_{\substack{i=1\\ i\neq j\neq\ldots\ldots\neq p}}^{n+2}\ \sum_{j=1}^{n+2}\ \cdots\ \sum_{p=1}^{n+2}\ \frac{1}{s+a_i+a_j+\ldots+a_p} \qquad \text{(B-3)}$$

which is of the same form as the result for n multipliers (Eq. (B-1)). In other words the result is true whenever we take a certain number of multipliers, whatever the number may be, it is true when we increase the number by one. This proves that the result of Eq. (B-1) is universally true.

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