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by

Adam Rivet

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LIST OF SYMBOLS

Introduced in Chapter 2

$\epsilon_{ss}(x, s, z)$	Strain in transverse direction
$\epsilon_{tt}(x, s, z)$	Strain in normal direction
$\epsilon_{zz}(x, s, z)$	Strain in longitudinal direction
$\epsilon_{sz}(x, s, z)$	In-plane shear strain
$\epsilon_{ts}(x, s, z)$	Shear strain in normal direction
$\epsilon_{zt}(x, s, z)$	Shear strain in normal direction
$\bar{\epsilon}_{ss}(s, z)$	Midplane (membrane) strain in transverse direction
$\bar{\epsilon}_{zz}(s, z)$	Midplane (membrane) strain in longitudinal direction
$\bar{\gamma}_{sz}(s, z)$	Midplane (membrane) shear strain
$\kappa_{ss}(s, z)$	Midplane curvature along transverse direction
$\kappa_{zz}(s, z)$	Midplane curvature along longitudinal direction
$\kappa_{sz}(s, z)$	Midplane twist
$\sigma_{ss}(x, s, z)$	Stress in transverse direction
$\sigma_{tt}(x, s, z)$	Stress in normal direction
$\sigma_{zz}(x, s, z)$	Stress in longitudinal direction
$\sigma_{ts}(x, s, z)$	Shear stress in normal direction
$\sigma_{zt}(x, s, z)$	Shear stress in normal direction
$\sigma_{sz}(x, s, z)$	In-plane shear stress
$\psi_i(s)$	i^{th} normal displacement shape function
$\Psi(s)_{1 \times N}$	Vector of normal displacement shape functions (i^{th} component is $\psi_i(s)$)
ν	Poisson ratio
Π	Potential energy
Π_B	Work potential from loads applied on the ends of a member
Π_D	Work potential energy from volumetric loads on a member

Π_s	Internal strain energy
b	Plate width
${}_0c_{ij}, {}_1c_{ij}, {}_2c_{ij}, {}_3c_{ij}, {}_5c_{ij}, {}_7c_{ij}$	Constants based on geometric and material properties of the plate as well as shape functions $\psi_i(s)$ and $\psi_j(s)$
${}_0\mathbf{c}, {}_1\mathbf{c}, {}_2\mathbf{c}, {}_3\mathbf{c}, {}_5\mathbf{c}, {}_7\mathbf{c}$	Matrix with $(i-j)^{\text{th}}$ component equal to ${}_0c_{ij}, {}_1c_{ij}, {}_2c_{ij}, {}_3c_{ij}, {}_5c_{ij}, {}_7c_{ij}$, dimensions $N \times N$
E	Modulus of elasticity
h	Plate thickness
i	Index
j	Index
k	Index
ℓ	Index
L	Plate length
$M_{ss}(s, z)$	Plate bending moment along transverse direction
$M_{zz}(s, z)$	Plate bending moment along longitudinal direction
$M_{sz}(s, z)$	Plate twisting moment
$m_i(z)$	Generalized distributed bending moment causing normal displacement (associated with $\psi_i(s)$)
$\mathbf{m}(z)_{N \times 1}$	Vector of generalized distributed moments (i^{th} component is $m_i(z)$)
${}_{z_0}M_i$	Generalized end moment causing normal displacement (associated with $\psi_i(s)$)
${}_{z_0}\mathbf{M}_{N \times 1}$	Vector of generalized end moments (i^{th} component is ${}_{z_0}M_i$)
$m_s(z)$	Resultant bending moment causing transverse displacement
${}_{z_0}M_s$	Resultant end moment causing transverse displacement
$N_{ss}(s, z)$	Membrane force in transverse direction
$N_{zz}(s, z)$	Membrane force in longitudinal direction

$N_{sz}(s, z)$	Membrane shear force
N	Number of shape functions
O	Origin of coordinate system
P	Point in coordinate system
$p_s(z)$	Resultant distributed load causing transverse displacement
$p_z(z)$	Resultant distributed load causing longitudinal displacement
${}_D p_s(x, s, z)$	Volumetric load in the transverse direction
${}_D p_x(x, s, z)$	Volumetric load in the normal direction
${}_D p_z(x, s, z)$	Volumetric load in the longitudinal direction
$z_0 p_s(x, s)$	Traction applied in the transverse direction to the plate ends
$z_0 p_x(x, s)$	Traction applied in the normal direction to the plate ends
$z_0 p_z(x, s)$	Traction applied in the longitudinal direction to the plate ends
$z_0 P_s$	Resultant end load causing transverse displacement
$z_0 P_z$	Resultant end load causing normal displacement
s	Transverse coordinate
$t_i(z)$	Generalized distributed load causing normal displacement (associated with $\psi_i(s)$)
$\mathbf{t}(z)_{N \times 1}$	Vector of generalized distributed loads (i^{th} component is $t_i(z)$)
$z_0 T_i$	Generalized end load causing normal displacement (associated with $\psi_i(s)$)
$z_0 \mathbf{T}_{N \times 1}$	Vector of generalized end loads (i^{th} component is $z_0 T_i$)
$u(x, s, z)$	Normal displacement
$\bar{u}(s, z)$	Normal midplane displacement
$U_i(z)$	Generalized normal displacement (associated with $\psi_i(s)$)
$\mathbf{U}(z)_{N \times 1}$	Vector of generalized normal displacements (i^{th} component is $U_i(z)$)
$v(x, s, z)$	Transverse displacement
$\bar{v}(s, z)$	Transverse midplane displacement

$V(z)$	Generalized transverse displacement for narrow plate
$w(x, s, z)$	Longitudinal displacement
$\bar{w}(s, z)$	Longitudinal midplane displacement
$W(z)$	Generalized longitudinal displacement for narrow plate
x	Normal coordinal
z	Longitudinal coordinate
z_0	Constant (is either 0 or L)

Introduced in Chapter 3

$\phi_i(s)$	i^{th} transverse displacement shape function
$\Phi(s)_{N \times 1}$	Vector of transverse displacement shape function (i^{th} component is $\phi_i(s)$)
$\varphi_i(s)$	i^{th} longitudinal displacement shape function
$\Phi(s)_{N \times 1}$	Vector of longitudinal displacement shape function (i^{th} component is $\varphi_i(s)$)
${}_0a_{ij}, {}_1a_{ij}, {}_2a_{ij}, {}_3a_{ij}, {}_4a_{ij}, {}_5a_{ij}, {}_6a_{ij}$	Constants based on geometric and material properties of the plate as well as shape functions $\varphi_i(s)$, $\varphi_j(s)$, $\phi_i(s)$, and $\phi_j(s)$
${}_0\mathbf{a}, {}_1\mathbf{a}, {}_2\mathbf{a}, {}_3\mathbf{a}, {}_4\mathbf{a}, {}_5\mathbf{a}, {}_6\mathbf{a}$	Matrices with $(i-j)^{\text{th}}$ component equal to ${}_0a_{ij}, {}_1a_{ij}, {}_2a_{ij}, {}_3a_{ij}, {}_4a_{ij}, {}_5a_{ij}, {}_6a_{ij}$, dimensions $N \times N$
$p_i(z)$	Generalized distributed transverse load (associated with $\phi_i(s)$)
$\mathbf{p}(z)_{N \times 1}$	Vector of generalized transverse loads (i^{th} component is $p_i(z)$)
${}_{z_0}P_i$	Generalized transverse end load (associated with $\phi_i(s)$)
${}_{z_0}\mathbf{P}_{N \times 1}$	Vector of generalized transverse end loads (i^{th} component is ${}_{z_0}P_i$)
$q_i(z)$	Generalized longitudinal distributed load (associated with $\varphi_i(s)$)
$\mathbf{q}(z)_{N \times 1}$	Vector of generalized longitudinal loads (i^{th} component is $q_i(z)$)
${}_{z_0}Q_i$	Generalized longitudinal end load (associated with $\varphi_i(s)$)

${}_{z_0}\mathbf{Q}_{N \times 1}$	Vector of generalized longitudinal end loads (i^{th} component is ${}_{z_0}Q_i$)
$V_i(z)$	Generalized transverse displacement (associated with $\phi_i(s)$)
$\mathbf{V}(z)_{N \times 1}$	Vector of generalized transverse displacements (i^{th} component is $V_i(z)$)
$W_i(z)$	Generalized longitudinal displacement (associated with $\varphi_i(s)$)
$\mathbf{W}(z)_{N \times 1}$	Vector of generalized longitudinal displacements (i^{th} component is $W_i(z)$)

Introduced in Chapter 4

α_{mn}	Cosine of θ_{mn}
β_{mn}	Sine of θ_{mn}
θ_{mn}	Angle between plates m and n

Deep plate formulation

$\xi_{(3NK-\tilde{N}) \times 3NK}$	Matrix based on continuity constraint between multiple plates
${}_1\xi_{(3NK-\tilde{N}) \times \tilde{N}}$	Submatrix of $\xi_{(3NK-\tilde{N}) \times 3NK}$
${}_2\xi_{(3NK-\tilde{N}) \times (3NK-\tilde{N})}$	Submatrix of $\xi_{(3NK-\tilde{N}) \times 3NK}$
${}_3\xi_{(3NK-\tilde{N}) \times \tilde{N}}$	Matrix based on ${}_1\xi_{(3NK-\tilde{N}) \times \tilde{N}}$ and ${}_2\xi_{(3NK-\tilde{N}) \times (3NK-\tilde{N})}$
$A_i(z)$	Generalized displacement
$\mathbf{A}(z)_{3NK \times 1}$	Vector of all generalized displacements (i^{th} component is $A_i(z)$)
$\tilde{A}_i(z)$	Non-redundant generalized displacement
$\tilde{\mathbf{A}}(z)_{\tilde{N} \times 1}$	Vector of non-redundant generalized displacements (i^{th} component is $\tilde{A}_i(z)$)
${}_0d_{ij}, {}_1d_{ij}, {}_2d_{ij}, {}_3d_{ij}, {}_4d_{ij}$	Constants based on geometric and material properties of the plates as well as their shape functions
${}_0\mathbf{d}, {}_1\mathbf{d}, {}_2\mathbf{d}, {}_3\mathbf{d}, {}_4\mathbf{d}$	Matrix with $(i-j)^{\text{th}}$ component equal to ${}_0d_{ij}, {}_1d_{ij}, {}_2d_{ij}, {}_3d_{ij}, {}_4d_{ij}$, dimensions $3NK \times 3NK$

${}_0\tilde{d}_{ij}, {}_1\tilde{d}_{ij}, {}_2\tilde{d}_{ij},$ ${}_3\tilde{d}_{ij}, {}_4\tilde{d}_{ij}, {}_5\tilde{d}_{ij},$ ${}_6\tilde{d}_{ij}, {}_7\tilde{d}_{ij}$	Constants based on geometric and material properties of the plates, their shape functions and connectivity
${}_0\tilde{\mathbf{d}}, {}_1\tilde{\mathbf{d}}, {}_2\tilde{\mathbf{d}}, {}_3\tilde{\mathbf{d}},$ ${}_4\tilde{\mathbf{d}}, {}_5\tilde{\mathbf{d}}, {}_6\tilde{\mathbf{d}}, {}_7\tilde{\mathbf{d}}$	Matrix with $(i-j)^{\text{th}}$ component equal to ${}_0\tilde{d}_{ij}, {}_1\tilde{d}_{ij}, {}_2\tilde{d}_{ij}, {}_3\tilde{d}_{ij}, {}_4\tilde{d}_{ij}, {}_5\tilde{d}_{ij}, {}_6\tilde{d}_{ij}, {}_7\tilde{d}_{ij}$ dimensions $\tilde{N} \times \tilde{N}$
${}_{1,\ell}\mathbf{d}, {}_{2,\ell}\mathbf{d}, {}_{3,\ell}\mathbf{d},$ ${}_{4,1}\mathbf{d}$	Submatrices of ${}_{\ell}\mathbf{d}_{\tilde{N} \times \tilde{N}}$, variable dimensions
${}_0f_i(z)$	i^{th} generalized distributed load
${}_0\mathbf{f}(z)_{3NK \times 1}$	Vector of generalized distributed loads (i^{th} component is ${}_0f_i(z)$)
${}_1f_i(z)$	i^{th} generalized distributed moment
${}_1\mathbf{f}(z)_{3NK \times 1}$	Vector of generalized distributed moments (i^{th} component is ${}_1f_i(z)$)
${}_0\tilde{\mathbf{f}}(z)_{\tilde{N} \times 1}$	Vector of non-redundant generalized distributed loads
${}_1\tilde{\mathbf{f}}(z)_{\tilde{N} \times 1}$	Vector of non-redundant generalized distributed moments
${}_{z_0}\tilde{\mathbf{F}}_{\tilde{N} \times 1}$	Vector of non-redundant generalized end loads
${}_{z_0}\tilde{\mathbf{G}}_{\tilde{N} \times 1}$	Vector of non-redundant generalized end moments
K	Number of interconnected plates
m	Index
n	Index
$\tilde{N}$	Number of non-redundant degrees of freedom

Narrow plate formulation

${}_0\tilde{c}_{ij}, {}_1\tilde{c}_{ij}, {}_2\tilde{c}_{ij}, {}_3\tilde{c}_{ij},$ ${}_5\tilde{c}_{ij}, {}_7\tilde{c}_{ij}, d_i, d_n$	Constants based on geometric and material properties of the plates, their shape functions and connectivity
---	--

$\tilde{m}_i(z)$	i^{th} non-redundant generalized distributed moment (associated with $\tilde{U}_i(z)$)
$_{z_0}\tilde{M}_i$	i^{th} non-redundant generalized end moment (associated with $\tilde{U}_i(z)$)
$p_w(z)$	Generalized distributed longitudinal load
$_{z_0}P_w$	Generalized longitudinal end load
$\tilde{f}_i(z)$	i^{th} non-redundant generalized distributed load (associated with $\tilde{U}_i(z)$)
$_{z_0}\tilde{T}_i$	i^{th} non-redundant generalized end load (associated with $\tilde{U}_i(z)$)
$\tilde{U}_i(z)$	i^{th} non-redundant generalized displacement
$\tilde{W}(z)$	Generalized longitudinal displacement

Lagrange multipliers formulation

Γ	Lagrange functional
${}_1f_{mn}(z), {}_2f_{mn}(z),$ ${}_3f_{mn}(z), {}_4f_{mn}(z)$	Lagrange multiplier functions
${}_1g_{mn}(z), {}_2g_{mn}(z),$ ${}_3g_{mn}(z), {}_4g_{mn}(z)$	Functions based on continuity constraints between plates m and n

Introduced in Chapter 5

ι	Imaginary constant
κ_i	Boundary condition constant
$\mathbf{\kappa}_{MN \times 1}$	Vector of boundary condition constants (i^{th} component is κ_i)
η	Number of zero generalized displacements
λ_m	m^{th} eigenvalue
${}_k\mathbf{\Theta}(\)$	Matrix differential operator expressing boundary condition equations
Υ_m	Dimension of Jordan block
${}_k\mathbf{\Phi}$	Vector of constants on right hand side of boundary condition equations
${}_m\mathbf{b}_{N \times N}$	Matrix of coefficients in SOLDE

${}_0\mathbf{B}_{MN \times MN}$	First companion matrix of SOLDE
${}_1\mathbf{B}_{MN \times MN}$	Matrix based on ${}_M\mathbf{b}_{N \times N}$
${}_2\mathbf{B}_{MN \times MN}$	Matrix based on ${}_0\mathbf{B}_{MN \times MN}$ and ${}_1\mathbf{B}_{MN \times MN}$ for non-singular SOLDE
$\mathbf{f}_{ij}$	Fourier coefficient of applied load
$\mathbf{f}_{N \times (\mu+1)}$	Matrix of Fourier coefficients of applied loads ($(i-j)^{\text{th}}$ component is $\mathbf{f}_{ij}$)
$\mathbf{H}(z)_{N \times MN}$	Homogeneous solution to SOLDE
$\mathbf{I}_{N \times N}$	$N \times N$ identity matrix
${}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)}$	Jordan matrix associated with singular SOLDE
${}_2\mathbf{J}_{MN \times MN}$	Jordan form of ${}_2\mathbf{B}_{MN \times MN}$
${}_{m,2}\mathbf{J}_{\Upsilon_m \times \Upsilon_m}$	m^{th} Jordan block of ${}_2\mathbf{J}_{MN \times MN}$
M	Number of terms in SOLDE
$\mathbf{N}_{\eta \times \eta}$	Shift matrix of nil-potency $\eta - 1$
$\mathbf{r}(z)_{N \times 1}$	Vector of functions on right hand side of SOLDE
${}_0\mathbf{r}(z)_{MN \times 1}$	Right hand side vector in linearized SOLDE
${}_0\mathbf{S}_{MN \times MN}$	Right hand transformation matrix for Weierstraß transformation
${}_1\mathbf{S}_{MN \times MN}$	Left hand transformation matrix for Weierstraß transformation
${}_2\mathbf{S}_{MN \times MN}$	Jordan transformation matrix
$\mathbf{X}(z)_{N \times 1}$	Vector of unknown functions
${}_p\mathbf{X}(z)_{N \times 1}$	Particular solution to SOLDE
$\mathbf{Y}(z)_{MN \times 1}$	Linearized vector of unknown functions
$Z_i(z)$	i^{th} transformed vector of unknown functions
$\mathbf{Z}(z)_{MN \times 1}$	Linearly transformed vector of unknown functions (i^{th} component is $Z_i(z)$)
$\tilde{\mathbf{Z}}(z)_{(MN-\eta) \times 1}$	Vector of non-zero unknown functions
$\bar{\mathbf{Z}}(z)_{\eta \times 1}$	Vector of unknown functions shown to be zero

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DISTORTIONAL ANALYSIS OF THIN WALLED BEAMS by ADAM RIVET

THESIS ABSTRACT

A general solution for the stress-deformation analysis of interconnected plates subjected to general loading conditions is developed. The solution is based on the assumptions of thin-walled plate theory and is limited to combinations of straight plates made of linearly elastic isotropic material.

The principle of stationary potential energy is used in conjunction with series expansion for the displacement fields to formulate the equilibrium conditions and boundary conditions. In general, the differential equilibrium equations are coupled. A quasi-closed form solution for the displacement fields is nonetheless provided.

The solution developed is successfully adopted to solve several example problems. Three of the problems involve individual plates, while six involve multiply connected plates. Comparisons with established finite element solutions demonstrate the ability of the model to accurately capture the behaviour with a remarkably small number of degrees of freedoms.

CHAPTER 1 - INTRODUCTION

Many thin walled structures, such as girders or folded plate roofing structures, are assembled from bolted or welded thin plates. Such structures may be subjected to a variety of loads causing bending and twisting of the assembly as a whole and may also experience localized deformation. The ability to accurately model the displacements and stresses in these structures is required to produce a safe, durable, and economical design. The model should be able to reproduce the three dimensional shape of the deformed plates while being simple enough for use in engineering practice.

Shell finite element models can be used to accurately model thin walled structures under general loading conditions. However, the time and resources required for such an analysis have been a major impediment in finite element being adopted as a design tool. There is a need to develop a simple model that can accurately predict the displacements and stresses.

In the current study, a simple and accurate model is introduced which can predict the behaviour under general static loading. Some of the assumptions in previous theories are relaxed. The model postulates a series of displacement modes and develops the field equations as a function of the longitudinal coordinate. The resulting system of equations is exactly solved. Particular consideration is given on implementing the solution as a computer code.

1.1 Terms and Concepts

In this section we will define a number of terms and explain a number of concepts critical to understanding subsequent chapters.

Local Plate Directions

Figure 1.1 shows an undeformed plate, local coordinate axes (x , s , z) and the labels that we will associate with specific orientations.

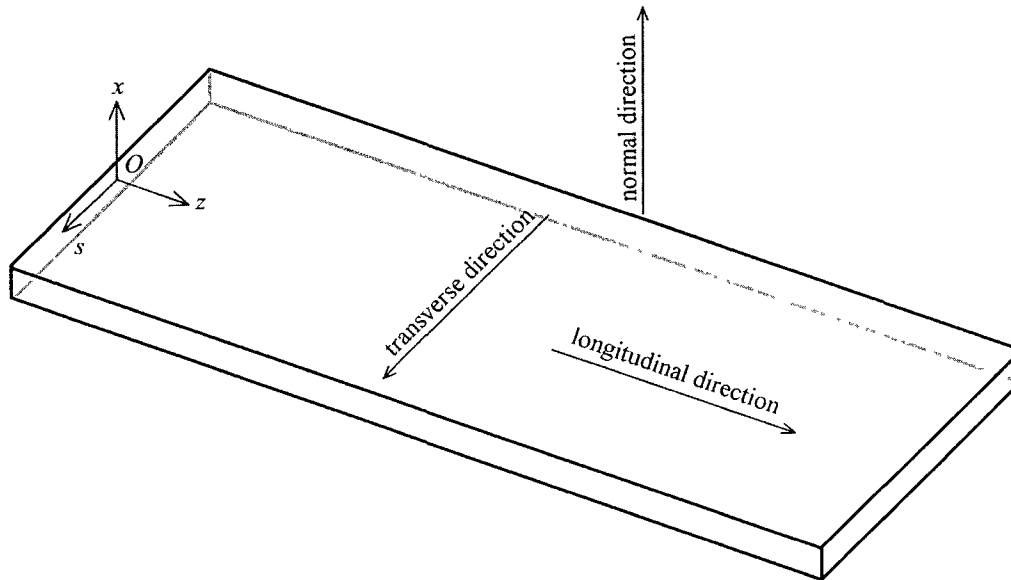


Fig. 1.1 - Plate directions

The normal, transverse, and longitudinal directions will be used to identify the displacements and stresses within an individual plate. For a straight beam assembled from several plates, the longitudinal direction is common to all plates in the assembly. In general, the normal and transverse directions will differ for each plate of the assembly. The transverse and longitudinal directions are collectively referred to as the local in-plane directions.

Cross-section Distortion

Distortion of a thin walled beam refers to any deformation pattern that cannot be decomposed into a combination of axial compression or extension, bending and torsion. Figure 1.2 shows the effect each mode of deformation has on the cross-section of a square box beam.

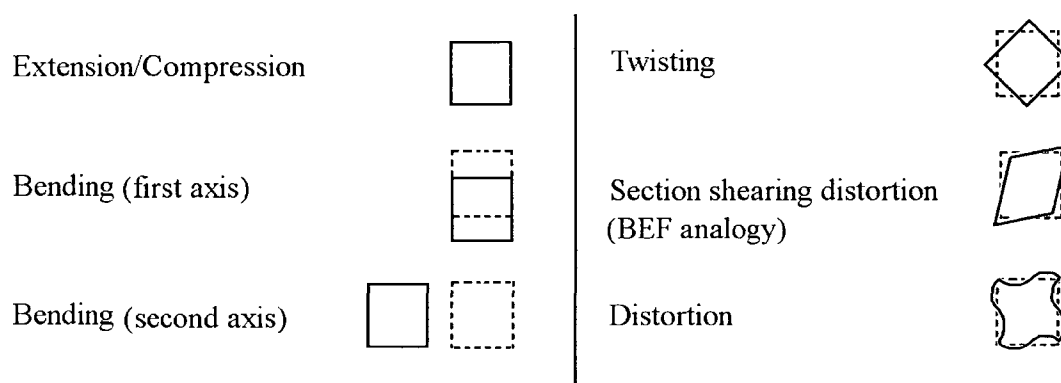


Fig. 1.2 - Deformation of a box beam cross section

The undeformed cross-section is shown as a dashed line and the deformed cross-section is a solid line. Euler-Bernoulli beam theory can take into account the first three modes: extension, compression, and bi-axial bending. Previous work on the distortion of thin-walled girders has centered on shearing of a closed box cross section. This shearing is considered to be a form of distortion but is specific to closed box sections. The deformed shape of a given cross-section depends on the applied loads and may not have any of the shapes shown in Fig. 1.2 or any combination thereof.

Cross-section Warping

A member consisting of any assembly of plates subject to torsion will, in general, undergo longitudinal displacements and the deformed cross-section will not remain plane. This mode of deformation is referred to as *warping* in the literature.

Shear Lag

Shear lag refers to the nonlinear distribution of longitudinal stress in a beam cross section. It occurs in members that are not stiff enough to uniformly transmit longitudinal stresses across

their width. Shear lag is not accounted for in the Euler-Bernoulli beam theory because the beam is assumed to be infinitely stiff in shear. In members with bolted or welded connections, shear lag will be most noticeable near connections where the load is unevenly transmitted to the cross-section. In girders without significant connections, shear lag is generally observed in excessively wide flanges. In such flanges, longitudinal stresses are largest at the connections to the web.

Shear Deformation

Shear deformation will affect a beam that is deep compared to its span. In this case, deflections and stresses due to shearing of the member may be significant compared to those due to bending. Significant shear stresses also cause a nonlinear distribution of longitudinal stresses through the cross-section depth. The Timoshenko beam captures the effect of shear deformation and yields better predictions of beam behaviour than the conventional Euler-Bernoulli theory.

1.2 Review of Past Methods

The purpose of this section is to summarize the analytical and numerical methods related to theories aimed at capturing the behaviour of thin plates or sets of connected thin plates. We will limit ourselves to *direct methods*, i.e. methods that use combinations of a priori known functions to describe the displacement field. The methods are divided into two categories: Kantorovich type methods and Ritz based methods.

1.2.1 Kantorovich Type Methods

Kantorovich and Krylov (1958) described a general method of reducing partial differential equations to sets of ordinary differential equation. The authors assumed that the unknown

multivariable function could be expressed as the product of one set of a priori known single-variable function in one coordinate and an a priori unknown set of functions in the other coordinate. The set of unknown functions were then determined using the either Galerkin's method or the principle of minimum potential energy. The method was applied to the Laplace and biharmonic equations and was shown to converge as more functions were included. The method has since become known as Kantorovich's method. When applied to plate bending problems in structural analysis, it is sometimes referred to as the Vlasov-Kantorovich method (Vlasov, 1964). Vlasov also extended the method to problems in general elasticity which he then named the method of initial functions (Babadzhanyan et al., 1975). Many well known models in structural engineering implicitly employ a similar approach by making assumptions about the displacements or stresses along one direction and deriving differential equations for the remaining fields. In this subsection we will summarize such models related to thin-walled structures.

Lévy's Solution

Lévy's solution can be applied to single rectangular plates subjected to normal loads (Reddy, 2007). The method assumes that the normal displacements can be expressed as the sum of a set of a priori unknown functions in the longitudinal direction multiplied by sinusoids of increasing frequency in the transverse direction. The resulting solution applies to plates with simply supported longitudinal edges and unsupported ends. The method leads to closed form series solutions if the loads are expressed as Fourier series.

Classical Theory of Thin Walled Beams

The modern theory of thin walled bars was developed by a number of authors at the turn of the 20th century and first published in a comprehensive monograph by Vlasov (1961). The theory accounts for compression, extension, bending, twisting and warping of thin walled structures of open cross-section. The theory can be generalized to closed cross-sections by adding constant shear stress around the cross section (Gjelsvik, 1981 or Slivker, 2006). Membrane stresses and stresses due to twisting are also taken into account. Beam cross-sections can be curved or straight. The bi-axial deflection and angle of twist of the cross section can be derived as a function of the longitudinal coordinate. Relatively simple closed form solutions are developed both for straight and curved girders. The theory captures the bending, twisting, and warping of shallow beams but does not account for distortion nor shear lag.

Beam on Elastic Foundation (BEF) Analogy

The BEF analogy was first developed by Wright et al. (1969). The method can be used to determine the distorted shape of a closed box girder under loads causing “shearing” of the cross section (see Fig. 1.2). The distorted shape of the section is described in terms of the change in the angle between the plates which are assumed to be pin connected. This distortion is shown to be orthogonal to other classical deformation modes. Thus, the displacements and stresses can be determined independently from distortion and then added to those from bending and twisting. The model takes into account longitudinal and shearing stresses. Relatively simple closed form solutions can be developed both for straight and curved girders under different boundary conditions.

Many authors have tried to modify or generalize the BEF analogy. Zhang and Lyons (1984), Dritsos (1991), Razaqpur and Li (1994), Pavazza and Blagojevic (2004), and Park et al. (2005), have extended the BEF analogy to closed multicell box girders. Boswell and Li (1995) added out-of-plane plates bending to the model. Kim and Kim (2000) developed a method for the analysis of distortion of single cell closed beams of arbitrary cross section. Some authors include the effect of transverse bending while others include only warping and shear stresses. Takahashi and Mizuno (1980) and later Saade et al. (2006) extended the method to beams with open cross sections. They retained the assumptions of Wright et al. but neglected midplane shear stresses. Goltermann (1992) derived the equations governing the distortion of a channel under a point load and a restraining spring. In a similar way, Jonsson (1999a and 1999b) introduced a model that takes into account transverse and longitudinal bending stresses as well as twist for a single distorted shape. The method is applicable both to open and closed cross sections. However it only allows for one distorted shape function. Hsu and Fu (2002) extended the method to open box girders and included the effects of transverse bending.

Folded plate method

Folded plate theory assumes that each plate in an assemblage behaves as an Euler-Bernoulli beam as it bends in the transverse direction (Kollbrunner and Basler, 1969). The plates are assumed to be pin connected. A set of linear equations governing the displacements at each joint between plates were derived based on Euler-Bernoulli beam theory and the static equilibrium of each joint. The theory can account for types of distortion that do not involve curvature of the plates in the plane of the cross section. The theory is limited to structures with open cross-sections and does not incorporate shear effects.

Generalized Beam Theory (GBT)

The basic formalism for GBT was presented by Schardt (1989) in German and then translated to English by Davies and Leach (1989) and Leach (1994). The three middle surface displacements of each plate are expressed as a series of transverse shape functions multiplied by longitudinal interpolation functions. Distinct shape functions are used for each direction. The same interpolation functions are used for both the normal and transverse directions. The Euler-Bernoulli assumption is conserved through appropriate choice of transverse shape function and by making the longitudinal interpolation functions equal to the derivatives of the transverse and normal interpolation functions. The first four shape functions for the section are chosen to represent the traditional compression, biaxial bending, and twisting of the cross section. The cross section is then discretized and higher order shape functions associated with normal displacements are made to vanish at all the nodes or at all but one of the nodes. The principle of minimum potential energy is used to derive the interpolation functions and boundary condition equations.

Shear Lag Models

Most analytical models for shear lag are based on work published by von Karman in 1923 and Reissner in 1941.

Von Karman (Timoshenko and Goodier, 1970) derived an equation for the stress in an infinitely wide and infinitely long flange under a single web. Plane stress was assumed in the flange and Euler-Bernoulli bending in the web. Stresses were expressed as an infinite trigonometric series. The principle of minimum potential energy was used to determine the coefficients of the series.

Reissner (Nakai and Yoo, 1988) developed a solution for shear lag in box beams by assuming a parabolic variation of stresses in the flanges. The stress field was determined using the principle of minimum potential energy.

Dowling and Burgan (1987) compiled an extensive review of both experimental and analytical work on shear lag in a variety of structures. Recently, several authors have published finite-element oriented shear lag models including Luo et al. (2004) and Wu et al. (2004).

1.2.2 Ritz based Models

Ritz based models differ from Kantorovich type models in that the interpolation functions for both directions are chosen a priori. The field variables (e.g. displacements) are expressed as the product of the two sets of functions. Each cross term is multiplied by unknown constants. The set of constants is usually determined using the principle of minimum potential energy, though other methods, such as least squares minimization and Galerkin's methods, can also be used.

Navier's Solution

Navier's solution is similar to Lévy's solution in that it can be applied to single rectangular plates subjected to normal loads (Reddy, 2007). The method assumes that the normal displacements can be expressed in a double sine series. The solution applies to plates which are simply supported along all four edges. The method leads to closed form series solutions if the loads are expressed as double sine series.

The Finite Element Method

The finite element method employs a Ritz's method in which the individual interpolation functions are non-zero only within a discrete area or volume. Because the structure must be discretized, the method is well adapted to problems with irregular geometry or boundary conditions. Non-linear effects such as plasticity can also be taken into account. Boundary conditions can be imposed exactly by the elements on the boundary. Since its inception in the 1960's, the method has been adapted to almost every field of engineering and can be used to solve a variety of beam, plate, and shell problems under various assumptions (Bathe, 1995).

The Finite Strip Method

The finite strip technique was developed primarily by Y. K. Cheung (Cheung and Tham, 1998) for use in the design of bridges and tall structures. The method prescribes dividing shell and plate structures into longitudinal strips. Each strip has an independent set of transverse shape functions and longitudinal interpolations functions. The transverse displacement of each strip is governed by the Euler-Bernoulli assumption in that longitudinal (warping) displacements are proportional to the lengthwise derivative of transverse displacements.

Spectral Methods

In spectral methods the solution is approximated by functions which are continuous over the entire domain. Sinusoids or Chebyshev polynomials are typically chosen as interpolation functions (Boyd, 2001). Typically boundary conditions are approximately enforced using a type of boundary collocation.

1.2.3 Summary of Past Methods

While some methods can be applied to a variety of structures and geometries, others are specific to certain types of structures. A given model will, in general, be geared towards certain features while neglecting others. Table 1.1 lists all the methods discussed in this chapter and summarizes their features. The table also indicates whether the method can be applied to single or multiple interconnected plates, or shells.

Methods of lower complexity, such as the BEF analogy or Ritz's method can be solved by hand calculation. The more complex methods, such as the current formulation or the finite element method, require the use of computers. The method introduced in the following chapters (i.e. the current formulation) is also shown.

Table 1.1 - Summary of direct methods in the static analysis of thin walled structures

	Type of displacement or stress							Type of structure		
	Compression/ Extension	Bending	Torsion	Shear Lag	Shear Deform.	Distortion	Plasticity	Single plate	Multiple plates	Shells
Lévy's method		✓	✓			✓		✓		
Euler-Bernoulli beam theory	✓	✓						✓	✓	✓
Classical beam theory (Vlasov)	✓	✓	✓					✓	✓	✓ (2)
Shear lag models	✓	✓	✓	✓				✓	✓	
Timoshenko beam theory	✓	✓			✓			✓	✓	
BEF analogy	✓	✓	✓			✓ (1)			✓ (1)	✓
Folded plate theory	✓	✓	✓			✓		✓	✓	
Generalized beam theory	✓	✓	✓			✓		✓	✓	
Current formulation	✓	✓	✓	✓	✓	✓		✓	✓	
Navier's method		✓	✓			✓		✓		
Ritz method	✓	✓	✓	✓	✓	✓		✓		
Finite strip method	✓	✓	✓			✓		✓	✓	✓ (2)
Spectral methods	✓	✓	✓	✓	✓	✓		✓	✓	✓
Finite elements	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

(1) - Specific to certain types of distortion in a box section

(2) - Assumes constant radius of curvatures in the longitudinal direction

1.3 Outline and Objective

The objective of the current study will be to extend the Kantorovich-type methods to complex sets of plates and to implement the method using a computer program. The model should account for shear effects as well as distortion in structures assembled using straight rectangular plates. Implementing such a model on a large scale would enable a designer to perform the detailed analysis of a complex structure without using the time and resources required for a finite element analysis. The governing equilibrium equations for single plates are established in Chapters 2 and 3 for narrow and deep plates respectively. The formulation in Chapter 2 neglects shear effects (shear lag and shear deformation) while the formulation of Chapter 3 includes them. The relations established in Chapters 2 and 3 are generalized to multiply-connected plates in Chapter 4. In all cases, the stationarity of the potential energy is evoked to derive the governing differential equations and boundary conditions. Chapter 5 presents some methods of solving the governing coupled differential equations. In Chapters 6 and 7, the results from the current method are compared to those resulting from finite element analysis using the commercial software program ABAQUS. Chapter 6 focuses on the solution for individual plates while Chapter 7 focuses on solutions for multiple connected plates.

CHAPTER 2 - EQUILIBRIUM EQUATIONS FOR A NARROW PLATE

The displacement field equations for thin and narrow plates are developed using the principle of stationary potential energy. The unknown displacement fields are expressed as a series of displacement shape functions modulated along the length of the plate. The strains and internal energy of the plates are derived within a set of assumptions specific to narrow plates. The total potential energy of the system is the sum of the internal strain energy and the work gained by the externally applied loads. Equilibrium is enforced by imposing stationarity of the potential energy with respect to the generalized displacements. The result is a set of differential field equations and boundary conditions.

2.1 Element Geometry

The undeformed and deformed configurations for the plate under consideration are shown in Fig. 2.1. A right hand coordinate system with origin O is selected as shown. The plate is rectangular with length L , width b , and thickness h . The location of a point, P , in the plate is given by coordinates (x, s, z) . Coordinate s is taken along the width of the plate from the center of the cross section. Coordinate z is taken along the length of the plate from one of the ends. The x coordinate is the distance from the midplane. The displacements of point P are $u(x, s, z)$, $v(x, s, z)$, and $w(x, s, z)$, in the x , s , and z directions, respectively.

The six independent components of stresses at a point (σ_{xx} , σ_{ss} , σ_{zz} , σ_{xs} , σ_{xz} , and σ_{sz}) are shown in Fig. 2.2. In general, each stress component is non-zero. For a plate or a beam, some of the stresses are neglected when formulating the displacement equations and strain energy.

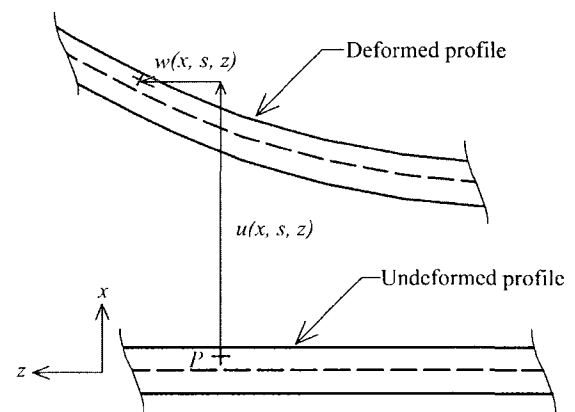
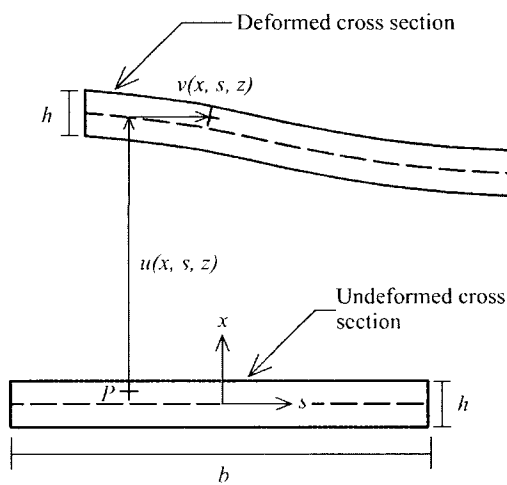
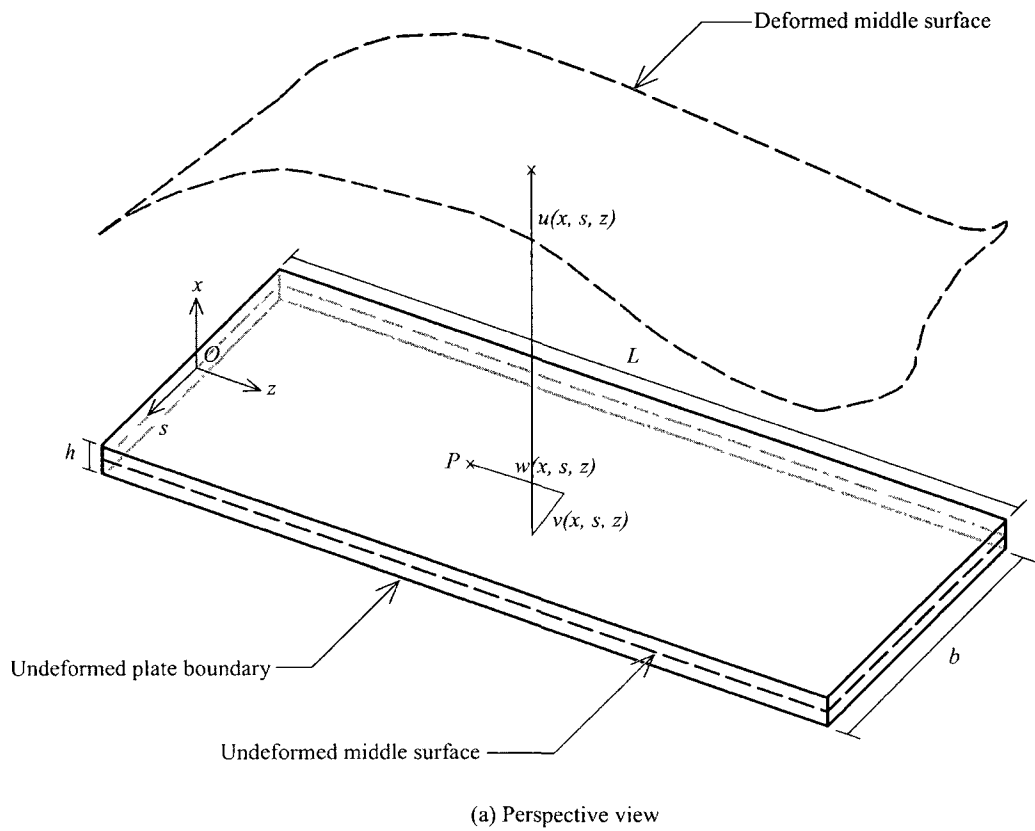


Fig. 2.1 - Deformation of a plate

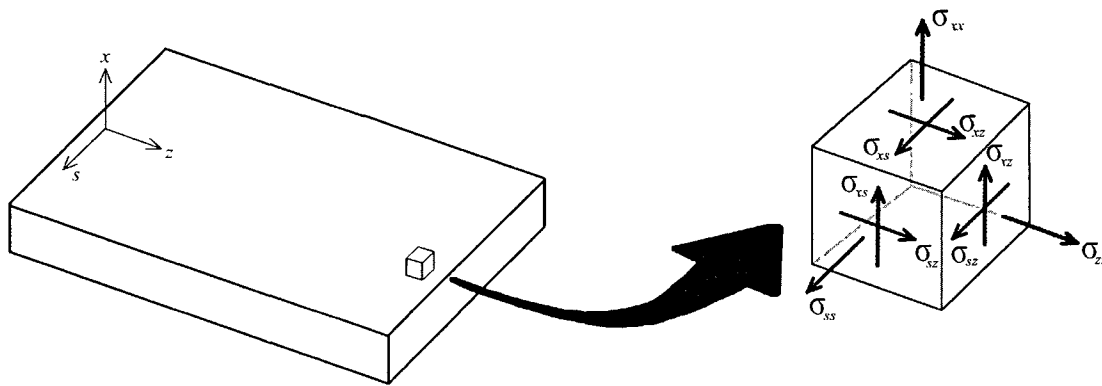


Fig. 2.2 - Stresses at a point

2.2 Assumptions

Various assumptions are made in the subsequent analysis. These assumptions are summarized in Table 2.1. We will limit ourselves to linear elastic analysis, that is, stresses are proportional to the strains in shear, extension, and compression (assumption 1.2). We will also assume that the strains are small enough that higher order strain terms are negligible compared to the linear terms (assumption 1.1). Since we are primarily interested in the analysis of steel beams, the material will be considered isotropic and homogeneous (assumptions 2.0). Assumptions 1.0 and 2.0 hold for steel structures deforming in their elastic range. We will adopt all of Kirchhoff's thin plate assumptions: transverse normals (fibres along the x directions) are inextensible, remain straight and perpendicular to the midplane after deformations (assumptions 3.0). We will further assume that the plate deflects as an Euler-Bernoulli beam in the s -direction and that there is no net extension or compression of the cross section in the s -direction (assumptions 4.0). Assumptions 3.3 and 4.3 assume plane stress in the plate and uniaxial membrane stress in the z -direction, respectively. It should be noted that assumptions 3.2 and 3.3 as well as 4.2 and 4.3 are contradictory (a plate cannot be both extensible and inextensible in the x -direction). However, relaxing assumptions 3.2 and 4.2 would significantly complicate the governing equations.

Table 2.1 - Assumptions

No.	Name	Description	Equations
1.0	Linear elastic assumptions		
1.1	Infinitesimal strains	Deformations are small.	--
1.2	Hook's Law	Stresses are proportional to strains.	2.7
2.0	Mechanical properties		
2.2	Isotropy	Material properties do not depend on direction.	2.7
2.3	Homogeneity	The mechanical properties of the material are the same everywhere.	2.7
3.0	Kirchhoff's thin plate assumptions		
3.1	Plane sections	Lines normal to the midplane remain straight and normal to the midplane after deformation.	2.2 2.2
3.2	Thin plate	The plate is inextensible in the x -direction.	2.2
3.3	Plane stress	Plate stresses are given by plane stress equations.	2.7
4.0	Euler-Bernoulli beam bending assumptions		
4.1	Vlasov's assumption	Midplane shearing stresses are negligible.	2.5
4.2	Constant plate width	The net plate width is assumed constant in the formulation of displacement equations.	2.4
4.3	Membrane stress	The transverse membrane stress is assumed zero in the calculation of the membrane stresses.	2.8
5.0	Geometric assumptions		
5.1	Boundaries	The plate has rectangular boundaries.	--
5.2	Uniform thickness	The plate has uniform thickness.	--
5.3	No buckling	The plate does not buckle.	--
6.0	Time dependent assumptions		
6.1	Static deformation	Deformations occur quasi-statically.	--
6.2	Constant temperature	The temperature is constant and uniform.	--

Relaxing assumptions 3.3 and 4.3 will result in an overly stiff model. In general, assumptions 3.0 are acceptable provided the plate thickness is small compared to its width ($b/h \gg 10$) and assumptions 4.0 are acceptable provided the plate width is small compared to its length ($L/b > 10$). Finally, the plate boundaries are rectangular (assumption 5.1) and the plate thickness is uniform (assumption 5.2). The plate does not undergo buckling (assumption 5.3). Only a static analysis will be performed and thermal effects will not be considered (assumptions 6.0).

2.3 Displacement Relations

From Kirchhoff's thin plate assumptions (assumptions 3.0) it is straightforward to show that (Timoshenko and Woinowsky-Krieger, 1959)

$$u(x, s, z) = \bar{u}(s, z) \quad (2.1)$$

$$v(x, s, z) = -x \frac{\partial}{\partial s} \bar{u}(s, z) + \bar{v}(s, z) \quad (2.2)$$

$$w(x, s, z) = -x \frac{\partial}{\partial z} \bar{u}(s, z) + \bar{w}(s, z) \quad (2.3)$$

where $\bar{u}(s, z)$, $\bar{v}(s, z)$, $\bar{w}(s, z)$ are the midplane displacements.

2.4 Assumed Forms of Midplane Displacement

If the plate deflects in the s -direction as an Euler-Bernoulli beam (assumption 4) one can show that (Gjelsvik, 1981)

$$\bar{v}(s, z) = V(z) \quad (2.4)$$

$$\bar{w}(s, z) = -s \frac{\partial}{\partial z} V(z) + W(z) \quad (2.5)$$

where $V(z)$ and $W(z)$ are the displacements at the center of the plate ($s = r = 0$) in the s -direction and z -direction respectively. Our final assumption is that the normal displacement, $u(x, s, z)$, can

be expressed as the sum of N shape functions, $\psi_i(s)$, whose amplitudes vary along the length of the plate, that is

$$\bar{u}(s, z) = \sum_{i=1}^N U_i(z) \cdot \psi_i(s) = \boldsymbol{\psi}(s)_{1 \times N}^T \cdot \mathbf{U}(z)_{N \times 1} \quad (2.6)$$

Vectors $\mathbf{U}_{N \times 1}(z)$ and $\boldsymbol{\psi}_{1 \times N}^T(s)$ contain the N generalized displacements $U_i(z)$ and N shape functions $\psi_i(s)$ respectively. The set of shape functions $\psi_i(s)$ should be (a) linearly independent and (b) able to accurately describe the deformed shape of the plate cross sections. No other restrictions will be placed on the shape functions until Chapters 3 and 4.

2.5 Strain Energy in terms of Midplane Displacement

The membrane forces and moments (Figure 2.3) are obtained by integrating the stresses in the plate across the plate thickness.

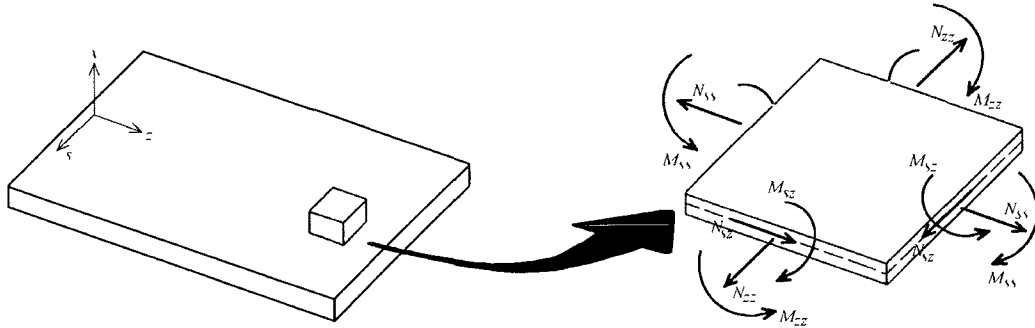


Fig. 2.3 - Thickness integrated forces and moments

Symbols N_{ss} and N_{zz} denote the axial forces per unit width in the s and z directions, respectively. Symbol N_{sz} denotes the internal shear force per unit width. Moments M_{ss} and M_{zz} are the plate bending moments per unit width in their respective direction. Moment M_{sz} is the twisting

moment per unit width. The thickness integrated forces and moments can be calculated from the midplane strains and curvatures as (Reddy, 2007)

$$\begin{Bmatrix} N_{ss}(s, z) \\ N_{zz}(s, z) \\ N_{sz}(s, z) \end{Bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{ss} \\ \bar{\epsilon}_{zz} \\ \bar{\gamma}_{sz} \end{Bmatrix} \quad (2.7a)$$

and

$$\begin{Bmatrix} M_{ss}(s, z) \\ M_{zz}(s, z) \\ M_{sz}(s, z) \end{Bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \kappa_{ss} \\ \kappa_{zz} \\ \kappa_{sz} \end{Bmatrix} \quad (2.7b)$$

where E is the modulus of elasticity and ν is the Poisson ratio. Symbols $\bar{\epsilon}_{ss}$ and $\bar{\epsilon}_{zz}$ denote the axial strains at the midplane, while κ_{ss} and κ_{zz} denote the curvatures along the s and z directions, respectively. Symbols $\bar{\gamma}_{sz}$ and κ_{sz} denote the shear strain and twist at the midplane. It should be noted that Equations 2.7 were derived under the assumptions of plane stress (assumption 3.3). Imposing zero shear stress and zero transverse membrane stress (assumptions 4.1 and 4.3), we can simplify the expressions for the membrane forces (Equation 2.7a) to

$$N_{ss}(s, z) = 0 \quad (2.8a)$$

$$N_{zz}(s, z) = Eh\bar{\epsilon}_{zz} \quad (2.8b)$$

$$N_{sz}(s, z) = 0 \quad (2.8c)$$

The strains and curvatures are defined in terms of the midplane displacements as

$$\bar{\epsilon}_{zz}(s, z) = \frac{\partial \bar{w}}{\partial z} \quad \kappa_{zz}(s, z) = -\frac{\partial^2 \bar{u}}{\partial z^2} \quad (2.9a)$$

$$\bar{\epsilon}_{ss}(s, z) = -\nu \frac{\partial \bar{w}}{\partial z} \quad \kappa_{ss}(s, z) = -\frac{\partial^2 \bar{u}}{\partial s^2} \quad (2.9b)$$

$$\bar{\gamma}_{sz}(s, z) = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial s} \quad \kappa_{sz}(s, z) = -2 \frac{\partial^2 \bar{u}}{\partial s \partial z} \quad (2.9c)$$

In accordance with assumption 4.3, $\bar{\epsilon}_{ss}$ was expressed using Equations 2.8a and 2.9a. The internal strain energy of the plate can be calculated by integrating the product of the thickness integrated forces and moments with their respective strains and curvatures, that is

$$\begin{aligned}\Pi_S &= \frac{1}{2} \int_0^L \int_{-b/2}^{b/2} (N_{ss} \bar{\epsilon}_{ss} + N_{zz} \bar{\epsilon}_{zz} + N_{sz} \bar{\epsilon}_{sz} + M_{ss} \kappa_{ss} + M_{zz} \kappa_{zz} + M_{sz} \kappa_{sz}) ds dz \\ &= \int_0^L \int_{-b/2}^{b/2} (N_{zz} \bar{\epsilon}_{zz} + M_{ss} \kappa_{ss} + M_{zz} \kappa_{zz} + M_{sz} \kappa_{sz}) ds dz\end{aligned}\quad (2.10)$$

Expanding Equation 2.10 using Equation 2.7, 2.8, and 2.9, we have

$$\begin{aligned}\Pi_S &= \frac{1}{2} \int_0^L \int_{-b/2}^{b/2} \left(Eh \left(\frac{\partial \bar{w}(s, z)}{\partial z} \right)^2 + \frac{Eh^3}{(1-\nu^2)} \left(\left(\frac{\partial^2 \bar{u}(s, z)}{\partial z^2} \right)^2 + \left(\frac{\partial^2 \bar{u}(s, z)}{\partial s^2} \right)^2 \right. \right. \\ &\quad \left. \left. + 2(1-\nu) \left(\frac{\partial^2 \bar{u}(s, z)}{\partial z \partial s} \right)^2 + 2\nu \frac{\partial^2 \bar{u}(s, z)}{\partial z^2} \frac{\partial^2 \bar{u}(s, z)}{\partial s^2} \right) \right) ds dz\end{aligned}\quad (2.11)$$

Equation 2.11 is similar to the classical equation for the strain energy of a plate under in-plane loads (Timoshenko and Woinowsky-Krieger, 1959) except that it lacks terms associated with midplane shear and transverse stress. These stresses will be incorporated in Chapter 3.

2.6 Variation of Total Potential Energy

2.6.1 Internal Strain Energy in terms of Generalized Displacements

Substituting Equations 2.4, 2.5, and 2.6 into 2.11, we can express the strain energy in terms of the generalized displacements $U_i(z)$, $W(z)$, and $V(z)$ as

$$\begin{aligned}\Pi_S &= \frac{Eh^3}{24(1-\nu^2)} \int_0^L \int_{-b/2}^{b/2} \left(\left(\sum_{i=1}^N U_i(z) \cdot \psi_i''(s) \right)^2 + 2(1-\nu) \left(\sum_{i=1}^N U_i'(z) \cdot \psi_i'(s) \right)^2 + \right. \\ &\quad \left. + 2\nu \left(\sum_{i=1}^N U_i(z) \cdot \psi_i''(s) \right) \cdot \left(\sum_{i=1}^N U_i''(z) \cdot \psi_i(s) \right) + \left(\sum_{i=1}^N U_i''(z) \cdot \psi_i(s) \right)^2 \right) ds dz \\ &\quad + \frac{Eh}{2} \int_0^L \int_{-b/2}^{b/2} (W'(z) - sV''(z))^2 ds dz\end{aligned}\quad (2.12)$$

Expanding, integrating with respect to s , and grouping like terms one obtains

$$\begin{aligned} \Pi_s = \int_0^L \left\{ \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^N \left({}_2c_{ij} U_i(z) U_j(z) + {}_1c_{ij} U_i'(z) U_j'(z) + {}_2{}_3c_{ij} U_i''(z) U_j(z) + {}_0c_{ij} U_i''(z) U_j''(z) \right) \right. \\ \left. + \frac{Eh}{2} \left(bW'(z)^2 + \frac{b^3}{12} V''(z)^2 \right) \right\} dz \end{aligned} \quad (2.13a)$$

with

$${}_0c_{ij} = \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i(s) \psi_j(s) ds \quad (2.13b)$$

$${}_1c_{ij} = \frac{Eh^3}{6(1+\nu)} \int_{-b/2}^{+b/2} \psi_i'(s) \psi_j'(s) ds \quad (2.13c)$$

$${}_2{}_3c_{ij} = \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i''(s) \psi_j(s) ds \quad (2.13d)$$

$${}_3c_{ij} = \nu \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i(s) \psi_j''(s) ds \quad (2.13e)$$

Equation 2.13a can also be expressed in matrix form as

$$\begin{aligned} \Pi_s = \frac{1}{2} \int_0^L \left(\mathbf{U}(z)_{1 \times N}^T \cdot {}_2\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} + \mathbf{U}'(z)_{1 \times N}^T \cdot {}_1\mathbf{c}_{N \times N} \cdot \mathbf{U}'(z)_{N \times 1} + 2\mathbf{U}''(z)_{1 \times N}^T \cdot {}_2{}_3\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} \right. \\ \left. + \mathbf{U}''(z)_{1 \times N}^T \cdot {}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}''(z)_{N \times 1} + Eh \left(bW'(z)^2 + \frac{b^3}{12} V''(z)^2 \right) \right) dz \end{aligned} \quad (2.14)$$

where the $(i-j)^{\text{th}}$ component of the matrices ${}_i\mathbf{c}_{N \times N}$ is ${}_ic_{ij}$.

2.6.2 Work Done by Volumetric Loads

For general volumetric loads ${}_D p_1(x, s, z)$, ${}_D p_2(x, s, z)$, and ${}_D p_3(x, s, z)$, acting along the x , s , and z directions respectively, the work done, Π_D , is

$$\Pi_D = \int_{-h/2}^{+h/2} \int_{-b/2}^{+b/2} \int_0^L \left({}_D p_1(x, s, z) \cdot u(x, s, z) + {}_D p_2(x, s, z) \cdot v(x, s, z) + {}_D p_3(x, s, z) \cdot w(x, s, z) \right) dz ds dx \quad (2.15)$$

Substituting expressions for $u(x, s, z)$, $v(x, s, z)$ and $w(x, s, z)$ using Equations 2.1 through 2.6 into Equation 2.15, grouping like terms and changing the order of integration, we have

$$\begin{aligned} \Pi_D = & \int_0^L \left(\sum_{i=1}^N U_i(z) \int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} {}_D p_v(x, s, z) \cdot \psi_i(s) dx ds + \sum_{i=1}^N U_i(z) \int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot {}_D p_s(x, s, z) \cdot \psi_i'(s) dx ds \right. \\ & + \sum_{i=1}^N U_i'(z) \int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot {}_D p_z(x, s, z) \cdot \psi_i(s) dx ds + V(z) \int_{-h/2-b/2}^{+h/2+b/2} {}_D p_s(x, s, z) dx ds \\ & \left. + V'(z) \int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} -s \cdot {}_D p_z(x, s, z) dx ds + W(z) \int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} {}_D p_z(x, s, z) dx ds \right) dz \end{aligned} \quad (2.16)$$

Integrals with respect to x and s can be replaced by resultant generalized distributed loads, that is

$$\Pi_D = \int_0^L \left(\sum_{i=1}^N \left(U_i(z) t_i(z) + U_i'(z) m_i(z) \right) + V(z) p_v(z) + V'(z) m_v(z) + W(z) p_z(z) \right) dz \quad (2.17)$$

Expressions and physical interpretations for $t_i(z)$, $m_i(z)$, $p_v(z)$, $m_v(z)$ and $p_z(z)$ are provided in Table 2.2.

Table 2.2 - Generalized distributed loads

Load	Expression	Description
$t_i(z)$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \left({}_D p_v(x, s, z) \cdot \psi_i(s) - x \cdot {}_D p_s(x, s, z) \cdot \psi_i'(s) \right) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
$m_i(z)$	$\int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot {}_D p_z(x, s, z) \cdot \psi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
$p_v(z)$	$\int_{-h/2-b/2}^{+h/2+b/2} {}_D p_s(x, s, z) dx ds$	Lateral load causing in-plane displacement.
$m_v(z)$	$\int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} -s \cdot {}_D p_z(x, s, z) dx ds$	Lateral moment causing in-plane displacement.
$p_z(z)$	$\int_{-b/2-h/2}^{b/2} \int_{-h/2}^{h/2} {}_D p_z(x, s, z) dx ds$	Axial load causing axial compression or extension.

2.6.3 Work Done by End Loads

The work performed by loads applied, Π_D , at the boundaries $z = 0$ and $z = L$ is

$$\begin{aligned} \Pi_B = & \int_{-h/2-b/2}^{+h/2+b/2} \int \left({}_0p_x(x,s) \cdot u(x,s,z) \Big|_{z=0} + {}_0p_s(x,s) \cdot v(x,s,z) \Big|_{z=0} + {}_0p_z(x,s) \cdot w(x,s,z) \Big|_{z=0} \right) ds dx \\ & + \int_{-h/2-b/2}^{+h/2+b/2} \int \left({}_Lp_x(x,s) \cdot u(x,s,z) \Big|_{z=L} + {}_Lp_s(x,s) \cdot v(x,s,z) \Big|_{z=L} + {}_Lp_z(x,s) \cdot w(x,s,z) \Big|_{z=L} \right) ds dx \end{aligned} \quad (2.18)$$

where ${}_0p_x(x,s)$, ${}_0p_s(x,s)$, ${}_0p_z(x,s)$ are externally applied pressures and tractions at the end

$z = 0$ acting along the x , s , and z directions and ${}_Lp_x(x,s)$, ${}_Lp_s(x,s)$, ${}_Lp_z(x,s)$ are those applied

at $z = L$. The work Π_B can be expressed in terms of resultant generalized loads and generalized coordinates as

$$\begin{aligned} \Pi_B = & \sum_{i=1}^N \left({}_0T_i \cdot U_i(z) \Big|_{z=0} + {}_0M_i \cdot U_i'(z) \Big|_{z=0} \right) + {}_0P_s \cdot V(z) \Big|_{z=0} + {}_0M_s \cdot V'(z) \Big|_{z=0} + {}_0P_z \cdot W(z) \Big|_{z=0} \\ & + \sum_{i=1}^N \left({}_LT_i \cdot U_i(z) \Big|_{z=L} + {}_LM_i \cdot U_i'(z) \Big|_{z=L} \right) + {}_LP_s \cdot V(z) \Big|_{z=L} + {}_LM_s \cdot V'(z) \Big|_{z=L} + {}_LP_z \cdot W(z) \Big|_{z=L} \end{aligned} \quad (2.19)$$

where ${}_0T_i$, ${}_0M_i$, ${}_0P_s$, ${}_0M_s$, ${}_0P_z$, ${}_LT_i$, ${}_LM_i$, ${}_LP_s$, ${}_LM_s$ and ${}_LP_z$ are the generalized loads on each

end. Expressions for the generalized loads are provided in Table 2.3.

2.6.4 Equilibrium Equations

The total potential energy, Π , is the difference of the internal strain energy and the work done by externally applied loads, i.e.,

$$\Pi = \Pi_s - \Pi_D - \Pi_B \quad (2.20)$$

Equilibrium field equations as well as boundary conditions are obtained by enforcing the stationarity of the functional Π , i.e.,

Table 2.3 - Generalized end loads

Generalized load	Expression	Description
T_{z_0}	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \left(z_0 p_x(x, s) \cdot \psi_i(s) - x \cdot z_0 p_s(x, s) \cdot \psi_i'(s) \right) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
M_{z_0}	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot z_0 q_z(x, s) \cdot \psi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
P_{z_0}	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z_0 q_s(x, s) dx ds$	Lateral load causing in-plane displacement.
M_{z_0}	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} -s \cdot z_0 q_z(x, s) dx ds$	Lateral moment causing in-plane displacement.
P_{z_0}	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z_0 q_z(x, s) dx ds$	Axial load causing axial compression or extension.

Note: $z_0 = 0$ or $z_0 = L$

$$\begin{aligned} \delta \Pi = \sum_{i=0}^N \left(\delta U_i \frac{\partial \Pi}{\partial U_i} + \delta U_i' \frac{\partial \Pi}{\partial U_i'} + \delta U_i'' \frac{\partial \Pi}{\partial U_i''} \right) + \delta W \frac{\partial \Pi}{\partial W} + \delta W' \frac{\partial \Pi}{\partial W'} \\ + \delta V \frac{\partial \Pi}{\partial V} + \delta V' \frac{\partial \Pi}{\partial V'} + \delta V'' \frac{\partial \Pi}{\partial V''} = 0 \end{aligned} \quad (2.21)$$

Details of the procedure, including performing integration by parts and grouping similar terms, are outlined in Appendix A.

2.6.4.1 Equilibrium Field Equations

Two of the equilibrium field equations arising after integration by parts of Equation 2.21 are

$$-EhbW''(z) = p_z(z) \quad (2.22)$$

$$E \frac{hb^3}{12} V''''(z) = p_s(z) - m_z'(z) \quad (2.23)$$

Equations 2.21 and 2.22 are the same as the equation of conventional beam theory (Timoshenko and Goodier, 1970) for a rectangular section with dimensions $h \times b$. This is because we assumed

that the plate deflected as an Euler-Bernoulli beam in the s -direction. The remaining equilibrium equations are coupled. They are best expressed in matrix form as

$${}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}^{\text{IV}}(z)_{N \times 1} + {}_5\mathbf{c}_{N \times N} \cdot \mathbf{U}^{\text{II}}(z)_{N \times 1} + {}_2\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} = \mathbf{t}(z)_{N \times 1} - \mathbf{m}^{\text{I}}(z)_{N \times 1} \quad (2.24a)$$

with

$${}_5\mathbf{c}_{N \times N} = {}_3\mathbf{c}_{N \times N} + {}_3\mathbf{c}_{N \times N}^T - {}_1\mathbf{c}_{N \times N} \quad (2.24b)$$

Matrix Equation 2.24 represents N coupled equations which can be fourth order differential, second order differential, or non-differential equations.

2.6.4.2 Boundary Conditions

The boundary conditions can also be derived by imposing stationarity of the potential energy.

Table 2.4 gives the boundary conditions at $z = 0$ for each of the generalized degrees of freedom

Table 2.4 - Boundary conditions at $z = 0$ (for $k = 1, \dots, N$)

Essential boundary condition	Natural boundary condition
$\delta W(z) _{z=0} = 0$	$EhbW'(0) = -{}_0P_z$
$\delta V(z) _{z=0} = 0$	$-E \frac{hb^3}{12} V'''(0) = -{}_0P_s$
$\delta V'(z) _{z=0} = 0$	$E \frac{hb^3}{12} V''(0) = -{}_0M_s$
$\delta U_k(z) _{z=0} = 0$	$-\sum_{j=1}^N {}_0c_{kj} \cdot U_j'''(0) + \sum_{j=1}^N {}_7c_{kj} \cdot U_j'(0) = -{}_0T_k$
$\delta U_k'(z) _{z=0} = 0$	$\sum_{j=1}^N {}_0c_{kj} \cdot U_j''(0) + \sum_{j=1}^N {}_3c_{kj} \cdot U_j(0) = -{}_0M_k$

In Table 2.8 we introduced ${}_7c_{ij}$ such that

$${}_7c_{ij} = {}_1c_{ij} - {}_3c_{ij} \quad (2.25)$$

For each degree of freedom, either the essential or the natural boundary condition must be satisfied. The natural boundary conditions correspond to the application of a generalized load on a free end. The displacement boundary conditions correspond to the fixity or settlement of a support. The conditions at the $z = L$ end are similar to the ones for the $z = 0$ end except that the signs of the applied loads have been inverted. The boundary condition equations at the $z = L$ end are given in Table 2.5.

Table 2.5 - Boundary conditions at $z = L$ (for $k = 1, \dots, N$)

Essential boundary condition	Natural boundary condition
$\delta W(z) _{z=L} = 0$	$EhbW'(L) = {}_L P_z$
$\delta V(z) _{z=L} = 0$	$-E \frac{hb^3}{12} V'''(L) = {}_L P_v$
$\delta V'(z) _{z=L} = 0$	$E \frac{hb^3}{12} V''(L) = {}_L M_s$
$\delta U_k(z) _{z=L} = 0$	$-\sum_{j=1}^N {}_0 c_{kj} \cdot U_j'''(L) + \sum_{j=1}^N {}_7 c_{kj} \cdot U_j'(L) = {}_L T_k$
$\delta U_k'(z) _{z=L} = 0$	$\sum_{j=1}^N {}_0 c_{kj} \cdot U_j''(L) + \sum_{j=1}^N {}_3 c_{kj} \cdot U_j(L) = {}_L M_k$

The boundary equations for $U_k(z)$ (last two lines of Tables 2.4 and 2.5) can be collectively expressed in matrix form as

$$-{}_0 \mathbf{c}_{N \times N} \cdot \mathbf{U}'''(0)_{N \times 1} + {}_7 \mathbf{c}_{N \times N} \cdot \mathbf{U}'(0)_{N \times 1} = -{}_0 \mathbf{T}_{N \times 1} \quad (2.26a)$$

$${}_0 \mathbf{c}_{N \times N} \cdot \mathbf{U}''(0)_{N \times 1} + {}_3 \mathbf{c}_{N \times N} \cdot \mathbf{U}(0)_{N \times 1} = -{}_0 \mathbf{M}_{N \times 1} \quad (2.26b)$$

and

$$-{}_0 \mathbf{c}_{N \times N} \cdot \mathbf{U}'''(L)_{N \times 1} + {}_7 \mathbf{c}_{N \times N} \cdot \mathbf{U}'(L)_{N \times 1} = {}_L \mathbf{T}_{N \times 1} \quad (2.26c)$$

$${}_0 \mathbf{c}_{N \times N} \cdot \mathbf{U}''(L)_{N \times 1} + {}_3 \mathbf{c}_{N \times N} \cdot \mathbf{U}(L)_{N \times 1} = {}_L \mathbf{M}_{N \times 1} \quad (2.26d)$$

for the $z = 0$ and the $z = L$ ends respectively. Just as the $(i-j)^{\text{th}}$ component of the matrices ${}_{\ell}\mathbf{c}_{N \times N}$ is ${}_{\ell}c_{ij}$, the i^{th} component of vectors ${}_0\mathbf{T}_{N \times 1}$, ${}_0\mathbf{M}_{N \times 1}$, ${}_L\mathbf{T}_{N \times 1}$, and ${}_L\mathbf{M}_{N \times 1}$ are ${}_0T_i$, ${}_0M_i$, ${}_LT_i$, and ${}_LM_i$ respectively. It should be noted that this boldface notation will be used throughout this work and for various symbols in order to define vectors or matrices from sets of scalars. The solution of the field and the boundary condition equations is discussed in Chapter 5.

CHAPTER 3 - EQUILIBRIUM EQUATIONS FOR A DEEP PLATE

In this chapter we will revise the derivation performed in Chapter 2 while relaxing the Euler-Bernoulli assumption for transverse displacement. The displacement field equations for the plate are developed using the principle of stationary potential energy. Equilibrium is found by imposing stationarity of the potential energy with respect to the new generalized displacements. The result is a new set of differential field equations and boundary conditions.

3.1 General

In Chapter 2 we derived the equations governing the displacements in a thin plate under the assumption that the plate will deform as an Euler-Bernoulli beam in the transverse direction. Though that formulation is somewhat simpler, it will not capture effects such as shear lag and shear deformation (as explained in Chapter 1). As such, it cannot accurately model plates that are wide compared to their length ($L/b > 10$) or plates forming part of a beam which has a closed cross-section (see Chapter 4).

In this chapter we will relax the Euler-Bernoulli assumptions in Chapter 2. We will adopt all the assumptions of Chapter 2 except those related to the transverse and longitudinal displacements (assumptions 4.0).

The geometric conventions and symbols of the previous chapter will be used here. Figure 2.1 presents most of the necessary information.

3.2 Thin Plate Displacement Relations

We will provide the displacement equations stemming from Kirchhoff's thin plate assumptions as they will be used to express the generalized loads:

$$u(x, s, z) = \bar{u}(s, z) \quad (3.1a)$$

$$v(x, s, z) = -x \frac{\partial}{\partial s} \bar{u}(s, z) + \bar{v}(s, z) \quad (3.1b)$$

$$w(x, s, z) = -x \frac{\partial}{\partial z} \bar{u}(s, z) + \bar{w}(s, z) \quad (3.1c)$$

where $\bar{u}(s, z)$, $\bar{v}(s, z)$, $\bar{w}(s, z)$ are the midplane displacements.

3.3 Assumed Forms of Midplane Displacement

We will express the midplane displacements as series of shape functions whose amplitudes vary along the length of the plate, that is

$$\bar{u}(s, z) = \sum_{i=1}^N U_i(z) \cdot \psi_i(s) = \boldsymbol{\psi}(s)_{1 \times N}^T \cdot \mathbf{U}(z)_{N \times 1} \quad (3.2a)$$

$$\bar{v}(s, z) = \sum_{i=1}^N V_i(z) \cdot \phi_i(s) = \boldsymbol{\phi}(s)_{1 \times N}^T \cdot \mathbf{V}(z)_{N \times 1} \quad (3.2b)$$

$$\bar{w}(s, z) = \sum_{i=1}^N W_i(z) \cdot \varphi_i(s) = \boldsymbol{\varphi}(s)_{1 \times N}^T \cdot \mathbf{W}(z)_{N \times 1} \quad (3.2c)$$

Individual sets of shape functions, such as $\{\psi_1(s) \ \psi_2(s) \ \dots \ \psi_N(s)\}$, should be linearly independent and able to accurately describe the deformed shape of the plate cross sections. No other restrictions will be placed on the functions until Chapter 4.

3.4 Internal Strain Energy in terms of Midplane Displacements

We will now restate Equations 2.7 and 2.9 which give the plate strains, curvatures, and forces in terms of the midplane displacements. The midplane strains and curvatures are given by

$$\bar{\epsilon}_{zz}(s, z) = \frac{\partial \bar{w}}{\partial z} \quad \kappa_{zz}(s, z) = -\frac{\partial^2 \bar{u}}{\partial z^2} \quad (3.3a)$$

$$\bar{\epsilon}_{ss}(s, z) = \frac{\partial \bar{v}}{\partial z} \quad \kappa_{ss}(s, z) = -\frac{\partial^2 \bar{u}}{\partial s^2} \quad (3.3b)$$

$$\bar{\gamma}_{sz}(s, z) = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial s} \quad \kappa_{sz}(s, z) = -2 \frac{\partial^2 \bar{u}}{\partial s \partial z} \quad (3.3c)$$

where all symbols are explained in Chapter 2. The membrane stresses and plate moments in terms of the midplane strains and curvatures are

$$\begin{Bmatrix} N_{ss}(s, z) \\ N_{zz}(s, z) \\ N_{sz}(s, z) \end{Bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{ss} \\ \bar{\epsilon}_{zz} \\ \bar{\gamma}_{sz} \end{Bmatrix} \quad (3.4a)$$

and

$$\begin{Bmatrix} M_{ss}(s, z) \\ M_{zz}(s, z) \\ M_{sz}(s, z) \end{Bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \kappa_{ss} \\ \kappa_{zz} \\ \kappa_{sz} \end{Bmatrix} \quad (3.4b)$$

The strain energy of the plate can be calculated by integrating the product of the thickness integrated forces with their respective strains and curvatures, that is

$$\Pi_s = \frac{1}{2} \int_0^L \int_{-b/2}^{b/2} (N_{ss} \bar{\epsilon}_{ss} + N_{zz} \bar{\epsilon}_{zz} + N_{sz} \bar{\gamma}_{sz} + M_{ss} \kappa_{ss} + M_{zz} \kappa_{zz} + M_{sz} \kappa_{sz}) ds dz \quad (3.5)$$

Expanding the integral using Equations 3.3 and 3.4 gives

$$\begin{aligned} \Pi_s = \frac{1}{2} \int_0^L \int_{-b/2}^{b/2} & \left(\frac{Eh}{(1-\nu^2)} \left(\left(\frac{\partial \bar{w}}{\partial z} \right)^2 + \left(\frac{\partial \bar{v}}{\partial s} \right)^2 + 2\nu \frac{\partial \bar{v}}{\partial s} \frac{\partial \bar{w}}{\partial z} \right) + \frac{Eh}{2(1+\nu)} \left(\left(\frac{\partial \bar{w}}{\partial s} \right)^2 + \left(\frac{\partial \bar{v}}{\partial z} \right)^2 + 2 \frac{\partial \bar{v}}{\partial z} \frac{\partial \bar{w}}{\partial s} \right) \right. \\ & \left. + \frac{Eh^3}{12(1-\nu^2)} \left(\left(\frac{\partial^2 \bar{u}}{\partial z^2} \right)^2 + \left(\frac{\partial^2 \bar{u}}{\partial s^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 \bar{u}}{\partial z \partial s} \right)^2 + 2\nu \frac{\partial^2 \bar{u}}{\partial z^2} \frac{\partial^2 \bar{u}}{\partial s^2} \right) \right) ds dz \end{aligned} \quad (3.6)$$

Equation 3.6 is the classical equation for the strain energy of a thin plate in terms of the midplane displacements. Unlike the expression for the strain energy of a narrow plate (Equation 2.11), it includes contributions from midplane shear and transverse stress.

3.5 Variation of the Total Potential Energy

3.5.1 Strain Energy in Terms of Generalized Displacements

Using Equations 3.2 and 3.6, we can express the strain energy (Equation 3.6) in terms of the generalized displacements $U_i(z)$, $V_i(z)$, and $W_i(z)$. Grouping like terms and integrating with respect to s one would obtain

$$\begin{aligned} \Pi_s = \frac{1}{2} \int_0^L & \left(\sum_{j=1}^N \sum_{i=1}^N \left({}_0a_{ij} \cdot W_i'(z) W_j'(z) + {}_1a_{ij} \cdot W_i'(z) V_j(z) + {}_2a_{ij} \cdot W_i'(z) W_j'(z) \right) \right. \\ & + \sum_{j=1}^N \sum_{i=1}^N \left({}_3a_{ij} \cdot V_i'(z) V_j'(z) + {}_4a_{ij} \cdot W_i(z) W_j(z) + {}_5a_{ij} \cdot W_i(z) V_j'(z) \right) \\ & \left. + \sum_{j=1}^N \sum_{i=1}^N \left({}_2c_{ij} U_i(z) U_j(z) + {}_1c_{ij} U_i'(z) U_j'(z) + {}_3c_{ij} U_i''(z) U_j(z) + {}_0c_{ij} U_i''(z) U_j''(z) \right) \right) dz \end{aligned} \quad (3.7)$$

with

$${}_0a_{ij} = \frac{Eh}{1-\nu^2} \int_{-b/2}^{+b/2} \phi_i(s) \phi_j(s) ds \quad (3.7b)$$

$${}_1a_{ij} = \nu \frac{Eh}{1-\nu^2} \int_{-b/2}^{+b/2} \phi_i(s) \phi_j'(s) ds \quad (3.7c)$$

$${}_2a_{ij} = \frac{Eh}{1-\nu^2} \int_{-b/2}^{+b/2} \phi_i'(s) \phi_j'(s) ds \quad (3.7d)$$

$${}_3a_{ij} = \frac{Eh}{2(1+\nu)} \int_{-b/2}^{+b/2} \phi_i(s) \phi_j(s) ds \quad (3.7e)$$

$${}_4a_{ij} = \frac{Eh}{2(1+\nu)} \int_{-b/2}^{+b/2} \phi_i'(s) \phi_j(s) ds \quad (3.7f)$$

$${}_5a_{ij} = \frac{Eh}{2(1+\nu)} \int_{-b/2}^{+b/2} \phi_i(s) \phi_j'(s) ds \quad (3.7g)$$

and expressions for constants ${}_i c_{ij}$ were given in Equations 2.13b-e. Equation 3.7a can also be

expressed in matrix form as

$$\begin{aligned} \Pi_s = \frac{1}{2} \int_0^L & \left(\mathbf{W}^I(z)_{1 \times N}^T \cdot {}_0 \mathbf{a}_{N \times N} \cdot \mathbf{W}^I(z)_{N \times 1} + \mathbf{V}^I(z)_{1 \times N}^T \cdot {}_3 \mathbf{a}_{N \times N} \cdot \mathbf{V}^I(z)_{N \times 1} + \mathbf{V}(z)_{1 \times N}^T \cdot {}_2 \mathbf{a}_{N \times N} \cdot \mathbf{V}(z)_{N \times 1} \right. \\ & + \mathbf{W}(z)_{1 \times N}^T \cdot {}_4 \mathbf{a}_{N \times N} \cdot \mathbf{W}(z)_{N \times 1} + 2 \mathbf{W}(z)_{1 \times N}^T \cdot {}_5 \mathbf{a}_{N \times N} \cdot \mathbf{V}^I(z)_{N \times 1} + 2 \mathbf{W}^I(z)_{1 \times N}^T \cdot {}_1 \mathbf{a}_{N \times N} \cdot \mathbf{V}(z)_{N \times 1} \\ & + \mathbf{U}(z)_{1 \times N}^T \cdot {}_2 \mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} + \mathbf{U}^I(z)_{1 \times N}^T \cdot {}_1 \mathbf{c}_{N \times N} \cdot \mathbf{U}^I(z)_{N \times 1} + 2 \mathbf{U}^{II}(z)_{1 \times N}^T \cdot {}_3 \mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} \\ & \left. + \mathbf{U}^{II}(z)_{1 \times N}^T \cdot {}_0 \mathbf{c}_{N \times N} \cdot \mathbf{U}^{II}(z)_{N \times 1} \right) dz \end{aligned} \quad (3.8)$$

3.5.2 Work Done by Volumetric Loads

For distributed volumetric loads ${}_D p_x(x, s, z)$, ${}_D p_y(x, s, z)$, and ${}_D p_z(x, s, z)$ acting along the x , s ,

and z directions respectively, the work done is

$$\Pi_D = \int_{-h/2}^{+h/2} \int_{-b/2}^{+b/2} \int_0^L \left({}_D p_x(x, s, z) \cdot u(x, s, z) + {}_D p_y(x, s, z) \cdot v(x, s, z) + {}_D p_z(x, s, z) \cdot w(x, s, z) \right) dz ds dx \quad (3.9)$$

Substituting expressions for $u(x, s, z)$, $v(x, s, z)$ and $w(x, s, z)$ from Equations 3.1 and 3.2,

grouping like terms and changing the order of integration we have

$$\Pi_D = \int_0^L \sum_{i=1}^N \left(V_i(z) p_i(z) + W_i(z) q_i(z) + U_i(z) t_i(z) + U_i'(z) m_i(z) \right) dz \quad (3.10)$$

We recognize here $t_i(z)$ and $m_i(z)$ already defined in Table 2.2. Expressions and physical interpretations for $t_i(z)$, $m_i(z)$, $p_i(z)$, and $q_i(z)$ are provided in Table 3.1.

Table 3.1 - Generalized distributed loads

Load	Expression	Description
$t_i(z)$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} ({}_D p_x(x, s, z) \cdot \psi_i(s) - x \cdot {}_D p_z(x, s, z) \cdot \psi_i'(s)) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
$m_i(z)$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot {}_D p_z(x, s, z) \cdot \psi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
$p_i(z)$	$\int_{-h/2}^{+h/2} \int_{-b/2}^{+b/2} {}_D p_x(x, s, z) \phi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\phi_i(s)$.
$q_i(z)$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} {}_D p_z(x, s, z) \phi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\phi_i(s)$.

3.5.3 Work Done by End Loads

The work, Π_B , done by loads applied at the boundaries $z = 0$ and $z = L$ is

$$\begin{aligned} \Pi_B = & \int_{-h/2}^{+h/2} \int_{-b/2}^{+b/2} \left({}_0 p_x(x, s) \cdot u(x, s, z) \Big|_{z=0} + {}_0 p_x(x, s) \cdot v(x, s, z) \Big|_{z=0} + {}_0 p_z(x, s) \cdot w(x, s, z) \Big|_{z=0} \right) ds dx \\ & + \int_{-h/2}^{+h/2} \int_{-b/2}^{+b/2} \left({}_L p_x(x, s) \cdot u(x, s, z) \Big|_{z=L} + {}_L p_x(x, s) \cdot v(x, s, z) \Big|_{z=L} + {}_L p_z(x, s) \cdot w(x, s, z) \Big|_{z=L} \right) ds dx \end{aligned} \quad (3.11)$$

where ${}_0 p_x(x, s)$, ${}_0 p_y(x, s)$, ${}_0 p_z(x, s)$ are externally applied pressures and tractions at the end

$z = 0$ acting along the x , s , and z directions respectively and ${}_L p_x(x, s)$, ${}_L p_y(x, s)$, ${}_L p_z(x, s)$ are

those applied at $z = L$. The work Π_B can be expressed in terms of resultant generalized loads and

generalized coordinates as

$$\begin{aligned} \Pi_B = & \sum_{i=1}^N \left({}_0T_i \cdot U_i(z) \Big|_{z=0} + {}_0M_i \cdot U_i'(z) \Big|_{z=0} + {}_0P_i \cdot V_i(z) \Big|_{z=0} + {}_0Q_i \cdot W_i(z) \Big|_{z=0} \right) \\ & + \sum_{i=1}^N \left({}_LT_i \cdot U_i(z) \Big|_{z=L} + {}_LM_i \cdot U_i'(z) \Big|_{z=L} + {}_LP_i \cdot V_i(z) \Big|_{z=L} + {}_LQ_i \cdot W_i(z) \Big|_{z=L} \right) \end{aligned} \quad (3.12)$$

where ${}_0T_i$, ${}_0M_i$, ${}_0P_i$, ${}_0Q_i$, ${}_LT_i$, ${}_LM_i$, ${}_LP_i$, and ${}_LQ_i$ are the generalized resultant loads on each end.

Expressions for the generalized end loads are provided in Table 3.2.

Table 3.2 - Generalized end loads

Load	Expression	Description
${}_0T_i$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \left({}_0p_i(x, s) \cdot \psi_i(s) - x \cdot {}_0p_s(x, s) \cdot \psi_i'(s) \right) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
${}_0M_i$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} -x \cdot {}_0p_z(x, s) \cdot \psi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\psi_i(s)$.
${}_0P_i$	$\int_{-h/2}^{+h/2+b/2} \int_{-b/2}^{b/2} {}_0p_s(x, s) \phi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\phi_i(s)$.
${}_0Q_i$	$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} {}_0p_z(x, s) \varphi_i(s) dx ds$	Generalized load causing normal displacement, associated with $\varphi_i(s)$.

Note: $z_0 = 0$ or $z_0 = L$

3.5.4 Equilibrium Equations

The total potential energy, Π is the difference of the internal strain energy and the work of the externally applied loads:

$$\Pi = \Pi_s - \Pi_D - \Pi_B \quad (3.13)$$

Equilibrium field equations as well as boundary conditions are obtained by imposing the stationarity of the total potential energy:

$$\delta \Pi = \sum_{i=0}^N \left(\delta U_i \frac{\partial \Pi}{\partial U_i} + \delta U_i' \frac{\partial \Pi}{\partial U_i'} + \delta U_i'' \frac{\partial \Pi}{\partial U_i''} + \delta V_i \frac{\partial \Pi}{\partial V_i} + \delta W_i \frac{\partial \Pi}{\partial W_i} + \delta V_i' \frac{\partial \Pi}{\partial V_i'} + \delta W_i' \frac{\partial \Pi}{\partial W_i'} \right) \quad (3.14)$$

Details of the procedure, including performing integration by parts and grouping similar terms, are outlined in Appendix B.

3.5.4.1 Equilibrium Field Equations

The equilibrium field equations for $W(z)$ and $V(z)$ are best expressed in matrix form as

$$-\begin{bmatrix} {}_3\mathbf{a} & \mathbf{0} \\ \mathbf{0} & {}_0\mathbf{a} \end{bmatrix} \begin{Bmatrix} \mathbf{V}(z)_{N \times 1} \\ \mathbf{W}(z)_{N \times 1} \end{Bmatrix}'' + \begin{bmatrix} \mathbf{0} & -{}_6\mathbf{a} \\ {}_6\mathbf{a} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{V}(z)_{N \times 1} \\ \mathbf{W}(z)_{N \times 1} \end{Bmatrix}' + \begin{bmatrix} {}_2\mathbf{a} & \mathbf{0} \\ \mathbf{0} & {}_4\mathbf{a} \end{bmatrix} \begin{Bmatrix} \mathbf{V}(z)_{N \times 1} \\ \mathbf{W}(z)_{N \times 1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}(z)_{N \times 1} \\ \mathbf{q}(z)_{N \times 1} \end{Bmatrix} \quad (3.15a)$$

where the following relation has been introduced

$${}_6\mathbf{a}_{N \times N} = {}_5\mathbf{a}_{N \times N} - {}_1\mathbf{a}_{N \times N} \quad (3.15b)$$

where all sub-matrices have $N \times N$ dimensions and each matrix is $2N \times 2N$. Also, the equilibrium equations for the generalized normal displacements, $U_i(z)$, are found identical to those derived in Chapter 2:

$${}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}^{IV}(z)_{N \times 1} + {}_5\mathbf{c}_{N \times N} \cdot \mathbf{U}^{II}(z)_{N \times 1} + {}_2\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} = \mathbf{t}(z)_{N \times 1} - \mathbf{m}^I(z)_{N \times 1} \quad (3.16)$$

Matrix Equation 3.15a represents $2N$ coupled equations which can be second order differential, first order differential or non-differential equations. Matrix Equation 3.16 represents N coupled equations which can be fourth order differential, second order differential or non-differential equations.

3.5.4.2 Boundary Conditions

Boundary conditions based on the imposed loads and displacements can also be formulated from the stationarity of the potential energy. Table 3.3 gives the boundary conditions at the ends for each of the generalized degrees of freedom.

Table 3.3 - Boundary condition equations

Essential boundary condition	Natural boundary condition
$\delta V_k(z) _{z=z_0} = 0$	$\sum_{j=1}^N {}_3a_{kj} \cdot V_j'(z_0) + \sum_{j=1}^N {}_5a_{kj} \cdot W_j(z_0) = \pm_{z_0} P_k$
$\delta W_k(z) _{z=z_0} = 0$	$\sum_{j=1}^N {}_0a_{kj} \cdot W_j'(z_0) + \sum_{j=1}^N {}_1a_{kj} \cdot V_j(z_0) = \pm_{z_0} Q_k$
$\delta U_k(z) _{z=z_0} = 0$	$-\sum_{j=1}^N {}_0c_{kj} \cdot U_j'''(z_0) + \sum_{j=1}^N {}_7c_{kj} \cdot U_j'(z_0) = \pm_{z_0} T_k$
$\delta U_k'(z) _{z=z_0} = 0$	$\sum_{j=1}^N {}_0c_{kj} \cdot U_j''(z_0) + \sum_{j=1}^N {}_3c_{kj} \cdot U_j(z_0) = \pm_{z_0} M_k$

Note: $z_0 = 0$ or $z_0 = L$

For each degree of freedom, either the essential or the natural boundary condition must be satisfied. The natural boundary conditions correspond to the application of a load on a free end. The displacement boundary conditions correspond to the fixity or settlement of a support. The boundary condition equations for $V_i(z)$ and $W_i(z)$ can be collectively expressed in matrix form:

$$\left[\begin{array}{c|c} {}_3\mathbf{a}_{N \times N} & \mathbf{0}_{N \times N} \\ \hline \mathbf{0}_{N \times N} & {}_0\mathbf{a}_{N \times N} \end{array} \right] \left\{ \begin{array}{c} \mathbf{V}^I(z_0)_{N \times 1} \\ \hline \mathbf{W}^I(z_0)_{N \times 1} \end{array} \right\} + \left[\begin{array}{c|c} \mathbf{0}_{N \times N} & {}_5\mathbf{a}_{N \times N}^T \\ \hline {}_1\mathbf{a}_{N \times N} & \mathbf{0}_{N \times N} \end{array} \right] \left\{ \begin{array}{c} \mathbf{V}(z_0)_{N \times 1} \\ \hline \mathbf{W}(z_0)_{N \times 1} \end{array} \right\} = \pm \left\{ \begin{array}{c} {}_{z_0}\mathbf{P}_{N \times 1} \\ \hline {}_{z_0}\mathbf{Q}_{N \times 1} \end{array} \right\} \quad (3.17)$$

The boundary conditions for $V_i(z)$ and $W_i(z)$ represent $2N$ coupled linear equations. For the normal displacements, $U_i(z)$, the equations are the same as those derived in Chapter 2:

$$-{}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}'''(z_0)_{N \times 1} + {}_7\mathbf{c}_{N \times N} \cdot \mathbf{U}'(z_0)_{N \times 1} = \pm_{z_0} \mathbf{T}_{N \times 1} \quad (3.18)$$

and

$${}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}''(z_0)_{N \times 1} + {}_3\mathbf{c}_{N \times N} \cdot \mathbf{U}(z_0)_{N \times 1} = \pm_{z_0} \mathbf{M}_{N \times 1} \quad (3.19)$$

In Table 3.3 and Equation 3.17 the positive sign applies to the $z_0 = L$ end while the negative sign applies to the $z_0 = 0$ end.

CHAPTER 4 - EQUILIBRIUM EQUATIONS FOR CONNECTED PLATES

The displacement fields for a combination of interconnected plates are developed using the results from the previous chapters. Continuity of displacement and slope are imposed along longitudinal plate joints. Two different methods are used to incorporate these constraints into the displacement field and boundary conditions equations.

4.1 Assumptions and Notation

All the assumptions of Chapter 2 or Chapter 3 (Tables 2.1) apply to this chapter. In general, sets of narrow plates and sets of deep plates are dealt with separately. In addition to the already stated assumptions, we will only deal with sets of interconnected plates that start and end at the same z -coordinate. The joint is located at the line of intersection of the plate midplanes. Continuity of slope and displacement are assumed across a joint. For simplicity, we will model the displacements in each plate using the same number of shape functions.

Left hand superscripts will be used to identify the plate to which are associated specific matrix and vector quantities (e.g., ${}^m U_i(z)$ and ${}^m c_{ij}$ are associated with the m^{th} plate). For scalar quantities, right hand subscripts will be used (e.g. h_m and b_m are thickness and width of the m^{th} plate, respectively). The only exceptions to this rule will be for the narrow plate displacements $V(z)$ and $W(z)$.

4.2 Continuity Equations at the Joints

Figure 4.1 shows a joint between two plates.

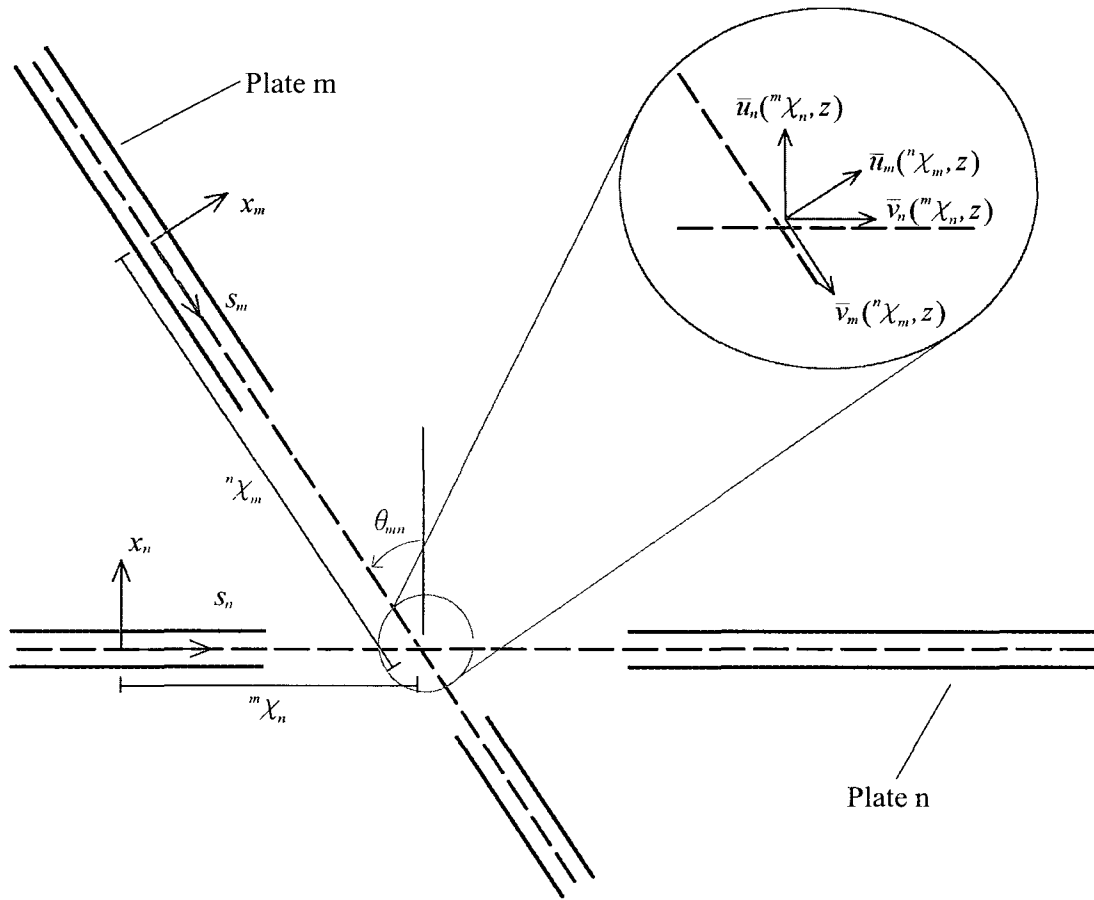


Fig. 4.1 - Joint between two plates (cross section)

The first plate will be referred to as plate m while the second plate will be identified as plate n . For both plates, the z -axis is oriented so that (x_m, s_m, z) and (x_n, s_n, z) form right hand coordinate systems. Since both plates start and finish at common edges in the longitudinal direction, $z_m = z_n = z$. The plates are joined at an angle θ_{mn} as shown. Each plate may continue past the joint. Plates m and n are joined at $s_m = ^n\chi_m$ and $s_n = ^m\chi_n$ respectively. The displacements and rotations in the plane of the cross-section for both plates must be identical at the joint. Projecting the normal displacements onto the local s -axes, the five continuity constraints can be written as

$$\alpha_{mn} \cdot \bar{v}_n(^m\chi_n, z) = \bar{u}_m(^n\chi_m, z) - \beta_{mn} \cdot \bar{u}_n(^m\chi_n, z) \quad (4.1a)$$

$$\alpha_{mn} \cdot \bar{v}_m(^n\chi_m, z) = \beta_{mn} \cdot \bar{u}_m(^n\chi_m, z) - \bar{u}_n(^m\chi_n, z) \quad (4.1b)$$

$$\bar{w}_m(^n\chi_m, z) = \bar{w}_n(^m\chi_n, z) \quad (4.1c)$$

$$\frac{\partial}{\partial s_m} \bar{u}_m(^n\chi_m, z) = \frac{\partial}{\partial s_n} \bar{u}_n(^m\chi_n, z) \quad (4.1d)$$

$$\frac{\partial}{\partial z_m} \bar{u}_m(^n\chi_m, z) = \frac{\partial}{\partial z_n} \bar{u}_n(^m\chi_n, z) \quad (4.1e)$$

where $\alpha_{mn} = \cos(\theta_{mn})$ and $\beta_{mn} = \sin(\theta_{mn})$. The last two equations relate the out of plane rotations of the plates in the z and s directions. It should be noted that Equation 4.1e is guaranteed if Equations 4.1a and 4.1b are met.

4.2.1 Continuity Equations for Narrow Plates

Using Equations 2.4 and 2.6, one can express Equations 4.1a and 4.1b in terms of the generalized displacement as

$$^nV(z) = \frac{1}{\alpha_{mn}} \sum_{i=1}^N {}^m\psi_i(^n\chi_m) \cdot {}^mU_i(z) - \frac{\beta_{mn}}{\alpha_{mn}} \sum_{i=1}^N {}^n\psi_i(^m\chi_n) \cdot {}^nU_i(z) \quad (4.2a)$$

$$^mV(z) = \frac{\beta_{mn}}{\alpha_{mn}} \sum_{i=1}^N {}^m\psi_i(^n\chi_m) \cdot {}^mU_i(z) - \frac{1}{\alpha_{mn}} \sum_{i=1}^N {}^n\psi_i(^m\chi_n) \cdot {}^nU_i(z) \quad (4.2b)$$

Also, from Equation 2.5 by substituting into Equation 4.1c, one has

$$-^n\chi_m \cdot {}^mV'(z) + {}^mW(z) = -^m\chi_n \cdot {}^nV'(z) + {}^nW(z) \quad (4.2c)$$

The remaining constraint at the joint (Equation 4.1d) can be expressed in terms of the generalized displacements, that is

$$\sum_{i=1}^N {}^m\psi_i(^n\chi_m) \cdot {}^mU_i(z) = \sum_{i=1}^N {}^n\psi_i(^m\chi_n) \cdot {}^nU_i(z) \quad (4.2d)$$

If the plates are pin connected Constraint 4.2d should be omitted.

4.2.2 Continuity Equations for Deep Plates

Using Equations 3.2, we can express Equations 4.1a and 4.1b in terms of the generalized displacement:

$$\alpha_{mn} \sum_{i=1}^N {}^n\phi_i({}^m\chi_n) \cdot {}^nV_i(z) = \sum_{i=1}^N {}^m\psi_i({}^n\chi_m) \cdot {}^mU_i(z) - \beta_{mn} \sum_{i=1}^N {}^n\psi_i({}^m\chi_n) \cdot {}^nU_i(z) \quad (4.3a)$$

$$\alpha_{mn} \sum_{i=1}^N {}^m\phi_i({}^n\chi_m) \cdot {}^mV_i(z) = \beta_{mn} \sum_{i=1}^N {}^m\psi_i({}^n\chi_m) \cdot {}^mU_i(z) - \sum_{i=1}^N {}^n\psi_i({}^m\chi_n) \cdot {}^nU_i(z) \quad (4.3b)$$

Also, for Equation 4.1c takes the form

$$\sum_{i=1}^N {}^m\phi_i({}^n\chi_m) \cdot {}^mW_i(z) = \sum_{i=1}^N {}^n\phi_i({}^m\chi_n) \cdot {}^nW_i(z) \quad (4.3c)$$

The final constraint at the joint is that the angle between the plates be maintained after deformation:

$$\sum_{i=1}^{N_m} {}^m\psi_i'({}^n\chi_m) \cdot {}^mU_i(z) = \sum_{i=1}^{N_n} {}^n\psi_i'({}^m\chi_n) \cdot {}^nU_i(z) \quad (4.3d)$$

If the plates are pin-connected, constraint 4.3d should be omitted.

4.3 Total Potential Energy

4.3.1 System of Narrow Plates

The strain energy of a system of plates is the sum of the strain energy of the individual plates.

Equation 2.13 can therefore be used to express the strain energy of a thin walled member

consisting of K narrow plates:

$$\begin{aligned}
\Pi_s = & \frac{1}{2} \sum_{k=1}^K \int_0^L \sum_{j=1}^N \sum_{i=1}^N \left({}^k c_{ij} \cdot {}^k U_i(z) \cdot {}^k U_j(z) + {}^k c_{ij} \cdot {}^k U_i'(z) \cdot {}^k U_j'(z) \right. \\
& \left. + 2 {}^k c_{ij} \cdot {}^k U_i(z) \cdot {}^k U_j''(z) + {}^k c_{ij} \cdot {}^k U_i''(z) \cdot {}^k U_j''(z) \right) dz \\
& + \frac{1}{2} \sum_{k=1}^K \int_0^L E_k h_k \left(b_k \cdot {}^k W'(z)^2 + \frac{b_k^3}{12} \cdot {}^k V''(z)^2 \right) dz
\end{aligned} \quad (4.4a)$$

Also, the work done by the external volumetric loads is

$$\begin{aligned}
\Pi_D = & \sum_{k=1}^K \int_0^L \left(\sum_{i=1}^N \left({}^k U_i(z) \cdot {}^k t_i(z) + {}^k U_i'(z) \cdot {}^k m_i(z) \right) + {}^k V(z) \cdot {}^k p_s(z) \right. \\
& \left. + {}^k V'(z) \cdot {}^k m_s(z) + {}^k W(z) \cdot {}^k p_z(z) \right) dz
\end{aligned} \quad (4.4b)$$

and that of the end loads is

$$\begin{aligned}
\Pi_B = & \sum_{k=1}^K \sum_{i=1}^N \left({}^k T_i \cdot {}^k U_i(z) \Big|_{z=0} + {}^k M_i \cdot {}^k U_i'(z) \Big|_{z=0} + {}^k T_i \cdot {}^k U_i(z) \Big|_{z=L} + {}^k M_i \cdot {}^k U_i'(z) \Big|_{z=L} \right) \\
& + \sum_{k=1}^K \left({}^k M_s \cdot {}^k V'(z) \Big|_{z=0} + {}^k M_s \cdot {}^k V'(z) \Big|_{z=L} + {}^k P_s \cdot {}^k V(z) \Big|_{z=0} + {}^k P_s \cdot {}^k V(z) \Big|_{z=L} \right) \\
& + \sum_{k=1}^K \left({}^k P_z \cdot {}^k W(z) \Big|_{z=L} + {}^k P_z \cdot {}^k W(z) \Big|_{z=0} \right)
\end{aligned} \quad (4.4c)$$

4.3.2 Potential Energy of a System of Deep Plates

Equation 3.7 can be used to express the strain energy of any set of interconnected thin plates. For K plates, that would be

$$\begin{aligned}
\Pi_s = & \frac{1}{2} \int_0^L \sum_{j=1}^N \sum_{i=1}^N \left({}^k a_{ij} \cdot {}^k W_i'(z) \cdot {}^k W_j'(z) + 2 {}^k a_{ij} \cdot {}^k W_i'(z) \cdot {}^k V_j(z) + {}^k a_{ij} \cdot {}^k V_i(z) \cdot {}^k V_j(z) \right. \\
& + \sum_{j=1}^N \sum_{i=1}^N \left({}^k a_{ij} \cdot {}^k V_i'(z) \cdot {}^k V_j'(z) + {}^k a_{ij} \cdot {}^k W_i(z) \cdot {}^k W_j(z) + {}^k a_{ij} \cdot {}^k W_i(z) \cdot {}^k V_j'(z) + {}^k a_{ij} \cdot {}^k U_i(z) \cdot {}^k U_j(z) \right. \\
& \left. \left. + \sum_{j=1}^N \sum_{i=1}^N \left({}^k c_{ij} \cdot {}^k U_i'(z) \cdot {}^k U_j'(z) + 2 {}^k c_{ij} \cdot {}^k U_i''(z) \cdot {}^k U_j(z) + {}^k c_{ij} \cdot {}^k U_i''(z) \cdot {}^k U_j''(z) \right) \right) dz
\end{aligned} \quad (4.5a)$$

Also, the work done by the external loads on all of the plates is

$$\Pi_D = \int_0^L \sum_{k=1}^K \sum_{i=1}^N \left({}^k V_i(z) {}^k P_i(z) + {}^k W_i(z) {}^k q_i(z) + {}^k U_i(z) {}^k t_i(z) + {}^k U_i'(z) {}^k m_i(z) \right) dz \quad (4.5b)$$

for the volumetric loads and

$$\begin{aligned} \Pi_B = & \sum_{k=1}^K \sum_{i=1}^N \left({}^k T_i \cdot {}^k U_i(z) \Big|_{z=0} + {}^k M_i \cdot {}^k U_i'(z) \Big|_{z=0} + {}^k P_i \cdot {}^k V_i(z) \Big|_{z=0} + {}^k Q_i \cdot {}^k W_i(z) \Big|_{z=0} \right) \\ & + \sum_{k=1}^K \sum_{i=1}^N \left({}^k T_i \cdot {}^k U_i(z) \Big|_{z=L} + {}^k M_i \cdot {}^k U_i'(z) \Big|_{z=L} + {}^k P_i \cdot {}^k V_i(z) \Big|_{z=L} + {}^k Q_i \cdot {}^k W_i(z) \Big|_{z=L} \right) \end{aligned} \quad (4.5c)$$

for the end loads.

4.4 Derivation of Equilibrium Equations using a Reduced Form of the Potential Energy

This section briefly explains how to derive the equilibrium equations for a system of interconnected plates by expressing the potential energy of the system in terms of the non-redundant generalized displacements. We will start with a treatment for sets of deep plates since the implementation of the continuity constraints is simpler than that for sets of narrow plates.

4.4.1 Assembly of Deep Plates

4.4.1.1 Reduction of the Internal Strain Energy

Because there are four continuity constraints at the joint (Equations 4.3), four of the generalized displacements used to express the potential energy (Equations 4.5) can be expressed as a linear combination of the others. For each additional plate connected at the joint, four additional generalized displacements can be expressed as a linear combination of the others. Generalized displacements that are eliminated from the expression for potential energy using the constraint equations will be referred to as *redundant*. For each of the continuity Equations 4.3, one of the functions (${}^k U_i(z)$, ${}^k V_i(z)$ or ${}^k W_i(z)$) becomes redundant. In the following we will present a

where

$$\mathbf{d}_{1NK \times 3NK} = \left[\begin{array}{ccc|ccc} \mathbf{1}_1 \mathbf{c}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & & & \\ \mathbf{0}_{N \times N} & \mathbf{1}_3 \mathbf{a}_{N \times N} & \mathbf{0}_{N \times N} & & & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{1}_0 \mathbf{a}_{N \times N} & \ddots & & \\ \hline & & & \ddots & & \\ & & & & \mathbf{1}_K \mathbf{c}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} \\ & & & & \mathbf{0}_{N \times N} & \mathbf{1}_3 \mathbf{a}_{N \times N} & \mathbf{0}_{N \times N} \\ & & & & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{1}_0 \mathbf{a}_{N \times N} \end{array} \right] \quad (4.6d)$$

$${}_2\mathbf{d}_{3NK \times 3NK} = \begin{bmatrix} {}^1\mathbf{c}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \end{bmatrix} \quad (4.6d)$$

$${}_3\mathbf{d}_{3NK \times 3NK} = \begin{bmatrix} \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & {}^1\mathbf{a}_{N \times N}^T & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & {}^1\mathbf{a}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \end{bmatrix} \quad (4.6e)$$

$${}_4\mathbf{d}_{3NK \times 3NK} = \begin{bmatrix} {}^1\mathbf{c}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \vdots & \vdots & \mathbf{0} \end{bmatrix} \quad (4.6f)$$

The set of all continuity equations at the joints may be written in matrix form as

$$\xi_{(3NK-\tilde{N}) \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1} = \begin{bmatrix} {}_1\xi_{(3NK-N) \times N} & | & {}_2\xi_{(3NK-\tilde{N}) \times (3NK-\tilde{N})} \end{bmatrix} \cdot \mathbf{A}(z)_{3NK \times 1} = \mathbf{0}_{(3NK-\tilde{N}) \times 1} \quad (4.7)$$

where $\xi_{(3NK-\tilde{N}) \times 3NK}$ contains the coefficients given by enforcing Equations 4.3 at each joints, and matrices ${}_1\xi_{(3NK-N) \times \tilde{N}}$ and ${}_2\xi_{(3NK-\tilde{N}) \times (3NK-\tilde{N})}$ respectively represent the coefficients associated with the non-redundant and redundant degrees of freedom. The strain energy can then be expressed

only in terms of the non-redundant generalized displacements (Appendix C). The reduced expression for the strain energy will be

$$\Pi_S = \frac{1}{2} \int_0^L \left(\tilde{\mathbf{A}}(z)_{1 \times \tilde{N}}^T \cdot {}_0 \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} + \tilde{\mathbf{A}}^I(z)_{1 \times \tilde{N}}^T \cdot {}_1 \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}^I(z)_{\tilde{N} \times 1} + \tilde{\mathbf{A}}^{II}(z)_{1 \times \tilde{N}}^T \cdot {}_2 \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}^{II}(z)_{\tilde{N} \times 1} \right. \\ \left. + 2\tilde{\mathbf{A}}^I(z)_{1 \times \tilde{N}}^T \cdot {}_3 \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} + 2\tilde{\mathbf{A}}^{II}(z)_{1 \times \tilde{N}}^T \cdot {}_4 \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} \right) dz \quad (4.8a)$$

where $\tilde{\mathbf{A}}(z)$ is a vector containing only the non-redundant degrees of freedom. The ${}_\ell \tilde{\mathbf{d}}$ matrices are the new system matrices given in Appendix C

$${}_\ell \tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} = {}_1 {}_\ell \mathbf{d}_{\tilde{N} \times N} + {}_2 {}_\ell \mathbf{d}_{\tilde{N} \times (3NK-N)} \cdot {}_3 \xi_{(3NK-\tilde{N}) \times \tilde{N}} + {}_3 \xi_{\tilde{N} \times (3NK-\tilde{N})}^T \cdot {}_3 {}_\ell \mathbf{d}_{(3NK-\tilde{N}) \times \tilde{N}} \\ + {}_3 \xi_{\tilde{N} \times (3NK-\tilde{N})}^T \cdot {}_4 {}_\ell \mathbf{d}_{(3NK-N) \times (3NK-\tilde{N})} \cdot {}_3 \xi_{(3NK-\tilde{N}) \times \tilde{N}} \quad (4.8b)$$

where

$$\left[\begin{array}{c|c} {}_1 {}_\ell \mathbf{d}_{\tilde{N} \times \tilde{N}} & {}_2 {}_\ell \mathbf{d}_{\tilde{N} \times (3NK-N)} \\ \hline {}_3 {}_\ell \mathbf{d}_{(3NK-\tilde{N}) \times \tilde{N}} & {}_4 {}_\ell \mathbf{d}_{(3NK-N) \times (3NK-\tilde{N})} \end{array} \right] = {}_\ell \mathbf{d}_{3NK \times 3NK} \quad (4.8c)$$

and

$${}_3 \xi_{(3NK-\tilde{N}) \times \tilde{N}} = -{}_2 \xi_{(3NK-N) \times (3NK-N)}^{-1} \cdot {}_1 \xi_{(3NK-N) \times N} \quad (4.8d)$$

4.4.1.2 Reduction of Load Potential

The expression for the work done by the external loads can be reduced in the same way as the strain energy. Equation 4.5b can be rewritten as

$$\Pi_D = \int_0^L \left({}_0 \mathbf{f}(z)_{1 \times 3NK}^T \cdot \mathbf{A}(z)_{3NK \times 1} + {}_1 \mathbf{f}(z)_{1 \times 3NK}^T \cdot \mathbf{A}^I(z)_{3NK \times 1} \right) dz \quad (4.9a)$$

with

$${}_0 \mathbf{f}(z)_{1 \times 3NK}^T = \left\langle {}^1 \mathbf{t}(z)_{1 \times N}^T \mid {}^1 \mathbf{p}(z)_{1 \times N}^T \mid {}^1 \mathbf{q}(z)_{1 \times N}^T \mid \dots \mid {}^K \mathbf{t}(z)_{1 \times N}^T \mid {}^K \mathbf{p}(z)_{1 \times N}^T \mid {}^K \mathbf{q}(z)_{1 \times N}^T \right\rangle \quad (4.9b)$$

$${}_1 \mathbf{f}(z)_{1 \times 3NK}^T = \left\langle {}^1 \mathbf{m}(z)_{1 \times N}^T \mid \mathbf{0}_{1 \times N}^T \mid \mathbf{0}_{1 \times N}^T \mid \dots \mid {}^K \mathbf{m}(z)_{1 \times N}^T \mid \mathbf{0}_{1 \times N}^T \mid \mathbf{0}_{1 \times N}^T \right\rangle \quad (4.9c)$$

If we eliminate $A_m(z)$ from Equation 4.9a using Equation 4.7, we have

$$\Pi_D = \sum_{\substack{i=1 \\ i \neq m}}^{3NK} \int_0^L \left(({}_0f_i(z) + \xi_{i-0} f_m(z)) A_i(z) + ({}_1f_i(z) + \xi_{i-1} f_m(z)) A_i'(z) \right) dz \quad (4.10)$$

Applying 4.10 for all the continuity constraints will yield an expression with the non-redundant generalized displacements only, that is

$$\Pi_D = \int_0^L \left({}_0\tilde{f}(z)_{1 \times N}^T \tilde{A}(z)_{N \times 1} + {}_1\tilde{f}(z)_{1 \times N}^T \tilde{A}'(z)_{N \times 1} \right) dz \quad (4.11)$$

where ${}_0\tilde{f}(z)$ and ${}_1\tilde{f}(z)$ are the new generalized distributed loads. The same procedure can be used to reduce the work from the boundary loads such that its final expression has the form

$$\Pi_B = {}_0\tilde{F}_{1 \times N}^T \tilde{A}(z)_{N \times 1} \Big|_{=0} + {}_0\tilde{G}_{1 \times N}^T \tilde{A}'(z)_{N \times 1} \Big|_{=0} + {}_L\tilde{F}_{1 \times N}^T \tilde{A}(z)_{N \times 1} \Big|_{=L} + {}_L\tilde{G}_{1 \times N}^T \tilde{A}'(z)_{N \times 1} \Big|_{=L} \quad (4.12)$$

where ${}_0\tilde{F}_{N \times 1}$, ${}_0\tilde{G}_{N \times 1}$, ${}_L\tilde{F}_{N \times 1}$, and ${}_L\tilde{G}_{N \times 1}$ are the new boundary load vectors

4.4.1.3 Equilibrium Equations

Equilibrium field equations and boundary conditions can be obtained by enforcing the stationarity condition of the total potential energy with respect to the non-redundant degrees of freedom. Details of the procedure, including performing integration by parts and grouping similar terms, are outlined in Appendix C. The resulting equilibrium field equations (Equation C.15a) take the form

$$\begin{aligned} {}_2\tilde{d}_{N \times N} \tilde{A}^{IV}(z)_{N \times 1} + {}_5\tilde{d}_{N \times N} \tilde{A}^{II}(z)_{N \times 1} + {}_6\tilde{d}_{N \times N} \tilde{A}'(z)_{N \times 1} + {}_0\tilde{d}_{N \times N} \tilde{A}(z)_{N \times 1} \\ = {}_0\tilde{f}(z)_{N \times 1} - {}_1\tilde{f}'(z)_{N \times 1} \end{aligned} \quad (4.13a)$$

with

$${}_5\tilde{\mathbf{d}}_{N \times N} = {}_4\tilde{\mathbf{d}}_{N \times N} + {}_4\tilde{\mathbf{d}}_{N \times N}^T - {}_1\tilde{\mathbf{d}}_{N \times N} \quad (4.13b)$$

$${}_6\tilde{\mathbf{d}}_{N \times N} = {}_3\tilde{\mathbf{d}}_{N \times N} + {}_3\tilde{\mathbf{d}}_{N \times N}^T \quad (4.13c)$$

The set of boundary condition equations related to the application of boundary loads is

(Equations C.15b-c)

$${}_2\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}''(z_0)_{N \times 1} + {}_4\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}(z_0)_{N \times 1} = \pm {}_{z_0}\tilde{\mathbf{G}}_{N \times 1} \quad (4.14a)$$

and

$$-{}_2\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}'''(z_0)_{N \times 1} + {}_7\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}'(z_0)_{N \times 1} + {}_3\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}(z_0)_{N \times 1} = \pm {}_{z_0}\tilde{\mathbf{F}}_{N \times 1} \quad (4.14b)$$

with

$${}_7\tilde{\mathbf{d}}_{N \times N} = {}_1\tilde{\mathbf{d}}_{N \times N} - {}_4\tilde{\mathbf{d}}_{N \times N} \quad (4.14c)$$

The plus sign is associated with $z_0 = L$ and the minus sign with $z_0 = 0$. Equations 4.14a and 4.14b can be divided so that different boundary conditions (fixity or load) can be applied to different plates and in different directions. Solving Equations 4.13 through 4.14 will only yield the non-redundant generalized displacements. The redundant displacement must then be calculated using Equations 4.3.

4.4.2 Assembly of Narrow Plates

4.4.2.1 Reduction of the Potential Energy

Although the narrow plate formulation uses simpler displacement fields, it results into a more complex formulation when combining multiple plates. This is because of the constraint associated with the longitudinal displacement (Equation 4.2c) which contains both derivatives and non-derivatives of the displacement functions. This contrasts with the equations for deep plates (Equations 4.3) which contain no derivatives. There are several methods of eliminating the redundant displacement fields from the potential energy expression. Focus here will be on the

method used to solve the sample problems in Chapter 7. The potential energy can be reduced using the following steps:

- All but one of the longitudinal displacement functions (${}^k W(z)$) are eliminated using constraints 4.2c.
- All the transverse displacement functions (${}^k V(z)$) are eliminated using constraints 4.2a and 4.2b.
- The remaining continuity constraints (Equations 4.2d and part of 4.2a and 4.2b) are used to eliminate redundant normal displacement functions (${}^k U(z)$).

Regardless of which generalized displacements are retained, the strain energy will have the form

$$\begin{aligned} \Pi_s = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} \left({}_2\tilde{c}_{ij} \tilde{U}_i(z) \tilde{U}_j(z) + {}_1\tilde{c}_{ij} \tilde{U}_i'(z) \tilde{U}_j'(z) + {}_3\tilde{c}_{ij} \tilde{U}_i(z) \tilde{U}_j''(z) \right. \\ \left. + {}_0\tilde{c}_{ij} \tilde{U}_i''(z) \tilde{U}_j''(z) \right) dz + \int_0^L \sum_{i=1}^{\tilde{N}} d_i \cdot \tilde{U}_i''(z) \cdot \tilde{W}'(z) dz + \int_0^L d_w \cdot \left(\tilde{W}'(z) \right)^2 dz \end{aligned} \quad (4.15)$$

where d_i , d_w , and ${}_i\tilde{c}_{ij}$ are constants stemming from the reduction, $\tilde{W}(z)$ is the generalized

longitudinal displacement of one of the plates and $\tilde{\mathbf{U}}_{\tilde{N} \times \tilde{N}}(z)$ is the set of non-redundant

generalized normal displacements for all the plates. The d_w term is the total longitudinal stiffness of the section. Similarly the expression for the load potential will be

$$\Pi_D = \int_0^L \left(\sum_{i=1}^{\tilde{N}} \left(\tilde{t}_i(z) \cdot \tilde{U}_i(z) + \tilde{m}_i(z) \cdot \tilde{U}_i'(z) \right) + p_w(z) \cdot \tilde{W}(z) \right) dz \quad (4.16a)$$

$$\Pi_B = \sum_{\substack{z_0=0 \\ z_0=L}} \sum_{i=1}^{\tilde{N}} \left({}_{z_0}\tilde{T}_i \cdot \tilde{U}_i(z) \Big|_{z=z_0} + {}_{z_0}\tilde{M}_i \cdot \tilde{U}_i'(z) \Big|_{z=z_0} \right) + \sum_{\substack{z_0=0 \\ z_0=L}} {}_{z_0}P_w \cdot \tilde{W}(z) \Big|_{z=z_0} \quad (4.16b)$$

where $\tilde{t}_i(z)$, $\tilde{m}_i(z)$, ${}_{z_0}\tilde{T}_i$, and ${}_{z_0}\tilde{M}_i$ are the generalized loads for the non-redundant degrees of

freedom. They depend on the loads applied to each of the plates as well as the geometry of the

section. The load $p_w(z)$ is the total distributed longitudinal load while ${}_0P_w$ are the total longitudinal loads at the boundaries.

A practical closed form solution for d_i , d_w , ${}_r\tilde{c}_{ij}$, $\tilde{t}_i(z)$, $\tilde{m}_i(z)$, ${}_0\tilde{T}_i$, and ${}_0\tilde{M}_i$ for a given assembly of plates and set of shape functions could not be found by the author. Closed form solutions for the specific geometries and shape functions used in Chapter 7 are provided in Appendix E.

4.4.2.2 Equilibrium Equations

Equilibrium field equations and boundary conditions can be obtained by imposing the stationarity condition of the total potential energy with respect to the non-redundant degrees of freedom. Details of the procedure, including performing integration by parts and grouping similar terms, are outlined in Appendix D. The resulting field equation associated with the variation of $\tilde{W}(z)$ is

$$d_w \tilde{W}''(z) + \sum_{j=1}^{\tilde{N}} d_j \cdot \tilde{U}_j'''(z) = \frac{p_w(z)}{2} \quad (4.17)$$

while the field equations associated with the variation of $\tilde{U}_k(z)$ are

$$\sum_{j=1}^N \left({}_0\tilde{c}_{kj} \tilde{U}_j^{IV}(z) + {}_r\tilde{c}_{kj} \tilde{U}_j''(z) + {}_2\tilde{c}_{kj} \tilde{U}_j(z) \right) + d_k \cdot \tilde{W}'''(z) = \tilde{t}_k(z) - \tilde{m}_k'(z) \quad (4.18)$$

The natural boundary conditions associated with $\tilde{W}(z)$ at $z = 0$ and $z = L$ are

$$2d_w \tilde{W}'(0) + \sum_{i=1}^{\tilde{N}} d_i \tilde{U}_i''(0) = -{}_0P_w \quad (4.19a)$$

$$2d_w \tilde{W}'(L) + \sum_{i=1}^N d_i \tilde{U}_i''(L) = {}_LP_w \quad (4.19b)$$

The boundary condition equations associated with $\tilde{U}_k(z)$ in the absence of fixity are

$$\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j''(z_0) + {}_3\tilde{c}_{kj} \cdot \tilde{U}_j'(z_0) \right) - d_k \tilde{W}'(z_0) = \pm_{z_0} \tilde{M}_k \quad (4.20a)$$

$$\sum_{j=1}^{\tilde{N}} \left(-{}_0\tilde{c}_{kj} \cdot \tilde{U}_j'''(z_0) + {}_7\tilde{c}_{kj} \tilde{U}_j'(z_0) \right) - d_k \tilde{W}''(z_0) = \pm_{z_0} \tilde{P}_k \quad (4.20b)$$

where the positive sign is associates with $z_0 = L$ and the negative sign with $z_0 = 0$. Equation 4.17 through 4.20 can also be expressed in matrix form. Solving them will yield the non-redundant generalized displacements. The redundant displacements must be calculated using Equations 4.2. It should be noted that if Equation 4.17 is used to express $\tilde{W}(z)$ then 4.18 and 4.20 can be written in terms of the normal displacements $\tilde{U}_j(z)$ only. The resulting system will have the same form as Equation 2.13 (see appendix D).

4.5 Formulating Equilibrium Equations using Lagrange Multipliers

The method of Lagrange multipliers provides a strategy for finding the maximum or minimum of a functional subjected to constraints. In our case we wish to minimize the potential energy subject to the constraints at each of the joints. In this section we will derive the equilibrium equations for a system of narrow plates using Lagrange multipliers. The equilibrium equations for deep plates will not be provided though they can be inferred from the ones for narrow plates. It is required to minimize functional $\Pi = \Pi_s - \Pi_D - \Pi_B$ (as given by Equations 4.4) subject to the constraints

$$\sum_{i=1}^N {}^m\psi_i({}^n\chi_m) \cdot {}^mU_i(z) - \beta_{mn} \sum_{i=1}^N {}^n\psi_i({}^m\chi_n) \cdot {}^nU_i(z) - \alpha_{mn} {}^mV(z) = 0 \quad (4.21a)$$

$$\beta_{mn} \sum_{i=1}^{N_m} {}^m\psi_i({}^n\chi_m) {}^mU_i(z) - \sum_{i=1}^{N_n} {}^n\psi_i({}^m\chi_n) {}^nU_i(z) - \alpha_{mn} {}^mV(z) = 0 \quad (4.21b)$$

$$-{}^m\chi_n {}^nV'(z) + {}^nW(z) + {}^m\chi_m {}^mV'(z) - {}^mW(z) = 0 \quad (4.21c)$$

$$\sum_{i=1}^{N_m} {}^m\psi_i ({}^n\chi_m) {}^mU_i(z) - \sum_{i=1}^{N_n} {}^n\psi_i ({}^m\chi_n) {}^nU_i(z) = 0 \quad (4.21d)$$

for each joint between plates m and n . This is achieved by defining a the functional, Γ , given by

$$\Gamma = \Pi + \int_0^L \sum_{\text{all joints}} ({}_1f_{mn}(z) {}_1g_{mn}(z) + {}_2f_{mn}(z) {}_2g_{mn}(z) + {}_3f_{mn}(z) {}_3g_{mn}(z) + {}_4f_{mn}(z) {}_4g_{mn}(z)) dz \quad (4.22a)$$

in which

$${}_1g_{mn}(z) = \sum_{i=1}^N {}^m\psi_i ({}^n\chi_m) \cdot {}^mU_i(z) - \beta_{mn} \sum_{i=1}^N {}^n\psi_i ({}^m\chi_n) \cdot {}^nU_i(z) - \alpha_{mn} {}^nV(z) \quad (4.22b)$$

$${}_2g_{mn}(z) = \beta_{mn} \sum_{i=1}^{N_m} {}^m\psi_i ({}^n\chi_m) {}^mU_i(z) - \sum_{i=1}^{N_n} {}^n\psi_i ({}^m\chi_n) {}^nU_i(z) - \alpha_{mn} {}^mV(z) \quad (4.22c)$$

$${}_3g_{mn}(z) = -{}^m\chi_n {}^nV'(z) + {}^nW(z) + {}^m\chi_m {}^mV'(z) - {}^mW(z) \quad (4.22d)$$

$${}_4g_{mn}(z) = \sum_{i=1}^{N_m} {}^m\psi_i ({}^n\chi_m) {}^mU_i(z) - \sum_{i=1}^{N_n} {}^n\psi_i ({}^m\chi_n) {}^nU_i(z) \quad (4.22e)$$

and ${}_1f_{mn}(z)$, ${}_2f_{mn}(z)$, ${}_3f_{mn}(z)$, ${}_4f_{mn}(z)$ are the Lagrange functions. The functional Γ must be minimized with respect to each of the generalized displacements (${}^kU_i(z)$, ${}^kV(z)$, and ${}^kW(z)$), their derivatives, and with respect to functions ${}_1f_{mn}(z)$, ${}_2f_{mn}(z)$, ${}_3f_{mn}(z)$, and ${}_4f_{mn}(z)$. The variation of the Lagrange function is equal to the variation of the potential energy plus the variation of the terms associated with the joint constraints, that is

$$\delta\Gamma = \delta\Pi + \sum_{\text{all joints}} \left(\delta \int_0^L {}_1f_{mn}(z) {}_1g_{mn}(z) dz + \delta \int_0^L {}_2f_{mn}(z) {}_2g_{mn}(z) dz \right. \\ \left. + \delta \int_0^L {}_3f_{mn}(z) {}_3g_{mn}(z) dz + \delta \int_0^L {}_4f_{mn}(z) {}_4g_{mn}(z) dz \right) \quad (4.23)$$

By studying the variation of the new terms associated with the Lagrange functions, we will be able to determine how they influence the equilibrium equations.

4.5.1 Variation of the Potential Energy

We recall that the variation of the potential with respect to all the generalized displacements for all the plates is

$$\begin{aligned} \delta\Pi = \sum_{k=1}^K \left(\sum_{i=1}^N \left(\frac{\partial\Pi}{\partial^k U_i} \delta^k U_i + \frac{\partial\Pi}{\partial^k U_i'} \delta^k U_i' + \frac{\partial\Pi}{\partial^k U_i''} \delta^k U_i'' \right) + \frac{\partial\Pi}{\partial^k W} \delta^k W + \frac{\partial\Pi}{\partial^k W'} \delta^k W' \right. \\ \left. + \frac{\partial\Pi}{\partial^k V} \delta^k V + \frac{\partial\Pi}{\partial^k V'} \delta^k V' + \frac{\partial\Pi}{\partial^k V''} \delta^k V'' \right) = 0 \end{aligned} \quad (4.24)$$

where Π is given by Equations 4.4. Minimizing the potential energy alone would lead to a set of equations identical to those given in Section 2.4 for each of the plates. Solving such a set of equations would yield the solution for a set of disconnected plates subjected to their own loads and boundary conditions. Imposing stationarity of the Lagrange function instead of the potential energy will couple the set of equations and take into account the connectivity of the plates.

4.5.2 Variation of the First Lagrange Term

Taking the variation of $\int_0^L f_{mn}(z) \cdot {}_1g_{mn}(z) dz$ with respect to the generalized displacements and the Lagrange multipliers we have

$$\begin{aligned} \delta \left(\int_0^L f_{mn}(z) \cdot {}_1g_{mn}(z) dz \right) = \int_0^L {}_1g_{mn}(z) \cdot \delta {}_1f_{mn} dz - \int_0^L f_{mn}(z) \cdot \delta {}^nV \cdot dz \\ + \int_0^L \sum_{i=1}^{N_m} f_{mn}(z) \cdot \frac{{}^m\psi_i({}^n\chi_m)}{\alpha_{mn}} \delta {}^mU_i \cdot dz \\ - \int_0^L \sum_{i=1}^{N_n} f_{mn}(z) \cdot \frac{\beta_{nm}}{\alpha_{mn}} {}^n\psi_i({}^m\chi_n) \cdot \delta {}^nU_i \cdot dz \end{aligned} \quad (4.25)$$

The first term indicates that imposing stationarity with respect to ${}_1f_{mn}(z)$ reproduces the original constraint (Equation 4.2a). The other lines of the equation give the terms that must be added to

the functional $\delta\Pi$. It should be noted that there are no boundary terms resulting from Equations 4.25, because ${}_1g_{mn}(z)$ contains only undifferentiated generalized displacements. Therefore constraint 4.2a does not influence the boundary condition equations.

4.5.3 Variation of the Second Lagrange Term

Taking the variation of $\int_0^L {}_2f_{mn}(z) \cdot {}_2g_{mn}(z) dz$ yields

$$\begin{aligned} \delta \int_0^L ({}_2f_{mn}(z) \cdot {}_2g_{mn}(z)) dz &= \int_0^L {}_2g_{mn}(z) \cdot \delta {}_2f_{mn} \cdot dz - \int_0^L {}_2f_{mn}(z) \cdot \delta {}^mV \cdot dz \\ &+ \int_0^L \sum_{i=1}^{N_m} {}_2f(z) \cdot \frac{\beta_{mn}}{\alpha_{mn}} {}^m\psi_i({}^n\chi_m) \cdot \delta {}^mU_i \cdot dz \\ &- \int_0^L \sum_{i=1}^{N_n} {}_2f_{mn}(z) \cdot \frac{1}{\alpha} {}^n\psi_i({}^m\chi_n) \cdot \delta {}^nU_i \cdot dz \end{aligned} \quad (4.26)$$

Again, the equation associated with $\delta {}_2f_{mn}$ will reproduce the original constraint. The other term must be added to the other field equations.

4.5.4 Variation of Third Lagrange Term

Taking the variation of $\int_0^L {}_3f_{mn}(z) \cdot {}_3g_{mn}(z) dz$ with respect to the generalized displacements and

the Lagrange multipliers we have

$$\begin{aligned} \delta \int_0^L ({}_3f_{mn}(z) \cdot {}_3g_{mn}(z)) dz &= \int_0^L {}_3g_{mn}(z) \cdot \delta {}_3f_{mn} \cdot dz \\ &- \int_0^L {}_3f_{mn}{}^l(z) \cdot \delta {}^mW \cdot dz + \int_0^L {}_3f_{mn}{}^l(z) \cdot \delta {}^nW \cdot dz \\ &+ \left[{}^n\chi_m \cdot {}_3f_{mn}(z) \cdot \delta {}^mV \right]_0^L - \int_0^L {}^n\chi_m \cdot {}_3f_{mn}{}^l(z) \cdot \delta {}^mV \cdot dz \\ &- \left[{}^m\chi_n \cdot {}_3f_{mn}(z) \cdot \delta {}^nV \right]_0^L + \int_0^L {}^m\chi_n \cdot {}_3f_{mn}{}^l(z) \cdot \delta {}^nV \cdot dz \end{aligned} \quad (4.27)$$

Equation 4.27 shows that, unlike the other constraints, imposing continuity of displacement in the longitudinal direction will modify the boundary condition equations at $z = 0$ and $z = L$. Only the boundary condition equations for ${}^mV(z)$ and ${}^nV(z)$ will be modified. The field equations for ${}^mV(z)$, ${}^nV(z)$, ${}^mW(z)$, and ${}^nW(z)$ will also be affected.

4.5.5 Variation of Fourth Lagrange Term

Taking the variation of $\int_0^L {}_4f_{mn}(z) \cdot {}_4g_{mn}(z) dz$ with respect to the generalized displacements and the Lagrange multipliers we have

$$\begin{aligned} \delta \int_0^L ({}_4f_{mn}(z) \cdot {}_4g_{mn}(z)) dz &= \int_0^L {}_4g_{mn}(z) \cdot \delta {}_4f_{mn} \cdot dz \\ &+ \int_0^L \sum_{i=1}^{N_m} {}^m\psi_i({}^n\chi_m) \cdot {}_4f_{mn}(z) \cdot \delta {}^mU_i \cdot dz \\ &- \int_0^L \sum_{i=1}^{N_n} {}^n\psi_i({}^m\chi_n) \cdot {}_4f_{mn}(z) \cdot \delta {}^nU_i \cdot dz \end{aligned} \quad (4.28)$$

The equation associated with $\delta {}_4f_{mn}$ reproduces the original constraint. Equation 4.28 is similar in form to Equations 4.25 and 4.26. The coefficients in Equation 4.27 are related to the derivatives of the shape functions, and Equation 4.28 does not contain the transverse displacements. Constraint 4.2d will not influence the boundary condition equations.

3.5.6 Equilibrium Equations

In this formulation, it is necessary to solve for all generalized displacements and the Lagrange multipliers simultaneously. The resulting set of coupled differential equations will have fourth order terms, second order terms, first order terms, and non-differential terms. The coefficients

can be determined by adding the terms associated with the variation of the Lagrange multipliers (Section 4.5.2 through 5.5.5) to the set of uncoupled equations for individual plates (Equation 2.13). The boundary condition equations for the generalized axial displacements of plate k are

$$-E_k h_k b_k \cdot {}^k W'(0) - {}_0 P_z = 0 \quad (4.30a)$$

$$E_k h_k b_k \cdot {}^k W'(L) - {}_L P_z = 0 \quad (4.30b)$$

for $z = 0$ and $z = L$, respectively. The boundary condition equations for the transverse displacements are

$$E_k h_k \frac{b_k^3}{12} {}^k V'''(0) \pm \sum_{\substack{\text{all joints} \\ k,n}} ({}^n \chi_k \cdot {}^{k,n} f(0)) - {}_0 P_s = 0 \quad (4.31a)$$

$$-E_k h_k \frac{b_k^3}{12} {}^k V''(0) - {}_0 M_s = 0 \quad (4.31b)$$

and

$$-E_k h_k \frac{b_k^3}{12} {}^k V'''(L) \mp \sum_{\substack{\text{all joints} \\ k,n}} ({}^n \chi_k \cdot {}^{k,n} f(L)) - {}_L P_s = 0 \quad (4.31c)$$

$$E_k h_k \frac{b_k^3}{12} {}^k V''(L) - {}_L M_s = 0 \quad (4.31d)$$

Care should be taken when choosing the sign of the ${}^{k,n} f$ terms. If a positive value is chosen for the boundary condition equation for plate k then a negative value must be chosen for plate n , and this is valid for every joint. The boundary condition equations for the generalized normal

displacements for plate k are

$$-{}_0 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}'''(0)_{N \times 1} + {}_7 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}'(0)_{N \times 1} = -{}_0 \mathbf{T}_{N \times 1} \quad (4.32a)$$

$${}_0 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}''(0)_{N \times 1} + {}_3 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}(0)_{N \times 1} = -{}_0 \mathbf{M}_{N \times 1} \quad (4.32b)$$

and

$$-{}_0 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}'''(L)_{N \times 1} + {}_7 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}'(L)_{N \times 1} = {}_L \mathbf{T}_{N \times 1} \quad (4.32c)$$

$${}_0 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}''(L)_{N \times 1} + {}_3 \mathbf{c}_{N \times N} \cdot {}^k \mathbf{U}(L)_{N \times 1} = {}_L \mathbf{M}_{N \times 1} \quad (4.32d)$$

for $z = 0$ and $z = L$ respectively.

CHAPTER 5 - CLOSED FORM SOLUTION TO THE EQUILIBRIUM EQUATIONS

The purpose of this chapter is to present methods of solving the system of differential equations derived in Chapters 2, 3, and 4. The methods discussed will include solutions both to the field equations and the boundary condition equations. Each of the field equations presented in the previous chapters have the same form: that of a coupled system of ordinary linear differential equations (SOLDE) with constant coefficients. Each of the systems is of order 4. Nonetheless, we will present here the solution for a system of order M since this adds no complication to the solution method.

5.1 System Linearization

Consider an M^{th} order SOLDE in N unknown functions

$$\sum_{m=0}^{M-1} {}_m\mathbf{b}_{N \times N} \cdot \frac{d^m}{dz^m} \mathbf{X}(z)_{N \times 1} = \mathbf{r}(z)_{N \times 1} \quad (5.1)$$

where ${}_m\mathbf{b}_{N \times N}$ are some set of M matrices, $\mathbf{X}(z)_{N \times 1}$ is the vector of unknown functions, $\mathbf{r}(z)_{N \times 1}$ is a vector of known functions, and $i = 1, \dots, N$ is the number of unknown functions $\mathbf{X}(z)_{N \times 1}$.

Equation 5.1 can be reduced (Gohberg et al., 2009) to the first order system

$${}_1\mathbf{B}_{MN \times MN} \cdot \mathbf{Y}^1(z)_{MN \times 1} - {}_0\mathbf{B}_{MN \times MN} \cdot \mathbf{Y}(z)_{MN \times 1} = {}_0\mathbf{r}(z)_{MN \times 1} \quad (5.2a)$$

with

$$\mathbf{Y}(z)_{1 \times MN}^T = \left\langle \mathbf{X}(z)_{1 \times N}^T \mid \mathbf{X}^1(z)_{1 \times N}^T \mid \dots \mid \mathbf{X}^{M-1}(z)_{1 \times N}^T \right\rangle \quad (5.2b)$$

$${}_1\mathbf{B}_{MN \times MN} = \left[\begin{array}{c|c} \mathbf{I}_{(M-1)N \times (M-1)N} & \mathbf{0}_{(M-1)N \times N} \\ \hline \mathbf{0}_{N \times (M-1)N} & \mathbf{M-1} \mathbf{C}_{N \times N} \end{array} \right] \quad (5.2c)$$

$${}_0\mathbf{B}_{MN \times MN} = \left[\begin{array}{c|c} \mathbf{0}_{(M-1)N \times N} & \mathbf{I}_{(M-1)N \times (M-1)N} \\ \hline -{}_0\mathbf{C}_{N \times N} \quad \dots \quad -\mathbf{M-3} \mathbf{C}_{N \times N} & \mathbf{M-2} \mathbf{C}_{N \times N} \end{array} \right] \quad (5.2d)$$

and

$${}_0\mathbf{r}(z)_{1 \times MN}^T = \left\langle \mathbf{0}_{1 \times (M-1)N} \mid \mathbf{r}(z)_{1 \times N}^T \right\rangle \quad (5.2e)$$

The last set of N lines of ${}_0\mathbf{B}_{MN \times MN}$, ${}_1\mathbf{B}_{MN \times MN}$, and ${}_0\mathbf{r}(z)_{MN \times 1}$ express Equation 5.1 while the other lines are used to relate $\mathbf{X}(z)$ to its derivatives. Reducing Equation 5.1 to a first order system is called a *linearization*. The linearization given here is not unique as the matrix blocks could have been arranged in a different way. This particular choice of linearization is referred to as a *first companion linearization*.

The solution to Equation 5.2 can be expressed as the sum of two parts: the particular part and the homogeneous part. The homogeneous part is the solution to the system in the absence of a right hand term. Constants that appear in the homogeneous solution must be determined from the boundary conditions. The particular part is a vector of functions that addresses the right hand side only (i.e. the distributed load) of Equation 5.2.

5.2 Homogeneous Solution

The homogeneous part of the solution must obey the condition

$${}_1\mathbf{B}_{MN \times MN} \cdot \mathbf{Y}^I(z)_{MN \times 1} = {}_0\mathbf{B}_{MN \times MN} \cdot \mathbf{Y}(z)_{MN \times 1} \quad (5.3)$$

Since no restrictions were placed on the system matrices, ${}_1\mathbf{B}_{MN \times MN}$ may be either singular or non-singular. We will treat the case in which ${}_1\mathbf{B}_{MN \times MN}$ is non-singular in Section 5.2.1 and the case in which ${}_1\mathbf{B}_{MN \times MN}$ is singular in Section 5.2.2.

5.2.1 The Non-Singular Case

If ${}_1\mathbf{B}_{MN \times MN}$ is non-singular, one can pre-multiply Equation 5.3 by ${}_1\mathbf{B}_{MN \times MN}^{-1}$ to get

$$\mathbf{Y}^I(z)_{MN \times 1} = \begin{pmatrix} {}_1\mathbf{B}_{MN \times MN}^{-1} & {}_0\mathbf{B}_{MN \times MN} \end{pmatrix} \mathbf{Y}(z)_{MN \times 1} = {}_2\mathbf{B}_{MN \times MN} \mathbf{Y}(z)_{MN \times 1} \quad (5.4)$$

Any square matrix such as ${}_2\mathbf{B}_{MN \times MN}$, can be expressed as the product of three matrices (Noble, 1969), i.e.,

$${}_2\mathbf{B}_{MN \times MN} = {}_2\mathbf{S}_{MN \times MN} {}_2\mathbf{J}_{MN \times MN} {}_2\mathbf{S}_{MN \times MN}^{-1} \quad (5.5)$$

in which both ${}_2\mathbf{J}_{MN \times MN}$ and ${}_2\mathbf{S}_{MN \times MN}$ are unique for any given matrix ${}_2\mathbf{B}_{MN \times MN}$. In Equation 5.5 ${}_2\mathbf{S}_{MN \times MN}$ is a transformation matrix and ${}_2\mathbf{J}_{MN \times MN}$ is referred to as the Jordan canonical form of ${}_2\mathbf{B}_{MN \times MN}$ composed of the Jordan blocks on the diagonal and zeros elsewhere, that is

$${}_2\mathbf{J}_{MN \times MN} = \begin{bmatrix} {}_{1,2}\mathbf{J}_{\Upsilon_1 \times \Upsilon_1} & \mathbf{0}_{\Upsilon_1 \times \Upsilon} & & \mathbf{0} \\ \mathbf{0}_{\Upsilon_1 \times \Upsilon_1} & {}_{2,2}\mathbf{J}_{\Upsilon \times \Upsilon} & & \\ \hline & & & \\ \hline & \mathbf{0} & & {}_{k,2}\mathbf{J}_{\Upsilon_k \times \Upsilon_k} \end{bmatrix} \quad (5.6)$$

where ${}_{m,2}\mathbf{J}_{\Upsilon_m \times \Upsilon_m}$ are square Jordan block matrices. The dimensions of the Jordan blocks may differ, however they must all have the form

$${}_{m,2}\mathbf{J}_{\Upsilon \times \Upsilon} = \begin{bmatrix} \lambda_m & 1 & 0 & 0 \\ 0 & \lambda_m & 1 & 0 \\ 0 & 0 & & \\ & & 0 & \lambda_m & 1 \\ 0 & 0 & 0 & 0 & \lambda_m \end{bmatrix} \quad (5.7)$$

with the same value, λ_m , on the diagonal, unity directly above the diagonal (i.e. on the superdiagonal), and zero elsewhere. If $\Upsilon_m = 1$ then ${}_{m,2}\mathbf{J}_{\Upsilon \times \Upsilon}$ is just the scalar λ_m . It is possible that several Jordan blocks have the same diagonal value, λ_m . From Equation 5.5 by substituting into Equation 5.4 and pre-multiplying both sides by ${}_2\mathbf{S}_{MN \times MN}^{-1}$ one has

$${}_2\mathbf{S}_{MN \times MN}^{-1} \cdot \mathbf{Y}^I(z)_{MN \times 1} = {}_2\mathbf{J}_{MN \times MN} \cdot {}_2\mathbf{S}_{MN \times MN}^{-1} \cdot \mathbf{Y}(z)_{MN \times 1} \quad (5.8)$$

Introducing the vector

$$\mathbf{Z}(z)_{MN \times 1} = {}_2\mathbf{S}_{MN \times MN}^{-1} \cdot \mathbf{Y}(z)_{MN \times 1} \quad (5.9)$$

one can express Equation 5.8 as

$$\mathbf{Z}^I(z)_{MN \times 1} = {}_2\mathbf{J}_{MN \times MN} \cdot \mathbf{Z}(z)_{MN \times 1} \quad (5.10)$$

Equation 5.10 is a nearly uncoupled set of equations for the unknown functions $Z_i(z)$. If

${}_2\mathbf{J}_{MN \times MN}$ is fully diagonal then all the equations are fully uncoupled. For each Jordan block of size $\Upsilon_m \times \Upsilon_m$, in which $\Upsilon_m > 1$ and starting at the m^{th} row of ${}_2\mathbf{J}_{MN \times MN}$, the partially coupled equations for the associated functions are

$$\begin{Bmatrix} Z_m(z) \\ Z_{m+1}(z) \\ \dots \\ Z_{m+\Upsilon-1}(z) \end{Bmatrix}_{\Upsilon} = \begin{bmatrix} \lambda_m & 1 & & \\ & \lambda_m & \dots & \\ & & \dots & 1 \\ & & & \lambda_m \end{bmatrix}_{\Upsilon \times \Upsilon} \begin{Bmatrix} Z_m(z) \\ Z_{m+1}(z) \\ \dots \\ Z_{m+\Upsilon-1}(z) \end{Bmatrix}_{\Upsilon} \quad (5.11)$$

Starting with the last row in Equation 5.11, one can show that the solution for the entry $Z_n(z)$ to Equation 5.11 (Spiegel, 1980) is

$$Z_n(z) = \sum_{j=0}^{(m+\Upsilon_m-1)-n} \left(\kappa_{n+j} \frac{z^j}{j!} \right) e^{\lambda_m z} \quad \text{with } m \leq n < m + \Upsilon_m \quad (5.12a)$$

where the set of constants κ_{n+j} are to be determined from the boundary condition equations.

Equation 5.12a is also valid when $\Upsilon_m = 1$ in which case

$$Z_m(z) = \kappa_m e^{\lambda_m z} \quad (5.12b)$$

Using Equations 5.12 and 5.9 one can express the unknowns vectors $\mathbf{Y}(z)_{MN \times 1}$ or, equivalently,

$\mathbf{X}(z)_{MN \times 1}$ in terms of the parameters of the Jordan transformation, ${}_2\mathbf{J}_{MN \times MN}$ and ${}_2\mathbf{S}_{MN \times MN}$.

Matrices ${}_2\mathbf{J}_{MN \times MN}$ and ${}_2\mathbf{S}_{MN \times MN}$ can be obtained using various numerical techniques and implementations exist for most programming libraries and software packages.

5.2.2 The Singular Case

If ${}_1\mathbf{B}_{MN \times MN}$ is singular, one cannot use the method described in Section 5.2.1. In this case, one can use Weierstraß decomposition to solve the initial equation (Kunkel and Mehrmann, 2006).

This type of decomposition takes the form

$${}_1\mathbf{B}_{MN \times MN} = {}_0\mathbf{S}_{MN \times MN} \cdot \left[\begin{array}{c|c} \mathbf{I}_{(MN-\eta) \times (MN-\eta)} & \mathbf{0}_{(MN-\eta) \times \eta} \\ \hline \mathbf{0}_{\eta \times (MN-\eta)} & \mathbf{N}_{\eta \times \eta} \end{array} \right] \cdot {}_1\mathbf{S}_{MN \times MN} \quad (5.13a)$$

$${}_0\mathbf{B}_{MN \times MN} = {}_0\mathbf{S}_{MN \times MN} \cdot \left[\begin{array}{c|c} {}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)} & \mathbf{0}_{(MN-\eta) \times \eta} \\ \hline \mathbf{0}_{\eta \times (MN-\eta)} & \mathbf{I}_{\eta \times \eta} \end{array} \right] \cdot {}_1\mathbf{S}_{MN \times MN} \quad (5.13b)$$

where both ${}_0\mathbf{S}_{MN \times MN}$ and ${}_1\mathbf{S}_{MN \times MN}$ are non-singular, ${}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)}$ is a matrix in the Jordan canonical form, and $\mathbf{N}$ is a shift matrix (i.e. unit values on the superdiagonal and zeros elsewhere). We can use the substitution

$$\mathbf{Z}(z)_{MN \times 1} = {}_1\mathbf{S}_{MN \times MN} \cdot \mathbf{Y}(z)_{MN \times 1} \quad (5.14)$$

to express Equation 5.3 as

$$\left[\begin{array}{c|c} \mathbf{I}_{(MN-\eta) \times (MN-\eta)} & \mathbf{0}_{(MN-\eta) \times \eta} \\ \hline \mathbf{0}_{\eta \times (MN-\eta)} & \mathbf{N}_{\eta \times \eta} \end{array} \right] \cdot \mathbf{Z}^1(z)_{MN \times 1} = \left[\begin{array}{c|c} {}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)} & \mathbf{0}_{(MN-\eta) \times \eta} \\ \hline \mathbf{0}_{\eta \times (MN-\eta)} & \mathbf{I}_{\eta \times \eta} \end{array} \right] \cdot \mathbf{Z}(z)_{MN \times 1} \quad (5.15a)$$

Equation 5.15a can be divided into two sub-problems:

$$\tilde{\mathbf{Z}}^{\mathbf{I}}(z)_{(MN-\eta) \times 1} = {}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)} \cdot \tilde{\mathbf{Z}}(z)_{(MN-\eta) \times 1} \quad (5.15b)$$

and

$$\mathbf{N}_{\eta \times \eta} \cdot \bar{\mathbf{Z}}^{\mathbf{I}}(z)_{\eta \times 1} = \bar{\mathbf{Z}}(z)_{\eta \times 1} \quad (5.15c)$$

where $\tilde{\mathbf{Z}}^{\mathbf{I}}(z)_{(MN-\eta) \times 1}$ contains the first $MN - \eta$ functions of $\mathbf{Z}^{\mathbf{I}}(z)_{MN \times 1}$ and $\bar{\mathbf{Z}}^{\mathbf{I}}(z)_{\eta \times 1}$ contains the last η functions. Equation 5.15b has the same form as Equation 5.10. The method described in Section 5.2.1 can be used to find $\tilde{\mathbf{Z}}(z)_{(N-\eta) \times 1}$. For a Jordan block of size $\Upsilon_m > 1$ and starting at the m^{th} line of ${}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)}$, the associated functions are

$$Z_n(z) = \sum_{j=0}^{(m+\Upsilon_m-1)-n} \left(\kappa_{n+j} \frac{z^j}{j!} \right) e^{\lambda_m z} \quad m \leq n < m + \Upsilon_m \leq N - \eta \quad (5.16)$$

Finally, using Equation 5.15c, it can be shown that

$$Z_n(z) = 0 \quad N - \eta < n \leq N \quad (5.17)$$

Knowing $\mathbf{Z}(z)_{MN \times 1}$ from Equations 5.16 and 5.17, one can obtain $\mathbf{Y}(z)_{MN \times 1}$ from Equation 5.14,

and subsequently $\mathbf{X}(z)_{N \times 1}$ from in terms of the parameter of the Weierstraß transformation,

${}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)}$ and ${}_1\mathbf{S}_{MN \times MN}$. Matrices ${}_0\mathbf{J}_{(MN-\eta) \times (MN-\eta)}$ and ${}_1\mathbf{S}_{MN \times MN}$ can be derived using different

numerical techniques though implementations that are not widely available in standard programming libraries.

5.2.3 Representation of the Solution in terms of Matrix Exponentials

It is convenient to express the solutions in matrix exponential form. The exponential of a square matrix $\mathbf{B}$ is

$$e^{\mathbf{B}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{B}^k \quad (5.18)$$

For the non-singular case Equation 5.12a can be expressed as

$$\mathbf{Y}(z)_{MN \times 1} = e^{z \mathbf{B}_{MN \times MN}} \mathbf{K}_{MN \times 1} = {}_2\mathbf{S}_{MN \times MN} e^{z \mathbf{J}_{MN \times MN}} \boldsymbol{\kappa}_{MN \times 1} \quad (5.19a)$$

where

$$\mathbf{K}_{MN \times 1} = {}_2\mathbf{S}_{MN \times MN}^{-1} \cdot \boldsymbol{\kappa}_{MN} \quad (5.19b)$$

The matrix under the exponential must in general be converted to its Jordan form for the series to converge. Using the Jordan form, it is fairly straight forward to show that Equation 5.19a is equivalent to 5.12. For the singular case (Section 5.2.2) the solution in exponential form is

$$\mathbf{Y}(z)_{MN \times 1} = {}_1\mathbf{S}_{MN \times MN}^{-1} \begin{bmatrix} \mathbf{I}_{(MN-\eta) \times (MN-\eta)} \\ \mathbf{0}_{\eta \times (MN-\eta)} \end{bmatrix} e^{z \mathbf{J}_{(N-\eta) \times (N-\eta)}} \cdot \boldsymbol{\kappa}_{N-\eta} \quad (5.20)$$

Finally, the vector of unknowns $\mathbf{X}(z)_{MN \times 1}$ and its derivatives can be obtained by pre-multiplying

$\mathbf{Y}(z)_{MN \times 1}$. The m^{th} derivative of $\mathbf{X}(z)_{N \times 1}$ is

$$\frac{d^m}{dz^m} \mathbf{X}(z)_{N \times 1} = \begin{bmatrix} \mathbf{0}_{MN \times (m-1) MN} & \mathbf{I}_{MN \times (m-1) MN} & \mathbf{0}_{MN \times (MN-m)} \end{bmatrix} \cdot \mathbf{Y}(z)_{MN \times 1} \quad (5.21)$$

Equation 5.21 is only valid up to the M^{th} derivative. For higher order derivatives, $\mathbf{X}(z)_{N \times 1}$ must be differentiated with respect to z using either 5.19a or 5.20.

5.2.4 Relation between the Jordan Form and the Eigenvalue Problem

If matrix $\mathbf{B}$ has a Jordan canonical form $\mathbf{J}$, then the diagonal of $\mathbf{J}$ contains all the eigenvalue of $\mathbf{B}$. The number of times a value is repeated on the diagonal corresponds to that eigenvalue's

algebraic multiplicity. The number of Jordan blocks associated with a specific value is that eigenvalue's geometric multiplicity. If $\mathbf{B}$ is fully diagonalisable (i.e. all Jordan blocks have dimension 1), the transformation matrix $\mathbf{S}$ contains the eigenvectors of $\mathbf{B}$ as row vectors.

If, on the other hand, $\mathbf{B}$ is non-diagonalisable, then $\mathbf{S}$ contains the *generalized eigenvector* of $\mathbf{B}$. The set of generalized eigenvectors includes the set of eigenvectors. A $\Upsilon_m \times \Upsilon_m$ Jordan block is associated with one eigenvector and $\Upsilon_m - 1$ other generalized eigenvectors. For example, assume that the first two Jordan blocks of $\mathbf{B}$ have the form

$${}_1\mathbf{J}_{2 \times 2} = {}_2\mathbf{J}_{2 \times 2} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (5.22)$$

and that the first four generalized eigenvectors are ${}_1\boldsymbol{\varsigma}_{MN \times 1}$, ${}_2\boldsymbol{\varsigma}_{MN \times 1}$, ${}_3\boldsymbol{\varsigma}_{MN \times 1}$, and ${}_4\boldsymbol{\varsigma}_{MN \times 1}$ such that

$$\mathbf{S}_{MN \times MN} = [{}_1\boldsymbol{\varsigma}_{MN \times 1} \mid {}_2\boldsymbol{\varsigma}_{MN \times 1} \mid {}_3\boldsymbol{\varsigma}_{MN \times 1} \mid {}_4\boldsymbol{\varsigma}_{MN \times 1} \mid \dots]_{MN \times MN}^T \quad (5.23)$$

If the other Jordan blocks have eigenvalues distinct from λ , then λ has an algebraic multiplicity of 4 and geometric multiplicity of 2. Vectors ${}_1\boldsymbol{\varsigma}_{MN \times 1}$ and ${}_3\boldsymbol{\varsigma}_{MN \times 1}$ are eigenvectors of $\mathbf{B}$ while ${}_2\boldsymbol{\varsigma}_{MN \times 1}$ and ${}_4\boldsymbol{\varsigma}_{MN \times 1}$ are strictly generalized eigenvectors. Using Equation 5.12a, we can write

$$Z_1(z) = (\kappa_1 + \kappa_2 z) e^{\lambda z} \quad (5.24a)$$

$$Z_2(z) = \kappa_2 z e^{\lambda z} \quad (5.24b)$$

$$Z_3(z) = (\kappa_3 + \kappa_4 z) e^{\lambda z} \quad (5.24c)$$

$$Z_4(z) = \kappa_4 z e^{\lambda z} \quad (5.24d)$$

Now, using Equation 5.9, we can solve for, say, the first parts of $Y_1(z)$ and $Y_2(z)$ given by the first two Jordan blocks

$$Y_1(z) = {}_1\varsigma_1 \cdot (\kappa_1 + \kappa_2 z) e^{\lambda z} + {}_1\varsigma_2 \cdot \kappa_2 z e^{\lambda z} + {}_1\varsigma_3 \cdot (\kappa_3 + \kappa_4 z) e^{\lambda z} + {}_1\varsigma_4 \cdot \kappa_4 z e^{\lambda z} + \dots \quad (5.25a)$$

$$Y_2(z) = {}_2\varsigma_1 \cdot (\kappa_1 + \kappa_2 z) e^{\lambda z} + {}_2\varsigma_2 \cdot \kappa_2 z e^{\lambda z} + {}_2\varsigma_3 \cdot (\kappa_3 + \kappa_4 z) e^{\lambda z} + {}_2\varsigma_4 \cdot \kappa_4 z e^{\lambda z} + \dots \quad (5.25b)$$

and similarly for the other components of $\{Y_i(z)\}$.

For the singular case (Section 5.2.2), the link to the eigenvalue problem is not as clear cut. In this case we are determining the “eigenvalues” and “eigenvectors” related to a system of two matrices. Some of these “eigenvalues” may be singular causing their associated functions to vanish. For example, if $\lambda_i \rightarrow -\infty$ and $Y_i(z) = \kappa_i e^{\lambda_i z}$ then $Y_i(z) \rightarrow 0$ which can be seen as an interpretation of Equations 5.15c and 5.17. To explain how the matrix exponential is related to the Jordan form we will consider the set of matrices $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{S}$ from earlier. Using Equation 5.18 and the properties of matrix multiplication we have

$$\begin{aligned}
 e^{\mathbf{B}z} &= e^{\mathbf{S}^{-1} \mathbf{J} \mathbf{S} z} \\
 &= \mathbf{S}^{-1} \cdot e^{\mathbf{J} z} \cdot \mathbf{S} \\
 &= \mathbf{S}^{-1} \cdot e^{\begin{bmatrix} \mathbf{J}_{2 \times 2} & 0 \\ 0 & \mathbf{J}_{2 \times 2} \end{bmatrix} z} \cdot \mathbf{S} \\
 &= \mathbf{S}^{-1} \cdot \begin{bmatrix} e^{\mathbf{J}_{2 \times 2} z} & 0 & \dots \\ 0 & e^{\mathbf{J}_{2 \times 2} z} & \vdots \\ \vdots & \dots & \ddots \end{bmatrix} \cdot \mathbf{S}
 \end{aligned}$$

We will now drive the exponential of individual Jordan blocks. For the first block

$$\begin{aligned}
 e^{\mathbf{J}_{2 \times 2} z} &= e^{\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) z} \\
 &= e^{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} z} e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z} \\
 &= \begin{bmatrix} e^{\lambda z} & 0 \\ 0 & e^{\lambda z} \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^0 + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^1 + \frac{z^2}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 + \dots \right) \\
 &= \begin{bmatrix} e^{\lambda z} & 0 \\ 0 & e^{\lambda z} \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{z^2}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \dots \right) \\
 &= \begin{bmatrix} e^{\lambda z} & 0 \\ 0 & e^{\lambda z} \end{bmatrix} + z \begin{bmatrix} 0 & e^{\lambda z} \\ 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} e^{\lambda z} & z e^{\lambda z} \\ 0 & e^{\lambda z} \end{bmatrix}
 \end{aligned}$$

Repeating this procedure for the second Jordan block, multiplying by the transformation matrix and an appropriate array of constants (see Equation 5.19), we will recover Equations 5.25.

5.3 Particular Solution using Fourier Series

If the right hand term of Equation 5.1 can be adequately approximated by a Fourier series consisting of $\mu+1$ terms, μ being an even integer, one has

$$\mathbf{r}(z)_{N \times 1} \approx \begin{Bmatrix} \mathbf{f}_{1,1} \\ \mathbf{f}_{2,1} \\ \dots \\ \mathbf{f}_{N,1} \end{Bmatrix}_N e^{i\left(1-\frac{\mu}{2}\right)z} + \begin{Bmatrix} \mathbf{f}_{1,2} \\ \mathbf{f}_{2,2} \\ \dots \\ \mathbf{f}_{N,2} \end{Bmatrix}_N e^{i\left(2-\frac{\mu}{2}\right)z} + \dots = \mathbf{f}_{N \times (\mu+1)} \cdot \left\{ e^{i(\iota-\mu/2)z} \right\}_{(\mu+1) \times 1} \quad (5.26)$$

where $\mathbf{f}_{N \times (\mu+1)}$ are Fourier coefficients for the loading functions, $\iota = \sqrt{-1}$ denotes the imaginary

constant, and $\left\{ e^{i(\iota-\mu/2)z} \right\}_{(\mu+1) \times 1}$ is the vector with i^{th} component equal to $e^{i(\iota-\mu/2)z}$. The exact

particular solutions, ${}_p\mathbf{X}(z)_{N \times 1}$, for the right hand side 5.26 can be expressed as a Fourier series

with the same number of terms. Using Equation 5.1 to solve for the Fourier series coefficients of

${}_p\mathbf{X}(z)_{N \times 1}$ leads to

$${}_p\mathbf{X}(z)_{N \times 1} = \left(\sum_{m=0}^{M-1} \left[\iota^m \left(i - \frac{\mu}{2} \right)^m \cdot {}_m b_{ij} \right]_{N \times N} \right)^{-1} \mathbf{f}_{N \times (\mu+1)} \left\{ e^{i(\iota-\mu/2)z} \right\}_{\mu} \quad (5.27)$$

5.4 General Solution

The general solution is the sum of the homogeneous and particular parts. Any such solution can be written as

$$\mathbf{X}(z)_{N \times 1} = \mathbf{H}(z)_{N \times MN} \cdot \boldsymbol{\kappa}_{MN \times 1} + {}_p\mathbf{X}(z)_{N \times 1} \quad (5.28)$$

where $\mathbf{H}(z)_{N \times MN}$ is given either by Equation 5.19 or 5.21 and ${}_p\mathbf{X}(z)_{N \times 1}$ is as in Equation 5.27.

5.5 Boundary Conditions

The boundary conditions discussed here refer to displacements and rotation along a transverse edge. It is also possible to impose boundary conditions along longitudinal edges by proper selection of shape functions. However, this will not be discussed here. All the boundary condition equations provided in Chapters 2, 3, and 4 can generically be expressed as

$${}_k\Theta(\mathbf{X}(z_0)_{N \times 1}) = {}_k\Phi \quad (5.29)$$

where ${}_k\Theta(\)$ is a linear differential operator and ${}_k\Phi$ is a vector of either specified displacements or generalized boundary loads. The dimensions of ${}_k\Theta(\)$ and ${}_k\Phi$ depend on the nature of the boundary condition. By substituting of Equation 5.28 into Equation 5.29

$${}_k\Theta(\mathbf{H}(z_0)_{N \times MN})\mathbf{\kappa}_{MN \times 1} = {}_k\Phi - {}_k\Theta({}_p\mathbf{X}(z_0)_{N \times 1}) \quad (5.30)$$

In general, enough boundary condition equations must be formulated to solve for all the boundary constants. For example, Table 5.1 gives typical boundary conditions for the single plate out-of-plane bending problem.

Table 5.1 - Boundary conditions for out-of-plane plate bending

z_0	Type of Displacement	Free Displacement		Fixed Displacement	
		${}_k\Theta(\)$	${}_k\Phi$	${}_k\Theta(\)$	${}_k\Phi$
0	Translation	${}_0\mathbf{c}_{N \times N} \cdot \frac{d^3}{dz^3} - {}_7\mathbf{c}_{N \times N} \cdot \frac{d}{dz}$	${}_0\mathbf{T}_{N \times 1}$	$\mathbf{I}_{N \times N} \cdot$	$\mathbf{0}_{N \times 1}$
0	Rotation	${}_0\mathbf{c}_{N \times N} \cdot \frac{d^2}{dz^2} + {}_3\mathbf{c}_{N \times N} \cdot$	${}_0\mathbf{M}_{N \times 1}$	$\mathbf{I}_{N \times N} \cdot \frac{d}{dz}$	$\mathbf{0}_{N \times 1}$
L	Translation	${}_0\mathbf{c}_{N \times N} \cdot \frac{d^3}{dz^3} - {}_7\mathbf{c}_{N \times N} \cdot \frac{d}{dz}$	$-{}_L\mathbf{T}_{N \times 1}$	$\mathbf{I}_{N \times N} \cdot$	$\mathbf{0}_{N \times 1}$
L	Rotation	${}_0\mathbf{c}_{N \times N} \cdot \frac{d^2}{dz^2} + {}_3\mathbf{c}_{N \times N} \cdot$	$-{}_L\mathbf{M}_{N \times 1}$	$\mathbf{I}_{N \times N} \cdot \frac{d}{dz}$	$\mathbf{0}_{N \times 1}$

In Table 5.1, the operators ${}_k\Theta(\)$ must be applied to the generalized normal displacement vector $\mathbf{U}(z)_{N \times 1}$. Each line in Table 5.1 will provide a set of N linear equations for constants κ_i . We know that for the out-of-plane plate bending problem there are $MN = 4M$ boundary condition constants. Therefore, we will need to choose 4 sets of equations from Table 5.1 to determine all the boundary constants. Table 5.2 gives typical boundary conditions associated with the in plane displacements problem for a single plate.

Table 5.2 - Boundary conditions for in-plane displacements

z_0	Direction of displacement	Displacement		Displacement Fixed	
		${}_k\Theta(\)$	${}_k\Phi$	${}_k\Theta(\)$	${}_k\Phi$
0	s-direction	$[_3\mathbf{a} \ \mathbf{0}]_{N \times 2N} \frac{d}{dz} - [_0 \ \mathbf{s}\mathbf{a}^T]_{N \times 2N}$	${}_0\mathbf{P}_{N \times 1}$	$[\mathbf{I} \ \mathbf{0}]_{N \times 2N}$	$\mathbf{0}_{N \times 1}$
0	z-direction	$[_0 \ \mathbf{a}]_{N \times 2N} \frac{d}{dz} - [_0 \ \mathbf{s}\mathbf{a}^T]_{N \times 2N}$	${}_0\mathbf{Q}_{N \times 1}$	$[_0 \ \mathbf{I}]_{N \times 2N}$	$\mathbf{0}_{N \times 1}$
L	s-direction	$[_3\mathbf{a} \ \mathbf{0}]_{N \times 2N} \frac{d}{dz} - [_0 \ \mathbf{s}\mathbf{a}^T]_{N \times 2N}$	${}_L\mathbf{P}_{N \times 1}$	$[\mathbf{I} \ \mathbf{0}]_{N \times 2N}$	$\mathbf{0}_{N \times 1}$
L	z-direction	$[_0 \ \mathbf{a}]_{N \times 2N} \frac{d}{dz} - [_0 \ \mathbf{s}\mathbf{a}^T]_{N \times 2N}$	${}_L\mathbf{Q}_{N \times 1}$	$[_0 \ \mathbf{I}]_{N \times 2N}$	$\mathbf{0}_{N \times 1}$

The operators $\Theta_k(\)$ must be applied to the generalized in-plane displacement vector

$\langle \mathbf{V}(z)_{1 \times N}^T \ \mathbf{W}(z)_{1 \times N}^T \rangle^T$ Each line in Table 5.2 will provide a set of N linear equations for the

constants κ_i . We know that for the in plane displacement problem, there are $4M$ boundary condition constants ($2N$ constants for each direction) Therefore, we will need to choose 4 sets of equations from Table 5.2 to determine all the constants

Possible boundary conditions are not limited to those provided in Tables 5.1 and 5.2 For example, we could impose specific displacements along the edges of the plate in order to calculate the resulting stresses in the rest of the plate

In a multiple plate problem, it is not as straight forward to state the boundary conditions. Different boundary conditions may be applied to different plates. If Lagrange multipliers are used, the solution to the field equation will yield fewer than MN constants. Our choice of boundary condition equations will depend on the conditions in the problem we are trying to solve. Regardless of the type of problem, we must formulate enough boundary conditions to solve for the boundary condition constants, that is

$$\mathbf{\kappa}_{MN \times 1} = \left[\begin{array}{c} \mathbf{1} \mathbf{\Phi} (H(z_0)_{N \times MN}) \\ \mathbf{2} \mathbf{\Phi} (H(z_0)_{N \times MN}) \end{array} \right]_{MN \times MN}^{-1} \left\{ \begin{array}{c} \mathbf{1} \mathbf{\Phi} - \mathbf{1} \mathbf{\Phi} (\mathbf{p} \mathbf{X}_1(z_0)_{N \times 1}) \\ \mathbf{2} \mathbf{\Phi} - \mathbf{2} \mathbf{\Phi} (\mathbf{p} \mathbf{X}_1(z_0)_{N \times 1}) \end{array} \right\}_{MN \times 1} \quad (5.31)$$

which would allow us to determine all the boundary condition constants

CHAPTER 6 - VERIFICATION PROBLEM FOR SINGLE PLATES

In this chapter, several sample problems are solved using the equations and methods developed in the preceding chapters. The methods were implemented using Mathematica. Scripts written by the author are provided in Appendix F. The validity and convergence of the method are demonstrated. The results are compared to results from a shell finite element analysis. The examples in this chapter involve single unconnected plates. Multiple plate problems are solved in Chapter 7.

6.1 Shell Finite Element Analysis

The finite element analysis was conducted using the commercial finite element analysis program Abaqus. All the models were created using the quadrilateral shell element S4R which is internal to the program. The S4R element is a general purpose quadrilateral shell element with 4 nodes (one at each corner). Each node has five independent degrees of freedom: three translations and two rotations. The element uses reduced integration to form the element stiffness with the integration point being at the center of the element. When associated with a linear elastic material, internal calculations are performed in terms of thickness integrated forces and moments. By default, the element employs hourglass control since reduced integration elements are prone to a membrane-mode and bending-mode hourglassing (a type of error associated with the finite element method).

Unlike the current formulation, the S4R elements do not implement thin shell theory exactly and account for some shear stress across the thickness. As the elements become thinner their behaviour approaches that of thin plates or shells. In addition, the elements can account for finite strains and large rotations whereas the current formulation assumes small displacements and

rotations. The material properties specified in Abaqus were identical to those used in the Mathematica scripts. Namely, the material (steel) is linearly elastic with modulus of elasticity $E = 200 \text{ GPa}$ and Poisson ratio $\mu = 0.3$. Elements were always square 10mm wide, 10mm long, and 10mm thick. For a 1.0m by 1.0m plate, the total number of elements was 10000. It was observed that in all cases, the number and distribution of elements produced a reliable solution. Point loads were applied as concentrated nodal loads. Line loads were applied as shell tractions.

6.2 Implementation of the Current Method

Mathematica 7.0 was used to implement the current method. Integrals of the shape functions were calculated by the program based on the functions entered by the user. The system matrices were assembled and stored in memory. Mathematica's Jordan decomposition routine was not able to perform the required matrix transformations for most cases (Chapter 5). Thus, the solution to the system of differential equations was determined using Mathematica's eigenproblem solver and subroutines written by the author. A symbolic form of the solution was saved in memory. Mathematica was able to determine the boundary constants based on the saved solution and a symbolic form of the boundary condition equations. Some post-processing of the data, including output of the field data in figures and tables, was also handled using the scripts. Because both the loads and shape functions were either symmetric or antisymmetric, it was necessary to increase the number of shape functions in increments of two in order to increase the accuracy of the solution. The number of shape functions was increased until the difference between a given solution and the previous one was negligible. In general this took no more than 10 shape functions. Computational times were typically a few seconds regardless of the number of modes.

A major complication which must be mentioned is that the system of linear equations for the boundary conditions (e.g. Equation 5.31) was generally identified by the linear equation solver as being ill-conditioned or singular. This problem was observed to be more severe as more shape functions were added and/or as the plate length increased. It is thought that this problem occurred because of the presence of extremely large and extremely small values (i.e. of the order $e^{\pm 300}$) in the system of linear equations. Because the problem occurred even when the system was fully diagonal, it is believed that the systems were never actually singular. To circumvent the problem calculations were performed using a high degree of arithmetic precision. It was nonetheless impossible to solve multiple plate problems involving very long plates or many shape functions with the available subroutines.

6.3 Shape Functions

The shape functions used for all the single plate problems were truncated power series centered on zero, that is

$$\phi_i(s) = \varphi_i(s) = \psi_i(s) = s^{i-1} \quad (6.1)$$

where i varies from 1 to the number of modes N . For a single plate, the first mode ($i = 1$) corresponds to bending in the normal direction and the second mode ($i = 2$) corresponds to twisting. The first mode is the only one that does not vanish at $s = 0$, while the second mode is the only one not to have nonzero slope at $s = 0$.

6.4 Example 1: Out-of-Plane Bending

Figure 6.1 shows the geometry and load acting on the plate under consideration.

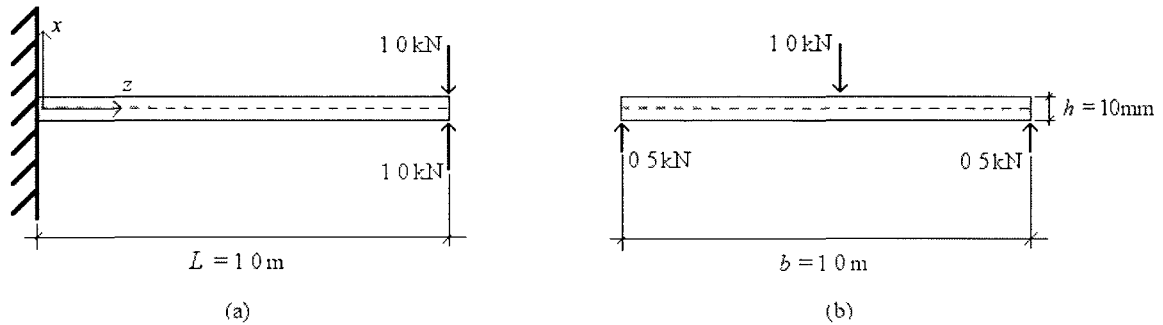


Fig. 6.1 - Dimensions and loading for example 1: (a) Elevation and (b) Cross-section

The plate is 1m long, 1m wide, and 10mm thick. It is clamped at one end and free at the other.

The z -coordinate is measured from the fixed end. Three point loads are applied at the free end: a downward 1.0kN load at the middle and upward 0.5kN loads on each corner. Under this load, the plate does not undergo in-plane deformation. Figure 6.2 shows the distorted shape of the plate as predicted by the current method.

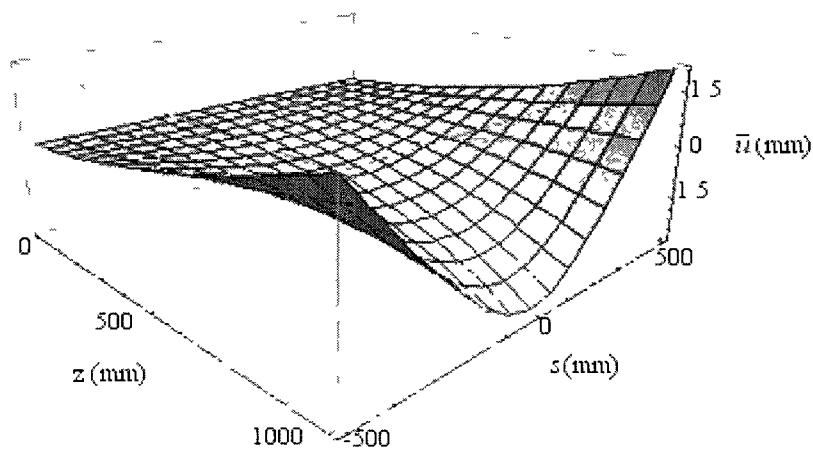


Fig. 6.2 - Normal displacement, $\bar{u}$, versus midplane coordinates, s and z

The plate takes on a U-shape at the free end, while no displacements nor rotations take place at the fixed end, consistent with the applied loads and boundary conditions. Figure 6.3 shows the deformed shape of the cross section at different distances, z , from the fixed edge.

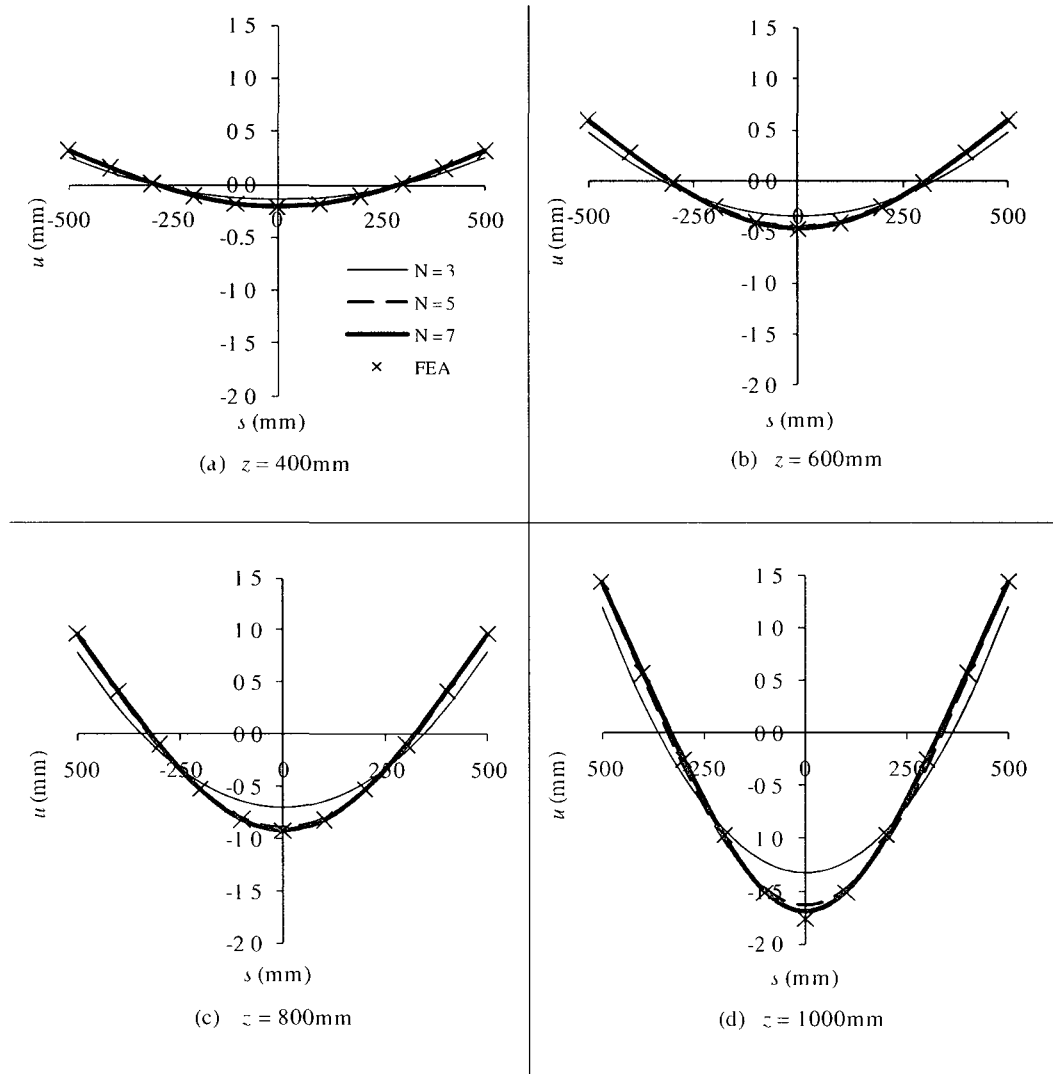


Fig. 6.3 - Normal displacement, $\bar{u}$, versus transverse coordinate, s , at (a) $z = 400$ mm, (b) $z = 600$ mm, (c) $z = 800$ mm, and (d) $z = 1000$ mm

In general, there is good agreement between the current method and the finite element results.

There are few changes in the predicted displacements for N larger than 5 except near the zone of

high curvature at $s = 0$. Figure 6.4 shows the transverse and longitudinal stresses at the point $(x = h/2, s = 0, z)$ versus z .

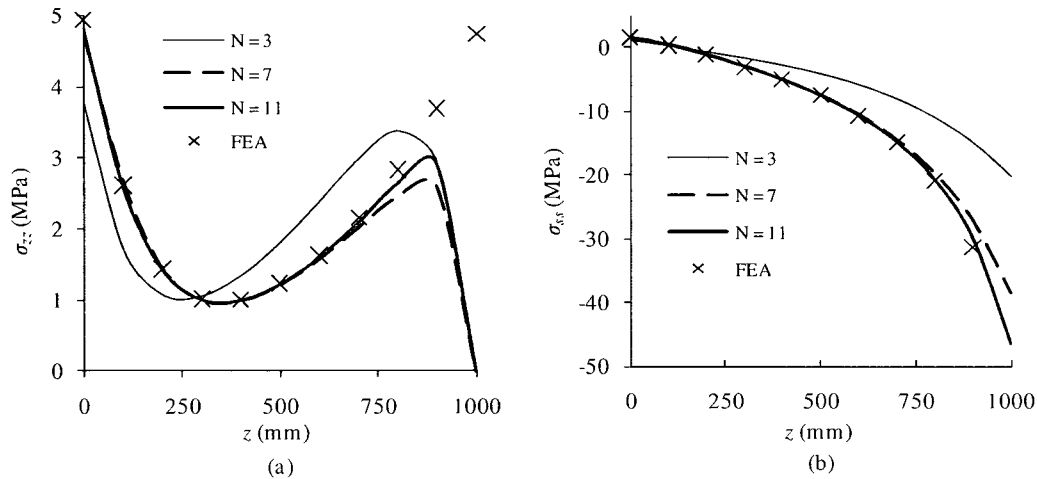


Fig. 6.4 - (a) Longitudinal stress, σ_{zz} , and (b) transverse stress, σ_{xs} , versus longitudinal coordinate, z , at the point $(x = h/2, s = 0, z)$

The transverse stress, σ_{xs} , is in general much larger than the longitudinal stress because the curvature in the transverse direction is larger. Away from the point of application of the loads, the computed stresses agree well with the results from the finite element analysis provided an adequate number of modes are used. It was observed that as the finite element mesh was refined the stresses at the point of application of the loads diverged. This is because the S4R element takes into account some through-thickness shear effects and, in principle, those shear stresses must be infinite under a normal point loads. Because the current model adopts thin plate theory, no through-thickness shear stresses are taken into account in the formulation of the equilibrium equations and the stresses converge under the point loads. Figure 6.5 shows the stresses at the point $(x = h/2, s = b/2, z)$ versus z .

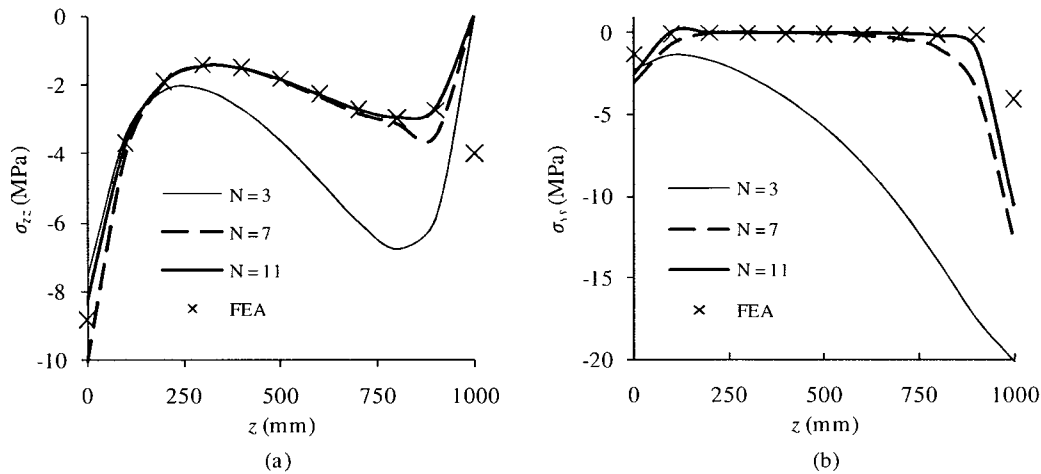


Fig. 6.5 - (a) Longitudinal stress, σ_{zz} , and (b) transverse stress, σ_{yy} , versus longitudinal coordinate, z , at the point $(x = h/2, s = b/2, z)$

Although, the results agree away from free end, some difference can be observed at the free end.

Figure 6.6 shows the percent difference between the results from the current method and the results from finite element analysis for the displacement and stress at the point

$(x = h/2, s = b/2, z = L/2)$. This point was chosen because it is away from externally applied loads.

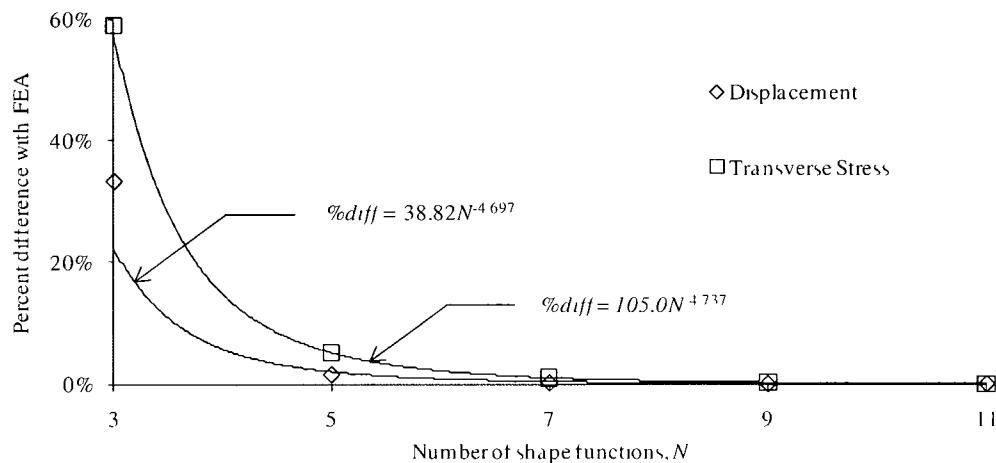


Fig. 6.6 - Percent difference with FEA versus number of shape functions, N

Because the finite element analysis used a very fine mesh, the error associated with the finite element solution relative to the “true” solution will be smaller than the error associated with the current method using a small number of modes. Best fit lines suggest that the current method converges sublinearly with N . The displacements converge more rapidly than the stresses because the stresses are related to the derivatives of the displacements. Figure 6.6 only provides information about the convergence of the results at one point. Results from the current method may or may not converge to the finite element results at other points in the plate. The rate of convergence may also be different at other points.

6.5 Example 2: In-Plane Bending

The plate under consideration in this section is shown in Fig. 6.7.

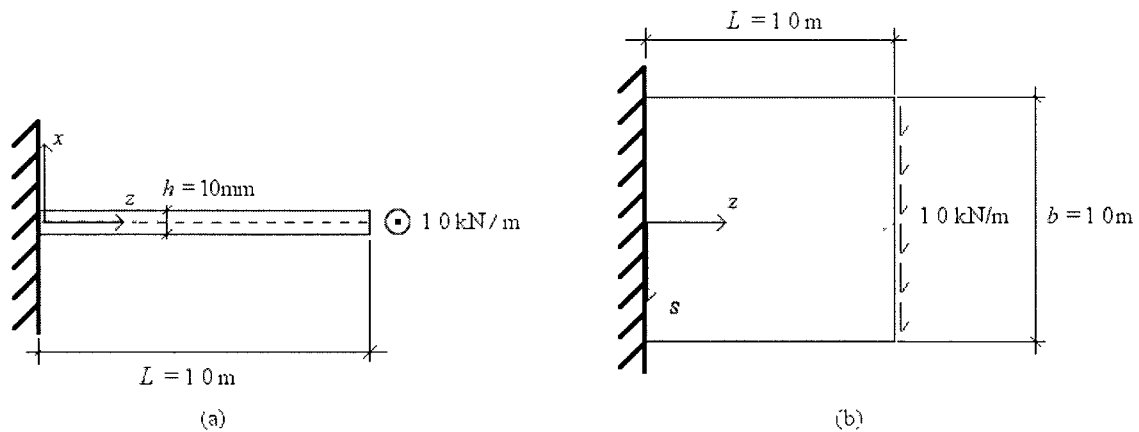


Fig. 6.7 - Dimensions and loading for example 2: (a) Elevation and (b) Plan view

The plate is 1m long, 1m wide, and 10mm thick. It is clamped at one end and free at the other. The z -coordinate is measured from the fixed end. A 1.0 kN/m load in the positive s -direction is applied at the free end along the midplane. Figure 6.8 shows the deformed shape of the plate as predicted by the deep plate bending formulation (Chapter 3).

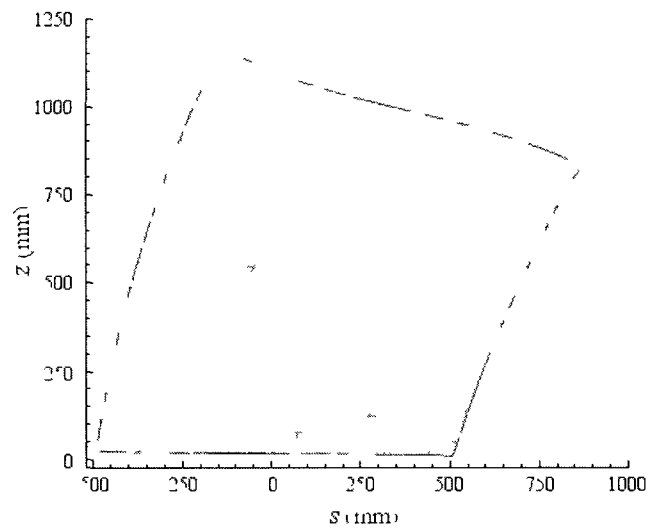


Fig 6.8 - Deformed shape of plate by current method (deformation scale = 10^5)

The plate takes on a shape that is consistent with the applied loads and boundary conditions. We can see from the grid lines (used for plotting) that transverse lines do not remain straight.

Therefore, the Euler-Bernoulli assumption would not be valid in this case. Figure 6.9 gives the transverse displacement along the line ($x = 0, s = 0, z$) versus z .

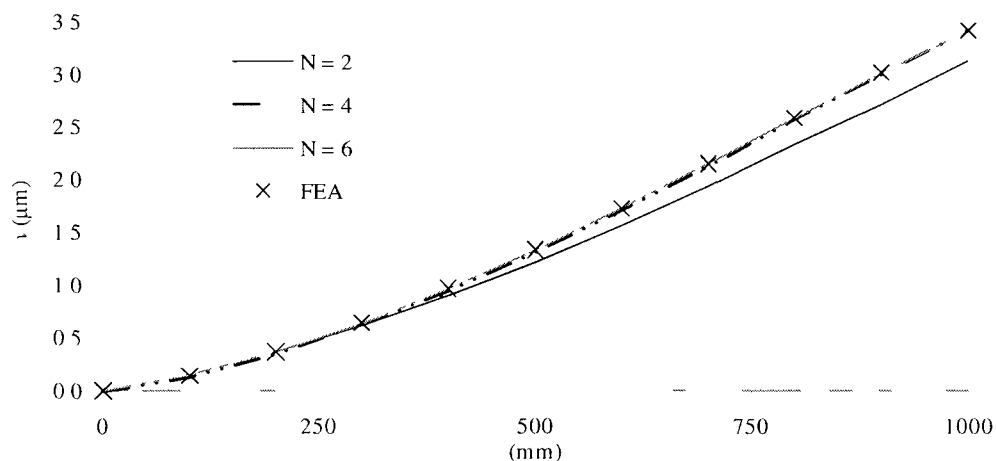


Fig 6.9 - Transverse displacement, v , versus longitudinal coordinate, z , at the point ($x = 0, s = 0, z$)

For $N = 4$ or more, the displacements agree well with the displacements predicted by the finite element analysis. Figure 6.10 shows the longitudinal stress profile in the cross section at different distances from the fixed end.

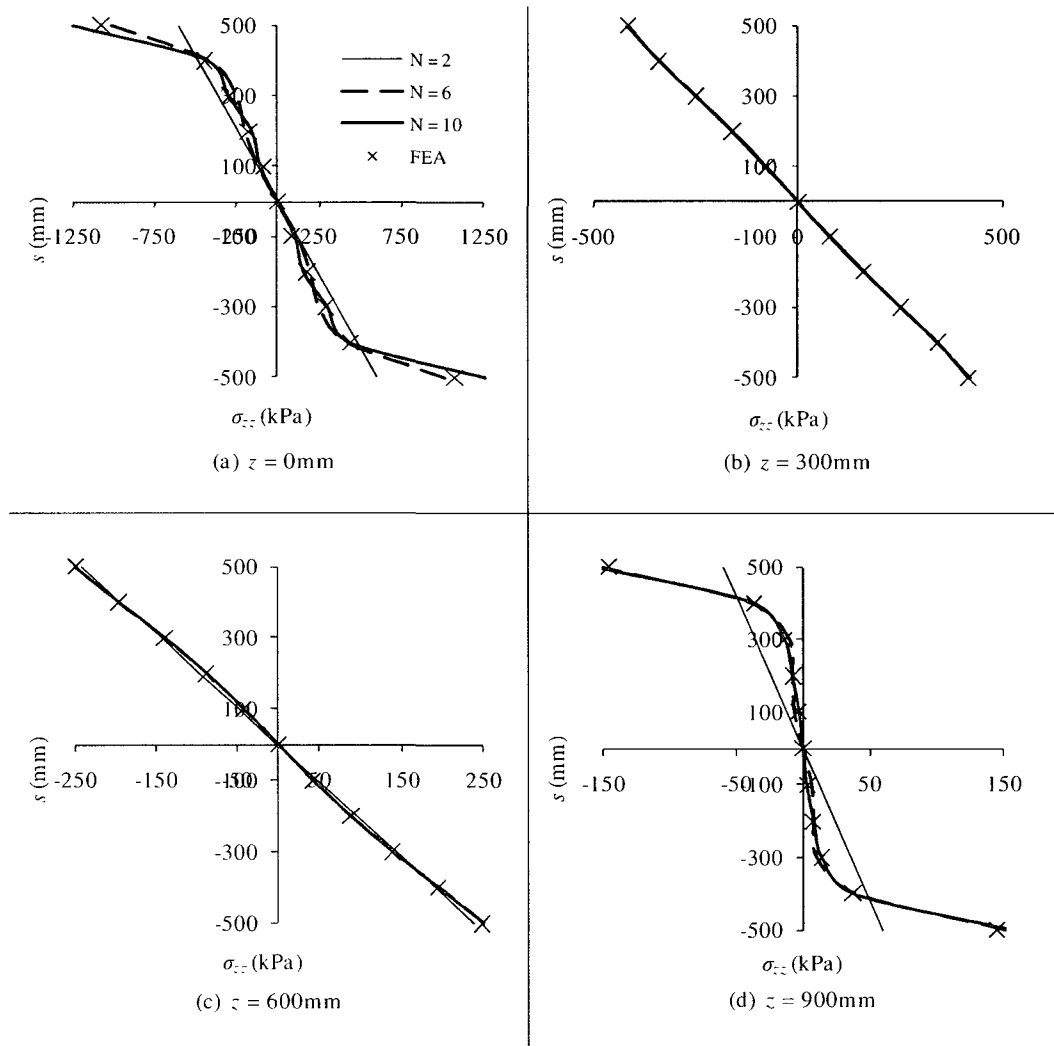


Fig. 6.10 - Longitudinal stress, σ_{zz} , versus, s , at (a) $z = 0\text{mm}$, (b) $z = 300\text{mm}$, (c) $z = 600\text{mm}$, and, (d) $z = 900\text{mm}$

Because the plate is subjected to in-plane loading only, the stresses are uniform through the plate thickness. In general, the stresses from the current method agree well with those from the finite

element analysis. Some differences can be observed near the plate ends for $N = 2$. Away from the ends, the stress distribution is linear implying constant shear stress. Near the ends, using more than two shape functions allows the model to more accurately fulfill the boundary conditions. At the fixed corners ($z = 0$ and $s = \pm 500\text{mm}$) the stresses predicted with $N = 11$ are larger than those predicted by the finite element analysis. This suggests that eleven shape functions provide greater accuracy than the finite element mesh. Figure 6.11 shows the percent difference between the current method and the finite element analysis for increasing N .

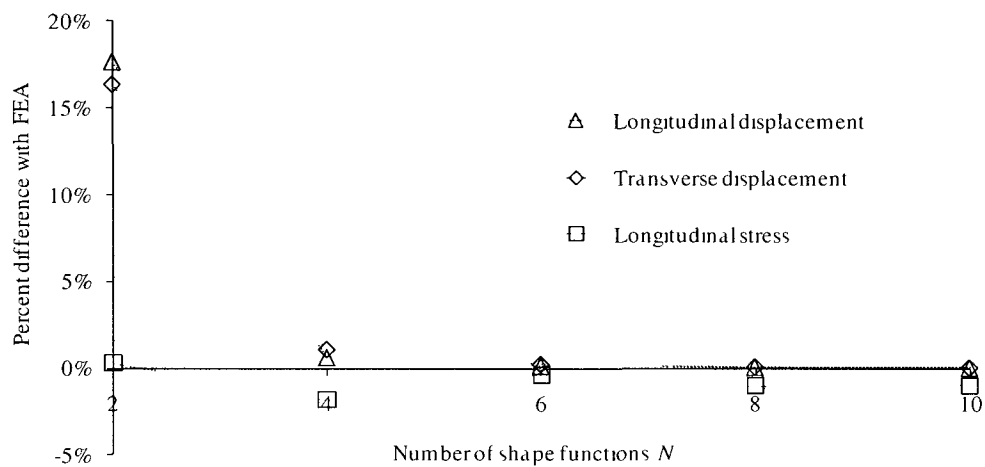


Fig. 6.11 - Percent difference with FEA versus number of shape functions, N

The stresses and displacements are compared near the center of the plate. At that point, the transverse stresses are zero. The longitudinal stresses oscillate slightly as they converge to the finite element results. The current method converges to the finite element results to within 1% at $N = 10$.

6.6 Example 3: Shear Lag in a Plate

The plate and loading under consideration in this section is shown in Fig. 6.12 below.

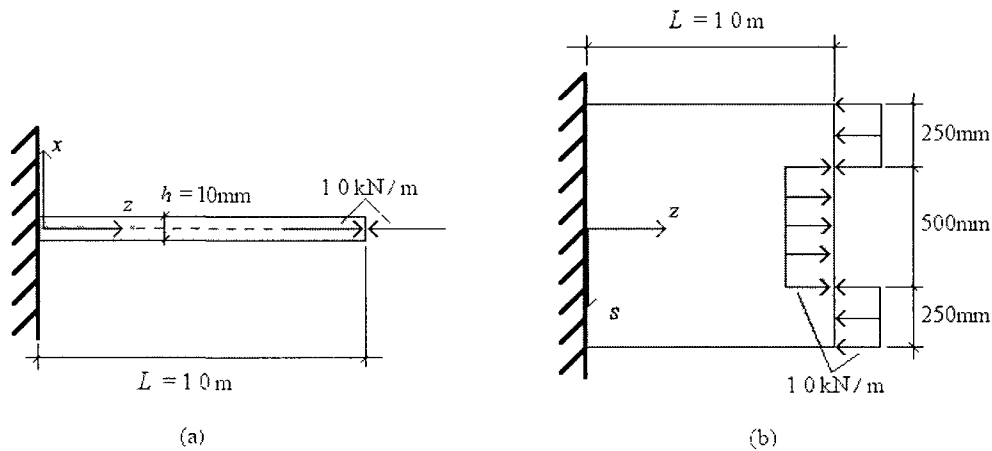


Fig. 6.12 - Dimensions and loading for example 3: (a) Elevation and (b) Plan view

The plate is 1m long, 1m wide, and 10mm thick. It is clamped along $z = 0$ and free along $z = L$. Longitudinal 1.0 kN/m loads are applied at the free end. Half the load is acting in the positive z -direction while the other half is acting in the negative z -direction (Fig. 6.12). Figure 6.13 shows the deformed shape of the plate as predicted by the current method.

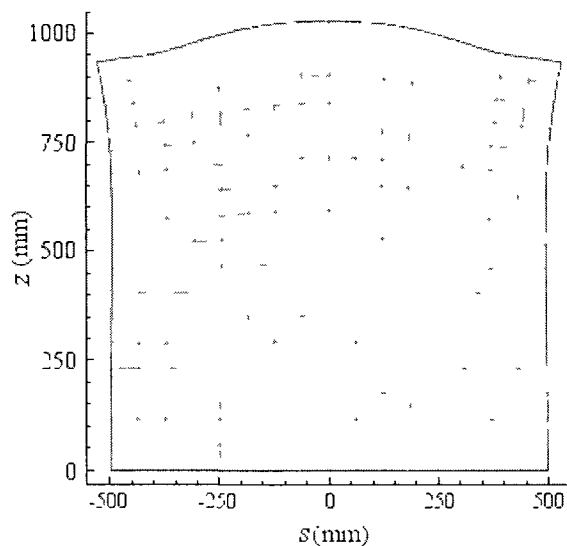


Fig. 6.13 - Deformed shape of plate as predicted by current method (deformation scale = 10^5)

The deformed shape is consistent with the applied load and boundary conditions. Figures 6.13a and 6.13b show the longitudinal displacements at the points $(x = 0, s = 0, z)$ and $(x = 0, s = b/2, z)$ versus z .

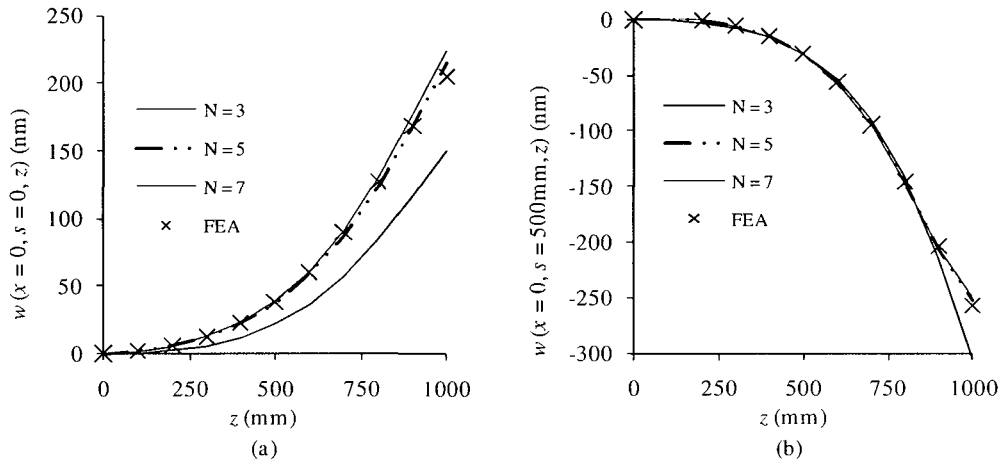


Fig. 6.13 - Longitudinal displacements, w , versus longitudinal coordinate, z , along lines
(a) $(x = 0, s = 0, z)$ and (b) $(x = 0, s = b/2, z)$

The maximum positive section displacements occurs at $s = 0$ while the maximum negative displacement occurs at $s = b/2$. The results from the current model are consistent with those based on finite element analysis. The current method converges with few modes and little change in the displacements occur with more than five modes. Figure 6.14 shows the longitudinal stress profile at different distances from the fixed end.

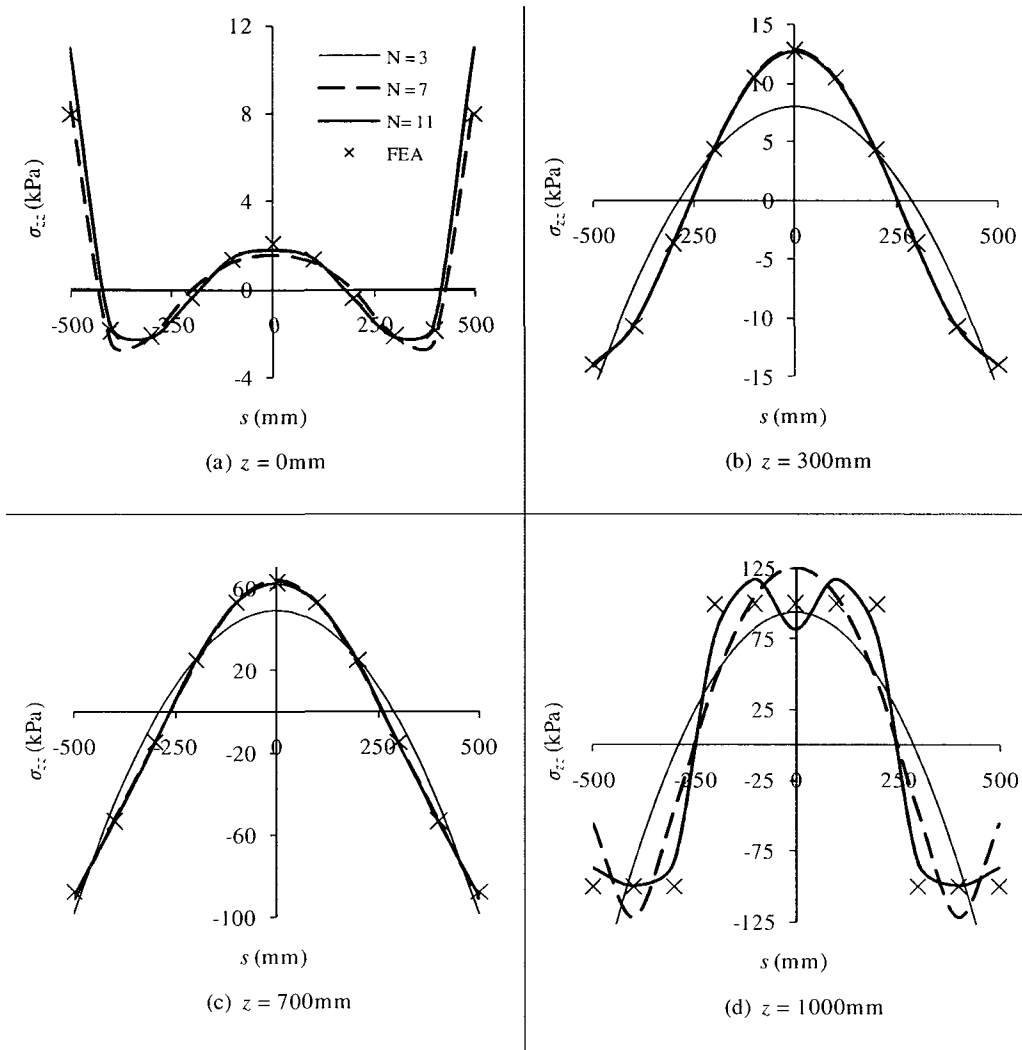


Fig. 6.14 - Longitudinal stress, σ_{zz} , versus transverse coordinate, s , at (a) $z = 0\text{mm}$, (b) $z = 300\text{mm}$, (c) $z = 700\text{mm}$, and (d) $z = 1000\text{mm}$

In general, the results from the current method agree well with those from the finite element analysis. At the fixed corners ($z = 0$ and $s = \pm 500\text{mm}$) the stresses predicted with $N = 11$ are larger than those predicted by the finite element analysis. This suggests that eleven shape functions provide greater accuracy than the finite element mesh. At the free end, the stress must be equal to the applied load divided by the plate thickness. The results from the current method converge fairly slowly to the step shape of the applied loading. Even when 11 modes are used

($N = 11$) significant oscillations are observed. This is a natural outcome when continuous displacement series (Polynomial, Fourier, etc) are used to behaviour under approximate discontinuous functions. In contrast, the finite element method is capable of accurately enforcing the traction boundary conditions. Therefore, no oscillations are observed in the finite element results. Away from the free end, the current method agrees well with the finite element analysis. Figures 6.15a and 6.15b show the longitudinal stress at the points $(x = 0, s = 0, z)$ and $(x = 0, s = b/2, z)$ versus z .

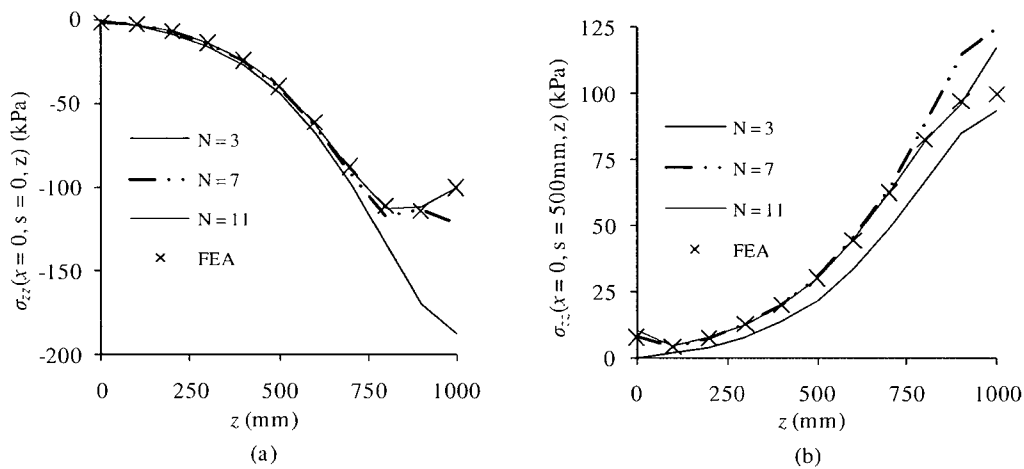


Fig. 6.15 - Longitudinal stress, σ_{zz} , versus longitudinal coordinate, z , at the points

(a) $(x = 0, s = 0, z)$ and (b) $(x = 0, s = b/2, z)$

Apart from the minor disagreements in stress at the free end, the two methods agree reasonably well. It can be verified that the net axial load vanishes at each cross-section. Transverse stresses in the plate are nearly zero and only occur because of the Poisson effect. Figure 6.16 shows the percent difference between the current method and the finite element analysis as a function of the number of modes.

The stresses and displacements were compared at the point $(x = 0, s = b/2, z = L/2)$. The results from the current method converge to the finite element results. The longitudinal stress predicted by the current method converges to a value 1% below the finite element results. This suggests that a refined finite element mesh would yield a smaller stress.

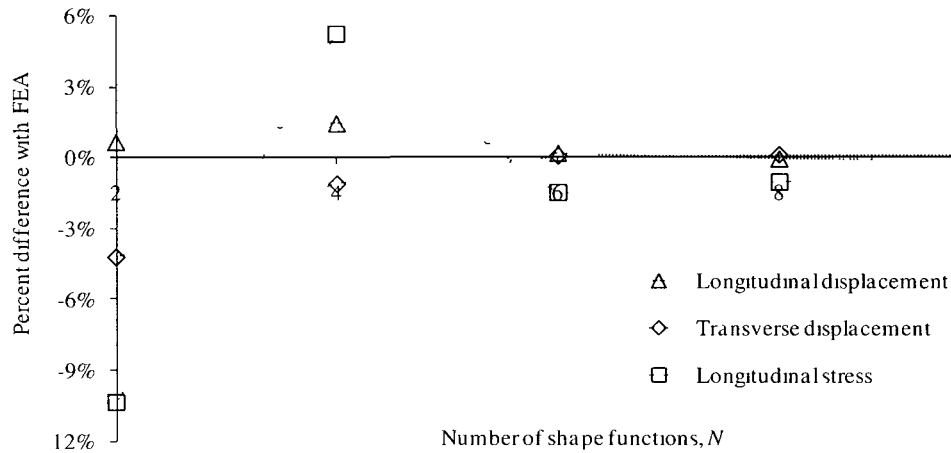


Fig. 6.16 - Percent difference with FEA versus number of shape functions, N

CHAPTER 7 - VERIFICATION PROBLEM FOR MULTIPLE PLATES

In this chapter several sample problems are solved using the equations and methods developed in the preceding chapters. The examples are meant to illustrate the behaviour of steel girders and involve multiply connected plates. The validity and convergence of the method are demonstrated through comparison to solutions based on shell finite element analysis.

7.1 Shell Finite Element Analysis

The parameters used for the finite element analysis are identical to those described in Section 6.1. Because multiple plate problems involved a larger surface area, the element dimensions were increased to 50mm by 50mm.

7.2 Implementation of the Current Method

The current method was implemented using Mathematica. Scripts written by the author are provided in Appendix F. Overall, the program steps were the same as those described in Section 6.2. Four of the examples presented in this chapter involve transverse bending of individual plates (Sections 7.4 through 7.7). For those examples, a narrow plate bending formulation was used (Section 4.4.2). Expressions for the final stiffness matrices (e.g. $\tilde{c}_k$, d_i , etc) are provided in Appendix F. Because of numerical difficulties encountered in solving the system of linear boundary condition equations (Section 6.2), it was not possible to fully implement the deep plate bending theory described in Section 4.4.1. The last two examples (Sections 7.8 and 7.9) which involve shear lag were implemented using the equations presented in Section 4.4.1 but neglect out-of-plane displacement of individual plates.

7.3 Shape Functions

The type of shape functions used to solve the examples problems depended on the end condition of the plates involved. For a plate connected along its middle ($s = 0$) we used the power series

$$\phi_i(s) = \varphi_i(s) = \psi_i(s) = s^{i-1} \quad (7.1)$$

For a plate connected along the edge $s = b/2$, we used the functions

$$\phi_i(s) = \varphi_i(s) = \psi_i(s) = \begin{cases} 1 & i = 1 \\ s - \frac{b}{2} & i = 2 \\ \bar{s}^{i-3} \left(s - \frac{b}{2} \right)^2 & i \geq 3 \end{cases} \quad (7.2)$$

For a plate connected along the edge $s = -b/2$, functions symmetric to those given by Equation 7.2 were used. For plates connected along both edges we used

$$\phi_i(s) = \varphi_i(s) = \psi_i(s) = \begin{cases} 2 \left(1 + \frac{s}{b} \right) \left(\frac{1}{2} - \frac{s}{b} \right)^2 & i = 1 \\ \left(s + \frac{b}{2} \right) \left(\frac{1}{2} - \frac{s}{b} \right)^2 & i = 2 \\ 2 \left(1 - \frac{s}{b} \right) \left(\frac{1}{2} + \frac{s}{b} \right)^2 & i = 3 \\ \left(\frac{b}{2} - s \right) \left(\frac{1}{2} - \frac{s}{b} \right)^2 & i = 4 \\ \bar{s}^{i-5} \left(\frac{s}{b} - \frac{1}{2} \right)^2 \left(\frac{s}{b} + \frac{1}{2} \right)^2 & i \geq 5 \end{cases} \quad (7.3)$$

For illustration purposes, the first eight shape functions for each type of plate are shown in Fig. 7.1.

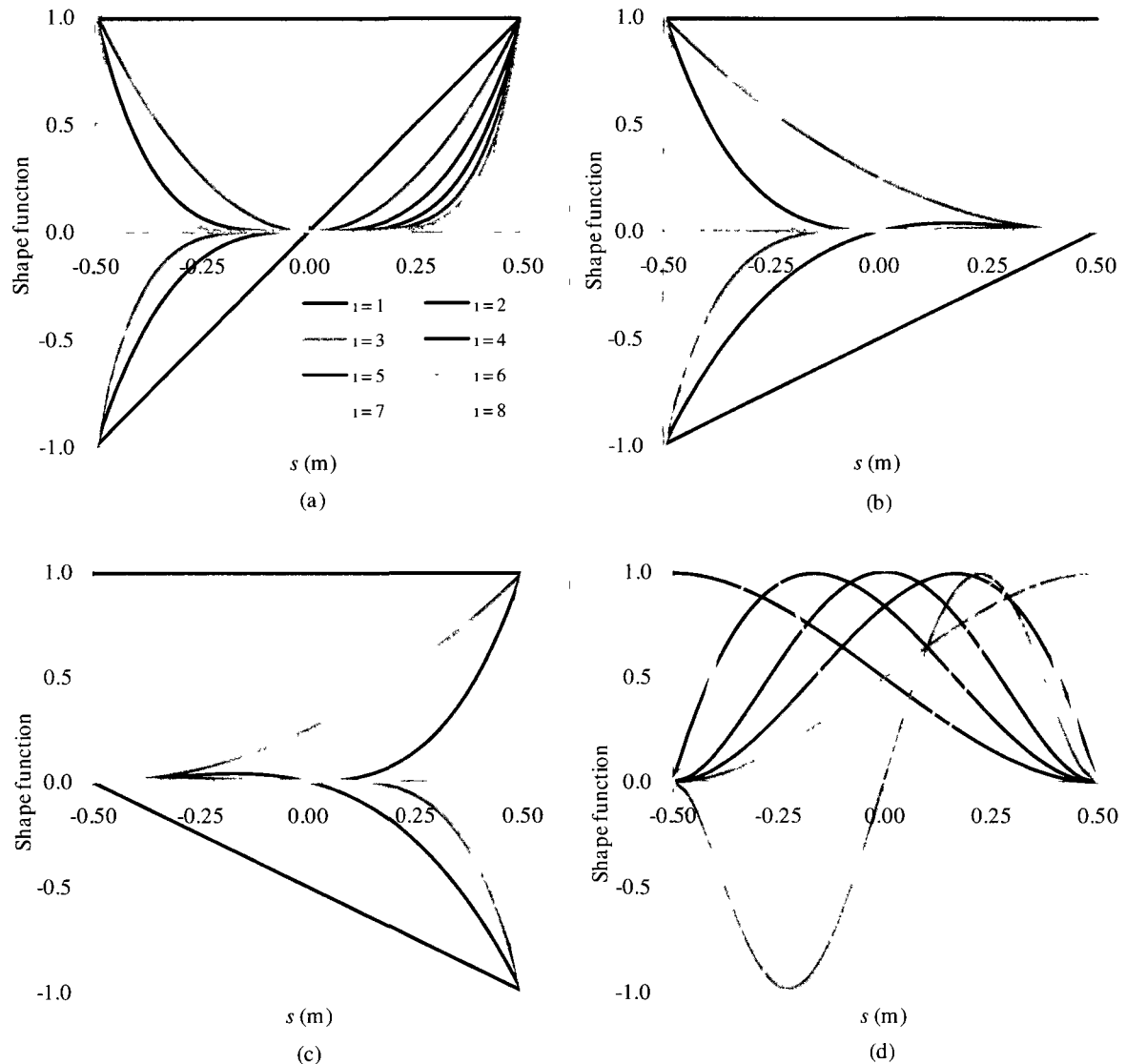


Fig. 7.1 - Shape functions for a 1.0m wide plate connected at (a) $s = 0\text{mm}$, (b) $s = 500\text{mm}$, (c) $s = -500\text{mm}$, and (d) $s = 500\text{mm}$ and $s = 500\text{mm}$

For each set of shape functions, only one function has a non-zero value along an edge connected to another plate. Also, only one function has a non-zero slope at an edge connected to another plate. These shape functions were used because they simplify the enforcement of the continuity equations at the plate joints (Chapter 4).

7.4 Example 1: Angle Subjected to Twist and Biaxial Bending

Figure 7.2 shows the geometry and load on the angle under consideration.

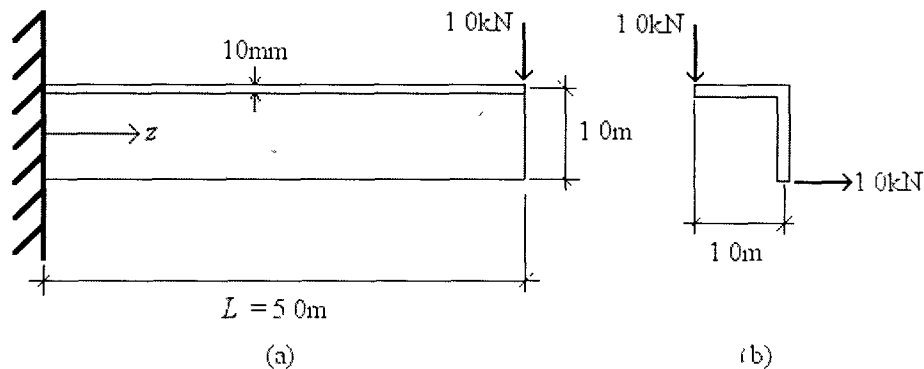


Fig. 7.2 - Dimensions and loading for Example 1: (a) Elevation and (b) Cross-section

The angle is 5.0m long, is free at one end and clamped at the other. The z coordinate is measured from the fixed end. Both the horizontal and vertical plates are 1m long and 10mm thick. It should be noted that typical hot-rolled angles have larger thickness-to-width ($b/h < 10$) ratios and may not behave according to classical thin plate theory. Two 1.0 kN loads are applied normal to the outside edge of each plate at the free end. These loads will cause twisting of the section along with some biaxial bending and distortion. The displacements and stresses in both legs will be numerically identical because of the symmetry of the applied load and boundary conditions.

Figure 7.3 shows the deformed shape of the cross section at different distances from the fixed end. The coordinate axes shown are not associated with a specific plate and are given for scale only.

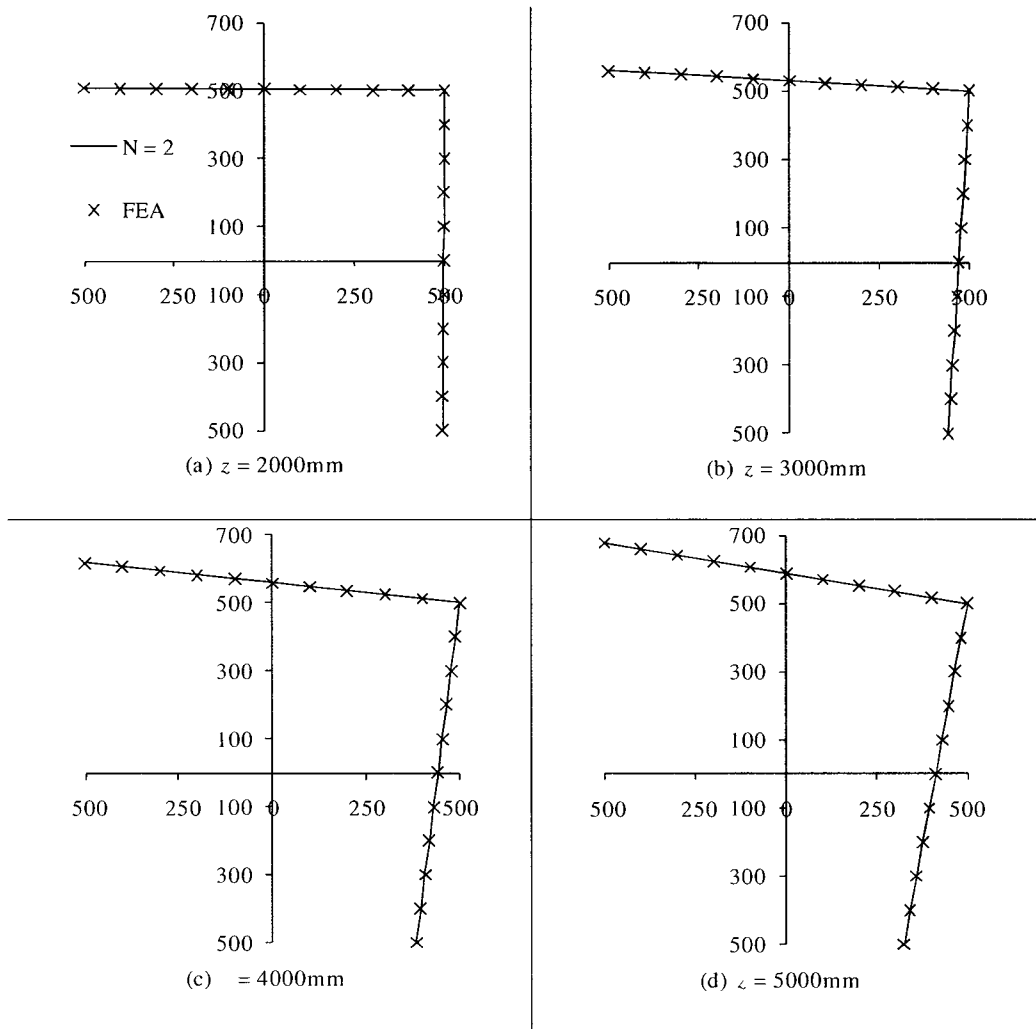


Fig 7.3 Deformed shape of cross section at (a) $z = 2000\text{mm}$, (b) $z = 3000\text{mm}$, (c) $z = 4000\text{mm}$, and (d) $z = 5000\text{mm}$

We can see from Fig 7.3 that the cross section undergoes twisting primarily and that adequate precision is obtained using $N = 2$ in the current formulation. Because curvature in the plane of the cross section is negligible, the addition of more shape functions does not increase the accuracy of the current method. Using $N = 2$ with the narrow plate formulation results in 3 independent generalized displacements: two displacements relevant to bending and the angle of twist. The displacements indicated by Fig 7.3 would therefore be identical to those given by

torsional beam theory. Figure 7.4 shows the largest longitudinal stress in the cross-section versus z .

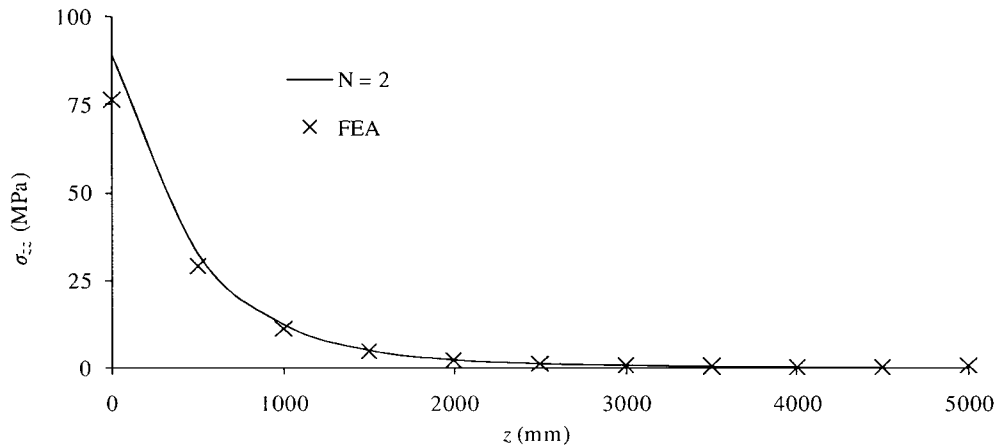


Fig. 7.4 - Largest longitudinal cross-section stress, σ_{zz} , versus longitudinal coordinate, z

Good agreement with the finite element analysis is again achieved by taking two modes ($N = 2$) in the current method. There is a 15% difference in stress at the support. This may be because of a singularity in the stress field not captured by the relatively coarse finite element mesh. Also, the narrow plate formulation does not accurately predict the stresses near a support providing transverse restraint.

7.5 Example 2: Angle Subjected to Distortion

The angle in Example 3 is subject to a different load combination.

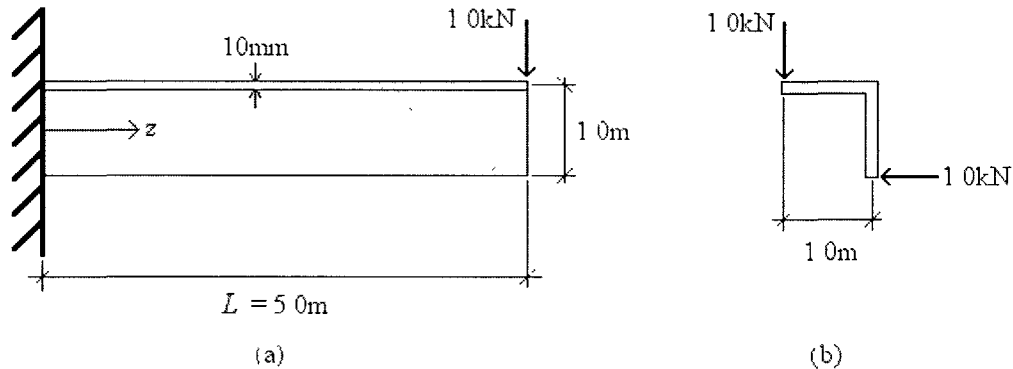


Fig. 7.5 - Dimensions and loading for Example 2: (a) Elevation and (b) Cross-section

Loads of 1.0 kN loads are applied at the free end as shown. These loads will produce distortion of the section and bending. Figure 7.6 shows a three dimensional view of the deformed angle generated using the current method.

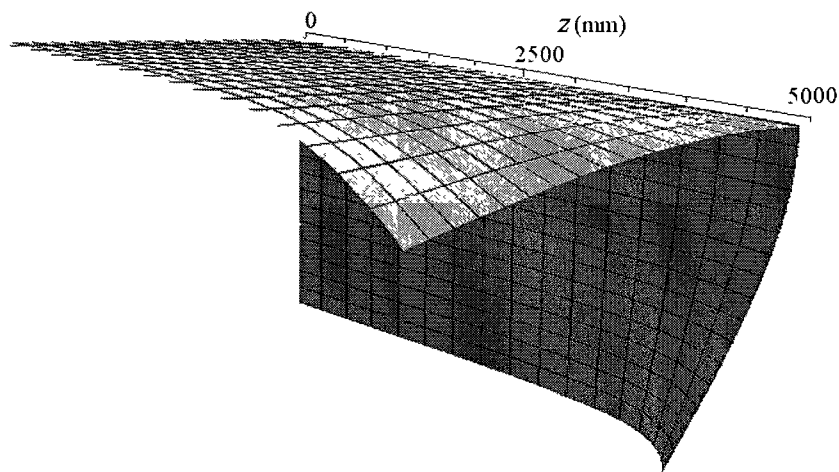


Fig. 7.6 - Deformed angle (deformation scale = 15)

Displacements were multiplied by a factor of 15. The displacements are consistent with the applied loading and boundary conditions. Distortion is most important near the free end and

vanishes at the support. Figure 7.7 shows the deformed shape of the cross section at different distances from the fixed end. The coordinate axes shown do not belong to a specific coordinate system and are given for scale only.

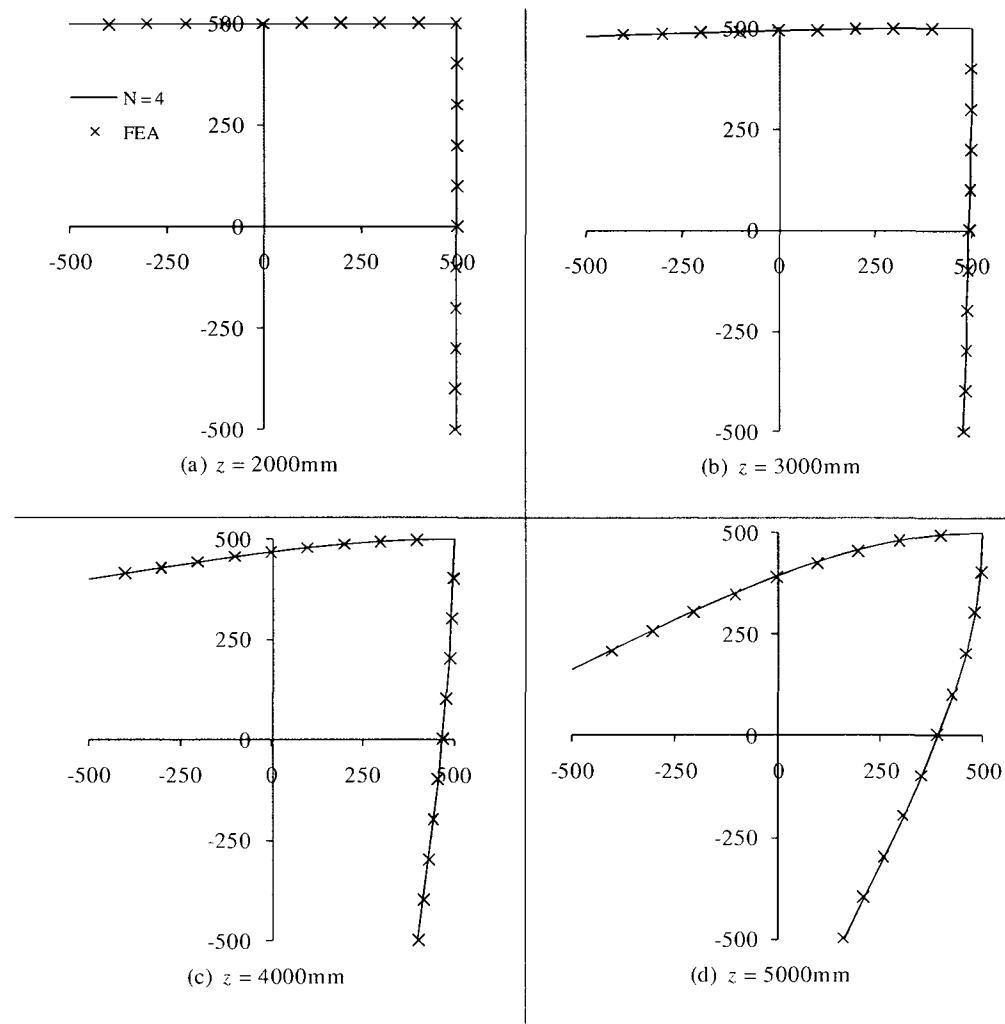


Fig. 7.7 - Deformed shape of cross section at (a) $z = 2000\text{mm}$, (b) $z = 3000\text{mm}$, (c) $z = 4000\text{mm}$, and (d) $z = 5000\text{mm}$ (deformation scale = 15)

Excellent agreement with finite elements is observed using four shape functions in the current method. The distortion of the section decreases quickly away from the free end and is almost negligible at $z = 3\text{m}$. For this example, transverse stresses are larger than longitudinal ones.

Figure 7.8 shows the transverse stress in the vertical plate at different distances from the fixed end. The horizontal plate is connected at $s = -500\text{mm}$.

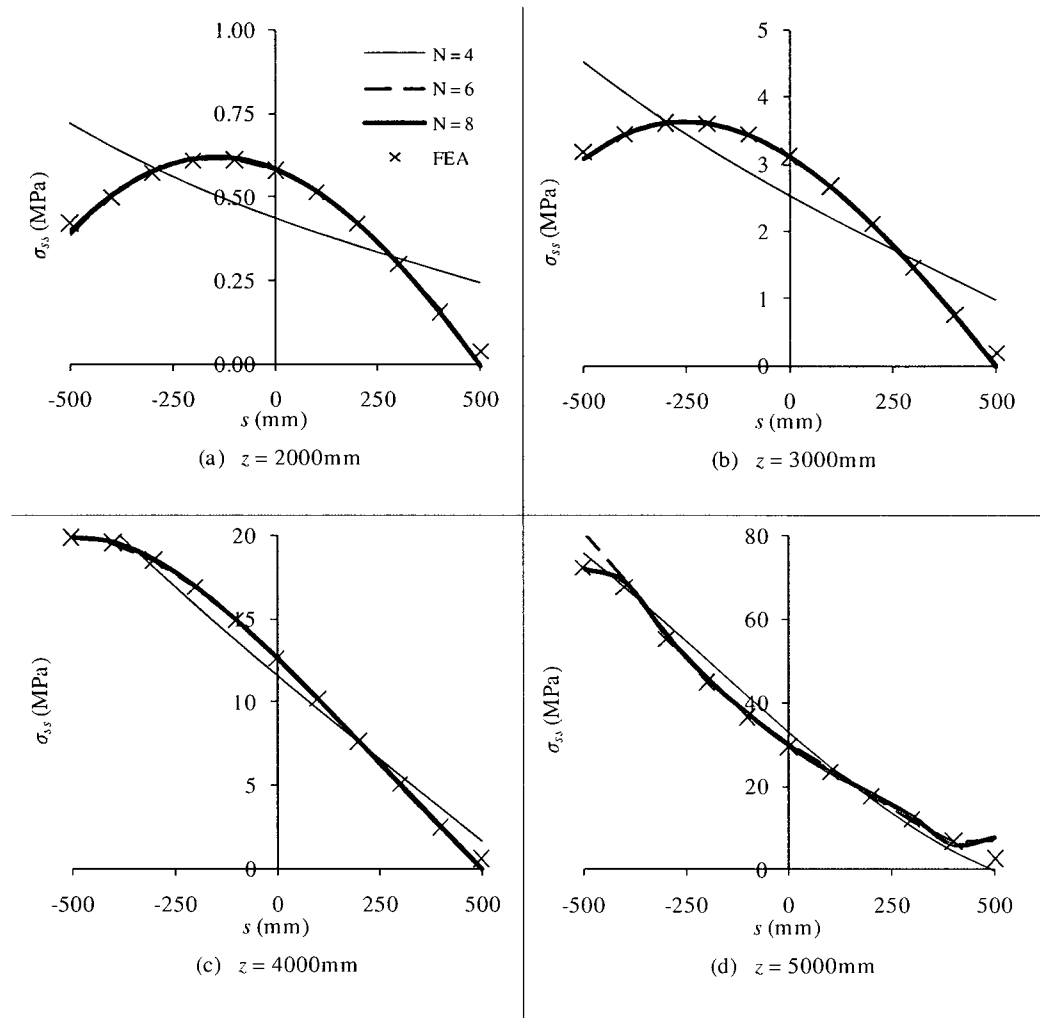


Fig. 7.8 - Transverse stress, σ_{ss} , versus transverse coordinate, s , in vertical plate at
(a) $z = 2000\text{mm}$, (b) $z = 3000\text{mm}$, (c) $z = 4000\text{mm}$, and (d) $z = 5000\text{mm}$

In general, there is good agreement of the stresses with the finite element method when six shape functions are used. Stresses are largest at the free end and rapidly decrease as one moves towards the fixed end. Near the free end, the stress distribution is roughly linear suggesting the transverse curvature also varies linearly.

7.6 Example 3: I-beam Subjected to Uniaxial Bending

Figure 7.9 shows the geometry and load on the I-beam under consideration in this section.

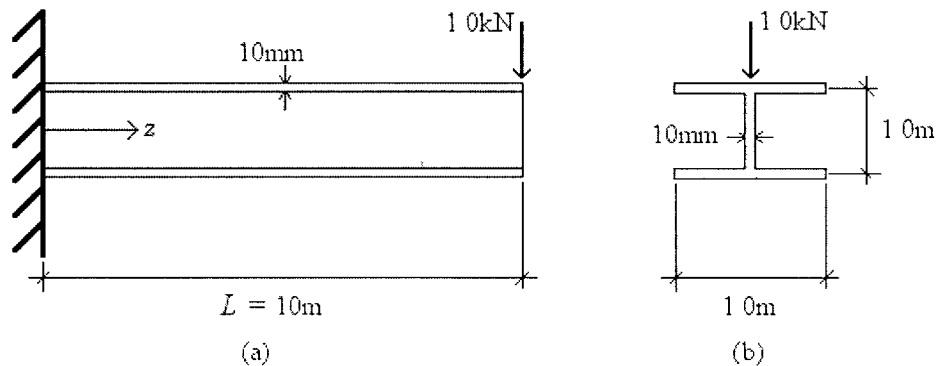


Fig. 7.9 - Dimensions and loading for Example 3: (a) Elevation and (b) Cross-section (NTS)

The beam is 10m long, is free at one end and clamped at the other. The web and flanges are 1m long and 10mm thick. Typical hot-rolled beams have larger thickness-to-width ratios and may not behave according to thin plate theory. A 1.0 kN load is applied at the free end as shown. The load will cause strong axis bending of the section with minimal distortion and no twist.

Downward displacements are almost uniform throughout the cross section. Figure 7.10 shows the downward displacement of the cross-section versus coordinate z .

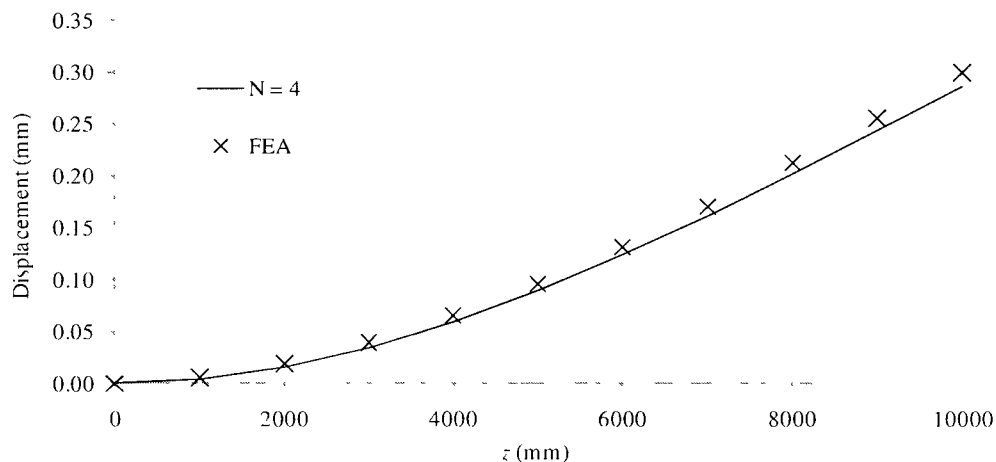


Fig. 7.10 - Displacement of cross-section versus longitudinal coordinate, z

The program written to model this problem uses a minimum of $N = 4$ shape functions. In general there is good agreement between the two methods. Displacements predicted by the current method are slightly smaller than those predicted by finite element method near the free end. This may be because of deflection from shearing of the web, which is not taken into account in the narrow plate formulation. Figure 7.11 shows the longitudinal stress in the top flange versus z .

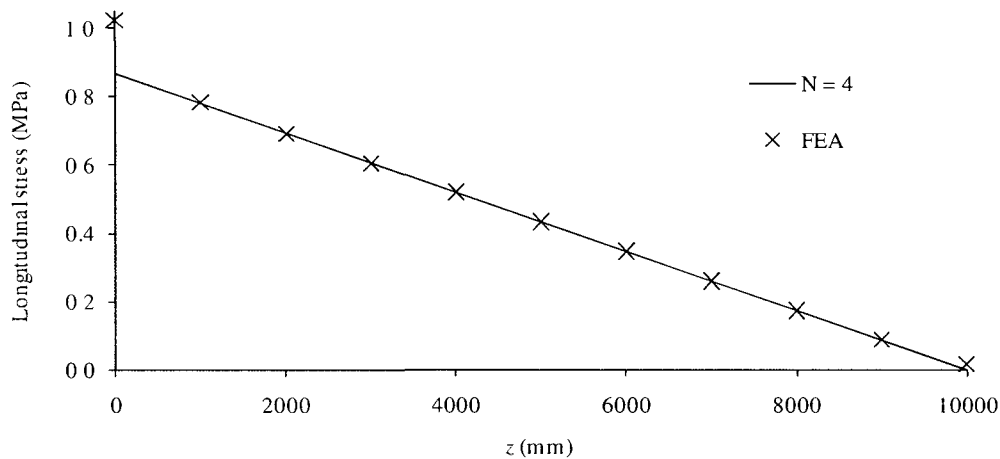


Fig. 7.11 - Longitudinal stress in the top flange versus longitudinal coordinate, z

The stress linearly decreases with z and vanishes at the free end as predicted by simple Euler-Bernoulli beam theory. There is good agreement between the current method and the finite element analysis except at the fixed end. This may be because of the transverse fixity imposed by the support which is not taken into account by the narrow plate formulation.

7.7 Example 4: I-beam Subjected to Distortion

The beam of Example 3 is shortened to 5.0m and subjected to a different load combination.

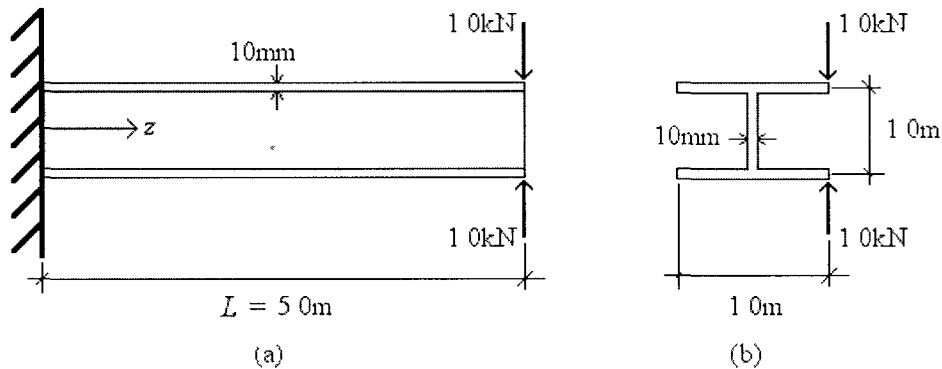


Fig. 7.12 - Dimensions and loading for Example 4: (a) Elevation and (b) Cross-section

Two 1.0 kN are applied at the free end to the outside edges of the flanges. This load will cause only distortion of the section with no bending, twisting nor shear lag. Figure 7.13 shows the deformed shape of the beam as predicted by the current method.

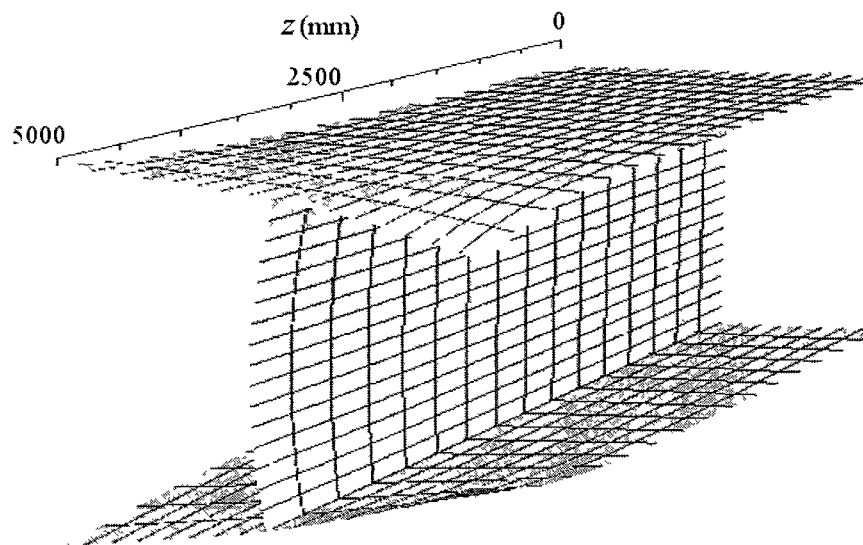


Fig. 7.13 - Deformed beam (deformation scale = 15)

Displacements were scaled by a factor of 15. The displacements are consistent with the applied load and boundary conditions. Bending of the flanges and the web are most pronounced at the free end. There is bending of the web on account of the rigid connection with the flanges. Figure 7.14 shows the deformed shape of the cross section at different distances from the fixed end.

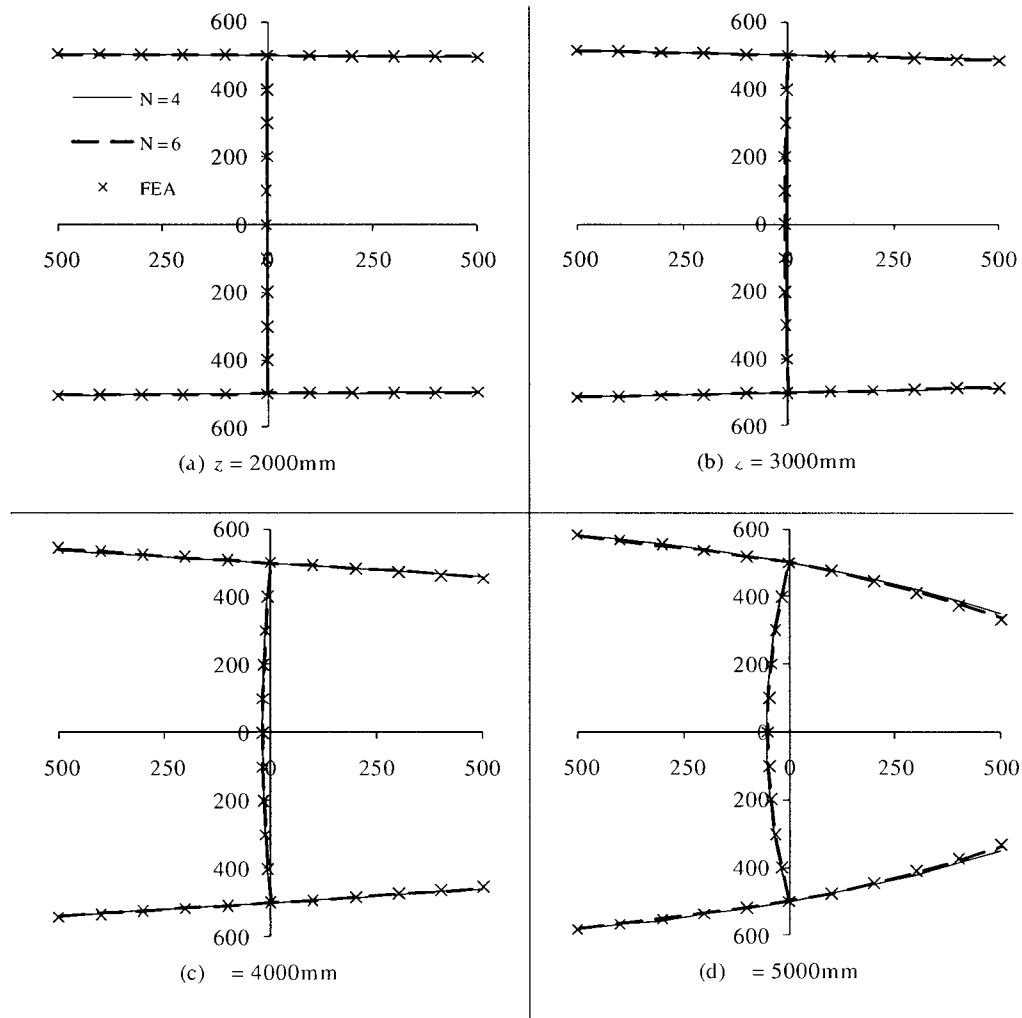


Fig 7.14 - Deformed shape of cross section at (a) $z = 2000\text{mm}$, (b) $z = 3000\text{mm}$, (c) $z = 4000\text{mm}$, and (d) $z = 5000\text{mm}$ (deformation scale = 15)

Excellent agreement with the finite element analysis is obtained using $N = 4$ in the current formulation. The distortion of the section decreases quickly away from the free end and is almost

negligible at $z = 3\text{m}$. For the loading and boundary conditions of this example, transverse stresses are typically larger than longitudinal stresses. Figure 7.15 shows the largest transverse stress both in the web and flanges versus z .

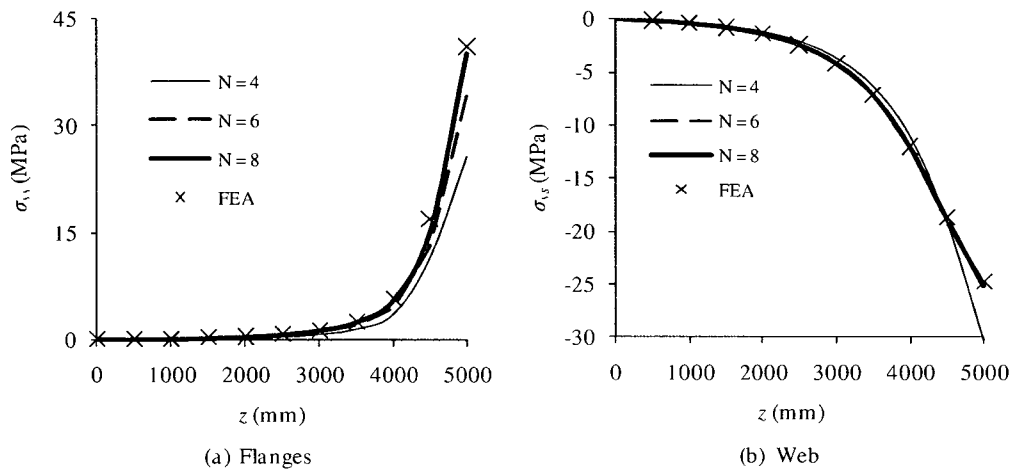


Fig. 7.15 - Largest transverse stresses, $\sigma_{x,y}$, in the cross section versus longitudinal coordinate, z , in (a) the flanges and (b) the web

Both stresses are largest at the free end and rapidly decrease towards the fixed end. Two meters from the free end, the stresses in the flanges and in the web are almost negligible. Very good agreement with finite elements is achieved using $N = 8$ shape functions.

7.8 Example 5: I-beam Subjected to Shear Lag

The beam in Example 4 is subject to a different loading.

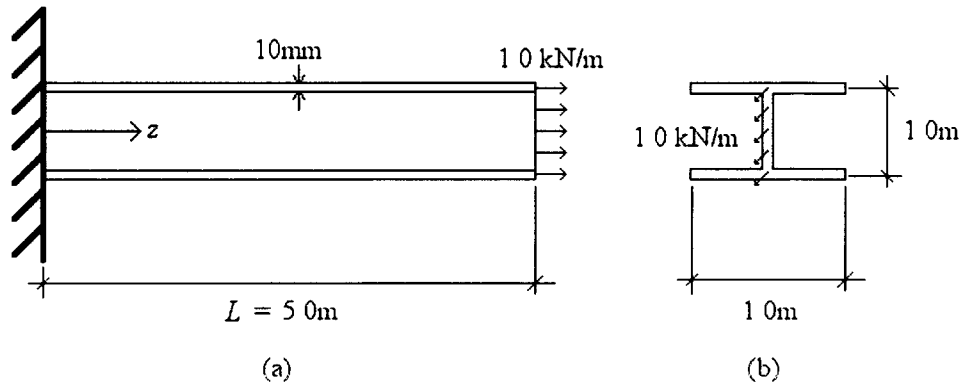


Fig. 7.16 - Dimensions and loading for Example 5: (a) Elevation and (b) Cross-section

A 1.0 kN/m longitudinal load is applied to the web at the free end. The load will cause extension and shear lag in both the web and flanges. Figure 7.17 shows longitudinal stress contours both in the flanges and web as predicted by the current method.

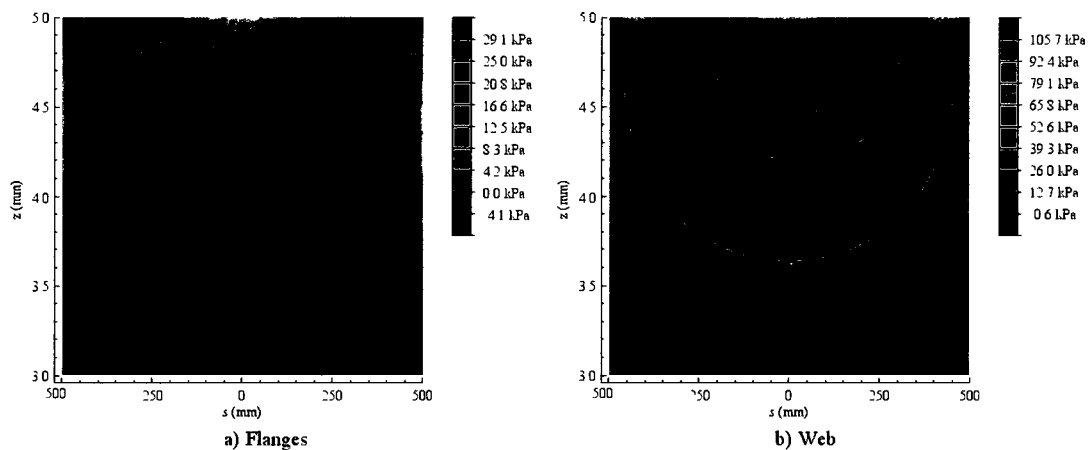


Fig 7.17 - Longitudinal stress, σ_{zz} , contour plots for (a) flanges and (b) web

In general, the stress field is consistent with the applied loading and boundary conditions.

Stresses in the web are largest at the free end and rapidly decrease away from the applied load.

Stresses in the flanges are smallest at the free end and increase as the end load is transmitted from the web. Past 2.0m from the free end, the stress field is a uniform 33.3 MPa both in the flanges and web. Stress concentrations in the flanges are most important at the connection with the web ($s = 0$). Oscillations in the stress field can be observed in web and flange at the free end. These fluctuations are discussed in more detail below. Figure 7.18 shows the stresses at the middle of the flanges and in the middle of the web versus z .

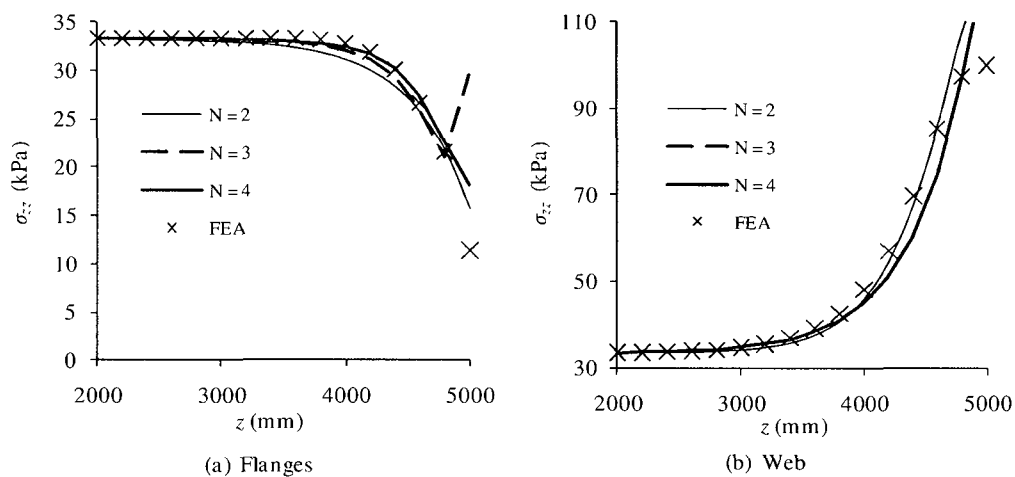


Fig. 7.18 - Longitudinal stress, σ_{zz} , versus longitudinal coordinate, z , in the middle of (a) the flanges and (b) the web

There is good agreement with the finite element analysis except at the free end. The discrepancy at the free may be due in part to the current method not taking into account out-of-plane bending of the plates. Distortion of the section is most significant at the free end because of transverse shortening of the plates under the applied load. Also, from static equilibrium, longitudinal stresses must be 0MPa in the flanges and 100MPa in the web. These conditions are only implemented approximately by the current method as can be seen in Fig. 7.19.

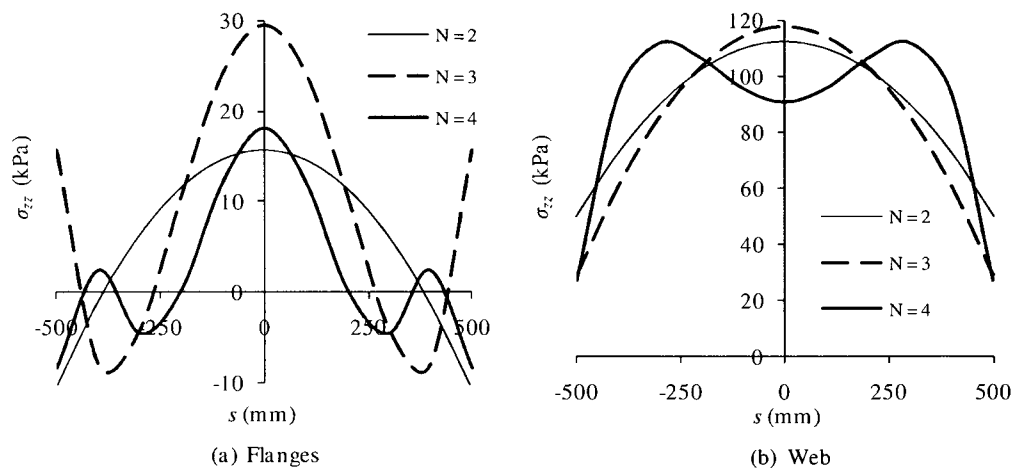


Fig. 7.19 - Longitudinal stress, σ_{zz} , versus transverse coordinate, s , at the free end in (a) the flanges and (b) the web

The stress distributions converge to their correct values (0 in the flanges and 100MPa in the web) as more shape functions are included. Oscillations in the edge stress are significant even though adequate accuracy is achieved away from the cantilever tip.

7.9: Example 6: Box Section Subjected to Shear Lag

Figure 7.20 shows the geometry and load on the box beam under consideration.

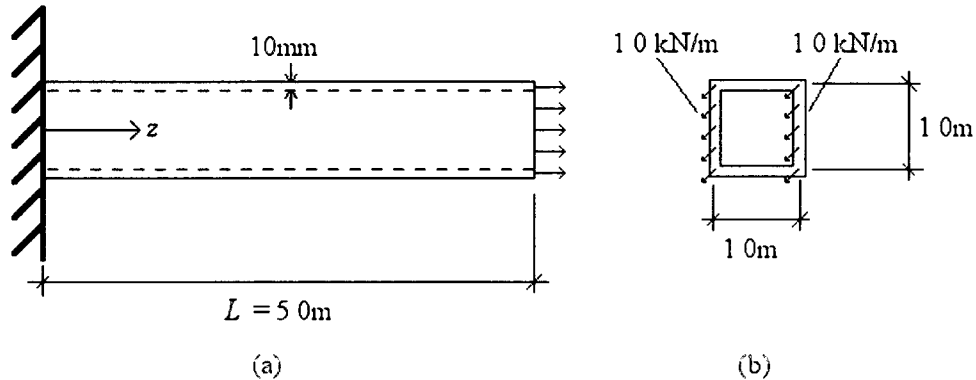


Fig. 7.20 - Dimensions and loading for Example 6: (a) Elevation and (b) Cross-section

The beam is 5.0m long, is free at one end and clamped at the other. The z coordinate is measured from the fixed end. The webs (vertical plates) and flanges (top and bottom plates) are 1m wide and 10mm thick. One kilonewton per meter tractions are applied to the webs as shown. This load causes extension and shear lag in the webs and flanges. Figure 7.21 shows the longitudinal stress contours both in the flanges and webs as predicted by the current method.

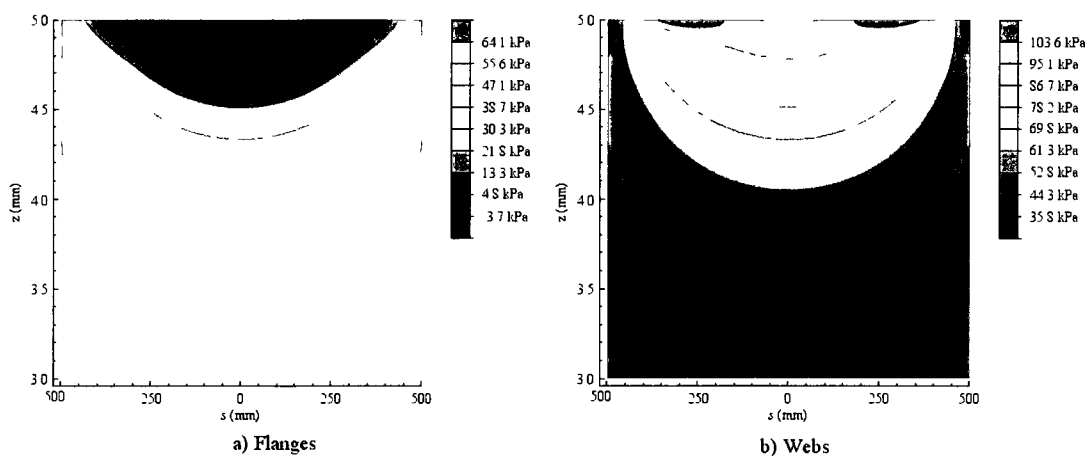


Fig. 7.21 - Longitudinal stress, σ_{zz} , contour plots for (a) flanges and (b) webs

In general, the stress field is consistent with the applied loading and boundary conditions.

Stresses in the webs are largest at the free end rapidly decrease away from the applied load.

Stresses in the flanges are smallest at the free end and increase as the end load is transmitted from the webs. Past 2.0m from the free end, the stress is uniform, and a 50MPa stress is observed in the webs and flanges. The stress fields in the flanges and webs are mirror images of one another. Adding a uniform 1.0kN/m compressive load to the free end would result in an equal and opposite stress distribution to the one observed. Figure 7.22 shows the stresses at the middle of the flanges and middle of the web versus z .

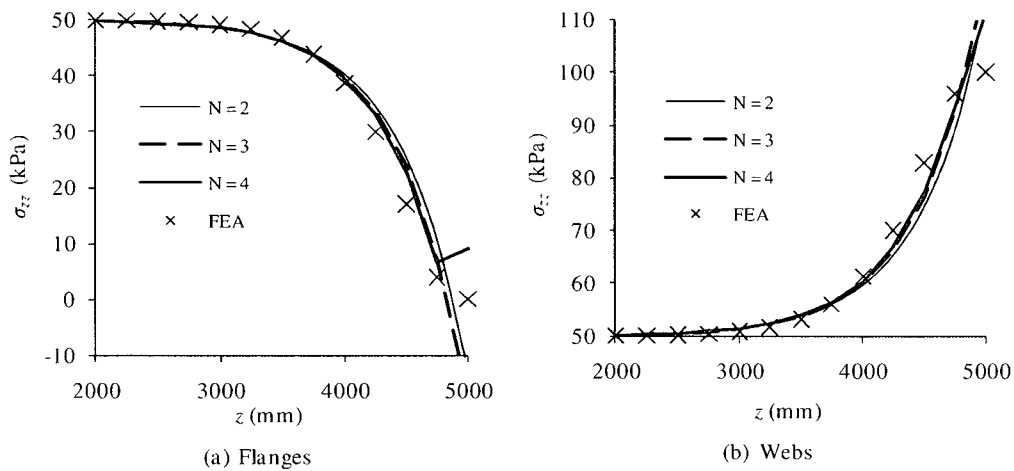


Fig. 7.22 - Longitudinal stress, σ_{zz} , versus longitudinal coordinate, z , in the middle of (a) the flanges and (b) the webs

Very good agreement with finite elements is observed except at the free end. The sources of disagreement at the free end were explained in Section 7.8.

CHAPTER 8 - CONCLUSIONS AND RECOMMENDATIONS

8.1 Summary

The principle of stationary potential energy was used with series expansions for the displacement fields to formulate the field and boundary conditions equations for elastic plates and thin-walled beams. The formulation was based on the assumptions of thin-walled plate theory. The solution to the coupled differential field equations was expressed in terms of matrix exponentials. The solution to the boundary condition equations was expressed in terms of the aforementioned solution. The method developed was successfully used to solve several problems. Comparisons with finite element solutions demonstrated the ability of the model to accurately capture the behaviour of thin walled beams and plates with a relatively small number of degrees of freedoms. In general, no more than ten shape functions were required to obtain a result within a few percent of the finite element solution.

8.2 Advantages and Disadvantages of the Current Method

A main advantage of the current method, in comparison with the shell FEA, is its computational efficiency. The method developed can very accurately predict the displacements and stresses with few degrees of freedom. Excellent results for both displacements and stresses are obtained using less than eight shape functions. In comparison, FEA required several thousand degrees of freedom to obtain results of comparable accuracy.

Within a robust and generalized computer program, the simplicity of the new method relative to a shell FEA has the potential to greatly reduce the modeling effort required by the analyst. The method could be used to model structures assembled from thin walled beams.

The principal shortcoming of the current method was the need to solve a set of linear equations containing both very large and very small values. Conventional solvers were generally ineffective for this type of system.

In addition, the current method involves solving an eigenvalue problem. Though determining the system eigenvalues and eigenvectors was straight forward using conventional solvers, some problems were encountered when using conventional solvers to determine the associated Jordan form.

Finally, it was observed that the current method behaves relatively poorly in the neighbourhood of line and point loads. Away from applied loads the method converged very rapidly.

8.3 Recommendation for Future Work

Further work on the current methodology should focus primarily on developing computer subroutines capable of calculating the Jordan form of the system matrices and capable of solving the boundary condition equations. Those were the principal numerical problems encountered in the present study and they should be solved before extensions to the theory are sought.

The methodology developed in the present research lends itself to a variety of possibilities for further research. Possible improvements to the model could include:

1. Orthogonalization of the differential equations through the selection of orthogonal shape functions.
2. Deriving approximate closed form solutions for common types of loading using appropriately chosen shape functions

It may be possible to orthogonalize the equilibrium differential equations for certain sections. It should also be possible to derive closed form solutions for specific loading on specific sections if a limited number of approximate shape functions are used. Possible generalizations to the current formulation include:

3. Modelling second order effects.
4. Analysis under dynamic loading.
5. Modelling thick plate behaviour.

Again, the equations derived for these generalizations may not be solvable though the dynamic equations should have constant coefficients. Possible modifications to the problem geometry include:

6. Modelling of curved girders, beams and plates.
7. Modelling plates of variable thickness or width.
8. Modelling girders with skewed ends.

Although formulating the equilibrium equations will be fairly straight forward, the equations associated with curved or variable width plates will have variable coefficient and may be solvable only using numeric techniques.

APPENDIX A - VARIATION OF STRAIN ENERGY FOR A NARROW PLATE

The purpose of this appendix is to derive in detail the field and boundary equations for a single narrow plate. This section uses the coordinate system and symbols introduced in Chapter 2.

A.1 Differentiation of a Double Series

In this section we will consider the derivative of

$$\tilde{\Pi} = \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} X_i Y_j \quad (\text{A.1})$$

with respect to X_i and Y_j . Differentiating with respect to X_k we have

$$\begin{aligned} \frac{\partial \tilde{\Pi}}{\partial X_k} &= \frac{\partial}{\partial X_k} \left(\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} X_i Y_j \right) \\ &= \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} Y_j \frac{\partial}{\partial X_k} X_i \\ &= \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} Y_j \delta_{ik} \\ &= \sum_{j=1}^N \gamma_{kj} Y_j \end{aligned} \quad (\text{A.2})$$

Similarly, we have

$$\frac{\partial \tilde{\Pi}}{\partial Y_k} = \sum_{i=1}^N \gamma_{ik} X_i \quad (\text{A.3})$$

If $Y_k = X_k$ for all k , then

$$\begin{aligned} \frac{\partial \tilde{\Pi}}{\partial X_k} &= \frac{\partial}{\partial X_k} \left(\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} X_i X_j \right) \\ &= \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \frac{\partial}{\partial X_k} (X_i X_j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \left(X_j \frac{\partial}{\partial X_k} X_i + X_i \frac{\partial}{\partial X_k} X_j \right) \\
&= \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} (X_j \delta_{ik} + X_i \delta_{jk}) \\
&= \sum_{i=1}^N (\gamma_{ik} + \gamma_{ki}) X_i
\end{aligned} \tag{A.4}$$

A.2 Variation of the Strain Energy

The expression for the internal strain energy is given by Equation 2.13:

$$\begin{aligned}
\Pi_s = \frac{1}{2} \int_0^L \left(\sum_{j=1}^N \sum_{i=1}^N ({}_2c_{ij} U_i(z) U_j(z) + {}_1c_{ij} U_i'(z) U_j'(z) + 2{}_3c_{ij} U_i''(z) U_j''(z) + {}_0c_{ij} U_i''(z) U_j''(z)) \right. \\
\left. + Eh \left(b W'(z)^2 + \frac{b^3}{12} V''(z)^2 \right) \right) dz
\end{aligned} \tag{A.5}$$

Using the identities A.2 and A.4, we can differentiate Π_s with respect to the generalized displacements yielding

$$\frac{\partial \Pi_s}{\partial U_k} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_2c_{jk} + {}_2c_{kj}) U_j(z) + 2{}_3c_{jk} U_j''(z) \right) dz \tag{A.6a}$$

$$\frac{\partial \Pi_s}{\partial U_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^N ({}_1c_{jk} + {}_1c_{kj}) U_j'(z) dz \tag{A.6b}$$

$$\frac{\partial \Pi_s}{\partial U_k''} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_0c_{jk} + {}_0c_{kj}) U_j''(z) + 2{}_3c_{kj} U_j(z) \right) dz \tag{A.6c}$$

$$\frac{\partial \Pi_s}{\partial V''} = \frac{Ehb^3}{12} \int_0^L V''(z) dz \tag{A.6d}$$

$$\frac{\partial \Pi_s}{\partial W'} = Ehb \int_0^L W'(z) dz \tag{A.6e}$$

The variation of the strain energy is

$$\delta \Pi_s = \sum_{k=0}^N \left(\frac{\partial \Pi_s}{\partial U_k} \delta U_k + \frac{\partial \Pi_s}{\partial U_k'} \delta U_k' + \frac{\partial \Pi_s}{\partial U_k''} \delta U_k'' \right) + \frac{\partial \Pi_s}{\partial W'} \delta W' + \frac{\partial \Pi_s}{\partial V''} \delta V'' \quad (\text{A.7})$$

From Equations A.6, by substituting into Equation A.7 we have

$$\begin{aligned} \delta \Pi_s = & \sum_{k=1}^N \left(\int_0^L \sum_{j=1}^N \left({}_2c_{kj} \cdot U_j(z) + {}_3c_{jk} \cdot U_j''(z) \right) \delta U_k dz + \int_0^L \sum_{j=1}^N {}_1c_{kj} \cdot U_j'(z) \delta U_k' dz \right. \\ & \left. + \int_0^L \sum_{j=1}^N \left({}_0c_{kj} \cdot U_j''(z) + {}_3c_{kj} \cdot U_j(z) \right) \delta U_k'' dz \right) + \frac{Ehb^3}{12} \int_0^L V''(z) \delta V'' dz + Ehb \int_0^L W'(z) \delta W' dz \end{aligned} \quad (\text{A.8})$$

where we have used the fact that ${}_l c_{ij} = {}_i c_{jl}$ for $\ell = 0, 1$, or 2 . The terms containing $\delta U_k'$ and $\delta W'$

must be integrated by parts once while the terms containing $\delta U_k''$ and $\delta V''$ must be integrated

twice by parts. Performing these integrations, the expression for $\delta \Pi_s$ becomes

$$\begin{aligned} \delta \Pi_s = & \int_0^L \sum_{j=1}^N \left({}_2c_{kj} \cdot U_j(z) + {}_3c_{jk} \cdot U_j''(z) \right) \delta U_k dz \\ & + \sum_{i=1}^N \left(\left[{}_1c_{kj} \cdot U_j'(z) \delta U_k \right]_0^L - \int_0^L {}_1c_{kj} \cdot U_j''(z) \delta U_k dz \right) \\ & + \sum_{j=1}^N \left(\left[({}_0c_{kj} \cdot U_j''(z) + {}_3c_{kj} \cdot U_j(z)) \delta U_k' \right]_0^L - \left[({}_0c_{kj} \cdot U_j'''(z) + {}_3c_{kj} \cdot U_j'(z)) \delta U_k \right]_0^L \right. \\ & \left. + \sum_{j=1}^N \int_0^L ({}_0c_{kj} \cdot U_j''''(z) + {}_3c_{kj} \cdot U_j''(z)) \delta U_k dz \right. \\ & \left. + \frac{Ehb^3}{12} \left[V''(z) \delta V' \right] - \frac{Ehb^3}{12} \left[V'''(z) \delta V \right] + \frac{Ehb^3}{12} \int_0^L V''''(z) \delta V dz \right. \\ & \left. + Ehb \left[W'(z) \delta W \right] - Ehb \int_0^L W''(z) \delta W dz \right) \end{aligned} \quad (\text{A.9})$$

Factoring δU_k , $\delta U_k'$, and $\delta U_k''$ reduces the expression to

$$\begin{aligned}
\delta \Pi_s = & \sum_{k=1}^N \left(\int_0^L \sum_{j=1}^N \left({}_0c_{kj} \cdot U_j^{(4)}(z) + {}_5c_{kj} \cdot U_j^{(3)}(z) + {}_2c_{kj} \cdot U_j(z) \right) \delta U_k dz \right. \\
& + \left[\sum_{j=1}^N \left(-{}_0c_{kj} \cdot U_j^{(4)}(z) + {}_7c_{kj} \cdot U_j'(z) \right) \delta U_k \right]_0^L \\
& + \left. \left[\sum_{j=1}^N \left({}_0c_{kj} \cdot U_j^{(3)}(z) + {}_3c_{kj} \cdot U_j(z) \right) \delta U_k' \right]_0^L \right) \\
& + \frac{Ehb^3}{12} \left[V^{(2)}(z) \delta V' \right]_0^L - \frac{Ehb^3}{12} \left[V^{(3)}(z) \delta V \right]_0^L + \frac{Ehb^3}{12} \int_0^L V^{(4)}(z) \delta V dz \\
& + Ehb \left[W'(z) \delta W \right]_0^L - Ehb \int_0^L W^{(2)}(z) \delta W dz
\end{aligned} \tag{A.10a}$$

in which we recall the definitions 2.13b-e

$${}_0c_{ij} = \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i(s) \psi_j(s) ds \tag{A.10b}$$

$${}_1c_{ij} = \frac{Eh^3}{6(1+\nu)} \int_{-b/2}^{+b/2} \psi_i'(s) \psi_j'(s) ds \tag{A.10c}$$

$${}_2c_{ij} = \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i''(s) \psi_j''(s) ds \tag{A.10d}$$

$${}_3c_{ij} = \nu \frac{Eh^3}{12(1-\nu^2)} \int_{-b/2}^{+b/2} \psi_i'(s) \psi_j''(s) ds \tag{A.10e}$$

A.3 Variation of the Load Potential due to Volumetric Loads

The expression for the work due to a volumetric load, as given by Equation 2.17, is

$$\Pi_D = \int_0^L \left(\sum_{i=1}^N \left(U_i(z) t_i(z) + U_i'(z) m_i(z) \right) + V(z) p_+(z) + V'(z) m_+(z) + W(z) p_-(z) \right) dz \tag{A.11}$$

The variation of the load potential is then

$$\begin{aligned}
\delta \Pi_D = & \sum_{k=1}^N \left(\frac{\partial \Pi_D}{\partial U_k} \delta U_k + \frac{\partial \Pi_D}{\partial U_k'} \delta U_k' \right) + \frac{\partial \Pi_D}{\partial W} \delta W + \frac{\partial \Pi_D}{\partial V} \delta V + \frac{\partial \Pi_D}{\partial V'} \delta V' \\
= & \int_0^L \left(\sum_{k=1}^N \left(t_k(z) \delta U_k + m_k(z) \delta U_k' \right) + p_+(z) \delta V + m_+(z) \delta V' + p_-(z) \delta W \right) dz
\end{aligned} \tag{A.12}$$

The terms containing $\delta U_k'$ and $\delta V'$ need to be integrated by parts. Performing the integrations yields

$$\begin{aligned} \delta \Pi_D = & \int_0^L \left(\sum_{k=1}^N (t_k(z) - m_k'(z)) \delta U_k + (p_s(z) - m_s'(z)) \delta V + p_z(z) \delta W \right) dz \\ & + \left[\sum_{k=1}^N m_k(z) \delta U_k \right]_0^L + [m_s(z) \delta V]_0^L \end{aligned} \quad (\text{A.13})$$

Since loads applied to the ends will be considered in Section A.4, all volumetric loads (including $m_k(z)$ and $m_s(z)$) can be assumed to vanish at boundaries $z = 0$ and $z = L$. The last terms of Equation A.28 can therefore be dropped and the variation of the energy from the distributed load becomes

$$\delta \Pi_D = \int_0^L \left(\sum_{k=1}^N (t_k(z) - m_k'(z)) \delta U_k + (p_s(z) - m_s'(z)) \delta V + p_z(z) \delta W \right) dz \quad (\text{A.14})$$

A.4 Variation of the Load Potential due to End Loads

The expression for the work done by pressures and tractions applied to the end $z = 0$ and $z = L$, as given by Equation 2.19, is

$$\begin{aligned} \Pi_B = & \sum_{i=1}^N \left({}_0T_i \cdot U_i(z) \Big|_{z=0} + {}_0M_i \cdot U_i'(z) \Big|_{z=0} \right) + {}_0P_s \cdot V(z) \Big|_{z=0} + {}_0M_s \cdot V'(z) \Big|_{z=0} + {}_0P_z \cdot W(z) \Big|_{z=0} \\ & + \sum_{i=1}^N \left({}_LT_i \cdot U_i(z) \Big|_{z=L} + {}_LM_i \cdot U_i'(z) \Big|_{z=L} \right) + {}_LP_s \cdot V(z) \Big|_{z=L} + {}_LM_s \cdot V'(z) \Big|_{z=L} + {}_LP_z \cdot W(z) \Big|_{z=L} \end{aligned} \quad (\text{A.15})$$

The variation of the load potential is then

$$\begin{aligned} \Pi_B = & \sum_{i=1}^N \left({}_0T_i \cdot \delta U_i(z) \Big|_{z=0} + {}_0M_i \cdot \delta U_i'(z) \Big|_{z=0} \right) + {}_0P_s \cdot \delta V(z) \Big|_{z=0} + {}_0M_s \cdot \delta V'(z) \Big|_{z=0} \\ & + \sum_{i=1}^N \left({}_LT_i \cdot \delta U_i(z) \Big|_{z=L} + {}_LM_i \cdot \delta U_i'(z) \Big|_{z=L} \right) + {}_LP_s \cdot \delta V(z) \Big|_{z=L} + {}_LM_s \cdot \delta V'(z) \Big|_{z=L} \\ & + {}_0P_z \cdot \delta W(z) \Big|_{z=0} + {}_LP_z \cdot \delta W(z) \Big|_{z=L} \end{aligned} \quad (\text{A.16})$$

A.5 Equilibrium Equations

The variation of the total potential energy, Π , is

$$\delta\Pi = \delta(\Pi_s - \Pi_D - \Pi_B) = \delta\Pi_s - \delta\Pi_D - \delta\Pi_B \quad (\text{A.17})$$

At equilibrium, the variation of the total potential energy is zero. Setting $\delta\Pi$ to zero and using Equations A.10, A.14, and A.17 a to express $\delta\Pi$ in terms of the generalized displacements, one obtains

$$\begin{aligned} & \sum_{k=1}^N \left\{ \int_0^L \sum_{j=1}^N \left({}_0c_{kj} \cdot U_j''''(z) + {}_5c_{kj} U_j''(z) + {}_2c_{kj} U_j(z) - (t_k(z) - m_k'(z)) \right) \delta U_k dz \right. \\ & + \left[\sum_{j=1}^N \left(-{}_0c_{kj} \cdot U_j'''(z) + {}_7c_{kj} U_j'(z) - {}_L T_k \right) \delta U_k \right]_{z=L} - \left[\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(z) + {}_7c_{kj} U_j'(z) + {}_0 T_k \right) \delta U_k \right]_{z=0} \\ & + \left[\sum_{j=1}^N \left({}_0c_{kj} U_j''(z) + {}_3c_{kj} U_j(z) - {}_L M_k \right) \delta U_k' \right]_{z=L} - \left[\sum_{j=1}^N \left({}_0c_{kj} U_j''(z) + {}_3c_{kj} U_j(z) + {}_0 M_k \right) \delta U_k' \right]_{z=0} \left. \right\} \\ & + \int_0^L \left(\frac{Ehb^3}{12} V''''(z) - (p_s(z) - m_s'(z)) \right) \delta V dz \\ & - \left[\left(\frac{Ehb^3}{12} V'''(z) + {}_L P_s \right) \delta V \right]_{z=L} + \left[\left(\frac{Ehb^3}{12} V'''(z) - {}_0 P_s \right) \delta V \right]_{z=0} \\ & + \left[\left(\frac{Ehb^3}{12} V''(z) - {}_L M_s \right) \delta V' \right]_{z=L} - \left[\left(\frac{Ehb^3}{12} V''(z) + {}_0 M_s \right) \delta V' \right]_{z=0} \\ & - \int_0^L (Ehb W''(z) + p(z)) \delta W dz + \left[(Ehb W'(z) - {}_L P_c) \delta W \right]_{z=L} - \left[(Ehb W'(z) + {}_0 P_c) \delta W \right]_{z=0} = 0 \end{aligned}$$

where ${}_5c_{ij} = {}_3c_{ij} + {}_3c_{ji} - {}_1c_{ij}$ and ${}_7c_{ij} = {}_1c_{ij} - {}_3c_{ij}$. Since δU_k , δV , and δW are arbitrary, their coefficients must vanish. The governing differential equations for the system are therefore

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''''(z) + {}_5c_{kj} U_j''(z) + {}_2c_{kj} U_j(z) \right) = t_k - m_k' \quad (\text{A.19a})$$

$$\frac{Ehb^3}{12} V''''(z) = p_s(z) - m_s'(z) \quad (\text{A.19b})$$

$$Ehb W''(z) = -p(z) \quad (\text{A.19c})$$

and the boundary condition equations for the system are

$$\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(L) + {}_7c_{kj} U_j'(L) \right) = {}_LT_k \quad \text{or} \quad \delta U_k(z) \Big|_{z=L} = 0 \quad (\text{A.20a})$$

$$\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(0) + {}_7c_{kj} U_j'(0) \right) = -{}_0T_k \quad \text{or} \quad \delta U_k(z) \Big|_{z=0} = 0 \quad (\text{A.20b})$$

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''(L) + {}_3c_{kj} U_j(L) \right) = {}_LM_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=L} = 0 \quad (\text{A.20c})$$

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''(0) + {}_3c_{kj} U_j(0) \right) = -{}_0M_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=0} = 0 \quad (\text{A.20d})$$

$$\frac{Ehb^3}{12} V'''(L) = -{}_LP, \quad \text{or} \quad \delta V(z) \Big|_{z=L} = 0 \quad (\text{A.20e})$$

$$\frac{Ehb^3}{12} V'''(0) = {}_0P, \quad \text{or} \quad \delta V(z) \Big|_{z=0} = 0 \quad (\text{A.20f})$$

$$\frac{Ehb^3}{12} V''(L) = {}_LM, \quad \text{or} \quad \delta V'(z) \Big|_{z=L} = 0 \quad (\text{A.20g})$$

$$\frac{Ehb^3}{12} V''(0) = -{}_0M, \quad \text{or} \quad \delta V'(z) \Big|_{z=0} = 0 \quad (\text{A.20h})$$

$$EhbW'(L) = {}_LP \quad \text{or} \quad \delta W(z) \Big|_{z=L} = 0 \quad (\text{A.20i})$$

$$EhbW'(0) = -{}_0P. \quad \text{or} \quad \delta W(z) \Big|_{z=0} = 0 \quad (\text{A.20j})$$

Equations A.19a and A.20a through A.20d must apply to all $U_k(z)$ from $k=1$ to $k=N$. For the boundary condition equations (Equations A.20) one must choose either the equation on the right or the equation on the left depending on the physical boundary conditions. Equation A.19a can be expressed for all k as

$${}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}^{IV}(z)_{N \times 1} + {}_5\mathbf{c}_{N \times N} \cdot \mathbf{U}^{II}(z)_{N \times 1} + {}_2\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} = \mathbf{t}(z)_{N \times 1} - \mathbf{m}^I(z)_{N \times 1} \quad (\text{A.21})$$

APPENDIX B - VARIATION OF STRAIN ENERGY FOR A DEEP PLATE

The purpose of this appendix is to derive in detail the field and boundary equations for a single deep plate. This section uses the coordinate system and symbols introduced in Chapters 2 and 3.

B.1 Variation of the Strain Energy

The expression for the internal strain energy (Equation 3.7) is

$$\begin{aligned} \Pi_s = & \frac{1}{2} \int_0^L \left(\sum_{j=1}^N \sum_{i=1}^N \left({}_0a_{ij} \cdot W_i'(z) W_j'(z) + 2 {}_1a_{ij} \cdot W_i'(z) V_j(z) + {}_2a_{ij} \cdot W_i'(z) W_j'(z) \right. \right. \\ & + \sum_{j=1}^N \sum_{i=1}^N \left. {}_3a_{ij} \cdot V_i'(z) V_j'(z) + {}_4a_{ij} \cdot W_i(z) W_j(z) + 2 {}_5a_{ij} \cdot W_i(z) V_j'(z) \right. \\ & \left. \left. + \sum_{j=1}^N \sum_{i=1}^N {}_2c_{ij} U_i(z) U_j(z) + {}_1c_{ij} U_i'(z) U_j'(z) + 2 {}_3c_{ij} U_i''(z) U_j(z) + {}_0c_{ij} U_i''(z) U_j''(z) \right) \right) dz \end{aligned} \quad (B.1)$$

Using Equations A.1 and A.4, we can differentiate Π_s with respect to the generalized displacements

$$\frac{\partial \Pi_s}{\partial V_k} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_2a_{kj} + {}_2a_{jk}) V_j(z) + 2 {}_1a_{jk} W_j'(z) \right) dz \quad (B.2a)$$

$$\frac{\partial \Pi_s}{\partial V_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_3a_{kj} + {}_3a_{jk}) V_j'(z) + 2 {}_5a_{jk} W_j(z) \right) dz \quad (B.2b)$$

$$\frac{\partial \Pi_s}{\partial W_k} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_4a_{kj} + {}_4a_{jk}) W_j(z) + 2 {}_5a_{jk} V_j'(z) \right) dz \quad (B.2c)$$

$$\frac{\partial \Pi_s}{\partial W_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_0a_{kj} + {}_0a_{jk}) W_j'(z) + 2 {}_1a_{jk} V_j(z) \right) dz \quad (B.2d)$$

$$\frac{\partial \Pi_s}{\partial U_k} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_2c_{jk} + {}_2c_{kj}) U_j(z) + 2 {}_3c_{jk} U_j''(z) \right) dz \quad (B.2e)$$

$$\frac{\partial \Pi_s}{\partial U_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_1c_{jk} + {}_1c_{kj}) U_j'(z) \right) dz \quad (B.2f)$$

$$\frac{\partial \Pi_s}{\partial U_k''} = \frac{1}{2} \int_0^L \sum_{j=1}^N \left(({}_0c_{jk} + {}_0c_{kj}) U_j''(z) + 2{}_3c_{kj} U_j(z) \right) dz \quad (\text{B.2g})$$

The variation of the strain energy is

$$\delta \Pi_s = \sum_{i=1}^N \left(\frac{\partial \Pi_s}{\partial U_i} \delta U_i + \frac{\partial \Pi_s}{\partial U_i'} \delta U_i' + \frac{\partial \Pi_s}{\partial U_i''} \delta U_i'' + \frac{\partial \Pi_s}{\partial V_i} \delta V_i + \frac{\partial \Pi_s}{\partial W_i} \delta W_i + \frac{\partial \Pi_s}{\partial V_i'} \delta V_i' + \frac{\partial \Pi_s}{\partial W_i'} \delta W_i' \right) \quad (\text{B.3})$$

Using Equations B.2 to express the partial derivatives we have

$$\begin{aligned} \delta \Pi_s = \sum_{k=1}^N \sum_{j=1}^N \int_0^L & \left(({}_2a_{kj} V_j(z) + {}_1a_{jk} W_j'(z)) \delta V_k + ({}_3a_{kj} V_j'(z) + {}_5a_{jk} W_j(z)) \delta V_k' \right. \\ & \left. ({}_4a_{kj} W_j(z) + {}_5a_{jk} V_j'(z)) \delta W_k + ({}_0a_{kj} W_j'(z) + {}_1a_{jk} V_j(z)) \delta W_k' \right. \\ & \left. ({}_2c_{kj} \cdot U_j(z) + {}_3c_{jk} \cdot U_j''(z)) \delta U_k + {}_1c_{kj} \cdot U_j'(z) \delta U_k' \right. \\ & \left. ({}_0c_{kj} \cdot U_j''(z) + {}_3c_{kj} \cdot U_j(z)) \delta U_k'' \right) dz \end{aligned} \quad (\text{B.4})$$

where we have used the fact that ${}_lc_{ij} = {}_lc_{ji}$ if ℓ is 0, 1, or 2 and ${}_la_{ij} = {}_la_{ji}$ if ℓ is 0, 2, 3, or 4.

Integrating once by parts the terms containing $\delta U_i'$, $\delta V_i'$, and $\delta W_i'$ and twice the term

containing $\delta U_i''$, we have

$$\begin{aligned} \delta \Pi_s = \sum_{k=1}^N \sum_{j=1}^N & \left(\int_0^L (-{}_3a_{kj} V_j''(z) + {}_6a_{jk} W_j'(z) + {}_2a_{kj} V_j(z)) \delta V_j dz \right. \\ & + \int_0^L ({}_0a_{kj} W_j''(z) - {}_6a_{kj} V_j'(z) + {}_4a_{kj} W_j(z)) \delta W_j dz \\ & + \int_0^L ({}_0c_{kj} \cdot U_j''''(z) + {}_5c_{kj} \cdot U_j''(z) + {}_2c_{kj} \cdot U_j(z)) \delta U_k dz \\ & + \left[({}_3a_{kj} \cdot V_j'(z) + {}_5a_{jk} \cdot W_j(z)) \delta V_j \right]_0^L + \left[({}_0a_{kj} \cdot W_j'(z) + {}_1a_{kj} \cdot V_j(z)) \delta W_j \right]_0^L \\ & \left. + \left[(-{}_0c_{kj} \cdot U_j'''(z) + {}_7c_{kj} \cdot U_j'(z)) \delta U_k \right]_0^L + \left[({}_0c_{kj} \cdot U_j''(z) + {}_3c_{kj} \cdot U_j(z)) \delta U_k' \right]_0^L \right) \end{aligned} \quad (\text{B.5})$$

where expressions for ${}_5c_{ij}$, ${}_7c_{ij}$, and ${}_6a_{ij}$ are provided in Equations 2.13 and 3.7.

B.2 Variation of the Load Potential due to Volumetric Loads

The expression for the load potential due to a volumetric load, as given by Equation 3.10, is

$$\Pi_D = \int_0^L \sum_{i=1}^N (V_i(z)p_i(z) + W_i(z)q_i(z) + U_i(z)t_i(z) + U_i'(z)m_i(z)) dz \quad (\text{B.6})$$

The variation of the load potential is then

$$\begin{aligned} \delta\Pi_D &= \sum_{k=0}^N \left(\frac{\partial\Pi_D}{\partial U_k} \delta U_k + \frac{\partial\Pi_D}{\partial U_k'} \delta U_k' + \frac{\partial\Pi_D}{\partial W_k} \delta W_k + \frac{\partial\Pi_D}{\partial V_k} \delta V_k \right) \\ &= \int_0^L \sum_{k=1}^N (t_k(z) \delta U_k + m_k(z) \delta U_k' + p_k(z) \delta V_k + q_k(z) \delta W_k) dz \end{aligned} \quad (\text{B.7})$$

The term containing $\delta U_k'$ can be integrated by parts yielding

$$\delta\Pi_D = \int_0^L \sum_{k=1}^N ((t_k(z) - m_k'(z)) \delta U_k + p_k(z) \delta V_k + q_k(z) \delta W_k) dz + \left[\sum_{k=1}^N m_k(z) \delta U_k \right]_0^L \quad (\text{B.8})$$

Since loads applied to the ends will be considered in Section B.3, all volumetric loads (including $m_k(z)$) can be assumed to vanish at the boundaries $z = 0$ and $z = L$. The last term of Equation B.7 can therefore be dropped and the variation of the load potential becomes

$$\delta\Pi_D = \int_0^L \sum_{k=1}^N ((t_k(z) - m_k'(z)) \delta U_k + p_k(z) \delta V_k + q_k(z) \delta W_k) dz \quad (\text{B.9})$$

B.3 Variation of the Load Potential due to End Loads

The expression for the work from pressures and tractions applied to the end $z = 0$ and $z = L$, as given by Equation 3.12, is

$$\begin{aligned} \Pi_B &= \sum_{i=1}^N \left({}_0T_i \cdot U_i(z) \Big|_{z=0} + {}_0M_i \cdot U_i'(z) \Big|_{z=0} + {}_0P_i \cdot V_i(z) \Big|_{z=0} + {}_0Q_i \cdot W_i(z) \Big|_{z=0} \right) \\ &\quad + \sum_{i=1}^N \left({}_LT_i \cdot U_i(z) \Big|_{z=L} + {}_LM_i \cdot U_i'(z) \Big|_{z=L} + {}_LP_i \cdot V_i(z) \Big|_{z=L} + {}_LQ_i \cdot W_i(z) \Big|_{z=L} \right) \end{aligned} \quad (\text{B.10})$$

The variation of Π_B is then

$$\begin{aligned} \Pi_B = & \sum_{i=1}^N \left({}_0T_i \cdot \delta U_i(z) \Big|_{z=0} + {}_0M_i \cdot \delta U_i'(z) \Big|_{z=0} + {}_0P_i \cdot \delta V_i(z) \Big|_{z=0} + {}_0Q_i \cdot \delta W_i(z) \Big|_{z=0} \right) \\ & + \sum_{i=1}^N \left({}_LT_i \cdot \delta U_i(z) \Big|_{z=L} + {}_LM_i \cdot \delta U_i'(z) \Big|_{z=L} + {}_LP_i \cdot \delta V_i(z) \Big|_{z=L} + {}_LQ_i \cdot \delta W_i(z) \Big|_{z=L} \right) \end{aligned} \quad (B.11)$$

B.4 Equilibrium Equations

The variation of the total potential energy, Π , is

$$\delta \Pi = \delta \Pi_s - \delta \Pi_D - \delta \Pi_B \quad (B.12)$$

At equilibrium, the variation of the total potential energy is zero. Setting it to zero and using

Equations A.5, A.9, and B.11 to express $\delta \Pi$ in terms of the generalized displacements we get

$$\begin{aligned} 0 = & \sum_{k=1}^N \left(\int_0^L \left(\sum_{j=1}^N \left(-{}_3a_{kj} V_j''(z) + {}_6a_{jk} W_j'(z) + {}_2a_{kj} V_j(z) \right) - p_k(z) \right) \delta V_j dz \right. \\ & + \int_0^L \left(\sum_{j=1}^N \left({}_0a_{kj} W_j''(z) - {}_6a_{kj} V_j'(z) + {}_4a_{kj} W_j(z) \right) - q_k(z) \right) \delta W_j dz \\ & + \int_0^L \left(\sum_{j=1}^N \left({}_0c_{kj} U_j'''(z) + {}_5c_{kj} U_j''(z) + {}_2c_{kj} U_j(z) \right) - (t_k(z) - m_k'(z)) \right) \delta U_k dz \\ & + \left[\left(\sum_{j=1}^N \left({}_3a_{kj} V_j'(z) + {}_5a_{jk} W_j(z) \right) - {}_LP_k \right) \delta V_j \right]_{z=L} + \left[\left(\sum_{j=1}^N \left({}_0a_{kj} W_j'(z) + {}_1a_{kj} V_j(z) \right) - {}_LQ_k \right) \delta W_j \right]_{z=L} \\ & - \left[\left(\sum_{j=1}^N \left({}_3a_{kj} V_j'(z) + {}_5a_{jk} W_j(z) \right) + {}_0P_k \right) \delta V_j \right]_{z=0} - \left[\left(\sum_{j=1}^N \left({}_0a_{kj} W_j'(z) + {}_1a_{kj} V_j(z) \right) + {}_0Q_k \right) \delta W_j \right]_{z=0} \\ & + \left[\left(\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(z) + {}_7c_{kj} U_j'(z) \right) - {}_LT_k \right) \delta U_k \right]_{z=L} + \left[\left(\left(\sum_{j=1}^N {}_0c_{kj} U_j''(z) + {}_3c_{kj} U_j(z) \right) - {}_LM_k \right) \delta U_k' \right]_{z=L} \\ & - \left[\left(\left(\sum_{j=1}^N -{}_0c_{kj} U_j'''(z) + {}_7c_{kj} U_j'(z) \right) + {}_0T_k \right) \delta U_k \right]_{z=0} - \left[\left(\left(\sum_{j=1}^N {}_0c_{kj} U_j''(z) + {}_3c_{kj} U_j(z) \right) + {}_0M_k \right) \delta U_k' \right]_{z=0} \Bigg) \end{aligned}$$

where ${}_6a_y$ is defined in Equation 3.15b. Since the variation of all the displacement fields are arbitrary, their coefficients must vanish. The governing equations for the system are therefore

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''''(z) + {}_5c_{kj} U_j''(z) + {}_2c_{kj} U_j(z) \right) = t_k(z) - m_k'(z) \quad (\text{B.14a})$$

$$\sum_{j=1}^N \left(-{}_3a_{kj} V_j''(z) + {}_6a_{jk} W_j'(z) + {}_2a_{kj} V_j(z) \right) = p_k(z) \quad (\text{B.14b})$$

$$\sum_{j=1}^N \left({}_0a_{kj} W_j''(z) - {}_6a_{kj} V_j'(z) + {}_4a_{kj} W_j(z) \right) = q_k(z) \quad (\text{B.14c})$$

and the boundary condition equations for the system are

$$\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(L) + {}_7c_{kj} U_j'(L) \right) = {}_LT_k \quad \text{or} \quad \delta U_k(z) \Big|_{z=L} = 0 \quad (\text{B.15a})$$

$$\sum_{j=1}^N \left(-{}_0c_{kj} U_j'''(0) + {}_7c_{kj} U_j'(0) \right) = -{}_0T_k \quad \text{or} \quad \delta U_k(z) \Big|_{z=0} = 0 \quad (\text{B.15b})$$

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''(L) + {}_3c_{kj} U_j(L) \right) = {}_LM_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=L} = 0 \quad (\text{B.15c})$$

$$\sum_{j=1}^N \left({}_0c_{kj} U_j''(0) + {}_3c_{kj} U_j(0) \right) = -{}_0M_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=0} = 0 \quad (\text{B.15d})$$

$$\sum_{j=1}^N \left({}_3a_{kj} V_j'(L) + {}_5a_{jk} W_j(L) \right) = {}_LP_k \quad \text{or} \quad \delta V(z) \Big|_{z=L} = 0 \quad (\text{B.15e})$$

$$\sum_{j=1}^N \left({}_3a_{kj} V_j'(0) + {}_5a_{jk} W_j(0) \right) = -{}_0P_k \quad \text{or} \quad \delta V(z) \Big|_{z=0} = 0 \quad (\text{B.15f})$$

$$\sum_{j=1}^N \left({}_0a_{kj} W_j'(L) + {}_1a_{kj} V_j(L) \right) = {}_LQ_k \quad \text{or} \quad \delta W(z) \Big|_{z=L} = 0 \quad (\text{B.15g})$$

$$\sum_{j=1}^N \left({}_0a_{kj} W_j'(0) + {}_1a_{kj} V_j(0) \right) = -{}_0Q_k \quad \text{or} \quad \delta W(z) \Big|_{z=0} = 0 \quad (\text{B.15h})$$

Equations B.14 and B.15 apply to all $U_k(z)$, $V_k(z)$, $W_k(z)$ from $k=1$ to $k=N$. For the

boundary condition equations (Equations B.15) one must choose either the equation on the right

or the one on the left depending on the physical boundary conditions. The field equations can be collectively expressed in matrix form as

$${}_0\mathbf{c}_{N \times N} \cdot \mathbf{U}^{\text{IV}}(z)_{N \times 1} + {}_5\mathbf{c}_{N \times N} \cdot \mathbf{U}^{\text{II}}(z)_{N \times 1} + {}_2\mathbf{c}_{N \times N} \cdot \mathbf{U}(z)_{N \times 1} = \mathbf{t}(z)_{N \times 1} - \mathbf{m}^{\text{I}}(z)_{N \times 1} \quad (\text{B.16})$$

$$-\left[\begin{array}{c|c} {}_3\mathbf{a} & \mathbf{0} \\ \hline \mathbf{0} & {}_0\mathbf{a} \end{array}\right] \left\{ \begin{array}{c} \mathbf{V}(z) \\ \mathbf{W}(z) \end{array} \right\}'' + \left[\begin{array}{c|c} \mathbf{0} & {}_6\mathbf{a} \\ \hline -{}_6\mathbf{a}^T & \mathbf{0} \end{array}\right] \left\{ \begin{array}{c} \{\mathbf{V}(z)\} \\ \{\mathbf{W}(z)\} \end{array} \right\}' + \left[\begin{array}{c|c} {}_2\mathbf{a} & \mathbf{0} \\ \hline \mathbf{0} & {}_4\mathbf{a} \end{array}\right] \left\{ \begin{array}{c} \mathbf{V}(z) \\ \mathbf{W}(z) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{p}(z) \\ \mathbf{q}(z) \end{array} \right\} \quad (\text{B.17})$$

APPENDIX C - VARIATION OF STRAIN ENERGY FOR MULTIPLE DEEP PLATES

The purpose of this appendix is to derive the field and boundary condition equations for the deep plate formulation. This section uses the coordinate system and symbols introduced in Chapters 2, 3, and 4.

C.1 Reduction of the Strain Energy

In this section, we will eliminate the redundant generalized displacements from the expression of strain energy. From Equation 4.6, the strain energy can be written as

$$\begin{aligned} \Pi_S = \frac{1}{2} \int_0^L & \left(\mathbf{A}(z)_{1 \times 3NK}^T \cdot {}_0\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1} + \mathbf{A}^I(z)_{1 \times 3NK}^T \cdot {}_1\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}^I(z)_{3NK \times 1} \right. \\ & + \mathbf{A}^{II}(z)_{1 \times 3NK}^T \cdot {}_2\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}^{II}(z)_{3NK \times 1} + 2\mathbf{A}^I(z)_{1 \times 3NK}^T \cdot {}_3\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1} \\ & \left. + 2\mathbf{A}^{II}(z)_{1 \times 3NK}^T \cdot {}_4\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1} \right) dz \end{aligned} \quad (C.1)$$

We will define ${}_{1,\ell}\mathbf{d}_{\tilde{N} \times N}$, ${}_{2,\ell}\mathbf{d}_{N \times (3NK-N)}$, ${}_{3,\ell}\mathbf{d}_{(3NK-N) \times N}$, and ${}_{3,\ell}\mathbf{d}_{(N-N) \times (3NK-N)}$ ($\ell = 0, 1, 2, 3, 4$) such that

$$\left[\begin{array}{c|c} {}_{1,\ell}\mathbf{d}_{\tilde{N} \times N} & {}_{2,\ell}\mathbf{d}_{N \times (3NK-N)} \\ \hline {}_{3,\ell}\mathbf{d}_{(3NK-N) \times N} & {}_{4,\ell}\mathbf{d}_{(N-N) \times (3NK-N)} \end{array} \right] = {}_{\ell}\mathbf{d}_{3NK \times 3NK} \quad (C.2)$$

An expression for the redundant displacements, $\bar{\mathbf{A}}(z)_{(3NK-N) \times 1}$, in terms of the non-redundant ones, $\tilde{\mathbf{A}}(z)_{(3NK-\tilde{N}) \times 1}$, follows from Equations 4.7 and 4.8d:

$$\bar{\mathbf{A}}(z)_{(3NK-N) \times 1} = {}_3\xi_{(3NK-N) \times N} \cdot \tilde{\mathbf{A}}(z)_{N \times 1} \quad (C.3)$$

Using Eq. C.2 into C.3, after some algebra the term $\mathbf{A}(z)_{3NK \times 1}^T \cdot {}_0\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1}$ becomes

$$\mathbf{A}(z)_{3NK \times 1}^T \cdot {}_0\mathbf{d}_{3NK \times 3NK} \cdot \mathbf{A}(z)_{3NK \times 1} = \tilde{\mathbf{A}}(z)_{N \times 1}^T \cdot {}_0\tilde{\mathbf{d}}_{N \times N} \cdot \tilde{\mathbf{A}}(z)_{N \times 1} \quad (C.4a)$$

in which

$$\begin{aligned} {}_0\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} = & {}_1{}_0\mathbf{d}_{\tilde{N} \times \tilde{N}} + {}_2{}_0\mathbf{d}_{\tilde{N} \times (3NK - \tilde{N})} \cdot {}_3\xi_{(3NK - \tilde{N}) \times \tilde{N}} + {}_3\xi_{\tilde{N} \times (3NK - \tilde{N})}^T \cdot {}_3{}_0\mathbf{d}_{(3NK - \tilde{N}) \times \tilde{N}} \\ & + {}_3\xi_{\tilde{N} \times (3NK - \tilde{N})}^T \cdot {}_4{}_0\mathbf{d}_{(3NK - \tilde{N}) \times (3NK - \tilde{N})} \cdot {}_3\xi_{(3NK - \tilde{N}) \times \tilde{N}} \end{aligned} \quad (\text{C.4b})$$

Similar manipulations can be performed to the terms of Equation C.1 (i.e. ${}_1\mathbf{d}_{3NK \times 3NK}$, ${}_2\mathbf{d}_{3NK \times 3NK}$, ${}_3\mathbf{d}_{3NK \times 3NK}$, and ${}_4\mathbf{d}_{3NK \times 3NK}$) in order to eliminate the redundant degrees of freedom from the strain energy.

C.2 Variation of the Internal Strain Energy

Once all the redundant displacements have been eliminated, the strain energy will have the form

$$\begin{aligned} \Pi_s = & \frac{1}{2} \int_0^L \left(\tilde{\mathbf{A}}(z)_{1 \times \tilde{N}}^T \cdot {}_0\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} + \tilde{\mathbf{A}}^1(z)_{1 \times \tilde{N}}^T \cdot {}_1\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}^1(z)_{\tilde{N} \times 1} \right. \\ & + \tilde{\mathbf{A}}^{\text{II}}(z)_{1 \times \tilde{N}}^T \cdot {}_2\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}^{\text{II}}(z)_{\tilde{N} \times 1} + 2\tilde{\mathbf{A}}^1(z)_{1 \times \tilde{N}}^T \cdot {}_3\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} \\ & \left. + 2\tilde{\mathbf{A}}^{\text{II}}(z)_{1 \times \tilde{N}}^T \cdot {}_4\tilde{\mathbf{d}}_{\tilde{N} \times \tilde{N}} \cdot \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} \right) dz \end{aligned} \quad (\text{C.5a})$$

$$\begin{aligned} = & \frac{1}{2} \int_0^L \left(\sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} {}_0\tilde{d}_{ij} \tilde{A}_i(z) \tilde{A}_j(z) + \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} {}_1\tilde{d}_{ij} \tilde{A}_i'(z) \tilde{A}_j'(z) + \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} {}_2\tilde{d}_{ij} \tilde{A}_i''(z) \tilde{A}_j''(z) \right. \\ & \left. + 2 \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} {}_3\tilde{d}_{ij} \tilde{A}_i'(z) \tilde{A}_j(z) + 2 \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} {}_4\tilde{d}_{ij} \tilde{A}_i'(z) \tilde{A}_j(z) \right) dz \end{aligned} \quad (\text{C.5b})$$

Differentiating Π_s with respect to the generalized displacements one has

$$\frac{\partial \Pi_s}{\partial \tilde{A}_k} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \left(({}_0\tilde{d}_{kj} + {}_0\tilde{d}_{jk}) \tilde{A}_j(z) + 2{}_3\tilde{d}_{jk} \tilde{A}_j'(z) + 2{}_4\tilde{d}_{jk} \tilde{A}_j''(z) \right) dz \quad (\text{C.6a})$$

$$\frac{\partial \Pi_s}{\partial \tilde{A}_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \left(({}_1\tilde{d}_{kj} + {}_1\tilde{d}_{jk}) \tilde{A}_j'(z) + 2{}_3\tilde{d}_{kj} \tilde{A}_j(z) \right) dz \quad (\text{C.6b})$$

$$\frac{\partial \Pi_s}{\partial \tilde{A}_k''} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \left(({}_2\tilde{d}_{kj} + {}_2\tilde{d}_{jk}) \tilde{A}_j''(z) + 2{}_4\tilde{d}_{kj} \tilde{A}_j(z) \right) dz \quad (\text{C.6c})$$

The variation of the strain energy is

$$\delta \Pi_s = \sum_{k=0}^{\tilde{N}} \left(\frac{\partial \Pi_s}{\partial \tilde{A}_k} \delta \tilde{A}_k + \frac{\partial \Pi_s}{\partial \tilde{A}_k'} \delta \tilde{A}_k' + \frac{\partial \Pi_s}{\partial \tilde{A}_k''} \delta \tilde{A}_k'' \right) \quad (C.7)$$

From Equations C.6 by substituting into Equation C.7 one has

$$\begin{aligned} \delta \Pi_s = \sum_{k=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} \int_0^L & \left(\left({}_0 \tilde{d}_{kj} \tilde{A}_j(z) + {}_3 \tilde{d}_{kj} \tilde{A}_j'(z) + {}_4 \tilde{d}_{kj} \tilde{A}_j''(z) \right) \delta \tilde{A}_k + \left({}_1 \tilde{d}_{kj} \tilde{A}_j'(z) + {}_3 \tilde{d}_{kj} \tilde{A}_j(z) \right) \delta \tilde{A}_k' \right. \\ & \left. \left({}_2 \tilde{d}_{kj} \tilde{A}_j''(z) + {}_4 \tilde{d}_{kj} \tilde{A}_j(z) \right) \delta \tilde{A}_k'' \right) dz \end{aligned} \quad (C.8)$$

where we have used the identity ${}_l \tilde{d}_{ij} = {}_i \tilde{d}_{jl}$ for ℓ is 0, 1, and 2. Integrating by parts the terms

containing $\delta \tilde{A}_k'$, and $\delta \tilde{A}_k''$, one has

$$\begin{aligned} \delta \Pi_s = \sum_{k=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} & \left(\int_0^L \left({}_2 \tilde{d}_{kj} \tilde{A}_j^{IV}(z) + {}_5 \tilde{d}_{kj} \tilde{A}_j''(z) + {}_6 \tilde{d}_{kj} \tilde{A}_j'(z) + {}_0 \tilde{d}_{kj} \tilde{A}_j''(z) \right) \delta \tilde{A}_k dz \right. \\ & + \left[\left(-{}_2 \tilde{d}_{kj} \tilde{A}_j'''(z) + {}_7 \tilde{d}_{kj} \tilde{A}_j'(z) + {}_3 \tilde{d}_{kj} \tilde{A}_j(z) \right) \delta \tilde{A}_k \right]_0^L \\ & \left. + \left[\left({}_2 \tilde{d}_{kj} \tilde{A}_j''(z) + {}_3 \tilde{d}_{kj} \tilde{A}_k(z) \right) \delta \tilde{A}_k' \right]_0^L \right) \end{aligned} \quad (C.9a)$$

where

$${}_5 \tilde{d}_{ij} = {}_4 \tilde{d}_{ij} + {}_4 \tilde{d}_{ji} - {}_1 \tilde{d}_{ij} \quad (C.9b)$$

$${}_6 \tilde{d}_{ij} = {}_3 \tilde{d}_{ij} + {}_3 \tilde{d}_{ji} \quad (C.9c)$$

C.3 Variation of the Load Potential

From Equations 4.12 and 4.13, the variation of the load potential is

$$\delta \Pi_D = \int_0^L \left({}_0 \tilde{f}(z)^T \cdot \delta \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} + {}_1 \tilde{f}(z)^T \cdot \delta \tilde{\mathbf{A}}^1(z)_{N \times 1} \right) dz \quad (C.10)$$

and

$$\begin{aligned} \delta \Pi_B = & {}_0 \tilde{\mathbf{F}}_{1 \times N}^T \cdot \delta \tilde{\mathbf{A}}(z)_{\tilde{N} \times 1} \Big|_{=0} + {}_0 \tilde{\mathbf{G}}_{1 \times N}^T \cdot \delta \tilde{\mathbf{A}}^1(z)_{N \times 1} \Big|_{=-0} \\ & + {}_L \tilde{\mathbf{F}}_{1 \times N}^T \cdot \delta \tilde{\mathbf{A}}(z)_{N \times 1} \Big|_{=L} + {}_L \tilde{\mathbf{G}}_{1 \times N}^T \cdot \delta \tilde{\mathbf{A}}^1(z)_{N \times 1} \Big|_{=-L} \end{aligned} \quad (C.11)$$

where we have neglected the boundary terms in Equation C.10.

C.4 Equilibrium Equations

The variation of the total potential energy, Π , is

$$\delta\Pi = \delta\Pi_s - \delta\Pi_D - \delta\Pi_B \quad (\text{C.12})$$

At equilibrium, the variation of the total potential energy is zero. Setting it to zero and using

Equations C.9, C.10, and C.11 a to express $\delta\Pi$ in terms of the generalized displacements, we get

$$\begin{aligned} & \sum_{k=1}^{\tilde{N}} \left(\int_0^L \left(\sum_{j=1}^{\tilde{N}} \left({}_2\tilde{d}_{kj} \tilde{A}_j^{IV}(z) + {}_5\tilde{d}_{kj} \tilde{A}_j''(z) + {}_6\tilde{d}_{kj} \tilde{A}_j'(z) + {}_0\tilde{d}_{kj} \tilde{A}_j''(z) \right) - {}_0\tilde{f}_i(z) + {}_1\tilde{f}_i'(z) \right) \delta\tilde{A}_k dz \right. \\ & + \left[\left(\sum_{j=1}^{\tilde{N}} \left(-{}_2\tilde{d}_{kj} \tilde{A}_j'''(z) + {}_7\tilde{d}_{kj} \tilde{A}_j'(z) + {}_3\tilde{d}_{kj} \tilde{A}_j(z) \right) - {}_L\tilde{F}_k \right) \delta\tilde{A}_k \right]_{z=L} \\ & + \left[\left(\sum_{j=1}^{\tilde{N}} \left({}_2\tilde{d}_{kj} \tilde{A}_j''(z) + {}_4\tilde{d}_{kj} \tilde{A}_j(z) \right) - {}_L\tilde{G}_k \right) \delta\tilde{A}_k' \right]_{z=L} \\ & - \left[\left(\sum_{j=1}^{\tilde{N}} \left(-{}_2\tilde{d}_{kj} \tilde{A}_j'''(z) + {}_7\tilde{d}_{kj} \tilde{A}_j'(z) + {}_3\tilde{d}_{kj} \tilde{A}_j(z) \right) + {}_0\tilde{F}_k \right) \delta\tilde{A}_k \right]_{z=0} \\ & - \left[\left(\sum_{j=1}^{\tilde{N}} \left({}_2\tilde{d}_{kj} \tilde{A}_j''(z) + {}_4\tilde{d}_{kj} \tilde{A}_j(z) \right) + {}_0\tilde{G}_k \right) \delta\tilde{A}_k' \right]_{z=0} \Bigg] = 0 \end{aligned} \quad (\text{C.13})$$

Since $\delta\tilde{A}_k$ are arbitrary functions, their coefficients must vanish. The governing field equations

for the system are

$$\sum_{j=1}^{\tilde{N}} \left({}_2\tilde{d}_{kj} \tilde{A}_j^{IV}(z) + {}_5\tilde{d}_{kj} \tilde{A}_j''(z) + {}_6\tilde{d}_{kj} \tilde{A}_j'(z) + {}_0\tilde{d}_{kj} \tilde{A}_j''(z) \right) = {}_0\tilde{f}_i(z) - {}_1\tilde{f}_i'(z) \quad (\text{C.14a})$$

and the boundary conditions are

$$\sum_{j=1}^{\tilde{N}} \left(-{}_2\tilde{d}_{kj} \tilde{A}_j'''(z) + {}_7\tilde{d}_{kj} \tilde{A}_j'(z) + {}_3\tilde{d}_{kj} \tilde{A}_j(z) \right) = {}_L\tilde{F}_k \quad \text{or} \quad \delta U_k(z)|_{z=L} = 0 \quad (\text{C.14b})$$

$$\sum_{j=1}^N \left(-{}_2\tilde{d}_{kj} \tilde{A}_j'''(z) + {}_7\tilde{d}_{kj} \tilde{A}_j'(z) + {}_3\tilde{d}_{kj} \tilde{A}_j(z) \right) = -{}_0\tilde{F}_k \quad \text{or} \quad \delta U_k(z) \Big|_{z=0} = 0 \quad (\text{C } 14\text{c})$$

$$\sum_{j=1}^N \left({}_2\tilde{d}_{kj} \tilde{A}_j''(z) + {}_4\tilde{d}_{kj} \tilde{A}_j(z) \right) = {}_L\tilde{G}_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=L} = 0 \quad (\text{C } 14\text{d})$$

$$\sum_{j=1}^N \left({}_7\tilde{d}_{kj} \tilde{A}_j''(z) + {}_4\tilde{d}_{kj} \tilde{A}_j(z) \right) = -{}_0\tilde{G}_k \quad \text{or} \quad \delta U_k'(z) \Big|_{z=0} = 0 \quad (\text{C } 14\text{e})$$

The equations apply for every k from 1 to N . For the boundary condition equations (all the equations except C 14a), one must choose either the equation on the right or the equation on the left depending on the physical boundary conditions. The equations can also be collectively expressed in matrix form as

$$\begin{aligned} &{}_2\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{IV}}(z)_{N \times 1} + {}_5\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{II}}(z)_{N \times 1} + {}_6\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{I}}(z)_{N \times 1} + {}_0\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}(z)_{N \times 1} \\ &= {}_0\tilde{\mathbf{f}}(z)_{N \times 1} - {}_1\tilde{\mathbf{f}}^{\text{I}}(z)_{N \times 1} \end{aligned} \quad (\text{C } 15\text{a})$$

$${}_2\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{II}}(z_0)_{N \times 1} + {}_4\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}(z_0)_{N \times 1} = \pm {}_{z_0}\tilde{\mathbf{G}}_{N \times 1} \quad (\text{C } 15\text{b})$$

$$-{}_2\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{III}}(z_0)_{N \times 1} + {}_7\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}^{\text{I}}(z_0)_{N \times 1} + {}_3\tilde{\mathbf{d}}_{N \times N} \tilde{\mathbf{A}}(z_0)_{N \times 1} = \pm {}_{z_0}\tilde{\mathbf{F}}_{N \times 1} \quad (\text{C } 15\text{c})$$

where the plus sign is associated with $z_0 = L$ and the minus sign with $z_0 = 0$

APPENDIX D - EQUILIBRIUM EQUATIONS FOR MULTIPLE NARROW PLATES

The purpose of this appendix is to derive the equilibrium equations for multiple interconnected plates. This section uses the coordinate system and symbols referred to in Chapters 2 and 4.

D.1 Variation of the Strain Energy

The strain energy for a set of connected narrow plates is (Equation 4.15)

$$\begin{aligned} \Pi_s = & \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} \left({}_2\tilde{c}_{ij} \tilde{U}_i(z) \tilde{U}_j(z) + {}_1\tilde{c}_{ij} \tilde{U}_i'(z) \tilde{U}_j'(z) + {}_3\tilde{c}_{ij} \tilde{U}_i(z) \tilde{U}_j''(z) \right. \\ & \left. + {}_0\tilde{c}_{ij} \tilde{U}_i''(z) \tilde{U}_j''(z) \right) dz + \int_0^L \sum_{j=1}^{\tilde{N}} d_j \cdot \tilde{U}_j''(z) \cdot \tilde{W}'(z) dz + \int_0^L d_w \cdot \left(\tilde{W}'(z) \right)^2 dz \end{aligned} \quad (D.1)$$

Differentiating Equation D.1 with respect to the generalized displacements, we have

$$\frac{\partial \Pi_s}{\partial \tilde{U}_k} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \left(({}_2\tilde{c}_{jk} + {}_2\tilde{c}_{kj}) \tilde{U}_j(z) + 2{}_3\tilde{c}_{jk} \tilde{U}_j''(z) \right) dz \quad (D.2a)$$

$$\frac{\partial \Pi_s}{\partial \tilde{U}_k'} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} ({}_1\tilde{c}_{jk} + {}_1\tilde{c}_{kj}) \tilde{U}_j'(z) dz \quad (D.2b)$$

$$\frac{\partial \Pi_s}{\partial \tilde{U}_k''} = \frac{1}{2} \int_0^L \sum_{j=1}^{\tilde{N}} \left(({}_0\tilde{c}_{jk} + {}_0\tilde{c}_{kj}) \tilde{U}_j''(z) + 2{}_3\tilde{c}_{kj} \hat{A}_j(z) \right) dz + \int_0^L d_k \cdot \tilde{W}'(z) dz \quad (D.2c)$$

$$\frac{\partial \Pi_s}{\partial \tilde{W}'} = \int_0^L \sum_{j=1}^{\tilde{N}} d_j \cdot \tilde{U}_j''(z) dz + \int_0^L 2d_w \cdot \tilde{W}'(z) dz \quad (D.2d)$$

The variation of the strain energy is

$$\delta \Pi_s = \sum_{k=1}^{\tilde{N}} \left(\delta \tilde{U}_k \frac{\partial \Pi_s}{\partial \hat{A}_k} + \delta \tilde{U}_k' \frac{\partial \Pi_s}{\partial \hat{A}_k'} + \delta \tilde{U}_k'' \frac{\partial \Pi_s}{\partial \hat{A}_k''} \right) + \delta \tilde{W}' \frac{\partial \Pi_s}{\partial \tilde{W}'} \quad (D.3)$$

From Equations D.2 by substituting into Equation D.3

$$\begin{aligned}
\delta \Pi_s = & \sum_{k=1}^{\tilde{N}} \left(\int_0^L \sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{IV}(z) + {}_5\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_2\tilde{c}_{kj} \cdot \tilde{U}_j(z) \right) \delta \tilde{U}_k dz \right. \\
& + \left[\sum_{j=1}^{\tilde{N}} \left(-{}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{III}(z) + {}_7\tilde{c}_{kj} \cdot \tilde{U}_j'(z) \right) \delta \tilde{U}_k - d_k \cdot \tilde{W}^{II}(z) \delta \tilde{U}_k \right]_0^L \\
& + \left. \left[\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_3\tilde{c}_{kj} \cdot \tilde{U}_j(z) \right) \delta \tilde{U}_k' + d_k \cdot \tilde{W}'(z) \delta \tilde{U}_k' \right]_0^L \right) \\
& - \int_0^L \left(2d_w \tilde{W}^{II}(z) + \sum_{j=1}^{\tilde{N}} d_j \cdot \tilde{U}_j^{III}(z) \right) \delta \tilde{W} dz + \left[\left(2d_u \tilde{W}'(z) + \sum_{j=1}^{\tilde{N}} d_j \cdot \tilde{U}_j^{II}(z) \right) \delta \tilde{W} \right]_0^L
\end{aligned} \tag{D.4}$$

D.2 Variation of the Load Potential

From Equations 4.16a and 4.16b the variation of the work from applied loads are

$$\delta \Pi_D = \int_0^L \left(\sum_{i=1}^{\tilde{N}} \left(\tilde{t}_i(z) - \tilde{m}_i'(z) \right) \delta \tilde{U}_i + p_u(z) \cdot \delta \tilde{W}(z) \right) dz \tag{D.5a}$$

$$\delta \Pi_B = \sum_{\substack{z_0=0 \\ z_0=L}}^N \sum_{i=1}^{\tilde{N}} \left(\tilde{T}_i \cdot \delta \tilde{U}_i(z) \Big|_{z=z_0} + \tilde{M}_i \cdot \delta \tilde{U}_i'(z) \Big|_{z=z_0} \right) + \sum_{\substack{z_0=0 \\ z_0=L}}^N P_u \cdot \delta \tilde{W}(z) \Big|_{z=z_0} \tag{D.5b}$$

for the distributed and boundary load.

D.3 Equilibrium Equations

Using Equations D.4 and D.5 we can express the variation of the total potential energy

$$\begin{aligned}
\delta \Pi = & \delta \Pi_s - \delta \Pi_B - \delta \Pi_D \\
= & \sum_{k=1}^{\tilde{N}} \left(\int_0^L \left(\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{IV}(z) + {}_5\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_2\tilde{c}_{kj} \cdot \tilde{U}_j(z) \right) + d_k \cdot \tilde{W}^{III}(z) - \tilde{t}_k(z) + m_k'(z) \right) \delta \tilde{U}_k dz \right. \\
& - \left[\sum_{j=1}^{\tilde{N}} \left(-{}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{III}(z) + {}_7\tilde{c}_{kj} \cdot \tilde{U}_j'(z) - d_k \tilde{W}^{II}(z) \right) \delta \tilde{U}_k \right]_0^L - \sum_{\substack{z_0=0 \\ z_0=L}}^N \left[{}_0\tilde{P}_k \cdot \delta \tilde{U}_k \right]_{z=z_0} \\
& + \left. \left[\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_3\tilde{c}_{kj} \cdot \tilde{U}_j(z) + d_k \tilde{W}'(z) \right) \delta \tilde{U}_k' \right]_0^L - \sum_{\substack{z_0=0 \\ z_0=L}}^N \left[{}_0\tilde{M}_k \cdot \delta \tilde{U}_k' \right]_{z=z_0} \right)
\end{aligned}$$

$$\begin{aligned}
& - \int_0^L \left(2d_w \tilde{W}''(z) + \sum_{j=1}^{\tilde{N}} d_j \tilde{U}_j'''(z) - p_w(z) \right) \delta \tilde{W} dz \\
& + \left[\left(2d_w \tilde{W}'(z) + \sum_{j=1}^{\tilde{N}} d_j \tilde{U}_j''(z) \right) \delta \tilde{W} \right]_0^L - \sum_{\substack{z_0=0 \\ z_0=L}} \left[z_0 P_w \cdot \delta \tilde{W}(z) \right]_{z=z_0}
\end{aligned}$$

Setting $\delta \Pi$ to zero for all arbitrary variations of the field variables, we can extract the field equation for $\tilde{W}(z)$:

$$\tilde{W}''(z) = \frac{1}{2} \left(\frac{p_w(z)}{d_w} - \frac{1}{d_w} \sum_{k=1}^{\tilde{N}} d_k \cdot \tilde{U}_k'''(z) \right) \quad (\text{D.7a})$$

We can also express the other derivatives of $\tilde{W}(z)$

$$\tilde{W}'(z) = \frac{1}{2d_w} \left(\int_0^z p_w(\varsigma) d\varsigma - \sum_{k=1}^{\tilde{N}} d_k \cdot \tilde{U}_k''(z) \right) + C \quad (\text{D.7b})$$

$$\tilde{W}'''(z) = \frac{1}{2} \left(\frac{p_w'(z)}{d_w} - \frac{1}{d_w} \sum_{k=1}^{\tilde{N}} d_k \cdot \tilde{U}_k'''(z) \right) \quad (\text{D.7c})$$

where C is some constant that can be determined from the boundary conditions. The load boundary condition equation at $z = 0$ associated with $\tilde{W}(z)$ can be extracted from second last line of Equation D.6:

$$2d_w \tilde{W}'(0) + \sum_{j=1}^{\tilde{N}} d_j \tilde{U}_j''(0) + {}_0P_w = 0 \quad (\text{D.8})$$

Using Equation D.7b to express $\tilde{W}'(0)$ in terms of the other generalized displacements yields

$$\left(\int_0^0 p_w(\varsigma) d\varsigma - \sum_k d_k \tilde{U}_k''(0) \right) + 2d_w C + \sum_{j=1}^{\tilde{N}} d_j \tilde{U}_j''(0) + {}_0P_w = 0 \quad (\text{D.9})$$

The $\hat{A}_j(0)$ terms vanish as does the integral of the distributed axial load. Equation D.9 then yields

$$C = -\frac{{}_0P_w}{2d_w} \quad (D.10)$$

Applying the boundary condition equation at the other end will yield the equation

$$\int_0^L p_w(z) dz = {}_0P_w + {}_L P_w \quad (D.11)$$

which is a statement of static equilibrium. In general, one cannot use the load boundary condition at both ends. The displacement $\tilde{W}(z)$ must be specified at one of the ends. The field equations for the generalized displacements $\tilde{U}_k(z)$ are

$$\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{IV}(z) + {}_5\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_2\tilde{c}_{kj} \cdot \tilde{U}_j(z) \right) + d_k \cdot \tilde{W}^{III}(z) - \tilde{t}_k(z) + m_k'(z) = 0 \quad (D.12)$$

Using Equation D.7c to express $\tilde{W}^{III}(z)$, expanding and grouping like terms we have

$$\sum_{j=1}^{\tilde{N}} \left(\left({}_0\tilde{c}_{kj} - \frac{d_k \cdot d_j}{2d_w} \right) \cdot \tilde{U}_j^{IV}(z) + {}_5\tilde{c}_{kj} \cdot \tilde{U}_j^{II}(z) + {}_2\tilde{c}_{kj} \cdot \tilde{U}_j(z) \right) = \tilde{t}_k(z) - \tilde{m}_k'(z) - \frac{d_k}{2d_w} p_w'(z) \quad (D.13)$$

The first boundary condition equation for $\tilde{U}_j(z)$ is

$$\sum_{j=1}^{\tilde{N}} \left(-{}_0\tilde{c}_{kj} \cdot \tilde{U}_j^{III}(z_0) + {}_7\tilde{c}_{kj} \cdot \tilde{U}_j'(z_0) \right) - d_k \tilde{W}^{II}(z_0) = \pm_{z_0} \tilde{P}_k \quad (D.14)$$

where the plus sign is associated with $z_0 = L$ and the minus sign with $z_0 = 0$. Using Equation D.7a

to express $\tilde{W}^{II}(z_0)$, expanding, and grouping like terms yields

$$\sum_{j=1}^{\tilde{N}} \left(-\left({}_0\tilde{c}_{kj} - \frac{d_k \cdot d_j}{2d_w} \right) \cdot \tilde{U}_j^{III}(z_0) + {}_7\tilde{c}_{kj} \cdot \tilde{U}_j'(z_0) \right) = \pm_{z_0} \tilde{P}_k \quad (D.15)$$

where we have assumed that the distributed axial load vanishes at the boundary. The second

boundary condition equation for $\tilde{U}_j(z)$ is

$$\sum_{j=1}^{\tilde{N}} \left({}_0\tilde{c}_{kj} \cdot \tilde{U}_j''(z_0) + {}_3\tilde{c}_{kj} \cdot \tilde{U}_j(z_0) \right) + d_k \cdot \tilde{W}'(z_0) = \pm_{z_0} \tilde{M}_k \quad (\text{D.16})$$

where the plus sign is associated with $z_0 = L$ and the minus sign with $z_0 = 0$. Using Equation D.7a to express $\tilde{W}''(z_0)$, expanding, and grouping like terms yields

$$\sum_{j=1}^{\tilde{N}} \left(\left({}_0\tilde{c}_{kj} - \frac{d_k \cdot d_j}{2d_w} \right) \tilde{U}_j''(z_0) + {}_3\tilde{c}_{kj} \cdot \tilde{U}_j(z_0) \right) = \pm \left({}_{z_0}\tilde{M}_k - \frac{d_k}{2d_w} {}_{z_0}P_w \right) \quad (\text{D.17})$$

where the plus sign is associates with $z_0 = 0$ and the minus sign with $z_0 = L$. Equations D.13, D.15, and D.17 are equilibrium field and boundary equations for the reduced set of generalized displacements $\tilde{U}_k(z)$. They do not contain $\tilde{W}(z)$ and have the same form as the single plate equations for $U_k(z)$.

APPENDIX E - SYSTEM MATRICES FOR NARROW PLATE EXAMPLE PROBLEMS

The purpose of this appendix is to provide expressions for the ${}_i\tilde{c}_{ij}$, d_i , $\tilde{t}_i(z)$, $\tilde{m}_i(z)$, ${}_0\tilde{T}_i(z)$ and ${}_0\tilde{M}_i(z)$ terms in the potential energy for angle of sections 7.4 and 7.5 and the I-beam of sections 7.6 and 7.7.

E.1 Two plates Connected along One Edge

Figure E.1 shows two plates connected along one edge.

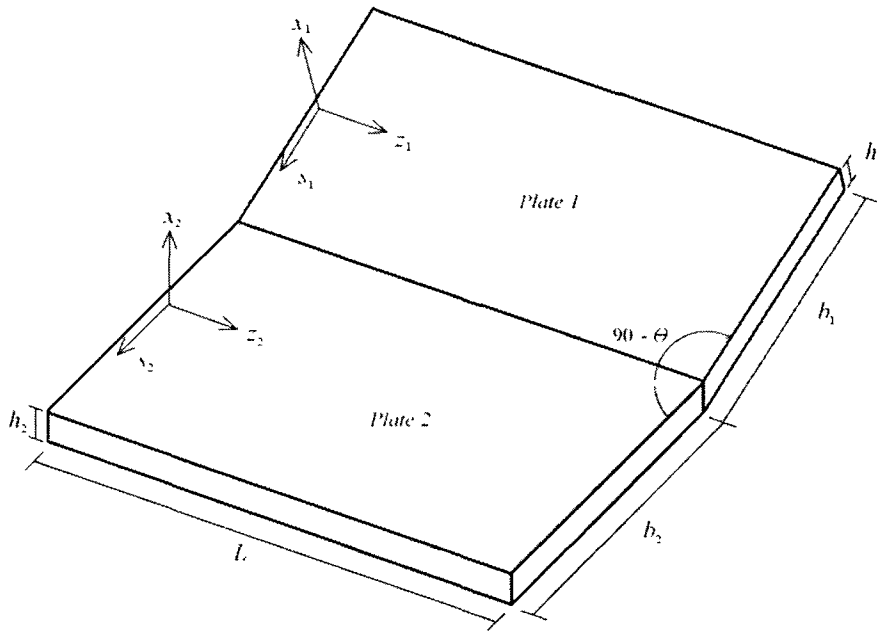


Fig. E.1 - Two plates connected along one edge

The first plate dimensions are $b_1 \times h_1 \times L_1$ and those of the second plate are $b_2 \times h_2 \times L_2$. The two plates are joined at an angle θ as shown. The plates are rigidly connected (i.e. welded or bent). The coordinate system is such that $z_1 = z_2 = z$. If the shape functions are taken from Section 7.3 then the continuity equations at the joint (Equations 4.2) become

$${}^1V(z) = \tan(\theta) {}^1U_1(z) - \sec(\theta) {}^2U_1(z) \quad (\text{E.1a})$$

$${}^2V(z) = (\cos(\theta) + \sin(\theta) \tan(\theta)) {}^1U_1(z) - \tan(\theta) {}^2U_1(z) \quad (\text{E.1b})$$

$${}^2U_2(z) = {}^1U_2(z) \quad (\text{E.1c})$$

$${}^1W(z) = {}^2W(z) + \frac{1}{2} (b_1 {}^1V'(z) + b_2 {}^2V'(z)) \quad (\text{E.1d})$$

E.1.1 Strain Energy

Using Equations E.1, we can eliminate ${}^1V(z)$, ${}^2V(z)$, ${}^2U_2(z)$, and ${}^1W(z)$ from the expression for the strain energy (Equation 4.4a). In this case, the vector of non-redundant generalized is

$$\{\tilde{U}_i(z)\}_{2N-1} = \{{}^1U_1(z) \ {}^1U_2(z) \ \dots \ {}^1U_N(z) \ {}^2U_1(z) \ {}^2U_3(z) \ {}^2U_4(z) \ \dots \ {}^2U_N(z)\}_{2N-1} \quad (\text{E.2})$$

By substitution of Equations E.1 into Equation 4.4a one has

$$\begin{aligned} U_s = & \frac{1}{2} \int_0^L \sum_{j=1}^N \sum_{i=1}^N \left(({}_2^1c_{ij} + \delta_{2,j} \cdot {}_2^2c_{i,2} + \delta_{i,2} \cdot {}_2^2c_{2,j}) {}^1U_i(z) {}^1U_j(z) \right. \\ & + ({}_1^1c_{ij} + \delta_{2,j} \cdot {}_1^2c_{i,2} + \delta_{i,2} \cdot {}_1^2c_{2,j}) {}^1U_i'(z) {}^1U_j'(z) \\ & + 2 ({}_3^1c_{ij} + \delta_{2,j} \cdot {}_3^2c_{i,2} + \delta_{i,2} \cdot {}_3^2c_{2,j}) {}^1U_i(z) {}^1U_j''(z) \\ & \left. + ({}_0^1c_{ij} + \delta_{2,j} \cdot {}_0^2c_{i,2} + \delta_{i,2} \cdot {}_0^2c_{2,j}) {}^1U_i''(z) {}^1U_j''(z) \right) dz \\ & + \frac{1}{2} \int_0^L \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq 2}}^N \left({}_2^2c_{ij} {}^2U_i(z) {}^2U_j(z) + {}_1^2c_{ij} {}^2U_i'(z) {}^2U_j'(z) + 2 {}_3^2c_{ij} {}^2U_i(z) {}^2U_j''(z) \right. \\ & \left. + {}_0^2c_{ij} {}^2U_i''(z) {}^2U_j''(z) \right) dz \\ & + \frac{1}{2} \int_0^L \frac{1}{6\beta^2} (E_1 h_1 b_1 (4\alpha^2 b_1^2 + 6\alpha b_1 b_2 + 3b_2^2) + E_2 h_2 b_2^3) ({}^1U_1''(z))^2 dz \\ & + \frac{1}{2} \int_0^L \frac{1}{6\alpha^2} (E_1 h_1 b_1^2 (4b_1 + \beta 6b_2) + \beta^2 b_2^2 (3E_1 h_1 b_1 + E_2 h_2 b_2)) ({}^2U_1''(z))^2 dz \\ & + \frac{1}{2} \int_0^L \frac{1}{24\alpha^2} (4E_1 h_1 b_1^3 + 6\beta E_1 h_1 b_1^2 b_2 + \beta^2 b_2^2 (3E_1 h_1 b_1 + E_2 h_2 b_2)) ({}^1U_1''(z) \cdot {}^2U_1''(z)) dz \\ & + \frac{1}{2} \int_0^L \frac{1}{24\alpha^2} (4E_1 h_1 b_1^3 + 6\beta E_1 h_1 b_1^2 b_2 + \beta^2 b_2^2 (3E_1 h_1 b_1 + E_2 h_2 b_2)) ({}^2U_1''(z) \cdot {}^1U_1''(z)) dz \end{aligned} \quad (\text{E.3})$$

$$\begin{aligned}
& + \int_0^L \frac{1}{2\alpha} E_1 h_1 b_1 (\beta b_1 + b_2) \left({}^1U_1''(z) \cdot {}^2W'(z) \right) dz \\
& - \int_0^L \frac{1}{2\alpha} E_1 h_1 b_1 (b_1 + \beta b_2) \left({}^2U_1''(z) \cdot {}^2W'(z) \right) dz \\
& + \int_0^L \frac{1}{2} (E_1 h_1 b_1 + E_2 h_2 b_2) \left({}^2W'(z)^2 \right) dz
\end{aligned}$$

from which ${}_i\tilde{c}_y$, d_i , and d_w can be determined by inspection.

E.1.2 Load Potential

By substituting Equations E.1 into Equation 4.4b the reduced expression for the work done by volumetric loads is

$$\begin{aligned}
U_D = & \int_0^L \sum_{i=1}^N \left({}^1U_i(z) \cdot \left({}^1t_i(z) + \delta_{i,2} \cdot {}^2t_2(z) \right) + {}^1U_i'(z) \cdot \left({}^1m_i(z) + \delta_{i,2} \cdot {}^2m_2(z) \right) \right) dz \\
& + \int_0^L \sum_{\substack{i=1 \\ i \neq 2}}^N \left({}^2U_i(z) \cdot {}^2t_i(z) + {}^2U_i'(z) \cdot {}^2m_i(z) \right) dz \\
& + \int_0^L \left(\frac{{}^2p_s(z) + {}^1p_s(z)\beta}{\alpha} {}^1U_1(z) - \frac{{}^1p_s(z) + {}^2p_s(z)\beta}{\alpha} {}^2U_1(z) \right) dz \\
& + \int_0^L \frac{1}{\alpha} \left({}^2m_s(z) + {}^1m_s(z)\beta + {}^1p_s(z) \frac{\beta b_1 + (z)b_2}{2} \right) {}^1U_1'(z) dz \\
& - \int_0^L \frac{1}{\alpha} \left({}^2m_s(z) + {}^1m_s(z)\beta + {}^1p_s(z) \frac{\beta b_1 + b_2}{2} \right) {}^2U_1'(z) dz \\
& + \int_0^L \left({}^1p_s(z) + {}^2p_s(z) \right) {}^2W(z) dz
\end{aligned} \tag{E.4}$$

from which ${}_i\tilde{t}_i(z)$ and ${}_i\tilde{m}_i(z)$ can be determined by inspection. Similarly, by substituting

Equations E.1 into Equation 4.4c we have for the work done by loads applied on the ends

$$\begin{aligned}
U_B = & \sum_{\substack{z_0=0 \\ z_0=L}}^N \sum_{i=1}^N \left(\left({}^1T_i + \delta_{i,2} \cdot {}^2T_2 \right) {}^1U_i(z) + \left({}^1M_i + \delta_{i,2} \cdot {}^2M_2 \right) {}^1U_i'(z) \right) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}}^N \sum_{\substack{i=1 \\ i \neq 2}}^N \left({}^2U_i \cdot {}^2T_i + {}^2U_i' \cdot {}^2M_i \right) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \left(\frac{{}^2P_s + {}^1P_s\beta}{\alpha} {}^1U_1(z) - \frac{{}^1P_s + {}^2P_s\beta}{\alpha} {}^2U_1(z) \right) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \frac{1}{\alpha} \left({}^2M_s(z) + {}^1M_s\beta + {}^1P_s \frac{\beta b_1 + (z)b_2}{2} \right) {}^1U_1'(z) \\
& - \sum_{\substack{z_0=0 \\ z_0=L}} \frac{1}{\alpha} \left({}^2M_s(z) + {}^1M_s\beta + {}^1P_s \frac{\beta b_1 + b_2}{2} \right) {}^2U_1'(z) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \left({}^1P_s + {}^2P_s \right) {}^2W(z)
\end{aligned} \tag{E.5}$$

from which ${}^2\tilde{T}_i(z)$ and ${}^2\tilde{M}_i(z)$ can be determined by inspection.

E.2 I-beam

The cross section of the beam under consideration is shown in Figure E.2.

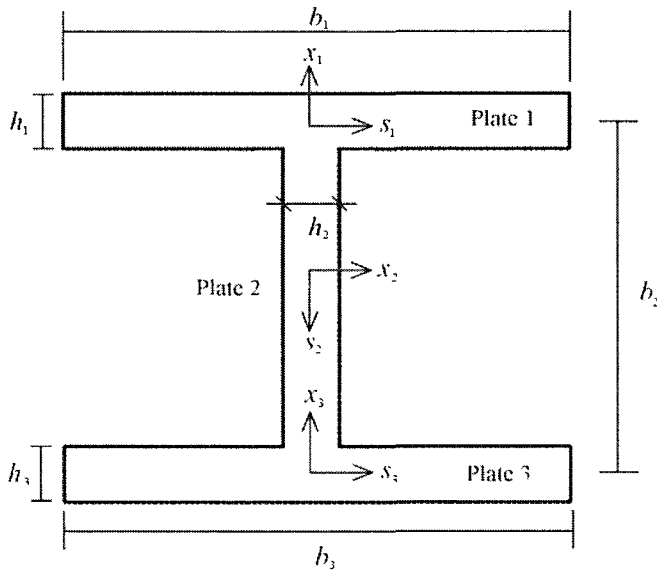


Fig. E.2 - I-beam cross-section

The upper flange, web, and lower flange have width b_1 , b_2 , and b_3 respectively. Their thicknesses are h_1 , h_2 , and h_3 . The plates are rigidly connected at right angles as shown. The z -axes for each of the plates point into the page such that $z_1 = z_2 = z_3 = z$. If the shape functions are taken from Section 7.3 then the continuity equations between the web and the top flange become

$${}^1V(z) = {}^2U_1(z) \quad (\text{E.6a})$$

$${}^2V(z) = -{}^1U_1(z) \quad (\text{E.6b})$$

$${}^1U_2(z) = {}^2U_2(z) \quad (\text{E.6c})$$

$${}^1W(z) = {}^2W(z) + \frac{b_2}{2} {}^2V'(z) \quad (\text{E.6d})$$

Similarly, continuity between the web and lower flange yields

$${}^3V(z) = {}^2U_3(z) \quad (\text{E.7a})$$

$${}^2V(z) = -{}^3U_1(z) \quad (\text{E.7b})$$

$${}^3U_2(z) = {}^2U_4(z) \quad (\text{E.7c})$$

$${}^3W(z) = {}^2W(z) - \frac{b_2}{2} {}^2V'(z) \quad (\text{E.7d})$$

If we use Equation E.6b to express ${}^2V(z)$, then Equation E.7b is only useful in that it yields

$${}^3U_1(z) = {}^1U_1(z) \quad (\text{E.7e})$$

which will allow us to eliminate ${}^3U_1(z)$ in favour of ${}^1U_1(z)$.

E.2.1 Strain Energy

Using Equations E.6 and E.7 we can eliminate ${}^1V(z)$, ${}^2V(z)$, ${}^3V(z)$, ${}^1U_2(z)$, ${}^3U_2(z)$, ${}^3U_1(z)$,

${}^1W(z)$ and ${}^3W(z)$ from the expression for the strain energy (Equation 4.4a). In this case, the

vector of non-redundant generalized is

$$\begin{aligned} \{\tilde{U}_i(z)\} = \{ & {}^1U_1(z) \quad {}^1U_2(z) \quad \dots \quad {}^1U_N(z) \quad \dots \\ & {}^2U_1(z) \quad {}^2U_3(z) \quad {}^2U_5(z) \quad {}^2U_6(z) \quad \dots \quad {}^2U_N(z) \\ & {}^3U_2(z) \quad {}^3U_3(z) \quad {}^3U_5(z) \quad {}^3U_6(z) \quad \dots \quad {}^3U_N(z)\} \end{aligned} \quad (E.8)$$

(3N-3) × (3N-3)

By substitution of Equations E.6 and E.7 into Equation 4.4a one has

$$\begin{aligned} U_s = & \frac{1}{2} \int_0^L \sum_{j=1}^N \sum_{\substack{j=1 \\ j \neq 2}}^N \left(({}^1c_{ij} + \delta_{2,j} \cdot {}^3c_{i,2} + \delta_{i,2} \cdot {}^3c_{2,j}) {}^1U_i(z) {}^1U_j(z) \right. \\ & + ({}^1c_{ij} + \delta_{2,j} \cdot {}^3c_{i,2} + \delta_{i,2} \cdot {}^3c_{2,j}) {}^1U_i'(z) {}^1U_j'(z) \\ & + 2({}^1c_{ij} + \delta_{2,j} \cdot {}^3c_{i,2} + \delta_{i,2} \cdot {}^3c_{2,j}) {}^1U_i(z) {}^1U_j''(z) \\ & \left. + ({}^1c_{ij} + \delta_{2,j} \cdot {}^3c_{i,2} + \delta_{i,2} \cdot {}^3c_{2,j}) {}^1U_i''(z) {}^1U_j''(z) \right) dz \\ & + \frac{1}{2} \int_0^L \sum_{j=1}^N \sum_{i=1}^N \left(({}^2c_{ij} + \delta_{2,j} \cdot {}^1c_{i,2} + \delta_{i,2} \cdot {}^1c_{2,j} + \delta_{4,j} \cdot {}^3c_{i,2} + \delta_{i,4} \cdot {}^3c_{2,j}) {}^2U_i(z) {}^2U_j(z) \right. \\ & + ({}^2c_{ij} + \delta_{2,j} \cdot {}^1c_{i,2} + \delta_{i,2} \cdot {}^1c_{2,j} + \delta_{4,j} \cdot {}^3c_{i,2} + \delta_{i,4} \cdot {}^3c_{2,j}) {}^2U_i'(z) {}^2U_j'(z) \\ & + 2({}^2c_{ij} + \delta_{2,j} \cdot {}^1c_{i,2} + \delta_{i,2} \cdot {}^1c_{2,j} + \delta_{4,j} \cdot {}^3c_{i,2} + \delta_{i,4} \cdot {}^3c_{2,j}) {}^2U_i(z) {}^2U_j''(z) \\ & \left. + ({}^2c_{ij} + \delta_{2,j} \cdot {}^1c_{i,2} + \delta_{i,2} \cdot {}^1c_{2,j} + \delta_{4,j} \cdot {}^3c_{i,2} + \delta_{i,4} \cdot {}^3c_{2,j}) {}^2U_i''(z) {}^2U_j''(z) \right) dz \quad (E.9) \\ & + \frac{1}{2} \int_0^L \sum_{j=3}^N \sum_{i=3}^N \left(({}^3c_{ij} {}^3U_i(z) {}^3U_j(z) + {}^3c_{ij} {}^3U_i'(z) {}^3U_j'(z) + 2{}^3c_{ij} {}^3U_i(z) {}^3U_j''(z) \right. \\ & \left. + {}^3c_{ij} {}^3U_i''(z) {}^3U_j''(z)) dz \right. \\ & + \frac{1}{2} \int_0^L \frac{b_2^2}{12} (3E_1h_1b_1 + E_2h_2b_2 + 3E_3h_3b_3) ({}^1U_1''(z))^2 dz \\ & + \frac{1}{2} \int_0^L \left(\frac{E_1h_1b_1^3}{12} ({}^2U_1''(z))^2 + \frac{E_3h_3b_3^3}{12} ({}^2U_3''(z))^2 \right) dz \\ & + \int_0^L \frac{1}{2} \left(b_2 (E_3h_3b_3 - E_1h_1b_1) ({}^1U_1''(z) \cdot {}^2W'(z)) + (E_1h_1b_1 + E_2h_2b_2 + E_3h_3b_3) ({}^2W'(z))^2 \right) dz \end{aligned}$$

from which ${}^0\tilde{c}_{ij}$, d_i , and d_u can be determined by inspection.

E.2.2 Load Potential

By substituting Equations E.6 and E.7 into Equation 4.4b the reduced expression for the work done by volumetric loads is

$$\begin{aligned}
 U_D = & \int_0^L \sum_{\substack{j=1 \\ j \neq 2}}^N \left(\left({}^1t_i(z) + \delta_{i,1} \cdot {}^3t_1(z) \right) {}^1U_i(z) + \left({}^1m_i(z) + \delta_{i,1} \cdot {}^3m_1(z) \right) {}^1U_i'(z) \right) dz \\
 & + \int_0^L \sum_{i=1}^N \left(\left({}^1t_i(z) + \delta_{i,2} \cdot {}^2t_2(z) + \delta_{i,4} \cdot {}^3t_2(z) \right) {}^1U_i(z) \right) dz \\
 & + \int_0^L \sum_{i=1}^N \left(\left({}^1m_i(z) + \delta_{i,2} \cdot {}^2m_2(z) + \delta_{i,4} \cdot {}^3m_2(z) \right) {}^1U_i'(z) \right) dz \\
 & + \int_0^L \sum_{i=3}^N \left({}^3t_i(z) \cdot {}^3U_i(z) + {}^3m_i(z) \cdot {}^3U_i'(z) \right) dz \\
 & + \int_0^L \left(-{}^2p_s(z) \cdot {}^1U_1(z) + {}^1p_s(z) \cdot {}^2U_1(z) + {}^3p_s(z) \cdot {}^2U_3(z) \right) dz \\
 & + \int_0^L \left(\left({}^2m_s(z) + \frac{b_2}{2} \left({}^3p_s(z) - {}^1p_s(z) \right) \right) {}^1U_1'(z) + {}^1m_s(z) \cdot {}^2U_1'(z) + {}^3m_s(z) \cdot {}^2U_3'(z) \right) dz \\
 & + \int_0^L \left({}^1p_s(z) + {}^2p_s(z) + {}^3p_s(z) \right) {}^2W(z) dz
 \end{aligned} \tag{E.10}$$

from which $\tilde{t}_i(z)$ and $\tilde{m}_i(z)$ can be determined by inspection. By substituting Equations E.6 and

E.7 into Equation 4.4c we have for the work done by loads applied on the ends

$$\begin{aligned}
 U_B = & \sum_{\substack{z_0=0 \\ z_0=L}} \sum_{\substack{j=1 \\ j \neq 2}}^N \left(\left({}^1T_i + \delta_{i,1} \cdot {}^3T_1 \right) {}^1U_i(z) + \left({}^1M_i + \delta_{i,1} \cdot {}^3M_1 \right) {}^1U_i'(z) \right) \\
 & + \sum_{\substack{z_0=0 \\ z_0=L}} \sum_{i=1}^N \left(\left({}^2T_i + \delta_{i,2} \cdot {}^1T_2 + \delta_{i,4} \cdot {}^3T_2 \right) {}^2U_i(z) \right) dz \\
 & + \sum_{\substack{z_0=0 \\ z_0=L}} \sum_{i=1}^N \left(\left({}^2M_i + \delta_{i,2} \cdot {}^1M_2 + \delta_{i,4} \cdot {}^3M_2 \right) {}^2U_i'(z) \right) dz
 \end{aligned} \tag{E.11}$$

$$\begin{aligned}
& + \sum_{\substack{z_0=0 \\ z_0=L}} \sum_{l=3}^N \left({}^3T_l \cdot {}^3U_l(z) + {}^3M_l \cdot {}^3U_l'(z) \right) dz \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \left(- {}^2P_s \cdot {}^1U_1(z) + {}^1P_s \cdot {}^2U_1(z) + {}^3P_s \cdot {}^2U_3(z) \right) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \left(\left({}^2M_s + \frac{b_2}{2} \left({}^3P_z - {}^1P_z \right) \right) {}^1U_1'(z) + {}^1M_s \cdot {}^2U_1'(z) + {}^3M_s \cdot {}^2U_3'(z) \right) \\
& + \sum_{\substack{z_0=0 \\ z_0=L}} \left({}^1P_z + {}^2P_z + {}^3P_z \right) {}^2W(z)
\end{aligned}$$

from which ${}_{z_0}\tilde{T}_l(z)$ and ${}_{z_0}\tilde{M}_l(z)$ can be determined by inspection.

APPENDIX F - SOURCE CODE

The current method was implemented using the Mathematica programming language. Programs were written to solve the example presented in Chapters 6 and 7. Included in this appendix are listings of those programs. There are a total of 6 programs that individually model

- a single plate subjected to in-plane loads (pages F- 2 to F-4),
- a single plate subjected to out-of-plane loads (pages F-5 to F-7),
- an angle made of two narrow plates (pages F-8 to F-12),
- an I-beam made of three narrow plates (pages F-13 to F-17),
- an I-beam made of three deep plates subjected to doubly-symmetric longitudinal loads (pages F-18 to F-22),
- and a box section made of four deep plates subjected to doubly-symmetric longitudinal loads (pages F-23 to F-26)

The geometric and material properties are hard coded into the programs. These can be changed and the program re-run to model different materials and geometries. The loading conditions are also hard coded into the programs. As with the material properties, the end loads and boundary conditions can be change by modifying the code and re-running the model. The number of shape functions and the numerical precision used in the calculations are specified with the rest of the input data. The programs can output a grid of displacements and stresses at specified coordinate intervals, two or three dimensional images of the deformed plates, and stress contour plots. The structure of each of the programs is the same. Figure F.1 is a flowchart showing the major steps for the programs.

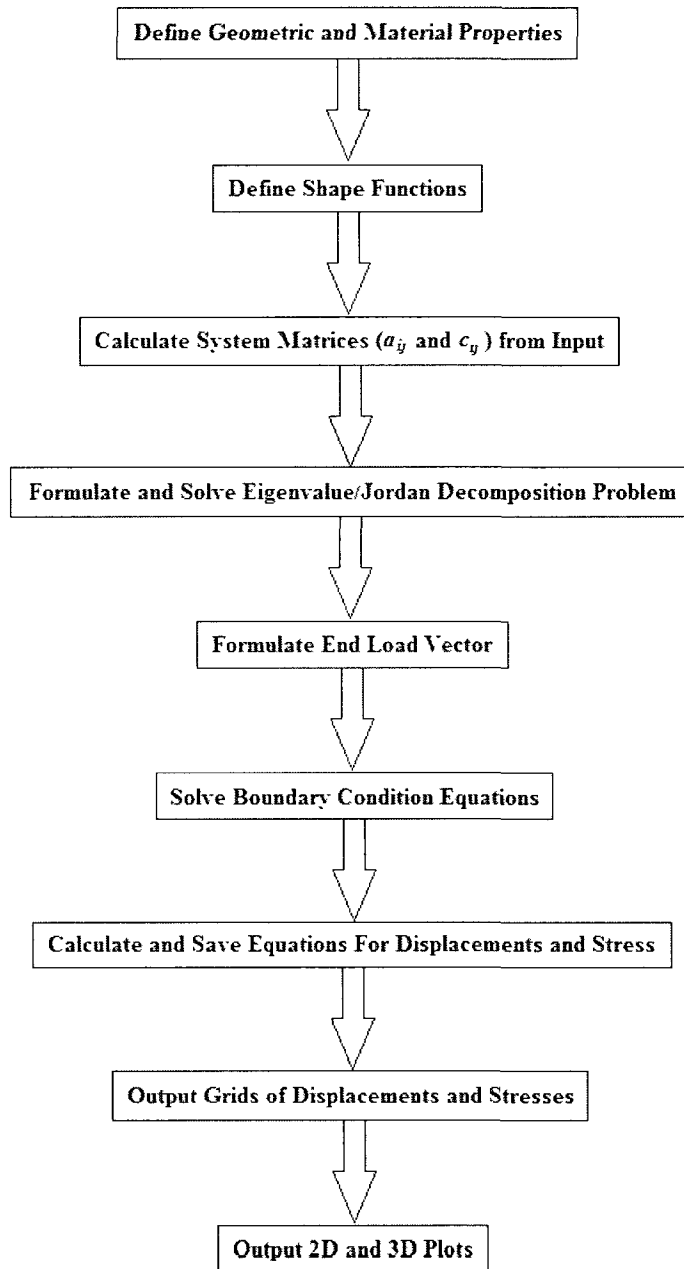


Fig. F.1 - Program Flowchart

```
(* This program computers the displacements in a plate subjected to in-
plane end loads *)
```

```
(* Basic input *)
```

```
M = 9,
Mn = 2 × M,
prec = 75,
$MinPrecision = 65,

b = SetPrecision[1 0, prec],
L = SetPrecision[1 0, prec]
v = SetPrecision[0 3, prec],
h = SetPrecision[0 01, prec],
E0 = SetPrecision[200 × 109, prec],
Λ = SetPrecision[(1 0) / (1 - v2), prec],
G = SetPrecision[(1 0) / (2 × (1 + v)), prec],
```

```
(* Shape functions *)
```

```
φ[s] = {},
For[i = 0, i < M, i++,
{φ, [s_] = SetPrecision[si, prec],
AppendTo[φ[s], φi[s]]}]
```

```
(* System matrices based on shape functions *)
```

```
For[i = 0, i < M, i++,
For[j = 0, j < M, j++,
{

$$a_0[i, j] = \Lambda \times \int_{b/2}^{b/2} \phi_i[s] \times \phi_j[s] ds,$$


$$a_1[i, j] = \Lambda \times v \times \int_{b/2}^{b/2} \phi_i[s] \times \phi_j'[s] ds,$$


$$a_2[i, j] = \Lambda \times \int_{b/2}^{b/2} \phi_i'[s] \times \phi_j'[s] ds,$$


$$a_3[i, j] = G \times \int_{b/2}^{b/2} \phi_i[s] \times \phi_j[s] ds,$$


$$a_4[i, j] = G \times \int_{-b/2}^{b/2} \phi_i'[s] \times \phi_j'[s] ds,$$


$$a_5[i, j] = G \times \int_{b/2}^{b/2} \phi_i'[s] \times \phi_j[s] ds,$$


$$a_6[i, j] = (a_5[i, j] - a_4[i, j])$$

}]]]
```

```
a0 = Array[a0, {M, M}, 0]
a1 = Array[a1, {M, M}, 0]
a2 = Array[a2, {M, M}, 0]
a3 = Array[a3, {M, M}, 0]
```

```

a4 = Array[a, {M, M}, 0],
a5 = Array[a, {M, M}, 0],
a6 = Array[a, {M, M}, 0],

b0 = ArrayFlatten[{{-a3, 0}, {0, -a0}}],
b5 = ArrayFlatten[{{0, -Transpose[a6]}, {a6, 0}}],
b2 = ArrayFlatten[{{a2, 0}, {0, a4}}];

B1 = ArrayFlatten[{{IdentityMatrix[2 x M], 0}, {0, b0}}],
B0 = ArrayFlatten[{{0, IdentityMatrix[2 x M]}, {-b2, -b5}}],
B3 = Inverse[B1] B0,

(* Determine solution to SOLDE *)
λ = Eigenvalues[N[B3, prec]],
B = Take[Eigenvectors[N[B3, prec]], 2 x Mn, Mn],

B4 = ArrayFlatten[{
  {0, 0, 0, 0, b2},
  {0, 0, 0, b2, 4 x b5},
  {0, 0, b2, 3 x b5, 12 x b0},
  {0, b2, 2 x b5, 6 x b0, 0},
  {b2, 1 x b5, 2 x b0, 0, 0}
}],
P = SetPrecision[NullSpace[B4], prec],

ZeroEigen = Dimensions[P][[1]]

X[z_] = SetPrecision[0, prec],
vars = {},
For[i = 1, i ≤ 4 x M - ZeroEigen, i++,
  {X[z_] = X[z] + Ki x Take[B[[i]], {1, Mn}] x eλ[[i]] x z,
  AppendTo[vars, Ki]}]

For[i = 1, i ≤ ZeroEigen, i++,
  {P0 = Take[P[[i]], {0 x Mn + 1, 1 x Mn}],
  P1 = Take[P[[i]], {1 x Mn + 1, 2 x Mn}],
  P2 = Take[P[[i]], {2 x Mn + 1, 3 x Mn}],
  P3 = Take[P[[i]], {3 x Mn + 1, 4 x Mn}],
  P4 = Take[P[[i]], {4 x Mn + 1, 5 x Mn}],
  X[z_] = X[z] + qi x (P0 + P1 x z + P2 x z2 + P3 x z3 + P4 x z4),
  AppendTo[vars, qi]}]

(* Load vector *)
LoadV = ConstantArray[SetPrecision[0, prec], M],
LoadW = ConstantArray[SetPrecision[0, prec], M],

For[i = 0, i < M, i++, LoadV[[i + 1]] = 0 x SetPrecision[ $\int_{h/2}^{b/2} \phi_1[s] ds$ , prec]]

```

```

For[i = 0, i < M, i++,
  LoadW[[i + 1]] = -1000 × SetPrecision[ $\int_{b/2}^{b/4} \phi_i[s] ds - \int_{b/4}^{b/4} \phi_i[s] ds + \int_{b/4}^{b/2} \phi_i[s] ds$ , prec]]
Load = Join[LoadV, LoadW];

(* Solve boundary condition equations *)
b1 = ArrayFlatten[{{0, Transpose[a5]}, {a1, 0}}];
eqns =
{
  X[0] == ConstantArray[0, 2 × M],
  -b0.X'[L] + b1.X[L] == Load / (E0 × h)
};
sol = Solve[eqns, vars, WorkingPrecision → prec];

V[z_] = Take[X[z], M] /. sol;
W[z_] = Take[X[z], -M] /. sol;

(* Stresses and displacements *)
v[s_, z_] = V[z].φ[s];
w[s_, z_] = W[z].φ[s];
v[s_, z_] = v[s, z][[1]],
w[s_, z_] = w[s, z][[1]];

σzz[s_, z_] =  $\frac{E_0}{1 - \nu^2} \times (\partial_z w[s, z] + \nu \times \partial_s v[s, z]) / 10^3$ ;
σss[s_, z_] =  $\frac{E_0}{1 - \nu^2} \times (\partial_s v[s, z] + \nu \times \partial_z w[s, z]) / 10^3$ ;

(* Output grid of displacements *)
TableForm[Table[Re[v[s, z] × 1000 000 000], {s, -b/2, b/2, 0.1}, {z, 0, L, 0.1}]]
(* NANOMETERS *)
TableForm[Table[Re[w[s, z] × 1000 000 000], {s, -b/2, b/2, 0.1}, {z, 0, L, 0.1}]]
(* NANOMETERS *)
TableForm[Table[Re[σzz[s, z]], {s, -b/2, b/2, 0.1}, {z, 0, L, 0.1}]] (* kPa *)
TableForm[Table[Re[σss[s, z]], {s, -b/2, b/2, 0.1}, {z, 0, L, 0.1}]] (* kPa *)

(* Plots *)
scale = 100 000;
ParametricPlot[{s + scale × v[s, z], z + scale × w[s, z]},
  {s, -b/2, b/2}, {z, 0, L}, Axes → False]
ContourPlot[Re[σzz[s, z]], {s, -b/2, b/2}, {z, 0, L}, ColorFunction → "Rainbow",
  Contours → Function[{min, max}, Range[min, max, (max - min) / 10]]]
ContourPlot[Re[σss[s, z]], {s, -b/2, b/2}, {z, 0, L}, ColorFunction → "Rainbow",
  Contours → Function[{min, max}, Range[min, max, (max - min) / 10]]]

```



```

(* This program calculates the out-of-
plane displacements in a plate subjected to arbitrary end loads *)

<< LinearAlgebra MatrixManipulation

(* Basic input *)
M = 11;
prec = 100;
$MinPrecision = prec - 25,
b = SetPrecision[1.0, prec];
L = SetPrecision[1.0, prec],
v = SetPrecision[0.3, prec],
h = 0.01;
E0 = 200 × 109;
Eb = SetPrecision[ $\frac{E_0 \times h^3}{12 \times (1 - v^2)}$ , prec];
A[z_] = SetPrecision[0, prec],
vars = {},
Iden = SetPrecision[IdentityMatrix[M], prec];
Zero = SetPrecision[ZeroMatrix[M], prec];
ZeroCol = Flatten[ZeroMatrix[1, M]];

(* Shape functions *)
G0[x_] = SetPrecision[1.0, prec],
G1[x_] = SetPrecision[x, prec],
G2[x_] = SetPrecision[x2, prec];
G3[x_] = SetPrecision[x3, prec],
G4[x_] = SetPrecision[x4, prec],
G5[x_] = SetPrecision[x5, prec],
G6[x_] = SetPrecision[x6, prec],
G7[x_] = SetPrecision[x7, prec];
G8[x_] = SetPrecision[x8, prec],
G9[x_] = SetPrecision[x9, prec],
G10[x_] = SetPrecision[x10, prec],

G[s_] = Take[
  {G0[s], G1[s], G2[s], G3[s], G4[s], G5[s], G6[s], G7[s], G8[s], G9[s], G10[s]}, M],

(* Load vector *)
F = ZeroCol,
For[i = 0, i < M, i++, F[[i + 1]] = SetPrecision[
  + 500 × G1[-b / 2] - 0.0 × G1[-b / 4] - 1000 × G1[0] + 0.0 × G1[b / 4] + 500 × G1[b / 2], prec]]

```

```

a4 = Array[a, {M, M}, 0],
a5 = Array[a, {M, M}, 0],
a6 = Array[a, {M, M}, 0],

b0 = ArrayFlatten[{{-a3, 0}, {0, -a0}}],
b5 = ArrayFlatten[{{0, -Transpose[a6]}, {a6, 0}}],
b2 = ArrayFlatten[{{a2, 0}, {0, a4}}],

B1 = ArrayFlatten[{{IdentityMatrix[2 × M], 0}, {0, b0}}],
B0 = ArrayFlatten[{{0, IdentityMatrix[2 × M]}, {-b2, -b5}}],
B3 = Inverse[B1] B0,

(* Determine solution to SOLDE *)
λ = Eigenvalues[N[B3, prec]],
B = Take[Eigenvectors[N[B3, prec]], 2 × Mn Mn],

B4 = ArrayFlatten[{
  {0, 0, 0, 0, b2},
  {0, 0, 0, b2, 4 × b5},
  {0, 0, b2, 3 × b5, 12 × b0},
  {0, b2, 2 × b5, 6 × b0, 0},
  {b2, 1 × b5, 2 × b0, 0, 0}
}],
P = SetPrecision[NullSpace[B4], prec],

ZeroEigen = Dimensions[P][[1]],

X[z_] = SetPrecision[0, prec],
vars = {},
For[i = 1, i ≤ 4 × M - ZeroEigen, i++,
  {X[z_] = X[z] + Ki × Take[B[[i]], {1, Mn}] × eλ[[i]] × z,
  AppendTo[vars, Ki]}]

For[i = 1, i ≤ ZeroEigen, i++,
  {P0 = Take[P[[i]], {0 × Mn + 1, 1 × Mn}],
  P1 = Take[P[[i]], {1 × Mn + 1, 2 × Mn}],
  P2 = Take[P[[i]], {2 × Mn + 1, 3 × Mn}],
  P3 = Take[P[[i]], {3 × Mn + 1, 4 × Mn}],
  P4 = Take[P[[i]], {4 × Mn + 1, 5 × Mn}],
  X[z] = X[z] + qi × (P0 + P1 × z + P2 × z2 + P3 × z3 + P4 × z4),
  AppendTo[vars, qi]}]

(* Load vector *)
LoadV = ConstantArray[SetPrecision[0, prec], M],
LoadW = ConstantArray[SetPrecision[0, prec], M],

For[i = 0, i < M, i++, LoadV[[i + 1]] = 0 × SetPrecision[ $\int_{h/2}^{h/2} \phi_1[s] ds$ , prec]]

```

```

For[i = 1, i ≤ 6, i++,
  {P0 = Take[P[[i]], {0 × M + 1, 1 × M}],
   P1 = Take[P[[i]], {1 × M + 1, 2 × M}],
   P2 = Take[P[[i]], {2 × M + 1, 3 × M}],
   P3 = Take[P[[i]], {3 × M + 1, 4 × M}],
   P4 = Take[P[[i]], {4 × M + 1, 5 × M}],
   P5 = Take[P[[i]], {5 × M + 1, 6 × M}],
   A[z_] = A[z] + qi × (P0 + P1 × z + P2 × z2 + P3 × z3 + P4 × z4 + P5 × z5),
   AppendTo[vars, qi],}]

(* Solve boundary condition equations *)
eqns =
  {A[0] == ZeroCol,
   A'[0] == ZeroCol,
   Transpose[c7].A'[L] - c0.A''[L] == F/Eb,
   c0.A''[L] + v × Transpose[c3].A[L] == ZeroCol};

(* Stresses and displacements *)
u[s_, z_] = A[z].G[s] /. Solve[eqns, vars];
u[s_, z_] = Re[u[s, z][[1]]];
u[s_, z_] = A[z].G[s] /. Solve[eqns, vars];
u[s_, z_] = u[s, z][[1]];


$$\sigma_{ss}[s_, z_] = -\frac{E_0}{1 - \nu^2} \times \frac{h}{2} \times (\partial_s \partial_s u[s, z] + \nu \times \partial_s \partial_z u[s, z]) / 10^6;$$



$$\sigma_{sz}[s_, z_] = -\frac{E_0}{1 - \nu^2} \times \frac{h}{2} \times (\partial_{sz} u[s, z] + \nu \times \partial_{zz} u[s, z]) / 10^6;$$


(* Output grid of displacements *)
TableForm[Table[Re[u[s, z] × 1000], {s, -b/2, b/2, 0.1}, {z, 0, 1.0, 0.1}]]
TableForm[Table[Re[σzz[s, z]], {s, -b/2, b/2, 0.1}, {z, 0, 1.0, 0.1}]]
TableForm[Table[Re[σss[s, z]], {s, -b/2, b/2, 0.1}, {z, 0, 1.0, 0.1}]]

(* Plots *)
Plot3D[u[s, z], {s, -b/2, b/2}, {z, 0, L},
  ViewPoint → {1.0, 1.0, 1.0}, AxesLabel → Automatic]
DensityPlot[σss[s, z], {s, -b/2, b/2}, {z, 0, L},
  AxesLabel → Automatic, ColorFunction → Hue, ClippingStyle → Automatic]
DensityPlot[σzz[s, z], {s, -b/2, b/2}, {z, 0, L},
  AxesLabel → Automatic, ColorFunction → Hue, ClippingStyle → Automatic]

```

```
(* This program calculates the displacements in an angle assembled from
two narrow plates of equal length subjected to arbitrary end loads *)
```

```
(* Subroutine for assembling system matrices *)
MakePerp[Locm1_, Locm2_, LocM_] := Module[{Loc},
  Loc = ArrayFlatten[{{Locm1, 0}, {0, Locm2}}];
  Loc[[2]] += Loc[[LocM + 2]];
  Loc[[All, 2]] += Loc[[All, LocM + 2]];
  Loc = Drop[Loc, {LocM + 2}, {LocM + 2}]
]
```

```
(* Basic input *)
M = 2;
Mn = 2 × M - 1;
prec = 100;
$MinPrecision = 90;

b = SetPrecision[1.0, prec];
L = SetPrecision[5.0, prec];
v = SetPrecision[0.3, prec];
h = 0.01;
E0 = 200 × 109;
Eb = SetPrecision[(E0 × h3) / (12 × (1 - v2)), prec];
D0 = SetPrecision[(E0 × h), prec];
```

```
(* Shape functions *)
1ψ0[s_] = SetPrecision[1.0, prec];
1ψ1[s_] = SetPrecision[s - 0.5, prec];
1ψ[s] = {1ψ0[s], 1ψ1[s]};
2ψ0[s_] = SetPrecision[1.0, prec];
2ψ1[s] = SetPrecision[s + 0.5, prec];
2ψ[s] = {2ψ0[s], 2ψ1[s]};
For[i = 0, i < M - 2, i++,
  {1ψ1,2[s_] = SetPrecision[si × (s - 0.5)2, prec],
2ψ1,2[s_] = SetPrecision[si × (s + 0.5)2, prec],
  AppendTo[1ψ[s], 1ψ1,2[s]],
  AppendTo[2ψ[s], 2ψ1,2[s]]
}]
```

```
(* System matrices based on shape functions *)
For[k = 1, k ≤ 2, k++,
  For[i = 0, i < M, i++,
    For[j = 0, j < M, j++,
```

$$\begin{aligned}
\left\{ \begin{aligned}
\dot{c}_0^k[i, j] &= \int_{-b/2}^{b/2} \psi_1^k[s] \times \psi_j^k[s] ds, \\
\dot{c}_1^k[i, j] &= \int_{-b/2}^{b/2} \psi_1'^k[s] \times \psi_j^k[s] ds, \\
\dot{c}_2^k[i, j] &= \int_{-b/2}^{b/2} \psi_1''^k[s] \times \psi_j^k[s] ds, \\
\dot{c}_3^k[i, j] &= \int_{-b/2}^{b/2} \psi_1''^k[s] \times \psi_j^k[s] ds, \\
\dot{c}_7^k[i, j] &= \left(2 \times (1 - \nu) \times \dot{c}_1^k[i, j] - \nu \times \dot{c}_3^k[i, j] \right), \\
\end{aligned} \right\}
\end{aligned}$$

```

For[i = 0, i < M, i++,
  For[j = 0, j < M, j++,
    {c5[i, j] = ν × c3[i, j] + ν × c3[j, i] - 2 × (1 - ν) × c1[i, j],
     c2[i, j] = ν × c3[i, j] + ν × c3[j, i] - 2 × (1 - ν) × c1[i, j]}]}]

c01 = Array[c0, {M, M}, 0]; c02 = Array[c0, {M, M}, 0]; c0 = MakePerp[c01, c02, M];
c21 = Array[c2, {M, M}, 0]; c22 = Array[c2, {M, M}, 0]; c2 = MakePerp[c21, c22, M];
c31 = Array[c3, {M, M}, 0]; c32 = Array[c3, {M, M}, 0]; c3 = MakePerp[c31, c32, M];
c51 = Array[c5, {M, M}, 0]; c52 = Array[c5, {M, M}, 0]; c5 = MakePerp[c51, c52, M];
c71 = Array[c7, {M, M}, 0]; c72 = Array[c7, {M, M}, 0]; c7 = MakePerp[c71, c72, M];

c0[[1, 1]] += (1/3 - 1/8) × b² × D0 / Eb;

c0[[M + 1, M + 1]] += (1/3 - 1/8) × b³ × D0 / Eb;

c0[[1, M + 1]] += (1/8 - 1/4) × b³ × D0 / Eb;

c0[[M + 1, 1]] += (1/8 - 1/4) × b³ × D0 / Eb;

(* Solve SOLDE *)
m1 = ArrayFlatten[{{c5, c0}, {IdentityMatrix[Mn], 0}}];
m2 = ArrayFlatten[{{c2, 0}, {0, -1.0 × IdentityMatrix[Mn]}}];
m = SetPrecision[Inverse[m1].m2, prec];
λ = Sqrt[-1 × Eigenvalues[m]];
B = Take[Eigenvectors[m], 2 × Mn, Mn];

```

```

m3 = ArrayFlatten[{
  {0, 0, 0, 0, 0, 0, 0, 0, 0, c2},
  {0, 0, 0, 0, 0, 0, 0, 0, c2, 0},
  {0, 0, 0, 0, 0, 0, 0, c2, 0, 72 x c5},
  {0, 0, 0, 0, 0, 0, c2, 0, 56 x c5, 0},
  {0, 0, 0, 0, 0, c2, 0, 42 x c5, 0, 3024 x c0},
  {0, 0, 0, 0, c2, 0, 30 x c5, 0, 1680 x c0, 0},
  {0, 0, 0, c2, 0, 20 x c5, 0, 840 x c0, 0, 0},
  {0, 0, c2, 0, 12 x c5, 0, 360 x c0, 0, 0, 0},
  {0, c2, 0, 6 x c5, 0, 120 x c0, 0, 0, 0, 0},
  {c2, 0, 2 x c5, 0, 24 x c0, 0, 0, 0, 0, 0}
}],
P = NullSpace[m3];

ZeroEigen = Dimensions[P][[1]] / 2,

A[z_] = SetPrecision[0, prec]
vars = {},
For[i = 1, i ≤ 2 x Mn - ZeroEigen, i++,
  {A[z_] =
    A[z] + Ki x Take[B[[i]], {1, Mn}] x eλ[[i]] x z + Ki x Take[B[[i]], {1, Mn}] x e-λ[[i]] x z,
    AppendTo[vars, Ki],
    AppendTo[vars, Ki],}]
For[i = 1, i ≤ ZeroEigen x 2, i++,
  {P0 = Take[P[[i]], {0 x Mn + 1, 1 x Mn}],
   P1 = Take[P[[i]], {1 x Mn + 1, 2 x Mn}],
   P2 = Take[P[[i]], {2 x Mn + 1, 3 x Mn}],
   P3 = Take[P[[i]], {3 x Mn + 1, 4 x Mn}],
   P4 = Take[P[[i]], {4 x Mn + 1, 5 x Mn}],
   P5 = Take[P[[i]], {5 x Mn + 1, 6 x Mn}],
   P6 = Take[P[[i]], {6 x Mn + 1, 7 x Mn}],
   P7 = Take[P[[i]], {7 x Mn + 1, 8 x Mn}],
   P8 = Take[P[[i]], {8 x Mn + 1, 9 x Mn}],
   P9 = Take[P[[i]], {9 x Mn + 1, 10 x Mn}],
   A[z_] =
    A[z] + qi x {P0 + P1 x z + P2 x z2 + P3 x z3 + P4 x z4 + P5 x z5 + P6 x z6 + P7 x z7 + P8 x z8 + P9 x z9},
   AppendTo[vars, qi],}]

(* Load vector *)
F1 = ConstantArray[SetPrecision[0, prec], M],
F2 = ConstantArray[SetPrecision[0, prec], M],
For[i = 0, i < M, i++,
  F1[[i + 1]] = SetPrecision[-1000 x  $\psi_1^1$ [-0 5] - 0 0 x  $\psi_1^1$ [0 0] + 0 0 x  $\psi_1^1$ [0 5], prec]]
For[i = 0, i < M, i++, F2[[i + 1]] =
  SetPrecision[0 0 x  $\psi_1^2$ [-0 5] + 0 0 x  $\psi_1^2$ [0 0] - 1000 x  $\psi_1^2$ [0 5], prec]]
F1[[2]] += F2[[2]],
F2 = Drop[F2, {2}],
F = Join[F1, F2],

```

```

(* Solve boundary condition equations *)
eqns =
  {A[0] == ConstantArray[0, Mn],
   A'[0] == ConstantArray[0, Mn],
   c0.A''[L] + v x Transpose[c3].A[L] == ConstantArray[0, Mn],
   Transpose[c7].A'[L] - c0.A'''[L] == F / Eb,
  };
sol = Solve[eqns, vars];

(* Stresses and displacements *)
A1[z_] = Take[A[z], M] /. sol;
A2[z_] =
  Join[ Take[A[z], {M+1, M+1}], Take[A[z], {2, 2}], Take[A[z], 2-M] ] /. sol;
u1[s_, z_] = A1[z]. $\psi^1[s]$ ;
u2[s_, z_] = A2[z]. $\psi^2[s]$ ;
u1[s_, z_] = u1[s, z][[1]];
u2[s_, z_] = u2[s, z][[1]];
V1[z_] = A2[z][[1, 1]];
V2[z_] = -A1[z][[1, 1]];
temp[s_, z_] =
  -E0 x  $\left( s \times \partial_z \partial_z (V2[z]) - \frac{b}{4} \times \partial_z \partial_z (A1[z][[1, 1]]) + \frac{b}{4} \times \partial_z \partial_z (A2[z][[1, 1]]) \right)$ ;
 $\sigma_{xz}[s_, z_] = -\frac{E_0}{1-\nu^2} \times \frac{h}{2} \times (\partial_s \partial_z u2[s, z] + \nu \times \partial_z \partial_s u2[s, z]) -$ 
 $E_0 \times \left( s \times \partial_z \partial_z (V2[z]) - \frac{b}{4} \times \partial_z \partial_z (A1[z][[1, 1]]) + \frac{b}{4} \times \partial_z \partial_z (A2[z][[1, 1]]) \right)$ ;
 $\sigma_{ss}[s_, z_] = -\frac{E_0}{1-\nu^2} \times \frac{h}{2} \times (\partial_{zs} u2[s, z] + \nu \times \partial_{zs} u2[s, z]);$ 

(* Output grid of displacements and stresses *)
TableForm[Table[Re[u2[s, z] x 1000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* mm *)
TableForm[Table[Round[V2[z] x 1000, .0001], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* mm *)
TableForm[Table[Re[ $\sigma_{xz}$ [s, z] / 1000000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* MPa *)
TableForm[Table[Re[ $\sigma_{ss}$ [s, z] / 1000000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* MPa *)

```

```

(* Plots *)
Plot3D[u1[s, z], {s, -b/2, b/2}, {z, 0, L}]
Plot3D[u2[s, z], {s, -b/2, b/2}, {z, 0, L}]

scale = SetPrecision[10 000, prec];
u1[s_, z_] = scale * u1[s, z];
u2[s_, z_] = scale * u2[s, z];
V1[z_] = scale * V1[z];
V2[z_] = scale * V2[z];
ParametricPlot3D[{ {u1[s, z], z, b/2 - s - V1[z]},
  {s + b/2 + V2[z], z, u2[s, z]} }, {s, -b/2, b/2}, {z, 0, L} ]
u1[s_, z_] = u1[s, z] / scale;
u2[s_, z_] = u2[s, z] / scale;
V1[z_] = V1[z] / scale;
V2[z_] = V2[z] / scale;

```

(* This program calculates the displacements in an I-beam assembled from three narrow plates of equal length subjected to arbitrary end loads *)

(* Subroutine for assembling system matrices *)

```
MakeW[Locm1_, Locm2_, LocM1_, LocM2_] = Module[{Loc},
  Loc = ArrayFlatten[{{Locm1, 0, 0}, {0, Locm2, 0}, {0, 0, Locm1}}],
  Loc[[LocM1 + 2]] += Loc[[2]];
  Loc[[All, LocM1 + 2]] += Loc[[All, 2]];

  Loc[[LocM1 + 4]] += Loc[[LocM1 + LocM2 + 2]],
  Loc[[All, LocM1 + 4]] += Loc[[All, LocM1 + LocM2 + 2]],

  Loc[[1]] += Loc[[LocM1 + LocM2 + 1]],
  Loc[[All, 1]] += Loc[[All, LocM1 + LocM2 + 1]],

  Loc = Drop[Loc, {2}, {2}],
  Loc = Drop[Loc, {LocM1 + LocM2 + 1}, {LocM1 + LocM2 + 1}],
  Loc = Drop[Loc, {LocM1 + LocM2}, {LocM1 + LocM2}]
]
```

(* Basic input *)

```
M1 = 4,
M2 = 4,
Mn = M1 + M2 + M1 - 3,
prec = 100,
$MinPrecision = 85,

b = SetPrecision[1 0, prec]
L = SetPrecision[10 0, prec],
v = SetPrecision[0 3, prec],
h = 0 01,
E0 = 200 x 109,
Db = SetPrecision[(E0 x h3) / (12 x (1 - v2)) prec],
D0 = SetPrecision[(E0 x h), prec],
```

(* Shape functions *)

```
1
ψ[s] = {},
For[1 = 0, 1 < M1, 1++,
  {1
    ψ1[s_] = SetPrecision[s1, prec],
    AppendTo[{1
      ψ[s], ψ1[s]}]}]

2
ψ0[s] = SetPrecision[2 x (1 + s) x (0 5 - s)2, prec]
2
ψ[s_] = SetPrecision[(s + 0 5) x (0 5 - s)2, prec],
2
ψ2[s_] = SetPrecision[2 x (1 - s) x (0 5 + s)2 prec],
2
ψ3[s_] = SetPrecision[(s - 0 5) x (0 5 + s)2, prec],
```

```


$$\psi[s] = \{\psi_0[s], \psi_1[s], \psi_2[s], \psi_3[s]\},$$

For[1 = 0, 1 < M2 - 4, 1++,
  { $\psi_{1..4}[s_] = \text{SetPrecision}[s^1 \times (s - 0.5)^2 \times (s + 0.5)^2, \text{prec}],$ 
  AppendTo[ $\psi[s], \psi_{1..4}[s]$ ]}]

(* System matrices based on shape functions *)
For[1 = 0, 1 < M1, 1++,
  For[j = 0, j < M1, j++,
    { $\bar{c}_0[1, j] = \int_{-b/2}^{b/2} \psi_1[s] \times \psi_j[s] ds,$ 
     $\bar{c}_1[1, j] = \int_{-b/2}^{b/2} \psi_1'[s] \times \psi_j[s] ds,$ 
     $\bar{c}_2[1, j] = \int_{-b/2}^{b/2} \psi_1''[s] \times \psi_j''[s] ds,$ 
     $\bar{c}_3[1, j] = \int_{-b/2}^{b/2} \psi_1''[s] \times \psi_j[s] ds,$ 
     $\bar{c}_7[1, j] = (2 \times (1 - \nu) \times \bar{c}_1[1, j] - \nu \times \bar{c}_3[1, j])$ 
    }]]

For[1 = 0, 1 < M2, 1++,
  For[j = 0, j < M2, j++,
    { $\bar{c}_0[1, j] = \int_{-b/2}^{b/2} \psi_1[s] \times \psi_j[s] ds,$ 
     $\bar{c}_1[1, j] = \int_{-b/2}^{b/2} \psi_1'[s] \times \psi_j'[s] ds,$ 
     $\bar{c}_2[1, j] = \int_{-b/2}^{b/2} \psi_1''[s] \times \psi_j''[s] ds,$ 
     $\bar{c}_3[1, j] = \int_{-b/2}^{b/2} \psi_1''[s] \times \psi_j[s] ds,$ 
     $\bar{c}_7[1, j] = (2 \times (1 - \nu) \times \bar{c}_1[1, j] - \nu \times \bar{c}_3[1, j])$ 
    }]]

For[1 = 0, 1 < M1, 1++,
  For[j = 0, j < M1, j++,
    { $\bar{c}_5[1, j] = \nu \times \bar{c}_3[1, j] + \nu \times \bar{c}_3[j, 1] - 2 \times (1 - \nu) \times \bar{c}_1[1, j]$ 
    }]]

```

```

For[i = 0, i < M2, i++,
  For[j = 0, j < M2, j++,
    
$$\bar{c}_5^2[1, j] = v \times \bar{c}_3^2[1, j] + v \times \bar{c}_3^2[j, 1] - 2 \times (1 - v) \times \bar{c}_1^2[1, j]]]$$


c01 = Array[ $\bar{c}_0^1$ , {M1, M1}, 0]; c02 = Array[ $\bar{c}_0^2$ , {M2, M2}, 0]; c0 = MakeW[c01, c02, M1, M2];
c21 = Array[ $\bar{c}_2^1$ , {M1, M1}, 0]; c22 = Array[ $\bar{c}_2^2$ , {M2, M2}, 0]; c2 = MakeW[c21, c22, M1, M2];
c31 = Array[ $\bar{c}_3^1$ , {M1, M1}, 0]; c32 = Array[ $\bar{c}_3^2$ , {M2, M2}, 0]; c3 = MakeW[c31, c32, M1, M2];
c51 = Array[ $\bar{c}_5^1$ , {M1, M1}, 0]; c52 = Array[ $\bar{c}_5^2$ , {M2, M2}, 0]; c5 = MakeW[c51, c52, M1, M2];
c71 = Array[ $\bar{c}_7^1$ , {M1, M1}, 0]; c72 = Array[ $\bar{c}_7^2$ , {M2, M2}, 0]; c7 = MakeW[c71, c72, M1, M2];

c0[[1, 1]] +=  $\frac{7}{12} \times b^3 \times \frac{D_0}{D_b}$ ;
c0[[M1, M1]] +=  $\frac{1}{12} \times b^3 \times \frac{D_0}{D_b}$ ;
c0[[M1 + 2, M1 + 2]] +=  $\frac{1}{12} \times b^3 \times \frac{D_0}{D_b}$ ;

(* Solve SOLDE *)
m1 = ArrayFlatten[{{c5, c0}, {IdentityMatrix[Mn], 0}}],
m2 = ArrayFlatten[{{c2, 0}, {0, -1.0 × IdentityMatrix[Mn]}}];
m = SetPrecision[Inverse[m1] m2, prec];
λ = Sqrt[-1 × SetPrecision[Eigenvalues[m], prec]];
B = Take[SetPrecision[Eigenvectors[m], prec], 2 × Mn, Mn];

m3 = ArrayFlatten[{
  {0, 0, 0, 0, 0, 0, 0, 0, 0, c2},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, c2, 0},
  {0, 0, 0, 0, 0, 0, 0, c2, 0, 72 × c5},
  {0, 0, 0, 0, 0, 0, c2, 0, 56 × c5, 0},
  {0, 0, 0, 0, 0, c2, 0, 42 × c5, 0, 3024 × c0},
  {0, 0, 0, 0, c2, 0, 30 × c5, 0, 1680 × c0, 0},
  {0, 0, 0, c2, 0, 20 × c5, 0, 840 × c0, 0, 0},
  {0, 0, c2, 0, 12 × c5, 0, 360 × c0, 0, 0, 0},
  {0, c2, 0, 6 × c5, 0, 120 × c0, 0, 0, 0, 0},
  {c2, 0, 2 × c5, 0, 24 × c0, 0, 0, 0, 0, 0}
}];
P = SetPrecision[NullSpace[m3], prec];

ZeroEigen = Dimensions[P][[1]] / 2,

A[z_] = SetPrecision[0, prec];
vars = {};
For[i = 1, i ≤ 2 × Mn - ZeroEigen, i++,

```

```

{A[z_] =
  A[z] + K1 × Take[B[[1]], {1, Mn}] × eλ[[1]]×z + K-1 × Take[B[[1]], {1, Mn}] × e-λ[[1]]×z,
  AppendTo[vars, K1],
  AppendTo[vars, K-1]]]
For[i = 1, i ≤ ZeroEigen × 2, i++,
  {P0 = Take[P[[i]], {0 × Mn + 1, 1 × Mn}],
   P1 = Take[P[[i]], {1 × Mn + 1, 2 × Mn}],
   P2 = Take[P[[i]], {2 × Mn + 1, 3 × Mn}],
   P3 = Take[P[[i]], {3 × Mn + 1, 4 × Mn}],
   P4 = Take[P[[i]], {4 × Mn + 1, 5 × Mn}],
   P5 = Take[P[[i]], {5 × Mn + 1, 6 × Mn}],
   P6 = Take[P[[i]], {6 × Mn + 1, 7 × Mn}],
   P7 = Take[P[[i]], {7 × Mn + 1, 8 × Mn}],
   P8 = Take[P[[i]], {8 × Mn + 1, 9 × Mn}],
   P9 = Take[P[[i]], {9 × Mn + 1, 10 × Mn}],
  A[z_] =
  A[z] + q1 × (P0 + P1 × z + P2 × z2 + P3 × z3 + P4 × z4 + P5 × z5 + P6 × z6 + P7 × z7 + P8 × z8 + P9 × z9),
  AppendTo[vars, q1]]]

(* Load vector *)
F1 = ConstantArray[SetPrecision[0, prec], M1];
F2 = ConstantArray[SetPrecision[0, prec], M2];
F3 = ConstantArray[SetPrecision[0, prec], M1];

For[i = 0, i < M1, i++,
  F1[[i + 1]] = SetPrecision[-0.0 × ψ11[-0.5] - 1000 × ψ11[0.0] + 0.0 × ψ11[0.5], prec]]
For[i = 0, i < M2, i++, F2[[i + 1]] =
  SetPrecision[0.0 × ψ12[-0.5] - 0.0 × ψ12[0.0] + 0.0 × ψ12[0.5], prec]]
For[i = 0, i < M1, i++, F3[[i + 1]] =
  SetPrecision[0.0 × ψ11[-0.5] + 0.0 × ψ11[0.0] + 0.0 × ψ11[0.5], prec]]

F2[[2]] += F1[[2]],
F2[[4]] += F3[[2]],
F1[[1]] += F3[[1]],

F1 = Drop[F1, {2}];
F3 = Drop[F3, {2}];
F3 = Drop[F3, {1}],

F = Join[F1, F2, F3],

(* Solve boundary condition equations *)
eqns =
{A[0] == ConstantArray[0, Mn],
 A'[0] == ConstantArray[0, Mn],
 c0 A''[L] + v × Transpose[c3] A[L] == ConstantArray[0, Mn],
 Transpose[c7] A'[L] - c0 A'''[L] == F / Db
},

```

```

sol = Solve[eqns, vars];

A1[z_] = Join[ Take[A[z], {1, 1}],
  Take[A[z], {M1+1, M1+1}], Take[A[z], {2, M1-1}] ] /. sol;
A2[z_] = Take[A[z], {M1, M1+M2-1}] /. sol;
A3[z_] = Join[ Take[A[z], {1, 1}],
  Take[A[z], {M1+3, M1+3}], Take[A[z], {2-M1}] ] /. sol;

(* Stresses and displacements *)
u1[s_, z_] = A1[z].1ψ[s];
u2[s_, z_] = A2[z].2ψ[s];
u3[s_, z_] = A3[z].1ψ[s];
u1[s_, z_] = u1[s, z][[1]];
u2[s_, z_] = u2[s, z][[1]];
u3[s_, z_] = u3[s, z][[1]];

V3[z_] = V1[z_] = A2[z][[1, 1]];
V2[z_] = -A1[z][[1, 1]];

u[s_, z_] = u1[s, z];
V[z_] = 0;


$$\sigma_{xz}[s_, z_] = -\frac{E_0}{1-\nu^2} \times \frac{h}{2} \times (\partial_x \partial_z u[s, z] + \nu \times \partial_s \partial_s u[s, z]) + E_0 \times (-s \times \partial_z \partial_z (V[z]));$$


$$\sigma_{ss}[s_, z_] = -\frac{E_0}{1-\nu^2} \times \frac{h}{2} \times (\partial_{s,s} u[s, z] + \nu \times \partial_{z,z} u[s, z]);$$


(* Output grid of displacements and stresses *)
TableForm[Table[Re[u[s, z] × 1000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* mm *)
TableForm[Table[Round[V[z] × 1000, .0001], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* mm *)
TableForm[Table[Re[σxz[s, z] / 1000000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* MPa *)
TableForm[Table[Re[σss[s, z] / 1000000], {s, -b/2, b/2, 0.1}, {z, 0, L, L/10}]]
(* MPa *)

(* Plots *)
Plot3D[u1[s, z], {s, -b/2, b/2}, {z, 0, L}]
Plot3D[u2[s, z], {s, -b/2, b/2}, {z, 0, L}]
Plot3D[u3[s, z], {s, -b/2, b/2}, {z, 0, L}]

```

```

scale = SetPrecision[ $0.5 \times 10^5$ , prec];
u1[s_, z_] = scale * u1[s, z];
u2[s_, z_] = scale * u2[s, z];
u3[s_, z_] = scale * u3[s, z];
V1[z_] = scale * V1[z];
V2[z_] = scale * V2[z];
V3[z_] = scale * V3[z];
ParametricPlot3D[{ {s + V1[z], z, u1[s, z] + b / 2}, {u2[s, z], z, -s - V2[z]},
  {s + V3[z], z, -b / 2 + u3[s, z]} } , {s, -b / 2, b / 2}, {z, 0, L}]
u1[s_, z_] = u1[s, z] / scale;
u2[s_, z_] = u2[s, z] / scale;
u3[s_, z_] = u3[s, z] / scale;
V1[z_] = V1[z] / scale;
V2[z_] = V2[z] / scale;
V3[z_] = V3[z] / scale;

```

```
(* This program calculates the displacements in an I-beam assembled from three
   deep plates of equal length subjected to longitudinal end loads *)
```

```
(* Subroutine for assembling system matrices *)
```

```
MakePerp[flange_, web_, M_] := Module[{Loc},
  Loc = ArrayFlatten[{{flange, 0, 0}, {0, web, 0}, {0, 0, flange}}];

  Loc[[3 × M + 1]] += Loc[[M + 1]];
  Loc[[All, 3 × M + 1]] += Loc[[All, M + 1]];
  Loc[[3 × M + 1]] += Loc[[5 × M + 1]];
  Loc[[All, 3 × M + 1]] += Loc[[All, 5 × M + 1]];

  Loc = Drop[Loc, {5 × M + 1}, {5 × M + 1}];
  Loc = Drop[Loc, {M + 1}, {M + 1}]
]
```

```
(* Basic input *)
```

```
M = 4;
Mn = 6 × M - 2;
prec = 120;
$MinPrecision = 110,

b = SetPrecision[1.0, prec],
L = SetPrecision[4.0, prec],
v = SetPrecision[0.3, prec],
h = SetPrecision[0.01, prec];
E0 = SetPrecision[200 × 109, prec];
Λ = SetPrecision[(1.0) / (1 - v2), prec];
G = SetPrecision[(1 0) / (2 × (1 + v)), prec];
```

```
(* Shape functions *)
```

```
1
ϕ[s] = {};
For[i = 0, i < M, i++,
  {1
  ϕi[s_] = SetPrecision[s2xi, prec],
  AppendTo[{1
  ϕ[s], ϕi[s]}]}]

2
ϕ0[s_] = SetPrecision[1.0, prec];
ϕ[s] = {2
  ϕ0[s]};
For[i = 1, i < M, i++,
  {2
  ϕi[s_] = SetPrecision[s4-1 × (s - 0.5) × (s + 0.5), prec],
  AppendTo[{2
  ϕ[s], ϕi[s]}]}]
```

```

(* System matrices based on shape functions *)
For[k = 1, k ≤ 2, k++,
  For[l = 0, l < M, l++,
    For[j = 0, j < M, j++,
      {

$$\bar{a}_0^k[l, j] = \Lambda \times \int_{-b/2}^{b/2} \phi_1^k[s] \times \phi_j^k[s] ds,$$


$$\bar{a}_1^k[l, j] = \Lambda \times \nu \times \int_{b/2}^{b/2} \phi_1^k[s] \times \phi_j^k[s] ds,$$


$$\bar{a}_2^k[l, j] = \Lambda \times \int_{-b/2}^{b/2} \phi_1'^k[s] \times \phi_j'^k[s] ds,$$


$$\bar{a}_3^k[l, j] = G \times \int_{b/2}^{b/2} \phi_1^k[s] \times \phi_j^k[s] ds,$$


$$\bar{a}_4^k[l, j] = G \times \int_{-b/2}^{b/2} \phi_1'^k[s] \times \phi_j'^k[s] ds,$$


$$\bar{a}_5^k[l, j] = G \times \int_{b/2}^{b/2} \phi_1'^k[s] \times \phi_j^k[s] ds,$$


$$\bar{a}_6^k[l, j] = \begin{pmatrix} \bar{a}_1^k[l, j] & \bar{a}_1^k[l, j] \end{pmatrix}$$

      }
    ]
  ]
]

a0p1 = Array[ $\bar{a}_0^1$ , {M, M}, 0], a0p2 = Array[ $\bar{a}_0^2$ , {M, M}, 0],
a1p1 = Array[ $\bar{a}_1^1$ , {M, M}, 0], a1p2 = Array[ $\bar{a}_1^2$ , {M, M}, 0],
a2p1 = Array[ $\bar{a}_2^1$ , {M, M}, 0], a2p2 = Array[ $\bar{a}_2^2$ , {M, M}, 0],
a3p1 = Array[ $\bar{a}_3^1$ , {M, M}, 0], a3p2 = Array[ $\bar{a}_3^2$ , {M, M}, 0],
a4p1 = Array[ $\bar{a}_4^1$ , {M, M}, 0], a4p2 = Array[ $\bar{a}_4^2$ , {M, M}, 0],
a5p1 = Array[ $\bar{a}_5^1$ , {M, M}, 0], a5p2 = Array[ $\bar{a}_5^2$ , {M, M}, 0],
a6p1 = Array[ $\bar{a}_6^1$ , {M, M}, 0], a6p2 = Array[ $\bar{a}_6^2$ , {M, M}, 0],

b0p1 = ArrayFlatten[{{-a3p1, 0}, {0, -a0p1}}],
b5p1 = ArrayFlatten[{{0, -Transpose[a6p1]}, {a6p1, 0}}],
b2p1 = ArrayFlatten[{{a2p1, 0}, {0, a4p1}}],
b7p1 = ArrayFlatten[{{0, Transpose[a5p1]}, {a1p1, 0}}],

b0p2 = ArrayFlatten[{{-a3p2, 0}, {0, -a0p2}}],
b5p2 = ArrayFlatten[{{0, -Transpose[a6p2]}, {a6p2, 0}}],
b2p2 = ArrayFlatten[{{a2p2, 0}, {0, a4p2}}],
b7p2 = ArrayFlatten[{{0, Transpose[a5p2]}, {a1p2, 0}}],

b0 = MakePerp[b0p1, b0p2, M],
b5 = MakePerp[b5p1, b5p2, M],
b2 = MakePerp[b2p1, b2p2, M],
b7 = MakePerp[b7p1, b7p2, M]

```



```

(* Solve SOLDE *)
B1 = ArrayFlatten[{{IdentityMatrix[Mn], 0}, {0, b0}}],
B0 = ArrayFlatten[{{0, IdentityMatrix[Mn]}, {-b2, -b5}}],
B3 = Inverse[B1] B0,

λ = Eigenvalues[N[B3, prec]],
B = Take[Eigenvectors[N[B3, prec]], 2 × Mn, Mn];
B4 = ArrayFlatten[{
  {0, 0, 0, 0, b2},
  {0, 0, 0, b2, 4 × b5},
  {0, 0, b2, 3 × b5, 12 × b0},
  {0, b2, 2 × b5, 6 × b0, 0},
  {b2, 1 × b5, 2 × b0, 0, 0}
}],
P = SetPrecision[NullSpace[B4], prec],

ZeroEigen = Dimensions[P][[1]],

X[z_] = SetPrecision[0, prec]
vars = {};
For[i = 1, i ≤ 2 × Mn - ZeroEigen, i++,
  {X[z_] = X[z] + K × Take[B[[i]], {1, Mn}] × eλ[[i]] × z,
   AppendTo[vars, Ki]}]

For[i = 1, i ≤ ZeroEigen, i++,
  {P0 = Take[P[[i]], {0 × Mn + 1, 1 × Mn}],
   P1 = Take[P[[i]], {1 × Mn + 1, 2 × Mn}],
   P2 = Take[P[[i]], {2 × Mn + 1, 3 × Mn}],
   P3 = Take[P[[i]], {3 × Mn + 1, 4 × Mn}],
   P4 = Take[P[[i]], {4 × Mn + 1, 5 × Mn}],
   X[z_] = X[z] + qi × (P0 + P1 × z + P2 × z2 + P3 × z3 + P4 × z4),
   AppendTo[vars, q]}]

(* Load vector *)
LoadV1 = ConstantArray[SetPrecision[0.0, prec], M],
LoadW1 = ConstantArray[SetPrecision[0 0, prec], M]
LoadV2 = ConstantArray[SetPrecision[0 0, prec], M]
LoadW2 = ConstantArray[SetPrecision[0 0, prec], M]
LoadV3 = ConstantArray[SetPrecision[0 0, prec], M]
LoadW3 = ConstantArray[SetPrecision[0 0, prec], M],

For[i = 0, i < M, i++ LoadV1[[i + 1]] = SetPrecision[0 0, prec]]
For[i = 0, i < M, i++ LoadW1[[i + 1]] = SetPrecision[0 0, prec]]
For[i = 0, i < M, i++ LoadV2[[i + 1]] = SetPrecision[0 0, prec]]
For[i = 0, i < M, i++ LoadW2[[i + 1]] = SetPrecision[1000 0 × ∫b/2b/2 ϕ1[s] ds, prec]]
For[i = 0, i < M, i++ LoadV3[[i + 1]] = SetPrecision[0 0, prec]]
For[i = 0, i < M, i++ LoadW3[[i + 1]] = SetPrecision[0 0, prec]]

```

```

LoadW2[[1]] += LoadW1[[1]];
LoadW2[[1]] += LoadW3[[1]];
LoadW1 = Drop[LoadW1, {1}];
LoadW3 = Drop[LoadW3, {1}];
Load = Join[LoadV1, LoadW1, LoadV2, LoadW2, LoadV3, LoadW3];

(* Solve boundary condition equations *)
eqns =
{
  X[0] == ConstantArray[SetPrecision[0.0, prec], Mn],
  -b0.X'[L] + b7.X[L] == Load / (E2 x h)
};
sol = Solve[eqns, vars];
V1[z_] = Take[X[z], {1, M}] /. sol;
W1[z_] = Join[ Take[X[z], {3 x M, 3 x M}], Take[X[z], {M + 1, 2 x M - 1}] ] /. sol;

V2[z_] = Take[X[z], {2 x M, 3 x M - 1}] /. sol;
W2[z_] = Take[X[z], {3 x M, 4 x M - 1}] /. sol;

(* Stresses and displacements *)
v1[s_, z_] = V1[z]. $\phi[s]$ ;
w1[s_, z_] = W1[z]. $\phi[s]$ ;
v2[s_, z_] = V2[z]. $\phi[s]$ ;
w2[s_, z_] = W2[z]. $\phi[s]$ ;
v1[s_, z_] = v1[s, z][[1]];
w1[s_, z_] = w1[s, z][[1]];
v2[s_, z_] = v2[s, z][[1]];
w2[s_, z_] = w2[s, z][[1]];

v[s_, z_] = v2[s, z];
w[s_, z_] = w2[s, z];
 $\sigma_1[s_, z_] = \frac{E_0}{1 - \nu^2} \times (\partial_z w1[s, z] + \nu \times \partial_s v1[s, z]) / 10^3$ ;
 $\sigma_2[s_, z_] = \frac{E_0}{1 - \nu^2} \times (\partial_z w2[s, z] + \nu \times \partial_s v2[s, z]) / 10^3$ ;

(* Output grid of displacements and stresses *)
TableForm[Table[Re[ $\sigma_1[s, z]$ ], {s, -b/2, b/2, 0.1}, {z, 1, 4, 0.2}]] (* kPa *)
TableForm[Table[Re[ $\sigma_2[s, z]$ ], {s, -b/2, b/2, 0.1}, {z, 1, 4, 0.2}]] (* kPa *)

(* Plots *)
ParametricPlot[{s + scale x v[s, z], z + scale x w[s, z]},
  {s, -b/2, b/2}, {z, 0, L}, Axes -> False]
ContourPlot[Re[ $\sigma_1[s, z]$ ], {s, -b/2, b/2}, {z, 2, 4},
  ColorFunction -> "Rainbow", FrameLabel -> {s, z},
  Contours -> Function[{min, max}, Range[min, max, (max - min) / 10]]]
ContourPlot[Re[ $\sigma_2[s, z]$ ], {s, -b/2, b/2}, {z, 2, 4},
  ColorFunction -> "Rainbow", FrameLabel -> {s, z},
  Contours -> Function[{min, max}, Range[min, max, (max - min) / 10]]]

```

```

(* This program calculates the displacements in an box section assembled from
four deep plates of equal length subjected to longitudinal end loads *)

(* Subroutine for assembling system matrices *)
MakePerp[a_, M_] = Module[{Loc},
  Loc =
    ArrayFlatten[{ {a, 0, 0, 0}, {0, a, 0, 0}, {0, 0, a, 0}, {0, 0, 0, a} }],

  Loc[[M+1]] += Loc[[7×M+1]];
  Loc[[All, M+1]] += Loc[[All, 7×M+1]];

  Loc[[M+1]] += Loc[[3×M+1]],
  Loc[[All, M+1]] += Loc[[All, 3×M+1]],

  Loc[[M+1]] += Loc[[5×M+1]];
  Loc[[All, M+1]] += Loc[[All, 5×M+1]];

  Loc = Drop[Loc, {7×M+1}, {7×M+1}],
  Loc = Drop[Loc, {5×M+1}, {5×M+1}],
  Loc = Drop[Loc, {3×M+1}, {3×M+1}]
]

(* Basic input *)
M = 4,
Mn = 8×M-3,
prec = 80,
$MinPrecision = 70,

b = SetPrecision[1 0, prec],
L = SetPrecision[4 0, prec],
v = SetPrecision[0 3, prec],
h = SetPrecision[0 01, prec],
E0 = SetPrecision[200×109, prec],
Λ = SetPrecision[(1 0) / (1 - v2), prec],
G = SetPrecision[(1 0) / (2×(1+v)), prec],

(* Shape functions *)
φ0[s_] = SetPrecision[1 0, prec],
φ[s] = {φ0[s]},
For[i = 1, i < M, i++,
  {φ1[s] = SetPrecision[s1 × (s - 0.5) × (s + 0.5), prec],
   AppendTo[φ[s], φ1[s]]}]

For[i = 0, i < M, i++,
  For[j = 0, j < M, j++,
    {a[i, j] = Λ × ∫b/2b/2 φ1[s] × φj[s] ds,
     a[i, j] = Λ × v × ∫b/2b/2 φ1[s] × φj'[s] ds,
     a[i, j] = Λ × ∫b/2b/2 φ1'[s] × φj'[s] ds,

```

$$\mathbf{a}_3[i, j] = G \times \int_{-b/2}^{b/2} \phi_2[s] \times \phi_j[s] ds,$$

$$\mathbf{a}_4[i, j] = G \times \int_{-b/2}^{b/2} \phi_1'[s] \times \phi_j'[s] ds,$$

$$\mathbf{a}_5[i, j] = G \times \int_{-b/2}^{b/2} \phi_1'[s] \times \phi_j[s] ds,$$

$$\mathbf{a}_6[i, j] = \left(\mathbf{a}_5[i, j] - \mathbf{a}_1[i, j] \right)$$

$$\left. \left. \left. \right. \right] \right]$$

```

(* System matrices based on shape functions *)
a0 = Array[a, {M, M}, 0];
a1 = Array[a, {M, M}, 0];
a2 = Array[a, {M, M}, 0];
a3 = Array[a, {M, M}, 0];
a4 = Array[a, {M, M}, 0];
a5 = Array[a, {M, M}, 0];
a6 = Array[a, {M, M}, 0];

b0 = ArrayFlatten[{{-a3, 0}, {0, -a0}}];
b5 = ArrayFlatten[{{0, -Transpose[a6]}, {a6, 0}}];
b2 = ArrayFlatten[{{a2, 0}, {0, a4}}];
b7 = ArrayFlatten[{{0, Transpose[a5]}, {a1, 0}}];

b0 = MakePerp[b0, M];
b5 = MakePerp[b5, M];
b2 = MakePerp[b2, M];
b7 = MakePerp[b7, M];

B1 = ArrayFlatten[{{IdentityMatrix[Mn], 0}, {0, b0}}];
B0 = ArrayFlatten[{{0, IdentityMatrix[Mn]}, {-b2, -b5}}];
B3 = Inverse[B1].B0;

(* Solve SOLDE *)
λ = Eigenvalues[N[B3, prec]];
B = Take[Eigenvectors[N[B3, prec]], 2 × Mn, Mn];
B4 = ArrayFlatten[{
  {0, 0, 0, 0, b2},
  {0, 0, 0, b2, 4 × b5},
  {0, 0, b2, 3 × b5, 12 × b0},
  {0, b2, 2 × b5, 6 × b0, 0},
  {b2, 1 × b5, 2 × b0, 0, 0}
}];
P = SetPrecision[NullSpace[B4], prec];

```

```

ZeroEigen = Dimensions[P][[1]];

X[z_] = SetPrecision[0, prec];
vars = {};
For[i = 1, i ≤ 2 × Mn - ZeroEigen, i++,
  {X[z_] = X[z] + Ki × Take[B[[i]], {1, Mn}] × eλi[z] × zi,
   AppendTo[vars, Ki]}]

For[i = 1, i ≤ ZeroEigen, i++,
  {P0 = Take[P[[i]], {0 × Mn + 1, 1 × Mn}],
   P1 = Take[P[[i]], {1 × Mn + 1, 2 × Mn}],
   P2 = Take[P[[i]], {2 × Mn + 1, 3 × Mn}],
   P3 = Take[P[[i]], {3 × Mn + 1, 4 × Mn}],
   P4 = Take[P[[i]], {4 × Mn + 1, 5 × Mn}],
   X[z_] = X[z] + qi × (P0 + P1 × z + P2 × z2 + P3 × z3 + P4 × z4),
   AppendTo[vars, qi]}]

(* Load vector *)
LoadV1 = ConstantArray[SetPrecision[0.0, prec], M];
LoadW1 = ConstantArray[SetPrecision[0.0, prec], M];
LoadV2 = ConstantArray[SetPrecision[0.0, prec], M];
LoadW2 = ConstantArray[SetPrecision[0.0, prec], M];
LoadV3 = ConstantArray[SetPrecision[0.0, prec], M];
LoadW3 = ConstantArray[SetPrecision[0.0, prec], M];
LoadV4 = ConstantArray[SetPrecision[0.0, prec], M];
LoadW4 = ConstantArray[SetPrecision[0.0, prec], M];

For[i = 0, i < M, i++, LoadV1[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadW1[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadV2[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadW2[[i + 1]] = SetPrecision[1000.0 × ∫-b/2b/2 φ1[s] ds, prec]]
For[i = 0, i < M, i++, LoadV3[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadW3[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadV4[[i + 1]] = SetPrecision[0.0, prec]]
For[i = 0, i < M, i++, LoadW4[[i + 1]] = SetPrecision[1000.0 × ∫-b/2b/2 φ1[s] ds, prec]]

LoadW1[[1]] += LoadW2[[1]];
LoadW1[[1]] += LoadW3[[1]];
LoadW1[[1]] += LoadW4[[1]];

LoadW2 = Drop[LoadW2, {1}],
LoadW3 = Drop[LoadW3, {1}],
LoadW4 = Drop[LoadW4, {1}],

Load = Join[LoadV1, LoadW1, LoadV2, LoadW2, LoadV3, LoadW3, LoadV4, LoadW4],

```

```

(* Solve boundary condition equations *)
eqns =
{
  X[0] == ConstantArray[SetPrecision[0.0, prec], Mn],
  -b0 X'[L] + b7 X[L] == Load / (E0 x h)
},
sol = Solve[eqns, vars],

V1[z_] = Take[X[z], {1, M}] / sol;
W1[z_] = Take[X[z], {M+1, 2xM}] /. sol;
V2[z_] = Take[X[z], {2xM+1, 3xM}] /. sol;
W2[z_] =
  Join[
    Take[X[z], {1, 1}]
    ,
    Take[X[z], {3xM+1, 4xM-1}]
  ] /. sol;

(* Stresses and displacements *)
v1[s_, z_] = V1[z].phi[s],
w1[s_, z_] = W1[z].phi[s],
v2[s_, z_] = V2[z].phi[s],
w2[s_, z_] = W2[z].phi[s],
v1[s_, z_] = v1[s, z][[1]],
w1[s_, z_] = w1[s, z][[1]],
v2[s_, z_] = v2[s, z][[1]],
w2[s_, z_] = w2[s, z][[1]],
sigma1[s_, z_] =  $\frac{E_0}{1-\nu^2} \times (\partial_z w1[s, z] + \nu \times \partial_s v1[s, z]) / 10^3$ ;
sigma2[s_, z_] =  $\frac{E_0}{1-\nu^2} \times (\partial_z w2[s, z] + \nu \times \partial_s v2[s, z]) / 10^3$ ,

(* Output grid of displacements and stresses *)
TableForm[Table[Re[sigma1[s, z]], {s, -b/2, b/2, 0.1}, {z, 1, 4, 0.25}]] (* kPa *)
TableForm[Table[Re[sigma2[s, z]], {s, -b/2, b/2, 0.1}, {z, 1, 4, 0.25}]] (* kPa *)

(* Plots *)
ContourPlot[Re[sigma1[s, z]], {s, -b/2, b/2},
  {z, 2, L}, ColorFunction -> "Rainbow", FrameLabel -> {s, z},
  Contours -> Function[{min, max}, Range[min, max, (max-min)/10]]]
ContourPlot[Re[sigma2[s, z] + 50], {s, -b/2, b/2}, {z, 2, L},
  ColorFunction -> "Rainbow", FrameLabel -> {s, z},
  Contours -> Function[{min, max}, Range[min, max, (max-min)/10]]]
(*ContourLabels->True*)

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REFERENCES

Babadzhanyan, V. G., Galin'sh, A. K., and Sachenkov, A. V. (1975), "Method of initial functions of V. Z. Vlasov", *International Applied Mechanics*, Vol. 11, No. 1, pp. 11-15

Bathe K.-J., (1995), "Finite Element Procedures", 2nd ed., Prentice Hall, Saddle River, New Jersey.

Boswell, L. F., and Li, Q., (1995), "Consideration of the Relationships Between Torsion, Distortion and Warping of Thin-Walled Beams", *Thin-Walled Structures*, Vol. 21, pp. 147-161

Boyd, J.P., (2001), "Chebyshev and Fourier Spectral Methods", 2nd ed., Dover Publications, Mineola, New York

Cheung, Y. K., and Tham, L.G., (1998), "Finite Strip Method", CRC Press, Boca Raton, Florida

Davies, J. M., and Leach, P., (1994), "First-Order Generalised Beam Theory", *Journal of Construction Steel Research*, Vol. 31, pp. 187-220.

Dowling, P. J., and Burgan, B. A., (1987), "Shear Lag in Steel and Composite Structures", in *Composite Steel Structures: Advances, Design, and Construction* (Narayanan R. editor), pp. 1-23, Spon Press, Abingdon, England.

Dritsos, S. E., (1991), "Distortion of Concrete Box Beams due to Eccentric Transverse Loads", *Journal of Structural Engineering, ASCE*, Vol. 117, No. 1, pp. 29-47

Gjelsvik, A. (1981), "The Theory of Thin Walled Bars", John Wiley and Sons, New York.

Gohberg, I., Lancaster, P., and Rodman, L., (2009), "Matrix Polynomials", Society for Industrial and Applied Mathematics, Philadelphia

Goltermann, P., (1992), "Linear Distortional Beam Theory", *Journal of Construction Steel Research*, Vol. 24, pp. 1-23.

Hsu, Y. T. and Fu, C. C., (2002), "Application of EBEF Method for the Distortional Analysis of Steel Box Girder Bridge Superstructure during Construction", *Advances in Structural Engineering*, Vol. 5, No. 4, pp. 211-221

Jonsson, J., (1999a), "Distortional warping functions and shear distributions in thin-walled beams", *Thin-Walled Structures*, Vol. 33, pp. 245-268

Jonsson, J., (1999b), "Distortional theory of thin-walled beams", *Thin-Walled Structures*, Vol. 33, pp. 269-303

Kantorovich, L. V. and Krylov V. I. (1958), "Approximate Methods of Higher Analysis", Translated from the fourth Russian edition by C. D. Benster, Interscience Publishers, New York.

Kim, J. H., and Kim, Y. Y., (2000), "One-dimensional analysis of thin-walled closed beams having general cross-sections", *International Journal for Numerical Methods in Engineering*, Vol. 49, pp. 653-668

Kollbrunner, C. F., and Basler, K., (1969), "Torsion in Structures", Translated from German by E. C. Glauser, Springer-Verlag, Berlin.

Kunkel, P., and Mehrmann, V., (2006), "Differential-Algebraic Equations: Analysis and Numerical Solution", European Mathematical Society, Zurich.

Leach, P., (1994), "The Calculation of Modal Cross-Section Properties for use in the Generalized Beam Theory", *Thin-Walled Structures*, Vol. 19, pp. 61-79.

Luo, Q., Z., Wu, Y., M., Li, Q., S., Tang, J., Liu, G., D., (2004), "A Finite Segment Model for Shear Lag Analysis", *Engineering Structures*, Vol. 26, pp. 2113-2124.

Nakai, H., and Yoo, C. H., (1988), "Analysis and Design of Curved Steel Bridges", McGraw-Hill, New York

Noble, B., (1969), "Applied Linear Algebra", Prentice Hall, Saddle River, New Jersey

Park, N.-H., Choi, S., and Kang, Y.-J., (2005), “Exact distortional behaviour and practical distortional analysis of multicell box girders using an expanded method”, *Computers and Structures*, Vol. 83, pp. 1607-1626

Pavazza, R. and Blagojevic, B., (2005), “On the cross-section distortion of thin-walled beams with multi-cell cross-section subjected to bending”, *International Journal of Solids and Structures*, Vol. 42, pp. 901-925

Razaqpur, A. G. and Li H. G., (1994), “Refined Analysis of Curved Thin-walled Multicell Box Girders”, *Computers and Structures*, Vol. 53, No. 1, pp. 131-142

Reddy, J. N., (2007), “Theory and Analysis of Elastic Plates and Shells”, 2nd ed., CRC Press, Boca Raton, Florida

Saadé, K., Warzée, G., and Espion, B., (2006), “Modeling distortional shear in thin walled elastic beams”, *Thin-Walled Structures*, Vol. 44, pp. 808-821

Schardt, R., (1989), “Verallgemeinerte Technische Biegetheorie” (Generalized Beam Theory). Springer-Verlag, Berlin

Slivker V. I., (2007), “Mechanics of Structural Elements: Theory and Applications”, Springer, Berlin

Spiegel, M. R., (1980), “Applied Differential Equations”, 3rd ed., Prentice Hall, Saddle River, New Jersey.

Takahashi, K., and Mizuno, M., (1980), “Distortion of Thin-Walled Open-Cross-Section Members (For the Case of Distortion of Multi-Degree-of-Freedom”, *Bulletin of the JSME*, Vol. 23, No. 182, pp. 1290-1296

Timoshenko, S. P., and Woinowsky-Krieger, S., (1959), “Theory of Plates and Shells”, 2nd ed., McGraw-Hill, New York

Timoshenko, S. P., and Goodier, J. N., (1970), “Theory of Elasticity”, 3rd ed., McGraw-Hill, New York

Vlasov, V. Z. (1961), “Thin-walled Elastic Beams”, 2nd ed., Israel Program for Scientific Translations, Jerusalem

Vlasov, V. Z. (1964) “Tonkostennyye prostranstvennyye sistemy (Izbrannyye trudy tom 3)”, (inRussian),

Wright, R. N., Abdel-Samad, S. R., and Robinson, A. R., (1968), “BEF Analogy for Analysis of Box Girders”, *Journal of Structural Division, ASCE*, Vol. 94, No. 7, pp. 1719–1743

Wu, Y., Lai, Y., Zhang, X., Zhu, Y., (2004), "A Fnite Beam Element for Analyzing Shear Lag and Shear Deformation Effects in Composite-laminated Box Girders", *Computers and Structures*, Vol. 82, pp. 763-771.

Zhang, S. H. and Lyons, L. P. R., (1984), "A Thin-walled Box Beam Finite Element for Curved Bridge Analysis", *Computers and Structures*, Vol. 18, No. 6, pp. 1035-1046