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A New Characterization of Topologically Amenable Groups

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A NEW CHARACTERIZATION OF TOPOLOGICALLY AMENABLE GROUPS

By

Yousef Al-Gadid

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To my uncle Ibraham

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Abstract

A countable group G is called topologically amenable if there exist a compact Hausdorff space X on which G acts by homeomorphisms and weak*-continuous maps b^n from X to the space, prob (G), of probability measures on G such that for every $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in X} ||gb_x^n - b_{gx}^n||_1 = 0.$$

For example, every amenable group is topologically amenable but not vice versa: The free group F_2 is topologically amenable without being amenable.

Inspired by a characterization of amenable groups due to Giordano and de la Harpe (a countable group G is amenable if and only if every continuous action of G on the Cantor set C admits an invariant probability measure), we give a new characterization of topologically amenable groups: A countable group G is topologically amenable if and only if it admits an amenable action on the Cantor set C.

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Introduction

The notion of amenability for groups has been studied under various aspects for more than half a century. A locally compact group is called amenable if there is a left-invariant mean on $L^\infty(G)$. The definition of amenability is simpler in the case of discrete groups: A countable group G is amenable if it has a finitely-additive left-invariant probability measure. In fact, there are many equivalent definitions of amenability for locally compact groups, in particular discrete groups. For instance, a locally compact group G is amenable if and only if every continuous action of G on a compact Hausdorff space admits an invariant probability measure. For countable groups, the result was refined by Giordano and de la Harpe: A countable group G is amenable if and only if any continuous action of G on the Cantor set has an invariant probability measure.

The notion of amenability for group actions was introduced by Zimmer in the measure space setting and imported to the topological spaces setting by Anantharaman-Delaroche (1987). A countable group G is called topologically amenable if there exist a compact Hausdorff space X on which G acts by homeomorphisms and weak*-continuous maps $b^n : X \longrightarrow \text{prob}\,(G)$ such that for every $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in X} ||gb^n_x - b^n_{gx}||_1 = 0.$$

Here prob (G) denotes the collection of all probability measures on G.

Topologically amenable groups are also known as amenable at infinity and Higson-Roe groups. It was showed that this class of groups coincides with the class of exact groups [19] and [4].

This notion is more general than amenability: Every amenable group is topologically amenable. On the other hand, the free group F_2 is topologically amenable but not amenable. In general not every group is topologically amenable, and a counter-example was outlined by Misha Gromov [14] and [15].

In this thesis, we are motivated by the result of Giordano and de la Harpe [18] mentioned above, and we obtain its analogue for topologically amenable groups. Our main result states that: A countable group G is topologically amenable if and only if it admits an amenable action on the Cantor set C.

In the first chapter we give some basic concepts in set theory and topology which will be used in chapter 3 and 4. Chapter 2 presents some concepts in functional analysis which help our understanding of chapter 4. Chapter 3 starts with group actions, then introduces amenability in terms of $L^{\infty}(G)$ and then in terms of probability measures for discrete groups going through known facts for amenability for discrete groups. We use limits along ultrafilters for proving some of those facts.

The final chapter which is chapter 4 gives an introduction to topologically amenable groups as well as contains new results. First we prove the following known result using a different technique from the one in [3]: A group G admits an amenable action on some compact metrizable Hausdorff space if and only if its action on the Stone-Čech compactification βG is amenable. This proof as well as the theorem of Giordano and de la Harpe inspires us to come up with the main result: A countable group G is topologically amenable if and only if it admits an amenable action on the Cantor set C.

This thesis is written so as to be readable by those who are familiar with group theory, functional analysis, and topological spaces. We assume familiarity with the algebraic properties of groups and basic properties of normed and topological spaces. Nevertheless we provide some basic facts in functional analysis and topology, including limits along filters, to help the reader in Chapter 3. This material may be found in [17] and [8].

Chapter 1

Set theory and topology

1.1 The Diagonal Product

Definition 1.1.1. Let X, X_i be topological spaces and let $f_i : X \longrightarrow X_i$. *The diagonal product* of the functions f_i is a function $\triangle_i f_i : X \longrightarrow \prod_i X_i$ such that $\triangle_i f_i(x) = (f_i(x))$.

Lemma 1.1.2. If each function f_i is continuous , then the diagonal product of f_i is continuous as well.

Proof. Let$(x_\lambda)_\lambda$ be a net in X such that $x_\lambda \longrightarrow x$, then $f_i(x_\lambda) \longrightarrow f_i(x)$ as f_i is continuous. Now, $\triangle_i f_i(x_\lambda) = (f_i(x_\lambda)) \longrightarrow (f_i(x)) = \triangle_i f_i(x)$. □

1.2 Quotient Space

Definition 1.2.1. Suppose X is a topological space and R is an equivalence relation on X. We define a topology on the quotient set X/R (the set consisting of all equivalence classes of R) as follows: a set of equivalence classes, $\mathfrak{A}$, in X/R is open if and only if the union $\bigcup \mathfrak{A}$ is open in X. This is the *quotient* topology on the quotient set

X/R.

Equivalently, the quotient topology can be characterized in the following manner: Let $q : X \longrightarrow X/R$ be the projection map which sends each element of X to its equivalence class. Then the quotient topology on X/R is the finest topology for which q is continuous.

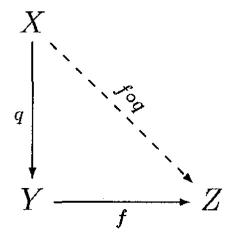
Given a surjective map $f : X \longrightarrow Y$ from a topological space X to a set Y we can define the quotient topology on Y as the finest topology for which f is continuous. This is equivalent to saying that a subset $V \subset Y$ is open in Y if and only if its pre-image $f^{-1}(V)$ is open in X. The map f induces an equivalence relation on X by saying $x_1 R x_2$ if and only if $f(x_1) = f(x_2)$. The quotient space X/R is then homeomorphic to Y (with its quotient topology) via the homeomorphism which sends the equivalence class of x to $f(x)$.

In general, a surjective, continuous map $f : X \longrightarrow Y$ is said to be a quotient map if the topology of Y coincides with the quotient topology determined by f.

Proposition 1.2.2. Quotient maps $q : X \longrightarrow Y$ are characterized by the following property: if Z is any topological space and $f : Y \longrightarrow Z$ is any function, then f is continuous if and only if $f \circ q$ is continuous.

Proof. Assume that $q : X \longrightarrow Y$ is such that for all Z and all $f : Y \longrightarrow Z$, f is continuous if and only if $f \circ q$ is continuous. Let O be a subset of Y, such that $q^{-1}(O)$

is open in X. We want to see that O is open.



Let Z be the space $\{0,1\}$ with topology $\{\{\},\{0,1\},\{0\}\}$ and define $f: Y \longrightarrow Z$ by $f(x) = 0$ for $x \in O$, and 1 otherwise. Then $f \circ q$ is continuous (as $(f \circ q)^{-1}(\{0\}) = q^{-1}(O)$, which is open, and $\{0\}$ is the only non-trivial open set of Z) so by the assumption on q : f is continuous, so $f^{-1}(\{0\}) = O$ is open in Y, which was to be shown. And taking $f = id_Y$ we see that q is continuous (as f is continuous so $q \circ f = q$ is continuous.) So q is a quotient map.

On the other hand, if q is quotient, let Z be any space and $f: Y \longrightarrow Z$ be any map. If f is continuous, so is $f \circ q$, as a composition of continuous maps. If $f \circ q$ is continuous, so is f: let O be open in Z, then $(f \circ q)^{-1} = q^{-1}(f^{-1}(O))$ is open in X. But as q is quotient, $f^{-1}(O)$ is thus open in Y, and so f is continuous. $\square$

1.3 Cantor set

Define inductively a sequence $(C_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$, where each C_n is a union of finitely many closed intervals, as follows: $C_0 = I$ $(= [0,1])$. Write C_n as a (finite) union of (disjoint) closed intervals. Divide each interval of C_n into three equal parts, and remove the (open) middle intervals to obtain C_{n+1}. The first terms in the sequence are

$$C_0 = I;$$

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right];$$

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

We see that C_n is a disjoint union of 2^n closed intervals of length 3^{-n} each. The *Cantor set* is the intersection $C = \cap_n C_n$.

We shall now describe the elements of C precisely, using their expansion in base 3. Recall that any number in I can be expanded in base 3 as $\sum_{n=1}^{\infty} d_n 3^{-n}$, where each digit d_i is 0, 1, or 2. Two such base 3 expansions may represent the same number because of the formula $\sum_{n=1}^{\infty} 3^{-n} = \frac{1}{2}$. We shall say that an expansion $\sum_n d_n 3^{-n}$ terminates if $d_n = 0$ for all $n >$ some n_0, and we shall agree we always take the non-terminating expansion when we have the choice. (The only $x \in I$ not having a non-terminating expansion is $x = 0$.) We shall prove the following

Proposition 1.3.1. 1. The number $x = \sum_{n=1}^{\infty} d_n 3^{-n}$ as above is in C if and only if $d_n \neq 1$ for all n.

2. The Cantor set C is compact, uncountable, and zero-dimensional without isolated points.

Proof. 1. From our definition of the C_n as obtained by removing the middle segments we see that C_n consists of those numbers whose first n digits are all $\neq 1$. In fact we see that C_n is the union of the 2^n intervals $I_d = [d, d + 3^{-n}]$ as d_n ranges over the 2^n numbers $d = \sum_{k=1}^{\infty} d_k 3^{-k}$, with each d_k being 0 or 2. (For $n = 0$ only $d = 0$ occurs, as the empty sum.) Intersecting the C_n's we obtain the asserted description of C.

2. The Cantor set C is compact as the intersection of closed set in the (compact) interval. It's uncountable because the digits of 3 base expansion give a bijection between C and $\prod_{n=1}^{\infty}\{0,1\}$, which is uncountable. Recall that $C = \cap_n C_n$, where each C_n is the union of 2^n many closed intervals, say $C_n = I_{n,i}, i = 1, 2, ...2^n$. The intersections $I_{n,i} \cap C$ are closed and open and form a base. Closedness is obvious, but the complement of each $I_{n,i} \cap C$ is the (finite) union of $I_{n,j} \cap C$ $(i \neq j)$ and so is closed in C as well. If O is some open interval in $[0, 1]$, that contains a point x of C, then x is in some intersection $\cap_n I_n$, where I_n are decreasing closed intervals of diameter tending to 0. When the diameter of I_n is smaller than that of O, all I_k with $k \geq n$ are a subset of O, and so all elements of the Cantor set from I_n are in O as well. (note that at least both endpoints of I_n are in C). So we have in each open set $O \cap C$ that contains x of the Cantor set has (even infinitely many) points of C unequal to x, so C has no isolated points.

□

Remark 1.3.2. The Cantor set C is a Hausdorff zero-dimentional space, so it is totally disconnected.

We state the following result without a proof. See, e.g., [17]

Theorem 1.3.3. The Cantor set C is homeomorphic with any uncountable compact metrizable zero-dimentional Hausdorff space without isolated points.

In the last Chapetr, we will use the fact that every compact metrizable space X is the continuous image of the Cantor set C. Every compact metrisable space X is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$. The interval [0,1] is a continuous image of C by the standard Cantor map. So, $[0, 1]^{\mathbb{N}}$ is a continuous image of $C^{\mathbb{N}}$. The space $C^{\mathbb{N}}$

is homeomorphic to C. Thus, every compact metrisable space X is the continuous image of a closed subset D of C. But every closed subset of C is a continuous image of C.

Done.

1.4 The Stone-Čech Compactification

Definition 1.4.1. Let X be a T_2 topological space. A *Stone-Čech compactification* of X is a compact T_2 topological space βX containing X so that:

1. The topology induced on X as a subset of βX is the original topology of X.

2. Whenever $f : X \longrightarrow Y$ is a continuous map of X into some compact T_2 space Y, there exists a unique continuous map $\tilde{f} : \beta X \longrightarrow Y$ whose restriction to X is f.

Remark 1.4.2. If βX Stone-Čech compactification of X, then X is dense in βX, namely, the closure of X in βX is all of βX. It follows easily from condition 2.

Theorem 1.4.3. Any two Stone-Čech compactifications, C_1 and C_2 of the same topological space X are homeomorphic. Moreover, there exists a homeomorphism $f : C_1 \longrightarrow C_2$ whose restriction to X is the identity map.

1.5 Filters and Ultrafilters

In this section, we will give a brief introduction to filters, ultrafilters, and limits along them. It's helpful to introduce some definitions and facts about limts along filters and ultrafilters since they will be used in the next section.

Definition 1.5.1. Let X be a non-empty set. A collection ξ of subsets of X is a *filter* if following conditions hold:

1. $\emptyset \notin \xi$;

2. If $A, B \in \xi$, then $A \cap B \in \xi$;

3. If $A \subset B$ and $A \in \xi$, then $B \in \xi$.

Example 1.5.2. Let X be a set, and $x \in X$. Then $\xi = \{F \subset X : x \in F\}$ is a filter on X.

Example 1.5.3. The collection $\xi = \{N : N \subset \mathbb{R}$ and N is a neighborhood of $0\}$ is a filter on $\mathbb{R}$. Where $\mathbb{R}$ endowed with the usual topology.

Definition 1.5.4. A filter ξ on a set X is said to be *free* if $\cap_{F \in \xi} F = \emptyset$.

Remark 1.5.5. Itis clear that the filter in Example 1.5.2 is not free. However, for every infinite set, we can define a free filter on it by taking the complements of the finite subsets.

Definition 1.5.6. Let X be a non-empty set. A collection ς of non-empty subsets of X is a *filter base* for a filter ξ on X if $\varsigma \subset \xi$ and $\xi = \{F \subset X : C \subset F$ for some $C \in \varsigma\}$.

Example 1.5.7. It's obvious that $\varsigma = \{\{x\}\}$ is a filter base for the filter ξ in Example 1.5.2

Example 1.5.8. One can see that $\varsigma = \{(-\epsilon, \epsilon) : \epsilon > 0\}$ is a filter base for the filter ξ in Example 1.5.3 .

Proposition 1.5.9. Let X be a set. A collection ς of subsets of X is said to be a filter base for some filter on X if the following two conditions hold:

1. $\emptyset \notin \varsigma$;

2. If $A_1, A_2 \in \varsigma$, then there exists $A_3 \in \varsigma$ such that $A_3 \subset A_1 \cap A_2$.

Example 1.5.10. Let $X = \mathbb{R}$. Then $\varsigma = \{(-n, n) : n = 1, 2, 3, ...\}$ is a base for some filter on $\mathbb{R}$ but it's not a filter itself.

Definition 1.5.11. A filter ξ is said to be an *ultrafilter* if there is no filter containing ξ and strictly finer than ξ. That is, ξ is an ultrafilter on X if for any filter ζ on X with $\xi \subset \zeta$ then $\xi = \zeta$.

Example 1.5.12. The filter ξ in Example 1.5.2 is an ultrafilter.

We shall state a proposition, without a proof, which is an equivalent definition of an ultrafilter.

Proposition 1.5.13. A filter ξ, on a set X, is an ultrafilter if an only if for every $E \subset X$ either $E \in \xi$ or $X \setminus E \in \xi$.

The following is a consequence of Zorn's lemma.

Proposition 1.5.14. Every filter is contained in an ultrafilter.

Corollary 1.5.15. For any infinite set X, there exists a free ultrafilter ξ on it.

This Corollary is an easy consequence of the Remark above and Proposition 1.5.14.

Proposition 1.5.16. If ξ is a free ultrafilter, then ξ contains no finite set.

Proof. Let ξ be a free ultrafilter on X, where X is an infinite set. And let A be any finite subset of $X, A = \{x_1, x_2, x_3, ..., x_n\}$. We shall show that $A \notin \xi$. It's clear $\{x_1\} \notin \xi$ because ξ is a free ultrafilter, otherwise x_1 would belong to every member of ξ (as a result, ξ does not contain any one point set). Since ξ is an ultrafilter and does not contain $\{x_1\}$, so it contains $X\backslash\{x_1\}$. Having $X\backslash\{x_1\} \in \xi$, then $\{x_1, x_2\} \notin \xi$; otherwise we would have $\{x_2\} = \{x_1, x_2\} \cap X\backslash\{x_1\} \in \xi$. We continue with the same process till we obtain that $A = \{x_1, x_2, x_3, ..., x_n\} \notin \xi$. □

Proposition 1.5.17. If ξ is an ultrafilter on X, and $f : X \longrightarrow Y$ is a function, then $f_*(\xi) = \{A \subset Y : f^{-1}(A) \in \xi\}$ is an ultrafilter on Y. It's sometimes called the *push-forward* ultrafilter of ξ along f.

Proof. We shall first show that it's a filter on Y.

1. It's clear that $\phi \notin f_*(\xi)$
2. If $A, B \in f_*(\xi)$, then $f^{-1}(A), f^{-1}(B) \in \xi$. Because ξ is a filter, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \in \xi$. Hence, $A \cap B \in f_*(\xi)$;
3. If $A \in f_*(\xi)$ and $A \subset B \subset Y$, then $f^{-1}(A) \subset f^{-1}(B)$, so $f^{-1}(B) \in \xi$(because $f^{-1}(A) \in \xi$). Hence, $B \in f_*(\xi)$.

We conclude from (1),(2), and (3) that $f_*(\xi)$ is a filter on Y. It's easy to verify that $f_*(\xi)$ is an ultrafilter because ξ is an ultrafilter and $f^{-1}(A^c) = (f^{-1}(A))^c$. □

1.6 Limits Along Filters

Definition 1.6.1. Let (a_n) be a bounded sequence in $\mathbb{R}$ and ξ be a filter on $\mathbb{N}$. Then the limit of (a_n) along ξ equals $a \in \mathbb{R}$, $\lim_{n \longrightarrow \xi} a_n = a$, if and only if for ever $\epsilon > 0$,

$$\{n : |a_n - a| < \epsilon\} \in \xi.$$

Example 1.6.2. Let $\xi = \{F \subset \mathbb{N} : 11 \in F\}$. Then $\lim_{n \longrightarrow \xi} 1/n^2 = 1/(11)^2$. Indeed, given $\epsilon > 0$, then $11 \in \{n : |1/n^2 - 1/(11)^2| < \epsilon\}$ and since $\{11\} \in \xi$, so $\{n : |1/n^2 - 1/(11)^2| < \epsilon\} \in \xi$.

Example 1.6.3. Let ξ be the *Fréchet* filter on $\mathbb{N}$, the collection of all co-finite subsets of $\mathbb{N}$. Then $\lim_{n\longrightarrow\xi} a_n = a$ if and only if $\lim_{n\longrightarrow\infty} a_n = a$.

Indeed,

$$\begin{aligned}
\lim_{n\longrightarrow\xi} a_n = a &\iff \forall\epsilon > 0, \{n : |a_n - a| < \epsilon\} \in \xi \\
&\iff \forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N} \ni \forall n \geq N_\epsilon, |a_n - a| < \epsilon \\
&\iff \lim_{n\longrightarrow\infty} a_n = a.
\end{aligned}$$

Proposition 1.6.4. Let ξ be a filter on $\mathbb{N}$ and (a_n) be a bounded sequence in $\mathbb{R}$. Then $\lim_{n\longrightarrow\xi} a_n$ is unique if it exists.

Proof. Suppose that $\lim_{n\longrightarrow\xi} a_n = a, b$ and $a > b$. Then

$$A = \{n : |a_n - a| < \frac{a-b}{2}\} \in \xi$$

$$B = \{n : |a_n - b| < \frac{a-b}{2}\} \in \xi$$

Now, for $n \in A \cap B$

$$\begin{aligned}
|a - b| &= |a - a_n + a_n - b| \\
&\leq |a_n - a| + |a_n - b| \\
&< \frac{a-b}{2} + \frac{a-b}{2} \\
&= a - b.
\end{aligned}$$

A contradiction. Therefore, $a = b$. □

Proposition 1.6.5. Let ξ be un ultrafilter on $\mathbb{N}$ and (a_n) be a bounded sequence in $\mathbb{R}$. Then $\lim_{n\longrightarrow\xi} a_n$ exists.

Proof. Since (a_n) is a bounded sequence, then it can be contained in a closed interval $[b, c]$. We have a function $a : \mathbb{N} \longrightarrow [b, c]$ such that

$$a(n) = a_n$$

By Proposition 1.5.17, $\zeta = \{A \subset [b, c] : a^{-1}(A) \in \xi\}$ is an ultrafilter on $[b, c]$. Now, we divide [b,c] in equal way to two closed intervals, and since ζ is an ultrafilter on $[b, c]$, so at least one of those intervals belongs to ζ, denote it by I_1. We continue the same process to the intervals that belong to ζ. So, we end up with a collection $\{I_n\}$ of decreasing closed subsets of $[b, c]$. Clearly, this collection has the finite intersection property. By compactness of [b,c], the collection has a non-empty intersection. One can see easily that $\cap I_n$ is a one point set, let us say $\{a\}$. Our claim is now that:

$$\lim_{n \longrightarrow \xi} a_n = a.$$

Given $\epsilon > 0$, we shall show that $K = \{n : |a_n - a| < \epsilon\} \in \xi$. For that $\epsilon > 0$, there exists an n such that $\frac{c-b}{2^n} < \epsilon$, i.e, there exists an interval $I_n \in \{I_n\}$ such that $I_n \subset B(a, \epsilon)$, and since ζ is an ultrafilter on $[b, c]$, so $B(a, \epsilon) \in \zeta$. Hence, $K = f^{-1}(B(a, \epsilon)) \in \xi$.

Hence, the limit along an ultrafilter exists and unique. (by Proposition 1.6.4) $\square$

Chapter 2

Some Concepts of Functional Analysis

2.1 Normed Spaces

Definition 2.1.1. A norm on a vector space E over a field $\mathbb{K}(\mathbb{K} = \mathbb{R}$ or $= \mathbb{C})$ is a function $||.|| : E \longrightarrow \mathbb{R}$ that satisfies the following properties:

1. $||x|| = 0$ iff $x = 0 \, \forall x \in E$;

2. $||\lambda \cdot x|| = |\lambda| \cdot ||x|| \, \forall x \in E$ and $\lambda \in \mathbb{K}$;

3. $||x + y|| \leq ||x|| + ||y|| \, \forall x, y \in E$.

We call $||x||$ the *norm* of the vector x.

Definition 2.1.2. A *normed space* is a pair $(E, ||.||)$, where E is a vector space and $||.||$ is a norm on E.

Note that any normed space can also be thought as a metric space by setting the distance between any two points $x, y \in E$ as $d(x,y) = ||x - y||$. It easy to verify that this distance function satisfies the definition of a metric. The induced metric in turn, defines a topology on X, the *norm topology.* By a *subspace* of X we mean a linear subspace Y of the underlying vector space, endowed with the norm on X (to be pedantic, with the restriction of the norm to Y). A subspace is *closed* if it is closed in the norm topology. A normed space is complete if it's complete as a metric space. A complete normed space is called a *Banach space.*

Example 2.1.3. The n-dimensional Euclidean space: the vector space is $\mathbb{R}^n$ or $\mathbb{C}^n$ and the norm $||x|| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$, where $x = (x_1, ..., x_n)$. The former is a real Euclidean space, the latter is a complex one.

Example 2.1.4. Let L be a topological space, let $X = C(L)$ be the vector space of all bounded continuous functions on L and set,

$$||f|| = \sup_{t \in L} |f(t)|$$

.

Example 2.1.5. This is a special case of the example 2.1.4. Let X be a compact Hausdorff space and let $C(X)$ be the space of continuous functions on X, with the supremum norm

$$||f|| = ||f||_\infty = \sup\{|f(x)| : x \in X\}$$

. Since X is compact, $|f(x)|$ is bounded on X and attains its supremum.

Example 2.1.6. Recall that

$$\mathcal{L}^p(X) = \{f : f \text{ is measurable and } (\int_X |f|^p d\mu)^{1/p}) < \infty\},\ 1 \leq p < \infty$$

$\mathfrak{L}^\infty(X) = \{f : f$ *is measurable and ess* $\sup |f| < \infty\}$. Let $(X, \sum, \mu)$ be a measure space. If $1 \leq p < \infty$ then $||f||_p = (\int_X |f|^p d\mu)^{1/p}$ is norm on $\mathfrak{L}^p(X)$ called the standard norm on $\mathfrak{L}^p(X)$. We do not distinguish between equivalent functions (functions which are different on a set with measure zero) in $\mathcal{L}^p$.

Example 2.1.7. $||f||_\infty = ess\ \sup\{|f(x) : x \in X\}$ is a norm on $\mathfrak{L}^\infty(X)$ called the standard norm on $\mathfrak{L}^\infty(X)$.

Recall that ℓ^p is the vector space of all sequences (x_n) in $\mathbb{C}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$ for $1 \leq p < \infty$ and ℓ^∞ is the vector space of all bounded sequences in $\mathbb{C}$. Therefore, if we take counting measure on $\mathbb{N}$ in (iv) and (v) we deduce that ℓ^p for $1 \leq p < \infty$ are normed spaces. For completeness we define the norms on these space in Example 2.1.8

Example 2.1.8. (1) If $1 \leq p < \infty$ then $||(x_n)||_p = (\sum_{n=1}^\infty |x_n|^p)^{1/p}$ is a norm on ℓ^p. called the standard norm on ℓ^p.

(2)$||(x_n)||_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$ is a norm on ℓ^∞ called the standard norm on ℓ^∞.

Definition 2.1.9. Let $||.||_1, ||.||_2$ be norms on a vector space E. Then $||.||_1$ and $||.||_2$ are said to be *equivalent* if there exists some constant $k > 0$ such that for all $x \in E$, $||x||_1 \leq k.||x||_2$ and $||x||_2 \leq k.||x||_1$.

2.2 Linear Operators

.

Definition 2.2.1. Let X and Y be normed spaces over $\mathbb{C}$. A *linear transformation* is a map $T : X \longrightarrow Y$ such that

1. $T(x+y) = T(x) + T(y)\ \forall x, y \in X$;

2. $T(ax) = a.T(x)\ \forall a \in \mathbb{C}$ and $x \in X$.

Lemma 2.2.2. Let X and Y be normed spaces and let $T : X \longrightarrow Y$ be a linear transformation. The following are equivalent:

1. T is uniformly continuous;

2. T is continuous at 0;

3. there exists a positive real number k such that $||T(x)|| \leq k$ whenever $x \in X$ and $||x|| \leq 1$;

4. there exists a positive real number k such that $||T(x)|| \ \leq \ k.||x||$ for all $x \in X$.

Example 2.2.3. Let $f \in \mathbb{C}_{\mathbb{F}}[0,1]$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be the space of all continuous functions on [0,1]. Then the linear map $T : \mathbb{C}_{\mathbb{F}}[0,1] :\longrightarrow \mathbb{F}$ defined by

$T(f) \ = \ f(0)$

is continuous. Indeed, let $f \in \mathbb{C}_{\mathbb{F}}[0,1]$. Then $|T(f))| \ = \ |f(0)| \ \leq \ \sup\{|f(x)| : x \in [0,1]\} = ||f||$. Hence T is continuous by condition(4) of Lemma 2.2.2 with $k = 1$.

Definition 2.2.4. Let X and Y be normal linear spaces and let $T : X \longrightarrow Y$ be a linear transformation. T is said to be bounded if there exists a positive real number k such that $||T(x)|| \leq k.||x||$ for all $x \in X$.

Notation 2.2.5. Let X and Y be normed spaces. Let us denote the set of all continuous linear transformation from X to Y by $B(X,Y)$. Elements of $B(X,Y)$ are also called bounded linear operators or linear operators or sometimes just operators. However, $B(X,Y)$ is a linear *subspace* of $L(X,Y)$, the space of all linear maps from X to Y, and so $B(X,Y)$ is a vector space.

2.3 The Norm of a Bounded Linear Operator

If X and Y are normed spaces we know that $B(X,Y)$ is a vector space. We now present a Lemma shows that $B(X,Y)$ is also *a normed space.* However, before presenting that Lemma we need the following

$$\sup\{||T(x)|| : ||x|| \leq 1\} = \inf\{k : ||T(x)|| \leq k||x||\}$$

for all $x \in X$ and so in particular $||T(y)|| \leq \sup\{||T(x)|| : ||x|| \leq 1\}||y||$ for all $y \in X$.

Lemma 2.3.1. Let X and Y be normed spaces. If $||.|| : B(X,Y) \longrightarrow \mathbb{R}$ is defined by $||T|| = sup\{||T(x)|| : ||x|| \leq 1\}$. Then $||.||$ is a norm on $B(X,Y)$.

Definition 2.3.2. Let X and Y be normed linear spaces and let $T \in B(X,Y)$. *The norm* of T is defined by

$$||T|| = \sup\{||T(x)|| : ||x|| \leq 1\}$$

.

Example 2.3.3. If $T : \mathbb{C}_{\mathbb{F}}[0,1] \longrightarrow \mathbb{F}$ is the bounded linear map defined by

$$T(f) = f(0)$$

then $||T|| = 1$. Indeed, it was shown in Example 1.1.13 that $|T(f)| \leq ||f||$ for all $f \in \mathbb{C}_{\mathbb{F}}[0,1]$. Hence $||T|| = \inf\{k : ||T(x)|| \leq k||x|| \text{ for all } x \in X\} \leq 1$. On the other hand, if $g : [0,1] \longrightarrow \mathbb{C}$ is defined by $g(x) = 1$ for all $x \in X$ then $g \in \mathbb{C}_{\mathbb{F}}[0,1]$ with $||g|| = \sup\{|g(x)| : x \in X\} = 1$ and $|T(g)| = |g(0)| = 1$. Hence $1 = |T(g)| \leq ||T|| \cdot ||g|| = ||T||$.

Definition 2.3.4. Let X and Y be normed linear spaces and let $T \in L(X,Y)$. If $||T(x)|| = ||x||$ for all $x \in X$ then T is called an *isometry.*

On every normed space there is at least one isometry.

Example 2.3.5. If X is a normed space and I is the identity linear transformation on X then I is an isometry. As another example of an isometry consider the following linear transformation.

Example 2.3.6. 1. If $x = (x_1, x_2, x_3, ...) \in \ell^2$ then $y = (0, x_1, x_2, x_3, ...) \in \ell^2$.

2. The linear transformation $S : \ell^2 \longrightarrow \ell^2$ defined by

$$S(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

is an isometry. Indeed,

Since $x \in \ell^2$,

$$|0|^2 + |x_1|^2 + |x_2|^2 + ... = |x_1|^2 + |x_2|^2 + |x_3|^2 + ... \leq \infty$$

and so $y \in \ell^2$.

$$||S(x)||^2 = |0|^2 + |x_1|^2 + |x_2|^2 + ... = |x_1|^2 + |x_2|^2 + |x_3|^2 + ... = ||x||^2$$

and hence S is an isometry.

Definition 2.3.7. If X and Y are normed linear spaces and T is an isometry from X onto Y then T is called an *isometric isomorphism* and X and Y are called *isometrically isomorphic.*

2.4 The Space $B(X, Y)$ and Dual space

Let us look at the space $B(X, Y)$ where X and Y are normed linear spaces.

Theorem 2.4.1. If X is a normed space and Y is a Banach space then the normed space $B(X, Y)$ is a Banach space. One case of the above which occurs sufficiently often to warrant separate notation is when $Y = \mathbb{F}$.

Definition 2.4.2. Let X be a normed space over $\mathbb{F}$. The space $B(X, \mathbb{F})$ is called the *dual space* of X and is denoted by X^*.

Corollary 2.4.3. If X is a normed vector space then X^* is a Banach space.

Indeed, the space $\mathbb{F}$ is complete so X^* is a Banach space by Theorem 2.4.1.

Example 2.4.4. Let C_0 be the linear subspace of ℓ^∞ consisting of all sequences which converges to 0.

1. If $a = (a_n) \in \ell^1$ and $(x_n) \in C_0$, then $(a_n x_n) \in \ell^1$, and that the linear transformation $f_a : C_0 \longrightarrow \mathbb{C}$ defined by

$$f_a((x_n)) = \sum_{n=1}^{\infty} a_n x_n$$

   is continuous with $||f_a|| \leq ||(a_n)||_1$.

2. For every $f \in C_0^*$ there exists a unique $a = (a_n) \in \ell^1$ such that $f = f_a$ and show also that $||(a_n)||_1 \leq ||f||$.

3. The linear transformation $T : \ell^1 \longrightarrow C_0^*$ defined by $T(a) = f_a$ is an isometry.

Proof. 1. Since $(x_n) \in C_0(\mathbb{C})$ and $(a_n) \in \ell^1(\mathbb{C})$ we have $\lambda = ||(x_n)||_\infty = \sup\{|x_n| : n \in \mathbb{N}\} < \infty$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. Since, for all $n \in \mathbb{N}$

$$|x_n a_n| \leq \lambda |a_n|,$$

$\sum_{n=1}^{\infty} |x_n a_n|$ converges by the comparison test. Thus $(x_n a_n) \in \ell^1(\mathbb{C})$. It's clear that f_a is bounded, so it's continuous. However, $||f_a((x_n))|| \leq ||(a_n)||_1$ for every $||(x_n)||_\infty \leq 1$. Hence, $||f_a|| \leq ||(a_n)||_1$

2. First note that if b and c are distinct elements of ℓ^1 then $f_b \neq f_c$ so there is at most one element a of ℓ^1 such that $f = f_a$. Let $a_n = f((e_n))$ for all $n \in \mathbb{N}$, where $(e_1) = (1, 0, 0, ...), (e_2) = (0, 1, 0, 0, ...)$...... If S is the linear subspace of C_0 consisting of sequences with only fintely many non-zero terms then S is dense in C_0. Let x be an element of S where $x = (x_1, x_2, x_2, ..., x_n, 0, 0, ...)$. Then

$$f(x) = f(\sum_{j=1}^{n} x_j e_j) = \sum_{j=1}^{n} x_j f(e_j) = (\sum_{j=1}^{n} x_j a_j).$$

Let $m \in \mathbb{N}$ and (b_n) be a sequence in S such that $b_n = 0$ if $n > m$ and $a_n b_n = |a_n|$ for $n \leq m$. Then

$$\sum_{j=1}^{m} |a_j| = |\sum_{j=1}^{m} a_j b_j| \leq ||f|| ||(b_n)||_\infty = ||f||.$$

Therefore $\sum_{j=1}^{\infty} |a_j| \leq ||f||$ so $(a_n) \in \ell^1$ and $||(a_n)||_1 \leq ||f||$. Finally as the contunuous function $f = f_a$ agree on the dense subset S it follows that $f = f_a$.

3. The map $T : \ell^1 \longrightarrow C_0^*$ defined by $T(a) = f_a$ is a linear transformation which maps ℓ^1 onto C_0^* by part (1). From the inequalities $||f_a|| \leq ||a||_1$ and $||a||_1 \leq ||f|| = ||f_a||$ it follows that $||T(a)|| = ||f_a|| = ||a||_\infty$. Hence T is an isometry.

□

The above example will be used in the first definition in the last chapter.

2.5 Weak*-Topology

Let X be a normed space. The dual space X^* is is itself a normed vector space by using the norm $||T|| = \sup\{||T(x)|| : ||x|| \leq 1\}$.

Definition 2.5.1. The *weak*-topology* on X^* is the weakest topology such that for every $x \in X$, the evaluation map

$$\Phi_x : X^* \longrightarrow \mathbb{C}$$

defined by

$$\Phi_x(f) = f(x) \text{ for all } f \in X^*$$

is continuous.

An important fact about the *weak**-topology is the Banach-Alaoglu theorem: the unit ball in X^* is compact in the *weak**-topology. The proof of this fact can be found in [8].

Chapter 3

Group Actions and Amenability

3.1 Group actions

If G is a group and X is a set, then a (left) group *action* of G on X is a binary map

$$. : G \times X \longrightarrow X$$

(where the image of $g \in G$ and $x \in X$ is written as $g.x$) which satisfies the following two axioms:

1. $(g \cdot h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$;

2. $e \cdot x = x$.

for every $x \in X$ (e denotes the identity element of G). From these two axioms, it follows that for every $g \in G$, the function which maps $x \in X$ to $g.x$ is a bijective map from X to X. Therefore, one may alternatively define a group action of G on X as a group homomorphism from G into the symmetric group S_X. If a group action is given, we also say that G acts on the set X or X is a *G-set.* In complete analogy, one can define a right group action of G on X as a function $g : X \times G \longrightarrow X$ by the two axioms:

1. $x \cdot (g \cdot h) = (x \cdot g) \cdot h$;

2. $x \cdot e = x$.

Note that the difference between left and right actions is only in the order in which a product like gh acts on x. For left actions h acts first followed by g, while for right actions g acts first followed by h. From a right action a left action can be constructed by composing with the inverse operation on the group. If r is a right action, then $f : G \times M \longrightarrow M : (g, m) \longmapsto r(m, g^{-1})$ is a left action, since
$f(gh, m) = r(m, (gh)^{-1}) = r(m, h^{-1}g^{-1}) = r(r(m, h^{-1}), g^{-1}) =$
$= r(f(h, m), g^{-1}) = f(g, f(h, m))$ and
$f(e, m) = r(m, e^{-1}) = r(m, e) = m$. Therefore in the sequel, we consider only left group actions, since right actions add nothing.

Examples 3.1.1. 1. Every group G acts on itself in two natural but essentially different ways: $g \cdot x = gx$ for all x in G, or $g \cdot x = gxg^{-1}$ for all $x \in G$.

2. The symmetric group S_n and its subgroups act on the set $\{1, ..., n\}$ by permuting its elements: $\sigma.k = \sigma(k)$.

Definition 3.1.2. Consider a group G acting on a set X. The *orbit* of a point x in X is the set of elements of X to which x can be moved by the elements of G. The orbit of x is denoted by Gx :

$$Gx = \{g.x : g \in G\}$$

The defining properties of a group guarantee that the set of orbits of X under the action of G form a partition of X.

Definition 3.1.3. If X and Y are two G-sets, we define a *morphism* from X to Y to be a function $f : X \longrightarrow Y$ such that $f(g.x) = g.f(x)$ for all g in G and all $x \in X$.

Morphisms of G-sets are also called *equivariant* maps or *G-maps.*

Continuous group actions.

One often considers continuous group actions: the group G is a topological group, X is a topological space, and the map $G \times X \longrightarrow X$ is continuous with respect to the product topology of $G \times X$. The space X is also called a G-space in this case. This is indeed a generalization, since every group can be considered a topological group by using the discrete topology. All the concepts introduced above still work in this context, however we define morphisms between G-spaces to be continuous maps compatible with the action of G.

3.2 Amenable Groups

Let G be a locally compact group and $L^\infty(G)$ be the Banach space of all essentially bounded functions $G :\longrightarrow \mathbb{R}$ with respect to the Haar measure.

Definition 3.2.1. A linear functional in $L^\infty(G)^*$ is called *a mean* if it maps the constant function $f(g) = 1$ to 1 and non-negative functions to non-negative numbers.

Definition 3.2.2. Let L_g be the left action of $g \in G$ on $f \in L^\infty(G)$, defind by $(L_g f)(h) = f(g^{-1}h)$ $({}^g f(h) = f(g^{-1}h))$ for every $h \in G$. Then, a mean m is said to be *left-invariant* if $m(L_g f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$.

Definition 3.2.3. A locally compact group G is *amenable* if there is a left-invariant mean on $L^\infty(G)$.

Example 3.2.4. Finite groups are amenable.

Proof. Let G be a finite group. If $f \in L^{\infty}(G)$, and $\mathrm{Im}(f) = \{a_1, \ldots, a_m\}$, where $m \leq n$, then $f = \sum_{i=1}^{m} a_i \chi_{A_i}$, where $A_i = f^{-1}(\{a_i\})$. Now, define $\mu : L^{\infty}(G) \longrightarrow \mathbb{C}$ by

$$\mu(f) = \sum_{i=1}^{m} a_i \frac{1}{|G|}$$

for every $f \in L^{\infty}(G)$. Then

1. $\mu(f) = 1$ if $f \equiv 1$.

2. $\mu(f) \geq 0$ if $f \geq 0$.

3.

$$\begin{aligned} \mu(L_x f) &= \mu(L_x(\sum_{i=1}^{m} a_i \chi_{A_i})) \\ &= \sum_{i=1}^{m} a_i \mu(L_x \chi_{A_i}) \\ &= \sum_{i=1}^{m} \mu(\chi_{xA_i}) \\ &= \sum_{i=1}^{m} a_i \\ &= \mu(f); \end{aligned}$$

for every $x \in Gf \in L^{\infty}(G)$.

□

Example 3.2.5. The group $\mathbb{Z}$ with the discrete topology is amenable.

Indeed, let $P(\mathbb{Z}) = \{f \in L^1(\mathbb{Z}), f \geq 0, \int f d\lambda = 1\}$. For every $f \in P(\mathbb{Z}), \hat{f}$ is a mean on $L^{\infty}(Z)$ where $\hat{f} : L^1(\mathbb{Z})' \longrightarrow \mathbb{C}$ such that $\hat{f}(\phi) = \phi(f)(L^1(\mathbb{Z})' \simeq L^{\infty}(\mathbb{Z}))$. Where

$L^1(\mathbb{Z})'$ is the dual space of $L^1(\mathbb{Z})$. It is well known that $P(\mathbb{Z})\hat{}$ is *weak**-dense in the set of means on $L^\infty(\mathbb{Z})$.

A typical element in $P(\mathbb{Z})$ is $\sum_{-\infty}^{\infty} a_r \chi_{\{r\}}$, where $a_r \geq 0$ for all r and $\sum_{-\infty}^{\infty} a_r = 1$. We need to construct a sequence $\{f_n\}$ in $P(\mathbb{Z})$ with at least one of its w^*-cluster points an invariant mean. Let

$$f_n = \frac{1}{2n+1} \sum_{r=-n}^{n} \chi_{\{r\}}$$

Then if $\phi \in \ell^1(\mathbb{Z})$ and $s \geq 0$ in $\mathbb{Z}$, we have

$$\begin{aligned}
|\hat{f}_n(\phi s) - \hat{f}_n(\phi)| &= |\frac{1}{2n+1}\left(\sum_{r=-n}^{n} (\phi(\chi_{\{r+s\}}) - \phi(\chi_{\{r\}}))\right)| \\
&= \frac{1}{2n+1}|\left(-\sum_{-n}^{-n+s-1} \phi(\chi_{\{r\}}) + \sum_{n+1}^{n+s} \phi(\chi_{\{r\}})\right)| \\
&\leq \frac{1}{2n+1}\left(|\sum_{-n}^{-n+s-1} \phi(\chi_{\{r\}})| + |\sum_{n+1}^{n+s} \phi(\chi_{\{r\}})|\right) \\
&\leq \frac{1}{2n+1}\left(\sum_{-n}^{-n+s-1} |\phi(\chi_{\{r\}})| + \sum_{n+1}^{n+s} |\phi(\chi_{\{r\}})|\right) \\
&\leq \frac{1}{2n+1}\left(\sum_{-n}^{-n+s-1} ||\phi|| + \sum_{n+1}^{n+s} ||\phi||\right) \\
&= \frac{2s||\phi||}{2n+1} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

A similar result holds if $s < 0$ and we see that every *weak**−cluster point of $\{\hat{f}_n\}$ in the set of means on $L^\infty(\mathbb{Z})$ is a left invariant.

Now, we shall restrict ourselves only to discrete groups, that is, groups with no topological structure.

Amenability of discrete groups

The definition of amenability is quite a lot simpler in the case of a discrete group.

Definition 3.2.6. A discrete group G is amenable if it has a finitely-additive left invariant probability measure.

Proposition 3.2.7. The definition 3.2.6 is equivalent to the definition in terms of $\ell^\infty(G)$.

Proof. If G is a amenable discrete group, then there exists a left-invariant mean $m : \ell^\infty(G) \longrightarrow \mathbb{C}$. Define $\mu : P(G) \longrightarrow [0,1]$ by

$$\mu(E) = m(\chi_E)$$

for every $E \subset G$. Then

1. $\mu(E) \geq 0$ for every $E \subset G$.

2. $\mu(G) = m(\chi_G) = 1$.

3. $\mu(\cup_{i=1}^n E_i) = m(\chi_{\cup_{i=1}^n E_i}) = m(\sum_{i=1}^n \chi_{E_i}) = \sum_{i=1}^n m(\chi_{E_i}) = \sum_{i=1}^n \mu(E_i)$. Where E_i are pairwise disjoint.

4. $\mu(xE) = m(\chi_{xE}) = m(L_x\chi_E) = m(\chi_E) = \mu(E)$ for ever $E \subset G$ and $x \in G$.

On the other hand, if G is a discrete group and it has a finitely-additive left invariant probability measure μ, then define $m^* : A \longrightarrow \mathbb{C}$ by

$$m^*(\sum_{i=1}^n a_i\chi_{E_i}) = \sum_{i=1}^n a_i\mu(E_i)$$

where $A = span\{\chi_E : E \subset G\}$ and $a_i \in \mathbb{C}$, $E_i \subset G$. First, we shall show m^* is a norm continuous.

$$\begin{aligned} |m^*(\sum_{i=1}^{n} a_i \chi_{E_i})| &= |\sum_{i=1}^{n} a_i \mu(E_i)| \\ &\leq \sum_{i=1}^{n} |a_i||\mu(E_i)| \\ &\leq |a_k| \sum_{i=1}^{n} |\mu|E_i| \\ &\leq |a_k|\mu(G) = |a_k|, \end{aligned}$$

where $|a_k| = \max_{i=1}^{n} |a_i|$. Now, since A is a norm-dense in $L^\infty(G)$, we can extend m^* to $m : L^\infty(G) \longrightarrow \mathbb{C}$ such that $m_{|A} = m^*$. Now,

(1) $m(f) > 0$ if $f > 0$;

(2) $m(f) = 1$ if $f \equiv 1$;

(3)

$$\begin{aligned}
m(L_g f) &= m(L_g(\lim_{k\longrightarrow\infty}\sum_{i=1}^{n_k} a_i\chi_{E_i})) \\
&= m(\lim_{k\longrightarrow\infty} L_g(\sum_{i=1}^{n_k} a_i\chi_{E_i})) \\
&= m(\lim_{k\longrightarrow\infty}(\sum_{i=1}^{n_k} a_i\chi_{gE_i})) \\
&= \lim_{k\longrightarrow\infty} m(\sum_{i=1}^{n_k} a_i\chi_{gE_i}) \\
&= \lim_{k\longrightarrow\infty}(\sum_{i=1}^{n_k} a_i\mu(gE_i)) \\
&= \lim_{k\longrightarrow\infty}(\sum_{i=1}^{n_k} a_i\mu(E_i)) \\
&= \lim_{k\longrightarrow\infty}(\sum_{i=1}^{n_k} a_i m(\chi_{E_i})) \\
&= m(\lim_{k\longrightarrow\infty}(\sum_{i=1}^{n_k} a_i\chi_{E_i})) \\
&= m(f) \text{ for every } f \in L^\infty(G) \text{ and } g \in G.
\end{aligned}$$

□

Example 3.2.8. Here is an example of non-amenable group. The free group on two (non-commuting) generators, say a and b. The group F_2 is the set of all (reduced, ie simplified) words of finite length constructed from the 'alphabet' a, a^{-1}, b, b^{-1}, and includes, as the identity, the empty word ϕ. The group operation is concatenation, that is, for any words $w, z \in F_2$, where

$$w = w_1 w_2 w_3 \cdots w_m,$$

and

$$z = z_1 z_2 z_3 \cdots z_n,$$

then $w.z$ is given by the concatenated word

$$w.z = w_1 w_2 w_3 \cdots w_m z_1 z_2 z_3 \cdots z_n,$$

which is then reduced if appropriate. Now, we will show that F_2 (equipped, of course, with the discrete topology) is not amenable.

Suppose F_2 is amenable, then it has a finitely-additive left invariant probability measure μ. For $x \in F_2$, let $E_x = \{y \in F_2 : y \text{ is a reduced word that starts with } x\}$. Then $F_2 = \{\phi\} \cup E_a \cup E_{a^{-1}} \cup E_b \cup E_{b^{-1}}$. Also, $aE_{a^{-1}} = E_{a^{-1}} \cup E_b \cup E_{b^{-1}} \cup \{\phi\}$. We notice that $F_2 = E_{a^{-1}} \cup a^{-1}E_a$. Similarly, $F_2 = E_{b^{-1}} \cup b^{-1}E_b$. Put $A = E_a \cup E_{a^{-1}}$ and $B = E_b \cup E_{b^{-1}}$. Using the invariance of μ, we obtain

$$\begin{aligned} \mu(A) &= \mu(E_a) + \mu(E_{a^{-1}}) \\ &= \mu(E_a) + \mu(aE_{a^{-1}}) = 1. \end{aligned}$$

Similarly, $\mu(B) = 1$. Now,

$$\begin{aligned} \mu(F_2) &= \mu(A \cup B) \\ &= \mu(A) + \mu(B) \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

which contradicts that $\mu(F_2) = 1$.

Example 3.2.9. Finite groups are amenable. If $|G| = n$. Then, $\mu : P(G) \longrightarrow [0,1]$ defined by

$$\mu(A) = \frac{|A|}{n}$$

for $A \subset G$, is a finitely-additive left invariant probability measure.

We shall give a different proof of $\mathbb{Z}$ being amenable from the one in 3.2.5.

Example 3.2.10. The group $\mathbb{Z}$ with the discrete topology is amenable.

Indeed, define $\mu_n : P(Z) \longrightarrow [0,1]$ by

$$\mu_n(A) = \frac{|A \cap [-n,n]|}{2n+1}$$

for every $A \subset \mathbb{Z}$. Then

1. $\mu_n(\phi) = 0$,
2. $\mu_n(A \cup B) = \frac{|(A \cup B) \cap [-n,n]|}{2n+1} = \frac{|A \cap [-n,n]|}{2n+1} + \frac{|B \cap [-n,n]|}{2n+1} = \mu_n(A) + \mu_n(B)$.
   if $A \cap B = \phi$,
3. $\mu_n(\mathbb{Z}) = 1$.

Thus, μ_n is a finitely-additive probability measure.

Now, let ξ be a free ultrafilter on $\mathbb{N}$ and define $\mu : P(\mathbb{Z}) \longrightarrow [0,1]$ by

$$\mu(A) = \lim_{n \longrightarrow \xi} \mu_n(A)$$

for every $A \subset \mathbb{Z}$. Then

1. $\mu(\phi) = 0$,
2. If $A, B \subset \mathbb{Z}$ and $A \cap B = \phi$, then $\mu(A \cup B) = \lim_{n \longrightarrow \xi} \mu_n(A \cup B) = \lim_{n \longrightarrow \xi}(\mu_n(A) + \mu_n(B)) = \lim_{n \longrightarrow \xi} \mu_n(A) + \lim_{n \longrightarrow \xi} \mu_n(B) = \mu(A) + \mu(B)$.
3. $\mu(\mathbb{Z}) = 1$.

Now, we shall show that μ is left-invariant, i.e, we need to show that

$$\mu(x+A) = \mu(A),$$

For every $A \subset \mathbb{Z}$ and $x \in \mathbb{Z}$.

In other words, we want to show that

$$\lim_{n \longrightarrow \xi} \mu_n(x+A) = \lim_{n \longrightarrow \xi} \mu_n(A),$$

Or equivalently

$$\lim_{n \longrightarrow \xi} |\mu_n(x+A) - \mu_n(A)| = 0.$$

Given $\epsilon > 0$. We want to show $C = \{n : |\mu_n(x+A) - \mu_n(A)| < \epsilon\} \in \xi$. Without loss of generality, consider $x > 0$ (if $x < 0$, consider $A' = A - x$ and $x' = -x$) and $n > x$. Put $|A \cap [-n-x, n-x]| = m$ and $|A \cap [-n, n]| = k$. It is clear that

$$A \cap [-n, n] = (A \cap [-n, n-x]) \cup (A \cap [n-x, n])$$

and

$$A \cap [-n-x, n-x] = (A \cap [-n-x, -n]) \cup (A \cap [-n, n-x]).$$

In other words,

$$k = |A \cap [-n, n-x]| + |A \cap [n-x, n]|$$

and

$$m = |A \cap [-n-x, -n]| + |A \cap [-n, n-x]|.$$

Then $m = k + |A \cap [-n-x, -n]| - |A \cap [n-x, n]|$.

Hence,

$$\begin{aligned} m-k &= |A\cap[-n-x,-n]|-|A\cap[n-x,n]| \\ &\leq |A\cap[-n-x,-n]| \\ &\leq \sharp[-n-x,-n]=x. \end{aligned}$$

However,

$$\begin{aligned} m-k &= |A\cap[-n-x,-n]|-|A\cap[n-x,n]| \\ &\geq -|A\cap[n-x,n]| \\ &\geq -x. \end{aligned}$$

Therefore, $|m-k|\leq x$. Now, we have

$$\begin{aligned} |\mu_n(x+A)-\mu_n(A)| &= \frac{1}{2n+1}|(x+A)\cap[-n,n]|-|A\cap[-n,n]| \\ &= \frac{1}{2n+1}|m-k| \\ &\leq \frac{|x|}{2n+1}. \end{aligned}$$

As ξ is an ultrafilter, by Proposition 1.5.10, $M=\{n : n\geq \frac{|x|}{2\epsilon}\}\in\xi$. Hence, as $M=\{n:\frac{|x|}{2n+1}\}\subset C$, we have $C\in\xi$.

We shall generalize Example 3.2.10 for the group $\mathbb{Z}^m$.

Example 3.2.11. The group $\mathbb{Z}^m$ is amenable.

Indeed, define $\mu_n : P(\mathbb{Z}^m)\longrightarrow[0,1]$ by

$$\mu_n(A)=\frac{|A\cap M|}{(2n+1)^m},$$

where $M=[-n,n]^m$.

It's obvious that μ_n is a finitely-additive probability measure for every n.

In order to obtain a finitely-additive left invariant probability measure μ, we will follow the procedure of the previous example.

Let ξ be a free ultrafilter on $\mathbb{N}$. Define $\mu : P(\mathbb{Z}^m) \longrightarrow [0,1]$ by

$$\lim_{n \longrightarrow \xi} \mu_n(A) = \mu(A)$$

Again, it's easy to see that μ is a finitely-additive probability measure. So, all we need to show is that it's left invariant. In order to show it's left invariant it's enough to show that

$$\lim_{n \longrightarrow \xi} |\mu_n(x + A) - \mu_n(A)| = 0.$$

Given $\epsilon > 0$. We want to show that $C = \{n : |\mu_n(x + A) - \mu_n(A)|\} \in \xi$.

$$\begin{aligned} |\mu_n(x + A) - \mu_n(A)| &= \frac{1}{(2n+1)^m}||(x + A) \cap M| - |A \cap M|| \\ &\leq \frac{m||x||}{2n+1}. \end{aligned}$$

As ξ is an ultrafilter, by Proposition 1.5.10, $M = \{n : n \geq \frac{m||x||}{2\epsilon}\} \in \xi$. Hence, as $M = \{n : \frac{m||x||}{2n+1}\} \subset C$, we have $C \in \xi$.

Proposition 3.2.12. Subgroups of amenable groups are amenable.

Proof. If G is an amenable group then it has a finitely-additive left-invariant probability measure μ. Now, let $H \leq G$ and let S be a selector of the family of cosets $\{Hg : g \in G\}$ (from each coset Hg we pick exactly one element). Let μ be left invariant mean on the group G; define for any subset A of H, $\mu'(A) = \mu(\cup_{s \in S} As)$. then

1. $\mu'(\emptyset) = 0$.

2. $\mu'(A \cap B) = \mu'(A) \cap \mu'(B)$. If $A \cap B = \emptyset$. (Beacuse $As \subset Ha$ and $Bs' \subset Hs'$).

3. $\mu'(hA) = \mu(\cup_{s\in S} hAs) = \mu(h(\cup_{s\in S} As)) = h\mu(\cup_{s\in S} As) = \mu'(A)$.

Thus, it defines a left invariant probability measure μ' on H. The point is that if A, B are disjoint subsets of H then $\cup_{s\in S} As$ and $\cup_{s\in S} Bs$ are disjoint too. □

Theorem 3.2.13.

1. Let G be a group and let N be a normal subgroup of G. Then G is amenable if and only if G/N and N are both amenable.

2. Let G be an amenable group, and let $\varphi : G \longrightarrow H$ be a homomorphism such that $\varphi(G) = H$. Then H is amenable.

Proof. 1. ($\Rightarrow$) Let G be amenable. By Proposition 3.2.12, N is amenable as well. Now, define $\Phi : \ell^\infty(G/N) \longrightarrow \ell^\infty(G)$ by

$$\Phi(\phi) = f, \text{ where } f(x) = \phi(xN)$$

If μ is a left-invariant mean on G, then $\mu \circ \Phi$ is a left invariant mean on G/N.

$$\ell^\infty(G/N) \xrightarrow{\Phi} \ell^\infty(G) \xrightarrow{\mu} \mathbb{R}$$

Indeed,

(a) If $\phi \equiv 1$, then $\Phi(\phi) = f \equiv 1$. So, $\mu \circ \Phi(\phi) = \mu(\Phi(\phi)) = 1$;

(b) If $\phi > 0$, then $\Phi(\phi) = f > 0$. So, $\mu \circ \Phi(\phi) = \mu(\Phi(\phi)) > 0$;

(c) If $gN \in G/N$ and $\phi \in \ell^\infty(G/N)$, then

$$\begin{aligned}\Phi({}^{gN}(\phi))(x) &= {}^{gN}\phi(xN) \\ &= \phi(g^{-1}xN).\end{aligned}$$

However, ${}^g f(x) = {}^{gN}\phi(xN) = \phi(g^{-1}xN)$. Hence, $\Phi({}^{gN}\phi) = {}^g f$. Therefore,

$$\begin{aligned}\mu \circ \Phi({}^{gN}(\phi)) &= \mu(\Phi({}^{gN}(\phi))) \\ &= \mu({}^g f) = \mu(f) \\ &= \mu(\Phi(\phi)) \\ &= \mu \circ \Phi(\phi).\end{aligned}$$

($\Leftarrow$) Since N is amenable, then there exists a left-invariant mean $\mu_N : \ell^\infty(N) \longrightarrow \mathbb{R}$. For every $g \in G$, define $m_g : \ell^\infty(gN) \longrightarrow \mathbb{R}$ by

$$m_g(R) = \mu_N({}^{g^{-1}}R).$$

Now, if $f \in \ell^\infty(G)$, define $\bar{f} : G/N \longrightarrow \mathbb{R}$ by

$$\bar{f}(gN) = m_g(f|_{gN}).$$

The function $\bar{f}$ is well defined. In indeed, if $gN = hN$(Thus, $h^{-1}g \in N$), then

$$\begin{aligned}
\bar{f}(gN) &= m_g(f|_{gN}) \\
&= \mu_N({}^{g^{-1}}(f|_{gN})) \\
&= \mu_N({}^{h^{-1}g}({}^{g^{-1}}(f|_{gN}))) \\
&= \mu_N({}^{h^{-1}}(f|_{gN})) \\
&= \mu_N({}^{h^{-1}}(f|_{hN})) \\
&= m_h(f|_{hN}) \\
&= \bar{f}(hN)
\end{aligned}$$

Define $\mu : \ell^\infty(G) \longrightarrow \mathbb{R}$ by

$$\mu(f) = \mu_{G/N}(\bar{f}),$$

where $\mu_{G/N}$ is a left-invariant mean on G/N (because G/N is amenable). This μ is a left-invariant mean on G. Indeed,

(a) If $f \equiv 1$, then $\bar{f} \equiv 1$. So, $1 = \mu_{G/N}(\bar{f}) = \mu(f)$.

(b) If $f \geq 0$, then $\bar{f} \geq 0$. So, $\mu(f) = \mu_{G/N}(\bar{f}) \geq 0$.

(c) Let $g \in G$ and $f \in \ell^\infty(G)$, then $\mu({}^g f) = \mu(f)$. In order to show that it's enough to show $\overline{{}^g f} = {}^{gN}\bar{f}$(because then $\mu({}^g f) = \mu_{G/N}(\overline{{}^g f}) = \mu_{G/N}({}^{gN}\bar{f}) = \mu_{G/N}(\bar{f}) = \mu(f)$). We shall start with $\overline{{}^g f}$.

$$\begin{aligned}
\overline{{}^g f}(hN) &= m_h({}^g f|_{hN}) \\
&= \mu_N({}^{h^{-1}}({}^g f|_{hN})).
\end{aligned}$$

On the other hand,

$$\begin{aligned} {}^{gN}\bar{f}(hN) &= \bar{f}(g^{-1}NhN) \\ &= \bar{f}(g^{-1}hN) \\ &= m_{g^{-1}h}(f|_{g^{-1}hN}) \\ &= \mu_N({}^{(h^{-1}g)}(f|_{g^{-1}hN})). \end{aligned}$$

Hence, once we show ${}^{h^{-1}}({}^{g}f|_{hN}) = {}^{(h^{-1}g)}(f|_{g^{-1}hN})$ we are done. Take $x \in N$, then

$$\begin{aligned} {}^{h^{-1}}({}^{g}f|_{hN})(x) &= ({}^{g}(f|_{hN})(hx) \\ &= f(g^{-1}hx), \end{aligned}$$

And

$$\begin{aligned} {}^{(h^{-1}g)}(f|_{g^{-1}hN})(x) &= f|_{g^{-1}hN}(g^{-1}hx) \\ &= f(g^{-1}hx). \end{aligned}$$

2. Define $m_H : \ell^\infty(H) \longrightarrow \mathbb{R}$ as follows

$$m_H(f) = m_G(f \circ \varphi),$$

where m_G is a left-invariant mean on G. Then

(a) If f $\equiv 1$, then $m_H(f) = m_G(f \circ \varphi) = 1$;

(b) If f ≥ 0, then $m_H(f) = m_G(f \circ \varphi) \geq 0$;

(c) In order to show m_H is left-invariant, we have to discuss the expression ${}^{h}f \circ \varphi$, where $h \in H$. Since φ is onto, there exists a $g_0 \in G$ such that

$\varphi(g_0) = h^{-1}$. Now,

$$\begin{aligned}({}^h f \circ \varphi)(g) &= {}^h f(\varphi(g)) \\ &= f(h^{-1}\varphi(g)) \\ &= f(\varphi(g_0)\varphi(g)) \\ &= f(\varphi(g_0 g))(\text{ because } \varphi \text{ is a homomorphism }) \\ &= (f \circ \varphi)(g_0 g) = ^{g_0^{-1}} (f \circ \varphi)(g).\end{aligned}$$

Hence, ${}^h f \circ \varphi = ^{g_0^{-1}}(f \circ \varphi)$. Now,

$$\begin{aligned}m_H({}^h f) &= m_G({}^h f \circ \varphi) \\ &= m_G({}^{g_0^{-1}}(f \circ \varphi)) \\ &= m_G(f \circ \varphi) \\ &= m_H(f).\end{aligned}$$

□

Corollary 3.2.14. If H and K are amenable groups, then $H \times K$ is amenable as well.

Proof. It follows from Theorem 3.2.13 part 1. □

From Corollary 3.2.14, we can conclude that $\mathbb{Z}^m$ is amenable.

Corollary 3.2.15. Let G be a finitely generated abelian group, then G is amenable.

Proof. Let G be a finitely generated abelian group, by The Fundamental Theorem of Finitely Generated Abelian Groups, G is isomorphic to a group of the form

$$\mathbb{Z}^m \oplus \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_t},$$

where $m \geq 0$, and the numbers $m_1, ..., m_t$ are powers of prime numbers. Hence, $G/N \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_1} \oplus ... \oplus \mathbb{Z}_{m_t}$, where $N \leq G$ such that $N \cong \mathbb{Z}^m$.

Now, G/N and N are amenable as G/N is a finite group and $N \cong \mathbb{Z}^m$. Therefore, by Theorem 3.2.13, G is amenable. □

Proposition 3.2.16. Let G be a group. Then G is amenable if and only if every finitely generated subgroup of G is amenable.

Proof. $\Rightarrow$ Let G be an amenable group and let H be a subgroup of G. By Proposition 3.2.12, H is amenable.

$\Leftarrow$ For every finite subset F of G, there exists a left-invariant mean $m_F : \ell^\infty(gp < F >) \longrightarrow \mathbb{R}$ (because $gp < F >$ is amenable, where $gp < F >$ denotes the subgroup generated by F). The collection ς of subsets of $P_w(G)$, where $P_w(G)$ is the set of all finite subsets of G, such that for every $F \in P_w(G)$,

$$\{E \subset G : |E| < \infty \text{ and } F \subset E\} \in \varsigma$$

is a filter base for some filter on $P_w(G)$. This filter is contained in an ultrafilter ξ. Now, define $m : \ell^\infty(G) \longrightarrow \mathbb{R}$ as follows

$$m(f) = \lim_{F \longrightarrow \xi} m_F(f|_{gp<F>}).$$

Then m is a left-invariant mean on G. Indeed,

1. If $f \equiv 1$, then $m(f) = \lim_{F \longrightarrow \xi} m_F(f|_{gp<F>}) = 1$;

2. If $f \geq 0$, then $m(f) = \lim_{F \longrightarrow \xi} m_F(f|_{gp<F>}) \geq 0$;

3. Let $g \in G$ and $f \in \ell^\infty(G)$, then $m({}^g f) = \lim_{F \longrightarrow \xi} m_F({}^g f|_{gp<F>})$.

   We shall show that $m({}^g f) = m(f)$. That is,

$$\lim_{F \longrightarrow \xi} m_F({}^g f|_{gp<F>}) = \lim_{F \longrightarrow \xi} m_F(f|_{gp<F>})$$

   an equivalently

$$\lim_{F \longrightarrow \xi} |m_F({}^g f|_{gp<F>}) - m_F(f|_{gp<F>})| = 0.$$

   Given $\epsilon > 0$. We want to show that $C = \{F : |m_F({}^g f|_{gp<F>}) - m_F(f|_{gp<F>})| < \epsilon\} \in \xi$. It's obvious that $C \neq \phi$ because $\{g\} \in C$.

   Now, for any finite subset E of G with $\{g\} \subset E$, we have $gp < \{g\} > \subset gp < E >$. Hence, $\{E \subset G : |E| < \infty$ and $\{g\} \subset E\} \subset C$ and since $\{E \subset G : |E| < \infty$ and $\{g\} \subset E\} \in \xi$, so $C \in \xi$.

□

Corollary 3.2.17. Abelian groups are amenable.

Proof. Simply combine Colloray 3.2.15 with Proposition 3.2.16 □

Corollary 3.2.18. Solvable groups are amenable.

Proof. Let G be a solvable group. Then there exists a anorm series

$$\{e\} = G^0 = G^1 \lhd G^2 \lhd \cdots G^n = G$$

such that G_{i+1}/G_i is abelian. The group G^0 is amenable because it is finite and G^1/G^0 is amenable as it is abelian, so by Theorem 3.2.12 G^1 is amenable as well. Now, by the finite induction, we conclude that $G^n = G$ is amenable. □

We shall end this chapter by presenting two statements equivalent to the definition of amenability. The proofs can be found in [1] and [11].

Theorem 3.2.19. Let p any real number such that $1 \leq p < \infty$.A group G is amenable if and only if G satisfies the Reiter's condition. That is, if for any compact $C \subset G$ and $\epsilon > 0$ there exists $f \in \{f \in L^p(G) : f \geq 0, ||f||_p = 1\}$ such that $||gf - f||_p < \epsilon$ $\forall g \in C$.

Theorem 3.2.20. A group G is amenable if and only if G satisfies the Følner condition. That is, for any finite collection of elements $g_1, ..., g_n$ and any $\epsilon > 0$, there is a finite subset $H \subset G$ such that for each g_i, $1 \leq i \leq n$,

$$\frac{|H\Delta(g_i.H)|}{|H|} < \epsilon.$$

Chapter 4

Topologically Amenable Groups

4.1 Basic definitions and properties

Throughout this note, Z will denote a countable infinite set.

Definition 4.1.1. Denote by $\operatorname{prob}(Z)$ the set of Borel probability measures on Z, or in other words, the set of functions $b : Z \longrightarrow [0,1]$ such that $\sum_{z\in Z} b(z) = 1$. We will view $\operatorname{prob}(Z)$ as a subset of $\ell^1(Z)$ and equip it with the *weak**-topology (recall that $\ell^1(Z)$ is the Banach space dual of $c_0(Z)$). We will denote by $||.||_1$ the usual norm on $\ell^1(Z)$.

Remark 4.1.2. It is clear that $\operatorname{prob}(Z)$ is not *weak**-compact. In fact, it is not even *weak**-closed in ℓ^1: take $b_n(z) = 1$ for $z = n$ and $b_n(z) = 0$ for $z \in Z\backslash\{n\}$.Then the sequence b_n is in $\operatorname{prob}(Z)$ and tends *weakly** to 0 (which dose not belong to $\operatorname{prob}(Z)$). The *weak** convergence of b_n means $< b_n, u >$ converges to $< b, u >$ for each u in $c_0(Z)$, where $< b, u > = \sum_{z\in Z} b(z)u(z)$. But $< b_n, u > = u(n)$ converges to 0 as $n \longrightarrow \infty$, because u is in $c_0(Z)$.

Definition 4.1.3 ([16] Definition 2.2). Let G be a countable discrete group and let X be a compact Hausdorff space on which G acts by homeomorphisms. The action is said to be *amenable if there exists a sequence of weak*-continuous maps* $b^n : X \longrightarrow \text{prob}\,(G)$ *such that for every* $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in X} ||gb_x^n - b_{gx}^n||_1 = 0.$$

Definition 4.1.4. A countable group G is called topologically amenable if there exist a compact Hausdorff space X on which G acts by homeomorphism and weak*-continuous maps $b^n : X \longrightarrow \text{prob}\,(G)$ such that for every $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in X} ||gb_x^n - b_{gx}^n||_1 = 0.$$

It is not clear to whom the following observation due, as it assumed to be known to everybody from the very beginning of the theory.

Theorem 4.1.5. For countable groups the following are equivalent:

1. G is amenable;

2. The trivial action on a one-point space X is amenable.

Proof. ($1 \Longrightarrow 2$) Let G be a countable amenable group. Since G is amenable, we can enumerate its elements, $G = \{g_1, g_2, g_3, ...\}$. By Reiter's Condition, for every n and $F_n = \{g_1, g_2, ..., g_n\}$ there exists a map $b^n : X \longrightarrow \text{prob}\,(G)$, where $X = \{x\}$, such that

$$\forall i \leq n, ||g_i b_x^n - b_x^n||_1 < \frac{1}{n}.$$

Now, for every $g \in G, g = g_i$ for some i, and therefore $g \in F_n$ for all $n \geq i$. This implies that for all such n, $||g_i b_x^n - b_x^n||_1 < \frac{1}{n}$, and therefore

$$\lim_{n \longrightarrow \infty} ||gb_x^n - b_x^n||_1 = 0.$$

($2 \Longleftarrow 1$) Let G be a countable group such that its action on a one point space $X = \{x\}$ is amenable. Then there exists a sequence (b^n) of maps from X to prob(G) as in Definition 4.1.3. Let F be a finite subset of G and let $\epsilon > 0$. Since the action is amenable, then there exists an N such that $\forall n > N$ and $\forall g \in F$,

$$||gb_x^n - b_x^n||_1 < \epsilon.$$

In other words, G satisfies Reiter's Condition for $p = 1$.

□

Proposition 4.1.6. Subgroups of topologically amenable groups are topologically amenable.

Proof. Let G be a topologically amenable group and H be a subgroup of G. Since G is topologically amenable, then there exist a compact Hausdorff space X on which G acts by homeomorphism and weak*-continuous maps $b^n : X \longrightarrow \text{prob}\,(G)$ such that for every $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in X} ||gb_x^n - b_{gx}^n||_1 = 0.$$

By axiom of choice, we can define a map $S : H\backslash G \longrightarrow G$ such that $\forall g \in G$, $S(Hg) \in Hg$. Define a map $\varpi : G \longrightarrow H$ as follows

$$\varpi(g) = g(S(Hg))^{-1}.$$

The map ϖ is onto: for $h \in H$, choose $g = hS(H)$. Then

$$\begin{aligned} \varpi(g) &= \varpi(hS(H)) \\ &= hS(H)(S(HhS(H)))^{-1} \\ &= hS(H)(S(H(S(H)))^{-1} \\ &= hS(H)(S(H))^{-1} \\ &= h. \end{aligned}$$

The map ϖ is H-equivariant: for $h \in H$, $\varpi(hg) = hg(S(Hhg)^{-1} = hg(S(Hg))^{-1} = h\varpi(g)$. Define a map $\pi : \ell^1(G) \longrightarrow \ell^1(H)$ as follows

$$\begin{aligned} \pi\left(\sum_{g\in G} f(g)g\right) &= \sum_{g\in G} f(g)\varpi(g) \\ &= \sum_{h\in H}\left(\sum_{g\in\varpi^{-1}(h)} f(g)\right)h. \end{aligned}$$

Then π is well defined and linear. Also,

$$||\pi\left(\sum_{g\in G} f(g)g\right)||_1 = \sum_{g\in G}|f(g)|.$$

So, it is an isometry with norm 1. Moreover, the map π is H-equivariant because ϖ is H-equivariant.

Now, restrict π on the probability spaces: $\pi_* : \operatorname{prob}(G) \longrightarrow \operatorname{prob}(H)$. This map is weak*-continuous. Now, consider the following composition,

$$X \xrightarrow{b^n} \operatorname{prob}(G) \xrightarrow{\pi_*} \operatorname{prob}(H)$$

Then, the map $\pi_* \circ b^n : X \longrightarrow \operatorname{prob}(H)$ is weak*-continuous. For $h \in H$,

$$\begin{aligned} ||h(\pi_* \circ b^n)_x - (\pi_* \circ b^n)_{hx}||_1 &= ||h\pi_*(b^n_x) - \pi_*(b^n_{hx})||_1 \\ &= ||h\sum_{g\in G} b^n_x\varpi(g) - \sum_{g\in G} b^n_{hx}(g)\varpi(g)||_1 \\ &= ||\pi_*(hb^n_x) - \pi_*(b^n_{hx})||_1 \\ &\leq ||\pi_*(hb^n_x - b^n_{hx})||_1 \\ &\leq ||hb^n_x - b^n_{hx}||_1||\pi_*||_1 \\ &= ||hb^n_x - b^n_{hx}||_1. \end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty}\sup_{x\in X}||h(\pi_* \circ b^n)_x - (\pi_* \circ b^n)_{hx}||_1 = 0.$$

□

It is clear (although, we give the proof for it at the next Proposition) from Definition 4.1.3 if G acts amenable on a compact Hausdorff space X and if $Y \longrightarrow X$ is any continuous map of G-spaces. Then the action of G on Y is amenable.

The following result appears in [16] without a proof.

Proposition 4.1.7 ([16] Proposition 2.3.). If G admits an amenable action on a compact Hausdorff space X then its action on the Stone-Čech compactification βG is amenable.

Proof. As X and βG are two G-spaces, there is a G-map $f : \beta G \longrightarrow X$, which can be obtained as follows: fix $x \in X$ and take its orbit map T
$T : G \longrightarrow X$ such that $g \longmapsto gx$, extend T to βG using the universal property of the Stone-Čech compactification to obtain a continuous map $f : \beta G \longrightarrow X$. We shall show that $gf(t) = f(gt)$ for every $g \in G$ and $t \in G$. For $t \in \beta G$, there exists a net (t_λ) in G such that $t_\lambda \longrightarrow t$. Now, $gf(t_\lambda) \longrightarrow gf(t)$ and $gf(t_\lambda) = f(gt_\lambda) \longrightarrow f(gt)$ and since X is Hausdorff, so $f(gt) = gf(t)$. Where,

$$\beta G \xrightarrow{\ f\ } X \xrightarrow{\ b^n\ } prob(G)$$

is continuous. Moreover,

$$\begin{aligned} \sup_{t\in\beta G} \|g(b^n \circ f)_t - (b^n \circ f)_{gt}\|_1 &= \sup_{t\in\beta G} \|g(b^n_{f(t)}) - b^n_{f(gt)}\|_1 \\ &= \sup_{t\in\beta G} \|g(b^n_{f(t)}) - b^n_{gf(t)}\|_1 \\ &\leq \sup_{x\in X} \|gb^n_x - b^n_{gx}\|_1. \end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup_{t\in\beta G} \|g(b^n \circ f)_t - (b^n \circ f)_{gt}\|_1 = 0.$$

□

Let us now reformulate the definition of toplogical amenability in the case of the action of a group G on its Stone-Čech compactification.

Lemma 4.1.8 ([16] Lemma 3.7.). A countable group G acts amenably on its Stone-Čech compactification if an only if there is a sequence of maps $b^n : G \longrightarrow \text{prob}\,(G)$ such that.

1. for each n, the image of the map b^n is contained within a *weak**-compact subset of $\text{prob}\,(G)$; and

2. for every $g \in G$,

$$\lim_{n \longrightarrow \infty} \sup_{h \in G} ||gb^n_h - b^n_{gh}||_1 = 0.$$

Proof. (⇒) Suppose that the action is amenable. There are then maps $b^n : \beta G \longrightarrow \text{prob}\,(G)$ as in Definition 4.1.3 and restricting them to $G \subset \beta G$ gives maps as in the statement of Lemma.

(⇐) If on the other hand maps $b^n : G \longrightarrow \text{prob}\,(G)$ are defined, as above, then by item (1) and the universal property of the Stone-Čech compactification the maps b^n extend to maps from βG into $\text{prob}\,(G)$. We note that

$$\sup_{h \in G} ||gb^n_h - b^n_{gh}||_1 = \sup_{x \in \beta G} ||gb^n_x - b^n_{gx}||_1$$

.

This is because G is dense in βG, because the map $x \longmapsto gb^n_x - b^n_{gx}$, from βG into $\ell^1(G)$, is *weak**-continuous, and because the norm function on $\ell^1(G)$ is *weak**-semicontinuous (norm-closed balls are weak*-closed). □

If $b \in \ell^1(Z)$ and F is a subset of Z then denote by $b|_F$ the function

$$b|_F(z) = \begin{cases} b(z) & \text{if } z \in F \\ 0 & \text{if } z \notin F \end{cases}$$

The following Lemma appears in [16] without a proof.

Lemma 4.1.9 ([16] Lemma 3.8.). Let Z be a discrete set. For every w^*-compact subset B of $\text{prob}\,(Z)$ and every $\varepsilon > 0$ there is a finite set $F \subset Z$ such that $||b - b|_F|| < \varepsilon$, for every $b \in B$.

Proof. Fix $\varepsilon > 0$, and for each finite set $H \subset Z$ let

$$U_H = \{b \in \text{prob}\,(Z) : ||b|_H||_1 > 1 - \varepsilon\}.$$

We shall prove that the sets form w^*-open cover of prob(Z).

Fix any $p \in Z$. Let g_p be the $c_0(Z)$ function which equals 1 at p and 0 everywhere else. Then $b(p) =< b, g_p >= \sum_{z \in Z} b(z) g_p(z)$. Hence by the definition of weak*-topology, the map $b \longmapsto b(p)$ (which is the same map as the map $b \longmapsto < b, g_p >$) is continuous for each fixed p in Z.

Now for a finite subset $H = \{p_1, ..., p_k\}$ of Z, the map $b \longmapsto b(p_i)$ form $\text{prob}\,(Z)$ into $\mathbb{R}$ is continuous for each $i = 1, 2, 3, ..., k$, and so the map $b \longmapsto b(p_1) + b(p_2) + ... + b(p_k)$ is continuous. But $b(p_1) + b(p_2) + ... + b(p_k) = ||b|_H||_1$ as $b(p) \geq 0$, so the map $b \longmapsto ||b|_H||_1$ is continuous. Hence for any real r, the set $\{b \in \text{prob}\,(Z) : ||b|_H||_1 > 1 - r\}$ is open.

The sets U_H defined in the proof of the lemma are of the above form, so U_H is open. Finally, for any $b \in \text{prob}\,(Z)$, $\sum_{z \in Z} b(z) = 1$ (and this series is non-negative, and so absolutely convergent), hence there is a finite subset H of Z such that $\sum_{z \in Z \setminus H} b(z) <$

ε, so $\sum_{z\in H} b(z) > 1-\varepsilon$, i.e., $||b|_H||_1 > 1-\varepsilon$, so $b \in U_H$. Thus the sets U_H cover prob (Z) as ε runs over positive reals and H over finite subsets of Z.

Therefore, the compact set B is covered by finitely many of the U_H. Take F to be the union of the finite sets H associated to the finite cover. □

The following is a new result.

Lemma 4.1.10. A countable group G acts amenably on its Stone-Čech compactification if an only if there is a sequence of maps $b^n : G \longrightarrow$ prob (G) such that.

1. for each n, there exists a finite set $F_n \subset G$ such that b^n_g is supported on F_n for every $g \in G$.

2. for every $g \in G$,

$$\lim_{n\longrightarrow\infty} \sup_{h\in G} ||gb^n_h - b^n_{gh}||_1 = 0.$$

Proof. ($\Rightarrow$) Suppose that the action is amenable. There are then maps $b^n : \beta G \longrightarrow$ prob (G) as in Definition 4.1.3 and restricting them to $G \subset \beta G$ gives maps as in the statement of Lemma 4.1.8 By Lemma 4.1.9, for every $\frac{\epsilon}{5} > 0$ there is a finite set $F \subset G$ such that $||b - b|_F||_1 < \frac{\epsilon}{5}$ for every $b \in b^n(G)$.

Define $w^n : G \longrightarrow$ prob (G) by

$$w^n_g = \frac{b^n_g|_{F_n}}{||b^n_g|_F||_1}.$$

then we notice that

1. $\text{supp}(w^n_g) \subset F_n$ for every $g \in G$; and

2. $\lim_{n\longrightarrow\infty}\sup_{h\in G}||gw_h^n - w_{gh}^n||_1 = 0$.

   Indeed, from $||b_g^n|_{F_n}||_1 > 1-\frac{\epsilon}{5}$, it follows that $||gw_h^n - gb_h^n|_{F_n}||_1 \leq \frac{\epsilon}{5}$, and then, taking n large enough, we obtain that $||gb_h^n - b_{gh}^n||_1 \leq \frac{\epsilon}{5}$(because $\lim_{n\longrightarrow\infty}\sup_{h\in G}||gb_h^n - b_{gh}^n||_1 = 0$ for every $g \in G$). Now,

$$\begin{aligned}
||gw_h^n - w_{gh}^n||_1 &= ||gw_h^n - gb_h^n|_{F_n} + gb_h^n|_{F_n} \\
&- gb_h^n + gb_h^n - b_{gh}^n + b_{gh}^n \\
&- b_{gh}^n|_{F_n} + b_{gh}^n|_{F_n} - gw_h^n||_1 \\
&\leq ||gw_h^n - gb_h^n|_{F_n}||_1 \\
&+ ||gb_h^n|_{F_n} - gb_h^n||_1 + ||gb_h^n - b_{gh}^n||_1 \\
&+ ||b_{gh}^n - b_{gh}^n|_{F_n}||_1 + ||b_{gh}^n|_{F_n} - gw_h^n||_1 \\
&\leq \frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5} = \epsilon.
\end{aligned}$$

($\Leftarrow$) Suppose there is a sequence of maps $b^n : G \longrightarrow \text{prob}\,(G)$ such that the conditions 1 and 2 hold. Define $w^n : G \longrightarrow \text{prob}\,(G)$ as follows

$$w_g^n = \frac{b_g^n|_{F_n}}{||b_g^n|_F||_1}.$$

We notice that $\text{supp}(w_g^n) \subset F_n$ for every $g \in G$. This implies that $w^n(G) \subset \delta_{\ell^2(F_n)}$. Thus, $w^n(G)$ is contained within a w^*-compact subset of prob$\,(G)$. □

The following result appears in [3]. The author does not give a proof, but he indicates that it follows from Schepin Spectral Theorem. Our proof here uses a different technique as well as leads to the proof of the main result.

Proposition 4.1.11 ([3] Assertion 1.). A countable group G admits an amenable action on some compact metrizable Hausdorff space if and only if its action on the Stone-Čech compactification βG is amenable.

Proof. $\Longrightarrow$It has been done in Proposition 4.1.7

$\Longleftarrow$Suppose G admits an amenable action on its Stone-Čech compactification. Then there is a sequence b^n of *weak**-continuous maps $b^n : \beta G \longrightarrow \text{prob}\,(G)$ such that for every $g \in G$,

$$\lim_{n\longrightarrow\infty} \sup_{x\in\beta G} ||gb^n_x - b^n_{gx}||_1 = 0.$$

For every $g \in G$, define $\bar{g} : \beta G \longrightarrow \beta G$

as follows

$$x \longmapsto gx.$$

Also, define the diagonal product $f = \triangle_{(g,n)\in G\times\mathbb{N}} b^n \circ \bar{g} : \beta G \longrightarrow \text{prob}\,(G)^{G\times\mathbb{N}}$ by

$$f(x) = (b^n_{gx})_{(g,n)\in G\times\mathbb{N}}.$$

It is clear that f is continuous because b^n and $\bar{g}$ are continuous. Now, define an equivalence relation R on βG as follows

$$(x, y) \in R \Longleftrightarrow b^n_{gx} = b^n_{gy} \text{ for every } n \in \mathbb{N} \text{ and } g \in G.$$

Also, define an action on $\beta G/R$ by G by $(g, [x]) \longmapsto [gx]$. This action is well-defined, in fact, if $[x] = [y]$, then $(g, [x]) = (g, [y])$ which means that $(x, y) \in R$ but that implies $b^n_{hx} = b^n_{hy}$ for every $h \in G$ and $n \in \mathbb{N}$. Now, for every $h \in G$ and $n \in \mathbb{N}$, $b^n_{h(gx)} = b^n_{(hg)x} = b^n_{(hg)y} = b^n_{h(gy)}$. Hence, $[gx] = [gy]$. Define $T : \beta G/R \longrightarrow f(\beta G)$ as follows

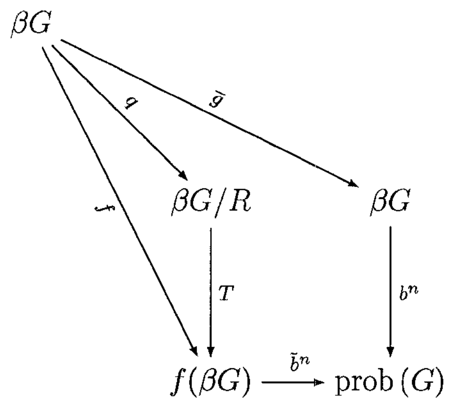
$$T([x]) = f(x).$$

The map q is a quotient map and $T \circ q = f$, so T is continuous, where we mean by f here the map $f : \beta G \longrightarrow f(\beta G)$. The map T is homeomorphism as βG is compact and T is bijective. Thus, $\beta G/R$ is metrizable as f is homeomorphism and $f(\beta G)$ is metrizabe (because $f(\beta G)$ is a subspace of a countable product of metrizable spaces of prob (G)). Now, define $\tilde{b}^n : f(\beta G) \longrightarrow \text{prob}\,(G)$ as follows

$$\tilde{b}^n = \pi_{(e,n)},$$

where $\pi_{(e,n)}$ the projection map restricted on $f(\beta G)$. Hence, $c^n = \tilde{b}^n \circ T : \beta G/R \longrightarrow \text{prob}\,(G)$ is a *weak**-continuous map.

Let $g \in G$, then



$$\begin{aligned}
\sup_{[x]\in\beta G/R} ||gc^n_{[x]} &- c^n_{g[x]}|| = \\
&= \sup_{[x]\in\beta G/R} ||g(\tilde{b}^n \circ T)_{[x]} - (\tilde{b}^n \circ T)_{g[x]}||_1 \\
&= \sup_{[x]\in\beta G/R} ||g(\tilde{b}^n(T([x]))) - \tilde{b}^n(T(g[x]))||_1 \\
&= \sup_{x\in\beta G} ||g(\tilde{b}^n(T([x]))) - \tilde{b}^n(T([gx]))||_1 \\
&= \sup_{x\in\beta G} ||g(\tilde{b}^n(f(x))) - \tilde{b}^n(f(gx))||_1 \\
&= \sup_{x\in\beta G} ||g(\tilde{b}^n((b^n_{hx}))_{(h,n)\in G\times\mathbb{N}}) \\
&- \tilde{b}^n(b^n_{hgx})_{(h,n)\in G\times\mathbb{N}})||_1 \\
&= \sup_{x\in\beta G} ||gb^n_x - b^n_{gx}||_1
\end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup_{[x]\in\beta G/R} ||gc^n_{[x]} - c^n_{g[x]}||_1 = 0.$$

□

4.2 The main result

The following result is new and forms the main result of the thesis.

Theorem 4.2.1. A countable group G is topologically amenable if and only if it admits an amenable action on the Cantor set C.

Proof. $\Longleftarrow$ By definition.

$\Longrightarrow$Suppose G has an amenable action on its Stone-Čech compactification. Then there is a sequence b^n of *weak**-continuous maps $b^n : \beta G \longrightarrow \text{prob}\,(G)$ such that for every $g \in G$,

$$\lim_{n\longrightarrow\infty} \sup_{x\in\beta G} ||gb^n_x - b^n_{gx}||_1 = 0.$$

For every $g \in G$, define $\bar{g} : \beta G \longrightarrow \beta G$ as follows

$$x \longmapsto gx.$$

The space $b^n(\beta G)$ is compact and metric, so it is the continuous image of the Cantor set C, i.e, there is a continuous map $f^n : C \longrightarrow b^n(\beta G)$ such that $f^n(C) = b^n(\beta G)$. Now, we shall define a continuous function T^n from βG onto C. We shall define it on G then extend it on βG. Let $g \in G$, then $b_g^n \in b^n(\beta G)$ and, as f^n is onto, we pick $c_g \in C$ such that $f^n(c_g) = b_g^n$(f^n is not necessarily injective). By sending g to c_g we obtain a map from G into C. By the the universal property of the Stone-Čech compactification, we can extend that map uniquely to a continuous map $T^n : \beta G \longrightarrow C$ as required. For every $g \in G$,

$$\begin{aligned} (f^n \circ T^n)(g) &= f^n(T^n(g)) \\ &= f^n(c_g) \\ &= b^n(g). \end{aligned}$$

Thus, $f^n \circ T^n = b^n$ on G. Since G is a dense subset of βG, so $f^n \circ T^n = b^n$. Define the diagonal product $R = \triangle_{(g,n)\times \in G\times\mathbb{N}} T^n \circ \bar{g} : \beta G \longrightarrow C^{G\times\mathbb{N}}$ as follows

$$R(x) = (T^n(gx))_{(g,n)\in G\times\mathbb{N}}.$$

By Proposition 1.2.2, R is a continuous map and so $R(\beta G)$ is compact metrizable zero-dimensional subspace of $C^{G\times\mathbb{N}}$. Also, define an action by G on $R(\beta G)$ as follows

$$(h, T^n(gx)) = (T^n(ghx))_{(g,n)\in G\times\mathbb{N}}.$$

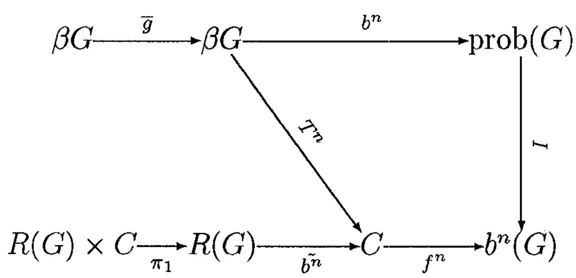
Now, define $\tilde{b}^n : R(\beta G) \longrightarrow C$ as follows

$$\tilde{b}^n = (\pi_{(e,n)})_{|R(\beta G)},$$

where $\pi_{(e,n)}$ the projection map restricted on $R(\beta G)$. Hence, $S^n = f^n \circ \tilde{b}^n : R(\beta G) \longrightarrow \text{prob}\,(G)$ is a *weak**-continuous map. The space $R(\beta G) \times C$ is uncountable compact metrizable zero-dimensional Hausdorff space without isolated points. Now, define an action by G on $R(\beta G) \times C$ as follows

$$(g, (R(x), c)) = (gR(x), c).$$

The map $S^n \circ \pi_1 : R(\beta G) \times C \longrightarrow \text{prob}\,(G)$ is *weak**-continuous, where is $\pi_1 : R(\beta G) \times C \longrightarrow R(\beta G)$ is the projection map.



For every $g \in G$,

$$\begin{aligned}
\sup_{(R(x),c)\in R(\beta G)\times C} ||g(S^n \circ \pi_1)_{(R(x),c)} - (S^n \circ \pi_1)_{g(R(x),c)}||_1 \\
&= \sup_{R(x)\in R(\beta G)} ||gS^n_{R(x)} - S^n_{gR(x)}||_1 \\
&= \sup_{R(x)\in R(\beta G)} ||g((f^n \circ \tilde{b}^n)(R(x))) - (f^n \circ \tilde{b}^n)(gR(x))||_1 \\
&= \sup_{x\in\beta G} ||g((f^n \circ \tilde{b}^n)((T^n(hx))_{(h,n)\in G\times\mathbb{N}})) \\
&- (f^n \circ \tilde{b}^n)((T^n(hgx))_{(h,n)\in G\times\mathbb{N}}))||_1 \\
&= \sup_{x\in\beta G} ||g(f^n(\tilde{b}^n((T^n(hx))_{(h,n)\in G\times\mathbb{N}}) - f^n(\tilde{b}^n((T^n(hgx))_{(h,n)\in G\times\mathbb{N}})||_1 \\
&= \sup_{x\in\beta G} ||g(f^n(T^n(x)) - f^n(T^n(gx))||_1 = \\
&= \sup_{x\in\beta G} ||gb^n_x - b^n_{gx}||_1
\end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup_{(R(x),c)\in R(\beta G)\times C} ||g(S^n \circ \pi_1)_{(R(x),c)} - (S^n \circ \pi_1)_{g(R(x),c)}||_1 = 0.$$

□

Remark 4.2.2. It was pointed out by one of the examiners, B. Sternberg, that an alternative proof of the main result can be obtained using the following obvservations: (Here G is a countable group).

1. If K is a totally disconnected compact metrizable amenable G-space and C is a Cantor set with diagonal trivial action of G, then $K \times C$ is a Cantor set and the diagonal action is amenable (the latter by Proposition 4.1.7 since the projection to the first coordinate is a G-morphism, the former for trivial reasons since we have removed isolated points).

2. Suppose X is a compact metrizable amenable G-space. Then there is a compact totally disconnected metric G-space K mapping to X (as a G-space). In light of the above, this will prove the result.

Using 1, we may assume without loss of generality that G has a dense orbit on X. That is, we have an onto continuous G-morphism $f : \beta G \longrightarrow X$. Now write X as a quotient $K : C \longrightarrow X$ of a Cantor set. Choose a section $s : X \longrightarrow C$ of K and extend $sf|_G$ to βG to obtain $S : \beta G \longrightarrow C$ such that $KS = f$. Now, G acts on the left of C^G by $gF(g') = F(g'g)$. Define a G-space map $h : \beta G \longrightarrow C^G$ by $h(t)(g) = S(gt)$. Then observe by evaluating at the identity e, we see $h(t) = h(t')$ implies $S(t) = S(t')$ and so $f(t) = KS(t) = KS(t') = f(t')$. Thus f factors as $q : h(\beta G) \longrightarrow X$, where q is the quotient map (since all spaces involved are compact Hausdorff). Moreover, this map must trivially be a G-morphism. Since $h(\beta G)$ is compact totally disconnected metric, this completes the proof.

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