HOW MATHEMATICS IS POLITICAL: AN ANALYSIS OF DISCOURSES AND ANTAGONISMS IN TWO UNDERGRADUATE MATHEMATICS COURSES

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Abstract

Sociopolitical issues of mathematics education have not been a priority in research conducted in post-secondary contexts (Adiredja & Andrews-Larson, 2017; le Roux, 2009). When sociopolitical lenses are utilised, the focus is usually on issues of equity and mathematics is not viewed critically. At the same time, although mathematics education research has highlighted and scrutinised political aspects of mathematics education (e.g., Pais, 2012; Valero, 2004), mathematics is not always thought of as political, while there is little research focusing on how it has political features. This study adopts a critical approach to mathematics, in order to examine how mathematics is political in post-secondary contexts. Deriving its theoretical lens from Laclau and Mouffe (1985/2001, 1987; Mouffe, 2005), it views mathematics as a space in which meanings are negotiated and antagonisms take place, while at the same time other ways of meaning making are excluded. More specifically, I identify three features of mathematics which make it inextricably political: the values of mathematical knowledge (Bishop, 1991; Ernest, 2018, 2020; Skovsmose, 2020), the rationality of mathematics (Kollosche, 2014; Walkerdine, 1988), and the formatting function of mathematics (Barwell, 2013; Skovsmose, 1994b, 2000). I have conducted an empirical investigation of the ways in which mathematics is political within two post-secondary courses, an *Introduction to Proofs* and a *Calculus* course. Through a discourse analysis perspective based on the works of Foucault (1972), and Laclau and Mouffe (1985/2001, 1987), I explore how political features of mathematics are negotiated and established through the ways in which people interact in the context of the two courses. Findings indicate that mathematics is not a final product that students need to absorb, but it is instead a nexus of interconnected discourses which are not fixed or stable entities but rather are constantly shifting and contested social constructions. In the "Introduction to Proofs" course, I identified four overarching discourses: the "formalising as a prototype of rigour" discourse, the "diverse participation in mathematics" discourse, the "imaginary practice of mathematics" discourse, and the "efficacy of mathematical work/labour" discourse. In the Calculus course, the overarching discourses I identified were the "humanity of mathematics" discourse, the "applying mathematics" discourse, and the "the mathematics of the class and the mathematics of mathematicians" discourse. These discourses entail ontological, epistemological, and axiological aspects; are specific to a given time and place; and sometimes fit together nicely while in other cases can enter antagonistic relations. This work contributes to mathematics education literature by problematising the neutrality of mathematics and its consideration as apolitical in post-secondary contexts.

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Chapter 0

In my early adult life, I spent more time than I would like to admit, explaining to my non-mathematician friends why my mathematical books start with a "Chapter 0". I strived to get them to see the beauty of the existence of Chapter 0. It does not mess with the numeration of the book's real content, but it still introduces the reader to concepts that they may not have encountered before, while it notifies them of all the agreements that the reader and the author make. For some reason, never did I manage to convince anybody about the elegance of this process.

My mathematical upbringing gave me a few other odd characteristics as well, that made me not fun at parties. Mathematical jokes could genuinely make me feel better during my undergraduate years. I have probably never laughed so hard about anything as I have for the "let epsilon be negative" joke. I have seldom felt as much satisfaction as when finally understanding the epsilon-delta definition of the limit of a function or when completing some proofs in calculus after hours of work and too much caffeine and nicotine in my body. My mathematical upbringing not only contributed to me learning some math, nor did it merely structure my way of thinking, looking at things, and understanding the world. It wove together what I learned to like and dislike, how I felt about things, and ultimately how I learned to be. It probably comes as no surprise that, sooner or later, I learned to look at things in terms of right and wrong, true and false, just and unjust.

But exiting the social bubble of an undergraduate mathematics program helped me to start realising the limits of this way of looking at the world. A key moment, or rather period of time, for this process was after I moved to Canada, where I started learning about colonisation and Indigenous peoples – a world which till then only included Columbus, Pocahontas, and Hollywoodian images of "Indian" people in the US for me. Apparently, these peoples did and do mathematics in ways that were different than the ones I had been learning at school. It took me a while to recognise these knowledges as mathematics and not as arithmetic or "ways to interact with the world". The constant need of Canadian mathematics education academia to insist on this knowledge being mathematical did not help. It felt like a distasteful version of orientalism; a forced effort to justify Indigenous peoples' knowledges as important – so important that they could even be named mathematical. And all this happened at the same time that Indigenous peoples' rights were clearly not respected and their knowledges had clearly not been legitimised. I still hold the view that there is some truth in my critique of this Canadian mathematics education discussion of orientalism. But this discussion also urged me to think of mathematics and my mathematical identity differently.

And through this process, I started doubting the limits of the mathematics that I knew, respected, and derived value from, while I started feeling uneasiness with the reservation of mathematics for a knowledge that only few can understand. I started to more deeply understand the role of mathematics in differentiating students who can and those who cannot – a differentiation that does not stay at school but reproduces the social division of labour. That very social division of labour which makes people live in poverty albeit working full time, while at the same time the wealthiest 1% of the people in the world own more than 50% of the world's wealth. It also helped me to more deeply realise why mathematics is such a gendered knowledge, that although the so-called gap between boys and girls has been closing, women continue to pursue academic mathematical careers in lower numbers than men, while women "choose" to follow applied mathematics more often than theoretical mathematics. Finally, it has helped me realise why this pure rationality that claims to be producing results beyond any doubt is at the core of a system in which humans attempt to control the world and other species through science and progress, leading to such detrimental results as the climate crisis or the current pandemic.

This thesis is an effort to rethink of the limits of the mathematics I have learned and like, while revisiting the context that taught me to develop this relationship with theoretical mathematics: undergraduate mathematics courses. It is a more conscious effort of looking at academic mathematics in a less romanticised and a more holistic, systematic, and critical way. And the way to do this has been to start thinking of mathematics as political.

Chapter 1: Scene Setting

This thesis examines what it means for mathematics to be political in the context of undergraduate mathematics education. In this introductory chapter, I discuss my motivation for this study and provide a brief overview of the thesis. The chapter is outlined as follows: I first outline a view claiming that "everything is political", which has acted as a motivation for this study. I then provide a rationale for the research project, followed by a brief description of the study. I continue with a snapshot from the study, which aims to offer a more tangible idea for what it means for mathematics to be political. I conclude this introductory chapter with an outline of the thesis.

1.1. If "Everything is Political", then so is Mathematics: Motivation for this Study

"Everything is political": this idea might sound strange, as the post-political vision is becoming increasingly hegemonic. As Mouffe (2005) notes:

> Sociologists claim that we have entered a 'second modernity' in which individuals liberated from collective ties can now dedicate themselves to cultivating a diversity of lifestyles, unhindered by antiquated attachments. The 'free world' has triumphed over communism and, with the weakening of collective identities, a world 'without enemies' is now possible. Partisan conflicts are a thing of the past and consensus can now be obtained through dialogue. Thanks to globalization and the universalization of liberal democracy, we can expect a cosmopolitan future bringing peace, prosperity and the implementation of human rights worldwide. (p. 1)

We thus live in a time, in which the focus on the political is weakening: grand discourses are deemed as de facto problematic, shared identities are mostly cultivated by liberal, non-radical discourses, and social/class interests are downplayed. This post-political society is characterised by the state's tendency to defend private interests and the interests of capital (Valero, 2018). Hence, the depoliticisation of the public sphere harms those who are in the more disadvantaged positions of society. The post-political vision pushes for an individualisation which makes it more difficult for political organisation to take place, exactly because we are all so "unique" and "different".

Counter to the post-political discourse, I claim that everything is political, including the ways we talk, the ways we travel, the ways we consume, the ways we entertain ourselves, the ways we make friends or fall in love, and so on. In the '60s and '70s, feminist scholars started developing this perspective by conceptualising politics as going beyond elections, representatives, and meetings, by focusing on the power relations entailed in gendered human relations and activities. For example, Millett (2000) in her book “Sexual Politics” shows that “sex is a status category with political implications” (p. 24), while Hanisch (2006) in her “The personal is political” paper highlights the connections between personal experiences and social and political structures. Today, the view that everything is political survives in and gets enriched by discourse theory as an approach which examines communicative instances in relation to wider ideologies and power systems. Discourse theory studies have focused on a wide range of issues, from movies (Doxiadis, 2011a) to humour (Archakis & Tsakona, 2020) and from credit card cardholder guides (Fairclough, 1992) to talk shows (Peck, 1995), highlighting their political nature.

If everything is political, then so is mathematics. Thinking of mathematics as political is a worthwhile endeavour exactly because of the naturalisation of the discourse that mathematics is a neutral (and hence a-political) form of knowledge both in mathematics education and beyond. This perspective has been instrumental to my interest in the work of scholars like Paola Valero, David Kollosche, Valerie Walkerdine, Anna Chronaki, Alexandre Pais, and others, who – one way or another – all explore political dimensions of mathematics education and mathematics. However, although I would nod in agreement, at the sound (or sight) of words proposing a variation of the phrase that “mathematics is political”, I have had difficulty understanding what this could mean in a satisfactory way. This thesis is my attempt to make sense of the idea that mathematics is political.

1.2. Mathematics is Political: Rationale for the Research Project

Mathematics is a special field of knowledge, for it is often assumed to be a universal and pure discipline, or in Russell’s (1907) words *pure thought’s natural home*. This assumption is related to the idea that mathematical truths do not rely on any empirical investigation and cannot be influenced by social, political, or ideological interests (Borba & Skovsmose, 1997). However, being a form of science, mathematics is embedded in the social and, thus it cannot be neutral (François & De Sutter, 2004). At the same time, mathematics and mathematics education play a central role in the fabrication and governance of the citizen by, for example, developing the idea of a “normal person” (Valero, 2018; Valero & Knijnik, 2015; Walkerdine & Lucey, 1989), the functioning of our economic systems (Kollosche, 2018b; Pais, 2012), as well as the shaping of particular ways of thinking about and acting in the world (Skovsmose, 2000; Walkerdine, 1988). In this sense, mathematics has a deeply political dimension because it both affects and is affected by wider sociopolitical circumstances. Therefore, there is great need to challenge the commonsensicality of mathematics’ neutrality and explore how mathematics relates to power (Valero, 2018).

Mathematics education has constituted itself as a research field in the intersection of mathematics and psychology, while more recently, social, political, and cultural insights from social sciences have been employed, signifying a "sociopolitical turn" (Valero, 2004). Critical views of mathematics have been developed within the wider field of mathematics education and especially through its sociopolitical turn. Critical mathematics education has problematised modernist discourses about mathematics, according to which mathematics is necessary for understanding nature, a powerful approach for technological invention, and a pure rationality divorced from other human activities (Skovsmose, 2011). Instead, Skovsmose (2011) counter-proposes that mathematics and mathematics education have no essential features: they can serve unjust socio-political roles, while they can also become tools for social justice. Moreover, the field of ethnomathematics has challenged the universality, neutrality, and intrinsic "goodness" of mathematics, by highlighting the diversity of practiced mathematics (e.g., d'Ambrosio, 1985). Finally, within gender and race research, mathematics classrooms have been viewed as racialised and gendered spaces and mathematics as a White and masculine discipline (e.g., Mendick, 2006). Nevertheless, mathematics is rarely conceptualised as political per se.

Within undergraduate mathematics education, which is a sub-field of mathematics education with some distinct conferences and journals, the socio-political turn has not been as popular (Adiredja & Andrews-Larson, 2017). When sociopolitical lenses are utilised in undergraduate mathematics research, the focus is primarily on issues of equity and inequity, as a response to the phenomenon of differentiated performance and differentiated learning experiences (Adiredja & Andrews-Larson, 2017). Nevertheless, within this scholarship, mathematics is not viewed critically. This study aims to contribute to undergraduate mathematics education scholarship, by offering a lens which conceptualises mathematical knowledge as political knowledge.

1.3. Overview of the Study and Research Questions

This study aims to inquire into how mathematics is political in the context of post-secondary mathematics education. Through a review of the literature, I argue that mathematics education research, while having taken a sociopolitical turn (Gutiérrez, 2013; Valero, 2004), does not thoroughly engage in a discussion of mathematics as political per se. Further, in undergraduate mathematics education contexts, mathematics is not viewed critically and sociopolitical questions are scarce. Through this empirical study, I aim to examine the political features of mathematics in two university mathematics courses, by investigating the discourses about mathematics in these courses.

Drawing upon Laclau and Mouffe (1985/2001, 1987; Mouffe, 2005), I construct a framework for mathematics being political. I argue that mathematics is political because it is a space in which meanings are negotiated and antagonisms take place, while at the same time other ways of meaning making are excluded. More specifically, we can see this in three axes: mathematics entails and reproduces values (Bishop, 1991; Ernest, 2018), mathematics puts forward a particular rationality (Kollosche, 2014; Walkerdine, 1988), and mathematics has a formatting function (Barwell, 2013; Skovsmose, 1994b, 2000).

In the empirical part of this study, I investigate how these political features of mathematics are manifested, negotiated, and established through the ways in which people interact in two undergraduate mathematics courses, an introductory calculus course and an introduction to proofs course. More specifically, the focus is on the courses' official discourses, which include the discourse of the syllabus, textbook, notes, professor, as well as on how the students and the professor engage in a process of negotiating these discourses.

Overall, the goal of the study is to explore what the statement that "mathematics is political" could mean. In this sense, the overall question that this study attempts to answer is:

> ***What does it mean for mathematics to be political in the context of undergraduate mathematics courses?***

This question gets specified in three research questions, which emerge from the three aforementioned political features of mathematics: the formatting function of mathematics, the rationality of mathematics, and the value given to mathematics. More specifically, the research questions are:

> ***1) What values are manifested in the two classes?***
>
> ***2) What mathematical rationality/rationalities is/are constructed in the two mathematics courses and how are they similar or different?***
>
> ***3) In what ways is the relation between mathematics and the physical/social world captured in the context of the two courses? In what ways is mathematics captured as having a formatting function?***

In the next section, I present a short piece of data, in order for the reader to be able to get a first sense of the data that I obtained and the ways in which I interacted with them[1]. It is intended to provide an initial

[1] Specifics of the data collection and analysis methods are presented in Chapter 4.

perspective for how this thesis attempts to connect the abstract notion of mathematics being political with ordinary discussions about mathematics in undergraduate classes.

1.4. A snapshot of the Study: *"The magic trick" example*

The following "magic trick" works as follows: a person chooses one number from 1 to 63. They then disclose in which of the following tables (1 – 6) this number is. It is then possible, for the person presenting the trick, the "magician", to identify what the number was, without having to go through all the tables.

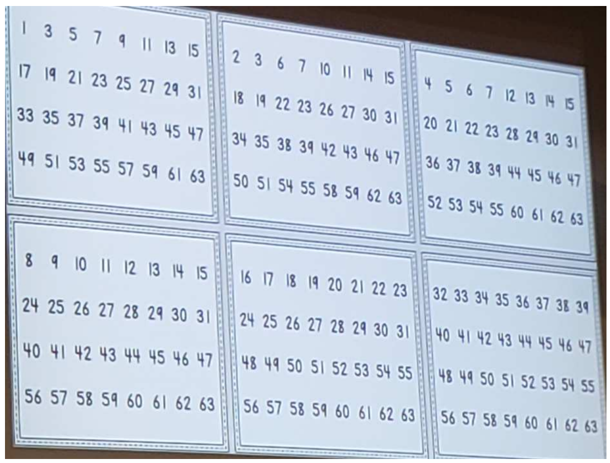

Figure 1.1: Magic Trick with Numbers 1-63

In the Introduction to Proofs Course, the professor, Chris, presents this trick to his students. The class will connect this trick to a mathematical theorem, which states that every natural number can be written as a sum of different powers of 2, and they will prove that theorem. But before this happens, the professor asks the students to observe the tables, identify patterns, and try to figure out how the trick works. As he gives them five minutes to do so and the students are initially silent, the professor makes the following joke:

> Chris: Oh, man. I just, I really hope that some like bureaucrat comes in and sees that our math class is just us staring at pages of numbers. [Some students are laughing] Uhm, that is what math is like, I guess. (Chris's course, p. 63; Lecture 7.2)

In what follows, I engage in analysis of this joke, which constitutes one possible interpretation of it based on the theoretical framework and the methodology of this thesis. On one level, this is just a joke about bureaucrats, numbers, and mathematics. The joke creates a dichotomy between "some bureaucrat" and "our class", as well as a dichotomy between (looking at) numbers and mathematics, while it also welcomes the event in which a bureaucrat would step in the classroom and misunderstand what is happening. The professor takes only a few seconds to tell it, the joke contributes to students engaging in what is happening in the class, while it also helps the professor build a more personal relationship with the students. It is a "good" joke, in the sense that it is clever and funny, for the sociocultural setting in which it is presented. From the perspective of incongruity theory (Oring, 1995/2017), we may argue that this joke is based on appropriate incongruities: its humour is the result of an implicit incongruity between mathematics as conceptualised by bureaucrats and mathematics as conceptualised in the class.

So, how does the joke relate to "mathematics being political"? What I wish to highlight is that this simple interaction – one of the many that might happen in a mathematics classroom – can showcase how mathematical knowledge is not a pure enterprise in which people engage far away from bureaucrats or worldly circumstances. Instead, it is a historical, cultural, and political product which happens in historically, culturally, and politically shaped institutions; it constructs "reality" and frames the possible actions within it; it introduces and cultivates particular ways for distinguishing between "truth" and "falsehood" or the "rational" and the "irrational"; and it reproduces values. In the following few paragraphs, I shortly elaborate on these elements.

First of all, I propose that the joke – as an interaction about mathematics – is a historical, cultural, and political product which happens in historically, culturally, and politically shaped institutions (Valero, 2018). The idea that a bureaucrat might enter the room "fits" the image of a North American university (and, thus, contributes to the success of the joke), exactly because mathematicians work and students learn mathematics in academic institutions, whose functions are shaped (also) by bureaucrats. Even if a bureaucrat did enter the room, they would not quite intervene in the lecture, as the professor and the bureaucrat have distinct professional roles which rarely intersect. The (historically, culturally and politically created) discourse of "academic freedom" is, thus, present here.

Second, I suggest that the joke – as an interaction about mathematics – constructs "reality" and frames the possible actions within it (Jørgensen & Phillips, 2011). In a sense, here, we have a joke about insiders and outsiders. A distinct role between the academic mathematical community and bureaucrats is painted: the word "some" in front the word "bureaucrat" (instead of, e.g., "a bureaucrat") shows the distance between the bureaucrat and the people in the mathematics classroom. This polarisation between the two is also reinforced by the initial phrase "Oh man", whose informality conceals the (hierarchical and other) differences between the professor and the students, thus constructing "our class" as a homogeneous, united group (Fairclough, 2003). The verbal welcoming of the possibility that a bureaucrat comes in (through the phrase "I really hope") is a jocularity. If the bureaucrat comes in, they will not be able to have a say on the level of the conducted class; they will merely misinterpret what is happening and, in a way, they will be fooled, thinking that all the math class is staring at numbers. However, this joke assumes that, contrary to popular discourses about mathematics, this is not what mathematics is. Overall, the joke draws upon a view of who “we” are, who others (i.e., bureaucrats) are, what is that “we” do, and what is that others think that “we” do. In this way, the joke paints a particular picture for reality, which in turn contributes to framing how the students and the professor (can) act in this course (Doxiadis, 2011a).

Third, I argue that the joke – as an interaction about mathematics – introduces and cultivates particular ways for distinguishing between “truth” and “falsehood” or the “rational” and the “irrational”. In other words, it creates a dichotomy between two different rationalities of mathematics. On the one hand, there is the version of the bureaucrat, in which making sense of mathematics consists of staring at it until an answer reveals itself; a practice which aligns with more solitary, disembodied, and mystic discourses about mathematics. This is in contrast with the practice manifested in the classroom, in which the students make observations, make conjectures, and prove the emerging theorem; a practice which aligns with more recent conceptualisations of mathematics as a social activity (e.g., de Villiers, 1990; Lakatos, 1976; Lerman, 2000). The two versions of mathematical rationality are brought in antithesis with each other through the joke.

Finally, I propose that the joke – as an interaction about mathematics – reproduces values about mathematics. From a discursive perspective, values may relate to how, through discourse, attitude is expressed (Thompson & Hunston, 2003), (un)desirability is constructed (Fairclough, 2003), and the discourse is related to ideology (Doxiadis, 2011a). While the bureaucrat would not be able to understand what is going on, the value of what the class engages in is not challenged. What the class does is important, regardless how useful or relatable it might appear to an outsider. This is further reinforced by it being exclusive.

With this description, my goal has been to explore what the statement that “mathematics is political” could mean. In this direction, I drew some possible connections of this short joke with a) mathematics constructing “reality” in a certain way, b) giving rise to different mathematical rationalities, and c) being related to values, while d) being a historical, cultural, and political product which manifests in historically, culturally, and politically produced institutions. In doing so, I did not mean that this joke is successful in capturing some elements of mathematics (which make it political), as if these elements were essentially or objectively related to mathematics. Instead, I aimed to suggest that it is through this joke (as well as other jokes and interactions that happen in the class) in which mathematics is constructed as a discursive (with political dimensions) practice. Through similar, everyday interactions – which at a surface level are just people doing, teaching, and learning mathematics – mathematics is attributed its character as an important activity which can lead to "rational", "certain", and "true" results.

1.5. Outline of the Thesis

This thesis is organised in eight chapters. In Chapter 2, I position this study in the mathematics education literature. By asking "where is the political?", I attempt to centre the notion of the political when looking at mathematics education literature. I review research which falls under three categories: a) sociopolitical research in mathematics education, with a focus on critical mathematics education, ethnomathematical and decolonising discourses, feminist and critical race theory studies, and cultural politics; b) sociopolitical research in undergraduate mathematics education, for which I distinguish between a liberal, a critical-sociological and a post-structuralist perspective; and c) undergraduate teaching and learning scholarship with a focus on the teaching and learning of calculus and the teaching and learning of proof and proving.

In Chapter 3, I outline the theoretical framework for this study, which is primarily based on the discourse theory work of Laclau and Mouffe (1985/2001). I focus on the notion of the political as developed by Mouffe (2005), Foucault's (1980) conceptualisation of power/knowledge, and discourse theory (Fairclough, 1995, 2003; Foucault, 1972; Laclau & Mouffe, 1985/2001, 1987). I then use these concepts to outline a framework for what it means for mathematics to be political, which is based on a) the values of mathematics, b) mathematics constructs and puts forward specific rationalities, and c) mathematics having a formatting function. Using this framework, I conclude the chapter with the presentation of the research problem and the research questions for this study.

In Chapter 4, I discuss the methodology of the empirical part of this study, which has been conducted in two undergraduate courses: an Introduction to Proofs course and a Calculus course. I talk about the critical ethnographic inspiration for this study, issues related to data collection and their rationale (i.e., recruitment of participants, characteristics of sites and participants, and data collection methods), the ways in which I worked with the data once collected (i.e., transcription of data, presentation of data, working across multiple discourse theories, and method of data analysis), ethical issues, and the rigour of the study.

In chapters 5 and 6, the findings of the project are presented. Chapter 5 refers to the proofs course, while Chapter 6 refers to the calculus course. Both chapters are organised around some overarching discourses that appear in each course. For the proofs course, these are the formalising as a prototype of rigour discourse, the diverse participation in mathematics discourse, the imaginary practice of mathematics discourse, and the efficacy of mathematical work/labour discourse. For the calculus course, the respective discourses are the humanity of mathematics discourse, the applying mathematics discourse, and the mathematics of mathematicians imaginary discourse. While presenting these discourses and their relations through examples, I refer to values of mathematics, the rationality of mathematics, and the formatting functions of mathematics, in order to address my research questions.

Chapter 7 is a discussion chapter. I use this space to articulate this thesis's thesis, respond to the study's research questions, make connections with literature, reflect on the theoretical and epistemological framework, discuss the limitations of the project, propose some political implications of this study, and articulate a few emerging questions.

Chapter 2: Where is the ~~math~~ Political?

This chapter positions this study in the wider field of mathematics education literature. Its title aims to capture and respond to a recent discussion conducted in the field of mathematics education: "where is the math?" (e.g., Wagner, 2017). This question constitutes a criticism to research which does not directly focus on the teaching and learning aspects of mathematics education. The criticism is addressed to texts and scholars who tend to focus on equity, identity, language, power, or the cultural politics of mathematics education, without exploring questions around best practices, or ways to teach and learn specific fields like algebra, calculus, or statistics. The "where is the math?" question is often posed in mathematics education conferences, while it has also been the topic of a 2010 editorial of Journal for Research in Mathematics Education (JRME), the biggest mathematics education journal in the United States. In that editorial, Heid (2010) mentions: "JRME publishes research in which mathematics is an essential component rather than being a backdrop for another area of inquiry. I encourage readers to continue to examine articles in JRME with the 'Where's the math?' question in mind" (p. 103)". In a response to the "where is the math?" criticism, Martin et al. (2010) maintain that Heid's editorial constitutes an exercise of power, which marginalises some scholarship and scholars.

From my perspective, the "where is the math?" question tends to miss the point of research which centres issues, other than mathematics. In the web of practices which constitute what we call mathematics education (e.g., Valero, 2010), there is a wide range of actors and factors that impact the teaching and learning of mathematics. The persistence on centring mathematics fails to address the complexity of schooling and students; by centring mathematics, there is other stuff that we simply cannot see. But, for me, there is also a bigger issue: why do leading figures in the field of mathematics education care so much for mathematics to stay in the centre, especially given that many of them are in fact interested in and sensitive towards sociopolitical issues like that of equity (e.g., Confrey, 2010)? And what does that have to do, if anything, with mathematics having such a privileged role in society and constituting an exemplar of rationality and ultimately truth? Overall, there seems to be something very political in the way in which (school) mathematics is centred, without much examination of the connections of this knowledge with sociopolitical issues. By changing the question to "where is the political?", I intend to highlight that the political aspects of mathematical knowledge and knowing are often overlooked in the context of teaching and learning. Hence, my goal in this chapter is to explore how the political can be centred in mathematics education research.

I start this chapter with a review of sociopolitical approaches in mathematics education literature which is the sub-area of mathematics education that is most relevant to this study. In doing so, I summarise some of the important discussions in this scholarship, while I also explore how mathematics has been conceptualised in it. Through this review, I maintain that mathematics education discourses have not widely conceptualised mathematics as political per se and I argue for the importance of taking up this kind of inquiry. I continue this chapter by exploring sociopolitical approaches in post-secondary education, which – as I will argue – primarily concern issues of equity and learner identity, without engaging in critical discussions about mathematics. Finally, I examine the literature focusing on the teaching and learning of undergraduate calculus and proof and proving, as the two courses in which the empirical study took place are a Calculus course and an Introduction to Proofs course. I argue that political perspectives in this literature are scarce and I, hence, engage in a political critique of this literature using sociopolitical mathematics education scholarship as a lens.

2.1. Sociopolitical Approaches in Mathematics Education

This section summarises some key discussions and concerns of sociopolitical approaches to mathematics education. Mathematics education has constituted itself as a research field in the intersection of mathematics and psychology, while more recently, social, political, and cultural insights from social sciences have been employed, signifying a "sociopolitical turn" in the field (Gutiérrez, 2013; Valero, 2004). Lerman's (2000) influential work "The social turn in mathematics education research" captures and accounts for the increasing emphasis on social aspects of the teaching and learning of mathematics – an emphasis which pinpoints a transition of mathematics education scholarship from mostly relying on cognitive explanatory theories towards an embracing of epistemological and theoretical frameworks which highlight the social character of learning. A few years later, Gutiérrez (2013) explores the idea of a "sociopolitical turn" in mathematics education; literature within the political turn in mathematics education is more directly concerned with issues of power and identity. The Mathematics Education and Society conference (MES), which started in 1998, epitomises this shift and signifies the existence of a community of mathematics education scholars who explore the social, political, cultural, and ethical dimensions of mathematics education.

This section is structured around four relatively distinct traditions in sociopolitical mathematics education scholarship: a) critical mathematics education, b) ethnomathematics and decolonising approaches, c) feminist and critical race theories approaches, and d) cultural politics of mathematics education. For each of these, I review some of their key contributions and I explore their conceptualisations of mathematics.

I argue that all this literature offers considerations of mathematical knowledge as political, but it is only in the cultural politics of mathematics education that the political is addressed per se.

2.1.1. Critical Mathematics Education

Critical mathematics education is a diverse pool of scholarship, which problematises the role of mathematics in society and explores ways, in which mathematics can be used as a tool for social justice. Within this approach, establishing social justice education in an unjust society is viewed as an ongoing challenge rather than a question that could have an a priori answer (Skovsmose, 2012). We may identify two major trajectories in critical mathematics education, one that has been mostly influenced by the Freirean framework and one by critical theory, with their leading figures being Frankenstein and Skovsmose respectively.

The first critical mathematics education trajectory has been influenced by the work of Paulo Freire. Freire, whose work was inspired by Marx (Andersson & Barwell, 2021), worked with illiterate adults in Brazil, developing ways in which the "oppressed" could learn to "read and write the world" and thereby break the bonds of their oppression (Freire, 1970/2005). Marilyn Frankenstein's work applies Freire's framework in mathematics education, by exploring how "critical mathematics education can develop critical understanding and lead to critical action" (Frankenstein, 1983, p. 327). Working with working-class adults, Frankenstein (1998) explores the potential held by mathematical tools for the project of social justice and focuses on statistics as a powerful tool with which to "read the world". Significantly influenced by this approach, Gutstein (2005, 2012) works with students of racialised and low-class backgrounds in Chicago and develops a framework for social justice mathematics education, through which he and his students identify sociopolitical issues that affect their lives and use mathematics to explore them. Overall, this trajectory is concerned with using mathematics as a tool for social justice, in order to "read and write the world with mathematics" (Gutstein, 2005).

A second critical mathematics education trajectory is what Andersson and Barwell (2021) call the Nordic school. The Nordic school refers to a tradition initially developed by Scandinavian researchers, mostly associated with the name of Ole Skovsmose. Drawing upon critical theory and the work of Wittgenstein, Skovsmose has become a leading figure in critical mathematics education. A central idea of his work is that although mathematics education often serves unjust socio-political roles, it can also empower students and act as a tool for social justice (Skovsmose, 2012). Skovsmose (1994a) distinguishes between three types of knowledge related to mathematics: mathematical knowing (knowledge of theorems, proofs, and algorithms), technological knowing (ability to apply mathematics and build models), and reflective knowing (competence to evaluate the use of mathematics) and suggests that the latter needs to also be developed in mathematics education in order for students to develop a critical mathemacy. Within this view, the formatting role of mathematics (i.e., its significant function of creating aspects of the world) creates the need for citizens to develop mathemacy in order to critically participate in society. By developing a wide range of theoretical concepts, Skovsmose's work has become influential in shaping critical mathematics education. In more recent years, the Nordic school and works influenced by this tradition have diversified, as, for example, is shown by a new focus on climate crisis and the role that mathematics education can play in relation to it (e.g., Hauge et al., 2017; Steffensen et al., 2021).

Critical mathematics education has problematised modernist discourses about mathematics. According to Skovsmose (2011), the modernist conception of mathematics entails three sets of ideas: "that mathematics is essential for understanding nature; that mathematics is a powerful resource for technological invention; and that mathematics is a pure rationality which operates almost as an intellectual game, divorced from other human activities" (p. 49). Critical mathematics education challenges the naturalisation of this discourse by exploring questions such as "what is mathematics in relation to society, what does mathematics do as part of the school curriculum, and what are the potentials of mathematics education to produce or challenge inequalities in society and among students" (Valero, 2018, p. 103). Research in this direction has explored how mathematics education often plays a role in perpetuating social inequalities through inclusion/exclusion of groups of students (e.g., Dowling, 1998; Jorgensen et al., 2013). However, mathematics can also be a tool which students can use to understand their social reality and to act for social emancipation (e.g., Frankenstein, 1983, 1998; Gutstein, 2012; Yasukawa & Brown, 2012). Under this perspective, mathematics education has no essential features by nature: it can serve unjust socio-political roles, while it can also become a tool for social justice (Skovsmose, 2012).

2.1.2. Ethnomathematics and Decolonising Approaches

A second distinct orientation in sociopolitical mathematics education research is ethnomathematical research and post-colonial approaches. In this section, I will first talk about the field of ethnomathematics, as developed through the work of Ubiratan d'Ambrosio and taken up in the work of many scholars who study the mathematics of distinct communities. I will then draw upon some critiques to ethnomathematical approaches, which claims that ethnomathematics can essentialise the culture of the Other and "fish" for mathematics in complex cultural practices which are thereby reduced to what can be seen by a Western[2] mathematical lens. I will then refer to decolonising approaches in mathematics education research which go beyond the discussed problematisation of ethnomathematics by building long-term relationships with communities, drawing upon Indigenous epistemologies, and treating mathematics education as a space of decolonising struggle.

The term ethnomathematics was coined by d'Ambrosio, who described ethnomathematics as "the mathematics which is practised among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (d' Ambrosio, 1985, p. 45). Many scholars who work within this trajectory explore the (ethno)mathematics of national-tribal groups, in an effort to identify the ways in which different peoples think about mathematics similarly and differently. This scholarship focuses on the study of the cultural practices of national-tribal groups and the identification of mathematics in them. For example, aiming to explore the mathematics of Indigenous people who have been excluded from dominant discussions about mathematics, Marcia Ascher (1994) presents a detailed documentation of mathematical concepts as they are encountered in the cultural practices of Indigenous peoples in South and North America, Oceania, and Africa. Ascher elaborates on mathematical ideas, such as number, logic, spatial configuration, and the organisation of these into systems or structures, as identified in practices like chance and strategy games, kinship relations, decorations, and the organisation of space.

There have been criticisms of this ethnomathematical approach which suggest that the process of trying to identify mathematics within complex cultural practices reduces these practices to their mathematical content, as seen from a "Western" point of view. For example, Doolittle (2007), a Mohawk mathematician and mathematics educator, writes about the oversimplification aspect of this sort of ethnomathematical research:

> An example of such oversimplification which I have encountered repeatedly in discussions with well-meaning people I call Cone on the Range. "The tipi is a cone," I have heard countless times. But that is surely wrong;

[2] By "Western mathematics", I refer to the mathematics that is valued in academia and in the increasingly globalised schooling. It is the paradigm of rigourous, proof-heavy mathematics developed in ancient Greece, later in Europe, and today can be found in the school curricula around the world and in universities. By contrast, the mathematics of Indigenous communities, the mathematics developed in ancient China or Babylonia, or the mathematics of carpenters are not "Western mathematics". I am using the term "Western mathematics" here, as it has been widely used in ethnomathematical and post-colonial scholarship and has been instrumental in facilitating understanding about a (universal) construct of (Western) mathematics. However, I consider the term to be somehow problematic, as it overlooks mathematics of groups within the West, which are not valued within a framework of inequality. For instance, carpentry involves mathematics, but – within a capitalistic framework – this mathematics is of lesser value and, hence, does not quite belong within the construct of universal Western mathematics.

> the tipi is not a cone. Just look at a tipi with open eyes. It bulges here, sinks in there, has holes for people and smoke and bugs to pass, a floor made of dirt and grass, various smells and sounds and textures. There is a body of tradition and ceremony attached to the tipi which is completely different from and rivals that of the cone. Similarly, there is a ceremonial and spiritual tradition connected with the peach pit bowl game that is completely lost in Ascher's treatment. (p. 20)

We can find another moment of this ongoing discussion about the role of ethnomathematics in mathematics education, in an exchange between Goossen Karssenberg and Pamela Rogers in the Canadian mathematics education journal "For the Learning of Mathematics". In his paper, Karssenberg (2014) explores how Persian mosaics can be a resource for learning geometry in a Dutch educational context, maintaining that it can activate recent immigrant children's (from Morocco and Turkey) cultural knowledge. Karssenberg's underlying position is that non-Western cultures have produced mathematics, which can act as an inspiration for everyone. In her response paper, Pamela Rogers (2015) critiques this approach as uncritical ethnomathematics which bears dangers. She proposes that the incorporation of non-Western content, no matter how well-intentioned, can endanger and further exclude students from minority backgrounds, as it oversimplifies historical content and essentialises students' experiences. We can, thus, observe that ethnomathematics and its incorporation in mathematics education comes with tensions, related to both the actual cultural practices of minoritised groups and the implications for minoritised students when implementing ethnomathematics in educational contexts.

Within these discussions about ethnomathematics, the dominant discourse that mathematics is an objectively "good" knowledge gets challenged and the role of mathematics in the project of cultural imperialism and the cultural hegemony of the West gets investigated (e.g., Bishop, 1990; Fasheh, 2012). Along the lines of the ideas proposed by Doolittle and Karssenberg, scholars around the world do decolonising work with Indigenous communities by developing long-term relationships with these communities and working alongside them, in an effort to treat the space of mathematics education as a decolonising space. Building long-term relationships is of tremendous importance as it breaks with an early anthropological tradition, in which Western researchers visited and exploited Indigenous communities and moves towards a way of doing research "in and with an Indigenous community" (Donald et al., 2011). It also offers space for researchers, Elders, community members, teachers, and students to draw upon Indigenous languages, systems of knowing, and traditions to explore how mathematics can be thought of from an Indigenous point of view and in ways that are relevant to the community (Parra & Valero, 2021). The struggles of this process are multiple. For example, regarding the differences in European languages' and Indigenous languages' structures; Lunney Borden (2011) explores the tensions of teaching mathematics in English to Mi'kmaw students, considering that English is a noun-based language while Mi'kmaq (like most Indigenous languages in Canada) is verb-based, and calls for a verbification of mathematics to support Indigenous students' learning of mathematics. Within a context of decolonising work, this sort of work can become instrumental in the revitilisation of Indigenous languages, as has been happening in New Zealand, where mathematics, as a core school subject, has been a space in which to think how a mathematical concept can make sense in Māori (as opposed to translating a (Western) mathematical concept from English to Māori), thereby contributing to the revitilisation of the language (Trinick, 2019).

Overall, ethnomathematics is widely understood as committed to the field of social justice. This view is not unproblematic, if we consider that, just as other ethnosciences, it often constitutes a colonial movement, in the sense that it does not challenge – but instead maintains – the unequal power dynamics of colonialism, through the exoticisation and objectification of the Other (Stathopoulou & Appelbaum, 2016). Nevertheless, through the critical examination of itself (similarly to the way in which anthropological traditions turn towards an examination of their biases), the field of ethnomathematical and decolonising mathematics education research has constituted a space of critical discussion about dominant views regarding what mathematics is and what it can do. As far as the social role of mathematics is concerned, the intrinsic "goodness" of mathematics has been challenged by research which has shed light into the role of mathematics in the project of cultural imperialism and the cultural hegemony of the West (e.g., Bishop, 1990; Fasheh, 2012). At the same time, ethnomathematics as has challenged the perception that mathematics is a universal and neutral body of knowledge (Pais, 2013) and the assumed supremacy of eurocentric approaches (Valero, 2004). Indigenous ways of knowing are becoming sources for thinking about both mathematics and mathematics education, opening up discussions of how mathematics can be rethought, including through its verbification (Lunney Borden, 2011) or even radical reimaginations of it which embrace the body, emotions, and harmony (Gutiérrez, 2017).

2.1.3. Feminist and Critical Race Theory Approaches

In mathematics education, the categories of gender and race have been taken up through the use of feminist and critical race theories respectively, in order to make sense of mathematical teaching and learning within the context of patriarchy and White supremacy. While these approaches are often treated as distinct, I am here reviewing them together for three reasons. First, there is an increasing number of studies which treat race and gender as interconnected (e.g., Bullock, 2018; Gholson & Martin, 2017; Leyva et al., 2021; Lubienski & Pinheiro, 2020; Przybyla-Kuchek, 2021). The second reason is more theoretical –

the concept of intersectionality pinpoints the need to stop looking at either identity categories (e.g., gender, race, class) or systems of oppression (e.g., patriarchy, White supremacy, capitalism) separately (Bullock, 2018). Intersectionality draws our attention to the not additive, but rather intersectional, interconnections of social categories like gender, race, class, ethnicity, ability, and sexual orientation, thereby resulting in social identities being interdependent and inequality/oppression being compounded (Bowleg, 2008). Finally, both the categories of race and gender have evolved in similar ways in scholarship, as they were – for a long time – considered to be fixed categories that a person gets born with and is determined by, but have evolved into being understood as flexible, performative social constructs (Lubienski & Pinheiro, 2020; Martin, 2009b). In this section, I review the gender and race scholarship in mathematics education. I start the section by reviewing some prevalent discussions in gender research, followed by a similar discussion on race research. I then explore how race and gender have been problematised in gender and race scholarship. Finally, I discuss how this scholarship has examined mathematics critically, by conceptualising it as a gendered and racialised space.

Gender starts being a focus of mathematics education scholarship in the 1970s-1980s when gender differences in students' performance, participation, and attitudes started to be documented (Hall & Norén, 2021). In her influential paper, Fennema (1974) reviews research on gender and mathematics education to explore whether there are sex differences in mathematics achievement and argues that differences in female/male achievement start manifesting as students enter the upper elementary and early high school years. Following the tenets of liberal feminism, during these years, gender scholarship has focused on supporting females to reach the standards that males attained (without critically examining these standards), making use of assimilation and deficit models for accounting for gender in mathematics education (Leder, 2019). In the years that followed, the feminist struggles, the changing position of women in labour and society, and the new questions posed by feminist scholarship (in light of post-structural and queer theories) contributed to gender research in mathematics education changing its orientation and starting to critically examine the assumptions of deficit and assimilating discourses. Walkerdine's (1998) book "Counting girls out: Girls and mathematics" constitutes a treatise which showcases that the hegemonic discourses about gender, within which research in the 1970s and 1980s has been produced, act as truths which both subjectify children and construct mathematics. Research following this tradition (e.g., Chronaki & Pechtelidis, 2012; Jaremus, 2021; Mendick, 2005, 2006; Walshaw, 2013) uses post-structural tools and insights from gender studies to engage in explorations of students' subjectification as well as the construction of mathematics as a field of gendered rationality.

Recently, influenced by the Black Lives Movement, race and racism in mathematics education have gained more attention, as organisations and universities commit to take action to address racism (e.g., Wagner et al., 2020). Research focusing on race has, however, deep roots in mathematics education scholarship. Similarly to research on gender, early and current research on race conceptualise students of colour as lacking something. Such deficit views try to identify the reasons that Black, Latinx, and Indigenous students do not perform to the standards of their White and Asian peers (e.g., Gutiérrez, 2008). Despite this scholarship and its associated efforts for equity, Martin (2009a) suggests that Black learners in the US continue to experience mathematics education as dehumanising and violent. The changes proposed and implemented are incremental and do not challenge the racial hierarchies; they are instead based on a white imaginary of equity. There are, however, scholars who deal with race and racism in mathematics education more deeply and treat mathematics classrooms as racialised spaces (e.g., Esmonde & Langer-Osuna, 2013; McGee & Martin, 2011). As a counter narrative and counter-movement to the "gap-gazing" practice, the focus on "excellence" has arisen (e.g., Gutiérrez, 2000; Matthews, 2008; McGee & Martin, 2011). McGee and Martin (2011) describe this choice of focusing on success rather than failure as an affirmation to Black students who pursue education despite institutional and structural obstacles. A key idea here is that students and communities of colour are not passive recipients of racism and racialised inequality – in McGee and Martin's (2011) words: "[r]acial stereotypes are powerful but they are not deterministic" (p. 1380). In one study within this orientation, Gutiérrez (2000) explores African American students' success in one mathematics department and identifies as contributing institutional factors a rigourous curriculum and the support to maneuver through it, commitment to students, commitment to a collective enterprise, a resourceful and empowering chairperson, and standards-based instructional practices. A similar direction taken up by research is the focus on racialised students' counterstories (e.g., Leyva, 2021), which focuses on the ways in which racialised students experience mathematics classrooms and develop their racial and mathematical identities. Overall, in the literature focusing on race and racism in mathematics education, we can distinguish between two trends: one approach which flags equity without engaging with structural and institutional racism and another approach which views classrooms as racialised and forecasts the experiences and stories of racialised students.

Within gender and race research, the categories of gender and race themselves have been problematised. Scholars operating within post-structural feminism, gender and queer studies, and critical race theory have highlighted the need to move beyond static notions of race and unitary gender categories (e.g., Esmonde, 2011; Martin, 2009b; Walkerdine, 1998). Esmonde (2011) describes the essentialisation of gender, as "a tendency to treat boys and girls as if they were monolithic groups with little internal variation", while Martin (2009b) maintains that static race categories continue to be used by most studies examining differential outcomes. However, despite the deep discussions about the constructs of gender and race as well as early calls on the issue, it is noteworthy that the majority of research continues to operate on early conceptual and methodological frames of gender and race, such as not distinguishing between gender/sex, considering gender to be a binary, or race to be biologically determined (Hall & Norén, 2021; Martin, 2009b). Yet, in the discussions that do attempt to unsettle categories as commonsensical as sex/gender[3] and race, the construct of mathematics is also taken into scrutiny.

In what follows, I discuss how gender and race research, following post-structural feminist and critical race theory orientations, opens up the discussion of examining the construct of mathematics per se. Within post-structural feminist and race research in mathematics education, not only are mathematics classrooms viewed as racialised and gendered spaces (e.g., Esmonde & Langer-Osuna, 2013), but mathematics is conceptualised as a racialised and gendered space (e.g., Battey & Leyva, 2016; Leyva et al., 2021; Martin, 2009b; Mendick, 2005, 2006). For example, Mendick (2005, 2006) proposes that doing mathematics is a form of doing masculinity, regardless of the gender of the person doing mathematics. Her work shows that mathematics is discursively constructed within gendered dichotomies, such as body/mind, hard/soft science, fast/slow, etc. (Mendick, 2006). Leyva (2021) develops a framework which explores how mathematics education is a white patriarchal space. Leyva's framework has three dimensions (ideological, institutional, and relational) and showcases the influences in race-gender intersectionality on mathematics experiences. In the ideological dimension, racial/gendered ideologies in mathematics, such as the hierarchy of ability, individualism and meritocracy, perpetuate educational inequities based on students' gendered and racialised identities. At the relational level, racialised and gendered interactions take place both inside and outside mathematics classrooms, while cognitive and emotional labour takes place for managing these and for developing positive mathematics identities. Finally, at the institutional level, racial-gendered exclusion takes place in advanced mathematics as well as in high-quality mathematics instruction and student support. These three dimensions are interrelated and "feed" one another.

Such theorisations which showcase the racialised and gendered character of mathematics are not only useful for informing the impact of mathematics and mathematics classrooms on the mathematical experiences of students, they are also instrumental in exploring how a racialised and gendered rationality is constructed at the ideological level. Ideologies that mathematics is a solitary and competitive activity

[3] The blurring of the two constructs, sex and gender, is intended here.

or that mathematical ability is innate (e.g., Battey & Leyva, 2016; Leyva, 2021) are both gendered and racialised. For example, the idea that some people have been born to be good at mathematics while others simply do not get it conceptualises White and Asian males as belonging to the former category and Latinx, Black, and Indigenous folks as well as women as belonging to the latter. Furthermore, not all mathematics is constructed to be of the same value. For example, Walkerdine (1990, 1998) captures the existence of higher and lower reasonings as categories which are disproportionately attached to girls and boys. So, gender and race research clarifies that there are ways of knowing mathematics that are more privileged than others.

Overall, research focusing on gender issues in mathematics education and research focusing on race issues in mathematics education have followed similar paths, while there is an increasing number of publications that attempt to deal with both gender and race through intersectionality lenses. The categories of gender and race in mathematics education have largely been discussed in relation to the achievement gap, the idea that some groups of students, be it girls or Black, Latinx and Indigenous students, perform at a lower level than their male or White/Asian peers (Esmonde, 2011; Gutiérrez, 2008). Although the majority of research on race and gender continues to use theoretical and methodological schemes that ignore contemporary conceptualisations of gender and queer studies and critical race theory, perspectives using insights from gender and queer studies and critical race theory have been influential in mathematics education scholarship. Moreover, within these latter perspectives, there is an ongoing discussion concerning the conceptualisation of mathematics as a gendered and racialised space and the discursive construction of a gendered and racialised rationality through and with mathematics.

2.1.4. Cultural Politics Approaches

In this subsection, I review the cultural politics of mathematics education scholarship. I start the subsection, by explaining the term "cultural politics" as it has been introduced to the field of mathematics education by Paola Valero. I then discuss how the cultural politics of mathematics education merges the borders of mathematics education by turning its attention to fields like social theory, curriculum studies, history, and cultural studies. I continue the section by elaborating on some discussions within the cultural politics of mathematics education, namely: the role of mathematics in governing the population, the problematisation of the "mathematics is for all" discourse, and the role of mathematics and mathematics education in the construction of the citizen/worker. I conclude this subsection by explaining how this body of research contributes to an understanding of mathematical knowledge as political knowledge.

The term “cultural politics” has been introduced to mathematics education by Paola Valero (2018) who describes it as follows:

> As I see it, the study of the cultural politics of mathematics education is a displacement—or even better, an expansion—of mathematics education research from the demand of the field to study the mathematical specificity of relationships of teaching and learning, toward the terrain of curriculum studies (e.g., Appelbaum & Stathopoulou, 2016) with the intention of tracing the specific role that mathematics education has played in the making of modern and contemporary culture and society. […] The cultural politics of mathematics education as an area of research is an attempt to understand the constitution of the practices of mathematics education as part of a larger cultural space where the meanings of mathematics in relation to education are constantly negotiated. […] [R]esearching the cultural politics of mathematics education allows revealing the way in which mathematics education generates concepts, distinctions, and categories that regulate the possibilities of thinking and being in/with mathematics as a privileged area. (pp. 105–108)

This means that cultural politics moves beyond the mathematics classroom as the major space in which to study human interactions with mathematics as well as beyond mathematical learning as the major product of mathematics education. Instead, the focus shifts towards the ways in which mathematics education has evolved into operating under “truths” that appear to be commonsensical. The cultural studies of mathematics education investigate these truths, not in order to deem them as rightfully true or false, but to understand them within the framework in which they were made possible as well as to critique their commonsensicality. For example, a prominent idea in mathematics education is that every child should have access to quality mathematics education. From a cultural politics perspective, the focus is not on arguing for or against this position, but rather to think how this position became widespread and what its functions are in terms of producing and regulating mathematics education practices. Furthermore, the products of mathematics in the manufacturing of rationality as well as the rational citizen are examined.

To this end, critical politics of mathematics education turn to other fields of study, including social theory, history, curriculum studies, and cultural studies. The work of philosophers like Foucault and Lacan can and has been used in mathematics education as a way of looking at the field as a discursive formation influenced by history and culture (Walshaw, 2016). Valero (2018) uses Picasso’s Guernica, as an example of how a painter managed to make it possible for people to see the world in different ways than the ones they were used to. For Valero, thinking about mathematics education *with* a philosopher like Foucault constitutes a similar process: it involves investigating the very norms, values, and rules in which we operate and with which we have learned to understand the world and conduct research. Other scholars have turned to history as a way of examining the practices of mathematics education. For example, Kollosche (2014) turns to history to explore how mathematics, from its very beginning, has been a knowledge that allows power, represents a specific worldview, and serves the interests of certain groups in society. Finally, the field of curriculum studies offers insights into the practices of mathematics education beyond the terms of curriculum development; it opens the discussion to the hidden and the neglected curricula, it pays attention to the dimensions that are absent from typical curriculum decision making, it asks what mathematical knowledge is of most worth and who gets to decide, it questions basic assumptions to generate alternatives, and it is open to discursive analysis attending to broad characteristics of human experience (Appelbaum & Stathopoulou, 2016).

In what follows, I elaborate on three discussions occurring within the cultural politics of mathematics education. First, within modernity, mathematics has evolved into a knowledge which partakes in the government of the population. Considered to be objective, mathematical descriptions, when applied to societies and humans, construct a reality about them which a) is beyond any doubt as it is described through numbers and b) constitutes the lens with which to think about these issues. The development of statistics is one such example. Walkerdine and Lucey (1989) describe the role of statistics in population management:

> One of the sciences of "population management" was statistics, which defined the characteristics of the population by finding the average or norm. The idea of the norm, which was an artifact of a technique, then became widely incorporated as the "normal"; such that patterns of development found across children were seen and told as "normal". This meant that any difference was a deviation from the norm and thus became abnormal, or pathological. (pp. 42–43)

Here, Walkerdine and Lucey propose that the mathematical construct of the average is related to the development of the idea of the "normal", thereby distributing the population to those who fall within and those who fall outside the normal. We can further explore this idea through examples of statistics usage in the specific context of body weight. Hall and Barwell (2021) examine the statistical construct of the Body Mass Index (BMI) and its role in prescribing obesity and normal body weight. Through a formula that requires a person's height and weight, the BMI is calculated and puts any person in one of four possible categories: underweight, normal weight, overweight, and obese. Quetelet, the person who developed the BMI, was concerned with identifying the characteristics of the "normal man". Quetelet's normal man, who was based on available data from White European population, constitutes the norm with which the BMI categorises individuals as normal or abnormal, thereby governing the population. Overall, as mathematics constitutes an exemplar of rationality and truth within modernity (e.g., Lerman, 2000), it has been critical in governing the population through mathematical models describing and prescribing the normal, the rational, the true, and the sensible.

Second, the “mathematics for all” discourse has been problematised by the cultural politics of mathematics education. The “mathematics for all” discourse entails the idea that everyone should learn mathematics and that educational systems should offer quality mathematics education to everyone. It has emerged as a slogan under which a significant amount of research and practice operates, responding to the realisation that mathematics education is an inequitable space with regards to gender, race, class, and culture. For cultural politics, the goal is not to argue for or against this discourse, but to defamiliarise it and explore how it operates, how it has evolved into becoming a commonsensical truth, and how it became possible to be articulated in the first place (Valero, 2018). From this perspective, “mathematics for all” has been criticised for being a device that effects inclusions and exclusions of individuals and countries (Valero, 2017), while “conceal[ing] the obscenity of a school system that year after year throws thousands of people into the garbage bin of society under the official discourse of an inclusionary and democratic school” (Pais, 2012, p. 58). The “mathematics for all” discourse is akin to discourses which advocate for the need to cultivate numeracy. Craig (2018) historicises the emergence of the numeracy discourse in mathematics education and shows that it constitutes a unification between three reform narratives about technological change, equity and justice, and social stability. Through this emergence, (in)numeracy makes a series of promises/assurances: numeracy promises to reflect modern reality, numeracy promises to empower, and innumeracy promises to have social costs. Overall, the “mathematics for all” discourse plays within the tension of mathematics on the one hand having a gatekeeping role and on the other hand its being considered a key to the gate. Thereby, mathematics preserves its privileged role in schooling and in discourses about schooling and education, while improving students’ performances in mathematics or elaborating on the reasons that certain groups of students appear to underperform become central in educational and public discourses.

Third, cultural politics explores the role of mathematics and mathematics education in the construction of the citizen/worker. An early study in this direction is Valerie Walkerdine’s (1988) Mastery of Reason, which examines meaning as part of social practices, thereby demonstrating that the “child” as a notion is discursively produced in language and particular social practices. Walkerdine’s work has been important as it shows that the result of mathematics education is not merely students’ mathematical learning, but it involves categories that might appear as natural and given as the notion of the “child”. More recently, scholarship has started elaborating how mathematics textbooks, curricula, and research do something similar by constructing specific notions of citizens as well as workers (Andrade-Molina & Valero, 2017; Andrade-Molina et al., 2018; Franco Neto & Valero, 2018; Llewellyn, 2020; Popkewitz, 2004; Valero & Knijnik, 2015). For example, Valero and Knijnik (2015) showcase how mathematics education research on the uses of ICT for learning mathematics contributes to the fabrication of the rational, Modern, self-regulated, entrepreneurial neoliberal child. Similarly, Franco Neto and Valero (2018) demonstrate how textbooks in rural Brasil conceptualise mathematics as necessary for the modernisation of rural work, thereby constructing the effective, industrialised, modernised farmer. Overall, this scholarship elaborates on the idea that mathematics education discourses generate ways of thinking about mathematics education and its participants (students/teachers/researchers), by demonstrating how the inclusion of mathematics in school curricula partakes in the creation of modern subjectivities.

Overall, the cultural politics of mathematics education scholarship shifts the attention from teaching and learning towards challenging the taken-for-granted truths about mathematics and mathematics education. By turning towards fields like social theory, history, cultural studies, and curriculum studies, the cultural politics of mathematics education explores how truths about mathematics are not natural but are instead contingent in time and space. Furthermore, it elaborates on the ways in which mathematics and mathematics education partake in governing the population and constructing the contemporary citizen/worker. In this way, the cultural politics of mathematics education conceptualises mathematics as political per se. In Valero’s (2018) words, “mathematics and mathematics education are political because the historical constitution of the knowledge and associated practices both have emerged and make part of the classifications and organisations that regulate social life and, within them, notions of who people are and should be” (p. 108).

2.1.5. Bringing it All Together

In this section, I reviewed sociopolitical mathematics education research, by distinguishing between critical mathematics education, ethnomathematical and decolonising approaches, race and gender research, and cultural politics of mathematics education. For each of these bodies of scholarship, I discussed how mathematics education research has provided valuable insights into how mathematics can be thought of in critical ways. Viewing mathematics critically does not mean criticising mathematics; it instead refers to analyzing mathematics, “to uncover and evaluate its hidden dimensions of meaning, and social and cultural significance” (Ernest, 2010, p. 66). These approaches have highlighted the roles that mathematics plays and can play in society, challenging its consideration as neutral or remote from human actions. In particular, critical mathematics education has challenged modernist discourses about mathematics and the conceptualisation of mathematics in essentialist ways; ethnomathematics and decolonising approaches have highlighted the role of mathematics in the project of cultural imperialism and the cultural hegemony; feminist and critical race theory scholarship has elaborated on how mathematics is a gendered and racialised space and how a gendered and racialised rationality is discursively constructed through and with mathematics; and the cultural politics of mathematics education explores how truths about mathematics are contingent in time and space, govern the population, and construct the contemporary citizen/worker. Thinking of mathematics as political can be an umbrella to think of it in relation to hierarchical systems of power, such as colonialism, neoliberal capitalism, patriarchy, and White superiority. Yet mathematics is conceptualised as political per se only within the cultural politics of mathematics education. In the next chapter (Chapter 3), I will turn to theoretical tools which I will use to try and further clarify the notion of mathematics being political.

2.2. Sociopolitical Approaches in Undergraduate Mathematics Education Research

Undergraduate mathematics education research is a relatively autonomous subfield of mathematics education, which has its own distinct journals and conferences (RUME Conference, “University Mathematics Education” Working Group in ICME, ERME Topic Conference on University Mathematics, International Journal of Research in Undergraduate Mathematics Education). While the sociopolitical turn in mathematics education has generated significant scholarship on issues of identity and power (Gutiérrez, 2013), it mostly focuses on primary and secondary education, while sociopolitical research within post-secondary contexts has been scarce (Adiredja & Andrews-Larson, 2017; le Roux, 2009). Sociopolitical undergraduate mathematics education research has primarily focused on exploring issues of equity and inequity in mathematics education, mainly with regards to gender and race. As a response to the phenomenon of differentiated performance and differentiated learning experiences, often described as the “achievement gap”, research has focused on understanding the experiences of women and racialised minorities and on identifying ways in which post-secondary mathematics education can become more equitable (Adiredja & Andrews-Larson, 2017). In this section, I review this research, by distinguishing between three kinds of approaches: a liberal approach, a critical sociological approach, and a post-structural approach[4].

[4] This distinction is similar –yet, not “one-to-one corresponding”– to Valero’s (2008) identification of three discourses of power (liberal, Marxist, and post-structural views of power).

2.2.1. The Liberal Approach

Most research about mathematics education in post-secondary contexts has been conducted in the US, within the wider field of science, technology, engineering, and mathematics (STEM) education. Within this framework, the need for equitable STEM education is not politically or ethically self-evident but is often treated as a requirement for the USA to maintain national security, economic competitiveness, as well as to ensure the survival of its mathematics departments (e.g., Espinosa, 2011; Fullilove & Treisman, 1990; Herzig, 2004; Ong et al. 2011; Palmer et al., 2011; Steele et al., 2008).

Within this context, many researchers follow a liberal orientation. Equity is framed in terms of diversity, which is desirable because of its assumed potential to bring about excellence in STEM. The core idea of this argument is that in a group of students who typically underachieve in mathematics, there should be some students who are "talented". These students need to be recognised at an early stage and receive suitable support in order to overcome the obstacles that they face. This approach captures a view about the possible pathways in STEM studies, which is often realised in literature through the pipeline metaphor (Ellington & Frederick, 2010; Espinosa, 2011; Harper, 2010). STEM is compared to a pipeline, in which the successful applicants start from the kindergarten and end up with PhD degrees. Those who do not make it to the end of the pipeline are described as "leaks". Therefore, within the liberal approach, the project of equity needs to ensure that a diverse group of people manage to go through the pipeline and exit as successful STEM scientists. A critique of this metaphor can be found later in this section.

Within this approach, there are three main orientations of empirical research in post-secondary contexts. First, there is research which attempts to identify which race and gender differences contribute to different performances in mathematics. For example, Treisman (1992) attributes the differences in performance between Chinese and Black students to differences in work and social patterns, while Ní Fhloinn et al. (2016) attempt to determine gender differences in the ways in which students engage in mathematics support opportunities. Second, there is research which focuses on successful women and racialised minorities, trying to determine the factors that have contributed to their success. Findings include faculty and peer support, access to resources, family support, honours programs, kind and caring teachers, small class sizes, and small communities (Brown, 2002; Ellington & Frederick, 2010; Espinosa, 2011; Ong et al., 2011; Palmer et al., 2011). This body of research aims to use these results in order to support the mathematical journeys of other minority students. Finally, there is research focusing on the effects that certain learning environments have on students' performance and personal empowerment. For example, Hassi and Laursen (2015) find that learning environments based on inquiry, collaboration, and problem solving contribute to students' personal empowerment, while Steele et al. (2008) study a women’s calculus course, in which the single-sex setting supports students to perform better and to continue enrolling in mathematics courses.

These studies seem to assume that discovering what has made some students successful in mathematics could directly result in practical knowledge which will support all students’ success. This view is problematic because it entails an essentialisation of underprivileged groups’ identities, by assuming that all students of a given group learn in the same way or need the same circumstances to succeed (Esmonde, 2011; Walkerdine, 1998). Moreover, assuming that all students of a certain racialised background can achieve success in similar ways overlooks the complexity of identities that cannot be reduced to race or gender but is realised in their intersectionality (Bullock, 2018).

Additionally, empirical research under the liberal approach accounts for inequity by identifying factors which influence students’ performances. Although this research is valuable, I argue that it ignores the impact of structural factors, which make access in advanced mathematics education difficult for students of lower social classes. For example, Espinosa (2011) identifies attending private institutions as a factor which promotes persistence. Although there might be a correlation between attending a private university and persisting in mathematics, there is no reason to assume that this correlation has causal characteristics. Instead, we could hypothesise that persistence is positively affected by belonging to higher social classes. In other words, research in this direction often does not pay enough attention to the institutional and societal complexity of post-secondary mathematics education.

This critique brings up some aspects, which problematise the widely used pipeline metaphor. The pipeline metaphor has been criticised, on the basis that it reproduces a linear progression which fails to account for the complexity of mathematics education (Reyes, 2011), represents students as not having agency, and is unable to explain why different groups perform differently (Herzig, 2004). Herzig (2004) counter-proposes the metaphor of a cross-country track event, which is assumed to share with STEM an individualistic, competitive, intensive, and lonely character, as well as a starting point, an ending point, and a predetermined path between them. Nevertheless, both metaphors attribute certain characteristics of STEM as essentially necessary to the field. The pipeline metaphor assumes that students *want* and *need* to get in and through the pipeline. This view is associated with the –highly controversial– discourse that being good in mathematics is a gift and those who have it should not waste it (Mendick, 2006). At the same time, capturing STEM as necessarily individualistic, competitive, and lonely inscribes the field in the work ethics of the neo-liberal paradigm and contributes to establishing its current inequitable character.

The above critique highlights some of the liberal perspective's limitations on the conceptualisation of equity. Adiredja and Andrews-Larson (2017) criticise the discourse of equity as diversity in order to benefit the US's economy and national security, for being capitalist, xenophobic, and ignoring students' agency and needs. In a similar vein, several scholars highlight the need for equity in post-secondary mathematics education as an ethical imperative (Adiredja & Andrews-Larson, 2017; Herzig, 2004; Ong et al., 2011). Moreover, I claim that the liberal approach not only fails to provide a paradigm of socially just education, but it also constructs mathematics as a necessarily individualistic and competitive field.

2.2.2. The Critical Sociological Approach

A stream of research has approached issues of inequity in undergraduate mathematics education by inquiring into the institutional characteristics of colleges, and universities. This scholarship uses perspectives from sociology of education, in order to explore how institutions reflect and reproduce inequitable aspects of our societies. Within primary and secondary contexts, this research has developed various foci, including aspects of the curricular organisation of schools, such as standardising, testing, or educational reforms (e.g., Apple, 1992), ability grouping settings and their effects (Boaler, 1997; Straehler-Pohl et al., 2014; Zevenbergen, 2003), and social class (Cooper & Dunne, 1998; Gates, 2019; Jorgensen et al., 2013).

With regards to post-secondary contexts, some research has focused on the creation and evaluation of programmes which aim to support minorities. Such programs have been the Mathematics Workshop Program (MWP), which has been successful in supporting Hispanic and African-American students (Fullilove & Treisman, 1990), the Emerging Scholars Program's (ESP) which aims to transform the institution in order to be more welcoming to minorities (Hsu et al., 2008), and an inquiry-based college learning programme (Laursen et al., 2014). A common element between these programs is that they do not attempt to "fix" women or racialised minorities, but rather focus on changing inequitable institutional practices. For example, Fullilove and Treisman (1990) attribute MWP's success to the workshop's culture, which values success, supports working efficiently and developing social and study skills, while Laursen et al. (2014) explore how the use of meaningful, ill-structured problems and the collaborative character of the program were crucial factors for women to develop a positive self-perception for their ability.

Research has also focused on the mainstream cultures of post-secondary institutions and their effects on the mathematical journeys of underprivileged or marginalised students (Herzig, 2004; Leyva et al., 2021; Reyes, 2011; Solomon, 2007). For example, Reyes (2011) examines how the different institutional cultures of colleges and universities create difficulties in transitioning from the former to the latter for women of colour. Focusing on gendered identities, Solomon (2007) explores what constitutes a functional identity in an undergraduate mathematics program. She finds that although women are less marginalised than men, they develop identities of exclusion, while men develop more functional identities based on their belief in their ability to succeed. In a similar vein, Herzig (2004) identifies social and academic integration as crucial factors for women and students of colour who persist in their doctoral studies.

These perspectives are insightful in that they do not "put the blame" on minority students for their collective underachievement, but instead cast light on structural and cultural characteristics of educational institutions. Through this lens, inequitable characteristics of universities and colleges can be more profoundly reflected upon and dealt with. By investigating the conditions of students' socialisation in the field of mathematics, this stream of research constitutes a space, in which mathematics can be thought of, not as a fixed, but rather as a flexible, social practice.

2.2.3. The Post-structural Approach

The third approach to sociopolitical undergraduate mathematics education research is what I call the "post-structural approach" and uses tools from poststructuralism to engage in a critical examination of both mathematical discourses and mathematical identities. Despite the multidimensional critique of the modernist conception of mathematics (e.g., Skovsmose, 2011), mainstream curricula largely reproduce a modernist view of the field. Based on the central role of dominant discourses about mathematics and performance in students' identity formation, Solomon (2007) maintains that undergraduate mathematics education needs to be more transparent about dominant discourses. Following a similar direction, some scholars explore how post-secondary curricula can incorporate alternatives to the mainstream discourses about mathematics. For example, Appelbaum et al. (2007) offer curriculum suggestions, through which students can reflect on the cultural and political implications of internationalisation, by exploring cultural aspects of mathematics. Similarly, Rauf (2002) has designed and implemented a university course about infinity, which combines a formal mathematical approach, artistic approaches, and an Indigenous approach to infinity.

Besides discourses about mathematics, the official mathematics discourse articulated in textbooks and curriculum documents is a significant area of research in mathematics education, as it frames the possibilities and identity categories made available to students (Morgan, 2009). Within primary and secondary educational contexts, research has also investigated classroom discourse, in relation to students' positioning (Esmonde & Langer-Osuna, 2013), the construction of identities (Andersson et al., 2015), and issues of authority (Wagner & Herbel-Eisenmann, 2014). Although research under this umbrella is scarce in post-secondary contexts, a significant exemption has been le Roux's (2009) work, which uses critical discourse analysis to examine a real-life problem offered to first year students and the students' discussion of it. Le Roux explores how the text and context both enable and constrain the ways in which students engage in classroom discourse.

Furthermore, within the post-structural approach, the issue of how students construct their mathematical identities in relation to the learning environments in which they participate has been taken up. For example, Oppland-Cordell and Martin (2015) investigate how diverse and collaborative learning environments contribute to transformations in participation patterns, while Povey et al. (2006) focus on how a mature female student has developed as a mathematics learner, through a pedagogy that supports the development of positive "disciplinary relationships". Studying the experiences of a successful woman in mathematics, Solomon et al. (2016) identify the need for creating hybrid spaces in which it is possible to be a "mathematical woman" in order to achieve a more inclusive mathematics education.

Some scholars have focused on minority students' experiences, including the role of experiences in identity formation (McClain, 2014), students' counterstories (Oppland-Cordell, 2014), and stereotype managing (McGee & Martin, 2011). This research is insightful, from a sociopolitical perspective, because it gives voice to students and values their (alternative to the mainstream academic) perspectives. Contrary to research in liberal approaches which focuses on what factors contribute to students' success in order to reproduce them, scholarship here looks for success stories in order to establish a counter-paradigm of research, as well as focuses on experiences rather than factors of success. This scholarship considers successful minority students' experiences to have the impetus to counter deficit discourses. The main argument is that, despite its best intentions, research focusing on structural factors that prevent racialised students from succeeding in mathematics education reproduces a discourse of deficiency for non-White, non-Asian students (Gutiérrez, 2008; Harper, 2010). Therefore, scholars in this approach advocate for changing the focus from the reasons why students of colour do not make it, to highlighting the stories of students who have made it.

Not treating students' identities as static and predetermined, this approach provides a valuable insight to students' creation of identities. For example, McGee and Martin (2011) show that although racial stereotypes are powerful within post-secondary contexts, they are not deterministic. However, this research orientation might overlook systemic problems and particularly the factor of social class (Lubienski, 2003), especially within post-secondary contexts in North America, an element of which is the existence of – often expensive – tuition fees. More specifically, by not focusing on economically disadvantaged racialised students, the role of universities as institutions which mainly serve the middle and upper classes remains intact. Therefore, I propose that studies focusing on minority students' experiences need to consider the role of social class, by exploring the intersectionality of students' identities.

With regards to mathematics, the post-structural approach – similarly to the critical sociological approach – gives space for a conceptualisation of mathematics as a flexible social practice. Through the emphasis on underprivileged students' experiences, it opens the discussion about how mathematics can be thought of and reflected upon, in relation to alternative epistemologies, such as Indigenous epistemologies, as well as feminist and critical race theories frameworks.

2.2.4. Bringing it All Together

In the previous sections, I described three approaches to sociopolitical research in post-secondary mathematics education research: a liberal, a critical-sociological, and a post-structural approach, all of which share an emphasis on equity. Naturally, the three approaches are not mutually exclusive, instead there are scholars whose discourses are hybrids of two or all three of them. Moreover, it should be noted that although there has been research in post-secondary contexts within all three approaches, the liberal approach has been significantly more prevalent. In this final bringing-it-all-together section, I discuss how mathematics is conceptualised within sociopolitical undergraduate mathematics education literature.

Within undergraduate mathematics education literature, mathematics is normally not the object of inquiry; mathematics is taken to be what professional mathematicians do. However, the possibilities of conceptualising mathematics differently varies between the three approaches. The liberal approach, through its metaphors of STEM as a pipeline and as a cross-country track event, conceptualises mathematics as an individualistic and competitive field. Within the critical-sociological approach, the emphasis is placed on students' socialisation in the field of mathematics, thereby allowing for conceptualisations of mathematics as an unfixed and flexible social practice. Similarly, within the post-structural approach, mathematics is not seen as a fixed practice, while there are connections with alternative epistemologies, such as Indigenous epistemologies, as well as feminist and critical race theories frameworks.

Furthermore, across the three approaches, certain assumptions about mathematics are treated as natural truths. This is particularly evident in the discussion about equity, in which, as discussed earlier, we can distinguish between equity being advocated so as to improve the US's national security and economic competitiveness (Espinosa, 2011; Fullilove & Treisman, 1990; Herzig, 2004; Ong et al., 2011; Palmer et al., 2011; Steele et al., 2008) and equity as an ethical imperative (Adiredja & Andrews-Larson, 2017; Herzig, 2004; Ong et al., 2011). Despite their vastly different political orientations, both sides of the debate presuppose a shared assumption: mathematics is considered to be a desirable knowledge for students. Moreover, within the framework of the second perspective, which is concerned with social justice issues, there is no questioning of whether students should or need to learn mathematics, nor is the goal of improving students' test results challenged. In this way, the naturalised truths about the sociopolitical roles of mathematics are not critically evaluated. This does not mean that these assumptions are necessarily problematic or wrong. However, if mathematics contributes to the US's (or any country's) national security and economic competitiveness, this means that mathematics plays social roles which serve unjust purposes. As such, it is important to think of mathematics under a more critical spotlight and to examine its political functions.

2.3. Undergraduate Teaching and Learning Research and the Sociopolitical

In this section, I review undergraduate teaching and learning research from a lens that emphasises the political. Given the wide variety of topics of undergraduate teaching and learning research, I will only focus on calculus research and proof and proving research, as examples of cognitive research which I will consider through an emphasis on the political. This means that other areas of research (e.g., teaching and learning of linear algebra, statistics, probability, computing, etc.) will not be tackled in this review. I chose these two research areas because the two courses in which I collected data are a Calculus course and an Introduction to Proofs course. This section is structured as follows. After a short discussion of the intersections between the cognitive and the sociopolitical, I review and critique research on calculus and then research on proof and proving. I conclude this section, by bringing teaching and learning research on calculus and proof together.

In undergraduate mathematics education research, there are very few intersections between the sociopolitical and the cognitive. One notable exception is the work of Adi Adiredja, who not only conducts research in both domains, but attempts to bring the two domains together. Adiredja (2019) develops an anti-deficit framework, with which to think about the mathematical thinking and learning of students of colour, which assumes that students of colour have resources which they can use to think and successfully reason about and with mathematics. For example, Adiredja and Zandieh (2020) investigate the sensemaking of women of colour in linear algebra, by asking them to extend the concept of the base in novel contexts. In this way, they construct sense-making counter stories. The choice of participants (i.e., women of colour), the choice of a theoretical perspective (i.e., anti-deficit approach), and the goal of this research (i.e., construction of sense-making counter stories) constitutes a sociopolitical lens in cognitive research.

A second intersection between the sociopolitical and the cognitive can be found in research on curricula, which illustrates how curricular changes are contested and adopted by different stakeholders. Researchers have worked on the intersection of calculus education and the political, in terms of curricular changes, their associated sociopolitical framework, and the categories of students served with particular curricular changes. Thompson and Harel (2021) locate social and political aspects of calculus education in the social and political pressures around the development and acceptance of reform curricula, including in the so-called “math wars” in the US, curricular changes in South Korea (Yoon et al., 2021) and Israel (Dreyfus et al., 2021). Bressoud et al. (2016) focus on the sociopolitical framework within which calculus in the US moved from being an exclusively university course to it being taught in high schools and demonstrate that this transition was, initially, only serving few elite students.

My sociopolitical lens, inspired by the cultural politics of mathematics education, adopts a different orientation. In my sociopolitical critique of calculus and proof and proving teaching and learning research, I am interested in the assumptions made by this scholarship and their political significance. Here, the political[5] does not refer to pressure from political groups, but rather the ways in which certain ways of talking about calculus/proofs allow for specific ways of thinking and acting in the world, in relation to calculus/proofs teaching and learning (Jørgensen & Phillips, 2011; Mouffe, 2005; Valero, 2018).

2.3.1. The Teaching and Learning of Calculus

In this section, I review research on calculus teaching and learning in undergraduate settings, by adopting a lens which focuses on the (often undiscussed) social and political dimensions of this research. This section is divided in two parts. In the first part, I attempt to summarise some key orientations and substantial findings in calculus teaching and learning research. In the second part, I engage in a critique of this research from a sociopolitical lens which conceptualises the political in terms of opening and closing possibilities for meaning making.

2.3.1.1. Calculus Research in Undergraduate Settings

Research on calculus has been reviewed by Rasmussen et al. (2014) as well as by Bressoud et al. (2016). Rasmussen et al. (2014) identify a pattern of four sequential “moments” or foci: identifying and exploring student difficulties and cognitive obstacles; investigating the processes by which students learn particular

[5] The concept of “the political” will be further strengthened in the following chapter.

concepts; classroom studies; and teacher knowledge, beliefs, and practices. These foci are evident in the different content-based subdomains of calculus, namely the teaching and learning of limit, continuity, and integral, while they share the underlying goal of improving the teaching and learning of calculus. Of course, the notion of "improving" is a political one; an idea on which I will further expand on the critique part of this section. Bressoud et al. (2016) engage in a more detailed review of the literature which includes a review of the theoretical frameworks used (that fall under two major categories, namely cognitive approaches and sociocultural, institutional, and discursive approaches[6]), a discussion of the epistemological underpinnings related to the notion of infinity, a detailed description of substantial findings regarding students' difficulties in content-based subdomains (including limits, functions, derivatives, integrals), analysis of calculus curricula, classroom practices, and theoretical and empirical perspectives on designing calculus materials. Bressoud et al.'s review explores the difficulties that students face in calculus, as a result of a divergence between their informal calculus thinking and the formal conceptualisation of calculus included in the curricula.

Early research on calculus focuses on exploring students' difficulties, learning obstacles, and (mis)conceptions on calculus concepts like the limit or the derivative (Rasmussen et al., 2014). A lot of research in this area aims to address where students' understanding of mathematical concepts is mistaken, by placing the emphasis on "misconceptions" or "false ideas" (e.g., Cangelosi et al., 2013; Cornu, 2002; Davis & Vinner, 1986; Eisenberg, 1991; Orton, 1983; Tall & Vinner, 1981). For example, Tall and Vinner (1981) study how students see limit as a dynamic process and have difficulties conceiving of it as a number. In a similar vein, Orton (1983) proposes that although his student participants are proficient with the calculation of derivatives, they have significant misunderstandings in the conceptualisation of the derivative as a rate of change and its graphical representation. Scholarship conceptualises these difficulties as instances of informal calculus thinking (Bressoud et al., 2016). For example, Cornu (2002) employs the concept of "spontaneous conceptions" to refer to students' ideas prior to formal teaching, while the chasm between concept definition and concept image (with the former being a concept's formal definition and the latter being an individual's mental images of this concept), introduced by Vinner and

[6] In this review, Bressoud et al. demonstrate that research in calculus teaching and learning with cognitive theoretical orientations primarily use the following frameworks: a) concept image – concept definition (Tall & Vinner, 1981; Vinner & Hershkovitz, 1980), b) register semiotic representation (Duval, 1995, as cited in Bressoud et al., 2016), c) Action-Process-Object-Schema theory (Dubinsky, 1991), d) Three Worlds of Mathematics (Tall, 2004). Research following sociocultural, institutional, and discursive perspectives use the following orientations: a) Theory of Didactic Situations (Brousseau, 1997), b) Anthropological Theory of Didactics (Chevallard, 1985, as cited in Bressoud et al., 2016), and c) Commognitive Framework (Sfard, 2008).

Hershkovitz (1980), is a popular theoretical framework in calculus teaching and learning literature (Bressoud et al., 2016). Scholarship on students' (mis)understanding of concepts is often combined with epistemological discussions about the nature of calculus concepts and the learning difficulties which are assumed to emerge as a result of the concepts' epistemological nature and/or their historical development (e.g., Eisenberg, 1991; Jayakody & Zazkis, 2015). For example, Jayakody and Zazkis (2015) interpret a first-year calculus student's claims about (dis)continuity through an epistemological discussion of available (dis)continuity definitions.

In more recent years, as constructivist and social constructivist frameworks started gaining more space in mathematics education literature, the notion of misconceptions started being criticised and reconceived (e.g., Smith III et al., 1994), while gradually the emphasis started shifting from misconceptions towards students' understanding (Rasmussen et al., 2014; Thompson & Harel, 2021). Through this transition, even when students' understanding does not fully align with the formal representation of a concept, there is merit to students' understanding, which is assumed to be valuable knowledge that can be formalised. We may get a snapshot of this transition by thinking, through today's mainstream discourse, about a trilogy of papers which explores students' ability to solve nonroutine calculus problems, written among Annie Selden, John Selden, Shandy Hauk, and Alice Mason around the 1990s. The first one of the three papers was titled "Can average calculus students solve nonroutine problems?" (Selden et al., 1989), a question that was answered in the title of the second paper: "Even good calculus students can't solve nonroutine problems" (Selden et al., 1994), and was further elaborated in the third paper "Why can't calculus students access their knowledge to solve nonroutine problems?" (Selden et al., 2000). The titles of the paper are revealing of the misconceptions trend, as the emphasis is on what students do not know and cannot do. We also notice that in the title of the third paper the existence of valuable student knowledge is assumed (although, according to the authors, it cannot be accessed). These papers use a discourse which has almost vanished from mathematics education literature and – today – it would have been criticised as deficit-based[7]. Overall, scholarship on students' (mis)understandings of calculus concepts has been prolific and we may notice a transition from an emphasis on the misunderstandings towards an emphasis on the understanding.

[7] This does not mean that deficit views on mathematics or calculus teaching and learning have vanished. Nevertheless, the enunciation of this deficit discourse is not as straightforward as it used to be, while there are also discussions which critically engage with deficit approaches in mathematics education per se (e.g., Adiredja, 2019; Adiredja & Louie, 2020).

Besides literature on individual students' difficulties and understanding of specific concepts, there is also literature which focuses on the institutional framework within which students learn calculus, most notably by presenting research on the transition that students undergo from secondary calculus to university calculus. According to Rasmussen et al. (2014), the secondary/tertiary differences in differential and integral calculus courses are not substantial if thought of from a cognitive, developmental point of view, but they can be important through a pedagogical or cultural lens. Research following an Anthropological Theory of Didactics lens focuses on the institutional factors of the transition and the existence of different praxeologies (Thomas et al., 2015). For example, Praslon (2000, as cited in Bressoud et al., 2016) suggests that the secondary/tertiary transition is not merely a transition towards formal calculus, but is rather a complex process, which involves "process-object duality, particular-general objects, algorithmic-conceptual techniques, the choice of semiotic registers and the conversions between them, autonomy given to solve problems and routinisation of practices, diversification of problems, etc." (p. 13). Nardi et al. (2014) further elaborate on the transitions that students undergo while learning calculus, by using a commognitive framework to capture discursive shifts that occur in the early stages of studying calculus; such discursive shifts include the instructors' and students' communicative practices, routines of constructing mathematical objects and ways of resolving commognitive conflicts.

While research on individual students' understanding of calculus concepts has been prolific and supported theoretically from the development of learning theories and methodologically from the development of clinical interview protocols, there are few classroom studies and little research focusing on the teaching practice. This reflects a direction evident more generally in undergraduate mathematics education literature, in which empirical research focusing on teaching practice within the context of actual classrooms is scarce (Speer et al., 2010). An exception to this is work conducted by Speer (2008), who engages in a detailed analysis of the relationship between a calculus instructor's beliefs and his teaching practice, finding that the instructor's beliefs about mathematics influenced his choices while interacting with students. Interested in students' persistence in STEM, Ellis et al. (2014) explore which perceived calculus instructor practices are associated with students taking a second calculus course. Such practices include perceived frequency of the instructor showing students how to work specific problems, preparing extra material to help students understand calculus concepts or procedures, holding a whole-class discussion, and requiring students to explain their thinking on exams. While research projects on teaching practices and classroom studies make use of different theoretical and methodological frameworks, they share the aim of examining the flexibility of mainstream teacher practices and their impact on student learning of calculus concepts (Bressoud et al., 2016).

Finally, there is research which develops and examines ways in which calculus could be taught differently than it currently and dominantly is. In the 1990s, a calculus reform movement took place in the US, led mainly by mathematicians, which attempted to change calculus curricula so that they are more engaging for students, enhance conceptual learning, utilise technology, and support instructors in the development and dissemination process (Ferrini-Mundy & Graham, 1991). It is as a result of this reform that graphical and numerical representations of derivatives and integrals are considered valid and included in calculus curriculum in addition to the algebraic ones (Bressoud et al., 2016). Similar approaches are also developed today. While the mainstream curricular approach is to teach calculus based on limits, there are alternative methods developed, at least theoretically, which use discrete (as opposed to continuous) mathematical constructs as the basis for the development of calculus and in particular sequences (Weigand, 2014) and infinitesimals and differentials (Ely, 2021; Moreno-Armella, 2014). Besides, calculus professors increasingly experiment with new techniques, which they incorporate in their classes, such as inquiry and other active learning methods (e.g., Keene et al., 2014; Liakos et al., 2021), flipped classes (e.g., Maciejewski, 2016), and online resources (e.g., Soares & Borba, 2014). Furthermore, universities differentiate their calculus courses and develop courses targeted to specific disciplines such as biology or economics. As Bressoud et al. (2016) put it, "[i]t is now the rare mathematics department that is not trying or at least thinking of trying one of these ideas" (p. 18).

2.3.1.2. Critique of Calculus Research

In what follows, I engage in a critique of calculus teaching and learning scholarship from a sociopolitical lens. A first observation that can be made is that research assumes that there is a single formal calculus knowledge, which can have multiple representations (e.g., visual representations) and applications in different fields. In its extreme version, this idea is evident in research which adopts a deficit view of student learning, by focusing on misconceptions (e.g., Cangelosi et al., 2013; Cornu, 2002; Davis & Vinner, 1986; Eisenberg, 1991; Orton, 1983; Selden et al., 1989; Tall & Vinner, 1981), thereby assuming that there is a correct form of calculus that needs to be learned by students. But even in research and practice which assumes more experimental ways of learning calculus or creates specific calculus courses for students of specific programs (e.g., biology, economics, education, etc.), the existence of a single formal calculus knowledge is assumed and students are expected to acquire it to a greater or lesser degree depending on the course. For example, Maciejewski's (2016) study of a flipped calculus course concentrates on the impact of the flipped classroom model on students' performance and understanding of calculus concepts, by comparing them to the respective performance and understanding of students in typical calculus courses; the conceptualisation of calculus and mathematics is taken as a given regardless the mode of instruction. Moreover, the prominent cognitive framework of the concept image, concept definition (Vinner & Hershkovitz, 1980) is an example of a theoretical framework which manifests the idea of the existence of a single formal calculus knowledge. For example, Tall and Vinner (1981) suggest that students' "right" and "wrong" answers "are determined by the strict mathematical formulation" (p. 166). Hence, within this view, the formal knowledge is entailed in a concept's definition, while an individual's learning is considered successful when their concept image resembles the concept definition.

The existence of a single formal mathematical knowledge has implications for the construction of a certain way of doing calculus as a superior form of mathematical knowledge. This means that students are expected to develop a particular way of conceptually understanding calculus. In prologuing a special issue on calculus in ZDM, Rasmussen et al. (2014) emphasise the commitment to conceptual learning by maintaining that the theme of the special issue's papers is "exploring the prospects and possibilities for more coherent and conceptual learning and teaching" (p. 510). The procedural/conceptual distinction is a prominent scheme for understanding students' learning of calculus (e.g., Ellis et al., 2015; Tallman et al., 2016) and is a hierarchical one: students are expected to not only develop procedural knowing, but also conceptual. Furthermore, the so-called students' intuitive ideas are viewed in research as lacking formality and needing to be transformed to formal knowledge (Keene et al., 2014).

Through the eclectic distinction between valid, formal calculus knowledge and informal, intuitive calculus knowledge, calculus is constructed as a superior form of mathematical knowledge. Calculus has status, at least in North America: students and parents often consider taking a calculus course to be a substantial intellectual achievement (Rasmussen et al., 2014). This contributes to the conceptualisation of calculus as a knowledge which is not for everyone as well as to calculus having a gatekeeping role into university programs.

Furthermore, the goal of improving the quality of teaching and learning is a political one. It implies that a certain form of calculus instruction is preferable, and that students and instructors should follow this form of teaching and learning. However, this practice conceptualises the possible teaching and learning in a narrow way. For example, it does not generally account for whether students know the circumstances under which this knowledge was produced and it does not engage them in critically evaluating the images calculus constructs for the world. In other words, using Skovsmose's (1994a) distinction between mathematical knowing (i.e., the knowing of traditional mathematical skills), technological knowing (i.e., the ability to use mathematical skill towards technological goals), and reflective knowing (i.e., the evaluation of the social and ethical consequences of using mathematics towards technological goals), the goal of enhancing teaching and learning identifies learning solely with the development of mathematical and technological knowing. This is further reinforced by the clinical interview (which typically includes cognitive tasks) being the predominant methodology in calculus research (Rasmussen et al., 2014), which focuses on neither the context of the classroom nor students' and their communities' sociopolitical contexts. In this way, calculus learning is constructed as a single formal knowledge with multiple representations and applications, autonomous from the wider sociopolitical context, and unrelated to critical ways of looking at the world and calculus knowledge. But also, more generally, should we consider the political aspects of prioritising STEM instruction (as discussed in section 2.2.), the concept of improving calculus instruction gets inscribed in the wider urge to create individuals who will be labouring in the STEM industry in a technocratic, capitalist, and nationalist framework (e.g., Adiredja & Andrews-Larson, 2017).

Overall, in calculus teaching and learning scholarship, there is an underlying assumption that students need to learn calculus and an underlying value that research needs to be aiming towards enhancing the quality of teaching and learning of calculus (Rasmussen et al., 2014). As Apple (2004a) maintains within research that centres around the issue of academic achievement, curricular knowledge is not problematised but is taken as given and neutral. Within this framework, calculus is conceptualised in particular ways, which assume that there is a single formal calculus knowledge, which students should learn at a level corresponding to the needs of their respective calculus courses. The construct of a single formal calculus knowledge constructs a single way of knowing calculus as exceptional, contributes to the construction of calculus as an exclusive knowledge, and divorces calculus knowledge from critical and reflective approaches towards it.

2.3.2. Proof and Proving in Undergraduate Settings

Research on proof, proving, and argumentation has been prolific both in undergraduate settings and in primary/secondary education. Often research in undergraduate contexts makes use of theoretical concepts in proof literature and findings which originate from research conducted in secondary contexts. For this reason, before discussing undergraduate literature per se, I will review some epistemological underpinnings and key theoretical ideas of proof/proving and argumentation literature in general. I conclude this section, by engaging in a political critique of this scholarship.

2.3.2.1. Proof and Proving Literature in Mathematics Education

In mathematics education research, proofs are assumed to constitute a central part of mathematical knowledge (e.g., Ross, 1998; Stylianides, 2007, 2009). This assumed centrality of proofs in mathematics has profound impact in the teaching and learning of mathematics. For example, Stylianides (2009) claims that there is “widespread agreement that the activity of reasoning-and-proving should be central to all students’ mathematical experiences” (p. 258). The teaching and learning of proofs is deemed important because of its central role in modern mathematics, the explanatory role of proofs, and because proofs, mathematical argumentation, and logical reasoning are expected to provide students with tools which are useful for active citizenship (e.g., Arzarello & Soldano, 2019; Ross, 1998; Stylianides et al., 2019). For example, Reid (1995) conceptualises proofs from a critical thinking perspective and argues for their potential to cultivate students’ ability to recognise the limits of numerical arguments. In undergraduate mathematics education, the teaching and learning of proof is deemed important for one extra reason; through their programs, students are expected to get socialised into the practices of professional mathematicians (Stylianides & Stylianides, 2009; Weber & Mejia-Ramos, 2014).

Yet, despite the widespread agreement about the need to teach students proofs and proving, there is no single answer to the question “what is a proof?” or even “what is a mathematical proof?”. Illustrating that a poof means different things for a judge and jury, a statistician, a scientist, and a mathematician, Tall (1989) writes:

> as Humpty Dumpty once said, “when I use a word, it means just what I want it to mean, and nothing else”. The term “proof” is just such a word. In different contexts it means very different things. (p. 28)

So, proof can mean different things depending on the context. What constitutes a proof depends on personal experience, mathematical area, and educational level (Stylianides et al., 2019). The consideration of an argument as a proof changes depending on the mathematical sub-discipline and time, while there is not always consensus regarding whether an argument constitutes a proof. For example, there is dispute in mathematics communities regarding whether computer generated proofs are indeed proofs beyond any doubt[8] (Krantz, 2011). With regards to mathematical proof within mathematics education research, its conceptualisation is also not universal, but depends on researchers’ epistemologies (Balacheff, 2008; Reid, 2006).

Cadwallader Olsker (2011) distinguishes between two widely held meanings of proof: the formal view of proof and the view of proofs as arguments – a distinction which relates to Reid’s (2006) distinction

[8] Perhaps the most famous example of a disputed computer generated proof is the following. In 1974, the “four colour theorem”, a theorem all of whose attempts for a proof had until then been considered false or incomplete, was proven using 1200 hours of computer time on the University of Illinois supercomputer, through the use of a technique which was beyond the ability of any human to reproduce. This proof has generated great dispute in the mathematical community regarding whether what the computer did could indeed be regarded as a proof, considering that no human could ever check it (Krantz, 2011).

between a traditional view on proof and a quasi-empirical view on proof. Within the formal view of proof, proof is conceptualised as a series of deductions – a proof requires clearly formulated definitions and statements, and agreed procedures to deduce the truth of one statement from another (Tall, 1989). Reid (2006) refers to this as the "traditional" concept of proof, which is assumed by its proponents to offer a priori universal validity. The idea of a "formal proof" in mathematics is inspired by formalism and refers to a proof which shows that a theorem follows from the axioms (e.g., that the Pythagorean theorem follows from Hilbert's axioms) (Cadwallader Olsker, 2011). Hence, formal proofs in this sense are often incredibly long and far from being a typical product of mathematicians' practice. In the context of mathematics education, where students are often expected to learn "formal proofs", a formal proof is conceptualised more loosely (Reid & Knipping, 2010). Similarly to how proof is a concept that depends on the context, the formality of a proof is also not fixed or uniform for all in mathematics education. A formal proof in an educational context, henceforth an "educationally formal proof", depends on the available "toolbox" (i.e., the things that are taken for granted in a specific course) and the local organisation of the mathematical knowledge (which differs from the global character of an axiomatic system) (Reid & Knipping, 2010). The relation between the categories of informal and formal has been explored in primary and secondary education and criticised for its frequent assumption of the a priori existence of the formal category (e.g., Barwell, 2005, 2016). This is not the case in undergraduate mathematics education, where the implicit aspects of the "toolbox" resemble what professional mathematicians do and, as such, are taken for granted by both scholars and instructors.

With regards to the second conceptualisation of proofs, proofs here are viewed as arguments that intend to convince the reader (Cadwallader Olsker, 2011). This perspective highlights the deep connection between proving and constructing an argument: for example, Krummheuer (2007) argues that "[l]earning mathematics is *argumentative* learning" (p. 62, emphasis in the original). Scholarship which focuses on the argumentative aspects of proof and proving highlights that proof relates to convincing. For example, from a cognitive perspective, Harel and Sowder (1998) conceptualise proof as the process to remove or create doubts about the proof of an observation. Relating proof and argumentation has also been conceptualised from a social lens, meaning that the social elements of proof and communication and their role within human communication are emphasised. For instance, Hersh (1993) suggests that in the mathematical community, proofs are "convincing argument[s], as judged by qualified judges" (p. 389). Moreover, Recio and Godino (2001) propose that the meanings of proof relate to the institutional context, which suggests that institutional meanings of proof impact and can explain personal proof schemes. This research orientation is reinforced by the study of the practice of professional mathematicians which highlights the social character and richness of proof and proving practices (Burton & Morgan, 2000; Parameswaran, 2010). However, while argumentation is deemed an important part of the teaching and learning of proof and proving, it is generally seen as something that eventually needs to be developed into formal proof (e.g., Pedemonte, 2007; Stylianides & Stylianides, 2009).

Placing the emphasis on argumentation rather than formal proof or vice versa is significant because it frames the epistemological conceptualisations of proof, the emerging research foci, and the teaching and learning of proof. First, regarding proofs' epistemological framing, the conceptualisation of proofs in terms of their ability to persuade others emphasises that "mathematics is a human endeavour, since proofs are written, read, understood, verified, and used by humans" (Cadwallader Olsker, 2011, p. 36). This stance relates to the view that rigourous mathematical proofs might hinder understanding, hence the emphasis shall not be on the form of proof but rather on the meaning of proof as part of mathematical activity (e.g., Hersh, 1993). On the other hand, emphasising formal proofs suggests that there is a common mathematical practice which shall be learned and adopted by students as such (e.g., Fischbein, 1982). Second, this distinction frames the various research foci that emerge. Researchers adopting a traditional view of proof examine students' difficulties in learning this practice (e.g., Azrou & Khelladi, 2019; Selden & Selden, 1995; Stylianou et al., 2015). In contrast, researchers adopting an argumentative perspective of proof are interested in how students make sense of proofs, engage in argumentation and proving, and convince each other (e.g., Inglis & Mejia-Ramos, 2008; Stylianides & Stylianides, 2009). Finally, the distinction has implications for how learning is conceptualised: while researchers adopting a traditional view of proof might propose that students can be motivated to prove by their need to verify or systematise, researchers adopting a quasi-empirical view of proof might emphasise the needs to explore, communicate, gain social acceptance, or explain (Reid, 2006).

Further delving into this distinction, it is significant to point out that the two discourses are not mutually exclusive: they co-exist and interact. In what follows, I refer to two examples of conversations in proof and proving literature, which highlight how the two discourses have different priorities and emphasise different aspects of the proving practice, but also get synthesised. The first conversation is about the status of visual proofs in mathematics and mathematics education, while the second conversation is about the functions of proof.

Visual proofs are an example of the dispute regarding what constitutes a proof. While there seems to be a consensus that visual representations are heuristic tools, their status as proofs or even as parts of proofs is contested both in the philosophy of mathematics and in mathematics education (Hanna & Sidoli, 2007).

Hanna and Sidoli (2007) talk about the different perspectives regarding the role of visualisation in proofs and distinguish between three conceptualisations of visual representations: as adjuncts to proofs, as an integral part of proofs, and as proofs themselves. We can see this tension (or antagonism between discourses about the status of visual proofs) in how Alsina and Nelsen's (2010) paper "An Invitation to Proofs without Words" refers to "visual proofs" as "pictures or diagrams that help the reader see why a particular mathematical statement may be true, and also to see how one might begin to go about proving it true" (p. 118). Although Alsina and Nelsen name these visual arguments "proofs" and argue about their importance in mathematical practice, they also describe them as a starting point for proving the theorem, indicating that the proof starts after the visual argument has taken place. Hence, we notice that different discourses about what a proof is interact with each other and co-exist in the discussion about visual proofs.

The second conversation is about the functions of proofs. Although there is a diversity of opinions of what constitutes a proof, there is widespread agreement that proofs have multiple functions. The most visible of these is the function of verification: proofs verify whether a mathematical result is true or false (Cadwallader Olsker, 2011; de Villiers, 1990). But proofs do much more than that. De Villiers (1990) elaborates on this and suggests that other functions of proof include explanation (i.e., a proof provides insight into why a statement is correct), systematisation (i.e., proof help organise results into a deductive system of axioms, concepts, and theorems), discovery (i.e., a proof contributes to the discovery/invention of new results), and communication (i.e., a proof contributes to the transmission of mathematical knowledge). This theorisation about multiple functions of proofs is significant because it breaks with a tradition, according to which proofs are merely formal texts that verify the truth of a mathematical statement. The notion of functions of proofs has been incredibly influential in proof and proving research: scholars have identified other functions of proof, including intellectual challenge (de Villiers, 1999), aesthetics (de Villiers, 1990; Lange, 2016; Rota, 1997), personal self-realisation (de Villiers, 1990, 1999), "theorem credits" function, which refers to the acknowledgement in the academic work that comes with publishing a theorem and its proof (Thurston, 1995), and epistemic value, which refers to the personal judgment regarding whether a proposition is believed (Duval, 1990, 2007, as cited in Reid & Knipping, 2010). All these different functions are emphasised differently by different scholars, depending on epistemological positioning and research interests: for example, while discovery is a function that focuses on a student's mathematical understanding and learning process, theorem credits is a function that highlights the characteristics of the social spaces in which mathematics takes place. Furthermore, different proofs might have different functions depending both on the proof text and the social context within which this proof is discussed, taught, or learned. For example, Hersh (1993) proposes that proofs aim to primarily convince in the mathematical community, while they need to primarily explain in the mathematical classroom.

2.3.2.2. Proof and Proving Literature in Undergraduate Settings

So far, I have discussed literature on proof and proving in general. In what follows, I focus on proof and proving in undergraduate settings. There are two transitions in literature which will help frame this scholarship. Mariotti (2006) describes a transition from early studies on students' conceptions of proof and difficulties in coping with proof and proving, towards teaching intervention studies that aim to overcome the difficulties associated with proof and proving. Furthermore, there is a transition from historical and epistemological conceptualisations of proof towards conceptualisations that inquire into the relation between the epistemological and the cognitive, focusing on the student as the subject who engages in proving (Mariotti & Balacheff, 2008). Both transitions underpin proof and proving scholarship as is outlined in what follows.

In the field of proof and proving, there is a distinct orientation which sets out to identify the difficulties that students face with proofs. Summarising existing literature, Moore (1994) suggests that the cognitive difficulties students face in relation to proof and proving concern perceptions of the nature of proof, logic and methods of proof, problem-solving skills, mathematical language, and concept understanding. As happens with early cognitive research on students' difficulties in mathematics education more generally, difficulties in proof and proving are often conceptualised in terms of misconceptions and obstacles faced by students. For example, Selden and Selden (1995) examine students' ability to "translate" sentences written in "mathematical English" into mathematical language and propose that students who fail to do that could not be expected to produce or validate proofs. Similarly, Azrou and Khelladi (2019) identify "weak meta-knowledge about proofs" as one of the main reasons for undergraduate students producing "poor proof texts" (p. 271). Here, we notice a parallel with calculus literature on students' difficulties: the focus of this research is what students' have failed to understand or master, while deficit language is used to describe this thesis.

The above orientation has also led to studies which examine students' perceptions and beliefs about proofs. The assumption of this research orientation is that students' understanding of proofs is related to their views, perceptions, and beliefs; hence, it is worthy to examine views, perceptions, and beliefs as a vehicle towards better inquiring into students' understanding (Moore, 1994; Stylianou et al., 2015; Weber & Mejia-Ramos, 2014). For example, Moore (1994) finds that, in his study, students' perceptions of mathematics and proofs influenced the kind of proofs that they produced. This pool of scholarship is diverse in terms of its epistemological assumptions. For example, Weber and Mejia-Ramos (2014) adopt a social view of beliefs by examining the differences between students' and their instructors' beliefs. Such differences include the expected needed time to understand a proof (students expect to be able to understand a proof in less time than their instructors do), the consideration of what it means to completely understand a proof (students believe that justifying each step of a proof means understanding it, while instructors think this is not enough), and the realisation of the proof understanding process as active/passive (students believe that understanding a proof is a passive process, while instructors think that this process involves constructing sub-proofs). On the other hand, Stylianou et al. (2015) evaluate positively students' understanding and appreciation of the rigour and central role of proof in mathematics – assuming in this way a perspective that advocates for, what I earlier called, "educationally formal proofs". The following question included in the questionnaire used to inquire into students' views is indicative:

> Proving a theorem in mathematics implies (a) one may need to check if it applies to all cases; (b) it may still be rejected at a future time; (c) one may be able to find an exception; (d) it will always be true. (p. 119)

This question, like all other questions of this questionnaire, has only one correct answer[9]. In other words, the authors claim that there are correct and incorrect views about what a proof is, at least in a university setting. Overall, research on beliefs, while adopting a psychological lens, is diverse in terms of its epistemological assumptions and specifically regarding the framework that defines students' desirable behaviours related to proofs and proving.

A related but distinct orientation in proof and proving scholarship theorises and inquires into students' proof schemes and argumentative styles. An individual's or community's "proof scheme" is a cognitive concept, which entails whatever is considered ascertaining and persuading for that person or community

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[9] I assume that the correct answer of this question is (d): "proving a theorem in mathematics implies it will always be true". However, I argue that this is a very narrow view on what mathematical proving and proof is. Here, proving and proof are treated as identical: while the question asks about proving, the correct answer (d) refers to the product of the proving process, something that will always be true. Furthermore, all other responses are also in a sense "true". A proof could include checking all cases, it could entail mistakes which have not been immediately identified in the peer-review process and, hence, it may still be rejected at a future time, while – in a similar vein of argumentation – one may be able to find an exception thus showing that the proof was not correct. It is noteworthy to mention that the authors of this paper refer to the functions of proofs (e.g., de Villiers, 1990) and Lakatos' (1976) work in their theoretical framework, a theoretical base which highlights the social character of proof and proving that is not reflected in the discussed question. This is important as it shows that while social perspectives on proofs are widely cited in proof and proving research, their epistemological underpinnings are not necessarily shared by scholarship citing them.

(Harel & Sowder, 2007). Harel and Sowder (1998) develop three categories (that are further developed in sixteen subcategories of proof schemes manifested by students in undergraduate teaching experiments), each of which corresponds to "a cognitive stage […] in students' mathematical development" (p. 244). These categories are a) external conviction proof schemes (ritual and form are considered the main elements of justification), b) empirical proof schemes (conjectures are evaluated by appeals to physical facts or sensory experiences), and c) analytical proof schemes (conjectures are validated using logical deduction). The idea of developing taxonomies has been taken up by other scholars as well, in an effort to categorise the ways in which students make sense of and engage in proof and proving in a variety of contexts. For example, scholars analyse the thinking styles when working on exam-type proving questions (e.g., Moutsios-Retzos, 2009; Moutsios-Rentzos & Kalozoumi-Paizi, 2017), when working on problems which entail notions that are familiar to all students (Recio & Godino, 2001), and when asked to assess the persuasiveness of an argument (Inglis & Mejia-Ramos, 2008). Another influential framework of analysing students' proof schemes is based on the distinction between syntactic proofs (i.e., proofs which are written by manipulating definitions and other facts in a logically permissible way) and semantic proofs (proofs that use instantiations of relevant mathematical objects). In a study about proving in algebra, Weber and Alcock (2004) show that undergraduate students rely on syntactic proving, while doctorate students make use of semantic arguments. Overall, these schemes and taxonomies are useful for understanding and describing students' ways of engaging in proof and proving.

Furthermore, there is research which focuses on teaching practices of proof and proving. In this direction, there is research which investigates traditional proof instruction and research which enriches this instruction. Traditional proof instruction follows a lecture-based "definition-theorem-proof" format – a mode which can entail multiple and diverse pedagogical techniques (Mills, 2011; Weber, 2004). This format has been the dominant type of instruction for many decades, but it has been criticised for not engaging students in authentic mathematical activity (Mesa et al., 2020). Within this zeitgeist, instructors and scholars engage in introducing alternative practices in the teaching of proof and proving. Stylianides and Stylianides (2009) design a classroom intervention which aims to help students transition from argumentation to formal proof and cultivate their meta-knowledge about mathematical proofs, by emphasising the limits in the verification power of empirical arguments. Technology is a tool used towards this goal especially in the field of geometry, as it is assumed to help students visualise geometrical objects, recognise patterns, form conjectures, test conjectures, develop counterexamples, and link the formal axiomatic system of geometry with its standard models (e.g., Cartesian system) (e.g., de Villiers, 1999; Komatsu & Jones, 2019; Quaresma & Santos, 2019;). Another instruction method adopted by instructors and examined in research is instruction around activities that engage students in meta-proof processes; for example, Pfeiffer and Quinlan (2015) engage students in proof evaluation tasks. Finally, inquiry-based courses and techniques are used as an alternative teaching approach, which aims to engage students in producing their own proofs and collaborating with peers during class time (Mesa et al., 2020). Overall, a double goal underlies this research. On the one hand, in line with social views of proof (e.g., de Villiers, 1990; Lakatos, 1976), this research attempts to enrich the in-class argumentation processes in which students engage. On the other hand, it tries to facilitate the transformation of students' argumentation into formal proof, as well as to help students see the need to do so.

Recognising proofs as central for the education of students in mathematics-related programs[10] and in response to the difficulties related to the learning of proofs, universities in North America have introduced courses focusing specifically on providing explicit instruction in writing and understanding proofs (Moore, 1994). These courses constitute a transition from high school to university curricula, given that typical high school curricula in North America do not prioritise the production of proof scripts[11]. In terms of curriculum, these courses vary in terms of the mathematical content that they cover and can be placed into two big categories: the standard courses[12] (which cover diverse mathematical content) and the topics-based courses (which can focus on algebra, analysis, number theory or discrete mathematics) (David & Zazkis, 2020). All these courses include in their curricula the following "standard" topics: "symbolic logic, truth tables, logical connectives and quantifiers, negation, set theory, direct proof, proof by contradiction, and proof by induction, and basic functions and relations" (David & Zazkis, 2020, p. 7).

2.3.2.3. Critique of Proof and Proving Literature

So far, I have discussed research on proof and proving in general and with particular focus on undergraduate courses. In what follows, I engage in a critique of this research from a political perspective. Political aspects of proof and proving have not been widely taken upon in the proof and proving research. Therefore, to critique the literature on proof, I use the sociopolitical literature reviewed earlier in this chapter and in particular the lens proposed by the cultural politics of mathematics education. I first position research on proof and proving within a particular historical narrative and discuss the

[10]For example, the Mathematical Association of America (MAA) has appointed a committee on the Undergraduate Program in Mathematics to develop a curriculum guide, which proposes that proofs, being the core of mathematical culture, should be a part of students' mathematical experiences (MAA, 2000, as cited in Stylianou et al., 2015).

[11] For example, the Ontario curriculum identifies "reasoning and proving" as one of the seven key mathematical processes along with problem solving, reflecting, selecting tools and computational strategies, connecting, representing, and communicating (Ministry of Education, 2007). However, in the overall and specific expectations of the courses' various strands, proving appears less often.

[12] The "Introduction to Proofs" course in which I collected data would fall under this category.

consequences of neglecting alternative historical narratives, in terms of generating a particular "reality" about proof and proving. I then discuss how proofs play specific social roles, including the reproduction of a specific scientific paradigm, the construction of the rational individual, and the maintenance of the capitalist educational system.

The above narrative about proofs and proving resides within what Reid and Knipping (2010) call the standard view of the history of mathematical proof. Within this view, the first occurrences of proof appeared in ancient Greece, where mathematics gets eventually systematised into the Euclidean axiomatic system. After a gap of several centuries, Descartes gets inspired by the Euclidean organisation of geometry bringing the axiomatic system and proofs to the centre of mathematical practices, a paradigm which continues until today (Reid & Knipping, 2010). This historical narrative about proof is evident in the historical references and/or sections about the history of proof in proof and proving scholarship (e.g., Harel & Sowder, 1998; Jahnke, 2008). This standard view is presented in a way that highlights the importance of proofs for mathematics, to the extent that mathematics without proofs cannot be conceived, implying that ever since proofs were discovered mathematics was synonymous with proving. However, there have been civilisations, like the Persians or the Arabs, who did develop mathematics without valuing and developing proofs (Hacking, 2014). In fact, alternative approaches to the history of mathematics show that the dominance of proof in mathematics is the result of a historical contingency and there could have been ample mathematics even if humans never conducted proofs (Hacking, 2014). The dominance of the standard view of the history of mathematical proof generates a particular "reality" about proof and proving, as it constructs the notions of proof and its importance as central for mathematics. The existence of alternative, compelling historical narratives challenges the idea that proof has always been central to mathematics or that proofs have always had the form that we conceive them to have[13].

One social role of proofs is that it operates within and reinforces a particular scientific paradigm. This paradigm conceptualises science as certain, rigourous and correct, and proofs as the (universal) arbiter of truth. In this sense, proofs are devices which can have the final say about a mathematical statement: if something has been proven mathematically, there is no doubt that it is true (e.g., Gutiérrez, 2013). This assumption is naturalised in proof and proving research. For example, the idea that proofs have the

[13] It is revealing for the changing nature of proof that in the 19th century, Euclidean proofs were to be learned verbatim, to the extent that if a student used different letters than Euclid but otherwise reproduced the exact proof, the proof was wrong (Tall, 1989).

function of convincing others beyond any doubt is emphasised in scholarship (e.g., Cadwallader Olsker, 2011; de Villiers, 1990; Hersh, 1993), while literature on students' difficulties, beliefs, and proof schemes on proof assesses students' socialisation into the idea that proofs are final and ensure a statement's truth (e.g., Harel & Sowder, 1998; Stylianou et al., 2015). In this way, the concept that proofs lead to air-tight science is cultivated and reinforces the imaginary of mathematics as a type of knowledge which cannot be challenged. This means that other forms of argumentation are deemed of lesser value. At the same time, the assumed certainty that this scientific paradigm offers reinforces the idea that humans can control the world through science.

The second social role of proofs is that they contribute to the construction of the rational individual. Literature within the cultural politics of mathematics explores how mathematics in general constructs the rational subject, by socialising students into forms of thinking that are typical in scientific rationality (Andrade-Molina & Valero, 2017; Popkewitz, 2004; Valero & Knijnik, 2015; Walkerdine, 1988). Proofs and especially deductive proofs are central in this process, as they are conceptualised in literature as the prototype of reasoning (e.g., Recio & Godino, 2001; Stylianou et al., 2015). Central to this construction is the "proof for all" discourse, the idea that mathematical proof and argumentation can help a person be more rational and argue effectively and, hence, all students need to be socialised into proofs and proving (Arzarello & Soldano, 2019; Ross, 1998; Stylianides, 2019; Stylianides et al., 2019). In this way, proofs play a key role in determining the (ir)rational behaviour and the (ir)rational individual.

The third social role of proofs is that it partakes in the functions of the capitalist educational system. By their very nature, as described in the previous historical analysis, proofs in/exclude. In the current scientific and educational paradigm, students are expected to master formal proof and proving (e.g., Pedemonte, 2007; Stylianides & Stylianides, 2009), while their so-called intuitive ideas about proofs are treated as insufficient and occasionally an obstacle to the mastery of formal reasoning (Recio & Godino, 2001; Selden & Selden, 1995; Stylianou et al., 2015). It is characteristic that formal proofs, following the Euclidean paradigm, are detached from visual clues (Reid & Knipping, 2010), while visual arguments rarely gain the status of a proof (Alsina & Nelsen, 2010; Hanna & Sidoli, 2007). Hence, the practice of teaching proof and proving in a detached way excludes specific cultures, such as cultures of lower social classes and Indigenous cultures. This exclusion can happen at multiple levels, like through the un-rootedness of the proving practice (e.g., Abtahi & Wagner, 2016) and the noun-based language of proof and proving (Lunney Borden, 2011). This practice is at the core of the capitalist educational system, which favours the (detached) cultures of higher social classes, rewarding the social capital of students whose home cultures are more like the culture of the scientific paradigm (Bourdieu & Passeron, 2014).

Overall, research on proof and proving in undergraduate settings makes use of concepts that have been developed in the context of secondary education and theoretical studies regarding the historical and epistemological aspects of proofs. Scholarship in undergraduate settings has inquired into the difficulties that students face with proofs, students' beliefs about proofs, students' proving schemes, and teaching methods. This literature does not account for the political role of proofs and proving. In the last part of this section, I engaged in a political critique of proof and proving scholarship, by discussing the historical narrative in which this scholarship operates as well as the social roles of proof namely the roles of proof within a specific scientific paradigm, the construction of the rational individual, and the capitalist educational system. These political roles of proof and proving practices are masked by undergraduate proof and proving literature which focuses on improving the teaching and learning of proofs.

2.3.3. Bringing it All Together

In this section, I reviewed some substantial findings of research in the teaching and learning of calculus and the teaching and learning of proof and proving in undergraduate settings. I adopted a lens which highlights the political aspects of this literature. I attempted to look at cognitive research from a political lens, by exploring which knowledge is conceptualised as important, what discourses are related to this knowledge, and what possibilities this opens/closes for critical and reflective evaluations of this knowledge.

Scholarship on undergraduate calculus and scholarship on undergraduate proof and proving follow similar trajectories, as is perhaps expected by the similar academic and educational context of these two bodies of literature. Early literature in undergraduate mathematics education has almost exclusively been conducted by mathematicians who teach undergraduate courses and have turned to research as a way of inquiring into students' learning and developing their own instruction practices (e.g., Selden & Selden, 1995; Tall, 1989). In this way, mathematics in undergraduate courses is conceptualised from a perspective which is heavily influenced by the academic view of mathematics. Moreover, a similar transition is evident in both pools of scholarship: literature has moved from its early focus on students' mistakes and misconceptions, towards an emphasis on students' sense-making. This transition can be thought of as an instance of what Lerman (2000) calls the "social turn" in mathematics education, within which social views of learning are becoming more common. Finally, a similarity of undergraduate calculus and proof and proving is related to the modality of these courses. Within a context of big classes, lecture-based instruction has been dominant for many years, but more recently, within the social turn of mathematics education, there has been a gradual turn towards incorporating other modes, such as inquiry-based methods.

The two pools of scholarship, calculus and proof and proving, diverge with regards to their theoretical frameworks and epistemological assumptions. With regards to literature on proof and proving, scholarship uses a wide range of theoretical frameworks, that include argumentation theory (e.g., Toulmin, 1958/2003), social theory (e.g., Habermas, 2003), and philosophy of mathematics (e.g., Lakatos, 1976). Reid (2006) and Balacheff (2008) explore researchers' epistemological views on the conceptualisation of proofs, showcasing the existence of a wide range with regards to what scholars consider to be proofs, their functions, and their position in mathematics. As a result of this diversity, a discussion takes place which has started as making one's epistemological assumptions explicit in order to improve mutual understanding and has evolved into examining theoretical tools and connecting them with specific epistemological stances (Mariotti et al., 2018). In calculus research, the picture is very different. The theoretical frameworks that are developed are mostly cognitive (Bressoud et al., 2016) and researchers do not engage in such deep epistemological discussions about calculus: epistemological discussions in calculus primarily search for historical reasons and facets of calculus concepts which can be misinterpreted – in other words, a correct epistemology is assumed in calculus research. Hence, research in calculus, more or less shares the position that valid, formal calculus exists and has a specific form. Overall, while calculus research is largely influenced by the academic view of theoretical mathematics, proof and proving research utilises a wider range of epistemologies and theoretical frameworks.

Overall, in both pools of scholarship, calculus and proofs and proving research, political aspects of the mathematical knowledge are not taken into consideration or at least do not constitute the object of inquiry. This observation is in line with Apple's (2004a) argument that:

> the language of learning tends to be apolitical and ahistorical, thus hiding the complex nexus of political and economic power and resources that lies behind a considerable amount of curriculum organization and selection. (p. 28)

Given this, there are two questions which can be posed and I am interested in exploring through this thesis. First, cognitive discourses about how students learn calculus or proof and proving entail aspects regarding what mathematics is and what it means to do mathematics. What discourses about mathematics are evident in cognitive discourses and how are they political? Second, in cognitive research, there is a transition taking place regarding what it means to learn mathematics: the notion of acquiring a pre-established knowledge starts fading and gets replaced by the notion of constructing a way of knowing as students interact with others. To what extent does this transition constitute a change of paradigm with regards to the relationship of individuals and mathematics? And does this transition also signify a transition with regards to the assumed ontology of mathematics?

2.4. Summary and Where to?

In this section, I engaged in a literature review of sociopolitical literature in mathematics education, sociopolitical research in undergraduate mathematics education, and literature on undergraduate teaching and learning in the areas of calculus and proof and proving. My initial question was about how mathematics is political in the context of undergraduate mathematics education. The literature review in this chapter suggests two things. First, research in undergraduate courses largely focuses on teaching and learning without addressing the sociopolitical, while research interested in the sociopolitical does not critically examine the construct of mathematics per se. Second, the notion of the political is not clearly defined in mathematics education scholarship. In fact, although mathematics is conceptualised as colonial, masculine, and White, it is not often analysed as political per se (with the exception of cultural politics literature). In the following section, the notion of the political will be explored theoretically.

Chapter 3: The Political, Discourse Theory, and Mathematics

In this chapter, I discuss the theoretical framework for exploring what it means to understand mathematics as political. While questions about the political aspects of mathematics education are becoming increasingly central for the field, there has been little discussion about the notion of "the political" per se. The political has been conceptualised in a variety of ways, including by asking questions about power (e.g., Valero, 2008), investigating connections between mathematics, mathematics education and the political system (e.g., Kollosche, 2014), examining the role of education in fabricating citizens (e.g., Valero, 2018), examining how mathematics partakes in organising our societies and lives (Barwell, 2020), developing mathematics as a tool for political action (e.g., Gutstein, 2012), and exploring how particular representations of "reality" are constructed thus excluding other possibilities (e.g., Chronaki & Kollosche, 2019). In other words, there is not necessarily a unified understanding of the notion of "the political" within mathematics education as a research field.

In this chapter, I set out to define a notion of the political which I will use in this thesis, drawing upon discourse theories. To do this, I draw upon three concepts: discourse as constitutive of reality (Jørgensen & Phillips, 2011; Laclau & Mouffe, 1985/2001, 1987); Foucault's conceptualisation of power and its relation with knowledge (Foucault, 1979, 1980); and the notion of "the political" as a space of power and antagonism (Mouffe, 2005). Building on these ideas, I conceptualise mathematics as political in terms of three axes: a) mathematics entails values (Bishop, 1991; Ernest, 2018, 2020; Skovsmose, 2020), b) mathematics puts forward particular rationalities (Kollosche, 2014; Walkerdine, 1988), and c) mathematics has a formatting function (Barwell, 2013; Skovsmose, 1994b, 2000). I conclude this chapter with a presentation of the study's research problem and research questions.

3.1. Discourse Theory

In this section, I present some key ideas from discourse analysis, which will enrich the discussion of what it means for mathematics to be political. The central discourse theory upon which I draw is the work of Laclau and Mouffe (1985/2001, 1987), while I also use ideas from other theorists including Foucault (1972), Fairclough (2003), and mathematics education scholars. As will be explored, these theories are sometimes in tension with each other. As an introductory comment regarding their synthesis, I derive my ontological understanding about discourse (i.e., what discourse is and how it relates to "reality") from the work of Laclau and Mouffe.

In recent years, discourse analysis has been widely used both in social sciences and education. Accounting for the increased academic interest in discourse, Mpoukala and Stamou (2020) discuss the importance of textual mediation in our societies:

> Today's post-modern societies are largely text-mediated, since we "learn" news about the world through the media, and "live" our social practices (e.g., work, family, love, friendship) by being invited to claim corresponding identities for ourselves through texts of mass culture (e.g., advertisements, TV series, movies, etc.). Given the availability of multiple and contradictory constructions of social reality, *what* and *how* something is said to be happening is of great importance. On the other hand, the interest is increasingly focused on the communicative handling of aspects of social life, such as elections, the economic crisis, or a pandemic. (pp. 12–13; my translation)

Within the context of mathematics education as a research field, the emphasis on discourse highlights the "learning" and "living" of mathematics as a complex process, which involves the ways in which one thinks about the world and about themselves. Further, within the public sphere, the big crises of our times, including climate change, the pandemic, and economic crises, all emerge in relation to heavily mathematical discourses, calling for critical examinations of public discourses which entail mathematics. In this sense, from the microcosm of the mathematics classroom to the public sphere, discourse becomes a useful tool with which to think about our interactions with mathematics and their societal and political implications.

3.1.1. Language Creates Reality

Language creates reality: "reality" does not exist in an a priori way – what we call "reality" is a discursive construct. It is more common to think of language as *representing* reality – an idea which supposes that reality pre-exists and the role of language, similar to the role of science, is to describe it. Let us first examine four ideas that challenge the pre-existence of reality and, thereby, set the scene for the idea that reality is discursively constructed: the Whorf hypothesis, Wittgenstein's later work, Austin's work on language acts, and Saussure's distinction between langue and parole.

An influential critique to the idea that language describes reality is the hypothesis of linguistic relativity (or Whorf hypothesis, or Sapir-Whorf hypothesis), which comes from anthropology. Formulated by Benjamin Lee Whorf working under the supervision of Edward Sapir, the Whorf hypothesis posits that cognition is influenced by language (Eriksen, 1995/2001). In other words, a person's or peoples' native language influences how they experience and think about the world. Hence, different languages classify items of experience in different ways (Whorf, 2012). This position has been supported by experiments which show that speaking different languages influences how individuals perceive colours as different or similar – for example, “green” and “blue” are different for an English speaker, but they may be similar for the speaker of a language which uses one word to describe both “blue” and “green”. The importance of the Sapir-Whorf hypothesis is that it challenges the idea that there can be “contact” with reality, which is unmediated by language. Instead, language is captured as shaping ideas and shaping the world itself (Skovsmose, 1994b). Whorf (2012) explicitly challenges the claim that mathematics and natural logic offer an unmediated access to reality, by maintaining that mathematics and natural logic have been developed through and with the use of natural languages that act as a background whose impact often goes unexamined.

From the field of philosophy, Wittgenstein’s later work emphasises the importance of the context in language. In his early work in the Tractatus, Wittgenstein, influenced by Frege’s sharp distinction between psychology and logic and Russell’s commitment to founding mathematics according to logicism, views formal language as providing pictures of reality (Duncan & Wittgenstein, 2021). However, in his later work, Wittgenstein (1953/1986) challenges his previous ideas, counterproposing that language consists of “games”, introducing in this way a theory of language in action (Skovsmose, 1994b). Within this view, a word or a sentence only have meaning within the context of the “game” that is being played and its “rules”. Wittgenstein’s theorisation of “language games” constitutes a break with his previous ideas, as he emphasises the active and performative part of language (Skovsmose, 2021). In the field of mathematics education, Wittgenstein’s ideas have been taken up to theorise that mathematics is not descriptive of a pre-existing reality, but instead mathematics prescribes reality (e.g., Skovsmose, 1994b, 2020, 2021).

Within the field of the philosophy of language, the work of Austin (1962) has been instrumental in showing how language not only describes, but also *does* stuff. Austin’s theory of *speech act* aims to show that language does not provide factual assertions, as positivist views of language suggest. Instead, speaking is an act. Austin uses the example of “I do (sc. take this woman to be my lawful wedded wife)” in the context of a marriage in some Christian dogmas to argue that there are statements which are neither true nor false; instead, they are *performative*: “the issuing of the utterance is the performing of an action” (p. 6). In academic mathematics, we might think of definitions as characteristic examples of performative statements. To illustrate, uttering that the absolute value of a real number x is defined to be the distance of x from 0 is neither a true nor a false statement: it is a statement which *constructs* the object of the absolute value and *determines* what shall be called the absolute value of a real number and what will not – it is a performative statement.

Conceptualising language as a system, the work of Saussure has been of great influence in establishing the field of linguistics and introducing a structuralist approach to language. Saussure distinguishes between langue and parole: langue refers to the network of signs which give meaning to one another, which for the structuralist Saussure is fixed and unchangeable; while parole is situated language use and refers to how people use specific signs in particular situations (Jørgensen & Phillips, 2011). While it has been challenged by poststructural discourse theories for the distinction between langue and parole as well as for the unchangeability of langue (Jørgensen & Phillips, 2011), Saussure's contribution has been of great influence to discourse theories as it proposes that and explores how language produces meaning through a system of differences which is not based on a pre-existing reality.

A common idea in the above contributions is that language partakes in the construction of reality. This position is an underlying idea of various (critical) discourse analysis methods, which embrace a social constructionist view about the world (Mpoukala & Stamou, 2020). Contrary to positivist approaches, which assume that reality is pre-determined, discourse theory suggests that and explores how the world is constructed through discourse. We can explore this idea, in its strong version, with an example. Let's consider a rose. A strong version of discursive constructionism would suggest that there is no essence about what a rose is, outside the ways in which we talk about it. This does not mean that anything goes – only that any essence or truth that we attribute to the rose is dependent upon a system of meaning. A mathematical discourse that "describes" a rose in terms of the golden ratio, a chemical discourse that "describes" a rose in terms of its vital chemical reactions, and a botanical discourse that "describes" a rose in terms of its genus, all in a sense construct a different image about the rose. These discourses format the rose in their particular ways, which in turn influence whether one can think and talk about proportions between adjacent petals, photosynthesis, or carolinae, gymnocarpae, and rosa.

While all discourse analysis theories agree that language is not reflecting a pre-existing reality, they do not all share the same view regarding the role of discourse in constituting reality (Jørgensen & Phillips, 2011). The view proposed in the previous example, which suggests that there is no essence of a rose, is a strong version of discursive constructionism, which we meet in the work of Laclau and Mouffe (1985/2001, 1987). However, this view is not universal for discourse theories; different theories assume different degrees of discursive constructionism (Jørgensen & Phillips, 2011). For example, Foucault (1972, Doxiadis, 2011a) would consider the rose as existing independently and discourse as referring to it. Laclau and Mouffe (1985/2001, 1987), with whose ontological view of discourse this thesis aligns, highlight how such an independent existence has no sense. This does not mean that the material existence of the rose is denied – it is only argued that the material existence has no meaning in itself. To put it differently, there is no objectively existing world, but this does not imply that "the world does not exist". In the following section, I explore how this ontological view unfolds into a conceptualisation of discourse.

3.1.2. What is Discourse and Discourse Analysis?

In this section, I engage in a discussion of what *discourse* means, followed by a discussion about what discourse analysis means. In different theories, discourse is conceptualised in different ways, which are often contradictory between theories and frameworks (Mpoukala & Stamou, 2020). In some theories, discourse represents objects which exist independently of it (e.g., Foucault, 1972), while for others this distinction makes no sense (e.g., Laclau & Mouffe, 1985/2001). Others think of discourse as abstract (e.g., Fairclough, 1995, 2003; Foucault, 1972), while others think of it as material (e.g.,Laclau & Mouffe, 1985/2001, 1987). While my understanding of discourse comes mostly from Laclau and Mouffe, I draw upon other theories as well because – although occasionally contradictory – they shed light to other aspects of discourse. When contradictions emerge, I point to them, explaining where I stand.

While Foucault's view of discourse differs from Laclau and Mouffe's, his writings are helpful in the context of this work, in that they offer a useful insight of how meaning is created through discourse and how discourse relates to issues of power. Foucault developed his discourse theory in his so-called archaeological phase, signified by *The Archaeology of Knowledge* (Foucault, 1972). For him, a central feature of discourses is that they play an active role in constituting the objects which they describe. For example, the ways in which we talk about school mathematics are not (and cannot be) a mere description of what happens in a mathematics classroom. Instead, they highlight certain aspects of this reality and ignore others, while they represent the objects about which they speak in non-neutral ways. Doxiadis's (2011a) analysis of Foucault's (1972) notion of discourse suggests that discourses refer to objects outside themselves (which would exist even if the discourse did not exist), but they do much more than that: they have a concrete role in the constitution of subjects, they produce knowledge, and they have a social role and are related with issues of power. To illustrate, let us consider the case of a student suggesting that 1 is the solution of the equation $x + 1 = 3$ while a mathematics class learn how to solve equations. Within the (socially constructed) system of this class, in which the equation $x + 1 = 3$ has a single solution ($x = 2$), this student response could be described as them not having mastered the equation solving algorithm, lacking previous knowledge, not being careful, experimenting with mathematical equations, learning by making a mistake, or it could even not be the focus of anyone's attention. All these different discourses refer to the very same event, but make different identities available for students, produce different kinds of knowledge about the student and their mathematical learning, and guide the teacher's, students', and researchers' actions in very different ways, thus creating different power relations.

For Laclau and Mouffe, the conceptualisation of discourse is related to questions of meaning, antagonisms, and hegemony. Their view starts from the point that meaning is not fixed. To illustrate the idea that meaning can never be fixed, Smith (2010) proposes the example of "Stalin" as a word whose meaning cannot be fixed: it could mean the heroic Soviet leader who defeated the Nazis, or the murderer who sent his opponents to Siberia. Meaning not being fixed means that there is no essence which assigns or can assign meaning to any sort of object or action. Illustrating this view, Laclau and Mouffe (1987) compare the actions of kicking a spherical object in the street and a ball in a football match and maintain that while the two physical acts might be the same, their meanings are different, since an object is a football only within a system of socially constructed relations with other objects. In their view, discourse is more than a combination of speech and writing; instead speech and writing are components of discursive totalities, which – to use our previous example – include the action of kicking.

Adopting the view about non-fixation of meaning, Laclau and Mouffe (1985/2001) conceptualise discourses as partial fixations of meaning. They maintain that "[a]ny discourse is constituted as an attempt to dominate the field of discursivity, to arrest the flow of differences, to construct a centre" (p. 112). Different discourses struggle to fix the meaning of privileged points called *floating signifiers*, while points whose meaning has been crystalised within a discourse are called *nodal points* (Jørgensen & Phillips, 2011). For example, learning is a floating signifier, in the sense that different learning theories, such as behaviourism and social constructivism, struggle to fix its meaning. Learning can also be a nodal point within a particular discourse, such as a constructivist discourse. Each discourse entails *moments* or, in other words, signs whose meanings have been fixed within the discourse. A sign whose meaning has not yet been fixed and has multiple potential meanings is called an *element* (Jørgensen & Phillips, 2011).

Another concept which is crucial in this conceptualisation of meaning is the concept of *hegemony*. Laclau and Mouffe (1985/2001) mention that hegemony reduces its distinct moments to the interiority of a closed paradigm. An example that could clarify this could be the idea that "mathematics should be relevant to students" as a moment which is hegemonic in Canadian mathematics education. The idea that mathematics should be relevant is almost commonsensical in the context of Canadian education. It takes its meaning from within a closed system, other moments of which would include the idea that mathematics should relate to experiences, that understanding something requires that it means something to the individual, as well as enactivist ideas. As this system is closed, it becomes hard to find space to think about it in ways that might problematise it.

The concepts of meaning not being fixed, antagonisms, and hegemony can be used to make sense of mathematics and mathematical truths. In doing so, I am disagreeing with the premise of dominant mathematics discourses which claim that mathematical truths are final as they are the results of rigidly produced arguments that go all the way back to a system's axioms. The claim which I am making is that mathematical discourses depend on a system of meaning and constitute historical products. This extends to mathematical ideas as commonsensical as "$1 + 1 = 2$": this very statement is a historical product. It might feel that the statement "$1 + 1 = 2$" is not really negotiable, in the sense that it is hard to see how it could be different. This is characteristic of ideas whose meaning has been crystallised. By saying that "$1 + 1 = 2$" is the result of antagonisms, I do not mean that thousands of years ago, people were disagreeing about whether one plus one equals two or three. I instead mean that this statement is the result of a system of meaning (which includes abstract constructs like 1, 2, and addition) which have been established through antagonisms. To illustrate, alternative systems of meaning, related to numbers and operations between them, could and do exist. For example, within a number system that employs a number other than 10 as its base, the statement "$1 + 1 = 2$" is wrong. Another example comes from some First Nations languages, in which "one" plus "one" is not always equal to "two" – there are multiple words used for each number, depending on what one is counting. The abstract notion of "1" or "2" regardless of what these numbers refer to, do not exist as such (Lunney Borden, 2013). In this sense, we see that natural languages partake in the construction of mathematical rationality, which suggests that rationality is a field in which the political operates – an idea which we will further explore later in this chapter.

So far, I have discussed what discourse is and I have explored concepts related to the notion of discourse – in what follows I engage in a discussion of discourse analysis as a theory and epistemology in the context of social studies and educational studies. One thing that discourse analysis is *not*, at least as I understand it, is the search for truths about societies through the study of how language is enunciated. Sometimes, reading discourse analysis studies may give this sense: the way in which grammar or syntax is used appears to reveal a truth about the intentions of the speaker, which is not visible in first sight. The discovery of subtle aspects in the way in which a discourse is enunciated might make us think that a truth about the speaker was there all along: they might have been trying to manipulate us one way or the other, or – even if they didn’t – some truth about society might be there that can explain why they spoke the way that they did. We can see this sort of approach in the work of Fairclough (e.g., 1995), who assumes the existence of a truth about society that can be approached by social theory. My view of discourse tries to leave aside this reductionism and reject the idea that something might be hidden behind the enunciated words that discourse analysis shall reveal. Instead, I view discourse analysis as an action which explores how reality is constructed through discourse, by examining discourses which are assumed to be in constant movement, in a constant becoming.

A major influence in this direction is the work of Foucault. Foucault (1972) talks about discourse analysis as:

> a task that consists of not –of no longer– treating discourses as groups of signs (signifying elements referring to contents or representations) but as practices that systematically form the objects of which they speak. Of course, discourses are composed of signs; but what they do is more than use these signs to designate things. It is this more that renders them irreducible to the language and to speech. It is this 'more' that we must reveal and describe. (p. 77)

As suggested in this extract, the task of discourse analysis is to examine what statements are accepted as meaningful or true at a particular time in history (Jørgensen & Phillips, 2011). The focus is not on whether specific statements are indeed true or false, but rather on why they are conceived of as such. Further, discourse analysis is not about the analysis of language itself, but rather about the relation between language and its sociopolitical context; the ways in which language designates “reality”.

A concept which derives from Foucault and is critical to discourse analysis is the concept of heterotopia. Jullien (2004) suggests that heterotopia assures that there is an alternative space of thought, which by being an “elsewhere”, can shake our thought. I see discourse analysis as doing exactly this: shaking our thought by exploring and showcasing how things can or could have been different. At the level of the analysis of a sentence, examining how it could have been enunciated differently shows how different discourses could have taken place. Similarly, when examining the relation between an enunciation and its cultural, social, and political context, there is space to examine how this relation could have been different and how it can be reimagined. Fairclough (1985) describes the task of critical discourse analysis in a similar way, when emphasising that interactions depend on naturalised ideologies and, hence, conducting critical discourse analysis on interactions involves denaturalising these ideologies by showing how social structures determine properties of discourse and vice versa.

In this subsection, I drew upon the previously discussed ideas about language creating reality to describe discourse as a partial fixation of meaning (Laclau & Mouffe, 1985/2001) and discourse analysis as an action which explores discourses in constant movement with the aim of a) positioning the enunciated words in relation to the social context in which they appear and b) exploring how these discourses could have been different. In the coming section, I refer to the notion of the imaginary (Castoriadis, 1975) and explore its connections with discourse theory.

3.1.3. Discourse Analysis and the Imaginary

In this section, I turn to the notion of "imaginary" (Castoriadis, 1975) to illustrate the importance of discourses in meaning making and structuring humans' conduct. I first explain the notion of the "imaginary" and I then turn to an example from literature, through which I illustrate how discourses about mathematics form an imaginary about mathematics, which in turn guides behaviour in relation to mathematics.

So, what is the "imaginary"? By "imaginary", I do not refer to the product of someone's imagination, in the sense that, for example, unicorns are imaginary. I rather refer to the images of mathematics and mathematical community, which are not explicitly painted through the class's practices at the symbolic level but are the product of the collective imagination which emerges through these practices (Castoriadis, 1975). Castoriadis' (1975) conceptualisation of the notion of "imaginary" refers to the attachment of meaning to social practices and institutions. For example, eating has the function of giving a person calories and nutrients; however, the specific eating habits of a society (e.g., which foods are consumed, how they are cooked) are not necessarily the result of some functionality – they relate to the ways in which a society gives meaning to its habits and labels animals as holy, dirty, or non-edible (Ktenas, 2021). With this in mind, the examination of imaginary mathematical community[14] and mathematics is concerned with the ways in which these images relate to a greater system of meaning.

In what follows, I turn to an example which concerns mathematics education and I discuss the notion of the imaginary in relation to it. In this extract from Davis and Hersh's (1981) book "The Mathematical Experience", a student and an ideal mathematician converse about the concept of mathematical proof:

[14] Imaginary communities are not unique to mathematics: for example, right now, I am writing these words having in mind what the mathematics education community looks like. My words emerge as a form of communication with my imaginary about the mathematics education community; an imaginary which has been formed by the subcommunities which I consider most familiar to me (e.g., the MES community) and the environments to which I have been mostly exposed (i.e., the Greek/Cypriot and the Canadian mathematics education communities). Poets, carpenters, comic-artists, cooks, scientists, and politicians – all do something similar.

> Ideal Mathematician: Well, this whole thing was cleared up by the logician Tarski, I guess, and some others, maybe Russell or Peano. Anyhow, what you do is, you write down the axioms of your theory in a formal language with a given list of symbols or alphabet. Then you write down the hypothesis of your theorem in the same symbolism. Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get the conclusion. That's a proof.
>
> Student: Really? That's amazing! I've taken elementary and advanced calculus, basic algebra, and topology, and I've never seen that done.
>
> Ideal Mathematician: Oh, of course no one really does it. It would take forever! You just show that you could do it, that's sufficient. (pp. 39–40)

Through this example, we see an instance of the discourse of logicism, the philosophical current which attempted to establish the foundation of mathematics in logic, thereby aiming to ensure the truth of all theorems (Hacking, 2014). This discourse creates an imaginary of a proof, according to which a proof consists of a deductive series of steps that start from the hypothesis and end to the conclusion, using the theory's axioms and the rules of logic. Through this image, meaning is attached to the practice of proving; proof is conceptualised as a vehicle towards *truth,* which manages to deductively connect a mathematical statement with the very axiomatic assumptions of a mathematical theory. Yet, as both the student and the ideal mathematician note no one really does this. But even so, this imaginary plays a very material role of constructing an image of mathematics and guiding individuals' behaviour by comparing their mathematical behaviours to this (unpractised) ideal.

In this section, I presented some ideas from discourse theory. More specifically, I discussed the discourse analysis premise that "language constructs reality", I described the notions of discourse and discourse analysis from the perspective of Laclau and Mouffe (1987), and I finally connected discourse analysis with the notion of the imaginary. In the coming section, I discuss the concept of power – which I have already tackled upon in relation to discourse – and I explore its relation to knowledge from a Foucauldian perspective.

3.2. Power and Knowledge

In this section, I discuss Foucault's conceptualisation of power and knowledge, exploring some implications of this view in the context of mathematics. Power is an important concept of this thesis for two reasons. First, as already discussed in the previous section, discourse analysis examines how discourse relates to issues of power, among other ways through the notion of antagonisms. Second, the notion of the political, which I will explore in more detail in the following section, is conceptualised as a space of power.

Foucault's conceptualisation of power and of the power-knowledge relation has tremendously influenced post-structural thinking by offering a radical alternative to previous conceptualisations, including liberal and Marxist discourses (Walshaw, 2007). In liberal discourses, power is equated with the rule of law (Walshaw, 2007) and is conceptualised as actors' intrinsic capacity to influence other people's behaviours (Valero, 2008). In Marxist discourses, power is held by certain classes (Walshaw, 2007) and is viewed as structural imbalance between groups of people (Valero, 2008). In both cases, power is considered to be negative, repressive, and acting on subjects which preexist the relations in which they participate (Walshaw, 2007). Foucault challenges these views by conceptualising power as relational, productive, and constitutive of subjectivities.

For Foucault, power is not "an intrinsic and permanent characteristic of social actors", but rather "a relational capacity of social actors to position themselves in different situations, through the use of various resources" (Valero, 2008, p. 53). For example, with regards to classroom power relations, power is not possessed by the teacher who can raise their voice, assign grades, or send a student to the principal's office; a student raising their hand, doing or not doing their homework, speaking, listening to, or staying silent while the teacher or a classmate speaks, and throwing their rubber when the teacher asks who has done their homework are all actions through which power is exercised. In other words, both the teacher and students exercise power by making use of resources that are available to each of them; power relations are not stable or exclusively depending on the actors' social positions but are instead in constant transformation. Moreover, power does not act on pre-constituted subjects; the teacher and students are constituted as subjects through the exercise of power.

A major idea in Foucauldian thought is that power is productive. This means that power is not merely or even primarily a repressive force but a "productive network which runs through the whole social body" (Foucault, 1980, p. 119). It "'conceals', 'masks', 'censors', and 'abstracts'. It also 'produces'; it produces reality; it produces domains of objects and rituals of truth" (Pinar, 1985, p. 212). In Foucault's (1980) words:

> What makes power hold good, what makes it accepted, is simply the fact that it doesn't only weigh on us as a force that says no, but that it traverses and produces things. It needs to be considered as a productive network which runs through the whole social body, much more than as a negative instance whose function is repression. (p. 119)

In other words, power is not simply oppressive, but it instead produces aspects of the world, knowledge, and subjectivities. Some philosophers, like Baudrillard and Donzelot, have criticised this approach for not being critical enough against power (Doxiadis, 2015). Doxiadis (2015) himself disagrees with this view and suggests that Foucault's conceptualisation of power is critical exactly because, through the notion of productivity, it rejects the "either for it or against it" view about power. In this sense, suggests Doxiadis, Foucault's critique is positive because it neither denies the limits of power nor their transgression; it constantly questions the need, universality, and mandatory character of these limits – a process which can *potentially* lead to their transgression. As I understand Foucault's theory, exploring the products of power enriches the political criticism that can be addressed to power. Examining and critiquing power on the basis of what it produces and how it produces it (and not on the basis of who has it) allows us to think what power does and how this makes it accepted. For example, let's think about the power in a classroom, in which the teacher is the one who decides how students spend their in-class time. Thinking about who "has the power" is not very interesting: it does not tell us much about why students accept this power relation, how this power relation influences the teacher's or students' subjectifications and experiences, nor does it help us think if this power imbalance should change (and if so, how it could). On the other side, paying attention to what this power relation produces (i.e., how children subjectify themselves; how children "learn things" in this process; how the teacher gets justified to exercise this kind of power for the children's own "good") can help us think about the nuances of this power exercising and can provide us with a richer space to think about its political implications.

Besides being productive, power does not always take the form of an external, oppressive force, which forces individuals to act in particular ways. The concept of governmentality illustrates this, by capturing how social control is achieved when individuals self-control their conduct according to rules which are presumed to be rational, commonsensical, and to serve the "common good" (Walshaw, 2007). Žižek's example about a boy having to visit his grandmother (e.g., Žižek, 2007) can illustrate the concept of governmentality (although Žižek uses this example to highlight different aspects of power and authority). Žižek distinguishes between the "old-fashioned authoritarian" father who tells his child "I don't care how you feel. Just do your duty, go to grandmother and behave there properly!" and the "'post-modern', non-authoritarian" father who says: "You know how much your grandmother loves you! But, nonetheless, I do not want to force you to visit her – go there only if you really want to!" (p. 92). In the latter case, we can see how power is exercised, in a way that the child learns to do what he is expected to, along with learning to want what is expected of him. "Visiting the grandmother", here, is not forced externally, but is gradually taken up by the child themselves.

A major implication of the Foucauldian notion of power is the way in which power relates to knowledge. Within the Foucauldian framework, the long tradition within which power and knowledge are treated as opposing categories is abandoned (Foucault, 1979). Doxiadis (2011b) maintains that most of the progressive critiques to power (e.g., humanism, enlightenment, Althusserian distinction between science and ideology) outline a negative relation between power and knowledge, through which knowledge is either discredited or utilised for the interests of power. Foucault (1979) challenges this view by counter-proposing that the relation between power and knowledge is positive: "power and knowledge directly imply one another" (Foucault, 1979, p. 27). This means that knowledge can only be constituted within power relations and that power always produces knowledge. To exemplify this cyclical relation, we can think of mathematical knowledge. Mathematical knowledge is produced within particular power relations crystalised in institutions such as university mathematics departments and structures such as academia. At the same time, within mathematical knowledge, particular problems emerge as important and particular sub-disciplines as promising, thus transforming the power relations related to knowledge production and distribution. Hence, knowledge is not a matter of true or false and knowing does not mean gaining access to an independent reality or truth; knowledge is constituted within power relations.

So, power, from this perspective, is a useful theoretical concept for investigating the political character of mathematics. The cyclical relation between power and knowledge can provide a framework for exploring how mathematics, as a form of knowledge, relates to issues of power and, as such, is also political. In the following section, I conceptualise the notion of "the political" in more detail.

3.3. The Political

In this section, I conceptualise the political, drawing upon a political theory trajectory, mainly attributed to theorists like Mouffe, Laclau, Beck, and Lefort, who do not reduce the political to politics but instead think of it as having ontological priority (Stavrakakis, 2000). As discussed, within mathematics education as a research field, there is not a unified understanding of the notion of "the political". This is not at all surprising if we consider the wide range of conceptualisations of "the political" in political theory itself. In a genealogy of the notion of "the political", Marchart (2007) identifies two trajectories in contemporary thought which conceptualise the "political" in different ways: on the one hand, the followers of Hannah Arendt treat the political as a space of freedom, while on the other hand the followers of Carl Schmitt conceptualise it as a space of power, conflict, and antagonism.

This thesis adopts the conceptualisation of the "political" proposed by Chantal Mouffe, whose work is positioned in the latter trajectory. Through a discussion of her following quote, I will explore some implications of her view which are important for this thesis and I will start exploring the difference between mathematics being political and mathematics relating to politics:

> by 'the political' I mean the dimension of antagonism which I take to be constitutive of human societies, while by 'politics' I mean the set of practices and institutions through which an order is created, organizing human coexistence in the context of the conflictuality provided by the political. (Mouffe, 2005, p. 9)

In this quote, Mouffe (2005) makes a distinction between the political and politics. Politics refers to what is widely understood by "conventional politics", for example the processes that relate to the parliament or elections. To illustrate this idea with regards to mathematics, the relation between mathematics and politics is evident in how during the 2018 Ontario elections, the conservative party of Ontario ran on a platform which declared that children and teachers in Ontario are not good enough at mathematics, while later – as the Ontario majority government – legislated that teachers need to pass a mathematics proficiency test in order to get their teaching licence. Another example of the relation between mathematics and politics at a non-parliamentary level is when a university mathematics department assembly decides which courses are going to be obligatory/elective for their mathematics program. On a different note, discussions about the mathematical underpinning of an electoral system also concern the intersection of mathematics and politics. In all these examples about the relation of mathematics and politics, the focus is on the practices and institutions which organise human coexistence. Barwell (2020) refers to this idea as "politics is mathematical". Mathematics being political refers to something more general.

The political is conceptualised as a space in which antagonisms occur and which constitutes a wider field than politics. Antagonisms occur when discourses collide, struggling to fix the meaning of floating signifiers (Jørgensen & Phillips, 2011). For example, in mathematics education discourses, mathematics is a floating signifier, in the sense that different discourses struggle to fix its meaning and its limits. An ethnomathematical discourse and a cognitive theory discourse would (typically) fill the meaning of mathematics differently, thus depicting mathematics education differently. Nevertheless, as different discourses antagonise each other, meaning can never be fixed (Laclau & Mouffe, 1985/2001). Sometimes, antagonisms can be resolved through a hegemonic intervention, or a perspective which has earned social consensus (Jørgensen & Phillips, 2011). However, this is only a temporary state. Laclau (2001) defines hegemony as "the type of political relation by which a particularity assumes the representation of an (impossible) universality" (p. 5).

This conceptualisation of the political has an ontological significance: Mouffe (2005) claims that the political is constitutive of human societies. This is an ontological statement which suggests that it is through discourses and the antagonisms between them that meaning is produced. Valero (2018) refers to something similar when writing:

> But also, when we are in society, we negotiate all the time what is right or wrong and what we take as true, as much as we negotiate what we take as mathematics education and why. And such struggle and negotiation are political because the privileging of certain values and ways of seeing the world—mathematics and mathematics education included—over others is connected to how power creates orders and classifications that, through knowledge, make us who we are and how we engage in the world. (p. 106)

In this way, the notion of the political (as distinct from politics) constitutes an embracing of the non-existence of an objective reality and signifies a turn towards post-foundationalism (Marchart, 2007). Stavrakakis (1997) describes this turn as a transition from a reductionist logic towards a logic of articulation:

> since social identities do not arise from the (class or other) essence of (individual or collective) subjects, they can only be a product of construction, a construction at the level of discourse, a construction that articulates heterogeneous elements and gives them new meaning. (p. 24; my translation)

In this sense, all things mathematical are constituted as such at the level of discourse. This means that, contrary to what Plato would argue, there is no essence in mathematical relations: mathematical truths – like all other kinds of truths – are constituted as such through the ways in which we talk about them and, thereby, their meaning is constructed.

Overall, the notion of the political, as discussed in this section, goes beyond reductionist logics and adopts a constructionist lens. It does so, by distinguishing between the political and politics and by putting antagonisms in the forefront. From this perspective, mathematics being political refers to it being a terrain, in which power operates, meanings are negotiated, and antagonisms (can) take place, while at the same time other ways of meaning making are excluded. As (academic) mathematics is to a large extent a stabilised form of knowledge (in the sense that, at least currently, its foundations are not the object of antagonisms), mathematics being political also signifies that this stabilisation is neither a neutral, nor a natural state. Mathematical knowledge and its associated practices are historically constituted: they are a product as well as part of classifications and organisations that regulate social life (Valero, 2018). In the coming section, I use the ideas discussed so far in this chapter to focus on mathematics and propose a framework for what it means for mathematics to be political.

3.4. Mathematics Being Political

With the exception of a few examples, the discussion so far on the political, power/knowledge, and discourse has not focused a lot on mathematics. In this section, after a short discussion of my ontological and epistemological assumptions with regards to mathematics, I use these notions to construct a framework for what it means for mathematics to be political. I propose a way of exploring the political features of mathematics by discussing three features of mathematics which are inextricably related to issues of power: a) mathematics entails values (Bishop, 1991; Ernest, 2018, 2020; Skovsmose, 2020), b) mathematics puts forward particular rationalities (Kollosche, 2014; Walkerdine, 1988), and c) mathematics has a formatting function (Barwell, 2013; Skovsmose, 1994b, 2000). These three axes are related to the axiology, epistemology, and ontology of mathematical knowledge respectively.

3.4.1. What I mean by mathematics

Considering the wide range of epistemological and ontological views on mathematics, in this section, I discuss what I mean when I refer to mathematics or mathematical discourse. In a nutshell, I propose that mathematics is a social practice in the sense that it does not exist autonomously, but it is performed by people. Moreover, I suggest that mathematics is not a single body of knowledge, but it is a practice that can have unique characteristics depending on time and place. Finally, I suggest that mathematics as an abstract concept also exists through the ways in which people discursively construct it.

I use the term "mathematical discourse" synonymously with "mathematics", thereby maintaining that mathematics is a discourse. This does not mean that mathematics is a language, like Hindi or Romani are. Instead, it is a discourse in the material sense that Laclau and Mouffe (1985/2001) write about it: mathematics is a discursive practice which may include sitting in a desk, reading a peer-reviewed article, using a software to make a calculation, and enunciating results in the form of theorems, in the case of a mathematical researcher; sitting in a desk, reading the textbook, using manipulatives, and solving mathematical exercises and problems on handouts in the case of a student. This conceptualisation of mathematics as a discourse is useful because it helps us think how mathematics is performative: "as discourses can form world-views and actions, so can mathematics" (Skovsmose, 2020). Moreover, contrary to a Platonic view that argues that mathematics has an independent existence, mathematics is conceptualised as something that people do, as a social practice.

In this sense, mathematics is also not *one* thing or *one* body of knowledge: while we often think of mathematics in particular ways, as if it included certain aspects of knowing, the options are unimaginable. As Valero (2008) puts it:

> [W]hat we understand by mathematics is far from being a unified body of knowledge determined by the practices of professional mathematicians, but rather a series of "knowledges" and "language games" bounded to a diversity of practices, all of which have a family resemblance. (pp. 14–15)

Drawing from Wittgenstein's ideas, Valero maintains here that mathematics is a series of knowledges. In fact, what we consider to be mathematics has changed and is changing depending on time and place. For example, the study of ethnomathematics has widened what we consider mathematics to be. As Pais (2013) puts it ‘"[e]thno" shifts mathematics from the places where it has been erected and glorified (university and schools) and spreads it to the world of people, in their diverse cultures and everyday activities.’ (p. 2).

While I consider mathematics to be a social practice that cannot have independent existence from the people doing, experimenting with, and talking about it, I also argue that mathematics is something more than whatever happens within a certain interaction that we may call mathematical. As mathematics has evolved into being a body of knowledge, people also talk about, analyse, or criticise "mathematics" at an abstract level. This means that mathematics also exists as a body of knowledge that is often disentangled from the people working with it. I argue that mathematics *exists* as an abstract concept, in the sense that people talk about it and thereby construct it at the discursive level. In this sense, mathematics is a floating signifier (Laclau & Mouffe, 1985/2001) whose meaning is in a constant negotiation as different discourses try to fix its meaning.

3.4.2. Values of Mathematical Knowledge

In this section, I examine how mathematics is political by exploring the values associated to it. Bishop (1991) conceptualises values as "the ideals and the principles lying behind the actual language or symbols developed by a culture" (p. 61). Following a discourse analysis approach and Bishop's conceptualisation, one way to think of values is as discourses which frame what is good, desirable, or important in relation to mathematics. The following discourses could be examples of values: short calculations are more desirable than time-consuming ones, only true (and not false) statements are the object of mathematics, mathematics is important, or mathematics is for all. Sometimes, these discourses are articulated in a way that assumes them being common ground, while other times they are more contested. Also, sometimes they are explicitly articulated, while other times they are assumed or go unnoticed. Besides as discourses in their own sake (e.g., the discourse of importance), values can also be conceptualised as nodal points for mathematics discourses or "privileged sign[s] around which other signs are ordered" (Jørgensen & Phillips, 2011, p. 26). For example, the "importance" of mathematics is a nodal point in mathematics curriculum discourses because mathematics being discussed as important is central in many discussions regarding what should be taught or how it should be taught.

The conceptualisation of values as nodal points makes a series of assumptions that require some discussion in their own sake: first, mathematics is neither value-free nor inextricably good. The idea that mathematics is value-free or harmless has been hegemonic within modernist discourses. However, it has been profoundly challenged by scholarship in mathematics education. For example, Bishop (1991) maintains:

> It is all too easy […] to become totally engrossed in the symbolic and manipulative aspects of mathematics through the technique curriculum, and thereby ignore values entirely. This of course does not mean that we don't teach values at present. On the contrary, I am convinced that we do teach them, unconsciously, implicitly and, of greater concern for education, uncritically" (p. 61).

Indeed, values are present when doing mathematics, from textbooks in Nazi Germany that ask students to calculate if there is enough poison gas to kill a city (Shulman, 2002) to optimisation problems that arise from "real-world situations" (Ministry of Education, 2007). Hence, if mathematical discourses entail values, mathematics cannot be inextricably good. Ernest (2018) elaborates on this idea by outlining a) how applying thinking from pure mathematics into social and human issues can be damaging, b) how the applications of mathematics to society can be detrimental, and c) how learning mathematics can negatively impact some students.

Second, values are integral parts of mathematical discourses. This position is in contrast with the idea that mathematics is pure and value-free and values may be externally attributed to it. François (2019) distinguishes between values which are intrinsic to pure and applied mathematics and values students and teachers add in mathematics. In this work, following the conceptualisation of discourse as explained earlier, I try to merge the distinction. When a mathematical utterance occurs, the utterance carries values or opens the possibility of certain values to be associated with it. To put it differently, I am arguing that, to examine mathematical values, we should not (and cannot) extract mathematics from the utterances within which it exists. After all, it is exactly through this process that mathematics is often considered value-free. Bishop (1991) makes a similar assumption about mathematics, when identifying "the principal values associated with Mathematics", while proposing that these values are neither universal nor associated with mathematics as it exists in any culture. Turning to historical and cultural manifestations of Western mathematics, Bishop categorises mathematical values as ideological (i.e., objectivism and rationalism), sentimental (i.e., control and progress), and sociological values (i.e., openness and mystery).

Third, mathematics can be practiced through different, and sometimes opposing, values. Even a Western mathematical rationality does not imply deterministic commitment to specific values. To illustrate, Fasheh (1982) proposes that while the study of deductive thinking has often supported a "dogmatic" belief in absolute truths, it can also be used to create awareness of mathematics as a human construct or to recognise similarities among objects that look different at first sight. Similarly, abstraction can imply valuing detachment (e.g., Abtahi, 2020; Ernest, 2018), but the development of (highly abstract) non-Euclidean geometries has also helped challenge the idea of an existing absolute truth (e.g., Panagiotatou, 2013). While mathematics may be practiced under different values, we may also go one step further and argue that mathematical values are also a space of antagonisms. Different discourses strive to establish different mathematical values and antagonise each other in doing so. For example, in STEM education literature, mathematics might be considered important because it can contribute to the US's national security and economic competitiveness (e.g., Espinosa, 2011; Palmer et al., 2011). However, within the context of an Indigenous community, the importance of mathematics might lie within its role in the revitilisation of languages and cultures (e.g., Trinick, 2019). These two discourses (among many possible others) bear different values in relation to mathematics, have emerged within different knowledge systems, and guide action in different ways.

Fourth, the hegemonic mathematical values are related to the specific social, political, and economic relations in which Western mathematics has been and is being developed. Through a genealogical analysis, Kollosche (2014) shows how bureaucracy and calculation developed together and share a common way of thinking, while logic and calculation have acted as technologies of governance. Within the current framework of Western, capitalist societies, mathematics continues to play the role of an important activity, by having been established as a powerful tool for making sense of the world and for distributing people in different positions in society. Compulsory mathematics education in schools in the "Western world" was imposed along with the proliferation of white-collar work (Kollosche, 2018b), while mathematics is closely related to maintaining the existing economic structures of our societies, by playing the role of the gate-keeper in prestigious and high-paying jobs (Stinson, 2004). Moreover, mathematics education has been central in reproducing a myth of meritocracy, while legitimising the excluding aspects of capitalist education (Pais, 2014). In this sense, Western, capitalist values are hegemonic in mathematics.

In conclusion, values are integral parts of mathematical discourses, but values are not universal. Mathematics is a space of antagonisms with regards to the ways in which its values are negotiated. At the same time, Western mathematics' values have become hegemonic as dominant mathematics is assumed to provide a pure way of reasoning, be powerful in modelling and solving problems of our world, and play a central role in the maintenance of our current economies. Therefore, using Laclau and Mouffe's (1985/2001) terminology, we can argue that there has been a fixation of meaning, which albeit partial and not permanent, is deeply rooted both in the way that (Western) mathematics has historically been developed and in the ways in which it is intrinsic to the current social, economic, and political circumstances.

3.4.3. Mathematical Rationality

A second way in which mathematics is political is that it constructs and puts forward a particular rationality. In Foucauldian thought, at a given time and space, some discourses are possible and some others are not (Foucault, 1972). Similarly, truth is neither transcendental nor independent of the society within which it is produced; it is instead related to power:

> Each society has its regime of truth, its "general politics" of truth; that is, the types of discourse which it accepts and makes function as true; the mechanisms and instances which enable one to distinguish true and false statements, the means by which each is sanctioned; the techniques and procedures accorded value in the acquisition of truth; the status of those who are charged with saying what counts as true. (Foucault, 1980, p. 131)

This perspective is useful for thinking about mathematical truths. Within any society, culture, or discipline, there are mechanisms, rules, norms, tools, techniques, and procedures with which we produce, judge, and accept mathematical truths. This produces various rationalities within which people do mathematics. For example, Wagner explains that "enough" potatoes is the quantity descriptor used by a Mi'kmaw elder's mother to explain how many potatoes are needed from the garden: not all potatoes have the same size, so asking for a particular number of potatoes is "irrational" (Abtahi & Wagner, 2016). This act implies a very different kind of mathematical rationality than the one evident in school mathematics where a number would be the expected answer. Similarly, different rationalities can be identified in different traditions of doing mathematics. For example, in the tradition of Indian mathematics indirect proofs are not considered valid (Joseph, 2019). Furthermore, even in what is labelled "Western mathematics", different epistemologies have given rise to different rationalities. Intuitionistic mathematics is a typical example of this: in the intuitionistic epistemology, methods like the law of excluding the middle and proof by contradiction are not valid (Anapolitanos, 2009). Hence, there is not one kind of rationality which can be identified as the only mathematical one: different mathematical rationalities can be identified in different (ethnical, disciplinary, and otherwise) cultures.

Nevertheless, mathematics is often related to a particular rationality, or a system of acceptable rules for creating and judging the truth of claims and arguments (henceforth, catachresticly referred to as Western mathematical rationality). In fact, mathematics is widely considered as a prototype of rationality, as is evident by the mainstream idea that "[t]he Other of mathematics is uncertainty, irrationality, out of control, madness, and so on" (Walkerdine, 1988, p. 199). In Gutiérrez's words (2013):

> Mathematics, as a rational, universal, and logical discipline is located in a unique position to be the ultimate arbiter of truth. Its ability to model the real world and to maintain a kind of internal certainty gives evidence of this privileged and earned position. Something proven with mathematics is seen to have final say. (p. 47)

A central element of Western mathematical rationality is abstraction, which guarantees the (desired) universality of mathematics (Joseph, 2019), by reducing a chaotic situation to a single, stable mathematical case (Kollosche, 2018b). Thereby, a regime of truth is constructed, within which whatever is proved with mathematics is irrefutable, because it does not depend on context-based information, facts, or methods: "decontextualization is an effect which renders this system of statements as statements of fact precisely because it claims a universal applicability – a rationality devoid of any content which can describe and therefore explain anything" (Walkerdine, 1988, p. 187). This is of course profoundly political: mathematics elevates itself above and beyond all others and claims objectivity and neutrality, while it is only a particular action of representing the world (François & De Sutter, 2004).

The rationality of mathematics can be explored through Francois Jullien's (2012) notion of "prototypical" thought. Jullien, a philosopher, hellinist, and sinologist, examines the role of the prototypes in Western thought and contrasts this with Chinese thought. Prototypical thought consists of setting up an ideal form, treating it as a goal, and trying to figure a plan to achieve it. Jullien (2004) talks about this idea as follows:

> We set up an ideal form (eidos), which we take to be a goal (telos), and we then act in such a way as to make it become fact. It all seems to go without saying – a goal, an ideal, and will: with our eyes fixed on the model that we have conceived, which we project on the world and on which we base a plan to be executed, we choose to intervene in the world and give a form to reality. And the closer we stick to that ideal form in the action that we take, the better our chances of succeeding. (p. 1)

For Jullien, mathematics is a typical example of looking at the world in a prototypical way. Mathematics models the world according to the tools that it has available, explains the world according to this model, and finally intervenes in the world executing the model. Modelling and acting upon climate change (e.g., Barwell, 2013) is an example of this prototypical practice.

Similarly, the kind of mathematics included in school curricula imply particular kinds of rationality. For instance, the German curricula include topics which “show the power of logic” and exclude others, like non-Euclidean geometries which “could threaten [mathematics’] status” (Kollosche, 2014, p. 1065). Moreover, any mathematics curriculum promotes a particular kind of reasoning that emerges from the way that the curriculum conceptualizes mathematics. At the same time, different curricula can assert different rationalities. For example, the one right answer of mathematics and the requirement for everybody to get the same result is connected to the existence of a path to this solution, any wrong turn out of which would lead to a wrong result (Kollosche, 2018b).

Overall, there are multiple mathematical rationalities, which can be identified in different cultures. However, (Western, academic) mathematics has been associated with the existence of a single “rationality”, to the extent that Western mathematical rationality has been elevated to a prototype of rigour. Of course, when dominant mathematics becomes a superior form of rational discourse, other ways of meaning making (either mathematical or otherwise) become of lesser importance, meaning that a hierarchy of epistemologies is being constructed.

3.4.4. The Formatting Function of Mathematics

A third way in which mathematics is political is that it has a formatting function, by which I mean that the ways in which we talk about mathematics and the ways in which we do mathematics formats the world. In discourse theories, descriptions not only capture elements of the world around us, but the ways in which we talk about things also form the social world (Jørgensen & Phillips, 2011). Similarly, mathematical descriptions and representations play a crucial role in formatting society: in attempting to explain or model the world around us, mathematics shapes it (Skovsmose, 2000). A mathematical description of a phenomenon shapes a framework which provides particular alternatives on what is possible to be said, thought, and done. Even in the case of relations between physical objects, mathematical representations consist of singling out certain elements of a relation and representing them in a generalised form through a particular discourse (Walkerdine, 1988).

To illustrate this idea, Skovsmose (1994b, 2000) has developed the concept of the formatting power[15] of mathematics, suggesting that “in many cases, social actions and decisions are prescribed by means of mathematics” (Skovsmose, 2000, p. 4). Skovsmose’s focus is mainly on mathematisation, modelling, and invisible mathematics in technology. For example, airline booking systems use mathematics in order to maximise profit by ensuring an optimal number of passengers per flight, profoundly influencing the ways in which society is organised with respect to air travel (Yasukawa et al., 2012). The mathematics of the booking systems, albeit invisible to most of us, structures parts of the air-travelling activity, including that often flights get overbooked and some people lose the option to fly with their booked ticket.

While Skovsmose’s formatting function mostly focuses on how mathematics is part of technology, we can think of the formatting function of mathematics in more general terms. A series of concepts with which we think and act, and which regulate our behaviours have been formatted through mathematics. The idea that people’s intelligence can be measured and described through a single number (IQ) has been the result of the proliferation of statistical research (Valero, 2017): we can say that through its constructs (e.g., IQ), statistics has formatted the ways in which we think, act, and organise our lives in relation to intelligence. Similarly, the idea of an ideal body has been mathematically formatted through the development and usage of the Body Mass Index (BMI) concept (Hall & Barwell, 2021). The climate crisis has also been formatted through mathematics, as mathematics occupies a central role in the description, prediction, and communication of climate change: the description of climate change as a system-wide phenomenon is accomplished through statistical descriptions about temperature recordings, sea level, glacial melting, etc.; the prediction of climate change is attempted with advanced mathematics, through which models for prediction are developed and evaluated; and the communication of climate change through the media entails mathematics and requires mathematical and statistical literacy both for the production and for the interpretation of climate change related texts (Barwell, 2013). Similarly, the understanding of the current COVID-19 pandemic as a pandemic is related to how the World Health Organisation categorises the spread of illnesses by using mathematics. The underlying idea in all these examples is that the mathematical practices of scientists format aspects of human activities, by describing them and by prescribing solutions to them (e.g., by categorising people according to the description).

2 In later writings, Skovsmose and colleagues stand critically to the connotations of this notion. For example, Yasukawa et al. (2012) maintain that: “The problem, however, with this notion is the ease with which we slip into a view that it is mathematics as such which exercises power in some autonomous way. We reject the idea that mathematics can exist or have influence on the social sphere independent of any social agency” (p. 268). Adopting the same orientation in this thesis, I use the term “formatting function of mathematics”.

The concept of formatting is crucial for exploring the complex role of mathematics and mathematisation in society. Straehler-Pohl (2017) writes:

> The usage of the word "formatting" is indicative of the profound effects that Skovsmose attributes to mathematisation. When a practice is mathematised, this does not simply affect a change in our behaviour, as we have an additional source of information that we can rely on when making decisions. Instead, mathematization intervenes more deeply in the very structure of practice, re-organising the basic conditions our actions refer to. (p. 37)

For Straehler-Pohl (2017), the concept of the formatting function of mathematisation challenges the artificial distinction between mathematics and reality itself. Through mathematisation, the described and the description get inextricably merged. To illustrate, going back to the case of the climate crisis, the description, prediction, and communication of it are inextricably linked processes. Similarly, the description of human weight in terms of normal and abnormal goes hand-in-hand with the categorisation of humans as underweight, normal, overweight, and obese as well as with the regulation of their behaviour accordingly (e.g., by being instructed to visit a dietitian).

In conclusion, just like any discourse, mathematics formats the world, as it defines limits about what can be said and done, what is reasonable and not, etc. Describing the world through mathematics, as opposed to for example through poetry, makes particular representations and ways of thinking possible. In this way, mathematics becomes embedded to reality, often in invisible ways, thereby shaping reality.

3.4.5. Bringing it All Together

In this section, I constructed a framework for what it could mean that mathematics is political. By using the theoretical tools that I introduced earlier on in this chapter (i.e., discourse, antagonisms, power, the political), I proposed that mathematics can be thought of as political because it entails values, it puts forward particular rationalities, and it has a formatting function. As already discussed in the beginning of the previous section, the three axes of the framework which conceptualises mathematics as political correspond to the ontological, epistemological, and axiological dimensions of doing mathematics in the context of the two undergraduate courses. More specifically, following Killam (2013), the axis of values corresponds to the axiological aspects of knowledge, the axis of rationality corresponds to the epistemological aspects of knowledge (i.e., how we come to know what we – claim to – know), and the axis of the formatting function corresponds to the ontological aspects of mathematical knowledge (i.e., the assumed nature of "reality"). In the following section, I introduce the research problem and the research questions which are based on this framework.

3.5. Research Problem and Research Questions

My research project aims to examine what it means for mathematics to be political in the context of two undergraduate courses. The framework of mathematics being political discussed in the previous section constitutes a theoretical approach of this problem. In order to explore the problem at the empirical level, in the context of the two courses, I use the following research questions which emerge from the three axes of mathematics being political:

1) What values are manifested in the two classes?

2) What mathematical rationality/rationalities is/are constructed in the two mathematics classes and how are they similar or different?

3) What are the formatting functions of mathematics in the two classes?

3.6. Summary

In this chapter, I discussed the theoretical framework of this study, which is based on three concepts: the notion of the political as a space of antagonisms; power as productive and entangled with knowledge; and discourse as constitutive of reality. Building upon these theoretical concepts, I presented a framework which conceptualises mathematics as political on the basis of three axes: a) it is a space in which values are at play, b) mathematics puts forward particular rationalities, and c) mathematics has a formatting function. In turn, this framework provides the ground on which the research problem and research questions are based.

Chapter 4: Methodology and Research Design

In this chapter, I discuss the methodology and research design of this discourse analysis study. I start the chapter with a discussion of the ways in which ethnography and critical ethnography have influenced the study and an explanation of why this study is nevertheless not an ethnographic one. I continue by presenting issues related to the recruitment of participants, the description of sites and participants, and the description and rationale of the data collection methods. The chapter continues with a presentation of the ways in which I worked with the data once collected from transcribing them to developing a data analysis method based on various discourse analysis approaches. I continue with an exploration of ethical issues that have arisen as part of the study and I conclude this chapter with a discussion of the study's rigour. But before I engage in a detailed discussion of the study's methodological aspects, I provide a short description of the empirical study, so that it is easier for the reader to follow the rest of the chapter.

The empirical part of this study was conducted in two undergraduate courses, a Calculus course and an Introduction to Proofs course. I attended the two courses in the 2020 Winter Semester for six and five weeks respectively, until in-person classes were suddenly interrupted due to the COVID-19 pandemic. Besides the lectures, I also collected the courses' textbooks and documents and conducted interviews with the professors and a few students. To answer the research questions, I used a variety of discourse analysis theories, combined under the umbrella of Laclau and Mouffe's (1985/2001) discourse analysis theory and epistemology.

4.1. A Critical Ethnographic Influence in a non-Ethnography

This study was designed with a critical ethnographic approach as a lens but is nevertheless not an ethnography. Pole and Morrison (2003) describe ethnography as "[a]n approach to social research based on the first-hand experience of social action within a discrete location, in which the objective is to collect data which will convey the subjective reality of the lived experience of those who inhabit that location" (p. 16). The focus on subjective experience is based on cultural relativism which became a central principle of anthropology through the work of Boas; the premise that each culture needs to be understood in its own terms and not in a universal manner (Eriksen, 1995/2001). Critical ethnography has developed as a branch of ethnography concerned with social inequalities in social sites, social processes, and cultural commodities (Carspecken, 2015). Influenced by critical theory, it responds to and goes beyond approaches which either focus on structures ignoring human actors or focus on human actors ignoring structures (Anderson, 1989). Critical ethnography attempts to capture social action in a distinct location, paying particular attention to issues of power.

In the initial design of the study, the goal was to depict how the various actors in the course (i.e., professor, teaching assistants, and students) negotiate the different discourses about mathematics. In my research proposal, I mentioned:

> Working with the data will aim to produce a "thick description" (Geertz, 1973) of how mathematics becomes a contested space within the context of the two courses, as well as how particular discourses about mathematics are accepted, declined, negotiated, and established. (p. 21)

However, the time spent on site was shorter than initially expected and the variety of the collected data was less rich. My time in the field was shorter both because I was not able to start visiting the course until the 4th week for one of the classes and the 5th week for the other (as the project had not yet received ethics approval) and because I only visited the two classes until the 9th week of the semester due to the COVID-19 pandemic. Moreover, I did not collect any data from tutorials taught by teaching assistants, while students' voices have been included in the study much less than initially designed, as not many interviews and no focus groups took place. For these reasons, this study is not an ethnography. This is reflected in the following shift: initially I was aiming to examine the participants' views on the political aspects of mathematics and the ways in which the participants negotiated the professors' privileged discourses as well as other discourses that appear in the classroom. However, the collected data led me to primarily focus on the professors' discourses mostly by using (my interactions with) theory and mathematics education literature.

Nevertheless, some ideas from anthropology, ethnography in education, and critical ethnography have played a significant role in shaping this project. Two of these influences are the most important and palpable. First, the interviews that I conducted were not designed before entering the field. After spending some time in the field doing mostly passive observation, keeping a field journal, and reflecting on the courses' practices, I started conducting interviews, in which I focused on asking participants about these practices (Carspecken, 1996). As the design of the interviews was based on the data collected in the field (and my interactions with them), I was able to include concepts which came up in the field and not considered at the design stage (e.g., beauty of proofs, knowledge production), following an ethnographic spirit of staying flexible and open to meanings and practices which emerge in the field.

The second way in which ethnographic literature significantly impacted the study was in helping me to stay critical to the affordances of research questions and stay open to adjusting them during the study. In fact, two of the research questions changed during the data collection and analysis. The research question about values ("What values are manifested in the two classes?") was initially focusing on the importance of mathematics and read as follows: “How is mathematics constructed as an important activity in the two courses?”. While collecting data, transcribing the audio-recordings, writing up the fieldnotes, and compiling my journal, I noticed that mathematics was not necessarily constructed in the classroom discourse as important per se, but in much richer ways, such as useful, beautiful, precise, and so on. Furthermore, while certain aspects of mathematics were constructed as important (e.g., mathematics being precise is constructed as important in one class), this importance was referring to specific manifestations or characteristics of mathematics rather than mathematics itself. For these reasons, transforming the research question to focus on values opened up the possibility of exploring richer aspects that make mathematics valued than merely its importance.

Similarly, the question about the formatting function of mathematics changed twice during the analysis. In its initial version, the question read: “In what ways is mathematics captured as having a formatting role in the settings of the two courses?”. With this wording, I mostly referred to the ways in which mathematics formats the physical/social world around it, closely following Skovsmose’s (1994b, 2000) concept of formatting. However, as the data were collected in an educational setting, they were not really about black-boxed mathematics, nor were the professors talking about mathematics in this way when referring to the relations between mathematics and the physical/social world. Therefore, the question was transformed and its second version was: “In what ways is the relation between mathematics and the physical/social world captured in the context of the two courses? In what ways is mathematics captured as having a formatting function?”. In this way, I aimed to include the exploration of different relations about mathematics and the physical/social world. However, as I started working with the data from the lens of this version, I felt (epistemologically) uncomfortable with the clear-cut distinction between “mathematics” and “the world”; a distinction that I needed to make when identifying data which referred to “the world”. To think about these epistemological tensions, I found useful the work of Laclau and Mouffe (1985/2001) and particularly their idea that everything is discursive and, hence, it is impossible to refer to a reality (or in my case a “world”) as if it existed independently of the discourses that talk about it. At the same time, I noticed that the idea of formatting about this “physical/social world” was very similar to the ways in which definitions and theorems formatted their mathematical objects as well as the ways in which the courses’ mathematical practices formatted the people doing mathematics. In this way, I started considering the notion of “formatting” in a wider sense, similarly to how discourse analysis frames formatting (which has been analysed in Chapter 3). This led to the third and final version of this research question which reads “What are the formatting functions of mathematics in the two classes?”.

4.2. Recruitment of Participants

In terms of identifying potential sites for the study (and thus professor-participants), the goal was to collect data from two Ontario undergraduate mathematics courses. The inclusion criteria for the professors were that they teach an introductory mathematics course in English, in an Ontario University. At the design stage, I planned for one of the courses to be intended for mathematics majors and the other one for non-STEM majors. However, this was not actualised because this was not the character of the courses whose instructors expressed interest to participate in the study.

In order to identify potential sites for the study, while awaiting ethics approval from the University of Ottawa, I emailed professors in three Ontarian Universities who, according to their institutions' websites were going to be teaching an introductory mathematics course during the Winter 2020 semester. The goal of this informal communication was to identify interested professors from multiple universities and, thus, have multiple alternatives for finding a site, in the case that I was not granted institutional access in one university. I emailed the professors with a brief overview of the study, asking them whether they would be potentially interested to participate. A first come/first serve basis selection method was planned to be adopted in the case that multiple professors expressed interest to participate in the study. Overall, I contacted twenty-one (21) professors, five (5) of whom declined, eleven (11) of whom did not respond, and five (5) of whom expressed an interest in the study and agreed to be contacted again provided that the study would receive ethical clearance in their university. A detailed distribution of these professors in the different universities is presented in table 4.1.

After the study received ethics approval from the University of Ottawa, I applied for organisational permission in the three universities, whose professors I had previously contacted. I also applied to one more university for which there were no publicly available data about the courses' instructors, which meant that a permission from their Research Ethics Board was required to contact professors. Once granted organisational permission by the first university, I contacted again the professors who had kindly agreed to be considered with the formal recruitment material (see Appendix 3). Of the three professors whom I contacted, two of them agreed to participate. These two professors were the participants of the study.

	INFORMAL COMMUNICATION				ETHICS	FORMAL RECRUITMENT	
	Professors contacted	Declined	Not answered	Agreed to be	Received Institutional Permit	Professors Declined	Professors Accepted

				contacted again			
University A	9	1	5	3	Jan 16	1	2
University B	4	1	2	1	Jan 24	Professors not contacted (participants already selected from University A)	
University C	8	3	4	1	Applications were never submitted[16]		
University D	Professors were not contacted because institutional access was needed first						

Table 4.1: Process of Professor-Participants Recruitment

Through the process described above, the sites and the professor-participants of this study were determined. The design of the study included the recruitment of teaching assistants (TAs) from each course. However, this was not achieved. In one of the courses, the professor did not feel comfortable with me contacting teaching assistants, considering that they were not working under his supervision, but under the supervision of the course coordinator. As a result, I did not make any effort to contact teaching assistants in that course. In the other course, it was not possible to contact all teaching assistants to invite them to participate (as was initially planned), because of the course's structure (the course included multiple professors, one course coordinator, and many teaching assistants answering to the course coordinator – a different person than my study's participant). In this case, the professor contacted a few teaching assistants whom he knew personally or had collaborated with in the past, passing my information to them. If they wished to participate, the teaching assistants could contact me directly (so that any pressure from responding to the professor would be mitigated). No teaching assistants contacted me and, as a result, the study ended up not including any teaching assistants.

Finally, the design of the study included student-participants to participate in interviews and focus groups. In order to recruit students, during my first visit in the class, I talked to them explaining what my study is about, that I will be visiting their class, and asking them to contact me if they wished to participate in an interview or a focus group. I also handed out this information to them during the break (see Appendix 3). On a later occasion, I made a second announcement at the beginning of the class asking them to contact

[16] Although I started the applications to Universities C and D, I never submitted them because it was becoming increasingly likely that I would be able to find participants in Universities A and/or B in time.

me if they wished to participate in an interview. No students expressed interest to participate in the focus groups, so no focus groups took place. Regarding the interviews, in the initial design, I aimed to conduct up to ten (one hour) interviews with ten individual students for each course. However, only three interviews took place between the two courses. All students who emailed me expressing an interest to participate, participated in an interview.

4.3. Sites and Participants

The study took place during the 2020 Winter Semester at two courses in one university in Ontario: an Introductory Calculus course and an Introduction to Proofs course. Alex[17] is the instructor of the Calculus course. This course is intended for students who intend to take a path in sciences and it is a mathematical requirement that many students have to fulfill as part of their diverse programs. The class is attended regularly by 60-80 students, who meet once a week for Alex's lecture. From Alex's class, I interviewed one student, Anthony. Although I had established contact with two more students who had expressed interest in potentially participating in an interview, the sudden interruption of in-person classes due to COVID-19 did not make this possible.

Chris is the instructor of the "Introduction to Proofs" course. The course is regularly attended by around 150 students. This course is selected mostly by students who wish to enter the computer science program, as to do so, it is required that they earn a good grade in the course compared to their classmates. Students can also attend office hours held by Chris once a week, some of which I attended too. From Chris's class, I conducted interviews with two students, Kevin and Vijin.

4.4. Data Collection Methods

I started attending Alex's class in the 4th week of the 2020 Winter Semester and Chris' class in the 5th week. Although data collection was planned to continue until the end of the semester, in-class meetings ended after the 9th week due to precautionary measures for the COVID-19 pandemic. This means that I attended the two classes for six and five weeks respectively.

4.4.1. Collected Data

While the classes were happening in-person, the collected data included audio-recordings, photos, and fieldnotes from my in-class visits, documents and other course materials, and interviews with the

[17] All names are pseudonyms for reasons of anonymity. Throughout the thesis, I refer to both professors on a first name basis as it was common for the students to refer or talk to them using their first name. Although it is quite possible that some students addressed both of them using a title (e.g., Doctor or Professor), doing so was not required or encouraged by the professors themselves, who made an effort to establish an authentic relationship with their students.

professors and some students. After the classes moved online, I was able to collect more data from Chris's class, but not from Alex's class. Overall, the collected data fall under two categories: the primary data (fieldnotes, transcriptions of lectures, and photos) and the secondary data (course documents, textbooks, interviews, online asynchronous discussions). The data from the two courses are summarised in table 4.2 and described in the following paragraphs.

COLLECTED DATA				
	Primary Data	Secondary Data		
	In-Class Visits	Documents & Other Material	Interviews	Other
Alex	*Audio-recordings, Photos & Fieldnotes:* 4 Visits x 3h 2 Visits x 2h *Fieldnotes only:* 2 Visits x 1h (audio-recording did not work)	Textbook Syllabus Online Info	Professor: 1 Students: 1	Informal Discussions
Chris	*Audio-recordings, Photos & Fieldnotes:* 5 visits x 2h 4 Visits x 1h *Fieldnotes only (Office Hours):* 3 Visits x 2h	Textbook Syllabus Online Info In class notes	Professor: 1 Students: 2	Informal discussions Asynchronous online discussions Pre-recorded lectures

Table 4.2: Summary of Collected Data

Data from Lectures: During the five and six weeks that I was visiting Chris's and Alex's course respectively, I attended all their lectures except for a 1-hour lecture of Chris because of personal circumstances. During these meetings, I was writing fieldnotes, taking photos of the board, and audio-recording the lectures. In my visits, I was sitting at the back of the room, alternating between chairs at the back-left, back-centre, and back-right from visit to visit, in order to have a panoramic view of the classroom. Furthermore, as most students appeared to occupy seats at the same part of the classroom in each lecture, alternating was helpful for sitting closer and observing different groups of students each time.

I kept fieldnotes about what happens in the classroom, including what the professor and the students say, non-verbal elements of instructor's discourse, and what the students do. The goal was to take as detailed notes as possible of whatever was considered to be relevant for the research project (Walford, 2009). Often while observing I had some reflections about what was happening. I was consistently writing these reflections in brackets in order to be able to distinguish between the data and my interpretations of them (Brown & Dowling, 1998). To protect students' anonymity, no identifiers (e.g., students' names, gender, race, language variety) are included in my notes.

I used an audio-recording device to record the instructor's words. The device was left on the professor's desk, so that they would always be able to stop the audio-recording if they wished to. When transcribing, any words of the students that were audible were not included in the transcription, as it was not possible for the students of this large classroom to consent to participating in the study[18]. Finally, regarding the photos, I took photos of whatever the professor in each class was writing on the board, taking care to not take a picture of anybody's face.

Data from Office Hours: Chris held office hours for his students once a week for two hours. During my data collection, I attended three of these office hour sessions. I was sitting in the back-left corner of the room and was keeping fieldnotes in the same manner as described for the classroom meetings. Because there were considerably fewer students in the room than there were in the lectures (i.e., the room had a capacity of 30 students and, during my visits, up to 10 students were there at any given time) I did not audio-record the sessions as any audio-recording would clearly capture students' voices. I also did not take any photos of these sessions, to not make students uncomfortable.

Course Documents and Other Materials: I collected the syllabi and textbooks of both courses as well as any other materials related to the courses that I had access to. This material includes online information about the courses on the university's website, information from a website associated with the course in Alex's class (including descriptions of lectures, a few extra resources, and homework questions), and Chris's course notes that were compiled during the lectures. In the presentation of the study's findings, I do not include any direct quotes from this material, in order for the courses to not be identifiable.

Online Pre-Recorded Lectures and Asynchronous Discussions: When COVID-19 was declared a pandemic by the World Health Organisation (WHO), the universities in Ontario stopped their in-person meetings. As a result, courses moved online. For the Introduction to Proofs course, each of the professors teaching different sections of the course recorded themselves presenting some of the remaining material. I got access to and transcribed Chris's pre-recorded lectures, which would correspond to one week of lectures, had the class not moved online. I also got access to the online asynchronous discussions that happened

[18] The rationale behind the choices of how to capture students' voices in a large classroom is explained in more detail in section *4.6. Ethical Issues*.

in the course. I collected the posts from this online forum, without including any student identifiers. The data that was collected after the course moved online were considered secondary. I did not analyse them per se, but I did consider them for developing a sense of the course.

Interviews with the Professors: I conducted one (one hour) interview with each professor. According to the initial plan, a second interview was going to take place towards the end of the semester. As classes moved online because of the pandemic, however, I did not attempt to conduct a next interview online (which would require a permission from the ethics board and consequently asking the professors to participate in one) because my sense of my interactions with the professors was that they were very busy adjusting to the new circumstances. Details about the design of the interviews that I conducted can be found in the following subsection.

Interviews with Students: I conducted one interview with a student from Alex's course (Anthony) and two interviews with students from Chris's class (Kevin and Vijin). The aim of these interviews was to investigate how students negotiate the courses' official discourses. A description of how these interviews were designed can be found in the following subsection.

Informal Discussions: During the semester, I had short informal discussions with the professors about the course, usually after the course ended. In these cases, I made as detailed notes of our discussion as possible after our talk and as soon as it was feasible to do so. I also engaged in some informal discussions with students during the breaks, at the end of some classes, and when Chris asked students to work amongst themselves.

4.4.2. Design of Interviews

Following Carspecken (1996), I spent the first weeks in the course observing, in order to construct a primary record of what happens in the course. This process was helpful for designing the interviews based on incidents and ideas that emerged as important from my observations. To design the interviews, I made notes of incidents that happened in the course which appeared to me to be closely related to my research problem and research questions. I wrote short descriptions of these incidents and questions about them, in order to ask participants to comment on them. In what follows, I first explain how interviews were structured, I then describe the kinds of questions that I planned, and I finally discuss how these questions were transformed during the interviews.

At the beginning of an interview, I asked students to sign the consent form and I let them know that they could withdraw their consent at any time. As I included pizza as an incentive for students, I also offered them pizza at the beginning of the interview. In the interviews with the professors, there was no need for them to sign a consent form during the interview, because they had already consented to participate in the interview when I started attending their classes. However, I did ask them if they were comfortable with me audio-recording the interview and I also reminded them that they can withdraw their consent, if they wish to do so.

After discussing these consent-related issues, I explained to the participants why I was conducting interviews and I asked them whether there was something specific that they wanted to discuss along these lines. When interviewing students, I also asked them a few general questions about their majors, why they are taking the course, and how it fits in their schedule. I then asked participants questions from the interview guide that I had designed, which contained some more specific and some more general questions. All questions that I designed can be found in Appendix 5. During the interviews, I chose questions on the spot that appeared to be more relevant to the emerging discussion with each participant.

In what follows, I give a few examples of questions. Some of the questions were quite general. An example of a more general question designed for interviews with students is the following:

> *What is a proof to you?*

However, most of the questions were specific to incidents that had happened in the class during my observations. Below are three examples of this sort. Questions 1 and 3 are for students in Chris's class and Question 2 is for Alex's class.

> <u>*Question 1:*</u> *Chris often talks about the reader of a proof. As I see it, he asks you to create your proofs in a way that you don't write them for yourselves, but also that you consider that somebody will read them. Is this something that you see as an important part of what Chris tries to teach you and is it important for you?*

> <u>*Question 2:*</u> *In one of the lectures that I attended, your professor suggested that machines can perform many tasks but it's important to have an intuition of what happens and how they do so, in order to be able to interpret the results correctly and to detect mistakes. What are your thoughts on that?*

> <u>*Question 3:*</u> *One day you proved Euclid's proof about there being an infinite number of primes. Chris made a point about this proof being considered beautiful. What do you think?*

The three questions vary in terms of how much I include my own interpretations in them. The first question is written in a way that includes my own take of why Chris refers to the reader of the proof and includes the phrase "As I see it" to make that clear. The second question is more descriptive and less evaluative of an idea shared by Alex, in the class. Being shorter, the third question is more open-ended and, thus, the most open-ended of the three. Of course, describing what has happened in a class constitutes a reconstruction of those events, based on my own perception and my focus on the research problem.

While conducting the interviews, the questions that I had designed acted as a guide and were transformed based on the emerging discussion with the participants. The following example, from Chris's interview, highlights this transformation. Below is the question that I had designed, which entails a series of sub-questions, not all of which were intended to be asked.

> *In one of your classes that I attended, you were talking about negations and as you were working on what it means to negate a negation, you gave the example of "I'm not not going to school tomorrow". You noted that a certain connotation that comes with saying I'm not not going to school is lost when you translate this sentence from English to mathematics. Can you talk a bit about this, maybe at a more general level? What is lost when we use mathematics to describe or interpret something around us? Does it matter?*

When conducting this interview, I asked the following:

> Dionysia: So, at some point I remember that you were doing negations and you were talking about what the negation of a negation is. The example that you gave is "I'm not not going to school". (Chris's class, p. 141; Interview with Chris)

Here, I did not actually ask Chris a specific question at all. Instead, I presented an incident of the course and Chris jumped into describing why he does that and how he thinks about it. This way of interacting emerged as a pattern in all interviews with both professors and students. In many cases, the questions that I had planned were not asked as such to the participants. Instead, my descriptions of an incident acted as prompts and the participants jumped in, offering their thoughts or feelings about them, without me having to articulate a specific question at the end of the prompt.

In this particular prompt, Chris responded with an explanation of why he does not like the "I'm not not going to school" example because of the English connotation in it. He explained how the phrase "I am not not going to school tomorrow" does not mean exactly the same as "I am going to school tomorrow". He also described to me a different example which he had encountered and was thinking of using the following year. As my interpretation of the "I'm not not going to school" example was different than the one Chris was describing, I continued with a follow-up question.

Dionysia: I thought it was very interesting- that presentation, exactly because you shared with the students that this connotation is actually lost in mathematics. So, to me, it was kind of a description of how there is limits in mathematics, or something.

Chris: Oh, there is a disconnect. Or, like, an essentialism, like it's cutting out some things.

Dionysia: Yeah, yeah. When you describe something with mathematics, something might be lost. (Chris's class, p. 141; Interview with Chris)

This led Chris to talk more about how he himself also sees ideas being lost when they are described with mathematics. At the same time, the way in which this interaction unfolded, allowed Chris to talk about what was most important to him in relation to the incident (i.e., the suitability of the example for communicating to students what the negation of a negation is about).

4.5. Working with the Data

Adopting an ethnographic philosophy for conducting research, the limits between data collection and data analysis have been loose in this study (Hammersley & Atkinson, 1983/2007). Preliminary analysis started while being at the site and it continued during the transcription of the data, in order to be utilised in the designing of the interviews. The deeper analysis of the data did not finish until several months after leaving the site. In what follows, I provide some detailed descriptions of the forms that data analysis took in this study.

4.5.1. Transcribing the Audio-recordings and Writing up the Fieldnotes

I transcribed the data myself as soon as possible after their collection. I did not use any software in order to gain greater familiarity with the text (Brown & Dowling, 1998), which was desirable both because English is not my first language and because the process of transcription was accompanied by a preliminary data analysis, the results of which would be used for further generation of data through interviews. This method entailed me pausing the recording every few seconds to transcribe the words that I had heard during that time. This was very helpful considering my discourse analysis methodology, as it gave me time to think about all words that were articulated and their intonation. It was also valuable because it provided space and time for interaction with pieces of data which did not necessarily appear very rich in a first reading.

For each class visit, I created one document which includes both my transcription of the audio-recording and my fieldnotes. After transcribing the audio-recordings of one visit, I wrote up the fieldnotes from that visit, using the same document. In this way, the transcript is enriched with my observations about what happens in the class (e.g., what the professors does, what students do) as well as with what students have said during it (I did not transcribe any students' words even if they were audible; section 4.6. entails some information on the rationale of this practice in relation to ethics). After the document for each visit was completed, anonymised, and proofread, I compiled one single document for each course that includes all the collected data.

At the same time as transcribing and writing up fieldnotes, I was keeping notes in a journal about everything that I considered important and relevant to the research problem. These notes include observations, patterns, comments, questions, connections between the two classes, and connections with theory. I used these notes in two ways. First, they acted as a guide for constructing the interview guides for the instructors and students. Second, as soon as I finished the data transcription and this phase of reflecting on the data, these notes were my starting point for working with the data.

4.5.2. Presentation of Data

Every time that I cite data in this thesis, I refer to one of the two documents that includes all data from the respective course. I indicate where the data is located by using a notation of the form (Class, page number; Type of Data). The class indicates which of the two classes the data is from. The page number refers to the page of the document which contains all data of that course. The type of data refers to whether the data are from a lecture, office hours, or an interview. If the data comes from a lecture or office hours, I also indicate which week and which visit it is from. For example, the notation "(Chris's class, p. 36; Lecture 6.1)" indicates that the presented data are from Chris's class, they are located on page 36 of the document with Chris's class data, and they are from the first lecture of the sixth week of the semester.

Furthermore, the examples that I use in the findings chapters (i.e., Chapters 5, 6) have been named using a characteristic phrase that appears in them. For instance, the "let's be good mathematicians" example is an example from Chris's class, in which Chris uses the phrase "let's be good mathematicians". Naming the examples is a helpful way for referring back to them, after they have been introduced. The examples have also been appendixed (see Appendices 1 and 2), in order for the reader to be able to find where an example is introduced and discussed.

Finally, when presenting data, I also use the notation in table 4.3. All examples in the table are from Alex's class.

Symbol	Example	Meaning

[…]	So, I have now the equation of the line, the equation of the tangent. […] In this case, our equation is very simple.	Some data have been skipped.
-	What it means geometrically is that the function has a change that is very sudden. It's so sudden that it's not- it's kind of unexpected.	The speaker interrupts their flow and starts a new phrase.
Italics and Square Brackets []	Example 1: The order is *[writes on the board]* logarithms grow smaller than polynomials and these grow smaller than exponentials. Example 2: So, that means that the new line should be more vertical than the first line. And it's happening, right? Like this has a slope 2, this one has a slope 3, so it's more vertical. *[As he talks, he uses his hands to show this kind of movement.]*	Comments about what happens.
Bold	So, we need **methods** that allow us to compute these limits without actually computing the limit itself.	The speaker stresses what they say.
[inaudible]	Now, if instead of writing $y = x^4$, I put $y = x^7$, for example, you would have a huge polynomial [inaudible] but forget it, it's a huge polynomial.	It is not possible to make out what was said.

Table 4.3 Transcription Notation

4.5.3. Working with Different Theories: Bringing Different Methodologies Together

For the data analysis, I have used the discourse analysis tools from three different discourse approaches. Laclau and Mouffe's (1985/2001, 1987) theory acts as a guiding theory which frames the ontological and epistemological conceptualisation of discourse, while I also draw upon the works of Fairclough (1985, 1992, 2003) and Doxiadis's reading of Foucauldian discourse analysis (Doxiadis, 2011a; Foucault, 1972). Furthermore, my practice has been influenced by the ways in which various scholars in mathematics education and other social sciences have used discourse analysis theories and their respective analyses (e.g., Abtahi & Barwell, 2022; Pais & Valero, 2012; Pechtelidis & Stamou, 2017; Silva & Valero, 2018; Valero & Knijnik, 2015; Walkerdine, 1998). In this section, I provide a brief explanation of the elements that I use from these theories and analyses, focusing on the more "practical" ideas which shaped my discourse analytic interactions with the data. But before embarking on this discussion about the different traditions and approaches, I will explain the reason for choosing to work with multiple discourse analysis methods in the first place, under the scope of "making the familiar strange".

4.5.3.1. Making the Familiar Strange with Discourse Analysis

The phrase "making the familiar strange" is quite popular (if not somewhat of a cliché) in several domains of social sciences and beyond, including in anthropology (Myers, 2011), sociology (Mills, 1959), and theatre (Brecht & Bentley, 1961). Within the vein of "making the familiar strange", Bertolt Brecht coins the term Verfremdungseffekt in theatre (translated as "the alienation effect" by Bentley), to refer to ways of acting and directing that make it possible to underline the historical nature of any given social condition, offering space for criticising society and reporting on accomplished changes (Brecht & Bentley, 1961). Following Brecht, I see discourse analysis as a vehicle for making the familiar strange, by examining taken-for-granted truths and looking at them from a place where they start looking strange. De-naturalising discourses is taken up in many discourse analysis approaches, whose scope is critical and investigative of power relations (e.g., Fairclough, 1985; Foucault, 1972; Mpoukala & Stamou, 2020).

It is within this context that bringing different approaches together is taken up in this study. Conducting multiperspectival inquiry by combining different approaches in discourse analysis opens up space to develop different forms of knowledge (Jørgensen & Phillips, 2011) and thus different paths for "estranging" the familiar. My choice to work with multiple discourse analysis approaches in this study is based on (and situated in) my previous engagement with different discourse analysis methods, through which I experienced Doxiadis's Foucauldian discourse analysis to be powerful in exploring the relations between knowledge and power (Pitsili-Chatzi, 2018), and Fairclough's critical discourse analysis to be useful in exploring how the linguistic aspects of a communicative event are critical in constructing particular ways of knowing about a phenomenon (Pitsili-Chatzi, 2019). In what follows, I describe how the works of Laclau and Mouffe, the axes method, Fairclough's theory, and inspirations by mathematics education discourse analysis studies are taken up in this work.

4.5.3.2. Laclau and Mouffe's Theory

Laclau and Mouffe's (1985/2001, 1987) work provides the epistemological foundation for this study. This means that it has played a role in terms of setting the overall scene, including in transforming other methodological tools, so that they are not in tension with each other. For example, I have transformed Doxiadis's (2011a) axes method so that it is not in tension with the ontological conceptualisation of discourse by Laclau and Mouffe. Furthermore, the primacy of Laclau and Mouffe's discourse theory has oriented my discourse analysis towards exploring how meaning is created by looking at discourses as attempts to fix meaning, while recognising that this fixation is only partial and temporary (Laclau & Mouffe, 1985/2001).

Besides its epistemological insight, the work of Laclau and Mouffe (1985/2001) has also been critical in offering conceptual tools for exploring how meaning is negotiated, challenged, and becomes hegemonic in the context of the two courses. In this direction, I use the complex of concepts *antagonisms, hegemony, fixation of meaning, floating signifiers,* and *nodal points*, whose meanings are explained in more detail in Chapter 3. As an example of how I have worked with these concepts, I use the concept of *antagonism* to discuss how the discourse of mathematics as a social activity and the discourse of mathematics as a product both emerge in Chris's class and participate in an attempt of *meaning fixation* of concepts like proof, formal proof, mathematics, and technical mathematical language. I conceptualise these concepts as *floating signifiers*, whose meaning is attempted to be fixed by the two discourses (i.e., the two discourses antagonise to fix the meaning of, for example, what a proof is). Within each discourse, these concepts constitute *nodal points* whose meaning has been fixed (e.g., the discourse of mathematics as a product conceptualises what a proof is in a relatively stable way). This discussion leads to posing and exploring questions regarding the hegemonic state of the two discourses and the space for them to be challenged.

4.5.3.3. The Axes Method (Foucault and Doxiadis)

Foucault (1972) proposes that discourses should be treated as practices that form the objects to which they refer. Working on Foucault's conceptualisation of discourse, Doxiadis (2011a) identifies four fundamental properties that every discourse has. Every discourse has the property of referentiality which means that the discourse points to something outside itself, which would exist even if the discourse did not exist; the property of subjectivity which is related to the discourse's conditions of enunciation; the property of knowledge which has to do with the concepts produced by the discourse; and lastly the property of ideology which emphasises the political dimension of the discourse and its relation to issues of power. This conceptualisation of discourse orients the discourse analysis task along four axes, which correspond to the four fundamental properties of a discourse: a) The axis of objects: the discourse's relation to what is outside of it is investigated. The focus is on the objects to which the discourse refers and which it constructs. b) The axis of enunciative modes: the discourse's relation to itself is investigated and two parts are identified. The first part is concerned with the external conditions of enunciation, the conditions within which the discourse is produced, such as its origin and address. The second part of this axis is the internal conditions of enunciation, which refer to the ways in which subjects are involved in the discourse. c) The axis of concepts: the discourse's relation to other discourses is investigated. The focus is on the construction of concepts, which is the main product of the discursive practice. d) The axis of thematics: the discourse's relation to power and the antagonism expressed within the discourse and as a result of the discourse are investigated. The properties of discourse, the axes of discourse analysis, and questions that guide this analysis in each axis can be found in table 4.4.

Properties of Discourse	**Axes of Discourse Analysis**	**Questions that Guide the Analysis**
Referentiality	Axis of objects	What does the discourse refer to, which would exist even if this discourse did not?
Subjectivity	Axis of enunciative modes • External conditions of enunciation • Internal conditions of enunciation	External Conditions: Who produces the discourse, to whom is it addressed, how is it enunciated (e.g., orally, written, which medium, etc.)? Internal Conditions: How are subjects involved in the discourse?
Knowledge	Axis of concepts	What concepts does the discourse produce?
Ideology	Axis of thematics	How does this discourse relate to power and how does it guide action?

Table 4.4: Doxiadis's (Foucauldian) Axes Method

When I started working with Doxiadis's discourse four axes analysis method, I experienced some practical and epistemological tensions related to the axis of objects and the axis of concepts. The practical tensions arose from the examined discourses emerging in an educational context. In this context, an "object" is often part of academic mathematical discourses, which is constructed as a concept in the classroom discourse. Not many concepts are produced for the first time in a mathematics classroom; almost any concept is an object of another discourse. Yet, students often have little "access" to these other – academic mathematical – discourses. Hence, the distinction between objects and concepts is not very helpful. The epistemological tensions arise from the conceptualisation of objects as being external to the discourse and existing even if the discourse did not exist, while concepts refer to what is constructed by the discourse. This distinction assumes the existence of a non-discursive space; an object is different than a concept because it has an existence, independently of the discourse in which it "appears". This Foucauldian idea is in tension with Laclau and Mouffe's (1985/2001, 1987) theory, in which it does not make sense to consider something as non-discursive; any object/concept does not have any "essence" or meaning beyond the discourses in which it appears, even if it has material existence[19].

[19] A methodological advantage of Doxiadis's distinction, which gets lost through my merging, is the focus on how the discourse under examination articulates "already said" ideas in new ways. However, Laclau and Mouffe's theory approaches the "mechanics" of "concept" production, by exploring how an articulation constructs nodal points and thereby fixes meaning. In doing so, it draws upon pre-existing floating signifiers.

Given this analysis and my positioning along Laclau and Mouffe's (1985/2001, 1987) epistemological conceptualisation of discourse, my use of Doxiadis's method merges the two axes. For a communicative event, I only work with three axes: a) objects and concepts, b) enunciation modes, and c) thematics. To explore the merged axis of objects and concepts, I explore the question: "Which objects/concepts does the discourse refer to and thereby construct?". The adjusted axes method is summarised in table 4.5.

Axes of Discourse Analysis	Questions that Guide the Analysis
Axis of objects and concepts	Which objects/concepts does the discourse refer to and thereby construct?
Axis of enunciative modes • External conditions • Internal conditions	External Conditions: Who produces the discourse, to whom is it addressed, how is it enunciated (e.g., orally, written, which medium, etc.)? Internal Conditions: How are subjects involved in the discourse?
Axis of thematics	How does this discourse relate to power and how does it guide action?

Table 4.5: Adjusted Axes Method

4.5.3.4. Fairclough's Textual Analysis Techniques

Fairclough's Critical Discourse Analysis attempts to bridge the gap between socially and linguistically oriented approaches to discourse analysis (Fairclough, 2003). Fairclough draws upon Halliday's linguistic tradition, while he also employs insights from social theory, including the work of Foucault (Blommaert, 2005). This approach is exemplified in his three-dimensional model (see Figure 4.1), according to which discourse analysis is conducted at three levels: the analysis of the linguistic features of the text, the examination of the processes through which the text has been produced and is consumed (i.e., the discursive practice), and the social practice which surrounds the communicative event (Fairclough, 1995).

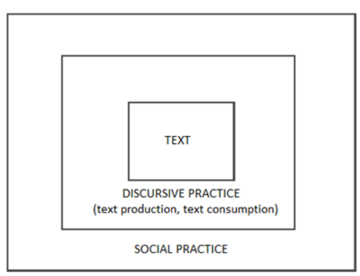

Figure 4.1: Fairclough's Three-Dimensional Model

For reasons related to my discomfort with how this whole package elevates theory as having the final say about the "truth" of a text, I do not use this three-dimensional model in all its complexity, but I instead focus on how Fairclough proceeds with the textual analysis of a communicative event (which he calls the description of the communicative event). More specifically, I use some of the linguistic tools employed by Fairclough (1995, 2003) including examining the choice of words, the use of pronouns, modality (modal verbs and hedges), nominalisation, and the syntactical structure of the sentences (e.g., active versus passive voice).

Exploring the values in the two courses is an example of how I used textual analysis techniques, by focusing on how evaluation happens at the micro-level of a statement through "evaluative statements". For instance, Fairclough (2003) frames evaluations as statements about un/desirability: what is good and what is bad. In a similar vein, Thompson and Hunston (2000) locate the core of evaluation to be about expressing an opinion: "evaluation is the broad cover term for the expression of the speaker or writer's attitude or stance towards, viewpoint on, or feelings about the entities or propositions that he or she is talking about. That attitude may relate to certainty or obligation or desirability or any of a number of other set of values" (p. 5).

In order to identify value-laden phrases, I focused on identifying value-laden words (e.g., good, bad, beautiful, etc.) and explicitly stated values mainly in interviews and lectures (e.g., introduced through a phrase like "We want"). For example, in the following articulation, part of what we will see as the "but I really don't like these words" example, there is a strong indication of like/dislike:

> Chris: There's actually fancy names for this. Like antecedent. And conclusion or whatever it is. But I really don't like these words. So, instead we'll say left side, right side. (Chris's class, p. 11; Lecture 5.1)

Here, the phrase "I really don't like these words" indicates a dislike, which manifests in the material practice that "instead we'll say left side, right side". In this way, the "fanciness" of terminology (or formal terminology) is coloured negatively. This example, combined with other similar examples, led to the identification of "simplicity" and "communication" as values.

4.5.3.5. Other Discourse Analysis Inspirations

Doing discourse analysis has not only been based on the basic theoretical texts that I have so far written about, but it has also been shaped by how scholars in mathematics education have employed discourse analysis in their work. Discourse analysis studies which I have read over the years (e.g., Abtahi & Barwell, 2022; Mendick, 2006; Pais & Valero, 2012; Silva & Valero, 2018; Valero & Knijnik, 2015) have played a critical role in shaping my understanding of discourse analysis in ways which are often subtle. Through these works, two approaches have been quite influential while also tangible enough to describe.

The first method which I follow inspired by mathematics education scholars consists of working with a theoretical concept in order to explore how a discourse represents social categories and constructs reality. Pais and Valero (2012) use Foucault's *biopolitics* and Žižek's *ideology critique* to analyse how the field of mathematics education fabricates learning and mathematics respectively as its objects of research; Silva and Valero (2018) write about financial mathematics and interdisciplinarity in Brazilian textbooks, using Foucault's *governmentality* and *subjectivity*; *governmentality* is also the core concept in Valero and Knijnik's (2015) investigation of ICT research; Abtahi and Barwell (2022) use the Foucauldian concept of *regime of truth* to analyse how media reporting of mathematics education constructs children. In all these studies, a theoretical construct is used to theorise the relation between texts and the social world. These approaches have oriented me to work on the relation between discourses and social "reality" through creative uses of theoretical concepts; using theoretical concepts as tools and inspirations to think with/through, without necessarily aiming to capture how the concept was initially developed. This approach is reflected in my data analysis with the use of concepts like Castoriadis' (1975) imaginary and Arendt's (1987) distinction between work and labour.

A second inspiration by discourse analysis studies in mathematics education has been the exploration of dichotomies constructed within a discourse. Working with Foucault's ideas, Mendick's (2006) discourse analysis method explores how mathematics is conceptualised within the masculinity/femininity dichotomy in the discourses articulated by secondary school students. Mendick's invitation to reflect on dichotomies has been influential on the way in which I think about discourses as it has been teaching me to pay attention to the articulation of oppositional constructs in discourses. More tangibly, I have used the main idea of this approach when working on how the notion of "formal" is conceptualised in Chris's class, in which I explore how the formal is conceptualised in opposition to what it is not (i.e., the informal).

4.5.4. Method of Data Analysis

My data analysis method emerged while working on the analysis of the data. This means that before engaging with the data analysis, I did not have a clear-cut way of working with the data, other than having compiled a toolbox with some data analysis techniques. The early stages of analysis consisted of a "messy" interaction between the data and myself, through which the goal was to familiarise myself with the data. In this process, I read the data multiple times, re-listened to all data, re-listened to parts of the data that I identified as particularly rich, thought about them through the research questions, made related notes, produced some preliminary writing about the data, and created a few "thinking maps" (like the ones in

Figure 4.2) in an attempt to make connections between the different pieces of data. The first one of these maps can also be find in Appendix 6, with better image resolution.

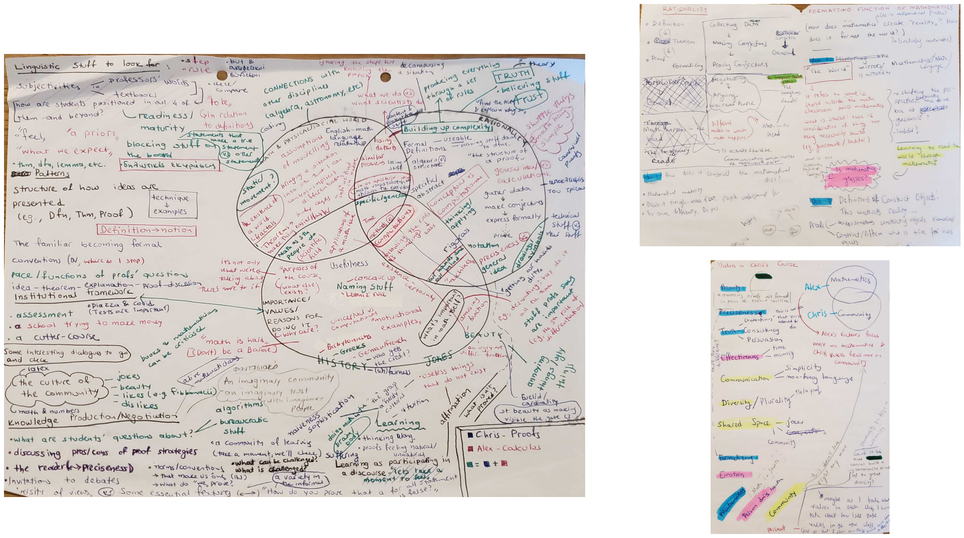

Figure 4.2: Examples of "Thinking Maps"

Through this process, my interaction with the data gradually fell into the pattern depicted in the following sketch (Figure 4.3). As shown in the sketch, the data analysis consisted of two main processes (see 1st main process and 2nd main process) which were happening parallel to each other and informed each other in a back-and-forth way. The first process consists of identifying and in-depth analysing "rich examples". The second process consists of looking across the different data and identifying themes from the lenses of the three research questions. This back-and-forth process led to the identification of overarching discourses and overarching antagonisms between discourses. In turn, this identification re-informed the two main processes, forming a back-and-forth way of working between the analysis of specific incidents of the course and the description of the course in terms of its overarching discourses. Finally, this way of working was applied in each one of the courses. In this way, some insights from one of the courses provided feedback to the analysis of the other course's data. In what follows the different aspects of this method are described in more detail.

The first process, focusing on the identification and analysis of "rich examples", aimed to explore what emerges from the instances that happen in the course at the micro level. In other words, I tried to explore what an otherwise every-day interaction in the class can tell us about the ways in which mathematics is political in the context of an undergraduate classroom. This process consisted of two sub-processes: the identification of rich examples and their analysis. I considered an example to be rich when it related to one (or more) of the research questions or when it related to the research problem of the political character of mathematics (even if its connection to one of the three research questions was not apparent at the time). For example, issues related to the knowledge production or the history of mathematics were not initially considered to be within the scope of the research questions, but were still considered "rich examples" because of their relation to the question of how mathematics is political. In this way, not only did the research questions take a more specific form compared to their initial theoretical, "on paper" conceiving, but also – as described in section 4.1. – two of the research questions changed. In terms of the discursive tools in this "1st main process", I mainly used the modified version of Doxiadis's Foucauldian four axes discourse analysis method (Doxiadis, 2011a; Foucault, 1972) and Fairclough's (1985, 1992, 2003) textual analysis techniques.

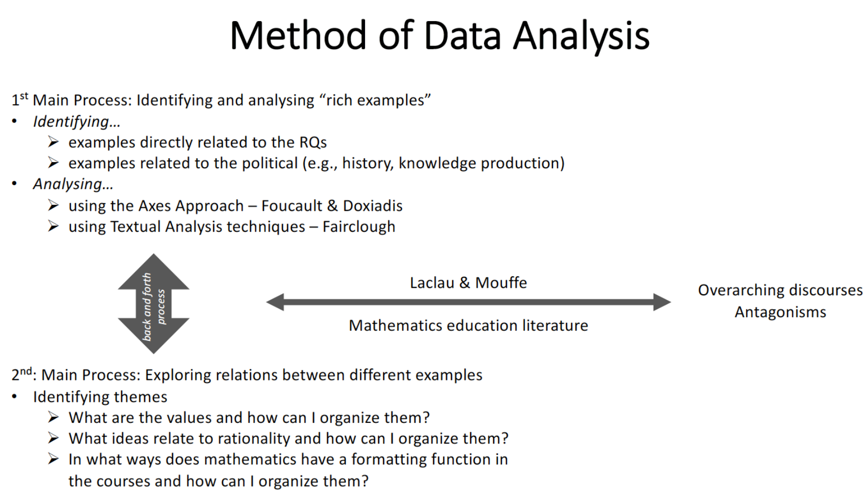

Figure 4.3: Sketch of Data Analysis Method

In the process of trying to identify "rich" examples, I approached the text thinking about ways in which discourses (as "natural" as the discourse that truth is important in mathematics) can be de-naturalised (Fairclough, 1985). To do so, I was particularly interested in situations which are typical for mathematics classrooms but are rather untypical for other settings. One of these practices is that the class only use stuff that has previously been covered by the professor. For example, in Alex's class, the students are not allowed to use L'Hospital's rule for calculating a limit until the class covers this topic (Alex's class, p. 15; Lecture 4), although most of them know this rule from high school. This process is different than, let's say, an English as a Second Language classroom, in which students are allowed to use words or grammatical phenomena which might not have been covered by the instructor. Furthermore, I considered examples to be "rich", when they were about situations that are considered natural. The rationale behind this strategy is that when something is described as if it were a natural truth, there has been a fixation of meaning around it, so ideology is at play there. For example, the sentence "the recursive definition requires a lot of work" (Chris's class, p. 57; Lecture 7.1), when compared to the explicit definition which "allows you to compute very large terms in a sequence, almost immediately" (Chris's class, p. 57; Lecture 7.1) implies that doing a computation as quickly and effortless as possible is desirable. This example, along with other similar examples, led to the emergence of "efficacy" as a value. In analysing these examples that appear to be "natural" in a mathematics classroom, my analysis has involved a great deal of contrasting the class discourses with examples from literature, the other course, or made-up alternative articulations, social arrangements, or proofs. Further, I try to put the data into discussion with macro ideologies (e.g., capitalism, individualism, etc.), in order to explore this relation between the micro and macro levels. In doing so, my goal is not to appraise or challenge the professors' discourses, but only to de-naturalise them by showing that, under different circumstances, they could have been different.

The second process, primarily consisting of the identification of themes, aimed to explore the relations between different examples. To do this, I worked across the specific examples exploring how they can be connected through the lenses of the research questions. In particular, in this process, I guided myself in the following directions: what the values are and how I can organise them; what ideas relate to rationality and how I can organise them; in what ways mathematics has a formatting function in the courses and how I can organise them. In some cases, I searched through the data systematically in order to identify all places where a certain concept appears. For example, in order to explore how the notion of formality is constructed in Chris's class, I searched for the word "formal" and its derivatives in all data. I then analysed each of these examples and compared the examples with each other.

I moved between these two processes (1st and 2nd main processes) in a back-and-forth way. This means two things. First, getting in depth in the analysis of a particular example helped me identify aspects of it which might relate to other examples, providing in this way feedback to the identification of themes. For example, my analysis of the "anything that the humankind does is important" example from Alex's class indicates that it is an example about the humanity of mathematics. This led me to explore how the humanity of mathematics emerges in other examples and later to identify humanity as a theme and value in this course. Second, the identification and elaboration on the emerging themes and the emerging responses to the research questions provided feedback to the process of identifying "rich examples" and analysing some less salient aspects of them. For example, although the professors do not explicitly discuss the importance of *truth* in mathematics, I have identified *truth* as a value in both courses (2nd main process). This guided me to think about how *truth* manifests itself in specific examples in less salient ways (1st main process) and enriched the analysis of many examples by adding the dimension of truth to them.

After some time working in this back-and-forth way between the two processes, overarching discourses started emerging. For example, in Chris's class, the *efficacy of mathematical work/labour* emerged as a meta-theme (or an overarching discourse) among the identified themes because it relates to the focus on quick computations, differentiated ways of working/labouring with mathematics, and the conceptualisation of mathematics as happening in the brain. A few points might clarify how these overarching discourses have emerged. First, this "emerging" has not been a natural process; it is rather the result of exploring how the various themes connect with each other under the (epistemological) assumption that "grand" discourses about mathematics are manifested in and get reproduced through every-day communicative events that happen at the micro level. Furthermore, Laclau and Mouffe's (1985/2001, 1987) discourse analysis concepts (e.g., antagonisms, floating signifiers, nodal points, (non-) fixation of meaning) and mathematics education literature have been critical in terms of drawing connections between the themes and articulating the overarching discourses and antagonisms of discourses. Finally, identifying overarching discourses has not linearly succeeded the two main processes described above; there has instead been a dialectic relation between framing the overarching discourses and working on and between the two main processes.

Moreover, while working in this back-and-forth way between the two processes, part of my strategy methodologically was to identify antagonisms between discourses, as a way of exploring how discourses interact with each other. Identifying antagonisms in Chris's class came as a result of the professor and students articulating different discourses as well as of Chris himself articulating multiple discourses while speaking. For example, Chris articulates both a discourse which conceptualises mathematics as a social practice and a discourse which conceptualises mathematics as the product of mathematicians' labour. Through my analysis, I discuss how these two discourses, both articulated by Chris himself, are in an antagonistic relation. While in Chris's class, identifying antagonisms was a more or less straight-forward task, this was not the case for Alex's class. Alex's class is based on the professor's lecture and as such it is a largely monophonic one. Moreover, Alex's discourse about mathematics is akin to a mainstream narrative that assumes mathematics to be a human endeavour, a powerful tool that can be applied in almost any problem, and a shared space. While analysing Alex's discourse, I struggled a lot to identify antagonisms of discourses and I often negotiated whether these antagonisms I was identifying were imposed by me or were negotiations that Alex himself was engaging in. Identifying antagonisms in Alex's class was difficult because Alex's narrative is akin to the narrative of modernity about mathematics – its potentially contradictory or scrappy elements have throughout the years been fit very tightly so that this assemblage has become normalised and acts as a regime of truth. In other words, finding antagonisms has been a hard task exactly because this discourse has been designed so as to not have any.

I followed the data analysis method described so far in each one of the two courses separately. While doing so, findings from one course sometimes provided an insight for the analysis of the other course. As an example of how working with both courses helped me "see" things, early on in my analysis I identified "shared space" as a value in Chris's class, which led me and enabled me to see it as a value in Alex's class as well (where the same value manifests itself, but in a less explicit way).

4.6. Ethical Issues

This study has been approved by the Office of Research Ethics and Integrity of the University of Ottawa (Ethics File Number: S-11-19-4924) and it has also obtained organisational approval by the University in which I collected the data. In this section, I discuss two ethical issues which emerged throughout the project. The first issue is about tensions related to the collection of data in large classrooms, especially regarding consent in this context. By discussing this issue, I wish to clarify the rationale behind some choices related to the design of this study. The second issue is related to using a discourse analysis lens and focuses on the tensions related to analysing decentred discourses which have been articulated by (very real) human beings.

Before embarking in the discussion of these two ethical issues, a note might be relevant regarding the ethical issues that I do not discuss in detail. Every – conscious or unconscious – decision made in the context of this study has been influenced by the ethical standards which frame educational research in Canada. This includes practices which are taught as part of the ethical standards of educational research courses and whose implementation is ensured by university ethics boards (e.g., participants need to consent to research, participants have the right to withdraw their consent, assessment of risks and benefits for the participants should be an essential and well-thought part of the research design, etc.). It also includes decisions which are not necessarily thought of as "decisions", but rather feel like commonsensical things to be doing as part of research. For example, observing a class by quietly sitting in the room and looking at what people around me do (as opposed to starting to scream that mathematics is political in the middle of a lecture to see how the people around me react) is only one example of how the unspoken ethicality of being respectful to the setting frames every aspect of the research design, data collection, and data analysis. Of course, this ethicality, spoken or unspoken, is the historical, cultural, and political product of this time and place. With these comments, I wish to acknowledge that this ethical commonsensicality – which I am essentially invoking by not deeply reflecting on the reasons that during my in-class visits I was in jeans (and not in a suit, but also not in my ripped jeans), that I use pseudonyms when writing about my participants, or that I tried to make sure that the questions that I asked during interviews do not make my participants uncomfortable – is situated at this time and place and might one day seem as obscure as the Stanford prison experiment does today.

4.6.1. Collecting Data in Large Classrooms

While designing the research project and as I was still unaware of the actual site in which the study was going to take place, I started designing the project with the goal of collecting data in large classrooms (i.e., classrooms that could have up to 250 students). Indeed, the classes in which I did collect data were regularly attended by approximately 150 students (for the proofs course) and about 60-70 students (for the calculus course). Moreover, I wanted the study to include students' voices, in order to explore how the courses' formal discourses (e.g., the professor's discourse, the textbook discourse, etc.) are negotiated, accepted, and/or challenged by students. This opened up ethical questions about students giving consent.

Essentially, it became impossible for students to give consent for their in-class voices to be included in the study. For students to give consent, I would need to collect a consent form from each one of them. But this would give rise to a series of practical problems. How would I know which students consented for their voices to be audio-recorded and which did not? How would I be able to associate a student's face and their name when the number of students is so large? How would I know if some students were not present on the day that I was collecting the consent forms? For all these reasons, I[20] decided to not audio-record any student voices. In case any student voices were audible during the transcription, I would (and did) ignore them. However, since I did not want to lose students' voices from the study, I included students' questions and comments in my fieldnotes. As students had not given consent for this, and considering the large number of them, I included no identifying information about any students (e.g, names, (apparent) gender, race, language variety, etc.). When multiple students participated in an interaction, I used "Student 1", "Student 2", etc. to differentiate between the different individuals. In this way, it becomes practically impossible for a question or comment to be traced back to the student who enunciated it. In other words, I treated the large classrooms as a semi-public space. Considering that

[20] The design of the study in order to deal with data collection in large classrooms has been shaped by my interactions with ethics protocol officer Kim Thompson, whose feedback and questions pushed me to think about the issues of collecting data in large classrooms.

classroom interactions are usually not sensitive in nature and the large numbers of students in each classroom, this practice was approved by the Research Ethics Board. At the same time, the interviews (and the planned but not actualised focus groups) aimed to capture students' voices in more depth.

Another implication working with a large class is that many individuals have interacted with it. For example, many students have been in Alex's and Chris's classes and, at least theoretically, they could read this thesis and get access to information that was shared with me during the interviews with the professors, knowing who those professors are. Considering the non-sensitive nature of the interviews and classroom observations, this has not been a major concern. However, to protect the anonymity of the professors as well as of the participating students, wherever possible, I have made an effort to not include information, which could reveal who the professors or the courses are. For this reason, the names given to the professors (Alex and Chris) are pseudonyms. Furthermore, I have not included the titles of the textbooks, any direct quotes from them, or other identifying information about them. For example, when I refer to the contents of a chapter in Alex's class, I do not mention which chapter this is (i.e., I mention chapter [n], instead of – for example – chapter 5) nor do I mention the chapter's title. When it is relevant to mention what a particular chapter entails, I avoid direct quotes. Similarly, I do not use any direct quotes from syllabi or publicly available information about the courses. Finally, I also do not include any board pictures, but I have instead recreated all board pictures to which I refer.

4.6.2. Analyzing Abstract Discourses, Enunciated by Very Real Human Beings

During the analysis of the discourses articulated by the two professors, a tension arose between analysing abstract, decentred discourses and the fact that these discourses have been articulated by very real human beings, who put labour, thought, and emotion into their work. Throughout this project, I have not been interested in judging the professors or evaluating their practices, but I have instead been interested in the discourses within which they talk and which they articulate. My interest is on how these discourses appear as a product of their time; what mathematical discourses emerge and how they are political. This focus takes my attention away from the goals and intentions of the professors and draws it towards discourses, which bear a notion of vagueness and intangibility.

However, these discourses are enunciated by very real people who enunciate the words that I am analysing in ways that are linked to their personal and professional identities. They probably take pride in their work and I can very clearly see why they would; sitting in their classrooms made me deeply respect both of them – both as people and practitioners – and gave me a lot of inspiration and many ideas for my work as a teacher. Furthermore, in doing this work, both professors are also very aware and reflective of issues which relate to education and social justice (e.g., students working parallel to their studies, racialised students learning mathematics, etc.); they think, act upon, and work within the kinds of injustices that this work starts from and inspires to be sensitive to. Yet, my focus is not on the unique and wonderful things that Alex and Chris do in their classrooms (which are many), but on what makes them ordinary, how they are products of their time, and how they participate in discourses and power relations greater than them.

I have not reached an equilibrium regarding this tension. It is a tension relating to the distinction between an autonomous and a decentred subject (e.g., Popkewitz, 2012) and, for me, it is a tension that poses the question of whether a theoretical conceptualisation of the subject as decentred has the "right" to intervene with a subject's potential conceptualisation of herself as autonomous. The tension also relates to whether and how abstract, intellectual exercises which aim at the disruption of discourses under the assumption that no social justice goal should be conceived of as a desirable telos (like this thesis) interact with actual struggles for social justice.

4.7. Rigour

Within the critical, post-structural, post-Marxist framework adopted in this thesis, the task of discourse analysis does not claim to offer access to an objective truth about texts. Texts are instead treated as being constructed by and at the same time constructing certain understandings of reality (Cheek, 2008). Fairclough (1992) uses the term "intertextuality" to highlight a text's function to both have history inserted into it (or in other words to be a product of pre-existing discourses) and to insert itself into history (or to contribute to the change of the existing discourses). In this sense, research is a product of pre-existing discourses and it simultaneously transforms the existing discourses in particular ways. Hence, research is political, because it favours certain understandings of "reality" over others and it is accountable to the research participants (Walshaw, 2007).

Within this framework, I think of this study's rigour in relation to Brown and Dowling's (1998) conceptualisation of research as a mode of interrogation, which applies a coherent, systematic, and reflexive way of questioning. Following Copland and Creese's (2015) vigorous invitation for transparency that "[d]escriptions of analysis should not be hidden in the appendices or assume an overly comfortable 'you know – that's how things are done' approach. Analysis is not an act of faith but a rigourous systematic activity" (p. 211), the trustworthiness of this study relies on being systematic, reflective, and transparent in the construction of both the data and the analysis. In this direction, throughout this chapter, I have tried to be transparent regarding the choices that I have made during data collection and their rationale, while I also discuss how unprecedented circumstances (e.g., non-interest of students to participate in focus groups, the pandemic, etc.) have influenced how the study has unfolded. Regarding the analysis, I have discussed in detail the tensions between the theories that I use and the ways in which I tried to resolve them in the context of the study. I have also tried to be as detailed and clear as possible about how I developed and followed my method of analysis, while I have taken care to not only present the methods that I followed in their final-product version, but to also explain how they have emerged (e.g., by describing how the research questions changed through the course of the study). Moreover, I have tried to be transparent regarding my position(ality) in the study, by being descriptive of my participation (e.g., my interactions on site, my role in interviews) and reflective of my presuppositions. Finally, thinking that the rigour of a study is ultimately judged by the community, I include detailed extracts of my transcripts and I attempt to make my analysis transparent by explaining how I have worked with the data and the theories, as opposed to merely presenting the conclusions that I have reached.

4.8. Summary

In this section, I discussed some methodological issues and the research design of this empirical study. The study was informed by critical ethnography and was conducted in two undergraduate courses, an Introduction to Proofs course and a Calculus course. I attended the lectures of the two courses for five and six weeks respectively, I collected the courses' textbooks and documents, and conducted interviews with the professors and a few students. To analyse the data, I used a variety of discourse analysis theories, combined under the umbrella of Laclau and Mouffe's (1985/2001) discourse theory and epistemology.

Chapter 5: The Introduction to Proofs Course

This chapter explores the findings from the "Introduction to Proofs" course. In subsection 2.3.2., I reviewed the literature on proof and proving with an emphasis on undergraduate settings. In that section, I argued that proof and proving scholarship not only does not address political aspects of proof and proving, but it actually masks the political aspects of proof and proving. In this chapter, I highlight the political aspects of the "Introduction to Proofs" course. To this end, I examine the discourses that are evident in the course and their political elements. I have identified four overarching discourses that weave through the mathematical practice of this class:

- formalising – through the use of definition-based arguments, the avoidance of visual arguments, and the utilisation of technical language – is conceptualised as ensuring rigour and being a catalyst for the production of mathematical results.
- diverse participation in mathematics is encouraged, including the participation of groups which have historically been underrepresented in academic mathematical practice and the consideration of individuals' personal mathematical ideas and interpretations.
- the mathematical community is constructed in an imaginary way, in which everyone has access to a mathematical knowledge shared across time and place, while mathematics is constructed as a system in which it is possible for every theorem to be traced back to axioms.
- developing ways which ensure the efficacy of mathematical work/labour is valued and encouraged.

The chapter is organised as follows. First, I provide a "sketch" of the course, which aims to give the reader a sense of what this course looks like and the kinds of mathematics that it studies. I continue with an exploration of each of the four overarching discourses (formalising as a prototype of rigour, diverse participation in mathematics, imaginary practice of mathematics, and efficacy of mathematical work/labour). Through this exploration of discourses and the relations between them, I attend to the research questions to discuss how these discourses are related to values, how they contribute to the construction of a particular rationality determining how the true statements are produced and distinguished by the false ones, and how mathematics has a formatting function. Through this discussion, I do not aim to criticise the mathematics discourses produced in the course, but I do wish to highlight how these discourses do not constitute a natural truth and hence could have been different.

5.1. A Sketch of the Course

In this section, I draw a sketch of the course. I am using the word “sketch” to highlight that the presentation of the findings is a reconstruction of the class which is influenced by my ideological presuppositions, theoretical understandings, and methodological undertakings. In this sense, what I produce resembles a painted picture of the class as opposed to somehow offering an objective description of it. To make this sketch, I first focus on the overall characteristics of this course. I then “draw the details” by attending to the class’s norms as they are evident in a typical classroom interaction, the “Happy Birthday” example. I then continue to “draw the big picture”, by highlighting the course’s curriculum and its gatekeeping role for the Computer Science program.

5.1.1. Drawing the Details: The Class

The Introduction to Proofs course is structured around lectures taught by one of the three course’s professors. In the lectures, the material is introduced and some exercises/problems are discussed. Every week, there are tutorial sessions, delivered by teaching assistants, who are typically graduate students in a mathematical field. In these sessions, the focus is primarily on solving exercises and problems related to the material. Finally, there are also office hours held by the professors, during which the students can come in on an as needed/wanted basis and they can discuss their questions with the professor. During my visits, I only attended lectures and office hours sessions.

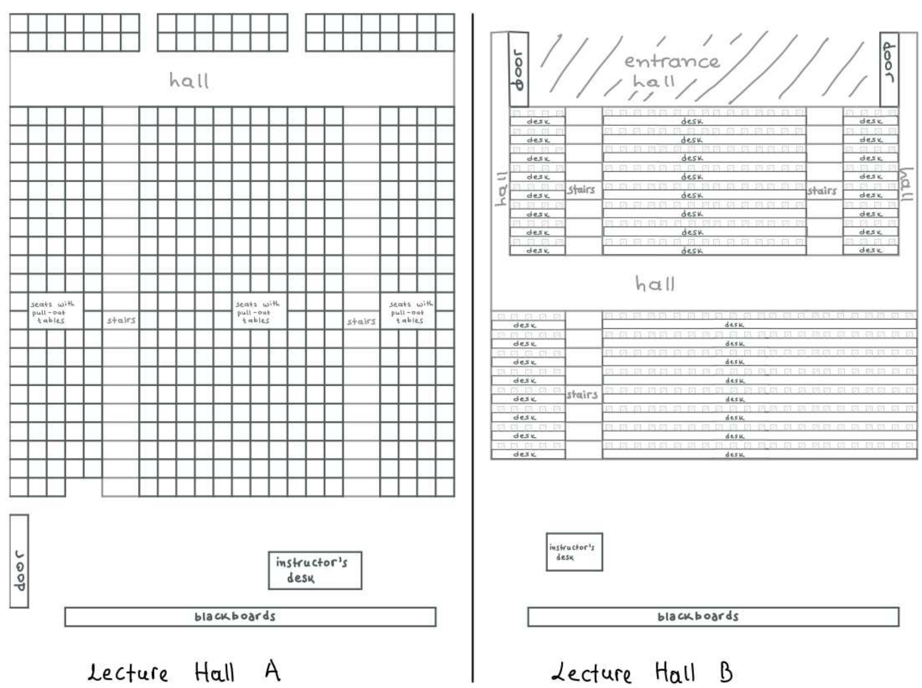

Figure 5.1: Sketch of the Proof Course Lecture Halls

Chris is one of the three professors who teach the course during the semester and is the professor whose lectures I attended. Chris is a mathematics professor, who participates in mathematics education conferences and follows mathematics education literature, being interested in issues of equity and the social functions of mathematics. He is very approachable and generally very beloved by his students. Chris's lectures happen twice a week: the first lecture of the week takes place in Lecture Hall A for two hours and the second lecture takes place in Lecture Hall B for one hour (see a sketch of the two rooms in Figure 5.1). In both rooms, students' desks and chairs cannot be moved around: they are screwed onto the floor. Lecture Hall A has the desks built in three groups separated by stairs, while there are also a few seats at the very back of the room behind a hall. Lecture Hall B is bigger and it shares a similar arrangement of desks and chairs with Lecture Hall A. In both rooms, there are blackboards behind the professor's desk and a projector on the professor's desk. During his lectures, Chris writes notes on sheets of paper which are in turn projected behind him on a white screen; the board is only used when discussing with students during the break. In this way, Chris always faces the students during class. In order to be heard in the whole room, he uses a microphone; the microphone is attached to the desk in Lecture Hall A, but it is wearable in the other room.

Shortly after COVID-19 was declared a pandemic by the World Health Organisation (WHO), the province of Ontario entered its first lockdown period. For the university, this meant that all in-person meetings and classes moved online. The course continued through pre-recorded lectures, while the online discussion boards started being active and hosting students' mathematical and house-keeping questions.

5.1.2. A Typical Interaction

In this section, I present the *"Happy Birthday" example* and I explain why I consider it to be a typical interaction in Chris's class. In this way, the reader can develop a sense of the culture and norms of this class. The "Happy Birthday" example takes place during the two-hour lecture, in Lecture Hall A. In the room, there are around 150 students, which is typical class attendance during my visits. During this lecture, Chris presents to students how a mathematical statement is formally negated. It has been around half an hour in the lecture and Chris has already explained how ANDs, ORs, negations, implications, and for-all statements are negated. For each type of statement, Chris first gives students an example of the type of statement that they try to negate, they negate the example, and then they generalise the idea that they followed. Now, the class is about to discuss how existential statements are negated (i.e., exploring what it means to negate that there exists an element x, so that a statement $P(x)$ is true, or $\neg(\exists x: P(x))$.

Chris: Existential statements.

[Chris pauses and writes on the paper that is projected on the screen behind him: "There is a person in this room who is more than a hundred years old".]

Chris: There is a person in this room who is more than a hundred years old. So, you can think, for a moment, do you think that the statement is true or false. I'm not super interested in that, but the question is how do you negate it? What's the negation of this? How'd you detect that it's false? So, write it down first. Write down the negation first. If you're trying to prove that it is false, what do you have to show?

[He gives them some time, around 45 seconds. Students start discussing the question with each other. Chris starts writing the answer, right below the point that is projected on the wall. Therefore, students cannot see what he writes.]

Chris: All right. So, what's the negation of this?

[Some students sitting at the front of the room raise their hands.]

Chris: Let's give someone else a chance. *[Addressing a particular student, who still has their hand up and probably insists on offering an answer through grimaces or words that I cannot see or hear from where I sit – with emphasis]* **No.** Sorry. *[Addressing another student who just raised their hand]* Yeah?

Student: Everyone in the class is less than a hundred years old.

Chris: Everyone in the class is less than *[Pauses, grimaces, and deepens his voice]* or equal to *[Pauses and starts talking again in his usual voice]* a hundred years old. Yeah. We might have someone whose is their hundredth birthday! Is there anyone's hundredth birthday today?

[Three students raise their hands.]

Chris: Three people! *[Students are laughing]* Wow! Happy Birthday! You know, you all have the same birthday!

[Chris pauses and writes down the answer.]

Chris: So, less than or equal to. Great. *[Addressing each one of the three students who raised their hands, by looking at each one of them]* Happy Birthday! Happy Birthday! Happy Birthday!

Chris: All right. What's the negation of an existential quantifier? So, if you want to express this symbolically. Well, write it down and then we'll check. (Chris's class, p. 6; Lecture 5.1)

This is a typical example in Chris's class in terms of the interaction between the professor and the students. It reveals that Chris has established a rapport with students. Chris is the one who controls classroom discourse; he asks students specific questions and gives them specific tasks to complete. Although Chris is the one who has the floor most of the time, students' voices are "audible" in a variety of ways. Students can take the floor by raising their hands, while there is care for different people to get the chance to participate in classroom discussions, as is evident by Chris insisting on not giving the floor to the student who has already shared their views with the class on a previous question. Besides asking questions very often, Chris dedicates class times for students to solve exercises or complete proofs working with a partner, and encourages them to discuss their ideas with each other[21]. Dedicating time for students to work amongst themselves is something that happens (usually multiple times) in every class. Each such period may take anywhere between 20 seconds and 5 minutes, depending on the nature of the task. During longer tasks, Chris moves among the students and answers their questions. Furthermore, in the example, we notice that the communication between Chris and the students occurs at several levels, including through jokes. Short jokes, like wishing "Happy Birthday" to students who claim to have their 100th birthday, contribute to the creation of a positive atmosphere, while they help the reinforcement of the material.

The "Happy Birthday" example is also indicative of the way in which mathematics is conceptualised in Chris's class. As happens usually in the course, the class navigates the construction of "formal" proofs, through connections with everyday situations. Chris uses the idea of birthdays, an idea that students are very familiar with, to give meaning to the negation of a statement. Overall, the "Happy Birthday!" example is indicative of how, in this course, mathematics is constructed as an activity which involves discussion, collaboration, and communication, while diverse ideas and interpretations are encouraged.

5.1.3. Drawing the Big Picture: The Curriculum and the Gatekeeping

The "Introduction to Proofs" course takes place in a big university in Ontario. Introduction to Proofs courses are typical 1st year courses in North America for students who wish to follow a mathematics-related discipline, as described in section 2.3.2. Chris's course is addressed mostly (but not exclusively) to students, who wish to enter the computer science program. The course acts as part of a fueling mechanism for entering into the program. In order to get a position, a student needs to achieve certain grades in a series of courses (including the "Introduction to Proofs" course); the specific passing grades change each semester, as they depend on how students have done as a group during that semester. Besides computer science students, the course is also selected by engineering, mathematics, and other

[21] Indicative of this is the fact that in the "Happy Birthday!" example, the students start engaging in discussions about Chris's question, without Chris explicitly asking them to.

interested students. In order to enter university in Ontario, these students were assessed on the basis of their high school grades and portfolio. The university charges tuition fees and entails other high costs (e.g., textbooks), for which many students need to work during the summer or throughout the year. This is important not only because people from lower socio-economic classes encounter economic obstacles to attend post-secondary education, but also because it can frame students' relationship with the university as a customer relationship.

The Introduction to Proofs course is advertised on the university's website as a course which aims to introduce students to proving techniques, logical thinking with regards to mathematical statements, and background topics that students may need throughout their studies in a mathematics related discipline. In the *"We're teaching them vocabulary and grammar" example* from Chris's interview, he talks about the goal of the course as follows:

> Chris: One of the ways that I think of this course in particular is that this is the course where we're teaching them vocabulary and grammar. In a math context. So, we're teaching them the- we're not so much teaching them the applications of a lot of these concepts. I mean, yeah, we're doing a couple big theorems, but really it's so that they feel comfortable either reading mathematics or writing mathematics. At a first year level. And we want them to be able to start, just, like evaluating their own types of statements and evaluating the statements of others mathematically. (Chris's class, p. 139; Interview with Chris)

Here, Chris offers a description of the course's goals. While in line with the website's description, Chris's description of the goals focuses on mathematical communication, which entails aspects like the development of students' mathematical language (e.g., "we're teaching them vocabulary and grammar"), the development of attitudes related to communicating mathematically (e.g., "it's so that they feel comfortable either reading mathematics or writing mathematics"), and the development of an ability to evaluate mathematical statements of selves and others (e.g., "we want them to be able to start, just, like evaluating their own types of statements and evaluating the statements of others mathematically"). To achieve these goals, the course goes through a series of topics. Before I started visiting the classroom, the class had talked about number systems, inequalities, sets, functions and informal logic (mathematical statements, truth tables). While I was visiting the class, they were working on negations, proof strategies, induction, bijections, and cardinality. When the class moved online due to the pandemic, divisibility and relations were covered.

5.2. The “*Formalising as a Prototype of Rigour”* Discourse

In this section, I discuss an overarching discourse in Chris’s class: the elevation of formalising to a prototype of rigour and a catalyst for the production of mathematical results. This discourse is manifested in the use of definition-based arguments, the avoidance of visual arguments, and the utilisation of technical language. The section is organised as follows. First, I present an example in which the class negotiates the formality of visual proofs, in order to discuss how the (in)formal is constructed through an antagonism of discourses that conceptualise mathematics as a social practice and as the product of mathematicians’ work. I then explore how the notion of the (in)formal gets constructed in Chris’s class. In the following three sections, I discuss how formalising contributes to the formatting of the world, mathematics, and mathematical objects. Finally, I explore how formality connects to the value of truth.

5.2.1. Is a Visual “Proof” a Formal Proof? Is it a Proof?

In this subsection, I start delving into the conceptualisation of the “formal” in Chris’s class, by focusing on an example, in which (what we may call) a visual argument/proof (e.g., Alsina & Nelsen, 2010; Hanna & Sidoli, 2007) is presented and discussed (although Chris himself does not use the word “visual” to describe it). I selected this example to discuss because it entails an explicit discussion about the formal, which is a central theme in Chris’s class. In this example, the conceptualisation of the “formal” is a product of the antagonism between two discourses: a discourse that conceptualises mathematics as a social practice and a discourse that conceptualises mathematics as the product of mathematicians’ work.

The interaction happens in the chapter on cardinality, as the class focuses on proving that the cardinality of the naturals is the same as the cardinality of the cartesian product of the integers with the integers (i.e., $Z \times Z$). To prove this statement, Chris presents the visual proof shown in figure 5.2, which is based on identifying a spiral representation of all elements of $Z \times Z$, so as to consider them as elements of a sequence (i.e., constructing a bijection from N to $Z \times Z$).

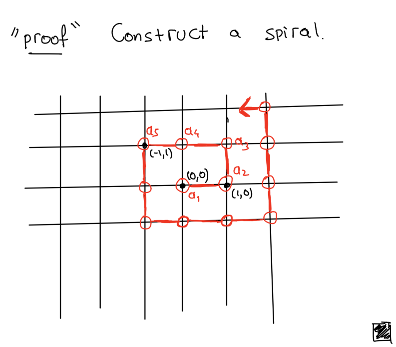

Figure 5.2: Visual Proof (Recreation of the picture that I took of the board)

Before the following interaction happens, the professor has orally suggested that the proof is not a formal one, while – as can be seen in the figure – the word proof itself has been written in quotation marks on the board (indicating that it might not be a "proper" proof). Regarding the reason that this (visual) proof is presented in the class, Chris mentions that "a formal proof of this is annoying" and that the way in which the proof is written in the textbook might be difficult to understand. After the presentation of this (visual) proof has been completed, Chris asks students if they have any questions and the following interaction, the *"is this considered a formal proof?" example* happens:

> Student: Is this considered a formal proof?
>
> Chris: This is not considered a formal proof, uhm, partly because it's- ok. Here's what [inaudible] Could you find the tenth element of this sequence? And to put it another way, if I asked you to find the tenth element and I asked someone else to find the tenth element in this room, would you come up with the same tenth element?
>
> Student: *[definitively]* Yes.
>
> Chris: So, that's pretty good. If the answer is "yes", that means that there is a pretty good common understanding of what's happening here. Uhm, ah, we could write that a little more formally, like, take one step to the right, then one step up. Then you go left one more than you've gone, go right one more than you've gone. And, like, I could explain that, like, robotically. But it's not clear that it's going to be easy to understand. Or that it will be, like, you have a better understanding of it. Uhm, yeah, we typically avoid presenting the complete formal proof of this, because it hides what's actually going on.

In this interaction, a student asks if the (visual) proof given by Chris is considered a formal proof. Chris categorically responds that it is not. As he proceeds to explain why this is the case, he asks the student back, whether the proposed sequence is presented in a way that any two people would find the same tenth element. The student responds with a definite "yes", which is a response that does not support the professor's argument regarding the informality of the proof. The professor accepts this as there being a "pretty good common understanding" and then suggests that this proof could be written more formally (through a more "robotic" description of the steps) but it then would hinder understanding.

This interaction can be read in light of two discourses being in an antagonistic relation: a discourse that conceptualises mathematics as a social practice and a discourse that conceptualises mathematics as the product of an established discipline. The discourse of mathematics as a social practice refers to mathematics being done in and through interactions between people. In the example, we can see the discourse of mathematics as a social practice in the following aspects: the discursive construction of a "good" proof as one which facilitates understanding (e.g., as shown in the sentence: "we typically avoid presenting the complete formal proof of this, because it hides what's actually going on"); the very fact that a visual proof is presented in order to make the idea of the argument clear and communicable; and the need for people to have a shared understanding of the argument (e.g., as shown in "would you come up with the same tenth element?").[22] The discourse of mathematics as the product of an established discipline is manifested in the clarity with which a visual proof does not quite qualify as a formal proof and maybe not even as a proof. This clarity implies that the standards of what constitutes a proof and a formal proof are created outside the walls of the classroom, independently of the professor's and students' interactions (e.g., the use of passive voice in "this is not considered a formal proof"). These standards are the result of the practices and norms of the mathematical community, which are in turn reflected in the academic requirements set for novice mathematicians and computer scientists by mathematics departments. In this context, it is the mathematical result (i.e., theorem, proof, etc.) expressed according to the traditional norms and rules, which is of most importance. A formal expression of this result can verify that the result is correct, can be tested, and is expressed in a way in which no misunderstanding may occur.

The first point which I wish to highlight about how the two discourses are in an antagonistic relation is that by antagonism, I do not refer to the disagreement between the professor and the student regarding the preciseness of the algorithm's presentation; in other words, here, we do not have two people (the professor and the student) antagonising each other. The two discourses are not each voiced by the professor and student respectively; instead, both discourses are voiced by the professor himself. For example, we can see both discourses present in the question "if I asked you to find the tenth element and I asked someone else to find the tenth element in this room, would you come up with the same tenth element?". Here, the idea that a formal proof leaves no space for ambiguity refers to proof as a product; however, the enunciation of it in terms of "actual" people having an "actual" conversation and agreeing or disagreeing highlights mathematics as a social practice.

[22] As shown in the following sections, the discourse of mathematics as a social practice not only appears in this interaction; it is also evident in multiple events, including when Chris encourages collaboration and communication between the students, invites students to share their disagreements and discomforts with the presented ideas, engages students in a "collecting data – making conjectures – proving the conjecture" process, emphasises the existence of a diversity of thinking about mathematics among different people, and emphasises the need for mathematical arguments to be understood by their audience(s). In all these events, mathematics is conceptualised as a social practice.

By suggesting that the two discourses are in an antagonistic relation, I intend to highlight that the concepts of proof and formal proof become floating signifiers, whose meaning is attempted to be fixed by the two discourses. Regarding "proof" as a floating siginifier, we see that the professor orally refers to the visual argument as being a proof, but he uses quotation marks when writing down the word proof in the notes. This difference between the oral and written practice can be interpreted in terms of the presence of the two aforementioned discourses. On the one hand, the discourse of mathematics as a social practice attempts to conceptualise proofs as including well-articulated visual arguments. On the other hand, the discourse of mathematics as the product of an established discipline attempts to conceptualise proofs as a formal construct in which each proposition follows from the previous one. The two discourses are differently privileged in the professor's written and oral speech. Mathematics as the product of mathematicians' work is privileged in Chris's written speech, in which the quotation marks indicate that the visual argument is similar to a proof but does not quite qualify as one. Mathematics as a social practice is privileged in Chris's oral speech, in which he never challenges the status of the proof as a proof. The differently privileged manifestation of the two discourses in the oral and written speech can be understood in terms of the different functions of the oral and written speech: in his oral speech, Chris aims to help students understand an argument, while his written speech aims (among other purposes) to act as notes for students to study. In the example, the two discourses collide with each other, while struggling to "fix" the meaning of "proof" in different ways.

With regards to "formal proof" as a floating signifier, the discourse of mathematics as a social activity tries to fix the meaning of the formality of a proof in terms of the existence of a "common understanding". In the interaction, the student suggests and Chris accepts that there is a common understanding regarding the tenth element of the sequence. Yet, this does not result in the recognition of the proof as formal. In terms of collision of discourses, we can argue that there seems to be "something more" about the formality of a proof which the discourse of mathematics as a social practice cannot quite challenge. This "more" is created by the discourse of mathematics as an established product, which is at a privileged position for defining what is formal. Formality – as developed historically – involves much more than the disappearance of the possibility of a mistaken representation, such as compliance with strict linguistic rules. To use Laclau and Mouffe's (1985/2001) concepts, the discourse of mathematics as an established practice is hegemonic, when it comes to formality of proofs. Playing a bit more with the metaphor of collision in a visual way, I would suggest that as the discourse of mathematics as a social practice attempts to fix the meaning of formality, it hits the "wall" of mathematics as an established practice. The wall is not unbreakable, but it is extremely hard to break.

As the two discourses are in an antagonistic relation in the professor's articulation, we can see the limits of each discourse. Laclau and Mouffe (1985/2001) write that "[a]ntagonism, far from being an objective relation, is a relation wherein the limits of every objectivity are shown – in the sense in which Wittgenstein used to say that what cannot be said can be shown" (p. 125). In the example above, we can see the limits of the two antagonising discourses. When universality of understanding is lost, we see the limits of the discourse of mathematics as a social activity, in the sense that this discourse loses its ability to fix the meanings of "proof" and "formal proof" as floating signifiers. We can also see the limits of the same discourse in the non-negotiability of whether a visual argument (in the context of cardinality) constitutes a formal proof. On the other side of the antagonistic relation, the limits of mathematics as an established practice discourse can be identified when the formal presentation of a proof hinders the meaning which it attempts to purify.

5.2.2. The Construction of the (In)Formal in Chris's Class

In this section, I discuss how the notion of the (in)formal is produced in Chris's class. An assumption of my analysis is that the notion of "formality" does not pre-exist uniformly in mathematics but is instead constructed in the contexts in which it appears. I propose that the "formal" is not explicitly defined in Chris's class, but it derives its meaning through the contrast between formal and informal situations. As I go through various examples in which formal and informal reasoning are opposed, I discuss some properties of the "(in)formal", as they emerge through dichotomies. Moreover, I propose that the categories of the "formal" and "informal" are determined externally to the classroom as is the categorisation of mathematical approaches, methods, and techniques into the two categories. In the end of this section, I explore the apparent contradiction of the formal being determined externally to the class's conversations, while also constructed within the classroom interactions.

Using Laclau and Mouffe's (1985/2001) theory about meaning, I propose that (in)formality in Chris's class is a nodal point; its meaning is to a great degree crystalised within the classroom discourse. While different discourses participate in the construction of the meaning of (in)formality, the "formal" is conceptualised in a way in which its meaning is quite stable, giving the sense that it could not have been constructed differently. Having the status of the product of mathematicians' labour and the promise to offer certainty and produce results, formal knowledge is conceptualised as not only powerful, but also as impossible to contest within the space of the classroom. Hence, while – as I argue – the distinction of the formal and the informal is not a natural truth, it is produced as if it were a natural truth. I suggest that this means that when learning about the formal, students also learn that they need to perform formality as if it were a natural truth.

5.2.2.1. The Construction of the (In)Formal through Dichotomies

In Chris's class, the notion of "formality" is not defined as such but it derives its meaning through its contrast with informal situations. I argue that the overarching opposition of the *informal* versus the *formal* gets specified in a series of oppositions: *i) Visual argument vs definition-based argument; ii) Simple language vs technical language; iii) Vagueness/fuzziness vs rigour; iv) Practices of mathematical labour vs product of mathematical labour; v) Key idea of proof vs textbook proof; vi) Key idea of proof vs "concrete jargon"; vii) Makes sense to oneself vs hard to understand; viii) Many possible approaches vs one concrete approach; Confusing vs productive; and ix) Intuition/instinct vs definition-based arguments*. Through these dichotomies, the formal gets constructed by being contrasted to the informal. In this way, the formal is, I argue, conceptualised as the product of mathematical labour that can be found in textbooks, entails definition-based arguments expressed in technical language and through "concrete jargon", is occasionally hard to understand, and constitutes a concrete, rigourous approach that has the potential to produce mathematical results. Further, the formal does not consist of visual arguments, key ideas, intuitive approaches, or the practices of mathematical labour themselves; it does not entail multiple approaches and it is not vague or confusing. In what follows, I provide evidence for the claims that I have made about this series of oppositions, using examples from the classroom.

In the class, the notion of the "formal" is not explicitly described. During my class visits, Chris never defines what formal is, although he often refers to formal proofs and definitions. Similarly, the textbook never defines the "formal", although it explicitly distinguishes between informal and formal arguments and definitions. In other words, although a definition of the formal is not given, the meanings of the "formal" and "informal" are constructed in several classroom conversations and in the textbook, through their contrast. I start illustrating this observation through the following two examples, in both of which an idea offered by students is deemed informal and the class subsequently works towards a formal one. The *"This is an informal reason" example* happens as Chris asks why a function is not onto and a student suggests that it fails the horizontal line test:

> Chris: Yeah. So, it fails the horizontal line test. This is an informal reason. So, can you give me a formal reason? […] So, why is this function not onto? Can someone give me a formal reason? Yeah? (Chris's class, p. 81; Lecture 8.1)

In a similar vein, the *"Infinitely many, but that's not really a number" example* happens as the class starts working on cardinality.

> Chris: How many number of elements does the natural have?
>
> *[A student responds "infinitely many".]*
>
> Chris: Infinitely many, but that's not really a number. So, instead- to be formal, we talk about bijections. (Chris's class, p. 106; Lecture 9.2)

Both of these examples are instances in which a student's suggestion is deemed informal and Chris moves on towards a more formal answer. In the first example, Chris expects an argument that uses the definition of an onto function, which has been presented to be the following: "Let $f: A \to B$ be a function. f is a surjection (or is surjective or is onto) if $\forall b \in B$ there is at least one $a \in A$ with $f(a) = b$". The visual argument of the "horizontal line test" is offered by a student and is characterised as "an informal reason". In the second example, to Chris's question "How many numbers of elements does the set of naturals have?", the response "infinitely many" is the response that students have learned throughout their mathematical education so far. In the context of this course, "infinitely many" is "not really a number". This is treated as a matter of formality and, for this reason, the class uses bijections to tackle the problem. Here, the distinction between "infinitely many" and "bijections" focuses on how rigourous the argument is, which is in turn based on the technical sophistication of the available tools ("infinity" versus "bijections"). This contrast is also depicted in the use of technical language, as a relatively simple term which exists both inside and outside of mathematics (i.e., infinity) is replaced by a more technical term (i.e., bijection). Similar examples can be identified in the textbook, where there exist statements which deem a visual argument to be informal as well as statements which suggest that a definition is informal when it contains not previously defined notions (e.g., rule, collection).

In both examples, I argue that there is a contrast between the informal and the formal. Through this contrast, the formal is defined in relation to what it is not: the "informal". The dichotomy *formal* vs *informal* takes the form of *visual argument* vs *definition-based argument* in the context of the first example and *simple language* vs *technical language* and *vagueness/fuzziness* vs *rigour* in the context of the second example. In this way, ambivalence is conceptualised as a "weakness" of the "informal" which is associated with the visual and the use of vague terms (e.g., infinity). Hence, formal mathematical practices claim to constitute a renunciation of this situation, by employing sophisticated definitions and thereby embracing rigour. It is also noteworthy that this renunciation is value-laden: characterising a response as informal and working on a formal alternative of it, shows that the expectation in this class is to develop formal arguments for the posed questions. In this way, the class embraces a particular epistemological paradigm of mathematics, in which rigour and certainty are of great importance.

So far, in the previous two examples, we have mostly seen the "transition" aspect of the informal – formal relation: students' informal contributions need to be transformed into formal mathematical approaches. A similar tendency is evident in the textbook, which is written in a way that ideas are often first "intuitively" described and consequently "formalised". However, the relation between the informal and the formal is more complex, as is discussed with the *"Read the textbook and you'll notice 'this is awful'" example* that follows:

> Chris: So, let me just write this down. $P(n)$ is $\{1, \dots, n\}$ has 2^n subsets. And see the textbook for a formal proof. The key idea is going from here to here, you take all of the old things and the new ones are just adding 3 to the old ones. So, clearly you doubled it. Now, that you noticed the idea, read the textbook proof and you'll be like "this is awful". Why is there a page of concrete jargon? Like technical jargon? Rewrite the proof. Make it make sense for you. (Chris's class, p. 43; Lecture 6.2)

In this example, students are encouraged to read the formal proof that can be found in the textbook[23] after they become familiar with its informal version; the element of transition is still present. However, Chris also characterises the formal version of the proof "awful" because, being very heavy on technical language, it is not easy to understand. Through this contrast between formal proofs being hard to understand and informal approaches making sense to oneself, I propose that the informal is elevated to something more than merely a stepping-stone towards the formal. Here, the informal is critical for making sense of the ideas related to a mathematical problem.

Furthermore, in this example, Chris presents to students the "key idea" of an inductive proof for the theorem that a set with n elements has 2^n subsets, which consists of comparing the number of sets with n and $n+1$ elements. This key idea is contrasted with the formal proof which can be found in the textbook and consists of "concrete jargon". The dichotomy between the "key idea" and the "textbook proof" can be understood as related to the process of producing mathematics and the result of this process or, in other words, the practices of mathematical labour and the product of mathematical labour.

[23] The proof that Chris refers to is a proof by induction. It defines a set with $k+1$ elements: $S = \{x_1, x_2, \dots x_{k+1})$ and subsequently defines the set $\tilde{S}$ by removing the element x_{k+1} from S. Assuming that $\tilde{S}$ has 2^k subsets that are denoted by $A_1, A_2, \dots, A_{2^k}$, S has 2^{k+1} subsets which include $A_1, A_2, \dots, A_{2^k}$ and $B_1 = A_1 \cup \{x_{k+1}\}$, $B_2 = A_2 \cup \{x_{k+1}\}$, $B_3 = A_3 \cup \{x_{k+1}\}$, …, $B_{2^k} = A_{2^k} \cup \{x_{k+1}\}$.

We have already encountered this contrast in the “is this considered a formal proof?” example, where the formal mathematical proof is contrasted to developing a visual argument for why the statement is true.

Another property of the formal, which is hinted at the previous examples’ focus on technical language and rigourous definitions, is, I argue, that a formal mathematical statement cannot be misinterpreted. This idea can be seen in more detail in the *“a negation is like a formal opposite of a math statement” example*, which happens as Chris introduces the negation of a mathematical statement:

> Chris: And a negation is like a formal opposite of a math statement. So, there’s lots of different ways of finding the opposite of something. But when we talk about it in this context, we mean a formal opposite. There are many ways to interpret opposite and here we are going to set a concrete “this is what it means to be the opposite of something”. (Chris’s class, p. 2; Lecture 5.1)

In my reading of this example, Chris presents the negation as the formal way to oppose a mathematical statement. The “opposite” is conceptualised as a general category that can be interpreted in many ways, while “negation” is one version of the “opposite”: the formal and concrete one. The last sentence of this example and its declarative syntax (i.e., “this is what it means to be the opposite of something) limits the meaning of “opposite” to a single interpretation of all the possible ones that could occur. In this way, the formal is constructed to not allow for multiple interpretations or ambiguity.

The formal, thus, becomes a vehicle for a concrete approach which is constructive of the imaginary common space. After this example, Chris presents a set of theorems which are then used to find *the* formal opposite of any mathematical statement. In other words, the practice of the “formal” that follows this example is not a single theorem or definition, but rather a group of theorems which together constitute a system. Within this system, the negation of any statement becomes the matter of a pre-paved path, on which any disagreements shall have an objective resolution determined by the path itself. Hence, I argue that the notion of the formal constructed in Chris’s class is intertwined with the discourse of imaginary mathematical practice, in which anyone can participate and collectively communicate the production of mathematical truths that do not depend on them.

5.2.2.2. The formal as productive

So far, I have discussed some properties of the (in)formal, which arise through the contrast of the formal and the informal. In what follows, I draw upon two more examples, to explore how the formal is conceptualised as productive and as having the potential to solve mathematical problems in a rigourous way. The *“let’s write it down a little more formally” example* happens as the class is about to use induction to prove the generalised triangle inequality (i.e., $\forall x_1, \ldots x_n \epsilon R\ |x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$).

> Chris: The thing that is also especially weird is that we have this “for all” statement here. So, it’s- I bring up this example, partly because when we’re proving that the triangle inequality is true for any number of terms, it’s- it kind of looks like it shouldn’t be-, proof by induction shouldn’t work. It feels like that’s the case. But let’s write it down a little more formally and we’ll see that we can prove the triangle inequality for any number of terms by induction. (Chris’s class, p. 44; Lecture 6.2)

I argue that this example shows how the formal is constructed as productive, in the sense that writing something like the generalised triangle inequality formally is captured as making it possible for a proof to be produced. Within Chris’s discourse, although the statement of the triangle inequality for n terms can be proven with the induction method, it is unclear how this proof would work if the statement is not written formally enough. Writing the statement formally is here presented as making clear how the induction method can be performed. In this way, formality is, I argue, conceptualised as enabling the production of proofs and theorems. Productivity is here opposed to the implied confusion which results from trying to prove a statement, expressed in a way that does not make clear how that could happen.

The same idea of formality being productive is evident in the *“We’ll come up with a formal way to answer this question” example*, which happens as the class is introduced to cardinality. Chris poses the problem of comparing two infinite sets in terms of which one has more elements.

> Chris: The second bit of motivation is much more fun. *[Pause]* So, functions can be used to formally measure how many elements are in a set. So, for example, which has more elements? The integers or the interval zero to two? So, we are going to use functions to measure things about how many elements a set has and compare how many elements sets have. And, so, one interesting question is: do the integers have more elements or does the closed interval zero to two have more elements? Maybe you have an intuition or instinct as to what you would guess for this. We’ll come up with a formal way to answer this question. (Chris’s class, p. 79; Lecture 8.1)

This example contrasts “an intuition or instinct” about comparing the elements of two infinite sets with a formal way to approach this question which is based on the definitions of functions. Like the previous example, formal mathematics is conceptualised here as opening up the possibility to solve problems. The discourse of formality being productive is also historically evident in the way in which formalism grew to be a very influential philosophical current in the history of mathematics. As infinity is a topic which has given rise to a series of paradoxes in the history of mathematics, Hilbert’s formalism and Cantor’s theory emerged as a response which managed to move the discipline beyond the paradoxes (Panagiotatou, 2013).

5.2.2.3. External Determination of the (In)formal

Finally, I argue that a key aspect of formality in Chris's class is that it is determined externally from the class and independently of the class's conversations. At first, this observation might appear to be contradictory with the so far proposed argument that the notions of "formality" and "informality" are constructed in the classroom conversations. The distinction that I am making here is between mathematical constructs (e.g., definitions, proofs, etc.) being considered formal and the notion of formality. I argue that it is external to the classroom discussions that it is determined whether a mathematical construct is formal or not. However, the notion of "formality", its nuances, complexity, and establishment as an important form of knowledge are all constructed in the classroom through the students' and professor's conversations.

Diving into the idea of the "external determination", the earlier discussed *"is this considered a formal proof?" example* shows how both the professor and student consider the formality of the proof to be objective. The passive voice in the student's articulation of the question "is this considered a formal proof?" and the professor's definite response "[t]his is not considered a formal proof" show that the formality is assumed to be external to the student and the professor. The characterisation of the proof as "informal" is assumed to be objectively determined according to an Invisible Third, which could be either the curriculum (i.e., the proof is not considered formal in relation to the standards of the course) or the perception of academic mathematics (i.e., the proof is not considered formal in relation to the standards set by mathematicians). While both of these levels are present in this interaction, it is noteworthy that Chris's consequent argumentation about the informality of the proof makes use of arguments related to mathematics (as opposed to the institution). This is why I suggest that the determination of the (in)formal is based on the construct of a mathematical imaginary.

Another example which shows that the formal is determined externally to the class is the *"formally, this isn't the negation of the definition of even" example.* In this example, Chris presents a proof for the statement "if x^2 is even, then x is even", after the students have had some time to work on some aspects of this theorem and proof amongst themselves. Chris presents this proof by using the contrapositive method, which consists of proving the (equivalent) statement that "if x is not even, then x^2 is not even".

> Chris: All right, so, let's go through the proof, *a* proof. So, starting with x is odd. So, there is an integer m with $x = 2m + 1$. So, formally, this isn't the negation of the definition of even, but this is a version of the negation

> of even. Is that- you can write it as $2m + 1$. We'll see a bit later in the course, by a little bit later, I mean near the end of the course, we'll see that this is equivalent to being odd, but for now, we'll use this. (Chris's class, p. 12; Lecture 5.1)

In this example, Chris uses the idea that if a number is not even, it is odd and can be written in the form $x = 2m + 1$, where m is an integer. However, he comments that formally this is not the negation of the definition of even, but only a version of it. Here, Chris suggests that for the proof to be formal, they would need to also prove that a natural number is either even or odd. This illustrates how the formal is defined externally to the class. The identification of the informality did not arise because someone wondered about the formality of moving from the statement "x is not even" to the statement "x is odd". It is instead the result of the imaginary of formal mathematics, in which every theorem arises from the previously-proven ones in a way that goes all the way back to the axioms.

Summing up this subsection, I have argued that in Chris's class, the (in)formality of a mathematical product (e.g., a statement or a definition) is determined externally to their interactions. In the class, a statement is considered to be formal or informal in an objective way, according to an imaginary mathematical construct, which includes definitions, theorems, lemmas, and proofs. However, I have argued that the conceptualisation of formality in all its complexity (e.g., its nuances, functions, and usefulness) is a product of the classroom interactions. Through their opposing relation, the informal and formal are viewed as two distinct categories that do not share any space: within this discourse, a mathematical construct shall be ("locally") either formal or informal, not both. Based on these findings, I propose that the explicitness of the dichotomy between the informal and the formal is of great importance because it contributes to the hierarchy between different mathematical knowledges. In other words, the value and "power" of formal mathematical knowledge depends on formality being a distinct characteristic of mathematical knowledge. Within this discourse, the informal is also important because it can facilitate understanding of the mathematical idea.

5.2.3. Formatting the World through Formalising

In this subsection, I discuss how formalising has a formatting function. In mathematics education, the work of Skovsmose (e.g., 1994b, 2000) and his concept of formatting has been very influential, highlighting the ways in which mathematics formats aspects of the world through modelling. Here, my goal is not to offer an ontological argument about the relation of mathematics with the world (as does the work of

Skovsmose[24]); I am instead interested in the classroom discourse about the relation of mathematics and the world and, more specifically, in the formatting aspects of this discourse. Hence, when, in the next paragraphs, I argue that the world is mirrored in mathematics, I do not claim that this is some sort of objective truth; I refer to how the classroom discourse produces an image of the world as reflected in mathematics, thereby formatting what the "world" is.

I propose that the mathematical practice in Chris's class formats the world by constructing it as "mirrored" in mathematics. Here, by "the world", I refer to what is shared between the professor and the students which is not directly related to mathematics. Formal mathematical language is assumed to offer a space in which the world can be translated. For example, sentences in a natural language, in this case English, (about basketball, grades, etc.) are mirrored in mathematics, by being "translated" in mathematics. I am going to elaborate and strengthen this position with three examples.

The *"I am tall and I play basketball" example*[25] happens as the class works on propositional logic and in particular on the negation of a mathematical statement.

> Chris: So, let's start up with a couple examples. So, let P be the statement "I am tall" and let Q be the statement "I play basketball". And so, someone says to you, someone says- well they don't quite say it like this, but they say "$P\ AND\ Q$". They come up to you and say "I am tall and I play basketball". So, how would you know, how would you detect that they are lying? They come up to you and they say "I am tall and I play basketball". How would you detect that they are lying?
>
> *[A hand rises.]*
>
> Chris: Take a moment to think about it. What would have to be true? What would you say?
>
> Student: Either they're not tall or they don't play basketball?
>
> Chris: Right, either they're not tall, so the first part that they said is a lie. Or they don't play basketball. If they are short and don't play basketball, you'd for sure know that they are lying, but you only need one of the two things to be false. So, in other words, the negation of "$P\ AND\ Q$" is either $not\ P\ OR\ not\ Q$.

In this example, Chris explains a theoretical, abstract idea through an example; a technique which he often uses in class. In particular, the negation of a statement of the form "$P\ and\ Q$" is explained through the

[24] I suggest that Skovsmose's argument about the formatting function of mathematics is ontological in nature because Skovsmose proposes something about the nature of the world by exploring how this world is shaped by mathematics. Of course Skovsmose's ontology is more relational than e.g. a Platonic ontology about the world.

[25] This example directly follows the *"a negation is like a formal opposite of a math statement" example*.

exploration of what it means to lie when saying "I am tall and I play basketball". I argue that the assumption that is made here is that English statements can be translated into mathematical language. In particular, in this example, lying corresponds to the negation, the statement "I am tall" corresponds to P, the statement "I play basketball" corresponds to Q, while the conjunction connective "AND" corresponds to the English "and". Wittgenstein's picture theory of language, as developed in the Tractatus (Duncan & Wittgenstein, 2021), is similar to how mathematics is framed in this course: mathematical language is assumed to picture reality. I suggest that this example assumes that the English language can be made to correspond to mathematical language, or more generally that a version of the world can be mirrored in mathematics.

I argue that this is not mere logic or common sense; it is a practice with strong ontological assumptions. To clarify through an example, a logical proposition like P or Q has a single truth value: P is either true or false and so is Q. But this is hardly the case for natural language statements like "I am tall" or "I play basketball". For example, there is no absolute answer to whether a person whose height is $170cm$ is really tall or whether a person who only plays basketball occasionally really plays basketball. In this sense, the mirroring of natural language sentences in propositional logic statements is not commonsensical but relates to a philosophical tradition which assumes the possibility of the existence of a mathematical language which can beyond any doubt formalise (part of) what can be said through natural languages (e.g., Duncan & Wittgenstein, 2021).

Another example is the *"Bad Guacamole" example*[26], in which making bad guacamole is corresponded to proof by contradiction.

> Chris: And the way that this starts is with a story. So, would you like to hear a story about guacamole?
>
> *[Some students reply yes.]*
>
> Chris: You know what guacamole is?
>
> *[Some students reply yes.]*
>
> Chris: Can someone tell me what the main ingredient of guacamole is?
>
> *[Someone says beans, many students say avocado.]*

[26] In the first part of this example, Chris explains what guacamole is, for those students who are not familiar with it. I do not expand on this aspect of the example here, but I will return to it in section 5.3. The *Diverse Participation in Mathematics* Discourse.

Chris: Avocados, yes. So, for those of you who don't know avocado is a fruit, is a vegetable, or a fruit. What is it? [inaudible] Well, for the purposes of the story it is a fruit and from avocados, you can make guacamole, which is a dip. So, my brother in law, this is a true story, is very good at making guacamole. He is very good. He makes extremely good guacamole. And follows the same recipe every time. And he makes it beautifully every time. So. one day, he calls me- this part is not so much true, it is for effect, but it could be true, he calls me over. He says I want you to come over for brunch. On your way could you pick up some avocados? Because I don't have any. I said, great, yeah. So, I did that. I bring the avocados, I give them to him, he makes guacamole as he does normally and it's ter-rible. It's just the worst guacamole. What can we conclude from this? What conclusions can we make? Take a moment to think about this. So, he follows the same recipe, he's really good at making guacamole. But the guacamole this time is terrible. All right, so what can we conclude for this?

Student: You picked bad avocados.

Chris: I picked bad avocados, right? That's the only thing that's changed. The initial input was wrong, was bad. This is how proof by contradiction works. If you believe every single one of your steps and at the end of everything, you get to a recipe that's terrible, it's false, it means that your input was false. So, this is the idea of proof by contradiction. So, we'll start off with the proof strategy and we'll see it in action a couple of times. (Chris's class, p. 16; Lecture 5.2)

I propose that in this example there is an analogy between preparing brunch and a proving technique, similarly to how we earlier saw an example about the mirroring between English and mathematics. In the guacamole example, there is an abstraction taking place which only looks at specific aspects of the situation. The guacamole is not looked at in terms of Latin American food culture, nutritional value, tastiness, alternative recipes, or discussions surrounding (un)ethical aspects of consuming avocados. Instead, the situation is formatted in terms of the need for the initial input (i.e., avocados) to be bad if the guacamole turns out to be bad, although a recipe (which always works) is followed – similarly to proof by contradiction.

The third example is the *"I'm not not going to school" example*. Similarly to the previous two examples, this one shows the assumption of a mirroring between the world and mathematics. However, it is a rare example that shows the limits of this mirroring.

Chris: What happens when you negate a negation?

[There are some whispers among the students.]

Chris: You have to think about that, right? *[He writes the answer in the notes]* How do you check this? Check the truth tables, if you want to be very formal about it. So, what does this actually mean? It's something like

> well "I'm not not going to class". What does that mean? It means you're going to class. Of course, in English, this has a sort of connotation that wouldn't be apparent in mathematics or wouldn't show up in math. Like, when you say I'm not, not doing this, it has its own meaning. But nots undo each other. (Chris's class, p. 4; Lecture 5.1)

In this example, we have the mirroring of an English statement to the negation of a negation in the context of propositional logic. What makes this example special (and richer than the "I am tall and I play basketball" example) is that it also showcases the limits of the mirroring. More specifically, Chris suggests that there is a connotation in the English phrase which is not shown in the mathematical representation of it. When I asked Chris about this example during the interview, he shared with me that "[he is] not so sure [he] like[s] that example" because of the lost connotation, while after being prompted by me, he mentions that "we use math to capture certain ideas and behaviours but it doesn't always-, it sort of misses some stuff. Or like it elevates certain things and hides other, not hides, but it avoids talking about some things" (Chris's class, p. 141; Interview with Chris). So, on the one hand, Chris adopts a critical lens regarding the limits of mathematics, while on the other hand, for his lecture, he searches for examples in which these limits are not highlighted. This suggests that the goal of the mathematical work in which the class engages is to fit the world into the mathematical format.

Overall, I argue that this "mirroring" is formatting because "the world" is described in a way which shapes how the world can be looked at. In this sense, formatting is important (and political) because it frames what it means to learn to read the world through mathematics. In particular, mathematics produces reality in a certain way which can be described through statements that fit propositional logic and mathematical reasoning, such as proof structures. Among other things, this means that any statement can be deemed true or false but not both. Even when it becomes clear that not everything fits this mathematical representation, which Skovsmose (2021) describes as the existence of a "similarity gap" (i.e., the difference between a situation and its mathematical representation), this does not change that this sort of mathematical practice tries to fit the world into a pre-determined form.

5.2.4. Formatting Mathematics through Formalising

In this section, I explore how the mathematical practice of this class (or in other words the doing and learning of mathematics in this particular way) formats/constructs mathematics as an abstract concept. An assumption which I am making here is that different conceptualisations of mathematics are possible. In what follows, I discuss how mathematics is conceptualised as a four-step process (i.e., a) gathering data, b) making a conjecture, c) expressing the conjecture formally, and d) proving the conjecture) and as a process of transition from specific cases to general arguments. I also discuss how the “general” is valued through the mathematical practice of the class.

Chris captures mathematical activity as a four-step process: a) gathering data, b) making a conjecture, c) expressing the conjecture formally, and d) proving the conjecture. This conceptualisation is evident in the class, as in several instances where a problem is posed, Chris invites students to engage in the aforementioned process. Below, we see one of these examples, *the “This is partly how we think mathematically” example,* which is about the sum of the first n odd numbers.

> Chris: But the question is, what is the sum of the first n odd natural numbers. So, we’re going to answer this question today formally, but before we answer it formally, we need to make a conjecture. And before we make a conjecture, we need to gather data. So, I want you to do that first. So, step 1, I want you to gather some data. Step 2, once you’ve gathered some data, I want you to make a conjecture. And step 3, I want you to express your conjecture in precise mathematical language.
>
> *[Students start conversing.]*
>
> Chris: So, your conjecture should say something like, for all natural numbers, something. Ok? So, I’m going to give you 3 minutes. I want you to gather this data, make a conjecture, express it precisely, and then we’ll talk about it. Ok? Three minutes. Any questions about what I’m asking you to do?
>
> *[Some students work on the problem by themselves, while others converse with their classmates. Chris walks around and chats with students about the problem: some students ask him a question, while he also approaches some other students himself.]*
>
> Chris: All right. So, you’ve all done some science. Gathered some data and made conjectures. This is partly how we think mathematically, when we get new problems. Someone asks a question, we gather some data, then we make a guess, and then we try to express that guess precisely, so that we can talk to other mathematicians, other scientists, and see what they think. (Chris’s class, p. 29; Lecture 6.1)

In this example, the professor invites students to work on a well-defined mathematical problem: the calculation of the sum of the first n odd numbers. He gives them three minutes to work on this problem, by suggesting that they first gather some data, then make a conjecture, and finally express their conjecture in precise mathematical language. After the three minutes have passed, Chris repeats the steps of the process that students were asked to follow and comments that students “[h]ave all done some science” and that “[t]his is partly how we think mathematically, when we get new problems”. Chris also emphasises the communicational aspect of mathematical practice by maintaining that the resulting guess is shared with mathematicians and other scientists to see what they think. This incident continues with a discussion among the students and the professor and, subsequently, with the proof of the conjecture.

Chris's four step process can be understood in relation to proof and proving research which emphasises the view of proofs as arguments and, hence, adopts a social conceptualisation of proofs. I argue that Chris's choice to engage students in a process of collecting data, making a conjecture, and expressing the conjecture formally is a manifestation of the discourse of mathematics as a social practice, as it captures mathematics as constructed through the communication that students have with their peers. It is notable that this discourse does not merely guide action towards engaging in exploratory activities as a way of doing mathematics; it actually conceptualises this practice as the way that mathematicians work. In the example, Chris proposes that "[t]his is partly how we think mathematically, when we get new problems", suggesting in this way that this is how mathematicians work. Here, an image of an "authentic" mathematical practice is created, which brings us back to the imaginary mathematics community, in which mathematicians talk with each other and through this process produce mathematics. Looking at the bigger picture of mathematics education, the discourse of authentic mathematical practice entailing reasoning-and-proving activities is common in the literature.

This element of authenticity of mathematical experience is important because it is through it that this discourse claims to define what mathematical practice is. This practice is notable from a political perspective for two reasons. First, other ways of doing mathematics are hereby losing (or not gaining) the status of "real" mathematical practice. Mathematics is framed to happen in a certain way, which is mostly exploratory, communicative, and disembodied. Second, the mathematical practice described here is a linear process; there is not much negotiation, disagreements, or antagonisms. So, although Chris invites disagreements in his class discussions[27], it is important to note that within the institutional limitations of the class, the process that is reproduced is empty of disagreements while the problems that are posed are so well-defined that it becomes difficult for disagreements to arise.

Another aspect of the four-step process is that the general is privileged. The four-step process constitutes a transition from the specific (i.e., data about specific cases) to the general (i.e., a general statement is proved). Through the following *"let's be good mathematicians" example*, this relation between the specific and the general is further discussed.

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[27] A more detailed description and evidence of the ways in which disagreements emerge or do not emerge in Chris's class follows in section 5.3.4.

> Chris: So, now, let's be good mathematicians. Uhm, we're gonna go through the example that we just did and we're going to see if there is any general phenomenon that happened. And we'd like to extract any interesting ideas, uhm, to be more general, lemmas or theorems or something. So, in this example, is there any general behaviour? Is there anything that's like an interesting idea that you can apply in other places? (Chris's class, p. 99; Lecture 9.1)

In this example, the introductory statement is revealing: Chris invites the class to "be good mathematicians"; in a very straightforward way, he performs an act of socialising students into the kind of mathematics that mathematicians do. Chris conceptualises being good mathematicians as searching for a general rule. In this way, the mathematics that mathematicians do is conceptualised as searching for general qualities. The specific case is useful for "extract[ing] any interesting ideas" that will eventually gain the status of a lemma or theorem. In other words, I argue that within this conceptualisation, the mathematics that mathematicians do consists of identifying interesting ideas from a specific case and being able to apply it to different cases. Working at a general level (e.g., through lemmas and theorems) is valued and treated as a goal of the mathematical practice.

In mathematics education more generally, the idea that mathematics is a form of general knowledge has been considered commonsensical. However, the "power" of general knowledge has been challenged by scholarship which conceptualises mathematics as a cultural product. For example, Abtahi talks about the "power of abstraction" which leads to "producing more, faster, and *the same*" (Abtahi & Wagner, 2016, p. 39), while Fasheh (2015) parallelises the process of extracting general principles from contextualised practices with the process of extracting sugar from food: the process of extracting turns the extract into poison. Epistemologically, general knowledge assumes that details are unimportant; instead, underlying common principles between different situations exist and it is desirable to study them. The general knowledge becoming of importance means that the particularities of different situations get lost, as they are (over)simplified.

Summing up this section, I have argued that in Chris's discourse, mathematics is conceptualised as a four-step process: a) gathering data, b) making a conjecture, c) expressing the conjecture formally, and d) proving the conjecture. Furthermore, mathematics is captured as a practice which moves from the specific to the general. Moving from the specific to the general through a rigourous process captures, in this discourse, the mathematics that mathematicians do. While this discourse does not reject the existence of other mathematics, the institutional space values the mathematics of mathematicians, namely rigourously identified and proven general truths.

5.2.5. Formatting Mathematical Objects through Formalising

In the previous section, I discussed how mathematics is formatted as an abstract concept. The goal of this section is to explore how mathematical objects themselves are constructed through the mathematical practices of the class. In particular, I focus on definitions and I explore how in the mathematical practice of Chris's class, definitions are a) the starting point of mathematical practice and b) an ending point of mathematical practice, in the sense that – contrary to the exploration of theorems and proofs – they are not negotiated in class. I argue that this practice establishes definitions as having a sense of finality and I end this subsection by pinpointing some political aspects of this finality as crucial for creating the illusion of things not being able to have been different.

Before exploring the role of definitions in classroom interactions, I will discuss the role of definitions as starting points in the textbook. Here, "starting points" can be understood quite literally. In the textbook, the exploration of a mathematical concept starts with definitions which are followed by theorems and proofs. We also encounter phrases like "the proof follows directly from definition", which show that definitions have the role of describing an object beyond ambiguity, resulting in all its properties to being derived from this description. In this sense, definitions have a prevailing position for mathematical knowing in the textbook because they constitute the beginning of an exploration.

To explore how definitions are both a starting point and a non-negotiable point in class interactions, we can briefly think about the *"here is our first formal definition" example,* an example which happens as the class starts working on cardinality.

> Chris: All right. So, here is our first formal definition and this is the one that we are going to come back to, over and over again. Ok? *[Pause]* So, let's take a look at these two definitions. When are the cardinalities of the two sets the same? How would you recognise this? How? With respect to functions. (Chris's class, p. 94; Lecture 9.1)

I propose that this example captures the definition as the beginning of the engagement with cardinality. In terms of enunciation, the notion of the beginning is reinforced through the inversed syntax of the phrase "here is our first formal definition", which places emphasis on this notion of "the first formal definition"; and the use of multiple phrases that indicates that this first definition is the point to which the class will be returning regularly (i.e., "come back", "over and over", "again"). At a more conceptual level, we can notice that this particular mathematical exploration, which is initiated through the "here is our first formal definition" example, takes the form of the presentation of a definition, followed by a discussion of what this definition means (as indicated by the question "When are the cardinalities of the two sets the same?") and how it can be used (as indicated by the question "How would you recognise this?"). One thing that this trajectory does not do is explore questions of the sort "does this definition make sense?" or "could it be otherwise?"; in other words, it constructs the definition and its consequences as given and unchangeable.

We see a similar case in the following example, the *"these exercises will lead us to a natural question" example:*

> Chris: So, these exercises will lead us to a natural question. Which is, are all sets countable? Right? The answer is no. And it's extremely weird. *[Pause]* So, definition. So, a set B is uncountable, if B is infinite and the cardinality of B is not equal to the cardinality of the naturals. (Chris's class, p. 109; Lecture 9.2)

Similarly to the previous example, here, definitions are conceptualised as starting points. The exploration of what is called "uncountable sets" starts with a comment about their existence (i.e., "are all sets countable? Right? The answer is no. And it's extremely weird") followed by their definition. Furthermore, although "it's extremely weird", the definition is not negotiated: it is articulated by the professor and written in the course notes. The characterisation of the question about the existence of uncountable sets as a "natural" question contributes to the conceptualisation of definitions as flowing from already existing mathematical knowledge. I argue that this creates the image of mathematical definitions and mathematical objects as if they could not have been otherwise.

This assumed determinism is evident in the *"how would we formally define the vertical line test?" example*, which captures the mechanics of formally defining an informal idea. This example happens as Chris shows how the idea of the "vertical line test" (i.e., a "test" to see if a graph represents a function) can be defined in a formal way:

> Chris: So, how would we formally define the vertical line test? Well, it says, there is an x that has two y's. But how do we make that make sense in terms of A's and B's? Well, it should say that there is an a in A and there are two b's such that, (a, b_1) and (a, b_2) are both in the relation. So, that's the sort of "not equal" version. And then, we can say it slightly differently. Is that if (a, b_1) and (a, b_2) are both in the relation, so our a coordinate doesn't change but out b coordinate might possibly change- If those pairs are in the relation, then actually their second coordinates are equal. You can't have an a that outputs two b's. That is in fact our vertical line test. (Chris's class, p. 121; Online Lecture)

In this example, Chris first suggests that they will formally define the vertical line test, a visual argument that is familiar to students since high school. To formally define it, Chris works in steps, in each of which he describes the idea of the vertical line test a bit differently than before. First, he refers to the visual representation of the vertical line test and describes it as “there is an x that has two y’s”. Then, he “translates” this idea in terms of sets A and B, suggesting that “there is an a in A and there are two b’s such that, (a, b_1) and (a, b_2) are both in the relation”. Finally, he offers a different representation which proposes that “if (a, b_1) and (a, b_2) are both in the relation […], then actually their second coordinates are equal”. In this way, the transition from the informal to the formal constitutes a transition from a visual argument towards an algebraic one. It is noteworthy that the “informal” situation is gradually transformed into the formal one, in a way that appears natural – as if no other options were possible. The grammar used is indicative of this: for example, the phrase “it should say” is strongly declarative, allowing no space for negotiation.

Overall, definitions (and sometimes theorems) construct mathematical objects, which are part of (mathematical) reality. Then, proofs construct/explore/confirm what is true about these objects. I suggest that the non-negotiability of definitions is crucial for creating the illusion of mathematical objects, and hence mathematical reasoning, not being able to have been different. Yet, there is no essential reason for mathematical objects to be the way they are. To give an example from the history of mathematics that pinpoints the legitimacy of this assumption, the development of cardinality on the basis of $1-1$ correspondence (i.e., defining that two sets have the same cardinality when a $1-1$ correspondence between them exists) was not historically inevitable; instead alternative theories could have predominated (e.g., based on the idea that when a set is a subset of another set, then the cardinality of the former is smaller than the cardinality of the latter) (Hacking, 2014). Further, the work of Lakatos (1976) shows the complexity and amount of negotiations that can go into defining an object: the process of defining an object in different ways means that different objects are produced and different statements are true about these objects.

To sum up, in this section I have argued that definitions play a central role in formatting mathematical objects. Definitions are conceptualised as being the starting and ending point of mathematical practice. They constitute beginnings, in the sense that tackling a new area of mathematical expertise in the course starts with defining the basic concepts. They also constitute ending points in the sense that they have finality – the definition of a concept is never the matter of disagreement or discussion; definitions are only discussed as a way to become understood, not to be contested. This gives the sense that mathematical objects could not be different, a thesis which can be contested by the fields of philosophy and history of mathematics.

5.2.6. Truth as a Value and Formality

In the construction of the formal as it has been discussed so far, there is a central assumption made, often implicitly, which is the existence of a single truth and the possibility of its discovery. In what follows, I explore how I see truth as being manifested as a value in Chris's class, in close connection to the notion of formality. First, I discuss how truth manifests in the class in implicit and explicit ways. I then explore how truth relates to the notion of consistency. I conclude this subsection by commenting on the political dimensions of truth.

One of the most central ways in which truth manifests as a value in Chris's course is in relation to proof and proving. The development and presentation of a proof provides verification for the truthfulness of a mathematical statement. In the *"well, we're gonna prove it" example*, which happens in the context of cardinality, we see that proving a statement signifies that the statement is true:

> Chris: This smallest interval from 0.1 to 0.2 has the same number of elements as the interval from 1 to 10 to the 10 to the 10. They have the same number of elements. How does this speak with people? Uncomfortable maybe? Well, we're gonna prove it. (Chris's class, p. 104; Lecture 9.2)

In this example, the professor introduces the mathematical statement that two line segments with different lengths have the same number of elements. This statement is considered counter-intuitive in mathematics education (e.g., Fischbein et al., 1979) and is also framed as such by the professor (through his asking how this "speak[s] with people" and by suggesting that it might feel "uncomfortable"). This counter-intuitiveness and uncomfortable feeling is "resolved" through the last sentence ("Well, we're gonna prove it"), through which Chris implies that the statement is true. Within this discourse, proving a statement signifies that the statement is undeniably true.

In the "well, we're gonna prove it" example, as is the case in most interactions, truth is an implicit value: Chris does not need to state that it is *true* that the two intervals have the same cardinality. One of the few occasions, in which truth is explicitly mentioned is in the context of truth tables, which are discussed early in the course. The *"it is true" example* showcases how truth is discussed in this context.

> Chris: Do we think this statement is true or false? This statement right here.
>
> Student: It is true.
>
> Chris: It is true. (Chris's class, p. 9; Lecture 5.1)

In this example, truth is discussed explicitly. Here, the discussed mathematical statement can be either true or false (as is indicated by Chris's question "do we think this statement is true or false?"), while its truth value is determined objectively (as is indicated by the fact that there is no hedging in both the student's and Chris's responses). Truth tables establish an understanding for how truth shall be understood in mathematics – which underlies all the course content. In this context, any statement shall necessarily have a truth value, which is also unique (i.e., a statement cannot be both true and false).

This notion of consistency of truth is discussed in an explicit way, in the *"you can only prove one of the two things" example*:

> Chris: Well, they're negations of each other, so only one of the two things- The question was "is it easier to prove a there exists than a for all statement?" I mean it is easier, but you can only prove one of the two things. Because one of the things is going to be true, the other thing is going to be false. (Chris's class, p. 8; Lecture 5.1)

Here, responding to a student's question regarding whether it is easier to prove a "for all" or a "there exists" statement, Chris maintains that only one of them can be proven because only one of them (i.e., either a "for all" statement or its "there exists" negation) is going to be true. The modality of Chris's phrasing is indicative here: Chris says that "one of the things *is going to be true"* – a phrasing which indicates that the truthfulness/falseness of the statement is independent of the person who enunciates it and proves it. Similarly, the phrasing "*you can only prove* one of the two things" suggests that it is only possible for the person who proves, following mathematical rules, to produce results that are true according to the mathematical system within which they work. Consistency, as an aspect of truth, is central here, as it is not possible within this discourse for a statement and its negation to be both true. Moreover, completeness emerges as a concept from Chris's discourse, as he implies that a statement or its negation can be proven. However, in academic mathematics, the famous incompleteness theorem of Goedel suggests that consistency and completeness are not both possible (Anapolitanos, 2009). This means that if a system is consistent, it is not complete: there are statements for which it is not possible to decide whether they are true or false (i.e., whether they can be derived from the system of already proven statements). The "you can only prove one of the two things" example seems to build an imaginary of mathematical theories, which contrary to Goedel's theorem, are both consistent and complete.

Finally, before exploring why I consider "truth" to be political, I present and discuss an example from a student's (Vijin) interview, the *"everything sort of comes down to true or false" example:*

> Vijin: I think that when a topic is well defined enough, it can be true or false. Because, maybe, maybe, you can't have -1 in the natural numbers. But if I say $x = -1$ in the natural numbers, that's false. But if I say $x = -1$, you don't really know. Because it could be the natural numbers. So, just by defining it more, you can

> figure out if something is true or false. I think everything sort of comes down to true or false. It just needs to boil down, hard enough. (Chris's class, p. 154; Interview with Vijin)

In this example, Vijin expresses that if defined well enough, any statement can be deemed true or false. This discourse of Vijin is in agreement with the imaginary of truth as described earlier. In this way, mathematics is conceptualised as a discipline which can decide about the truthfulness of anything, as long as it has been defined suitably[28].

In the rest of this subsection, I discuss why I consider the focus on truth to be political. But first, I go deeper into the political aspects of truth, as a way of denaturalising the commonsensicality of mathematics being related to truth; I wish to highlight that the mathematical focus on truth is not inevitable. Truth is undoubtedly a value in academic mathematics in general, in the sense that mathematics is assumed to be dealing with the identification, study, proof, and application of true statements. However, the focus on truth is not inevitable when doing mathematics. For example, when mathematics is conceptualised in relation to art[29], it might be possible to ask questions which highlight truth or falsehood, but this can become irrelevant or even unnatural, as questions about patterns or aesthetics are more prevalent.

I argue that mathematical truth, conceptualised in an absolutist way, contributes to the construction of mathematics as a special kind of knowledge. Within the discourse of Chris's class, mathematical truth is objective, it is consistent (a statement may not be both true and false), and it is independent of individuals' opinions, preferences, and labour. Moreover, mathematical meaning is final and it pre-exists the people who do the mathematics – any negotiation that might happen shall be in relation to reaching that truth and not the truth itself. Although truth technically depends on the rationality and the axioms of a mathematical system, this does not become visible. The development of non-Euclidean geometries shows exactly this – that no absolute mathematical truth exists. Yet, when working within a certain paradigm, as happens for example in Chris's class, this does not seem to be the case. The professor and students operate within a regime that conceptualises truth as absolute. Despite the significant amount of scholarship that challenges the absolutist version of truth in mathematics and highlights the socially constructed character of mathematical knowledge (e.g., Lakatos, 1975; Skovsmose, 1994b), the notion of

[28] This view of Vijin's does not mean that Chris's discourse is or should necessarily be interpreted in the way that I do through my analysis or the way that Vijin does. It is, however, an indication that Chris's discourse *could* be interpreted by students in similar ways to mine, among of course other possible (and potentially even contradictory) interpretations.

[29] The Bridges conference is an example of a community which highlights and inquires into the connections of mathematics with art, music, architecture, and culture.

an absolute, objective truth in mathematics remains appealing (Mendick, 2006). Thus, mathematics is elevated to something that gives access to an independent, absolute truth. In this way, mathematics is elevated to a special kind of knowledge.

Moreover, mathematics becomes a tool for abstracting the world in terms of truth and non-truth. As truth is conceptualised in an objective and absolutist way, mathematics is constructed as an ideology-free construct. In this way, mathematics becomes a system of knowledge which claims to produce true statements, in a final and value-free way, while it also claims that these true statements can speak for the world outside mathematics. The assumption of a system which can talk about the world, by offering "truth" in final ways (which shall never change) is very political.

5.2.7. Bringing it All Together

In this section, I explored the "formalising as a prototype of rigour" discourse in Chris's class. I discussed how the notion of the formal gets produced through the antagonistic relation between two discourses: a discourse that conceptualises mathematics as a social practice and a discourse that conceptualises mathematics as the product of mathematicians' work. I proposed that the notion of the formal is constructed in the classroom through its opposition with what it is not: the informal. Yet, the consideration of what is and is not formal is derived from outside the classroom walls: from academic mathematics. I also argued that mathematical formalising contributes to formatting the world, the formatting of mathematics as an abstract concept, and the formatting of mathematical objects through formal definitions. Finally, I explored how truth is a value that is implied in the mathematical practice of formalising.

Mathematics education has put a lot of emphasis on assisting students to transition from informal ways of knowing mathematics to formal ones. For example, from a mainstream mathematics education perspective, learning mathematical language constitutes a transition from "everyday" ways of speaking about mathematical objects towards more conventional, formal mathematical language (Barwell, 2016). In undergraduate contexts, the idea that students shall get socialised into formal, academic mathematics is commonsensical and, as such, less articulated and discussed. The assumption is, however, evident in many theoretical frameworks of undergraduate mathematics education. For example, the three worlds of mathematics theory (Tall, 2004) assumes the idea that learning university mathematics is about moving towards a formal way of doing mathematics. Contrary to this mainstream assumption, there have been criticisms of this view which propose that the notion of formality is not defined in a uniform way in mathematics, but it is instead constructed through the class interactions (e.g., Barwell, 2016). Through my data analysis, I tried to highlight that this is also evident in Chris's class. The discursive construction of the formal through the classroom interactions is a view which highlights that what is considered formal mathematics in one context might be informal in a different context. To illustrate, a formal proof in Chris's class is probably neither a formal proof in a Grade 12 calculus course nor a formal proof in a graduate real analysis course.

Finally, formal knowledge and formalising are of particular importance from a political perspective because through formalising, one form of knowledge becomes more accurate, valuable and trustworthy than others. In this way, formal mathematics becomes powerful as it is constructed in relation to truth as well as through value-ladden oppositions. This means that alternative ways of doing mathematics, either at a scientific or a cultural space, are viewed and evaluated in relation to their relation to formal mathematics.

5.3. The "*Diverse Participation in Mathematics"* Discourse

The second overarching discourse concerns the diverse participation in mathematics. Diversity of thinking is a prevalent value in Chris's class. It refers to a view of mathematics education which strives for equity, diverse participation, and inquiry approaches and assumes the existence of multiple ways of thinking and intuitions. This section is organised as follows. I first discuss how historically underrepresented groups in mathematics become visible in Chris's course. I continue by examining how Chris's course is a space in which diversity in ways of doing mathematics as well as of speaking about mathematics are manifested. In what follows, I turn to what I call "disagreements that never emerge" and discuss how they showcase different possible relations with mathematics. I then discuss the implications of the diverse participation in mathematics discourse for mathematics rationality, by proposing that this discourse conceptualises mathematics as happening through communication between real people. I conclude this section by summarising the findings associated with the diverse participation in mathematics discourse.

5.3.1. Historically Underrepresented Groups in Mathematics

Chris's discursive practice takes into consideration the participation in mathematics of groups who have historically been underrepresented in academic mathematics (e.g., Indigenous students, Black students). Entering Chris's office on campus, the visitor comes across a book about Cree numbers, placed at a visible space. In his interview, Chris talks about reconciliation in a way that shows his active involvement with reconciliation in mathematics education, through participation in relevant conferences and engagement with relevant ideas, such as Gutiérrez's (e.g., 2017) notion of mathematx. Moreover, during class time, Chris structures his practice in a way through which he gives individual attention to students who are visible minorities. When asking the class to work amongst themselves, Chris often moves around the room to answer students' questions. At that time, he also approaches students who are visible minorities sitting by themselves, to check in how they are doing. Through this kind of actions, Chris attempts to perform a "counteraction" to "math [being] historically dominated by white men of privilege" (Chris's class, p. 144; Interview with Chris).

In what follows, I present two examples from Chris's class, through which, I argue that, Chris challenges the mainstream history of mathematics, within which white men of privilege have been the ones who predominantly do mathematics. In the *"Fibbonacci didn't invent these numbers" example,* Chris starts from the so-called "Fibbonacci numbers" to highlight that even though this sequence is named after Fibbonacci, the Italian mathematician was not the one who invented them.

> Chris: So, Fibonacci numbers are historically- mathematicians tend to like them. And especially, like they show up in a lot of strange places. Places that you wouldn't expect. And they're quite simply defined. They're defined from literally just 1 and itself. And they're quite nice. Do, you know why they were named after Fibbonacci? Did he invent them?
>
> Student: Who's Fibbonacci?
>
> Chris: Who's Fibbonacci? That's a good question. Uhm, Fibbonacci was a mathematician in the 1600s. He's an Italian mathematician. And he is famous for writing an accounting textbook. So, he introduced double entry book keeping to Europe. Uhm.
>
> *[A student says something that I cannot hear.]*
>
> Chris: Yes, yes *[laughs]* It's an extremely big deal! Uhm, in that book, he also explained the Fibbonacci numbers.
>
> *[A student says something that I cannot hear.]*
>
> Chris: Yeah, I'll talk to you, I'll tell you. But the thing is that Fibbonacci didn't invent these numbers. He brought them over from, not Europe, from India. Yeah. He introduced them to Europe. So, historically, these numbers were well known before this. (Chris's class, p. 58; Lecture 7.1)

In this example, Chris comments on the Fibbonacci numbers being named after the Italian mathematician Fibbonacci. Chris highlights that Fibbonacci did not discover these numbers but introduced them to Europe from India where they were well known before Fibbonacci. Although Chris does not explicitly critique this event, I suggest that he implies that the action of naming a mathematical object after a mathematician is associated with them having discovered/invented the object. This is evident by Chris's initial question “Did he invent them?” as well as the use of antithesis in the phrase “But the thing is that Fibbonacci didn’t invent these numbers”. Through this example, Chris highlights that the possible expectation that Fibbonacci discovered the Fibbonacci numbers is untrue, thereby implicitly showing, I argue, how the history of mathematics has been generous in terms of attributing fame to European mathematicians.

The following *“how unfair mathematics is” example* follows a similar line, exploring the idea that mathematical credit is not always given fairly to those who develop a mathematical result.

> Chris: Yeah?
>
> Student 1: $arctan$.
>
> Chris: $arctan$, yeah!
>
> *[Chris pauses. There are some laughs and chatting among students.]*
>
> Chris: $arctan(x)$ is a great example. Uhm, it passes the horizontal line test. *[turning to another student]* You were going to say arctan?
>
> Student 2: Yeah.
>
> Chris: Aaa. This is- there is a life lesson in here, which is about how unfair mathematics is. Uhm, sometimes we give credit to not the person who came up with it, but the person who was loudest or the person that the publisher decided on. So, unfortunately, [inaudible]. Would it be even worse, if I named it after [them] *[uses the pronoun of Student 1]*?
>
> *[Students are laughing and some say “yes”.]*
>
> Chris: That would just [inaudible]. That happens all the time in math. We name things after people who didn’t invent them. (Chris’s class, p. 83; Lecture 8.1)

In this example, Chris draws upon an interaction that happens among some students to discuss how credit is attributed to mathematicians. More explicitly than in the “Fibbonacci didn’t invent these numbers” example, Chris refers to the expectation that mathematical objects be named after those people who invented them. Chris highlights the social framework of mathematical discovery, by proposing that sometimes being louder or the publisher deciding on one person rather than another can mean that a person gets credit for an idea, although they might not have been the one who came up with it.

The reading of the “Fibbonacci didn’t invent these numbers” example and the “how unfair mathematics is” example that I have proposed is not the only possible interpretation of these examples. For example, the students in Chris's class who participated in or observed these interactions might have interpreted Chris's claims in different ways. However, I propose that these examples were intended as counter-discourses that emphasise diversity by Chris himself, as Chris shares more thoughts about this issue in his interview with me, as part of the *"Then we're like, ok, I'll take it for granted" example*:

> Chris: So, reflecting on some of the questions that you asked. One of the things that I would like to do a little bit differently is modify, ahm- […] We attribute theorems to the wrong people. And like not necessarily on purpose. It's just that we don't often have time to think about the history of things. And so, if someone said that this theorem was due to so and so. Then we're like, ok, I'll take it for granted. This is like, and this is like quite bad. […] Certainly in number theory this is not great. We, we attribute a lot of things to like the French and Germans of the 1700s. Whereas, in reality a lot of these things were known well before that. Or they were known like to Indian mathematicians or Islamic mathematicians or Chinese mathematicians. And we just never bothered to figure out whether they also knew them. And we presented- like at some point we have to go through and fix these things. And I understand why you might want it, like you might still want to call it Pythagoras's theorem. But- for like ease of reading reasons. But we should probably go through and give it a better name. Or something that's more reflective of who came up with these things. Or even like, not even necessarily, you don't necessarily have to decide who came up with it, but I don't know. It feels like the status quo is making a decision to continue emphasizing one particular thing and we could probably go through to correct those. (Chris's class, p. 150; Interview with Chris)

In this example, Chris explicitly states that theorems are sometimes attributed to the wrong people and suggests that one activity in which he would like to engage is going through the theorems of his courses and modifying the names of the theorems so that it becomes clearer who invented them or who was familiar with them before Europeans. Chris also refers to the status quo "emphasising one particular thing", referring to the attribution of mathematical knowledge to, as he elsewhere mentions "white men of privilege". At the same time, he emphasises the need that "we" (probably referring to mathematics instructors) engage in a practice that does not take this historical knowledge for granted. In this way, Chris offers a counter-discourse with regards to the history of mathematics which emphasises the achievements of peoples other than the Europeans of the 1700s including Indian, Islamic, and Chinese mathematicians.

Overall, the idea that mathematics does not fairly depict the people and peoples who contributed to its development but presents a narrative that privileges the contributions of White, Europeans, men of privilege is becoming more mainstream in mathematics education (e.g., Bishop, 1990; Joseph, 2019). In his class, Chris presents aspects of this discussion as an action that attempts to counter the dominant narrative which is responsible for marginalising non-privileged people and peoples and reproducing the discourse that being acknowledged in mathematics is only for some.

5.3.2. Diversity in Ways of Doing Mathematics

Besides an effort to include more (traditionally excluded) people in mathematics, diversity in Chris's class also manifests through multiple ways of thinking about and working on mathematics. In this section, I explore how this idea manifests in Chris's class. First, I discuss how the classroom itself is organised around the idea that there are diverse ways of thinking about mathematics. Then, I examine how Chris's discourse constructs the work of mathematicians in terms of diversity. I conclude this subsection, by outlining how this idea of diversity is not limited to thinking about mathematics, but also extends to preferences and emotions related to mathematics.

5.3.2.1. Diversity of Thinking in the Classroom

Chris emphasises the existence of diverse ways of mathematical thinking and structures the class in a way in which this diversity is utilised. For example, often, Chris mentions that he is going to present "*the* proof" of a theorem, only to immediately correct himself and suggest that they are going to see "*a* proof". The *"Or a proof I should say" example* is an example of this.

> Chris: So, we're going to prove that for-all statement. So, let's go through the proof. Or *a* proof I should say. (Chris's class, p. 9; Lecture 5.1)

This self-correction of phrasing indicates that multiple proofs exist or may exist in the future about a particular mathematical statement – an approach which assumes the existence of multiple, valid ways of thinking about a mathematical concept or statement. The fact that Chris does not use the indefinite article "a" all along is in line with the course's curricular practice. In this class, working on a theorem involves a specific proof that shall be presented and subsequently the class's focus will be on understanding the proof rather than, for example, searching to construct alternatives of it. In this sense, this verbalised distinction between the two articles "the/a" constitutes a facet of an antagonism between two discourses: mathematics as a singularity and mathematics as a plurality.

When, during the interview with Chris, I ask him about this practice of his, he mentions the following, which I have labelled as the *"it makes sense to…let people give their perspective on stuff" example*:

> Chris: In a very, like, practical sense, in a day-to-day sense, the more voices you have the- the- the more different ideas you can come up with. I'm learning things in this course, all the courses I teach all the time, there's lots of things I don't know and students especially in this university come from lots of different backgrounds. So, there's lots of like different tools and techniques and ideas and perspectives that are new

> to me and will surely be new to the rest of the students. So, it makes sense to talk about them and to let people give their perspective on stuff. (Chris's class, p. 144; Interview with Chris)

In this example, Chris makes a connection between the diversity of students' backgrounds and the diversity of ways in which people think about mathematics. Chris constructs the notion of diversity in terms of *tools, techniques, ideas, and perspectives* which students come to class with and others (himself included) are unfamiliar with. The construction of this diversity in positive terms, as something that other students and he can learn from, leads to structuring the class accordingly. In a naturalised way (as is evident by the phrase "it makes sense"), Chris indicates that talking about the different ways of thinking about a mathematical idea needs to be part of mathematical practice.

Diversity of thinking is also stated in more explicit ways by Chris, as is the case with the *"the way people think of mathematics is totally different" example*. This example happens after some students express multiple views and disagree on how intuitive a mathematical statement is.

> Chris: Uhm, it's somewhat challenging when you want to discuss mathematics. Because the way people think of mathematics is totally different. Uhm, and it's hard to get in someone's head. So, just because someone has a different approach or something is intuitive or isn't intuitive, it doesn't mean that it's necessarily wrong. It's just that like you don't understand the way that they're thinking. And so, it's really, like, history has been very bad about, like, people thinking in a different way and it's like, it really presses them down. Uhm, so, you got to be careful. (Chris's class, p. 54; Lecture 7.1)

Here, Chris emphasises that people think differently about mathematics. A distinction is made between Chris's construction of diverse thinking and mathematical history in which diverse thinking has been pressed down. In this way, Chris creates a counter-narrative to the mainstream mathematical practice in the same line of the examples discussed in the previous subsection. This counter-narrative is not a mere theoretical account about the history of mathematics, but it has a direct connection to students' mathematical practices, as it encourages them to consider that when they do not understand a different approach, it does not mean that it is wrong.

This concept of diversity that is constructed by Chris's discourse is reflected in the way in which Chris structures his lectures. Students' own sense-making process is often shared with the whole class, as students are asked to share their own examples of a mathematical idea. For example, in one of my visits, the professor asks students to give an example of why "$P\ implies\ Q$" is not the same as "$Q\ implies\ P$". Students are given time to think and discuss about it and three different students give three different accounts of it. One of the students offers an explanation using truth tables, a second student suggests that the statement “If it rains my shoes will get wet” is different than its converse “If my shoes get wet, then it rains”, and a third student proposes that the statement “If the entire planet explodes, I’m not going to class tomorrow” does not mean that if they do not go to class tomorrow, the planet will have exploded (Chris’s class, p. 13; Lecture 5.1).

Moreover, in Chris’s class there is consideration of the existence of a diversity of ways in which students interact with the world. For instance, in the “bad guacamole” example, discussed in section 5.2.3., Chris uses a (partly fictional) story about guacamole to describe the proof by induction method. In this example, Chris asks students if they know what guacamole is and explains that avocados are the basic ingredient of guacamole. Not assuming that all students are familiar with a Latin American food that has made its way to mainstream North American food culture is an instance of accepting and navigating the diversity of students’ diverse experiences with the world. Similarly, in the context of proof by induction, Chris mentions that a common parallelisation of induction is the process of climbing a ladder. To clarify this parallelisation, Chris explains the two skills required to actually climb a ladder (i.e., getting on the ladder and once on the ladder going one step up each time) and asks students how many of them have actually climbed a ladder themselves (Chris’s class, p. 35; Lecture 6.1).

5.3.2.2. Diverse Practices among Mathematicians

Besides the consideration for the diversity of ways in which students interact with the world and mathematics, Chris’s discourse also addresses diversity in the ways in which mathematicians engage with mathematics. The *“they all gave six differently worded proofs” example* highlights this diversity.

> Chris: Last term, there were six instructors. And I asked the instructors to all write a proof by induction for this theorem and they all gave six differently worded proofs. So, there’re lots of ways to describe things, but there’re only a couple essential features, like they need to say they’re using induction, they need to tell you what the induction hypothesis is. Things like this. Prove the basis case. Blah blah blah. (Chris’s class, p. 42; Lecture 6.2)

In this example, Chris refers to six professors providing six differently worded proofs for the same theorem. All these proofs followed the same method (proof by induction) and entailed a few common elements but were otherwise different. Through this example, Chris constructs the notion of diversity as existing among the practices of professional mathematicians and thereby objectively as is indicated by his phrase “there’re lots of ways to describe things”. At the same time, this diversity is bounded by what Chris calls “a couple essential features” that any proof by induction needs to entail. In this sense, the constructed diversity does not imply that “anything goes”, but it instead surrounds an intersubjectively constructed mathematical core.

Another example that pinpoints the diversity of professional mathematicians’ practice is the *“Richard K. Guy” example.* This example refers to a Canadian mathematician and highlights his work from a non-conventional point of view.

> Chris: Before we actually get started in the material today, I want to mention something. So, yesterday, a very important and famous mathematician died.
>
> *[Chris holds a red book of Guy’s.]*
>
> Chris: Uhm, he was 103 years old. His name was Richard K. Guy. Uhm, he was faculty of the University of Calgary. So, I’ll give you the- that’s Richard K. Guy in the bottom there. So, this is one of his two very famous books. “Winning ways for your mathematical plays”. Uhm, we have a copy at [University’s Name]. I encourage you to take a look at this book because it’s unlike any math book that you have probably ever seen. So, it’s about, like, how to play games and using math to answer them and play around a little bit. It’s very playful, very fun. I encourage you to look at it. (Chris’s class, p. 104; Lecture 9.2)

Through this example, Chris constructs a notion of what it means to be a mathematician. Through the example of Richard Guy, he conceptualises the notion of the mathematician in ways that significantly differ from public discourses about mathematics and mathematicians. By highlighting a book that is an unusual mathematical one (“it’s unlike any math book that you have probably ever seen”) and focuses on an area that is usually not considered mathematical in the first place (“how to play games”), Chris expands the notion of mathematics and diversifies what mathematicians do and look like.

5.3.2.3. Diversity is also about Preferences and Emotions

In this part, I explore how Chris’s discourse does not merely conceptualise diversity in terms of mathematical techniques and understandings, but it also extends to preferences and emotions. The relation between mathematics and emotion/affect has been a complicated one, considering that the rationality of mathematics has been discursively constructed as opposing emotion (Walkerdine, 1988). Nevertheless, a lot of mathematics education research has focused on the affect and its relation to cognition as well as its role in the creation of mathematics identities (Liljedahl & Hannula, 2016). In Chris’s class, preferences and emotions about mathematics are discussed and their exploration is encouraged. My interest in these articulations does not lie with whether and how addressing one’s emotions when doing mathematics can help become better at mathematics. Instead, my interest lies with how the incorporation of emotion is an aspect of enculturation into mathematics and the political aspects of this process.

The following *"Would you like to see the exact same proof but one you will like?" example* shows how Chris encourages students to reflect on whether they like a proof and how students engage in this reflection. This example happens after Chris has presented students with a proof for the sum of the first n odd numbers that uses the method of the induction.

> Chris: Ok. There is a- So, one of the things that is true now, that was not true when I was growing up: in Sesame Street, at the end of- like when they're talking, like one of the things they do as they talk is they reflect on whether they enjoyed the activity. And so, the Sesame Street character would say, oh I had fun doing this thing, or I liked this, or whatever. And, it's an interesting thing because it gets you to think about like your own feelings about things. Like, did you enjoy the thing you just did? And it tells you that like the answer can sometimes be no. Like maybe you didn't enjoy yourself. So, it's worth thinking about proofs when you see them. Did you like them? Did you not like them? Are there better ways?
>
> Chris: Do you like this proof?
>
> *[Students respond no.]*
>
> Chris: Would you like to see the exact same proof but one you will like?
>
> *Students respond yes.*
>
> Chris: Ok. […] So, it's worth thinking about proofs when you see them. Did you like them? Did you not like them? Are there better ways?
>
> *[Chris starts presenting the alternative proof, which is a visual proof of the sum of the first n odd numbers]*
>
> *[A student sitting in the front of the room says something, which I cannot hear.]*
>
> Chris: You don't like this?
>
> Student: Go back to induction!
>
> Chris: Go back to induction? So, there's another thing that's hidden inside this proof, which makes the algebra go through. *[Chris continues with the presentation of the visual proof]* (Chris's class, pp. 36–37; Lecture 6.1)

In this example, Chris invites students to reflect on whether they like a proof or not through a reference to the television program "Sesame Street". In this way, Chris maintains that "sometimes the answer can be no". This example conceptualises mathematics and in particular proofs as something that relates to preferences and affect. Doing mathematics is conceptualised as a process that is not merely about solving a problem or developing a proof, but it entails emotions and it can extend to involve a person's reflections about these emotions and preferences. Hence, the process of learning is conceptualised not as a merely cognitive process of acquiring a pre-established knowledge, but as a process that engages oneself in the interaction with this knowledge.

By highlighting the issues of preference and emotion related to proof production, Chris draws upon a non-mainstream mathematics education discourse in which affect is greatly valued. Maintaining an alliance with non-mainstream mathematics discourses, in this example, Chris foreshadows that the visual proof is " a proof that [students] will like". It is noteworthy that, after Chris presents the visual proof, one student engages by proposing that he does not like it and asks Chris to "go back to induction". This example is an incident in which a student is the one who advocates for the more formal, traditional, and mainstream mathematics discourses in the class.

The following *"people tend to have strong opinions about proof by contradiction" example* draws upon similar ideas, but focuses specifically on proof by contradiction.

> Chris: As you're gaining more experience, you'll come up with this list and one thing you'll find is that people tend to have strong opinions about proof by contradiction. And as you go, through like your second or third year, you'll find either people trying to avoid it completely and never use it, and other people are just like, yeah, yeah, use it whenever you want. (Chris's class, p. 26; Lecture 6.1)

Similarly to the previous example, here Chris refers to individuals having personal preferences regarding different kinds of proof. In this example, he refers to proof by contradiction – a proof method which has been contested throughout the history and philosophy of mathematics. More specifically, the 20th century philosophical current called "intuitionistic mathematics" rejects proof by contradiction (Anapolitanos, 2009). In this way, preferences and emotions are not merely personal, but they implicitly relate to currents of thought, hence having a social character.

Another way in which the pattern of focusing on emotions and personal preferences emerges in the Introduction to Proofs course is manifested in the *"any… complaints?" example*, which happens as Chris finishes a proof and then askes students to express any complaints that they may have about the proof.

> Chris: Ok, any other questions, observations, complaints, about this exercise? (Chris's class, p. 85; Lecture 8.1)

In this example, Chris creates time and space in the class, in which it is possible for students to engage in a discussion about their likes and dislikes regarding the produced proof. Nevertheless, students do not take up this opportunity (or other similar opportunities provided by Chris), suggesting, I argue, the hegemony of the mainstream mathematical culture, which does not entail complaints or discussions by the amateur about not liking a proof.

Overall, Chris's discourse places emphasis on the diversity of ways in which students may think about, work on, or feel about mathematics as well as mathematicians themselves may engage with mathematics. In the coming subsection, I discuss how diversity also manifests in Chris's class in terms of a multiplicity of ways with which to speak about mathematics.

5.3.3. Diversity in Ways of Speaking Mathematically

In this section, I explore the diversity in ways of speaking mathematically, as they manifest in Chris's class as different ways for using mathematical terminology. I refer to the different ways of using mathematical terminology as distinct language varieties. In sociolinguistics, language variety signifies a form of a language that is systematically distinct from others and it has emerged as a relatively neutral term, which avoids the hierarchical distinction between "dialect" and "language" (Meyerhoff, 2006). I am using this term to refer to relatively distinct ways of speaking about mathematics in the class. As the professor and the students navigate the use of mathematical terminology, different degrees and ways of speaking using this terminology emerge. I differentiate between two language varieties that appear in the course on the basis of their usage of terminology: a variety which adopts the terminology and attaches value to it (henceforth, the terminology-oriented variety) and a variety which stands critically towards the terminology considering its role in terms of facilitating understanding (henceforth, the terminology-critical variety).

The terminology-oriented variety entails using a lot of mathematical terms when talking about mathematics. This does not only refer to the kinds of words that a person uses, but also to valuing this terminology and portraying it as charming and worth of using. We can see a manifestation of this language variety in the *"it's called idempotent" example*, in which the professor introduces the term "idempotent" and the class engage in a short discussion about it and express interest in its meaning.

> Chris: not-s undo each other. You know what the fancy term for this is? It has a really sweet name. It's called idempotent. Negations are idempotent. No, wait, that's not true, sorry, sorry. Scratch what I just said. It's not true, it's self-duals, it's not quite right.
>
> *[A student says that they now need to know what idempotent is.]*
>
> Chris: Yeah, you need to know what idempotent- Yeah, do we have an example in this class?

[Another student says absolute value.]

Chris: Right, absolute values are idempotents. It's something that if you do it once, it's the same as you do it two times, or three times, or four times. So, absolute value, once you do it once, it doesn't change. Great example. (Chris's class, p. 4; Lecture 5.1)

In this example, Chris talks about an operation's property of undoing itself, as is the case with the negations (i.e., the proposition "$not - not - P$" is the same as P; "not-s" undo each other). After he mistakenly introduces the term "idempotent" to capture this property and immediately corrects himself introducing the term "self-dual" instead, a student expresses interest to learn what "idempotent" means. Chris explains it through the example of the absolute value provided by another student. In this interaction, the terminology-oriented variety manifests itself in two ways. Firstly, a technical term is introduced and discussed in the class, although it is not part of the technical language that students are required to learn for the course. Secondly, and perhaps more importantly, the term is captured as interesting and charming, with both the student and the professor talking highly about it. Chris talks about "a really sweet name", while the student says that they "need to know" the meaning of the term. The choice of the word "need" is an intentionally excessive way of expressing intense interest to learn what the term means. The student obviously does not "need" to learn the meaning of the term; they rather perform, with this word choice, a persona that embraces the use of technical language and treats it as being meaningful and charming on its own. In this way, the example is a manifestation of using a language variety which values technical language, while it also shows how this way of speaking is related to a mathematical culture, in which formalism and technical language are important.

A different way in which the terminology-oriented language variety manifests itself in the Introduction to Proofs class is through using terminology without explaining its meaning. In the *"so, we could write down what contrapositive is" example*, we can see some tension around explaining the meaning of the technical terms used. As the class is about to prove their first statement using the proof by contrapositive method (after the method has been described and explained by the professor), a student suggests that the term "proof by contrapositive" needs to be explained, while the professor argues against it, although suggesting that "maybe this is unfair".

Chris: So, prove by contrapositive. So, what's the first line of our proof gonna be? Yeah?

Student: Write down what the contrapositive is?

> Chris: Oh, so, we could write down what the contrapositive is! Interesting! Uhm, I think, maybe this is unfair, but I think that our audiences are going to be- we're going to assume that our audiences know what contrapositive is and other mathematical words like this. So, we're not going to write down the contrapositive, although we could if we wanted. What would be the first line of our proof? (Chris's class, p. 12; Lecture 5.1)

Here, we see that the use of some terms, such as "proof by contrapositive", is mentioned without being explained on paper. In the mathematical culture of this first-year undergraduate course, it is assumed that there exists a shared base of terms. Chris's comment that "maybe this is unfair" is of importance, as it shows that not explaining a term might not make the mathematical writing accessible to everyone. In other words, Chris recognises the potentially inclusionary/exclusionary role of the usage of particular words and by extension language varieties. Although this idea is not particularly profound per se (e.g., it is self-evident that not knowing French excludes an individual from a conversation conducted in French), it is very significant that it is expressed in this space. Identifying inclusion/exclusion as an aspect of the academic mathematical practice constitutes a critical reading of it (even if only a mild one), as its culture is not treated merely as one which students need to be socialised into, but as a multifaceted one that needs to be reckoned.

Furthermore, this example offers a picture of the antagonistic relation between the two language varieties. Using a term without explaining it can be seen as a facet of the terminology-oriented variety, whereas proposing the explanation of a term and considering the inclusionary/exclusionary role of technical language usage are facets of the terminology-critical variety. The two varieties are here brought into an antagonistic relation, while – notably – both are enunciated by Chris himself (much like the "is this considered a formal proof?" example). In terms of the result of the antagonism, or its (temporary) equilibrium point, the construction of a shared space through the previous introduction and explanation of the term is critical. In other words, the shared space contributes to the creation and maintenance of a balance between the two discourses.

While the previous examples primarily show a manifestation of the terminology-oriented language variety on the part of the professor, we also see an opposing practice in terms of what it means to speak mathematically: often the professor advocates for using a simple language variety. For example, he often asks students to express a statement in a way that someone in Grade 5 would understand (e.g., Chris's class, p. 9; Lecture 5.1). More generally, the professor is critical towards using very dense and terminology-heavy mathematical varieties, when they hinder meaning[30]. The following *“I’ve never used the word ‘base case’ in my life” example* shows a manifestation of this terminology-critical variety. A student’s use of the term “base case” in the context of proof by induction is discouraged by the professor, because the term has not yet been introduced to the class.

> *[A student asks a question which I cannot hear from where I sit. The question includes the term “base case”.]*
>
> Chris: Ahm. So, I’ve never used the word ‘base case’ before in my life. I don’t know what you mean. What we’re trying to prove was $P(n)$ for all n, implies $P(n+1)$. And I think that’s what we did. *[Pause; Students sitting at the front of the room discuss using the term “base case”]* People keep talking about base case. I’ve no idea what you mean. *[People are laughing]* That’s something else. Any other questions about this proof specifically? (Chris’s class, pp. 36–37; Lecture 6.1)

Similarly to the “so, we could write down what contrapositive is” example, this one shows that the choice to use a technical term is related to the construct of the shared space; using a technical term is constructed to be acceptable or even desired when it has been introduced to everyone. In the previous example, the term “proof by contrapositive” is recognised as a term whose use can include and exclude – distinguish the insider from the outsider. In this example, the “base case” does not play this role, until it is formally introduced by the professor. So, we notice that the attitude towards technical language is not objectively determined in academic mathematics but is rather defined in the classroom through the classroom practices and the construct of a shared space, which is assumed to capture what has been agreed upon.

Through the examples so far, we have seen that the professor is the one who introduces, advocates, and decides when a technical term should be used. Nevertheless, the use of formal and technical language is not advocated only by the professor. In the *“but I really don’t like these words” example*, a student prompts the professor to use the term “antecedent”, which leads to a discussion in which the student advocates for the term being “cool” and the professor against it.

> Chris: How do you prove something implies something? Yeah?
>
> Student: we should start by assuming the- *[pauses as they do not know how to call the “left side”, leans their body towards the left, and finally says:]* left side.

[30] This reading of Chris’s discourse has been influenced by Chris’s emphasis on the need for mathematical communication to take into consideration the reader and aim at the reader’s understanding of the presentation of an idea. This idea is further discussed in subsection 5.3.5.

Chris: Yeah, I know. There's actually fancy names for this. Like antecedent. And conclusion or whatever it is. But I really don't like these words. So, instead we'll say left side, right side.

Student: antecedent sounds cooler.

Chris: Antecedent. Sounds cooler? OK, that's fine! So, we'll assume the not Q part and we'll conclude the not P. (Chris's class, p. 11; Lecture 5.1)

Here, the student advocates for a more terminology-heavy, dense mathematical variety, while the professor is the one who challenges it. While Chris bluntly says "I really don't like these words", the student proposes that "antecedent sounds cooler". The culture of mathematics as entailing an in-group language variety is, here, introduced in the classroom by students. This is in tension with one of the mainstream hermeneutic schemes in mathematics education, according to which one goal of education is to transform students' informal language varieties into a/the formal mathematical variety (a Bakhtinian criticism of this scheme is developed in Barwell, 2016). In the last two examples, students do not lack the formal variety. They are socialised into mathematics through a variety of channels (outside the classroom) and some of them bring up formal norms and values in the classroom, even when this is not the goal of the course's curriculum or the discourse promoted by the professor.

Overall, the explored examples depict varying relations with mathematical terminology, which suggests that navigating technical mathematical language in a mathematics classroom is a complex process, in which antagonisms between different language varieties are often at play. When the professor and the student disagree about the "coolness" of the term, they do not merely express their (dis)like for a particular word or practice. They bring two discourses in collision. Each of these discourses bears values, ideological assumptions, and assertions about the status of a practice. Through this collision, language varieties include and exclude. These antagonisms are more complex than the instructor being on the side of the terminology-heavy language variety and the students on the side of a terminology-critical variety. In fact, in some examples, it is the students who enunciate a discourse which is associated with a traditional, "hardcore", academic mathematical culture, while it is the professor the one who challenges it. Similarly, when the professor talks about how "it might be unfair", but there is no need to explain what proof by contrapositive is, we see two perspectives being enunciated by the same person (the professor) in the same utterance. These observations reinforce the idea that the discussed antagonism does not manifest between distinct actors each defending one of the antagonism's poles but is rather an antagonism of discourses.

5.3.4. Disagreements that Never Surface

A pattern manifesting in Chris's course is that Chris often invites students to express their disagreement or discomfort about an idea which is considered counter-intuitive, subtle, or historically contested. However, students do not respond to these kinds of invitations and these disagreements or discomforts do not get expressed. In this section, I frame this pattern in terms of an antagonism between different discourses about relations with mathematics. In the spirit of mathematics as a social practice discourse, Chris adopts a discourse which cultivates an active relation with mathematics. Students' (assumed) non-surfacing disagreements and discomforts capture a passive relation with mathematics, in the spirit of the discourse that frames mathematics as the result of professional mathematicians' labour. I conclude this subsection, by arguing that the passive relation with mathematics that is taken up by students constitutes a resistance to the discourse of the diverse participation in mathematics.

The *"did we ever assume the thing we want to prove?" example* is a manifestation of this pattern. As he always does after a proof is presented, Chris asks students to reflect on the proof by induction for the statement $1 + 3 + \cdots + (2n - 1) = n^2$ that they just worked on. Although students typically contribute questions, observations, and comments at this phase of discussing a proof, they do not do so in this example.

> Chris: Did we justify every one of our equalities? Or are there any equalities that we wrote, we would like this to be true? Did we ever assume the thing we want to prove? *[Pauses. No students raise their hands, ask any questions, or talk with each other.]* I hope not. (Chris's class, p. 36; Lecture 6.1)

In this example, Chris draws upon the principle of deductive reasoning that the statement that should be proven shall not be assumed. His question "Did we ever assume the thing that we want to prove?" refers to the idea that to prove the statement $P(n)$ for every natural number n, the class needs to prove that $P(n)$ implies $P(n + 1)$ for a random number n. To do so, Chris's solution has assumed $P(n)$ for a random number n at the induction hypothesis stage. Confusing the two ideas and thinking that proof by induction assumes what it needs to prove has been identified as a common "obstacle" or "misconception" among students (e.g., Ernest, 1984) and relates to the distinction between the variable n representing a random number (in the case of the induction hypothesis) versus all numbers (in the case of the general statement).

Here, Chris's questions urge students to respond that the final statement has been assumed at the induction hypothesis stage. Although Chris pauses after his question about whether they have assumed the thing that they want to prove, no student takes the floor to refer to this apparent peculiarity of proof by induction. Chris's invitation to students to think about this idea constitutes an invitation to converse with the mathematical idea of proof by induction; it entails a consideration of doing mathematics as a social practice. However, students do not respond to this invitation. I argue that this reflects a more passive relation with mathematics, in which any confusion is not brought up into discussion.

A similar instance happens as the class goes through the study of cardinality. The cardinality of infinite sets has been defined on the basis of the idea of bijections: two sets have the same cardinality as long as there exists a bijection from one to another. This means that some infinite sets (e.g., the evens and the naturals) have the same cardinality even if one is a subset of the other. In mathematics education literature, this idea is considered counter-intuitive because at the intuitive level a set is expected to have "more elements" than any of its proper subsets (e.g., Mamolo & Zazkis, 2008; Monaghan, 2001). The *"this should be unsettling" example* captures how Chris invites the class to "confront" the counter-intuitiveness of the idea that a set might have the same cardinality as a proper subset of it. As is the case in the previous example, the class do not take up this opportunity: nobody engages in the discussion of this idea being unsettling, uncomfortable, or counter-intuitive.

> Chris: So, if you have a bijection, what do you know? What do you know about their cardinalities?
>
> Student: They are equal.
>
> Chris: They are equal. This should be unsettling. This should be a little bit uncomfortable. Uhm, we've shown that the evens and the naturals have the same cardinality. Yeah?
>
> Student: Because they are both infinite?
>
> Chris: Uhm, this is part of the reason why this works. Is because they are both infinite. Yeah. It's not the only reason, it's like, it's a necessary condition, but it's not sufficient. So, the naive thing that you might think is that the naturals should be strictly smaller than the evens, right? Because it really looks like, sorry- Other way around. There are more naturals than there are evens. But in terms of cardinality, there is a way to biject the two to each other. So, that they have the same number of elements. Ok. This is unsettling. Do you have any questions, do you reject the notion completely? You're like, "this is not how it should be!" *[Pauses. Students do not engage]* Now, is the time to object because it gets weirder as we move on. (Chris's class, p. 101; Lecture 9.1)

Chris urges his students to experience and express discomfort with the idea that the evens are as many as the naturals and proposes that this idea "should be unsettling", "should be uncomfortable", and asks students if they "reject the notion completely". I argue that in this way, the professor adopts a discourse in which a conversational relation with mathematical knowledge is proposed. On the other hand, the students do not engage in this discussion. The student who asks the question “because they are both infinite?” tries to make sense of the idea, without evaluating it in terms of it being potentially unsettling. In other words, I argue that the students perform a discourse (through the question and their silence) in which a more passive absorption of mathematical knowledge is proposed: students need to understand and learn whatever mathematics has been fixed to be.

Cardinality, as a theory of equinumerosity developed by Cantor, is based on definitions which *assert* when two sets are equinumerous. However, an alternative theory of equinumerosity (and contradictory to that of Cantor’s), based on the idea that a set is larger than its proper subsets, has also been developed in the history of academic mathematics (Hacking, 2014). In this sense, engaging in a discussion about whether a theory is “reasonable”, “successful”, or “acceptable” are not irrelevant to the mathematical practice of professional mathematicians. They are, however, contradictory to a discourse about mathematics which considers the developed mathematical theory as “true” in a vacuum.

In a nutshell, in the two discussed examples, I argued that we saw two different discourses about possible relations with mathematics. Through the two antagonising discourses, the learner and their role are constructed. On the one hand, Chris advocates for an active, conversational relationship with mathematics, in which the student may express their dislikes, complaints, or counter-arguments about the mathematics discussed in class. On the other hand, the students (who need to pass the class, for which they have paid tuition fees, in order to get their degree and/or enter their desired program) adopt a discourse which constructs a more passive relation with mathematics. These two discourses are related to the overarching antagonising discourses about mathematics as a social practice and as the product of an established discipline respectively.

These examples show a resistance against the discourse of diverse participation in mathematics. In the previous subsections, I tried to make the argument that the discourse of diverse participation that is taken up by Chris opens up space for students to express their various ways of thinking about, talking about, and experiencing mathematics. Overall, through the examples discussed in these subsections, students appear to take up this discourse themselves (or perhaps more accurately, *there are* students in the class who appear to do so), by actively engaging in peer conversation about mathematical ideas, offering their own examples on mathematical concepts, asking questions, contributing their views on the discussed terminology, etc.[31]. The passivity with which students react to Chris’s invitation to express their

[31] The coming subsection returns to this idea of mathematics happening through human communication and discusses the implications of it for the construction of a mathematical rationality.

disagreements and discomforts pinpoints the limits of the diverse participation discourse as they emerge in this class. In other words, using Doxiadis's (2011a) terminology, the discourse of diverse participation loses its ability to "guide students' behaviour" when it comes to the point of expressing discomforts. The point where this happens and a more passive relationship with mathematics (i.e., with the mathematics as they exist within the institutional framework of the course) starts emerging is perhaps when the disagreement or discomfort come from a place of "naiveness" and not mathematical mastery. Hence, I argue that the relation with mathematics that is being built becomes more passive, when students are less familiar or less enculturated in the social practice of professional mathematicians.

5.3.5. Implications for Rationality

The diverse participation in mathematics discourse has implications for mathematical rationality, as it conceptualises mathematics as a space which happens through communication between real people, who sometimes disagree and argue. In this subsection, I explore this idea by first discussing how Chris's class is organised around the discourse that mathematics happens through communication and conversations between real people. I then outline how this constitutes a transition about the truth of a mathematical proof: from it being objective towards it being decided intersubjectively.

In Chris's class, mathematics is conceptualised in terms of communication between real people. This is evident both through Chris's interview and through the norms developed in the class. In his interview, Chris talks about the importance of students learning how to communicate with others because mathematics is increasingly produced through collaboration among mathematicians and other scientists. This is evident in the *"math transitioned to being a very collaborative thing" example*:

> Chris: I don't know when this changed. But it was around, I would guess like 90s, I don't know. At some point, math transitioned from being a solitary activity to being a very, like, collaborative thing. So, in one sense, it's like how do you measure this? You measure it by like how many single authored papers are there. And so, versus, co-authored papers. That's one way of doing it. And what we see is that like for big results, it's very rare for us, for there to be just a person.

Here, Chris conceptualises academic mathematics as a collaborative activity which in the past used to be a solitary one. He provides evidence for this conceptualisation, which include the existence of more co-authored papers (as opposed to single authored ones) when it comes to "big results". This conceptualisation "guides action", through the idea that students learn how to effectively communicate with mathematicians and other scientists.

This discourse frames Chris's practice and is evident in the class norms. Indeed, Chris very often encourages communication between students, as discussed in the "sketch of the course" in section 5.1. For example, in every class, he dedicates some time in which students work on a problem and are encouraged to talk "with the person sitting next to them". This manifests in the words of a student during an interview in the *"I think you've also probably heard him say" example*:

> Kevin: like, sometimes he'll ask a question and then, I think you've also probably heard him say, like "talk amongst yourselves". Like "turn to your neighbour". (Chris's class, p. 175; Interview with Kevin)

In the following *"if you haven't already, talk to the person next to you" example*, we can see one of these occasions, in which Chris asks students to turn to their neighbours and discuss a problem.

> Chris: Do you think this statement is true or false? So, take a moment to decide for yourself. Read it, first of all. And decide whether you think it's true or false, and how you'd prove it.
>
> *[People start discussing.]*
>
> Chris: If you haven't already, talk to the person next to you. (p. 7)

Here, it is noteworthy that my fieldnotes (in italics) indicate that students start discussing the problem immediately, before Chris asks them to. This indicates that discussing mathematical problems with peers during class time has been established as a norm in this class. Indeed, it is a practice that usually happens multiple times during a single lecture.

One more aspect of the discourse that mathematics happens through communication among people is that sometimes people doing mathematics disagree and argue. The following *"I'm particularly interested in the conversation that was happening right here" example* happens on one of the occasions when students are given some time to think about and discuss a mathematical problem among themselves. On this particular occasion, Chris notices that a couple of students seem to disagree. When the class reconvenes, Chris asks this group to share what their discussion was about and frames their disagreement in terms of negotiation of a conjecture.

> Chris: Let x be an integer. If x squared is even, then what do you know about x? If you have an integer whose square is even, what do we know about the original number? Take, like, a minute. Think about it, write it down if you want, what can we conclude?

[Students start discussing. A couple of students in the first row discuss very lively. Although they sit very far away from me, I notice them and they seem to disagree about something. This is the group of students that the professor talks about in the next line.]

Chris: All right, wrap up your conversations. *[Pause]* All right. So, I'm particularly interested in the conversation that was happening right here. Would someone want to summarise the conversation?

[Students say something that I cannot hear.]

Chris: You don't know what happened! You're just like "we've been talking about it for 10 minutes, now I blocked out and now I'm here". So, what was the-? It looked like there was a back and forth between like someone made a conjecture and someone else said "no! think about six squared". What was that about?

[A student responds.]

Chris: If x squared is 6. Could x squared be 6? I don't mean to have anyone against you. *[Addressed to the class]* Could x squared in this case be 6?

[A student responds.]

Chris: No, because the square root of 6 is not an integer. So, did someone come up with a conclusion for what x should have to do? Yeah? (Chris's class, p. 11; Lecture 5.1)

In this example, a couple of students have a disagreement, while they work by themselves on the problem "Let x be an integer. If x squared is even, then what do you know about x?". Chris asks them to share their discussion with the classroom and frames the disagreement in terms of a conjecture being made by one student and challenged by another. The students have been disagreeing about the special case of x^2 being 6. Chris asks the class to evaluate whether x^2 can be *6*, which is deemed wrong on the basis that they work with integers (and not real numbers). Immediately after an argument is given for this, Chris returns to the question that the class had been asked to work on by themselves or with their peers. In this example, Chris gives space to a disagreement developed in a group of students to be discussed in front of the whole class, thereby conceptualising mathematics as an activity in which people may disagree and argue.

The discourse that mathematics happens through communication between real people can be explored in relation to the social turn in mathematics education (Lerman, 2000): for a long time, mathematics has been taught as a solitary activity in classrooms where the students have been almost silent; this is, however, changing as research emphasises the importance of communication in mathematics. For example, Sfard (2008) has coined the term commognition to suggest that communication and cognition are the two sides of the same coin; a term and approach which have been very popular in the conceptualisation of mathematics classroom activity and discourse. In the context of literature in mathematical argumentation, this turn has taken the form of a shift of emphasis from proof as a product towards the processes that are related to proving (e.g., identifying patterns, making conjectures, etc.) and has been significantly influenced by the classic works of Lakatos (1976) and Polya (1945/1957) who have both been pioneers in examining heuristic aspects of mathematical activity.

This "turn" is related to the development of a "democratic" program in mathematics education: the instructor talks less and gives more space to students; the students have more space to make their own connections and have mathematics make sense to *them*; mathematics taught at school becomes more similar to what mathematicians do. In mathematics education literature, it has become almost commonsensical that these ideas are part of what progressive/democratic education is about and what progressive/democratic educators do. However, I wish to problematise this idea, on the basis that mathematical communication, as it takes place in educational institutions, is a peculiar kind of communication since its product is determined before the communication even starts.

In the context of mathematics in the university, the goal of the conversation is predetermined: there are specific things to be learned by the students. For instance, in the "I'm particularly interested in the conversation that was happening right here" example, the professor poses an open-ended, explorative question. However, answers like "x is a letter", "x represents a number", or "x cannot be 3" are – although technically correct – either irrelevant or not general enough. Through his question, Chris seems to ask for the most general statement that can be derived about x, which is of interest to *this* mathematical culture. To this question, the answer is pre-determined; as if the truth exists and waits to be discovered. If different students or a different professor were to have this discussion, different ideas might have emerged; however, the answer to the problem would have been the very same one: x is an even number. And the class would have then moved on to prove this, just like Chris's class did.

My goal here is not to criticise Chris's teaching strategies; I am not suggesting that he should have been posing more general or open-ended questions. I am instead trying to highlight how in this particular setting of a first-year proofs course (which involves a large number of students in the same room who shall take an exam at the end of the term to determine who will get into their desired academic program and who will not), acting includes the reproduction of often unstated norms and expectations that comply with the discipline's culture. This happens even when the professor, as is the case with Chris, adopts a critical view of mathematics and mathematics education and makes a deliberate effort to make their norms and expectations very explicit to students. Further, I argue that the project of focusing on the conversational aspect of doing mathematics entails a peculiarity: within this institutional context, mathematical conversations are conversations with a pregiven telos. This idea relates to Jullien's (2004) conceptualisation of European mathematics as a prototypological field. Indeed, in the example, an ideal form is set up (i.e., the theorem "if x^2 is even, then x is even") and is taken to be the goal that shall be achieved (by students conjecturing the theorem).

Within the context of mathematics happening through the communication between real people, the truth of a statement is decided at the intersubjective level. In other words, a statement can be deemed true or false through communication that persuades others. The *"Hey, Chris. Is this a proof?" example* encapsulates this discourse. The example happens after Chris has asked the class to work on a problem amongst themselves. During that time, he moves among the students, answering questions and occasionally approaching students and asking them how they are doing. Once students are done working amongst themselves, Chris presents a proof for the problem, and once that is done, he says the following:

> Chris: There is one thing that I want to bring up, which happened over here *[points to a group of students to his right]*, which is nice. What often happens is that someone might be talking with their peers, with their friends, and they'll come and say "Hey, Chris. Is this a proof?" And my answer is: It's not up to me to judge. It's up to you to figure out. If your peers don't believe your proof, your argument, it's not a proof. Because who are you writing proofs for? You're writing it for yourself or for your peers? And if they don't agree with it, you haven't convinced them. It's not a proof. So, that's the first bar: If your peers don't believe it, it's not a proof. If they do believe it, well that's a good step. (Chris's class, p. 13; Lecture 5.1)

In this example, Chris deals with the question of what constitutes a proof. He conceptualises something as being a proof when it can persuade others. Being able to convince others is a necessary (although not sufficient) condition. In this way, Chris's discourse guides students' action towards evaluating the proofs that they produce, by discussing them with their peers and checking whether these proofs can convince them. This view of proof breaks with traditional discourses about mathematics and proofs, for which the focus is on the truth of mathematical knowledge, as a product largely irrelevant to the people who create it or talk about it. Instead, it aligns with discourses that are more human-centred and conceptualise mathematics as a social practice and truth as intersubjective.

The *"Do we believe this?" example* below focuses on how the idea of "believing" an argument manifests in Chris's discourse as the class works on a specific mathematical problem.

> Chris: What is the only way of multiplying two integers together to get 1?

> Student: they are both 1 or -1.
>
> Chris: Either they're both 1 or they're both -1. Right? It's the only way, if we're multiplying two integers together, to get 1. So the only way for $(x + y)(x - y)$ to be equal to 1 is if both factors are 1 or -1. Do we believe this? Does anyone not believe it? (Chris's class, p. 19; Lecture 5.1)

In this example, a student proposes a statement and Chris invites students to reflect on whether they believe it. This action shifts the focus from truth as an objective state towards the evaluation of an argument in terms of whether it is able to persuade others. Nevertheless, it is notable that in this example both discourses about proof are present. Before Chris asks students if they believe the result, he presents the statement that is being discussed in a very demonstrative way, which leaves no space for its truth to be negotiated. This is evident by the repetition of the phrase "the only way" as well as by the rhetorical question "Right?". Hence, although the focus is shifted towards a more social conceptualisation of truth, the objectivity of the truth is not totally challenged.

In this subsection, I discussed the implications of the diverse participation in mathematics discourse for the construction of a mathematical rationality. Using examples from the class, I proposed that a rationality is constructed which conceptualises mathematics as happening through communication between real people who sometimes disagree and argue and mathematical truth as intersubjective. Nevertheless, I outlined how this rationality is not hegemonic but rather co-exists with a more traditional rationality (as described in section 5.2 with regards to formalising).

5.3.6. Bringing it All Together

In this section, I explored the discourse of diverse participation in mathematics as it emerges in Chris's class. This discourse manifests in Chris's efforts to support historically marginalised groups in mathematics and challenge the mainstream history of mathematics within which white men of privilege have benefitted, as well as in the emphasis on the diverse ways in which a person can do, think about, and speak about mathematics. I suggested that this discourse manifests in the organisation of the class and that students participate in this discourse themselves by offering their own examples and views on the studied mathematical content and mathematical terminology. I argued that a glimpse on the limits of this discourse can be caught when the class works on counter-intuitive mathematical results, during the study of which Chris encourages students to develop a more active relationship with mathematical knowledge and express their disagreements and discomforts, but students assume a more passive relationship with mathematical knowledge. Finally, I discussed the implications of the discourse of diverse participation in mathematics for the construction of a mathematical rationality and I suggested that this discourse conceptualises mathematics as happening through communication between real people who sometimes disagree and argue while resolution to these disagreements is provided through the establishment of a common ground.

The discourse of diverse participation in mathematics can be positioned in the sociopolitical turn of mathematics education (Gutiérrez, 2013; Lerman, 2000) which has put emphasis on the social aspects of learning mathematics and issues of equity in mathematics education. In this sense, the discourse of diverse participation in mathematics assumes equity to be important and suggests that everyone shall be given the opportunity to actively engage with mathematics. Without partaking in the discussion about the desirability of these goals, I engage in a critical exploration of their function in producing mathematical work/labour which is efficient and diverse. As the academic world and the mathematical field in particular are changing so as to be more collaborative, a new challenge is placed on universities so as to continue being competitive and produce the workers who will be able to adjust to this new environment. This means that the often celebrated character of communicative and collaborative ideas in the teaching and learning of mathematics are a way of producing the new kind of specialised academic and industry workers.

Hence, I argue that the focus on equity and collaborative learning is not merely an action that aims to disrupt an inequitable mathematics education. It is also about adjusting to a changing system that requires new competencies from its workers; it is about the reproduction of the system itself. It is indicative that the emphasis on peer-learning, problem-based, and project-based learning which developed as demands from the students' movements after the 1960s found an unexpected ally in the industry sector which demanded workers who were able to manage projects and cooperate (Kolmos, 2009). In this sense, I argue that the emphasis on diversity and inquiry discourses, as part of the contemporary mainstream mathematics education paradigm, constitutes a moment in the aforementioned trajectory of mathematical knowledge as opposed to a challenge against it.

5.4. The "*Imaginary Practice of Mathematics"* Discourse

In this section, I argue that in the context of Chris's class, mathematical community is conceptualised in an image-based way, as a space in which everyone has access to a mathematical knowledge shared across time and place (e.g., through books and papers). Then, doing mathematics entails communication with this community of people. However, this communication does not always involve an interchange between "real people" or "flesh-and-blood people", but it might take the form of responding to the ideas of a mathematician who has died or thinking about what an "impersonal reader" of a proof would understand.

In turn, mathematics is conceptualised as a system in which it is possible for every theorem to be traced back to the axioms. This image is not necessarily a description of the mathematical practice in which the class or mathematicians participate; it is rather a phantasy of the mathematics community as a whole and the structure of its mathematical products. In this section, I elaborate on this imaginary way of conceptualising mathematics and the mathematics community. The section in organised as follows: I first analyse the imaginary construct of mathematics and mathematical community, I then elaborate on the notion of the "impersonal reader", and I finally refer to the value of beauty in relation to the imaginary.

5.4.1. Mathematics Happens within an Imaginary Community

In this subsection, I discuss how mathematics and the mathematics community are constructed in an imaginary way. At the imaginary level, mathematics is conceptualised as a system of theorems, each of which follows from the axioms and previously proven theorems. The imaginary mathematical community is constructed as a community of specialists, it is characterised by democratic access, and its members communicate with each other in a precise manner. I elaborate on the characteristics of this imaginary through a discussion of examples, in which I suggest that this imaginary manifests.

The *"gold standard for proofs" example* provides an introduction to the concept of the imaginary mathematics community. It happens as, during the interview with Chris, I ask him about his practice of presenting proofs of statements that some students might consider self-evident, such as the statement that an even number is not odd.

> Chris: For proving basic facts about like evens and odds, the goal here is to show off that- to show off like a sort of gold standard for proofs. Not in the sense of, like, this is the socially best proof for socially like- like, in terms of what's actually done. But in terms of like how far we can push things down to the axioms. And so long as we all agree there, we can all agree on the steps, and if we all agree on the steps then we can all agree on the argument. And to do something simple like "why is the sum of two evens an even number?" is to do that, is to break things down into the definitions that we all agree on. And if we all agree on the steps, then we all agree on the argument. And, of course, as we progress through the course, we'll start to like not use this level of formality. But the understanding that we want the students to have is that you can always go back to that level of formality. Even, even if it's just like one layer. (Chris's class, p. 142; Interview with Chris)

This example refers to a "gold standard of proofs" in an imaginary way. Chris makes a distinction between "what is actually done" and a level of greater formality, in which things are pushed down to the axioms. The practice, in which an argument is pushed down to the system's axioms, does not capture what mathematicians actually do, but is rather an image-based description of what they could do if they wished.

Within this imaginary of mathematics, any argument shall emerge from the axioms in a series of clearly defined steps. A similar pattern is evident more generally in Chris's class. For instance, the non-negotiability of definitions, described earlier in this chapter, contributes to the constructions of the imaginary of mathematics in a way in which the mathematical objects are defined in a final way so that theorems about them emerge naturally.

Besides an imaginary of mathematics, the "gold standard for proofs" example also paints the mathematics community in an imaginary way. The use of the pronoun "we" refers to the community of the class and the mathematics community, in the imaginary context in which everyone engages in a practice of examining and agreeing on the validity of the steps for a given argument. Within this context, antagonisms are non-existent as disagreements are resolved by going back to the agreed upon definitions.

The imaginary mathematics community is a community of specialists, as is elaborated through the *"we're going to assume that our audiences know" example*. This example happens after the class has been introduced to "proof by contrapositive" and as they are working on their first example of producing such a proof:

> Chris: So, prove by contrapositive. So, what's the first line of our proof gonna be? Yeah?
>
> Student: Write down what the contrapositive is?
>
> Chris: Oh, so we could write down what the contrapositive is! Interesting! Um, I think, maybe this is unfair, but I think that our audiences are going to be- we're going to assume that our audiences know what contrapositive is and other mathematical words like this. So, we're not going to write down the contrapositive, although we could if we wanted. What would be the first line of our proof? (Chris's class, p. 12; Lecture 5.1)

Here, a student suggests that the – newly introduced – term "proof by contrapositive" needs to be explained when writing a proof using this method. Chris argues that, although they could if they wanted, they are not going to do so. Chris suggests that "we're going to assume that our audiences know what contrapositive is and other mathematical words like this", introducing the idea of "our audiences". This reference to audiences, here, signifies an imaginary community as opposed to a "flesh-and-blood" group of people. We might see this through the use of the word "assume": the audiences exist as part of a community, whose members are not known to the class – the class needs to assume what their characteristics are. By way of contrast, Chris could have said that the audience knows this term because it is explained in the book, limiting the audience to the individuals who are, in some way, participating in the class (students, TAs, professors); but he does not do that. Instead, "the audiences" are constructed as an *image* of people, who have some sort of specialisation in the field of mathematics and are knowledgeable of mathematical terms. We, thus, have a community of – at least to some degree – specialists; not everyone fits this image.

Besides it being a community of specialists, the imaginary mathematics community entails an aspect of democratic access: there is an image of everyone being able to enter this community if they wished to. We can see this nuance of Chris's discourse in the phrase "maybe this is unfair". Chris does not explicitly describe what makes this practice "unfair". Although he suggests that "we" could write what contrapositive is – a practice that is presumably fairer – he does not do that, by instead reposing his initial question in a categorical manner "What would be the first line of our proof?". I suggest that the unfair aspect of this practice lies with the potential existence of people in the audience who do not know what "proof by contrapositive is" or other related terms[32]. From this point of view, the recognition of the possible unfairness of this practice signifies the premise of mathematics being accessible to everyone. The imaginary of mathematical community as both a community of specialists and a community which offers democratic access relates to what Bishop (1991) calls the sociological value of openness: "[m]athematical truths, propositions and ideas generally, are open to examination by all" (p. 75) – as long as they have the necessary background.

Another characteristic of this imaginary mathematics community is that its members communicate with each other. The *"Do you not want me to understand?" example* happens during Chris's interview, when I ask him about his practice of drawing students' attention to the reader of a proof:

> Chris: Part of this is for like a personal reason, which is that as a grad student when you're reading through papers, you're like, why is this all written so, so densely? Do you not want me to understand what's going on? So, I kind of, I have that in mind. Make sure you can write so that other people would understand. And I understand that sometimes you might want to like- be a little bit purposefully vague, so that people struggle a little bit with something and then have an insight. But that's, uh, that should be a choice, not something that accidentally happens. (Chris's class, p. 143; Interview with Chris)

Here, Chris highlights mathematical communication as a key aspect of what I have so far called an imaginary mathematics community. Papers are written for people to read and, thereby, communicate

[32] This interpretation of Chris's phrase "maybe this is unfair" is also based on Chris's emphasis that a proof should be written in such a way that the reader understands what is going on. This idea is further discussed through the following examples.

about their mathematical work. For communication to be achieved, understanding what a person means is key. The phrase "Do you not want me to understand?" itself is indicative of this type of communication as it assumes a (hypothetical) dialogue between Chris as the reader and the writer of a paper. By way of contrast, subject-free enunciations about papers being often badly written and hindering understanding would have been possible here. However, Chris chooses a way of speaking that highlights the intent of the writer of a paper (through the word "want"), which needs to be that their paper gets understood by its readers. Moreover, the emphasis on the word "not" (as opposed to the alternative "Don't you want me") positions understanding as a commonsensically shared goal in mathematical communication. In this way, this example also constitutes a criticism of traditional formalist discourses[33]. Dense writing, as the prototype of formalist mathematics is here captured as hindering understanding.

The way in which the imaginary community operates through specialists communicating with each other on the basis of democratic access is described in the words of a student in the *"it's sort of, like, you need to be in Chris's class" example*. In this example, we see how students get socialised in this imaginary of mathematics and mathematics community.

> Vijin: Ahm, well, just because, like, it's sort of, like, you need to be in Chris's class. Because he gets you to agree on the basic facts. He's like any even number can be expressed as $2n$. And you're like, yeah, of course. That makes sense. And then from there, he can prove such complex things. So, it's like, agree on the basic facts. Because you as a person, would end up agree with him. Because you can't disagree to these basic facts. And there from that point, just by manipulating the basic facts, he can get to the total answer. (Chris's class, p. 156; Interview with Vijin)

Here, during the interview with Vijin, he explains how Chris persuades them about the truth of a mathematical proposition, by starting from "basic facts" on which everyone can agree and continuing to prove "complex things". This description of Vijin's depicts mathematics in an imaginary way, similarly to the "gold standard for proofs" example. At the same time, we see that reaching this agreement happens within a communication-heavy context. The production of these proofs is based on the professor (as the expert of the mathematical practice) "get[ting] [others] to agree" with the statements that he uses. Vijin's phrase "it's sort of, like, you need to be in Chris's class" indicates that there is a culture of communication and mathematical interaction, which cannot be fully described in words, but constitutes a particular atmosphere.

[33] In this sense, it could be argued here that we have an antagonism taking place between a discourse that values communication and a more formalist discourse as described in section 5.2.

Through the previous examples, I have tried to draw the reader's attention to how the imaginary mathematics community is a specialised community, it is accessible to all, and it functions through the communication of its members who need to understand each other. The *"dot, dot, dot" example* captures another aspect of this imaginary community, which is the mathematical striving for preciseness. The example takes place as Chris introduces the sigma and pi notations, which are compressed notations for expressing a long sum or product respectively. As he often does when introducing new ideas, Chris starts with a motivating example:

> Chris: So, we want to express things like this. Things like $1 + 2 + 3 + \cdots + n$, in a more compact way. More precise way. So, part of the issue is that we have this dot, dot, dot in there. And dot, dot, dots are not super precise. They leave stuff up to the imagination of the reader. And you can imagine that if your reader was particularly dense, they might misinterpret what you mean. Maybe for this one, it's hard to misinterpret, but there are definitely cases where you put a dot, dot, dot and it's not at all clear what you mean. (Chris's class, pp. 28–29; Lecture 6.1)

In this example, Chris presents students with a motivating problem which asks for a precise, compact representation of a sum. The example highlights a need for preciseness in mathematical discourse, in order to not be misinterpreted when speaking or writing mathematics. Let us first think about the notion of misinterpretation. Being potentially misinterpreted assumes that the writer wants to express a very specific idea, that does not allow for a multiplicity of meanings or interpretations: the meaning of an idea is unique. This meaning should be understood by the reader because otherwise the communication has failed. In this way, preciseness is elevated as a key aspect of mathematical enunciation as it enables the (imaginary) mathematical community to communicate effectively. In this way, preciseness contributes to the imaginary of a shared space. Through precise communication, everyone is assumed to have access to what is said and there is no space for misunderstandings and misinterpretations.

In this imaginary community, the role of the reader is central. In the example, Chris argues for the need for a precise writing on the basis that details should not be left up to the reader's imagination. Here, "the reader" is referred to, as if the reader is always there. The way that "the reader" is included in the sentence "They leave stuff up to the imagination of the reader" (as opposed to, for example, "If someone reads what you have written, they might imagine something different than what you mean") implies that the reader is an essential part of the writing process. The reader is present, even if no one (or at least before someone) actually performs the material act of reading this proof. Writing a mathematical proof becomes writing with the *Other* in mind: the impersonal, ever-present reader.

In this section, I explored how the imaginary mathematical community is constructed as a community of specialists, characterised by democratic access, and the members of which communicate with each other in a precise manner. We have already seen that this communication do not only happens between flesh-and-blood people, but involves (imaginary) interactions with an impersonal reader. The following subsection examines the regulatory function of the impersonal reader.

5.4.2. The Impersonal Reader and the Imaginary

In this section, I will go deeper into the idea of the "impersonal reader" and explore their regulatory role. In the various examples so far, a focus on the reader of a proof is evident. In the "we're going to assume that our audiences know" example, there are explicit references to what mathematical terms the audiences know. In the "do you not want me to understand?" example, there is a distinction between two kinds of readers: the "flesh-and-blood reader" and the "impersonal reader". The flesh-and-blood reader refers to Chris himself, in the context of him reading papers as a graduate student. The "impersonal" reader takes the form of "other people" reading his work, when Chris becomes the writer of a proof. Similarly to that, the "dot, dot, dot" example refers to *any* reader and how mathematical writing can be produced in such a way that *anybody* does not misinterpret what is intended to be communicated. To describe this very notion of "any reader", I use the phrase "impersonal reader", which hints at Lacan's idea of the Big Other. As Žižek (2007) describes:

> When I talk about other people's opinions, it is never only a matter of what I, you, or other individuals think, but also of what the impersonal 'one' thinks. When I violate a certain rule of 'decency', I never simply do something that the majority of others do not do – I do what 'one' doesn't do. (p. 11)

Similarly to Žižek's impersonal 'one', the impersonal reader, as a representative of the imaginary mathematical community, is "present" when students produce a mathematical text and are required to identify every possible mistake or impreciseness. In other words, mathematical texts are produced and addressed to the impersonal reader, while the impersonal reader plays the role of regulating the writing of a mathematical text.

This means that the impersonal reader is not merely the addressee of a proof. Scholarship examining the characteristics of the mathematical register has focused on the addressees of mathematical texts, who do not have personal characteristics (e.g., Pimm, 2017). If, for example, we consider the expression "let x be a natural number", which is a common way of starting a mathematical argument, we notice that this phrase has an addressee, yet there are no references to their individual characteristics. Rotman (1980, as cited in Walkerdine, 1988, p. 186) describes the addressee of a proof:

> Mathematical addresses are theoretical and impersonal: mathematicians prohibit their codes from making any sorts of reference to the individual characteristic of the reader; or to his [sic] subjectivity or to his physical presence in the world. The addressee is merely the repository of conceptual shells that is necessary to decode mathematical signifiers. His psychology is transcendental: independent of cultural variation and the differences between one individual and another.

The addressee of the proof and the impersonal reader share some characteristics: they are impersonal, while their individual characteristics, subjectivities, and physical presence in the world are irrelevant. However, the construct of the impersonal reader is a bit different, as they are not merely the addressee: instead, they are the person who (even if solely at the imaginary level) goes through the very material process of reading the proof. The following example elaborates a bit further on this idea.

The *"How does the reader know?" example* happens as the class works on proof by induction as a proof technique. After Chris has presented a proof and has addressed students' questions about the steps of the proof, he starts a meta-discussion about what a proof by induction should look like, by putting emphasis on the reader of the proof.

> Chris: Ok. So, let's ask a couple of questions. There is many different ways to present a proof by induction. This is the one that I prefer and I'll walk you through the why in this presentation of it. So, first off, if you're reading this, do you know that we're using proof by induction? If so, how? How do you know?
>
> Student: because it says "Proof by Induction"
>
> Chris: Yeah, because it says proof by induction. Great? Awesome. How do we know what the- so, the last line of the proof is $P(n+1)$ is true. How do we know what $P(n)$ is? Did the reader have to be in this class where I said here's the proof strategy? *[Some students say no]* No. How will they know what $P(n)$ is?
>
> Student: because we state that we assume $P(n)$?
>
> Chris: Right. And how do they know what $P(n)$ is? So, I said assume $P(n)$. How do they know what $P(n)$ is?
>
> Student: At the top, you defined what $P(n)$ is.
>
> Chris: Great. I actually defined what particular statement I'm talking about. Good. That's awesome. Does the reader know where we used the induction hypothesis? How do they know? How are they aware where I used the induction hypothesis?
>
> *[A student responds, but I can't hear what they say.]*
>
> Chris: Yeah, we told the reader. We told the reader where we used it. So, all these things are important. Tell the reader we're using induction, define all your terms including $P(n)$. And make sure you tell them, where

> you ‘re using the induction hypothesis. So, those are- I’m saying this not just because this is a useful thing to tell your reader. This is also a useful thing to put on, let’s say, a problem set or that kind of thing. (Chris’s class, pp. 41–42; Lecture 6.2)

In this example, Chris is transparent about the requirements of writing a proof. As indicated at the end of this example, this is mentioned by Chris both in terms of socialising students in the practice of producing mathematical text and in terms of meeting the standards of grading in problem sets and similarly graded activities. The impersonal reader, referred to as “the reader” by Chris plays a crucial role in assessing the quality of the proofs produced by students. While diversity is valued in proof presentations (e.g., “There is many different ways to present a proof by induction”), some elements of proof by induction are treated as essential (i.e., mentioning that the method of induction will be followed, explicitly stating the statement $P(n)$ to be proved, and writing where the induction hypothesis is used). The presentation of these essential elements is assessed through the “impersonal reader”: Chris asks whether the reader would be able to know about each of these when reading the proof. Moreover, as the example unfolds, the reader becomes increasingly impersonal. In the assessment of the first essential element, the reader is any student, as indicated by the second person pronouns used in the sentence “if you’re reading this, do you know that we’re using proof by induction? If so, how? How do you know?”. For the second essential element, Chris initially uses an unspecific first-person pronoun (“we”), followed by a reference to “the reader”. (“How do we know what $P(n)$ is? Did the reader have to be in this class where I said here’s the proof strategy?”). The impersonality of the reader is constructed in this sentence through the need for the proof to be self-contained and not referring to the specifics of their class interactions. Finally, in the third essential element, the reader remains impersonal and is referred to through the pronoun “they”. In this example, the emphasis lies with the reader not being able to misinterpret any element of the proof. In this way, the particularity of the imaginary mathematical community in Chris’s discourse lies with the assumption, expectation, and objective that everyone interprets the same piece of mathematical writing in the same way.

In the rest of this subsection, I will maintain that the impersonal reader has a regulatory role in terms of the writing of mathematical texts. During his interview, Kevin, a student in Chris’s class proposes a similar idea, by comparing the impersonal reader with the “watchtower”. In this way, Kevin suggests that the practice of writing mathematics with an impersonal reader in mind entails a form of surveillance and regulation. Generally, there are multiple ways in which the practice of mathematics is regulated in undergraduate courses. The grades in the course, the tests, the “academic integrity” rules, the expectation to come to class, as well as the positioning of students as customers who buy their education by the university are all institutional practices which regulate students' behaviours. However, a more mathematical in nature regulation is also evident. The existence of the impersonal reader and, more generally, the imaginary community constitutes a "covert" regulatory practice[34].

Let us focus on the example in which Kevin assigns a regulatory role to the impersonal reader. When I ask Kevin about Chris suggesting that they consider the reader of the proof, Kevin replied that this reminded him of "the watchtower". In the *"the watchtower" example*, he elaborates on this idea.

> Dionysia: Ok, let me- ok I think that I have a last thing that I want to ask you. And it's about a different thing that I've seen coming up in the class quite often. And it's when [Chris] talks about the reader of a proof. So, he kind of-
>
> Kevin: yeah. Like the reader is someone out there. Like, ahm, it's like, actually it reminds me a little bit of that thought experiment. I think it's called the watchtower by the English Bentham. I think, but it's like, like you behave differently when someone is watching you. Like, this. That's the thought experiment. But the point I want to make is like, there is always the other, right? Like, so when you are doing- like the other in this case will be the reader of the proof. So, ahm, like, bringing up, you always want to think about the reader it's like you want to do the proof in such a way where, if someone, not you is reading, (like that's a crazy thing to try and do anyway, right?) But if someone, not you, is reading this proof, is it going to make sense to them? Like, I think that is the point he is trying to make. […] But like, for this, for the proofs course, like, I mean if you're, like, yeah, is that strategy effective? It's probably going to help you be conscious of the fact that like don't just do anything. But I think all you are really doing is actually just critically thinking of your own work anyway. And having, thinking of somebody else reading it is not that's all it is. It's that thinking of somebody else reading it ultimately you must make those decisions. If that thinking of somebody else reading it, motivates you to be more meticulous, which it very well might, which is the whole premise of the watchtower, that it affects your behaviour. Ahm, you know, you obey the rules more, when you think someone is looking, maybe you obey the rules more, when you think that somebody is going to be reading it, judging your work. (Chris's class, pp. 174–176; Interview with Kevin)

In this description, we see that for Kevin doing mathematics is a practice which is regulated through the existence of the imaginary reader and more generally the imaginary mathematics community. Unpacking

[34] By way of contrast, in the calculus course, the professor explicitly states mathematical actions that students are not supposed to do. For example, when doing differentiation, students are taught the "rules of differentiation", followed by common mistakes that they are instructed to avoid. We could say that this is a quite overt form of regulating the thinking and doing that is related to differentiation: there are rules to be followed and there are "tips" of how to not "misfollow" them by accident.

this example, the first thing that might be noticed is that, when I ask Kevin, he interrupts me to immediately talk about this practice. This indicates that this practice of Chris, asking students to consider the reader of the proof is a practice that feels very familiar to Kevin. Similarly, his immediate parallelism with the “watchtower” indicates that it might even be a practice on which he has spent some time reflecting.

The “watchtower” or panopticon is a system of control analysed by Foucault (1979) in his book “Discipline and Punish: The Birth of Prison”. The panopticon is a glass tower located at the centre of a prison, from which a guard can watch the inmates. The prisoners cannot see whether the guard watches them at any particular moment. In this way, the panopticon makes them feel that they are being watched at any moment, which makes them act as if they are being watched at all moments. Hence, inmates get disciplined by the suspicion that they may be watched, as opposed to them being tortured or punished by a guard. Foucault uses the panopticon as a metaphor to discuss how it is a disciplinary mechanism of power, which emerged in the 18^{th} century but finds its place in contemporary societies, in spaces like hospitals, factories, schools, and universities. Kevin extends this idea to suggest that the impersonal reader regulates mathematical practice: a person writing a mathematical text needs to do so in such a way that it makes sense to the impersonal reader.

I argue that in Chris’s class, there are two sources which have a function of controlling the production of mathematical writings. First, students do mathematics in this particular way because this is what they are graded on; their grades, in turn, determine whether they will be able to enter the academic program of their choice or not. In other words, for a student to achieve their academic goals, they need to learn how to mathematically express themselves in a particular way. Second, a regulatory role is played by the imaginary of a mathematics community in which everyone has access, specialists communicate with each other, and precise writings can enable effective communication. The impersonal reader plays a role of controlling the individual’s behaviour so that it aligns with the characteristics of this imaginary community.

Overall, the “impersonal reader” manifests in Chris’s frequent references to the “audience” and the “reader of a proof”. In this way, Chris guides students’ mathematics writing behaviour so as to comfort with the standard academic practices. The notion of the impersonal reader constitutes a covert form of regulating one’s mathematical behaviour, in the sense that it “teaches” students to behave in a specific way, by having been socialised in this practice and not by fear of being punished if they do not (e.g., through their grades). Hence, here, we have an instance of governmentality (Foucault, 2004), in the sense that through the impersonal reader, students learn to self-control their conduct.

5.4.3. Beauty as a Value and the Imaginary

A value that manifests in Chris's class and relates to the imaginary of mathematics is the value of beauty. In his lectures, Chris makes comments about proofs being beautiful or ugly. More generally, in mathematics and mathematics education, mathematical beauty is considered to be an important aspect of mathematical practice and it is considered to be objective – just as truth is (Rota, 1997). In this section, I will first elaborate on how beauty manifests as a value in Chris's class, contributing to the imaginary of mathematics. I will then discuss three points that make this focus on beauty political.

The beauty of a proof is discussed in various incidents in Chris's class – all references to beauty are made by the professor himself and not by the students. The textbook also mentions that some people see beauty in proofs and that beauty is associated with the discovery and creation of proofs. All references to mathematical beauty in Chris's class are about the beauty of proofs: theorems, definitions, or algorithms are not characterised as beautiful or not beautiful. This seems to be a pattern in academic mathematical practice, where proofs are the most common mathematical object to be described as being beautiful (Inglis & Aberdein, 2015). This contributes to the conceptualisation of proof as a multidimensional object or process, which in turn contributes to proof being constructed as the most important and profound mathematical process.

In what follows, I use examples from the class to maintain that within the discourse of this class, beauty (or the lack of beauty) is a feature of mathematical proofs and to explore the characteristics of mathematical beauty within this discourse. The *"So, good, what a beautiful proof!" example* pinpoints beauty being a dimension of proofs and contrasts beautiful mathematical proofs to technical proofs.

> Chris: What y will you pick? *[A student responds x]* x will work; $x + 1$ will also work. There are many choices. And now what do we need to do with our y? Well, we need to prove that it has this property, right? Does our y have this property with x? Yes, great. So, we say note $y = x + 1$ is always greater or equal to x. That's our proof. Ok. So, good, what a beautiful proof! (Chris's class, p. 9; Lecture 5.1)

Here, the proof is ironically characterised as beautiful. Chris presents a proof whose key idea is that a y needs to be constructed so that it is always greater or equal to x. This proof involves some technical steps, but it is a trivial one for this class as is indicated by Chris only asking rhetorical questions during its presentation. The proof is considered to lack any form of beauty. In this way, two concepts are constructed by this enunciation. First, the very comment that the proof is not beautiful indicates that beauty is something to be sought, when studying proofs. Second, although the characteristics of beauty here are not elaborated upon, beauty is conceptualised in comparison to what it is not: trivial and technical.

A similar, but more explicit, example is the *"weird and counter-intuitive and complicated and beautiful" example*, which contributes to the conceptualisation of mathematical beauty.

> Chris: So, there are some sizes of infinity that are bigger than others. Uhm, which is a little weird and counter-intuitive and complicated and beautiful at the same time. (Chris's class, p. 110; Lecture 9.2)

In this example which happens as the class works through the cardinality chapter, Chris talks about the different sizes of infinity and uses a series of adjectives to describe this mathematical result. In this way, beauty is by association related to weird, counter-intuitive, and complicated results. In this way, beauty is related to more profound mathematical ideas. The following example, the *"what makes the cardinality in this course so beautiful" example*, further reinforces the characteristics of beauty discussed so far.

> Chris: Uhm, cardinality. So, throughout the course, we teach you lots of different things about, like, like, functions and about induction and all sorts of stuff. And a lot of it is cool, it's interesting, but it's, it's quite technical. Cardinality is the thing that, like, we most enjoy. Like the mathematicians who teach this class, like this is the reason we teach this class, is because it contains a bunch of very cool deep, interesting, counter-intuitive ideas. And, for many of us, this is one of the two things that, like, influenced us to study more mathematics. Uhm, the ideas of cardinality are very interesting. Very strange. Very captivating. And part of, part of what makes the cardinality in this course so beautiful is that as we progress through the course, we learn how to prove things. And we learn, like, what does it mean to provide an air-tight proof of something. Partly so that when we get to results like in cardinality, which are very counter-intuitive, we can say, this breaks my intuition. However, I know that this is still a proof. So, that you believe something, even though it's extremely counter-intuitive. Uhm, and this is part of the power of mathematics. Is that it allows us to understand things that aren't intuitive even though they're true. (Chris's class, p. 91; Lecture 9.1)

In this example, Chris discusses more explicitly why cardinality is considered to be a beautiful area of mathematics. As Chris conceptualises mathematical beauty in this example, four characteristics are prominent. First, similarly to the "So, good, what a beautiful proof!" example, beauty is contrasted to technical mathematical results: cardinality is contrasted to other concepts of the course which, albeit interesting, are "technical". In this way, beauty is evident in non-technical results. Second, similarly to the "weird and counter-intuitive and complicated and beautiful" example*,* beauty is associated with counter-intuition. Counter-intuitive results, like the ones appearing in cardinality, are conceptualised as beautiful. In this way, beauty relates to the element of surprise and captivation that can occur when intuition breaks.

Third, beauty is related to mathematical truth and the "power of mathematics". The counter-intuitive character of some mathematical results gets complimented with the certainty that this result is true, as it has been proven mathematically. Finally, beauty is not merely a characteristic of mathematics, but it is an element of the process of doing mathematics: Chris associates the beauty of cardinality with the enjoyment that a person experiences when working on or teaching this mathematical area. In this way, mathematical beauty gets inscribed in the discourse of mathematics being a social practice between actual people.

The last example about beauty from Chris's lectures that I will discuss here is the *"this is widely regarded as one of the most beautiful proofs" example*, in which Chris engages in an explicit and (compared to other instances) lengthy discussion about the beauty of Euclid's proof that there are infinite prime numbers. The analysis of this example contributes to the conceptualisation of the characteristics of mathematical beauty.

> Chris: let's see the most classic example of the proof by contradiction and the most beautiful, one of the most beautiful. This is widely regarded as one of the most beautiful proofs. […] So, there's a proof that shows up in the textbook "Euclid's Elements" that says there are infinitely many prime numbers. There are many, many proofs of this. But this particular proof is thought to be quite beautiful. We think it's beautiful. […] I have a question for you. Why do people think this proof is beautiful? What's the beautiful part of the proof? I mean that's a kind of strange question to ask, because you can't just pick a part or one thing and say this is the beautiful part. It's in its totality beautiful. That's sort of the nature of the beauty. But, what is about this proof that people point at it and say, oh, yeah, this is nice, I like this? [Pauses] It's a hard question, eh? So, there's a couple of answers to this and it's something that I'd like you to think as you go through it. Like, this is a proof that you should [inaudible] a couple of times and think what is it that draws people to it? I can give you a couple of things that I think about it, but you should on your own try to discover [inaudible]. First off, could you explain this proof to someone in high school? Do you need serious technology to prove this?
>
> *[A person signs no with their head.]*
>
> Chris: No, it's pretty simple. There's primes involved, division, but these are like pretty basic concepts. So, one of the reasons that this proof is so nice is that it doesn't involve any high, abstract constructs [inaudible]. The second thing is, so if you were to write down the proof by contradiction, you would write down the first three lines, where on earth did this third thing come from? Right? Where did this N, this capital n come from? It's like "what?", it's like seeing this out of nowhere. You can imagine Euclid or whoever is who actually first came up with this proof, had some deep insight, oh, yeah this is the N. So, there is this element of mystery. And finally the mystery is a bit big because it's not that they generated a bunch of data, right? If they

generated a bunch of data, they'd see that these types of N-s don't produce primes. So, they had some nice insight about it. So, I think people partly like it because this new prime that they find is quite natural, it's just defined from the things you know [inaudible] (Chris's class, pp. 20–23; Lecture 5.2)

In this example, mathematical beauty is conceptualised within the context of what Chris calls "one of the most beautiful proofs". There are four elements of this conceptualisation that I wish to highlight. First, one of the reasons that Euclid's proof is considered beautiful is that it is a simple proof that involves "pretty basic concepts". In this way, – and in agreement with previous examples in which beauty was dissociated from heavily technical proofs – mathematical beauty is associated with simplicity. Second, the notions of insight and mystery are related to mathematical beauty. In this way, mathematical beauty is related to a proof being intellectually challenging, by entailing an idea that was not foreseeable. Third, Chris encourages students to reflect (and be reflecting) on what makes a proof beautiful. In this way, beauty is deemed to be an important part of students' socialisation in mathematics, which not only includes learning some mathematical theorems, proofs, and methods, but involves learning to appreciate them in similar ways to those that the mathematical community does. Finally, there is a consensus in the mathematical community regarding what constitutes a beautiful proof. Hence, beauty is not subjective: people doing mathematics not only experience the same true mathematical construct, but they share similar appreciation about its aesthetic elements. In this way, the imaginary mathematical construct is not only true and axiomatically structured but is also beautiful in an almost objective way.

Finally, before engaging in a discussion of what makes mathematical beauty political, I present two examples from the interviews with students, in which the two students, Vijin and Chris, propose two very different ways of socialising themselves in the discourse of mathematical beauty enunciated by Chris. In the *"I sort of see it through his eyes" example,* Vijin talks about his enculturation in mathematical beauty.

Dionysia: when I come and observe Chris's course, he often talks about, he often refers to emotions, how a proof feels, how a proof is beautiful or not beautiful.

Vijin: Yeah. I like how he says that proofs are beautiful. I, I, because that's not my initial reaction upon seeing it. But then what he says it, I sort of see it through his eyes. And I'm like oh yeah, it is, it is neat, beautiful. (Chris's class, p. 153; Interview with Vijin)

In this example, Vijin manifests the enculturation aspect of mathematical learning: he socialises himself into mathematics, by seeing mathematics "through [the professor's] eyes". It is in this way that the mathematical practice of the class formats the new mathematicians. As is shown in this example, and more generally in Vijin's interview, Vijin is very open to accepting the professor's discourse. The other student, Kevin, has a more critical attitude towards Chris's discourse, as is shown in the *"the beauty thing, that is just tough" example:*

> Kevin: Yeah, I don't know. The beauty thing, that is just tough. Like. If I were the prof, I probably wouldn't even like bring it up. Like about beauty, like, because, ahm, like, how many people are going to be able to relate to that? And then, if they can relate to that and then the authority figure is saying, like, this is beautiful, then, ahm, like, they are inclined to think, you know because the students, they trust Chris as the professor, like, they are inclined to think, oh this is beautiful, like, ahm, but is it actually beautiful? (Chris's class, p. 174; Interview with Kevin)

In this example, Kevin manifests a very different attitude towards Chris's discourse than Vijin does. Kevin indirectly criticises Chris's choice to refer to beauty in the class, by suggesting that if he were the professor, he wouldn't bring it up on the basis of students not being able to relate to mathematical beauty. He suggests that when the professor, being an authority figure, talks about certain proofs being beautiful, students are inclined to think that this is the case without being in the position to judge if "it [is] actually beautiful". The case of Kevin shows that the class discourse is not directly translated to students' socialisation in it. However, as Kevin maintains, the authority of the professor is significant in terms of "persuading" students about the "accuracy" of his discourse.

So far, we have seen how beauty is manifested as a value in Chris's class. In what follows, I wish to highlight why the discussion of proofs as beautiful is important and how it relates to the political. I will discuss three aspects of the intersection between beauty and the political. First, ideology in general and looking at the world through mathematics in particular are not mere collections of ideas; they also have aesthetic aspects. Bourdieu and Passeron (2014), focusing on reproduction through education, highlight that reproduction does not merely happen based on the pieces of knowledge that one acquires through school; instead, it is also through the cultivation of "good taste". A similar argument can be made regarding mathematics education and the importance of beauty. Learning to look at the world through mathematics and learning to appreciate the world being seen through mathematics involves socialising oneself in the aesthetic culture that is related to mathematics and science.

Second, in(ex)clusion in mathematics and mathematics-related disciplines does not merely happen based on who can produce a proof or who is able to decide whether a proof is formal or not. While these are important elements of becoming mathematically competent, in(ex)clusion also happens through being able to understand the aesthetics of a proof. The aesthetics of proofs and more generally the aesthetics of mathematics are part of the mathematical culture, into which novice mathematicians and computer scientists are expected to socialise. This process of socialisation is a process of in(ex)clusion. As the literature proposes, the sense of not belonging in mathematics is a common reason why students, including women and racialised minorities, get excluded from mathematics (Lahdenperä & Nieminen, 2020; Solomon, 2007). By explicitly discussing aesthetic elements of proofs, Chris makes visible some aspects of mathematical culture that are often not spelled out in university classes – an act which is advocated for as enabling students to belong in mathematics (e.g., Selling, 2016). Making the “untold” visible might be a technique for working towards equity; it does not however cancel the otherwise in(ex)clusive functions of mathematical fields.

Third, aesthetics contributes to a differentiation of mathematical knowledge. Since certain ways of doing mathematics are beautiful, while others are not, the various proofs and mathematical approaches are valued differently. For example, in Chris’s class, “insightful” proofs which have an element of mystery are of greater beauty, and thus, I argue, of greater value, compared to constructive proofs. Furthermore, through the legitimisation of particular ways of doing mathematics as beautiful, the associated values of insightfulness, smartness, and intellectuality are also legitimised. In this way, learning “beauty” in the context of the course establishes this particular mathematics and the practices which surround it as not only beautiful but also valuable.

Overall, in this section, I argued that beauty is a value in Chris’s discourse and is conceptualised as being not trivial or technical, as related to counter-intuition, simplicity, insight, mystery, and enjoyment - elements which are also identified as characteristics of mathematical beauty in literature (e.g., Ernest, 2015). Furthermore, beauty is not subjective and is related to the value of truth. In this way, beauty relates to the imaginary of the “power of mathematics”. Furthermore, I suggested that the aesthetics of proofs relates to the political because aesthetics is constitutive of what it means to look at the world through mathematics, aesthetics is a key aspect of the mathematical culture and hence partakes in the in(ex)clusionary function of mathematics, and the aesthetics contributes to a differentiation of mathematical knowledge.

5.4.4. Bringing it All Together

In this section, I discussed how the discourse of imaginary mathematics manifests in Chris’s class. Through a discussion of examples from the class, I argued that mathematics is conceptualised at the imaginary level as a system of theorems, each of which follows from axioms and previously proven theorems. Related to this powerful image of mathematics which produces results that are true beyond any doubt is the notion of beautiful mathematical proofs which compliments the image of mathematics as a superior form of knowledge. In this context, a central concept is the imaginary mathematics community, which is specialised, accessible to all, functions through the communication of its members, prioritises understanding, and employs preciseness as a mode of writing to achieve that. Furthermore, this discourse constructs an "impersonal reader", as the audience with whom in mind, proofs need to be written. This impersonal reader plays a role of regulating mathematical behaviour.

The imaginary of mathematical practices constructs strict boundaries for mathematical practice. As any disagreement or misunderstanding is able to be resolved by going back to the axioms or a previously proven theorem, everything has been agreed upon prior to it being discussed in class. This means that any personal images of mathematical concepts do not quite matter, except maybe to the extent that they enable the production of mathematical knowledge. They are, however, irrelevant to the process of achieving an agreement. In this way, the mathematical practice of the class is defined by the imaginary of mathematics.

The discourse of the imaginary practice of mathematics shapes several aspects of the course and relates to many aspects of the research questions (the course's values, its rationality, and the formatting role of mathematics in the context of the course). With regards to values, the values of truth and beauty are part of the imaginary picture of mathematics, while the value of the shared space is a version of the image of a mathematical community in which everyone has access and can contribute. As far as rationality is concerned, one of the ways in which mathematics is developed is through interactions within an imaginary mathematics community. Finally, the imaginary practice of mathematics is evident in the ways in which both mathematics and (novice and experienced) mathematicians are constructed as the product and members respectively of the imaginary community.

5.5. The *"Efficacy of Mathematical Work/Labour"* Discourse

The fourth overarching discourse concerns valuing and encouraging the efficacy of mathematical work/labour. Efficacy is a value in Chris's course and refers to the way in which a person does mathematics. More specifically, efficacy relates to the time spent and the labour/work required to complete a mathematical task. Efficacy is important in terms of how mathematics is political because it shapes a particular way of doing labour/work with/in mathematics. This section is organised as follows. First, I start with a discussion of how time is brought up in the classroom discourse. I continue with the exploration of the work/labour distinction. I then explore the idea that "mathematics happens in the brain", a theme which emerges in Chris's class, in relation to this discussion by situating it in discourses about the mind/body dualism. Finally, I discuss the notion of efficacy within the institutional context of a first year undergraduate course and its emerging importances.

5.5.1. Time

In this section, I discuss the parameter of time as a starting point in the exploration of the efficacy of mathematical work/labour discourse. The observation that time has not been widely examined in mathematics education literature (Makrakis, 2019; Otten & Pimm, 2018) has drawn my interest in Chris's in-class direct and indirect references to time and mathematical speed. Before going into two specific examples, it is notable that time conventions play a crucial role in shaping this course as they do with mathematics education more generally. For example, class meetings are strictly-timed: the meetings happen at a specific time and for a specific duration, showing how time conventions play a key regulatory role which both make the class possible (e.g., people know when to meet) and exclude those who fail to follow the convention (e.g., if a student does not hear their alarm clock in the morning, they miss the class meeting). The regulatory role of time is perhaps more visible in the strictly-timed manner that assessments take place; coming in late or missing the exam can mean that a student fails the course. Finally and in a more direct relation to the idea of efficacy, the institutional emphasis on speed when completing a task (e.g., quiz, exam, homework problem) is of tremendous importance for students to do well in the class and get through the course's selective function for the desired computer science program.

I argue that time also plays an important role in formatting Chris's class by formatting the discussions during the lecture. Chris gives students a specific amount of time to work on a problem and, when this time is over, students' work is interrupted and a whole-class discussion begins. Moreover, sometimes students ask a question, but the question is not immediately addressed; instead Chris asks them to discuss the topic with him during the break or office hours. Through this practice, we see the emergence of a notion of efficacy: in-class time is limited and needs to be used efficiently.

Furthermore, mathematical practice is conceptualised in a way in which doing things fast is desirable. I illustrate this point through two examples from the class. In the "*let's race" example*, Chris invites students to write the formal negation of a mathematical statement in the class by themselves, while he is doing the same, by jokingly proposing that they race:

> Chris: All right, I want you to compute the negation of this. The negation of "$\exists x \in R\ \forall Y \in R, y < x$". Remember to pull in the negation one step at a time. So, let's race, I'm going to write down every single step of pulling into this one step at a time, but you can do this. You don't need to watch me do it. (Chris's class, p. 9; Lecture 5.1)

Here, Chris does not literally invite students to race. However, he invites them to work on the question quickly, as they will only have time to do so until he finishes writing the answer. The invitation to race is indicative of the way in which time and speed are valued in mathematics. First, this example makes explicit the speed in which students are required to think and work during the classroom: students are asked to work on this problem in a similar amount of time needed by the professor to complete the same task. Moreover, in a race, it is clear that the winner is the one who finishes first; in other words, here, it is clear that doing things fast in mathematics is important.

This position becomes more explicit in the following *"how long did it take you?" example*, in which the professor asks one student to compute the tenth term of a sequence by using the sequence's explicit definition and another student to compute the same term by using the recursive definition. After the two students have completed their tasks, the class compares the two methods:

> Chris: And how long did it take you for the explicit definition?
>
> Student 1: Pi seconds.
>
> Chris: Like around pi seconds? Ok, so, around 3 seconds to compute this. How long did it take you?
>
> Student 2: [Inaudible]
>
> Chris: It's like 90 seconds, right? 90 seconds. And let's think for a moment, why it took you so long to compute it. It wasn't particularly hard. But, what- was it that took you so long?
>
> Student 2: [Inaudible]
>
> Chris: Yeah, you had to start from a_1. And you had to keep plugging in each individual thing. So, you actually did like a lot of computation. Whereas for you, how many, like, times did you like punch things into your calculator? Like, I don't know? You pressed like a total of six buttons or something. So, the explicit definition allows you to compute very large terms in a sequence, almost immediately. But for the recursive definition, you have to build your way up from the bottom every single time. So, the recursive definition requires a lot of work. (Chris's class, pp. 56–57; Lecture 7.1)

In this example, there is a direct reference to time, as the two students are asked to estimate the time that they needed to complete the two tasks. In this process, it is clear that the less time used to do a computation, the better. In this way, time contributes to framing the meaning of efficacy as a desired characteristic of mathematics. This practice resonates with a political economy, in which time shall not be wasted. In this way, I argue that the students, who throughout this process learn to *labour/work* with mathematics, also learn how to be efficient workers who focus on producing a result fast.

Another aspect of the "how long did it take you?" example is related to the kind of knowledge which offers a solution quickly. In the example, saving time is achieved through the explicit definition, which has a global power: an individual can calculate the n-th term of the sequence simply by substituting *n* into the definition. On the other hand, wasting time is associated with the recursive definition which has a more local power: in order to calculate the n-th term of the sequence, someone would need to know the (n-1)-th term. In this way, the knowing which is constructed as more valued is the one that can be applied globally and can save time. This observation relates to Abtahi's comment on the "power" and potential violent aspect of abstraction, which emphasises that within mainstream mathematics education, less context-based mathematics is considered to be more "powerful" than context sensitive tools, just like a chicken factory has more power than a chicken farm, by "producing more, faster, and *the same*" (Abtahi & Wagner, 2016, p. 39).

5.5.2. Work/Labour

In this section, I explore how the notion of work/labour emerges in Chris's class. As an introductory comment, spending time with a mathematical task or activity can be conceptualised in a multitude of ways: a person can *play* with a mathematical problem, *study* a mathematical concept, *learn* a mathematical theory, *entertain themselves* with a mathematical concept, or *do work/labour* on a mathematical problem. In other words, references to work are not the only way to conceptualise mathematical activity and, as a result, it is important to explore how mathematics is conceptualised as work. The *"Well, you're skipping a bunch of work" example* conceptualises proving a theorem as doing some work:

> Chris: So, to maybe explain a little bit, when we were doing equivalences, I sort of made this big thing between implications and bidirectional arrows. A bunch of you ask me, can't I put bidirectional arrows everywhere? And I prickled because if you're trying to prove if an only if, you have to prove two different implications. Roughly, when we prove some implications, it took us some work to do, right? And then to do the other way implication, we have to basically do the same amount of work. So, what you're asking me to do, like by saying "Am I allowed to put bidirectional arrows?" is "Can I skip basically the same amount of work?" And that's why I prickled. Well, you're skipping a bunch of work. [inaudible] All right, that was to get off my chest. (Chris's class, pp. 13–14; Lecture 5.1)

In this example, the (in)correctness of a mathematical task is justified in terms of how much work/labour an individual has put in. This means that proving a proposition presupposes doing some work; skipping the work means not having proved the theorem. This articulation draws upon a "culture of effort". As has already been suggested elsewhere in this thesis, mathematics curricula do not only aim at the learning of mathematical concepts and techniques, but also at the production of a certain citizen and worker (e.g., Valero, 2018). I suggest that, here, students as prospective citizens/workers “learn” to espouse this culture of effort and that success is discursively constructed as being the result of having completed the required work. This relates to an educational system as well as a capitalistic economic system which claim to be meritocratic and to reward the work and effort contributed (e.g., Kollosche, 2018a, Rafailidis, 2000). However, in capitalism, doing the work by no means ensures “success”; yet, this culture serves to frame the (capitalistic) world as meritocratic, concealing how it is unfair (e.g., Pais, 2014).

This discourse also relates to the educational institution. In order to get into the computer science program, students need to compete with their peers. The conceptualisation of doing mathematics as performing work/labour acts as an ideological framework which justifies the selective function of the course: doing work means solving exercises correctly (e.g., proving an equivalence), which in turn means doing well in the course, and finally leads to entering the computer science program. This narrative draws the attention away from there only being a specific number of positions in the program and, thus, the fact that some will inevitably fail even if they have completed the required work. It also draws the attention away from the complex cultural practices which weave what it means to do mathematics (e.g., appreciating mathematical beauty, understanding mathematical humour, etc.) and reduces the potential richness of mathematical practice to completing some work. However, the selective function of educational institutions also occurs through students’ different socialisations into the complex cultures of schooling and mathematics (Bourdieu & Passeron, 2014; Jorgensen et al., 2013).

A significant aspect of mathematical work/labour as conceptualised in Chris’s discourse is that it is not uniform; there are different versions of mathematical work/labour which are valued differently. The *“Be a hero” example*, which helps to illustrate this idea, happens as the class tries to to show that the function $h(x) = \frac{x}{x+1}$ is onto. In this process, the class encounters the quantity $\frac{\frac{y}{1-y}}{\frac{y}{1-y}+1}$ and needs to simplify it and show that it equals y. Chris does not engage in this computation, but instead states the following:

> Chris: Be a hero, simplify this, you’ll get that it’s y. (Chris’s class, p. 89; Lecture 8.1)

In this example, we see a reference to some computational work, which is typically conducted in mathematics classrooms by doing some algebraic manipulations. What is notable regarding these computations is that the professor does not do them, but asks the students to do so. The phrase “be a hero” indicates that these computations are undesirable; although not particularly difficult, they might be tedious.

This conceptualisation of mathematical work/labour is very different than the mathematical work/labour that is associated with the production of a theorem and its proof, that is occasionally beautiful, and that relates to the "collecting data – making conjectures – proving a statement" triptych during which people discuss and occasionally disagree. Thus, Chris's discourse conceptualises two different types of work/labour present in mathematics. I use the terms work and labour, drawing upon Arendt's distinction between work and labour. Arendt (1987) proposes that labour "produce[s] whatever is necessary to keep the human organism alive", whereas work "create[s] whatever is needed to house the human body" (p. 29). Arendt derives the distinction between work and labour from Locke's distinction who talks about the labour of our body and the work of our hands. In my transfer of this idea in the context of Chris's class, we may talk about the work of our mind and the labour of our hands – on the one hand, there is the conceptual work that is associated with the production of definitions, theorems, and proofs and on the other hand, there is the computational labour that is associated with the production of the necessary computations in the above process. In this way, contrary to work, labour entails repetition, happens to meet a necessity that derives from the production of a theorem or proof, and its products do not last but are rather immediately consumed, in the sense that these calculations do not have any value outside the initial context of their production (as in contrast does a theorem). Furthermore, contrary to the collaborative and discursive character of work, mathematical labour consists of completing computations which do not require discussion or collaboration: the professor and students merely conduct this labour but they do not need to discuss it.

This practice of differentiating between the work and labour aspects of producing a proof can be seen in the way in which Chris frames the class's discussions about the presented proofs. Any time Chris presents a proof to the class (which might or might not follow students' engagement with the proof by themselves or with their peers), he afterwards invites students to ask questions or make observations about the proof. In doing so, Chris usually organises the discussion by first asking if there are any questions about the steps of the proof and then about the "zoomed out" picture. We can see an instance of this in the following *"the zoomed out version of this" example*:

> Chris: So, before we get into the zoomed out version of this, are there any questions about any of the steps?
> (Chris's class, p. 62; Lecture 7.1)

Through this organisation of the classroom discussion, two ways of doing mathematics become evident, corresponding to labour and work: the first one focuses on the mechanics of a proof, while the second one focuses on the big picture of the proof and the theorem. Chris's approach of organising the discussion by differentiating between the “steps” and the “zoomed out version” constructs and highlights the two forms of mathematical engagement as distinct and separable. Through this distinction (which is by no means natural), the main focus of the course becomes the logic of proofs as opposed to the computations which are part of proofs.

5.5.3. Implications for Rationality

In this section, I discuss the implications that the discourse of the efficacy of mathematical work and labour has for the construction of a mathematical rationality, specifically through the construct that mathematics is something that happens in the brain. So far in this chapter, mathematics has been conceptualised in terms of how it happens through communication between real people and how it happens within an imaginary mathematics community. In the idea that mathematics happens in the brain of an individual person, we encounter a third facet of mathematical rationality in Chris’s class.

Often, Chris refers to students’ brains, as the ones engaging when students do mathematics. I present three examples which support this idea and I then discuss them and explore how the relations between mathematics and brains are conceptualised. The *“let’s dig deep into our brain boxes” example* happens as Chris asks students to recall “the contrapositive of Q implies P”:

> Chris: So, let’s dig deep into our brain boxes, if you try to prove Q implies P, what’s the contrapositive of Q implies P? (Chris’s class, p. 14; Lecture 5.1)

In a similar vein, Chris refers to what happens in the class in terms of brains working. We can see this in *the “brain juices” example*, which happens right when the class meets after the break.

> Chris: All right. Let’s get started. *[Pause]* All right, so now that you got a chance to, for your brain juices to circulate and shuttle, are there any questions? No? Ok. (Chris’s class, p. 31; Lecture 6.1)

In this example, taking a break is parallelised with brain juices circulating and shuttling. A third example is the *“brain melting” example*:

> Chris: So, now we’re going to do a slightly weird example. This, I like to describe as a brain melting exercise. Or brain melting example. What is the sum from i equals 1 to 0, of a_i? (Chris’s class, p. 32; Lecture 6.1)

This example is about figuring out the value of an empty sum or in other words of a sum of no terms. As this sum is discussed through the lens of sigma notation, it requires the addition of the terms a_i, where i takes all values from 1 all the way “up” to 0. This example is a subtle one and is characterised as “weird” and “brain melting” by Chris. Finally, it should also be noted that it is not only Chris who refers to mathematics as happening in the brain; students sometimes do, too. For instance, as shown in the *“give* *me your brain" example*, during an office hours session, a student asks help from a classmate and friend by requesting their brain:

> S2 (to S4): give me your brain (Chris's class, p. 114; Office Hours Week 9).

In these examples, mathematics is conceptualised as something that happens in people's heads: information is stocked in "brain boxes", thinking happens through movement of "brain juices", some examples are "brain melting", and asking for help means "borrowing someone's brain". Contrary to the other two previously discussed ways of doing mathematics (i.e., through communication between real people and within an imaginary mathematics community), this way of doing mathematics positions mathematics at a more individualistic level. Conceptualising mathematics as something that happens in the brain aligns – although not necessarily intentionally – with two interrelated dominant discourses about mathematics: first, the discourse that mathematics is something done by smart people and second the capitalist (hierarchical) distinction between intellectual and manual labour. I discuss these two discourses in the following paragraphs.

First, the conceptualisation of mathematics as something that happens in the brain relates mathematics to smartness or mathematical ability (both of which discursively refer to some ability of the mind/brain). Mathematical ability, conceived of as an innate characteristic of a person positions some people as smart or talented in this type of work and others as not. However, research in mathematics education has shown that smartness and mathematical ability are not natural characteristics of a person; instead, they constitute gendered, racialised, and classist properties (e.g., Martin, 2009b; Mendick, 2006; Walkerdine 1998). When smartness and mathematical ability are conceptualised as naturally produced, their relation to the systems of patriarchy, white supremacy, and capitalism is hidden. Hence, smartness and mathematical ability become an inclusion/exclusion mechanism which not only operates within these oppressive and exploitative systems but also ultimately reproduces them.

Second, the conceptualisation of mathematics as happening in the brain occurs within the prevalent mind/body dualism, in the first pole of which mathematics is positioned (Mendick, 2006). The mind/body dualism is prevalent in Western cultures and philosophies and is related to the distinction between manual and intellectual labour which is prevalent in capitalist economies. Poulatzas (2008) highlights that "[t]his division can by no means be understood in an empirical-naturalistic manner, as if there were a split between those who work with their hands and those who work with their brains: it comes down directly to the politico-ideological relations as they exist in given production relations" (p. 77). The manual/intellectual labour division is reflected in the field of mathematics through the prevalent hierarchical distinction in mathematics education between “procedural” and “propositional knowledge” (Walkerdine, 1998). In Chris’s class this distinction takes the form of the computational/conceptual knowledge and the labour/work distinctions discussed in the previous subsection. Walkerdine (1998) challenges this distinction, by arguing that it has significant effects on the ways in which mathematics is taught. Procedural knowledge is advocated for equipping students to complete every-day or non-academic tasks related to mathematics, while propositional knowledge is viewed as necessary for students who are going to enter mathematics related disciplines. Similarly, in Chris’s class, students are not only expected to conduct labour and master the computational aspects of the course, but to succeed in the course and enter mathematics related disciplines such as computer science, they are required to be successful in what I called work and conceptual knowledge. In this sense, the dichotomies built in Chris’s class are not innocent since they reflect and reproduce the existing divisions of labour at the societal level.

My point is not that the professor aims to include/exclude people on the basis of their smartness; on the contrary, he never talks about smart people/students/mathematicians in his lectures and he does not associate cleverness with doing well in mathematics. For example, when a student informally asks him during office hours about finding an academic job after a PhD, he challenges the importance of cleverness by suggesting that “You certainly don’t have to be the smartest person. You have to work hard and you have to work towards networking with people” (Chris’ class, p. 117; Lecture 9). Similarly, I do not suggest that the professor deliberately positions mathematics as intellectual labour, in an effort to reproduce the social division of labour that supports capitalist production relations. My intention is to pinpoint how otherwise “innocent” references to mathematics as something that happens in individual peoples’ minds reflect the relation of mathematics with the two discourses, each one of which has a long history of conceptualising mathematics as an activity for the few.

5.5.4. Institutional Context and Institutional Importances

In this section, I discuss the efficacy discourse in Chris’s class in light of the university’s institutional context and its emerging institutional importances. Here, institutional importance refers to the acknowledgment of the institutional framework, its restrictions, and priorities. I start with the presentation of an example from Kevin’s interview, which positions the course within the institutional framework of the university. I continue with three examples from Chris’s class, which highlight institutional aspects of the course, in relation to exams as well as the memory needed by students in order to remember proofs. I conclude this subsection with a short discussion of the institutional framework as it emerges through the discussed examples and its relation to the notion of efficacy.

The institutional context of the course is highlighted in the *"it's the cycle it goes" example*, which happens during the interview with Kevin as I ask him how the course fits his schedule:

> Kevin: I think, it's just like, it's just barely relevant to what I want to do and how I even think about math. Uhm, because I have no intention of being, like, a mathematician. And I think, even the computer science, like, you can always make that argument. Like, you know, people often say, it's a way of thinking. And that is true. It's a way of thinking, but like what is exactly that way of thinking? Like, uhm, so there is a little bit of relevance to computer science. But yeah, I think, I know, I don't think it's as valuable as the school makes it seem by making people take it. And making them get an 80 or something - it depends on the year - but, fortunately at that mark, in this one specific course, if they want to go in this specific major. I think it's a little bit, I'm not sure what the word will be, like, I don't like speaking badly of the university. But like, a professor's written a textbook. They made a course, they're charging students money. Like, it's the cycle it goes. So, it seems to be working very well for them. I'm not sure if that was the main motivating factor, but I don't know where it fits in. Like well. I mean it fits in ok, but not really well. Not for me and not for probably many other students. (Chris's class, pp. 164–165; Interview with Kevin)

In this example, Kevin criticises the assumed value of the course. Kevin argues that his own take regarding how valuable the course is diverges from how valuable the university makes it look. He suggests that the value of the course for the university is related to the department having created a course, written a textbook for it, and charging students money in order for them to take it. Moreover, he positions within this economic function of the course, the condition that students need to take this course and achieve a very high grade in it, in order to follow their desired academic path. The enunciation of this idea is indicative, as Kevin uses the phrase "I don't think it's as valuable as the school makes it seem by making people take it". Here, students are conceptualised as being very restricted by the context, while the university in an active way forces students to take the course, parallel to conceptualising it as academically valuable. Overall, this example brings to the forefront two aspects of the political economy of the course and the university. First, the university has the function of distributing its students to different career paths by putting in place measures that ensure that only a handful of students will be able to follow certain academic paths. Second, the university is not an institution with a merely educational function, but it also operates as a company that wishes to be profitable and reproduce itself at the economic level. These aspects, although a central part of students' experience (e.g., students pay tuition fees, sometimes they need to re-take courses, many students go in debt in order to acquire university education, etc.) are not often studied or taken into consideration by mathematics education research, as it focuses on either aspects of teaching and learning or issues of equity that do not address the institutional context of the university (Allais, 2014).

The institutional context becomes central in some every-day aspects of the course, too. The institutional context affects the content of the course, in the sense that Chris makes an effort to draw connections between mathematical proofs and computer science. For example, when working on the calculation of the sum of the first n natural numbers, Chris presents his students with two solutions, a "math solution" and a "CS[35] solution" (p. 30). Moreover, at the beginning of each lecture, Chris always dedicates some time to bureaucratic aspects of the course (e.g., deadlines, tests, etc.). The *"if you want to give feedback about the quiz" example* is such an instance that happens the day after the class have written an hour-long quiz, the results of which contribute to the course's final grade.

> Chris: So, before we get started today, I'd like to take this opportunity to talk a little bit about the quiz, if you want to talk about it. So, yeah, it's up to you. Uhm, how did the quiz go?
>
> *[Some students respond.]*
>
> Chris: It was good? Someone said plentiful? That's what I heard. Any comments, feedback, questions, observations? Do you wanna write a quiz every-
>
> Some students: No
>
> Chris: You got no feedback at all?
>
> *[A student jokes that they want to write a quiz very often.]*
>
> Chris: Yeah, yeah. Ok, so if you want to give feedback about the quiz, you're welcome to come and talk to me during the break, or after class, or send me an email, all that's fine. (Chris's class, p. 24; Lecture 6.1)

In this example, Chris invites students to provide feedback about the quiz that they had taken the day before. This example is of significance, as quizzes, tests, and exams are in place in order to assess students' performance. Traditionally, the professor is the one who sets the framework for these assessments (e.g., by posing the questions and determining the time and conditions under which these questions shall be answered), the students are required to respond to this framework by showing what they have learned, and the professor is the one who has the final say regarding the level to which students succeeded in learning the required material. Here, Chris challenges this process, by inviting students to provide

[35] CS stands for computer science here.

feedback about the quiz itself. In this way, Chris acts towards a democratisation of the classroom, as he seems open to adjusting his practice based on students' take of it. Nevertheless, the institutional aspects highlighted by Kevin are not challenged in any way by this action of Chris's, a process that indicates the restrictive character of the institutional framework.

Another way in which the institutional becomes central is through in-class examples that relate to how students' exam answers are graded. One such example is the *"you get to be the person who's in charge of grading" example*:

> Chris: Ok. So, now, I typically ask you do you have any questions. But [inaudible] if this was my proof that I gave on the final exam, this would not worth full marks. Find two places, where you would remove marks from me. You get to be the person who's in charge of grading. Why is this a 3/5 solution? Find two marks that are gone. (Chris's class, p. 44; Lecture 6.2)

This example happens after Chris has presented a proof to the students. As indicated by Chris and in accordance with my own observations of the class, after completing the presentation of a proof, Chris typically asks students if they have any questions. However, in this particular example, Chris has presented a proof which contains a couple of flaws that would result in students losing some grades, had they presented it in an exam. Chris invites students to reflect on what these flaws are and how they would fix them. The framing of this example within the context of a final exam indicates the importance of the context of the exams for both Chris and his students. The goal of these lectures is not merely for students to "learn the material", but also to be able to perform in an exam in such a way that shows that they have learned the material in accordance to the course's standards.

Finally, I turn to an example, in which the concept of memory is highlighted. Besides time and work/labour (which I discussed earlier in this section), memory is a third factor which relates to the emergence of efficacy as a value. In the *"how much memory it takes to remember this proof" example*, memory is described in terms of remembering a proof:

> Chris: Uhm, in terms of how much, how like, how much memory it takes to remember this proof, uhm, it basically has like, the only thing you need to remember is the compositions of injections are injections. There is not really a lot to remember. You need to, I guess, remember the definitions. Once you write down all the definitions, there is really only one or maybe two choices for what you can do. That's it. So, this shouldn't feel like a hard proof to remember. It's write down the definitions. (Chris's class, p. 96; Lecture 9.1)

This example refers to the memory that “it takes to remember [a] proof” and conceptualises memory as important in the context of an undergraduate course. The concept of memory described here by Chris is discussed in literature through a term coined by Gowers (2007, as cited in Hanna, 2014) as “the width of proof”. Within a syntactic proof (Weber & Alcock, 2004) as the one presented by Chris, the width of the proof or the memory needed to remember how to reproduce the proof is quite small. In the example, there is not a reason cited as to why students need to remember this (or any other) proof. In a naturalised way, students are expected to remember proofs, as they might be asked to reproduce them during exams. In this way, the institutional, exam-oriented framework shapes the ways in which students are expected to interact with mathematics including by being able to remember what they have been taught.

Overall, in this section, I focused on the institutional context and the institutional importances as they emerge through the course. These are mostly related to the exam-oriented framework of the course. Although the institutional context frames the way in which the course can happen as well as shapes students’ experiences, the more structural elements of this framework are not quite evident in the course in the sense that they are not brought up during the lectures by either the professor or the students. Although the professor introduces a democratic framework within which students are encouraged to provide feedback to the quizzes, the structure of the course in an exam-oriented way is not really challenged. In this way, the institutional context frames students’ experiences with mathematics in an exam-oriented way, within which students’ efficacy relates to them being able to remember the covered material as well as them being able to perform in the exams according to the course’s standards.

5.5.5. Bringing it All Together

In the previous subsections, I explored how efficacy is weaved in Chris’s class through a focus on time and work/labour. Doing things fast is important in the political economy of neoliberal capitalism. In a context in which a worker sells their labour, the more that they can do in a given amount of time, the better for capital (Rafailidis, 2000). Mathematics education shapes students (and current or future workers) in ways which align with the values of this workspace and economy. A second element that we see here is that mathematics is constructed as a cultural product which has two distinct forms of engaging with it: work and labour. This teaches students to see things through a similar distinction, in which some forms of labour/work are of greater value and depth than others. This is important in terms of justifying and reproducing the ideology of manual and intellectual work, as well as labour/work which is paid differently despite being performed for the same amount of time. Finally, the discourse of the efficacy of mathematical work/labour manifests in the exam-oriented institutional context of the university, within which the course plays a gatekeeping role for the popular computer science program.

5.6. Summary of Main Findings

In this chapter, I presented the findings from Chris's "Introduction to Proofs" course. I organised the findings in terms of four overarching discourses that are evident in Chris's class. First, formalising is conceptualised as ensuring rigour and being a catalyst for the production of mathematical results. Second, diverse participation in mathematics is encouraged and celebrated. Third, the mathematical community is constructed in an imaginary way, in which everyone has access to a mathematical knowledge shared across time and place, while mathematics is constructed as a system in which it is possible for every theorem to be traced back to axioms. Finally, developing ways which ensure the efficacy of mathematical work/labour is valued.

In this process, I have used resources from the history of mathematics, philosophy of mathematics, and mathematics education to make the point that there is no essential reason for these discourses to be the way that they are. In an attempt to situate them, I have explored how the course's construct of mathematics is a product of the university institution and of a specific trajectory of mathematical knowledge (from ancient Greece to the European 17th century, all the way to today's globalised academic landscape). Moreover, I have discussed how this mathematical construct supports a political economy that entails a form of work/labour in the academic sector and beyond, which prioritises efficacy and diversity. Students in the course learn to perform these mathematical discourses, while discursive constructs, like the imaginary mathematics community and formality as a prototype of rigour, act as regulatory mechanisms. In Chapter 7, I return to these ideas and combine them with the findings from Alex's calculus course (discussed in Chapter 6), in order to address the study's research questions.

Chapter 6: The Calculus Course

In this chapter, I present the major findings related to the Calculus course. In subsection 2.3.1., I reviewed the literature on calculus teaching and learning with an emphasis on undergraduate settings. In that section, I suggested that literature on calculus teaching and learning does not consider the political, but instead constructs a single formal calculus knowledge as the norm, thereby constructing calculus as an exclusive knowledge and divorcing it from critical and reflective approaches towards it. In this chapter, I present the findings from the "Calculus" course, by highlighting its political aspects. To do so, I focus on the discourses that are evident in the course and I attempt to highlight their political elements. I have identified three overarching discourses that manifest themselves in the mathematical practice of this class:

- the "humanity of mathematics" discourse, which conceptualises mathematics as something that humans have always been doing and will always be doing.
- the "applying mathematics" discourse, which conceptualises mathematics as a knowledge that can be applied to everything.
- the "mathematics of the class and the mathematics of mathematicians" discourse, which distinguishes between the mathematical practice of the class and the mathematical practice of the mathematicians, with the latter acting as the source of rigour for the former.

This chapter is organised as follows. First, I present a "sketch" of the course, in which I give an overall idea of how the course is organised and of the mathematical content covered in the course. I continue the presentation of the findings by exploring each of the three overarching discourses through examples from the class. I conclude this chapter, by summarising the main findings discussed in the previous sections.

6.1. A Sketch of the Course

In this section, I draw a sketch of the Calculus course. The section is organised as follows. I first provide some general information about the course and describe how it looks at an everyday level. I then provide a short description of the contents covered in the course.

6.1.1. Drawing the Details of the Course

The calculus course consists of lectures and tutorial sessions. It has multiple lecture sessions, which means that the professor whose lectures I attended, Alex, is not the only professor who teaches the course during the semester. Students also attend tutorials once a week, which are run by graduate students who work as teaching assistants (TAs). During the lectures, the professors deliver new material, while the tutorial sessions focus on exercises and problems on the already delivered material. Sometimes, the professors and the TAs hold office hours, which students can attend on an as-needed/as-wanted basis. During my visits, I only attended some of Alex's lectures.

Alex is an instructor who cares about his students, their backgrounds, and their foregrounds. In his interview, he shared with me his understanding of the educational and mathematical background of his students, considering how they also live within a complex wider social context, which for example includes several of them having to work in parallel to their studies.

Alex's lecture takes place in a big, amphitheatric lecture hall in an old university building. The room has about 250 seats (see a sketch of the room in Figure 6.1). In the front of the room, there are three blackboards, arranged one behind the other. The boards can be moved vertically, so that two boards may be visible to the students at any time. The professor uses all boards while lecturing. The room also has a projector. During my visits, Alex used this option once to complete two tasks: graphing two functions using a digital mathematics tool and using an assessment task during which the professor projects some questions and students answer them using their personal electronic devices.

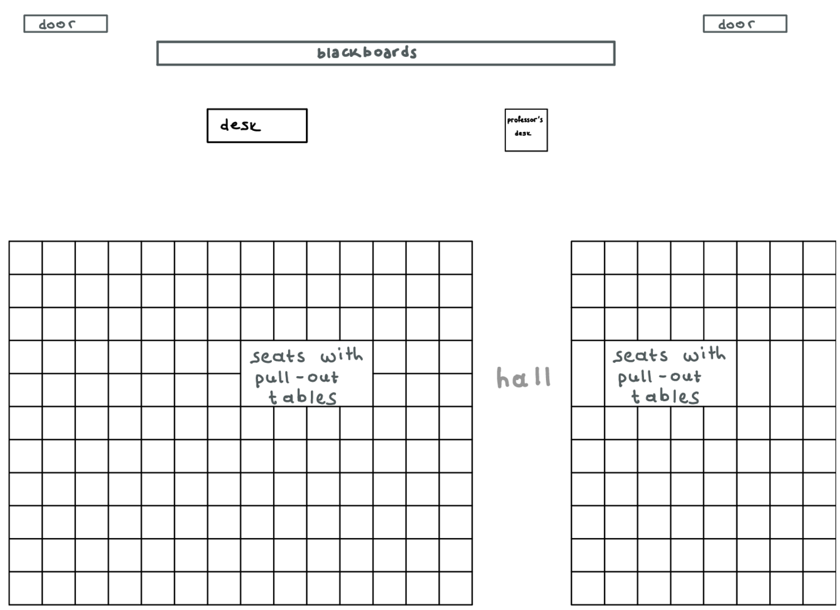

Figure 6.1: Sketch of the Calculus Lecture Hall

The class meets once a week in the evening and starts a quarter of an hour after its scheduled time, as is the norm in this university. The class is attended regularly by about 60-70 students. Before the class starts, students move towards the front of the room and leave their homework assignment (which students are required to submit every week) on a pile for the professor to take at the end of the class. About ten minutes before class starts, the professor arrives and sets up. Some students approach the professor to chat individually about issues related to the course. Sometimes, they talk about the homework assignments, which students are just about to submit. Sometimes, they discuss more bureaucratic or house-keeping issues. Apart from before class, students also approach the professor during the break and after class. Often, they ask clarification on something that Alex presented during class (referring to what is written on the board), while sometimes a few students just casually chat with the professor.

At the beginning of the class, after individual discussions between students and the professor have been completed and when the class is about to start, I approach the professor. I turn the recording device on and I leave it on the professor's desk. I go back to my seat, which is towards the back of the room and near the hall. I have a notebook with me, which I use to keep notes. Rarely do students sit near me and my interactions with them are limited. If the professor has announcements, these happen at the beginning of the class and are succeeded by the material that the professor has planned to teach. After Alex starts lecturing, students keep entering the room quietly. They sit alone or in small groups. They do not appear to have formed big groups, while many students come and leave the class without talking with any of their classmates.

The professor delivers the material in the following way. The mathematical content which is shared with students is typically written on the board, including graphs, calculations of limits, solving equations, processes for describing the behaviour of a function, rules, and theorems. Alex talks about the mathematical material by referring to what is written on the board; the board and the written words are the extension of the voice and vice versa. The board is well organised, while the use of various chalk colours signifies important aspects of the material or aspects that the students need to be cautious about. Furthermore, the professor is lively and, when lecturing, often moves around the front part of the class or makes hand gestures to depict the graph of a function or to depict how change occurs in relation to a function (e.g., rate of change, increasing/decreasing, concavity). The words, symbols, and graphs on the board, the professor's words, and the professor's movements and gestures are attuned; each of them contributes to a lively lecture. During lectures, Alex often pauses to ask students if they have questions and sometimes students respond by posing a question. More rarely, students raise their hand themselves to take the floor in order to either ask a question or correct a computational mistake on the board.

Often, Alex pauses the delivery of the curriculum material to tell, in Alex's own words, a "tale" (Alex's class, p. 108; Lecture 9). During these tales, Alex stops using the board and tells students a story that relates to the topic that they have been discussing. Tales can be about the history of mathematics, applications of mathematics, or ideas that relate to the philosophy or epistemology of mathematics. For example, Alex talks to the students about how Newton and Leibniz developed calculus, how Neptune was discovered, how conic sections can be used to illuminate a room, Weierstraß's function which is continuous everywhere and differentiable nowhere, or how in nature we are able to see a derivative but not the function itself (i.e., not the exact behaviour of a phenomenon, but how changes occur). Alex shares these stories in a very lively way and often seems excited. Students do not have an outward response to these tales: more specifically, they not ask questions about these tales, nor do they engage in any discussion about them (at least not during class time).

During class, students sit in their seats, which are fixed to the floor and each one of which has a pull-out table. In general, students look at the board, but focus on other things as well. A few students have their computers on the table and check their emails or surf on the internet, while others spend some class time on their smartphones. Some take notes on a tablet, while others take notes using pen and paper. Some students just look at the board without taking out their pull-out tables. Finally, there are some students who attend the class from start to finish, but do not look at the board or the professor at all. It is significant to note here that although no one checks which students are in class, in some lectures, students are asked to connect on their electronic devices and complete some assessment tasks. Their participation in these assessment tasks contributes to their final grade.

6.1.2. The Mathematical Content of the Course

In this subsection, I discuss the mathematical content covered in Alex's course. To do so, I draw upon the description of the course in the university's website and the syllabus as well as my own in-class visits. But before focusing on the content covered in the course, for reasons of context, I position Alex's course in relation to the other calculus courses offered by the university and to the Ontario secondary school curriculum as part of which students have covered material that is similar to the content covered in this course.

Alex's course is not the only calculus course offered by the university. It is the first part of a pair of calculus courses, the combination of which constitutes the default introduction-to-calculus course for science students. The university also offers other calculus courses, which are more proof-oriented and are addressed to students whose programs are more mathematically-heavy. Alex's course is advertised, through the university's website, as teaching the understanding of concepts, not including proofs, and focusing on modelling and applications in science and social sciences.

Moreover, the secondary Ontario curriculum is relevant, not only because it provides a context for the previous mathematical experiences of domestic students, but also because international students need to prove equivalent high school credits, in order to enter the university and be able to participate in this course. High school calculus is a pre-requisite for this course. With regards to the secondary Ontario curriculum, followed by the students who completed their high school education in Ontario, calculus is taught for the first time in Grade 12. According to the curriculum document published by the Ministry of Education (2007), Grade 12 students choose between three strands in mathematics: workplace, college, and university. Calculus is only studied by students who take the university strand. Grade 12, University oriented students have three mathematics courses: Mathematics of Data Management (MDM4U), Advanced Functions (MHF4U), and Calculus and Vectors (MCV4U). The last two courses are relevant to the course discussed in this chapter. "Advanced Functions" needs to be taken prior to or concurrently with "Calculus and Vectors" and provides an overview of the following families of functions, several of which have been introduced in previous grades: polynomial, rational, trigonometric, exponential, and logarithmic. Calculus and Vectors covers rate of change, derivatives, applications of derivatives, and vectors. Therefore, as we will see, these two courses cover most of the material discussed in the university Calculus course explored in this chapter.

In the description of the course on the university's website, two ideas related to the course's curriculum are prominent. First, this course is not proof-oriented. This orientation is in agreement with my own observations and understanding of the course: no proof is ever presented, nor are students asked to produce any proofs themselves. Second, this course is presented as being about understanding concepts and not about memorising or mimicking techniques, which is (presented as being) in opposition with high school practices. This is a more complicated assertion, as the distinction between conceptual and computational mathematical knowing is present in many aspects of the course. I discuss this distinction in more detail later in the chapter. For now, I should state that my own observations agree that during lectures, the professor emphasises the conceptual understanding of the concepts. Nevertheless, the procedural element of the course is also present and students are required to develop some mastery over computational methods.

Furthermore, both the syllabus and the online description of the course mention that this course focuses on applications of calculus in other sciences. While the online description refers to science and social sciences, the syllabus refers to life sciences and physical sciences. From the perspective of my observations, students are asked to explore problems which are applications of calculus in other scientific contexts. Often, this happens at an abstract level, when the professor offers an oral description of the kinds of applications that a concept might have and refers to the process of modelling. Sometimes, problems with a scientific context are stated in a mathematical way and solved by the professor on the board. Further, students are also asked to work on applications of calculus for their weekly homework assignments.

The mathematical concepts that are explored in this course, according to its syllabus, are the following[36]: functions (including: trigonometric functions, inverse trigonometric functions, logarithms, exponentials), limits, continuity, derivatives (including: rules of differentiation, implicit differentiation, higher derivatives), linear approximations, mean value theorem, graphing functions, minimising/maximising problems, l'Hopital's rule, and anti- derivatives. Table 6.1 shows the material covered in the classes which I attended:

Lecture 4	Introduction to derivative as the limit of slopes, Review of some techniques for calculating limits
Lecture 5	The notion of the derivative, Calculating the derivative of powers
Lecture 6	Studying a function through its first and second derivative
Lecture 7	Rules of differentiation, Examples of differentiating functions, Implicit differentiation
Lecture 8	Rules of differentiation, Examples of differentiating functions, Drawing the graph of a function
Lecture 9	Revision of differentiation techniques and concepts, Optimisation and rate of change problems

Table 6.1: Topics Covered in each Lecture during my Visits

6.2. The *"Humanity of Mathematics"* Discourse

An overarching discourse that manifests itself in Alex's class is the humanity of mathematics. The humanity of mathematics encompasses the idea that mathematics is quintessentially human: it is something that humans do, have always been doing, and will always be doing. As such, mathematics is attributed value. This discourse is often articulated along with highlighting the historicity of mathematics, which draws attention to the long (European) history of mathematics and the importance of this history for doing mathematics today.

[36] The order and wording of these concepts have been changed from the way in which they appear in the syllabus, in order to not make the course identifiable.

This humanistic discourse of mathematics can be explored in relation to wider discussions that happen in mathematics education. Regarding the connections between mathematics and humans, in mathematics education literature, a trajectory of three important moments, namely platonism, humanism, and post-humanism, can be identified. More specifically, platonism centres mathematics and considers mathematical concepts as existing independently of the humans who work with them; humanism centres people and conceptualises mathematics as an activity performed by human beings (e.g., Lakatos, 1976; Mukhopadhyay & Greer 2015); finally, post-humanism criticises the central position attributed to humans, turns to materialist epistemologies, and proposes that mathematics can also be an activity of non-humans (e.g., de Freitas & Sinclair, 2014). In the presentation and discussion of Alex's class findings that follows, I draw upon the discussions related to these moments and the transitions from one to the other.

This section is structured as follows. I first discuss how mathematics is conceptualised as something that humans do as part of Alex's philosophy about mathematics and its teaching and learning. I then discuss humanity through the prism of mathematics conceptualised as something that humans solve problems with, before I focus on Alex's attention to the history of mathematics. I continue by discussing how the notion of mathematical importances (as part of the discourse of humanity) enter in an antagonistic relationship with the institutional importances that emerge from the institutional framework of this calculus course. I conclude this subsection with a summary of the findings that are related to the discourse of humanity.

6.2.1. Mathematics as Something that Humans do

In this subsection, I engage in the analysis of an example from Alex's interview, which highlights some aspects of Alex's philosophy about mathematics and its teaching and learning that relate to the humanity of mathematics. The *"anything that the humankind does is important" example* is centred around a parallelism between art and mathematics. Here, Alex constructs a thought experiment of what would happen if the two, art and mathematics, disappeared from the world:

> Alex: I think that mathematics is very important just as any area. Like, anything that the humankind does is important. […] Maybe let's mention it with artists. Just to make the point, but... Like, artists have been persecuted through ages. All the time. […] And the point is that… let's suppose that for a moment, the world succeeds and eliminates all artists. Suddenly, there are no more artists in the world. And people, like, the new generations come. What will happen? Naturally, what will happen is that there will be people that will start doing art by themselves. They do not need the previous generations to decide, there are people, they just want to draw. But there are people that want to write poetry and there are people who want to tell

> stories. Because these are what- and that's what I mean when I say like, like, history has taught us that these things are important.
>
> So, math is in a certain sense, like, the same. Like, if you were to disappear all mathematicians from the world, maybe no mathematician gets to know nothing and maybe you'll ask the mathematicians of the newer generation to read this book and no one can say anything. But the point is that they would start from zero again. Like, and that's the point. Because to do these questions, to ask the questions of what is our role in the universe, how to represent the things, or simply the question of, I cannot solve this thing and I am [inaudible] because I cannot believe I cannot solve this thing. It's a very human thing. But there are certain things that we cannot give a deep answer, in the sense, like, why is this human? And for some reasons, it's like, well it is human because humans do it. […] Like, one thing is- it's important because humans do it in very deep ways. They have been doing it for ages, it's something that we know how to do. It's something that it's important in many regards. It's something that is human. Like- mathematics belongs to the humankind, even to those that are not mathematicians. But that's one thing. But the other thing is that, it keeps place of, I don't know how to explain this one, but it, there will always be those that will never feel at home doing something else. […] If we're going to accept that everybody has a place in the world, we have to accept that everybody, not everybody, we have to accept that there will be those that will never be able to flourish in other places. That maybe this is the only thing they have. (Alex's class, pp. 136–137; Interview with Alex)

In this example, Alex creates a parallelism between art and mathematics, in order to explain that mathematics is important, just like any other human activity. In doing so, Alex conceptualises mathematics as quintessentially human. This idea is related to mathematics being something that humans do in deep ways and that there are mathematicians who would not be able to flourish doing something different than mathematics.

With regards to the internal conditions of enunciation, Alex engages two kinds of subjects in his discourse. First, he refers to mathematicians. Mathematicians are captured as having an intrinsic urge to do mathematics and as being passionate about doing mathematics. Second, Alex constructs "humans" or the "humankind" as an abstract subject that does mathematics (and art); a subject which encompasses every human being, although lacking flesh and bones itself. These two subjects, although different in Alex's discourse, are merged in a way that constructs what mathematicians do as something that they do because they are humans. This is evident, in the following extract: "humans do [mathematics] in very deep ways. They have been doing it for ages, it's something that we know how to do". In this way, the focus is not on the individual human beings who do mathematics at a particular place and at a particular time, but rather an abstract subject who does mathematics throughout history.

Regarding the concepts that this discourse constructs, the humanity of mathematics is central. The analogy between mathematics and art serves to establish that doing mathematics is a *natural* and *human* thing. By creating a thought experiment, within which art disappears from the world, Alex argues that some people will naturally start producing art again, contesting in this way that doing art happens just because the previous generations have been doing it. He then argues that the same is true about mathematics: if mathematics disappeared, although there would be no connection with what the previous generations have done, people would start doing mathematics again. In this way, the humanity of mathematics is constructed as a concept and is being strengthened by the parallelism with art, which is hegemonically conceptualised as something both important and natural. To elaborate this last point, the "negation" of the discourse, the proposition that "people should not be doing art because it is unnatural", is almost unthinkable. This signifies the hegemony of the discourse that art is naturally human. This humanity of art is "transferred" to mathematics through the analogy.

Furthermore, the construction of the concept of humanity in Alex's discourse centres the human, as opposed to an abstract notion of mathematics. In other words, mathematics is not primarily constructed as the product of mathematicians' labour, but as a human activity. This is significant because it aligns with a historical break in the philosophy and sociology of mathematics. Contrary to the hegemonic Platonic framework, according to which mathematics is part of the world of ideas and has an ideal form that can only be approached by humans, Alex's discourse aligns with a more human-centred approach to what mathematics is. Similarly to the philosophical current associated with the work produced by Lakatos (1976) and Polya (1945/1957), Alex's discourse conceptualises mathematics as an activity that humans undertake in order to solve problems and make sense of the world around them.

The references to nature, the parallelism with art, as well as enunciative elements of this communicative event (e.g., the abstract subject of the "humankind" and the declarative style of the text) contribute to restricting any space in which this discourse could be negotiated. To put it differently, this discursive instance strives to *fix* what we understand mathematics to be. One element of the text that contributes to this direction is that the arguments employed through the thought experiment are not falsifiable. Here, I am borrowing the concept of "falsification" by Popper (1959/2005), who proposes that a theory needs to be falsifiable in order to be deemed scientific. Skovsmose (1994b) offers a sociological twist to the idea of falsifiability by suggesting that a theory "must be available to a group of persons for whom it is possible to articulate a critique” (pp. 112–113), thus complimenting the logical dimension with a sociological one. Here, my interest does not lie with exploring whether Alex’s discourse is “true” or “false”, “scientific” or “unscientific”. However, I want to highlight that since the hypothetical character of Alex’s argument captures a situation that is a mind experiment never expected to happen, there is no space for this discourse to be in antagonistic relation with other discourses on the basis of their truth or falsehood. So, a totalising *truth* exists in this example (mathematics being human), which also cannot be falsified. Another element of this discourse, that contributes to this truth-conserving function, is the apparent absence of discontinuities (Foucault, 1972). Alex’s narrative about the history of mathematics as well as its nature (as constructed through this thought experiment) focuses on “mathematics being human” and people “start[ing] from zero again”. In this way, the possibility of shifts or breaks is excluded. The very use of a hypothetical discontinuity (mathematics disappearing from the world) establishes a narrative of continuity. Things would follow the same rules and patterns as they do now. As I argued through the idea of falsifiability, the absence of discontinuities establishes a discourse which allows little space for negotiation.

I suggest that this contributes to conserving certain ideological assumptions about mathematics. One of these “truths” relates to how mathematics is conceptualised. In this example, mathematics is constructed as done by mathematicians, it gets communicated and is passed down through books, and it is tied to academia. Further, a hint of the a-historic and timeless element of mathematics is to be found. More specifically, regarding art, Alex says “They do not need the previous generations to decide, there are people, they just want to draw”. This utterance conceptualises art as relatively independent of time and space, an idea which – through the parallelism – is extended as also being true for mathematics. As Alex continues speaking, this a-chronic element (of the thought experiment) is diffused into the diachronic element of his discourse (knowledge surviving through books, previous-newer generations doing mathematics). In this way, mathematics is conceptualised as natural; yet it is a certain way of doing mathematics, to which Alex refers: the mathematics that people are doing now, within academic institutions of Western origin. This conceals the existence of other types of mathematics ((e.g., non-academic mathematics or other types of academic mathematics), the existence and importance of which is highlighted by ethnomathematical scholarship (e.g., d’Ambrosio, 1985). Further, this process not only naturalises a type of mathematics, but also a certain social, economic, and political framework within which doing mathematics in this way occurs. For example, academia (including universities, books, journals, conferences, etc.) is also conceptualised as the necessary and natural place, in which mathematics is done.

Finally, I argue that another “truth” about mathematics reproduced in this discourse is that mathematics is harmless: it constitutes a space in which people can flourish (and thus do not get harmed). In particular, this nuance of Alex’s discourse manifests itself through the way in which mathematicians as subjects are presented and constructed through the discourse. The existence of people naturally inclined to do mathematics, who “would rather die” if they couldn’t do it any more is weaving an almost romantic, egalitarian framework, within which everyone should be given space to flourish. This discourse tries to support people who do mathematics within academia, thus not relating mathematics with any harm that could be done to people. In this sense, Alex’s discourse is constituted in a way that excludes the possibility of mathematics being related to signs that are in contrast with the well-being of humans or mathematics students.

In this subsection, I have argued that enunciatively speaking, Alex produces a discourse about mathematics, which is free from discontinuities and is not falsifiable. My interest here does not lie with its historical accuracy nor with whether it is indeed “natural” for humans to do mathematics or not. Rather, I want to focus on how this discourse interacts with other discourses and what this discourse enables. For the first part, I argued that this discourse leaves little space to be in antagonism with other discourses. For the second part, I argued that through the “humanity of mathematics”, certain “truths” about mathematics are conserved. A particular way of doing mathematics, in the Western academic tradition and within academic institutions, is conceptualised as harmless and natural, while it also frames what mathematics is. To put it differently, using Laclau and Mouffe’s terminology, mathematics is fixed as “Western mathematics that happens in academia”.

6.2.2. Mathematics as Something that Humans Solve Problems with

A central aspect of the humanity discourse, already encountered in the previous subsection in the “anything that the humankind does is important” example*,* is the idea that mathematics is developed so that humans can solve problems and understand the world around them. In this subsection, I elaborate on this idea through three examples from Alex’s class.

The *“the discovery of Neptune” example* refers to the way in which scientists used mathematics to discover Neptune.

> Alex: Like, one of the biggest successes of calculus, if you want to say it that way, is the discovery of Neptune. Neptune was the planet that was not discovered just by observation but was discovered by mathematics, by mathematical calculation that told us where to look. Because the point is that what you could see was not another planet, you could see a planet that was moving not in the right way. So, they would check how it was

> moving and then it started to move faster or it started to move slower. And they were asking like, why does this happen? Why is this planet not moving the right way? And then they started to plot the things, in the way that I am doing there *[shows the board]*. And what they predicted is, is there must be a planet that due to gravity is moving and shifting the orbit that we are supposed to see. So, they solved for these things in this way, like where you should have been and where you actually are, they computed the differences, they solved a more complicated type of thing but they were based on these ideas and they concluded that at a particular point, somewhere in a certain region of the space, and this is not a trivial thing because all the regions of the space that you might be looking for are tiny, they did this for big amounts of precision and they looked in one direction and Neptune was there! Like, they found it! It was, certainly, a very interesting thing. Very interesting how they found Neptune, that's the way they did it, in summary. These kinds of things were developed to understand the movement of planets in general. It's kind of interesting but it turns out to be a very important idea. (Alex's class, pp. 9–10; Lecture 4)

This is a "tale" in the sense discussed in subsection 6.1.1: Alex describes as tales the short stories that he sometimes tells students, which are related to the material delivered in class. In this example, Alex presents a simplified, popularised version of the story of how Neptune was discovered as well as of how the mathematics used in this process is connected to the mathematics studied by the class. We notice a distinction here between the mathematics of mathematicians/scientists (which is evident in the way in which the practice of the people who discovered Neptune is described) and the mathematics of the class (which is evident in the parts where Alex talks about the calculations performed by himself on the board, e.g., "in the way that I am doing there [shows the board]").

In this example, Alex refers to doing mathematics as an activity which aims to solve a problem in astronomy and connects his class's mathematics with this historical mathematical activity. In this way, learning mathematics is conceptualised in terms of getting to know what mathematicians have developed over the centuries. Alex provides a popularised narrative of the process followed for discovering Neptune, providing his students with a glance of what working with mathematics looks like, from the point of view of the discourse within which he operates. This process entails identifying that some observations are not in agreement with the existing theory, studying the movements, asking questions, plotting and computing, drawing conclusions, and verifying if the conclusion was correct; until "Neptune was there!". I argue that this narrative conceptualises the process of doing mathematics in a linear way, in the sense that there are no backs and forths or disagreements regarding the method that shall be followed, the questions asked, or the meaning of the performed calculations. This linear conceptualisation of mathematical practice is of significance because it contributes to the construction of the humanity of mathematics as a concept. Humanity is constructed as concerning and referring to all humans regardless of their particular social contexts, the institutions they operate in, or their interests.

Enunciatively, Alex makes a peculiar word choice in the example, when proposing that "Neptune was discovered […] by mathematics, by mathematical calculation that told us where to look". This word choice is peculiar in the sense that, contrary to Alex's discourse as a whole, it assumes a self-existent nature of mathematics. Mathematics, in this enunciation, "discovers" and "tells". The linear conceptualisation of mathematical practice as manifested in this example relates to this: it contributes to the reification of mathematics, as there appear to be no diverse views, disagreements, or antagonisms within the mathematical practice performed by humans. This autonomation of mathematics leads to conceptualising mathematics as something that has a diachronic/achronic character and acts by itself. This example shows that although mathematics is conceptualised by Alex within a discourse that emphasises mathematics as a human practice, other discourses are also present and contribute to the attempted fixation of meaning.[37]

Another example that conceptualises mathematics in relation to solving problems is the *"Imagine you wanted to luminate the place" example*, which happens after Alex has presented the notion of convexity in relation to a graph that he has plotted on the board:

> Alex: For example, another place where this idea of convexity appears is like if these things were not, like, drawings. Imagine that these things were walls, right? And you wanted to do lamps or something to luminate the place, if the thing is convex you need less light. You can put just a very bright light in the centre, for example, and that light will arrive everywhere. But if I have to illuminate this thing and I put a lamp here, that lamp, like the light of that lamp is not going to arrive here. Right? Because it bounces, it crashes with this one. So, that kind of problem is the thing that are measured by convexity. And it's a very important concept in not only math, but in sciences in general, like convexity, we like things that are convex. (Alex's class, p. 47; Lecture 6)

In this example, Alex refers to the process of illuminating a room by utilising the concept of convexity. It is another example of popularising a scientific/mathematical problem and presenting some main ideas in class by making connections with the mathematical material covered in class. Similarly to the "the discovery of Neptune" example, here there is the underlying distinction between the mathematics of the

[37] It is worth-mentioning here that the story of how Neptune was discovered has also been captured in relation to the sociopolitical zeitgeist as well as a long story of disagreement and conflicts among scientists (e.g., Smith, 1989); an approach which significantly diverges from the linear narrative presented by Alex.

class and the mathematics of scientists. The mathematics of the class is evident in the discussion of convexity in relation to the graph on the board that precedes Alex's above description. The mathematics of scientists relate to the process of using convexity to solve the problem of illuminating a room. Hence, the material delivered in the class (or the mathematics of the class) is conceptualised as a simple version of the mathematics used by scientists in order to solve the problem of illuminating a room. More generally, mathematics is conceptualised in relation to its humanity as it is constructed as solving a problem faced by humans.

Finally, a third example which conceptualises mathematics in relation to solving problems is the *"it is something that matters, that you need to know what is going on" example*:

> Alex: What happens? What happens is the following. In this case, our equation is very simple. Like, our function is $y = x^2$. But the point is we are going to find things that are not that simple. That's the whole thing.
>
> So, imagine. Imagine that this function represents something. Something that is very important. Maybe it is the weather. Maybe it is the amount of money in, I don't know something. It is something that matters, that you need to know what is going on.
>
> But the function is so complicated that you do not know how to produce the values for it. But you need them. You might need the values of this thing at $(2,4)$ for whatever reason. Like you might need that. But you don't need them exactly at $(2,4)$. You need to know more or less what happens around $(2,4)$. Now for this question, I know exactly what happens. Like you give me the point, I just square it and that's it. But again, like, if you give me a more complicated question, I cannot do this. (Alex's class, p. 2; Lecture 4)

As with the previous two examples, mathematics is conceptualised here as an activity that aims to solve problems, which are either problems that humanity faces or problems that arise from humanity's curiosity. In this example, Alex presents a method for calculating the limits of a function, using the example of $y = x^2$. Alex asks the class to *imagine* that the function represents something important, operating within a discourse that conceptualises mathematics in relation to solving problems, although there is not a specific problem that is being discussed. Alex's phrasing is similar to the "Imagine you wanted to luminate the place" example, in which he asks students to *imagine* that the lines of the graph represent walls: in both instances, students are asked to imagine the problem that mathematics can solve. Similarly, Alex asks students to engage in a similar process of imagining, when he proposes that they think of the function as potentially more complicated. This process of imagining the context of a problem or imagining more complicated aspects of a problem is significant because it reinforces the relation between the mathematics of the class and the mathematics of mathematicians/scientists as discussed earlier. The mathematics of the class constitutes a sort of a mathematical prototype, while the mathematics of mathematicians/scientists is more complicated and related to problems that can arise in the world which can be modelled through mathematics.

Overall, in this subsection, I highlighted that Alex's discourse conceptualises mathematics as an activity through which humans can solve problems and understand the world around them. This process is a human-centric one, which constructs mathematics as a sort of linear, occasionally self-existing knowledge which can be applied not only to understand but also to control the world around us. In this sense, not only does it centre humans, but it paints a particular type of relation between humans and the world around them, within which humans are conceptualised as the masters who can dominate the world.

6.2.3. Mathematics through its History

In this subsection, I draw upon two examples from Alex's class to explore how the discourse of the humanity of mathematics manifests itself in the classroom discourse though connections with history. Such connections happen often in Alex's classroom and are about European mathematicians. Indeed, Alex's connections with history include the discovery of Neptune, Newton and Leibniz's development of calculus, Euler's development of differentiation techniques, Descartes' development of the plane and its relation with geometry developed in ancient Greece, and Weierstraß's definition of a function that is continuous everywhere and differentiable nowhere. In this subsection, I present the analysis related to Newton and Leibniz's development of calculus and Euler's development of differentiation techniques.

The *"the people who started doing these were Newton and Leibniz" example* happens as Alex presents the class with the "rules of differentiation". After he has introduced the derivative of the sum and difference of two functions, he starts talking about the product, as follows:

> Alex: And we can imagine, when they were discovering these things. Like the people who started doing these were Newton and Leibniz, they were the first ones to study derivatives. And, of course, they were trying to simplify them by understanding the rules. And, of course, they got this one *[shows the rule for the sum]* and they got this one *[shows the rule for the difference]*, these two are pretty simple. But then, what other thing can you do with functions? well, you can multiply them. […] Now, this one is the first one that is important to be careful, is the product. So, this is important because it is not what you expect – in the sense that maybe you expect that the product of derivatives is the derivative of the product. And that's not true. So, this is also called Leibniz's rule. *[Alex starts writing on the board]* So, the derivative of a product is derivating one by one. And leaving the other functions alone. So, this is the derivative of f times g plus f times the derivative of g [$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$]. Now, this is called Leibniz's rule because it was Leibniz the one that proved this. But if you actually go and check the notes, like Leibniz had some

correspondence with other mathematicians, he actually made a mistake. Like, for a while he computed derivatives by saying the following, like- He thought fg prime, was f prime g prime for a while. *[*$(fg)'(x) = f'(x)g'(x)$*]* This is wrong. This one is wrong. But it's a mistake that Leibniz did when he was discovering all this calculus. It took him a while to actually realise that he was wrong. So, don't make this mistake. Be careful. This is wrong. This is wrong. *[Writes "this is wrong" on the board]* (Alex's class, p. 65; Lecture 7)

In this example, Alex refers to the "discovery" of "the rules of differentiation" by Leibniz and Newton. He offers a narrative in which he explains how the two mathematicians came up with each of the three rules (derivative of addition, subtraction, and product). He also describes a mistake that Leibniz used to make regarding the derivative of a product, thereby warning students not to make the same mistake themselves.

As Alex talks about the production of this knowledge, he creates an analogy between the development of calculus by Newton and Leibniz on the one hand and students' learning of calculus on the other. This analogy is constructed through deictic expressions (e.g., "*they* were discovering *these things*"), while it is framed as natural both enunciatively (through the use of words/phrases like "of course", "discovery") and conceptually (Leibniz's mistake is parallelised to students' making the same mistake). In this way, Alex puts in the forefront the humans who have worked on calculus, while he further "humanises" these great mathematicians by talking about their mistakes.

At the same time, the rules of differentiation are constructed as having an independent existence of the people working with them. Within Alex's discourse, the rules of differentiation could not be different than what they are: the derivative of a sum is the sum of the derivatives and this is unnegotiable. To discuss the notion of inevitability in mathematics, Hacking (2014) distinguishes between the long-run level (when according to Hacking, nothing is inevitable in mathematics) and the short-run case, in which a person works within a certain paradigm (when, according to Hacking, inevitability is the reality). Absorbed as the class may be in differentiating functions, the rules of differentiation appear to be pre-existing and inevitable. It is important to note that this pre-existence (and its naturalness) is the product of an educational social setting and of educational interactions; in other words it is socially determined. In turn, this socially constructed inevitability contributes to mathematics gaining the status of existing independently of the people working with it.

The educational setting, unlike a research setting, for example, has the characteristic that the knowledge taught pre-exists the interaction between the instructor and the students. While different degrees of variation regarding this statement might exist (depending on – among other things – the object of teaching, the teaching methods, etc.), this observation is particularly salient in Alex's class. To put it differently, students are not expected to construct their own understandings, opinions, or views about derivatives (as could, for example, happen in a social science class, where students might be required to develop their own position about a concept like, e.g., power). Instead, students are expected to absorb the knowledge about derivatives, taught by the professor. This condition gives a sense of naturalness regarding how mathematical knowledge comes to respond to the problems and questions raised in the course. For example, the question of "the rule of the product of two functions" appears naturally in class (as a natural question following what the class has been discussing) and, there is a very straight-forward response to this question, which constitutes part of common mathematical knowledge. Had the question been different, already established mathematical knowledge would not necessarily have such a straight-forward answer to provide. I am making this distinction in order to highlight two aspects about how the notion of "knowledge" is constructed here: first, the cohesiveness is socially and institutionally constructed: the knowledge, which appears to be waiting there for the class to ask the right questions to uncover it, is the result of a particular language game set up in a particular institutional framework. Second, this cohesiveness of knowledge is not necessary for all mathematical language games; it is contingent and the result of deliberate efforts (manifesting – not exclusively – but also in educational settings) for knowledge to take a form of cohesiveness.

There is some tension, here, between mathematics being something that humans do and the existence of mathematics independently of what humans do. Although as already discussed, it appears that the knowledge and truth about differentiation rules pre-exists people doing mathematics, Alex's discourse also conceptualises mathematics as something that humans do. In particular, Alex talks about Leibniz being in correspondence with other mathematicians, which weaves a picture of doing mathematics as a social endeavour (contrary to many popular discourses). The discourse of the humanity of mathematics as developed by Alex resolves this tension through the idea that the inevitability of mathematical results occurs and becomes visible as people interact with mathematics and mistakes repeat themselves when the same questions are posed (by humans engaging with mathematics) at different times throughout history, i.e., through a diachronic/achronic aspect of mathematical knowledge.

Within the humanity of mathematics discourse, the idea that mathematics is constructed as an individual's endeavour becomes salient through Alex's phrase "this is called Leibniz's rule because it was Leibniz the one that proved this". Here, a causal relation is established between proving a rule and having the rule being named after oneself. The phrase "proving the rule" denotes proving it *for the first time.*

Thousands, maybe millions, of people ever since have also proved this rule but have not had the rule named after them. So, the production of mathematical knowledge is hereby conceptualised in terms of providing a proof for a rule or theorem for the first time, thereby showing that a proof exists[38]. Moreover, the references to Newton and Leibniz frame mathematics as the endeavour of individuals (and not, for example, the product of historic or social circumstances), while it is noteworthy that although reference to Leibniz's communications is made, it is Leibniz himself who realised and corrected his mistake; the communications or the collective work do not appear to have contributed to this "discovery".

From a discourse analysis perspective, it is important to consider what kind of power relations emerge from this discourse, as well as what this narrative about mathematics allows as a possibility and what it excludes. In this discourse, calculus is a construct of a particular culture. I argue that the normalisation of a connection between the learning that happens in the course and history makes calculus a thing that shall be done only through/within this culture. This encompasses ideologies (manifested in practices) which frame the ontological character of knowledge (here mathematics exists independently of what people do), the way in which knowledge is produced (individual contributions are valued), and the ways that this knowledge is commemorated (through references to the actions of those individuals who first proved a theorem). Humanity emerges as a discourse, within the idea that when people interact with mathematics, things could not have happened otherwise, while they tend to repeat themselves when the same questions are posed at different times throughout history.

The second example that I discuss in this subsection is the "*it makes you appreciate the problems" example,* which happens during a lecture in which Alex has brought Euler's book "Foundations of Differential Calculus" to class. Alex chooses some examples of function differentiation from the book and calculates the derivatives of these functions on the board. The following refers to finding the derivative of $f(x) = \ln \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$.

> Student: Is this going to be on the midterm?
>
> *[Some students laugh.]*
>
> Alex: We're not going to put something like- You have to learn the rules of differentiation. But probably we're not going to put something that is so… lengthy, let's say. Like, something like- But, for example, the logarithm of the square root of something, maybe that's ok. Something like that. But something this lengthy,

[38] The same idea/expectation appears in Chris's discourse, as discussed in Chapter 5 in relation to the *"Fibbonacci didn't invent these numbers" example* and the *"how unfair mathematics is" example.*

> no. But the only reason I want to do this is just to see that- the rules in action. But- that you can actually see, that ok, this is happening here, this is happening here. This sort of thing. Because it's an important thing to see for two reasons. And the first one is, you learn to recognise how different rules look like. And the other one is it gives meaning to the problems that we have to do next. Like, you actually start appreciating the technology. That you can actually go through life without having to do this. Like, you know, this is an example in one of the main books of analysis of calculus, done by, like, one of the most important mathematicians in the world. Uhm, his way of doing it was like another thing. But- uhm, and we could decide, like, oh, this is just his kind of mastery. We can just ignore it and put it in the calculator. It makes you appreciate the problems, which is an important thing to do. Because, eventually, probably in 400 years, they are going to see the things that we are doing and they will be: oh, thank God we don't need to do these things anymore. It gives that feeling. (Alex's class, p. 88; Lecture 8)

This example relates to an artefact: it concerns the presentation of a differentiation example discussed in Euler's book. The choice of Euler's book as an artefact to bring into class is significant, in that it contributes to a particular discourse about knowledge and its production. It connects the mathematical content of the class with some of its historical roots, thus highlighting a thread through which calculus has been developed. It is also a statement on whose knowledge has value, what roots the knowledge that we learn has, where it stems from, and how it is produced. In this case, it is produced by a famous European mathematician, one whose name is often mentioned when speaking about mathematics. There is thus an alignment with an ancient Greek-European tradition of doing mathematics. This alignment is evident in the explicit references to the book and its author made by the professor: "one of the main books of analysis of calculus, done by, like, one of the most important mathematicians in the world".

With regards to the external conditions of enunciation, it is important to consider the way in which the book is brought into the discussion. Students never interact with the actual extract from the book. Alex mentions that Euler uses a different notation than the one he uses when presenting the problem to the students. Later during the lecture, Alex uses the book again to check the result that he found when calculating the derivative of the function on the board. In this way, students do not have a direct interaction with the artefact, except for a presentation of the problems that Euler was trying to solve and a confirmation of the result. The humanity of mathematics discourse emerging in the context of this classroom highlights the mathematical problem without the linguistic or notational context in which the problem originated. We could say that what is being discussed in the class is the *problem itself*. By this, I do not mean to imply that the "problem itself" or some essence of the problem exist regardless of the circumstances in which people work, talk, or write about this problem. On the contrary, I wish to argue that what we might consider as “the problem itself” is constructed in this way; by ignoring the particular (linguistic, social, etc.) characteristics of the problem’s occurrences. Solving it in this class and solving it back in the 1600s becomes the same thing. Different notation and potentially different emerging meanings do not count as different. This process disentangles the problem from the social contexts in which it was initially produced and in which it re-appears. While Alex refers to the different ways of handling the problem (e.g., notation changes with time, technology changes), the problem itself is a-chronic as it is expected to continue making sense in 400 years.

Another nuance of the humanity of mathematics discourse is the appreciation of a mathematical problem and, more generally, the appreciation of knowledge, brought up by Alex in the above example. Alex’s discourse conceptualises mathematics and the appreciation of mathematical problems through a historical continuity which starts from Euler to today and continues from today to 400 years in the future. Referring to mathematics as something that occupies humanity over the course of so many years conceptualises mathematics as deep and important.

Overall, Alex’s historical references contribute to the emergence of the humanity of mathematics discourse in this class. References to the history of calculus as developed in Europe construct mathematics through a connection between the mathematics that is discussed in the course and calculus as historically developed. Mathematics is thereby conceptualised as inevitable and unnegotiable knowledge, as an individual’s endeavour, and as having an independent existence of the people working with them (as well as their social, political, and institutional frameworks) and thereby having a diachronic/achronic character.

6.2.4. Institutional Importance in Antagonism with Mathematical Importance

So far, I have painted a picture of the discourse of the humanity of mathematics in which hardly any antagonisms take place. In this sense, Alex’s discourse in general and the discourse of the humanity of mathematics in particular are constructed in a way that offers a totalising narrative regarding mathematics as an endeavour of humans. However, Alex’s discourse is sometimes challenged by the interactions happening between him and the students, mostly in relation to what is deemed important at a particular moment. I refer to this challenging as an antagonism between institutional importances (which are enunciated both by students and Alex himself) and mathematical importances (which are primarily being enunciated by Alex).

By mathematical importances, I refer to discursive instances, primarily enunciated by Alex, which entail the use of examples that are not directly part of what students are examined on as part of this course.

Such discursive instances include mathematical examples that are far more complicated than the ones that would be on a test, even though they address the content of the syllabus and they contribute to learning the material that will be on the test. An example of this is finding the derivative of $f(x) = \ln \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$, as discussed in relation to the "it makes you appreciate the problems" example. Similarly, Alex's "tales" are another example of discursive instances that do not directly address the material that shall be on a test or exam.

By institutional importances, I refer to how knowledges/ideas/practices are assigned importance because of or in relation to the specific institutional framework in which this course unfolds. Institutional importances are articulated in the discourse of the professor himself as he delivers the material in a particular way: following the textbook, dedicating time in the beginning of each lecture for homework or test-related issues, covering specific sections of the textbook in each lecture, etc. With regards to students, we see that institutional importances are what brings students into class in the first place: students are allowed and obligated to be in class because of the institution. Classes are obligatory and part of students' final grade depends on their participation in in-class activities.

I claim that institutional importances and mathematical importances are in an antagonistic relation in this course. To discuss the characteristics of this antagonistic relation, it is useful to reemphasise the norms of Alex's class. Alex uses most of the time to present the material and show examples on the board. Sometimes, students ask clarifications on the material that has been presented. When, during my visits, Alex tells a "tale", students, never ask any sort of questions nor do they make any comments. This silence – in contrast to the clarifying questions when the official course material is presented – can be read as a non-engagement, which can be a form of exercising power. It is a way of waiting for the professor to finish his "tale", in order to continue with the material that is more important to them.

The *"I feel like that really doesn't need to be" example* is an example from an interview with Alex's student, Anthony. Although the example does not depict the norms of the class, as it happens within the setting of an interview, it highlights the existence of a tension around the practice of "tales" as well as it shows that the priorities of Anthony might be different than those of Alex.

> Anthony: So, there is the required teachings, right? And apart from that, there isn't really much else except through the occasional tangent, where he'll go on to give like a very-very brief summary or background, of you know how this happened or something like that. […] What I expect with regards to mathematical discourse is definitely something more. Though we are limited by time and material, all right. In lower level

> math courses, first, second year for example, I feel like that really doesn't need to be. It's not a requirement with regards to discussing things outside of the realm of the material that needs to be taught, right? Questions can be asked, clarifications can be made. But, doing a dive into some kind of historical background or the emergence of this idea, I don't think it's really necessary. In high level courses, where things start to get a little bit more specific, I think it's pretty important to know who developed this, how they developed it, who influenced it, right? And how this very brilliant idea came into fruition, right? Because it can, it can give people in higher level math courses a motivation, so to speak to pursue it a little bit farther. […] Because this is very, like you want to build a foundation, right? And a strong one. So, that requires a lot of practice and a lot of understanding. So, I would appreciate it if in lower level courses, the discourse were to revolve around examples, continuing to work on proofs and stuff like that, making sure the students can understand every step of the process and its changes if necessary. Rather than giving kind of a like conceptual, big picture view. I do think that is important, but not at the sacrifice of really like knowing how to deal with a problem that's given to you, right? (Alex's class, p. 143; Interview with Anthony)

In this example, Anthony highlights his perception that lower level courses should aim at building a foundation and talks about his expectation that they focus on the material itself. At the same time, he identifies Alex's practice as having a "conceptual, big picture view". Using the constructs that I have talked about here, Anthony leans to the institutional importances of the course. Nevertheless, during class time, Anthony does not enunciate this discourse. In what follows, I discuss two incidents, in which students proceed in an articulation which indicates the existence of antagonism between a discourse about institutional importances and a discourse about mathematical importances.

An instance in which the institutional importances and the mathematical importances collide is evident in the "it makes you appreciate the problems" example. This example is initiated when a student asks Alex if the example Alex has been presenting is going to be on the midterm. In Fairclough's (2003) terms the "is this going to be in the midterm?" question is an interrogative question as it expects a yes/no answer. However, the question of whether something is going to be on the test does not necessarily or solely request information. Instead, it shows some values of (some of) the student population. The question establishes that what is important is what is going to be on the test. Moreover, this question is an indirect exercise of power, as it reframes the discussion within the institutionalised framework. In other words, it "tells" the professor to focus on the test and to present exercises similar to those that will be on the test. In this sense, the discourse of mathematical importances and the discourse of institutional importances are in antagonism in this example. The institutional importances discourse is enunciated in the beginning of the interaction, by the student who asks Alex if the discussed example is going to be on the midterm;

the students who laugh; and Alex himself as he answers the question and explains that this particular example is too lengthy for an exam. The mathematical importances discourses, enunciated by Alex as he presents the mathematical problem and talks about its value, broadens up the content of the course, by drawing upon the historical roots of calculus and the problems that calculus can approach. The institutional importances discourse antagonises the broadening of the context, as this potentially broadens the material of the content that students are examined on. Through this interaction, we get a glimpse of the limits of the discourse of humanity as they emerge in the institutional context of this university class.

Another example of this form of antagonism appears in Lecture 9. It is worth mentioning that this lecture takes place one day after COVID-19 was declared a pandemic by W.H.O. and on the same day that the premier of Ontario announced that schools will remain closed after the coming week's March break. Within this context, there is a general sense of uncertainty, which is evident by Alex's introduction to the class (the "*we are in a complicated situation" example*):

> Alex: Aaah, so, we are in a complicated situation. So, we need to discuss several things of the course and- So, let's proceed. Quite probably is that this one will be the last physical class. That's not for sure. (Alex's class, p. 98; Lecture 9)

As a result of the uncertainty of the situation and in order to respond to the possibility that this class might be the last physical class of the semester (which indeed it was), Alex spends this lecture summarising the main ideas of the course and working on some examples that students might encounter in the final exam. I refer to this context because students appear to be more tense in this particular class, likely in response to the situation, and I regard this tension to be a catalyst for the next incident that I will discuss.

The *"they create a hyperbola" example* happens after Alex has presented an implicit differentiation example and, referring to the graph that he has made for it, presents the following "tale" that is about conic sections.

> Alex: Have you ever seen a lamp, have you ever seen this lamp, sometimes they are in the coffee shops or something like that. But like, the lamp is here, and they have like a lit. Where, they, the lit is actually a cone, like this. And it creates two shadows. It creates two shadows. One shadow is like the light going up and the other shadow is like the light going down. I don't know if you have seen this. But if not, the next time you go to a coffee shop and see this, look at the shadows. And the shadows create these figures. They create a hyperbola. […] It relates with how murals, that are created in this way reflect the light. And because of that, they were studied by the Greeks. Actually, there is a story of Archimedes […] So, those figures were very

important in ancient Greece. Uhm, this is like 2000 years ago. But, later when equations appeared into the picture of mathematics, they realised that these figures correspond to equations of degree two. Like, two variables with two, powers of two. And problems like this. And it became an important thing to know how do they appear in the plane. And they found out [inaudible] and things that maybe some of you have studied at a certain point in analytic geometry in school. What is the form of a circle? What is the form of an ellipse. And these sort of things.

[A student raises their hand.]

Alex: Yes?

Student: Just a quick question. Like, I'm trying to read through and understand how you got to that first part, but I don't get it.

Alex: Like, this, this equation?

Student: $2x + \cdots$

Alex: Ok. But let me finish my tale and then *[students laugh]* and then I'll explain this one. So, all I want to say is that […] (Alex's class, pp. 107–108; Lecture 9)

During this tale, Alex refers to the conic sections in the case of the shadows created by lamps, continues with the ancient Greeks' study of the conic sections, and explains how the study of conic sections evolved through the development of analytic geometry. Alex's discourse here is paradigmatic of the discourse of the humanity of mathematics as has been discussed so far. However, this example is unusual because it entails an unusual interaction with a student. Indeed, a student raises their hand to interrupt Alex's tale and ask him to talk more about the technical aspects of the problem that Alex demonstrated earlier and which are likely to appear in the final exam. In this sense, the act of politely interrupting Alex to ask something about the material that could appear on the final exam is indicative of the antagonism between the discourse of mathematical importances enunciated by Alex and the discourse of institutional importances enunciated by the student.

Overall, in this subsection I argued that in Alex's class, an antagonism takes place between a discourse that focuses on mathematical importances and a discourse that focuses on the institutional importances. Although students do not often take the floor in Alex's class, they sometimes do take the floor or sometimes use their silence to enunciate a discourse that collides with Alex's mathematical importances discourse.

6.2.5. Bringing it all Together

Through the previous examples and their discursive analysis, I explored some aspects of how the humanity of mathematics emerges as a discourse in Alex's course. The humanity of mathematics refers to a discourse in which mathematics is constructed as quintessentially human; as something that humans have been and will be doing. This discourse about mathematics is framed through a thought experiment in Alex's interview, as well as in the classroom discourse through the conceptualisation of mathematics as being developed to solve problems faced by humans and through connections with history.

Regarding the ways in which the humanity of mathematics is woven through the narrative that mathematics is a knowledge that can solve problems faced by humans, the examples that I presented extend from figuring out the movement of planets to illuminating a room and from meteorology to economy. It is significant that these problems are presented as objectively constituting problems that shall be solved, while they also constitute problems for humanity as a whole. Yet, as Marxism teaches us, the solution to an economical problem cannot ever be a solution that benefits all (struggling) classes, thereby all humanity (e.g., Marx & Engels, 1984/1848). In this sense, the very notion of humanity as constructed by Alex's discourses does not encompass the different social, political, and institutional contexts of different people across time and space.

Through the connections with history, Alex's discourse conceptualises mathematics as a-chronic and independent of social circumstances. The history of mathematics that the professor draws upon is the ancient Greek and the European. In the discussion of the previous examples, I argued that the references to the history of mathematics do not situate mathematics in the historical conditions of its occurrence, but they rather – somehow oddly – contribute to a conceptualisation of mathematics and mathematical problems as achronic. This is achieved exactly by not highlighting the historical conditions in which mathematics has appeared or appears, but rather by using historical references as a way of showing that mathematics has been there and will be there. In other words, the humanity of mathematics is related to an achronic conceptualisation of mathematical practice.

In its appearance as quintessentially human, doing mathematics is also conceptualised as natural. As mathematics is constructed in terms of academic mathematics that follows an ancient Greek – European tradition, this assumed naturalness of doing mathematics entails specific characteristics, which albeit historically contingent are essentialised. In the discussed examples, we can identify a discourse about the history of mathematics, from which discontinuities are absent or not highlighted. In this way, mathematical practices are conceptualised as naturally and a priori having the characteristics that are manifested in academia and in the specific calculus classroom. For example, "space discontinuities", or the idea that people at different places in the world and in different societies do different mathematics, is excluded from this discourse. More generally, the discourse of the humanity of mathematics constitutes a discursive formation which does not appear to be in an antagonistic relationship with other aspects of Alex's discourse. Hence, I claim that this discourse, with its strong historical roots in modernity, gives the impression of "*moments* of a closed and fully constituted totality where every moment is subsumed from the beginning under the principle of repetition" (Laclau & Mouffe, 1985/2001, p. 106). Yet, as Laclau and Mouffe propose, this fixation is never complete or final. The conceptualisation of mathematics both as a reified product and as a human practice as well as the antagonism between mathematical importances and institutional importances discussed in this section pinpoints the absence of this completeness. As I continue with the discussion of the other overarching discourses in the following sections, I try to pinpoint other elements of Alex's discourse that fail to be fully transformed into moments.

Overall, the discourse of humanity as manifested in Alex's class breaks with a more platonic discourse about mathematics as it centres humans as opposed to mathematics. Nevertheless, as discussed, Platonic ideas "survive" in Alex's discourse, especially as mathematics is assumed to have an independent existence of the people who work with it. In presenting and critically discussing the discourse of the humanity of mathematics, I derived aspects from critical epistemologies to criticise the notion of humanity itself, namely the idea that there is something that humans quintessentially share regardless of their interests or sociopolitical contexts.

6.3. The *"Applying Mathematics"* Discourse

The applying mathematics discourse is central in Alex's class, a calculus course for emerging scientists who will not choose mathematics as their field of study. The notion of applying was briefly tackled in the previous subsection, when I discussed mathematics as a tool for solving problems faced by humans. In this section, I focus further on the notion of applying with an emphasis on the use of mathematics (as opposed to what humans do, which was the focus of the previous subsection). To this end, I use examples in which mathematics is viewed as *applied*, I discuss the ontological assumptions associated with this notion of application as emerging in Alex's discourse, and I explore the rationality of the application process. To critically discuss this discourse, I turn to and expand upon an idea that I briefly mentioned when presenting the humanity of mathematics discourse: mathematics is conceptualised as a tool to master and dominate the world.

This section is organised as follows. I first discuss how the idea of mathematics being applied to reality emerges in Alex's discourse and I discuss the ontological assumptions of this discourse. In the coming three subsections, I analyse two rationality techniques, through which the applying mathematics discourse is realised, namely simplifying problems and developing prototypical solutions. I continue with a discussion of how developing technical mastery is conceptualised within the applying mathematics discourse. I conclude this section with a critique of the applying mathematics discourse and a discussion of its relation with the humanity of mathematics discourse.

6.3.1. Mathematics as Applied to Reality: Ontological Assumptions

In this subsection, I discuss how Alex's discourse conceptualises mathematics as being applied in every-day situations and I explore the emerging ontological assumptions about the relation between mathematics and "the world". I argue that Alex's discourse conceptualises mathematics as applicable to everything in the world, a process through which mathematics is attributed value and gains power. Moreover, Alex's discourse merges the studied object/phenomenon with its mathematical representation (function).

The *"if you make mistakes" example* refers to the notion of applying mathematics to almost any field, while positioning students as the individuals who perform the application process. It happens as the class works on a problem of maximising/minimising the surface area of a cylinder:

> Alex: When you obtain things that you are going to maximise or minimise, you can imagine that a lot of things in the real world, from sciences, to commerce, to finances to whatever, you have to do this. Maximising and minimising are very important problems. If you make mistakes, you obtain wrong answers and that has very important consequences. Like, a different result, it can mean that you predicted something in the market wrong or it can mean that you constructed a machine that does not work and then exploded. The things can have consequences. So, it's very important to do all this correctly. Ok, so the speech is: derivate correctly. (Alex's class, p. 117; Lecture 9)

The notion that mathematics can be applied in almost any field (i.e., "from sciences, to commerce, to finances, to whatever") is evident here, similarly to the "it is something that matters, that you need to know what is going on" example. This universal potential applicability of mathematics conceptualises mathematics as a very powerful tool. In this way, a relation between mathematics and other sciences is discursively constructed: mathematics is used by scientists to solve problems in their respective fields. (Mathematical) knowledge is conceptualised in relation to its universal applicability and this applicability attributes value to mathematics. Applicable knowledge is desired in the context of a capitalist, modernist discourse that values human activity that is purposeful and productive. Overall, the notion of applicability is important because, in a capitalist framework, high status is given to knowledge which has use-value and is technical (Apple, 2004a). Moreover, making mistakes when differentiating a function is conceptualised as having very important consequences – exactly because the mathematics taught in the class is not just supposed to be applied, but it is supposed to be applied in important and critical sectors. In this context, students are positioned as the individuals who undertake the process of applying mathematics. Here, it is important to consider the external conditions of enunciation of this articulation: the students of this class are expected to follow a scientific field but not mathematics.

The next example, the *"So, this kind of things are actually used" example* provides some more insight into the idea of mathematics as applicable. It happens as Alex talks to the class about derivatives and proposes that one application of them is the ways in which a car's speed is measured by a camera:

> Alex: Where do we use these types of things? It is like that this is the process that is done by the computer programs that check the speed with which you drive. As you are driving, the driver drives and there is a camera that's going to take a picture and tell you your speed. What is happening, the camera is pointing at the street and is sending a laser and the laser bounces and according to the speed that you have, it detects what was your speed at one point, the distance like it takes the distance, like you go back and forth, the closer you go, it starts doing that and it just approximates something. That's what it's been doing. So, this kind of things are actually used. Of course it is used by a machine that does these things very fast but that's what they do. (Alex's class, p. 13; Lecture 4)

Here, we have an "application" of mathematics: Alex explains a way in which mathematics is used in the everyday world. This example is similar to the "the discovery of Neptune" example and the "Imagine you wanted to luminate the place" example discussed in the previous section, where I explored how Alex's discourse conceptualises mathematics as something that humans solve problems with. However, the "So, this kind of things are actually used" example is different than them in that it privileges the notion of applying mathematics as opposed to the notion of engaging with mathematics while humans solve a problem. This is evident by the way in which the word "use" is employed in this example (as in "Where do we use these types of things?", "So, this kind of things are actually used").

This notion of "using" or "applying" mathematics is significant because, as will be discussed in more detail through the coming examples, it has ontological implications for mathematics and the world. For now, I want to highlight that in Alex's discourse there are two distinct practices: the first one is mathematics (which was explained right before the articulation of the discussed example and referred to in this example through the phrase "these types of things") and the second one is "the world" (where the technological problem of measuring a car's speed manifests). The already developed mathematical knowledge (in this case, calculus) is applied in the context of the technological problem or more generally the world. In this way, mathematics is constructed as an autonomous domain that is applied at the everyday, scientific, or technological level.

The notion of applicability in Alex's class makes assumptions regarding what we *can* see in the world. On various occasions during the course, the professor suggests that most of the time what we can see in the world is the derivative, but not the function. Such an occasion is the *"what we actually can see in nature is the derivative" example*:

> Alex: Remember that the whole point is that what we actually can see in nature is the derivative. We don't see the function. We see the derivative. But suppose I know that a function, its derivative behaves like this, but I don't know the function. So, even though I don't know the function I can say, when is the thing increasing, when is the thing decreasing. And these things, increasing and decreasing, have meaning. So, for example, in the example that we mentioned the other time, when they discovered these planets, again they could not see the planet. They could see the disturbances that the planet created in the near objects. But at some points, that disturbances were very big and at some points the disturbances were smaller. So, what does that mean? When the disturbances were very big, it means that the derivative is bigger, so somehow the distance they were seeing was increasing. So, they'd say, the planet, that we cannot see, is closer to the objects in a way, because it's disturbing them more. Whereas when we see that the disturbance being smaller, we can expect that the planet is farther away. Because it's disturbing them in lower quantities. So, that's usually what happens. What we have is the derivative and we have to deduce everything that we have about the object without having the object, only by having the derivative of the object. And this is the kind of thing that we can discuss. And we say, ok I don't know the object, but I know at which moment, the whatever property this object represents is increasing. And that's important for certain reasons. For example, there are many models in nature that are described by derivatives and that we cannot solve. We cannot actually say what is the function. I cannot just go and plot these, I cannot go and ask google, plot this thing. Because I don't know this thing. I just know the derivatives. For many, many things. But the point is that I don't need those things. If I know how to interpret the derivative. And that is what is actually done. (Alex's class, pp. 35–36; Lecture 5)

In this example, Alex makes an assumption with regards to what the world looks like, what it is that we can see in the world, and what the nature of the relation between mathematics and "reality" is. Alex proposes that "in nature", it is not possible to see a function, but it is possible to see the derivative – meaning that it is not possible to see how an object behaves as such, but it is possible to see how its behaviour changes. In Alex's discourse, the object/phenomenon that is being studied and the function that describes it become one thing. For instance, we notice this through the phrase "we have to deduce everything that we have about the object without having the object, only by having the derivative of the object". This means that a phenomenon is not simply described by a function, but it in fact *is* the function. Similarly, the derivative *is* a property of the function[39], which concerns how the object/function is changing its behaviour.

Here, we notice a difference with Chris's class. As discussed for Chris's discourse, a mirroring of the world in mathematics is assumed. In Alex's discourse, the merging of an object and its function, or more generally of the world and mathematics is assumed. Moreover, while Chris recognises limits in this process of mirroring, as described in relation to the "I'm not not going to school" example, Alex does not. Alex recognises the world as complicated and identifies limits regarding what aspects of the world can be seen in the world (i.e., a thing itself or its reality cannot always be seen) but its properties are supposed to be seeable.

Another example, in which the ontological assumptions of Alex's discourse are evident is the *"Implicit differentiation" example*:

> Alex: So, let me talk first about implicit differentiation. Just to make sure that everybody has seen these things. *[erases board]* So, implicit differentiation. *[Starts writing]* So, what is implicit differentiation. Is what we said. Like, we do not have a variable solving [inaudible]. And usually our variable is going to represent something. But for whatever reason, we have an equation that relates it to something else. But we do not know, how to solve for it. And that's a, that is a very common situation that happens in many, in many natural aspects of the world. We can't actually solve for one thing in terms of another. We just take a relationship of several things all together. So, let's see. (Alex's class, p. 77; Lecture 9)

Similarly to the previous example, here a phenomenon is described by an equation without it being possible to solve for the variable that represents the studied object; in other words, it is not possible to know the function. The method followed here is implicit differentiation. Although the object/function itself is unknowable, by implicitly differentiating, it becomes possible to learn properties of this object. Through the mathematical knowledge developed so far in this course, knowing an object gets merged with knowing its function. For example, a planet gets conceptually merged with the function that

[39] The notion that an object itself might not be knowable, except for through its properties is very common in academic mathematics. For example, in algebra, objects like groups are defined through their properties, while "different" objects all constitute a group as long as they have a group's properties (Fraleigh, 2009).

describes its movement. In this framework, mathematics has a formatting function as it frames the object in terms of the function and the things that can be done with it (e.g., the object/function can be graphed, the object/function can be differentiated to see how it changes and how fast these changes occur, etc.). In this sense, the mathematical model and the object are both assumed to be knowable and the model becomes a representation of the object, to the extent that the object and its model are considered to be the same thing (e.g., Skovsmose, 1994b). Through implicit differentiation, the formatting of the object happens in a different way. The object/function is itself unknowable but it can be sufficiently described by a relation that combines the function and its derivatives.

The following *"this is something because we're in the first course" example* also concerns the relation between mathematics and the world, while it refers to the limits of the mathematics of the class.

> Alex: What does this mean? This means that for all the functions that we're going to be working with, are going to have derivatives everywhere. So, for us, computing a derivative is going to become eventually an algebraic situation. You can do the computations, and you evaluate what you're going to evaluate and [inaudible] on whether it is or it is not differentiable. It will be only the first exercises around this topic. So, this is something because we're in the first course.
>
> The reality [inaudible] usually we assume when we're modelling something that it is differentiable. And this is something that was done for a very long time. Because differentiable in a certain sense means the idea of, if you change something that changes, like if you move upside a parameter that changes in a function that depends on this parameter, they do not change in a very sudden way. Like the change is so nice that it allows this. It doesn't suddenly go up. And in a certain sense this is kind of a natural assumption. But the point is that it is not always true in real life, but that was not obvious. This is a remark that it's important to have in mind. *[Writes on the board and reads]* It is not true in general that functions have to be differentiable. And this is an important thing. (Alex's class, p. 28; Lecture 5)

In this example, the distinction between the mathematics of the class and the mathematics of mathematicians (earlier discussed in relation to the "the discovery of Neptune" example) becomes evident. Within the mathematics of the class, every function is differentiable, but within the mathematics of mathematicians (which is in this example described as "the real life") this is not the case. The limits of the mathematics of the class become evident here. Nevertheless, mathematics in general (i.e., seen as the mathematics of mathematicians) does not appear to have the same limits. The imaginary constructed here is that mathematical tools, others than those studied in class, possibly more sophisticated, are available to tackle situations where a function is not differentiable. This is significant because mathematics (i.e., the mathematics of mathematicians) is hereby constructed as having no limits.

The *"The derivative is the mathematical tool to measure change" example* further elaborates on this idea. It happens as Alex introduces derivative as a concept to the class.

> Alex: So, the derivative, uhm, it's a tool to measure change. That's the main idea of it. So, *[writes on the board and reads out loud]* The derivative is the mathematical tool to measure change. But, like, what do we mean by that? The point that you should keep in mind is the following. Usually, when you want to measure change it means that you want to understand how something is evolving with respect to time in general. And the point is that the things that you want to understand are usually complicated. You can understand this in many perspectives: you maybe want to understand the weather, you maybe want to understand substances, you maybe want to understand like how systems of populations behave, like, almost anything. Almost anything can be put into this perspective. And the point is that to understand how it evolves is complicated. And most of the times, the best that we can do is to make predictions about what will happen, but these predictions very rarely are going to be very precise after a certain amount of time. So, why is that? The reason is that to make computations that appear in these things is complicated because the system of equations and the problems that we need to do and to solve in order to understand these phenomena are very complicated. Like, we cannot actually solve them. (Alex's class, p. 2; Lecture 4)

In this example, the idea that derivative (and hence also calculus) is powerful because it can be applied to almost any situation ("almost everything can be put into this perspective") is put forward, similarly to the "it is something that matters, that you need to know what is going on" example and the "if you make mistakes" example. The claim of a tool that can explain almost everything entails a totalising assumption. One aspect of this is that the tool does not depend on the problems that it intends to solve. Another aspect relates to the "power" attributed to mathematics, as mathematics is conceptualised as a knowledge that can solve *any* problem and be helpful in understanding *any* situation.

Furthermore, in this example, Alex refers to predictions not always being precise, as desired. This impreciseness is not the tool's fault, the model's fault, or the mathematician's fault. It is not that "our models" are not suitable or are not good enough; it is that the world is complicated and so are the equations that describe it. The academic mathematical background to which Alex refers here is the description of situations through differentiable equations which are often not analytically solvable. However, the point with regards to how this discourse is political is that, within this discourse, the internal mathematical complexity of this type of differentiable equation coincides with the complexity of the world. Hence, not only does this discourse assert a derivative or a differentiable equation to be a description or a model of almost everything in the world, but it further claims that it bears something of its inner complexity. The formatting role of mathematics is evident here. Skovsmose (1994b) describes the formatting role of mathematics through the thesis: “mathematics produces new inventions in reality, not only in the sense that new insights may change interpretations, but also in the sense that mathematics colonises part of reality and reorders it-just as the Indian social reality was changed after the English conquered the country and took power” (p. 42).

Moreover, in this example, Alex refers to predictions not being precise enough after a certain amount of time – also brought up in a similar way in his interview, in the *“the point is not that our models are terrible” example*.

> Alex: the predictions that it’s like, tomorrow there is a terrible snowfall and instead the sun is there all the time. The point is not that our models are terrible. But is that the period in which they are trustable is very brief because the amount of degrees of freedom of actual weather is huge. (Alex’s class, p. 129; Interview with Alex)

If our predictions are not going to be very precise after a certain amount of time, if they are only going to be “acceptable” for a very short period, it is a (politically charged) “stretch” to consider them predictions. Similarly to the joke about searching for the needle where there is light, this discourse assumes great trust in the method of calculus.

Another thing that we notice, here, is related to the modality of the “The derivative is the mathematical tool to measure change” example. In the phrase “The derivative is the mathematical tool to measure change, the derivative is described in a declarative way: not only is that expressed grammatically (not much space is allowed to contest the statement) but it is also a statement that the professor writes on the board. Later (in the sentences “Usually, when you want to measure change it means that you want to understand how something is evolving with respect to time in general. And the point is that the things that you want to understand are usually complicated. You can understand this in many perspectives: you maybe want to understand the weather, you maybe want to understand substances, you maybe want to understand like how systems of populations behave, like, almost anything”), modality is introduced through “usually”, “maybe”, and “almost [anything]”. This difference between the declarative statement about the derivative and the hedged utterances indicates the conceptualisation of a difference between what a derivative “is” and how people work with it as part of the course. Through this utterance and this enunciative mode, “understanding” is created as a concept: measuring happens because of desire to understand. This desire to understand becomes “strong” and “important” as it refers to a very wide range of things that can be understood through this method. I argue that the concept of “understanding” holds the status of a value (in this sense, “knowledge” is a value). This is related to how “understanding” is introduced in this utterance in a way that is taken for granted. For example, Alex mentions “you want to understand”, without there being any reference to why a student might want to understand. This emphasis on understanding is of significance because it creates a link between the humanity of mathematics discourse and the applying mathematics discourse.

I have already discussed that Alex’s discourse proposes that mathematics can be used to interpret any situation; a discourse that is not being challenged in any way in the class. Nevertheless, the *“mathematics is considered royalty” example* from the interview with a student from Alex’s class, Anthony, suggests that this narrative of Alex is not necessarily hegemonic in the class:

> Anthony: Uhm, so, it’s I think it’s safe to say that mathematics is considered royalty, even among the scientific community. And this has been, like, a long-standing idea. Even, like, from secondary education and stuff like that. Right? So. But, the thing is that it can only grow so far, because, in my personal opinion, you have so many different things, that aren’t quantifiable, aren’t governed by mathematics. Like, I mean, we are still searching, right? So, for instance, emotions or, uhm, very complex emotions. So, one could say that they are governed by chemical processes and those chemical processes can be quantified using mathematics. But, *[laughs]* it’s like, it’s not that simple, right? It’s not, like. This materialist worldview, right?, can’t have the strong and far reach of mathematics, you know, influence everything. But, yeah, like I said. People generally think that mathematics is the tippy top of objective thinking. (Alex’s class, p. 141; Interview with Anthony)

In this example, Anthony enunciates a discourse which is critical about the extent to which mathematics can describe and interpret everything. Anthony’s discourse is not aligned with Alex’s discourse regarding the extent to which mathematics can describe everything. Nevertheless, Anthony does not enunciate this discourse in the class itself. It is in this sense that I propose that Alex’s discourse (within the institutional framework of this undergraduate course) does not leave space for negotiation and, hence, eliminates the possibility of antagonisms taking place.

Overall, in this subsection, I discussed how applying mathematics emerges in Alex’s discourse and I discussed some ontological assumptions related to this discourse. A prominent idea is that mathematics can be applied to tackle almost any problem – an idea which attributes great value and “power” to mathematics. Furthermore, Alex’s discourse assumes a clear-cut distinction between mathematics and the world and merges the studied object/phenomenon with its mathematical representation (function). This happens through a second distinction realised in Alex’s discourse between the mathematics of the class and the mathematics of mathematicians.

6.3.2. Simplifying Problems

In this subsection, I explore a theme emerging in Alex's course, in which a complicated problem is transformed into a simpler problem which is considered solvable. After solving these simpler problems, the solution is translated back into the context of the initial problem. This theme relates to the rationality of mathematics as constructed in Alex's discourse.

The derivative, the main concept of the course according to Alex, is an example of this approach. Describing a complicated function whose equation is often unknown is captured as a complicated problem. Locally linearising the function by finding its tangent line is a simplification technique that translates the problem into a problem of lines, which is a solvable problem. Alex explores this idea in the *"we try to transform the problem into another problem" example*, which happens in his introductory lecture to derivatives.

> Alex: The reason is that to make computations that appear in these things is complicated because the system of equations and the problems that we need to do and to solve in order to understand these phenomena are very complicated. Like, we cannot actually solve them.
>
> Like, so how do we do? We try to transform the problem into another problem. And this idea, is precisely what the derivative is going to do, in which, the derivative will allow us to make a simpler version of the problem. Now, of course, because we are simplifying the problem, we are going to lose some of the properties of the original problem, but the point is that we are not going to lose so much information that we lose everything. So, the main point is that we are going to transform the problem into another problem that we can solve and we expect that whatever we find in the simpler problem, can be brought back somehow to the original problem. That's the main idea of the derivative, that's what the derivative allows us to do. And the idea that I just said, going from a hard problem to an easy problem, has its own name and it's in a certain sense the main concept of this part of calculus and it's called linearisation. So, the way *[writes on the board]* The way in which it allows us to do this is via linearisation.
>
> Now, this is a very big concept and as you are going to continue to study other things in math, you are going to find this idea a lot. The idea of linearisation. In a certain sense, it's the most important thing because linear problems, whatever it is that we mean by that, are the ones that we can actually solve precisely. So, we want to transform those that we can into linear problems which we can solve them. […] So, what do we do? Like the idea is that if we want to understand what happens with a function around a point – I'll pick a point like this one – we see what happens to the function, but the function itself might be complicated, like the function, there is something about it that we don't understand precisely. so, we changed the function for a line, like for its tangent line: that's the main idea. So, the point is *[writes]* change the function around the point, change

> the function around the point for its tangent line. And the point is that we already know everything about lines. We know that lines have a particular equation like a line is y=mx+b, where m is the slope and b is a particular parameter that we can find, we have already seen that. (Alex's class, pp. 2–3; Lecture 4)

In this example, Alex talks about the tangent line as the local approximation of a function and he conceptualises the derivative as a "tool" that transforms a given problem into a simpler one. This rationality method consists of reframing a problem into a solvable one, by (locally) transforming curves into lines. The idea of transforming a problem into another problem encapsulates a method through which mathematical knowledge and truth are derived. It is in this sense that it relates to rationality. This process is associated to what Bishop (1991) calls control: the urge to frame a problem in such a way, so that there is complete control over it. Transforming the problem into a linear one allows for it to be solved "precisely", as "we already know everything about lines".

I argue that handling a problem in such a way so that there is complete control over it is of political significance. The premise of mathematics as a knowledge that can offer precise results while also being able to handle and tame any kind of problem is constructed through such instances. This promise for the ability of mathematics to tame any kind of problem and control any situation has been critiqued in the light of climate crisis and biodiversity crisis (e.g., Mukhopadhyay & Greer, 2021). The desire and urge for control is not only part of the problem (i.e., how humans have created a relationship with other species and natural elements that exploits the ecosystem and appears to be in the core of the climate crisis), but it is also quite implausible that humans will be able to handle the climate crisis in a way that allows them to remain intact as a species.

In what follows, I present a few more examples which show how the idea of simplifying problems manifests in the more technical aspects of the course. I will not engage in a deep analysis of each one of these examples, as my goal is to show the range in which Alex's discourse makes use of this idea, thus reinforcing the ideology of control. But before looking at a new example, it is worth mentioning that the notion of simplifying problems is also manifested in the "the people who started doing these were Newton and Leibniz" example. In that example, Alex mentions "And of course they were trying to simplify them by understanding the rules". Here, the notion of simplifying derivatives is conceptualised as natural (which is evident by the use of "of course"). In this way, the method of simplifying is constructed as a commonsensical method which has already been followed by great mathematicians centuries ago.

The *"I have reduced the problem to a similar problem" example* is another instance, in which the notion of simplification manifests:

Alex: And now I have reduced the problem to a similar problem, that is technically simpler. And now I evaluate again. The whole question is whether I get zero over zero or not. So, let's see. (Alex's class, p. 18; Lecture 4)

In this example, Alex works on a limit that is of "zero over zero" form and, thus, simplifies the numerator and the denominator of the fraction in order to compute it. Alex conceptualises this algebraic simplification as a "reduction" of the problem to a similar but simpler one. Hence, the notion of simplifying a problem so as to solve it also entails and incorporates the technical manipulations performed over a mathematical object. Something similar happens in the *"but with something that looks way easier" example*:

Alex: So, that fraction over there, is the same as $\frac{t+3-1}{t+1}$. Right? This is what the long multiplication means. Long division, sorry. So, it's the same function. This quotient here, this quotient here is exactly the same as this thing here. They are the same function. So, they have to have the same derivative. If I derivate directly here, I have to use the quotient rule. And I obtain this. But, maybe, I say, well maybe, if I divide, I obtain something simpler. I obtain the polynomial, which is a sum of power functions. And I still have to use the quotient rule but with something that looks way easier. So, let's do it. (Alex's class, p. 110; Lecture 9)

Here, Alex works on finding the derivative of a function, which has the form of a quotient between two polynomials. Alex proceeds with dividing the two polynomials, creating in this way a "simpler" version of the function's equation, which can therefore be differentiated more easily. The aspect of difficulty is used here as the reason for why this simplification process is taking place. Hence, avoiding difficult work and working on simple problems is constructed as ideal in a naturalised way.

Finally, the *"in order to simplify the computations" example* also draws upon the idea of simplification:

Alex: We are going to learn some cases by memory, some basic formulas, like what happens with the power functions, what happens with trigonometric functions, with exponential functions, with logarithms, like the basic ones that we saw. And then everything else is built out of these ones. So, we are going to use the rules that we are going to learn in order to simplify the computations and reduce them to those. (Alex's class, p. 14; Lecture 4)

Here, Alex refers to the rules of differentiation, which the class is going to work on. Alex refers to the rules as simplifying the computations and suggests that any question of differentiation shall be reduced to these rules. This example conceptualises simplification as a rationality technique which forms mathematical knowledge as a kind of knowledge that is able to transform any problem so as to solve it.

Overall, through the examples discussed in this subsection, I argued that the simplification of problems emerges as a rationality technique in Alex's class. This technique concerns transforming problems into "simpler" versions of them, so that they are solvable. I critiqued this rationality technique on the grounds of it manifesting within an ideology of control of mathematics, in the sense that it conceptualises mathematics as a tool that can tame any sort of problem.

6.3.3. Developing Prototypical Methods

This subsection explores how developing prototypical methods is a rationality technique in Alex's class which is part of the applying mathematics discourse. In Alex's course, developing prototypical methods emerges as a theme, in the sense of developing and using algorithms that deal with a multitude of similar problems, instead of treating every problem as unique. I derive the term "prototypical" by Jullien (2012) whose insight into mathematics as a prototypical field was discussed in Chapter 3. In what follows, I proceed with the presentation of my findings, first by exploring how the idea of prototypical solutions emerges within the distinction between "rules" over "tricks" and then by exploring how "rules in action", mistakes, and making sense of solutions constitute part of carrying a prototypical plan.

Developing prototypical solutions in Alex's class manifests through an emphasis on the development and use of "rules" and "methods" over "tricks". The development of "rules" or "methods" as opposed to "tricks" or "inspection" is highlighted often by the professor. Here, I present two examples in which this happens. In the *"please learn the quadratic formula" example*, Alex presents the computation of the limit of a rational function, for which the limit of the numerator and the limit of the denominator are both zero. The professor factors the numerator and the denominator. In doing so, he uses the quadratic formula and discusses the reasons that he does do.

> Alex: So, we need to factor. For quadratic polynomials, we can use the quadratic formula, so let's make the example. And I make the example in all its glory. Because one of the most common things that I see when I check math is that you know how to do everything except writing the quadratic formula. So, please, learn the quadratic formula. I've seen everything with the quadratic formula. So, let's write it. It's- *[He writes the formula on the board]* So, make sure you know this. This is an important formula. And the reason it's important is because it's a formula. You don't have to think to obtain the result. You just have to substitute correctly. But, if you have the wrong formula, of course, you'll get the wrong result. And this is a very common mistake that you don't know the formula. This is a very nice formula. The formula was known by the Babylonians, you can learn it by now, 8000 years later, so please learn it. […]

[Alex continues with the computation of the limit, by applying the quadratic formula, dividing the two polynomials, simplifying, and substituting. After he finishes the computation, a student raises their hand and is given the floor:]

Student: Why do we have to use the quadratic formula?

Alex: You do not have to. I am just making the example, that it works with the quadratic formula. You can do this thing of factorising by looking for two numbers whose product is this one and the sum is this one or you can do it by inspection. It is not that you **need to** use the quadratic formula. Basically, what I am saying is if you- the point is that the other things are somehow by inspection. If you see it by inspection, fine. But if you are trying to find the numbers for a while, the quadratic formula gives them to you, like, immediately. Ok? (Alex's class, pp. 17–19; Lecture 4)

Here, Alex talks about the use and importance of the quadratic formula for factoring a quadratic expression. He identifies the importance of it, in it being a formula, which means that "You don't have to think to obtain the result. You just have to substitute correctly". After he has finished the computation of the limit – which involved a division of polynomials, a simplification, and a substitution – a student gets back to the idea of the quadratic formula, by asking why they have to use it. The professor responds that students do not *need* to do that. He refers to the method of finding two numbers with a given sum and a given product as an alternative (which, due to the structure of the high school curriculum, is the most standard method for students in Ontario to solve quadratic equations). In comparing these methods, Alex mentions that the sum-product method is "by inspection" and is something that students may or may not *see*. On the contrary, the quadratic formula can be used for any case and gives the solution "like, immediately".

In this example, we can see the contrast between two alternative approaches of solving a quadratic equation, both introduced by the professor: the quadratic formula and the sum/product method. It is noteworthy that both approaches are merely algebraic. For example, graphing the quadratic could be another way for obtaining its solutions, but it is not mentioned as an option. The algebraisation of calculus and pre-calculus knowledge in this class has the function of producing precise results quickly and uniformly, thereby conceptualising desired mathematical knowledge as having these characteristics.

But, also, between the two algebraic methods that are mentioned, the formula one is praised more than the sum/product one, on the basis of its algorithmic nature. As Alex talks about the two approaches of solving a quadratic equation, he constructs two different *doers* of mathematics. The doer of the sum-product method conducts "inspection" or investigation. By doing so, the doer may or may not *see* the solution. This implies that the solution of the mathematical problem exists, it is embedded in the problem itself, while finding the solution becomes a matter of *seeing* it. Seeing the solution might not happen. On the other hand, solving by the formula, conceptualises a doer of mathematics who without needing to think, more surely, can find the solution to the problem by applying a method which works quickly, does not require them to think, and works for every case. The latter "doer of mathematics", the one who makes use of prototypical methods for solving problems, is the desired doer of mathematics in this class.

The construction of the desired student as a doer of mathematics in this example is evident by the enunciative choices of Alex, which contribute to the directness with which students are encouraged to use the quadratic formula (e.g., through the use of imperative along with the word "please" in the phrase "so please learn it") as well as with which the desired student (as doer of mathematics) is constructed. The use of pronouns is significant in the process of the construction of the desired student. In the beginning of this example, Alex uses the pronoun "we" as he presents the computation of the limit. But since the third sentence of the presented extract, he takes up the use of pronouns "I" and "you". The roles of the professor and the student become more distinct here. On the one hand, the professor (through the pronoun "I") is the one who will write down the example in a very detailed manner ("I make the example in all its glory") and corrects and grades mathematical assignments or tests ("when I check math"). In other words, the use of the first person pronoun here is used to capture very typical actions done by a professor (showing something to students, grading) and, thus, contributes to a positioning, at this time, in which Alex is talking to students as their professor (and not e.g., as a mathematician or a representative of the mathematical community). The students, in turn, captured through the second person pronoun ("you") here, are those who produce work that is to be graded (and not, for example, for their own pleasure).

A second example in which the notion of prototypical solutions is evident is the *"we need rules" example*. Alex introduces the notion of derivative and uses the definition to compute the derivative at a given point of the $y = x^2$ function and subsequently of the $y = x^4$ function.

> Alex: Now, if instead of writing $y = x^4$, I put $y = x^7$, for example, you would have a huge polynomial [inaudible] but forget it, it's a huge polynomial. […] Let's do another example *[writes on the board]* What happens if instead of doing something like this, I do "repeat", but instead of that, just say, y is equal to the square root of x. […]
>
> *[Alex explains how the limit would look like and suggests that it is complicated.]*

> So, the limits start getting complicated. In this case, you can try, I am not going to do it, but you can try the $u = \sqrt{1+h}$ and you solve for h and the point is that you substitute in this kind of things. The idea when you have square roots like that one is to eliminate the square root and transform everything into polynomials, that's the kind of idea.
>
> But, we could have more complicated things. Let's say, what happens if, for example, we were looking at the y = $sinx$, let's say if I do this one, what is the limit that I get. Well, again, it is something like this *[draws the graph]*. I know how the sine looks, we do at $(0,0)$, so we are here and what do we have to do? We pick another point for (h, sinh) and what is the limit? Well, the limit is *[writes the limit]*. And this is one of the limits that we actually, like, solve by looking at the graph. We haven't actually said much about it. Or if you have seen the section of the squeezing theorem, you have to use a theorem. Let alone computations or tricks, you have to actually *[emphasises]* go and look for a **theorem** *[emphasises the word]* that solves this thing.
>
> And the point is now, imagine that I start mixing these things. Imagine if I put $y = sinx + x$ divided by the exponential, I don't know, like-. Its something possible [inaudible] it's just undoable. So, undoable from the perspective of actually coming up with tricks. So, we need something better than tricks, we need **rules** that help us simplify derivatives, in this case [inaudible] So, basically that's what we are going to do for the following week. (Alex's class, pp. 10–11; Lecture 4)

In this example, Alex works on finding the derivative of some functions using the definition. Through a gradual escalation in complexity, he argues about the need of rules to simplify the process of computing derivatives. Starting from limits that the students know how to compute, they move to limits which are practically "undoable" to solve. In this way, the need for "rules" emerges. To illustrate, the "rules of differentiation" include rules about the derivatives of common functions (e.g., polynomial, exponential, trigonometric, etc.) and about operations with these functions (sum, difference, product, quotient, and composition). Although limits are used for the development of these rules (both in the mathematical tradition from which Alex talks and – in some cases – in the course itself), using the rules means not working explicitly with limits and their algebra. Working with derivatives without having developed rules is conceptualised as a difficult process, that not only requires the use of tricks, but occasionally requires the use of "theorems". Alex emphasises the word "theorem", thus conceptualising the use of theorems as a hard process for students. Here, it is important to consider that Alex's class emphasises the technical aspects of calculus and, as such, students do not spend much time working with theorems per se. In other words, Alex makes an appeal to the hardship of more formal versions of mathematics, with which students are less familiar.

Through this example, Alex constructs avoiding a hard mathematical method as desirable and instead advocates for the development of methods that will allow students to avoid the hard thing. Avoiding hardship is kind of common space in mathematics. Methods are constructed as desirable for their general character and for being able to solve every problem efficiently. These methods, which take the form of the rules of differentiation here, are based on the repeatedness of the same idea. Instead, hard mathematical methods are conceptualised as the ones that are based on the development and employment of "tricks", which are often unique for every different case.

Alex presents the "rules of differentiation" (e.g., derivative of a polynomial, derivative of exponential, derivative of sum, etc.) as rules that allow the calculation of *any* derivative, whereas finding a derivative by calculating the limit relies on tricks. In this way, Alex's discourse constructs the development of rules as a much-needed process, exactly because it is prototypical: it involves the development and use of a method which is the same for every example that might be encountered. In this way, similarly to the "please learn the quadratic formula" example, students' mathematical involvement in developing a strategy to solve a problem is limited to carrying-out a predeveloped plan.

So far, I have discussed how the notion of developing prototypical solutions emerges in Alex's class as a rationality technique in contrast to the use of "tricks". In what follows, I focus on the way in which Alex's discourse conceptualises the process of carrying out a plan. The following example shows the "rules in action", as part of applying the plan. I am borrowing the phrase "rules in action" from Alex, who uses it in the "it makes you appreciate the problems" example, to refer to the completion of a mathematical task through the suitable application of a series of rules. We can see this idea in the *"we are going to derivate with respect to x" example*, in which Alex works on the curve with equation $x^2 + y^2 = xy + 1$.

> Alex: Basically, we are going to derivate with respect to x. So, this one is x squared. It's a power function. That's fine. But this one is not a power function, this one is a composition. y is a function of x. I don't know what function, but it's a function of x composed with the square. So, this is a composition, so I need the chain rule. This one, x is a- x is a function of x. It's a power function. And y is a function of x, but I don't know. So, this is a product. I'm going to use the product rule here. This is a constant. So, we derivate. *[writing]* and what do we obtain?
>
> We derivate in respect to x and the first thing is $2x$. That's the derivative there. Then is y squared. This is a composition. Is a composition of the square with y. So, what do we obtain? Is that, is $2y$, times the derivative of y. That's the chain rule. It's the composition of the square with y. The outer function is this x squared. So, it's the derivative of the outer function times the derivatives of the function. So, I get $2y$, y prime. *[Pause]*

> Now, we derivate the other side. The other side is a product. It's a, the derivative of x is 1. So, 1 times y. Plus x times the derivative of y. And then, plus the derivative of 1, that's zero. So, what I obtain is this one. (Alex's class, pp. 104–105; Lecture 9)

As Alex talks about and performs the differentiation of the equation, we get a glimpse of "the rules in action". This example has two parts (and for this reason, I have separated the text in two paragraphs). The first part is about recognising: Alex examines the equation and recognises its parts in relation to the rules that are available in the class. He, for example, recognises x^2 as a power function, y^2 as a composition of functions, etc. The second part is about carrying out the plan. Based on what he has already recognised, Alex performs the differentiations.

In this example, the rationality that emerges in Alex's class becomes evident. It is a rationality which is based on rules that students need to learn and subsequently apply. Through the use of deictic words (e.g., this, that) and short sentences, Alex exemplifies how this rationality works and what the doer of mathematics (as constructed in this class) is expected to do. As Alex applies the rules, there is almost no choice that the doer of mathematics is required to make; there is no creativity required for handling the problem. The doer of mathematics is merely required to recognise the rules that can be applied and subsequently perform this application.

Another theme in Alex's class that relates to carrying out a plan is the handling of mistakes. There are two important aspects of how mistakes are conceptualised by Alex. On the one hand, Alex conceptualises mistakes as a natural part of doing mathematics and, on the other hand, he makes clear that mistakes should be avoided. When Alex makes a computational mistake on the board, he comments on it in a joyful and jocular way. The *"You have to substitute correctly of course" example* is such an instance.

> Alex: So, how do we do? x is 2, y is 4. Substitute that thing, so I get four equals two times two plus b. [*He writes* $4 = 2 \cdot 2 + b$, but *makes a mistake]* You have to substitute correctly of course, so 4. And this is an equation. (Alex's class, p. 7; Lecture 4)

Similarly, Alex refers to the mistakes that famous mathematicians have made, which also contributes to conceptualising mathematical mistakes as natural. For instance, in the "the people who started doing these were Newton and Leibniz" example, Alex refers to the mistakes that Leibniz has made. In these examples, mistakes are conceptualised as a natural part of doing mathematics.

On the other hand, it is also clear that mistakes need to be avoided. Alex cautions the students about possible mistakes that they might make, trying to support them in solving mathematical exercises in a way that is mistake free. The comments about the mistakes of Alex himself as well as the references to Leibniz's have a character of warning students what mistakes not to make and how to avoid making them. Avoiding mistakes can be understood as essential within the context of the institutional framework, within which students lose marks if they make mistakes on an exam or homework assignment. This is also reflected in Alex's class discourse. For example, in the "please learn the quadratic formula" example, Alex refers to the need that students learn the quadratic formula because it is a common mistake that he encounters when marking students' work. Moreover, avoiding mistakes is also important within the rationality framework of developing prototypical methods. Indeed, there is little value in using a method that can be applied to a multitude of problems, if this method is used in a wrong way. So, avoiding mistakes is significant because it contributes to the success of automation.

Finally, the rationality technique of developing prototypical methods entails making sense of the solutions that are put forward. The *"it doesn't mean that they have meaning for the problem" example* relates to this issue:

> Alex: So, these are mathematical solutions. But it doesn't mean that they have meaning for the problem. So, I have to go and see. Does this have meaning? Well, the negative one does not. Because I am taking x to be here. Particularly it is positive. So, you don't know. Mathematically I have solutions, some of them do not [inaudible] to the problem. I can ignore these ones. It doesn't mean anything for the problem. A priori. Like, if I get- maybe I could say something, if I really think hard what does it mean. Well, it means that because of the symmetry, I could have said that instead of my variable going to this side, maybe my variable goes to this side. And the minus three gives as the same solution but going in this side. But for us- (Alex's class, p. 121; Lecture 9)

In this example, Alex talks about how the "mathematical solution" has meaning for the problem that he set up to solve. Even though one of the solutions does not initially appear to have meaning for the problem, Alex suggests that "maybe I could say something, if I really think hard what does it mean". Here, there is an ontological assumption made that the result produced by mathematics can have meaning even if it initially appears not to. This is significant because it constructs faith in the mathematisation of a problem and, thus, it attributes strength to the prototypical approach.

Overall, Alex's discourse conceptualises prototypical solutions as a rationality technique. The "need" to develop prototypical solutions is constructed as natural and commonsensical. The educational context is important in the construction of this commonsensicality. Not only do students need to learn specific things, but students need to perform these things efficiently in the context of their homework assignments and exams. Curriculum-wise, students are expected to efficiently and quickly be able to compute limits, which in turn, prepares them for working with derivatives. In this pre-determined path, learning goals have been set in a strict way and every task has pre-determined sub-tasks. Within a political economy of solving problems, prototypical solutions are a very "economical" way to do so. Solving similar problems multiple times by employing different tricks might be fun and intellectually engaging to someone who has nothing to do. Yet, this is not what the class prepares students for. In this way, calculus in this setting is constructed as a rather instrumental thing.

6.3.4. Developing Technical Mastery

The "applying mathematics" discourse also captures the desired characteristics of the technical mastery that students are expected to develop. In this subsection, I discuss the notion of efficiency as well as the notion of practice, as being desirable for students' mathematical development.

I argue that being efficient is important in Alex's discourse. There is a series of examples in Alex's class, in which the undesirability of long calculations and the importance of efficacy are constructed. We have already encountered such examples in relation to the rationality technique of developing prototypical solutions. For instance, in the "please learn the quadratic formula" example and the "we need rules" example, efficacy takes the form of a method which is universal, in the sense that it does not depend on the characteristics of a specific mathematical problem but it can be used to solve a multitude of problems.

Another characteristic of efficiency is speed in completing computations. When evaluating the methods used to solve a differentiation problem using limits (i.e., through the definition of derivatives), Alex often comments on how lengthy these computations are. For instance, we see the undesirability of lengthy computations in the *"it took us, like, 20 seconds" example*:

> Alex: So, before we computed the whole limit and it was a whole drama. But now, each one of these is a limit. And each one of these, it's a power function. And each one of these, it's a power function. So, each one of these has a limit that exists. I can write these. And now I know my formula. The coefficient, exponent goes down and the exponent goes one down. So is $3x^2 - 1$. And that's it. No limit, no nothing. It took us, like, 20 seconds. Whereas the other thing took us like very long. (Alex's class, p. 41; Lecture 5)

In this example, lengthy calculations are constructed as undesirable, as is evident by phrases like "it was a whole drama" and "the other thing took us like very long". On the contrary, the use of differentiation rules is constructed as easier ("No limit, no nothing") as well as quicker ("it took us, like, 20 seconds"). Within the political economy of this course, time is viewed as something that shall be spared.

The *"There is nothing like doing these things to appreciate mathematica" example* is another communicative event, in which efficiency is brought up, this time in relation to the condition of doing mathematics with and through machines.

> Alex: There is nothing like doing these things to appreciate mathematica. Like these things, as long as you can actually write these things correctly in mathematica, which is something like writing the square roots with code. Aha. That's it. I'm terrible with mathematica, you couldn't believe it. Ok, anyway, so let's do this. I'm just rewriting. So, I haven't done anything new. Ok, and this is over that thing squared. Oof. (Alex's class, p. 86; Lecture 8).

This example was preceded by lengthy computations. Software is used here to complete these same calculations. The undesirability of lengthy computations is evident in the way in which Alex proposes that doing them leads someone to "appreciate mathematica". In this way, the development of software that can quickly and universally complete computations is valued in this discourse.

A third example on efficacy is the *"it's simpler computationally" example*. As the class works on implicit differentiation, Alex finds the tangent line of a curve in two ways: first, he identifies a function (which turns out to have a complicated equation) and finds its derivative and second, identifies the derivative at a point through implicit differentiation. He discusses the relation between these two methods, commenting on the implicit differentiation, as follows:

> Alex: Which is exactly the same with what we got before. But this is simpler. It's simpler computationally. It's harder conceptually. We have made a trade, right? Like, we have obtained simpler computations as long as you understand what you are doing. (Alex's class, p. 80; Lecture 7)

In this example, Alex draws upon a distinction which is frequently used in mathematics education: the conceptual/computational distinction. Here, this distinction is discussed in relation to efficacy. Achieving computational efficacy is here possible through the development and the understanding of a deep conceptual idea. Similarly, achieving computational efficacy regarding differentiating a function is the result of complex conceptual steps, which are not entirely visible to students. It is significant here that, similarly to the previous examples, obtaining simpler and quicker computations is not negotiated: it is constructed as a commonsensical value.

Another example which connects the notion of struggling through mathematical problems with a notion of progress is the "it makes you appreciate the problems" example, discussed earlier in this chapter. In that example, Alex proposes that today, humans do not have to do the lengthy computations that people performed 400 years ago, while in 400 years from now, people will probably not have to do the lengthy computations that mathematicians do now. The concept of progress is thereby constructed, in terms of increasing effectivity as time goes by.

Overall, Alex's discourse constructs lengthy calculations as undesirable and developing technology that allows quicker computations as desirable. Efficacy is, thus, constructed as a value here. However, efficacy is not a neutral value. Time and the effort to minimise the required time to complete a task are inscribed into a particular economy. Indeed, the computation is performed because its result is needed and the quickest way for this to happen is the most desirable. In this sense, efficacy is a value which must be understood in relation to neoliberalism. Within a capitalistic framework, efficacy is highly valued: time should not be wasted (Rafailidis, 2000).

Along with efficacy, the notion of practicing and struggling through problems is also constructed as desirable in Alex's discourse. In the *"repetition is very important" example*, the importance of struggle in individual work is outlined. This example happens as Alex computes the derivatives of some functions, using the "rules of differentiation". After he has finished finding the derivative of a function and before he presents a new example, he says the following[40].

> Alex: [1] I, when I studied calculus, that was [n] years ago, I had a teacher and she would, like, write the chain rule every day for the first month. [2] So, that you make sure that you actually learn it. [3] It's very important to actually repeat the same thing at the beginning for several times so that it becomes a natural thing. [4] Like repetition is very important. [5] Probably you don't get told that in modern perspectives but repetition, it's important. [6] Because it's the thing that actually makes the connections in the brain to work. [7] So, if you want to become good at this, it's very important that you practice. [8] Even if you, if you are having problems with learning the rules, write them down several times. [9] It's an important thing. [10] Like, it's fine that you understand what's going on, but it's also important to know the thing technically. [11] So, write the rules and make sure you learn them. (Alex's class, p. 82; Lecture 8)

In this extract, Alex introduces a distinction between his own narrative and "modern perspectives". This distinction is important, because it is within it that practising through repetition is constructed as a concept. In line [1], there is a reference to what his teacher used to do and a (rare) reference to Alex himself learning/doing mathematics. Through this reference, the narrative about practicing through repetition gains importance, as Alex has (obviously) learned the chain rule and he has been successful in

[40] As the analysis of this example is more linguistically heavy than other examples, sentences have been numbered.

mathematics. This gives authority to the narrative and introduces a connection with more traditional discourses.

In sentences [5, 6], through the hedging [“probably”], some distancing from “modern perspectives” is expressed, while the passive voice of “don’t get told” takes the focus away from students as agents and moves it to “modern perspectives” and people who talk within those. Alex, thus, conducts a dialogue with “modern perspectives” at this moment. Two things are important in this dialogue. First, Alex very clearly distances himself from “modern perspectives” and positions his repetition narrative in a minority state within the current academic and educational discourses. Second, he does not merely express disagreement or content-related objections to these perspectives. Instead, by expressing that modern perspectives do not tell students about repetition, he introduces suspicion and distrust against them.

The value of repetition is not something that is constructed by Alex’s narrative as being contested. The declarative syntax (it’s) and the use of the word “actually” leave no space for it to be negotiated. Further, the syntax “it’s the thing that” establishes the need for creating brain connections to be a common ground: this is what is desirable. The contested part is about *telling* which is the thing that makes it happen (which also uncontestably is repetition). In this way, the honesty of modern discourses is brought into question.

Another distinction created and reproduced by Alex here is the one between “understand[ing] what’s going on” versus “know[ing] the thing technically”. This distinction is a different facet of the distinction between conceptual vs computational, which is prominent in the course’s curriculum. Students’ engagement with mathematics is guided within this distinction, in a way that students are strongly encouraged to learn the technical elements of the course (such as the chain rule) by repetition. This is supported by a series of expressions that add emphasis, including: “make sure” [2, 11], actually [2, 6], (very) important [3, 4, 5, 7, 9, 10].

The second pole of this distinction (developing technical mastery) is associated with “actually” learning. Rules are given and are unnegotiable. “Actually learning them” means them becoming natural for the person. If we see learning as taking up a discourse, the unnegotiability of rules and the goal of them to become natural signifies that learning mathematics or taking up the discourse of mathematics is a rather strict, not very flexible, thing. Further, mathematics happens on paper and in the brain, while practice through repetition is itself associated with the physical act of writing [1, 11].

Finally, the elevation of technical mastery and mathematical knowledge to values can be seen in statement [7], viewed from a governmentality perspective. Obviously, for students to pass the course or get a good grade, the rules of differentiation are important. Yet, Alex does not choose to refer to the external ways that secure students' consent to learn the rules. Alex rather conceptualises practicing the rules as important on the grounds of whether students want to become good at this. Why wouldn't anyone want to become good at this? A reinforcement of how power is exercised through this passage (by which I mean that [students'] actions are guided in a particular way) are lines [8, 11], where the imperative is used. The assumed commonsensically of "becoming good" as well as of how this can happen is reinforced through the use of the imperative. In this sense, becoming "good at this" (in the sense of developing technical mastery and obtaining mathematical knowledge) is an implicit value here.

Overall, in Alex's discourse, a lot of focus is placed on the importance of individual work, as well as on working efficiently. Struggle has emerged as a theme in undergraduate mathematics education literature. We might also encounter struggle in popular discourses, often in contrast to giving up after making a mistake. Alex's discourse values appreciating the struggle. Efficacy, on the other hand, is about avoiding struggle and it relates to developing methods which can help treat various problems in a pre-determined, algorithmic way. Convincing students of the need for an efficient method is a task in which Alex engages. The interplay between struggle and efficacy becomes a Catch-22: each of these discourses reinforces the other: Efficacy is viewed as a way to overcome struggle and, at the same time, appreciating efficient methods comes through struggle. To avoid struggling, one needs to develop efficient methods, while to develop efficient methods, one needs to struggle. According to Jullien (2012), struggle and heroism are the natural outcome of a prototypical way of thinking and acting. This interplay inscribes itself into an economy of neoliberal values, where it is desirable to work quickly, efficiently, and universally while at the same time individuals are expected to work hard.

6.3.5. Bringing it All Together

In this section, I presented the "applying mathematics" discourse as an overarching discourse in Alex's class. Within the context of this discourse, mathematics is conceptualised as able to be applied in every field and to solve (almost) any kind of problem. Alex also develops a couple of rationality techniques in relation to this discourse: simplifying problems and developing prototypical solutions. These rationality techniques are significant in the construction of the desired technical mastery, which entails being efficient and practicing hard to develop the required mastery.

Within the “applying mathematics” discourse, the notion of control is central. Control has two levels here. First, the “applying mathematics” discourse reproduces an ideology of control on the world through the application of mathematics. Second, it reinforces the notion of control within the mathematical practice, as rationality techniques like the ones of simplifying the problem and developing prototypical solutions aim at establishing total control over the mathematical problem at hand. The notion of control, reproduced here in the ways in which students learn to compute limits and derivatives, is significant because it is a central premise of mathematics within modernity and as such contributes to the mathematical formatting of the world (Mukhopadhyay & Greer 2015).

The two overarching discourses of Alex’s discussed so far, namely the humanity of mathematics and the applying mathematics discourse, are both deeply ontological in the sense that they conceptualise the nature of mathematics and its relation to the “world”. The meaning of mathematics (seen as a floating signifier for mathematical discourses) is crystallised differently (and in contradictory ways) in the two discourses, resulting in mathematics being a nodal point with a significantly different content for each discourse: it is a human activity for the humanity of mathematics discourse and a tool that can be applied in the world for the applying mathematics discourse. Both discourses coexist in Alex’s discourse, yet they are not in an antagonistic relationship with each other. This is because the two discourses are rarely both articulated in the same utterances. The result is a discourse with heterogeneous discourses as part of it, though these discourses co-exist and are not brought into collision by the professor himself; the discussion of how they conceptualise mathematics as a floating signifier is brought up at the level of the analysis.

Within Alex’s discourse and the discourse of the class, the coexistence of the two discourses constitutes a hegemonic entity. One function of this hegemonic modernist articulation is that it makes other possible representations unthinkable or unimaginable. For example, borrowing representations from Chris’s class, it is unthinkable for Alex’s discourse that mathematics may be unfair or that there are diverse ways of doing, thinking about, and talking about mathematics. This does not mean that Alex himself never thinks of these discourses or disagrees with these discourses per se – frankly, there is no way to tell from the data and, from a discourse analysis point of view, this is also a meaningless question. It means that these discourses are unthinkable within the limits set by the discourse articulated in the class.

6.4. The *"Mathematics of the Class and the Mathematics of Mathematicians"* Discourse

The third overarching discourse in Alex's course concerns a distinction between the mathematics of the class and the mathematics of mathematicians. Mathematicians' mathematics takes the form of formal mathematical knowledge, which is cohesive and entails definitions, theorems, and proofs. The knowledge of this class is a distinct mathematical knowledge which derives its cohesion from formal mathematical knowledge. This section is organised as follows. I first focus on the mathematics of the class by discussing how Alex's discourse is based on a conceptual/computational distinction and consequently by presenting how Alex's discourse constructs a shared space. I continue by focusing on the mathematics of mathematicians by exploring first how Alex's discourse conceptualises truth as a value and then the ways in which Alex's discourse conceptualises formalising.

6.4.1. The Conceptual-Computational Distinction

The distinction between conceptual and computational learning and knowing of mathematics is often brought up in Alex's class and is significant because it epistemologically frames what (academic) mathematical knowledge looks like. The computational aspect of mathematics entails technical aspects of a phenomenon and is related to solving problems. The conceptual aspect is about what a concept is or means and is related to developing intuition, interpreting results, and developing theoretical arguments. The distinction between the two is manifested both in the course's curriculum and in Alex's discourse and it is often articulated in relation to the existence and increased availability and use of software that can solve computational problems.

Before examining the conceptual/computational distinction in Alex's class, it is worth mentioning that the antithesis between the conceptual and computational ways of doing mathematics is not new in mathematics education discourses. In discussing its long history, Walkerdine (1998) talks about the conceptualisation of two different ways of knowing and thinking:

> This distinction is a recognition of the fact that when people wish to complete some practical task successfully they may do so simply by following rules, by applying a procedure, but still have little idea of why the rules are effective, or of their range of application. On the other hand, people who apply a procedure and at the same time know its rationale may have a deeper understanding of the meaning of what they are doing and why the procedure works. (p. 33)

This distinction has appeared under various terminologies in mathematics education discourses. Walkerdine (1998) herself mentions "basic skills" or "computational techniques" being counterposed to "understanding", Buxton's (1978) distinction between different levels of understanding, and Skemp's

(1976) contrast between "instrumental understanding" and "relational understanding". In this section, my focus will be on exploring how the *elements* of conceptual and computational ways of doing mathematics become *moments* in Alex's discourse: I will explore how these moments are fixed through the course's various communicative events.

The course's curriculum entails the distinction between conceptual and computational learning: the way in which the textbook and the lectures are organised relates to this distinction. In particular, the concept of the derivative is studied in two chapters of the textbook. The first chapter defines the derivative as a limit, provides an interpretation of the derivative as a rate of change, introduces the derivative of power functions, introduces the derivative of second and higher orders, and discusses (non-) differentiability. The second chapter introduces the rules of differentiation (sum, product, quotient, and chain rules), the derivatives of standard functions (polynomials, exponential, and trigonometric functions), and implicit differentiation. In his lectures, Alex follows the organisation of contexts as introduced by the book. In doing so, he refers to these two chapters as focusing on the conceptual and computational aspects of the derivative respectively. The *"make sure that you understand these two chapters" example* takes place when Alex introduces the derivative:

> Alex: Ok, so we're starting the derivative, so I want to tell you a little bit about what comes next in chapter [n] and chapter [n+1], so that you know what is the plan that we're following. Chapter [n] *[Alex starts writing on the board]* is, like, the derivative conceptually. So, in this one, what happens is we're going to talk about what the derivative *is* and what does it *mean* in general. But the rules on how to use it and how to compute it is going to be chapter [n+1]. So, they come, like, together somehow. So, we're going to have a lot of mixture between these things. Is like computing; this is… *[Refers to what is already written on the board]* computational aspect. *[He writes it on the board]* So, these two chapters are very important in the sense of- this is the main topic of the course to learn, the derivative. So, make sure that you understand these two chapters; they're the core of the course what happens in these two. (Alex's class, p. 24; Lecture 5)

In this example, Alex introduces the structure of the textbook (and part of the course's outline) to students and conceptualises it in terms of the distinction between the conceptual and the computational knowing of mathematics. While doing so, the conceptual aspect involves "what the derivative is and what it means", while the computational aspect refers to "the rules on how to use it and how to compute it". In this example, the derivative as a mathematical object is constructed as entailing both aspects, which are presented as complementary with regards to learning the derivative.

The importance of the conceptual/computational distinction as a moment in Alex's discourse becomes evident also with regards to the assessment of students' learning. In the last physical class, as there was the expectation that everything (including exams) would soon move online, Alex proposed students a plan for moving forward. Among other things, this plan aimed to deal with the situation that no in-person final exam was expected to take place. Alex proposed that students vote for and consequently take an emergency in-person midterm exam before all activities move online. This exam would entail computational questions, while the final exam (which would be a take-home exam) would focus on conceptual questions. The rationale behind this proposed emergency midterm was that the computational questions could be answered with the use of some mathematics software, like mathematica, and would therefore be unsuitable for the final take-home exam. Below, in the *"the questions that we want to put in this midterm" example*, Alex explains the rationale of this exam to students:

> Alex: The questions that we want to put in this midterm are the questions that you would be able to put in mathematica. And the point is that, because it's not conceptual, there is no way, in which we cannot- in which we can avoid that someone does that. So, we want to make sure that the computations, you actually do by yourself, without the aid of calculators or all this sort of things that we would not allow in our usual test. (Alex's class, p. 99; Lecture 9)

The changes, as a response to the pandemic, were more rapid than initially expected and everything moved online, earlier than expected. Therefore, this emergency midterm exam never happened, but I do not have data regarding how the professors of this course handled this situation. Nevertheless, the idea of this emergency midterm and the rationale behind it are important with regards to how the distinction between the conceptual and the computational mathematical knowing is articulated in this course. In particular, the distinction between conceptual and computational emerges in a situation, which constitutes a sudden – and violent – break with the "normal". The unprecedented circumstances related to the first wave of the COVID-19 pandemic in Canada led to rapid changes in the lives of students and academic institutions. In this context, in which many things became fluid and uncertain, the computational/conceptual distinction remained intact. We can, thus, see the plan for an emergency midterm as an attempt to maintain aspects of the "normal", which are being evaluated as important. Ensuring that students can *themselves* complete computational tasks is, therefore, evaluated as important in this course.

Furthermore, in this communicative event, the dimension of surveillance related to the assessment of knowledge emerges. The surveillance in showcasing conceptual understanding is assumed to be unnecessary (or at least easier): there are no concerns raised regarding students cheating during the final exam, while working on conceptual questions. However, the need to surveil the assessment of computational knowledge is maintained because students could use software. This signifies that, in this discourse, it is easier to cheat when computing but not when showing understanding. Thus, there is a noteworthy parallel between the surveillance related to the assessment of students' learning (making sure that they *really* know what they are supposed to know) and the conceptual/computational distinction as discussed by Walkerdine (1998), who maintains that the "conceptual" knowing is conceptualised as "deeper knowing" by mathematics education discourses.

In particular, this event is of importance because it shows how technologies and software like mathematica perplex and challenge the norms of instruction in calculus (e.g., Borba & Villarreal, 2005). Alex often talks about "machines", in order to refer to software (e.g., mathematica) which can, among other things, compute a limit, find a derivative, or graph a function – tasks, which students are often asked to complete as part of their homework assignments and exams, and which Alex himself presents in the lectures. On several occasions, Alex talks to his students about why the class works on completing computational tasks, although the same tasks can be completed almost immediately with the use of software. The *"just to understand how it happens" example* is such an instance:

> Alex: Now, these kinds of things here are things that computer programs can do very fast. Like in modern life, you wouldn't be given something like this to graph. But we're gonna do it, just to understand how it happens. But most of these things, you just plug and you can see, actually like, the graph itself. (Alex's class, p. 35; Lecture 5)

Here, Alex talks about the process of sketching the graph of a function by hand. He mentions that this task will not be asked by students in "modern life", as this task can be completed through mathematics software. Nevertheless, the class will work on it, in order for students to understand how it happens. There are two different practices, which are compared here: drawing a graph by hand and through a software. Drawing a graph with the computer is enunciatively conceptualised as an easy task, through the use of the adverb "just" ("you just plug"). Further, it is a task that is not completed by hand "in modern life". The phrase "in modern life" introduces an implicit comparison with how "life" was in the past, by drawing attention to how technology can facilitate or expedite complicated tasks. On the other hand, the practice of drawing the graph by hand is conceptualised as being able to facilitate understanding ("we're gonna do it, just to understand how it happens"). Therefore, through the comparison of the two ways to draw the graph of a function, we have a distinction between understanding (what one is doing) and doing things easily and fast through a computer.

In a previous lecture, as shown in the *"it is always important to know what you are doing" example*, Alex elaborates in more detail about the reasons why the class works on computational tasks that can be completed by a computer:

> Alex: Now, of course from the perspective of today's technology and all this kind of things, this kind of things can be done by mathematica and all this sort of things. But, it is always important to know what you are doing. Even if the only thing that you are going to do is to put it into the machine so that the machine does it for you, it is important that you know what is happening, so at least you have the intuition to know what you are expecting to find. Because the point is that when you do programing, or when you do things with machines, when there is a mistake, the person who is looking for a mistake are people. So, if you have absolutely no idea how to do it, you are never going to find the mistake. Or worse, sometimes things that are mistakes do not look like mistakes. You have to have certain intuition, what are you expecting, so that maybe you can suspect that something is wrong. Even if it doesn't look wrong. So, even though, I agree, you can do all these things by machines or by computers and that's a really important thing, do not disregard this aspect; not for you to become experts and not use the machines, but to always be able to suspect that something looks suspicious. (Alex's class, pp. 11–12; Lecture 4)

In this example, Alex frames his discourse in antagonism against a discourse that claims that mathematical tasks can be completed exclusively with the use of software. However, this antagonistic discourse is not articulated by a different voice in the course. For example, during my visits to the course, no student argued that machines can do everything or that certain aspects of the curriculum should not be taught because they can be completed by a machine. Instead, the two antagonistic discourses are both articulated by Alex himself. The most salient dialogical relation between the two discourses in this communicative event is the phrase "I agree" in the last sentence of this extract. The phrase "I agree", usually employed when two or more people talk with each other, is here used in a monologue to show partial acceptance with an antagonistic discourse introduced by Alex himself. This event shows how Alex's discourse is polyphonic. It is also an example of how the political has ontological priority as antagonisms between discourses are productive (Laclau & Mouffe, 1985/2001). In particular, Alex's (intended) discourse becomes what it is, through the polyphony and by responding to a discourse which it antagonises.

As Alex's discourse conceptualises doing mathematics through and with mathematics software, it also frames what conceptual knowing is. While the computational task might be completed through the machine, students as emerging professionals are expected to "know what [they] are doing", "know what is happening", and have an intuition to "suspect that something is wrong" – all conceptual processes.

While in the "it is always important to know what you are doing" example, the notion of identifying a mistake is prominent, the notion of interpreting a result is also important in Alex's discourse as shown in the *"the interpretation of the computations, that, cannot be done by a machine" example*:

> Alex: Even if you do not follow every step of the computations, that's not the most important thing. The most important thing is to understand what these computations mean. The computations now can be done by a machine. But the interpretation of the computations, that, cannot be done by a machine. That has to be done by a person that does this thing.
>
> So, if you don't understand what the computations mean, even if you get to the computations, you cannot give an interpretation which is very important. And it is very important because actually one of the most important problems that we have now in many experiments that happen is that people can do many computations but they are unable to give interpretations. The problem is that if you are unable to give interpretations, fine. But sometimes it's even worse. Sometimes you give wrong interpretations. I mean the interpretations are wrong because we don't understand what all these things actually mean. (Alex's class, p. 36; Lecture 5)

Here, the idea of interpreting a result is introduced as part of conceptual knowing. In this way, the concept of "understanding" that we saw above (having intuition about the problem, being able to identify mistakes) is expanded to include the ability to interpret results. Interpreting results is introduced as an activity which takes place in scientific contexts, as the example of experiments is used. In this way, doing science is conceptualised in terms of the computational/conceptual distinctions, by prioritising the conceptual: the (good) scientist is the person who has a thorough conceptual mathematical ability.

Another reason why these two examples, the "it is always important to know what you are doing" example and the "the interpretation of the computations, that, cannot be done by a machine" example, are important is that they contribute to creating available subject positions for students. Students are positioned in the discourse, in ways in which they use mathematics to solve a problem that might or might not be mathematical. The role of the students as future professionals is to be able to use software in order to solve specific tasks; an activity which includes being able both to identify mistakes in the process and to be able to interpret the results correctly.

Although in the two previous examples, the conceptual is prioritised over the computational, Alex also attempts to fix the meaning of the computational as important in various incidents. For example, after having introduced the rules of differentiation to students, he makes a point about the importance of repeating the rules in order for students to learn them and be able to use them in computations. This is evident in the "repetition is very important" example, in which developing computational skills through constant practice is also deemed as important. In this example, the relation between the two aspects of mathematical knowing is shifted, attaching the adjectives "fine" and "important" to conceptual and the computational knowing respectively. As this class focuses on the computational aspects of differentiation, we notice that the discourse also shifts to conceptualise this way of knowing as important.

The distinction between conceptual/computational appears in a later class under the name "computational-theoretical". In this class, Alex introduces the following definition and theorem, which he writes on the board:

> For a function a critical point is a point where the derivative is zero.
>
> Theorem: Suppose $f: [a, b] -> R$ is continuous and differentiable in (a, b). The maximum and minimum exist and they occur within the critical points or in the borders.

Talking about the importance of this theorem, Alex mentions in the *"the maximums and minimums exist, that's the first thing" example*:

> Alex: So, the maximums and the minimums, maybe they are more than one, but the maximums and minimums *exist*, that's the first thing. So, from a theoretical point of view, this is important too because, well, we know that it exists. From a computational point of view, this means that if I do the algorithm that I'm going to explain, we're gonna solve the problem. Like a priori there is no reason for why we are going to solve the problem; but the point is that this theorem tells me that if I am looking for maximum and minimums, they will exist. As long as I am looking for them in closed intervals. (Alex's class, p. 111; Lecture 9)

The importance of this theorem is conceptualised in two ways: theoretically and computationally. Theory provides certainty: it offers the certainty that the problem does have a solution (the maximum and minimum of the function do exist). Computationally, the theorem offers an algorithm, which acts as a guide for how to solve the problem of finding these objects that exist.

This communicative event is important because it adds a new layer in the previous conceptualisation of the distinction: not only does the conceptual/computational distinction describe two aspects of doing mathematics, but it also depicts two aspects of mathematical knowledge. In the three previous examples, we saw the conceptual being prioritised over the computational, by referring to the importance of having an intuition and of being able to interpret the result, as well as the computational being prioritised, through a reference to the importance of technical competency. Both of these kinds of examples refer to people doing mathematics. Here, it is mathematical knowledge that is depicted as having both these elements: references are made to the existence of mathematical objects and the existence of algorithms that can be used to find these objects. It is not only what *people* do with this knowledge; it is what the knowledge itself is about. The *"the chain rule" example* is similar in that sense:

> Alex: the chain rule has two sides: it's a computational tool but also a theoretical tool (Alex's class, pp. 90–91; Lecture 8)

Here, the focus is on the chain rule (i.e., on mathematical knowledge), which is conceptualised as an object with two purposes: a computational and a theoretical one.

Overall, in Alex's discourse, the conceptual is about what a concept is or means and is related to developing intuition, interpreting results, and developing theoretical arguments, while the computational focuses on technical aspects of a phenomenon and is related to solving problems. These two elements constitute what knowledge is: mathematical knowledge is conceptualised through these two aspects. Both aspects of mathematical knowing are conceptualised as necessary. Nevertheless, it is also clear that in this discourse, students are positioned on the side of the computational; they are the ones that will use the conceptual ideas to do and evaluate or interpret computations. Although this antithesis might not be very hierarchical, it guides student learning in a particular way.

Although the conceptual/computational distinction is very prevalent both in Alex's class and more generally in mathematics education, following the idea of Laclau and Mouffe (1985/2001) regarding the contingency of meaning, I argue that there is nothing essential about the distinction between conceptual and computational aspects of mathematics learning and doing. In other words, there is no a priori way why doing mathematics needs to be separated in conceptual and computational aspects. Ethnomathematical discourses, which conceptualise mathematics as an internal and organic part of human activities (e.g., Abtahi & Wagner, 2016; Lunney Borden, 2011), are an example of how the distinction between conceptual/computational aspects of mathematics is not the only way of thinking about mathematics.

The focus on these two aspects can be understood by reflecting on the characteristics of the social practice in which this knowledge study takes place. This calculus course (typical of Canadian universities) aims to teach calculus to a diverse (as far as their academic profiles are concerned) group of undergraduate students. These courses are designed with increased input from non-mathematical faculties, while still offered by the mathematics faculties. In this sense, it could be argued that the conceptual/computational distinction is a facet and result of the (perceived) interactions between different social actors: mathematics professors and professors of other departments who find it important that their first year students learn calculus. Hence, the distinction articulated in the course and reflecting the social practice of collaboration between different faculties in a university, relates to the reproduction of a particular political economy in relation to mathematical knowledge.

6.4.2. The Construction of a Shared Space

In this section, I refer to the way in which Alex's discourse constructs a notion of a shared space in which the class works. This refers to the discourse that the calculus class constitutes a shared space, not only physically but also at the imaginary level. This shared space is based on the knowledge that the class shares and, as such, is allowed to use.

As an introduction to the discourse and value of shared space, we can see the *"so far, we don't know this" example*, which takes place when the class works on limits:

> Student: For questions like this, are we allowed to use L'Hospital's rule if we know how to do it?
>
> Alex: Not so far, because we haven't seen- At the point, we haven't seen derivatives even. Once, we have seen derivatives, you'll be allowed to do L'Hospital's rule. And it will become very easy. But so far, we don't know this, so, so far you have to rely on this kind of things. (Alex's class, p. 15; Lecture 4)

In this example, a student asks if they are allowed to use L'Hospital's rule for calculating limits, a rule which uses the notion of derivatives that the class has not yet worked on but students are familiar with from high school calculus. Through their question, the student creates a distance between the professor and the students and probably refers to situations in which students are evaluated by the professor. Indeed, the enunciation of the student's question (bluntly asking if they are *allowed* and not, for example, if they can) positions the professor as the one (not) allowing specific (mathematical) student behaviour. On the other hand, the professor responds with what "we" have seen, but now this "we" does not refer to the students who become a group because they share an identity and situation. Instead, the "we" refers to the class and its activities, in which the professor also participates. Fairclough (2003) describes how the inclusive "we" "reduces hierarchy and distance by implying that all of "us" are in the same boat" (p. 76). This is something that we can see in this shared space. The "we" of doing mathematics in the shared space includes the professor, the students, Euler and Laplace, Euclid and Thales. In this way, the shared space of the class is constructed.

Another example of similar nature happens in the last physical class, when the professor presents aspects which are relevant to the emergency midterm and the final exam. The *"if you know it, it is useful and you can use it" example* happens as a student asks what they need to know in relation to trigonometry:

> Alex: That's always, that's always a delicate point. Uhm, in the sense, like, what would you need to know. Well, you need to know that trigonometry that we have more or less talked about. Which is basically how do you define the sine and the cosine. And the implications that that has. So, like, for these ones, it may be useful, like, we do make an effort that you don't need to use these things. But, that doesn't mean that it's not useful that you know the, like, basically this one. *[Writing down the trigonometrical identity $sin^2 x + cos^2 x = 1$]* Maybe this one. Eeh. Like this kind of things. Like these things are useful to know. They might make your life easier. And they, they can be seen from the drawing. This is because they are in the circle, and you can see these from the graphs. Then, there are all these double angle formulas and these sort of things. That, if you know them, you can use them, but we are not going to put things that necessarily need that. So, you should be able to do without these ones. I'll write it just in case. But- *[Starts writing]* Like, probably, you have seen these in high school or something. If you know it, it is useful and you can use it. But we make an effort to not put things that you need to use these by force. Like- So, if you don't know them, and don't want to get in the trouble to learn them, it's fine. The only one that you should know is this one. This one is, this one is the real important one. *[Shows the $sin^2 x + cos^2 x = 1$ identity which he has written on the board]* (Alex's class, pp. 101–102; Lecture 9)

This example is similar to the "so far, we don't know this" example, as in both examples students are unsure and ask about the mathematical knowledge that they can and cannot use in an exam. Moreover, in both examples, there is an appeal to the shared space by Alex, as he focuses on the kind of material that the class has studied and discussed. At the same time, this example is different to the previous one. A significant difference with the previous example is that while students were not allowed to use L' Hospital's theorem because the class has not studied it, they are allowed to use trigonometric identities which they have seen in high school but have not discussed in the current course. So, a question that arises is how the two situations are different. Trigonometric identities are considered to be a part of pre-calculus; a body of knowledge which students are generally expected to be familiar with when studying calculus. When the class operates within a shared space, where only the discussed mathematical knowledge is "allowed" to be used, this refers to calculus knowledge. In this way, the in(ex)clusive characteristics of the "shared space" become visible, as to be able to participate in this shared space a person needs to meet certain requirements.

The construct of the "shared space" also relates to truth and progress as can be seen in the *"for the moment, we haven't seen how to do that" example*. In this example, Alex has just presented the class with an exercise: graphing the function e^{sinx}. However, Alex has not yet presented the derivatives of trigonometric and exponential functions; they have only seen the derivatives of polynomial functions.

> [1] So, you can use that $f'(x) = e^{sinx}cosx$ and $f''(x) = e^{sinx}\cos^2 x - e^{sinx}sinx$.
>
> [2] So, the point is that later on in the course, I'm just gonna give you this one *[He points to the original function]*
>
> [3] and you're gonna have to do these by yourselves.
>
> [4] But, for the moment, we haven't seen how to do that.
>
> [5] So, we're gonna believe this thing. (Alex's class, p. 55; Lecture 6)

In this example, Alex presents the class with the first and second derivative of the function $f(x) = e^{sinx}$ as the class has not already seen the derivatives of exponential and trigonometric functions. Alex lets the class know that later on, after they have learned all derivatives, they will need to complete this step themselves. For now, they are going to "believe this thing". As finding the derivative of exponential and trigonometric functions is a typical part of the calculus university course in Ontario and around the world, it is reasonable to assume that the majority of students (if not all of them) have been taught how to differentiate exponential and trigonometric functions in high school. However, they all consent to *believe* that the first and second derivative of the function e^{sinx} are the ones proposed by the professor.

The professor here discursively constructs a shared space. One source of this shared space could be claimed to be the pedagogical condition. The very educational setting is constructed in a way that certain things are assumed to be known and certain things have to be learned. This can be interpreted as being part of a relatively democratic discourse about learning, within which not too much student background knowledge is assumed; the prerequisites are kept to a minimum.

However, there is something more than this, that is more particular to doing mathematics. Often in mathematics courses, students are only allowed to use what has been taught or agreed upon. But this is not common practice across academic disciplines. For instance, in an ESL class, the instructor would not ask students to not use present perfect continuous because they have not learned it yet. Thus, I suggest that this shared space relates to doing mathematics (and not merely to learning mathematics).

In the "for the moment, we haven't seen how to do that" example, the construction of this shared space is enunciatively enabled through the use of pronouns. In the first couple of sentences [1-3], first person singular and second persons are used. The roles of the professor ("I") and students ("you") are distinct and clear: the professor gives mathematical tasks to the students [2] and the students have to complete these tasks [1, 3]. The professor also defines the limits of what the students can and cannot use when doing mathematics. In [1], the professor extends the range of things that students can use. Later [4, 5], the first plural person is used ("we"), suggesting a (differently than before) shared space. Within this space, the "class" perform(s) more abstract tasks: they "have seen" and "believe". This enunciative turn signifies an imaginary element in this shared space: it is not strictly the space that is created by the educational framework (within which all these people are in the same room and perform activities related to this framework, such as assigning homework, completing homework, writing on the board, asking questions, etc.). Instead, there is a shared space which emphasises a common, shared attitude towards mathematics, what is regarded as common knowledge, and what is regarded as true.

In the shared space that is constructed here as a concept (and value), everybody has seen the same things and *believes* the same things to be true. The shared space relates to the production of knowledge and truth: doing mathematics, in this discourse, means believing the same things and having a common frame of reference. In [1], articulating a mathematical statement and suggesting that students can use it, implies that the statement is true.

In line [5], Alex uses the verb "believe", introducing in this way "believing" as a concept that relates to truth. Up to that moment, the parameter of truth is not articulated (although the derivatives in [1] are assumed to be true, this is only assumed and not explicitly articulated). This is supported both lexically and grammatically. No words are used that relate to truth or its examination; there is no hedging; the phrase "We haven't seen how to do that" in [4] implies that differentiation exists outside this class and as such it is probably true. To put it differently, lines [1-4] do not leave any room for negotiation of whatever has been said on the basis of truth.

Through the use of "believe", the parameter of truth enters the utterance. However, the emphatic tone of the phrase "We're gonna" continues to not allow space for negotiation, while the verb "believe" itself further builds on this idea being true (regardless of what the class thinks about it). The non-existence of space in which to negotiate whether something is true or not (or whether to believe that something is true) relates to power. Truth is not negotiated anywhere in the extract, nor is there space to negotiate whether the proposed mathematical statements are true or false. The truth of these statements is constructed as guaranteed, regardless of what happens in this particular classroom. Seeing “how we do this thing” in the future will, in no way, challenge the truth value of it, nor will it change its content in any way.

Although the statement is true, lines [4-5] imply that using something without “having seen it” is a break from the norm. As far as values are concerned, the idea is that accepting that something is true because someone tells us so, is not the desirable method, even if this someone is the professor themselves. Instead, it is within the shared space that truth is decided, negotiated, and accepted.

Finally, another value that manifests itself in this example is progress. A continuity within the course is constructed, through the phrases “later in the course” [2] and “for the moment” [4]. Within this continuity, there is progress in students’ knowledge. Students see new things in the course and once this happens, they perform these things themselves (instead of for example, being given the answers by the professor or the textbook). Progress here refers to the progress of the individual students, but it also connects with the progress of humanity in mathematics. Moreover, progress might be seen as a facet of a particular discourse about learning and doing mathematics, which: is solitary (the individual person needs to know and complete by themselves all tasks required to reach a result); closely relates knowing something and performing it to *show* that you know it; and sees learning as progressively adding up new knowledge to what one already knows.

The notion of progress and the idea that mathematical knowledge progresses smoothly are also evident in an example discussed earlier in this chapter, the “the people who started doing these were Newton and Leibniz” example. The process of “discovering” these rules is assumed to have happened through a kind of pre-determined path; there is something natural about how things have been discovered. This is manifested in three utterances. First, Alex talks about the discovery of the sum and difference rules as being natural and self-evident (“of course they got this one”, “these two are pretty simple”). Second, wanting to find the rule about the product is also conceptualised as natural, both through the use of a rhetorical question (“But then, what other thing can you do with functions? well, you can multiply them”) and through the cohesion that is implied when multiplication follows addition and subtraction (which in turn is a historical product of the theory of didactics and curricula around the world, which have conceptualised the existence of four operations that come in the order addition – subtraction –

multiplication – division). Third, this example constructs the concept of a shared expectation regarding how the multiplication rule would look ("it is not what you expect, in the sense of maybe you expect that the product of derivatives is the derivative of the product"), which is shared not only by the students but also by Leibniz himself. I argue that through these three utterances, knowledge – seen here as the journey towards truth – is constructed as well-organised and cohesive.

Overall, I propose that the shared space is a construct which has the function of eliminating any antagonisms, exactly because it constructs an agreement about the kind of mathematical discourses that can and cannot be expressed. Nevertheless, a tension might be identified between a shared space on the one hand and authority lying with the professor on the other. In the "for the moment, we haven't seen how to do that" example, the professor is the one who gives permission regarding what tools can be used [1], assigns the exercises [2], determines what tasks students have to perform [3], establishes what is considered to be common knowledge within the shared space [4], and decides what is going to be believed within the shared space [5]. Similarly, in the "so far, we don't know this" example, the student asks if they are "allowed" to use L'Hospital's rule, assigning to the professor the role of deciding which mathematical knowledge students are (not) allowed to use. In this sense, the construction of a "shared space" regulates the behaviour of students, while it is also a construction that does not happen collectively but is imposed by the professor as the representative of the institutional framework of the course.

6.4.3. Truth as a Value in Mathematics

In Alex's course, truth is a central, yet implicit, value. By truth, here, I refer to a discourse, within which mathematical knowledge is both given the status of being true and is concerned with truth. Further, I argue that in Alex's discourse, mathematical truth is unnegotiable and independent of those who prove it or articulate it. Truth is a very central value in this course, although it is not always explicitly articulated. In the lectures that I attended, there were not many explicit references to truth or any discussion about what it means for a mathematical statement to be true or false. As Ernest (2016) comments about truth in mathematics, "[t]he superiority of truth over falsehood and knowledge over mistaken belief are presumed to be self-evident" (p. 189). In this section, I discuss how the value of truth manifests in Alex's class in an implicit way.

The notion of truth is evident in the way in which Alex talks about mistakes. The notion of mistakes assumes that there are right and wrong ways of performing mathematical actions, while it is unquestionable that right ways are the desired ones. We saw this discourse in examples discussed previously in this chapter, such as the "the people who started doing these were Newton and Leibniz" example, the “if you make mistakes” example, the “please learn the quadratic formula” example, the “You have to substitute correctly of course” example, and the “it is always important to know what you are doing” example.

A similar discourse about truth is articulated in the textbook used in the class. In the textbook, the word “truth” never appears but the word “true” appears in 127 instances. References to truth are primarily to be found in textbook exercises, where truth is in direct opposition to falsehood. More specifically, in many exercises, students are presented with mathematical statements and are asked to decide if they are true or false, explaining their answer. This type of question suggests that truth in mathematics has an absolute character: any statement in mathematics is either true or false.

Generally, in Alex’s discourse, mathematics is about truth. Although Alex himself does not use the word “true”, he uses the word “real” to signify a similar notion. For instance, in *“the real number” example*, he mentions:

> Alex: And the idea is that as I keep going, this number will start approaching something. Like, I obtained 2, then I obtained 3, now 3.5. And if I pick a closer number, I should get something, even closer to whatever the name, not the name, the real number is. (Alex’s class, p. 5; Lecture 4).

In this example, “the real number” signifies that the question that is being discussed has a single, right answer that can be approximated through the process followed by Alex. This notion of a single right answer (along with the class’s engagement with problems that have a single right answer) produces a certain narrative about truth, within which truth is associated with the existence of a single true answer to a problem.

In Alex’s discourse, there is trust that the mathematical system within which the class works is consistent and produces true statements. This is evident in the *“That’s why it’s called implicit differentiation” example*.

> Alex: Is this clear? Like what I did? Like, what I’m doing is derivating as if I knew that y is a function of x. Which it is. But the point is that I don’t know which one. I don’t know its specific form. That’s why it’s called implicit differentiation. Like, the way in which y depends on x is implicitly given by that equation but I might not be able to extract it in an explicit point. Bit still, I can, I can derivate. And the whole point is that I need the slope. The thing is that I can solve for the slope. That means I can solve for y prime. So, let’s solve for slope. (Alex’s class, p. 105; Lecture 9)

In this example, Alex explains the process of implicit differentiation. In his description, there is an assumption that there is something "more" behind the equation that is known: this "more" is a differentiable function. Of course the existence of such a function is not a mere assumption, in the sense that its existence can be proven through calculus although the class does not engage in this process. Nevertheless, within the discourse that is articulated in the class, a notion of truth about the equation and the function is constructed, which is a truth that cannot be fully checked by the students themselves. Hence, truth is also constructed in an imaginary way, through the belief that it is ensured by the "mathematics of mathematicians".

Overall, truth is a value in Alex's class, although Alex does not explicitly refer to truth itself. Nevertheless, the class focuses on the study of *true* statements, while "mistakes" signify the absence of truth and need to be avoided. More generally, in Alex's class, all mathematical statements are true, unless otherwise stated – in which case they are discussed as mistakes. I argue that through these communicative events, mathematical truth is constructed as being single and absolute.

6.4.4. Formalising Ideas

Formalising ideas is a rationality technique manifested in Alex's class. I argue that formalising happens in two ways in Alex's class. First, it manifests as the geometrical representations of calculus ideas are formalised into algebraic representations. Second, it manifests in relation to the distinction between the mathematics of the class and the mathematics of mathematicians (discussed earlier in this chapter) and more specifically as the mathematics of mathematicians are constructed as a formalised version of the mathematics of the class.

First, I will discuss how formalising happens in Alex's class, i.e., in relation to the mathematics of the class. When talking about mathematical concepts, Alex employs multiple representations such as drawings and representations of function movement using his arms. These representations, that constitute a more geometrical way of thinking about calculus, are formalised into algebraic representations. The *"the slope we are looking for is the limit!" example* is such an instance:

> Alex: So, the slope, the slope we are looking for is the limit! *[Writes on the board]* Is the limit as I do this, is the limit of the slopes. So, that's why we wanted to compute a lot of limits. Because those limits would appear. They appear when we're computing this sort of things. (Alex's class, p. 5; Lecture 4)

This example constitutes a transition from geometrically approaching a point on the graph of a function to algebraically describing it using the concept of the limit. This is a process of formalisation within the mathematics of the class, in the sense that the algebraic description is briefer, it contains almost exclusively mathematical language, it avoids ambiguity, and it is more precise.

Another example in which a geometrical idea is formalised through the use of algebra is the *"let's try to write the limit in all its glory" example*:

> Alex: So, let's try to write the limit in all its glory. So, let's do the drawing again, but now, let's do it like this. So, we are like this, we are going to say that this is $(2,4)$. *[Marks the point on the graph]* My scale is like terrible, but never mind. Here, $(1,1)$. *[Marks the point on the graph]* So, what are we doing? What we are doing is, this is 2 and what are we picking? It's a point. Some point. So, how do I describe this point? I want to describe this point according to how close it is from 2. So, you have to imagine that you are at 2 and your other point is 2 plus/minus something. Like, it is plus something if I am to the right or minus something if I am to the left. Right? Like. And that's what I want to write. So, if I am here, how do I describe this thing? Well, this is two minus this thing, whatever this is. So, this case is like minus some h. But, if I was here, well this number is 2 plus this thing. Right? Something like this. So, that's how I want to describe the thing. I want to be centred at the point that I am interested and how I move, I describe it by how far I am. So, this "$+h$" is exactly how I am moving. So, I want to erase this one here. *[erases the board]*
>
> So, a general point will be described as *[writes on the board]* "$2 + h$" and I allow h be either positive or negative according to whether I am to the right or to the left. And then, the value of the function. So, it will be $(2 + h)$ squared, in this case. Right? So, in my drawing if I say that this is "$2 + h$", in this case, h would be negative, this point here is $(2 + h)$ squared.
>
> And I do the same thing. So, for example, for our first, point, h was two. For the one I erased over there, h was one. For the one I erased here, h was one half. Right, that's the, the, … negative, right? Minus one, minus two, minus one, minus one half, that's what I was doing. So, our general point is described like that.
>
> And now I compute the slope. So, it's the slope of the line that goes through this point and the point I care about, that is $(2, 4)$. We just write the formula of the slope, that's the formula *[Writes the formula on the board]*. (Alex's class, pp. 5–6; Lecture 4)

In this example, the geometrical process of approaching a curve through its secant lines is formalised as the limit of the slopes of the secant lines. Thereby, the derivative is described algebraically. Formalising is also achieved through the generalised formula for the coordinates of a random point. Alex engages in an algebraisation of the random point by algebraising its distance to the point at which the slope of the function is studied. This happens by noticing geometric aspects of these points and describing them in algebraic language (e.g., whether a point is to the right or to the left). This process continues until a formula for the slope of the function is derived. In this way, the definition of the derivative that is going to be used by the class is produced. This process of formalising through algebraisation produces general descriptions of the studied phenomenon; this generality is valued as is signified by the fact that these general definitions, after being derived in this way, are the ones that the class uses to work on slope/derivative problems.

In the previous two examples, the rationality technique of formalising was applied using the mathematics of the class. However, in Alex's discourse, the mathematics that the class has not studied as such, the "mathematics of mathematicians", are conceptualised as a formalised version of the mathematics of the class. This idea of formalising relates a lot to definitions, theorems, and proofs. Starting from proofs, Alex's class is a proof-free course. The major function of proofs in mathematics (at least according to more traditional academic mathematical discourses) which refers to verifying truth is derived by other argumentation techniques (e.g., identifying a pattern). Nevertheless, the "mathematics of mathematicians" along with its definitions, theorems, and proofs is evident in the class as a body of knowledge that formalises the "mathematics of the class".

An instance of this is the "the people who started doing these were Newton and Leibniz" example, discussed earlier in this chapter. In that example, Leibniz's rule derives its name from Leibniz being the one who (first) proved the rule. In this way, proving is constructed as a vital part of mathematics, especially considering that the rule is named after the person who first proved it. Moreover, the rule that students have learned and use in the course is conceptualised in relation to its existence as part of the "mathematics of mathematicians", which entails more aspects of the rule, not known to students, such as its proof.

Theorems are also seldom used in the class. Sometimes, the professor refers to theorems and sometimes, when writing on the board, he uses this word to label an important result. The *"the conclusion is an important theorem" example* is an instance of theorems that are mentioned as such in Alex's class:

> Alex: So, this is an important thing. So, the conclusion of all this that we can make. So, the conclusion is an important theorem. We're not going to prove the theorem, rigourously or anything, but it's making this drawing precise. Is the following. The theorem is. If f is differentiable at A, then it's continuous at A. So, what does that mean? It means that if I know that a function is differentiable at the point A, I can evaluate the function at the point A, by taking limits. (Alex's class, p. 45; Lecture 6)

Similarly, the *"The answer is a theorem" example* takes place in the following lecture and refers to the same theorem as the "the conclusion is an important theorem" example did.

Alex: how do we know when the function is continuous? The answer is a theorem. (Alex's class, p. 9; Lecture 7)

In these examples, Alex refers to the theorem that a differentiable function is continuous. The very labelling of this proposition as a theorem elevates its mathematical importance, exactly because the word "theorem" is seldom used in this class but does belong to the imaginary of the "mathematics of mathematicians (i.e., students know that mathematicians work with theorems). Moreover, when presenting the theorem, in the "the conclusion is an important theorem" example, Alex refers to its proof; another aspect of mathematical knowledge which is not studied in the class but belongs to the "mathematics of mathematicians". Hence, the "mathematics of the class" is constructed as being formalised into the "mathematics of mathematicians", which involves theorems and proofs.

Definitions are also mentioned in the class as part of the mathematics of mathematicians. In the *"we do not actually have a good definition" example*, Alex introduces the derivatives of e^{x}, $sinx$, and $cosx$ and refers to how the definitions available to the class (unlike the definitions available to mathematicians) are unsuitable for proving the formulas for the derivatives of these functions.

Alex: The derivative of x^r is rx^{r-1}. This one, we discussed a little bit about why it is true. We did it for particular values of r. And we said it is true always.

Now, the other two, at least for the purposes of what we're doing in this course, we're gonna state that, but we can't actually prove it. The reason for that is because we do not actually have a good definition of what the sine and the cosine is. Like, we know what the sine and the cosine are; we can compute an angle for triangles, or we can see that the unit circle and we know that the sine and the cosine correspond to certain entries in that unit circle. Those are valid definitions, but they are not good definitions to compute the derivative. So, the [inaudible] which is here to compute the derivatives of the trigonometric functions is once you see integration. Which is the, like, the topic of the next course. Not this one. But because of that, we're gonna state the rules, but we're not really going to prove them. We're gonna believe that it is true. So, the derivative of the sine *[Starts writing]* the derivative of the $sinx$ is the cosine.

[He continues writing the other derivatives]

Now, again, we do not know… Ah, not, it's not that we do not know. We do not have yet a good way to define the exponential function that will allow us to see why this is true. The reason why [inaudible] is because for the exponential, for the sine, for the cosine, like, if you want to give a good definition, like meaning a mathematical sensible definition, of the sine, cosine, or the exponential, you need to use the inverse functions. Not… So, you would not, you would not use the sine; you would use the arcsin. And you would not

> use the cosine; you would use the arccosine. You do not use the exponential; you use the logarithm. And you need to use certain definitions of these three functions that require integration. So, we're not ready for that. We're going to accept these ones as rules. But the main point is that you've seen these four rules here, we can compute everything else. (Alex's class, p. 64; Lecture 7)

In this example, Alex talks about how the rules of the derivatives of sine and cosine are justified through definitions that the class is not yet ready for. This example draws connections between definitions and truth; it conceptualises the "good definition"; and it creates a distinction between the mathematics done by the students and the existing mathematical knowledge or the "mathematics of mathematicians".

As often done in mathematical discourses, here, Alex connects the proof of a mathematical statement (that the derivatives of the exponential, sine, and cosine are the exponential, cosine, and minus sine respectively) with the truth of this statement. Here, Alex says that not having a "good definition" is the reason why they cannot proceed with proving these statements (although it is notable that no "proof" has been presented in this class). Therefore, here, we see that in this (otherwise very common) connection between proof and truth, definitions play an instrumental role (in a way that might be different than school mathematics). Definitions are crucial for enabling the proof of a "fact".

In this communicative event, Alex also sketches two different practices. On the one hand, there is the mathematical practice of this particular class. Here, Alex uses the first person plural pronoun (we) to refer to the class. The class is not ready to go through the "good" definitions and, thus, see why the proposed derivatives are true. Nevertheless, the class knows that they are true; the tools that they use are valid. This suggests that the truth about the objects that the class uses and the existence of these objects (in this case, the derivatives of the exponential and trigonometric functions) do not depend on the definitions that the class uses or has access to. These are guaranteed by the existence of the second practice.

The second practice is the "existing mathematical knowledge" or the "mathematics of mathematicians". To refer to the subject of this practice, Alex uses the second person pronoun ("you"), without addressing a particular student or group of students: it is a generic "you". He, thus, refers to what an imaginary person would do: a "good mathematician" perhaps. In this sense, the second practice sketches the existing mathematical knowledge, much more as an existing body of knowledge and less as a practice that "real people" with material bodies do. Within the "existing mathematical knowledge", there exist "good definitions" for all three functions. I am not arguing that Alex sees mathematical knowledge as something that is not practiced by "real people". Instead, I argue that in this communicative event, by sketching these two practices, mathematical knowledge is constructed within the imaginary field. Further, the mathematical practice of the class is constantly related to this imaginary.

In this example, Alex also constructs the concept of the “good definition”. Good definitions are “mathematically sensible” and they are “good” for performing a specific task, that of showing that the derivatives of the three functions are the ones that are claimed to be. Good definitions play a crucial role in distinguishing the two practices.

Overall, formalising ideas happens in Alex’s class in two distinct ways. First, geometrical ideas are formalised into algebraic ones as part of the mathematics in which the class engages. Second, the mathematics of the class are constructed as formalised by the mathematical practice of mathematicians, a practice which involves definitions, theorems, and proofs that the class does not directly engage with.

6.4.5. Bringing it All Together

In this section, I proposed that the “mathematics of the class and the mathematics of mathematicians” discourse is an overarching discourse in Alex’s course. This means that, in Alex’s discourse there is an overarching distinction between two practices: the mathematics performed in class and the mathematics performed by professional mathematicians. The two practices are not hierarchical: Alex’s discourse values both. Regarding the former practice, mathematics is assumed to consist of a conceptual and computational aspect, both of which are significant for the practice. Moreover, the mathematics of the class occur within a construct of a shared space of the class. Within this discourse, no antagonisms take place; eliminating the existence of antagonisms is exactly one function of the shared space. On the other hand, the value of truth is important with regards to the “mathematics of mathematicians”, which constitutes a formalised version of the mathematics of the class. The construct of the “mathematics of mathematicians” provides rigour and prestige to the “mathematics of the class”, eliminating space for alternative discourses to pop up and get enunciated.

6.5. Summary of Main Findings

In this chapter, I presented the findings from Alex’s “Calculus” course. I argued that three overarching discourses manifest themselves in the course. First, the “humanity of mathematics” discourse proposes that mathematics is a quintessentially human activity. Second, the “applying mathematics” discourse suggests that almost anything in the world can be described through mathematics and, hence, mathematical applications have great power and value. Finally, the “mathematics of the class and the mathematics of mathematicians” discourse draws upon the existence of academic mathematics as a cohesive body of knowledge that constitutes a formalised version of the mathematics of the class and gives rigour and prestige to the mathematics of the class.

Earlier in this chapter, I discussed how a trajectory of three important moments, namely platonism, humanism, and post-humanism, can be identified in mathematics education literature. All three overarching discourses come together neatly under a humanistic framework of mathematics, as they all consider the relation between humans and mathematics. Moreover, the three discourses fit together in a way that hardly any antitheses take place. Overall, Alex's discourse appears to constitute a totality and not entail any antagonisms. This is the premise of the modernist tradition, upon which Alex's discourse draws: within this tradition, mathematics is constructed as an objective body of knowledge which reaches its results through pure rationality and not through the interaction of multiple ideologies, worldviews, or perspectives. Indeed, Alex's discourse does not leave much space for alternative discourses to be enunciated. A few antagonisms do nevertheless emerge in Alex's class, including the antagonism between institutional importances and mathematical importances, the antagonism regarding the role of technology in doing mathematics, and the tension between a shared space and authority lying with the professor.

Chapter 7: Mathematics as Political in the two Courses and Beyond

In the previous two chapters, I presented the findings of this project for each one of the two courses. In this chapter, I engage in a discussion of the findings along with a connection of the findings with mathematics education literature. First, I outline the thesis of this thesis, which I consequently support by bringing the two courses together discussing their similarities and differences and by addressing the three research questions. I then make some connections with mathematics education literature that focuses on the sociopolitical aspects of mathematics education and consequently with undergraduate teaching and learning literature. I continue this chapter by reflecting on the theoretical and epistemological framework of Laclau and Mouffe. I then explore some political implications of this work, I discuss the limits of this project's scope, and I finally pose some questions that arise from this thesis.

7.1. The Thesis of this Thesis

Undergraduate mathematics courses not only teach students how to produce a mathematical proof or calculate the derivative of a function, but the mathematical knowledge of undergraduate courses also entails ontological, epistemological, and axiological aspects. This thesis proposes that mathematics is a nexus of interconnected discourses which are not fixed or stable entities but rather are constantly shifting and contested social constructions. In an undergraduate course, certain discourses manifest themselves weaving the political characteristics of mathematics. In the "Introduction to Proofs" course, I identified four overarching discourses: the formalising as a prototype of rigour discourse, the diverse participation in mathematics discourse, the imaginary practice of mathematics discourse, and the efficacy of mathematical work/labour discourse. In the Calculus course, the respective discourses were the humanity of mathematics discourse, the applying mathematics discourse, and the mathematics of the class and the mathematics of mathematicians discourse. These discourses are specific to the norms that develop in each class, its mathematical content, and its institutional characteristics. Sometimes, these discourses fit nicely together and crystalise the meaning of floating signifiers (like mathematics, proof, or formality). In this way, a totality of the modernist tradition is established, making it hard for alternative discourses to get enunciated. In other cases, the various discourses collide with each other and enter antagonistic relations, through which the content of floating signifiers gets fixed in a partial, non-final way. In the following two sections, I support this thesis, first by bringing the two classes together and discussing their similarities and differences, and consequently by addressing the study's research questions.

7.2. Bringing the two Classes Together: Similarities and Differences

In this section, I bring the two classes together, highlighting some of their similarities and differences, which are significant in the context of this study. I first summarise the findings of the two courses by emphasising the overarching discourses that I identified in each one of them. Subsequently, I discuss the similarities and differences of the two courses, first by focusing on their institutional framework and then by focusing on the discourses that manifest in each one of them.

In the two courses, I identified some overarching discourses, which I argued play a primary role in terms of conceptualising mathematics and, hence, weaving its political characteristics. In the *Introduction to Proofs* course, these discourses were a) the formalising as a prototype of rigour discourse which conceptualises formalising as ensuring rigour and being a catalyst for the production of mathematical results; b) the diverse participation in mathematics discourse, which values diversity and refers both to the participation of groups which have historically been underrepresented in academic mathematical practice and the consideration of individuals' personal mathematical ideas and interpretations; c) the imaginary practice of mathematics discourse, which constructs the mathematical community in an imaginary way, in which everyone has access to a mathematical knowledge shared across time and place, while mathematics is constructed as a system in which it is possible for every theorem to be traced back to axioms; and d) the efficacy of mathematical work/labour discourse, within which developing ways that ensure the efficacy of mathematical work/labour is valued and encouraged. In the *Calculus* course, the respective discourses were a) the humanity of mathematics discourse, which conceptualises mathematics as something that humans have always been doing and will always been doing; b) the applying mathematics discourse, which conceptualises mathematics as a knowledge that can be applied to everything; and c) the mathematics of the class and the mathematics of mathematicians discourse, which distinguishes between the mathematical practice of the class and the mathematical practice of mathematicians, with the latter acting as the source of rigour for the former.

The two courses are both first year undergraduate courses. While the *Introduction to Proofs* course taught by Chris is a course that is aimed for prospective Computer Science students as well as some mathematics students, the *Calculus* course taught by Alex is the standard calculus course for science students. In other words, the *Proofs* course is aimed at students who will have a mathematically much heavier program of study compared to the students who take the *Calculus* course. The two courses are both about mathematics, which at the university level has the particularity of lying in-between the social practice of school mathematics and the social practice of professional mathematicians.

Both courses share a similar institutional framework as they are both taught in the same Ontarian university. This means that students pay to participate in the program that they are in and will have both economic and academic cost if they do not pass the class. In both cases, there are mechanisms at place which govern the way in which students participate in the course. There is a similarity of the grading systems which requires students to complete weekly assignments, a midterm, and a final exam. Besides, students are required to attend Alex's class and sometimes participate in graded in-class activities. In the case of Chris's class, the vast majority of students who intend to enter the Computer Science program are required to complete the course with a high grade while competing with their peers for a position in the program.

The mode of instruction is different in the two courses. In the Proofs course, students are asked to complete some tasks during class, while to do so they are encouraged to collaborate with their peers. These tasks require students to collect data, make conjectures, evaluate conjectures, and produce proofs. Moreover, Chris often asks questions and students think about and answer these questions during class time. In the Calculus course, Alex delivers the material, while students' participation entails asking questions about the ideas presented by Alex. The Calculus course prepares students to develop an understanding of calculus concepts as well as to develop a technical mastery.

Both professors, in their own way, strive to capture mathematics as a social practice; not merely as the product of mathematicians' labour, but as something that people do. In Chris's class, this is evident in the diverse participation in mathematics discourse as well as the imaginary practice of mathematics discourse, while in Alex's course the same idea primarily manifests in the humanity of mathematics discourse. Nevertheless, in both cases, the discourse that mathematics is a social practice has limits and the discourse that mathematics is a product also emerges. In Chris's class, I explored the manifestation of mathematics as a product in the advocacy for visual arguments as not being (formal) proofs, the adoption of a terminology-oriented language variety, and the development of a passive relation with mathematics. On the other hand, I explored the manifestation of mathematics as a social practice through the construction of arguments which prioritise the facilitation of understanding (including through the use of visual arguments), the adoption of a terminology-critical language variety, and the construction of an active, conversational relation with mathematics. These two discourses are in an antagonistic relation in Chris's class as they attempt to crystallise the meaning of floating signifiers, such as proof, formal proof, mathematics, and mathematical language. In Alex's class, mathematics is conceptualised as a human activity within the humanity of mathematics discourse, but as a tool that can be applied in the world within the applying mathematics discourse. Although the two discourses are not brought into collision in Alex's class, their co-existence shows the limits of the discourse that mathematics is a human/social practice.

Another similarity between the two courses is that in both classrooms, the mathematics of mathematicians is constructed as a practice which ensures rigour. This is evident in the formalising as a prototype of rigour discourse in the proofs course and in the mathematics of the class and the mathematics of mathematicians discourse in the calculus course. This construction of the mathematics of mathematicians as rigourous relates to the discourse of mathematics as a product, as it emphasises the result of mathematicians' labour over the social practice of doing mathematics. It also constructs mathematical meaning as final, in the sense that it does not allow space for the negotiation of the truthfulness or falsehood of mathematical propositions.

Furthermore, both courses have a similar political economy of mathematics, which is constructed mainly through the applying mathematics discourse in Alex's class and the efficacy of mathematical/work labour discourse in Chris's class. More specifically, completing tasks as fast and effectively as possible and not wasting time, energy, or resources is desirable in both courses. This is significant because undergraduate courses not only teach students mathematical material, but they also contribute to the construction of future STEM professionals. The emphasis on speed and effectiveness is, thus, inscribed in and reproduces a neoliberal framework, in which labour should be flexible and competitive.

A major difference between the two courses is the way in which the overarching discourses come together in each class to constitute what we may call a nexus of discourses that is specific to each class. In the case of Chris's class, there is more space for alternative discourses to be articulated while antagonisms between discourses are evident. For example, in Chris's class, the discourse of mathematics as a social practice collides with the discourse of mathematics as the product of mathematicians' labour. On the contrary, Alex's discourse is more totalising, as the overarching discourses in the class co-exist harmoniously and rarely do any discourses collide. This difference relates to how multivocal the two courses are. In Chris's class, students have more space to express their views which contributes to the emergence of a multiplicity of discourses. Moreover, Chris himself also draws upon different discourses which he attempts to bring into discussion within his practice.

Another difference between the two courses is the kind of knowing of mathematics which is prioritised in each class. Here I draw upon Skovsmose's (1994a) distinction between mathematical knowing, technological knowing, and reflective knowing. Alex's class prioritises mathematical and technological knowing, as the emphasis is on the understanding of concepts as well as on the ways in which calculus can be applied in everyday problems. In contrast, Chris's class does not focus on technological knowing and applications of mathematics but emphasises mathematical knowing as well as reflective knowing. Reflective knowing in Chris's class takes the form of reflecting upon the social practices of mathematics in critical ways, such as by considering who usually takes the credit for a mathematical result.

Finally, another difference between the two courses is the way in which diversity is brought up in the two classroom discourses. In Chris's class, diversity is valued both with regards to the participation of groups which have historically been underrepresented in academic mathematics and the consideration of individuals' personal mathematical ideas and interpretations. This emphasis on diversity in Chris's discourse can be read within the current zeitgeist of mathematics education, which emphasises the need for equity as well as differentiated teaching that addresses the needs of all students. On the other hand, in Alex's course, diversity in mathematical practice is not evident. On the contrary, Alex emphasises the need for practice which conceptualises learning more uniformly for all students.

Overall, within the overarching discourses of the two courses, there are common ideas which emerge in both courses. This is not surprising, if we consider that both courses share a similar framework, as they are both offered at the undergraduate level in the same university. At the same time, there are also differences with regards to the discourses that manifest in the two courses, which relate both to the different teaching styles of the two professors (that result in different kind and quantity of in-class student participation) as well as different epistemological views on mathematics and mathematics teaching in the sense that the two professors adopt different reactions to current mainstream mathematics education discourses such as the discourse of diversity in mathematical teaching and learning.

7.3. Addressing the Research Questions

In this section, I address the study's research questions, which focus on the axiology, epistemology, and ontology of mathematics in the two courses. More specifically, axiology relates to the first research question, which is about the values that manifest in the course. Epistemology, in the sense of the study of what is true and how truth claims are made, relates to the second research question which is about rationality. Finally, ontology relates to the third research question, which is about the ways in which mathematics creates the "world" or "reality".

7.3.1. Addressing Research Question 1: Values

In this subsection, I discuss how the findings of this thesis address the first research question which is about the values that emerge in the two courses. In mathematics education discourses, mathematics is often conceptualised as neutral and value-free. However, this position has been contested by research that explore values in mathematics (e.g., Bishop, 2008; Ernest, 2018; François, 2019) as well as by approaches which highlight the social, political, and economic relations in which academic mathematics has been and is being developed (e.g., Kollosche, 2014; Pais, 2014). The findings of this thesis contribute to this critique by conceptualising mathematics as a value laden knowledge and by exploring how these values can highlight the ideological aspects of mathematical knowledge(s), showcasing how it is a knowledge with political aspects. This section is organised as follows. I first present the values in Chris's course and I elaborate on some connections and tensions between them. I then do the same for the values that emerge in Alex's course. Finally, I discuss how values, as an inherent aspect of mathematical knowledge are situated and how they play a key role in weaving the political aspects of mathematics.

Table 7.1. includes the values identified in Chris's course. I have organised these values in three categories: values related to mathematics (beauty, truth, simplicity, and generality), values related to doing mathematics individually (efficacy and affect), and values related to doing mathematics as a community (communication, diversity, shared space, and institutional importances). In what follows, I elaborate on these values, highlighting how they emerge in Chris's class and how they frame mathematics as political knowing.

Values Related to Mathematics	Values Related to Doing Mathematics Individually	Values Related to Doing Mathematics as a Community
Beauty	**Efficacy**	**Communication**
Truth	**Affect**	**Diversity**
Simplicity		**Shared Space**
Generality		**Institutional Importances**

Table 7.1: Values in Chris's Class

The four values related to mathematics are significant because they capture the characteristics of the mathematical knowledge that is taught in this class and is valued at the academic level. When I talk about values that relate to mathematics, I refer to the desired characteristics of the product of mathematical practice. However, these values also relate to doing mathematics, as they shape mathematical practice so as to produce results with these desired characteristics. The class discourse constructs as important the mathematics that is beautiful, true, simple, and general. Nevertheless, not all mathematics has these characteristics. For example, the mathematics of Indigenous communities does not value generality and unrootedness to a specific culture (Abtahi & Wagner, 2016). Even more importantly, mathematics does not need to have these characteristics. In other words, there is no a priori reason why mathematics needs to be associated with these values and not others. Through the emergence of these specific values, mathematical proofs are constructed as entailing these characteristics. In this way, other ways of mathematical meaning making are excluded.

Regarding the values related to doing mathematics individually, efficacy and emotion manifest as important in Chris's class. Historically, these two values have their roots in different discourses. On the one hand, efficacy is related to, what I called the "efficacy of mathematical labour/work" discourse and, in this sense, is part of a wider political economy about mathematical practice in which time and labour should not be wasted. Efficacy as a value has its roots in the ideals of capitalism, which is interested in the product of individuals' labour within the context of a desired never-ending economic growth (Apple, 2004b). On the other hand, affect manifests as part of a more humanitarian discourse, which values the experience of the person doing mathematics (Liljedahl & Hannula, 2016). This set of values contributes to the production of graduates with particular characteristics, who both learn to work efficiently and learn to take their affect into consideration. Although these values depart from different discourses, a capitalist and a humanitarian discourse, they do not collide with each other but coexist smoothly. This is not a surprise considering the ways in which capitalist discourses have evolved in ways that absorb alternative and competitive discourses.

Finally, in Chris's course, I have also identified values related to doing mathematics as part of a community, which include communication, diversity, shared space, and institutional importances. This set of values is central in Chris's course as doing mathematics in collaboration with each other while respecting and striving for diversity are important for this class. Diversity refers both to diversity of thinking among individuals as well as diversity in representation of groups of people who are traditionally underrepresented in mathematics. Communication and shared space are values which attend to providing the conditions within which mathematics can be done at the level of the community. Communication refers to ensuring that mathematical results are enunciated in ways that enable and facilitate their being read by, understood, and discussed with others. Shared space is assumed to capture not only the mathematical community of the class, but it also operates at an imaginary level that includes mathematicians who do not live any more as well as mathematicians whom the student shall never meet in person. Finally, institutional importances refer to doing mathematics within the specific context of an undergraduate mathematics course, which both structures the mode in which the course can happen and frames the students' objective of passing the course with a good grade.

With regards to values in Alex's course, my findings suggest values that can be organised in terms of a micro, meso, and macro level. The micro level refers to values that are relevant to the specific interactions in the course (i.e., people doing mathematics at this place and time) and include the existence of a shared space, the significance of individual work, and institutional importances. The meso level is about interactions with mathematics which refer to applications of mathematics and its role in society. Values at the meso level include the applicability, generalisability, and efficacy of mathematics, as well as control through mathematics. Finally, the macro level refers to those elements of interacting with mathematics which refer to its history (dating back centuries) or claim a diachronic existence. The values at this level include truth, progress, the conceptualisation of mathematics as a quintessentially human activity, and the production of cohesive mathematical knowledge. A schematic representation of the values in Alex's course can be found in Figure 7.1.

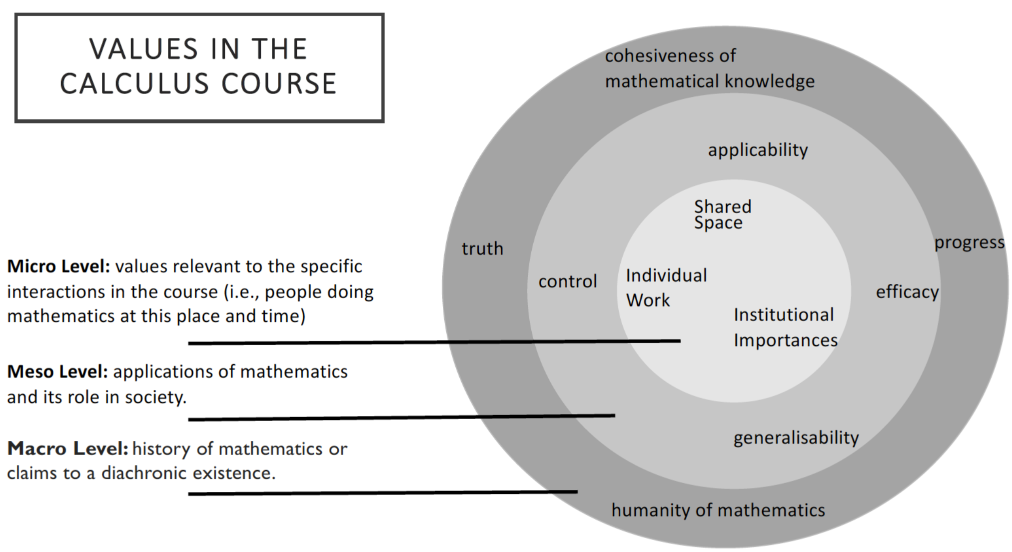

Figure 7.1: Values in Alex's Class

As mathematics is constructed as a cohesive, true knowledge which is always progressing (at the macro level) and as generalisable, applicable, efficient knowledge which offers control over nature (at the meso level), mathematical knowledge bears a hegemonic status – it cannot be challenged and other discourses cannot antagonise it. This hegemonic status is further legitimised by the notion that mathematics is a quintessentially human activity as well as the construct of the shared space which regulates the kind of mathematical actions that can happen in the class both at a real and an imaginary level.

Moreover, the values of this calculus course weave a technocratic image of mathematics. This is evident in all three levels of values. At the macro level, there is the promise of progress. This promise feeds the technocratic framework by constructing the expectation of an ever-developing mathematical knowledge that can solve all kinds of problems that it could not before. At the meso level, there is generalisability, applicability, efficacy, and control. The values of generalisability, applicability, and control construct (the expectation of) a mathematical knowledge that is always general enough and relevant enough to solve almost any kind of problem. Efficacy regulates the actions of the individuals involved in the process of producing and applying the mathematical knowledge, by demanding that their tasks are completed as fast and efficiently as possible. Finally, at the micro level, the value of individual work contributes to the production of scientists who desire to work hard and struggle in order to excel. This nexus of values inscribes calculus in a technocratic framework, in which calculus shall be oriented to solving all kinds of problems and individuals performing calculus shall facilitate that.

In what follows, I draw some connections between the values of the two courses. But before doing so, I need to make a comment on having used different ways of organising the values in the two courses. In Chris's course, I differentiate between values related to mathematics, values related to doing mathematics individually, and values related to doing mathematics in a community. In Alex's course, I discuss values in relation to their emergence at a micro, meso, and macro level. This choice makes it more difficult to compare the values in the two courses. Nevertheless, I made the choice of adopting different ways of organising the values because my goal is not to compare the courses per se. Instead, as I am interested in inquiring into the values as they are situated in the two courses, I decided to adopt a way of organising and presenting the values that is more specific to the particular characteristics of each course.

There are some values which are similar in the two courses (truth, efficacy, and institutional importances). There are also differences between the courses, as in the Proofs course there is an emphasis on communicative aspects of mathematical knowledge and diversity, while in the Calculus course there is an emphasis on applicability and control. In the following paragraphs, I account for these similarities and differences by situating these values in the current academic mathematical framework and the current mainstream political economy of mathematical labour, as well as by emphasising that values are situated in the particular social and cultural framework of each course.

Values emerge from the particular academic mathematical framework as it has been formed throughout the years. For example, truth is a value in both courses, which is a product of this framework. Academic mathematics is framed by the need to produce results that are true, to the extent that truth is so commonsensical that it does not even need to be mentioned explicitly. Furthermore, the emphasis on applicability and control in Alex's class emerges within the context of calculus which is a mathematical discipline that is widely used in other scientific disciplines, but not in Chris's class whose content is more abstract and theoretical.

Furthermore, values are both the product and a key component of a particular political economy of mathematical labour. For example, generalisability relates to a capitalist economy both because general knowledge that can produce algorithms is valued in capitalism and because the construction of mathematics as a general knowledge reinforces rootless, applicable knowing. Moreover, doing things quickly was desirable in both courses, with no particular discussion of why this is preferred. This suggests that a mathematical method or tool that enables saving time is desirable, which is a value rooted in a capitalist paradigm within which labour needs to be fast-paced (Castoriadis, 2000).

Finally, the differences between the two courses can be accounted for if values are considered as situated. Values are a central aspect of both the social framework in which they occur and the specific mathematical practices performed. The social norms of each class are different as Alex's class is more lecture-oriented class, while Chris's class is more discussion/participation-oriented. This leads to different values regarding communication. Alex's class entails values that reflect mathematics as a sole activity (e.g., individual work), while Chris's class entails values that reflect mathematics as an activity done within a community.

Overall, in this section I summarised the values that I have identified in the two courses. Through the discussion, I highlighted three main points. Values emerge from the current academic mathematical framework as it has been formed throughout the years, they are part of the current mainstream political economy of mathematical labour, and they are situated in the particular social and cultural framework of each course.

7.3.2. Addressing Research Question 2: Rationality

The second research question which I set out to examine focuses on the rationality manifested in the two classes. While mathematics has been treated as a synonym to rationality, I have adopted a more open understanding of the term rationality to explore the mechanics of how mathematical truths are produced in each classroom. To do so, I explore questions, such as who can decide what is mathematically true, how, on what premises, etc. This subsection is organised as follows: I first describe the rationality techniques in Chris's class, I continue with the rationality techniques in Alex's class, and I conclude with a discussion of the similarities and differences between the two courses' rationalities.

Through my analysis, I have identified three ideas related to rationality as being prevalent in Chris's class: first, the existence of various "spaces" of doing mathematics (i.e., in the brain, through human communication, within an imaginary community); second, mathematical activity is framed as the process of "gathering data – making conjectures – expressing conjectures in formal mathematical language – proving conjectures" in an effort to move from specific cases towards general statements; and third, formalising is a central process which produces definitions and theorems. In the following paragraphs, I discuss each of these in more detail.

My analysis of the data from Chris's class indicates that there are three "spaces" in which mathematics happens and mathematical truths are produced and negotiated: mathematics happens in the brain of an individual person, mathematics happens through communication between real people (who sometimes disagree and argue), and mathematics happens within the imaginary of a mathematics community. These spaces are complementary, in the sense that Chris conceptualises mathematics as happening in all three of them. Generally, in this course, mathematics is not a bodily experience, nor is it a material experience; mathematics happens through the mind and discursive communication. The imaginary mathematics community plays a regulatory role which leads individuals towards behaving in a specific way.

Second, a rationality technique in Chris's course is the methodology of "gathering data – making conjectures – expressing conjectures in formal mathematical language – proving conjectures". This process is employed multiple times in Chris's class and is constructed as a prototypical way for producing and proving mathematical results. In this way, Chris's discourse aligns with recent currents of thought and practice in proof and proving literature, which have moved the focus from the mere understanding and production of proofs to socialising students into the processes that are related to proof and proving.

The third rationality principle in Chris's course is formalising. Formalising is a central process in this class which acts as a catalyst for the production of mathematical results, while it plays a key role in constructing mathematics as a rigourous activity. Formalising in this class occurs in the use of definition-based arguments, the avoidance of visual arguments, and the utilisation of technical language. Overall, it can be argued that a major goal of this Proofs course is to socialise students into the process of formalising mathematics.

In Alex's course, there are three major rationality techniques which I have identified: a) simplifying problems, b) developing prototypical methods to work on problems, and c) formalising ideas. Moreover, the distinction between conceptual and computational mathematical knowledge sets the framework and colours both the rationality techniques and the roles of theorems, definitions, and proofs. In the following paragraphs, I discuss these ideas in more detail.

Simplifying problems refers to the technique through which a complicated problem is translated into simpler problems that are considered solvable. After solving these simpler problems, the solution is translated back into the context of the initial problem. The derivative itself is an example of this technique, in the following sense. Describing a complicated function whose equation is unknown is captured as a complicated problem. Locally linearising the function by finding its tangent line is a simplification technique that translates the problem into a problem of lines, which is a solvable problem.

Developing prototypical solutions refers to identifying ways in which an algorithm is developed and preferred for solving similar problems, instead of treating each problem as unique. The development of "rules" instead of "tricks" is often highlighted by the professor. For example, Alex asks his students to use the quadratic formula when solving a quadratic equation, instead of trying to find two numbers with a given sum and a given product. Similarly, he presents the "rules of differentiation" (e.g., derivative of a polynomial, derivative of exponential, derivative of sum, etc.) as rules that allow the calculation of *any* derivative, whereas finding a derivative by calculating the limit relies on tricks. Finally, often when solving a problem, the goal becomes to reach a point for which the class has developed a method for solving it (e.g., when working with limits, the goal might be to transform a given limit into the limit of a rational function, because they know how to proceed from there).

Formalising ideas refers to the technique in which a mathematical idea is first described geometrically, verbally, or through examples and is later formalised by being "translated" into symbolic language and/or theorems. For example, Alex calculates the derivative of various functions of the form $f(x) = x^r$. He then generalises and formalises the results into a theorem. Similarly, often when presenting an idea, a geometrical or visual explanation is followed by the rule (normally expressed with the use of technical language and symbols). Furthermore, although the class does not engage with exploring formal definitions, theorems, and proofs, these are all key elements for "guaranteeing" the methods that lead to "true" results. This course tries to not be very theoretical, is not proof oriented, and does not require students to learn or reflect on definitions and theorems as such. However, the professor sometimes refers to mathematical ideas as "theorems" or "definitions" and (more rarely) uses these terms when writing on the board. Definitions and theorems are used as connecting the mathematics of this classroom with a "more theoretical" mathematical knowledge, in a way that also implies that the theoretical mathematical knowledge can confirm the truth of the course's mathematics.

Finally, the distinction between conceptual and computational knowledge is prevalent in this course and is often verbalised as such by the professor. This distinction functions in showing the limits between the knowledge produced in this course and other kinds of mathematical knowledge (e.g., while the existence of theorems is mentioned, studying/proving theorems is conceptualised as something not done in this course, but by other people). Moreover, this distinction functions in positioning the knowledge produced in this course as a combination of the conceptual and the computational.

Both courses have their distinct rationality methods, principles, and techniques, which are summarised in table 7.2. However, mathematics does not necessarily need to happen in any of these ways. Mathematics can happen in many different manners: by proving a theorem while attempting to present it as understandably as possible, by proving a theorem while trying to make the proof as obscure and difficult to understand as possible, by rope-making and dancing (Gerofsky et al., 2017), or by exploring the notion of infinity in Indigenous cosmologies (Rauf, 2002). So, the question becomes what makes people in the two classes do mathematics in the particular ways that they do.

<table>
<tr><th colspan="3">Rationality in the two Courses</th></tr>
<tr><th>Proofs Course</th><th colspan="2">Calculus Course</th></tr>
<tr><td>various "spaces" of doing mathematics
• in the brain,
• through human communication
• within an imaginary community</td><td>simplifying problems</td><td rowspan="3">conceptual vs computational mathematical knowledge</td></tr>
<tr><td>gathering data – making conjectures – expressing conjectures in formal mathematical language – proving conjectures</td><td>developing prototypical methods to work on problems</td></tr>
<tr><td>formalising</td><td>formalising</td></tr>
</table>

Table 7.2: Rationality in the Two Courses

A common aspect in both courses is formalising. This refers both to formalising the mathematical results that the class produces and to the notion of there being formal mathematics performed by mathematicians that ensure rigour. The notion of formalising is, thus, related to the existence of the practice of mathematics of mathematicians and to the prestige that this practice has within an undergraduate setting. At a more philosophical level, formalising is related to the prototypical framework in which both mathematics and (more generally) Western thought are inscribed. Jullien (2004) suggests that European mathematics is a prototypical field, in the sense that, from ancient Greece to Galileo, and then Descartes and Newton, mathematics has been conceptualised as the language in which the world is written; in turn, learning mathematics offers a way to understand the world. Jullien agitates the commonsensicality of this discourse (which is of course a commonsensicality only from a Western perspective) by highlighting how Chinese mathematics are transformations of algorithmic nature which were never conceived of as a language and, hence, mathematics was never understood to be able to describe nature. However, there is no a priori reason why mathematics or mathematical communication need to have a telos. For example, we might consider Gerofsky et al.'s (2017) project of "dancing braid into being", in which a group of people dance and record their movement into an emerging braid. In this way, they explore the mathematics that emerge (which in fact falls under the discipline of group theory, a quite abstract field of academic mathematics) without there being a telos. Lakatos's (1976) "Proofs and Refutations" describes an imaginary discussion in the context of a classroom, in which a definition is negotiated, without there being an already pre-determined "truth" for what the definition should be. These are both examples of mathematical communications which do not have a telos. So, why does mathematics in Chris's and Alex's class (and in so many classes around the world) have a telos? My analysis has focused on two aspects. First, the "telos" supports the distributive function of the institutional framework: the existence of a certain thing to be learned enables the assessment of this thing, which in turn decides who enters the academic program of their choice and who does not. Second, mathematics in the Western world has been an extremely efficient method for describing and acting in the world (Jullien, 2004); a process which has supported the emergence of the world as we know it (with its factories, online shops, credit cards, stock market, airplane travelling, weapons, etc.) including the distribution of power within it.

With regards to the rationality techniques through which mathematical results are produced, there is a significant difference between the two courses. Chris's class employs a model of "gathering data – making conjectures – expressing conjectures in formal mathematical language – proving conjectures", whereas Alex's course focuses on simplifying problems and developing prototypical methods to work on problems. This difference in rationality approaches can be accounted for in two ways. First, proof and proving literature emphasises the need to prioritise argumentation and proving practices over the learning of the results of the proving process. However, in calculus research this orientation is not as mainstream. Second, calculus is an area of mathematics which focuses much more on modelling the world and exploring applications of mathematics than the theory of proofs. This becomes more intense in Alex's course, which is the general course for first year science students, as it focuses on the applications that calculus can have in a wide range of disciplines. Overall, the rationality techniques that are employed in each course are influenced by the social framework, the mathematical area, and the mathematics education literature related to the mathematical area of each course.

Another difference between the two courses with regards to rationality is the spaces within which mathematics happens. In Chris's course, mathematics happens in the brain, through human communication, and within an imaginary community. In Alex's class, the spaces of human communication and imaginary community are not as prominent and, instead, mathematics is constructed as a sole activity that happens in the brain. The distinction between conceptual and computational mathematical language contributes to this conceptualisation of mathematics as a mental activity exactly because it considers mathematics in relation to concepts that need to be understood and computations that need to be performed.

7.3.3. Addressing Research Question 3: Formatting Function

The third research question which I set out to examine focuses on the formatting function in the two courses. In a nutshell, my argument constitutes an extension of Skovsmose's notion of the formatting function of mathematics and goes as follows. The notion of formatting function in Skovsmose suggests that mathematics not only describes but also prescribes the world. The context of the two courses is educational, so prescription becomes different than Skovsmose's sense exactly because the mathematical knowledge produced in an undergraduate course is not actualised. However, I suggest that mathematics in the context of a class not only describes an external world but also constructs the world in a specific way. This construct is taught to people who will become scientists and develop algorithms in the future as well as to people who, as citizens, will accept these algorithms and their effects as politically desirable. So, when I ask about the formatting role of mathematics in this educational context, I focus on how mathematics constructs the world in ways that are actually or potentially prescriptive. We encounter "actually prescriptive formatting" in the formatting of mathematics and mathematical objects: the construction of a certain mathematical theory through the manifestation of a certain mathematical social practice. "Potentially prescriptive formatting" refers to the construction of the world through the mathematical practice of the class in a certain way which in turn educates future scientists and citizens in order to produce and accept mathematics that describes and prescribes the world in a certain way. This section is organised as follows. I first elaborate on some aspects of this argument before summarising the findings that support it. I conclude this section with a discussion of how it compares to Skovsmose's ideas. This discussion includes an exploration of the epistemological similarities and differences between my work and Skovsmose's theory.

In mathematics education, the formatting function of mathematics has its roots in Skovsmose's (1994b, 2000) concept of the "formatting power of mathematics, which has been very influential in shaping the field's discussions about the role of mathematics in society. Skovsmose's contributions (as well as the contributions of several scholars who work in the same vein) help us explore the formatting function of mathematics in modelling and technology: the ways in which mathematics is at play and – often in invisible ways – is instrumental in shaping the world. My engagement with the concept of formatting in this thesis has been influenced by Skovsmose, but I mainly draw upon some ideas of formatting as they have been discussed in discourse studies. In this way, I refer to a concept of formatting that is wider than Skovsmose's and attempts to describe how the practice of doing mathematics contributes to the shaping of the world, mathematics, and mathematical objects. In discourse theories, or at least the discourse theories that this thesis epistemologically aligns with[41], reality does not pre-exist, but it is constructed through discourse. This means that a discursive practice, like doing mathematics, constructs the reality of doing mathematics as well as formats concepts and objects that are related to the doing of mathematics.

The "formatting function" which I have been describing in this thesis manifests as part of the unfolding of a pedagogical discourse. In a sense, the role of the pedagogical discourse is to socialise students into a mathematical discourse. As many aspects of the mathematical discourse are new to students, the goal of the pedagogical discourse is to familiarise them through detailed descriptions, which are usually more explicit than experts enunciating or producing mathematical discourse. Hence, I propose that the pedagogical aspect of the mathematical discourses in the two courses makes the formatting aspects of mathematical discourse more visible. In other words, I argue that pedagogical mathematical discourse is a privileged space for examining the formatting function of mathematical discourse.

Through my data analysis, I have discussed how the practice of doing mathematics constructs the world, mathematics, and mathematical objects. Each of these is constructed in similar, but not identical, ways in the two courses, which points to the idea that these do not pre-exist but are constructed through the class interactions. In other words, the practice of doing mathematics formats a series of mathematics-related

[41] Sections 3.3.1. and 4.5.3. have tried to address what these theories are, what their epistemological assumptions are, and how they are related to other discourse theories.

concepts. In the following few paragraphs, I describe of how the mathematical practices of the two courses construct the world, mathematics, and mathematical objects.

Regarding the formatting function of the world through mathematics, similar but not identical discourses are developed in the two courses. In Chris's class the world is being constructed as being mirrored in mathematics. For example, English statements (as common as "If it rains tomorrow, I will bring an umbrella") are translated into mathematics while everyday actions (such as unsuccessfully preparing guacamole) are formatted as corresponding to proof methods (like proof by contradiction). Through this methodology, a way of looking at the world through mathematics is crystallised. In Alex's class, calculus formats natural and social situations as mathematical/ mathematised problems. For example, the notion of the derivative is used to format the world by capturing/constructing almost any situation that is changing as fitting into the model of the derivative. Similarly, almost any phenomenon is constructed as being described by an (often unknowable or unknown) function. Alex's discourse proposes that models have the ability of prediction in almost any context, including substances, the weather, and the ways in which systems of population behave. The example of Neptune, whose existence according to Alex's discourse, was mathematically predicted before the planet was directly observed, is indicative and constructs the "power" of mathematics in *describing* the universe. In this way, because – within the professor's discourse – mathematics is so powerful in *describing* the world, the ways in which mathematics is formatting the world are not always transparent/visible and, thus, the formatting happens in deeper, less conscious ways.

The mathematical practice of each class also formats mathematics. Here, my argument is that mathematics is not a discipline with finalised characteristics that does stuff and formats the world, but mathematics is formatted in various ways depending on the various social mathematical practices. In Chris's class, mathematics is formatted as a four-step social practice, which entails gathering data, making a conjecture, expressing the conjecture formally, and proving the conjecture. Moreover, the notion of (in)formal mathematics is formatted through the interactions and negotiations that happen in the class, such as discussions regarding whether a visual proof is indeed a proof. In Alex's class, mathematics is constructed as a discipline that is used to understand and solve problems in the world around us. Moreover, the conceptualisation of mathematics as something quintessentially human, as something that people have always been doing and will always be doing, paints a very particular picture about humanity whose engagement with mathematics is not incidental. In this way, a historically and geographically specific way of doing mathematics (which can be traced back to ancient Greece and Europe) is inscribed as a natural activity, thus formatting both mathematics and humanity (i.e., what it means to be human). Furthermore, mathematical practice creates and formats a shared space, for the professor and students. In this space, certain tools and methods are allowed, while others are not (e.g., de L'Hospital is not allowed because the class has not studied it collectively yet). Further, doing mathematics in this particular way that is followed in Alex's class formats communication (that includes a specific kind of language to be used, a deicticly heavy communication with constant references to what is written on the board, and certain ways in which communication and arguments are structured). This shared space is significantly different than the shared space formatted by the mathematical practice in Chris's class, where the development and negotiation of arguments is valued.

Finally, the mathematical practice of each class also formats the mathematical objects studied by each class. In Chris's class, the development of formal definitions is captured as enabling the production of proofs. In this way, formal mathematical definitions format the kind of statements that are true for each mathematical object that is studied. For example, definitions about cardinality format the notion of infinity in a way that is final and non-negotiable. Similarly, definitions in Alex's class – although not studied in the same detail as in Chris's class – are captured as crucial for the development of different mathematical results: different definitions of the same mathematical concept can produce different statements about the concept.

In the rest of this section, I discuss how the concept of formatting as developed so far compares to Skovsmose's. Through his concept of formatting, Skovsmose makes an ontological claim about the world. I consider Skovsmose's argument to be ontological because it provides an image of what the world is and how it gets constructed through mathematics. Of course, this ontology suggested by Skovsmose is more relational than traditional understandings of ontology which provide a solid view about the world. Straehler-Pohl (2017) explains the difference between Skovsmose's ontological framework and other, more traditional ontologies about mathematics, as follows:

> where mathematics is proposed as a means for structuring a problem of reality […] mathematics is sharply distinct from reality itself. Generated mathematical solutions can then be transferred back into reality and evaluated by reference to reality. Circular approximation brings the description closer to what is described. Such a view portrays mathematics as a means for the description and prediction of reality. The focus on formatting undermines this artificial dualism: through the availability of the mathematical model, the original problem of reality changes itself. Description and the described approximate each other, the described object itself also transforming towards the description. (p. 37)

In terms of illustrating how the way in which I have employed the notion of formatting differs from Skovsmose's, I turn to a divergence in perspectives that is encountered in discourse analysis. In discourse analysis, we might talk about a bifurcation in terms of the subjective/objective character of discourse. On the one hand, that is represented among others by Fairclough (1995) and his Marxist epistemology, discourse has objective effects and functions regardless of how these functions are being experienced by the individuals who participate in this discourse. This does not mean that Fairclough is not interested in or does not take into account individuals' perspectives, but in the last instance it is the social theory that shall "decide" about the effects and functions of a discourse. On the other hand, that is represented in its more "extreme" version by discursive psychology and conversational analysis (e.g., Reis & Barwell, 2013) it is the participants' views and how they participate in the discourse or interpret it that matters. This bifurcation can be thought of as a spectrum on which different discourse analysis theorists and scholars position themselves differently.

I consider Skovsmose's view of formatting to correspond to Fairclough's view of discourse analysis. Skovsmose, while interested in how people interact with mathematics and the mathematical modelling of the world, focuses on the material aspects of this formatting. For example, he examines how the mathematics of airline travelling format airline travelling itself (Yasukawa et al., 2012). In other words, he explores how mathematisation alters our lives mostly at the material level or how "mathematization intervenes more deeply in the very structure of practice, re-organising the basic conditions our actions refer to» (Straehler-Pohl, 2017, p. 37). My interest lies with combining an insight from social theory with an account for how people socialise themselves into this discourse. Hence, when I talk about formatting of the world, I do not merely refer to the organisation of social activities as a result of using mathematics; I also refer to the ways in which certain representations of reality and narratives regarding reality become available.

To strengthen the argument I am making, I wish to make a claim about the material effects of mathematics. A strong element of Skovsmose's theory is that Skovsmose examines the very material effects of modelling and technology. For example, Ravn and Skovsmose (2019) refer to G.D.P. and how it has a formatting power. I propose that this materiality of the effects of mathematics is not limited to the realised abstractions, thought of as the cases in which a mathematical model gets actualised. An example is that the Greek ex-minister's of economics, Varoufakis's, thinking abstractions about how the Greek economic crisis could be handled, although never realised, still played an important role in formatting the Greek political landscape as well as the European Left in very material ways (Jones, 2016; Varoufakis, 2017). The representations of mathematicians in textbooks or in the popular culture are another example of the same idea – they format the way in which mathematics and mathematicians can be imagined.

Summing up, the formatting role of mathematics is evident in the data, as the practice of mathematics shapes the world, mathematics, and mathematical objects. I argue that mathematical discourse might format reality, regardless of whether this discourse becomes materialised (e.g., through modelling or technology), as it de facto has a function of shaping an understanding of the world, making certain concepts and actions possible while overshadowing others.

7.4. Contributions of this Study

In this section, I discuss the contributions of the findings of this thesis to mathematics education literature. More specifically, I first explore connections with (undergraduate) teaching and learning literature and consequently with sociopolitical mathematics education literature. To do so, I go back to the conclusions and questions of my literature review and I briefly explore how this thesis contributes to these discussions.

7.4.1. Contributions to Sociopolitical Mathematics Education Literature

In the literature review of sociopolitical mathematics education scholarship, I explored how mathematics is conceptualised as an object in critical mathematics education, ethnomathematics and decolonising approaches, feminist and critical race theories approaches, and cultural politics of mathematics education. I suggested that mathematics is not thought of as political per se except within the cultural politics of mathematics education scholarship.

My conceptualisation of mathematics as political is based on the idea that mathematics is a space in which meanings are negotiated and antagonisms take place: mathematics does not have a fixed meaning but multiple discourses operate in an effort to fix the meaning of floating signifiers. At the same time, within the social practice of mathematics, certain ways of working, thinking, and interacting are accepted and prioritised resulting in the exclusion of other ways of meaning making. More specifically, I have identified three features of mathematics which make it inextricably political: the values of mathematical knowledge (Bishop, 1991; Ernest, 2018, 2020; Skovsmose, 2020), the rationality of mathematics (Kollosche, 2014; Walkerdine, 1988), and the formatting function of mathematics (Barwell, 2013; Skovsmose, 1994b, 2000). Furthermore, through the analysis of the findings, I have proposed that mathematics is not a final product that students need to absorb but is a nexus of discourses into which students socialise themselves. In each mathematics course, some discourses manifest themselves weaving the political characteristics of mathematics in a situated way. In what follows, I will not discuss the characteristics of these discourses as I have already done this earlier in this discussion chapter; I will instead discuss the implications of this view for the conceptualisation of mathematics as a central object of inquiry for mathematics education.

A central idea that emerges from this way of conceptualising mathematics as political is that mathematics is not an objectively pre-determined body of knowledge. Mathematics is what it *becomes* when discourses attempt to fix its meaning. As Pais (2014) maintains "[m]athematics itself is nothing outside the different places where it is used" (p. 1091). Another example that illustrates this view of mathematics acquiring its meaning within the contexts in which it is used and developed is the following. A common argument that advocates for the universality of mathematics is that "$1 + 1$ is equal to 2, no matter what". However, following Laclau and Mouffe's (1985/2001) framework, "1", "$+$", and "2" only make sense within a system from which these words gain their meaning. There are civilisations, like the Pirahã people's, in which people count with a numerical system that only includes "one", "two", and "many" (Wagner & Davis, 2010). Moreover, in Mi'kmaw, the counting words differ based on whether animate or inanimate objects are counted (Lunney Borden, 2013). But even within academic mathematics, $1 + 1 = 10$ if we work in base 2. Hence, the universality of "$1 + 1$ is equal to 2" is not as commonsensical as we may think.

This view has certain implications regarding how the mathematics of a class is constructed. What mathematics gets to be in the context of a classroom is not what is described in the curriculum or textbook; it is rather the result of antagonisms between discourses that take place in the classroom. These discourses do not start and end in the classroom; they do not necessarily coincide with the narrative/discourse of an individual enunciator (e.g., professor, textbook, ministry, student, etc.) and they have their roots in society. For example, such discourses might be that "mathematics is the product of mathematicians' work", "mathematics happens as people talk with each other", "mathematics helps us see if a statement is true or not", "mathematics is the queen of science with applications in every domain of the human experience". There is no a priori reason why any of these discourses is more mathematical than any other one of them.

The conceptualisation of mathematics as a space in which antagonisms take place opens different possibilities than the ones offered from a-political conceptualisations, since it avoids treating antagonisms and power as external to mathematics. Within this framework, the characteristics of mathematics are defined within the power relations which constitute the field itself. Therefore, following Skovsmose (e.g., 2011), mathematics is treated as not entailing any characteristics that are essentially necessary to the field. At the same time, this approach recognises that mathematics has been developed, exercised, and has derived its sociopolitical significance within a-symmetrical power relations and, therefore, has historically been mainly serving the interests of those "in power" (Kollosche, 2014).

7.4.2. Contributions to Undergraduate Teaching and Learning Literature

In the literature review of undergraduate mathematics teaching and learning scholarship, I suggested that political aspects of the mathematical knowledge do not constitute the object of inquiry of this scholarship. Concluding my overview of the literature on proofs and calculus learning, I posed two questions. The first question was "What discourses about mathematics are evident in cognitive discourses and how are they political?" – a question which I have already addressed through my findings and in this discussion chapter. The second question that I posed, which I discuss in this subsection, read as follows: "in cognitive research, there is a transition taking place regarding what it means to learn mathematics: the notion of acquiring a pre-established knowledge starts fading and gets replaced by the notion of constructing a way of knowing as students interact with others. To what extent does this transition constitute a change of paradigm with regards to the relationship of individuals and mathematics? And does this transition also signify a transition with regards to the assumed ontology of mathematics?"

I discuss this question through the discussion of the antagonism between mathematics as a product and mathematics as a social practice. I have explored this antagonism in my findings in Chris's class[42]. As the professor and the students in this course draw upon the two discourses, meaning for some floating signifiers (like proof, formal proof, mathematics, way of speaking mathematically, and an individual's relation with mathematics) is built. This means that in the classroom, it is not pre-determined which language variety will be adopted or which relation with mathematics will be built. Instead, these meanings are constructed through the interaction between the two discourses. In this way, the antagonistic relation between the two discourses is political because it conceptualises various aspects of doing mathematics within the possibilities opened up by the two discourses and as a result of their interaction. However, this does not mean that anything can emerge as a result of this collision. As the two discourses are historically situated in mathematics education practices, they also bear a different status. Although Chris makes a great effort to actualise social aspects of mathematical practice in the class, adopt a terminology-critical language variety, and construct an active relationship between the class and mathematics, the discourse of mathematics as a social practice cannot be fully actualised as it collides with the discourse of mathematics as a product. The hegemonic status and impact of mathematics as a product discourse is

[42] For Alex's class, I discussed a similar distinction: the conceptualisation of mathematics as a reified product and as a human practice

manifested throughout the various examples from the class. In this way, the discourse of mathematics as a social practice (within institutions in which students are expected to master a specific material, be examined in it, and be accepted or not in a program based on their performance) falls very short in terms of challenging the existence of a knowledge which will not change, regardless of what the people talking about it think or do.

Besides this course, the antagonism between the discourse of mathematics as a social practice and mathematics as a product is also present more generally in mathematics education without being necessarily framed as antagonistic. In what follows, I argue that the concept of antagonism is able to capture the relation between the two discourses, in a way which highlights its political aspects. My argument runs as follows: I describe how the two discourses (social practice and product) are evident in mathematics education; I continue by examining and critiquing an alternative narrative which captures their relation as a relation of progress; I then explore how this relation can be understood as an antagonistic one; and I conclude by suggesting that the notion of antagonism is useful for highlighting political aspects of discourses.

For quite some time, in mathematics education, the idea that mathematics is what mathematicians do has been hegemonic. Within what might be called "the social turn" in mathematics education, we see the gradual prevalence of a counter-discourse that suggests that mathematics is a social practice; a discourse that conceptualises meaning, thinking, and reasoning (both in the classroom and in mathematicians' work) as products of a social activity (Lerman, 2000). Contrary to Platonic views which claim that mathematics is somewhere out there (not in our tools, actions, or discussions), the discourse of mathematics as a social practice conceptualises mathematics as happening through people's communication and actions. At the same time, however, the discourse of mathematics as a product preserves much of its hegemony in mathematics education. We can see this discourse in two aspects: first, academic mathematics bears a hegemonic status: its truth, once produced, is not negotiable (Gutiérrez, 2013). Second, as schooling has a function of distributing students into different social positions (e.g., Pais, 2013), the institutional framework is instrumental in reinforcing the discourse of mathematics as a product. For example, "the kinds of mathematics that we teach" is described in curricula across the world through specific outcomes and objectives that the students are required to meet (Abtahi, 2020). Similarly, standardised assessment is enabled through a discourse of mathematics as a product: every student is assessed on the same piece of knowledge, which has been created outside the classroom walls.

One way of describing the relation between the two discourses is through a story of progress: mathematics education used to focus on mathematics as a product, but it has gradually turned towards focusing on the social elements of doing mathematics. Mathematics as a social practice is framed as a set of more recent moments in a linear progressive path followed by a mathematics education which constantly becomes better, more participatory, and more equitable. In other words, the discourse of mathematics as a social practice contributes to mathematics education constantly becoming "better", more participatory, and more inclusive, mitigating or challenging the impersonal, inequitable character of ("old") mathematics. This narrative of progress runs through mathematics education which, both as research and practice, appears to be committed to straightforward progress (Llewellyn, 2015). Deriving from Laclau and Mouffe's (1985/2001) view on the impossibility of meaning fixation, we can see the fragility of this approach: even if we agree that making mathematics more activity based contributes to a betterment of mathematics education, there is no reason to assume that this will continue to be the case until mathematics education becomes fully participatory or equitable. In my view, this narrative of progress fails to address how the "old practice" is still here, hegemonic, trying to absorb the "new" practice.

The concept of antagonisms (Laclau & Mouffe, 1985/2001) offers an alternative reading of this relation which suggests that both discourses co-exist in mathematics education, while each of them tries to fix the meaning of mathematics; the result of this collision is unfixed. The conceptualisation of antagonism can be helpful for addressing the limits of both hegemonic discourses and counter-discourses, as well as the space for challenging hegemonic discourses. In the "is this considered a formal proof?" example, the discourse of mathematics as an established practice was hegemonic but we can recognise its limits (e.g., it sometimes hinders understanding) as well as the ways in which the discourse might be challenged, through discourses which value people's understanding and interactions. At the same time, an antagonism-oriented analysis makes visible the limits of the ability of the counter-discourse (i.e., mathematics as a social practice) to challenge the established practice of mathematics as a product.

While there seems to be a wide agreement in mathematics education about mathematics being a "social practice", this discourse seems to often oversee the hegemony and impact of mathematics as a product. In other words, I suggest that the discourse of mathematics as a social practice (within institutions in which students are expected to master a specific material, be examined in it, and be accepted or not in a program based on their performance) falls short in terms of challenging the existence of a knowledge which will not change, regardless of what the people talking about it think or do.

7.5. Reflections on the Theoretical/Epistemological Framework

In this section, I reflect on the affordances of Laclau and Mouffe's theoretical/epistemological framework with a focus on the notion of antagonism. Antagonisms were crucial for the generation of my theoretical framework. In particular, the three political features of mathematics (i.e., values, rationality, and formatting function) were generated in relation to the concept. This section is organised as follows. I first turn to mathematics education and comment on the ways in which the notion of antagonism fits with the existing literature. I then elaborate on how the notion of antagonisms can provide insights to that respect. I finally discuss how the political can be addressed through the work of Laclau and Mouffe, by considering the insights of this discursive theory in relation to Marxist approaches which are the starting point of Laclau and Mouffe's epistemology.

Laclau and Mouffe's discourse theory has not been widely used in mathematics education, with the exception of Chronaki and Kollosche (2019) who have used Laclau and Mouffe's discourse analysis framework in order to examine the political aspects of identity work. In doing so, they examine how within one discursive articulation "discursive truths come into existence, relate with identity categories, stand in conflict with each other or prevail over others and become hegemonic" (p. 458). Through the examples of mathematics and mathematics education, I tried to propose that Laclau and Mouffe's framework can also be valuable for investigating how antagonisms take place among different discourses. Following Mouffe (2005), I suggested that it is fruitful to think of the political as a space of antagonisms. In this sense, mathematics can be thought of as political since it is a space in which discourses collide and other ways of meaning making are excluded. From this perspective, education is not about the absorption or construction of a pre-existing knowledge, but it is instead a political process of negotiating meaning. Knowing means accepting, adopting, and acting within a discourse, while learning refers to the process of enculturating oneself into one discourse. This process is political, as it happens within a framework of antagonistic discourses and results in exclusion of other ways of meaning making.

Albeit not hegemonic, the idea that mathematics is a space of antagonisms is not new to mathematics education. Within formalist philosophy, which has profoundly influenced both the development of mathematics and mathematics education, mathematics is identical to formalised mathematics and there is no interest in the processes of mathematical discovery or the methodology of mathematics (Lakatos, 1976). However, at the epistemological level, this direction has been greatly challenged by Lakatos (1976), who argued that mathematics progresses through social interactions among mathematicians. In his book "Proofs and Refutations", Lakatos shows how the definition of a concept is neither pre-given by the framework nor a natural, uncontested discovery; it is instead a development that is the result of contest and interactions among mathematicians. In turn, the definition of a concept one way or another constructs different mathematical objects and, hence, different theorems, and different questions that lead the development of mathematical knowledge.

This epistemological turn has been taken up in mathematics education by scholars who work within what Lerman calls the social turn in mathematics education. Lerman (2000) attributes the interest of mathematics education scholarship in the work of Lakatos to “the humanistic image of mathematics it presents, as a quasiempiricist enterprise of the community of mathematicians over time rather than a monotonically increasing body of certain knowledge” (p. 22). Indeed, through the social turn, there is a transition from the way in which the individual learns mathematics by overcoming epistemological obstacles and encountering learning difficulties, towards the way in which the community of the class (i.e., the teacher, students, manipulatives, technology, etc.) interact towards the production of mathematical meaning. Within this direction, there is great focus on the ways in which different meanings are produced and interact with each other. Examples of this are literature on multiple representations of mathematical concepts (e.g., Özgün-Koca, 1998) and language diversity (e.g., Barwell, 2012; Planas, 2018).

The concept of antagonism of discourses and, more generally the discourse theory of Laclau and Mouffe’s tradition, provide theoretical tools to take up the epistemological turn signified by the work of Lakatos at the political level in mathematics education. More specifically, it allowed me to inquire into the macro discourses about mathematics that manifest in a mathematics classroom and consider their interactions. This inquiry can happen in such a way that the antagonistic and hegemonic relations of the discourses are highlighted. In this way, the political character of mathematics discourses becomes the focus.

While other discursive theoretical frameworks can and do offer similar theoretical spaces to think about political aspects, I consider the advantages of Laclau and Mouffe’s tradition to be twofold. First, Laclau and Mouffe’s work focuses on macro discourses and in this sense allows me to go beyond the interactions of a mathematical classroom and – through them – see the “bigger” discourses about mathematics that manifest. Something similar also happens with the work of Foucault as has been taken up in mathematics education by scholars like Valero (e.g., 2018) and Kollosche (e.g., 2014). Second, the notion of antagonisms allows us to specifically focus on the way in which discourses collide with each other in the mathematical space, which is hegemonically constructed as lacking any antagonisms and being the result of a rationally constructed consensus.

A point that needs to be highlighted here is that antagonisms of discourses do not necessarily imply antagonisms between actors. In fact, in my findings, I identified examples in which different discourses were articulated and brought into collision by the same individual. We also encounter this idea in Bakhtin's (1981) notion of polyphony which refers to one individual using multiple voices. This idea is crucial because it allows us to disentangle a discourse by the person who articulates it and, hence, focus on the characteristics and interactions of discourses themselves. Moreover, the reading of the data through this understanding of antagonism between discourses helps us challenge the view that mathematical culture lies with the professor and that the students are the ones who need to get socialised into it. As discussed through the examples in Chris's class, students are sometimes socialised into academic mathematics through a variety of channels and often advocate for more "hardcore" practices of academic mathematical culture (e.g., using a lot of advanced terminology) than the professor or the institution.

Working within the position that mathematics and mathematics education are political, a question that arises is how Laclau and Mouffe's framework can help us think about the available space of (political) action from this epistemological standpoint. In my view, the value of Laclau and Mouffe's post-Marxist, post-structural theory lies with the ways in which it deals with (what I consider to be) the blind spots of more traditional Marxist approaches. In the following paragraphs, I discuss how Laclau and Mouffe's work goes beyond the limitations of traditional Marxist approaches regarding what can/needs to be done in terms of (political) action in relation to mathematics education, while I argue that this is related to how the notion of reality and truth are taken up in Marxist approaches, often leaving little space for action.

While Laclau and Mouffe use Marxism as a starting point to think about the social, their approach is differentiated from classical Marxism, in that they do not assume the existence of a stable or unambiguous structure of the "reality" around us. For example, this means that their goal is not to describe how classes constitute divisions in society in an objective way or beyond any ambiguity (Jørgensen & Phillips, 2011). Instead, their focus is on how meaning is constructed and how objects are constituted as objects of discourse. This does not mean that the existence of these objects is disputed; while objects might exist objectively (in the sense that their existence does not depend on human thought), "they could [not] constitute themselves as objects outside any discursive condition of emergence" (Laclau & Mouffe, 1985/2001, p. 108). In my findings, the notion of formal mathematics is such an example of an object that does not exist objectively, but is instead constituted within discourse.

Drawing upon Derrida's work, Laclau and Mouffe argue that meaning is never fixed and conceptualise the political within this context. Discourses are partial fixations of meaning: "[a]ny discourse is constituted as an attempt to dominate the field of discursivity, to arrest the flow of differences, to construct a centre" (Laclau & Mouffe, 1985/2001, p. 112). However, this is never achievable once and for all. Universality is merely "political universality", or to put it differently, every universality is hegemonically mediated: "antagonisms are not *objective* relations, but relations which reveal the limits of all objectivity" (Laclau & Mouffe, 1985/2001, p. xiv). In this sense, the political becomes "an ontology of the social".

This perspective problematises the ways in which political action has been conceptualised within mathematics education by more traditionally critical or Marxist discourses, by highlighting some of their limitations. More specifically, a discourse about political action in mathematics education focuses on first understanding and then acting upon a situation that is considered to be inherently or objectively problematic, in order to overturn it. We can see this idea being evident in both critical mathematics education that has its roots in a Freirian framework, in which education is viewed as a tool to write and read the world (e.g., Gutstein, 2012) as well as in traditionally Marxist approaches in which the role of schooling and education is conceptualised in terms of a system that plays a very mechanistic role within capitalism (e.g., Cabral & Baldino, 2019). Both discourses (which despite their ontological similarity are logically incompatible as they cannot account for change within school and capitalism), can account for aspects of what we conceive to be "reality" in relation to education. Nevertheless, epistemologically there is no reason for assuming that one discourse more accurately describes reality than another (as these discourses do). Instead, any discourse constructs the objects that it describes and the interests that it represents (as opposed to first recognising them and then representing them) (Laclau & Mouffe, 1985/2001).

Laclau and Mouffe's framework addresses the historicity and systemic nature of discursive articulations, while by not treating meaning as fixed and accepting that "things could always be otherwise" (Mouffe, 2005, p. 18), it allows us to identify spaces for action, towards transforming mathematics education and, more generally, our societies. In this sense, an articulation, a "practice establishing a relation among elements such that their identity is modified as a result of the articulatory practice" (Laclau & Mouffe, 1985/2001, p. 105) is meaningful although it cannot establish a hegemony once and for all. In this sense, the goal is not to step outside of power- this is not possible; nor is it to get rid of power relations. Instead, political action can be thought of in terms of ongoing articulations, with the goal of establishing chains of equivalence between different ideas, practices, and movements, towards transforming social relations related to mathematics education.

7.6. Political Implications of this Work

This thesis focuses on the notion of the political, but it employs a conceptualisation of the political which might appear odd to a big part of the education community on the grounds of the political being associated with a certain form of social practice. In this section, I attempt to partially bridge this gap of different understandings, by explicitly discussing the kind of political action that I see as emergent from my so far theoretical discussion. More specifically, I propose some political implications of this thesis in research, teaching/curriculum, and the social sphere. In particular, I explore the importance of disrupting discourses as a political struggle in academia, making antagonisms visible as a political struggle in teaching, and reconceptualising mathematics as a political struggle in the sociopolitical sphere. But before exploring each of these three topics, I start with a more general point of view which has acted as a main motivation for this thesis from the start and which provides some context for the discussed political implications of this work.

My motivation for this work starts from the observation that formal mathematical knowledge is considered of higher order than informal knowledge. For example, adding two fractions algebraically is valued more than using manipulatives to add fractions. This does not only happen in the classroom. It happens more generally in the educational system, when mathematics becomes the gatekeeper to higher education and prestigious and high-paying jobs (e.g., Stinson, 2004). It also happens in society, when mathematical knowledge is assumed to provide a path to an absolute truth and its opposite is irrationality (e.g., Walkerdine, 1988). In a nutshell, mathematics is constructed as a special kind of knowledge, as the "Queen of Sciences". This relates to societal injustices, because a particular way of knowing and being (the mathematical, scientific, Western, etc.) is escalated to being more "reasonable", "efficient", "advanced", etc. than other ways of being and knowing. In what follows, I discuss how this regime of truth can be challenged within academia, teaching, and the sociopolitical sphere through the notion of antagonisms.

A first political implication of this thesis is disrupting discourses as a political struggle in academia. Academia is a social space and, hence, its discourses do not provide access to an ever-existing truth, but are instead constructs that are influenced by the current zeitgeist. This means that academia acts within regimes of truth, outside of which it is hard to think and produce research. The theory of Laclau and Mouffe urges us to accept the inevitability of antagonisms and welcome them as a part of the research practice. In this way, it becomes possible to consider how things could have been different and produce work that challenges the very framework in which we work. Such work is of course already produced in mathematics education (e.g., Kollosche, 2018b; Valero, 2018), but I argue that more research and thought in this direction is needed.

A second political implication of this thesis is the idea of making antagonisms visible in teaching. Mathematics has constantly been constructed by curricula as a knowledge with finality, which cannot be challenged in any way. This is a practice which hinders the political aspects of mathematical knowledge and mathematics education. Here it is useful to think of knowledge not as a matter of true or false and learning not as gaining access to an independent reality or truth. Instead, knowing can be understood as taking up a particular discourse. In this sense, knowing mathematics means having adopted, incorporated, or following a particular discourse about mathematics, where by discourse I refer to a totality that exceeds the linguistic and also contains material elements (Laclau & Mouffe, 1985/2001). For example, knowing first year calculus in a proof-oriented course not only means having memorised and understood the ε-δ definition of a function's limit, but also includes having spent time working with paper and pencil on the definition and its applications while sitting on a chair, reading about it from a textbook or notes that have been written in Latex and getting used to Latex's font, using one's hands to represent the process of "approaching", considering the "let ε<0" joke funny, or a variation of these depending on time and place. Similarly, learning can be thought of as a (political) negotiation of meaning; a process of enculturating oneself in one discourse, which is in an antagonistic relation with other discourses. My point here is that a course that struggles to reproduce only one discourse, such as the one just described, reinforces the construction of mathematics as a final, non-negotiable knowledge.

Nevertheless, mathematics cannot escape antagonisms as has been shown by the work of Lakatos (1976) as well as is supported by my findings. Mathematics classrooms are a space in which various discourses enter, struggling to fixate meaning. Highlighting how these discourses enter in antagonistic relationships can be a way of exploring the negotiation of meaning happening in mathematics classrooms. Hence, a political struggle in teaching would be to bring "antagonisms" to the forefront of a classroom in order to challenge the finality of mathematical knowledge and consider its ethical implications.

The third political implication of this thesis refers to reconceptualising mathematics as a political struggle in the sociopolitical sphere. There are mathematical practices in the public sphere, while we may also think of the public sphere as a site of mathematical learning. This is particularly evident in the examples of climate change and the COVID-19 pandemic; two global crises whose articulation as such is heavily mathematical. Regarding climate change, mathematics is used in the public sphere in order to describe, explain and communicate the phenomenon (Barwell, 2013). As technocratic discourses increasingly struggle to dominate the field of discursivity, individuals are increasingly required to interact with mathematics, act as knowing mathematics, and learn mathematics. This frames the public sphere as a space in which citizens are being educated (in a wide sense of the word) in and through mathematics. In another example from mathematics education scholarship, Chassapis (2017) shows how the proliferation of numerical and mathematical descriptions in the newspapers' front pages during the three first year of the economic crisis in Greece contributed to a technocratic framing of the economic-socio-political situation at the time. I wish to suggest that reflecting on this situation in terms of antagonisms of discourses and meaning negotiation can help us see how mathematics intersects with the political as well as how mathematics education (i.e., people's interactions with mathematics) is political. Moreover, acting within such a situation can mean navigating these antagonisms in ways that can serve social justice. In particular, this can happen by focusing our analyses on how alternative mathematical discourses might emerge as well as on producing alternative mathematical discourses. In the example of Greece, the Left produced numerical discourses that aimed at politicising politics. For example, a popular argument against neoliberal politics has been that youth unemployment almost reached 60%. Furthermore, producing discourses that actively reject the mathematical in political discourses is another form of political action that attempts to make discourses that are different than the mainstream ones hegemonic. For instance, this was also evident in Greece's social movements, for example through the slogan that people are not numbers.

7.7. Limits of the Study's Scope

In this section, I discuss the limits of this project. As would any study, my thesis is constructed by the discourses that I participate in as well as the nexus of discourses that the project is taking place in. It is not a journey to discovering an objective truth; such an objective truth, at least from the perspective of my theoretical and epistemological framework, does not exist. Hence, this work is a product of the data that I have collected, the theoretical and epistemological tools that I have been using, the mathematics education discourses which inform the study, and my own subjectivity. These surrounding discourses cannot be escaped and, hence, this study is a construct of these discourses and it is bound by them.

This research project happened in undergraduate education. While this choice has had certain advantages, such as that sociopolitical undergraduate mathematics education is scarce, undergraduate education is a space to which not everyone has access. This is a result of universities not being free of charge in North America. Two things are important here: first, not everyone can afford post-secondary education in Ontario and many people who do start education end up in debt. As a result (and considering that being able to and/or willing to be in debt to acquire education relates to social class cultures), the university population has certain class and social characteristics. Second, the institutional framework itself might often be an obstacle for facilitating a negotiation of the proposed knowledge, in the sense that university students are expected to socialise themselves into the culture of a certain discipline. Nevertheless, the data that I have collected and my analysis of them reflect the social organisation of the two classrooms, in which students do not talk much and rarely engage in negotiations of the taught mathematical knowledge. This institutional context has provided more space for my project to look into the political aspects of the formal academic mathematical discourses put forward by the professors themselves but it also masks the ways in which this knowledge is or can be negotiated.

Furthermore, the findings focus on the professors' discourses and the antagonisms within these discourses, while they only offer a glimpse of the ways in which students in the two classes interact, take up, and oppose these discourses. This is a result of the extent to which students participate in an undergraduate mathematics course as well as of the limited access that I had to students' voices and the absence of teaching assistants' voices. Inclusion of other spaces in which the courses took place (e.g., tutorials, peer groups) would allow for a deeper insight into the interactions among different discourses. Nevertheless, the difficulties in collecting data from these kinds of spaces reflect the everyday life and workload of students, as well as the organisational division of labour in the university.

7.8. Questions arising

In this section, I pose a few questions that arise from this thesis. These questions concern the norms and characteristics of a classroom discourse that encourages the development of antagonisms; the political characteristics of North America educational classroom discourses; and the affordances of Laclau and Mouffe's framework for inquiring into mathematics education research discourses.

The first arising question starts from the observation that there is a noticeable difference between the two courses in which I collected data: antagonisms occur much more in Chris's course than in Alex's. I have interpreted this finding in relation to two aspects: first, the amount of interactions between the professor and students is greater in Chris's class and second Chris's discourse is more polyphonic than Alex's in the sense that Chris often articulates discourses that collide with each other and discusses these antagonisms. A question that arises is how the interactions in a class allow for the manifestations of antagonisms and the kind of norms that can be established so that this can happen.

The second arising question refers to North American educational calculus discourse. In undergraduate programs, calculus is the standard mathematical area studied by students with many different academic foci. Universities have developed several kinds of calculus courses (e.g., proof-oriented, non-proof oriented, calculus for life sciences, etc.) in order to differentiate instruction considering the wide range of characteristics and needs of the student population. In this way, calculus education constitutes a wide range of practices which all have some common characteristics. Hence, a question that arises is what the political characteristics of North American educational calculus discourses are.

Finally, the third arising question refers to the theoretical and epistemological framework of Laclau and Mouffe. In this thesis, I used this framework to inquire into the discourses that emerge in two undergraduate courses. I propose that this framework can also be used to examine academic discourses themselves in order to consider how academic discourses construct the objects of their study in hegemonic ways as well as how antagonisms occur within academia. This inquiry is in the spirit of work already conducted in mathematics education (e.g., Valero & Knijnik, 2015; Walkerdine, 1988, 1998). Hence, the question that arises is what the affordances of Laclau and Mouffe's framework are for inquiring into mathematics education academic discourses.

7.9. Summary

This thesis aimed to explore what it can mean for mathematics to be political in the context of undergraduate courses. At the theoretical level, I suggested that mathematics can be thought of as political because it entails values, puts forward particular rationalities, and has a formatting function. At the empirical level, the exploration of the nexus of discourses that are evident in the two courses showed that multiple discourses are present in mathematics courses and weave the political character of mathematics either by fitting well together within a modernist framework or by antagonising each other in an effort to fix the meaning of mathematics.

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Appendix 1: Index of Examples from Chris's Class

Appendix 2: Index of Examples from Alex's Class

Appendix 3: Recruitment Documents

Recruitment Email to Professors

Dear Professor [Professor's Last Name],

My name is Dionysia Pitsili-Chatzi and I am a PhD student in mathematics education at the University of Ottawa, working under the supervision of Professor Richard Barwell. I am contacting you in order to request that you consider participating in my research study, titled "Political Functions of Mathematics in two Undergraduate Mathematics Courses".

The goal of my study is to explore how mathematics might be thought of as political in undergraduate university classes, or in other words as a space in which meanings are negotiated (meanings about the relation of mathematics with the physical and social world, the rationality of mathematics and the importance of mathematics). I am going to conduct this study in two introductory mathematics courses, one intended for mathematics majors and one intended for non-STEM majors. I was hoping that you consider participating in the study through the course [Course's Name] that you will be teaching in the [Semester].

Please see below for details about what will happen in the study.

<u>What will happen in the study:</u>

The project is an ethnographic study in two undergraduate mathematics courses. If you choose to participate, I will observe your class for the duration of one semester and will conduct some interviews with you, your TAs and some of your students. The interviews will be about features of mathematics (e.g., their role in describing the physical and social world, the mathematical rationality/rationalities, the importance of mathematics, etc.) and I will include some incidents from the course or quotes from the textbook/notes/syllabus to discuss with you, your teaching assistants, and students, in order to better understand your views about mathematics and its social roles.

More specifically, if you agree to participate:

- I will be attending all your classroom sessions during the semester. In my visits, I will keep notes about what happens in the classroom and will take pictures of the board. I will not take any pictures of anybody's face. If you agree, I will also audio-record you but I will not be audio-recording the students.

- If you hold meetings with your teaching assistants about the content of the course (e.g., training meetings), I will attend these meetings, if you are comfortable with this. I will keep notes, in which I will not include any information that is not about the content of the course (e.g., information about individual students).
- I will conduct and audio-record two interviews with you during the semester of approximately one hour each.
- If you are available after or before a class, I will sometimes hold short discussions with you (of maximum fifteen minutes each) about something that happened during the course. With your permission, I might audio-record these discussions.
- If there is an electronic aspect in your course, I will keep notes about what happens there.
- collect the course's syllabus and any notes or other materials that you distribute and/or use with the students.

Please, note that if some of the above do not apply to your course, if you are not comfortable, or do not have time for some of the above, you can still participate as long as you agree that I can visit your classes and keep notes about what is happening (in relation to mathematics).

Moreover, I will do the following, which do not involve your (direct) participation. I will:

- attend some tutorials held by teaching assistants. I will keep notes about what happens, take pictures of the board), and I might audio-record those sessions.
- conduct interviews with teaching assistants (up to 1-hour interview with each TA).
- conduct interviews with up to 10 students (1-hour interview with each of the students)
- conduct focus group discussions with students (three 1-hour meetings with a group of 5-8 students).

In conducting these activities, I will respect students' and TAs' consent. I will only attend the tutorial sessions and take interviews from TAs who consent to participate in the research project. I will only conduct interviews and focus group discussions with students who give consent to participate in these activities. When I take notes about what happens in the classroom, I will not include any identifiers (e.g., names) about the students.

After the project is finished, I will use these data to write my dissertation and publish papers. To protect your anonymity, I will not disclose the name and exact place of the university, the course's code and name, your name, nor any other participants' names.

Please, note that besides you, I have also requested from some other professors to participate in the study. However, I can only include two courses in my project. If multiple professors express interest to participate in the project, I will select the professors who participate on a first come/first serve basis.

Thank you for reading this information. Please, do not hesitate to contact me if you have any questions or concerns. I hope that you consider participating in the study. If you are interested in participating, please contact me by [deadline].

Kind regards,

Dionysia Pitsili-Chatzi

Letter of Invitation – Distributed to Students

Letter of Invitation (Students)

Title of the study: Political Functions of Mathematics in two Undergraduate Mathematics Courses

Researcher: Dionysia Pitsili-Chatzi (PhD Candidate)
Faculty of Education, University of Ottawa
[Dionysia's institutional email]
[Dionysia's personal phone number]

Supervisor: Prof. Richard Barwell
Faculty of Education, University of Ottawa
[Richard's institutional email]
[Richard's office phone number]

Dear Student,

Hello. My name is Dionysia Pitsili-Chatzi and I am a PhD student at the University of Ottawa. My research focus is on mathematics education. In order to conduct my research project, your professor has kindly agreed that I can do my study at your course during this semester. The goal of my study is to explore how mathematics might be thought of as political, or in other words as a space in which meanings are negotiated.

Please see below for details about what will happen in the study.

<u>What will happen in the study:</u>

The project is an ethnographic study. I will attend and take notes from the course's lectures and some tutorial sessions. I will sometimes audio-record your instructor, but I will never audio-record you. If a student's voice is heard indirectly via the instructor's audio-recording, I will not transcribe that part nor will I use it in any other way. I will also take pictures of the board, but I will never take pictures of anybody's faces.

During my visits, I will also be keeping notes on what happens in the classroom. I will keep notes about what some students say, but I will not include anybody's names or any personal or identifying information. Moreover, I will not focus on you individually, but I will be interested in and keeping notes about what is being told about mathematics.

I will also be monitoring the electronic platform. I will be keeping notes on what is being discussed there about mathematics (e.g., what material your instructor shares, what kind of questions are asked, etc.). When I keep notes, I will never keep notes of anybody's names or any other personal or identifying information.

Finally, during my visits, I might chat with you before or after a class. I might make a note of something you told me, but without including your name or any personal information about you. If you do not want to chat with me, just let me know. There will be no consequences regarding your participation in the course.

Similarly, if you are asked to work as a group at some point during class time (either at a lecture or during tutorials), I might approach your group and ask if it is ok to join you and listen in. I will keep some notes about what your group discusses, but I will not include any information about who says something or any identifying information (e.g., names) about you or any of your classmates. If you do not feel ok with that, please let me know. Your participation (or not) in the study as well as your interactions (or not) with me will not in any way impact your participation in the course or your grades (neither positively nor negatively).

How you can participate:

If you would like to participate:

- You can sign up for an interview with me. The interview will be approximately one hour long and will take place somewhere on the university campus, during the semester. During that time, I will ask you some questions about mathematics and its roles in society. I will audio-record the interview.
- You can sign up to participate in a focus group with 4-7 of your classmates. The focus group will meet three times during the semester for an hour each time, somewhere on the university campus. During that time, we will discuss about mathematics and its roles in society. I will audio-record the session.

In both cases, there will be pizza lunch.

If you would like to participate, please send me an email, stating your name and whether you would like to participate in a focus group or in an interview (or any of them).

I will conduct 10 interviews and one focus group with up to 8 students. **If more students express interest to participate, I will randomly select 10 students for interviews and 8 students for the focus group.**

After the project is finished, I will use this data to write my dissertation and publish papers. To protect your anonymity, I will not disclose the name and exact place of the university, the course's code and name, the professor's name, your name, nor any other participants' names.

Please note that your participation (or not) in the study as well as your interactions (or not) with me will not in any way impact your participation or involvement in the course (neither positively nor negatively). More specifically, there will be no consequences regarding your grades in the course. Also, your professor will not know whether you participate in the study or not.

Moreover, even if you agree to participate, you can change your mind anytime.

Thank you for reading this information. Please do not hesitate to contact me, if you have any questions or concerns.

Dionysia Pitsili-Chatzi

Appendix 4: Consent Forms

Professor Consent Forms

Researcher: Dionysia Pitsili-Chatzi (PhD Candidate) Faculty of Education, University of Ottawa [Dionysia’s institutional email] [Dionysia’s personal phone number]	Supervisor: Prof. Richard Barwell Faculty of Education, University of Ottawa [Richard’s institutional email] [Richard’s office phone number]

Consent Form (Professors)

Title of the study: Political Functions of Mathematics in two Undergraduate Mathematics Courses

I am invited to participate in the abovementioned research study conducted as part of Ms Pitsili-Chatzi’s PhD thesis, under the supervision of Professor Richard Barwell.

Purpose of the Study: The purpose of the study is to explore how mathematics is political in the context of undergraduate mathematics education. More specifically the goal is to examine how discourses about some features of mathematics (its social roles, its rationality, and its importance) are negotiated in undergraduate mathematics courses.

Participation:

My participation entails that Dionysia Pitsili Chatzi will be attending all my lectures during the semester. In her visits, she will keep notes, and take pictures of the board. In her notes, she will not include and personal or identifying information about the students (e.g., names) nor will she take a photo of anybody’s faces. If at any time during my classes students are asked to work collaboratively, she will approach a group of students and observe them working together. She might ask questions, as long as they do not distract the students from what they are asked to do. She will only approach a group if all of the students in the group agree that she does so.

If I agree, Dionysia Pitsili Chatzi will audio-record my classes; will attend meetings with the teaching assistants, which are about the content of the course (e.g., training meetings); will conduct and audio-record two interviews with me during the semester of approximately one hour each; might hold short discussions with me (of maximum fifteen minutes each) before or after a class during the semester about something that happened during the course; will have access to the course’s electronic platform and keep notes about what happens there; and will collect the course's syllabus and any notes or other materials that I distribute and/or use with the students. **Please see the check boxes at the end of this form.**

Dionysia Pitsili Chatzi will also conduct the following activities related to the course that I teach, which do not directly involve my participation. She:

Will attend tutorials held by teaching assistants. She will keep notes from the sessions, take pictures of the board, and might audio-record those sessions.
Will conduct interviews with teaching assistants
Will conduct interviews with students
Will conduct focus group discussions with students

Risks: While the researcher will make every effort to protect my anonymity, this research project engages a large number of participants and, therefore, there is always a risk that identifying information about the university, the course, or myself become public. I have received assurance from the researcher that every effort will be made to minimize these risks (please see below).

Benefits: My participation in this study will engage me in a process where I will have space to reflect upon and discuss about my professional practice. Moreover, this project will contribute to mathematics education scholarship, by providing an empirical account that offers a critical view on mathematics in post-secondary contexts.

Confidentiality and anonymity: I have received assurance from the researcher that the information I will share will remain strictly confidential. I understand that the contents will be used only for the research project and that my confidentiality will be protected.

Anonymity will be protected in the following manner: The name and exact place of the university will not be disclosed in any presentations, publications, or the researcher's dissertation; instead a general description of the location will be employed (e.g., a university in Ontario). Similarly, the name of the course and direct quotes of any publicly available data (e.g., the syllabus if published online) will not be mentioned. All participants will be given pseudonyms for the purposes of analyzing and publishing.

Conservation of data: The data collected (notes, audiorecordings, transcripts, photos, photocopies of course materials) will be kept in a secure manner. All computer files will be password-protected in a password-protected computer, while all hard copies of data and research documents will be kept at a locked cabinet. Data will only be shared with the thesis supervisor. Anonymized data will be kept indefinitely, but all research documents will be retained for a period of five years.

Voluntary Participation: I am under no obligation to participate and if I choose to participate, I can withdraw from the study at any time and/or refuse to answer any questions, without suffering any negative consequences. If I choose to withdraw, all data gathered until the time of withdrawal will be destroyed.

Moreover, at any time, I can ask Dionysia Pitsili Chatzi to leave the classroom that I teach or the room in which I hold meetings with the teaching assistants. At any time, I can turn off the audio-recording device.

Acceptance: I, ______________________, agree to participate in the above research study conducted by Dionysia Pitsili Chatzi of the Faculty of Education, University of Ottawa, which research is under the supervision of Professor Richard Barwell.

I hereby agree that Dionysia Pitsili Chatzi

	YES/NO
Can audio-record my classes. If a student's voice is heard indirectly via my audio-recording, she will not transcribe that part nor will she use it in any other way.	
Will attend meetings with the teaching assistants, which are about the content of the course (e.g., training meetings). She will keep notes, but not about anything that is not about the content of the course (e.g., information about individual students),	
Will conduct two interviews with me during the semester of approximately one hour each. The interviews will take place on campus at a place and time that is convenient for me.	
Will audio-record the interviews.	

Might hold short discussions with me (of maximum fifteen minutes each) before or after a class during the semester about something that happened during the course.	
might audiotape these discussions, with my permission.	
Will have access to the course's electronic platform and keep notes of what happens there.	
Will collect the course's syllabus and any notes or other materials that I distribute and/or use with the students	

If I have any questions about the study, I may contact the researcher or her supervisor.

If I have any questions regarding the ethical conduct of this study, I may contact the Protocol Officer for Ethics in Research, University of Ottawa, Tabaret Hall, 550 Cumberland Street, Room 154, Ottawa, ON K1N 6N5
Tel.: (613) 562-5387
Email: ethics@uottawa.ca

There are two copies of the consent form, one of which is mine to keep.

Participant's name: Participant's signature: Date:

______________________ ______________________ ____________________

Researcher's signature: Date:

______________________ ____________________

Student Consent Form

Researcher: Dionysia Pitsili-Chatzi (PhD Candidate) Faculty of Education, University of Ottawa [Dionysia's institutional email] [Dionysia's personal phone number]	Supervisor: Prof. Richard Barwell Faculty of Education, University of Ottawa [Richard's institutional email] [Richard's office phone number]

Consent Form (Students- Interviews)

Title of the study: Political Functions of Mathematics in two Undergraduate Mathematics Courses

I am invited to participate in the abovementioned research study conducted as part of Ms Pitsili-Chatzi's PhD thesis, under the supervision of Professor Richard Barwell.

Purpose of the Study: The purpose of the study is to explore how mathematics is political in the context of undergraduate mathematics education. More specifically the goal is to examine how discourses about some features of mathematics (its social roles, its rationality, and its importance) are negotiated in undergraduate mathematics courses.

Participation:

My participation in the study entails participating in an interview with Dionysia Pitsili-Chatzi. The interview will be approximately one hour long and will take place somewhere on the university campus, during the semester. We will discuss about issues related to mathematics in general as well as the mathematics that we do in the course.

Risks: There is no foreseeable risk related to my participation in the study.

Benefits: My participation in this study will engage me in a process where I will have space to reflect upon and discuss about my professional practice. Moreover, this project will contribute to mathematics education scholarship, by providing an empirical account that offers a critical view on mathematics in post-secondary contexts.

Confidentiality and anonymity: I have received assurance from the researcher that the information I will share will remain strictly confidential. I understand that the contents will be used only for the research project and that my confidentiality will be protected.

Anonymity will be protected in the following manner: The name and exact place of the university will not be disclosed in any presentations, publications, or the researcher's dissertation; instead a general description of the location will be employed (e.g., a university in Ontario). Similarly, the name of the course and direct quotes of any publicly available data (e.g., the syllabus if published online) will not be mentioned. All participants will be given pseudonyms for the purposes of analyzing and publishing.

Conservation of data: The data collected (notes, audio-recordings, transcripts, photos, photocopies of course materials) will be kept in a secure manner. All computer files will be password-protected in a password-protected computer, while all hard copies of data and research documents will be kept at a locked cabinet. Data will only be shared with the thesis supervisor. Anonymized data will be kept indefinitely, but all research documents will be retained for a period of five years.

Voluntary Participation: I am under no obligation to participate and if I choose to participate, I can withdraw from the study at any time and/or refuse to answer any questions, without suffering any negative consequences. If I choose to withdraw, all data gathered until the time of withdrawal will be destroyed.

Acceptance: I, ____________________, agree to participate in the above research study conducted by Dionysia Pitsili Chatzi of the Faculty of Education, University of Ottawa, which research is under the supervision of Professor Richard Barwell.

I hereby agree that Dionysia Pitsili Chatzi

	YES/NO
Can audio-record the interview.	
Can take pictures of my work and/or notes, that I show her during the interview.	

If I have any questions about the study, I may contact the researcher or her supervisor.

If I have any questions regarding the ethical conduct of this study, I may contact the Protocol Officer for Ethics in Research, University of Ottawa, Tabaret Hall, 550 Cumberland Street, Room 154, Ottawa, ON K1N 6N5
Tel.: (613) 562-5387
Email: ethics@uottawa.ca

There are two copies of the consent form, one of which is mine to keep.

Participant's name:	Participant's signature:	Date:
______________________	_______________________	____________________

Researcher's signature:	Date:
_______________________	____________________

Appendix 5: Interview Guides

Interview with Chris

The following questions were used as a guide for the interview with Chris:

- This class is often abstract, by which I mean that proofs are often abstract. My sense from your classes is that you constantly try to make connections between the world of mathematics, so to speak, and ideas or concepts that mean something to your students. Could you tell me a little bit about how you try to do that and what the rationale behind it is.
- In one of your classes that I attended, you were talking about negations and as you were working on what it means to negate a negation, you gave the example of "I'm not not going to school tomorrow". You noted that a certain connotation that comes with saying I'm not not going to school is lost when you translate this sentence from English to mathematics. Can you talk a bit about this, maybe at a more general level? What is lost when we use mathematics to describe or interpret something around us? Does it matter?
- Sometimes, when you work on some proofs, you talk about how you've moved from simpler stuff to more complicated stuffs. Building up complexity in a sense.
- Sometimes, in class you prove something that some people would think of as being self-evident. For example, you've proven that an even number is not odd. Why do you choose to present these proofs to your students and how do you think they are received?
- Sometimes you talk about the reader of a proof. As I see it, you ask them to create proofs in a way that they don't write them for themselves, but also to consider that somebody will read them.
- When you showed Euclid's proof about there being an infinite number of primes, you made a point about this proof being considered beautiful. What I wanted to ask is why you decided to spend some time opening this discussion and how you think that ideas about the beauty of mathematics are received by students.
- What is the position of proofs within mathematics? Once you mentioned "Proofs are the language of mathematics" and then corrected yourself saying "a language of mathematics".
- It appears to me that you try to create room for diversity in your class. You say that people have different intuitions about mathematical ideas and this does not mean that one is necessarily correct and another wrong. You encourage students to construct and share their own proofs. In

multiple occasions, you have said that this is “a” proof for a theorem, versus this is the proof. Could you talk about that? What is it that you try to do?

- You often ask students to reflect and express their thoughts about a proof. Sometimes, you ask if a proof feels natural or unnatural. When you finish a proof, you do not only ask them if they have any questions, but also if they have any observations, comments, or even complaints. Why do you do that and how do students react?

Interview with Chris's Students

Consent Form.

- Description of the consent form
- Making sure that any questions/concerns that they have about the process are answered.

A bit about themselves

- Tell me a bit about yourself.
- What are you majoring in?
- How does this course fit into your schedule and/or academic plan?

Introduction

- My study is about what happens in undergraduate mathematics classes. More specifically, I am interested to see what kind of ideas about mathematics are brought into your classroom by your professor, the textbook, the syllabus etc. But I am also interested to see how students like you receive these kinds of ideas: do they make sense, are they somehow weird, are there alternative ideas about mathematics that resonate more with you.
- I have several topics that I would be interested to explore, but if there is something in particular that you would like to talk about in relation to what I just said, or the handout I had given you, something that you've been thinking about mathematics after I introduced the study to you, bring it on.

Math in general-Prompt Question:

- What is this class about?
- What is a proof to you?
- What are some characteristics of a math professor? What stands out to you as being very characteristic of [Chris]?
- What is the role of proofs in mathematics?

Relation between Mathematics and aspects of the social/physical world

- How do you see the relation between mathematics and the social/physical world in the context of the course?

- Does this resonate with you?
- Is there space for alternative ways of thinking or talking about this relationship in the classroom?
- This class is often abstract, by which I mean that proofs are often abstract. How do you see what you do in the class as being connected with the world around you? And by asking this, I want you to both consider and reflect on how you think that the course describes this kind of relationship and what you personally find that this relationship looks like.
- In one of your classes that I attended, [Chris] was talking about negations. And as he was working on what it means to negate a negation, he gave the example of "I'm not not going to school tomorrow" and he said that this means that I am going to school tomorrow. But he also noted that a certain connotation that comes with saying I'm not not going to school is lost when you translate this sentence from English to mathematics. I want you to think about and tell me about this part that is "being lost". Not necessarily in relation to this particular example, but maybe a little bit more generally: Is something lost when we use mathematics to describe or interpret something around us? If yes, what could that be? Does it matter?
- A mathematical statement can either be true or false. It can't be anything inbetween or maybe a little bit of both. How's that influencing the way that we make sense of stuff that is not mathematical through mathematics?

Mathematical Rationality

- What kind(s) of mathematical logic is (are) present in the course?
- Does this resonate with you? Is this logic relevant anywhere else in your life, outside this math course? Is there space for alternative mathematical rationalities/logic in the classroom?
- How do you make sense of methods and proofs? Do they mean something to you and if so, does that have to do with experiences you've had in mathematics or with experiences you've had outside of mathematics? (related to [Chris's] effort to make math mean something to them)
- Sometimes, when you work on some proofs, Chris talks about how you've moved from simpler stuff to more complicated ones. Building up complexity in a sense. What do you think of that as a part of a mathematical rationality?
- Sometimes, in class you prove something that some people would think of as being self-evident. For example, you've proven that an even number is not odd. Most people I know would say that there's no argument you need to make to support this statement, it's just the way it is. What are your thoughts on this?

- Sometimes your prof asks you if a proof feels natural or unnatural. What is that about?

Importance of mathematics

- In what ways mathematics is captured as important in the context of the course?
- How do you see this idea/discourse being rooted in society? Where does it come from?
- What value does mathematics have for you? Is it important? Is it something else?
- One way that mathematics can be said to be important is because it is something that humanity has been doing for thousands of years.
- Another way that mathematics can be said to be important is because it can model and solve any problem.
- Mathematics can be said to be important because it is pure logic. What is your take on that?
- Why are proofs important? How are proofs captured as important in the class and how are they important for you?
- Chris often talks about the reader of a proof. As I see it, he asks you to create your proofs in a way that you don't write them for yourselves, but also that you consider that somebody will read them. Is this something that you see as an important part of what Chris tries to teach you and is it important for you? One day you proved Euclid's proof about there being an infinite number of primes. Chris made a point about this proof being considered beautiful. What do you think?

Interview with Alex

Introduction

- The idea behind the interview is that I can get a better insight into the professor's rationale about certain topics that are related to mathematics, I am basically interested to see what kind of discourses about mathematics you bring into the course. And then also, how students negotiate these discourses, by this I mean do they accept them, do they refute them, do they bring alternative discourses about mathematics into the course? That's why I am also doing some interviews with students.

Audience and Context of the Course

- I know that the math department website it says that this course is the default calculus course for science students. But, I was hoping that you could talk a little bit more about which students take this course. (in terms of backgrounds and academic aspirations).
- How diverse is the audience?
- How does that affect the kind of math taught in the course?
- Can you describe the kind of mathematics (i.e., mathematical topics, mathematical methods, etc.) that the course focuses on and why this approach has been chosen?

Mathematics and its relation to the Physical and Social World

- Can you describe how you try to capture the relation between mathematics and the social/physical world in your class?
- There are multiple references about the weather, economy, science (gravity, astronomy)
- Do you find that there are any aspects of the physical/social world that cannot be captured through mathematics? How does that influence what/how you teach in your class?
- How do students react to the ways in which you and this course in general try to capture a relation between mathematics and the social/physical world? Do they bring in their own insights/ do they ever negotiate the discourses offered by you or the book, do they accept it, do they refute it?
- At some point, as you were teaching about the relation of a function and its tangent line, you were describing how we often try to tackle some complex problem and we do that by trying to solve a simpler problem. In this process, we might lose some details, but we hope to be able to get back to the original problem. Can you talk about that? So, what do we lose by using mathematics to understand or describe the world? And how does it matter?

- In one of your classes, as you were talking about the derivative, you were saying that what we can detect in nature is how things change: so the derivatives rather than the functions themselves. Yet, my impression is that the way it works in your class is the opposite: you give students the function, they calculate the derivative and they figure stuff out.

Mathematical Rationality

- In what aspects of mathematical rationality do you try to enculturate the students? How do students negotiate the kind of rationality that you try to enculturate them into through the course.
- For example, there are not many proofs in the course.
- You sometimes tell your students how important it is to "get the idea" of something. And it seems to me that this somehow is contrasted to being able to carry out a procedure in all its glory or be super precise about something. Can you talk more about that?
- As you introduced derivatives, you strongly tried to make a point about why tricks (when computing limits) do not always work and for this reason methods are necessary. Similarly, for the simpler example of how to solve a quadratic equation. Is the idea of methods something that's central in the kind of rationality that the course promotes and, if so, how?

Importance of Mathematics

- In what ways do you consider mathematics to be an important human activity? And how do you try to promote that view to your students?
- How do students negotiate this idea about the importance of mathematics?
- In what ways do you think mathematics is valued by your students? As important, non-important, or otherwise?
- In your classes, you often make the point that machines can perform many tasks but it's important to have an intuition of what happens and how they do so, in order to be able to interpret the results correctly and to detect mistakes. Can you talk a bit about this?
- I get the sense from your classes that you consider learning mathematics important, also because this is something that humans have been doing for many years. For instance, you talk about the quadratic formula in relation to the Babylonians, or in relation to astronomy, etc. Can you talk about this?

Interview with Alex's Students

Consent Form.

- Description of the consent form
- Making sure that any questions/concerns that they have about the process are answered.

A bit about themselves

- Tell me a bit about yourself.
- What are you majoring in?
- How does this course fit into your schedule and/or academic plan?

Introduction

- My study is about what happens in undergraduate mathematics classes. More specifically, I am interested to see what kind of ideas about mathematics are brought into your classroom by your professor, the textbook, the syllabus etc. But I am also interested to see how students like you receive these kinds of ideas: do they make sense, are they somehow weird, are there alternative ideas about mathematics that resonate more with you.
- So, I have several topics that I would be interested to explore, but if there is something in particular that you would like to talk about in relation to what I just said, or the handout I had given you, something that you've been thinking about mathematics after I introduced the study to you, bring it on.

Math in general-Prompt Question:

- What is this class about?
- What does it mean to do mathematics?

Relation between Mathematics and aspects of the social/physical world

- How do you see the relation between mathematics and the social/physical world in the context of the course?
- Does this resonate with you?
- Is there space for alternative ways of thinking or talking about this relationship in the classroom?
- How do you see this idea/discourse being rooted in society? Where does it come from?

- In the class, there are sometimes references about the weather, economy, science (gravity, astronomy). How do these resonate with you?
- Are there things that we cannot describe/interpret with mathematics?
- What does it mean to use mathematics for understanding the social/physical world?

Mathematical Rationality

- What kind(s) of mathematical logic is (are) present in the course?
- Does this resonate with you? Is this logic relevant anywhere else in your life, outside this math course?
- Is there space for alternative mathematical rationalities/logic in the classroom?
- How do you see this idea/discourse being rooted in society? Where does it come from?
- Three possible ways to think about mathematics follow – What do you think about them?
    a. Developing methods as opposed to trying each case separately is all mathematics is about.
    b. Proofs is all mathematics is about.
    c. Figuring out a way to solve a problem is all mathematics is about.
- As your professor introduced the derivatives, he strongly tried to make a point about why tricks (when computing limits) do not always work and for this reason methods are necessary. Similarly, regarding how to solve a quadratic equation: once he made a point about how using the quadratic formula is better vs searching for roots through tricks. What do you think?
- Your professor often tells that it is important to “get the idea” of something. Is this what math is about?
- At some point, when teaching about the relation of a function and its tangent line, your professor was describing how we often try to tackle some complex problem and we do that by trying to solve a simpler problem. In this process, we might lose some details, but we hope to be able to get back to the original problem. What are your thoughts on that?

Importance of mathematics

- In what ways mathematics is captured as important in the context of the course?
- How do you see this idea/discourse being rooted in society? Where does it come from?
- What value does mathematics have for you? Is it important? Is it something else?

- One way that mathematics can be said to be important is because it is something that humanity has been doing for thousands of years.
- Another way that mathematics can be said to be important is because it can model and solve any problem.
- A third way in which mathematics can be said to be important is because it is pure logic.
- What are your thoughts on these?
- In one of the lectures that I attended, your professor suggested that machines can perform many tasks but it’s important to have an intuition of what happens and how they do so, in order to be able to interpret the results correctly and to detect mistakes. What are your thoughts on *that?*

Appendix 6: Example of a Thinking Map