Motivic Decomposition of a Hyperplane Section of a Milnor Hypersurface Twisted by a Crossed Product Algebra

Evan Marth

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

Let A be a central simple algebra over an arbitrary field F. Associated to A, there is a twisted Milnor hypersurface $X(A)$. Given an element $\alpha \in A$ which generates a Galois extension L of F with $[L : F] = \deg A$, we construct a hyperplane section $Y(A, \alpha)$ of $X(A)$ and give a motivic decomposition for $Y(A, \alpha)$. This generalises work of Xiong and Zainoulline ([XZ]).

Dedications

To my parents

Acknowledgements

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Contents

Preface

As the title indicates, this thesis is concerned with the motivic decomposition of a hyperplane section of a Milnor hypersurface twisted by a crossed product algebra. This requires some elaboration.

Motives

To any smooth projective variety X, one may associate to it a *motive* $\mathbf{M}(X)$. Much like the relation of vectors to vector spaces, there is not much sense in talking about a motive in isolation. Rather, there is an additive category of motives $\mathbf{Mot}_F$, and a contravariant functor from smooth projective F-schemes to $\mathbf{Mot}_F$ taking X to $\mathbf{M}(X)$. In some sense, $\mathbf{Mot}_F$ is a "linearisation" of smooth projective varieties. In particular, there is a good notion of decomposition of objects into direct summands; this is what is meant by a motivic decomposition.

From another perspective, much like singular cohomology with $\mathbf{Z}$ coefficients takes values in graded abelian groups, we can think of the aforementioned functor from smooth projective F-schemes to motives as a cohomology theory taking values in $\mathbf{Mot}_F$. For various cohomology theories $H^\bullet$ (e.g., Chow groups, ℓ-adic cohomology), there are then "realisation functors" on $\mathbf{Mot}_F$, sending $\mathbf{M}(X)$ to $H^\bullet(X)$. In this sense, motives are a "universal cohomology theory", and a motivic decomposition of $\mathbf{M}(X)$ translates into a "geometrically meaningful" decomposition of the graded abelian groups $H^\bullet(X)$, as it would have an incarnation in any suitable cohomology theory. For more on the philosophy of motives, see for example [An], [Kl] or [Ma].

Twisted Milnor hypersurfaces

Following [LM, §2.5.3], we define Milnor hypersurfaces to be effective Cartier divisors $H_{n,m} \subseteq \mathbf{P}^n_F \times \mathbf{P}^m_F$ given by a generic section of $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. In our case, we are concerned with the Milnor hypersurfaces $H_{n,n}$ for $n \geq 1$, which can be interpreted as a partial flag variety $\mathbf{Fl}(V, 1, n)$, $\dim V = n+1$, that is

$$\mathbf{Fl}(V, 1, n) = \{W_1 \subseteq W_n \subseteq V : \dim_F W_i = i\},$$

by identifying one copy of $\mathbf{P}$ with $\mathbf{P}(V)$, the other with $\mathbf{P}^\vee(V)$ and taking the section to be that induced by the canonical pairing $V \otimes V^\vee \to F$. To a central simple algebra A of degree $n+1$, we associate a twisted form $X(A)$ of $H_{n,n}$ by making an analogous partial flag construction for A using right ideals. Informally,

$$X(A) = \{I_1 \subseteq I_n \subseteq A : \dim_F I_k = k(n+1)\}$$

In the case $A = \mathrm{End}(V)$, this gives $\mathbf{Fl}(V, 1, n)$.

Hyperplane sections

As in [XZ, §1.2], we define a smooth hyperplane section of $X(A)$ as follows: let $\alpha \in A$ be an element such that $F[\alpha]$ is an étale F-algebra of degree $n+1$. We define

$$Y(A, \alpha) = \{I_1 \subseteq I_n \subseteq A : \dim_F I_k = k(n+1), \alpha I_1 \subseteq I_n\}$$

In [XZ], only the case where A is a cyclic algebra is considered, and α is assumed to generate a cyclic Galois extension. In this thesis, we consider crossed product algebras (see [GS, Remarks 2.2.13]) of this kind more generally. Namely, we assume that $L = F[\alpha]$ is a Galois extension of F. This is the sense in which the Milnor hypersurface is "twisted by a crossed product algebra". The main result of the thesis is the following:

Theorem (2.3.1)**.** *With the notations as above, the motive of $Y(A, \alpha)$ decomposes as*

$$\mathbf{M}(Y) \cong \bigoplus_{i=0}^{n-2} \mathbf{M}(\mathrm{SB}(A))(i) \oplus \mathbf{M}(\mathrm{Spec}\, L)(n-1)$$

where $\mathrm{SB}(A)$ *is the Severi-Brauer variety of* A*.*

This is a generalisation of the decomposition obtained in [XZ] and holds exactly, not "up to phantoms" – that is, motives which vanish after base change to a larger field.

Overview of the proof

The core of the proof is to relate the motive of a projective bundle to that of a smooth effective divisor on it of class $\mathcal{O}(1)$. This is undertaken in Section 2.2. The end result of this work is

Corollary (2.2.1)**.** *Let B be a smooth projective variety and $\mathcal{E}$ a locally free sheaf on B of rank $r+1$. Let $X = \mathbf{P}(\mathcal{E})$ and $s \in H^0(X, \mathcal{O}_X(1))$ such that the effective divisor*

$Y \subseteq X$ is smooth and $Z \subseteq B$, the zero locus of s viewed as a global section of $\mathcal{E}^\vee$, is also smooth of pure codimension $r+1$.

Let $\pi : X \to B$ be the projection map, $\pi|_Z : \pi^{-1}(Z) \to Z$ its restriction and $i : \pi^{-1}(Z) \hookrightarrow Y$ the inclusion map. If for all smooth projective F-schemes S, the homomorphisms $(i \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^* : \mathrm{CH}^\bullet(Z \times_F S) \to \mathrm{CH}^\bullet(Y \times_F S)$ are injective, then the motive of Y decomposes as

$$\mathbf{M}(Y) \cong \bigoplus_{i=0}^{r-1} \mathbf{M}(B)(i) \oplus \mathbf{M}(Z)(r)$$

To apply this to $Y(A, \alpha)$, it is proven that $X(A)$ is a projective bundle over $\mathrm{SB}(A)$ of relative dimension n (Theorem 2.1.1), and Y is a divisor of class $\mathcal{O}(1)$ (Proposition 2.1.1). The zero locus $Z \subseteq B$ can be interpreted as the locus of B over which the fibres of $Y(A, \alpha)$ are the same as those of $X(A)$.

Proposition (2.3.1)**.** *The zero locus Z is isomorphic to* $\operatorname{Spec} L$*.*

Thus, if one can prove the injectivity hypothesis of corollary 2.2.1, one has proven theorem 2.3.1. A key observation is that all of the points of Z are defined over L since L is Galois. Effectively, this allows one to reduce the injectivity problem to showing that, over L, the $n+1$ fibres E_i in $Y(A, \alpha)_L$ over the points $\{z_0, \ldots, z_n\} \in Z_L$ give linearly independent classes in $\mathrm{CH}^{n-1}(Y_L)$. This is accomplished via

Proposition (2.3.2)**.** *For $0 \leq i, j \leq n$, the degree of $[E_i] \cdot [E_j] \in \mathrm{CH}^{2n-2}(Y_L) =$ $\mathrm{CH}_0(Y_L)$ is $(-1)^{n-1}$ if $i = j$, and 0 otherwise.*

In the case $n = 2$, this is just the classical theory of exceptional curves on surfaces. This computation is in fact already implicit in [XZ], using localisation results for equivariant Chow groups.

Overview of the thesis

This thesis is separated into two parts. The first is a review of necessary background. Section 1.1 introduces the language of functors of points, and applies it to defining and constructing various types of partial flag varieties. Along the way, important classes of morphisms (flat, smooth, etc.) are introduced, as well as the theory of faithfully flat descent. The main references are [EH1] and [BLR].

This is followed in Section 1.2 by an account of Galois descent, which is used to explain the relationship between central simple algebras and Severi-Brauer varieties. The main reference is [GS]. Section 1.3 explains intersection theory using the Chow ring, and its equivariant enhancement in the case of vareities with torus actions.

The main references are [Fu], [EG] and [To]. The final section, Section 1.4, briefly introduces the category of effective Chow motives, following [Ma].

Chapter 2 presents the main result of the thesis. In Section 2.1, the varieties $X(A)$ and $Y(A, \alpha)$ are defined and the fundamental results about their geometry are proven. Section 2.2 studies the Chow groups of $\mathcal{O}(1)$-class divisors on projective bundles, and ultimately arrives at the criterion of Corollary 2.2.1. In Section 2.3, the decomposition theorem for $Y(A, \alpha)$ is proven.

Chapter 1

Preliminaries

1.1 Fundamental algebraic geometry

In this section, we define various partial flag varieties, which are moduli spaces for partial flags (ascending chains of subspaces of fixed dimensions) in a vector space. The natural setting for this is the language of functor of points, which we briefly review. We also review important classes of morphisms and faithfully flat descent, highlighting connections to the functor of points perspective.

1.1.1 The functor of points

We mostly follow the presentation of [EH1, Chapter VI].

Let S be a scheme. We denote by $\mathbf{Sch}_S$ the category of S-schemes. To any S-scheme X, one can attach the functor of points $h_X : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$ given by $T \mapsto \mathrm{Hom}_S(T, X)$. Given a morphism of S-schemes $f : X \to Y$, one gets a natural transformation $h_f : h_X \to h_Y$ given by the maps $\mathrm{Hom}_S(T, X) \ni g \mapsto f \circ g \in \mathrm{Hom}_S(T, Y)$. The fundamental result about this construction is the following:

Lemma 1.1.1 (Yoneda's Lemma)**.** *For X, Y S-schemes, the map $\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ given by $f \mapsto h_f$ is bijective.*

Proof. See [EH1, Lemma VI-1]. □

We call the assignment $X \mapsto h_X$ the Yoneda embedding of $\mathbf{Sch}_S$ into the functor category $\mathbf{Fun}(\mathbf{Sch}_S^\circ, \mathbf{Sets})$. A functor $F : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$ is said to be represented by an S-scheme X if F is naturally isomorphic to h_X. Such functors are called representable. The representable functors are exactly the essential image of the Yoneda embedding.

Example 1.1.1. Recall that we define $\mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}[x]$. The functor of points $h_{\mathbf{A}^1}$ is isomorphic to $T \mapsto \mathcal{O}_T(T)$ by the natural transformation $\operatorname{Hom}_S(T, X) \ni f \mapsto f^\#(x)$.

Example 1.1.2. Let $T \to S$ be a morphism of schemes. Given a functor $F : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$, one has the restriction $F_T : \mathbf{Sch}_T^\circ \to \mathbf{Sets}$ defined by $F_T(U) = F(U)$, where U is considered as an S-scheme via the composition $U \to T \to S$. If F is represented by the S-scheme X, F_T is represented by $X \times_S T$, viewed as a T-scheme by projection onto the second factor. Indeed, to give an S-morphism $U \to X$ is equivalent to giving a T-morphism $U \to X \times_S T$ by the universal property of fibred products.

Definition 1.1.1. Let $F : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$ be a functor and for any open immersion $i : U \hookrightarrow X$ of S-schemes and $s \in F(X)$, write $s|_U$ for $F(i)(s)$. We say that F is a sheaf for the Zariski topology if for any S-scheme T and open cover $\{U_i\}_{i \in I}$ of T the sequence

$$F(T) \xrightarrow{\rho} \prod_{i \in I} F(U_i) \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \cap U_j)$$

where

$$\rho(s) = (s|_{U_i})_{i \in I}, \quad \rho_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i,j \in I}, \quad \rho_2((s_i)_{j \in I}) = (s_j|_{U_i \cap U_j})_{i,j \in I}$$

is exact in the sense that ρ is injective and its image is the equaliser of the ρ_i.

Unpacking the definition, it says that for any collection of elements $s_i \in F(U_i)$, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique element $s \in F(T)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

Example 1.1.3. The most important example of sheaves for the Zariski topology are representable functors. Indeed, this follows from the fact that given schemes X, Y, an open cover U_i of X and morphisms $f_i : U_i \to Y$ which agree on overlaps (i.e., $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$), they glue together uniquely into a morphism $f : X \to Y$ with $f|_{U_i} = f_i$.

We write $\mathbf{Aff}_S$ for the full subcategory of $\mathbf{Sch}_S^\circ$ consisting of the affine schemes. Note that when S is affine, say $S = \operatorname{Spec} A$, $\mathbf{Aff}_S$ is equivalent to the category of (commutative) A-algebras $\mathbf{Alg}_A$. By abuse of notation, we will often use the same symbol for an object in $\mathbf{Aff}_S$ and its ring of global sections.

Proposition 1.1.1. *The natural restriction functor* $\mathbf{Fun}(\mathbf{Sch}_S^\circ, \mathbf{Sets}) \to \mathbf{Fun}(\mathbf{Aff}_S, \mathbf{Sets})$ *is a fully faithful embedding on the full subcategory of sheaves for the Zariski topology.*

Proof. This is a corollary of the stronger [SP, Lemma 020W]. □

Definition 1.1.2. A functor $F : \mathbf{Aff}_S \to \mathbf{Sets}$ is a sheaf for the Zariski topology if for any object A of $\mathbf{Aff}_S$ and elements $f_1, \ldots, f_n \in A$ such that $1 \in (f_1, \ldots, f_n)$, the sequence

$$F(A) \longrightarrow \prod_{1 \leq i \leq n} F(A_{f_i}) \rightrightarrows \prod_{1 \leq i,j \leq n} F(A_{f_i f_j})$$

induced by the canonical localisation maps $A \to A_{f_i}$, $A_{f_i} \to A_{f_i f_j}$, $A_{f_j} \to A_{f_i f_j}$ is exact.

Example 1.1.4. If a functor $F : \mathbf{Sch}_S^{\circ} \to \mathbf{Sets}$ is a sheaf for the Zariski topology in the sense of Definition 1.1.1, then its restriction to $\mathbf{Aff}_S$ is a sheaf for the Zariski topology in the sense of Definition 1.1.2.

We call objects of $\mathrm{Fun}(\mathbf{Aff}_S, \mathbf{Sets})$ which are sheaves for the Zariski topology S-spaces, and refer to the full subcategory formed by the objects as the category of S-spaces, $\mathbf{Esp}_S$. By abuse of notation, we will write h_X for the S-space obtained via restriction, and write $h_X(A)$ for $h_X(\operatorname{Spec} A)$ (we may also simply use $X(A)$). The notion of an S-scheme representing an S-space is thus defined in the same way.

Definition 1.1.3. Let F, G, H be S-spaces with morphisms $\alpha : F \to H$, $\beta : G \to H$. The fibre product $F \times_H G$ is given by

$$(F \times_H G)(A) = \{(f, g) \in F(A) \times G(A) : \alpha(f) = \beta(g)\}$$

for objects A of $\mathbf{Aff}_S$.

It is easy to check that the resulting functor is indeed a sheaf for the Zariski topology.

If F, G, H are representable, say by X, Y, Z, then clearly $F \times_H G$ is represented by the corresponding fibre product $X \times_Z Y$. In particular, consider a closed (resp. open) immersion of S-schemes $Y \to Z$ and let $X = \operatorname{Spec} A$ for some ring A. Then the fibre product is represented by a closed (resp. open) subscheme of $\operatorname{Spec} A$. Conversely, since Z can be covered by open affines and the property of being a closed (resp. open) immersion is Zariski local on the base, if as X varies over all affine schemes and morphisms to Z, all fibre products are represented by closed (resp. open) immersions, then $Y \to Z$ is a closed (resp. open) immersion.

Definition 1.1.4. An S-subspace of an S-space F is a morphism $\alpha : G \to F$ such that $\alpha(A) : G(A) \to F(A)$ is injective for all objects A of $\mathbf{Aff}_S$. A subspace $\alpha : G \to F$ is said to be open if for all S-space morphisms $h_X \to F$ with $X = \operatorname{Spec} A$, there exists an ideal $I \subseteq A$ such that $H = G \times_F h_{\operatorname{Spec} A}$ is given by

$$H(B) = \{f \in h_X(B) : f(I)B = B\}$$

A subfunctor $\alpha : G \to F$ is said to be closed if instead there exists ideals I such that H is given by

$$H(B) = \{f \in h_X(B) : f(I) = 0\}$$

The two conditions on the fibre product are precisely the characterisations of the S-subspaces of $h_{\operatorname{Spec} A}$ represented by open and closed subschemes respectively. Then by the above argument, they coincide with the notion of open and closed immersions for representable S-spaces.

Proposition 1.1.2. *An S-space F is representable (by an S-scheme) if and only if there is a family of open subfunctors of F, $\alpha_i : U_i \to F$, which are representable by affine schemes, such that for any field K, $F(K) = \bigcup_i \alpha_i(U_i(K))$.*

Proof. See [EH1, Theorem VI-14]. □

More examples

Example 1.1.5. Let F be a field. For any F-scheme X and point $x \in X$, the residue field $\kappa(x)$ has a natural structure of field extension of F. For a field extension $F \subseteq L$, $X(L)$ is in bijective correspondence with pairs of points $x \in X$ and F-algebra homomorphisms $\kappa(x) \to L$. The bijection can be given by sending such a pair to the compisition of the induced morphism of S-schemes $\operatorname{Spec} L \to \operatorname{Spec} \kappa(x)$ with $\operatorname{Spec} \kappa(x) \hookrightarrow X$ ([Ha2, Exercise II.2.7]).

In particular, $X(F)$ is in one-to-one correspondence with the set

$$R_X = \{x \in X : [\kappa(x) : F] = 1\}$$

We call both sets the set of F-rational points of X. Moreover, this correspondence is natural, in that for a morphism of F-schemes $f : X \to Y$, the diagram

$$\begin{array}{ccc} R_X & \xrightarrow{f|_{R_X}} & R_Y \\ \downarrow \cong & & \downarrow \cong \\ X(F) & \xrightarrow{f(F)} & Y(F) \end{array}$$

commutes.

Example 1.1.6. Let X, Y be F-schemes. Let $F[\varepsilon] := F[x]/(x^2)$ be the ring of dual numbers over F and denote by $\pi : F[\varepsilon] \to F$ the unique F-algebra homomorphism sending ε to 0. For any F-rational point $x \in X(F)$, define $\operatorname{Tan}_x X$ by the fibre of x for the map $\pi_* : X(F[\varepsilon]) \to X(F)$.

Let $\mathcal{O}_x$ be the local ring of the actual point in X corresponding to x and let $\mathfrak{m}_x$ be its maximal ideal. Since we can identify F with $\mathcal{O}_x/\mathfrak{m}_x$, we can view $\mathfrak{m}_x/\mathfrak{m}_x^2$ as an F-vector space. For any F-morphism $f : X \to Y$, with $f(x) = y$, there is an induced F-linear map $f_* : (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee$ coming from the homomorphism of local rings $\mathcal{O}_y \to \mathcal{O}_x$.

For any $x \in X(F)$, there is a natural bijection $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \to \operatorname{Tan}_x X$ ([Ha2, Exercise II.2.8]) in the sense that for any morphism of F-schemes $f : X \to Y$, the diagram

$$\begin{array}{ccc} (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee & \xrightarrow{f_*} & (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Tan}_x X & \xrightarrow{f(F[\varepsilon])|_{\operatorname{Tan}_x X}} & \operatorname{Tan}_y Y \end{array}$$

commutes. Therefore, the sets $\operatorname{Tan}_x X$ are endowed with a natural structure of F-vector space.

Example 1.1.7. Let X be an S-scheme, given a factorisation of $h_X : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$ through the category of groups, X is equipped with the structure of a group object in the category of S-schemes. The multiplication map $m : X \times_S X \to X$ comes from the natural transformation $h_X(T) \times h_X(T) \to h_X(T)$ and the inversion map is defined similarly.

Example 1.1.8. Let V be a finite dimensional vector space over F. Define the functor $\mathrm{GL}(V) : \mathbf{Sch}_F^\circ \to \mathbf{Grp}$ by $S \mapsto \operatorname{Aut}_{\mathcal{O}_S}(V \otimes_F \mathcal{O}_S)$. Fixing a basis $e_1, \ldots, e_n$ of V, we get natural isomorphisms $\mathrm{GL}(V)(S) = \mathrm{GL}_n(\mathcal{O}_S(S))$, where $\mathrm{GL}_n(A)$ is the group of $n \times n$ matrices with entries in A and unit determinant. Forgetting the group structure, it is clear that there is a natural bijection $\mathrm{GL}_n(A) = \operatorname{Hom}_F(F[x_{ij}, \det(x_{ij})^{-1}], A)$, so we see $\mathrm{GL}(V)$, as a functor to sets, is representable. Thus, $\mathrm{GL}(V)$ is an F-groups scheme.

1.1.2 Some classes of morphisms

Flat morphisms

Recall that an A-module M is said to be flat if the functor $- \otimes_A M$ is exact. We generalise this to schemes as follows:

Definition 1.1.5. Let $f : X \to S$ be a morphism of schemes and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is flat, if for all $x \in X$, $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{S,f(x)}$-module. The morphism $f : X \to S$ is said to be flat if $\mathcal{O}_X$ itself is flat.

Flat morphisms are closed under composition and stable under base change, see [Ha2, Proposition III.9.2].

Example 1.1.9. Any scheme X over a field F is flat over $\operatorname{Spec} F$. By stability of flatness under base change, this implies that the projections $p_1 : X \times_F Y \to X$ and $p_2 : X \times_F Y \to Y$ are flat for any F-schemes X, Y.

Example 1.1.10. Let C be a non-singular curve over a field F. Any dominant morphism of F-varieties $f : X \to C$ is flat. Indeed, for any $x \in X$ the local ring of $\mathcal{O}_{C,f(x)}$ is regular, hence principal since $\dim C = 1$. Since f is dominant, $\mathcal{O}_{X,x}$ is torsion-free as a $\mathcal{O}_{C,f(x)}$-module, hence flat ([Ei, Corollary 6.3]).

A key property of flat morphisms is that the fibres of such morphisms are "well-behaved". Here is one such example of good behaviour which we will need to develop the theory of Chow groups later:

Proposition 1.1.3. *Let $f : X \to Y$ be a flat morphism of finite type F-schemes with Y irreducible. The following are equivalent:*

i. *Every irreducible component of X has dimension* $\dim Y + n$.

ii. *For any $y \in Y$, each irreducible component of the fibre $X_y := f^{-1}(y)$ has dimension n.*

Proof. See [Ha2, Corollary III.9.6]. □

Definition 1.1.6. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. The relative dimension of f at $x \in X$ is defined to be $\dim_x X_{f(x)}$.

If $f : X \to Y$ is a flat morphism of irreducible finite type F-schemes, the relative dimension is constant by Proposition 1.1.3.

Smooth morphisms

We use [BLR, Chapter 2] as our main reference.

Definition 1.1.7. A morphism of schemes $f : X \to S$ is formally smooth (resp. formally étale) if for any absolutely affine S-scheme Y and closed subscheme Y' of Y defined by a nilpotent ideal, the natural map

$$\operatorname{Hom}_S(Y, X) \to \operatorname{Hom}_S(Y', X)$$

is surjective (resp. bijective).

Proposition 1.1.4. *Let $f : X \to S$ be of locally finite presentation. The following are equivalent:*

i. *f is formally smooth.*

ii. *f is flat and for any geometric point $\bar{s} : \operatorname{Spec}\Omega \to S$, the geometric fibre $X_{\bar{s}}$ is a regular scheme.*

Proof. This follows by combining Proposition 2.2/6, Proposition 2.4/8, and Proposition 2.2/15, c) of [BLR]. □

We say that $f : X \to S$ is smooth if it is locally of finite presentation and satisfies either of the equivalent conditions from the proposition.

Remark 1.1.1. Condition **ii.** makes it clear that smoothness is a local condition: say that f is smooth at a point $x \in X$ if there exists a neighbourhood $U \subseteq X$ of x and an open subset $V \subseteq Y$ with $f(U) \subseteq V$ such that the restriction $f|_U : U \to V$ is smooth, then f is smooth if and only if it is smooth at all points $x \in X$.

For any smooth morphism $f : X \to S$, the relative dimension of f is locally constant [BLR, p. 35-36]. We say that a morphism is étale if it is smooth and everywhere of relative dimension 0. As one might expect, formally étale is equivalent to étale for morphisms which are locally of finite presentation (this follows from [BLR, Proposition 2.2/2] and [BLR, Proposition 2.2/6]).

Example 1.1.11. It is easy to see from Proposition 1.1.4 that for any field F, $\mathbf{A}^n_F$ is smooth over $\operatorname{Spec} F$. Hence by Remark 1.1.1, $\mathbf{P}^n_F$ is also smooth over $\operatorname{Spec} F$.

We briefly recall the definition of the relative cotangent sheaf of a morphism $f : X \to S$ (for details see [BLR, §2.1] and [Ha2, §II.8]). Since the diagonal map $\Delta : X \to X \times_S X$ is an immersion, there is an open subset $U \subseteq X \times_S X$ which is maximal among open subset in which $\Delta(X)$ is closed. The map Δ factors through U by a closed immersion (which we also call Δ by abuse of notation) corresponding to an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_U$. The relative cotangent sheaf $\Omega_{X/S}$ is defined to be $\Delta^*\mathcal{I}/\mathcal{I}^2$. There is a universal $\mathcal{O}_S$-linear derivation $d : \mathcal{O}_X \to \Omega_{X/S}$ given by $f \mapsto p_1^*f - p_2^*f$, p_i the projection maps of $X \times_S X$. Given a morphism of S-schemes $f : X \to Y$, there is a natural induced homomorphism $f^*\Omega_{Y/S} \to \Omega_{X/S}$ of $\mathcal{O}_X$-modules. These maps give rise to the "conormal sequence":

Proposition 1.1.5. *Let $i : Y \to X$ be a closed immersion of S-schemes and $\mathcal{J} \subseteq \mathcal{O}_X$ the associated ideal sheaf. Let $\delta : \mathcal{J}/\mathcal{J}^2 \to i^*\Omega_{X/S}$ be the map induced from d, and $\epsilon : i^*\Omega_{X/S} \to \Omega_{Y/S}$ the canonical map induced by i. Then the resulting sequence of $\mathcal{O}_Y$-modules*

$$\mathcal{J}/\mathcal{J}^2 \xrightarrow{\delta} i^*\Omega_{X/S} \xrightarrow{\epsilon} \Omega_{Y/S} \to 0$$

is exact.

Proof. See [BLR, Proposition 2.1/1]. □

Proposition 1.1.6 (Jacobi Criterion)**.** *Let X be a smooth S-scheme of relative dimension n and $i : Y \to X$ be a closed immersion which is locally of finite presentation. Let $\mathcal{I}$ be the ideal sheaf defining Y in X. For a point $y \in Y$ (letting $x = i(y)$), the following are equivalent:*

i. *$Y \to S$ is smooth and has relative dimension r at y.*

ii. *The extended conormal sequence*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{X/S} \to \Omega_{Y/S} \to 0$$

is split exact at y, and $\dim_{\kappa(y)} \Omega_{Y/S} \otimes \kappa(y) = r$.

iii. *There exist sections $g_{r+1}, \ldots, g_n \in \mathcal{O}_X(U)$ of a neighbourhood U of x which generate $\mathcal{I}_x$ such that the image of the differentials $dg_{r+1}, \ldots, dg_n$ in the fibre $\Omega_{X/S} \otimes \kappa(x)$ are linearly independent.*

Proof. See [BLR, Proposition 2.2/7]. □

Example 1.1.12. For any scheme S, $\Omega_{\mathbf{A}^n_S/S}$ is freely generated by the global sections $dx_1, \ldots, dx_n$, where the x_i are the standard coordinate functions for $\mathbf{A}^n_S$. See [Ei, Proposition 16.1] for the local case, from which the claim follows immediately.

Remark 1.1.2. Let $X \to S$ be smooth. Since X is locally of finite presentation over S, every point $x \in X$ has a neighbourhood which admits a closed immersion over S into an open subscheme of $\mathbf{A}^n_S$. The freeness of $\Omega_{\mathbf{A}^n_S/S}$ then implies that $\Omega_{X/S}$ is locally free since the sequence of Proposition 1.1.6, ii) is split exact.

Definition 1.1.8. An algebraic scheme X over a field F is called non-singular if its structural morphism is smooth.

Example 1.1.13. The Jacobi criterion and Example 1.1.12 give a method of showing that a given algebraic scheme X over a field F is non-singular. Since smoothness is local, we reduce to the case where X is affine, and hence is a closed subscheme of an affine space $\mathbf{A}^n_F$ with defining equations $f_1, \ldots, f_m$. By linearity and the Leibniz rule, each df_i can be written in terms of the basis dx_j as $\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$. The Jacobi criterion then implies that if the Jacobian matrix $J = (\frac{\partial f_i}{\partial x_j})$ has rank m at every point of X, then X is non-singular of dimension $n - m$. In fact, since this amounts to determining if the closed subscheme defined by the f_i and the $m \times m$ minors of J is empty, it suffices to check on only the closed points.

If X is already known to be equidimensional, then X is non-singular iff for all closed points $x \in X$, the rank of the Jacobian matrix at x is the codimension of X.

1.1.3 Faithfully flat descent

Definition 1.1.9. A morphism of schemes $f : X \to Y$ is fpqc if it is flat, surjective and quasi-compact.

Remark 1.1.3. The term "fpqc" is an initialism for the French "fidèlement plat et quasi-compacte", which translates to "faithfully flat and quasi-compact". Faithful flatness in this case is coming from surjectivity and flatness. Another characterisation of this property will be given in the affine case later.

This class of morphism turns out to be incredibly useful as many properties of morphisms of schemes are "fpqc local on the base", i.e., a morphism $f : X \to S$ has a given property if the base change $f : X' \to S'$ by an fpqc morphism $S' \to S$ has the property. For example:

Theorem 1.1.1. *The conditions that a morphism is affine, separated, proper, flat, finite, smooth, an open immersion, a closed immersion or an isomorphism are fpqc local on the base.*

Proof. See [SP, Section 02YJ]. □

There is also a theory of descent for fpqc morphism which, given an fpqc morphism $S' \to S$, describes the quasicohorent sheaves on S' which are the pullback of a quasicoherent sheaf on S. It can also be generalised to quasicoherent algebras and even schemes, as we shall see. Our treatment this topic will follow that of [BLR, Chapter 6].

Definition 1.1.10. Let $p : S' \to S$ be a morphism of schemes and let $p_i : S'' := S' \times_S S' \to S'$ $(i = 1, 2)$ be the projection maps. Given a quasicohorent sheaf $\mathcal{F}'$ on S', a covering datum for $\mathcal{F}'$ (with respect to p) is an isomorphism $\varphi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$. Given quasicohorent sheaves on S' $\mathcal{F}'$ and $\mathcal{G}'$ with covering data φ and ψ, a morphism $f : (\mathcal{F}', \varphi) \to (\mathcal{G}', \psi)$ is a morphism of quasicoherent sheaves such that $\psi \circ p_1^* f = p_2^* f \circ \varphi$.

It is clear that quasicoherent sheaves with covering data and their morphisms form a category. If $\mathcal{F}' = p^*\mathcal{F}$ for some quasicoherent sheaf on S, then $p_1^*\mathcal{F}' = p_1^*p^*\mathcal{F} \cong p_2^*p^*\mathcal{F} = p_2^*\mathcal{F}$ since by definition $p \circ p_2 = p \circ p_1$. Moreover, the isomorphism is natural in $\mathcal{F}$ so we get a functor from quasicoherent sheaves on S to quasicoherent sheaves on S' with covering data by $\mathcal{F} \mapsto p^*\mathcal{F}$ with the canonical isomorphism as the covering datum.

Consider $S''' := S' \times_S S' \times_S S'$ with the various projection morphisms $p_{ij} : S''' \to S''$, $(i, j = 1, 2, 3$ and $i < j)$, where $p_1 \circ p_{ij} = p_i$ and $p_2 \circ p_{ij} = p_j$. We obtain the following diagram where the dashed arrows denote canonical isomorphisms

$$\begin{array}{ccccc} p_{12}^*p_1^*\mathcal{F}' & \xrightarrow[p_{12}^*\varphi]{} & p_{12}^*p_2^*\mathcal{F}' & \dashrightarrow & p_{23}^*p_1^*\mathcal{F}' \\ \downarrow & & & & \downarrow {\scriptstyle p_{23}^*\varphi} \\ p_{13}^*p_1^*\mathcal{F}' & \xrightarrow{p_{13}^*\varphi} & p_{13}^*p_2^*\mathcal{F}' & \dashrightarrow & p_{23}^*p_2^*\mathcal{F}' \end{array}$$

A morphism of quasicoherent S'-modules with covering data gives a corresponding morphism of such diagrams. In particular, given an isomorphism of quasicoherent S'-modules with covering data $(\mathcal{F}', \varphi) \cong (\mathcal{G}', \psi)$, the diagram commutes for $\mathcal{F}'$ and φ iff it commutes for $\mathcal{G}'$ and ψ. Therefore, for $(\mathcal{F}', \varphi)$ to be in the essential image of the functor p^*, it is necessary for the above diagram to commute, since it will for any $p^*\mathcal{F}$ given that all the isomorphisms in the digram will be the canonical ones.

Definition 1.1.11. A covering datum φ is said to be a descent datum if the above diagram commutes, or equivalently, after identifying canonically isomorphic objects $p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$. This equality is referred to as the cocycle condition.

We refer to the full subcategory of quasicoherent S'-modules with covering data whose covering datum are descent datum simply as the category of quasicoherent S'-modules with descent data. We can now formulate faithfully flat descent for quasicoherent sheaves:

Theorem 1.1.2. *Let $p : S' \to S$ be an fpqc morphism. The functor from quasicoherent S-modules to quasicoherent S'-modules with descent data $\mathcal{F} \mapsto p^*\mathcal{F}$ is an equivalence of categories.*

This is an analogue of the elementary fact that S-modules (and morphisms between them) can be glued from modules defined on an open cover $\{U_i\}_{i\in I}$. In this case, take the morphism $p : S' := \coprod_{i\in I} U_i \to S$ to be the open covering. A quasicoherent S'-module is then nothing but a family of quasicoherent U_i-modules $\{\mathcal{F}_i\}_{i\in I}$. We have

$$S'' = \coprod_{i\in I} U_i \times_S \coprod_{i\in I} U_i = \coprod_{i,j\in I} U_i \times_S U_j = \coprod_{i,j\in I} U_i \cap U_j$$

and similarly $S''' = \coprod_{i,j,k\in I} U_i \cap U_j \cap U_k$. With these identifications, a covering datum is a family of isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ and the condition to be a descent datum becomes $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$ for all $i, j, k \in I$.

In fact, gluing for Zariski open covers allows one to reduce Theorem 1.1.2 to the affine case by first taking an open affine cover $U_i = \operatorname{Spec} R_i$ of S, then for each U_i taking a finite open affine cover $U'_{ij} = \operatorname{Spec} R'_{ij}$ of $p^{-1}(U_i)$ (which exists by quasi-compactness) and considering the composite morphism $\bar{p} : \coprod U'_{ij} \to U_i$, where each

U'_{ij} is mapped to U_i by p (see [BLR, §6.1] for details). This corresponds to a ring homomorphism $R_i \to \prod R'_{ij}$. This is the reason behind the quasi-compactness assumption, since only finite disjoint unions of affine schemes are still affine.

Definition 1.1.12. Let $R \to R'$ be an R-algebra. R' is faithfully flat if it is flat and for all R-modules M, $R' \otimes_R M = 0$ iff $M = 0$.

In other words, this says that the exactness of a complex of R-modules and its base change to R' are equivalent. Indeed, since R' is flat over R, for a complex of R-modules $F^\bullet$, $H^i(R' \otimes_R F^\bullet) = R' \otimes_R H^i(F^\bullet)$, and so by the second condition $H^i(F^\bullet) = 0$ if and only if $H^i(R' \otimes_R F^\bullet) = 0$.

Example 1.1.14. If R' is free as an R-module it is faithfully flat over R. This follows from the commutativity of direct sums and tensor products.

Example 1.1.15. A flat local homomorphism of local rings $R \to R'$ is faithfully flat. Indeed, let M be a non-zero R module, then it has a non-trivial finitely generated submodule $N \subseteq M$. Since R' is flat over R, if $R' \otimes_R N \neq 0$, then we must have $R' \otimes_R M \neq 0$. By Nakayama's lemma, $R' \otimes_R N = 0$ iff $K \otimes_R N = 0$, where K is the residue field of R'. Since $R \to R'$ is local, we have an induced map $k \to K$, where k is the residue field of R. This is free hence faithfully flat. So we have $K \otimes_R N = K \otimes_k (k \otimes_R N) = 0$ iff $k \otimes_R N = 0$ iff $N = 0$ by Nakayama's lemma, completing the proof.

Example 1.1.15 shows that an fpqc morphism of affine schemes induces a faithfully flat homomorphism on global section: if $M \neq 0$, then for some $\mathfrak{p} \in \operatorname{Spec} R$, $M_\mathfrak{p} \neq 0$. Since there is a $\mathfrak{q} \in \operatorname{Spec} R'$ with $f(\mathfrak{q}) = \mathfrak{p}$, we have a flat local homomorphism $R_\mathfrak{p} \to R'_\mathfrak{q}$, and so $(M \otimes_R R')_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_\mathfrak{q} \neq 0$, hence $M \otimes_R R' \neq 0$. Conversely, the induced map on schemes from a faithfully flat morphism $R \to R'$ is automatically quasi-compact and flat. It is also surjective since for any prime $\mathfrak{p} \subseteq R$, the residue field $\kappa(\mathfrak{p}) \neq 0$, so $R' \otimes_R \kappa(\mathfrak{p}) \neq 0$, hence there is a prime of R' above $\mathfrak{p}$.

Thus, it makes sense to translate the descent problem to the case of faithfully flat ring extensions $R \to R'$. Let M be an R' module. $p_1^* \tilde{M}$ and $p_2^* \tilde{M}$ correspond to the two ways in which $M \otimes_R R'$ is an $R' \otimes_R R'$-module: by $(a \otimes b)(m \otimes r) = am \otimes br$ or $(a \otimes b)(m \otimes r) = bm \otimes ar$. A covering datum is thus an R-linear isomorphism $\varphi : M \otimes_R R' \to M \otimes_R R'$ which twists the $R' \otimes_R R'$ multiplication. For a covering datum φ, the cocycle condition translates as follows: let $\varphi(m \otimes a) = \sum m_i \otimes a_i$, then the maps

$$\varphi_{12}(m \otimes a \otimes b) = \sum m_i \otimes a_i \otimes b$$
$$\varphi_{23}(m \otimes b \otimes a) = \sum m_i \otimes b \otimes a_i$$
$$\varphi_{13}(m \otimes b \otimes a) = \sum m_i \otimes a_i \otimes b$$

express the cocycle condition as $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ ([Wa, §17.1]). We still call covering data satisfying the cocycle condition descent data, and retain the same notion of morphisms between R'-modules with covering (resp. descent) data.

If N is an R-module, then $N \otimes_R R'$ has the canonical descent datum $\varphi_N : N \otimes_R R' \otimes_R R' \to N \otimes_R R' \otimes_R R'$, $\varphi_N(n \otimes a \otimes b) = n \otimes b \otimes a$. So, fpqc descent in the affine case is reformulated as:

Proposition 1.1.7. *The functor $N \mapsto (N \otimes_R R', \varphi_N)$ from R-modules to R'-modules with descent data is an equivalence of categories.*

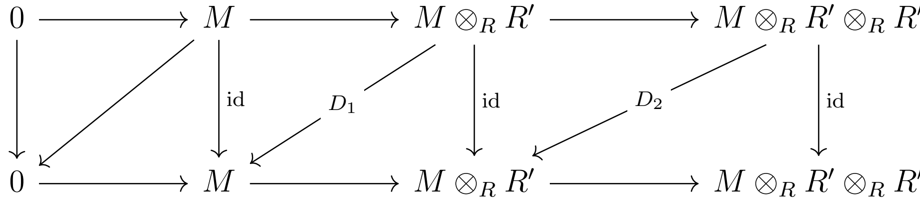
The main property of faithfully flat morphisms used to prove Proposition 1.1.7 is the following:

Lemma 1.1.2. *Let $f : R \to R'$ be a faithfully flat ring homomorphism, and let M be an R-module. The sequence of R-modules*

$$0 \to M \to M \otimes_R R' \xrightarrow{\delta} M \otimes_R R' \otimes_R R'$$

with $\delta(m \otimes a) = m \otimes a \otimes 1 - m \otimes 1 \otimes a$ is exact.

Proof. It is clear that the composition of any two adjacent maps is 0. Since R' is a faithfully flat R-algebra, we may verify exactness after a base change to R'. But this is just the analogous sequence given by $f' : R' \to R' \otimes_R R'$. Therefore, we may assume that f has a retraction g, i.e. $g \circ f = \mathrm{id}_R$. One verifies that



with $D_1(m \otimes a) = g(a) \cdot m$ and $D_2(m \otimes a \otimes b) = m \otimes af(g(b))$ commutes, therefore id is null-homotopic so the sequence is exact. □

We now sketch the proof of Proposition 1.1.7, following the proof of [Wa, Theorem 17.2]. First, to prove full faithfulness, for R-modules M, N, let $\psi : M \otimes_R R' \to N \otimes_R R'$ be an R'-linear map compatible with descent data, i.e. $(\psi \otimes 1) \circ \varphi_M = \varphi_N \circ (\psi \otimes 1)$. By the definition of φ_M and φ_N, this implies that

$$\begin{array}{ccc} M \otimes_R R' & \xrightarrow{\delta} & M \otimes_R R' \otimes_R R' \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi\otimes 1} \\ N \otimes_R R' & \xrightarrow{\delta} & N \otimes_R R' \otimes_R R' \end{array}$$

commutes. Therefore ψ induces an R-linear map on kernels $\psi|_M : M \to N$, and clearly we have $\psi = \psi|_M \otimes 1$.

For essential surjectivity, consider an R'-module M' with descent datum φ. Let $M = \{m \in M' : \varphi(m \otimes 1) = m \otimes 1\}$. It is enough to show that the natural map $M \otimes_R R' \to M'$ is an isomorphism to conclude that $(M', \varphi) \cong (M \otimes_R R', \varphi_M)$. Due to the cocycle condition, the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_R R' & \longrightarrow & M' \otimes_R R' & \xrightarrow{(\varphi-1)\otimes 1} & M' \otimes R' \otimes R' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_{23}} \\ 0 & \longrightarrow & M' & \longrightarrow & M' \otimes_R R' & \xrightarrow{\delta} & M' \otimes_R R' \otimes_R R' \end{array}$$

Since φ and φ_{23} are isomorphisms and the rows are exact, this implies that $M \otimes_R R' \to M'$ is an isomorphism, as desired.

Example 1.1.16 (Descent for locally free sheaves)**.** Let $\mathcal{E}'$ be a locally free S'-module of finite rank. By Theorem 1.1.2, for any descent datum φ for $\mathcal{E}'$, there is a quasi-coherent S-module $\mathcal{E}$ such that $p^*\mathcal{E}$ is isomorphic to the pair $(\mathcal{E}', \varphi)$. To show that descent is effective for locally free sheaves, we need then only prove that $\mathcal{E}$ is already locally free of finite rank. The question is local on S, and by quasicompactness we may reduce to the situation where S' is also affine. Thus, one must prove the following: let $R \to R'$ be a faithfully flat homomorphism, and M an R-module such that $M' := M \otimes_R R'$ is free of finite rank, then M is projective and finitely generated, or equivalently, flat and of finite presentation.

The flatness of M is immediate from the flatness of M' by faithful flatness. As for finite presentation, first let $\sum_i m_{ij} \otimes r_{ij}$ be generators of M' indexed by $0 \leq j \leq n$. Then $\pi' : R'^N \to M'$, mapping each canonical basis element e_k to one of the distinct m_{ij} is surjective and descends to a R-linear homomorphism $\pi : R^N \to M$. By faithful flatness, π is surjective, so M is finitely generated. Since M' is free, the surjection π'

splits and hence has a finitely generated kernel N'. By flatness, $N' = \ker \pi \otimes_R R'$, and applying the same argument as before, we conclude that $\ker \pi$ must also be finitely generated.

Multiplicative structures and schemes

Following the idea of [Wa, §17.3], we extend our theory of descent to quasicoherent S-algebras.

Definition 1.1.13. Let $p : S' \to S$ be an fpqc morphism. Let $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \psi)$ be quasicoherent S'-modules with covering data. We define the tensor product $(\mathcal{F} \otimes \mathcal{G}, \varphi \otimes \psi)$, where $\varphi \otimes \psi$ is considered as covering datum via the canonical isomorphisms $p_i^*(\mathcal{F} \otimes \mathcal{G}) = p_i^*\mathcal{F} \otimes p_i^*\mathcal{G}$, $(i = 1, 2)$.

First, note that the tensor product of covering data which are descent data is also a descent datum. Second, note that if $\mathcal{F}$ is a quasicoherent S-module, the canonical isomorphism $p^*(\mathcal{F} \otimes \mathcal{G}) \to p^*\mathcal{F} \otimes p^*\mathcal{G}$ is compatible with the induced descent data.

Proposition 1.1.8. *The functor $\mathcal{F} \mapsto p^*\mathcal{F}$ from quasicoherent S-algebras to quasicoherent S'-algebras with descent data that are S''-algebra homomorphisms is an equivalence of categories.*

Proof. To show that it is fully faithful, suppose $\mathcal{F}$ is a quasicoherent S-algebra. This S-algebra structure is equivalent to a morphism of S-modules $m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$. Likewise, let $\mathcal{G}$ be a quasicoherent S-algebra, with multiplication map n. A morphism of S-modules $f : \mathcal{F} \to \mathcal{G}$ is an algebra homomorphism iff $f \circ m = n \circ (f \otimes f)$. By the full faithfulness of p^* for quasicoherent S-modules and the compatibility with tensor products explained above, this latter condition is equivalent to that for $p^*f : p^*\mathcal{F} \to p^*\mathcal{G}$, p^*m, and p^*n. Thus, the image of the S-algebra homomorphisms of $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ in $\mathrm{Hom}_{S'}(p^*\mathcal{F}, p^*\mathcal{G})$ is exactly the S'-algebra homomorphisms.

For essential surjectivity, one needs to show that, for a quasicoherent S'-algebra $\mathcal{F}'$, if a descent datum φ for $\mathcal{F}'$ is an algebra homomorphism, then the multiplication morphism $m' : \mathcal{F}' \otimes \mathcal{F}' \to \mathcal{F}'$ descends. This is equivalent to being a morphism of covering data, which is equivalent to the diagram

$$\begin{array}{ccc} p_1^*(\mathcal{F}' \otimes \mathcal{F}') & \xrightarrow[\varphi\otimes\varphi]{} & p_2^*(\mathcal{F}' \otimes \mathcal{F}') \\ \Big\downarrow {\scriptstyle p_1^* m'} & & {\scriptstyle p_2^* m'} \Big\downarrow \\ p_1^*\mathcal{F}' & \xrightarrow{\varphi} & p_2^*\mathcal{F}' \end{array}$$

commuting, and this is nothing but the condition of φ being an algebra homomorphism. □

Remark 1.1.4. If $\mathcal{F}'$ is unital, then there is a canonical S'-algebra map $1 : \mathcal{O}_{S'} \to \mathcal{F}$, which descends iff we also require φ to be unital. Hence the property of being unital descends. The commutativity of the multiplication in $\mathcal{F}'$ can be expressed by the equality $m \circ s = m$, where $s : \mathcal{F}' \otimes \mathcal{F}' \to \mathcal{F}' \otimes \mathcal{F}'$ is the morphism of S'-modules swapping the factors. Since s descends for any descent datum of $\mathcal{F}'$, the property of being commutative also descends. A similar argument can be applied to show that associativity also descends.

Recall that for a scheme S, the functor $\mathcal{A} \mapsto \operatorname{Spec} \mathcal{A}$ gives an anti-equivalence of categories between the category of unital, commutative quasicoherent S-algebras and (relatively) affine S-schemes; see [Gr1, §1], in particular Proposition 1.2.7 and Proposition 1.3.1. Moreoever, for a morphism of schemes $p : S' \to S$, there is a canonical isomorphism $\operatorname{Spec} p^*\mathcal{A} = \operatorname{Spec} \mathcal{A} \times_S S'$. Therefore, translating everything to the equivalent statements in the algebra case, one shows that for $p : S' \to S$ fpqc, the functor $X \mapsto X \times_S S'$ gives an equivalence of categories between affine S-schemes and affine S'-schemes with analoguously defined descent datum.

Concretely, for $p : S' \to S$ a morphism of schemes and $f : X \to Y$ a morphism of S-schemes, write p^*X for $X \times_S S'$, p^*Y for $Y \times_S S'$ and p^*f for the induced morphism $f \times \mathrm{id}_{S'} : X \times_S S' \to Y \times_S S'$. If p is an fpqc morhism, a covering datum for and S'-scheme X' is an S''-isomorphism $\varphi : p_1^*X' \to p_2^*X'$, and a covering datum is a descent datum iff $p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$ (as in the case of quasicoherent sheaves, we surpress canonical isomorphisms in the notation). The previous remarks show that all descent data on affine S'-schemes are effective. This is not true in general, however in the case S' and S affine, there is a simple criterion:

Proposition 1.1.9. *Let $p : S' \to S$ be an fpqc morphism of affine schemes.*

i. *The functor $X \mapsto p^*X$ of S-schemes to S'-schemes with descent data is fully faithful.*

ii. *A descent datum φ on an S'-scheme X' is effective if and only if X' can be covered with affine open subschemes $\{U_j\}$ which are stable under φ, i.e. for each U_j, φ restricts to a descent datum $\varphi' : p_1^*U_j \to p_2^*U_j$.*

Proof. See [BLR, Theorem 6.1/6]. □

One consequence of this theorem is a useful necessary condition for a functor to be representable by a scheme:

Proposition 1.1.10. *Let T be a scheme. Let $p : S' \to S$ be an fpqc morphism of T-schemes. If $F : \mathbf{Sch}_T^\circ \to \mathbf{Sets}$ is a representable functor, then the sequence*

$$F(S) \xrightarrow{F(p)} F(S') \underset{F(p_2)}{\overset{F(p_1)}{\rightrightarrows}} F(S'')$$

is exact.

Proof. See [BLR, Proposition 8.1/1]. □

This fact is often refered to by saying that representable functors are "sheaves for the fpqc topology".

1.1.4 Relative Grassmannians and related constructions

We roughly follow [Ka, Part 2] with a few differences: the relevant representable functors are defined on all schemes over an arbitrary base, instead of algebras over a field. This allows for somewhat cleaner constructions of flag varieties.

Let F be a field and S an F-scheme.

Definition 1.1.14. For a locally free sheaf $\mathcal{E}$ of rank n on S, and a positive integer k, define the relative Grassmannian functor $\mathbf{Gr}(\mathcal{E}, k) : \mathbf{Sch}_S^\circ \to \mathbf{Sets}$ by

$$\mathbf{Gr}(\mathcal{E}, k)(T) = \{\text{quotients } f^*\mathcal{E} \twoheadrightarrow \mathcal{Q}, \mathcal{Q} \text{ locally free of rank } n-k\}/\sim$$

for an S-scheme $f : T \to S$ and for any morphism of S-schemes $g : T \to U$, define $g^* : \mathbf{Gr}(\mathcal{E}, k)(U) \to \mathbf{Gr}(\mathcal{E}, k)(T)$ by mapping the class of $f^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ to the class of $g^*(f^*\mathcal{E}) \to g^*\mathcal{Q}$.

Note that this is well-defined since pullbacks of locally free sheaves of rank m are locally free sheaves of rank m, and the pullback functor is right exact. We will show that $\mathbf{Gr}(\mathcal{E}, k)$ is representable by a smooth, projective S-scheme.

Remark 1.1.5. The condition that $\mathbf{Gr}(\mathcal{E}, k)$ is a sheaf in the Zariski topology amounts to the statement that subsheaves over an open cover (in this case the kernels of the isomorphism classes of quotients) can be glued if they agree on intersections, which is true.

Lemma 1.1.3. *If $S = \operatorname{Spec} F$ and $\mathcal{E} = \mathcal{O}_{\operatorname{Spec} F}^{\oplus n}$, $\mathbf{Gr}(\mathcal{E}, k)$ is representable, and we write it $\mathbf{Gr}_F(n, k)$.*

The following proof is based on that of [SP, Tag 089T].

Proof. By the remark, it is enough to show that the associated $\operatorname{Spec} F$-space is representable. On an affine scheme $\operatorname{Spec} A$, we will use freely the identification of quasi-coherent sheaves with A-modules. In particular, locally free sheaves (always of finite rank) correspond to finitely generated projective modules. We already know that $\mathbf{Gr}_F(n, k)$ is a sheaf for the Zariski topology, so we need only give a covering by representable open subspaces.

Let I be the set of subsets of size $n-k$ of $\{1,\dots,n\}$. For A an F-algebra, let $f_1,\dots,f_{n-k}$ be the standard basis of A^{n-k} and $e_1,\dots,e_n$ the standard basis of A^n. For $i \in I$, we define maps $s_i : A^{n-k} \to A^n$ sending f_j to e_{i_j} where $i_1 < i_2 < \cdots < i_{n-k}$ are the elements of i. We define $\operatorname{Spec} F$-subspaces U_i for each $i \in I$ by

$$U_i(A) = \{[\pi : A^n \to Q] \in \mathbf{Gr}_F(n,k)(A) : \pi \circ s_i \text{ is surjective}\}$$

Since a surjective homomorphism between finitely generated projective modules of the same rank is an isomorphism, all elements of $U_i(A)$ are represented by surjections $\pi : A^n \to A^{n-k}$. The image of $\pi \circ s_i$ does not affect the isomorphism class, however two surjections π_1, π_2 agreeing on the image of s_i give the same isomorphism class iff $\pi_1(e_j) = \pi_2(e_j)$ for all $j \in \{1,\dots,n\}\backslash i$. Thus we have defined an isomophism (of sets) $U_i(A) \to A^{k(n-k)} = \mathbf{A}_F^{k(n-k)}(A)$. By construction, this isomorphism is natural in the sense that over all F-algebras A, it yields an isomorphism $U_i \cong h_{\mathbf{A}_F^{k(n-k)}}$ of functors. Moreover, it is clear that for any field extension K of F, $\mathbf{Gr}_F(n,k)(K) = \bigcup_{i\in I} U_i(K)$.

To finish the proof, we must show that each U_i is an open subspace of $\mathbf{Gr}_F(n,k)$ (Proposition 1.1.2). Let A be an F-algebra, $f : h_{\operatorname{Spec} A} \to \mathbf{Gr}_F(n,k)$ a map of $\operatorname{Spec} F$-spaces and $\pi : A^n \to Q$ a surjective homomorphism representing the isomorphism class corresponding to f. We must show that there is an ideal $J \subseteq A$ such that for any F-algebra homomorphism $\varphi : A \to B$, $(\pi \otimes 1) \circ s_i : B^{n-k} \to Q \otimes_A B$ is surjective iff $\varphi(J)B = B$. We can take the annihilator of $\operatorname{coker}(\pi \circ s_i)$ for J. Indeed, $B^{n-k} \to Q \otimes B$ is surjective iff for all primes $\mathfrak{p} \subseteq B$, the induced maps $\kappa(\mathfrak{p})^{n-k} \to Q \otimes_A \kappa(\mathfrak{p})$ are. However, since

$$\begin{array}{ccc} A & \longrightarrow & \kappa(\varphi^{-1}(\mathfrak{p})) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow \\ B & \longrightarrow & \kappa(\mathfrak{p}) \end{array}$$

commutes, $\kappa(\mathfrak{p})^{n-k} \to Q \otimes_A \kappa(\mathfrak{p})$ is just the base change of $\kappa(\varphi^{-1}(\mathfrak{p}))^{n-k} \to Q \otimes_A \kappa(\varphi^{-1}(\mathfrak{p}))$ by $\kappa(\varphi^{-1}(\mathfrak{p})) \to \kappa(\mathfrak{p})$. Field extensions are faithfully flat, so we have that $B^{n-k} \to Q \otimes B$ is surjective iff $\kappa(\varphi^{-1}(\mathfrak{p}))^{n-k} \to Q \otimes_A \kappa(\varphi^{-1}(\mathfrak{p}))$ is surjective for all $\mathfrak{p}$, which is equivalent to $A^{n-k}_{\varphi^{-1}(\mathfrak{p})} \to Q_{\varphi^{-1}(\mathfrak{p})}$ being surjective for all $\mathfrak{p}$. By the definition of J, this last condition is equivalent to J not being contained in any $\varphi^{-1}(\mathfrak{p})$, i.e. $\varphi(J)B = B$, as desired. $\square$

By the proof of the lemma, $\mathbf{Gr}_F(n,k)$ is covered by open affines F-isomorphic to $\mathbf{A}_F^{k(n-k)}$. We thus conclude that $\mathbf{Gr}_F(n,k)$ is smooth over F of relative dimension $k(n-k)$. By looking at F-points, we can also see that each of the U_i intersect non-trivially. Indeed, using the same notations as the proof, for any $i,j \in I$, take a representative $\pi : F^n \to F^{n-k}$ of a point $x \in U_i(F)$. Create a modified map

$\pi' : F^n \to F^{n-k}$ by keeping it the same image for e_ℓ when $\ell \notin j$ or $\ell \in i$, then, choose a bijection $\psi : j \setminus i \to i \setminus j$ and set $\pi'(e_\ell)$ to $\pi(e_{\psi(\ell)})$ for $\ell \in j \setminus i$. The class of π' is then both in $U_i(F)$ and $U_j(F)$. Therefore we conclude that $\mathbf{Gr}_F(n,k)$ is connected (geometrically connected, in fact, since the same argument works after arbitrary field extension) and since all its local rings are (geomtrically) integral, it must be a (geometrically) integral scheme.

Moreover, by applying the valuative criterion for properness ([Gr1, Théorème 7.3.8]), we see that $\mathbf{Gr}_F(n,k)$ is proper. The criterion can be stated in the following form: for any F-algebra A which is a discrete valuation ring, the map $\mathbf{Gr}_F(n,k)(A) \to \mathbf{Gr}_F(n,k)(K)$ induced by the inclusion of A into its fields of fractions K is bijective. For surjectivity, suppose $x \in \mathbf{Gr}_F(n,k)(K)$ be represented by $\pi : K^n \to K^{n-k}$. Then the image $\pi(A^n)$ is torsion-free and such that $\pi(A^n) \otimes_A K \cong K^{n-k}$, so $\pi(A^n) \cong A^{n-k}$ since A is principal. Therefore the isomophism class of $\pi' : A^n \to \pi(A^n)$, defined by $\pi'(y) = \pi(y)$, is a lift of x. For injectivity, suppose that $\pi_1 : A^n \to A^{n-k}$ and $\pi_2 : A^n \to A^{n-k}$ are quotients which become isomorphic over K, say by $\psi \in \mathrm{Aut}_K(A^{n-k} \otimes_A K)$. By since the π_i are surjective, ψ must restrict to an automorphism of A^{n-k} and so π_1 and π_2 are equivalent over A. In sum, we conclude that $\mathbf{Gr}_F(n,k)$ is a complete, smooth algebraic variety over F.

Remark 1.1.6. Let V be a vector space over F of dimension n. The choice of an isomorphism $V \cong F^n$ induces an isomorphism of functors $\mathbf{Gr}(\tilde{V},k) \cong \mathbf{Gr}_F(n,k)$. Thus $\mathbf{Gr}(\tilde{V},k)$ is representable and has the same properties listed above. Similarly, for any F-scheme X and trivial locally free sheaf $\mathcal{E}$ of rank n (i.e., $\mathcal{E} \cong \mathcal{O}_X^{\oplus n}$), $\mathbf{Gr}(\mathcal{E},k)$ is represented by $\mathbf{Gr}_F(n,k) \times_F X$.

Lemma 1.1.4. *Let S be an F-scheme and $\mathcal{E}$ be locally free sheaf of rank n on S. For any $k \in \mathbf{N}$, $\mathbf{Gr}(\mathcal{E},k)$ is representable.*

Proof. Cover S with open subschemes U_i such that all $\mathcal{E}|_{U_i}$ are trivial. We define open S-subspaces G_i of $\mathbf{Gr}(\mathcal{E},k)$ by taking a fibre product over h_S with h_{U_i}. Then $G_i(T)$ is the same if $T \to S$ factors through U_i and empty otherwise, i.e. it is represented by $\mathbf{Gr}(\mathcal{E}|_{U_i},k)$ viewed as an S-scheme (it exists as a scheme because $\mathcal{E}|_{U_i}$ is trivial). It is clear that the G_i cover $\mathbf{Gr}(\mathcal{E},k)$ since for any field K, any morphism $\mathrm{Spec}\, K \to S$ factors through one of the U_i since they form an open cover. Since $\mathbf{Gr}(\mathcal{E},k)$ is a sheaf for the Zariski topology, we conclude that it is representable. □

Definition 1.1.15. Let S be an F-scheme and $\mathcal{E}$ be locally free sheaf $\mathcal{E}$ of rank n on S. Let $p : \mathbf{Gr}(\mathcal{E},k) \to S$ be the structural morphism. We define the universal quotient bundle $p^*\mathcal{E} \to \mathcal{Q}$ to be the quotient (up to isomorphism) corresponding to the identity map $\mathrm{id}_{\mathbf{Gr}(\mathcal{E},k)}$. We define the tautological subbundle to be the kernel $\mathcal{S} \subseteq p^*\mathcal{E}$ associated to this isomorphism class. Note that $\mathcal{S}$ is locally free of rank k.

Remark 1.1.7. With the same assumptions as the definition, let $q : X \to S$ be an S-scheme and $f : X \to \mathbf{Gr}(\mathcal{E}, k)$ a morphism over S. Since $f = f \circ \mathrm{id}_{\mathbf{Gr}(\mathcal{E},k)}$, we have that f is uniquely determined by the isomorphism class of $q^*\mathcal{E} \to f^*\mathcal{Q}$. In turn, since

$$0 \to \mathcal{S} \to p^*\mathcal{E} \to \mathcal{Q} \to 0$$

is an exact sequence of locally free sheaves, the induced map $f^*\mathcal{S} \to q^*\mathcal{E}$ is an injection, and defines a subsheaf of $q^*\mathcal{E}$ which is the kernel of $q^*\mathcal{E} \to f^*\mathcal{Q}$. Therefore, $f^*\mathcal{S}$, considered as a subsheaf of $q^*\mathcal{E}$, also determines f uniquely. This gives another characterisation of the universal property of $\mathbf{Gr}(\mathcal{E}, k)$. More precisely, any locally free subsheaf $\mathcal{F}$ of rank k of $q^*\mathcal{E}$ which is locally a direct summand gives rise to a unique S-morphism $f : X \to \mathbf{Gr}(\mathcal{E}, k)$ such that $f^*\mathcal{S}$ gives the same subsheaf as $\mathcal{F}$.

Example 1.1.17. Let $f : T \to S$ be an S-scheme. Define a map $\mathbf{Gr}(\mathcal{E}, k)(T) \to \mathbf{Gr}(\mathcal{E}^\vee, n-k)(T)$ by sending the class of $\pi : f^*\mathcal{E} \to \mathcal{Q}$ to the dual map of $\ker \pi \subseteq f^*\mathcal{E}$, i.e. $f^*\mathcal{E}^\vee \to (\ker \pi)^\vee$. Since the kernel is locally free, it is preserved by pullback so this construction gives a natural transformation. Moreover, using the identification of $\mathcal{E}$ with $(\mathcal{E}^\vee)^\vee$, we see that applying it again to $\mathbf{Gr}(\mathcal{E}^\vee, n-k)$ gives an inverse, so $\mathbf{Gr}(\mathcal{E}, k) \cong \mathbf{Gr}(\mathcal{E}^\vee, n-k)$. In particular, $\mathbf{Gr}_F(n, k) \cong \mathbf{Gr}_F(n, n-k)$.

Example 1.1.18. By Remark 1.1.7, we can also interpret $\mathbf{Gr}_F(n, k)(F)$ as parameterizing k-dimensional subspaces of F^n, and similarly for any field extension $F \subseteq E$. For this reason, we view $\mathbf{Gr}_F(n, k)$ as the moduli space of k-dimensional subspaces of F^n. One may then expect that $\mathbf{Gr}_F(n+1, 1) \cong \mathbf{P}^n_F$. This is indeed the case. By Example 1.1.17, this is the same as showing that $\mathbf{Gr}_n(n+1, n) \cong \mathbf{P}^n_F$. This follows from the fact that for any F-scheme S,

$$\mathbf{P}^n(S) = \{\mathcal{O}_S^{\oplus n+1} \to \mathcal{L} : \mathcal{L} \text{ invertible sheaf on } S\}/\sim$$

Theorem 1.1.3. *Let S be an F-scheme and $\mathcal{E}$ be locally free sheaf of rank n on S. $p : \mathbf{Gr}(\mathcal{E}, k) \to S$ is smooth and projective.*

Proof. First, note that taking an open cover U_i of S on which $\mathcal{E}$ is trivial, we have $p^{-1}(U_i) = \mathbf{Gr}(\mathcal{E}|_{U_i}, k) \to U_i$ is smooth and proper since it is isomorphic as a U_i-scheme to $U_i \times_F \mathbf{Gr}_F(n, k)$, and $\mathbf{Gr}_F(n, k)$ is smooth and proper. Hence we conclude that $\mathbf{Gr}(\mathcal{E}, k)$ is smooth and proper over S since both properties are Zariski local on the base.

To show that it is projective, we must construct an S-morphism $\varphi : \mathbf{Gr}(\mathcal{E}, k) \to \mathbf{P}(\mathcal{F})$ ($\mathcal{F}$ a locally free sheaf on S of some rank r) which is a closed immersion, or what is the same an S-morphism $\varphi : \mathbf{Gr}(\mathcal{E}^\vee, n-k) \to \mathbf{Gr}(\mathcal{F}^\vee, r-1)$ with the same property. For an S-scheme $f : T \to S$, define

$$\varphi(T) : \mathbf{Gr}(\mathcal{E}^\vee, n-k)(T) \to \mathbf{Gr}(\wedge^k \mathcal{E}^\vee, r-1)(T),$$

$$[f^*\mathcal{E}^\vee \to \mathcal{Q}] \mapsto \left[f^*\left(\bigwedge^k \mathcal{E}^\vee\right) = \bigwedge^k (f^*\mathcal{E}^\vee) \to \bigwedge^k \mathcal{Q}\right]$$

where the second map is the one induced on k-th exterior powers and $r = \binom{n}{k}$. Note that this is well-defined since $\mathcal{Q}$ is of rank k, hence $\bigwedge^k \mathcal{Q}$ is of rank 1.

First, we show that it is true for S is $\operatorname{Spec} F$ and $\mathcal{E} = \tilde{V}$, V a vector space over F of dimension n. We may assume that F is algebraically closed since closed immersions are fpqc local on the base (Theorem 1.1.1). By [Ha2, Proposition II.7.3], it is enough to show that:

i. $\varphi(F) : \mathbf{Gr}(V^\vee, n-k)(F) \to \mathbf{Gr}(\wedge^k V^\vee, r-1)(F) = \mathbf{P}(\wedge^k V)$ is injective.

ii. For all $x \in \mathbf{Gr}(V^\vee, n-k)(F)$, there is a neighbourhood U of x such that $\varphi|_U(F[\varepsilon]) : U(F[\varepsilon]) \to \mathbf{Gr}(\wedge^k V^\vee, r-1)(F[\varepsilon])$ is injective, where $F[\varepsilon]$ are the dual numbers over F (this implies that φ "separates tangent vectors" by Example 1.1.6).

Taking duals, i) is equivalent to the claim that a k-dimensional subspace $W \subseteq V$ is determined by the subspace $\bigwedge^k W \subseteq \bigwedge^k V$. This is true, since for any non-zero $\eta \in \bigwedge^k W$, the kernel of $V \to \bigwedge^{k+1} V, v \mapsto v \wedge \eta$ is exactly W. For ii), it follows from Lemma 1.1.5 below, as the surjectivity condition define an open subscheme in the same way as the U_i of the proof of Lemma 1.1.3.

Since $\mathbf{Gr}(\mathcal{E}, k)$ is proper over S, φ being monic already implies it is a closed immersion ([Gr2, Corollaire 18.12.6]). By [Gr2, Proposition 17.2.6], φ is monic if, for all $x \in \mathbf{Gr}(\wedge^k \mathcal{E}^\vee, r-1)$, the scheme-theoretic fibre $\varphi^{-1}(x)$ is the empty scheme or isomorphic to $\operatorname{Spec} \kappa(x)$. Now, x lies over some $s \in S$. Note that the canonical map $\operatorname{Spec} \kappa(x) \to \mathbf{Gr}(\wedge^k \mathcal{E}^\vee, r-1)$ factors through the closed subscheme $\mathbf{Gr}(\wedge^k \mathcal{E}^\vee, r-1) \times_S \kappa(s) = \mathbf{Gr}((\wedge^k \mathcal{E}^\vee) \otimes \kappa(s), r-1) = \mathbf{Gr}(\wedge^k (\mathcal{E} \otimes \kappa(s))^\vee, r-1)$ And hence computing the fibre reduces to the case over a field, in this case $\kappa(s)$, since the naturality of the definition of φ means it respects base change. However, over a field φ is a closed immersion, hence the fibre is either empty or $\operatorname{Spec} \kappa(x)$. □

Remark 1.1.8. The embedding given in the proof is a relative version of the classical Plücker embedding. See, for example, [Ha1, §6].

Lemma 1.1.5. *Let A be a local F-algebra, V a vector space of dimension n over F and $W \subseteq V$ a subspace of dimension k. The isomophism class of a quotient $\pi : V \otimes A \to A^k$ where $W \otimes A$ surjects onto A^k is determined by the isomophism class of $\wedge^k \pi : \bigwedge^k (V \otimes A) \to \bigwedge^k A^k$.*

Proof. Decompose V as $W \oplus W'$ for some $W' \subseteq V$. The the isomorphism class of π is exactly determined by its restriction in $\operatorname{Hom}_A(W' \otimes A, A^k)$, which we will call f.

$\bigwedge^k(V \otimes A)$ decomposes as

$$\bigoplus_{i+j=k} \bigwedge^i (W \otimes A) \otimes_A \bigwedge^j (W' \otimes A)$$

Restricting $\wedge^k \pi$ to the $i = k-1, j = 1$ summand, we get a map g given by

$$g((w_1 \wedge \cdots \wedge w_{k-1}) \otimes w') = \wedge^k \pi(w_1 \wedge \cdots \wedge w_{k-1} \wedge w') = \wedge^{k-1}\pi(w_1 \wedge \cdots \wedge w_{k-1}) \wedge f(w')$$

Choosing some isomorphism $\bigwedge^k A^k \cong A$, we get an isomorphism of $\bigwedge^{k-1} A^k \to \mathrm{Hom}_A(A^k, A)$ by $\eta \mapsto (x \mapsto \eta \wedge x)$. Since $\wedge^{k-1}\pi$ is surjective, this means g determines the value of f on all linear forms, hence determines f. □

As an application of relative Grassmannians, we show that certain partial flag varieties are representable by smooth, projective varieties.

Definition 1.1.16. Let V be an n dimensional vector space over F. For integers $0 < k < \ell < n$, define the functor $\mathbf{Fl}(V, k, \ell) : \mathbf{Sch}_F^\circ \to \mathbf{Sets}$ by

$$\mathbf{Fl}(V, k, \ell)(S) = \{(\mathcal{F}_k, \mathcal{F}_\ell) \in \mathbf{Gr}(V, k)(S) \times \mathbf{Gr}(V, \ell)(S) : \mathcal{F}_k \subseteq \mathcal{F}_\ell\}$$

for an F-scheme S (here the $\mathcal{F}_i$ are considered as subbundles of $V \otimes \mathcal{O}_S$, as in Remark 1.1.7) and for $f : S \to T$, let $\mathbf{Fl}(V, k, \ell)(f) : \mathbf{Fl}(V, k, \ell)(T) \to \mathbf{Fl}(V, k, \ell)(S)$ be the map induced $\mathbf{Gr}(V, k)(f) \times \mathbf{Gr}(V, \ell)(f)$.

Proposition 1.1.11. *The functor $\mathbf{Fl}(V, k, \ell)$ is represented by $\mathbf{Gr}(\mathcal{Q}, \ell - k)$ where $\mathcal{Q}$ is the universal quotient bundle of $\mathbf{Gr}(V, k)$.*

Proof. To give a map of F-schemes from S to $\mathbf{Gr}(\mathcal{Q}, \ell - k)$ is to give

i. A map of F-schemes $f : S \to \mathbf{Gr}(V, k)$, or equivalently choosing a subbundle $\mathcal{F}_k \subseteq V \otimes \mathcal{O}_S$ with the relation that $\mathcal{F}_k = f^*\mathcal{S}_k$ where $\mathcal{S}_k$ is the tautological bundle.

ii. A choice of a rank $\ell - k$ subbundle of $f^*\mathcal{Q}$, which is equivalent to a rank ℓ subbundle of $V \otimes \mathcal{O}_S$ containing $\mathcal{F}_k$, since

$$0 \to \mathcal{F}_k = f^*\mathcal{S}_k \to V \otimes \mathcal{O}_S \to f^*\mathcal{Q} \to 0$$

is exact. This defines a bijection $\mathbf{Fl}(V, k, \ell)(S) \to \mathbf{Gr}(\mathcal{S}, \ell - k)(S)$ which is easily seen to be natural in S. □

Corollary 1.1.1. *$\mathbf{Fl}(V, k, \ell)$ is a smooth, projective variety.*

1.2 Some Galois theory

A *twisted form* is informally an algebraic or geometric object defined over a field F which becomes isomorphic to some fixed object after base change to some field extension of F. Two examples of such objects are central simple algebras, which are twisted forms of matrix algebras, and Severi-Brauer varieties, which are twisted forms of projective spaces. In fact, there is an explicit geometric construction showing that the two objects are equivalent. The underlying reason for this is that $n \times n$ matrices and projective $n-1$-space both have PGL_n as their automorphism group. The connection is then provided by the first Galois cohomology group and 1-cocycles. After reviewing the relevant facts from Galois theory and group cohomology, we explain in detail this theory.

1.2.1 Étale algebras

Definition 1.2.1. Let F be a field with algebraic closure $\bar{F}$. A finite F-algebra A is said to be (finite) étale if $A \otimes_F \bar{F} \cong \prod_{1 \leq i \leq \dim_F A} \bar{F}$ as an $\bar{F}$-algebra.

Proposition 1.2.1. *Let A be a finite F-algebra. The following are equivalent:*

i. *A is an étale F-algebra*

ii. *The structural morphism* $\operatorname{Spec} A \to \operatorname{Spec} F$ *is étale.*

iii. *A is isomorphic to a direct product of finite separable field extensions of F.*

Proof. Recall that a finite étale morphism $X \to S$ is nothing but a finite smooth morphism. Both properties are fpqc local on the base (Theorem 1.1.1), so in the case of $S = \operatorname{Spec} F$, may be verified after extension to $\bar{F}$. It is then clear that i) implies ii).

ii) $\implies$ iii) amounts to the claim that a connected finite, smooth F-scheme is the spectrum of a finite separable field extension. A connected and finite scheme over F is the spectrum of a local ring, and as smoothness implies regularity, it must be the spectrum of a field. Call it L. By [Ei, Corollary 16.17], $\Omega_{L/F} = 0$ if and only if L is separable, so the claim follows.

To see i) from iii), note that a finite separable extension L/F is isomorphic to $F[x]/(p(x))$, where $p(x)$ is a separable polynomial, say with roots α_i in $\bar{F}$. Then $L \otimes_F \bar{F} \cong \bar{F}[x]/(p(x)) \cong \bar{F}[x]/(\prod_i (x - \alpha_i))$. Of course, this is just isomorphic to a direct product of copies of $\bar{F}$. □

Recall that for a field F, the absolute Galois group of F, written Γ_F, is defined as the projective limit $\varprojlim \operatorname{Gal}(L/F)$, where the inverse system is defined by associating to an inclusion $K \subseteq L$ of finite Galois extensions of F within a fixed separably closed overfield the canonical restriction map $\mathrm{res}^L_K : \mathrm{Gal}(L/F) \to \mathrm{Gal}(K/F)$ ([Se2, p. 2]). It obtains the natural topology of an inverse limit of topological groups (the finite Galois groups are given the discrete topology) which gives it the structure of a profinite group.

Definition 1.2.2. Let G be a topological group. A discrete G-set is a set X with a G-action such that, when X is equiped with the discrete topology, the action map $\varphi : G \times X \to X$ is continuous.

Let Ω be a separably closed field extension of F. To such a field extension is associated the fibre functor $\mathrm{Fib}_\Omega : \mathbf{F\acute{E}t}_F^\circ \to \mathbf{Sets}$, where $\mathbf{F\acute{E}t}_F$ is the full subcategory of finite étale F-algebras, defined by $A \mapsto \mathrm{Hom}_F(A, \Omega)$. This is in fact a functor to discrete Γ_F-sets in the following way: let $\sigma \in \Gamma_F$, and $\varphi \in \mathrm{Hom}_F(A, \Omega)$. Then the image of φ is contained in a finite Galois extension of F, say L. Then define $(\sigma \cdot \varphi)(x) = \sigma_L(\varphi(x))$, where σ_L is the image of σ under the projection to $\mathrm{Gal}(L/F)$. This definition is independent of the choice of L and gives a continuous group action (since it factors through a finite quotient) with respect to which the functorial maps $\mathrm{Hom}_F(A, \Omega) \to \mathrm{Hom}_F(B, \Omega)$ are Γ_F-equivariant.

Remark 1.2.1. By Proposition 1.2.1, the opposite category of finite étale F-algebras are finite étale F-schemes, and the fibre functor corresponds to taking the Ω-points of a finite étale F-scheme X. These points can be identified with the points of the underlying topological space of the scheme $X \times_F \Omega$, hence the name "fibre functor", since it comes from the fibre over the "point" $\mathrm{Spec}\,\Omega \to \mathrm{Spec}\,F$.

Theorem 1.2.1. *The fibre functor gives a contravariant equivalence of categories between finite étale F-algebras and finite discrete Γ_F-sets.*

Proof. Taking into account Remark 1.2.1, the fibre functor is equivalent to that defined in [GR, Exp. V, §7] in the case of the spectrum of a field. The result the follows by applying Théorème 4.1 and Proposition 8.1 of [GR, Exp. V]. □

Example 1.2.1. One can recover the usual Galois correspondence for this by taking stabilisers. Indeed, for a fixed finite Galois extension L and subfield $F \subseteq E \subseteq L$, applying the fibre functor one gets a Γ_F-set whose action factors through $\mathrm{Gal}(L/F)$, is transitive, and comes equipped with a distinguished point coming from the inclusion $E \subseteq L$. The stabiliser in $\mathrm{Gal}(L/F)$ of the distinguished point is $\mathrm{Gal}(L/K)$. Since one can reverse this construction by taking a subgroup $H \subseteq \mathrm{Gal}(L/F)$ to the discrete Γ_F-set $\mathrm{Gal}(L/F)/H$, Theorem 1.2.1 gives the usual Galois correspondence.

Definition 1.2.3. Let $\mathcal{G} : \mathbf{F\acute{E}t}_F \to \mathbf{Grp}$ be a functor. We say that $\mathcal{G}$ is a sheaf for the étale topology if it preserves products, and for any finite separable extensions $F \subseteq E \subseteq L$, with L/E Galois, the induced map $\mathcal{G}(E) \to \mathcal{G}(L)$ is an isomorphism onto $\mathcal{G}(L)^{\mathrm{Gal}(L/E)}$, where the Galois action on $\mathcal{G}(L)$ is the one induced by that on L. A morphism of sheaves for the étale topology is just a natural transformation.

Example 1.2.2. The functor assigning a finite étale F-algebra A to its group of units is a sheaf for the étale topology.

For an inclusion of Galois extensions $E \subseteq L$ of F, let $r_E^L : \mathrm{Gal}(L/F) \to \mathrm{Gal}(E/F)$ and $i_E^L : \mathcal{G}(E) \to \mathcal{G}(L)$ be the canonical restriction and inclusion maps. Then for any $\sigma \in \mathrm{Gal}(L/F)$, and $g \in \mathcal{G}(E)$. We have that $i_E^L(r_E^L(\sigma) \cdot g) = \sigma \cdot i_E^L(g)$. The r_E^L form an inverse directed system with limit Γ_F, and the i_E^L form a directed system whose limit we will call $\bar{\mathcal{G}}$, which gains the structure of a Γ_F-module by the above compatibility. Moreover, when $\mathcal{G}$ is considered with the discrete topology, the action of Γ_F is continuous and acts by group homomorphisms.

Theorem 1.2.2. *The functor $\mathcal{G} \mapsto \bar{\mathcal{G}}$ from étale sheaves of (abelian) groups over F to discrete (abelian) groups with continuous Γ_F-action is an equivalence of categories.*

Proof. First, using the equivalence of categories of Theorem 1.2.1, we translate the sheaf condition for the corresponding contravariant functor $\mathcal{G}' : \Gamma_F\text{-}\mathbf{Sets} \to \mathbf{Grp}$. The condition of $\mathcal{G}$ preserving products corresponds to $\mathcal{G}'$ sending coproducts to products. Up to isomorphism, a Galois extension $E \subseteq L$ in $\mathbf{F\acute{E}t}_F$ corresponds to a quotient $\pi : \Gamma_F/N \to \Gamma_F/H$, with N normal in H. The (right) action of $\sigma \in \mathrm{Gal}(L/E)$ on Γ_F/N is then given by taking an extension σ' in Γ_F and multiplying the cosets of Γ_F/N by the right. So the second part of the sheaf condition corresponds to $\pi^* : \mathcal{G}'(\Gamma_F/H) \to \mathcal{G}'(\Gamma_F/N)$ being an isomorphism onto the fixed points of the induced (left) H/N action on $\mathcal{G}'(\Gamma_F/N)$ for all finite index subgroups H, N of Γ_F, with N normal in H.

Thus, we have reduced the problem to constructing an inverse to the functor

$$\mathcal{G} \mapsto \varinjlim_{H \subseteq \Gamma_F, [\Gamma_F : H] < \infty} \mathcal{G}(\Gamma_F/H)$$

of the full subcategory of contravariant functors of finite Γ_F-sets to groups satisfying the above "sheaf condition" to discrete groups with continuous Γ_F actions. We claim that the functor assigning a discrete Γ_F-group G to the contravariant functor $X \mapsto \mathrm{Hom}_{\Gamma_F}(X, G)$ is such an inverse. The hom-sets are given a group structure by point-wise multiplication. For the sheaf condition, the condition on coproducts is clearly satisfied. Let $N \subseteq H \subseteq \Gamma_F$ and π be as before. It is clear that π^* is injective, so we must show that the H/N fixed points in $\mathrm{Hom}_{\Gamma_F}(\Gamma_F/N, G)$ are precisely the Γ_F-equivariant maps which factor through Γ_F/H. One direction is clear, and noting that by normality we have $hN = Nh$ for any $h \in H$, it follows that if $f \in \mathrm{Hom}_{\Gamma_F}(\Gamma_F/N, G)$ is invariant, then $f(gN) = f(ghN)$ for all $h \in H$ and so factors through Γ_F/H.

If $\mathcal{G}$ satisfies the sheaf conditions, than letting $G = \varinjlim \mathcal{G}(\Gamma_F/H)$, for any finite index subgroup $H \subseteq \Gamma_F$, there are isomorphisms (natural in G) $\mathrm{Hom}_{\Gamma_F}(\Gamma_F/H, G) = G^H = \mathcal{G}(\Gamma_F/H)$. Since both functors send coproducts to products, this extends to a natural isomorphism $\mathcal{G} = \mathrm{Hom}_{\Gamma_F}(-, G)$. Starting with a discrete Γ_F-group G, we may identify the directed system $\mathrm{Hom}_{\Gamma_F}(\Gamma_F/H, G)$ over subgroups H of finite index with G^H as before. Since the action of Γ_F is continuous, all $x \in G$ have finite index stabiliser, and so $\varinjlim \mathrm{Hom}_{\Gamma_F}(\Gamma_F/H, G)$ is naturally isomorphic to G as desired. □

Remark 1.2.2. The category of discrete abelian groups with continuous $\mathbf{Z}$-linear Γ_F-action and equivariant homomorphisms is an abelian category. From this we conclude that the category of étale sheaves of abelian groups over F is an abelian category.

1.2.2 Group cohomology

We follow the exposition of [Se1, Chapter VII].

Definition 1.2.4. Let G be a group. Let A be an abelian group. A G-module structure on A is a homomorphism $\varphi : G \to \mathrm{Aut}(A)$. For $g \in G$ and $a \in A$, we write $g \cdot a$ or simply ga for $(\varphi(g))(a)$.

Definition 1.2.5. Let G be a group. The group ring is a pair consisting of a ring $\mathbf{Z}[G]$ and a group homomorphism $G \to \mathbf{Z}[G]^\times$ which is universal in the sense that any homomorphism of G to the unit group a ring R factors through $\mathbf{Z}[G]^\times$ via a ring homomorphism $\mathbf{Z}[G] \to R$.

For any group G, $\mathbf{Z}[G]$ exists and as an abelian group is $\bigoplus_{g \in G} \mathbf{Z}$. Let e_g denote 1 in the g-th summand, then we set $e_g \cdot e_h = e_{gh}$ and define multiplication on $\mathbf{Z}[G]$ by extending $\mathbf{Z}$-linearly. One easily checks this gives a ring with the above universal property. Given a G-module A, one clearly gets a $\mathbf{Z}[G]$-module by the universal property. In the sequel we shall use this to treat G-modules as modules over a ring, and apply the usual notions from categories of modules. In particular, we get an abelian category with enough injectives (see [Ha2, §III.1]).

Let A be a G-module. We write $A^G = \{x \in A : \forall g \in G, gx = x\}$ for the submodule of fixed points of A. For any G-module homomorphism $f : A \to B$, there is an induced map $f_* : A^G \to B^G$. This makes the assignment $A \to A^G$ into a functor from G-modules to abelian groups. Moreover, letting $\mathbf{Z}$ be a G-module via the trivial action, we have an isomorphism $A^G \cong \mathrm{Hom}_G(\mathbf{Z}, A)$ natural in A. This implies that $A \mapsto A^G$ is left-exact, with derived functors $\mathrm{Ext}^i_G(\mathbf{Z}, A)$ for $i \geq 0$.

Definition 1.2.6. We define the i-th cohomology group of a group G with coefficients in a G-module A to be $H^i(G, A) := \mathrm{Ext}^i_G(\mathbf{Z}, A)$. In particular, $H^0(G, A) = A^G$.

Although the cohomology groups are defined as a right-derived functor, computations using injective resolutions are not practical. However, $\mathrm{Ext}^i_R(M, N)$ can be computed with a projective resolution of M. We define the following free resolution of G-modules of $\mathbf{Z}$:

For $i \geq 0$, let F_i be the free abelian group generated by $i+1$-tuples $(g_0, \dots, g_i) \in G^{i+1}$. We define a G-module structure by setting $h \cdot (g_0, \dots, g_i) = (hg_0, \dots, hg_i)$ and extending by $\mathbf{Z}$-linearity. Clearly, each F_i is a free $\mathbf{Z}[G]$-module, with basis given by, say, $\{(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_i)\}_{(g_1,\dots,g_i)\in G^i}$. For $i \geq 1$, we define $d_i : F_i \to F_{i-1}$ by

$$d_i(g_0, \dots, g_i) = \sum_{k=0}^{k=i} (-1)^k (g_0, \dots, \hat{g}_k, \dots, g_i)$$

One easily checks that the d_i are G-homomorphisms, and $d_i \circ d_{i+1} = 0$. Moreover, the chain complex (F_i, d_i) is exact for $i \geq 1$. At $i = 0$, define a G-homomorphism $F_0 \to \mathbf{Z}$ by $\sum n_g \cdot (g) \mapsto \sum n_g$. This is surjective and the kernel is generated by elements of the form $(g) - (h)$, $g, h \in G$. But this is none other than the image of d_1, so (F_i, d_i) is a free resolution of $\mathbf{Z}$, which we call the *bar resolution*.

Applying $\mathrm{Hom}_G(-, A)$ to the bar resolution, we obtain a complex consisting of the abelian groups $\mathrm{Hom}_G(F_i, A)$. Using the basis elements $(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_i)$ for F_i, we can identify the elements of $\mathrm{Hom}_G(F_i, A)$ with the groups of maps $f : G^i \to A$, by letting $f(g_1, \dots, g_i)$ be the image of $(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_i)$ in A. We call these *cochains*. Under this identification, one checks that the differential maps $d_i^* : \mathrm{Hom}_G(F_{i-1}, A) \to \mathrm{Hom}_G(F_i, A)$ (which we simply write as d, with the source and target being understood) are given by

$$\begin{aligned} df(g_1, \dots, g_i) =& g_1 f(g_2, \dots, g_i) + \sum_{k=1}^{k=i-1} (-1)^k f(g_1, \dots, g_k g_{k+1}, \dots, g_i) \\ &+ (-1)^i f(g_1, \dots, g_{i-1}) \end{aligned}$$

The homology of this complex will compute the groups $H^i(G, A)$. Since the identification is natural in A, the complex will also compute the cohomology as a δ-functor as well. More explicitely, we say $f : G^n \to A$ is an n-cocycle if $df = 0$ and is an n-coboundary if there exists $g : G^{n-1} \to A$ such that $f = dg$. Both of these form subgroups, which we denote respectively $Z^n(G, A)$ and $B^n(G, A)$, and we have $H^n(G, A) = Z^n(G, A)/B^n(G, A)$. We say that two cocycles which represent the same element in cohomology are *cohomologous*. For a morphism of G-modules $\varphi : A \to B$, the induced map on cohomology is given in terms of cocyles by $\varphi_*([f]) = [\varphi \circ f]$.

First cohomology

With notation as above, let $f : G \to A$ be a 1-cochain. We have $df(g, g') = gf(g') - f(gg') + f(g)$, so $df = 0$ iff for any $g, g' \in G$, $f(gg') = gf(g') + f(g)$. This describes $Z^1(G, A)$. Let f be a 0-cochain, then we have $df(g) = ga - a$ for a fixed element

$a \in A$. This provides a description of $B^1(G, A)$. Notice that when the action of G on A is trivial, $Z^1(G, A) = \mathrm{Hom}(G, A)$ and $B^1(Z, A) = 0$, so $H^1(G, A) = \mathrm{Hom}(G, A)$. These concrete definitions of 1-cocycles and coboundaries can just as easily apply to the case A a (not necessarily abelian) group with a G-action (i.e. homomorphism $\varphi : G \to \mathrm{Aut}(A)$). To fix notation, we write cochains as $G \ni \sigma \mapsto \mathfrak{a}_\sigma \in A$. A cochain $\mathfrak{a}$ is a cocycle if for all $\sigma, \tau \in G$, $\mathfrak{a}_{\sigma\tau} = \mathfrak{a}_\sigma \sigma \cdot \mathfrak{a}_\tau$ and two cocycles $\mathfrak{a}, \mathfrak{b}$ are cohomologous, and we write $\mathfrak{a} \sim \mathfrak{b}$, if there exists an element $a \in A$ such that $\mathfrak{b}_\sigma = a^{-1}\mathfrak{a}_\sigma(\sigma \cdot a)$ for all $\sigma \in G$.

Inflation map

Let $\varphi : G \to H$ be a group homomorphism and A an H-module. We define the G-module $\phi^* A$ to be A as an abelian group with G-action $g \cdot a = \varphi(g) \cdot a$. This association is functorial, since H-homomorphisms $f : A \to B$ will be also be G-homomorphisms. It is clearly an exact functor. If an element is fixed by H, it is then necessarily fixed by the induced G-action, so we have a natural inclusion $A^H \to (\varphi^* A)^G$, i.e. we have a natural transformation $\Phi^0 : H^0(H, -) \to H^0(G, \varphi^*(-))$. Since φ^* is exact, the functors $H^i(G, \varphi^*(-))$ form a δ-functor, so by the universal property of the $H^i(H, -)$, there is a unique morphism of δ-functors $\Phi^i : H^i(H, -) \to H^i(G, \varphi^*(-))$ extending Φ^0. Computing cohomology groups with cocycles, the maps $\Phi^i(A) : H^i(H, A) \to H^i(G, \varphi^* A)$ can be realised by $[f] \mapsto [f \circ \varphi]$. Indeed, such a map on cochains induce morphisms of complexes which induce a δ-functor morphism which in degree 0 is the inclusion $A^H \to (\varphi^* A)^G$. If we have another group homomorphism $\psi : G' \to G$, then we have $\psi^* \varphi^* A = (\varphi \circ \psi)^* A$ and a composite map $A^H \to (\varphi^* A)^G \to ((\varphi \circ \psi)^* A)^{G'}$, of which the induced map of δ-functors as described above is the composite of those induced by φ and ψ.

Consider the special case $\varphi : G \to G/N$ the quotient map for N a normal subgroup of G. Let A be a G-module, then A^N is naturally a G/N, and the map $i : \varphi^* A^N \to A$ is G-linear. We thus get a composite map $i_* \circ \Phi^i(A^N) : H^i(G/N, A^N) \to H^i(G, \varphi^* A^N) \to H^i(G, A)$ which we call the inflation map for N, written $\inf_{G/N}$, which is natural in A. For normal subgroups $N \subseteq N' \subseteq G$, we have, $\inf_{(G/N)/(N'/N)} \circ \inf_{G/N} = \inf_{G/N'}$ after identifying G/N' and $(G/N)/(N'/N)$. This follows from composing the quotient maps and inclusion maps of fixed point modules. The inflation map is given explicitly by $\inf_{G/N}([f]) = [i \circ f \circ \varphi]$. This can be seen by combining the two previous descriptions of how cocycles are mapped.

1.2.3 Galois descent

Our main references will be [GS, §2.3] and [BLR, §6.2].

Proposition 1.2.2. *For a finite Galois extension $F \subseteq L$ with Galois group Γ, the F-algebra homomorphism*

$$L \otimes_F L \to \prod_{\sigma \in \Gamma} L, \qquad a \otimes b \mapsto (\sigma(a) \cdot b)_{\sigma \in \Gamma}$$

is an isomorphism.

Proof. See [BLR, Example 6.2/B]. □

Example 1.2.3. If G is an F-group scheme, then h_G restricted to $\mathbf{F\acute{E}t}_F$ is a sheaf of groups for the etale topology. This follows from Proposition 1.1.10 and that Proposition 1.2.2 gives an isomorphism of exact sequences

$$\begin{array}{ccccc} K & \longrightarrow & L & \overset{x\mapsto 1\otimes x}{\underset{x\mapsto x\otimes 1}{\rightrightarrows}} & L \otimes_K L \\ \downarrow & & \downarrow & & \downarrow \wr \\ K & \longrightarrow & L & \overset{x\mapsto (x)_\sigma}{\underset{x\mapsto (\sigma x)_\sigma}{\rightrightarrows}} & \prod_{\sigma\in\Gamma} L \end{array}$$

for a finite Galois extension $K \subseteq L$.

Additionally, since for $F \subseteq E \subseteq F^s$, we can identify $G(E)$ with its image in $G(F^s)$, we have that $\varinjlim G(E) = G(F^s)$.

This also gives a way of interpreting descent datum for the faithfully flat extension $F \subseteq L$. For simplicity, we explain the case of L-modules, but the same remains true with an algebra structure, or for schemes (see [BLR, Example 6.2/B]).

Let M be an L-module. A Galois action on M is an F-linear Γ-action on M such that for any $x \in L$, $m \in M$ and $\sigma \in \Gamma$, $\sigma(x \cdot m) = \sigma(x) \cdot \sigma(m)$. A Galois action on M gives a descent datum as follows:

One can identify $M \otimes_F L$ with $\oplus_{\sigma\in\Gamma} M$ by $m \otimes a \mapsto (\sigma(m))_{\sigma\in\Gamma}$, under which the $L \otimes_F L$-multiplication and the $\prod_{\sigma\in\Gamma} L$ multiplication correspond. Under the identification $L \otimes_F L = \prod_{\sigma\in\Gamma} L$, the "swap factors" homomorphism corresponds to the map $s : \prod_{\sigma\in\Gamma} \to \prod_{\sigma\in\Gamma}$, $(a_\sigma)_{\sigma\in\Gamma} \mapsto (\sigma(a_{\sigma^{-1}}))_{\sigma\in\Gamma}$. The analogously defined map for $\oplus_{\sigma\in\Gamma}$ then gives a covering datum. Continuing with similar identifications for $M \otimes_F L \otimes_F L$, one sees that the cocycle condition is equivalent to the condition that for all $\sigma, \tau \in \Gamma$ and $m \in M$, $\sigma(\tau(m)) = (\sigma\tau)(m)$. Then by Proposition 1.1.7, we have that:

Proposition 1.2.3. *Let $F \subseteq L$ be a finite Galois extension. The category of L-modules (resp. algebras) with Galois actions and equivariant homomorphisms is* *equivalent to the category of L-modules (resp. algebras) with descent data. Under this correspondence, for an F-module M, the canonical descent datum on $M \otimes_F L$ corresponds to the trivial Galois action $\sigma(m \otimes a) = m \otimes \sigma(a)$.*

For the case of schemes, the same holds with the right definition of Galois action. We first set the convention that Γ acts on the left on $\operatorname{Spec} L$ by the maps $(\sigma^{-1})^* : \operatorname{Spec} L \to \operatorname{Spec} L$. A Galois action on an L-scheme X is a Γ-action over F on X such that

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \xrightarrow{\sigma} & \operatorname{Spec} L \end{array}$$

commutes. The trivial Galois action for $X \times_F L$ (X any F-scheme) is simply $\mathrm{id}_X \times \sigma$.

Proposition 1.2.4. *The functor $X \mapsto X \times_F L$ from algebraic F-schemes to algebraic L-schemes with Galois actions is fully faithful. Its essential image is those algebraic L-schemes X' such that the Γ-orbit of every closed point $x \in X'$ is contained in an open affine subscheme of X'.*

Proof. Fully faithfulness is clear. For essential surjectivity, by Proposition 1.1.9, we need to show that a covering by Γ-stable affine open subschemes is equivalent to to orbits being contained in affine opens. One direction is clear. Let $x \in X'$ be a closed point and U an affine open containing its Γ-orbit. Then the affine open (X' is separated over $\mathbf{Z}$) $\bigcap_{\sigma\in\Gamma} \sigma U$ is Γ-stable and contains x. Finally, since X' is an algebraic scheme, any open cover containing all closed points is in fact an open cover. □

Example 1.2.4. Let X be an algebraic F-scheme. Then a closed subscheme $Y' \subseteq X \times_F L$ is defined over F, i.e. of the form $i \times \mathrm{id}_L : Y \times_F L \hookrightarrow X \times_F L$ for some $i : Y \hookrightarrow X$, if and only if Y' is Γ-stable.

Lemma 1.2.1. *Let X be a projective scheme over a field F and let $x_1, \ldots, x_k \in X$ be a finite set of closed points. There exists an affine open subscheme $U \subseteq X$ containing the x_i.*

Proof. Consider a closed embedding $X \hookrightarrow \mathbf{P}^N_F$. We are then reduced to the case $X = \mathbf{P}^N_F$. Let $F \subseteq E$ be a finite extension such that all the x_i are defined over E. Then each x' over an x_i defines a hyperplane in $H \subseteq (\mathbf{P}^N_E)^\vee$ such that $H(E)$ correspond exactly to the hyperplanes with coefficients in E in $\mathbf{P}^N_E$ containing x'. The complement of all such H is an open subscheme of $(\mathbf{P}^N_E)^\vee$, say V. If F is infinite then, considering $V(E)$ as a subset of $(\mathbf{P}^N_F)^\vee(E)$, $V(E) \cap (\mathbf{P}^N_F)^\vee(F) \neq \emptyset$, i.e. there is a hyperplane defined over F which does not contain any of the x_i. Its complement is our desired open subset U. If F is finite, then E is Galois, and up to taking a further finite extension, we may assume that $V(E) \neq \emptyset$. Let $Y \subseteq \mathbf{P}^N_E$ be a hyperplane corresponding to an element of $V(E)$. Then $Y' = \bigcup_{\sigma \in \Gamma} \sigma(Y)$ is a Γ-stable closed subvariety, so descends to a closed subscheme $Y'' \hookrightarrow \mathbf{P}^N_F$ by Example 1.2.4. The complement $U = \mathbf{P}^N_F \setminus Y''$ is clearly open, and affine since $U \times_F E$ is isomorphic to the complement of Y' in $\mathbf{P}^N_E$, which is affine since it is the intersection of affine opens in a separated scheme. □

Corollary 1.2.1. *The functor $X \mapsto X_L := X \times_F L$ of projective schemes over F to projective schemes over L with Galois action is an equivalence of categories.*

Proof. First, we show that all Γ-actions give effective descent data. The Γ-orbit of any closed point is finite, so by the lemma they are contained in an open affine subscheme. All that is left to show is that an algebraic F-scheme X is projective if X_L is. This is proven in [Gr1, Corollaire 6.6.5]. □

Galois cohomology

Let $\mathcal{G}$ be a sheaf of commutative groups for the etale topology over a field F. Then for any finite Galois extension L/F, $\mathcal{G}(L)$ is a $\mathrm{Gal}(L/F)$-module. If $F \subseteq E \subseteq L$ is a tower of finite Galois extensions, then the inclusion of groups $\mathrm{Gal}(L/E) \to \mathrm{Gal}(L/F)$ give an inflation maps $H^i(\mathrm{Gal}(E/F), \mathcal{G}(E)) \to H^i(\mathrm{Gal}(L/F), \mathcal{G}(L))$ by the canonical isomorphisms $\mathrm{Gal}(L/F)/\mathrm{Gal}(L/E) = \mathrm{Gal}(E/F)$ and $\mathcal{G}(E) = \mathcal{G}(L)^{\mathrm{Gal}(L/E)}$. These inflation maps, for any degree i, form a directed system over the Galois subfields of a fixed separable closure of F.

Proposition 1.2.5. *For all $i \geq 0$, there is an isomorphism*

$$H^i(\Gamma_F, \bar{\mathcal{G}}) \to \varinjlim H^i(\mathrm{Gal}(L/F), \mathcal{G}(L)).$$

More precisely, it gives an isomorphism of δ-functors of etale sheaves of abelian groups.

Proof. This is a special case of [Se2, Proposition I.8]. □

As before, one takes the cochain definition for H^1 in the non-abelian case, although only the continuous cochains, in which case the same isomorphism holds; see [Se2, §5.1]. In either case, we will also use the notation $H^i(F, \mathcal{G}) = H^i(\Gamma_F, \bar{\mathcal{G}})$ for an etale sheaf $\mathcal{G}$.

Classifying twisted forms

Definition 1.2.7. Let X_0 be an F-scheme (resp. F-module or F-algebra). For a finite separable extension L/F, we say that an F-scheme (resp. F-module or F-algebra) is an L/F-form of X_0 if $X_L \cong (X_0)_L$ over L (resp. $X \otimes_F L \cong X_0 \otimes_F L$). If X is an L/F form for some finite separable extension L, it is called a twisted form of X_0, and L is a splitting field for X with respect to X_0.

We will write the set of isomorphism classes of L/F-forms of X_0 as $A(L/F)$, and the set of isomorphism classes of all twisted forms as $A(F)$. Clearly, there are inclusions $A(L/F) \subseteq A(E/F)$ whenever one has an inclusion $L \subseteq E$. There are also functorial maps $A(F) \to A(K)$ for any field extension $F \subseteq K$, by $[X] \mapsto [X_K]$. All these sets have a canonical structure of a pointed set, with base point $[X_0]$ or base change thereof, and all the maps defined above are base point preserving.

Proposition 1.2.6. *Let F be a field. The following functors are sheaves for the etale topology:*

i. *M an F-module,* $\mathrm{Aut}(M) : \mathbf{F\acute{E}t}_F \to \mathbf{Grp}$*, $A \mapsto \mathrm{Aut}_A(M \otimes_F A)$, the group of A-linear automorphisms*

ii. *B an F-algebra,* $\mathrm{Aut}(B) : \mathbf{F\acute{E}t}_F \to \mathbf{Grp}$*, $A \mapsto \mathrm{Aut}_A(B \otimes_F A)$, the group of A-algebra automorphisms*

iii. *X an F-scheme,* $\mathrm{Aut}(X) : \mathbf{F\acute{E}t}_F \to \mathbf{Grp}$*, $A \mapsto \mathrm{Aut}_A(X \times_F A)$, the group of automorphisms over A*

Proof. It is clear that these all preserve products. In all cases, the Galois action is given by conjugation of the automorphism: $\sigma \cdot \alpha = \sigma \circ \alpha \circ \sigma^{-1}$. So we see that the automorphisms fixed by this action are exactly the automorphisms which are morphisms of descent data, hence by full faithfulness descend to a unique automorphism over the subfield, so the second sheaf condition is satisfied. □

We now describe how to related 1-cocycles with values in such automorphism groups to twisted forms following [GS, p. 28].

Let $F \subseteq L$ be a finite Galois extension. Let X be an L/F-form of X_0. Let $\Gamma = \mathrm{Gal}(L/F)$ and let $G = \mathrm{Aut}(X_0)(L)$. To an isomorphism $f : (X_0)_L \to X_L$, one associates a 1-cocyle $a \in Z^1(\Gamma, G)$ defined by $a_\sigma = f^{-1} \circ \sigma \circ f \circ \sigma^{-1}$. Indeed,

$$a_{\sigma\tau} = f^{-1} \circ \sigma \circ \tau \circ f \circ \tau^{-1} \circ \sigma^{-1} = (f^{-1} \circ \sigma \circ f \circ \sigma^{-1}) \circ \sigma \circ (f^{-1} \circ \tau \circ f \circ \tau^{-1}) \circ \sigma^{-1} = a_\sigma(\sigma \cdot a_\tau)$$

Essentially, this is an obstruction to f being a morphism of descent data. If one were to choose a different isomorphism, say $g : (X_0)_L \to X_L$, then $g^{-1} \circ f = \alpha$ for some $\alpha \in G$, and we have that

$$f^{-1} \circ \sigma \circ f \circ \sigma^{-1} = \alpha^{-1} \circ (g^{-1} \circ \sigma \circ g \circ \sigma^{-1}) \circ (\sigma \cdot \alpha)$$

i.e., the 1-cocycles defined from f and g are cohomologous. Now, consider two isomorphic L/F-twisted forms X and X'. There is then a Γ-equivariant L-isomorphism $\psi : X_L \to X'_L$. Letting $f : (X_0)_L \to X_L$ be an isomorphism and $g = \psi \circ f$, we have that

$$g^{-1} \circ \sigma \circ g \circ \sigma^{-1} = f^{-1} \circ \psi^{-1} \circ \sigma \circ \psi \circ \sigma^{-1} \circ \sigma \circ f \circ \sigma^{-1} = f^{-1} \circ \sigma \circ f \circ \sigma^{-1}$$

Thus, we have a well-defined map $\varphi : A(L/F) \to H^1(\Gamma, G)$. It is also base point preserving since $\mathrm{id} : (X_0)_L \to (X_0)_L$ gives the trivial cocycle.

Theorem 1.2.3. *The map φ is an isomorphism for the cases where X_0 is a module, algebra or an affine or projective scheme.*

Proof. The cases are precisely those where all descent data are effective (see Proposition 1.1.8, Corollary 1.2.1). They can be treated simultaneously, just as in the definition of φ.

For injectivity, we must show that for two twisted forms X, X', if 1-cocycles attached to isomorphisms $f : (X_0)_L \to X_L$ and $g : (X_0)_L \to X'_L$ are cohomologous, then $X \cong X'$, or what is the same, there exists a Γ-equivariant isomorphism $\psi : X_L \to X'_L$. By hypothesis, we have an $\alpha \in G$ such that

$$f^{-1} \circ \sigma \circ f \circ \sigma^{-1} = \alpha^{-1} \circ g^{-1} \circ \sigma \circ g \circ \alpha \circ \sigma^{-1}$$

which then implies that

$$(g \circ \alpha \circ f^{-1}) \circ \sigma = \sigma \circ (g \circ \alpha \circ f^{-1})$$

so $g \circ \alpha \circ f^{-1}$ gives the desired isomorphism.

For surjectivity, let $a \in Z^1(\Gamma, G)$ and define the twisted Galois action on $(X_0)_L$ by $\tilde{\sigma} = a_\sigma \circ \sigma$. The cocycle condition shows that this is indeed a group action, and it is a Galois action by construction. By Galois descent, this is Γ-equivariantly isomorphic to some X_L, X an L/F-form of X_0. It is clear that the cohomology class associated to X is $[a]$. □

Proposition 1.2.7. *Let X be an L/F form of X_0 and $L \subseteq E$ be a Galois extension. Let $c \in H^1(\mathrm{Gal}(L/F), \mathrm{Aut}(X_0)(L))$ and $c' \in H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(X_0)(E))$ be the classes associated to X. We have that $c' = \inf(c)$.*

Proof. Let $X_L \cong (X_0)_L$ be an isomorphism. It defines a cocycle representing c, which we will also call c. By extending to E, ones gets an isomorphism $X_E \cong (X_0)_E$. The cocycle defined by this isomorphism $c' \in Z^1(\mathrm{Gal}(E/F), \mathrm{Aut}(X_0)(E))$ then takes values $c'_{\sigma'} = c_\sigma \times \mathrm{id}_E$, where σ is the restriction of σ' to L. This is precisely the inflation of c. □

Corollary 1.2.2. *There is a base point preserving isomorphism* $A(F) \to H^1(F, \mathrm{Aut}(X_0))$.

Proof. The first equality of the proposition shows that

$$\begin{array}{ccc} A(L/F) & \hookrightarrow & A(E/F) \\ \downarrow \wr & & \downarrow \wr \\ H^1(\mathrm{Gal}(L/F), \mathrm{Aut}(X_0)(L)) & \xrightarrow{\inf} & H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(X_0)(E)) \end{array}$$

commutes, i.e. that $A(L/F)$ and $H^1(\mathrm{Gal}(L/F), \mathrm{Aut}(X_0)(L))$ are isomorphic directed systems over the finite Galois extensions of F. This induces an isomorphism of the inductive limits $A(F) = \varinjlim A(L/F) \to \varinjlim H^1(\mathrm{Gal}(L/F), \mathrm{Aut}(X_0)(L)) = H^1(F, \mathrm{Aut}(X_0))$, which is basepoint preserving as all the maps involved were. □

Example 1.2.5 (Hilbert 90)**.** We compute the cohomology group $H^1(F, \mathbf{G}_m)$ by interpreting it as classifying twisted forms. In this case, taking X_0 to a one-dimensional F-vector space. Indeed, then the automorphisms over any extension L are the units $L^\times$. Of course, there is only one vector space of each dimension up to isomorphism, so $H^1(F, \mathbf{G}_m) = 0$. A similar argument shows that $H^1(F, \mathrm{GL}_n) = 1$ for all $n \geq 1$.

1.2.4 Severi-Brauer varieties

We mostly follow [GS, Chapter 5] and [Ar].

Definition 1.2.8. An algebraic scheme P over F is a Severi-Brauer variety if there is a finite separable field extension L/F such that $P \times_F L \cong \mathbf{P}^n_L$ for some $n \geq 0$.

Remark 1.2.3. A Severi-Brauer variety P is automatically a smooth, projective variety. Smoothness is clear since the property is fpqc local on the base. To show that it is a variety is then reduced to connectedness, and P must be connected since $P \times_F L$ is.

There is also an "obvious" projective embedding of P. The anti-canonical divisor $\omega_P^\vee := \bigwedge^n \Omega_{P/F}^\vee$ pulls back to the anti-canonical divisor of $\mathbf{P}^n_L$ under the morphisms $\mathbf{P}^n_L \xrightarrow{\sim} P \times_F L \to P$, which is very ample (it has degree $n+1$). The map to $\mathbf{P}^N_L$ (up to linear automorphism) given by the complete linear system $H^0(P_L, \omega_{P_L}^\vee)$ is thus a closed immersion. It is also the base change by $\mathbf{P}^N_L \to Pn^N_F$ of the analogous morphism over F defined by $H^0(P, \omega_P^\vee)$. Since the property of being a closed immersion is fpqc local, it follows that $H^0(P, \omega_P^\vee)$ gives an embedding $P \hookrightarrow \mathbf{P}^N_F$ over F.

Definition 1.2.9. Let V be a vector space over a field F of dimension n. We define the functor $\mathrm{PGL}(V) : \mathbf{Aff}_F \to \mathbf{Grp}$ by $\mathrm{PGL}(V)(A) = \mathrm{Aut}_A(V \otimes_F A)/\sim$, where two linear automorphisms $\varphi, \psi \in \mathrm{Aut}_A(V \otimes_F A)$ are equivalent iff $\varphi \circ \psi^{-1}$ is multiplication by a unit of A. When $V = F^n$, we simply write PGL_n.

For any Galois extension L/F, the automorphism group of $\mathrm{Aut}_L(\mathbf{P}^n_L)$ is isomorphic as a group to $\mathrm{PGL}_{n+1}(L)$. Indeed, as all automorphisms must pullback $\mathcal{O}(1)$ to $\mathcal{O}(1)$, they are all linear changes of coordinates, in the sense that they are induced by a linear automorphism $\varphi : L^{n+1} \to L^{n+1}$. It is clear that two automorphisms induce the same map iff they differ by scaling. Thus we get an isomorphism $\alpha(L) : \mathrm{PGL}_{n+1}(L) \to \mathrm{Aut}_L(\mathbf{P}^n_L)$ which is $\mathrm{Gal}(L/F)$-equivariant and natural in the sense that:

$$\begin{array}{ccc} \mathrm{PGL}_{n+1}(L) & \xrightarrow{\alpha(L)} & \mathrm{Aut}_L(\mathbf{P}^n_L) \\ \uparrow & & \uparrow \\ \mathrm{PGL}_{n+1}(E) & \xrightarrow{\alpha(E)} & \mathrm{Aut}_E(\mathbf{P}^n_E) \end{array}$$

commutes for any inclusion of Galois extensions $E \subseteq L$. Then, it is enough to remark that for any fields L, L', $\mathrm{PGL}_{n+1}(L \times L') = \mathrm{PGL}_{n+1}(L) \times \mathrm{PGL}(L')$ to see that $\mathrm{Aut}(\mathbf{P}^n)$ and PGL_{n+1} give isomorphic sheaves of groups on $\mathbf{F\acute{E}t}_F$.

We introduce the notation that, for a finite separable extension $F \subseteq L$, $\mathfrak{S}_n(L/F)$ is the set of isomorphism classes of Severi-Brauer varieties over F of dimension n which are split by L. The set of all isomorphism classes of Severi-Brauer varieties over F of dimension n is $\mathfrak{S}_n(F)$.

Corollary 1.2.3. *There is a isomorphism $\mathfrak{S}_n(F) \to H^1(F, \mathrm{PGL}_{n+1})$ which is base-point preserving.*

Theorem 1.2.4. *Let P be a Severi-Brauer variety over F. P is isomorphic to $\mathbf{P}^n_F$ for some n if and only if P has an F-rational point.*

This theorem has an amusing proof using the notion of dual Severi-Brauer varieties which we reproduce from [Ar, §3].

Definition 1.2.10. Let P be a Severi-Brauer variety, whose isomorphism class corresponds to $c \in H^1(F, \mathrm{PGL}(V))$ for some finite dimensional vector space V. Define a group homomorphism $\varphi : \mathrm{PGL}(V) \to \mathrm{PGL}(V^\vee)$ by $\alpha \mapsto (\alpha^{-1})^\vee$. It is in fact an isomorphism. The dual of P, $P^\vee$, is defined up to isomorphism as the Severi-Brauer variety corresponding to the class $c' := \varphi_*(c) \in H^1(F, \mathrm{PGL}(V^\vee))$.

Definition 1.2.11. A twisted linear subspace of dimension k of a Severi-Brauer P is a closed subscheme $P' \hookrightarrow P$ such that over a splitting field L, $P'_L \hookrightarrow P_L \cong \mathbf{P}(V)_L$ is a projecitve k-plane in $\mathbf{P}(V)_L$, i.e. corresponding to a $k+1$-dimensional subspace $W \subseteq V$.

Note that this is independent of the chosen isomorphism since isomorphisms of projective spaces are linear.

Lemma 1.2.2. *Let P be an n-dimensional Severi-Brauer variety and $P^\vee$ be its dual. For $k \geq 1$, the $k-1$-dimensional twisted linear subspaces of P are in natural one-to-one correspondence with the $n-k$-dimensional twisted linear subspaces of $P^\vee$.*

Proof. By Galois descent, a twisted linear subspace of P is a linear subspace $P' \subseteq \mathbf{P}(V)_L$, where P corresponds to a class $c \in H^1(\mathrm{Gal}(L/F), \mathrm{PGL}(V))$ and P' is stable under the twisted Galois action coming from any representative of c. By duality and the definition of $P^\vee$, there is a corresponding linear subpace $P'' \subseteq \mathbf{P}(V^\vee)_L$ which is stable under the twisted Galois action(s) associated to $P^\vee$. Repeating this again gives an inverse map by duality. It is clear that this construction sends $k-1$-dimensional subspaces to $n-k$ dimensional ones. $\square$

To prove Theorem 1.2.4, it suffices to show that P corresponds to the class of the trivial cocycle, which is equivalent to the same being true for $P^\vee$ since φ_* is a basepoint preserving bijection. In this case, the F-rational point of P gives a 0-dimensional twisted linear subspace, and so $P^\vee$ has a $n-1$-dimensional twisted linear subspace D. Let L be a splitting field of P and let $\pi : \mathbf{P}^n_L \cong P \times_F L \to P$ be the composition of an isomorphism followed by projection onto the first factor. D is a divisor and by its definition we have that $\pi^*\mathcal{O}(D) \cong \mathcal{O}(1)$. Since cohomology "commutes" with flat base change (see [Ha2, Proposition III.9.3]), $\dim_F H^0(P, \mathcal{O}(D)) = \dim_L H^0(\mathbf{P}^n_L, \mathcal{O}(1))$, so choosing $n+1$ linearly independent sections $s_0, \ldots, s_n \in H^0(P, \mathcal{O}(D))$, we define an F-morphism $\varphi : P \to \mathbf{P}^n_F$. Similarly, define $\varphi' : \mathbf{P}^n_L \to \mathbf{P}^n_L$ by the sections $\pi^*(s_0), \ldots, \pi^*(s_n) \in H^0(\mathbf{P}^n_L, \pi^*\mathcal{O}(D))$. This gives a fibre square

$$\begin{array}{ccc} \mathbf{P}^n_L & \xrightarrow{\varphi'} & \mathbf{P}^n_L \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ P & \xrightarrow[\varphi]{} & \mathbf{P}^n_F \end{array}$$

φ' is clearly an isomorphism, and $\mathbf{P}^n_L \to \mathbf{P}^n_F$ is fpqc, so we conclude that φ is an isomorphism, as desired.

Central simple algebras

In what follows, an F-algebra is an associative algebra over F which is of finite, positive dimension. as an F-vector space. These properties clearly descend under arbitrary field extensions. We will frequently identify F with its image in an algebra.

Definition 1.2.12. An algebra A over F is central iff F is the entire centre of A. An algebra A is simple if it has no non-trivial two-sided ideals. It is a division algebra if all non-zero elements have multiplicative inverses.

We call F-algebras satisying both of these properties *central simple algebras* (CSA).

Example 1.2.6. The algebra of $n \times n$ matrices $M_{n\times n}(F)$ in a central simple algebra. If a central simple algebra is isomorphic to a matrix algebra, we say that it is split.

Theorem 1.2.5 (Artin-Wedderburn)**.** *Let A be a simple algebra over F. Then there exists a division F-algebra D such that $A \cong M_n(D)$, the $n \times n$ matrix algebra over D. Moreover, the integer n and division algebra D are unique. In particular, D is isomorphic to the ring of A endomorphisms of any non-trivial minimal right ideal of A.*

Proof. See [GS, Theorem 2.1.3]. □

Theorem 1.2.6 (Skolem-Noether)**.** *Let A be a central simple algebra over F. All F-algebra automorphisms of A are inner.*

Proof. See [GS, Theorem 2.7.2]. □

Let $F \subseteq K$ be a field extension. Define a group homomorphism $\mathrm{PGL}_n(K) \to \mathrm{Aut}_A(M_{n\times n}(K))$ by $[C]$ maps to conjugation by C. This is easily seen to be injective and is surjective by the Skolem-Noether theorem. Thus, twisted forms of projective space correspond to twisted forms of matrix algebras, which we shall see are exactly the central simple algebras.

Proposition 1.2.8. *Let $F \subseteq K$ be a finite field extension and suppose A is an algebra over F. $A \otimes_F K$ is a central simple algebra over K if and only if A is a central simple algebra over F.*

Proof. See [GS, Lemma 2.2.2]. □

Corollary 1.2.4. *Let A be a central simple algebra over F, then $\dim_F A$ is a square.*

Proof. Since $\dim_F A = \dim_{\bar{F}} A \otimes_F \bar{F}$, it suffices to prove the result with F algebraically closed. By the Artin-Wedderburn theorem, it is enough to show that all division algebras over an algebraically closed field F are just F itself. Suppose to the contrary that there was $x \in D$ which was not contained in F. Then $F[x] \subseteq D$ is a finite integral domain, and hence a finite field extension of F. But then $F = F[x]$, so $x \in F$. □

We call the square root of the dimension of a central simple algebra A its *degree*, and write it $\deg A$.

Proposition 1.2.9. *Let A be a central simple algebra of degree n. There exists a finite separable field extension $F \subseteq L$ such that $A \otimes_F L \cong M_{n\times n}(L)$. If F is finite, we can take $L = F$. Any subfield $L \subseteq A$ of degree n is a splitting field for A, i.e. $A \otimes_F L \cong M_{n\times n}(L)$.*

Proof. See Theorem 2.2.7, Remark 2.28 and Proposition 2.2.9 of [GS]. □

Corollary 1.2.5. *Central simple algebras over F of degree n are twisted forms of $M_{n\times n}(F)$.*

Remark 1.2.4. It follows from Corollary 1.2.5 that Proposition 1.2.8 is true for arbitrary field extensions.

Therefore, we have a natural correspondence between central simple algebras of degree n and Severi-Brauer varieties of dimension $n-1$ through the $H^1(F, \mathrm{PGL}_n)$. For a central simple algebra A, we write $\mathrm{SB}(A)$ for the corresponding Severi-Brauer variety (up to isomorphism).

Varieties of ideals

Let A be an F-algebra (the same convetions as last section apply). For an F-scheme X, we write $\mathcal{A}_X$ for the $\mathcal{O}_X$-algebra $A \otimes_F \mathcal{O}_X$. Following [Ar, 1.2] we make the defintion:

Definition 1.2.13. Let A be a central simple algebra of degree n over F. Define the functor $\mathrm{SB}(A) : \mathbf{Sch}_F^\circ \to \mathbf{Sets}$

$$\mathrm{SB}(A)(S) = \{\mathcal{I} \in \mathbf{Gr}(n, A)(S) : \mathcal{I} \text{ is a right } \mathcal{A}_S\text{-module}\}$$

with pullbacks the same as in $\mathbf{Gr}(n, A)$.

To see that the pullback maps are indeed well-defined, i.e. to see that $\mathrm{SB}(A)$ is really a subfunctor of $\mathbf{Gr}(n, A)$, note that since in any case, for an F-morphism $f : T \to S$ the pullbacks $f^*\mathcal{I}$ are $\mathcal{O}_T$-modules, it suffices to check that for a basis $e_1, \ldots, e_{n^2}$ of A, multiplication by the $e_i \otimes 1$ on $\mathcal{A}_T$ send $f^*\mathcal{I}$ into $f^*\mathcal{I}$. However, by definition multiplication by each $e_i \otimes 1$ on $\mathcal{A}_S$ sends $\mathcal{I}$ into $\mathcal{I}$, and the aforementioned multiplication maps on $\mathcal{A}_T$ are merely the pullback by f of the multiplication maps on $\mathcal{A}_S$ under the identification $f^*\mathcal{A}_S = \mathcal{A}_T$.

As being a right-ideal can be checked locally, $\mathrm{SB}(A)$ is also a sheaf for the Zariski topology.

Remark 1.2.5. The set $\mathrm{SB}(A)(F)$ is simply the set of right ideals of A which are $\deg A$-dimensional as F-vector spaces.

Theorem 1.2.7. *For any central simple algebra A, $\mathrm{SB}(A)$ is representable and agrees (up to isomorphism) with $\mathrm{SB}(A)$, the Severi-Brauer variety associated to A by means of first Galois cohomology.*

We will need a lemma:

Lemma 1.2.3. *Let V be an F-vector space of dimension n and R an arbitrary commutative F-algera. For $k \geq 1$, the map assigning to right ideals I of $\mathrm{End}_R(V \otimes_F R)$ the submodule $W_I = \bigcup_{\varphi \in I} \mathrm{Im}\,\varphi$ gives a bijection between right ideals of $\mathrm{End}_R(V \otimes_F R)$ which as R-modules are direct summands of rank nk and direct summands of $V \otimes_F R$ of rank k.*

Proof. First, let us prove that if I is a direct summand, then W_I is as well. By [Sw, Proposition 3.1], I is also a direct summand as a $\mathrm{End}_R(V \otimes R)$-module. Therefore, there is an idempotent $\varphi \in A := \mathrm{End}_R(V \otimes R)$ with $\varphi A = I$. Then, it follows that $W_I = \mathrm{Im}\,\varphi$, which is a direct summand of $V \otimes R$. Moreover, this shows that I is of the form $\mathrm{Hom}_R(V \otimes R, W_I)$. This gives the inverse map, and from this it is clear that the ranks correspond as described. □

Corollary 1.2.6. *For $n \geq 1$, $\mathrm{SB}(M_{n\times n}(F)) \cong \mathbf{P}^{n-1}$ by a PGL_n-equivariant natural isomorphism.*

Remark 1.2.6. Implicit in Corollary 1.2.6 is that the correspondence of Lemma 1.2.3 globalises to F-schemes. By checking locally, one sees that this globalized correspondence is given by a similar construction, namely the subsheaf generated by all local sections which are images of all local sections of the ideal sheaf.

Remark 1.2.7. Supposing $\mathrm{SB}(A)$ is representable, we have that for a field extension $F \subseteq L$, $\mathrm{SB}(A \otimes_F L) \cong \mathrm{SB}(A) \times_F L$, so $\mathrm{SB}(A)$ will be a Severi-Brauer variety by the corollary. Moreover, since the PGL_n actions correspond, the class in $H^1(F, \mathrm{PGL}_n)$ associated to $\mathrm{SB}(A)$ is the same as that associated to A.

Proof of Theorem 1.2.7. In view of the remark, we need only prove representability. The corollary already shows representability in the case where F is finite by Proposition 1.2.9, so we may assume F infinite. Write $\rho : A \to \mathrm{End}_F(A)$ for the map $\rho(a)(b) = ba$. Then for any $a \in A$ such that $\det \rho(a) \neq 0$, $\rho(a)$ induces an automorphism $\rho(a)_* : \mathbf{Gr}(\deg A, A) \to \mathbf{Gr}(\deg A, A)$. Since $\mathbf{Gr}(\deg A, A)$ is separated over F, the fixed-point subfunctor is represented by a closed subscheme. Therefore, if $S = \{a \in A : \det \rho(a) \neq 0\}$ spans A, we will have realized $\mathrm{SB}(A)$ as a closed subscheme of $\mathbf{Gr}(\deg A, A)$, since the scheme-theoretic intersection of closed subschemes corresponds exactly to the intersection of their functor of points. Let W be the span of S in A. Suppose, for the sake of contradiction, that W is a proper subspace. By choosing a basis $e_1, \ldots, e_k$ of A, we identify A with $\mathbf{A}^k(F)$ such that W corresponds to the F-points of a proper linear subspace $L \subsetneq \mathbf{A}^k$ and $\det \rho(a)$ comes from a regular function on $\mathbf{A}^k$. But then, since at least $\det \rho(1) \neq 0$, S is identified with the rational points of an open subset $U \subseteq \mathbf{A}^k$. Since F is infinite, the topology induced on the F-rational points is irreducible, therefore S cannot be contained in W. □

Example 1.2.7. Let A be a central simple algebra. Let K be the function field of $\mathrm{SB}(A)$. Then $\mathrm{SB}(A) \times_F K = \mathrm{SB}(A \otimes_F K)$ has a rational point, so is isomorphic to some $\mathbf{P}^n$ by Theorem 1.2.4. By Theorem 1.2.7, for any central simple algebra B, we have that B and $\mathrm{SB}(B)$ give the same class in $H^1(F, \mathrm{PGL}_m)$, so we conclude that $A \otimes_F K \cong M_{m\times m}(K)$.

Definition 1.2.14. Let $i : \mathrm{SB}(A) \hookrightarrow \mathbf{Gr}(\deg A, A)$ be the canonical inclusion. We define the tautological sheaf of ideals $\mathcal{I}_{\mathrm{SB}(A)} \subseteq \mathcal{A}_{\mathrm{SB}(A)}$ on $\mathrm{SB}(A)$ by $i^*\mathcal{S}$, where $\mathcal{S}$ is the tautological bundle of $\mathbf{Gr}(\deg A, A)$.

Since i is a monomorphism, one deduces a universal property for the pair $(\mathrm{SB}(A), \mathcal{I}_{\mathrm{SB}(A)})$ from that of $(\mathbf{Gr}(\deg A, A), \mathcal{S})$, namely that the map $\mathrm{Hom}_F(S, \mathrm{SB}(A)) \to \mathrm{SB}(A)(X)$ defined by $f \mapsto [f^*(\mathcal{I}_{\mathrm{SB}(A)}) \subseteq \mathcal{A}_X]$ is a bijection.

It is clear that all the above results will hold with the rank of the ideal sheaves changed to any multiple $k \deg A$ for $1 \leq k < \deg A$.

Example 1.2.8. For $k = \deg A - 1$, we call the construction $\mathrm{SB}^\vee(A)$. The variety obtained is isomorphic to the dual Severi-Brauer variety $\mathrm{SB}(A)^\vee$.

1.3 (Equivariant) intersection theory

In this section, we introduce the basics of intersection theory, in particular the Chow ring. After a review of principal bundles, the equivariant Chow ring for torus actions is defined. Finally, localisation theorems for torus equivariant Chow groups are stated.

Notation and conventions: A scheme over F is assumed to be an *algebraic* scheme over F, that is finite type and separated. A variety X is an algebraic scheme which is integral i.e. irreducible and reduced. We also adopt the convention that smooth and flat morphisms are of constant relative dimension.

1.3.1 Chow groups

We give a short exposition of Chow groups, following [Fu].

Definition 1.3.1. Let X be a scheme over F. For any integer $k \geq 0$, an algebraic k-cycle on X is a formal sum of k-dimensional subvarieties $V_i \subseteq X$, $\sum n_i V_i$, $n_i \in \mathbf{Z}$. The abelian group of such formal sums is denoted $Z_k(X)$.

Clearly, $Z_k(X) = Z_k(X_{red})$ and if $\dim X = n$, $Z_n(X)$ is the free abelian group generated by the irreducible components of X of dimension n. Any subvariety $V \subseteq X$ induces an inclusion $Z_k(V) \subseteq Z_k(X)$ as k-dimensional subvarieties V will also be k-dimensional subvarieties of X.

Now let $W \subseteq V$ be subvarieties of X There is an associated generic point $\eta_W \in V$ whose local ring we write $\mathcal{O}_{V,W}$. V has a generic point $\eta = \eta_V$ whose local ring in V (which is in fact a field) we write as $K(X)$. This is the *function field* of V. Then there is an inclusion $\mathcal{O}_{V,W} \to K(V)$ as η is contained in any non-empty open subset of V. If W has codimension 1 in V, then for any $0 \neq f \in \mathcal{O}_{V,W}$, $\mathcal{O}_{V,W}/(f)$ is Artinian and we define the order of f at W by $\mathrm{ord}_{V,W}(f) = \mathrm{length}_{\mathcal{O}_{V,W}} \mathcal{O}_{V,W}/(f) \in \mathbf{N}$. This function is multiplicative, and hence extends to a well-defined group homomorphism $\mathrm{ord}_{X,V} : K(X)^\times \to \mathbf{Z}$ by setting $\mathrm{ord}_{V,W}(f/g) = \mathrm{ord}_{V,W}(f) - \mathrm{ord}_{V,W}(g)$, $f, g \in \mathcal{O}_{V,W} - \{0\}$. Clearly, $\mathrm{ord}_{V,W}(f)$ is only non-zero for the W along which f vanishes or has a pole; that is to say for only finitely many. Hence we can define an element $(f) \in Z_k(V)$, $(f) = \sum \mathrm{ord}_{V,W}(f) \cdot W$, where $\dim V = k+1$. As explained above, (f) may also be regarded as an element of $Z_k(X)$.

Definition 1.3.2. Let X be a scheme. For an integer $k \geq 0$, we define $R_k(X)$ to be the subgroup of $Z_k(X)$ generated by cycles $(f) \in Z_k(X)$, where $f \in K(V)^\times$, $V \subseteq X$ a $k+1$-dimensional subvariety. We define the k-th Chow group of X to be the quotient $\mathrm{CH}_k(X) := Z_k(X)/R_k(X)$. Two k-cycles with the same image in $\mathrm{CH}_k(X)$ are said to be rationally equivalent.

We form the graded abelian group $Z_\bullet(X) = \bigoplus_{k\geq 0} Z_k(X)$, and similarly for $R_\bullet(X)$ and $\mathrm{CH}_\bullet(X)$. We make the obvious identification $\mathrm{CH}_\bullet(X) = Z_\bullet(X)/R_\bullet(X)$.

Example 1.3.1. If X is an n-dimensional variety, $Z_{n-1}(X)$ is the classical group of Weil divisors on X, and $R_{n-1}(X)$ is the subgroup of divisors linearly equivalent to zero. In particular, there is a homomorphism $\operatorname{Pic} X \to \mathrm{CH}_{n-1}(X)$ sending $[\mathcal{L}]$ to the class of its "zero section in X": given that $\mathcal{L}$ is invertible, it is a flat $\mathcal{O}_X$-module, so there is an inclusion $\mathcal{L} \subseteq \mathcal{L} \otimes_{\mathcal{O}_X} K(X)$, where $K(X)$ is understood as a constant sheaf on X. Due to the local triviality of $\mathcal{L}$, $\mathcal{L} \otimes_{\mathcal{O}_X} K(X)$ is locally isomorphic to $K(X)$, but the obstruction to a global isomorphism is $H^1(X, K(X)^\times) = 0$, so up to a choice of element in $K(X)^\times$, we get an inclusion $\mathcal{L} \subseteq K(X)$. For any codimension 1 subvariety $V \subseteq X$, we define the order of $\mathcal{L}$ (with respect to the previously defined inclusion) at V to be $\operatorname{ord}_{X,V}(\mathcal{L}) := \operatorname{ord}_{X,V}(s)$, where s is a free generator of $\mathcal{L}$ in a neighbourhood of η_V. This is clearly independent of the choice of s, and we define the cycle $\sum \operatorname{ord}_{X,V}(\mathcal{L}) \cdot V \in Z_{n-1}$ which we call a zero section of $\mathcal{L}$.

The construction makes it clear that the zero section is well-defined up to linear equivalence. When X is smooth (in fact, local factoriality suffices, [Ha2, Corollary II.6.16]) this map is an isomorphism. We write $c_1(\mathcal{L})$ for the image of $[\mathcal{L}]$, and conversely for $D \in Z_{n-1}$, $\mathcal{O}(D)$ for the unique (up to isomorphism) invertible sheaf with $c_1(\mathcal{O}(D))$ rationally equivalent to D.

Definition 1.3.3. Let $Y \subseteq X$ be a closed subscheme with irreducible components $Y_1, \ldots, Y_m$. Let n_{Y_i}, $1 \leq i \leq m$, be the length of the Artinian ring $\mathcal{O}_{X,Y_i}$. We define the class of Y in X, to be the cycle $[Y] = \sum_{1\leq i\leq k} n_{Y_i} \cdot Y_i \in Z_\bullet(X)$.

By abuse of notation, we will also write $[Y]$ for the class of the cycle in $\mathrm{CH}_\bullet(X)$. For V a subvariety, $[V] = 1 \cdot V$. We will prefer this notation.

Example 1.3.2. If X is smooth of dimension n, then $D \mapsto [D]$ gives an isomorphism between the group of effective Cartier divisors (subschemes with invertible ideal sheaf) and $Z_{n-1}(X)$ ([Ha2, Proposition II.6.11]). We will simply write $\mathcal{O}(D)$ for $\mathcal{O}([D])$.

Functorial properties

Let $f : X \to Y$ be a proper morphism of schemes over F. for any subvariety $V \subseteq X$, we have that $W := f(V) \subseteq Y$ is a subvariety since f is closed. The induced dominant map of V onto W gives an inclusion $K(W) \subseteq K(V)$. If $\dim W = \dim V = k$, then this is a finite field extension since $K(W)$ and $K(V)$ are finitely generated and have the same transcendence degree over F.

Definition 1.3.4. Let $f : X \to Y$, V and W be as above. We define a graded homomorphism $f_* : Z_\bullet(X) \to Z_\bullet(Y)$ by $f_*([V]) = 0$ if $\dim W < \dim V$, and $f_*([V]) = [K(V) : K(W)] \cdot [W]$ if $\dim W = \dim V$.

This assignment is functorial in the sense that for two proper morphisms $f : X \to Y$ and $g : Y \to Z$ we have $g_* \circ f_* = (g \circ f)_*$. Indeed, with V a subvariety of X, $W = f(V)$, $U = g(W) = (g \circ f)(V)$, if any are of different dimension, then $g_*(f_*([V])) = (g \circ f)_*([V]) = 0$. Otherwise, we have $g_*(f_*([V])) = [K(W) : K(U)] \cdot [K(V) : K(W)] \cdot [U] = [K(V) : K(U)] \cdot [U] = (g \circ f)_*[U]$.

Proposition 1.3.1. *If $\alpha \in Z_k(X)$ is rationally equivalent to zero, then $f_*(\alpha) \in Z_k(Y)$ is rationally equivalent to zero. In particular, f_* induces a well-defined graded homomorphism $f_* : \mathrm{CH}_\bullet(X) \to \mathrm{CH}_\bullet(Y)$.*

Proof. See [Fu, Theorem 1.4]. □

Example 1.3.3. Let $i : Y \to X$ be a closed immersion. Since closed immersions are proper, we get a map $i_* : \mathrm{CH}_\bullet(Y) \to \mathrm{CH}_\bullet(X)$. Thinking of Y as a closed subscheme, this says that the inclusion on cycles defined above induces a map on Chow groups.

Example 1.3.4. If X is a proper scheme over F, then we call the pushforward of the structure morphism on CH_0 the degree map $\deg : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} F) = \mathbf{Z}$. This map is easily described: for a closed point $P \in X$, $\deg([P]) = [\kappa(P) : F]$, where $\kappa(P)$ is the residue field of the local ring $\mathcal{O}_{X,P}$. If F is algebraically closed, then in particular we have $\deg(\sum n_i[P_i]) = \sum n_i$ and this "point count" is invariant under rational equivalence.

Definition 1.3.5. Let $f : X \to Y$ be a flat morphism of relative dimension n. For $k \geq 0$, we define homomorphisms $f^* : Z_k(Y) \to Z_{n+k}(X)$ by $[V] \mapsto [f^{-1}(V)]$, where V is a k-dimensional subvariety of Y and $f^{-1}(V)$ denotes the scheme-theoretic inverse image.

Note that the dimensions work out due to the flatness hypothesis (Proposition 1.1.3).

Proposition 1.3.2. *Let $f : X \to Y$ be flat of relative dimension n. For any closed subschemes $Z \subseteq Y$, $f^*[Z] = [f^{-1}(Z)]$. If $\alpha \in Z_k(X)$ is rationally equivalent to zero, then $f^*(\alpha)$ is as well.*

Proof. See Lemma 1.7 and Theorem 1.7 of [Fu]. □

These two facts imply that f^* induces a map $\mathrm{CH}_k(Y) \to \mathrm{CH}_{n+k}(X)$ and that if $g : Y \to Z$ is flat of relative dimension m, $f^* \circ g^* = (g \circ f)^*$.

Example 1.3.5. Any scheme over F of pure dimension n is flat of relative dimension n over $\operatorname{Spec} F$. Hence given a scheme Y, we have that the projection $\pi : X \times_F Y \to Y$ is flat of relative dimension n, and $\pi^*[Z] = [X \times Z]$ for any closed subscheme Z of Y (here we apply π^* on each graded piece of $\mathrm{CH}_\bullet(Y)$).

Example 1.3.6. Let $\pi : E \to X$ be a vector bundle of rank n. This is flat of relative dimension n, and all pullback maps $\pi^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{n+k}(E)$ are isomorphisms and $\mathrm{CH}_k(E) = 0$ for $0 \leq k < n$ ([Fu, Theorem 3.3]). In particular, $\mathrm{CH}_\bullet(\mathbf{A}^n)$ is generated by $[\mathbf{A}^n]$.

Example 1.3.7. Let $j : U \to X$ be an open immersion. j is flat of relative dimension 0 so we get a graded homomorphism $i^* : \mathrm{CH}_\bullet(X) \to \mathrm{CH}_\bullet(U)$ sending $[V]$ to $[j^{-1}(V)]$. In particular, cycles whose subvarieties are contained in $X - j(U)$ are sent to 0.

Proposition 1.3.3. *Let $i : Z \to X$ be a closed immersion, with $j : U \to X$ the open immersion given by the inclusion of $U = X \setminus i(Z)$ into X. The sequence*

$$\mathrm{CH}_\bullet(Z) \xrightarrow{i_*} \mathrm{CH}_\bullet(X) \xrightarrow{j^*} \mathrm{CH}_\bullet(U) \to 0$$

is exact.

Proof. See [Fu, Proposition 1.8]. □

Affine stratifications

We explain a nice class of schemes for which it is easy to determine generators for the Chow groups, following the exposition of [EH2, §1.3.5].

Definition 1.3.6. Let X be a scheme. A stratification of X is a finite collection of locally closed subvarieties of X, $\{U_i\}$, such that $\bigcup U_i = X$ is a disjoint union and the closure of any U_i is a union of some U_j. We call the elements of the collections cells. We call a stratification affine if each U_j is isomorphic to an affine space.

If any two U_i have the same closure then they are equal. Indeed, letting Y be the shared closure of cells U_1 and U_2, we have by local closure that U_1 and U_2 are open subsets of Y. Since Y is irreducible, we must have that $U_1 \cap U_2 \neq \emptyset$, hence $U_1 = U_2$. Thus we get a partial order on cells by setting $U_j \leq U_i$ if $U_j \subseteq \overline{U}_i$.

Proposition 1.3.4. *Let X be a scheme. If X admits an affine stratification with cells U_i $(i = 1, \ldots, n)$, then $\mathrm{CH}_\bullet(X)$ is generated by the classes $[\overline{U}_i]$.*

Proof. We proceed by induction on the number of cells. If there is only one, then this follows from the computation of $\mathrm{CH}_\bullet(\mathbf{A}^m)$. Suppose $n > 1$. Since the cells form a finite poset, there is a minimal element U_j, which by minimality must be closed in X. $X \setminus U_j$ has an affine stratification with $n-1$ cells, so by induction $\mathrm{CH}_\bullet(X \setminus U_j)$ is generated by the classes of their closures. Then, the exact sequence $\mathrm{CH}_\bullet(U_j) \to \mathrm{CH}_\bullet(X) \to \mathrm{CH}_\bullet(X \setminus U_j) \to 0$ implies that $\mathrm{CH}_\bullet(X)$ is generated by $[U_j]$ and the $[\overline{U}_i]$, $i \neq j$. □

Example 1.3.8. Using a complete coordinate flag, we have an ascending sequence of closed subvarieties $\mathbf{P}^0 \subseteq \mathbf{P}^1 \subseteq \cdots \subseteq \mathbf{P}^n$, with $U_i = \mathbf{P}^i \setminus \mathbf{P}^{i-1} \cong \mathbf{A}^i$ for $i \geq 1$ and $U_0 := \mathbf{P}^0 = \mathbf{A}^{\mathbf{0}}$. These are all disjoint and $\overline{U}_i = \mathbf{P}^i = \bigcup_{0 \leq j \leq i} U_j$ for $0 \leq i \leq n$, so we have an affine stratification. By Proposition 1.3.4, $\mathrm{CH}_\bullet(\mathbf{P}^n)$ is generated by the classes $[\mathbf{P}^i]$. In fact, these form a $\mathbf{Z}$-basis of $\mathrm{CH}_\bullet(\mathbf{P}^n)$.

1.3.2 Intersection products

Normal Cones and Gysin maps

Let X be of pure dimension n. We adopt the notation that $\mathrm{CH}^k(X) := \mathrm{CH}_{n-k}(X)$ and set $\mathrm{CH}^\bullet(X) = \bigoplus_{k \geq 0} \mathrm{CH}^k(X)$. Notice that with this grading, flat pullbacks are graded homomorphisms. We want to generalise the class of morphisms which have well-behaved pullbacks.

Definition 1.3.7. Let $\pi : E \to X$ be a vector bundle of constant rank, X pure dimensional. For any section $s : X \to E$ we define the Gysin map $s^* : \mathrm{CH}^\bullet(E) \to \mathrm{CH}^\bullet(X)$ to be $(\pi^*)^{-1}$ (see Example 1.3.6).

Definition 1.3.8. Let $i : Y \to X$ be a closed immersion with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. The normal cone of Y in X is defined to be $C_Y X := \underline{\mathrm{Spec}} \bigoplus_{k \geq 0} \mathcal{I}^k / \mathcal{I}^{k+1}$.

Note that the quasicoherent X-algebra $\bigoplus_{k \geq 0} \mathcal{I}^k / \mathcal{I}^{k+1}$ descends to a quasicoherent Y-algebra. In this way, $C_Y X$ is given the structure of a Y-scheme. If X has pure dimension n, then so does $C_Y X$ for any closed subscheme Y of X ([Fu, B.6.6]).

Definition 1.3.9. Let $i : Y \to X$ be a closed immersion. We define the specialisation to the normal cone $\sigma : Z_k(X) \to Z_k(C_Y X)$ by $[V] \mapsto [C_{Y \times_X V} V]$, V a k-dimensional subvariety of X.

Proposition 1.3.5. *Let $i : Y \to X$ be as above. If $\alpha \in Z_k(X)$ is rationally equivalent to zero, then so is $\sigma(\alpha)$.*

Proof. See [Fu, Proposition 5.2]. □

If X is equidimensional, we get a well-defined graded homomorphism $\sigma : \mathrm{CH}^\bullet(X) \to \mathrm{CH}^\bullet(C_Y X)$, since $C_Y X$ is also equidimensional with $\dim C_Y X = \dim X$.

Definition 1.3.10. A closed immersion $i : Y \to X$ is said to be a regular embedding of codimension d if the ideal sheaf is locally generated by a regular sequence of length d.

Example 1.3.9. The inclusion of an effective Cartier divisor is a regular embedding of codimension 1.

Proposition 1.3.6. *Let S be a scheme and X, Y smooth S-schemes of relative dimensions n and m respectively. A locally finite presentation closed immersion $i : Y \to X$ over S is a regular embedding of codimension $n - m$.*

Proof. See [Gr2, Theorem 17.12.1]. □

Example 1.3.10. Let X, Y be varieties over F, with Y smooth of dimension n. For any morphism of varieties $f : X \to Y$, the graph map $\Gamma_f : X \to X \times_F Y$ is a closed immersion. It is also a morphism over X, and since $X \times_F Y$ is smooth over X of relative dimension n, Γ_f is a regular embedding of codimension n by Proposition 1.3.6.

Proposition 1.3.7. *If $i : Y \to X$ is a regular embedding with ideal sheaf $\mathcal{I}$, then $C_Y X$ is Y-isomorphic to the normal bundle of Y in X, i.e. the vector bundle $N_Y X$ with sheaf of sections $(\mathcal{I}/\mathcal{I}^2)^\vee$.*

Proof. See [Fu, B.6.2]. □

We can now formulate the following pullback homomorphism:

Definition 1.3.11. The Gysin homomorphism for a regular embedding $i : Y \to X$ is a graded homomorphism $i^* : \mathrm{CH}^\bullet(X) \to \mathrm{CH}^\bullet(Y)$ defined to be the composite of σ with the Gysin map of a section s of the normal cone/bundle of Y in X.

The Chow ring of smooth varieties

Let X be smooth over F. It follows that the diagonal embedding $\delta : X \to X \times_F X$ is a regular embedding since both schemes are smooth over F. Suppose Y is smooth and we have a morphism $f : X \to Y$. The graph morphism $\Gamma_f = \mathrm{id} \times f : X \to X \times_F Y$ is a closed immersion (since Y is separated over F, $\delta : Y \to Y \times_F Y$ is a closed immersion; Γ_f is the base change of δ by $f \times \mathrm{id} : X \times_F Y \to Y \times_F Y$), and hence a regular embedding since $X \times_F Y$ is smooth.

For X, Y schemes and $n, m \geq 0$, there is a bilinear map $\times : Z_n(X) \times Z_m(Y) \to \mathrm{CH}_{n+m}(X \times_F Y)$ given by $([V], [W]) \mapsto [V \times_F W]$ for subvarieties $V \subseteq X$, $W \subseteq Y$ of dimensions n and m respectively. For a fixed subvariety $V \subseteq X$, the induced homomorphism $Z_m(Y) \to \mathrm{CH}_{n+m}(X \times_F Y)$, $\alpha \mapsto [V] \times \alpha$ corresponds to flat pullback by $V \times_F Y \to Y$ and hence sends rationally equivalent cycles to the same class in $\mathrm{CH}_{n+m}(X \times_F Y)$. By symmetry, the same is true for the homomorphism induced by selecting a subvariety $W \subseteq Y$. Thus, we get a bilinear map $\times : \mathrm{CH}_n(X) \times \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_F Y)$. If X, Y are equidimensional, we can also write this map as $\times : \mathrm{CH}^i(X) \times \mathrm{CH}^j(Y) \to \mathrm{CH}^{i+j}(X \times_F Y)$.

Definition 1.3.12. Let $f : X \to Y$ be a morphism of smooth varieties. For $i, j \geq 0$, we define multiplication maps $\mathrm{CH}^i(X) \times \mathrm{CH}^j(Y) \to \mathrm{CH}^{i+j}(X)$ by composing $\times$ with $(\Gamma_f)^*$.

In the special case of $Y = X$ and $f = \mathrm{id}_X$, the multiplication map defined above is just $\times$ followed by δ^*. We refer to this as multiplication on $\mathrm{CH}^\bullet(X)$. It does in fact give $\mathrm{CH}^\bullet(X)$ the structure of a graded commutative ring:

Theorem 1.3.1. *Let X, Y, Z be non-singular varieties over F and $f : X \to Y$, $g : Y \to Z$ morphisms:*

i. *The map $m : \mathrm{CH}^\bullet(X) \times \mathrm{CH}^\bullet(X) \to \mathrm{CH}^\bullet(X)$ defined from the multiplication maps on graded pieces makes $\mathrm{CH}^\bullet(X)$ into a commutative graded ring with unit $1_X = [X]$. We will write the product of two classes as $x \cdot y$ or simply xy.*

ii. *The multiplication map induced by f makes $\mathrm{CH}^\bullet(X)$ into a graded $\mathrm{CH}^\bullet(Y)$-module. This defines a graded ring homomorphism $f^* : \mathrm{CH}^\bullet(Y) \to \mathrm{CH}^\bullet(X)$. If f is flat, f^* agrees with the flat pullback homomorphism. If f is a regular embedding, it agrees with the Gysin homomorphism.*

iii. *The pullback homomorphism is functorial, i.e. $f^* \circ g^* = (g \circ f)^*$.*

iv. *If f is proper, the so-called projection formula holds: for any $x \in \mathrm{CH}^\bullet(X)$ and $y \in \mathrm{CH}^\bullet(Y)$, $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$, i.e. f_* is a $\mathrm{CH}^\bullet(Y)$-homomorphism.*

Proof. See Proposition 8.3 and Example 8.3.1 of [Fu]. □

We briefly explain a situation in which the intersection product has an easy interpretation.

Definition 1.3.13. Let X be a scheme of pure dimension n. Let $\alpha, \beta \in Z_\bullet(X)$. The cycles α and β are said to intersect properly if for any two subvarieties $V, W \subseteq X$ with the coefficients of $[V]$ in α and $[W]$ in β non-zero, $V \cap W$ is of pure dimension $\dim V + \dim W - n$ if $\dim V + \dim W - n \geq 0$, and empty otherwise. Two cycles intersecting properly intersect transversely if moreover all irreducible components of $V \cap W$ have multiplicity one.

Proposition 1.3.8. *Let X be a smooth variety with $V, W \subseteq X$ closed subvarieties. If $[V]$ and $[W]$ intersect transversely, then $[V] \cdot [W] = \sum [Z_i]$ in $\mathrm{CH}^\bullet(X)$, where Z_i are the subvarieties appearing as the irreducible components of $Z = V \cap W$.*

Proof. See [Fu, Proposition 8.2]. □

Example 1.3.11 (Chow ring of $\mathbf{P}^n$)**.** Fix homogeneous coordinates $x_0, \ldots, x_n \in H^0(\mathbf{P}^n, \mathcal{O}(1))$. We have already seen that $\mathrm{CH}^k(\mathbf{P}^n)$ is generated by the class of the codimension k linear subspace $\mathbf{P}^{n-k} \subseteq \mathbf{P}^n$ defined by $x_0 = \cdots = x_{k-1} = 0$. For

$k = 0$ or $k = n$ the class is a free generator: this is obvious for $k = 0$ and follows from the degree map for $k = n$, since $\deg([P]) = 1$, where P is the single rational point with coordinates $[0 : \cdots : 1]$. For $k = 1$, we can describe the generator as $x = c_1(\mathcal{O}(1))$, hence any coordinate hyperplane $x_i = 0$ defines the same generator. We now argue by descending induction on $k = n, \ldots, 0$ that $[\mathbf{P}^{n-k}]$ is a basis for $\mathrm{CH}^k(\mathbf{P}^n)$. We have already done this for $k = n$. For the inductive case, we express $x = [H]$, with H defined by $x_k = 0$. H intersects $\mathbf{P}^{n-k}$ transversely, with intersection class $[\mathbf{P}^{n-k-1}]$. But this gives a basis for $\mathrm{CH}^{k+1}(\mathbf{P}^n) \cong \mathbf{Z}$ by induction, so the map $\mathrm{CH}^k(\mathbf{P}^n) \to \mathrm{CH}^{k+1}(\mathbf{P}^n)$ given by multiplication by x is an isomorphism, hence $[\mathbf{P}^{n-k}]$ must be a basis for $\mathrm{CH}^k(\mathbf{P}^n)$. In fact, it follows from this argument that $x^k = [\mathbf{P}^{n-k}]$ and $\mathrm{CH}^\bullet(\mathbf{P}^n) \cong \mathbf{Z}[\zeta]/(\zeta^{n+1})$ by a map sending x to ζ.

The above computation is in fact a special case of a much more general result:

Theorem 1.3.2. *Let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on a smooth scheme X. Let $\pi : \mathbf{P}(\mathcal{E}) \to X$ be the associated projective bundle, with twisting sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.*

i. *$\mathrm{CH}^\bullet(\mathbf{P}(\mathcal{E}))$ is a free $\mathrm{CH}^\bullet(X)$-module (via π^*) of rank $r+1$ with basis $1, x, \ldots, x^r$, $x = c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$.*

ii. *For $\alpha = \sum_{0 \le k \le r} \alpha_k \cdot x^k \in \mathrm{CH}^\bullet(\mathbf{P}(\mathcal{E}))$, $\pi_*(\alpha) = \alpha_r$.*

Proof. See Theorem 9.6 and Lemma 9.7 of [EH2]. □

1.3.3 Torus equivariant Chow groups

Throughout this section, F will denote an *algebraically closed* ground field and all schemes are separated, finite type F-schemes.

Definition 1.3.14. An affine algebraic group over F is an F-group scheme which is a finite type affine F-scheme. A homomorphism of affine algebraic groups is an F-morphism $\varphi : G \to H$ such that for each F-algebra A, $\varphi(A) : G(A) \to H(A)$ is a group homomorphism.

Example 1.3.12. As noted in Example 1.1.8, GL_n is an affine algebraic group. In particular, $\mathbf{G}_m := \mathrm{GL}_1$ is an affine algebraic group which is also commutative.

Definition 1.3.15. A torus is an affine algebraic group T which is isomorphic to $\prod_{1 \le i \le n} \mathbf{G}_m$ for some $n \ge 1$.

Remark 1.3.1. Over an arbitrary field F, this definition corresponds to the notion of a *split* torus.

Definition 1.3.16. Let G be an affine algebraic group. The character group of G, $X(G) := \mathrm{Hom}_F(G, \mathbf{G}_m)$, is the group of homomorphisms from G to $\mathbf{G}_m$ with multiplication defined pointwise, i.e., for $\chi, \chi' \in X(G)$, $\chi \cdot \chi'$ is given by the composition

$$G \times_F G \xrightarrow{\chi \times \chi'} \mathbf{G}_m \times \mathbf{G}_m \xrightarrow{m} \mathbf{G}_m$$

Since $\mathbf{G}_m$ is commutative, it is clear that character groups are abelian.

Proposition 1.3.9. *For $n \geq 1$, let $T = (\mathbf{G}_m)^n$. The homomorphism $\mathbf{Z}^n \to X(T)$ defined by sending the i-th standard basis element e_i to the projection $p_i : T \to \mathbf{G}_m$ onto the i-th factor is an isomorphism.*

Proof. Injectivity is clear by looking at F-points. For surjectivity, see [Mi2, Lemma 12.4], noting that $\mathcal{O}_T(T) \cong k[\mathbf{Z}^n]$ as Hopf algebras. □

Definition 1.3.17. For an affine algebraic group G, a G-scheme is an algebraic scheme X and an action map $\varphi : G \times_F X \to X$ such that for all F-algebras A, $\varphi(A) : G(A) \times X(A) \to X(A)$ is a group action of $G(A)$.

Remark 1.3.2. This condition can be expressed without the functor of points through various identities combining the multiplication and inverse morphisms on G and φ. Indeed, each axiom of the group action gives some identity on natural transformations of the functor of points. As a consequence, we see that is X and G are reduced, it suffices to check that $\varphi(F) : G(F) \times X(F) \to X(F)$ gives a group action.

A morphism of G-schemes $f : X \to Y$ is said to be equivariant if the maps $f(A) : X(A) \to Y(A)$ are $G(A)$-equivariant for all F-algebras A. Again as in the above remark, if all G, X and Y are reduced, we need only verify this is the case $A = F$.

Digression on principal bundles

Following [To, p. 12], we make the following definition:

Definition 1.3.18. Let G be an affine algebraic group. Let $\pi : X \to Y$ be a flat, surjective morphism of schemes such that X is a G-scheme and the action morphism $\varphi : G \times_F X \to X$ is a Y-morphism. We say that $\pi : X \to Y$ is a principal G-bundle if $\mathrm{id}_X \times \varphi : G \times_F X \to X \times_Y X$ is an isomorphism.

Example 1.3.13. For any base Y, the trivial G-bundle $\pi : G \times_F Y \to Y$ with the G-action given by left multiplication on the first factor of $G \times_F Y$ is a principal bundle. More generally, a *Zariski locally trivial* G-bundle is a G-scheme X and projection $\pi : X \to Y$ commuting with the group action such that there exists an open cover U_i

of Y for which the restricted projections $\pi^{-1}(U_i) \to U_i$ are trivial G-bundles. All the properties of a principal bundle can be proved locally on the base, so these are also principal G-bundles.

Proposition 1.3.10. *Let G be GL_n or a torus, and Y a scheme. Every principal G-bundle over Y is Zariski locally trivial.*

We briefly sketch a proof as it is instructive on the nature of principal bundles; for proper references see [EG, p. 37].

Sketch of proof. By the very definition of a prinicipal bundle, it is "fppf locally trivial", fppf being the condition that a morphism is flat, surjective and of finite presentation. Namely, $X \to Y$ itself gives such a trivialization (recall our assumptions on X and Y). In analogy with the classification of twisted forms obtained in Section 1.2.3, fppf locally trivial G-bundles over Y are classified by a suitably defined first cohomology group whose elements are represented by descent data. In this case, the automorphism "sheaf" of the trivial G-bundle is merely the functor of points of G restricted to the so-called "small fppf site" on Y. Now, in the case of $G = \mathrm{GL}_n$, as in Example 1.2.5, fpqc descent for locally free sheaves (cf. Example 1.1.16) shows that one need only consider Zariski open covers to represent all possible elements of $H^1(Y_{fppf}, \mathrm{GL}_n)$, which is in fact exactly the content of the proposition for $G = \mathrm{GL}_n$. For tori, it then suffices to reduce to $\mathbf{G}_m$ via the direct sum decomposition. □

Definition 1.3.19. Let X be a G-scheme. A categorical quotient for the action of G on X is a scheme Y and a G-invariant morphism $\pi : X \to Y$ such that for any other G-invariant morphism of schemes $f : X \to Z$, there is a unique morphism $g : Y \to Z$ such that $g \circ \pi = f$.

Of course, it follows by standard universal property arguments that categorical quotients are unique up to unique isomorphism.

Definition 1.3.20. Let X be a G-scheme. We say that X admits a principal bundle quotient X/G if there is a scheme X/G and a morphism $\pi : X \to X/G$ which makes X into a principal G-bundle over X/G.

Lemma 1.3.1. *Let G be GL_n or a torus. A principal bundle quotient for a reduced G-scheme X is a categorical quotient for the action of G on X.*

Proof. Let $f : X \to Z$ be a G-invariant morphism and let $\pi : X \to X/G = Y$ be a principal bundle quotient. The uniqueness of any factorization of f through Y follows from the fact that π is fpqc (Proposition 1.1.10). To see existence, we first consider the case where X is the trivial bundle. Then there is a section $s : Y \to X$ of π, and for any closed point $x \in X$, $s(\pi(x))$ is in the orbit of x. Since X is reduced and Z is separated, f is determined by how it maps closed points, and hence $f = (f \circ s) \circ \pi$. In the general case, by Proposition 1.3.10, $\pi : X \to Y$ is locally Zariski trivial, hence for an open cover U_i of Y, we have maps $g_i : U_i \to Z$ such that $f|_{\pi^{-1}(U_i)} = g_i \circ \pi$. Uniqueness then shows that these maps glue to give the desired factorization. □

Corollary 1.3.1. *Let G be GL_n or a torus. Let X be a reduced G-scheme. For any two principal bundles quotients $\pi : X \to Y$ and $\pi' : X \to Y'$ of X, there exists an isomorphism $\varphi : Y \to Y'$ such that $\varphi \circ \pi = \pi'$.*

Example 1.3.14. The projection $\pi : U = \mathbf{A}^{n+1} \setminus \{O\} \to \mathbf{P}^n$ given by the standard coordinates $t_0, \ldots, t_n \in \mathcal{O}_U(U)$ is a principal $\mathbf{G}_m$-bundle. Indeed, it is trivial on the open subschemes $U_i \subseteq \mathbf{P}^n$ given by $x_i \neq 0$ for $0 \leq i \leq n$. If $X \subseteq \mathbf{P}^n$ is a closed subscheme, then $U \times_{\mathbf{P}^n} X \to X$ is also a principal $\mathbf{G}_m$-bundle, and is exactly the affine cone of X with the vertex removed.

Proposition 1.3.11. *Let G be an affine algebraic group. For every $n \geq 1$, there exists an affine space V with a linear G-action and a G-stable open subscheme $U \subseteq V$ with* $\mathrm{codim}(V \setminus U) > n$ *such that there exists a principal bundle quotient $U \to U/G$.*

Proof. See [EG, Lemma 9]. □

Proposition 1.3.12. *Let G be GL_n or a torus. Let U be a reduced G-scheme with a principal bundle quotient U/G. If X is any reduced G-scheme, then $X \times U$ with the diagonal G-action admits a principal bundle quotient $(X \times U)/G$ such that the induced diagram*

$$\begin{array}{ccc} X \times U & \longrightarrow & U \\ \downarrow & & \downarrow \\ (X \times U)/G & \longrightarrow & U/G \end{array}$$

is a fibre square.

We give essentially the same proof as [EG, Proposition 23].

Proof. First, we consider the case of a trivial principal bundle $G \times Y \to Y$, with Y reduced. Then the morphism $\pi : X \times G \times Y \to X \times Y$ given (on closed points) by $\pi(x, g, y) = (g^{-1}x, y)$ is easily seen to give a principal G-bundle. Moreover, it fits into the fibre square

$$\begin{array}{ccc} X \times G \times Y & \longrightarrow & G \times Y \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X \times Y & \longrightarrow & Y \end{array}$$

Cover U/G with open subschemes V_i such that $p : U \to U/G$ becomes trivial over each V_i. Then by the above argument, the open subschemes $X \times p^{-1}(V_i)$ have morphisms $\pi_i : X \times p^{-1}(V_i) \to (X \times p^{-1}(V_i))/G$. The uniqueness of quotients identifies the open subschemes $\pi_i(X \times (V_i \cap V_j))$ and $\pi_j(X \times (V_i \cap V_j))$, so we may glue the $(X \times p^{-1}(V_i))/G$ into a scheme Z, and glue the morphisms $(X \times p^{-1}(V_i))/G \to V_i$ and π_i into $Z \to U/G$ (this is why Z is separated over F) and $\pi : X \times U \to Z$, respectively. Checking locally shows that

$$\begin{array}{ccc} X \times U & \longrightarrow & U \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ Z & \longrightarrow & U/G \end{array}$$

is a fibre square and that $\pi : X \times U \to Z$ is a principal G-bundle. □

Lemma 1.3.2. *Let G be a GL_n or a torus and X a smooth G-scheme. A principal bundle quotient Y is smooth if it exists.*

Proof. Since in any case G is connected, we may assume X is connected and hence a non-singular variety of some dimension n. Let $m = \dim G$, then we have by Proposition 1.1.3 that Y is irreducible of dimension $\dim X/G = n - m$. By [Ha2, Theorem II.8.15], it suffices to show that $\Omega_{Y/F}$ is locally free of this rank. This can be checked Zariski locally, in which case by Proposition 1.3.10, we may assume we have a trivial bundle $p : Y \times G \to Y$. As in any case G is smooth, it follows that $p^*\Omega_{Y/F}$ is locally free of rank $\dim X/G$ using the direct sum decomposition of $\Omega_{Y \times G/F}$ ([BLR, Proposition 2.1/4]). Since p is fpqc, it follows that $\Omega_{Y/F}$ is also a locally free sheaf of rank $\dim X/G$ by Example 1.1.16. □

Definition 1.3.21. Let G be an affine algebraic group and X a reduced G-scheme. A vector bundle $p : E \to X$ together with a G-action on E is a G-equivariant vector bundle if the diagram

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & E \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\;g\;} & X \end{array}$$

commutes for all $g \in G(F)$, and the action on fibres $E_x \to E_{gx}$ is linear.

Example 1.3.15. Let $\pi : X \to Y$ be a principal G-bundle, with both X and Y reduced. Let E be a vector bundle on Y. Then for $g \in G(F)$, we have the obvious maps $\mathrm{id} \times g : E \times_Y X \to E \times_Y X$. This gives π^*E the structure of a G-equivariant vector bundle on X.

Proposition 1.3.13. *Let $\pi : X \to Y$ be a principal G-bundle with X and Y reduced. The functor $E \mapsto \pi^* E$ from vector bundles on Y to G-equivariant vector bundles on X is an equivalence of categories.*

Proof. This is a special case of [Vi, Theorem 4.46] (note that $\pi : X \to Y$ is a G-torsor by [Vi, Proposition 4.43] where we take coverings to be fppf morphisms. That the stack condition is satisfied for vector bundles is essentially the content of Example 1.1.16). □

Remark 1.3.3. If X is a T-scheme admitting a principal bundle quotient Y, and E' is an equivariant vector bundle on X, then the associated vector bundle E on Y is such that there is a fibre square

$$\begin{array}{ccc} E' & \xrightarrow{p} & X \\ \downarrow & & \downarrow \\ E & \longrightarrow & Y \end{array}$$

Since by hypothesis p is T-equivariant, this implies that $E' \to E$ is a principal T-bundle. Thus, equivariant vector bundles on X have quotients E/T on Y with a vector bundle structure.

Definition of the equivariant Chow groups

Throughout this section, T is a torus. A representation V of T will mean an affine space on which T acts by linear automorphisms. Since the notation can get rather cumbersome, we will sometimes write $\mathrm{CH}^\bullet X$ for $\mathrm{CH}^\bullet(X)$.

We follow the presentation of [To, §2.2-2.3].

Definition 1.3.22. Let X be a non-singular variety with a T-action. Choose a representation V of T and a closed subset $S \subseteq V$ such that $U := V \setminus S$ is T-stable and has a principal bundle quotient for the action of T. For $i < \mathrm{codim}(S)$, define the T-equivariant codimension i Chow group of X by $\mathrm{CH}^i_T(X) := \mathrm{CH}^i((X \times U)/T)$.

Proposition 1.3.14. *Definition 1.3.22 gives well-defined equivariant Chow groups in all codimensions.*

Proof. By Proposition 1.3.11, we can indeed find representations V and closed subsets S which define $\mathrm{CH}^i_T(X)$ for all $i \geq 0$, so we must show that the groups are independent of this choice.

First, we keep the representation fixed and vary the closed subset. In particular, we shall show that for $S_1, S_2 \subseteq V$ with the desired properties, there are natural restriction maps $\mathrm{CH}^j(X \times (V \setminus S_i))/T \to \mathrm{CH}^j(X \times (V \setminus S_1 \cup S_2))/T$ are isomorphisms for $j < \mathrm{codim}(S_1 \cup S_2)$ and $i = 1, 2$. To do so, it is sufficient to show that if a closed subset S is such that $V \setminus S$ is T-stable with principal bundle quotient, then for any closed subset S' containing S, the induced open immersion $i : (X \times (V \setminus S))/T \to (X \times (V \setminus S'))/T$ is such that $i^* : \mathrm{CH}^i(X \times (V \setminus S))/T \to \mathrm{CH}^i(X \times (V \setminus S'))/T$ is an isomorphism for $i < \mathrm{codim}(S')$. Let $U = V \setminus S$, and $\pi : X \times U \to (X \times U)/T$ be the quotient map. Since $S' \setminus S$ is T-stable in U, hence $X \times (S' \setminus S)$ is T-stable in $X \times U$, we have that it is the inverse image of $\pi(X \times (S' \setminus S))$, and so by the flatness of π,

$$\mathrm{codim}(\pi(X \times (S' \setminus S))) = \mathrm{codim}(X \times (S' \setminus S)) \geq \mathrm{codim}(S').$$

By the exact sequence of Proposition 1.3.3, i^* is an isomorphism in codimensions $i < \mathrm{codim}(S')$.

Now suppose V and W are representations with chosen closed subsets S_V and S_W, respectively, with $n = \min(\mathrm{codim}(S_V), \mathrm{codim}(S_W))$. We may identify $X \times (V \times W \setminus S_V \times W)$ with the vector bundle $W \times (X \times (V \setminus S_V)) \to X \times (V \setminus S_V)$. The T-action gives this the structure of T-equivariant vector bundle, so Remark 1.3.3 shows that $p_W : (W \times (X \times (V \setminus S_V)))/T \to (X \times (V \setminus S_V))/T$ is a vector bundle. This gives isomorphisms $p_W^* : \mathrm{CH}^i(X \times (V \setminus S_V))/T \to \mathrm{CH}^i(W \times (X \times (V \setminus S_V)))/T$ for all i by Example 1.3.6, and reversing the role of V and W, gives an isomorphism $p_V^* : \mathrm{CH}^i(X \times (W \setminus S_W))/T \to \mathrm{CH}^i(V \times (X \times (W \setminus S_W)))/T$. Then by the first part, $\mathrm{CH}^i(X \times (V \times W \setminus V \times S_W))/T$ and $\mathrm{CH}^i(X \times (V \times W \setminus S_V \times W))/T$ agree for $i < n$, so we get an identification of $\mathrm{CH}^i(X \times (V \setminus S_V))/T$ and $\mathrm{CH}^i(X \times (W \setminus S_W))/T$ for $i < n$. □

Example 1.3.16. Let X be a non-singular T-variety and $Y \subseteq X$ a T-stable subscheme. For any U as above, we get fibre square

$$\begin{array}{ccc} Y \times U & \hookrightarrow & X \times U \\ \downarrow & & \downarrow \\ (Y \times U)/T & \hookrightarrow & (X \times U)/T \end{array}$$

and so can define a class $[(Y \times U)/T] \in \mathrm{CH}^\bullet(X \times U)/T$. These classes are compatible with the identifications of the above proof for different choices of U, and so we get a well-defined class in $\mathrm{CH}_T^\bullet(X)$ which we call $[Y]$. If Y is a subvariety of codimension d, then $[Y] \in \mathrm{CH}_T^d(X)$.

Intersection product and functoriality

Let $f : X \to Y$ be an equivariant morphism of non-singular T-varieties. Choosing a representation V and a closed subset S, it follows from Proposition 1.3.12 that the induced diagram, with $U = V \setminus S$

$$\begin{CD} X \times U @>{f\times \mathrm{id}}>> Y \times U \\ @VVV @VVV \\ (X \times U)/T @>{\tilde{f}}>> (Y \times U)/T \end{CD}$$

is a fibre square. Since $Y \times U \to (Y \times U)/T$ is fpqc, $\tilde{f}$ is proper (resp, a regular embedding) if f is. Thus for proper f of relative codimension d, we get induced homomorphisms $\tilde{f}_* : \mathrm{CH}^i(X \times U)/T \to \mathrm{CH}^{i+d}(Y \times U)/T$. Since in the proof of Proposition 1.3.14 the identifications of the groups $\mathrm{CH}^i_T(X)$ for various choices of representation V and closed subset $S \subseteq V$ are given by flat pullbacks, we can extend this to a well-defined operation $f_* : \mathrm{CH}^i_T(X) \to \mathrm{CH}^{i+d}_T(Y)$ for all i. In particular, we obtain fibre squares of the form

$$\begin{CD} (X \times U')/T @>{\tilde{f}'}>> (Y \times U')/T \\ @V{p}VV @VV{q}V \\ (X \times U)/T @>{\tilde{f}}>> (Y \times U)/T \end{CD}$$

where the vertical maps are flat and induce isomorphisms on Chow groups below a certain codimension. By [Fu, Proposition 1.7], we have that $\tilde{f}' \circ p^* = q^* \circ \tilde{f}$, so the choice of representation will not change the pushforward.

Since $(X \times U)/T$ and $(Y \times U)/T$ are also non-singular (Lemma 1.3.2), we have arbitrary pullbacks and multiplication in the Chow ring. The former is defined analogously to proper pushforwards, and can be seen to be well-defined by the analogous diagram and the fact that pullbacks are functorial. To multiply classes $\alpha \in \mathrm{CH}^i_T(X)$ and $\beta \in \mathrm{CH}^j_T(X)$, choose a representation V such there is a closed subset S of codimension $i + j$ with the necessary properties on $U = V \setminus S$. Then α, β are represented in $\mathrm{CH}^\bullet(X \times U)/T$ and their product is defined by $\alpha\beta \in \mathrm{CH}^{i+j}(X \times U)/T$. This is well-defined by a similar argument to the first two cases, noting that a flat pullback is a ring homomorphism.

By construction, it is clear that the functor from non-singular T-varieties and equivariant morphisms to graded rings $X \mapsto \mathrm{CH}^\bullet_T(X)$ also enjoys the properties of

Theorem 1.3.1. Indeed, it suffices to apply these properties on ordinary Chow rings after making a suitable choice of U such that all the involved classes can be defined.

Example 1.3.17. Let $* = \operatorname{Spec} F$. We construct a homomorphism $\varphi : X(T) \to \mathrm{CH}^1_T(*)$ as follows: given a character $\chi \in X(T)$, one gets a T-representation on $\mathbf{A}^1_F$ via

$$T \times \mathbf{A}^1_F \xrightarrow{\chi \times \mathrm{id}} \mathbf{G}_m \times \mathbf{A}^1_F \xrightarrow{m} \mathbf{A}^1_F$$

where m gives the usual action of $\mathbf{G}_m$ on $\mathbf{A}^1_F$. Call it V_χ. This gives a T-equivariant line bundle $V_\chi \times U$, and hence a line bundle L_χ on U/T, and we define $\varphi(\chi) = c_1(\mathcal{L}_\chi)$, where $\mathcal{L}$ is the invertible sheaf associated to L_χ. Note the T-equivariant isomorphism $V_{\chi\chi'} \cong V_\chi \otimes V_{\chi'}$ (we view the tensor product as a tensor product of line bundles over $*$). It follows that $L_{\chi\chi'} \cong L_\chi \otimes L_{\chi'}$, so φ is a homomorphism. This defines a homomorphism of graded rings $\operatorname{Sym}^\bullet X(T) \to \mathrm{CH}^\bullet_T(*)$.

Corollary 1.3.2. *Let X, Y be non-singular T-varieties. Let $R = \operatorname{Sym}^\bullet X(T)$. Then $\mathrm{CH}^\bullet_T(X)$ and $\mathrm{CH}^\bullet_T(Y)$ are both graded R-algebras. If $f : X \to Y$ is a T-equivariant proper morphism, then f_* is R-linear. For arbitrary T-equivariant f, f^* is also R-linear.*

Proof. The R-algebra structure comes from the pullback of the structural morphism $s : X \to \operatorname{Spec} F$, i.e. for $c \in R$ and $x \in \mathrm{CH}^\bullet(X)$, $c \cdot x = s^*(\varphi(x))x$. Letting $f : X \to Y$ be a proper T-equivariant morphism, where s_X and s_Y are the structural morphisms of X and Y respectively, it follows that for $x \in \mathrm{CH}^\bullet_T(X)$ and $c \in R$,

$$c \cdot f_*(x) = s_Y^*(\varphi(c))f_*(x) = f_*(f^*(s_Y^*(\varphi(c)))x) = f_*(s_X^*(\varphi(c))x) = f_*(c \cdot x).$$

For pullbacks, R-linearity follows from the functoriality of pullbacks. □

Proposition 1.3.15. *For $* = \operatorname{Spec} F$, $\mathrm{CH}^\bullet_T(*)$ is isomorphic to R as an R-algebra.*

Proof. Fix an isomorphism $T \cong (\mathbf{G}_m)^r$ and let $t_1, \ldots, t_r$ be the induced homomorphisms to each factor. Define the representation V_i, which is given by scaling by t_i the $n+1$-dimensional affine space V_i with origin O. It follows from Example 1.3.14 that $U = \prod_{i=0}^r V_i \setminus \{O\} \subseteq \bigoplus_{i=0}^r V_i = V$ with the diagonal T-action admits a principal bundle quotient $U/T = \prod_{i=0}^r \mathbf{P}^n$ and we have that $\operatorname{codim}(V \setminus U) < n$. Repeated applications of the projective bundle theorem show that for any $n \geq 1$, $\mathrm{CH}^\bullet(U/T)$ is generated as a $\mathbf{Z}$-algebra by $x_i = \pi_i^*([H]) \in \mathrm{CH}^1(U/T)$, $i = 1, \ldots, r$, where $H \subseteq \mathbf{P}^n$ is a hyperplane and π_i is the i-th projection map. Considering the t_i as elements of $X(T)$, we now need only show that $t_i \cdot 1_{U/T} = x_i$ for $i = 1, \ldots, r$.

The line bundle associated to x_i is $\pi_i^* O(1)$, so the pullback to U is T-equivariantly isomorphic to $\pi^* O(1) \times \prod_{j \neq i} V_i \setminus \{O\}$, where $\pi : V_i \setminus \{O\} \to \mathbf{P}^n$ is the quotient for the $\mathbf{G}_m$-action on V_i. We are then reduced to showing that $\pi^* O(1)$, as a $\mathbf{G}_m$-equivariant line bundle, is isomorphic to $L = V_i \setminus \{O\} \times \mathbf{A}^1$, where $\mathbf{A}^1$ is given the usual $\mathbf{G}_m$-action. The line bundle $\pi^*O(1)$ is trivial on the open subschemes W_i defined by $x_i \neq 0$, with transition functions x_i/x_j. For each i, define isomorphisms of vector bundles $f_i : W_i \times \mathbf{A}^1 \to W_i \times \mathbf{A}^1$ by $(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n, x_i y)$ It is easily seen that these glue to an isomorphism $f : \pi^*O(1) \to L$ which is $\mathbf{G}_m$-equivariant. $\square$

Following [EG, §2.2], we define the "forgetful homomorphism" to ordinary Chow groups. Let U be an open subscheme of a T-representation such that $\mathrm{CH}^j(X \times U)/T$ defines the T-equivariant Chow groups of X for $j < n$. By Proposition 1.3.12, the natural morphism $(X \times U)/T \to U/T$ has fibres isomorphic to X. Choosing a closed point $x \in U/T$, this defines a closed immersion $i : X \hookrightarrow (X \times U)/T$. Then for $j < n$, we define the forgetful homomorphism by $i^* : \mathrm{CH}^j(X \times U)/T \to \mathrm{CH}^j(X)$. This does not depend on the choice of x (see [EG, p. 6]). It is also independent of the choice of U since pullbacks are functorial, so in total we get a homomorphism of graded rings $\mathrm{CH}^\bullet_T(X) \to \mathrm{CH}^\bullet(X)$.

Proposition 1.3.16. *The forgetful homomorphism is natural with respect to proper pushforwards and arbitrary pullbacks.*

Proof. Let $f : X \to Y$ be an equivariant morphism of non-singular T-varieties. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow & & \big\downarrow \\ (X \times U)/T & \xrightarrow{\quad \tilde{f} \quad} & (Y \times U)/T \end{array}$$

clearly commutes, and so functoriality of pullbacks implies naturality for arbitrary pullbacks. If f is proper, then naturality of pushforwards follows from the fact that the diagram is a fibre square and both closed immersions are regular embeddings of codimension $\dim U/T$ by applying [Fu, Theorem 6.2] part a) and c), and Theorem 1.3.1 part ii). $\square$

Localisation

We present two results from [Br] which elucidate the structure of T-equivariant Chow groups.

Let X be a non-singular T-variety. We write X^T for the fixed-point subscheme of X. By [Mi2, Theorem 13.1], it is a smooth, closed subscheme of X.

Theorem 1.3.3. *Let $i : X^T \hookrightarrow X$ be the inclusion map. The homomorphism $i_* : \mathrm{CH}^\bullet_T(X^T) \to \mathrm{CH}^\bullet_T(X)$ becomes an isomorphism after localisation to Q, the field of fractions of $R := \mathrm{Sym}^\bullet X(T)$.*

Proof. [Br, Corollary 2.3.2]. □

For a (closed) point $x \in X^T$, we have that for all $t \in T(F)$, $t \cdot x = x$. Thus one can associate to x a representation $\rho : T(F) \to \mathrm{Aut}_F(\mathrm{Tan}_x X)$, which we will often simply refer to by the underlying vector space $\mathrm{Tan}_x X$. Let V be an F-linear representation of $T(F)$. For a character $\chi \in X(T)$, we say that a representation V has weight χ, if, considering $\mathbf{G}_m(F)$ as $F^\times$, $t \cdot v = \chi(t)v$ for all $t \in T(F)$ and $v \in V$.

Definition 1.3.23. Let X be a non-singular T-variety. A (closed) point $x \in X^T$ is non-degenerate if $\mathrm{Tan}_x X$ decomposes as $\bigoplus_{i=0}^n V_{\chi_i}$, where each V_{χ_i} is a one-dimensional subrepresentation with non-zero weight χ_i.

If V and V' are one-dimensional representations of weight χ and χ' respectively, then $\mathrm{Hom}_T(V, V') \cong F$ if $\chi = \chi'$ and is trivial otherwise. It follows that for a non-degenerate point $x \in X$, the list of characters χ_i appearing in any decomposition of $\mathrm{Tan}_x X$ as in Definition 1.3.23 is unique up to order.

Theorem 1.3.4. *Let X be a non-singular T-variety and $x \in X^T$ a non-degenerate fixed point with weights $\chi_1, \dots, \chi_n$.*

i. *There exists a unique R-linear homomorphism $e_{x,X} : \mathrm{CH}^\bullet_T(X) \to Q$ such that $e_{x,X}([x]) = 1$ and $e_{x,X}([Y]) = 0$ for any T-stable closed subscheme $Y \subseteq X$ with $x \notin Y$.*

ii. *For all smooth T-stable subvarieties $Y \subseteq X$ containing x, $e_{x,X}([Y]) = e_{x,Y}([Y])$*

iii. $e_{x,X}([X]) = 1/(\chi_1 \dots \chi_n)$.

Proof. See [Br, Theorem 4.2]. □

Corollary 1.3.3. *Let X be a non-singular T-variety such that all closed points $x \in X^T$ are non-degenerate. For any $\alpha \in \mathrm{CH}^\bullet_T(X)$, we have that*

$$\alpha = \sum_{x \in X^T} e_{x,X}(\alpha) \cdot [x]$$

in $\mathrm{CH}^\bullet_T(X) \otimes_R Q$.

Proof. See [Br, Corollary 4.2]. □

Note that the sum makes sense since the non-degeneracy hypothesis requires that $\mathrm{Tan}_x X^T = 0$ for all closed points $x \in X^T$, hence X^T is the union of finitely many closed points.

1.4 Pure motives

For a field F, we write $\mathbf{SmProj}_F$ for the full subcategory of smooth projective F-schemes. The aim of this section is to introduce the category of effective Chow motives, which is essentially a "linearisation" of $\mathbf{SmProj}_F$. Our exposition follows [Ma], to which we will also defer most proofs.

1.4.1 Definition and first properties

First we fix our categorical terminology. A pre-additive category is a category $\mathcal{C}$ whose hom-sets are endowed with the structure of an abelian group, and where composition is bilinear with respect to this. An additive category is a pre-additive category where all finite products exist. See [SP, Section 09SE].

Category of correspondences

We start by defining the category of correspondences, $\mathbf{Corr}_F$. The objects of this category are the same as $\mathbf{SmProj}_F$. To make the distinction of categories clear, if X is a smooth projective F-scheme, we will write $\overline{X}$ for the object in $\mathbf{Corr}_F$. The set of morphisms in $\mathbf{Corr}_F$ $\mathrm{Hom}(\overline{X},\overline{Y})$ is defined to be $\mathrm{CH}^\bullet(X \times_F Y)$ (from here on, unspecified products are over F). An element of this set is called a correspondence. By definition, correspondences have the structure of an abelian group. Two correspondences $\alpha \in \mathrm{Hom}(\overline{X},\overline{Y})$ and $\beta \in \mathrm{Hom}(\overline{Y},\overline{Z})$ are composed by the convolution product

$$p_{13*}(p_{12}^*(\alpha)\cdot p_{23}^*(\beta)) \in \mathrm{CH}^\bullet(X\times Z) = \mathrm{Hom}(\overline{X},\overline{Z}) \tag{1.4.1}$$

where p_{ij} are the projection maps onto the i-th and j-th factors of $X \times Y \times Z$. A correspondence $\alpha \in \mathrm{Hom}(\overline{X},\overline{Y})$ is said to be homogeneous of degree d if $\alpha \in \mathrm{CH}^{\dim X + d}(X\times Y)$ (see Remark 1.4.1). The subgroup of degree d correspondences is denoted by $\mathrm{Hom}^d(\overline{X},\overline{Y})$.

Remark 1.4.1. So far, we have only defined a grading on the Chow groups by codimension in the case of varieties. However, the connected components of smooth projective schemes X, Y are varieties. Let $X_1,\dots,X_n$ be the connected components of X and $Y_1,\dots,Y_m$ those of Y. Then, $\mathrm{CH}_\bullet(X)$ canonically decomposes as $\bigoplus_i \mathrm{CH}_\bullet(X_i)$ and likewise $\mathrm{CH}_\bullet(X\times Y)$ as $\bigoplus_{i,j}\mathrm{CH}_\bullet(X_i\times Y_j)$. We make the convention that $\mathrm{CH}^k(X) = \bigoplus_i \mathrm{CH}_{\dim X_i - k}(X_i)$ and $\mathrm{CH}^{\dim X + d}(X\times Y) = \bigoplus_{i,j}\mathrm{CH}_{\dim Y_j - d}\mathrm{CH}(X_i\times Y_j)$ for all $d,k\in\mathbf{Z}$. In the case of $\mathrm{CH}^\bullet(X)$, it is given a graded ring structure as the direct product of the $\mathrm{CH}^\bullet(X_i)$. This is how we make sense of the multiplication in Equation (1.4.1).

Example 1.4.1. Let X, Y be smooth projective F-schemes and write $X \sqcup Y$ for their disjoint union. This is still a smooth projective F-scheme, and the object $\overline{X \sqcup Y}$ is the direct sum $\overline{X} \oplus \overline{Y}$

Proposition 1.4.1. *Composition of correspondences is associative and additive. For any smooth projective F-scheme X, the correspondence $\delta_*([X]) \in \mathrm{Hom}(\overline{X}, \overline{X})$ is the identity morphism and homogeneous of degree zero. Degree is additive in the sense that for $\alpha \in \mathrm{Hom}^d(\overline{X}, \overline{Y})$ and $\beta \in \mathrm{Hom}^e(\overline{Y}, \overline{Z})$, $\beta \circ \alpha \in \mathrm{Hom}^{d+e}(X, Z)$.*

Proof. Additivity is immediate from the definition. Associativity of composition and the claimed identity morphism are treated in the lemma of [Ma, p. 446]. That the identity is homogeneous of degree zero is immediate. Additivity of degrees under composition is proven in the lemma of [Ma, p. 452]. □

Corollary 1.4.1. *The category $\mathbf{Corr}_F$ and its subcategory $\mathbf{Corr}_F^0$ consisting of only the degree zero correspondences are both well-defined pre-additive categories.*

Proposition 1.4.2. *The assignment*

$$X \mapsto \overline{X}, \qquad (Y \xrightarrow{f} X) \mapsto (f \times \mathrm{id}_Y)_*([Y]) \in \mathrm{Hom}^0(\overline{X}, \overline{Y})$$

gives a contravariant functor from $\mathbf{SmProj}_F$ to $\mathbf{Corr}_F^0$.

Proof. By definition, it is clear these correspondences are homogeneous of degree zero. The functoriality is the content of the proposition of [Ma, p. 447]. □

Proposition 1.4.3. *Both $\mathbf{Corr}_F$ and $\mathbf{Corr}_F^0$ have finite products, hence are additive. In particular, the product of objects $\overline{X_i}$, $1 \leq i \leq n$, is given by $\overline{\coprod_{i=1}^n X_i}$.*

Proof. See [Ma, p. 448]. □

As usual, a product $A \times B$ in an additive category can be given the structure of a coproduct by defining coprojections $\mathrm{id}_A \times 0 : A \to A \times B$ and $0 \times \mathrm{id}_B : B \to A \times B$. We will call the (co)product the direct sum and write it $A \oplus B$, writing p_A, p_B for the projections and i_A, i_B for the coprojections. Much the same holds for any finite products and we will retain the same conventions.

Category of effective Chow motives

Definition 1.4.1. An additive category $\mathcal{C}$ is idempotent complete if for any object X in $\mathcal{C}$ and idempotent element $\rho \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, there is a direct sum decomposition $X = X_1 \oplus X_2$ corresponding to ρ. That is, there are objects X_1 and X_2 in $\mathcal{C}$ such that $X = X_1 \oplus X_2$ and that $i_{X_1} \circ p_{X_1} = \rho$.

Definition 1.4.2. The idempotent completion $\widehat{\mathcal{C}}$ of an additive category $\mathcal{C}$ is a pseudo-additive category whose objects consist of pairs (X, ρ), X an object of $\mathcal{C}$ and ρ and idempotent endomorphism of X, with

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}((X, \rho_X), (Y, \rho_Y)) := \rho_Y \circ \mathrm{Hom}_{\mathcal{C}}(X, Y) \circ \rho_X$$

By construction, $\widehat{\mathcal{C}}$ is additive and idempotent complete, and there is a fully faithful additive functor $\iota : \mathcal{C} \to \widehat{\mathcal{C}}$ sending X to (X, id_X).

Definition 1.4.3. The category of effective Chow motives $\mathbf{Mot}_F$ is defined to be the idempotent completion of $\mathbf{Corr}^0_F$. The motive of a smooth projective F-scheme X is defined as $\mathbf{M}(X) := (\overline{X}, \mathrm{id}_{\overline{X}})$.

Remark 1.4.2. Since it cannot cause confusion, given an idempotent $p \in \mathrm{Hom}^0(\overline{X}, \overline{X})$, we will simply write (X, p) for $(\overline{X}, p)$.

By Proposition 1.4.2, the assignment $X \mapsto \mathbf{M}(X)$ gives rise to a contravariant functor $\mathbf{M} : \mathbf{SmProj}_F \to \mathbf{Mot}_F$.

Example 1.4.2. The motive of $\mathbf{P}^1_F$ decomposes as $\mathbf{M}(\mathbf{P}^1_F) \cong \mathbf{M}(\mathrm{Spec}\, F) \oplus \mathbf{L}$, where $\mathbf{L} := (\mathbf{P}^1_F, 1_{\mathbf{P}^1_F} \times [P])$. Indeed, by the projective bundle theorem, $\mathrm{CH}^1(\mathbf{P}^1_F \times \mathbf{P}^1_F)$ is generated by the classes $H_1 = [\mathbf{P}^1_F \times \{P\}]$ and $H_2 = [\{P\} \times \mathbf{P}^1_F]$, for some rational point $P \in \mathbf{P}^1_F$. Thus the class of the diagonal is of the form $\delta_{Y*}(1_Y) = aH_1 + bH_2$. But it is clear looking at intersections that $\deg(H_i \cdot H_j) = \delta_{ij}$. Similarly, one sees that $\deg(H_i \cdot \delta_{Y*}(1_Y)) = 1$, therefore we conclude that $\delta_{Y*}(1_Y) = H_1 + H_2$.

Now consider the idempotent morphism given by the composition $\mathbf{P}^1_F \xrightarrow{p} \mathrm{Spec}\, F \xrightarrow{i} \mathbf{P}^1_F$, with p the structural morphism and i the inclusion of P. Since $i \circ p$ is idempotent, its associated correspondence $f \in \mathrm{Hom}(\mathbf{M}(\mathbf{P}^1_F), \mathbf{M}(\mathbf{P}^1_F))$ is as well. It is easy to see that the correspondences c_i and c_p associated to i and p give mutually inverse isomorphisms between $\mathbf{M}(\mathrm{Spec}\, F)$ and $(\mathbf{P}^1_F, f)$. It is a straighforward calculation that $f = H_2$, and hence $H_1 = \delta_{Y*}(1_Y) - H_2$ is idempotent and such that $\mathbf{M}(\mathbf{P}^1_F) = (\mathbf{P}^1_F, H_2) \oplus (\mathbf{P}^1_F, H_1)$, whence $\mathbf{M}(\mathbf{P}^1_F) \cong \mathbf{M}(\mathrm{Spec}\, F) \oplus \mathbf{L}$.

Tensor products

At the level of $\mathbf{Corr}_F$, we define the tensor product $\overline{X} \otimes \overline{Y}$ by $\overline{X \times Y}$. As with the tensor product in the category of modules, given $f \in \mathrm{Hom}(\overline{X}, \overline{X'})$ and $g \in \mathrm{Hom}(\overline{Y}, \overline{Y'})$, we define a correspondence

$$f \otimes g = s_{23*}(p_{12}^*(f) p_{34}^*(g)) \in A^\bullet(X \times Y \times X' \times Y') = \mathrm{Hom}(\overline{X} \otimes \overline{Y}, \overline{X'} \otimes \overline{Y'})$$

where $s_{23} : X \times X' \times Y \times Y' \xrightarrow{\sim} X \times Y \times X' \times Y'$ is the isomorphism interchanging the 2nd and 3rd factor. This construction is functorial in the following sense:

Lemma 1.4.1. *For* $f_1 \in \mathrm{Hom}(\overline{X}, \overline{X'})$, $f_2 \in \mathrm{Hom}(\overline{X'}, \overline{X''})$, $g_1 \in \mathrm{Hom}(\overline{Y}, \overline{Y'})$, $g_2 \in \mathrm{Hom}(\overline{Y'}, \overline{Y''})$, *we have that* $(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1)$.

Proof. This is the first lemma of [Ma, p. 448]. □

It is clear that if f and g are correspondences of degree 0, then $f \otimes g$ is as well, so tensor products also make sense in $\mathbf{Corr}^0_F$.

Definition 1.4.4. Let $M = (X, p)$ and $N = (Y, q)$ be motives. We define the tensor product $M \otimes N = (X \times Y, p \otimes q)$.

Let $M' = (X', p')$ and $N' = (Y', q')$. If $f \in \mathrm{Hom}(M, M')$ and $g \in \mathrm{Hom}(N, N')$, then

$$f \otimes g = (p' \circ f \circ p) \otimes (q' \circ g \circ q) = (p' \otimes q') \circ (f \otimes g) \circ (p \otimes q) \in \mathrm{Hom}(M \otimes N, M' \otimes N'),$$

so the functorial properties transfer to motives as well, namely the identity $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.

1.4.2 Tate motives

Definition 1.4.5. Let $n \geq 0$, we define the motive $\mathbf{Z}(n)$ to be the n-th fold tensor product $\mathbf{L}^{\otimes n}$, where $\mathbf{L}$ is the motive defined in Example 1.4.2. If M is a motive, then we write $M(n)$ for $M \otimes \mathbf{Z}(n)$.

Proposition 1.4.4. *The abelian group of homomorphisms* $\mathrm{Hom}(\mathbf{Z}(n), \mathbf{Z}(m))$ *is isomorphic to* $\mathbf{Z}$ *and generated by the identity if* $n = m$*, and is trivial otherwise.*

Proof. See [Ma, p. 454]. □

There is the following vast generalization of Example 1.4.2:

Theorem 1.4.1. *Let* Y *be a smooth projective* F*-scheme and* $\mathcal{E}$ *a locally free sheaf of rank* $r + 1$ *on* X*. The motive of the projective bundle* $X = \mathbf{P}(\mathcal{E})$ *decomposes as*

$$\mathbf{M}(X) \cong \bigoplus_{i=0}^{r} \mathbf{M}(Y)(i)$$

Proof. See [Ma, §7]. □

Definition 1.4.6. Let M, N be motives. For any $n \geq 0$ and $f \in \mathrm{Hom}(M, N)$, we define $f_n = f \otimes \mathrm{id}_{\mathbf{Z}(n)} \in \mathrm{Hom}(M(n), N(n))$.

It is clear that $(f \circ g)_n = f_n \circ g_n$ and that $(\mathrm{id}_M)_n = \mathrm{id}_{M(n)}$. Thus, the assignment $M \mapsto M(n)$ gives an endofunctor of $\mathbf{Mot}_F$. It is in fact fully faithful, see the lemma of [Ma, p. 458].

Definition 1.4.7. Let M and N be motives and $i \in \mathbf{Z}$. We define the group $\mathrm{Hom}^i(M, N)$ by $\mathrm{Hom}(M(n+i), N(n))$ for n sufficiently large. Given $f \in \mathrm{Hom}^i(M_1, M_2)$ and $g \in \mathrm{Hom}^j(M_2, M_3)$, we define $g \circ f \in \mathrm{Hom}^{i+j}(M_1, M_3)$ by composing representatives in $\mathrm{Hom}(M_1(n+i), M_2(n))$ and $\mathrm{Hom}(M_2(n), M_3(n-j))$ for n sufficiently large.

This gives well-defined abelian groups and a well-defined composition law by full faithfulness.

Proposition 1.4.5. *Let X, Y be smooth projective F-schemes and $\mathrm{Hom}^i(X, Y)$ the group of degree i homogeneous correspondences. There is an isomorphism*

$$\mathrm{Hom}^i(X, Y) \xrightarrow{\sim} \mathrm{Hom}^i(\mathbf{M}(X), \mathbf{M}(Y))$$

which is compatible with composition.

Proof. See the proposition of [Ma, p. 459]. □

Remark 1.4.3. Proposition 1.4.5 shows that even though $\mathbf{Mot}_F$ is defined from $\mathbf{Corr}_F^0$, no information about correspondences is lost.

1.4.3 Identity principle

There is a natural anti-equivalence on $\mathbf{Corr}_F$ called the transpose which fixes objects and maps correspondences by the isomorphism $\mathrm{Hom}(\overline{X}, \overline{Y}) = \mathrm{CH}^\bullet(X \times Y) \to \mathrm{CH}^\bullet(Y \times X) = \mathrm{Hom}(\overline{Y}, \overline{X})$ induced by the factor switching isomorphism $X \times Y \xrightarrow{\sim} Y \times X$. We write α^t for the image of a correspondence under this isomorphism.

Following [Ma, §3], for $\varphi : Y \to X$ be a morphism in $\mathbf{SmProj}_F$ and $x \in \mathrm{CH}^\bullet(X)$, we define the correspondences

$$\begin{aligned} c_\varphi &= (\varphi \times \mathrm{id}_Y)_*(1_Y) \in \mathrm{Hom}(\overline{X}, \overline{Y}) \\ c_x &= \delta_*(x) \in \mathrm{Hom}(\overline{X}, \overline{X}) \end{aligned} \tag{1.4.2}$$

Note that c_φ is homogeneous of degree zero and that if $x \in \mathrm{CH}^i(X)$ for some $i \geq 0$, then c_x is homogeneous of degree i. For any smooth projective F-scheme S, and correspondence $\alpha \in \mathrm{Hom}(\overline{X}, \overline{Y})$, one has the natural induced map

$$\alpha(\overline{S}) : \mathrm{Hom}(\overline{S}, \overline{X}) \to \mathrm{Hom}(\overline{S}, \overline{Y}), \qquad \beta \mapsto \alpha \circ \beta$$

Lemma 1.4.2. *With the notation as above, one has that*

$$c_\varphi(\overline{S}) = (\varphi \times \mathrm{id}_S)^*, \quad c^t_\varphi(\overline{S}) = (\varphi \times \mathrm{id}_S)_*, \quad c_x(\overline{S}) = \textit{multiplication by } x \times 1_S$$

Proof. See the corollary of [Ma, p. 450]. □

Applying Yoneda's lemma, Lemma 1.4.2 enables one to think of identities among certain correspondences as "universal identities" among functorial maps on Chow groups.

Proposition 1.4.6. *Consider the full subcategory of the category of effective Chow motives whose objects are finite direct sums of motives* $\mathbf{M}(X)(i) = \mathbf{M}(X) \otimes \mathbf{Z}(i)$*, with* X *a smooth projective* F*-scheme. Let* M, N *be objects in this subcategory, and* $\psi \in \mathrm{Hom}(M, N)$*. If for all smooth projective* F*-schemes* S*,* $\psi_S : \mathrm{Hom}^\bullet(\mathbf{M}(S), M) \to \mathrm{Hom}^\bullet(\mathbf{M}(S), N)$ *is an isomorphism, then* $M \cong N$*.*

Proof. Since we have a natural isomorphism $\mathrm{Hom}(\mathbf{M}(S)(i), -) \cong \mathrm{Hom}^i(\mathbf{M}(S), -)$ (Proposition 1.4.5), the condition that ψ_S be an isomorphism implies that the induced morphism $\mathrm{Hom}(\mathbf{M}(S)(i), M) \to \mathrm{Hom}(\mathbf{M}(S)(i), N)$ is an isomorphism. For a finite direct sum $M' = \bigoplus_k \mathbf{M}(S_k)(i_k)$, the universal property of direct sums

$$\mathrm{Hom}(\bigoplus_k \mathbf{M}(S_k)(i_k), -) \cong \bigoplus_k \mathrm{Hom}(\mathbf{M}(S_k)(i_k), -)$$

similarly shows that the induced morphism by ψ, $\mathrm{Hom}(M', M) \to \mathrm{Hom}(M', N)$, is an isomorphism. It follows from Yoneda's lemma that $M \cong N$. □

Chapter 2

Main Result

2.1 Construction of the hyperplane section

2.1.1 Twisted Milnor hypersurfaces

Definition 2.1.1. Let A be a central simple algebra over F of degree $n+1$. The twisted Milnor hypersurface associated to A is defined as the scheme-theoretic intersection $X(A)$ of the closed subschemes

$$\mathrm{SB}(A) \times_F \mathrm{SB}^\vee(A), \mathbf{Fl}(n+1, n(n+1), A) \subseteq \mathbf{Gr}(n+1, A) \times_F \mathbf{Gr}(n(n+1), A)$$

In other words, we have an F-scheme $X(A)$ representing the functor

$$X(A)(S) = \{\mathcal{I}_1 \subseteq \mathcal{I}_n \subseteq \mathcal{A}_S : \mathcal{I}_1, \mathcal{I}_n \text{ right ideal subbundles of } \mathcal{A}_S, \mathrm{rk}\, \mathcal{I}_k = k(n+1)\}$$

In the case $A = \mathrm{End}(V)$, the correspondence of Remark 1.2.6 shows that $X(A)$ is naturally isomorphic to $\mathbf{Fl}(V, 1, n)$. The functor of points of $X(A)$ also shows that for any field extension K/F, $X(A \otimes_F K) = X(A) \times_F K$. This shows that $X(A)$ is a twisted form of $\mathbf{Fl}(n+1, 1, n)$. We thus conclude that the $X(A)$ are smooth, projective and geometrically integral and have dimension $2n-1$.

Remark 2.1.1. The variety $\mathbf{Fl}(n+1, 1, n)$ is called a Milnor hypersurface here since it is embedded as a codimension-one subvariety of $\mathbf{P}^n_F \times (\mathbf{P}^n_F)^\vee$ by the canonical inclusion.

Theorem 2.1.1. *Define the* $\mathrm{SB}(A)$*-module*

$$\mathcal{F} = \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathrm{SB}(A)}}(\mathcal{A}_{\mathrm{SB}(A)}/\mathcal{I}_{\mathrm{SB}(A)}, \mathcal{I}_{\mathrm{SB}(A)}),$$

where $\mathcal{I}_{\mathrm{SB}(A)}$ *is the tautological sheaf of ideals of Definition 1.2.14. $\mathcal{F}$ is locally free and $X(A)$ is isomorphic to the projective bundle $\mathbf{P}(\mathcal{F})$ over* $\mathrm{SB}(A)$.

We will need some lemmas:

Lemma 2.1.1. *Let S be an F-scheme and V an m-dimensional vector space over F. Let $\mathcal{E}, \mathcal{E}' \subseteq V \otimes \mathcal{O}_S$ be local direct summands with corresponding right ideal sheaves (c.f. Remark 1.2.6) $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{M} = \underline{\mathrm{End}}(V \otimes \mathcal{O}_S)$. There is a canonical isomorphism of $\mathcal{O}_S$-modules $\underline{\mathrm{Hom}}_{\mathcal{O}_S}((V \otimes \mathcal{O}_S)/\mathcal{E}, \mathcal{E}') \to \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}/\mathcal{I}, \mathcal{I}')$.*

Proof. Consider $\mathcal{M}$ as a right $\mathcal{M}$-module. We have the canonical isomorphism $\mathcal{M} \to \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}, \mathcal{M})$ of $\mathcal{O}_S$-modules, where $f \in \mathcal{M}(U)$ maps to left-multiplication by f on $\mathcal{M}(U)$. Then we have commuting diagrams

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V \otimes \mathcal{O}_S, \mathcal{E}') & \hookrightarrow & \mathcal{M} \\ \downarrow \sim & & \downarrow \sim \\ \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}, \mathcal{I}') & \hookrightarrow & \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}) \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{O}_S}((V \otimes \mathcal{O}_S)/\mathcal{E}, V \otimes \mathcal{O}_S) & \hookrightarrow & \mathcal{M} \\ \downarrow \sim & & \downarrow \sim \\ \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}/\mathcal{I}, \mathcal{M}) & \hookrightarrow & \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}) \end{array}$$

To conclude the proof, we note that the intersections of the sub $\mathcal{O}_S$-modules given by the images of the inclusions in the diagrams correspond to the images of the inclusions $\underline{\mathrm{Hom}}_{\mathcal{O}_S}((V \otimes \mathcal{O}_S)/\mathcal{E}, \mathcal{E}') \hookrightarrow \mathcal{M}$ and $\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}/\mathcal{I}, \mathcal{I}') \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}, \mathcal{M})$. □

Lemma 2.1.2. *Let R be an F-algebra and A a central simple F-algebra of degree m. Let $A_R = A \otimes_F R$, and let $I \subseteq A_R$ be a right ideal which is a direct summand of rank m.*

i. *The R-module homomorphism $f : R \to \mathrm{End}_{A_R}(I)$ given by $f(1) = \mathrm{id}_I$ is an isomorphism.*

ii. *Let P be a rank one direct summands (as R-modules) of $\mathrm{Hom}_{A_R}(A_R/I, I)$. There is a well-defined "kernel" of P in A_R/I. Let I_P be the corresponding right ideal in A_R. As an R-module, I_P is a rank $m(m-1)$ direct summand. If $I_P = I_Q$ for two rank one direct summands $P, Q \subseteq \mathrm{Hom}_{A_R}(A_R/I, I)$, then $P = Q$.*

iii. *Let $R \to R'$ be an extension of F-algebras. For a rank one direct summand $P \subseteq \mathrm{Hom}_{A_R}(A_R/I, I)$, let*

$$P' = P \otimes_R R' \subseteq \mathrm{Hom}_{A_R}(A_R/I, I) \otimes_R R' = \mathrm{Hom}_{A_{R'}}(A_{R'}/A_{R'}I, I \otimes_R A_{R'})$$

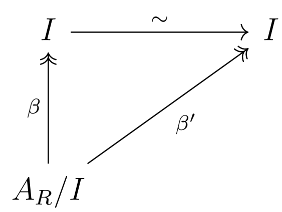
Then $I_P \otimes_R R' \subseteq \mathrm{Hom}_{A_{R'}}(A_{R'}/A_{R'}I, I \otimes_R A_{R'})$ is equal to $I_{P'}$.

Proof. For i), by localization we can reduce to the case where R is a local ring. In this case, I is free as an R-module so injectivity is clear. Let κ be the residue field of R and consider the homomorphism

$$\kappa \to \mathrm{End}_{A_R}(I) \otimes_R \kappa = \mathrm{End}_{A_\kappa}(I \otimes_R \kappa)$$

This is simply f in the case $R = \kappa$, so by Nakayama's lemma it suffices to prove this map is surjective. This is the case since $I \otimes_R \kappa$ is a right ideal of A_κ of dimension m, hence by Theorem 1.2.4, A_κ is split and so by Theorem 1.2.5 the endomorphism ring of $I \otimes_R \kappa$ must be κ itself.

For ii), replace R with a localization R_f such that $P \otimes_R R_f$ is free. Let $\beta \in P$ be a generator, then β is surjective. To shows this, we may assume R is local with residue field κ. Since P is a direct summand, $\overline{\beta} : A_\kappa/A_\kappa I \to I \otimes_R \kappa$ is non-trivial and hence surjective since $I \otimes_R \kappa$ is a simple A_κ-module. It then follows from Nakayama's lemma that β is surjective. Let $J = \ker \beta$. J is a direct summand of A_R/I of rank $m(m-2)$ and independent of the choice of β, therefore these locally defined J glue to give an ideal over the original ring with the same properties. It is then clear that I_P is a right ideal of A_R which is a direct summand of rank $m(m-1)$. Now, suppose for $P, Q \subseteq \mathrm{Hom}_{A_R}(A_R/I, I)$ we have that $I_P = I_Q$. Passing to localizations on which both P and Q are free, we get a commutative diagram



where β and β' are generators of P and Q respectively. By i), $\mathrm{Aut}_{A_R}(I) = R^\times$, so it follows that $P = Q$.

For iii), since all involved R-modules are projective, kernels are preserved by arbitrary base change. The equality is then immediate from the construction. □

Proof of Theorem 2.1.1. First, we verify that $\mathcal{F}$ is in fact locally free. Let $\bar{F}$ be an algebraic closure of F, then since $p : \mathrm{SB}(A) \times_F \bar{F} \to \mathrm{SB}(A)$ is fpqc, it is enough to show that

$$p^*\mathcal{F} \cong \underline{\mathrm{Hom}}_{p^*\mathcal{A}_{\mathrm{SB}(A)}}(p^*\mathcal{A}_{\mathrm{SB}(A)}/p^*\mathcal{I}_{\mathrm{SB}(A)}, p^*\mathcal{I}_{\mathrm{SB}(A)})$$

is locally free. Since $A \otimes_F \bar{F} \cong \mathrm{End}(V)$, with V an $n+1$ dimensional vector space over $\bar{F}$, setting $\mathcal{M} = \underline{\mathrm{End}}(V \otimes \mathcal{O}_{\mathrm{SB}(A)\times_F \bar{F}})$, we have that $p^*\mathcal{F} \cong \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{M}/\mathcal{I}, \mathcal{I}')$ for local direct summand right ideals $\mathcal{I}, \mathcal{I}'$ of $\mathcal{M}$. Lemma 2.1.1 shows this is isomorphic to a locally free $\mathcal{O}_{\mathrm{SB}(A)\times_F \bar{F}}$-module. It also follows that the rank of $\mathcal{F}$ is n, so in particular $\mathbf{P}(\mathcal{F})$ is a variety of dimension $2n-1$.

Let R be an F-algebra. By the universal property of $\mathrm{SB}(A)$, an F-morphism $f : \operatorname{Spec} R \to \mathrm{SB}(A)$ is equivalent to a choice of right ideal $I \subseteq A_R$ which is a direct summand of rank $n+1$. Considering $\mathbf{P}(\mathcal{F})$ and $X(A)$ as $\mathrm{SB}(A)$-spaces, define morphisms $\varphi(R) : \mathbf{P}(\mathcal{F})(R) \to X(A)$ by $\varphi(R)([P]) = [I \subseteq I_P \subseteq A_R]$, with I_P as in Lemma 2.1.2 (here we are identifying quasi-coherent $\mathcal{O}_{\operatorname{Spec} R}$-modules with their global sections). By Lemma 2.1.2 ii) and iii) this gives a monomorphism $\varphi : \mathbf{P}(\mathcal{F}) \to X(A)$. Since both schemes are proper over $\mathrm{SB}(A)$, φ is proper and hence a closed immersion ([Gr2, Corollaire 18.12.6]). However, both schemes are F-varieties of dimension $2n-1$, so φ must be an isomorphism. □

Corollary 2.1.1. *The motive of $X(A)$ decomposes as*

$$\mathbf{M}(X(A)) \cong \bigoplus_{i=0}^{n-1} \mathbf{M}(\mathrm{SB}(A))(i)$$

Proof. Apply Theorem 1.4.1 to $X(A) \cong \mathcal{F}$. □

2.1.2 The hyperplane section

Let A be a central simple algebra of degree $n+1$ over F and $\alpha \in A$ an element such that $F[\alpha]$ is an étale F-algebra of degree $n+1$. To such a pair (A, α), we associate a closed subscheme $Y(A, \alpha)$ of $X(A)$, defined by

$$Y(A,\alpha)(S) = \{[\mathcal{I}_1 \subseteq \mathcal{I}_n \subseteq \mathcal{A}_S] \in X(A)(S) : (\alpha \otimes 1)\mathcal{I}_1 \subseteq \mathcal{I}_n\}$$

To see that this gives a closed subscheme, note that the F-linear automorphism $m_\alpha : A \to A$, $x \mapsto \alpha x$ induces an automorphism α_* of $\mathbf{Gr}(A, n+1)$. $Y(A, \alpha)$ is then just the scheme-theoretic intersection

$$X(A) \cap (\alpha_* \times \mathrm{id}_{\mathbf{Gr}(A,n(n+1))})^{-1}(\mathbf{Fl}(A, n+1, n(n+1)))$$

in $\mathbf{Gr}(A, n+1) \times_F \mathbf{Gr}(A, n(n+1))$.

Lemma 2.1.3. *$Y(A,\alpha)$ is a smooth projective F-scheme of pure dimension $2n-2$.*

Proof. By Theorem 1.1.1, it is enough to show that $Y(A,\alpha) \times_F \bar{F} = Y(A \otimes_F \bar{F}, \alpha \otimes 1)$ is smooth over $\bar{F}$ and connected. $A \otimes_F \bar{F} \cong \mathrm{End}_{\bar{F}}(V)$ for some $n+1$-dimensional vector space V over $\bar{F}$. Under any such isomorphism, $\alpha \otimes 1$ maps to an invertible endomorphism β with $n+1$ distinct eigenvalues. Using Lemma 1.2.3, we have the following commutative diagram

$$\begin{array}{ccc} Y(A\otimes_F \bar{F},\alpha\otimes 1) & \hookrightarrow & \mathrm{SB}(A\otimes_F\bar{F})\times_{\bar{F}}\mathrm{SB}^\vee(A\otimes_F\bar{F}) \\ \downarrow\sim & & \downarrow\sim \\ Y(\mathrm{End}(V),\beta) & \hookrightarrow & \mathrm{SB}(\mathrm{End}(V))\times_{\bar{F}}\mathrm{SB}^\vee(\mathrm{End}(V)) \\ \downarrow\sim & & \downarrow\sim \\ Y & \hookrightarrow & \mathbf{P}(V)\times_{\bar{F}}\mathbf{P}^\vee(V) \end{array} \tag{2.1.1}$$

where Y is given by the closed subfunctor

$$Y(S)=\{(\mathcal{F}_1,\mathcal{F}_n)\in\mathbf{P}(V)(S)\times\mathbf{P}^\vee(V)(S):\mathcal{F}_1\subseteq\mathcal{F}_n,(\beta\otimes 1)\mathcal{F}_1\subseteq\mathcal{F}_n\},$$

that is the scheme-theoretic intersection of $X=\mathbf{Fl}(V,1,n)\hookrightarrow\mathbf{P}(V)\times_{\bar{F}}\mathbf{P}^\vee(V)$ and its image X' under the automorphism of $\mathbf{P}(V)\times_{\bar{F}}\mathbf{P}^\vee(V)$ by applying β^{-1} to the first factor. Both of these are reduced subschemes so are determined by their $\bar{F}$-points, from which it is easy to see that, letting $y_0,\dots,y_n$ be an eigenbasis for β with associated eigenvalues $\lambda_0,\dots,\lambda_n$, and $x_0,\dots,x_n$ the corresponding dual basis, X and X' are given by the homogeneous equations $\sum_{i=0}^n x_iy_i=0$ and $\sum_{i=0}^n\lambda_ix_iy_i=0$ respectively. Note that the λ_i are distinct since α (and hence β) generates an étale algebra of degree $n+1$. Consider the principal $\mathbf{G}_m\times_{\bar{F}}\mathbf{G}_m$-bundle

$$\pi:(\mathbf{A}^{n+1}_{\bar{F}}\setminus\{O\})\times_{\bar{F}}(\mathbf{A}^{n+1}\setminus\{O\})\to\mathbf{P}(V)\times_{\bar{F}}\mathbf{P}^\vee(V),$$

where the x_i pullback to standard coordinates of the first factor, and y_i those of the second (c.f. Example 1.3.14). By Lemma 1.3.2, it is enough to show that $Y'=\pi^{-1}(Y)$ is smooth of dimension $2n$. Retaining the same notations for coordinates, by Example 1.1.13, it is enough to show that the Jacobian matrix of the polynomials $\sum_{i=0}^n x_iy_i$ and $\sum_{i=0}^n\lambda_ix_iy_i$ has rank two at all closed points of Y'. For any $0\le i\ne j\le n$, the Jacobian has a 2×2 minor of the form

$$\begin{pmatrix} x_i & y_j \\ \lambda_ix_i & \lambda_jy_j\end{pmatrix}$$

The distinctness of the λ_k shows that this minor has rank 2 at a point P if $x_i(P),y_j(P)\neq 0$. For any closed point $P\in Y'$, the equation $\sum_{i=0}^n x_iy_j=0$ implies that there must be a pair (i,j) with $i\neq j$ such that $x_i(P),y_j(P)\neq 0$, thus the Jacobian matrix has rank 2 as desired. □

Since $X(A)$ is smooth and hence regular, this implies that $Y(A,\alpha)$ is an effective Cartier divisor in $X(A)$.

Lemma 2.1.4. *Let S be a complete variety over a field F, and let $\bar{F}$ be an algebraic closure of F. An invertible sheaf $\mathcal{L}$ on X is trivial if and only if $\pi^*\mathcal{L}$ is, where $\pi : X \times_F \bar{F} \to X$ is the projection onto the first factor. In particular, $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X \times_F \bar{F})$ is injective.*

Proof. See [Mi1, Lemma 6.2]. □

Proposition 2.1.1. *Let $\varphi : \mathbf{P}(\mathcal{F}) \to X(A)$ be the isomorphism from the proof of Theorem 2.1.1. $Y(A, \alpha)$ is of class $(\varphi^{-1})^*\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$.*

Proof. To begin, suppose F is algebraically closed and $A = \mathrm{End}(V)$, V an $n+1$ dimensional vector space over F. Then we may make the identifications $\mathrm{SB}(A) = \mathbf{P}(V)$ and $X(A) = \mathbf{Fl}(V, 1, n)$ by Lemma 1.2.3. Under these identifications, $Y(A, \alpha) = Y$ as in (2.1.1). By Lemma 2.1.1, we have an isomorphism $\mathcal{F} \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbf{P}(V)}}(\mathcal{Q}, \mathcal{S})$, where $\mathcal{S}$ is the tautological sub-bundle of $\mathbf{P}(V)$ and $\mathcal{Q}$ is the universal quotient bundle $\mathcal{Q} = (V \otimes \mathcal{O}_{\mathbf{P}(V)})/\mathcal{S}$. Under the identification $\mathbf{P}(\mathcal{F}) = \mathbf{P}^\vee(\mathcal{F}^\vee)$ (c.f. Example 1.1.17), $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is the universal quotient bundle of $\mathbf{P}^\vee(\mathcal{F}^\vee) = \underline{\mathrm{Proj}}(\mathrm{Sym}^\bullet \mathcal{F}^\vee)$. As $\mathcal{F}^\vee = \mathcal{S}^\vee \otimes \mathcal{Q}$, by [Ha2, Lemma II.7.9], there is an isomorphism $\psi : \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{Q}^\vee)$ such that

$$\psi^*\mathcal{O}_{\mathbf{P}(\mathcal{Q}^\vee)}(1) \otimes \pi^*\mathcal{S}^\vee = \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1) \tag{2.1.2}$$

where $\pi : \mathbf{P}(\mathcal{F}) \to \mathbf{P}(V)$ is the projection morphism. Under the identification $\mathbf{P}(\mathcal{Q}^\vee) = \mathbf{Fl}(V, 1, n)$ (see Proposition 1.1.11), ψ gives φ. Indeed, let $f : S \to \mathbf{P}(V)$ be a morphisms of F-schemes and $g \in \mathbf{P}(\mathcal{F})(S) = \mathbf{P}^\vee(\mathcal{F}^\vee)(S)$ be given by a quotient $f^*(\mathcal{Q} \otimes \mathcal{S}^\vee) \twoheadrightarrow \mathcal{L}$.

The computation of $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ in 2.1.2 shows that $\psi \circ g$ is given by the quotient $f^*\mathcal{Q} \twoheadrightarrow \mathcal{L} \otimes f^*\mathcal{S}$. From duality, it follows that the kernel of this quotient agrees locally with with the kernels of generators of $\mathcal{L}^\vee \hookrightarrow f^*(\mathcal{Q}^\vee) \otimes f^*\mathcal{S} = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(f^*\mathcal{Q}, f^*\mathcal{S})$. Since this is universal, φ agrees with ψ under our identifications.

Let $i : \mathbf{Fl}(V, 1, n) \hookrightarrow \mathbf{P}(V) \times_F \mathbf{P}^\vee(V)$ be the inclusion map, and let p_1, p_2 be the projection maps. Once more identifying $\mathbf{P}(\mathcal{Q}^\vee)$ with $\mathbf{Fl}(V, 1, n)$, $\mathcal{O}_{\mathbf{P}(\mathcal{Q}^\vee)}(1)$ corresponds to $i^*(p_2^*\mathcal{O}_{\mathbf{P}^\vee(V)}(1))$. The proof of Lemma 2.1.3 shows that Y is of class $i^*(p_1^*\mathcal{O}_{\mathbf{P}(V)}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^\vee(V)}(1))$, so using (2.1.2) we have that

$$\varphi^*\mathcal{O}(Y) \cong \varphi^*(p_1^*\mathcal{O}_{\mathbf{P}(V)}(1)) \otimes \psi^*\mathcal{O}_{\mathbf{P}(\mathcal{Q}^\vee)}(1) \cong \pi^*\mathcal{O}_{\mathbf{P}(V)}(1) \otimes \psi^*\mathcal{O}_{\mathbf{P}(\mathcal{Q}^\vee)}(1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$$

as desired.

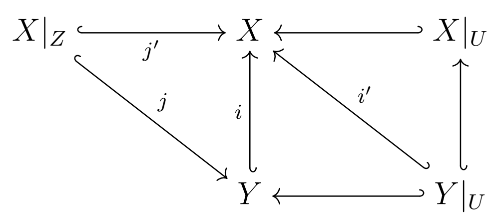
For the general case, by Lemma 2.1.4 it is enough to show it after extension to $\bar{F}$. Then we may fix a isomorphism $A \otimes_F \bar{F} \cong \mathrm{End}(V)$ and proceed as above. □

2.2 Chow groups of $\mathcal{O}(1)$-class divisors on projective bundles

In this section, we examine the situation of Proposition 2.1.1 and prove results on Chow groups strong enough to give a conditional motivic decomposition formula for $Y(A, \alpha)$.

2.2.1 Definitions and notations

Let F be an arbitrary field. We consider the following situation: B is a smooth projective variety over F with a locally free sheaf $\mathcal{E}$ of rank $r+1$. Set $X = \mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}^\vee)$ with projection map π. We will be concerned with sections $s \in H^0(B, \mathcal{E}^\vee) = H^0(X, \mathcal{O}(1))$ such that the zero locus Z of s is smooth of codimension $r+1$ in B and Y, the divisor corresponding to s, is smooth. Letting $U = B - Z$, $Y|_U := Y \times_B U$ is a projective bundle of rank $r-1$ over U (corresponding to the kernel $\mathcal{F}$ of $\mathcal{E} \xrightarrow{s} \mathcal{O}_B$, restricted to U) and $Y|_Z = X|_Z$ is a projective bundle of rank r over Z (corresponding to $\mathcal{E} \otimes \mathcal{O}_Z$). The following commutative diagram summarises our notation for the inclusion maps:



By the projective bundle theorem (Theorem 1.3.2), we have that $\mathrm{CH}^\bullet(X)$ is a free $\mathrm{CH}^\bullet(B)$-module generated by basis elements H_X^i, $H_X = c_1(\mathcal{O}(1))$ and $i = 0, \ldots, r$. The same holds for $\mathrm{CH}^\bullet(X|_Z)$, $\mathrm{CH}^\bullet(Z)$ and $H = j'^* H_X$. For $\mathrm{CH}^\bullet(Y|_U)$, $\mathrm{CH}^\bullet(U)$ and $H_Y = i'^* H_X$, it is true with $i = 0, \ldots, r-1$. The latter requires some explanation. By construction, $Y|_U$, as a U-scheme, is $\mathbf{P}(\mathcal{F}|_U)$. The inclusion into $X|_U = \mathbf{P}(\mathcal{E}|_U)$ comes from the surjective homomorphism $\mathrm{Sym}^\bullet \mathcal{E}^\vee|_U \twoheadrightarrow \mathrm{Sym}\, \mathcal{F}^\vee|_U$ induced by the inclusion $\mathcal{F} \subseteq \mathcal{E}$. Hence the invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{F}|_U)}(1)$ on $Y|_U$ is the pullback of $\mathcal{O}_{\mathbf{P}(\mathcal{E}|_U)}(1)$ by this inclusion (this follows from the local case given in [Ha2, Proposition II.5.12] part c). Since $\mathcal{O}_{\mathbf{P}(\mathcal{E}|_U)}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_U$, it follows that the H_Y^i $(i = 0, \ldots, r-1)$ give a $\mathrm{CH}^\bullet(U)$-basis of $\mathrm{CH}^\bullet(Y|_U)$.

Given a smooth projective F-scheme S, we can take the data $B, \mathcal{E}, s$ and associate to it the data $B \times_F S, p_1^*\mathcal{E}, p_1^*(s)$, where $p_1 : B \times_F S \to B$ is the projection onto the first factor. Then, applying the above constructions to $B \times_F S, p_1^*\mathcal{E}, p_1^*(s)$, we see that the schemes X, Y, Z are obtained from those constructed from $B, \mathcal{E}, s$ by taking a product with S. The same holds for morphisms and classes in the Chow rings. All of these operations will simply be called "base change by S".

Remark 2.2.1. It is harmless to take Z_{red} in this setup instead of Z. Indeed, U remains the same, and all other data are unaffected. The description of the "base change" of Z will still work, since a smooth variety S is geometrically reduced, hence $Z_{\mathrm{red}} \times_F S = (Z \times_F S)_{\mathrm{red}}$ by [SP, Lemma 035Z]. We will denote both by Z is the sequel.

2.2.2 Main result on the Chow groups

Define a group homomorphism $\varphi : \mathrm{CH}^\bullet(X) \oplus \mathrm{CH}^\bullet(X|_Z) \to \mathrm{CH}^\bullet(Y)$ by

$$\varphi = (i^* \ j_*)$$

Proposition 2.2.1. *With the same notations as above:*

i. *φ is surjective*

ii. *For every class $\gamma \in \mathrm{CH}^\bullet(Y)$, there exist $\alpha_0, \ldots, \alpha_{r-1} \in \mathrm{CH}^\bullet(B)$ and $\beta \in \mathrm{CH}^\bullet(Z)$ such that*

$$\varphi(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i, \pi|_Z^*\beta) = \gamma$$

iii. *If $j_* \circ (\pi|_Z)^*$ is injective, then such elements are unique.*

Proof. **i.** By the right exact sequence $\mathrm{CH}^\bullet(X|_Z) \xrightarrow{j_*} \mathrm{CH}^\bullet(Y) \to \mathrm{CH}^\bullet(Y|_U) \to 0$ of Proposition 1.3.3, we are reduced to showing that i'^* is surjective. $\mathrm{CH}^\bullet(Y|_U)$ is generated (as a ring) by $(\pi|_U)^*\mathrm{CH}^\bullet(U)$ and H_Y. Clearly H_Y is in the image of i'^* and the commutativity of

$$\begin{array}{ccc} \mathrm{CH}^\bullet(X) & \xrightarrow{i'^*} & \mathrm{CH}^\bullet(Y|_U) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi|_U^*} \\ \mathrm{CH}^\bullet(B) & \longrightarrow & \mathrm{CH}^\bullet(U) \end{array}$$

and the surjectivity of the restriction $\mathrm{CH}^\bullet(B) \to \mathrm{CH}^\bullet(U)$ shows that $(\pi|_U)^*\mathrm{CH}^\bullet(U)$ is also in the image.

ii. That we can eliminate positive powers of H in the $\mathrm{CH}^\bullet(X|_Z)$ argument of φ follows from the equality $j_*(H \cdot \alpha) = i^*(j'_*(\alpha))$. This identity holds since $i^* \circ j'_* = (i^* \circ i_*) \circ j_*$ and since Y is a divisor we have $(i^* \circ i_*)(\beta) = i^*(H_X) \cdot \beta$ ([Fu, Proposition 2.6] part c). But by the projection formula, $i^*H_X \cdot j_*(\alpha) = j_*(j^*(i^*(H_X)) \cdot \alpha) = j_*(H \cdot \alpha)$.

Now let $\alpha \in \mathrm{CH}^\bullet(B)$, then $i'^*(\pi^*(\alpha) \cdot H_X^r) = \sum_{i=0}^{r-1} (\pi|_U)^*\gamma_i \cdot H_Y^i$ by the projective bundle theorem, hence by the proof of **i.** there are elements $\alpha_0, \ldots, \alpha_{r-1}$ such that $i'^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i) = \sum_{i=0}^{r-1} (\pi|_U)^*\gamma_i \cdot H_Y^i$. Hence $i'^*(\pi^*\alpha \cdot H_X^r - \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i) = 0$, so $i^*(\pi^*\alpha \cdot H_X^r) - i^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i) \in \operatorname{Im} j_*$. So there are $\beta_0, \ldots, \beta_r$ such that $\varphi(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i, \sum_{j=0}^{r} (\pi|_Z)^*\beta_j \cdot H^i) = i^*(\pi^*\alpha \cdot H_X^r)$. Eliminating the positive powers of H as above will then give an element of the desired form.

iii. Elements of the form given in **ii.** form a subgroup in $\mathrm{CH}^\bullet(X) \oplus \mathrm{CH}^\bullet(X|_Z)$, so we just need to prove $\ker\varphi$ meets this subgroup trivially. If $\varphi(x) = 0$, then $i_*(\varphi(x)) = 0$. We have $i_*(i^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i)) = \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^{i+1}$. Let $\hat{j} : Z \hookrightarrow B$ denote the inclusion of Z in B. We have $i_*(j_*((\pi|_Z)^*\beta)) = j'_*((\pi|_Z^*\beta)) = \pi^*(\hat{j}_*\beta)$ since π is flat and

$$\begin{array}{ccc} X|_Z & \xhookrightarrow{\;j'\;} & X \\ \Big\downarrow{\scriptstyle \pi|_Z} & & \Big\downarrow{\scriptstyle \pi} \\ Z & \xhookrightarrow{\;\hat{j}\;} & B \end{array}$$

is a fibre square (by definition!). Putting these two facts together, we find that

$$\left(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^i, (\pi|_Z)^*\beta \right) \in \ker\varphi \implies \pi^*(\hat{j}_*\beta) + \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_X^{i+1} = 0$$

Since $1, H_X, \ldots, H_X^r$ are a $\mathrm{CH}^\bullet(B)$-linear basis, this implies $\alpha_i = 0$ for $i = 0, \ldots, r-1$ and $j_*(\pi|_Z^*\beta) = 0$, hence by hypothesis $\beta = 0$, so the intersection with the kernel is trivial as desired. □

2.2.3 Criterion for motivic decomposition

Let $x = i^*(H_X) \in \mathrm{CH}^1(Y)$, $f = \pi \circ i$ and $g = \pi|_Z$. Then we have correspondences:

$$\begin{aligned} &c_x \in \mathrm{Hom}^1(\mathbf{M}(Y), \mathbf{M}(Y)), \qquad && c_f \in \mathrm{Hom}(\mathbf{M}(B), \mathbf{M}(Y)), \\ &c_g \in \mathrm{Hom}(\mathbf{M}(Z), \mathbf{M}(X|_Z)), \qquad && c_j^t \in \mathrm{Hom}^r(\mathbf{M}(X|_Z), \mathbf{M}(Y)) \end{aligned}$$

(c.f. Proposition 1.4.5) and we define for $0 \le i \le r-1$ correspondences

$$f_i = c_x^{(i)} \circ c_f \in \mathrm{Hom}(\mathbf{M}(B)(i), \mathbf{M}(Y)), \qquad f' = c_j^t \circ c_g \in \mathrm{Hom}(\mathbf{M}(Z)(r), \mathbf{M}(Y))$$

and a morphism

$$\psi : M := \bigoplus_{i=0}^{r-1} \mathbf{M}(B)(i) \oplus \mathbf{M}(Z)(r) \to \mathbf{M}(Y), \qquad \psi = (f_0 \ \ldots \ f_{r-1} \ f')$$

For S a smooth projective F-scheme, let $\psi_S : \mathrm{Hom}^\bullet(\mathbf{M}(S), M) \to \mathrm{Hom}^\bullet(\mathbf{M}(S), \mathbf{M}(Y))$.

There are canonical isomorphisms

$$\mathrm{Hom}^\bullet(\mathbf{M}(S), M) = \bigoplus_{i=0}^{r-1} \mathrm{CH}^\bullet(B \times_F S) \oplus \mathrm{CH}^\bullet(Z \times_F S), \qquad \mathrm{Hom}^\bullet(\mathbf{M}(S), \mathbf{M}(Y)) = \mathrm{CH}^\bullet(Y \times_F S)$$

Then, by Lemma 1.4.2, ψ_S factors through the φ of Proposition 2.2.1 defined from the data after "base change" by S, such that $\mathrm{Hom}^\bullet(\mathbf{M}(S), M)$ maps isomorphically onto the distinguished subgroup $C \subseteq \mathrm{CH}^\bullet(X \times_F S) \oplus \mathrm{CH}^\bullet(X|_Z \times_F S)$ of elements of the form described in Proposition 2.2.1, part ii). Combining part iii) of Proposition 2.2.1 and Proposition 1.4.6, we obtain

Corollary 2.2.1. *If $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ is injective for all smooth projective F-schemes S, then ψ is an isomorphism.*

2.3 Motivic decomposition of the hyperplane section

In this section we prove the main theorem of this thesis:

Theorem 2.3.1. *Let A and α be as in Section 2.1.2. Additionally, assume that $L = F(\alpha)$ is a Galois extension of F of degree $n+1$. The motive of $Y(A, \alpha)$ decomposes as*

$$\mathbf{M}(Y(A,\alpha)) \cong \bigoplus_{i=0}^{n-2} \mathbf{M}(\mathrm{SB}(A))(i) \oplus \mathbf{M}(\operatorname{Spec} L)(n-1)$$

The proof amounts to an application of Corollary 2.2.1, proved in the previous section. We retain the notations of the theorem for the rest of this section, but not the extra assumption on L unless specified.

2.3.1 Determining the zero locus

By Proposition 2.1.1, $Y(A, \alpha)$ is a smooth effective Cartier divisor on the projective bundle $\pi : X(A) \to \mathrm{SB}(A)$ for a locally free sheaf $\mathcal{E}$ of rank n on $\mathrm{SB}(A)$. Write $\mathcal{O}(1)$ for the twisting sheaf on $X(A)$. Let $s \in H^0(X(A), \mathcal{O}(1)) = H^0(\mathrm{SB}(A), \mathcal{E}^\vee)$ be a section corresponding to $Y(A, \alpha)$. The choice does not matter since they only differ by multiplication by a unit of F.

Proposition 2.3.1. *The zero locus of s, $Z \subseteq \mathrm{SB}(A)$, is isomorphic to* $\operatorname{Spec} L$.

Proof. First, assume $A \cong \mathrm{End}(V)$ is split and let α' be the image of α in $\mathrm{End}(V)$. Extending to the closure $\bar{F}$, we have a commutative diagram

$$\begin{array}{ccccc}
X(A) \times_F \bar{F} & \xrightarrow{\sim} & \mathbf{Fl}(\bar{V}, 1, n) & \hookrightarrow & \mathbf{P}(\bar{V}) \times_{\bar{F}} \mathbf{P}^\vee(\bar{V}) \\
\downarrow{\scriptstyle \pi \times \mathrm{id}_{\bar{F}}} & & \downarrow & \swarrow{\scriptstyle p_1} & \\
\mathrm{SB}(A) \times_F \bar{F} & \xrightarrow{\sim} & \mathbf{P}(\bar{V}) & &
\end{array}$$

where $\bar{V} = V \otimes_F \bar{F}$. The section s then corresponds to $\sum_{i=0}^n \lambda x_i y_i$, where the λ_i are the eigenvalues of α', the y_i are the corresponding eigenbasis, and the x_i are the dual basis (c.f. Lemma 2.1.3). The equation $\sum_{i=0}^n x_i y_i = 0$ cuts out the embedding of $\mathbf{Fl}(\bar{V}, 1, n)$ in $\mathbf{P}(\bar{V}) \times_{\bar{F}} \mathbf{P}^\vee(\bar{V})$. Thus the closed points of the zero locus of s are those $P \in \mathbf{P}(\bar{V})(\bar{F})$ where $\sum_{i=0}^n \lambda_i x_i y_i$ and $\sum_{i=0}^n x_i y_i$ become colinear. Since the λ_i are distinct, these are the finitely many points P such that $x_i = 0$ for all but one $0 \le i \le n$.

We conclude that Z is a finite F-scheme, whose $\bar{F}$-points are correspond to the eigenspaces of α' in $\mathbf{P}(V)(\bar{F})$. By Remark 2.2.1, it is harmless to assume that Z is reduced, and so $Z \cong \operatorname{Spec} R$, R an étale F-algebra of degree $n+1$. The action of $\operatorname{Gal}(F^{sep}/F)$ on $Z(\bar{F})$ by an element σ sends the eigenspace of λ_i to that of $\sigma(\lambda_i)$. Therefore we have a $\operatorname{Gal}(F^{sep}/F)$-equivariant bijection between $\operatorname{Hom}_F(L, \bar{F})$ and $\operatorname{Hom}_F(R, \bar{F})$, so Theorem 1.2.1 implies that $L \cong R$, and so $Z \cong \operatorname{Spec} L$.

If A is not split, we still have that Z is finite and reduced. Let K be the function field of $\operatorname{SB}(A)$. Then $A \otimes_F K$ is split (Example 1.2.7), and applying the same argument to $K \otimes_F L \cong K(\alpha \otimes 1) \subseteq A \otimes_F K$, we have that $Z \times_F K \cong \operatorname{Spec}(L \otimes_F K)$. Since $\operatorname{SB}(A)$ is geometrically integral, K is a geometrically integral F-algebra ([SP, Lemma 054Q] and [SP, Lemma 04KN]), so Lemma 2.3.1 concludes the proof. □

Lemma 2.3.1. *Let F be a field and K a geometrically integral field extension of F. Let R, R' be two finite reduced F-algebras. $R \otimes_F K \cong R' \otimes_F K$ as K-algebras if and only if $R \cong R'$ as F-algebras.*

Proof. One direction is clear. Suppose that $R \otimes_F K \cong R' \otimes_F K$. Finite reduced algebras over a field are direct products of field extensions. Since K is geometrically integral, direct factors of R and R' are in one-to-one correspondence with the direct factors of $R \otimes_F K$ and $R \otimes_F K$. Finite direct products of field extensions are isomorphic if and only if they have the same number of factors in each isomorphism class, so we are reduced to the case where R and R' are fields.

Since an isomorphism $R \otimes_F K \cong R' \otimes_F K$ of K-algebras is also an isomorphism of F-algebras, the algebraic closures of F in both of these fields are isomorphic over F. It then suffices to show that for any finite field extension E of F, the image of the inclusion $E \hookrightarrow E \otimes_F K$ is algebraically closed. Suppose it were not, then there is a non-trivial finite extension $E \subsetneq E' \subseteq E \otimes_F K$. Let $\bar{E}$ be an algebraic closure of E. The algebra $\bar{E} \otimes_E E'$ is not an integral domain, and we have an inclusion $\bar{E} \otimes_E E' \hookrightarrow \bar{E} \otimes_F K$. As $\bar{E} \otimes_F K$ is an integral domain by geometric integrality, we obtain a contradiction. □

2.3.2 Verifying the criterion

To begin, suppose $A = \operatorname{End}(V)$ is split and all the eigenvalues of α are in F. In this case, we will simply write X for $X(A)$ and Y for $Y(A, \alpha)$. The proof of Proposition 2.3.1 shows that $Z = \coprod_{i=0}^{n} \{z_i\}$, where the z_i are the F-rational points in $\mathbf{P}(V)$ corresponding to the eigenspaces of α. Let $\pi : X \to \mathbf{P}(V)$ be the projection map, and let $E_i = \pi^{-1}(z_i)$ be the fibres. By the definition of Z, $E_i \subseteq Y$ for $0 \leq i \leq n$. Since $\pi : X \to \mathbf{P}(V)$ is a projective bundle of relative dimension $n-1$, each E_i is isomorphic to $\mathbf{P}_F^{n-1}$ and $[E_i] \in \operatorname{CH}^{n-1}(Y)$.

Proposition 2.3.2. *For $0 \leq i, j \leq n$, $[E_i] \cdot [E_j] \in \mathrm{CH}^{2n-2}(Y) = \mathrm{CH}_0(Y)$, and $\deg([E_i] \cdot [E_j]) = (-1)^{n-1}\delta_{ij}$, with δ_{ij} the Kronecker delta function.*

We defer the proof of Proposition 2.3.2 to the next section.

Definition 2.3.1. Let S be a smooth projective F-scheme and U a smooth projective variety. Let $p : U \times_F S \to S$ be the projection map. Define a pairing $\langle \cdot, \cdot \rangle_S : \mathrm{CH}^\bullet(U \times_F S) \times \mathrm{CH}^\bullet(U \times_F S) \to \mathrm{CH}^\bullet(S)$ by $\langle \alpha, \beta \rangle_S = p_*(\alpha\beta)$.

Lemma 2.3.2. *This pairing is $\mathrm{CH}^\bullet(S)$-bilinear with respect to the $\mathrm{CH}^\bullet(S)$-module structure on $\mathrm{CH}^\bullet(U \times_F S)$ given by p^*. Additionally, for $\alpha, \beta \in \mathrm{CH}^\bullet(U)$, $\langle \alpha \times 1_S, \beta \times 1_S \rangle_S = p'_*(\alpha\beta) \times 1_S$, where $p' : U \to \operatorname{Spec} F$ is the structural morphism.*

Proof. Bilinearity follows from the projection formula. By definition, we have the fibre square,

$$\begin{array}{ccc} U \times_F S & \xrightarrow{\;p\;} & S \\ \downarrow & & \downarrow \\ U & \xrightarrow{\;p'\;} & \operatorname{Spec} F \end{array}$$

with p' proper and $S \to \operatorname{Spec} F$ flat. By [Fu, Proposition 1.7], $p'(\gamma) \times 1_S = p_*(\gamma \times 1_S)$ for any $\gamma \in \mathrm{CH}^\bullet(U)$. The desired identity follows from the case $\gamma = \alpha\beta$. □

Proposition 2.3.3. *Let $j : X|_Z \hookrightarrow Y$ denote the inclusion map. The homomorphism $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ is injective for any smooth projective F-scheme S.*

Proof. We have that $\mathrm{CH}^\bullet(Z \times_F S) = \bigoplus_{0 \leq i \leq n} \mathrm{CH}^\bullet(S)$, with the images of the classes of $\{z_i\} \times_F S$ under $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ being the classes of $E_i \times_F S$ in $\mathrm{CH}^\bullet(Y \times_F S)$ $(i = 0, \ldots, n)$. Indeed, this follows from the definitions of flat pullbacks and proper pushforwards. Let $\gamma_i = [E_i \times_F S] = [E_i] \times 1_S$ for $0 \leq i \leq n$. Since $j \times \mathrm{id}_S$ and $\pi|_Z \times \mathrm{id}_S$ are S-morphisms, the functoriality of pullbacks and the projection formula show that the homomorphism $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ is $\mathrm{CH}^\bullet(S)$-linear. Therefore, to show injectivity it suffices to show that the γ_i are $\mathrm{CH}^\bullet(S)$-linearly independent in $\mathrm{CH}^\bullet(Y \times_F S)$. For $0 \leq i, j \leq n$, $\langle \gamma_i, \gamma_j \rangle_S = \deg([E_i] \cdot [E_j]) \cdot 1_S$ by Lemma 2.3.2. This is nothing but $\delta_{ij}(-1_S)^{n-1}$ by Proposition 2.3.2. This shows linear independence of the γ_i. □

Proof of Theorem 2.3.1. In view of Proposition 2.1.1 and Proposition 2.3.1, it suffices to verify the criterion given in Corollary 2.2.1. Let S be a smooth projective F-scheme.

Applying base change by L/F, we obtain the commutative diagram of Cartesian squares:

$$\begin{array}{ccccc} Z_L \times_L S_L & \longleftarrow & (X(A)|_Z)_L \times_L S_L & \longrightarrow & Y(A,\alpha)_L \times_L S_L \\ \downarrow & & \downarrow & & \downarrow \\ Z \times_F S & \xleftarrow[\pi|_Z \times \mathrm{id}_S]{} & X(A)|_Z \times_F S & \xrightarrow{j \times \mathrm{id}_S} & Y(A,\alpha) \times_F S \end{array}$$

where we write S_L for $S \times_F L$ and so forth. This induces the commutative diagram on Chow groups

$$\begin{array}{ccccc} \mathrm{CH}^\bullet(Z_L \times_L S_L) & \longrightarrow & \mathrm{CH}^\bullet((X(A)|_Z)_L \times_L S_L) & \longrightarrow & \mathrm{CH}^\bullet(Y(A,\alpha)_L \times_L S_L) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{CH}^\bullet(Z \times_F S) & \xrightarrow{(\pi|_Z \times \mathrm{id}_S)^*} & \mathrm{CH}^\bullet(X(A)|_Z \times_F S) & \xrightarrow{(j \times \mathrm{id}_S)_*} & \mathrm{CH}^\bullet(Y(A,\alpha) \times_F S) \end{array}$$

The left-hand vertical map is injective. Indeed, by Proposition 2.3.1, $Z \cong \operatorname{Spec} L$. Since $L \otimes_F L \cong \prod_{0 \le i \le n} L$, we need only check that the induced map $\mathrm{CH}^\bullet(S_L) \to \bigoplus_{0 \le i \le n} \mathrm{CH}^\bullet(S_L)$ is injective, which is clear. It follows that if the composite of the top row is injective, then the composite of the bottom row is as well, which is $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$. So we are reduced to showing that the composite of the upper row is injective.

The upper row is precisely the situation of Corollary 2.2.1 for $X(A \otimes_F L)$ and $(Y(A \otimes_F L, \alpha \otimes 1))$ in the case where S_L is the smooth projective L-scheme. By Proposition 1.2.9, $A \otimes_F L$ is split. Fix an isomorphism $A \otimes_F L \cong \mathrm{End}_L(W)$. Then by the definition of L, the image of $\alpha \otimes 1$ in $\mathrm{End}_L(W)$ has all of its eigenvalues in L. Thus, injectivity follows from Proposition 2.3.3. □

2.3.3 Localisation under the torus action

In this section, we use the localisation theorems of Section 1.3.3 to prove Proposition 2.3.2.

As before, let $A = \mathrm{End}(V)$ be split and let all eigenvalues of α lie in F. We use the same notations $X = X(A)$, $Y = Y(A,\alpha)$, E_i, etc. Since we are interested only in computing the degree of a zero cycle, we may first make a base change to $\bar{F}$. Thus, for the rest of this section we assume that F is algebraically closed. Let $V = \bigoplus_{i=0}^n V_i$ be the decomposition of V into the one-dimensional eigenspaces of α. For such a decomposition, there is a unique torus $T \subseteq \mathrm{GL}(V)$ such that for all $t \in T(F)$, $t(V_i) \subseteq V_i$ for each $0 \le i \le n$. Let $t_i \in X(T)$ denote the weight of V_i.

Through $\mathrm{GL}(V)$, T has a left action on $\mathbf{P}(V)$ and $\mathbf{P}^\vee(V)$, namely on closed points these are given by, for $W_1, W_n \subseteq V$ ($\dim_F W_j = j$), $t \cdot [W_1] = [t(W_1)] \in \mathbf{P}(V)(F)$ and $t \cdot [W_n] = [t(W_n)] \subseteq \mathbf{P}^\vee(V)(F)$. Note that this second action, when identifying $\mathbf{P}^\vee(V)$ with $\mathbf{P}(V^\vee)$, is the first action but with the elements of $T(F)$ acting by their transpose inverse.

It is clear that the embedding of X as a subvariety of $\mathbf{P}(V) \times_F \mathbf{P}^\vee(V)$ is T-stable. Moreover, since α commutes with all $t \in T(F)$ (by definition of T), we have that Y is T-stable for the induced T-action on X. Similarly, as the E_i are nothing but $\pi^{-1}([V_i])$ and by definition the $[V_i]$ are the T-fixed points of $\mathbf{P}(V)$, the E_i are also T-stable and together (i.e., $X|_Z$) contain all the T-fixed points of X. In particular, these fixed points are precisely those corresponding to the points $([V_i], [V^j]) \in \mathbf{P}(V)(F) \times \mathbf{P}^\vee(V)(F)$, where $V^j = \bigoplus_{0 \leq k \neq j \leq n} V_k$, with $i \neq j$. We shall denote them by $z_{ij} \in X$.

Let $R = \mathrm{Sym}^\bullet X(T) = \mathbf{Z}[t_0, \cdots, t_n]$ and Q be its field of fractions. Let $\chi_{ij} = t_j - t_i \in X(T)$.

Lemma 2.3.3. *For each $0 \leq i \neq j \leq n$, the points z_{ij} are non-degenerate T-fixed points for both Y and $X|_Z$. In particular,* $\mathrm{Tan}_{z_{ij}}(Y)$ *has weights χ_{ki} and χ_{ik}, $0 \leq k \leq n$, $k \neq i, j$, and* $\mathrm{Tan}_{z_{ij}}(X|_Z)$ *has weights χ_{kj}, $0 \leq k \neq j \leq n$.*

Proof. We may consider Y and $X|_Z$ are closed subvarieties of $\mathbf{P}(V) \times_F \mathbf{P}^\vee(V)$. For fixed $i \neq j$, for any $k \neq i, j$, the codimension 2 subspace $V^k \cap V^j$ corresponds to a T-stable projective line $L_{kj} \subseteq \mathbf{P}^\vee(V)$. Clearly, $C_{kj} = \{[V_i]\} \times L_{kj} \subseteq Y$ is T-stable and it is an easy computation that $T(F)$ acts on $\mathrm{Tan}_{z_{ij}}(C_{kj})$ by χ_{kj}. Indeed, in affine local coordinates the T-action on C_{kj} is given by

$$\left(t, \frac{x_j}{x_k}\right) \mapsto \frac{t_j^{-1} x_j}{t_k^{-1} x_k} = \frac{t_k}{t_j} \cdot \frac{x_j}{x_k}$$

Similarly, one defines a line $L_{ik} \subseteq \mathbf{P}(V)$ corresponding to $V_i \oplus V_k$, and sets $C_{ik} = L_{ik} \times \{[V^j]\} \subseteq Y$. Once again, using affine local coordinates, it is easily verified that $T(F)$ acts on $\mathrm{Tan}_{z_{ij}}(C_{ik})$ by χ_{ik}. All of these curves are subvarieties of Y, so $\mathrm{Tan}_{z_{ij}}(Y)$ contains $2n-2$ one-dimensional subrepresentations via the inclusions of the tangent spaces of the curves. As they all have distinct weights, they give a direct sum decomposition of $\mathrm{Tan}_{z_{ij}}(Y)$ as a $T(F)$-representation. This proves the claim for Y. For $X|_Z$, only the curves C_{kj} are contained in it. Since $X|_Z$ is of pure dimension $n-1$, these $n-1$ curves are sufficient to make the same argument as for Y and give the claimed weights. □

Definition 2.3.2. Let U be a projective non-singular T-variety over F with structural morphism p. Define the pairing $\langle \cdot, \cdot \rangle_T : \mathrm{CH}_T^\bullet(U) \times \mathrm{CH}_T^\bullet(U) \to R$ by $p_*(\alpha\beta) = \langle \alpha, \beta \rangle_T \cdot 1 \in \mathrm{CH}_T^\bullet(\mathrm{Spec}\, F)$.

Recall the forgetful homomorphism $\mathrm{CH}^\bullet_T(U) \to \mathrm{CH}^\bullet(U)$ from Section 1.3.3. We write the image of a class $\alpha \in \mathrm{CH}^\bullet_T(U)$ by this map as $\overline{\alpha}$.

Lemma 2.3.4. *For $\alpha, \beta \in \mathrm{CH}^\bullet_T(U)$, $\overline{\langle \alpha, \beta \rangle_T} = p_*(\overline{\alpha}\overline{\beta})$. In particular, if $\dim U = 2r$ and $\alpha, \beta \in \mathrm{CH}^r_T(U)$, $\overline{\langle \alpha, \beta \rangle_T} = \deg(\alpha\beta)$.*

Proof. Immediate from Proposition 1.3.16 and the definition of the degree map. □

Lemma 2.3.5. *For $\alpha \in \mathrm{CH}^\bullet_T(Y)$ and $0 \leq i \neq j \leq n$, let α_{ij} be the pullback of α by the inclusion $\{z_{ij}\} \hookrightarrow Y$. We have the following identities:*

$$e_{z_{ij},Y}(\alpha) = \frac{\alpha_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \tag{2.3.1}$$

$$\langle \alpha, \beta \rangle_T = \sum_{0 \leq i \neq j \leq n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \tag{2.3.2}$$

Proof. For (2.3.1), let $\iota_{ij} : \{z_{ij}\} \hookrightarrow Y$, $\iota : Y^T \hookrightarrow Y$ be the inclusion maps. By Theorem 1.3.4, Corollary 1.3.3 and Lemma 2.3.3,

$$[Y] = \sum_{0 \leq i \neq j \leq n} \frac{1}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}}[z_{ij}], \qquad \alpha = \sum_{0 \leq i \neq j \leq n} e_{z_{ij},Y}(\alpha)[z_{ij}]$$

in $\mathrm{CH}^\bullet_T(Y) \otimes_R Q$. Using the identification $\mathrm{CH}^\bullet_T(Y^T) \otimes_R Q = \bigoplus_{0 \leq i \neq j \leq n} Q$ coming from the inclusion of each fixed point into Y^T (c.f. Theorem 1.3.3 and Proposition 1.3.15), we can rewrite these equalities as

$$[Y] = \iota_* \left(\frac{1}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \right)_{ij}, \qquad \alpha = \iota_* \left(e_{z_{ij},Y}(\alpha) \right)_{ij}$$

But $\alpha = \alpha \cdot [Y]$, so by the projection formula we have

$$\iota_* \left(\frac{\alpha_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \right)_{ij} = \iota_* \left(e_{z_{ij},Y}(\alpha) \right)_{ij}$$

By Theorem 1.3.3, ι_* is an isomorphism after tensoring with Q, so the equality follows.

For (2.3.2), since $R \subseteq Q$, it is enough to compute after localising. By (2.3.1),

$$\alpha\beta = \sum_{0 \leq i \neq j \leq n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \iota_{ij*}(1)$$

Now, $p_* \circ \iota_{ij*}$ is the identity on Q since $p \circ \iota_{ij}$ is a map of a point to itself. Thus, by linearity we have

$$\langle \alpha, \beta \rangle_T = p_*(\alpha\beta) = \sum_{0 \leq i \neq j \leq n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}}$$

□

Proof of Proposition 2.3.2. By Lemma 2.3.4, it is enough to show that for $0 \leq i, j \leq n$, $\langle [E_i], [E_j] \rangle_T = \delta_{ij}(-1)^{n-1}$. By Lemma 2.3.3, $e_{z_{ij},Y}([E_i]) = (\prod_{l \neq i,j} \chi_{lj})^{-1}$. Hence by Lemma 2.3.5, $\langle [E_i], [E_j] \rangle_T = 0$ when $i \neq j$ (since E_i and E_j share no T-fixed points and $\alpha_{ij} = 0 \iff e_{z_{ij},Y}(\alpha) = 0$ by (2.3.1)) and

$$\langle [E_i], [E_i] \rangle_T = \sum_{s \neq i} \frac{\prod_{l \neq i,s} \chi_{il}}{\prod_{l \neq i,s} \chi_{ls}}$$

This is seen to be $(-1)^{n-1}$ by the following observation in [XZ, Lemma 4.2]: treating R as a polynomial ring in t_i over $\mathbf{Z}[t_0, \ldots, \hat{t}_i, \ldots, t_n]$, by Lagrange interpolation it is enough to show that the polynomial

$$f(t_i) = \sum_{s \neq i} \frac{\prod_{l \neq i,s} \chi_{il}}{\prod_{l \neq i,s} \chi_{ls}}$$

of degree at most $n-1$ evaluated at t_j for each $j \neq i$ is $(-1)^{n-1}$. Clearly, $\prod_{l \neq i,s} \chi_{il}$ evaluated at t_j is 0 if $j \neq s$, thus

$$f(t_j) = \frac{\prod_{l \neq i,j} \chi_{jl}}{\prod_{l \neq i,j} \chi_{lj}} = (-1)^{n-1}$$

for $j \neq i$. □

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