Formal and Non-Formal Homogeneous Spaces of Small Rank

Paul-Eugène Parent

A M.Sc. Thesis

submitted to the School of Graduate Studies and Research

in partial fulfilment of the requirements for

the Master's degree in Mathematics*

University of Ottawa

Ottawa, Ontario

Canada

1996

*The M.Sc. Program is a joint program with

Carleton University, administered by the Ottawa-Carleton

Institute of Mathematics and Statistics

National Library of Canada
Bibliothèque nationale du Canada

Acquisitions and Bibliographic Services Branch
Direction des acquisitions et des services bibliographiques

395 Wellington Street
Ottawa, Ontario
K1A 0N4

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

ISBN 0-612-15751-2

Canada

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

M.Sc. Thesis

Paul-Eugène Parent

Abstract

Formal and Non-Formal Homogeneous Spaces of Small Rank

The aim of this thesis is to determine which 2-tori T make G/T formal for a compact connected Lie group G of rank 3. We show that the only time there is a possibility of a non-formal homogeneous space G/T is when the Lie algebra E of G is semisimple and contains three simple ideals. In such a case, the Koszul complex is given by

$$(\Lambda(y_2, z_2, x_3, x_3', x_3''), d)$$

and

$$dx_3 = -y_2^2, \quad dx_3' = -z_2^2, \quad \text{and} \quad dx_3'' = -(\alpha y_2 + \beta z_2)^2,$$

where $\alpha, \beta \in \mathbf{Q}$. We prove

Theorem 5.5.4: *This minimal c.g.d.a is formal if and only if $\alpha = 0$ or $\beta = 0$.*

This, as we will see, indicates, in the case of non-formality, a special mixing of the 2-torus inside G.

University of Ottawa, June 1996.

À grand-papa Jean-Paul,

"...mais à qui d'autre voulez-vous qu'il l'ait donnée?"

Jean-Paul Vandal, 1986.

Introduction

The aim of algebraic topology is to find enough algebraic invariants of a given topological space (i.e. homotopy, homology,...) such that ultimately one could classify those spaces up to homeomorphism. This, for example, was done in the case of compact connected 2-manifolds with empty boundaries.

One such invariant is the *real homotopy type* of a differentiable manifold M. One starts by first associating to M its *commutative graded differential algebra* ($c.g.d.a$) of smooth forms $(\mathcal{A}^*(M), d)$: the de Rham complex. Then, two smooth manifolds M and N are said to have the same real homotopy type, if there exists a chain of $c.g.d.a$'s and homomorphisms

$$(\mathcal{A}^*(M), d) \longrightarrow (A_1, \delta_1) \longleftarrow \ldots \longrightarrow (A_n, \delta_n) \longleftarrow (\mathcal{A}^*(N), d)$$

which, at each stage, induce an isomorphism on the level of cohomology. Thus two manifolds of the same real homotopy type must have isomorphic cohomology. The converse is in general false.

A smooth manifold M is *formal* if its real homotopy type is a formal consequence of its rational cohomology. To be more precise, M is formal if there is a $c.g.d.a$ homomorphism $(\mathcal{A}^*(M), d) \xrightarrow{\phi} (\mathcal{H}(M), 0)$ such that ϕ induces the identity on cohomology.

Many interesting spaces (Lie groups, spheres, projective spaces and their products) are formal, but not all homogeneous spaces are. For homogeneous spaces of compact connected Lie groups, one knows that G/H is formal iff G/T is, where T is a maximal torus in H.

Recall that the *rank* of a compact connected Lie group is the dimension of an imbedded maximal torus. The aim of this thesis is to determine all tori T of an arbitrary compact connected Lie group G of rank 3 for which G/T is formal. One knows that if $dim\, T = 1$ or 3, this is automatic, so it is a question of determining all the possible imbedded 2-tori T which make G/T formal.

Throughout the exposition we will be working over $\mathbf{R}$. In Chapter 1, we introduce the basic notions on graded algebras, Hopf algebras, and Lie algebras. We will show that if E is a Lie algebra, then the exterior algebra over E^* can be given a *c.g.d.a* structure. Thus we can define the cohomology of such an algebra. Finally, for E reductive, we obtain

$$\mathcal{H}(E) \cong \Lambda P_E,$$

where P_E is a graded vector space concentrated in odd degrees called the *primitive subspace* associated to E. We will see that this vector space is crucial in solving our problem.

We present the de Rham complex in Chapter 2, and also give some information on tangent bundles and vector fields.

In Chapter 3, we describe the important properties of our main object of study: Lie groups. We will compute their cohomology in the compact connected case. Compact connected Lie groups are formal spaces par excellence.

We return in Chapter 4 to the notions of real homotopy type and formality. Then, given a compact connected Lie group G and a closed Lie subgroup H with associated Lie algebras E and F respectively, one can construct a *c.g.d.a*, called the Koszul complex,

$$(\Lambda Q_F \otimes \Lambda P_E, d)$$

where P_E is the primitive subspace associated to the Lie algebra E of G, $Q_F \cong P_F$ but where the gradation is suspended up by one, and d is given by

$$d(Q_F) = 0, \quad \text{and} \quad d(P_E) \subset \Lambda Q_F.$$

The crucial results are that there is a homomorphism $(\Lambda Q_F \otimes \Lambda P_E, d) \xrightarrow{\phi} (\mathcal{A}^*(G/H), \delta)$ which induces an isomorphism on the level of cohomology, and that

$$dim\, P_E = dim\, Q_F + dim\, \hat{P}_E \quad \Rightarrow \quad G/H \text{ is formal},$$

where $\hat{P}_E$ is a subspace of P_E consisting of elements that can be made into cocycles by an appropriate transformation. We will thus define $\hat{P}_E$ and show how to construct the Koszul complex's differential.

In Chapter 5, we determine which 2-tori T make G/T formal. One can show that in the compact connected case $dim\, P_E = rk\, G$. Thus in our case $dim\, P_E = 3$ and $dim\, Q_F = 2$, and so for G/T to be formal we must construct a nonzero element of $\hat{P}_E$. We finally show that the only time there is a possibility of a non-formal G/T is when the Lie algebra E of G is semisimple and contains three simple ideals. In such a case, the Koszul complex is given by

$$(\Lambda(y_2, z_2, x_3, x_3', x_3''), d)$$

and

$$dx_3 = -y_2^2, \quad dx_3' = -z_2^2, \quad \text{and} \quad dx_3'' = -(\alpha y_2 + \beta z_2)^2,$$

where $\alpha, \beta \in \mathbf{Q}$. We prove

Theorem 5.5.4: *This minimal c.g.d.a is formal if and only if $\alpha = 0$ or $\beta = 0$.*

This, as we will see, indicates, in the case of non-formality, a special mixing of the 2-torus inside G.

Acknowledgment

I would like to thank my supervisor Barry Jessup for his support and good advice. He is a great teacher and a good friend. These past two years were very important in solidifying my basic knowledge of mathematics. I also want to thank my wife Céline to whom I am very greatful for her moral support and understanding. I also acknowledge the support of the department of mathematics and school of graduate studies and research of the University of Ottawa, OGS, and FCAR.

Paul-Eugène Parent
University of Ottawa
1996

Contents

This thesis was typesetted using Plain TEX.

Chapter I

Algebraic Preliminaries

In this chapter we will give an overview of the definitions, properties and important results concerning the three following kind of algebras: *graded algebras*, *Hopf algebras*, and *Lie algebras*. The information for graded algebras can be found in [5], that for Hopf algebras in [12], and the material on Lie algebras in [8] and [10].

1.1 Graded Algebras

By a *graded vector space* we will mean a vector space V together with distinguished subspaces $\{V^k\}_{k\in\mathbb{Z}^+}$ such that

$$V = \bigoplus_{k\in\mathbb{Z}^+} V^k.$$

An element in V^k is called *homogeneous of degree k*. A linear map $\phi : V \to W$ between two graded vector spaces is said to be *homogeneous of degree r* if its restriction to V^k is such that

$$\phi(V^k) \subset W^{k+r}.$$

Definition 1.1.1: Let V be a graded vector space. We define sV, the *suspension of* V, by

$$(sV)^{k+1} = V^k, \quad \text{and} \quad (sV)^0 = 0.$$

We thus get a canonical isomorphism $V \xrightarrow{s} sV$ of vector spaces which is homogeneous of degree +1, i.e., for $v \in V^k$ we have $sv \in (sV)^{k+1}$.

Definition 1.1.2: By a *commutative graded algebra(c.g.a)*, we will mean a graded vector space A equipped with a bilinear map $\mu : A \times A \to A$ such that

(1) $\mu(A^k.A^l) \subset A^{k+l}$ with $1 \in A^0$ as an identity element, and

(2) $ab = (-1)^{deg\, a \cdot deg\, b} ba$.

If $A^0 = \mathbf{R}$, we say that A is *connected*.

Let A be a *c.g.a.* The notation A^+ will designate the following ideal

$$\bigoplus_{k>0} A^k.$$

Definition 1.1.3: Let A and B be two *c.g.a*'s. We define their *tensor product* to be the *c.g.a*

$$A \otimes B = \bigoplus_k (A \otimes B)^k,$$

where $(A \otimes B)^k = \oplus_{i+j=k} A^i \otimes B^j$, together with multiplication defined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{deg\, b \cdot deg\, a'} aa' \otimes bb'.$$

Let $V = \oplus_{k \geq 1} V^k$ be a graded vector space.

Definition 1.1.4: Let ΛV denote the *free commutative graded algebra generated by* V. It is obtained via the following construction.

(1) Consider the following graded vector spaces

$$T^0(V) = \mathbf{R}, \quad T^n(V) = \overbrace{V \otimes V \otimes ... \otimes V}^{n \text{ times}}, \ (n \geq 1),$$

and form the graded algebra $T(V) = \oplus_{n=0}^{\infty} T^n(V)$ where the product is juxtaposition, i.e.,

$$(x_1 \otimes ... \otimes x_k)(x'_1 \otimes ... \otimes x'_l) = x_1 \otimes ... \otimes x_k \otimes x'_1 \otimes ... \otimes x'_l.$$

Note that $T(V)$ is *bigraded*, i.e., $x_{i_1} \otimes ... \otimes x_{i_k} \in T^k(V)$, $x_{i_j} \in V^{i_j}$, has *length* k, and degree $i_1 + ... + i_k$.

(2) Let J denote the ideal in $T(V)$ generated by

$$xy - (-1)^{deg\, x \cdot deg\, y} yx, \quad x, y \in T(V).$$

Then ΛV is defined to be

$$\boxed{T(V)/J}.$$

The notation $\Lambda^k V$ will refer to the subspace consisting of words of length k, while $(\Lambda V)^k$ is the subspace of elements of degree k.

- If V is oddly graded, i.e., $V^{2k} = 0$, then ΛV is an *exterior algebra*. Note that for $x \in V^k$, $y \in V^l$ we have $xy = -yx$, and thus $x^2 = 0$.
- If V is evenly graded, i.e., $V^{2k+1} = 0$, then ΛV is a *polynomial algebra*. Note that for $x \in V^k$, $y \in V^l$ we have $xy = yx$.

We note the following fact: (for a proof see [5; p.125])

Proposition 1.1.1: *Let ΛV be a free c.g.a, and let $V_{odd} \equiv \oplus_k V^{2k+1}$ and $V_{even} \equiv \oplus_k V^{2k}$. Then*

$$\Lambda V \cong \Lambda V_{odd} \otimes \Lambda V_{even}.$$

Definition 1.1.5: Let A and B be *c.g.a*'s. A *homomorphism of algebras* is a linear map $\phi : A \to B$ homogeneous of degree 0 such that

$$\phi(1) = 1, \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b).$$

Definition 1.1.6: Let A and B be *c.g.a*'s and $\phi : A \to B$ a homomorphism of algebras. A linear map $\delta : A \to B$ homogeneous of degree r is called a *ϕ-derivation of degree r* if

$$\delta(ab) = \delta a \cdot \phi(b) + (-1)^{r \cdot \deg a}\phi(a) \cdot \delta b.$$

In the case where $\phi = \iota_A$, we simply call δ a derivation. Note that if we want to show the equality of two ϕ-derivations or two homomorphisms on the *c.g.a* ΛV, it is sufficient to show the equality on V, i.e., on a generating set.

The next results can be found in [5]. They give us a way of extending linear maps to homomorphisms and derivations.

Let V be a graded vector space concentrated in degree 1.

Proposition 1.1.2: *Let $\theta : V \to V$ be a linear map. Then θ can be uniquely extended to*

(1) a homomorphism of algebras $\theta : \Lambda V \to \Lambda V$;

(2) a derivation of degree 0 on ΛV such that $\theta(\mathbf{R}) = 0$;

(3) a homomorphism of algebras $\theta : \Lambda sV \to \Lambda sV$; or

(4) a derivation of degree 0 on ΛsV such that $\theta(\mathbf{R}) = 0$.

The next result is basic to all that follows and the proof can be found in [5; p.114].

Proposition 1.1.3: *Let $\partial : V \to \Lambda^2 V$ be a linear map. Then it can be extended uniquely to a derivation of degree +1 on ΛV such that $\partial(\mathbf{R}) = 0$.*

The following construction will be the main object of our study.

Definition 1.1.7: A *c.g.a.* A, equipped with a derivation δ of degree +1 such that $\delta^2 = 0$, is called a *commutative graded differential algebra*(*c.g.d.a*).

If (A, δ_A) and (B, δ_B) are two *c.g.d.a*'s, their tensor product is the *c.g.d.a* $(A \otimes B, \delta)$ where

(1) $A \otimes B$ is the tensor product of *c.g.a*'s, and

(2) $\delta(a \otimes b) = \delta_A a \otimes b + (-1)^{deg\, a} a \otimes \delta_B b$.

Definition 1.1.8: Let (A, δ) be a *c.g.d.a.* We define the *subspace of cocycles* to be $Z^*(A) \equiv ker\, \delta$ and the *subspace of coboundaries* as $B^*(A) \equiv Im\, \delta$.

Note that $Z^*(A)$ and $B^*(A)$ are naturally graded. Moreover, because $\delta^2 = 0$ we have $B^*(A) \subset Z^*(A)$. Note also that by the derivation properties of δ, $Z^*(A)$ is a subalgebra of A, while $B^*(A)$ is an ideal in $Z^*(A)$. Thus we have

Definition 1.1.9: Let (A, δ) be a *c.g.d.a.* The *cohomology algebra of A* is defined to be the following quotient *c.g.a*

$$\mathcal{H}(A, \delta) = Z^*(A)/B^*(A).$$

We will usually denote it by $\mathcal{H}(A)$ when δ is understood.

Definition 1.1.10: Let (A, δ_A) and (B, δ_B) be *c.g.d.a*'s, and let $\phi : A \to B$ be a homomorphism of algebras. If $\phi \circ \delta_A = \delta_B \circ \phi$ then we call ϕ a *c.g.d.a homomorphism.*

A $c.g.d.a$ homomorphism $\phi : A \to B$ induces a homomorphism of algebras on the level of cohomology by

$$\phi^{\#} : \mathcal{H}(A) \to \mathcal{H}(B)$$
$$[a] \mapsto [\phi(a)].$$

Theorem 1.1.4(Künneth Theorem [5; p.57]): *For two c.g.d.a's, (A, δ_A) and (B, δ_B), there is an isomorphism of c.g.a's*

$$\mathcal{H}(A \otimes B) \cong \mathcal{H}(A) \otimes \mathcal{H}(B).$$

1.2 Hopf Algebras

Definition 1.2.1: A *graded connected Hopf algebra* is a 4-tuple $(A, \mu, \Delta, \epsilon)$ where

(1) (A, μ) is a connected $c.g.a$,

(2) $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbf{R}$, called the *comultiplication* and *co-unit* respectively, are homomorphisms of algebras such that

(i) Δ is co-associative with co-unit ϵ, i.e.,

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta, \quad \text{and} \quad (\epsilon \otimes \iota) \circ \Delta = \iota = (\iota \otimes \epsilon) \circ \Delta;$$

(ii) for any $a \in A^+$ we have

$$\Delta(a) = a \otimes 1 + 1 \otimes a + b, \quad b \in A^+ \otimes A^+.$$

Definition 1.2.2: An element $a \in A^+$ is called *primitive* if

$$\Delta(a) = a \otimes 1 + 1 \otimes a.$$

The set of primitive elements in A forms a graded vector space P_A.

Lemma 1.2.1: *If $(A, \mu, \Delta, \epsilon)$ is a graded finite dimensional Hopf algebra, then P_A is oddly graded.*

Proof: Let $a \in P_A$ be of even degree. Since A is finite dimensional, there is a least integer $m > 0$ such that $a^m = 0$ (for degree reasons). But then, since Δ is a homomorphism of algebras and a is of even degree, we get

$$0 = \Delta(a^m) = \Delta(a)^m = (a \otimes 1 + 1 \otimes a)^m = \sum_{k=0}^{m} \binom{m}{k} a^k \otimes a^{m-k}.$$

But the right hand side is nonzero unless $a = 0$. Q.E.D.

The next result, due to Hopf, is a very important structure theorem. Its proof can be found in either [12] or [3].

Theorem 1.2.2(Hopf Theorem): *If $(A, \mu, \Delta, \epsilon)$ is a graded connected finite dimensional Hopf algebra, then the inclusion $P_A \hookrightarrow A$ extends to an isomorphism*

$$\Lambda P_A \xrightarrow{\cong} A$$

of Hopf algebras.

1.3 Lie Algebras

Definitions and Properties

Definition 1.3.1: A *Lie algebra* E is a vector space equipped with a bilinear map $[\,,\,] : E \times E \to E$, called the *Lie bracket*, such that

(1) $[x, y] = -[y, x]$ for all $x, y \in E$, and

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in E$.

The identity (2) is called the *Jacobi identity*. Note that over $\mathbf{R}$, (1) is equivalent to the requirement that $[x, x] = 0$ for all $x \in E$.

A subspace $I \subset E$ is called an *ideal of E* if $[x, y] \in I$ for all $x \in E$, $y \in I$. We call the subspace of elements that commute with all the others,

$$Z_E \equiv \{z \in E \,|\, [z, x] = 0,\ x \in E\},$$

the *center of E*. It is an ideal in E, as follows from the Jacobi identity. Clearly E is abelian if and only if $E = Z_E$. The *derived algebra* E' is the subspace generated by all products

$[x, y]$, $x, y \in E$. E' is an ideal in E. A Lie algebra is called *simple* if it is nonabelian and contains no proper nontrivial ideals.

A *homomorphism* $\phi : E \to F$ *of Lie algebras* is a linear map such that

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

Let V be a finite dimensional vector space and consider the set of linear endomorphisms $End(V)$ on V together with the following bracket

$$[\phi, \psi] \equiv \phi \circ \psi - \psi \circ \phi, \quad \phi, \psi \in End(V).$$

Then $(End(V), [\,,\,])$ is a Lie algebra. It is denoted by $gl(V)$.

Definition 1.3.2: A *representation* of a Lie algebra E in a finite dimensional vector space V is a Lie algebra homomorphism

$$\theta : E \to gl(V).$$

Given a representation θ, we say that a subspace $W \subset V$ is *E-stable* if $\theta(x)W \subset W$ for all $x \in E$. The set

$$V^E \equiv \{v \in V \,|\, \theta(x)v = 0, \, x \in E\},$$

forms an E-stable subspace of V. It is called the *invariant subspace.* Another E-stable subspace of V is the subspace generated by all the vectors of the form $\theta(x)v$, $x \in E$, $v \in V$. We denote it by $\theta(V)$.

Definition 1.3.3: The *adjoint representation* of a Lie algebra E is the representation $ad : E \to gl(E)$, given by

$$ad\, x(y) = [x, y], \quad x, y \in E.$$

Using the Jacobi identity one easily shows that ad is indeed a representation.

Definition 1.3.4: Let θ be a representation. We can associate to E a symmetric bilinear form $T_\theta : E \times E \to \mathbf{R}$ called the *trace form of* θ given by

$$T_\theta(x, y) = tr\, \theta(x) \circ \theta(y).$$

It is symmetric since in general $tr\,AB = tr\,BA$, for $A, B \in End(V)$. If we combine this fact with the Jacobi identity we find

$$T_\theta([x,y],z) + T_\theta(y,[x,z]) = 0, \quad x,y,z \in E.$$

In the special case where $\theta = ad$, the trace form is called the *Killing form of E* and is denoted by $(x,y) \longmapsto K(x,y)$.

We record here a technical result associated to the Killing form which can be found in [10].

Lemma 1.3.1: *Let I be an ideal of E. If K is the Killing form of E and K_I is the Killing form of I considered as a Lie algebra, then*

$$K_I = K_{|_{I\times I}}.$$

Definition 1.3.5: A representation θ is called *semisimple* if, for every E-stable subspace $W \subset V$, there is an E-stable subspace $W' \subset V$ such that

$$V = W \oplus W'.$$

A direct consequence of the definition is

Lemma 1.3.2: *Let θ be a semisimple representation. Then $V = V^E \oplus \theta(V)$.*

The following result can be found in either [10] or [2].

Proposition 1.3.3: *Let E be a finite dimensional Lie algebra. Then the following conditions are equivalent:*

(1) The Killing form of E is nondegenerate.

(2) E is the direct sum of simple ideals.

(3) Every representation of E in a finite dimensional vector space is semisimple.

Definition 1.3.6: A *semisimple* Lie algebra is such that it satisfies at least one of the conditions above.

We notice that because of (2) in the last proposition if E is semisimple, then $Z_E = 0$ and $E' = E$.

Definition 1.3.7: A Lie algebra is called *reductive* if

$$E = Z_E \oplus E',$$

and E' is semisimple as a Lie algebra.

In fact, one can show using Proposition 1.3.3 that E is reductive if and only if the adjoint representation is semisimple.

Definition 1.3.8: A Lie algebra E is called *compact* if there exists a negative definite inner product $<,>$ on E such that

$$<[x,y],z> + <y,[x,z]>= 0, \quad x,y,z \in E.$$

Clearly, if E is compact then it is reductive.(Given an E-stable subspace $W \subset E$ under the *ad* representation then $W^{\perp}$ is also an E-stable subspace.)

The next result is a very useful criterion to determine if a given representation is semisimple. Its proof can be found in [2].

Proposition 1.3.4: *A representation θ of a reductive Lie algebra E in a finite dimensional vector space V is semisimple if and only if its restriction to Z_E is semisimple.*

Multilinear Algebra

For the rest of this section, we will work with the adjoint representation of a finite dimensional Lie agebra E. We will essentially follow the exposition in [8; Chapter 5]. Note that we will denote the duality between the two vector spaces V and V^* by $<,>$ and we will consider E as a graded vector space concentrated in degree 1.

From the adjoint representation we can construct new representations.

Definition 1.3.9:

(1) **Contragredient:** We obtain a representation ad^* on E^* by letting

$$ad^* a \equiv -(ad\, a)^*, \quad a \in E.$$

(2) **Suspension:** We obtain a representation on sE by the following rule:

$$a \longmapsto s \circ ad\, a \circ s^{-1}.$$

We still denote it by ad. The context should make it clear.

(3) **Multilinear(exterior algebras):** For $a \in E$, extending respectively each $ad\, a$ and $ad^* a$ to derivations of degree 0 on the algebras ΛE and ΛE^*, we get two new representations that we still denote by ad and ad^* (Cf. Proposition 1.1.2).

(4) **Multilinear(polynomial algebras):** Again by extending respectively $ad\, a$ and $ad^* a$ to derivations of degree 0 on the algebras ΛsE and ΛsE^* we get two new representations that we still denote by ad and ad^*.

For $a \in E$, consider the linear map $\mu(a) : \Lambda E \to \Lambda E$ defined by $\mu(a)b = a \wedge b$, $b \in \Lambda E$, and its dual, the *substitution operator*, $i(a) : \Lambda E^* \leftarrow \Lambda E^*$.(Note that there is a counterpart operator to μ on the algebra ΛE^* and a counterpart to i on the algebra ΛE. We still denote them by μ and i.) One can prove

Lemma 1.3.5: *Let $h \in E$. Then*

(1) $i(h)$, is a derivation of degree -1 in the algebra ΛE^*.†

(2) $i(h)x^* = \langle x^*, h \rangle$, $x^* \in E^*$.††

Since the bracket on E is a skew-symmetric bilinear map it induces a linear map $\partial_E : \Lambda^2 E \to E$ given by

$$\partial_E(x \wedge y) = [x, y], \quad x, y \in E.$$

By Proposition 1.1.3 we can extend the negative dual $-(\partial_E)^* : \Lambda^2 E^* \leftarrow E^*$ to a derivation $\delta_E : \Lambda E^* \leftarrow \Lambda E^*$ of degree $+1$.

Theorem 1.3.6: *We have*

$$(1)\quad i(x) \circ \delta_E + \delta_E \circ i(x) = ad^* x,$$

$$(2)\quad \delta_E^2 = 0,$$

$$(3)\quad (ad^* x) \circ \delta_E = \delta_E \circ (ad^* x), \quad x \in E.$$

† Cf. [5; p.118]
†† Cf. [5; p,117]

Proof: (3) follows by applying (2) to (1). For (1), we have that

$$< (i(x)\delta_E + \delta_E i(x))x^*, y > =< \delta_E x^*, x \wedge y >=< x^*, -[x,y] >$$

$$=< ad^* x(x^*), y >, \qquad x, y \in E, \, x^* \in E^*,$$

holds in E^*. Thus, since both sides are derivations in ΛE^*, it must be true in general. Now to proove (2) we use (1), the fact that ad is a derivation on ΛE, and the following relation

$$< \delta_E(x^* \wedge y^*), x \wedge y \wedge z >$$

$$=< i(x)\delta_E(x^* \wedge y^*), y \wedge z >$$

$$= - < x^* \wedge y^*, ad\, x(y \wedge z) > - < \delta_E i(x)(x^* \wedge y^*), y \wedge z >$$

$$= - < x^* \wedge y^*, [x,y] \wedge z + y \wedge [x,z] - x \wedge [y,z] >,$$

where $x, y, z \in E$, and $x^*, y^* \in E^*$. Thus for any $\Phi \in \Lambda^2 E^*$ we have

$$< \delta_E \Phi, x \wedge y \wedge z >= - < \Phi, [x,y] \wedge z + y \wedge [x,z] - x \wedge [y,z] > .$$

In particular, by the Jacobi identity, we have for $x^* \in E^*$

$$< \delta_E^2 x^*, x \wedge y \wedge z >=< x^*, [[x,y],z] + [[z,x],y] + [[y,z],x] >= 0.$$

Finally, since δ_E^2 is a derivation, (2) must hold in general.

Q.E.D.

Corollary 1.3.7: *$(\Lambda E^*, \delta_E)$ is a c.g.d.a.*

Definition 1.3.10: The *cohomology of a Lie algebra E* is defined to be the cohomology algebra of the *c.g.d.a* $(\Lambda E^*, \delta_E)$. We will denote it by $\mathcal{H}(E)$.

Given a pair of dual basis $\{e_\nu\}$ and $\{e^{*\nu}\}$ of E and E^* respectively, one can prove the *Koszul formula**, i.e.,

$$\delta_E = \frac{1}{2} \sum_\nu \mu(e^{*\nu}) ad^* e_\nu .$$

* Cf. [8; p.177]

Thus the restriction of δ_E to the invariant subalgebra $(\Lambda E^*)^E$ is 0. Thus the inclusion $i : ((\Lambda E^*)^E, 0) \hookrightarrow (\Lambda E^*, \delta_E)$ is a *c.g.d.a* homomorphism. One can prove*

Proposition 1.3.8: *If E is a reductive Lie algebra, then i induces an isomorphism of graded algebras on the level of cohomology, i.e.,*

$$(\Lambda E^*)^E \cong \mathcal{H}(E).$$

We will present a proof of this fact in Chapter 3.

Since $dim\, E < \infty$ we must have $dim\, \mathcal{H}(E) < \infty$. Thus the following integers are defined.

Definition 1.3.11: The integers $b_p \equiv dim\, \mathcal{H}^p(E)$ are the *Betti numbers of E.*

The next result gives us some information on the structure of the cohomology of a compact semisimple Lie algebra. We refer the reader to [8; p.204] for the proof.

Proposition 1.3.9: *Let $E = E_1 \oplus ... \oplus E_m$ be the decomposition of a compact semisimple Lie algebra E into simple ideals. Then*

(1) $b_1 = b_2 = 0$, *and*

(2) $b_3 = m$.

Finally we wish to show that $(\Lambda E^*)^E$ can be given the structure of a Hopf algebra. It will give us an insight on the structure of $(\Lambda E^*)^E$, i.e., by using Theorem 1.2.2, we obtain

$$(\Lambda E^*)^E \cong \Lambda P_E.$$

Let E be a reductive Lie algebra.

By Proposition 1.3.4, ad^* is a semisimple representation of E in E^* since $ad^*x \equiv 0$ for $x \in Z_E$. Thus we have by Lemma 1.3.2

$$\Lambda E^* = (\Lambda E^*)^E \oplus ad^*(\Lambda E^*).$$

Consider the Lie algebra $E \oplus E$ where the bracket is defined component wise. Then $E \oplus E$ is still reductive with

$$Z_{E\oplus E} = Z_E \oplus Z_E, \quad \text{and} \quad (E \oplus E)' = E' \oplus E'.$$

* Cf. [8; p.189]

Since the operators $ad(x \oplus y) \equiv 0$ on $E \oplus E$ for $x, y \in Z_E$, their extensions to the algebra $\Lambda(E^* \oplus E^*)$ are also 0. Thus by Proposition 1.3.4 ad extends to a semisimple representation ad^* of $E \oplus E$ in $\Lambda(E^* \oplus E^*)$. So by Lemma 1.3.2 we have

$$\Lambda(E^* \oplus E^*) = (\Lambda(E^* \oplus E^*))^{E \oplus E} \oplus ad^*(\Lambda(E^* \oplus E^*)).$$

Lemma 1.3.10(Cf. [5; p.120]): *Let V and W be two finite dimensional vector spaces. Then there is a canonical c.g.a isomorphism between $\Lambda(V \oplus W)$ and $\Lambda V \otimes \Lambda W$, given by*

$$f : x \otimes y \mapsto i_1(x) \wedge i_2(y),$$

where i_1 and i_2 are the extensions to homomorphisms of the usual inclusions.

One can easily prove $ad^*(x \oplus y) \circ f = f \circ (ad^* x \otimes \iota + \iota \otimes ad^* y)$. Noticing that the difference of the two sides is an f-derivation, it suffices to show the equality on $E^* \otimes 1$ and $1 \otimes E^*$. For now on, we will identify $ad^*(x \oplus y)$ with $ad^* x \otimes \iota + \iota \otimes ad^* y$. It is now clear that

$$ad^*(\Lambda(E^* \oplus E^*)) = ad^*(\Lambda E^*) \otimes \Lambda E^* + \Lambda E^* \otimes ad^*(\Lambda E^*).$$

Theorem 1.3.11: *If E is a finite dimensional reductive Lie algebra, then*

$$(\Lambda(E^* \oplus E^*))^{E \oplus E} = (\Lambda E^*)^E \otimes (\Lambda E^*)^E.$$

Proof: By Proposition 1.3.10 we have

$$\begin{aligned}(\Lambda(E^* \oplus E^*))^{E \oplus E} &= (\Lambda E^* \otimes \Lambda E^*)^{E \oplus E} \\ &= \left((\Lambda E^*)^E \otimes \Lambda E^* \oplus ad^*(\Lambda E^*) \otimes \Lambda E^*\right)^{E \oplus E} \\ &= ((\Lambda E^*)^E \otimes \Lambda E^*)^{E \oplus E} \oplus (ad^*(\Lambda E^*) \otimes \Lambda E^*)^{E \oplus E}\end{aligned}$$

since each subspace is ad^*-invariant. We also have

$$(ad^*(\Lambda E^*) \otimes \Lambda E^*)^{E \oplus E} \subset (ad^*(\Lambda(E^* \oplus E^*)))^{E \oplus E} = 0.$$

On the other hand, $v \in ((\Lambda E^*)^E \otimes \Lambda E^*)^{E \oplus E}$ can be written as $v = \sum e_\alpha \otimes b_\alpha$ with $\{e_\alpha\}$ a basis of $(\Lambda E^*)^E$ and $b_\alpha \in \Lambda E^*$. The fact that $0 = ad^*(x \oplus y)(v) = \sum e_\alpha \otimes ad^* y(b_\alpha)$, $x, y \in E$, implies that $ad^* y(b_\alpha) = 0$.

Q.E.D.

We can thus define the projection $\eta : \Lambda(E^* \oplus E^*) \to (\Lambda E^*)^E \otimes (\Lambda E^*)^E$ with kernel $ad^*(\Lambda(E^* \oplus E^*))$.

Now consider the following linear map $E \oplus E \to E : x \oplus y \mapsto x + y$. Then by Proposition 1.1.2 we can extend this map to a homomorphism μ of algebras, and using Lemma 1.3.10 we get

$$\Lambda E \otimes \Lambda E \xrightarrow{\cong} \Lambda(E \oplus E) \xrightarrow{\mu} \Lambda E$$

which is simply multiplication since

$$u \otimes v \mapsto \mu(i_1(u) \wedge i_2(v)) = \mu(i_1(u)) \wedge \mu(i_2(v)),$$

and for $u = a_1 \wedge ... \wedge a_k$ we have

$$\mu(i_1(u)) = \mu(a_1, 0) \wedge ... \wedge \mu(a_k, 0) = a_1 \wedge ... \wedge a_k = u.$$

We will denote the multiplication map by μ.

Let $\Delta \equiv \eta \circ \mu^* \circ i$, where $i : (\Lambda E^*)^E \hookrightarrow \Lambda E^*$ denotes the inclusion map, and let $\epsilon : (\Lambda E^*)^E \to (\Lambda^0 E^*)^E = \mathbf{R}$ be the natural projection.

Lemma 1.3.12: *Let $\Phi \in (\Lambda^+ E^*)^E$. Then*

$$\Delta(\Phi) = \Phi \otimes 1 + 1 \otimes \Phi + \Psi, \quad \Psi \in (\Lambda^+ E^*)^E \otimes (\Lambda^+ E^*)^E.$$

Proof: We have $\Delta(\Phi) = \Phi_1 \otimes 1 + 1 \otimes \Phi_2 + \Psi$. Using the duality* between $(\Lambda E^*)^E$ and $(\Lambda E)^E$ we get

$$< \Phi_1, a > = < \Delta(\Phi), a \otimes 1 > = < \Phi, \mu(a \otimes 1) > = < \Phi, a >$$

for $a \in (\Lambda E)^E$. Thus by duality $\Phi_1 = \Phi$.

Q.E.D.

Theorem 1.3.13: *If E is a reductive Lie algebra, then the 4-tuple $((\Lambda E^*)^E, \wedge, \Delta, \epsilon)$ is a graded connected finite dimensional Hopf algebra.*

* Cf. [8; p.175]

Proof: Since ad^* is a derivation $((\Lambda E^*)^E, \wedge)$ is a *c.g.a.* Using the duality between $(\Lambda E^*)^E$ and $(\Lambda E)^E$ we get that Δ^* is simply multiplication in $(\Lambda E)^E$ and is associative. Thus Δ is co-associative. Using Lemma 1.3.12 one can easily show that ϵ is a co-unit for Δ. Next, define two automorphisms T and T^* on $(\Lambda E)^E \otimes (\Lambda E)^E \otimes (\Lambda E)^E \otimes (\Lambda E)^E$ and $(\Lambda E^*)^E \otimes (\Lambda E^*)^E \otimes (\Lambda E^*)^E \otimes (\Lambda E^*)^E$ respectively by

$$T(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = (-1)^{deg\, x_2 \cdot deg\, x_3} x_1 \otimes x_3 \otimes x_2 \otimes x_4, \quad \text{and}$$
$$T^*(\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4) = (-1)^{deg\, \Phi_2 \cdot deg\, \Phi_3} \Phi_1 \otimes \Phi_3 \otimes \Phi_2 \otimes \Phi_4,$$

and consider the 4-tuple $((\Lambda(E^* \oplus E^*))^{E \oplus E}, \wedge_{E\oplus E}, \Delta_{E\oplus E}, \epsilon_{E\oplus E})$ obtained from the reductive Lie algebra $E \oplus E$. Then clearly we have $\mu_{E\oplus E} = (\mu \otimes \mu) \circ T$ and $\wedge_{E\oplus E} = (\wedge \otimes \wedge) \circ T^*$. By dualizing the first one, we obtain

$$\Delta_{E\oplus E} = T^* \circ (\Delta \otimes \Delta).$$

Using the natural relation* $\Delta \circ \wedge = (\wedge \otimes \wedge) \circ \Delta_{E\oplus E}$, one finally gets

$$\begin{aligned} \Delta \circ \wedge &= (\wedge \otimes \wedge) \circ \Delta_{E\oplus E} = (\wedge \otimes \wedge) \circ T^* \circ (\Delta \otimes \Delta) \\ &= \wedge_{E\oplus E} \circ (\Delta \otimes \Delta). \end{aligned}$$

Q.E.D.

Definition 1.3.12: Let E be a reductive Lie algebra. The *primitive subspace* P_E for E is defined as the subspace of primitive elements $P_{(\Lambda E^*)^E}$ with respect to this Hopf algebra structure.

Remark: P_E is not a subspace of E!

Corollary 1.3.14: $(\Lambda E^*)^E \cong \Lambda P_E$.

Corollary 1.3.15: *$\mathcal{H}(E)$ is an exterior algebra over an oddly graded vector space, in particular*

$$\mathcal{H}(E) \cong \Lambda P_E.$$

* Cf. [8; p.200]

Chapter II

The de Rham Complex

We will present in this chapter the main tool that will permit us to associate to any differentiable manifold M a characteristic *c.g.d.a*, $\mathcal{A}^*(M)$, called *the de Rham Complex*. Indeed, this will be a *contravariant functor* $\mathcal{A}^*(\cdot)$ *from the category of differentiable manifolds and smooth maps to the category of c.g.d.a's and their homomorphisms.*

We start by constructing this functor in the special case of $\mathbf{R}^n$ and its open sets and then generalize to abstact differentiable manifolds. We will essentially follow very closely the exposition in [1].

2.1 Euclidean space

Let $x_1, ..., x_n$ be the linear coordinates on $\mathbf{R}^n$ and consider the following exterior algebra over $\mathbf{R}$

$$\mathcal{A}^* = \Lambda(dx_1, ..., dx_n)$$

where $deg\, dx_i = 1$. Thus $\mathcal{A}^*$ is naturally graded by the length of words.

Now if we consider the $\mathbf{C}^\infty$ functions on $\mathbf{R}^n$ as the usual algebra concentrated in degree 0, we can then form the following commutative graded algebra

$$\mathcal{A}^*(\mathbf{R}^n) = \mathbf{C}^\infty(\mathbf{R}^n) \otimes_{\mathbf{R}} \mathcal{A}^*.$$

Elements of this *c.g.a* are called $\mathbf{C}^\infty$ *differentiable forms* on $\mathbf{R}^n$. Thus if ω is such a form, then ω can be written uniquely as $\sum f_{i_1...i_q} dx_{i_1}...dx_{i_q}$, where the coefficients $f_{i_1...i_q}$ are

C^∞ functions. By construction, the natural grading on $\mathcal{A}^*$ induces one on $\mathcal{A}^*(\mathbf{R}^n)$, i.e.,

$$\mathcal{A}^*(\mathbf{R}^n) = \oplus_{q=0}^{n} \mathcal{A}^q(\mathbf{R}^n),$$

where $\mathcal{A}^q(\mathbf{R}^n)$ consists of the C^∞ q-forms on $\mathbf{R}^n$. We will write $\omega = \sum f_I dx_I$ for $\omega = \sum f_{i_1 \ldots i_q} dx_{i_1} \ldots dx_{i_q}$.

Consider the following linear map

$$d : \mathcal{A}^q(\mathbf{R}^n) \to \mathcal{A}^{q+1}(\mathbf{R}^n),$$

defined by

1. if $f \in \mathcal{A}^0(\mathbf{R}^n)$, then $df \equiv \sum \partial f / \partial x_i dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

Notice that this construction is consistent with the notation dx_i since if we consider $x_i \in \mathcal{A}^0(\mathbf{R}^n)$ then $d(x_i) = dx_i$. We call d the *exterior differentiation.*

Proposition 2.1.1(Cf. [1; p.14]):

1) d is a derivation of degree +1, i.e.,

$$d(\tau \cdot \omega) = (d\tau) \cdot \omega + (-1)^{\deg \tau} \tau \cdot d\omega.$$

2) $d^2 = 0$.

Definition 2.1.1: The algebra $\mathcal{A}^*(\mathbf{R}^n)$ together with the derivation d is called the *de Rham complex* on $\mathbf{R}^n$. We see, by Proposition 2.1.1, that $(\mathcal{A}^*(\mathbf{R}^n), d)$ is a *c.g.d.a.*

Note that all the definitions and constructions so far are equally valid for an arbitrary open set U of $\mathbf{R}^n$. Thus we can speak of the de Rham complex on U, i.e.,

$$\mathcal{A}^*(U) = C^\infty(U) \otimes_{\mathbf{R}} \mathcal{A}^*,$$

with the appropriate derivation d.

2.2 The de Rham Cohomology

Let $U \subset \mathbf{R}^n$ be an open set. We have seen in Chapter 1 that to a given *c.g.d.a* we can consider the subalgebra of cocycles, the ideal of coboundaries, and finally the associated cohomology algebra. It is customary in the case of the de Rham complex to call a cocycle ω a *closed form*, and to call a coboundary τ an *exact form*.

We will denote the subalgebra of closed forms by $Z^*(U)$ and the ideal of exact forms by $B^*(U)$.

Definition 2.2.1: The cohomology algebra $\mathcal{H}(U)$ associated to the *c.g.d.a* $(\mathcal{A}^*(U), d)$ is called the *de Rham cohomology algebra of* U.

As we have seen, the de Rham cohomology algebra is naturally graded, i.e.,

$$\mathcal{H}(U) = \oplus_{q=0}^{n} \mathcal{H}^q(U),$$

where $\mathcal{H}^q(U) = Z^q(U)/B^q(U)$.

Note that we can look at the de Rham complex as a set of differential equations, whose solutions are the closed forms. Thus, for example, finding a closed 1-form $f dx + g dy$ on $\mathbf{R}^2$ is equivalent to solving the differential equation $\partial g/\partial x - \partial f/\partial y = 0$. The trivial solutions are the exact forms which are automatically closed. Thus a measure of the size of the space of "interesting" solutions is the de Rham cohomology algebra.

2.3 Pullbacks and Functoriality

Let $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_n\}$ be the standard coordinates on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Every smooth map $f : \mathbf{R}^m \to \mathbf{R}^n$ induces a map $f^* : \mathcal{A}^0(\mathbf{R}^m) \leftarrow \mathcal{A}^0(\mathbf{R}^n)$ via

$$f^*(g) = g \circ f, \qquad g \in \mathcal{A}^0(\mathbf{R}^n).$$

We call f^* the *pullback*. Notice that $f^*(g \cdot h) = f^*(g) \cdot f^*(h)$. We now want to extend f^* to all forms such that f^* is a *c.g.d.a* homomorphism.

Let $f^* : \mathcal{A}^*(\mathbf{R}^m) \leftarrow \mathcal{A}^*(\mathbf{R}^n)$ be defined by

$$f^*(\sum g_I \, dy_{i_1} ... dy_{i_q}) = \sum (g_I \circ f) df_{i_1} ... df_{i_q},$$

where $f_i = y_i \circ f$ is the i-th component of the function f.

Proposition 2.3.1(Cf. [1; p.19]):

1) f^* is an algebra homomorphism.

2) f^* commutes with d.

3) d is independent of the coordinate system on $\mathbf{R}^n$.

We thus get: $\mathcal{A}^*(\cdot)$ *is a contravariant functor from the category of Euclidean spaces* $\{\mathbf{R}^n\}_{n\in\mathbf{Z}^+}$ *and smooth maps* $\mathbf{R}^m \to \mathbf{R}^n$ *to the category of c.g.d.a's and their homomorphisms.*

2.4 Differentiable manifolds

We will now extend the functor $\mathcal{A}^*(\cdot)$ to the category of differentiable manifolds. For more detailed explanations on manifolds, we refer the reader to [11].

A differentiable structure on a real manifold M of dimension n is given by an *atlas*, i.e., a collection of pairs $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ such that $\{U_\alpha\}_{\alpha\in I}$ is an open cover for M and each open set U_α is homeomorphic to $\mathbf{R}^n$ via $\phi_\alpha : U_\alpha \xrightarrow{\cong} \mathbf{R}^n$; and on the overlaps $U_\alpha \cap U_\beta$ the transition functions

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are diffeomorphisms of open sets in $\mathbf{R}^n$. We will require the atlas to be maximal with respect to inclusions and all manifolds to be Hausdorff and have a countable basis.

Let $\{u_1, ..., u_n\}$ be the standard coordinate system on $\mathbf{R}^n$. We can write $\phi_\alpha = (x_1, ..., x_n)$ where $x_i \equiv u_i \circ \phi_\alpha$ are a coordinate system on U_α. A function f on M is *differentiable* if for each U_α the function $f \circ \phi_\alpha^{-1}$ is a differentiable function of $\mathbf{R}^n$.

Let $p \in M$ and f a differentiable function on M. Choose a chart (U_α, ϕ_α) such that $p \in U_\alpha$. Then we define the *i-th partial derivative at p*, $\partial f/\partial x_i(p)$, with respect to the chart (U_α, ϕ_α) to be the i-th partial of the pullback $f \circ \phi_\alpha^{-1}$ on $\mathbf{R}^n$, i.e.,

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p)).$$

A *differential form* ω on M, is a collection of forms ω_α on U_α in the atlas defining M such that we have compatibility on the overlaps, i.e., if i and j denote the inclusions

$$U_\alpha \xleftarrow{\;i\;} U_\alpha \cap U_\beta \xrightarrow{\;j\;} U_\beta$$

then $i^*(\omega_\alpha) = j^*(\omega_\beta)$ in $\mathcal{A}^*(U_\alpha \cap U_\beta)$. Thus, we can extend the definitions of exterior differentiation and product of forms on $\mathbf{R}^n$ in a compatible way on forms on an arbitrary manifold. We thus obtain the following *c.g.d.a*: $(\mathcal{A}^*(M), d)$.

Moreover, just as for $\mathbf{R}^n$, a smooth map of differentiable manifolds $f : M \longrightarrow N$ induces in a natural way a pullback map on forms $f^* : \mathcal{A}^*(M) \longleftarrow \mathcal{A}^*(N)$ which is a *c.g.d.a* homomorphism. Thus $\mathcal{A}^*(\cdot)$ becomes a contravariant functor on the category of differentiable manifolds.

We can now talk about the de Rham cohomology algebra $\mathcal{H}(M)$ associated to a differentiable manifold M. As we have seen in Chapter 1, given $f : M \longrightarrow N$, the pullback f^* induces a homomorphism of algebras on the level of cohomology which is denoted by

$$f^\# : \mathcal{H}(M) \longleftarrow \mathcal{H}(N).$$

If $g : N \longrightarrow Q$ is another smooth map, then clearly we have

$$f^\# \circ g^\# = (g \circ f)^\#, \quad \text{and} \quad \iota_M^\# = \iota_{\mathcal{H}(M)}.$$

Thus, the de Rham cohomology algebra is a *smooth invariant* of the differentiable manifold M.

We will now give some classical results on cohomology and refer the reader to [6] for their proofs.

Let M be a differentiable n-manifold.

Proposition 2.4.1: *If M is connected, then $\mathcal{H}^0(M) \cong \mathbf{R}$.*

This justifies why we call a *c.g.a* A connected if $A^0 = \mathbf{R}$.

Proposition 2.4.2: *If M is any compact manifold, then*

$$dim\, \mathcal{H}(M) < \infty.$$

Our manifolds M will always be compact, since we will be studying compact connected Lie groups and their homogeneous spaces.

Corollary 2.4.3: *If M is compact, then the Betti numbers $b_p = \dim \mathcal{H}^p(M)$ are defined. If, in addition, M is orientable, then*

$$b_p = b_{n-p}, \qquad 0 \le p \le n \quad \textit{(Poincaré duality).}$$

We will show in Chapter 3 that Lie groups are parallelizable, and thus orientable. Thus, they enjoy Poincaré duality.

2.5 Geometric Interpretation of Forms

We will start with $\mathbf{R}^n$. The extension to an arbitrary differentiable n-manifold will be straightforward since the work is done locally. We will follow very closely the exposition in [15].

Definition 2.5.1: To each $p \in \mathbf{R}^n$ we associate

$$T_p(\mathbf{R}^n) \equiv \{(p, v) \mid v \in \mathbf{R}^n\}.$$

It can be made into a vector space in the obvious way, by defining

$$(p, v) + (p, w) = (p, v + w),$$
$$a \cdot (p, v) = (p, av).$$

We call $T_p(\mathbf{R}^n)$ the *tangent space of* $\mathbf{R}^n$ *at* p. We will denote the elements $(p, v) \in T_p(\mathbf{R}^n)$ by X_p and call then *tangent vectors at* p.

Let $\{(e_1)_p, ..., (e_n)_p\}$ be the canonical basis of $T_p(\mathbf{R}^n)$ and consider a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$. Then its differential $Df(p)$ at p is a linear map from $T_p(\mathbf{R}^n)$ to $T_{f(p)}(\mathbf{R}) \cong \mathbf{R}$. Thus $Df(p)$ is a linear functional on $T_p(\mathbf{R}^n)$, i.e.,

$$Df(p) \in T_p^*(\mathbf{R}^n).$$

Take $f = x_i$, the i-th coordinate on $\mathbf{R}^n$. Then, if $X_p = \sum \alpha_j (e_j)_p$, $\alpha_j \in \mathbf{R}$, we have

$$Dx_i(p)(X_p) = \alpha_i.$$

So $\{Dx_1(p), ..., Dx_n(p)\}$ is just the dual basis to $\{(e_1)_p, ..., (e_n)_p\}$.

Now using the isomorphism* between $\Lambda^k T_p^*(\mathbf{R}^n)$, and the space of skew-symmetric k-linear maps on $T_p(\mathbf{R}^n)$, $k = 1, ..., n$, we have that the expression

$$\sum_{i_1<...<i_k} f_{i_1,...,i_k}(p) \cdot [Dx_{i_1}(p) \wedge ... \wedge Dx_{i_k}(p)] \quad \in \quad \Lambda^k T_p^*(\mathbf{R}^n),$$

can be interpreted as a skew-symmetric k-linear map on $T_p(\mathbf{R}^n)$, where $f_{i_1,...,i_k} \in \mathbf{C}^\infty(\mathbf{R}^n)$. Thus letting $dx_i \equiv Dx_i$ we see that we can associate to any form $\omega \in \mathcal{A}^*(\mathbf{R}^n)$ a map on $\mathbf{R}^n$ such that

$$\omega(p) \in \Lambda T_p^*(\mathbf{R}^n).$$

All this discussion can be readily extended to general differentiable n-manifolds since when we evaluate a form $\omega \in \mathcal{A}^*(M)$ at $p \in M$ everything becomes local by construction of ω. Thus by taking an appropriate chart, we can reduce the case to $\mathbf{R}^n$.

Thus for $X_p^1, ..., X_p^k \in T_p(M)$ and $\omega \in \mathcal{A}^k(M)$ the expression

$$\omega(p; X_p^1, ..., X_p^k)$$

makes sense and characterises completely ω.

Note that tangent vectors can be thought naturally as tangent vectors to smooth curves on a manifold. Let $\phi : M \to N$ be a smooth map between manifolds. Let $p \in M$ and $x(t)$ a curve on M such that $x(0) = p$ and $\dot{x}(0) = X_p \in T_p(M)$. Then we have

Definition 2.5.2: The *differential at* p of ϕ is the linear mapping

$$\phi_p' : T_p(M) \to T_{\phi(p)}(N)$$

such that $\phi_p'(X_p)$ is the tangent vector to the curve $\phi(x(t))$ at $\phi(p) = \phi(x(0))$.

If $\psi : N \to O$ is another smooth map, then the differentials satisfy naturally the chain rule

$$(\psi \circ \phi)_p' = \psi_{\phi(p)}' \circ \phi_p'.$$

* Cf. [5; p.145]

Thus, given a smooth map $\phi : M \to N$, its pullback ϕ^* is determined by the expressions

$$\phi^*(\omega)(p; X_p^1, ..., X_p^k) = \omega(\phi(p); \phi_p'(X_p^1), ..., \phi_p'(X_p^k)),$$

where $\omega \in \mathcal{A}^k(N)$ and ϕ_p' is the differential at p of ϕ.

2.6 Vector Fields

In this section we will give some complementary information on *tangent bundles* and *vector fields*. For more details, we refer the reader to [6].

Let M be a differentiable n-manifold and consider the disjoint union

$$T_M \equiv \bigcup_{p \in M} T_p(M),$$

and define a map $\pi_M : T_M \to M$ by the natural projection

$$\pi_M(X_p) = p, \qquad X_p \in T_p(M).$$

On can show* that there is a unique smooth manifold structure on T_M such that T_M is locally diffeomorphic to $M \times \mathbf{R}^n$ and makes π_M smooth.

Definition 2.6.1: T_M is known as the *tangent bundle* of M.

Remark: The 4-tuple $(T_M, \pi_M, M, \mathbf{R}^n)$ is an example of a *vector bundle*.

With the set of differentials $\{\phi_p'\}_{p \in M}$ of a smooth map $\phi : M \to N$ we can define a *bundle map* $D\phi : T_M \to T_N$ by

$$D\phi(X_p) = \phi_p'(X_p), \qquad X_p \in T_p(M).$$

We call this map *the derivative of* ϕ.

Definition 2.6.2: A *vector field on* M is a smooth map $X : M \to T_M$ such that

$$X(p) \in T_p(M), \quad p \in M.$$

Remark: The set of vector fields $\chi(M)$ on M forms a $\mathbf{C}^\infty(M)$-module.

* Cf. [6; p.94]

Let $\{(U_\alpha, \phi_\alpha)\}$ be a coordinate cover of M. Recall the definition of the operators $\{\partial/\partial x_i(p)\}_{i=1}^n$ from Section 2.4. One can show that they span $T_p(M)$. Thus in the neighborhood of $p \in U_\alpha \subset M$, a vector field X can be written as

$$X = \sum f_i \frac{\partial}{\partial x_i}, \qquad f_i \in \mathbf{C}^\infty(U_\alpha).$$

Given another vector field Y one can construct a new vector field $[X, Y]$ which in the neighborhood of p has the form

$$[X,Y] = \sum_{j,k} \left(f_k \left(\frac{\partial g_j}{\partial x_k}\right) - g_k \left(\frac{\partial f_j}{\partial x_k}\right) \right) \frac{\partial}{\partial x_j},$$

where $Y = \sum g_i \frac{\partial}{\partial x_i}$, $g_i \in \mathbf{C}^\infty(U_\alpha)$.

Proposition 2.6.1(Cf. [6; p.108]): *$\chi(M)$ equipped with the bracket operation $[\,,\,]$ is a real Lie algebra. In particular it satisfies the Jacobi identity:*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0, \qquad X,Y,Z \in \chi(M).$$

Let $\phi : M \to N$ be a diffeomorphism. Then $D\phi$ is an isomorphism since each ϕ'_p is a linear isomorphism. Thus there is a map $\phi_* : \chi(M) \to \chi(N)$ defined by

$$\phi_* X = D\phi \circ X, \qquad X \in \chi(M),$$

with $(\phi_* X)(\phi(p)) = \phi'_p(X(p))$.

Proposition 2.6.2(Cf. [6; p.110]): *ϕ_* is a Lie algebra isomorphism, i.e.,*

$$\phi_*[X,Y] = [\phi_* X, \phi_* Y], \qquad X,Y \in \chi(M).$$

Chapter III

Lie Groups

In this chapter we will describe some important properties of our main object of study: *Lie groups.* We will present the computation of their cohomology in the compact connected case. Such Lie groups are formal spaces par excellence.

We will essentially follow the development in [7], and will use the notation and some results from differential geometry. We refer the reader to Section 2.6.

3.1 Definitions and Properties

Definition 3.1.1: A *Lie group* is a topological group equipped with a smooth manifold structure, such that the multiplication map μ and the inversion map ν

$$\begin{aligned} \mu : G \times G \to G, \qquad & \nu : G \to G \\ (a,b) \mapsto ab \qquad & a \mapsto a^{-1} \end{aligned}$$

are smooth. We will denote the unit element of a Lie group by e.

A *homomorphism of Lie groups* $\phi : G \to H$ is a smooth homomorphism of groups. An *isomorphism of Lie groups* is a map that is both a homomorphism and a diffeomorphism.

For each $a \in G$ we have two automorphisms on G defined by

$$\lambda_a(x) = ax, \quad \text{and} \quad \rho_a(x) = xa.$$

They are called *left* and *right translation* by a. We will denote their derivatives by

$$L_a : T_G \to T_G, \quad \text{and} \quad R_a : T_G \to T_G$$

respectively, where T_G denotes the tangent bundle over G.

Recall that if $\phi : M \to N$ is a diffeomorphism then it induces a Lie algebra isomorphism

$$\phi_* : \chi(M) \xrightarrow{\cong} \chi(N).$$

Thus $(\lambda_a)_*$ and $(\rho_a)_*$ are automorphisms of the Lie algebra $\chi(G)$ of vector fields on G.

We will now show that the tangent bundle of a Lie group G is isomorphic to the trivial bundle $G \times T_e(G)$, i.e., G is parallelizable. We will denote the elements in $T_e(G)$ by lower case letters, $h, k, \ldots$

Definition 3.1.2: A vector field $X \in \chi(G)$ is called *left invariant* if

$$(\lambda_a)_* X = X, \quad a \in G.$$

Note that the left invariant vector fields form a subalgebra $\chi_L(G)$ of $\chi(G)$.

The derivatives of the multiplication and inversion maps are bundle maps,

$$D\mu : T_{G\times G} \to T_G, \quad \text{and} \quad D\nu : T_G \to T_G,$$

and an application of the identification $T_{G\times G} \cong T_G \times T_G$ gives

Lemma 3.1.1: *Let $X_a \in T_a(G)$, $Y_b \in T_b(G)$. Then*

(1) $D\mu(X_a, Y_b) = R_b X_a + L_a Y_b$, *and*

(2) $D\nu(X_a) = -L_{a^{-1}} \circ R_{a^{-1}} X_a$.

Proposition 3.1.2(Cf. [7; p.26]): *T_G is isomorphic to the trivial bundle $G \times T_e(G)$ via*

$$\alpha : (a, h) \mapsto L_a(h) = D\mu(0_a, h), \quad a \in G,\ h \in T_e(G).$$

Thus there is a canonical isomorphism $\chi_L(G) \xrightarrow{\cong} T_e(G)$ given by $X \mapsto X(e)$. Moreover

$$\dim \chi_L(G) = \dim G.$$

Definition 3.1.3: The unique left invariant vector field generated by $h \in T_e(G)$ will be denoted by X^h, and is defined by

$$X^h(e) = h.$$

The Lie algebra structure on $\chi_L(G)$ induces a Lie algebra structure on $T_e(G)$ via

$$[h,k] \equiv [X^h, X^k](e), \quad h,k \in T_e(G).$$

Suppose that $\phi : G \to H$ is a Lie group homomorphism. Then we must have $\phi(e) = e$ and so the differential at e restricts to

$$\phi'_e : T_e(G) \to T_e(H).$$

We will denote this map by ϕ'.

Noticing that $\phi \circ \lambda_a = \lambda_{\phi(a)} \circ \phi$ we get $\phi'(X^h(a)) = X^{\phi' h}(\phi(a))$, $h \in T_e(G)$, $a \in G$. One can then show

Proposition 3.1.3: *ϕ' is a Lie algebra homomorphism.*

3.2 Representations

We call $(T_e(G), [,])$ the *associated* Lie algebra to G and it will be denoted by E.

Definition 3.2.1: A *representation* of a Lie group G in a finite dimensional vector space V is a Lie group homomorphism

$$\Theta : G \to GL(V),$$

where $GL(V)$ is the Lie group of linear automorphisms of V.

By Proposition 3.1.3 and the fact that $T_e(GL(V)) = gl(V)$, we obtain a Lie algebra representation

$$\theta \equiv \Theta' : E \to gl(V).$$

Definition 3.2.2: The *invariant subspace of V* with respect to Θ is defined by

$$V^G \equiv \{v \in V \mid \Theta(a)v = v,\ a \in G\}.$$

We have seen in Chapter 1 that there is an invariant subspace associated to a Lie algebra representation. If $\Theta' = \theta$, then

Proposition 3.2.1(Cf. [7; p.40]): *If G is a Lie group and E is its Lie algebra, we have*

$$V^G \subset V^E.$$

Moreover, if G is connected, then $V^G = V^E$.

We will present a very important representation: the *adjoint representation.* Details of the construction can be found in [7].

Each $a \in G$ defines an *inner automorphism* τ_a of G by

$$\tau_a(g) = aga^{-1}, \quad g \in G,$$

i.e., $\tau_a = \lambda_a \circ \rho_{a^{-1}}$. Thus τ_a' is an automorphism of the Lie algebra E. We will denote τ_a' by $Ad\,a$. So we have

$$Ad\,a = L_a \circ R_{a^{-1}}.$$

Proposition 3.2.2: *The correspondence*

$$Ad : a \longmapsto Ad\,a$$

defines a representation of G in E. It is the adjoint representation of G.

Recall from Chapter 1 that we had an adjoint representation of the Lie algebra E on itself given by

$$ad\,h(k) = [h, k], \quad h, k \in E.$$

The following is natural:

Proposition 3.2.3: $Ad' = ad$.

Just as for the *ad* representation we can construct new representations from *Ad.* We have included all the corresponding one from Chapter 1 for completeness.

Definition 3.2.3:

(1) **Contragredient:** We get a representation on E^* by letting

$$Ad^* a \equiv ((Ad\,a)^{-1})^*, \quad a \in G.$$

Clearly, we have that $(Ad^*)' = ad^*$.

(2) **Suspension:** We get a representation on sE by considering the following correspondence

$$a \longmapsto s \circ Ad\, a \circ s^{-1}.$$

We still denote it by Ad. The context should make it clear. Proposition 3.2.3 is clearly valid.

(3) **Multilinear(exterior algebras):** By extending respectively $Ad\, a$ and $Ad^* a$ to homomorphisms on the algebras ΛE and ΛE^* we get two new representations that we still denote by Ad and Ad^*. Note that because of the product rule, Proposition 3.2.3 is still valid in these cases.

(4) **Multilinear(polynomial algebras):** Again by extending respectively $Ad\, a$ and $Ad^* a$ to homomorphisms on the algebras ΛsE and ΛsE^* we get two new representations that we still denote by Ad and Ad^*. Proposition 3.2.3 is still valid in these cases.

The next result justifies our restriction to reductive Lie algebras.

Proposition 3.2.4: *The Lie algebra of a compact Lie group is reductive.*

Proof: Let G be a compact Lie group with Lie algebra E. Since G is compact there is an Ad-invariant inner product* $<,>$ on E, i.e.,

$$< Ad\, a(h), Ad\, a(k) > = < h, k >, \quad a \in G,\ h, k \in E.$$

Thus by differentiating we get

$$< ad\, j(h), k > + < h, ad\, j(k) > = 0, \quad h, k, j \in E,$$

which shows that E is a compact Lie algebra, and hence is reductive.

Q.E.D.

* Cf. [7; p.54]

3.3 Invariant Forms

In this section we will look at the dual notion to invariant vector fields: *invariant forms.*

Definition 3.3.1: A *left* invariant form Φ on G is a form such that

$$\lambda_a^* \Phi = \Phi, \quad a \in G.$$

Since pullbacks commute with d and are homomorphisms, the set of left invariant forms $\mathcal{A}_L^\bullet(G)$ on G is a subalgebra of $\mathcal{A}^\bullet(G)$ stable under d. We can thus consider its cohomology algebra $\mathcal{H}_L(G)$. Note that we could have done our analysis with right invariant forms and get $\mathcal{A}_R^\bullet(G)$ and $\mathcal{H}_R(G)$.

Another application of the fact that a Lie group G is parallelizable is the result dual to $\chi_L(G) \cong E$, i.e.,

Proposition 3.3.1: *The correspondence $\Phi \mapsto \Phi(e)$ defines an isomorphism*

$$\varrho : \mathcal{A}_L^\bullet(G) \xrightarrow{\cong} \Lambda E^\bullet,$$

of c.g.a's. In particular, the left invariant functions are constant, and $\mathcal{A}_L^\bullet(G)$ is generated by the vector space $\mathcal{A}_L^1(G)$ as an exterior algebra.

Proof: Here is ϱ^{-1}: Given a skew-symmetric k-linear map ω on E we associate to it the following left invariant form

$$\Phi(a; X_a^1, ..., X_a^k) \equiv \omega(L_{a^{-1}} X_a^1, ..., L_{a^{-1}} X_a^k),$$

where $a \in G$ and $X_a^j \in T_a(G)$, $j = 1, .., k$.

Q.E.D.

Using this isomorphism we can define a unique operator on $\Lambda E^\bullet$ by

$$\delta \equiv \varrho \circ d \circ \varrho^{-1}.$$

By construction, δ satisfies

(1) $\delta^2 = 0$, and

(2) δ is a derivation of degree $+1$.

Thus $(\Lambda E^*, \delta)$ is a *c.g.d.a* and indeed* it is isomorphic to the *c.g.d.a* $(\Lambda E^*, \delta_E)$ of Chapter 1, i.e.,

$$< \delta h^*, h \wedge k >= - < h^*, [h,k] >, \quad h,k \in E,\ h^* \in E^*.$$

Definition 3.3.2: A differential form $\Phi \in \mathcal{A}^*(G)$ is called *bi-invariant* if

$$\lambda_a^* \Phi = \Phi, \quad \text{and} \quad \rho_a^* \Phi = \Phi, \quad a \in G.$$

Again, the set of bi-invariant forms $\mathcal{A}_I^*(G)$ on G is a subalgebra of $\mathcal{A}^*(G)$ which is stable under d. To see that they are closed, we prove

Lemma 3.3.2: $\nu^* \Phi = (-1)^k \Phi, \quad \Phi \in \mathcal{A}_I^k(G)$.

Proof: Using Lemma 3.1.1 we get for $a \in G$, $h_1, ..., h_k \in E$

$$\begin{aligned}(\nu^*\Phi)(a; R_a h_1, ..., R_a h_k) &= \Phi(a^{-1}; -L_{a^{-1}} h_1, ..., -L_{a^{-1}} h_k) \\ &= (-1)^k (\lambda_{a^{-1}}^* \Phi)(e; h_1, ..., h_k) \\ &= (-1)^k \Phi(e; h_1, ..., h_k) \\ &= (-1)^k \Phi(a; R_a h_1, ..., R_a h_k).\end{aligned}$$

Q.E.D.

Proposition 3.3.3: *The bi-invariant forms on G are closed, and so the inclusion*

$$\mathcal{A}_I^*(G) \hookrightarrow \mathcal{A}_L^*(G)$$

induces a homomorphism of graded algebras $\mathcal{A}_I^*(G) \to \mathcal{H}_L(G)$.

Proof: By the preceding lemma and the fact that $\mathcal{A}_I^*(G)$ is stable under d, we have

$$(-1)^{k+1} d\Phi = \nu^* d\Phi = d\nu^* \Phi = (-1)^k d\Phi, \quad \Phi \in \mathcal{A}_I^k(G).$$

Thus $d\Phi = 0$.

Q.E.D.

* Cf. [7; p.156]

Lemma 3.3.4: *If $\Phi \in \mathcal{A}_L^{\bullet}(G)$, then $\rho_a^* \Phi \in \mathcal{A}_L^{\bullet}(G)$ and*

$$\varrho(\rho_a^* \Phi) = Ad^* a(\varrho \Phi). \quad a \in G.$$

Proof: $\mathcal{A}_L^{\bullet}(G)$ is stable under ρ_a^* since $\rho_a^* \circ \lambda_a^* = \lambda_a^* \circ \rho_a^*$. If $\Phi \in \mathcal{A}_L^{\bullet}(G)$, then

$$\varrho(\rho_a^* \Phi) = (\rho_a^* \lambda_{a^{-1}}^* \Phi)(e) = (\tau_{a^{-1}}^* \Phi)(e),$$

where $\tau_{a^{-1}}$ is conjugation by a^{-1}. So for $\Phi \in \mathcal{A}_L^k(G)$ and $h_1, ..., h_k \in E$ we have

$$\begin{aligned}(\tau_{a^{-1}}^* \Phi)(e; h_1, ..., h_k) &= \Phi(e; \tau_{a^{-1}}' h_1, ..., \tau_{a^{-1}}' h_k) \\ &= \Phi(e; Ad\, a^{-1}(h_1), ..., Ad\, a^{-1}(h_k)) \\ &= (Ad^* a(\Phi(e))(h_1, ..., h_k).\end{aligned}$$

Q.E.D.

By the preceding lemma, ϱ restricts to an isomorphism between the invariant subalgebras. Thus the diagram

$$\begin{array}{ccc} A_I(G) & \longrightarrow & A_L(G) \\ \varrho_I \big\downarrow \cong & & \cong \big\downarrow \varrho \\ (\Lambda E^*)^G & \longrightarrow & \Lambda E^* \end{array}$$

commutes, where the horizontal maps are inclusions and $(\Lambda E^*)^G$ is the invariant subalgebra with respect to the Ad^* representation.

Note that by Proposition 3.2.1 and the Koszul formula we have $(\Lambda E^*)^G \subset (\Lambda E^*)^E \subset ker\, \delta_E$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} A_I(G) & \longrightarrow & \mathcal{H}_L(G) \\ \varrho_I \big\downarrow \cong & & \cong \big\downarrow \varrho^{\#} \\ (\Lambda E^*)^G & \longrightarrow & \mathcal{H}(E). \end{array}$$

3.4 Cohomology of Lie Groups

In this section we will compute the cohomology of compact connected Lie groups.

Let $T : G \times M \to M$ be an action of G on a smooth manifold M. Then, for each $a \in G$, T determines a diffeomorphism $T_a : M \to M$ by

$$T_a(x) = T(a, x).$$

If we let $\mathcal{A}_T^{\bullet}(M)$ represent the set of forms $\Phi \in \mathcal{A}^{\bullet}(M)$ such that

$$T_a^{*}\Phi = \Phi, \quad \text{for all } a \in G,$$

then $\mathcal{A}_T^{\bullet}(M)$ is a subalgebra of $\mathcal{A}^{\bullet}(M)$ stable under d. Again we can consider its cohomology algebra, $\mathcal{H}_T(M)$.

The cornerstone of the computation is the following theorem and we will refer the reader to [7; p.151] for its proof.

Theorem 3.4.1: *Let $i : \mathcal{A}_T^{\bullet}(M) \hookrightarrow \mathcal{A}^{\bullet}(M)$ denote the inclusion. If G is a compact Lie group then $i^{\#}$ is injective. Moreover, if G is connected, then i induces an isomorphism*

$$i^{\#} : \mathcal{H}_T(M) \xrightarrow{\cong} \mathcal{H}(M).$$

Note that it is independent of the nature of the action.

Theorem 3.4.2: *If G is a compact connected Lie group then the diagram*

$$\begin{array}{ccccc}
A_I(G) & \xrightarrow{\cong} & \mathcal{H}_L(G) & \xrightarrow{\cong} & \mathcal{H}(G) \\
{\scriptstyle \varrho_I}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \varrho^{*}} & & \\
(\Lambda E^{\bullet})^{E} & \xrightarrow{\cong} & \mathcal{H}(E) & &
\end{array}$$

commutes and all maps are isomorphisms.

Proof: Note that since G is connected we have $(\Lambda E^{*})^{G} = (\Lambda E^{*})^{E}$ where the invariant subspaces are taken with respect to the Ad^{*} and ad^{*} representations respectively. By Section 3.3, we thus have to show that the inclusions

$$\mathcal{A}_I^{*}(G) \hookrightarrow \mathcal{A}^{*}(G). \quad \text{and} \quad \mathcal{A}_L^{*}(G) \hookrightarrow \mathcal{A}^{*}(G)$$

induce isomorphisms $\mathcal{A}_I^{*}(G) \xrightarrow{\cong} \mathcal{H}(G)$, and $\mathcal{H}_L(G) \xrightarrow{\cong} \mathcal{H}(G)$. By Theorem 3.4.1 $\mathcal{H}_L(G) \to \mathcal{H}(G)$ is an isomorphism, where the action of G on itself is left translation. Now, define an action T of the compact connected Lie group $G \times G$ on G by

$$T_{(a,b)}(g) = a^{-1}gb, \qquad a, b, g \in G.$$

Then $\mathcal{A}_I^{*}(G)$ is exactly the algebra of differential forms on G which are invariant under this action. Since the forms in $\mathcal{A}_I^{*}(G)$ are closed, then by Theorem 3.4.1 $\mathcal{A}_I^{*}(G) \to H(G)$ is an isomorphism.

Q.E.D.

Corollary 3.4.3: $\mathcal{H}(G) \cong \Lambda P_E$.

Proof: By Theorem 3.4.2, $\mathcal{H}(G) \cong (\Lambda E^{*})^{E}$; and by Corollary 1.3.14, $(\Lambda E^{*})^{E} \cong \Lambda P_E$.

Q.E.D.

Chapter IV

Models, Formality, and Homogeneous Spaces

In this chapter we will make precise the idea of *real homotopy type of a manifold* M. This will lead us to the question of *formality*, i.e., when does the cohomology of a manifold M determine its real homotopy type?

Our aim is to determine when the homogeneous space G/T^2 is formal, where G is a compact connected Lie group of rank three and T^2 is an imbedded 2-torus. To begin, we will need to construct a model for this homogeneous space. We describe a general procedure to construct such a model.

The exposition comes essentially from [8] and [3].

4.1 Real Homotopy Type

Definition 4.1.1: Let (A, δ_A) and (B, δ_B) be two *c.g.d.a*'s. A *quasi-isomorphism* $\phi : (A, \delta_A) \rightarrow (B, \delta_B)$ is a *c.g.d.a* homomorphism such that the induced map $\phi^{\#}$ is an isomorphism.

Definition 4.1.2: Let (A, δ_A) be a *c.g.d.a*. A *model for* A is a *c.g.d.a* (B, δ_B) together with a quasi-isomorphism

$$\phi : (B, \delta_B) \rightarrow (A, \delta_A).$$

A *model* for a manifold M is a model of its de Rham complex $(\mathcal{A}^*(M), d)$.

Definition 4.1.3: Two *c.g.d.a*'s (A, δ_A) and (B, δ_B) are said to have the same *real homotopy type* if there exists a sequence of *c.g.d.a*'s $\{(D_k, \delta_k)\}_{k=1}^{n}$ and a chain of quasi-isomorphisms

$$(A, \delta_A) \longrightarrow (D_1, \delta_1) \longleftarrow \ldots \longrightarrow (D_n, \delta_n) \longleftarrow (B, \delta_B).$$

Moreover, the *real homotopy type of a manifold M* is the real homotopy type of its de Rham complex.

Recall that two smooth maps $\phi, \psi : M \to N$ are said to be homotopic, $\phi \sim \psi$, if there exists a smooth map $H : I \times M \to N$ such that

$$H(0, x) = \phi(x), \quad \text{and} \quad H(1, x) = \psi(x).$$

In this case, one can show* in general that

$$\phi^{\#} = \psi^{\#} : \mathcal{H}(M) \leftarrow \mathcal{H}(N).$$

The relation $\sim$ gives rise to the usual homotopy relation between manifolds, i.e.,

Definition 4.1.4: We say that two manifolds M and N have the same *homotopy type* if there exists two smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f \sim \iota_M$ and $f \circ g \sim \iota_N$. We call f and g *homotopy equivalences*.

By the preceding remarks it is clear that two manifolds with the same homotopy type must have the same *real* homotopy type.(The homotopy equivalences induce quasi-isomorphisms between the de Rham complexes.) The converse is false. For example let $\pi : \mathbf{S}^3 \to \mathbf{R}P^3$ be the canonical projection onto the real projective 3-space $\mathbf{R}P^3$. One can show** that

$$\mathcal{H}^0(\mathbf{S}^3) = \mathcal{H}^0(\mathbf{R}P^3) = \mathbf{R}, \quad \mathcal{H}^p(\mathbf{S}^3) = \mathcal{H}^p(\mathbf{R}P^3) = 0, \quad 1 \le p \le 2,$$

$$\text{and} \quad \mathcal{H}^3(\mathbf{S}^3) = \mathcal{H}^3(\mathbf{R}P^3) \cong \mathbf{R},$$

and that π^* is a quasi-isomorphism, i.e., $\mathbf{S}^3$ and $\mathbf{R}P^3$ have the same real homotopy type. On the other hand, two spaces with the same homotopy type must have isomorphic fundamental groups [13; p.371]. In this case we have

$$\pi_1(\mathbf{S}^3) = 0, \quad \text{but} \quad \pi_1(\mathbf{R}P^3) \cong \mathbf{Z}_2.$$

* Cf. [6;p.179]
** Cf. [6;p.187]

For the rest of this exposition we will concern ourselves only with the real homotopy type of a manifold M.

It is clear that two manifolds which have the same real homotopy type must have isomorphic cohomology. The converse is false. So, when does the cohomology of a manifold M determine its real homotopy type?

Definition 4.1.5: A *c.g.d.a* A is said to be *formal* if its cohomology $\mathcal{H}(A)$ considered as a *c.g.d.a* with 0 differential is a representative of its real homotopy class.

A differentiable manifold M is said to be *formal* if its de Rham complex is formal.

Remark: If (A, d_A) and (B, d_B) are two formal *c.g.d.a*'s, then their product $(A \otimes B, \delta)$ is also formal.

Here are some examples of formal manifolds.

(1) One knows that

$$\mathcal{H}(\mathrm{S}^n) \cong \begin{cases} \Lambda x_n & n \text{ odd} \\ \Lambda x_n/(x_n^2) & n \text{ even} \end{cases}, \quad n \geq 1,$$

where $deg\, x_n = n$. Let $\omega \in \mathcal{A}^n(\mathrm{S}^n)$ be an orientation form for S^n.

Suppose n is odd and consider the following map

$$\phi : (\Lambda x_n, 0) \to (\mathcal{A}^\bullet(\mathrm{S}^n), d)$$

$$1 \mapsto 1$$

$$x_n \mapsto \omega.$$

Then, since $\phi^2(x_n) = 0$ and $d\omega = 0$, ϕ is clearly a model for S^n. Thus the odd spheres are formal.

Now suppose n is even and consider

$$(\Lambda(a_n, b_{2n-1}), \delta), \quad \text{with} \quad \delta a_n = 0, \quad \text{and} \quad \delta b_{2n-1} = a_n^2.$$

One can easily see that $\mathcal{H}(\Lambda(a_n, b_{2n-1})) \cong \mathcal{H}(\mathrm{S}^n)$, and that the following map

$$\alpha : (\Lambda(a_n, b_{2n-1}), \delta) \to (\mathcal{A}^\bullet(\mathrm{S}^n), d)$$

$$1 \mapsto 1$$

$$a_n \mapsto \omega$$

$$b_{2n-1} \mapsto 0$$

is a model for S^n. In the same way, we can construct the following model β : $(\Lambda(a_n, b_{2n-1}), \delta) \to (\Lambda x_n/(x_n^2), 0)$ of $\mathcal{H}(S^n)$ by letting

$$1 \mapsto 1, \quad a_n \mapsto x_n, \quad \text{and} \quad b_{2n-1} \mapsto 0.$$

Putting α and β together, we get a chain

$$(\mathcal{A}^*(S^n), d) \longleftarrow (\Lambda(a_n, b_{2n-1}), \delta) \longrightarrow (\Lambda x_n/(x_n^2), 0)$$

of quasi-isomorphisms between $(\mathcal{A}^*(S^n), d)$ and $(\mathcal{H}(S^n), 0)$. So even spheres are formal.

(2) Let G be a compact connected Lie group. Using Theorem 3.4.2 to show that i : $((\Lambda E^*)^E, 0) \hookrightarrow (\mathcal{A}^*(G), d)$ is a quasi-isomorphism, makes $(\mathcal{H}(G), 0)$ a representative of the real homotopy type of G. Thus G is formal. Recall from Corollary 3.4.3 that $\mathcal{H}(G)$ is an exterior algebra over an oddly graded vector space, the primitive subspace P_E associated to the Hopf algebra $(\Lambda E^*)^E$.

(3) Other examples of formal spaces are the projective spaces, symmetric spaces*, simply connected *Kähler* manifolds**, and some homogeneous spaces; eg. G/T where G is a compact connected Lie group and T is a maximal torus (see Section 5.1).

4.2 Minimal Models

From the above examples we see that some representatives of the real homotopy type of a manifold are "simpler" in the sense that they are free as *c.g.a*'s (eg. α). In 1970, Sullivan made the following

Definition 4.2.1: A *c.g.d.a* (A, d) is said to be *minimal* if

(1) A is free as a *c.g.a* on a graded vector space V, i.e., $A \cong \Lambda V$, and

(2) V admits a homogeneous basis $\{x^\alpha\}_{\alpha \in J}$ indexed by a well ordered set J such that $\alpha < \beta \Rightarrow deg\, x^\alpha \leq deg\, x^\beta$, and $dx^\alpha \in \Lambda^{\geq 2}(x^\beta)_{\beta < \alpha}$.

A *minimal model* of a *c.g.d.a* (A, d_A) is a model $\phi : (\Lambda V, d) \to (A, d_A)$ where $(\Lambda V, d)$ is minimal.

* Cf. [3; p.41]

** Cf. [3; p.172]

Sullivan showed in [16] that minimal *c.g.d.a*'s are unique in the following sense.

Proposition 4.2.1: *A quasi-isomorphism between minimal c.g.d.a's is an isomorphism.*

Remark: If $V = \oplus_{i>1} V^i$, then

$$(\Lambda V, d) \text{ is minimal} \iff d(V) \subset \Lambda^{\geq 2} V.$$

The converse is false. Here is an example. Consider $\Lambda(x_1, y_1, z_1)$ and define

$$dx_1 = y_1 z_1, \quad dy_1 = z_1 x_1, \quad \text{and} \quad dz_1 = x_1 y_1.$$

Now consider the minimal *c.g.d.a* $(\Lambda\alpha_3, 0)$ and the following model

$$(\Lambda\alpha_3, 0) \to (\Lambda(x_1, y_1, z_1), d)$$

$$1 \mapsto 1$$

$$\alpha_3 \mapsto x_1 y_1 z_1.$$

It is clearly not an isomorphism, but it is a quasi-isomorphism. Note that by Corollary 1.3.7 $(\Lambda(x_1, y_1, z_1), d) \cong (\Lambda so(3)^*, \delta)$.

If (A, d_A) is such that $\mathcal{H}^0(A) = \mathbf{R}$ and that $\mathcal{H}^1(A) = 0$, then it is said to be 1-*connected*. If $dim\, \mathcal{H}^p(A) < \infty$ for all p, then $\mathcal{H}(A)$ is said to be of *finite type*.

Sullivan also showed that

Proposition 4.2.2: *If a c.g.d.a (A, d_A) is 1-connected and its cohomology is of finite type, then it has a minimal model.*

4.3 Pure Models

We introduce next a particular class of models: *pure models*. Their importance lie in the fact that they arise as models of homogeneous spaces in a natural way (Section 4.6).

Let $(\Lambda V, d)$ be a *c.g.d.a* where $V = Q \oplus P$ is a graded vector space with $Q = V_{even}$ and $P = V_{odd}$.

Definition 4.3.1: $(\Lambda V, d)$ is said to be a *pure c.g.d.a* if

$$d(Q) = 0, \quad \text{and} \quad d(P) \subset \Lambda Q.$$

A *pure model* of a *c.g.d.a* (A, d_A) is a model $\phi : (\Lambda V, d) \longrightarrow (A, d_A)$ such that $(\Lambda V, d)$ is a pure *c.g.d.a.*

Remark: Given a linear map $\delta : Q \oplus P \longrightarrow \Lambda Q \otimes \Lambda P$ homogeneous of degree $+1$ together with $\delta(Q) = 0$ and $\delta(P) \subset \Lambda Q$, one can extend δ to a derivation such that $(\Lambda Q \otimes \Lambda P, \delta)$ becomes a *c.g.d.a.* It is called the *Koszul complex.* Clearly it is a pure *c.g.d.a.* On the other hand, it is easy to see that if $\delta|_P = d|_P$, then

$$(\Lambda V, d) \cong (\Lambda Q \otimes \Lambda P, \delta).$$

Next we introduce the *Samelson subspace.* Its dimension will determine when a given homogeneous space is formal or not. Consider the map $\kappa : (\Lambda Q \otimes \Lambda P, d) \longrightarrow (\Lambda P, 0)$ given by

$$\kappa(1 \otimes \omega + \Phi \otimes \nu) = \omega, \qquad \omega, \nu \in \Lambda P,\ \Phi \in \Lambda^+ Q.$$

It is a *c.g.d.a* homomorphism.

Definition 4.3.2: The homomorphism $\kappa^\#$ induced by κ is called the *Samelson projection,* and the graded subspace

$$\hat{P} = P \cap Im\,\kappa^\#$$

is called the *Samelson subspace of P.*

We will call a graded subspace $P^c \subset P$ a *Samelson complement* if

$$P = \hat{P} \oplus P^c.$$

Lemma 4.3.1: *Let $\omega \in P$. Then*

$$\omega \in \hat{P} \iff d\omega \in \Lambda^+ Q \cdot d(P).$$

Proof: It follows from the observation that $\hat{P} = \kappa(Z^\bullet(\Lambda Q \otimes \Lambda P) \cap (\Lambda Q \otimes P))$.

Q.E.D.

Proposition 4.3.2(Reduction Theorem): *Let $(\Lambda Q \otimes \Lambda P, d)$ be a pure c.g.d.a with Samelson subspace $\hat{P}$ and Samelson complement P^c. Then there is a c.g.d.a isomorphism*

$$f : ((\Lambda Q \otimes \Lambda P^c) \otimes \Lambda \hat{P}, d \otimes \iota) \xrightarrow{\cong} (\Lambda Q \otimes \Lambda P, d).$$

Proof: Choose a linear map $\rho : \hat{P} \to Z^{\bullet}(A) \cap \Lambda Q \otimes P$ homogeneous of degree 0 such that $\kappa \circ \rho = \iota$. Since $\hat{P}$ is oddly graded we must have

$$\rho(x)^2 = 0, \quad x \in \hat{P}.$$

Thus we can extend ρ to a *c.g.a* homomorphism $\rho : \Lambda\hat{P} \to Z^{\bullet}(A)$. Define f by setting

$$f((\Phi \otimes \nu) \otimes \omega) \equiv (\Phi \otimes \nu) \cdot \rho(\omega), \qquad \Phi \in \Lambda Q,\ \nu \in \Lambda P^c,\ \omega \in \Lambda\hat{P}.$$

Clearly f is a *c.g.a* homomorphism. Moreover, since $d \circ \rho = 0$, we have

$$\begin{aligned}(d \circ f)((\Phi \otimes \nu) \otimes \omega) &= d\big((\Phi \otimes \nu) \cdot \rho(\omega)\big) \\ &= d(\Phi \otimes \nu) \cdot \rho(\omega) \\ &= \big(f \circ (d \otimes \iota)\big)((\Phi \otimes \nu) \otimes \omega).\end{aligned}$$

Thus f is a *c.g.d.a* homomorphism. Now consider the isomorphism

$$\Gamma : \Lambda Q \otimes \Lambda P^c \otimes \Lambda\hat{P} \xrightarrow{\cong} \Lambda Q \otimes \Lambda P$$

given by $\Phi \otimes \nu \otimes \omega \mapsto \Phi \otimes \nu \cdot \omega$ (see Proposition 1.3.10). Since $\kappa \circ \rho = \iota$, we must have $\rho(\omega) = 1 \otimes \omega + \Psi$ for $\omega \in \hat{P}$ and $\Psi \in \Lambda^+ Q \otimes P$, which implies that

$$f - \Gamma : \Lambda^k Q \otimes \Lambda P^c \otimes \Lambda\hat{P} \longrightarrow \bigoplus_{j>k} \Lambda^j Q \otimes \Lambda P. \tag{1}$$

Now filter the algebras $\Lambda Q \otimes \Lambda P^c \otimes \Lambda\hat{P}$ and $\Lambda Q \otimes \Lambda P$ by the ideals $\bigoplus_{j \geq k} \Lambda^j Q \otimes \Lambda P^c \otimes \Lambda\hat{P}$ and $\bigoplus_{j \geq k} \Lambda^j Q \otimes \Lambda P$ respectively. Then, since Γ is a filtration preserving isomorphism, it induces an isomorphism $\overline{\Gamma}$ between the associated algebras. Moreover, because of (1), f is a filtration preserving homomorphism and $\overline{f - \Gamma} \equiv 0$. Thus $\overline{f}$ is an isomorphism. It implies* that f is an isomorphism.

Q.E.D.

* Cf. [8; p.40]

Corollary 4.3.3: *The pure c.g.d.a* $(\Lambda Q \otimes \Lambda P), d)$ *has the same real homotopy type as*

$$(\Lambda Q \otimes \Lambda P^c, d) \otimes (\Lambda \hat{P}, 0).$$

Let $(\Lambda Q \otimes \Lambda P, d)$ be a pure *c.g.d.a.* If Q and P are finite dimensional then we can define the following integer

$$I \equiv dim\, P - dim\, \hat{P} - dim\, Q.$$

Remark: There is another gradation on $\Lambda Q \otimes \Lambda P$, called the *lower gradation*, given by

$$\Lambda Q \otimes \Lambda P = \bigoplus_k \Lambda Q \otimes \Lambda^k P.$$

Evidently d is homogeneous of degree -1 with respect to the lower gradation. Thus a lower gradation is induced on $\mathcal{H}(\Lambda V)$ and is denoted by

$$\mathcal{H}(\Lambda V) = \bigoplus_k \mathcal{H}_k(\Lambda V), \quad \text{where} \quad \mathcal{H}_k(\Lambda V) = Z_k(\Lambda V)/B_k(\Lambda V).$$

Note that $\mathcal{H}_0(\Lambda V) = \Lambda Q/(dP)$.

The next theorem is a technical result and we refer the reader to [8; p.78] for its proof.

Theorem 4.3.4: *Let* $(\Lambda Q \otimes \Lambda P, d)$ *be a pure c.g.d.a with finite dimensional cohomology. Then* $I \geq 0$, *i.e.,*

$$dim\, P \geq dim\, \hat{P} + dim\, Q.$$

Moreover, given a Samelson complement P^c, *the following conditions are equivalent*

(1) $dim\, P = dim\, \hat{P} + dim\, Q$,

(2) $\mathcal{H}_+(\Lambda Q \otimes \Lambda P^c) = 0$.

Remark: $dim\, \mathcal{H}(\Lambda Q \otimes \Lambda P) < \infty$ if and only if $dim\, \Lambda Q/(dP) < \infty$. This can be seen by noting that since $ker\, d$ is a ΛQ-submodule of the finitely generated ΛQ-module $\Lambda Q \otimes \Lambda P$ and ΛQ is Noetherian, then $ker\, d$ is also finitely generated. Thus $\mathcal{H}(\Lambda Q \otimes \Lambda P)$ is a finitely generated ΛQ-module. Hence it is also a finitely generated $\Lambda Q/(dP)$-module.

The following theorem is key in that it gives us a criterion to determine if a given pure *c.g.d.a* is formal.

Theorem 4.3.5: *Let* $(\Lambda Q \otimes \Lambda P, d)$ *be a pure c.g.d.a with finite dimensional cohomology. If* $dim\, P = dim\, \hat{P} + dim\, Q$, *then* $(\Lambda Q \otimes \Lambda P, d)$ *is formal.*

Proof: Assume $dim\, P = dim\, \hat{P} + dim\, Q$ and let P^c be a Samelson complement. Clearly $(\Lambda \hat{P}, 0)$ is formal. Thus, using Corollary 4.3.3, we just have to show that

$$(\Lambda Q \otimes \Lambda P^c, d)$$

is formal. Define $\psi : (\Lambda Q \otimes \Lambda P^c, d) \to (\Lambda Q/(dP^c), 0)$ by

$$\psi(\Phi \otimes 1 + \Psi \otimes \nu) = [\Phi], \qquad \Phi, \Psi \in \Lambda Q,\ \nu \in \Lambda^+ P^c.$$

Applying Theorem 4.3.4 we get $\mathcal{H}_+(\Lambda Q \otimes \Lambda P^c) = 0$. Thus $(\Lambda Q \otimes \Lambda P^c, d)$ is formal since $\mathcal{H}(\Lambda Q \otimes \Lambda P^c) = \mathcal{H}_0(\Lambda Q \otimes \Lambda P^c) = \Lambda Q/(dP^c)$ and ψ is a quasi-isomorphism.

Q.E.D.

Remark: The converse is also true*.

To conclude, we include here the notion of *formal dimension*. It will be crucial in Section 5.4.

Definition 4.3.3: A finite dimensional *c.g.a* A is said to be of *formal dimension* n if $A^n \neq 0$ and $A^k = 0$ for $k > n$.

Halperin gave the following formula to determine the formal dimension of the cohomology algebra of a pure minimal *c.g.d.a*.[9] It is also known as the *degree of the top cohomology class*.

Theorem 4.3.6(Halperin's formula): *Let* $(\Lambda Q \otimes \Lambda P, d)$ *be a pure minimal c.g.d.a with finite dimensional cohomology. Given a homogeneous basis* $\{x_\alpha\}$ *of* $Q \oplus P$, *the degree* m *of the top cohomology class is given by*

$$m = dim\, Q - \sum_\alpha (-1)^{deg\, x_\alpha} deg\, x_\alpha.$$

* Cf. [8; p.152]

4.4 The Weil Algebra

Our next step is to construct a model for a homogeneous space G/H. We present here the *Weil algebra*. It is a key step in constructing the *Cartan map* which in turn is a key ingredient in the construction of the differential of our model.

Let E be a finite dimensional Lie algebra with basis $\{e_\nu\}$ and dual basis $\{e^{*\nu}\}$, and consider the following free *c.g.a* generated by $sE^* \oplus E^*$ (again we consider E^* as a graded vector space concentrated in degree 1)

$$W(E) = \Lambda sE^* \otimes \Lambda E^*,$$

together with the following derivations on $W(E)$

$$\begin{aligned} d &= 1 \otimes \delta_E + d_s + h, \\ d_s &= \sum_\nu ad^* e_\nu \otimes \mu(e^{*\nu}), \\ h &= \sum_\nu \mu(se^{*\nu}) \otimes i(e_\nu), \\ k &= \sum_\nu i(se_\nu) \otimes \mu(e^{*\nu}), \qquad \text{and} \\ \theta_W(x) &= ad^* x \otimes 1 + 1 \otimes ad^* x, \quad x \in E, \end{aligned}$$

(recall the operators μ, i, and δ_E from Section 1.3). One easily shows that d, d_s, and h are derivations of degree +1, that k is a derivation of degree −1, and that θ_W is a derivation of degree 0. Note that θ_W is a Lie algebra representation of E in $W(E)$. The invariant subspace $W(E)^E$ will always be taken with respect to θ_W. The next proposition* is similar to Theorem 1.3.6, i.e.,

Proposition 4.4.1: *Let $x \in E$. Then*

$$1 \otimes i(x) \circ d + d \circ 1 \otimes i(x) = \theta_W(x), \qquad \theta_W(x) \circ d = d \circ \theta_W(x),$$

$$\text{and} \qquad d^2 = 0.$$

---

* Cf. [8; p.226]

Definition 4.4.1: The *c.g.d.a* $(W(E), d)$ is called the *Weil algebra of the Lie algebra E.*

Remark: Combining the Koszul formula with the definition of d_s one gets

$$2(1 \otimes \delta_E) + d_s = \sum_\nu 1 \otimes \mu(e^{*\nu}) \circ \theta_W(e_\nu).$$

Thus d restricts to $\frac{1}{2}d_s + h$ on $W(E)^E$.

Theorem 4.4.2: $\mathcal{H}(W(E)) \cong \mathcal{H}(W(E)^E) \cong \mathbf{R}$.

Proof: If we consider the derivation $B \equiv d \circ k + k \circ d$ of degree 0 one can show* that it restricts to linear isomorphisms on $W^r(E)$, $r \geq 1$. Fix an $r \geq 1$. Then the minimal polynomial of the restriction must have a nonzero constant term. Thus for $\Omega \in W^r(E)$ we have

$$\Omega = \sum_{j\geq 1} c_j B^j \Omega, \qquad c_j \in \mathbf{R}.$$

Combining this fact with $k^2 = 0$ and $k\theta_W(x) = \theta_W(x)k$, one obtains the result.

Q.E.D.

4.5 Cartan map and Transgressions

Throughout this section E and F will denote finite dimensional reductive Lie algebras.

Let $\phi : F \to E$ be a Lie algebra homomorphism. Then its dual can be extended to homomorphisms of algebras

$$\phi_1 : \Lambda F^* \leftarrow \Lambda E^*, \quad \text{and} \quad \phi_2 : \Lambda sF^* \leftarrow \Lambda sE^*.$$

The relation $\phi \circ (ad\, y) = (ad\, \phi y) \circ \phi$ for $y \in F$, yields

$$(ad^* y) \circ \phi_i = \phi_i \circ (ad^* \phi y), \quad i = 1, 2,$$

by dualizing and noting that both sides are ϕ_i-derivations. Thus they both restrict to the invariant subspaces and we will denote their restrictions by

$$\phi_* : (\Lambda F^*)^F \leftarrow (\Lambda E^*)^E, \quad \text{and} \quad \phi^* : (\Lambda sF^*)^F \leftarrow (\Lambda sE^*)^E.$$

---

* Cf. [8; p229]

The Cartan map

We need two maps in the construction of the differential for the model of G/H: *the Cartan map* and a *transgression.* We now build the first one.

Consider the projection $\pi_E : W(E) \to \Lambda E^*$ given by

$$\pi_E(1 \otimes \omega + \Phi \otimes \nu) = \omega, \qquad \omega, \nu \in \Lambda E^*, \Phi \in \Lambda^+ sE^*.$$

Clearly π_E is a *c.g.a* homomorphism. Moreover we have

$$\delta_E \circ \pi_E = \pi_E \circ d \quad \text{and} \quad \pi_E \circ \theta_W(x) = ad^* x \circ \pi_E, \qquad x \in E.$$

In particular, π_E restricts to a homomorphism on the invariant subalgebras, i.e.,

$$\pi_E : W(E)^E \to (\Lambda E^*)^E.$$

Now, since $d = \frac{1}{2} d_s + h$ on $W(E)^E$, on this subalgebra

$$\pi_E \circ d = 0.$$

And finally, if $\phi : F \to E$ is a Lie algebra homomorphism, then

$$\phi_1 \circ \pi_E = \pi_F \circ (\phi_2 \otimes \phi_1).$$

Proposition 4.5.1: *For all $\Phi \in (\Lambda^+ sE^*)^E$, there is a unique element $\omega \in (\Lambda^+ E^*)^E$ such that for some $\Omega \in W^+(E)^E$*

$$\pi_E(\Omega) = \omega, \quad \text{and} \quad d\Omega = \Phi \otimes 1.$$

Proof: Clearly $d(\Phi \otimes 1) = 0$ since Φ is invariant. Thus, by Theorem 4.4.2, there exists an element $\Omega \in W^+(E)^E$ such that

$$d\Omega = \Phi \otimes 1.$$

We claim that $\omega = \pi_E(\Omega)$ is unique. If $\Omega' \in W^+(E)^E$ is another such element with $d\Omega' = \Phi \otimes 1$, then

$$d(\Omega - \Omega') = 0.$$

And so, again by Theorem 4.4.2,

$$\Omega - \Omega' = d\hat{\Omega}, \quad \text{for some} \quad \hat{\Omega} \in W^+(E)^E.$$

It follows that

$$\pi_E\Omega - \pi_E\Omega' = \pi_E d(\hat{\Omega}) = 0.$$

Thus ω is independent of the choice of Ω.

Q.E.D.

Definition 4.5.1: The correspondence $\Phi \longmapsto \omega$ defines a linear map

$$\varrho_E : (\Lambda^+ sE^*)^E \longrightarrow (\Lambda^+ E^*)^E$$

homogeneous of degree -1 called the *Cartan map for E*.

The Cartan map is canonical in the following sense:

Lemma 4.5.2: *If $\phi : F \longrightarrow E$ is a homomorphism of Lie algebras, then*

$$\phi_* \circ \varrho_E = \varrho_F \circ \phi^*.$$

Proof: We will use the fact* that $(\phi_2 \otimes \phi_1) \circ d = d \circ (\phi_2 \otimes \phi_1)$. Let $\Phi \in (\Lambda^+ sE^*)^E$ with $\Omega \in W^+(E)^E$ such that

$$d\Omega = \Phi \otimes 1.$$

Then $\phi_2\Phi \otimes 1 = \phi_2 \otimes \phi_1(\Phi \otimes 1) = \phi_2 \otimes \phi_1(d\Omega) = d \circ (\phi_2 \otimes \phi_1)(\Omega)$, and hence

$$\varrho_F\phi_2\Phi = \pi_F \circ (\phi_2 \otimes \phi_1)(\Omega) = \phi_1\pi_E\Omega = \phi_1\varrho_E\Phi.$$

Q.E.D.

Using the isomorphism between $\Lambda^q sE^*$ and the symmetric q-linear forms on E, and the isomorphism between $\Lambda^{2q-1}E^*$ and the skew-symmetric $(2q-1)$-linear forms on E, one can give** an explicit description of the Cartan map:

* Cf. [8; p.228]
** Cf. [8; p.234]

Proposition 4.5.3: *The Cartan map for a Lie algebra E is given by*

$$(\varrho_E\Phi)(x_1,\dots,x_{2q-1}) = \frac{(-1)^{q-1}(q-1)!}{2^{q-1}(2q-1)!}\sum_{\sigma\in S^{2q-1}}\epsilon_\sigma\Phi(x_{\sigma(1)},[x_{\sigma(2)},x_{\sigma(3)}],\dots,[x_{\sigma(2q-2)},x_{\sigma(2q-1)}]),$$

where S^{2q-1} is the symmetric group, ϵ is the signature, and $\Phi\in(\Lambda^q sE^*)^E$, $x_i\in E$.

The following properties of the Cartan map will be needed in Chapter 5 to show that in some cases G/T^2 is formal by exhibiting a basis of P_E: the subspace of primitives of the associated Lie algebra E of G.

Proposition 4.5.4: *If E is a semisimple Lie algebra, then the restriction of the Cartan map to $(\Lambda^2 sE^*)^E$ is injective.*

Proof: If we restrict ourselves to $q=2$, since $\Phi\in(\Lambda^2 sE^*)^E$ is invariant, we have

$$\Phi([x,y],z)=\Phi([y,z],x)=\Phi([z,x],y),\qquad x,y,z\in E.$$

Thus, using Proposition 4.5.3, we have $\varrho_E(\Phi)(x,y,z)=-\frac{1}{2}\Phi([x,y],z)$. Now suppose that $\Phi\in ker\,\varrho_E$. Then, we must have

$$\Phi([x,y],z)=0,\quad \text{for all } x,y,z\in E.$$

But, since E is semisimple, $E'=E$ thus $\Phi\equiv 0$.

Q.E.D.

Corollary 4.5.5: *If E is a semisimple Lie algebra, then*

$$\varrho_E(K)\neq 0,$$

where K is the Killing form of E.

Proof: Recall from Section 1.3 that K is a symmetric bilinear form and satisfies

$$K([x,y],z)+K(y,[x,z])=0,\qquad x,y,z\in E.$$

Thus $K\in(\Lambda^2 sE^*)^E$. Moreover, since E is semisimple, K is nondegenerate. In particular $K\neq 0$. The result follows from Proposition 4.5.4.

Q.E.D.

Transgressions

Next we introduce a *transgression.* It is the heart of the differential of the model for G/H. Recall the Definition 1.3.12 of the primitive subspace P_E of E.

Definition 4.5.2: A linear map $\tau : P_E \longrightarrow (\Lambda sE^{*})^E$ homogeneous of degree $+1$ is a *transgression* if for every $\omega \in P_E$, there is an element $\Omega \in W^{+}(E)^E$ with

$$d\Omega = \tau\omega \otimes 1, \quad \text{and} \quad 1 \otimes \omega - \Omega \in ker\, \pi_E.$$

A simple application of the definition of the Cartan map gives

Lemma 4.5.6: *Let $\tau : P_E \longrightarrow (\Lambda sE^{*})^E$ be a linear map homogeneous of degree $+1$. Then*

$$\tau \text{ is a transgression} \iff \varrho_E \circ \tau = \iota.$$

Using spectral sequences and Theorem 4.4.2, Koszul proved the next result, which is known as the *Symmetric Hopf Theorem.* We refer the reader to [8; p.242] for its proof.

Theorem 4.5.7: *Let E be a reductive Lie algebra. Then there exists a transgression $\tau : P_E \longrightarrow (\Lambda sE^{*})^E$. Moreover, we can extend τ to a c.g.a isomorphism*

$$\tau : \Lambda sP_E \xrightarrow{\cong} (\Lambda sE^{*})^E.$$

Thus $(\Lambda sE^{*})^E$ is a polynomial algebra.

Corollary 4.5.8: *Let E be a reductive Lie algebra. Then*

$$ker\, \varrho_E = (\Lambda^{+} sE^{*})^E \cdot (\Lambda^{+} sE^{*})^E, \quad \text{and} \quad Im\, \varrho_E = P_E.$$

Proof: Let $\Phi_1, \Phi_2 \in (\Lambda^{+} sE^{*})^E$ with $\Omega \in W^{+}(E)^E$ such that $d\Omega = \Phi_1 \otimes 1$. Since $d(\Phi_2 \otimes 1) = 0$, we have

$$\Phi_1\Phi_2 \otimes 1 = (\Phi_1 \otimes 1) \cdot (\Phi_2 \otimes 1) = d\Omega \cdot (\Phi_2 \otimes 1) = d(\Omega_1 \cdot (\Phi_2 \otimes 1)).$$

And so

$$\varrho_E(\Phi_1\Phi_2) = \pi_E(\Omega_1 \cdot (\Phi_2 \otimes 1)) = \pi_E\Omega \cdot \pi_E(\Phi_2 \otimes 1) = 0.$$

We thus have $(\Lambda^+ sE^*)^E \cdot (\Lambda^+ sE^*)^E \subset ker\,\varrho_E$. By Theorem 4.5.7, there is a transgression τ, and, by Lemma 4.5.6, it satisfies $\varrho_E \circ \tau = \iota$. Clearly we have

$$\Lambda^+ sP_E = sP_E \oplus \Lambda^+ sP_E \cdot \Lambda^+ sP_E,$$

and applying the extension of τ to $\Lambda^+ sP_E$ leaves us with

$$(\Lambda^+ sE^*)^E = \tau(sP_E) \oplus (\Lambda^+ sE^*)^E \cdot (\Lambda^+ sE^*)^E.$$

Q.E.D.

Important for our later work is the next result.

Corollary 4.5.9: *If E is a semisimple Lie algebra, then*

$$0 \neq \varrho_E(K) \in P_E.$$

4.6 Homogeneous Spaces

Let G be a compact connected Lie group and H a connected closed Lie subgroup. There is a well defined smooth manifold structure* on the homogeneous space G/H. We can thus ask ourselves: "*When is G/H formal?*" We will tackle the problem by first constructing a pure model for G/H, then use the characterisation of formality given in Theorem 4.3.5.

We will now indicate a way of constructing such a model. Note that, by Proposition 2.4.2, we must have

$$dim\,\mathcal{H}(G/H) < \infty.$$

Denote the imbedding of H into G by

$$J : H \to G.$$

Let E and F denote the Lie algebras of G and H respectively. Then, by Proposition 3.2.4, they are both reductive. By Proposition 3.1.3, we have a Lie algebra homomorphism

$$j \equiv J' : F \hookrightarrow E$$

* Cf. [7; p.77]

which is simply the inclusion of F into E. Choose a transgression $\tau : P_E \longrightarrow (\Lambda sE^*)^E$ and let $Q_F \equiv sP_F$. Then, using the Hopf Symmetric Theorem, we get a linear map

$$P_E \xrightarrow{\tau} (\Lambda sE^*)^E \xrightarrow{j^*} \Lambda Q_F$$

homogeneous of degree +1. Thus, we can form the Koszul complex, i.e.,

$$(\Lambda Q_F \otimes \Lambda P_E, \delta).$$

The cornerstone of this thesis is the following result. We refer the reader to [8; p.462] for the proof.

Theorem 4.6.1: *There is a quasi-isomorphism*

$$\phi : (\Lambda Q_F \otimes \Lambda P_E, \delta) \longrightarrow (\mathcal{A}^*(G/H), d).$$

Corollary 4.6.2: $dim\, \mathcal{H}(\Lambda Q_F \otimes \Lambda P_E) < \infty$.

Corollary 4.6.3: *If* $dim\, P_E = dim\, \hat{P}_E + dim\, Q_F$, *then* G/H *is formal.*

Remark: When it comes to actually computing the Koszul complex, one chooses a homogeneous linearly independent set $\{\Omega_i\} \subset (\Lambda^+ sE^*)^E$ such that $\{\omega_i = \varrho_E(\Omega_i)\}$ is a basis for P_E (recall that $Im\, \varrho_E = P_E$). Then, we define

$$\tau(\omega_i) \equiv \Omega_i,$$

and extend linearly. By Lemma 4.5.6, τ is a transgression. It remains to compute $\{j^*\Omega_i\}$. But this is done by computing the restriction of the symmetric form Ω_i to F. These kinds of calculations were first studied by Koszul. We present an example in Section 5.5.

Chapter V

Formal and Non-Formal Homogeneous Spaces

Let G be a compact connected Lie group and $J : T^2 \to G$ a Lie group imbedding of a 2-torus T^2 in G.

The aim of this chapter is to classify all homogeneous spaces of the type G/T^2, where G is of *rank* 3, according to whether or not they are formal. We will denote the Lie algebras of G and T^2 by E and F respectively and $J' = j : F \hookrightarrow E$ the natural inclusion. Note that since F is abelian we have

$$(\Lambda sF^*)^F = \Lambda sF^*, \quad \text{and} \quad P_F = F^*.$$

Thus $(\Lambda sF^*)^F$ is a polynomial algebra in two variables(of degree 2). We will denote this algebra by

$$\Lambda Q_F = \Lambda(y_2, z_2).$$

5.1 Maximal Tori

Let T be a closed connected abelian subgroup of G. Then T is compact and a Lie subgroup*. It follows that T is a torus**. Note that the Lie algebra of a torus is abelian.

Definition 5.1.1: We will say that T is a *maximal torus* in G if it is not properly contained in another torus.

---

* Cf. [7; p.63]
** Cf. [7; p.45]

Clearly the automorphisms $\tau_g(x) = gxg^{-1}$, $g \in G$, carry a maximal torus onto a maximal torus. Recall that an element a in G is called a *generator of* G if

$$\overline{\{a^k \mid k \in \mathbf{Z}\}} = G.$$

Lie group theory tells us that ([7; p.46 and p.92])

Proposition 5.1.1: *Let G be a compact connected Lie group and T a torus. Then*

(1) T has a generator,

(2) every element in G is contained in a maximal torus, and

(3) any two maximal tori are conjugate.

This result permits us to define

Definition 5.1.2: Let T be a maximal torus in G. Then the *rank* of G is

$$rk\, G \equiv dim\, T.$$

Given a Lie algebra E, recall that the *normalizer* of a Lie subalgebra F is given by

$$N(F) \equiv \{x \in E \mid ad\, x(F) \subset F\}.$$

Definition 5.1.3: A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra E is a nilpotent subalgebra such that

$$N(\mathfrak{h}) = \mathfrak{h}.$$

In particular, when E is compact, one can show the existence of Cartan subalgebras, and that they are in fact the maximal abelian Lie subalgebras of E. Note that they always contain the center Z_E of E. Moreover, one can show that they are in fact conjugate. Thus we can define the *rank* of a compact Lie algebra E by

$$rk\, E \equiv dim\, \mathfrak{h}.$$

Given a compact connected Lie group G, we see that the Lie subalgebra of a maximal torus is in fact a Cartan subalgebra (since E is compact). Thus, we have

$$rk\, G = rk\, E.$$

For E compact, this is another way of seeing that $rk\,E$ is independent of the chosen Cartan subalgebra since all maximal abelian Lie subalgebras have the same dimension (being the Lie algebras of the maximal tori in G).

Remark: Since E is reductive, we have $E = Z_E \oplus E'$. Thus

$$rk\,E = dim\,Z_E + rk\,E'.$$

We have shown that (Theorem 3.4.2 and Corollary 3.4.3)

$$\Lambda P_E \cong (\Lambda E^*)^E \cong \mathcal{H}(G).$$

Moreover, one can show* that

$$dim\,P_E = rk\,G.$$

Thus we obtain

Theorem 5.1.2: *If G is a compact connected Lie group and T is a maximal torus in G, then G/T is formal.*

Proof: We have that

$$dim\,P_E = dim\,T = dim\,Q_F.$$

Hence, by Theorem 4.3.4 and 4.3.5, G/T is formal.

Q.E.D.

5.2 The primitive elements

We are interested here in the case of a compact connected Lie group G of rank 3 and an imbedded 2-torus T^2. Notice that the model of G/T^2 described in Chapter 4 dependents only on the nature of the corresponding Lie algebras and the imbedding. We will see that in most cases the knowledge of the primitive elements solves the problem of determining when is G/T^2 formal.

Recall from Chapter 3 that if G is compact, its Lie algebra is reductive, and thus can be written as $E = Z_E \oplus E'$ with E' semisimple. Hence, $E^* = (Z_E)^\perp \oplus (E')^\perp$ with

$$dim\,(E')^\perp = dim\,Z_E.$$

* Cf. [7; p.167]

Lemma 5.2.1: *If E is a reductive Lie algebra, then $(E')^\perp = P^1_E$. Moreover*

$$dim\, Z_E = dim\, P^1_E.$$

Proof: We first show that $(E')^\perp \subset P^1_E$. Let $x^* \in (E')^\perp$ and $z \in E$. Then for all $x \in E$ we have

$$< ad^* x(x^*), z > \; = \; - < x^*, ad\, x(z) >$$
$$= - < x^*, [x,z] >= 0.$$

Thus x^* is invariant. On the other hand recall that the comultiplication Δ on $(\Lambda E^*)^E$ from Theorem 1.3.13 is a *c.g.a* homomorphism. In particular it is homogeneous of degree 0. Now, since $deg\, x^* = 1$, we must have

$$\Delta(x^*) \; = \; x^* \otimes 1 + 1 \otimes x^*.$$

Hence $x^* \in P^1_E$. Next we show $P^1_E \subset (E')^\perp$. Let $x^* \in P^1_E$. We have $ad^* x(x^*) = 0$ for all $x \in E$ since $P^1_E \subset (\Lambda E^*)^E$. In other words we have

$$0 =< ad^* x(x^*), y > \; = \; - < x^*, ad\, x(y) >$$
$$= - < x^*, [x,y] >, \qquad x, y \in E.$$

Thus $x^* \in (E')^\perp$.

Q.E.D.

When $Z_E = \{0\}$ we have $E = E'$ and E a compact, semisimple Lie algebra. And so, by applying Theorem 1.3.9 we get $b_3(E) = dim\, P^3_E$, which is the number of simple ideals in E. We still have some grip on P^3_E when E is a nonabelian reductive Lie algebra.

Lemma 5.2.2: *If E is a reductive Lie algebra such that $E \neq Z_E$, then $dim\, P^3_E \geq 1$.*

Proof: First, suppose $E = E'$. Then E is semisimple. Thus, by Theorem 1.3.3, the Killing form K_E is nondegenerate. Using the Cartan map and Corollary 4.5.9 we have

$$P^3_E \ni \varrho_E(K_E) \neq 0.$$

Recall that the Cartan map is homogeneous of degree -1. Now consider the general case $E \neq E'$, and consider the inclusion $i : E' \hookrightarrow E$. Then, by Lemma 1.3.1, we have

$$K_{E'} = K_{E|_{E' \times E'}} = i^*(K_E).$$

Now combining the first result with Lemma 4.5.2, i.e.,

$$i_* \circ \varrho_E = \varrho_{E'} \circ i^*,$$

shows that $P_E^3 \ni \varrho_E(K_E) \neq 0$.

Q.E.D.

Using Lemmas 5.2.1 and 5.2.2 we can now enumerate all the possible primitive subspaces associated to a rank 3 compact connected Lie group's Lie algebra. We have divided the possible P_E's into six cases. We will show in Section 5.4 that the first five cases are always formal. On the other hand, we will see in Section 5.5 that the last case depends on the imbedding whether it is formal or not.

Now, since P_E is oddly graded and $dim\, P_E = 3$, we have

Generic format of the generators

of the possible P_E's

Cases	dim Z_E	$b_3(E)$	P_E
a)	3	1	$< x_1, x_1', x_1'' >$
b)	2	1	$< x_1, x_1', x_3 >$
c)	1	1 or 2	$< x_1, x_3, x_{2j+1} >, j \geq 1$
d)	0	1	$< x_3, x_{2j+1}, x_{2k+1} >, j, k \geq 2$
e)	0	2	$< x_3, x_3', x_{2j+1} >, j \geq 2$
f)	0	3	$< x_3, x_3', x_3'' >$

Table 1

where the subscripts represent the degree of the generators and the occasional primes or double primes distinguish two generators of the same degree.

5.3 The Killing form

The Killing form K plays a central role in our problem. As we will see, its importance lies in the fact that it is a "well-behaved", canonical element of $(\Lambda^2 sE^*)^E$ (Corollary 4.5.9). Every nonabelian reductive Lie algebra comes equipped naturally with this non-trivial symmetric, bilinear form.

The next result is key. Recall that $j^*(K)$ is a homogeneous polynomial of degree 2(i.e. *polynomial* degree) in ΛsF^*, where F is the 2-dimensional abelian Lie algebra of T^2 and j denotes the usual inclusion of F into E.

Theorem 5.3.1: *Let G be a compact connected semisimple Lie group of rank at least 2 and T^2 an imbedded 2-torus. Then $j^*(K)$ is an irreducible polynomial in ΛsF^*.*

Proof: Since the Lie algebra E of G is compact, there exists a negative definite inner product on E such that

$$< ad\,\alpha(x), y > + < x, ad\,\alpha(y) > = 0, \qquad \alpha, x, y \in E.$$

That is $(ad\,\alpha)^\dagger = -ad\,\alpha$. Thus the Killing form has value

$$K(\alpha,\alpha) = tr\,(ad\,\alpha)^2 = -tr\,ad\,\alpha \cdot (ad\,\alpha)^\dagger \leq 0.$$

But, since E is semisimple and K is nondegenerate, K must be negative definite. Thus the signature of K is $-dim\,E$. Now $j^*(K)$ is the restriction of K to a 2-dimensional subspace $F \subset E$. It must still be negative definite, thus, by the *Principal-Axis Theorem*, there is an orthonormal basis for F such that

$$K|_F = -(a(sx^*)^2 + b(sy^*)^2), \qquad a, b \in \mathbf{R}^+,$$

where $F =< x, y >$ and s is the suspension map (see Definition 1.1.1). Hence $j^*(K)$ is irreducible in ΛsF^*.

Q.E.D.

5.4 The formal cases

In this section we will show that the first five cases of Table 1 are all formal. The important result that we will use is Theorem 4.3.5 that asserts that formality is implied by

$$dim\,P_E = dim\,F + dim\,\hat{P}_E,$$

where $\hat{P}_E$ is the Samelson subspace. Note that in our case, this means that we have to construct a nonzero element in $\hat{P}_E$. Thus, we will apply the criterion of Lemma 4.3.1, i.e., for $x \in P_E$ (recall from p.53 that $\Lambda sF^* = \Lambda(y_2, z_2)$)

$$x \in \hat{P}_E \iff d(x) \in \Lambda^+(y_2, z_2) \cdot d(P_E).$$

Finally we will need the following notion from commutative algebra.

Definition 5.4.1: Let R be a commutative ring with unity. A sequence $\{r_1, r_2, \ldots, r_n\} \subset R$ is said to be *regular* if r_1 is not a zero divisor in R and the class $[r_i]$ is not a zero divisor in

$$R/(r_1, \ldots, r_{i-1}), \qquad \text{for } 2 \leq i \leq n.$$

One can then show*

Lemma 5.4.1: *Consider the polynomial ring $\Lambda(y_2, z_2)$ and two homogeneous polynomials f_1 and f_2. Then*

$$dim_{\mathbf{R}} \Lambda(y_2, z_2)/(f_1, f_2) < \infty \quad \Longleftrightarrow \quad \{f_1, f_2\} \text{ is a regular sequence.}$$

Theorem 5.4.2: *The five cases, (a) through (e), of Table 1 are all formal.*

Proof:

(a): This is the abelian case. Being connected, G is generated by its abelian Lie algebra through the exponential map** and thus must be abelian***. Now, since T^2 is a closed normal subgroup of G, G/T^2 is a Lie group, and by Section 4.1 it must be formal. Note that this result is true for any n-torus G with an imbedded k-torus T^k, $1 \leq k \leq n$. The resulting Lie group must be an $(n-k)$-torus. In our case G/T^2 is a circle.

(b): Notice that any odd cocycle of P_E is in the Samelson subspace. Thus if $dx_1 = 0$ or $dx_1' = 0$ we are done. If $dx_1 = \alpha dx_1'$ for $\alpha \in \mathbf{R}$, then clearly $x_1 - \alpha x_1' \in \hat{P}_E$ and we are done. Now if they are linearly independent, then without lost of generality we can assume that $dx_1 = y_2$ and $dx_1' = z_2$. Since the model is pure, we have

$$\begin{aligned} dx_3 &= \alpha y_2^2 + \beta y_2 z_2 + \gamma z_2^2, \quad \alpha, \beta, \gamma \in \mathbf{R}, \\ &= \alpha y_2 dx_1 + \beta y_2 dx_1' + \gamma z_2 dx_1' \in \Lambda^+(y_2, z_2) \cdot d(P_E), \end{aligned}$$

and so $x_3 \in \hat{P}_E$.

* Cf. [17; Appendix 6]

** Cf. [7; p.35]

*** Cf. [7; p.44]

(c): If $dx_1 = 0$ we are done. Else, without lost of generality we can assume that $dx_1 = y_2$. Again we have

$$dx_3 = \alpha y_2^2 + \beta y_2 z_2 + \gamma z_2^2, \qquad \alpha, \beta, \gamma \in \mathbf{R},$$

and if $\gamma = 0$, then $x_3 \in \hat{P}_E$ and we are done. But, if $\gamma \neq 0$, then y_2^2, y_2z_2, and z_2^2 can be written as

$$y_2^2 = y_2 dx_1,$$

$$y_2 z_2 = z_2 dx_1, \text{ and}$$

$$z_2^2 = \gamma^{-1}(dx_3 - \alpha y_2 dx_1 - \beta z_2 dx_1).$$

Now we know that $dx_{2j+1} \in \Lambda^{j+1}(y_2, z_2)$, $j \geq 1$. Thus it can be written as

$$dx_{2j+1} = p y_2^2 + q y_2 z_2 + r z_2^2,$$

where $p, q, r \in \Lambda^{j-1}(y_2, z_2)$. Clearly if $r = 0$ then we are done. Else , if $j > 1$, then $x_{2j+1} \in \hat{P}_E$. And if $j = 1$, denoting x_{2j+1} by x_3', then $p, q, r \in \mathbf{R}$ and thus $\gamma^{-1}x_3 - r^{-1}x_3' \in \hat{P}_E$.

(d): This case arises when E is a simple Lie algebra. We know, by Theorem 5.3.1, that dx_3 can be chosen as an irreducible polynomial in $\Lambda^2(y_2, z_2)$. Thus $\Lambda(y_2, z_2)/(dx_3)$ has no zero divisors and so the class $\overline{dx_{2j+1}} \in \Lambda(y_2, z_2)/(dx_3)$ is either zero or it is not a zero divisor. In the first case we must have

$$dx_{2j+1} = p(y_2, z_2) \cdot dx_3, \qquad p \in \Lambda^{j-1}(y_2, z_2),$$

i.e., $x_{2j+1} \in \hat{P}_E$. In the other case $\{dx_3, dx_{2j+1}\}$ is a regular sequence. Moreover, applying Theorem 4.3.4 and its associated remark together with Lemma 5.4.1 to the model $(\Lambda(y_2, z_2, x_3, x_{2j+1}), d)$, one gets

$$dim\, \mathcal{H}(\Lambda(y_2, z_2, x_3, x_{2j+1}), d) = dim\, \mathcal{H}_0(\Lambda(y_2, z_2, x_3, x_{2j+1}), d)) < \infty.$$

Now, since this model is minimal (every differential is quadratic and there are no generators in degree 1), we can apply Halperin's formula and find that the degree of its top cohomology class is

$$m = 2 - 4 + 3 + 2j + 1 = 2j + 2.$$

So for $k > j$, dx_{2k+1} must be a boundary in $(\Lambda(y_2, z_2, x_3, x_{2j+1}), d)$, i.e., there exists an element $u \in (\Lambda(y_2, z_2, x_3, x_{2j+1}))^{2k+1}$ such that $du = dx_{2k+1}$. Notice that u must be a linear combination of monomials of the form

$$p(y_2, z_2) \cdot x_3 \quad \text{or} \quad q(y_2, z_2) \cdot x_{2j+1}, \quad \text{with} \quad p, q \in \Lambda^+(y_2, z_2),$$

since we want u to be of odd degree. Thus we have $x_{2k+1} \in \hat{P}_E$. Now, since E is simple, using the classification of real compact simple Lie algebras, we can thus enumerate* the possible P_E of dimension 3. There are only two cases

$$P_E = \langle x_3, x_5, x_7 \rangle \qquad E = so(6), su(4)$$

$$\text{and} \quad P_E = \langle x_3, x_7, x_{11} \rangle \qquad E = so(7), sp(3).$$

And so we always have $k > j$.

(e): Using the same arguments as for case (d), we can construct a regular sequence $\{dx_3, dx_3'\}$. The degree of the top cohomology class is given by

$$m = 2 - 4 + 3 + 3 = 4,$$

which is strictly less than $2j + 2$ when $j > 1$. Thus $x_{2j+1} \in \hat{P}_E$.

Q.E.D.

5.5 A non-formal example

In this section we will show that, under certain circumstances, the case (f) of Table 1 exhibits non-formal homogeneous spaces. We will make precise those circumstances.

Let G be a Lie group of rank n and T an imbedded k-torus, $1 \leq k \leq n$. Consider the conjugation map $\tau_g(x) = gxg^{-1}$, $x, g \in G$, and define $\alpha_g : G/T \to G/gTg^{-1}$ by

$$[x] \mapsto [\tau_g(x)].$$

Clearly, α_g is well defined and bijective. (Inverse: $[x] \mapsto [g^{-1}xg]$)

The next result shows that not only are two conjugate tori diffeomorphic as Lie groups but that they produce diffeomorphic homogeneous spaces.

* Cf. [14]

Remark: By construction of α_g, the diagram

$$\begin{array}{ccccc} T & \hookrightarrow & G & \xrightarrow{\pi_1} & G/T \\ \tau_g \Big\downarrow \cong & & \tau_g \Big\downarrow \cong & & \alpha_g \Big\downarrow \cong \\ gTg^{-1} & \hookrightarrow & G & \xrightarrow{\pi_2} & G/gTg^{-1} \end{array}$$

commutes, where π_1 and π_2 are the canonical projections. Since G/T is a quotient manifold of G, and π_2 and τ_g are smooth so is α_g. Moreover, all vertical maps are diffeomorphims. Thus we get the following well-known result

Theorem 5.5.1: *Let T and T^k be respectively a maximal torus and a k-torus in G. Then all homogeneous spaces G/T are diffeomorphic. Moreover, there is a k-torus S in T such that G/S is diffeomorphic to G/T^k.*

Proof: The first statement is clear since all maximal tori are conjugate. For the second, let $a \in G$ be a generator of T^k. Then, there exists $g \in G$ such that the maximal torus gTg^{-1} contains a. Thus $T^k \subset gTg^{-1}$. Now let $S \equiv g^{-1}T^k g \subset T$.

Q.E.D.

Thus, to determine which G/T^2 are formal it is sufficient to pick an arbitrary maximal torus T and consider the homogeneous spaces G/T^2 for all $T^2 \subset T$.

Recall that in the last case of Table 1, the Lie algebra E of G is compact, semisimple and contains three simple ideals. These ideals are necessarily compact and of rank 1. By [4; p.436], these ideals are isomorphic to $so(3)$.

Thus $E = I_1 \oplus I_2 \oplus I_3$ where $I_i = < a_i, b_i, c_i >$, $i = 1, 2, 3$, together with

$$[a_i, b_i] = c_i, \quad [b_i, c_i] = a_i, \quad \text{and} \quad [c_i, a_i] = b_i.$$

If we consider the I_i's as Lie algebras, then their Killing forms are given by $K_i = -2I$. The K_i's can be extended to elements of $(\Lambda^2 sE^*)^E$ by

$$(K_i)_{|I_j \times I_k} = -2I \cdot \delta_{ij} \cdot \delta_{ik},$$

where δ_{ij} is the Kronecker delta.

Now choosing an abelian subalgebra of E of dimension three is equivalent to choosing a maximal torus in G since G is connected (use the exponential map). Let

$$W = < a_1, a_2, a_3 >$$

which is such a subalgebra. Denote by T the maximal torus generated by W. Then any 2-torus $T^2 \subset T$ has its Lie algebra in W, so we only have to study the 2-dimensional subspaces of W.

Let $F \subset W$ be a 2-dimensional subspace.

Remark: There are in general more 2-dimensional subspaces in W then there are 2-tori in T because under some circumstances $\overline{exp(F)} = T$ (see example 1 below and use α or $\beta \notin \mathbf{Q}$). We will assume that indeed F generates a 2-torus.

Without lost of generality, we can assume that this plane, i.e. F, has a normal of the following form:

$$n = \alpha a_1 + \beta a_2 + a_3, \qquad \alpha, \beta \in \mathbf{R}.$$

Thus a basis of F is given by $\{y \equiv a_1 - \alpha a_3, z \equiv a_2 - \beta a_3\}$, and so

$$Q_F = sF^* = < y_2 \equiv sy^*, z_2 \equiv sz^* >,$$

where $\{y^*, z^*\}$ is the dual basis for F^*.

Now, the Killing forms $\{K_i\}$ are homogeneous linearly independent elements of $(\Lambda^2 sE^*)^E$ and since $dim\, P_E = 3$, by Proposition 4.5.4, we have that

$$\{x_3 = \varrho_E(K_1), x_3' = \varrho_E(K_2), x_3'' = \varrho_E(K_3)\}$$

is a basis for $P_E^3 = P_E$.

We are now in a position to build the Koszul complex associated to G/T^2. Let $j : F \hookrightarrow E$ denote the natural inclusion and define a transgression $\tau : P_E \to (\Lambda sE^*)^E$ by

$$\tau(x_3) = K_1, \quad \tau(x_3') = K_2, \quad \text{and} \quad \tau(x_3'') = K_3.$$

Since $j^*(K_i)$ is just the restriction of K_i to F, one can show easily that

$$j^*(K_1) = -y_2^2, \quad j^*(K_2) = -z_2^2, \quad \text{and} \quad j^*(K_3) = -(\alpha y_2 + \beta z_2)^2.$$

Thus the Koszul complex is given by $(\Lambda(y_2, z_2, x_3, x_3', x_3''), d)$ where

$$dx_3 = -y_2^2, \quad dx_3' = -z_2^2, \quad \text{and} \quad dx_3'' = -(\alpha y_2 + \beta z_2)^2.$$

Theorem 5.5.4: *This minimal c.g.d.a is formal if and only if $\alpha = 0$ or $\beta = 0$.*

Proof: If $\alpha = 0$ or $\beta = 0$ then it is clearly formal since either $\alpha^2 x_3 - x_3''$ or $\beta^2 x_3' - x_3''$ is a cocycle. Conversely, suppose both are nonzero and let $V = < y_2, z_2, x_3, x_3', x_3'' >$. The cohomology of this minimal *c.g.d.a* is given by

$$\mathcal{H}(\Lambda V) \cong \Lambda(a_2, b_2, c_5, d_5)/(a_2^2, b_2^2, a_2 b_2, a_2 c_5, b_2 d_5, c_5 d_5, a_2 d_5 - b_2 c_5).$$

Now, by Proposition 4.2.2, $(\mathcal{H}(\Lambda V), 0)$ has a minimal model

$$\phi : (\Lambda W, \delta) \to (\mathcal{H}(\Lambda V), 0).$$

This model must have two generators $\gamma_2, \beta_2 \in W^2$ such that $\delta\gamma_2 = \delta\beta_2 = 0$, $\phi(\gamma_2) = a_2$, and $\phi(\beta_2) = b_2$. Since ΛW is free, there must be three generators $\eta_3, \eta_3', \eta_3'' \in W^3$ such that $\delta\eta_3 = \gamma_2^2$, $\delta\eta_3' = \beta_2^2$, $\delta\eta_3'' = \gamma_2\beta_2$, and $\phi(\eta_3) = \phi(\eta_3') = \phi(\eta_3'') = 0$. Now, as it stands, $\phi((\Lambda W)^5) = 0$ since ϕ is a homomorphism of algebras. So there must be two generators $\Phi_5, \Phi_5' \in W^5$ such that $\delta\Phi_5 = \delta\Phi_5' = 0$, and $\phi(\Phi_5) = c_5$, $\phi(\Phi_5') = d_5$. But then we see that $(\Lambda W, \delta)$ is not isomorphic to $(\Lambda V, d)$ contradicting Proposition 4.2.1. Thus $(\Lambda(y_2, z_2, x_3, x_3', x_3''), d)$ is not formal.

Q.E.D.

Remark: This process of adding generators to represent cohomology classes of the original *c.g.d.a*, and adding extra ones to kill other classes which are not present in the original cohomology, is exactly the process of building the minimal model for a given *c.g.d.a*.

Example 1: Let $G = SU(2) \times SU(2) \times SU(2)$ and

$$T^2 \equiv \left\{ \left[\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{pmatrix}, \begin{pmatrix} e^{-i(\alpha t + \beta s)} & 0 \\ 0 & e^{i(\alpha t + \beta s)} \end{pmatrix} \right] \middle| s, t \in \mathbf{R} \right\}$$

with $\alpha, \beta \in \mathbf{Q}$, and

$$F = \left\langle \begin{array}{c} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -i\alpha & 0 \\ 0 & i\alpha \end{pmatrix}\right], \\ \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -i\beta & 0 \\ 0 & i\beta \end{pmatrix}\right] \end{array} \right\rangle.$$

By Theorem 5.5.4, G/T^2 is formal if and only if $\alpha = 0$ or $\beta = 0$. Thus we see, by considering $T^2 = S^1 \times S^1$, that if one of the circles of T^2 is not intertwined in at least two copies of $SU(2)$, then the model is formal. All other cases are non-formal.

References

[1] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York, 1982.

[2] N. Bourbaki, *Éléments de Mathématique, Groupes et Algèbres de Lie I*, Hermann, Paris, 1960.

[3] Y. Félix, D. Tanré, and J.-C. Thomas, *Minimal models and geometry*, preprint, Lille University, France, 1993.

[4] W. Fulton, J. Harris, *Representation Theory, A first course*, Springer-Verlag, New York, 1991.

[5] W. Greub, *Multilinear algebra 2nd Ed.*, Springer-Verlag, New York, 1978.

[6] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature, and Cohomology I*, Academic Press, Inc., New York, 1972.

[7] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature, and Cohomology II*, Academic Press, Inc., New York, 1972.

[8] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature, and Cohomology III*, Academic Press, Inc., New York, 1972.

[9] S. Halperin, *Finiteness in the Minimal Models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977) p.173-199.

[10] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[11] S. Kobayashi, and K. Nomizu, *Foundations of differential geometry I*, Interscience Publishers, New York, 1963.

[12] J. C. Milnor, J. W. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81**(1965), p.211-264.

[13] J. R. Munkres, *Topology A first course*, Prentice-Hall, New Jersey, 1975

[14] H. Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. **58**, (1958), p.2-37.

[15] M. Spivak, *Calculus on Monifolds*, W. A. Benjamin, Inc., New York, 1965.

[16] D. Sullivan, *Infinitesimal Computations in Topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1978) p.269-331.

[17] O. Zariski, P. Samuel, *Commutative algebra*, vol. II, Van Nostrand, Princeton, NJ, 1958.