IMBEDDING METHODS IN THE NUMERICAL SOLUTION OF OPTIMIZATION PROBLEMS

by

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ABSTRACT

Two Point Boundary Value Problems (TPBVP) are frequently encountered in almost every branch of engineering and physical sciences. The conventional numerical solution of TPBVP is based on a trial and error procedure, in order that the missing initial conditions be obtained. This procedure not only has a relatively slow convergence rate but, moreover, the starting or guessed missing initial conditions must be very close to the correct ones before the procedure will converge. Imbedding Techniques, such as Invariant Imbedding and the Continuation method, have the major advantage of reducing TPBVP into initial-value problems. In this thesis the two imbedding methods will be presented and discussed as to their absolute and relative merits and disadvantages. Applications of these techniques to the solution (numerical) of optimal control problems will be presented, with examples. In addition a method for employing invariant imbedding to the design of optimal non-linear feedback controllers will be presented.

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CHAPTER 1

INTRODUCTION

Recent advances in Engineering and Science have stimulated much interest in determining optimal solutions for the control and the trajectory in any control problem where minimum cost criteria are desired. There is no doubt that the availability of modern high-speed general purpose computers makes it feasible to consider the actual computation of these optimal trajectories and optimal controls. Several computational methods for obtaining the trajectory and the control for systems whose motions can be described by a set of ordinary linear or nonlinear differential equations are now available.

Since the early fifties much work has been done in the field of Optimal Control Theory by mathematicians and engineers in the United States and the Soviet Union. The outstanding works of Bellman and Pontryagin [15, 35] during the time 1955-1960 created a new era in this field. Classical variational theory could not readily handle the typical constraints usually encountered in optimal control problems. This difficulty was completely removed after the enunciation of Pontryagin's celebrated maximum principle [15] in 1956. This pioneering work of Pontryagin actually has established the basis of modern control theory. While the maximum principle may be viewed as an outgrowth of the Hamiltonian approach to variational problems, the method of dynamic programming, which was developed by Bellman [35] may be viewed as an outgrowth of the Hamilton-Jacobi approach.

The necessity of actual computation of the solutions of control-problems of physical systems gave rise to a variety of computational techniques. The basic methods employed in these procedures vary widely. They include, for example non-linear and dynamic programming, gradient methods in function space, solution of the associated two-point boundary-value problems (TPBVP) by function - minimization methods and initial value solutions of the imbedding equations [3,8,11,26,27,28,29,35].

The Dynamic Programming method and the steepest descent techniques have applications in many control problems. Because of the large storage capacity and long computation times, the application of dynamic programming is limited to control problems of low dimension. The gradient methods in function spaces (Steepest Descent techniques) have many excellent applications to optimal control problems. But there is also limitation of these methods because of restricted convergence.

Solution of the associated TPBVP seems to be more promising for many optimization problems, as compared to other techniques discussed above. It has to be mentioned, however, that not all control problems can be converted to TPBVP and the usual iterative means of solving TPBVP need some arbitrary starting "guess". For some "guess" the TPBVP might not be solved at all, because, in general, the "convergence region" of the iterative procedures is narrow.

In this thesis the application of imbedding techniques, such as invariant imbedding and the continuation method, to the numerical solution of TPBVP will be examined. The major advantage of these techniques is reduction of TPBVP to Initial-value problems with global, in most problems, convergence.

In Chapter 2, an introduction to TPBVP of Ordinary Differential equations is given. The application of iterative methods to their solution is discussed. Invariant Imbedding and the Continuation Method as applied to the solution of TPBVP are introduced, with some historical review also given.

In Chapter 3, an introduction to the Optimal Control problem and Pontryagin's Maximum Principle is given. The Optimal Control problem is reduced there to a TPBVP.

In Chapter 4, the numerical solution of TPBVP by the two imbedding methods is examined. The invariant imbedding approach requires solution of a system of quasi-linear, hyperbolic, partial-differential equations. The Continuation method introduces a system of ordinary differential and algebraic equations. Two examples are explicitly solved by each of the two methods.

In Chapter 5, the Kalaba-Shridhar approach to the solution of optimal control problems is introduced. This is also an invariant imbedding technique, but it does not require prior formulation of a TPBVP. An application of this approach to the derivation of optimal nonlinear feedback controllers is presented. A scheme for efficient storage of the data required for the synthesis of the feedback controller is introduced. An example demonstrates the approach.

In the Conclusion of the thesis, the numerical results obtained are compared and discussed.

CHAPTER 2

BOUNDARY VALUE PROBLEMS AND SOLUTION METHODS

In mathematical physics and engineering the solution of many optimization problems reduces to either Two Point or Multi-Point Boundary Value Problems. We will be discussing the solution of Two Point Boundary Value Problems (TPBVP) only, though there is no basic difference between the methods employed to solve either TPBVP or Multi Point Boundary Value Problems.

2.1 TWO POINT BOUNDARY VALUE PROBLEMS (TPBVP) OF ORDINARY DIFFERENTIAL EQUATIONS.

The Two Point Boundary Value Problems obtained in engineering and physical sciences are mostly non-linear and thus associated with various analytical and numerical difficulties. Analytically, it is very difficult to give any general proof for the existence and uniqueness of their solutions. Numerically there is no widely accepted convenient technique for obtaining the numerical solutions on modern digital computers. Whereas, if the complete initial conditions are given modern digital computers are efficient tools to solving differential equations describing dynamical systems. Now let us consider the system of n first order non-linear ordinary differential equations :

$$\frac{dx^{(i)}}{dt} = f^{(i)}(x^{(1)}, x^{(2)}, \ldots x^{(n)}, t), \; i = 1, 2, \ldots n \qquad (2.1.1)$$

where $f^{(i)}$ are sufficiently smooth functions. In addition to satisfying (2.1.1), the solution $x^{(i)}(t)$ must satisfy the two point boundary conditions :

$$x^{(i)}(a) = \sigma_1^{(i)} \qquad i = 1, 2, \ldots J \tag{2.1.2}$$

$$x^{(i)}(b) = \eta_1^{(i)} \qquad i = J+1, \ldots n \tag{2.1.3}$$

Equations (2.1.1) - (2.1.3) define a Two Point Boundary Value Problem, where the initial state at $t = a$ as well as the final state at $t = b$ are partly given. Usually one cannot find a closed form solution of this problem, and has to resort to digital or analog techniques. To get the solution by digital techniques we can use either one of the following methods.

(1) Iterative Methods for the Numerical solution of TPBVP

(2) Initial Value Methods for the Numerical solution of TPBVP

The Initial value methods to be considered here are the following

(i) Invariant Imbedding

(ii) The Continuation Method.

2.2 ITERATIVE METHODS FOR THE NUMERICAL SOLUTION OF TPBVP.

These are so named because the solution is started with some guess and after some iterations it converges to the actual solution. There are numbers of ways of approaching the solution from this guess. One such way is to define a multi-dimensional error function. Minimization of this function is equivalent to the solution of boundary value problems [27]. There are many methods of minimizing a multidimensional function. Most of them are based on gradient information. One such method is due to Davidon [28] and this was refined by Fletcher and Powell [29]. This is now known as DFP algorithm and has been widely accepted as one of the most powerful tools in n dimensional minimization problems. It has been shown [27] that the DFP algorithm can find the minimum of a quadratic function of n variables, in atmost n steps. The implementation of the DFP technique requires the solution of a one dimensional minimization problem at each stage of the iterative process. This can be achieved either by Cubquad or Fibonacci search technique .

To show the initiation of the problem let us consider the system of equations (2. 1. 1) - (2. 1. 3) with an error function defined as:

$$E = \left[\sum_{i=J+1}^{n} \{ x^{(i)}(b) - \eta_1^{(i)} \} \right]^2 \qquad (2.2.1)$$

where $x^{(i)}(b)$ is the final value of the state obtained by solving the system of differential equations with some initial guess such as

$$x^{(i)}(a) = \omega^{(i)} ; \qquad i = J+1, \ldots n \qquad (2.2.2)$$

for the initial conditions that are not given.

Let Ω represent that set of n-J vectors ω, for which the differential system (2. 1. 1) - (2. 1. 3) has a unique solution in the interval $[a, b]$. Then, for each $\omega \in \Omega$ there corresponds a unique non-negative value for the scalar function E; that is the function E (the terminal error) can be viewed as a function of the initial guess

$$\omega^{(i)} = x^{(i)}(a) \quad , \qquad i = J + 1, \ldots n \qquad (2.2.3)$$

Now if one could find $\omega^* \in \Omega$ such that $E(\omega^*) = 0$, then the resultant solution would be the required solution. It may here be noted that ω^* renders E its minimum and so the problem of finding ω^* (and hence of solving (TPBVP)) is equivalent to the problem of minimizing the terminal error function, $E = E(\omega)$. But this minimization is quite complex, because at each stage of iteration the gradient of the function has to be found along with one dimensional minimization. The analytical method of obtaining the gradient is quite involved and thus the gradient of $E(\omega)$ may be obtained by a finite difference method having perturbation size of $10^{-7} - 10^{-10}$, usually.

2.3 INVARIANT IMBEDDING

Invariant Imbedding, previously known as invariance principles, has been first introduced by Ambarzumian in the study of transport phenomena [1]. Later on these principles have been studied extensively by Chandrashekhar, Bellman, Kalaba, Wing and co-workers [13, 14] in the study of neutron transport theory, radiative transfer, random walk, scattering etc. - In this work we will consider the use of this principle for the solution of TPBVP. Let us consider the scalar TPBVP :

$$\pi : \begin{cases} \dot{x} = f(t, x, \psi), & x(0) = x_0 \\ \ddot{\psi} = g(t, x, \psi), & \psi(T) = \psi_1 \end{cases} \tag{2.3.1}$$

for $0 \leq t \leq T$, T given.

where the functions f and g are continuous and continuously differentiable.

This problem π is imbedded into the family of TPBVP with more general initial conditions:

$$\pi(a,c) : \begin{cases} \dot{x} = f(t,x,\psi) , & x(a) = c \\ \dot{\psi} = g(t,x,\psi) , & \psi(T) = \psi_1 \end{cases} \tag{2.3.2}$$

Now let

$$\psi(a) = r(a,c) \tag{2.3.3}$$

be the missing initial condition of π. To find an expression for $r(a,c)$ let $x(t)$ and $\psi(t)$ be the solution trajectories as shown in Fig. 1.

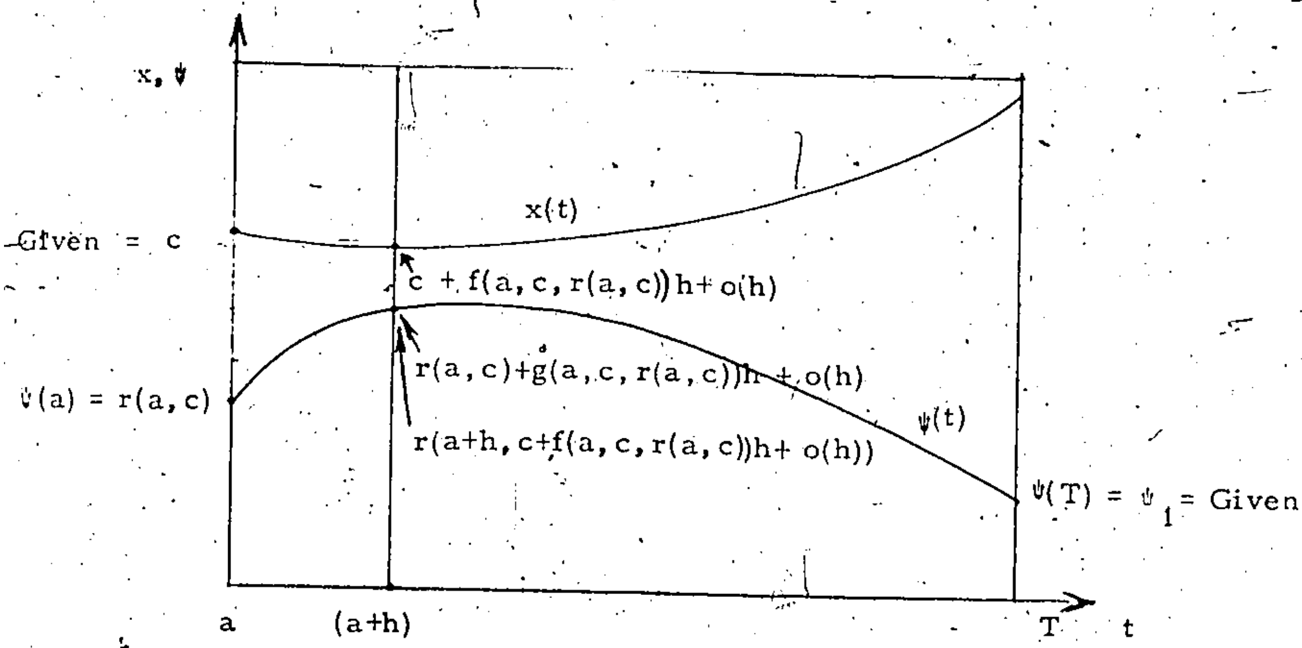

Fig. 1

At $t = (a+h)$, the values of $r(a,c)$ can be expressed in two different ways. Equating these two, i.e., finding the relationship between the neighboring values of the process, we get

$$\begin{aligned} & r(a+h,\ c+f(a,c,r(a,c))h+0(h)) \\ & \quad = r(a,c) + g(a,c,r(a,c))\,h + 0(h) \end{aligned} \tag{2.3.4}$$

Expanding in Taylor's series on both sides of the equation we get the following Partial Differential Equation (P.D.E) :

$$r_a(a,c) + r_c(a,c)f(a,c,r(a,c)) = g(a,c,r(a,c)) \qquad (2.3.5)$$

For $0 \leq a \leq T$ and $-\infty < c < \infty$

The initial condition of this P.D.E. may be obtained from the consideration that when $a = T$ we have

$$\psi(T) = r(T,c) = \psi_1 .$$

(2.3.5) should now be solved backward from $a = T$ to $a = 0$.

Then for $c = x_o$ and $a = o$ we get $\psi(o) = r(0,x_o)$ which is the required initial condition of π. Now π can be integrated forward from 0 to T for the required trajectory.

It has been shown in [1] by Lee that instead of finding the missing initial condition $\psi(a)$, we could find the missing final condition $x(T)$, T being fixed. The missing final condition $x(T)$ for the family of problems with different starting time values, a, depends on the given initial condition, c, and the starting time a . Let

$$s(a,c) = x(T) = \begin{array}{l}\text{The missing final condition for the}\\ \text{system } \pi(a,c)\text{: starting at}\\ t = a \text{ with } x(a) = c.\end{array} \qquad (2.3.6)$$

By finding the relationship between $s(a,c)$ and its neighboring process, with Taylor's series and making use of the semi-group property, the following P.D.E is obtained for $s(c,a)$:

$$\frac{\partial s(a,c)}{\partial a} + f(a, r(a,c), c)\,\frac{\partial s(c,a)}{\partial c} = 0 \qquad (2.3.7)$$

with initial condition $s(c,T) = c$.

This means that the missing final condition would be equal to the given initial condition if the duration of the process is zero or $a = T$. After finding $x(T)$ the original system π, may be integrated backwards. If now instead of considering the scalar case we take the system of equations (2.1.1) - (2.1.3) the following system of P.D.E. is obtained for the missing initial conditions,

$$r_i(a,D) = x^{(i)}(a) \quad \text{For } i = J+1, \ldots n \tag{2.3.8}$$

where D is the initial vector with components $\delta^{(i)}$, $i = 1, 2, \ldots J$.

$$\frac{\partial r_i}{\partial a}(a,D) + \sum_{k=1}^{J} f^{(k)}(a, R(a,D), D) \frac{\partial r_i(a,D)}{\partial \delta^{(k)}}$$

$$= f^{(i)}(a, R(a,D), D) \tag{2.3.9}$$

For $i = J + 1, \ldots n$.

with $R(a,D) = [\, r_{J+1}(a,D), \ldots, r_n(a,D) \,]$, the missing initial vector. It has been shown by Bailey and Wing [14] that if the original system of ordinary differential equations becomes linear then the P.D.E obtained either in (2.3.5), (2.3.7) or (2.3.9) can be reduced to a system of ordinary Ricatti differential equations and the solution becomes very easy.

Indeed, let us consider the case :

$$f(x, \psi, t) = q_{11}(t)\, x + q_{12}(t)\, \psi \tag{2.3.10}$$

$$g(x, \psi, t) = q_{21}(t)\, x + q_{22}(t)\, \psi \tag{2.3.11}$$

Because of this linearity the missing initial condition will be proportional to the starting state c. Thus

$$r(a,c) = R(a)\,c \tag{2.3.12}$$

Substituting into (2.3.5) we get

$$\frac{dR(a)}{da} = q_{21}(a) + [q_{22}(a) - q_{11}(a)]\,R(a) - q_{12}(a)R^2(a) \tag{2.3.13}$$

with boundary condition

$$r(T,c) = R(T)\,c = \psi_1 \tag{2.3.14}$$

Equation (2.3.13) is of the Riccatti type. Not only can the missing initial or missing final conditions be obtained from the invariant imbedding equations, and the original boundary-value problem thus become an initial value problem, but also the solution of the original equations, x and ψ, can be obtained in terms of the functions r and s. The relationships between r and s, and x and ψ have been shown [1] to be

$$\psi(t) = r(t, x(t)) \tag{2.3.15}$$

$$s(t, x(t)) = d \tag{2.3.16}$$

where d = missing final condition of the original problem (2.3.1).

2.4 THE CONTINUATION METHOD

Let π be the problem to be solved. The principle of the continuation method is to imbed this problem into the one-parameter family of problems

$$\{\pi(s) : 0 \leq s \leq 1\} \tag{2.4.1}$$

having the property that

i) $\pi(o)$ is a problem which can be easily solved ("base" problem)

ii) $\pi(1) \equiv \pi$

Symbollically it can be represented as shown in Fig. 2.

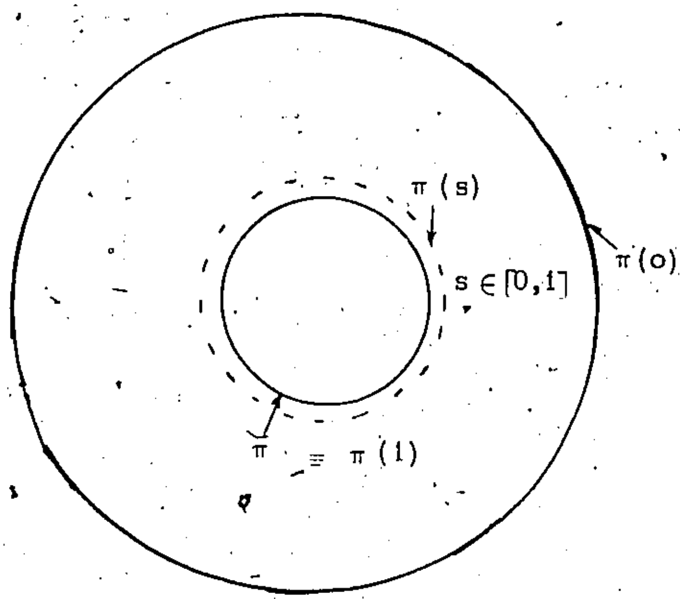

Fig. 2

The solution of π is obtained as the "continuation" of the solution of the "base" problem $\pi(0)$ as s varies from $s = 0$ to $s = 1$.

Continuation methods have been fruitful over the last century in the proof of existence and uniqueness of various problems [22,30]. However the application of these methods to numerical calculations became practical only with the advances in the technology of digital and analogue computers. - These methods have successfully been applied to different problems, such as : finding the roots of a polynomial, solving eigenvalue problems, TPBVP., etc. - There are different variants of the continuation method [24].

We are going to describe here a variant of this method [23] for the numerical solution of the TPBVP (2.1.1), - (2.1.3). The merit of the method of continuation is that it naturally lends itself to the reformulation of such problems as initial value problems. Now we have :

$$\pi \quad = \quad \{\text{TPBVP } (2.1.1) - (2.1.3)\}.$$

We imbed π into the one-parameter family of problems $\pi(s)$ [23]:

$$\pi(s) : \begin{cases} \dfrac{\partial x^{(i)}}{\partial t}(s,t) = f^{(i)}(s, x^{(1)}, \ldots, x^{(n)}, t) \\ \qquad\qquad i = 1,2,\ldots n & (2.4.2) \\ x^{(i)}(s,a) = \sigma^{(i)}(s), i = 1,2,\ldots J & (2.4.3) \\ x^{(i)}(s,b) = \eta^{(i)}(s), i = J+1,\ldots n & (2.4.4) \end{cases}$$

for $0 \le s \le 1$. The boundary-value problem (2.4.2) - (2.4.4) will be denoted by $\pi(s)$. The functionals $f^{(i)}$ and the functions $\sigma^{(i)}(s)$ and $\eta^{(i)}(s)$ are chosen so that they are continuously differentiable with respect to s and that a) at s = 0 the base problem $\pi(o)$ can be solved and its known solution is :

$$\pi(o) : \begin{cases} x^{(i)}(o,t) = x_o^{(i)}(t) \quad i = 1,2,\ldots n & (2.4.5) \\ x^{(i)}(o,a) = \sigma^{(i)}(o) = \sigma_o^{(i)} \quad i = 1,2,\ldots n & (2.4.6) \\ x^{(i)}(o,b) = \eta^{(i)}(o) = \eta_o^{(i)} \quad i = 1,2,\ldots n & (2.4.7) \end{cases}$$

b) at s = 1 the problem $\pi(1)$ coincides with the original problem π, i.e.,

$$f^{(i)}(1,\ x^{(1)}(1,t)\ldots,t) = f^{(i)}(x^{(1)}(t)\ldots,\ t) \qquad (2.4.8)$$

and similarly for the boundary conditions.

Here we identify at $s = 1$

$$x^{(i)}(1,t) = x^{(i)}(t) \qquad (2.4.9)$$

Condition b) is satisfied by choosing

$$\sigma^{(i)}(s) = \sigma_1^{(i)}, \quad i = 1,2,\ldots J \quad \text{for all } s \qquad (2.4.10)$$

and the functions $\eta^{(i)}(s)$, $i = J+1,\ldots n$, so that at $s = 0$ they pass through the base (2.4.7) and at $s = 1$ they coincide with

$$\eta^{(i)}(1) = \eta_1^{(i)}, \quad i = J+1,\ldots n \qquad (2.4.11)$$

A simple choice is

$$\eta^{(i)}(s) = \eta_0^{(i)} + (\eta_1^{(i)} - \eta_0^{(i)})s, \quad i = J+1,\ldots n \qquad (2.4.12)$$

Let the functions $\sigma^{(J+1)}(s)$, $\sigma^{(J+2)}(s)\ldots\sigma^{(n)}(s)$ be the unknown initial conditions of $\pi(s)$, i.e.,

$$x^{(i)}(s,a) = \sigma^{(i)}(s), \quad i = J+1,\ldots n \qquad (2.4.13)$$

The initial values, at $s = o$, of these functions are known from (2.4.6) (an arbitrary selection) : We shall now derive an ordinary initial value problem that describes the behaviour of $\sigma^{(i)}(s)$, (2.4.13) as s varies from $s = o$ to $s = 1$. By differentiating (2.4.4) with respect to s we get :

$$\frac{\partial x^{(i)}}{\partial s} + \sum_{k=J+1}^{n} Z_{ik}\frac{d\sigma^{(k)}}{ds} = \frac{d\eta^{(i)}}{ds}, \quad i = J+1,\ldots n \qquad (2.4.14)$$

where we have defined

$$z_k^{(i)}(s,t) \equiv \frac{\partial x^{(i)}(s,t)}{\partial \sigma^{(k)}} \qquad (2.4.15)$$

and

$$Z_{ik} \equiv z_k^{(i)}(s,b) \qquad (2.4.16)$$

The right-hand side of (2.4.14) is known; for example, if we choose the linear variation (2.4.12) then

$$\frac{d\eta^{(i)}}{ds} = \eta_1^{(i)} - \eta_o^{(i)}, \quad i = J+1, \ldots n \qquad (2.4.17)$$

For simplicity we shall restrict ourselves to the case where the functionals $f^{(i)}$ in (2.4.2) do not depend explicitly on s. Therefore, the first term in (2.4.14) $\frac{\partial x^{(i)}}{\partial s}$ vanishes.

We shall show below how to compute the elements of the matrix $\{Z_{ik}\}$ in (2.4.14). If the matrix is not singular, this linear system can be solved to give :

$$\frac{d\sigma^{(k)}}{ds} = \sum_{i=J+1}^{n} [Z^{-1}]_{ik} \frac{d\eta^{(i)}}{ds} = G_k(s) ; \; k = J+1, \ldots n \qquad (2.4.18)$$

Since the initial values of these equations are known [see (2.4.6)], they can be integrated from s = o until s = 1. Because of our choice of (2.4.8) - (2.4.10) we will have :

$$\sigma^{(k)}(1) = \sigma_1^{(k)}, \quad k = J+1, \ldots n \qquad (2.4.19)$$

It remains to show how to compute the elements of the matrix Z (s). Differentiating (2.4.6) - (2.4.7) with respect to $\sigma^{(k)}$ yields :

$$\frac{\partial z_k^{(i)}(s,t)}{\partial t} = \sum_{r=1}^{n} \frac{\partial f^{(i)}}{\partial x^{(r)}} z_k^{(r)} \qquad (2.4.20)$$

and

$$z_k^{(i)}(s,a) = \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad (2.4.21.)$$

where $i = 1, 2, \ldots n$, $k = J+1, \ldots n$.

The integration of this initial value problem from $t = a$ to $t = b$ produces the desired matrix [see (2.4.16)]. In general, this linear initial - value problem for the $z_k^{(i)}$ cannot be solved independently since the co-efficients in (2.4.20) depend on the unknown $x^{(i)}$. Therefore, the solution of (2.4.20) and (2.4.21) has to be carried out together with (2.4.2) and (2.4.3). However, the initial values (2.4.3) are known for $0 \leq s \leq 1$ only for $i = 1, 2, \ldots J$ [see (2.4.10)]. The remaining $\sigma^{(k)}(s)$ must be obtained from the solution of the initial-value problem (2.4.18), (2.4.6).

We see that two initial-value problems are coupled. The first one, consisting of (2.4.2), (2.4.3), (2.4.20), and (2.4.21) cannot be integrated without knowing $\sigma^{(k)}(s)$ and the second one (2.4.18) and (2.4.6) cannot be integrated without knowing the matrix Z which is obtained from the solution of the first problem. The solution of this problem can be carried out on a hybrid computer with a repetitive mode as follows. At $s = 0$ we know [from (2.4.6)] all the initial values $\sigma^{(i)}(o)$ and therefore we can integrate (2.4.2), (2.4.20) and (2.4.21) from $t = a$ to $t = b$. This can be done by fast analog integrators; see Box A in the flow chart in Fig. 3. At $t = b$ the quantities $z_{ik}(o,b)$, $i, k = J+1, \ldots n$, are delivered to the digital part of the computer.

The derivatives, $\frac{d\sigma^{(k)}}{ds}$ at $s = 0$ are evaluated in the digital computer by solving the linear system (2.4.18) in Box B of the flow chart. We now take a small increase Δs and compute (in Box C)

$$\sigma^{(k)}(\Delta s) = \sigma^{(k)}(o) + \Delta s\, G_k(o) \quad k = J+1, \ldots n \qquad (2.4.22)$$

THE DOUBLE LOOP SCHEME OF THE CONTINUATION METHOD

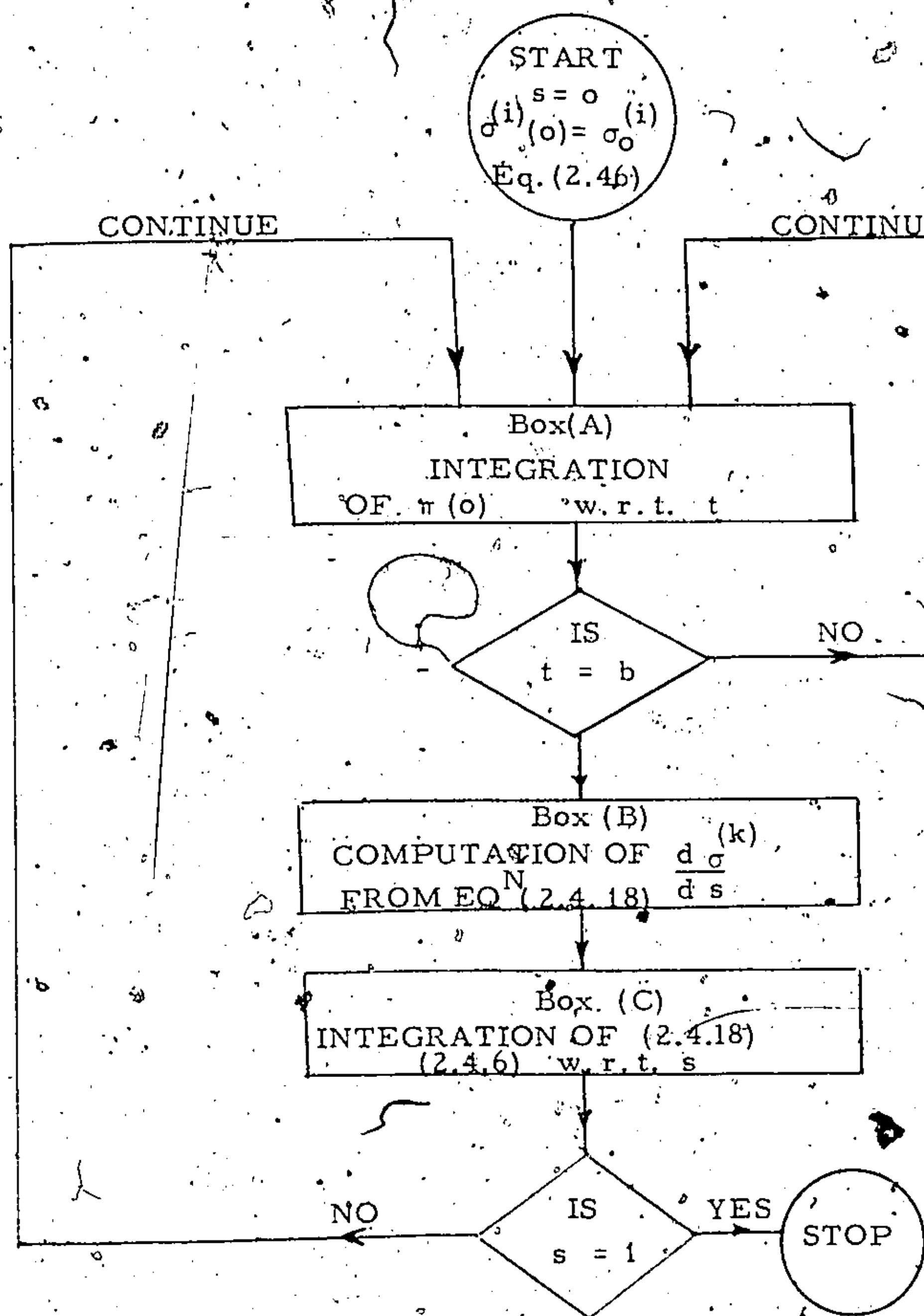

Fig. 3.

Now the whole cycle is repeated for Δs, $2 \Delta s$ ...

$$k \Delta s, \text{ until } k \Delta s = 1.0$$

The whole process can be executed either on a pure analog computer or digital computer. Two-time scaling is needed if the whole process is to be done on an analog computer.

The main advantage of the continuation method is its superior convergence properties even from a distant base. The short comings of the continuation method are its relative complexity and its slowness. The main attraction of analog and hybrid computers in our context is that they can substantially speed up the continuation method, which make it useful for real time solutions. The variant of the continuation method described here is different from the one described in [24], but this variant is better for its fast (repetitive) mode. In the other variant, the number of equations is greater than here.

In this work we shall carry out the continuation method on a digital computer.

CHAPTER 3

OPTIMAL CONTROL AND TPBVP

The difficulties arising in the solution of two-point boundary-value problems limit the use of both the calculus of variations and Pontryagin's maximum principle in obtaining the numerical solution of optimization problems. Dynamic programming, though, seems to avoid these difficulties by employing completely different concepts, has dimensionality problems. - The classical variational approach to optimal control problems can not handle several such problems because of certain types of control constraints. The use of the maximum principle is a very advantageous tool for the solution of many optimal control problems, if we can solve the problem of handling the associated TPBVP. Before proceeding further in converting the optimal control problem to a TPBVP, we shall give a brief review of Pontryagin's Maximum Principle [15].

3.1. THE OPTIMAL CONTROL PROBLEM AND PONTRYAGIN'S MAXIMUM PRINCIPLE.

Basic Assumptions

Assume that the controlled object's law of motion can be described by a system of differential equations :

$$\begin{aligned} \frac{d}{dt} x^{(i)} &= f^{(i)} (x^{(1)} \ldots x^{(n)}, u^{(1)} \ldots u^{(r)}) \\ &= f^{(i)} (x, u) \qquad (3.1.1) \end{aligned}$$

For, $i = 1, 2, \ldots n$.

or, in the vector form

$$\frac{d}{dt} x = f(x,u) \tag{3.1.2}$$

where $f(x,u) = \{ f^{(i)}(x,u) \}$, $i = 1,2,\ldots n$, $x = (x^{(1)}, x^{(2)}, \ldots x^{(n)})$ is the state vector and $u = (u^{(1)}, u^{(2)} \ldots u^{(r)})$ is the control vector. The functions of $f^{(i)}$ are defined for $x \in X$ and for $u \in U$ where U is the control region, an bounded subset of E^r. It is also assumed that $\frac{\partial}{\partial x^{(i)}} f^{(i)}(x^{(1)} \ldots x^{(n)}, u)$; $i, j = 1, 2, \ldots n$ is continuous on the direct product $X \times U$.

In technical problems the control region U is not arbitrary but a closed and bounded set in E^r.

Formulation of the Optimal Problem: Let M^o denote the set of all bounded measurable functions defined on $I = [a,b]$ to E^r such that $u(t) \in U$ a.e. This set will be called the class of admissible controls. The functional to be minimized is

$$J(u) = \int_a^b f^o(x,u)\, dt, \ldots \tag{3.1.2(a)}$$

where it is assumed that $f^o(x,u)$ satisfies the same conditions as the functions $f^{(i)}(x,u)$, $i = 1,2,\ldots n$

Define a new state coordinate $x^{(o)}(t) \triangleq \int_a^t f^o(x(s),u(s))ds$.

By adjoining this new coordinate, we obtain the (n+1) dimensional state space $\overline{X}$. Then the equation (3.1.2) takes the form

$$\frac{d\overline{x}}{dt} = \overline{f}(x,u), \ldots \tag{3.1.3}$$

where, $\overline{x} = (x^{(o)}, x^{(1)} \ldots x^{(n)})$.

and $\overline{f}(x,u) = (f^o(x,u), f^{(1)}(x,u) \ldots f^{(n)}(x,u))$

The problem may then be posed as follows :

Statement of the Fundamental Problem :

In the (n+1) dimensional state space $\overline{X}$ the point $\overline{x}_o = (0, x_o)$ and the line π are given. The line π is assumed to be parallel to the $x^{(o)}$ axis, and to pass through the point $(0, x_1)$. Among the admissible controls $u = u(t)$, having the property that the corresponding solution $\overline{x}(t)$ of equation (3.1.3) with initial conditions $\overline{x}(a) = \overline{x}_o$ intersects π, find one whose point of intersection with π has the smallest coordinate $x^{(o)}$.

Fundamental Theorems of Pontryagin et al :

Before presenting the theorem that solves the fundamental problem, we consider the following : The Hamiltonian function $\overline{H}$ is written as

$$\overline{H} = < \overline{\psi} , \overline{f}(x,u) > = \sum_{\nu=o}^{n} \psi_\nu f^{(\nu)}(x,u)$$

where ψ_i, $i = 0, 1, \ldots, n$ are auxiliary variables (called adjoint or co-state variables) satisfying the differential equations

$$\frac{d\psi_i}{dt} = - \sum_{\nu=o}^{n} \frac{\partial}{\partial x_i} f^{(\nu)}(x,u)\, \psi_\nu, \quad i = 0, 1, \ldots, n \qquad (3.1.4)$$

It can be seen that the equations (3.1.3) and (3.1.4) may be rewritten with the help of the function $\overline{H}$ in the form of the Hamiltonian system :

$$\left\{ \begin{aligned} \frac{d}{dt} x^{(i)} &= \frac{\partial}{\partial \psi_i} \overline{H}, \ldots && (3.1.5) \\ \frac{d}{dt} \psi_i &= -\frac{\partial}{\partial x^{(i)}} \overline{H} \ldots && (3.1.6) \end{aligned} \right.$$

For, $i = 0, 1, 2, \ldots n$.

Theorem 3.1.1 (Pontryagin's Maximum Principle [15]

Let $u(t)$, $a \le t \le b$, be an admissible control such that the corresponding trajectory $\bar{x}(t)$ (see 3.1.5) which begins at the point $\bar{x}_o$ at the time $t = a$ is defined on the interval $a \le t \le b$, and passes, at the time b through a point on the line π.
In order that $u(t)$ and $\bar{x}(t)$ be optimal it is necessary that there exist a non-zero absolutely continuous vector functions $\bar{\psi}(t) = (\psi_o(t), \psi_1(t) \dots \psi_n(t))$ corresponding to the functions $u(t)$ and $\bar{x}(t)$ (see 3.1.6) such that :

1) The function $\bar{H}(\bar{\psi}(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$ almost everywhere in the interval $a \le t \le b$,

$$\bar{H}(\bar{\psi}(t), x(t), u(t)) = \max_{v \in U} \bar{H}(\bar{\psi}(t), x(t), v) = \Xi(\bar{\psi}(t), x(t)) \qquad (3.1.7)$$

2) At the terminal time b the relations

$$\psi_o(b) \le 0 \;;\; \Xi(\bar{\psi}(b), x(b)) = 0 \qquad (3.1.8)$$

are satisfied. Furthermore, it turns out that if $\bar{\psi}(t)$, $\bar{x}(t)$, and $u(t)$ satisfy systems (3.1.5) and (3.1.6) and condition (1), the time functions $\psi_o(t)$ and $\Xi(\bar{\psi}(t), x(t))$ are constant.
Thus (3.1.8) may be verified at any time t, $a \le t \le b$ and not just at b.

For the time optimal case, $f^o(x,u) \equiv 1$, and we consider the Hamiltonian function,

$H = \psi_o + \sum_{\nu=1}^{n} \psi_\nu f^{(\nu)}(x,u)$, where ψ is an 'n' dimensional vector i.e $\psi = (\psi_1, \psi_2 \dots \psi_n)$. Here we consider the

n dimensional state space only. Since ψ_o is constant, the Hamiltonian system becomes

$$\begin{cases} \dfrac{d}{dt} x^{(i)} = \dfrac{\partial}{\partial \psi_i} H , & i = 1, 2, \dots n \qquad (3.1.9) \\[2ex] \dfrac{d \psi_i}{dt} = \dfrac{-\partial}{\partial x^{(i)}} H , & i = 1, 2, \dots n \qquad (3.1.10) \end{cases}$$

The theorem for time optimality is given below :

Theorem 3.1.2 (Pontryagin et al [15])

Let $u(t)$, $a \le t \le b$, be an admissible control which transfers the state point from x_o to x_1, and let $x(t)$ be the corresponding trajectory (see 3.1.9) so that $x(a) = x_o$, $x(b) = x_1$. In order that $u(t)$ and $x(t)$ be time-optimal it is necessary that there exist a nonzero, continuous vector function $\psi(t) = (\psi_1(t), \psi_2(t), \dots \psi_n(t))$ corresponding to $u(t)$ and $x(t)$ (see 3.1.10) such that :

1) For all t, $a \le t_0 \le b$, the function $H(\psi(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$:

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) \qquad (3.1.11)$$

2) At the terminal time b, the relation

$$M(\psi(b), x(b)) \ge 0 \qquad (3.1.12)$$

is satisfied. Furthermore it turns out that if $\psi(t)$, $x(t)$, and $u(t)$ satisfy system (3.1.9, 3.1.10) and condition (1), the time function $M(\psi(t), x(t))$ is constant . Thus (3.1.12) may be verified at any time t, $a \le t \le b$ and not just at b.

Let us now pass to the formulation of the control problem with variable endpoints. Let S_o and S_1 be smooth manifolds in X of arbitrary (but less than n) dimension r_o and r_1. Let us pose the problem: find an admissible control u(t) which transfers the state point from some (not previously given) position $x_o \in S_o$ to x_1 S_1 which in so doing imparts a minimum value to the functional (3.1.2(a)). This problem is called the optimal problem with variable end-points. If both S_o and S_1 degenerate into points, the problem with variable end-points becomes the problem with fixed end-points. It is clear that if the points x_o and x_1 were known, we would have a problem with fixed end-points. Therefore, the control u(t), optimal in the sense of the problem with variable endpoints is also optimal in the sense of free-end i.e., the maximum principle (Theorems 3.1.1 and 3.1.2) remains valid for problem with free end points. However in this case it is also necessary to have additional relations from which one can determine the positions of x_o and x_1 on S_o and S_1. As a matter of fact, the transversality conditions are such conditions. These conditions allow us to write $r_o + r_1$ relations involving the coordinates of the end-points x_o and x_1. On the other hand, since the number of unknown parameters (in comparison with the problem with fixed end-points) is also increased by $r_o + r_1$ (since the position of x_o on the r_o dimensional manifold S_o is characterized by r_o parameters, and the position of $x_1 \in S_1$ is characterized by r_1 parameters), the maximum principle, together with the transversality conditions, form a sufficient system of relations for solving the given optimal problem with variable end-points.

Let us now give the formulation of the transversality conditions. Let $x_o \in S_o$ and $x_1 \in S_1$ be certain points, and let T_o and T_1 be the tangent planes of S_o and S_1 which pass through these points. The planes T_o and T_1 are in $\overline{X}$, and have dimensions r_o and r_1, respectively. Furthermore, let $u(t)$, $\overline{x}(t)$, $a \leq t \leq b$, be the solution of the optimal problem with fixed end-points x_o and x_1. Finally let $\overline{\psi}(t)$ be a vector whose existence is asserted in Theorem 3.1.1. We shall say that the vector $\overline{\psi}(t)$ satisfy the transversality condition at the right-hand end-point of the trajectory $\overline{x}(t)$ [i.e., at the point $\overline{x}(b)$] if the vector $\psi(b) = (\psi_1(b), \psi_2(b)\ldots \psi_n(b))$ is orthogonal to T_1. Making use of transversality conditions, the following theorem may be stated for the solution of control problem with variable end-points.

<u>Theorem 3.1.3</u> (Pontryagin et al [15])

Let $u(t)$, $a \leq t \leq b$, be an admissible control which transfers the state point from some position $x_o \in S_o$, and let $\overline{x}(t)$ be the corresponding trajectory starting at the point $\overline{x}_o = (0, x_o)$. In order that $u(t)$ and $\overline{x}(t)$ yield the solution of the optimal problem with variable end-points, it is necessary that there exist a nonzero continuous vector function $\overline{\psi}(t)$ which satisfies the conditions of Theorem 3.1.1, and in addition, the transversality condition at both end-points of the trajectory $\overline{x}(t)$. Of course, if either S_o, or S_1 degenerates into a point, the transversality condition at the corresponding end-point of the trajectory $\overline{x}(t)$ is replaced by the condition that the trajectory $\overline{x}(t)$ pass through this point. For the case of time optimality, we replace $\overline{x}(t)$ by $x(t)$ in the statement Theorem 3.1.3, and the reference to Theorem 3.1.1 is replaced by a reference to Theorem 3.1.2. The above theorems are, in general, applicable to both linear and nonlinear processes with any linear or non-linear cost function.

3.2. REDUCTION OF THE OPTIMAL CONTROL PROBLEM TO A TPBVP.

If we now take any control problem satisfying all the conditions in the previous section, we can define the Hamiltonian function and optimize it with respect to the control to find an expression for the optimal control in terms of states and co-states. After that the Hamiltonian function may be differentiated with respect to the states first and then with respect to the co-states to obtain a TPBVP. Initial conditions of the states are given and the final conditions of the co-states may be obtained from Theorem 3.1.3. To have a clear picture let us consider an example in the next section.

3.2.1. EXAMPLE (BRACHISTOCHRONE PROBLEM)

In 1696 Johann Bernoulli published a paper, in which he suggested to mathematicians the problem of determining the path of quickest descent - the brachistochrone. This problem consists in finding which curve joining two given points A and B, not lying on the same vertical line has the property that a massive particle sliding down along this curve from A to B reaches B in the shortest possible time (Fig. 4). It is easy to see that a path of quickest slide down is not a straight line joining the points A and B even though the straight line is the shortest line joining these points.

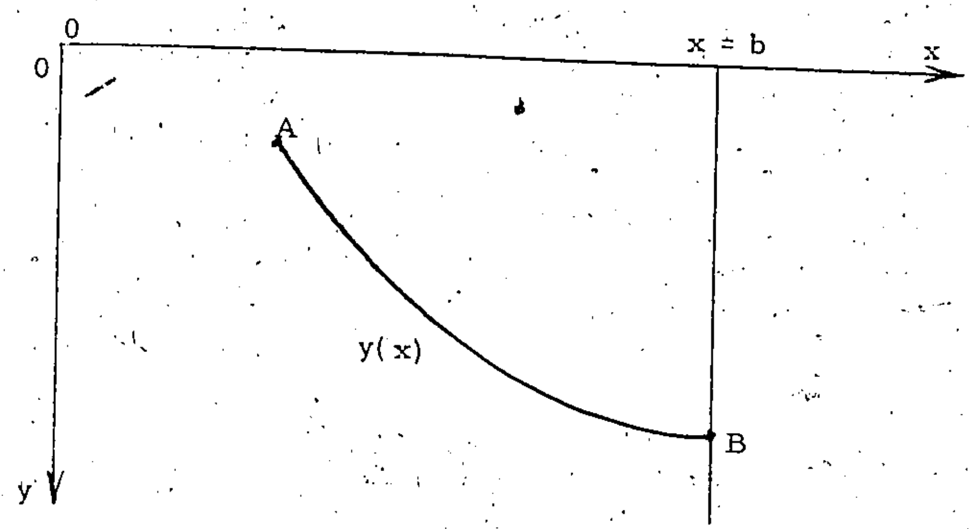

Fig. 4

When moving down a straight line, a particle picks up speed comparatively slowly. If the line is steeper near the start point A, then its length will increase, but greater part of it will be done with more speed. Mathematically the time function may be expressed as [31] (if the origin of the coordinate system is at the point A, the x axis being the horizontal and the y-axis, vertical directed downwards).

$$t(y(x)) = \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \sqrt{\frac{1+y'^2}{y}}\, dx, \quad y(0) = 0, \quad y(x_B) = y_1 \qquad (3.2.1.1)$$

We shall consider a variant of the brachistochrone problem where the point B is not fixed, but it lies on a given vertical line $x = b$.

Let $y(o) = +3$ and $b = \pi$. We can solve the variational problem as the following free-endpoint optimal control problem :

Minimize the functional

$$J(u) = \int_0^{\pi} \sqrt{\frac{1+u^2}{y}}\, dx \qquad (3.2.1.2)$$

subject to the constraint

$$\frac{dy}{dx} = u(x); \quad y(o) = +3.0 \qquad (3.2.1.2)$$

The Hamiltonian Function is

$$\bar{H} = f^o \psi_o + f^1 \psi$$

$$= \psi_o \sqrt{\frac{1+u^2}{y}} + u\,\psi \qquad (3.2.1.3)$$

For optimal u,

$$\frac{\partial \bar{H}}{\partial u} = 0$$

or $$u = \psi \sqrt{\frac{y}{(1-y\,\psi^2)}} \qquad (3.2.1.5)$$

The adjoint system is :

$$\frac{d\psi}{dx} = -\frac{\partial \bar{H}}{\partial y} = -\frac{1}{2} \frac{1}{y\sqrt{y(1-y\,\psi^2)}} \qquad (3.2.1.6)$$

$$\text{with, } \psi(\pi) = 0.0 \qquad (3.2.1.7)$$

Thus the Brachistochrone problem reduces to the following TPBVP :

$$\begin{cases} \dfrac{dy}{dx} = \psi \sqrt{\dfrac{y}{(1-y\,\psi^2)}}, & y(o) = 3.00 \qquad (3.2.1.8) \\ \dfrac{d\psi}{dx} = -\dfrac{1}{2} \dfrac{1}{y\sqrt{y(1-y\,\psi^2)}}, & \psi(\pi) = 0 \qquad (3.2.1.9) \end{cases}$$

Numerical solution of this TPBVP by the Davidon-Fletcher-Powell Iterative method with one-dimensional Fibonacci search gives the optimal trajectory tabulated below :

TABLE 3.2.1

x	0	.314	.628	.942	1.257	1.57	1.885	2.198	2.512	2.826	3.14
y	+3.0	+3.143	+3.267	+3.375	+3.466	+3.542	+3.603	+3.65	+3.686	+3.704	+3.71

Initial Condition $\psi(o) = 0.2526128$

Terminal Error $= 10^{-17}$

CHAPTER 4

NUMERICAL SOLUTION OF SOME OPTIMIZATION PROBLEMS BY IMBEDDING TECHNIQUES

In Chapter 3 we have shown how optimal control problems are converted to TPBVP. Solution methods for TPBVP have been described in Chapter 2. As we have seen there the solution of the TPBVP by invariant imbedding techniques requires the solution of a system of coupled Quasilinear, Hyperbolic, partial-Differential Equations. A brief discussion on the solution of these partial differential equations (PDE) will be presented in this Chapter, together with some comments on numerical stability of the solution and the choice of the step size and grid size of the scheme.

4.1. NUMERICAL SOLUTION OF THE QUASI-LINEAR HYPERBOLIC, PARTIAL DIFFERENTIAL EQUATIONS OF INVARIANT IMBEDDING.

The peculiarity of the PDE's we face is that they are only of the initial value type. There are many powerful, unconditionally stable, difference schemes for the solution of initial-boundary value type PDE [18, 19, 34]. Those unconditionally stable schemes for initial-boundary value problems can not be employed in the solution of initial-value type problems. There are however many conditionally stable schemes [13, 33] for the solution of the initial-value type problem. One such scheme used for the solution of our problems [13] can be explained by the following mesh for the one dimensional case :

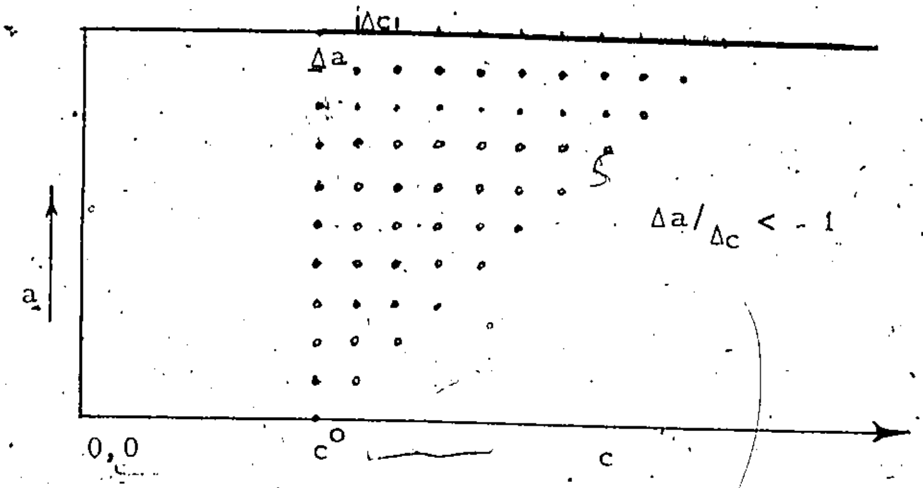

Fig. 5

Suppose we have the following scalar equation to be solved :

$$\frac{\partial f}{\partial a} = G(a,c) + F(a,c) \frac{\partial f}{\partial c} \qquad (4.1.1)$$

with initial condition

$$f(T,c) = 0.0 \text{ for all } c \qquad (4.1.2)$$

Our aim is to solve (4.1.1), (4.1.2) for the value of $f(a,c)$ corresponding to the particular values $c = c^o$ and $a = 0.0$. Here no boundary condition is prescribed. We first note that $f(T,c)$ is known to be zero at the points $c = k\delta$ for the final time T, where $k = 0, 1, 2, \ldots K$. In addition $f_c(T,c)$ is zero at these points. Next, we evaluate the given function $F(a,c)$, $G(a,c)$ at the points $c = \pm k\delta$ and $a = T$. These results are used to determine approximations to the function $f(a,c)$ for $a = T - \Delta a$ by using the formula [13].

$$f(T - \Delta a, c) = f(T,c) - \Delta a \; f_a(T,c) \qquad (4.1.3)$$

In this way an approximate solution is obtained at points $(T-\Delta a,(k-1)\delta)$. In the lower level we always get the value of the function at grid points less one than in the previous level, because we use the following relationsship for the derivative approximation of the function :

$$f_c(T,k\delta) = \frac{f(T,(k+1)\delta) - f(T,k\delta)}{\delta} \tag{4.1.4}$$

Then by repeating this operation the solution for $a = T - 2\Delta a$ can be obtained for certain set of c values. If we are interested the value of f at $c = c^{\circ}$ and $a = 0$ and divide the time level in 10 steps, we calculate the values of the function for all grid points as shown in Fig. 5, in the c direction as we proceed solving the equation from $a = T$ to $a = 0.0$. It can be easily shown that this scheme can be employed for solving the system of PDE obtained in (2.3.9) [33, 34]. But as the dimension of the system increases, the solution also becomes a more and more difficult with regards to selecting appropriate grid size and step size. Because there is an critical ratio of $\Delta a / \Delta c_i$ and also of $\Delta c_i / \Delta c_j$, $i, j = 1, 2, \ldots n$, but $i \neq j$. A slight deviation from this critical ratios may cause error propagation trouble for the numerical stability of the solution. In general $\Delta a / \Delta c_i < 1$. In addition to that, individual problems may present their own special difficulties with regards to the F and G functions of (4.1.1). There have been several studies for such cases [33, 34, 18].

A flow chart for the general solution of the Quasi-linear Hyperbolic PDE, is presented in Appendix A.

4.1.1 SOLUTION OF THE BRACHISTOCHRONE PROBLEM.

The solution of the Brachistochrone problem, formulated in section 3.2.1, can be obtained by solving the TPBVP of equations

(3.2.1.8) and (3.2.1.9). For this, we have to obtain the initial condition of (3.2.1.9). This can be obtained by solving the associated Invariant Imbedding Equation (as discussed in section 2.3.) which is as follows :

$$\frac{\partial r(a,c)}{\partial a} = - r(a,c)\sqrt{\frac{c}{1-cr^2(a,c)}} \cdot \frac{\partial r}{\partial c}(a,c) - \frac{1}{2c\sqrt{c(1-r^2(a,c))}} \tag{4.1.1.1}$$

For, $0 \leq a \leq \pi = 3.14$ and $3.0 \leq c \leq 4.0$

We are interested for $\psi(o) = r(o,3)$. Solution of (4.1.1.1) gives $\psi(o) = 0.25464$. Now solving (3.2.1.8) and (3.2.1.8) by RUNGE-KUTTA method, the following trajectory was obtained.

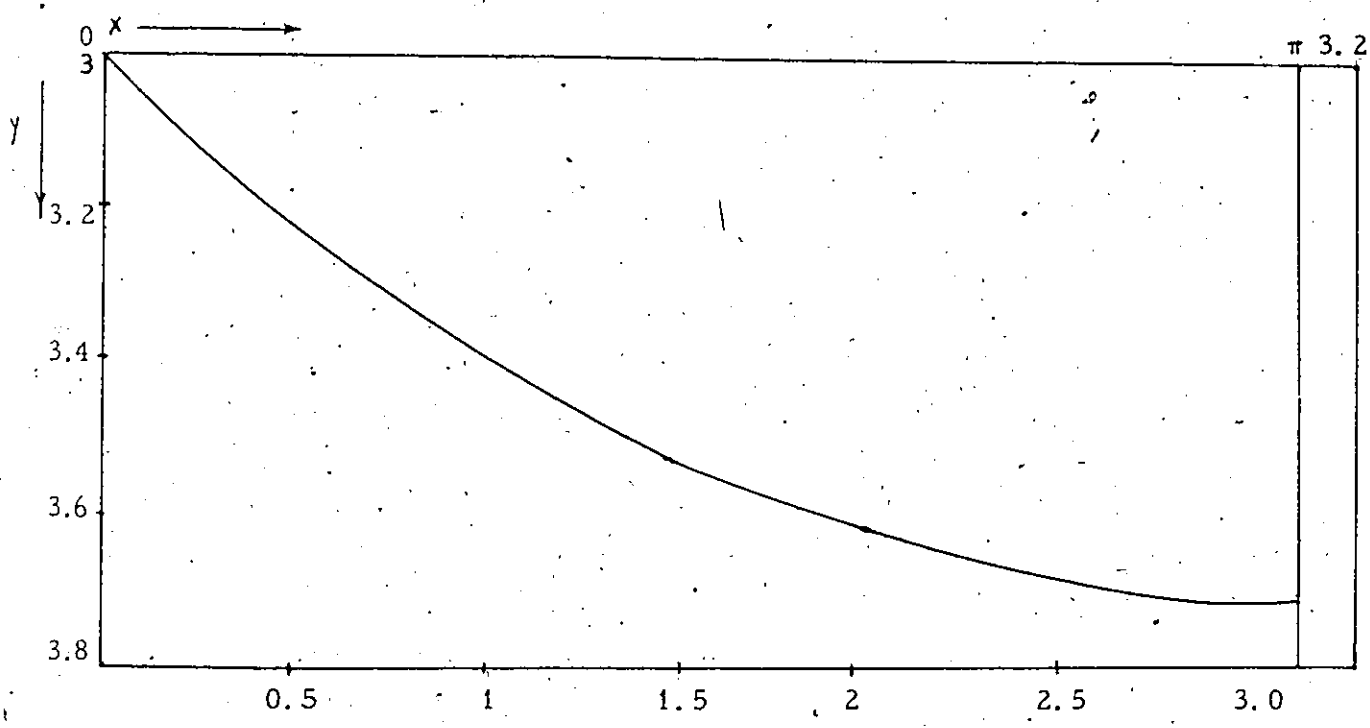

Fig. 6

owing

$$x \quad 2$$

(1.2.8)

(.1),

(1.2.9)

TABLE 4.1.1.

x	0	.314	.628	.942	1.257	1.570	1.885	2.198	2.512	2.826	3.142
y	+3.0	+3.143	+3.267	+3.374	+3.4646	+3.539	+3.6002	+3.6464	+3.6787	+3.6975	+3.7029

Initial Condition $\psi(o) = 0.25464$.

4.1.2 SOLUTION OF THE SECOND ORDER VAN DER POL SYSTEM

Minimize the functional

$$J(u) = \int_o^1 (x_1^2 + x_2^2 + u^2)\, dt \tag{4.1.2.1}$$

Subject to the constraint

$$\begin{cases} \dot{x}_1 = x_1(1-x_2^2) - x_2 + u, & x_1(o) = 0 \quad (4.1.2.2) \\ \dot{x}_2 = x_1, & x_2(o) = 1.0 \quad (4.1.2.3) \end{cases}$$

Treating this problem with Pontryagin's Maximum Principle the following TPBVP is obtained :

$$\begin{cases} \dot{x}_1 = x_1(1-x_2^2) - x_2 - \frac{1}{2}\psi_1, & x_1(o) = 0.0 \quad (4.1.2.4) \\ \dot{x}_2 = x_1, & x_2(o) = 1.0 \quad (4.1.2.5) \\ \dot{\psi}_1 = -2x_1 - \psi_1(1-x_2^2) - \psi_2, & \psi_1(1) = 0.0 \quad (4.1.2.6) \\ \dot{\psi}_2 = -2x_2 + 2\psi_1 x_1 x_2 + \psi_1, & \psi_2(1) = 0.0 \quad (4.1.2.7) \end{cases}$$

Now to get the initial condition of (4. 1. 2. 5) and (4. 1. 2. 6), the following coupled PDE's are derived

$$\begin{bmatrix} \frac{\partial r_1}{\partial a}(a, c_1, c_2) \\ \frac{\partial r_2}{\partial a}(a, c_1, c_2) \end{bmatrix} = - \begin{bmatrix} -2c_1 - r_1(1-c_2^2) - r_2 \\ -2c_2 + 2r_1 c_1 c_2 + r_1 \end{bmatrix} - \begin{bmatrix} \frac{\partial r_1}{\partial c_1} & \frac{\partial r_1}{\partial c_2} \\ \frac{\partial r_2}{\partial c_2} & \frac{\partial r_2}{\partial c_2} \end{bmatrix} \times \begin{bmatrix} c_1(1 - c_2^2) - c_2 - \frac{1}{2} r_1 \\ c_1 \end{bmatrix} \qquad (4.1.2.8)$$

where $0 \le a \le 1$, $x_1(a) = c_1$, $x_2(a) = c_2$ and $\psi_1(o) = r_1(0,0,1)$, $\psi_2(o) = r_2(0,0,1)$.

The solution of (4. 1. 2. 8) gives the following values :

$$\begin{aligned} \psi_1(o) &= -0.04974 \\ \psi_2(o) &= 1.92852 \end{aligned} \qquad (4.1.2.9)$$

With these initial conditions equations (4. 1. 2. 1) - (4. 1. 2. 7) were solved for x_1 and x_2 and the following trajectories were obtained.

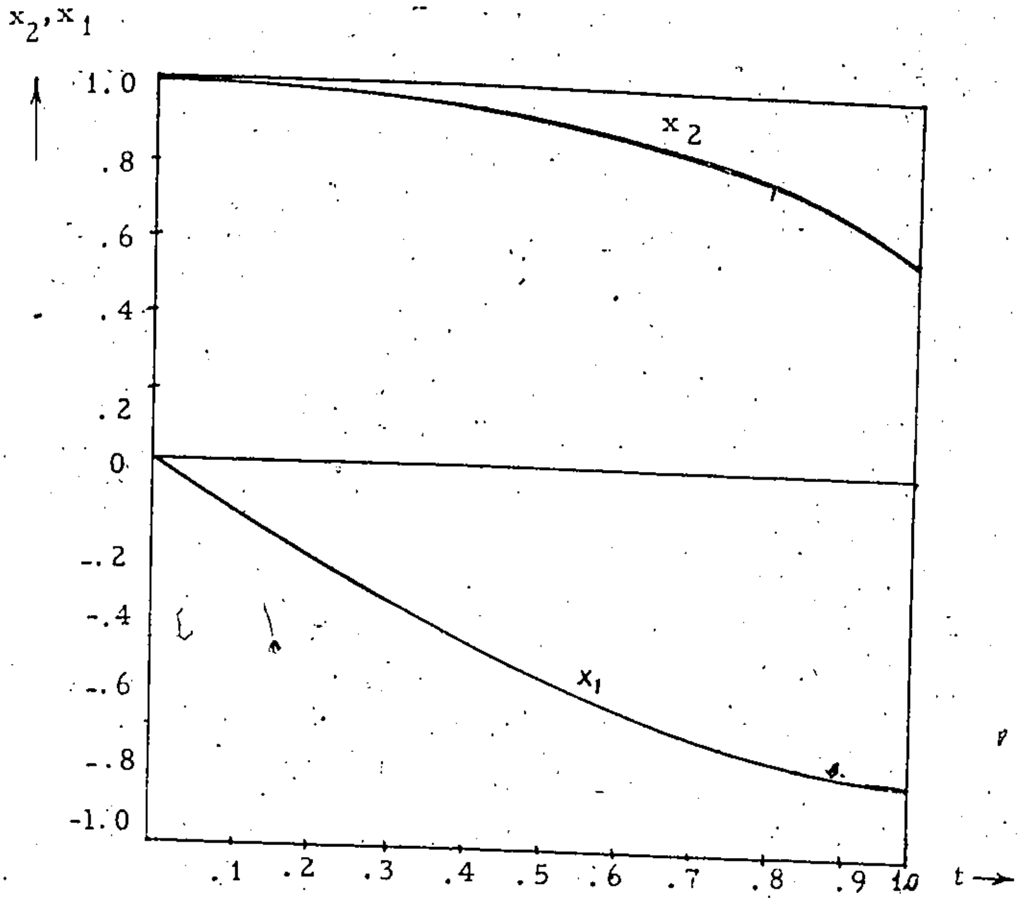

Fig. 7 a

The optimal control is given in Fig. 7b.

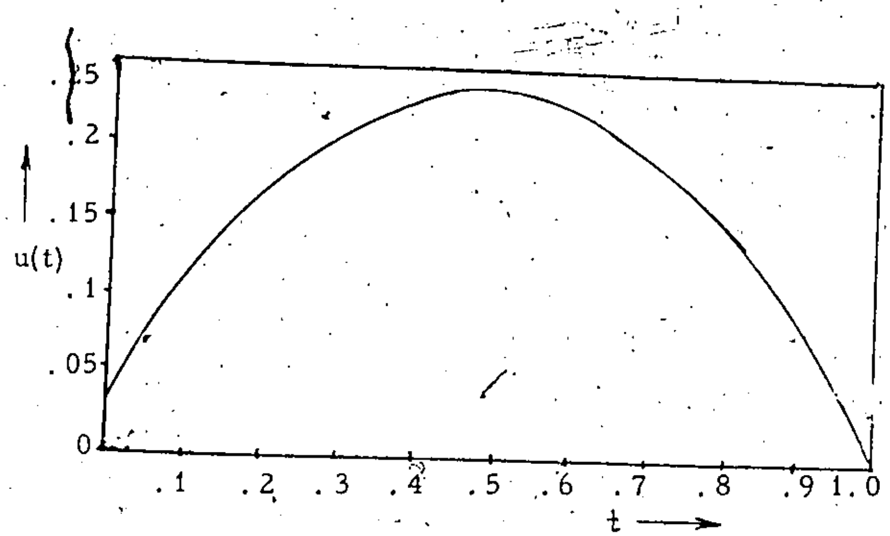

Fig. 7 b

TABLE 4.1.2.1

Invariant Imbedding Solution

t	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$x_1(t)$	0.0	-.09289	-.1775	-.2556	-.3294	-.4012	-.4734	-.5488	-.63004	-.70025	-.8219
$x_2(t)$	1.0	.9943	.9817	.9599	.9307	.8942	.8505	.7994	.7405	.6731	.5961

Initial Conditions : $\psi_1(o) = -0.04974$, $\psi_2(o) = 1.92852$

For comparison, the solution of this problem by the Davidon-Fletcher-Powell method with one-dimensional Fibonacci search is given in Table 4.1.2.2 .

TABLE 4.1.2.2

Iterative Method Solution

t	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$x_1(t)$	0.0	-.0927	-.1775	-.2563	-.3313	-.4047	-.4791	-.5571	-.6417	-.7357	-.8422
$x_2(t)$	1.0	.9953	.9817	.96	.9301	.8938	.8496	.7979	.7380	.6692	.5904

Initial Conditions : $\psi_1(o) = -0.05881676$, $\psi_2(o) = 1.836459$

Initial "guess" : $\psi_1(o) = 0.5875$, $\psi_2(o) = 5.8119$

Terminal error $= 10^{-19}$

4.2. NUMERICAL SOLUTION OF THE SYSTEM OF DIFFERENTIAL AND ALGEBRAIC EQUATIONS OF THE CONTINUATION METHOD.

By using the continuation method we can avoid the laborious process of solving partial differential equations. The only disadvantage of this method is that we have to solve n-J (J < n) algebraic and n + (n+1)(n-J) ordinary differential equations, if the original system of equations is n and the initial conditions of (n-J) equations are missing. The programming of the continuation method on a digital computer is very simple. All the examples we have considered, have been solved by CSMP and the results are very comparable.

4.2.1. SOLUTION OF THE BRACHISTOCHRONE PROBLEM.

As this problem has only two equations, (3.2.1.8), (3.2.1.9) to be solved, the problem will consist of 2 + (2+1) (2-1) = 5 ordinary differential equations and (2-1) = 1 algebraic equation.

We have now

$$\pi : \begin{cases} \dfrac{dy}{dt} = \psi \sqrt{\dfrac{y}{1 - y\psi^2)}} , \; y(0, 0) = 3.0 & (4.2.1.1) \\[2ex] \dfrac{d\psi}{dt} = -\dfrac{1}{2} \cdot \dfrac{1}{y\sqrt{y(1-y\psi^2)}} , \; \psi(3.14) = 0 . & (4.2.1.2) \end{cases}$$

This problem is imvedded into the one-parameter family of problems:

$$\pi(s) : \begin{cases} \dfrac{\partial y}{\partial t}(s,t) = \psi(s,t) \sqrt{\dfrac{y(s,t)}{1-y(s,t)\psi^2(s,t)}} ; \; y(s,0)=3.00 & (4.2.1.3) \\[2ex] \dfrac{\partial \psi}{\partial t}(s,t) = -\dfrac{1}{2} \dfrac{1}{y(s,t)\sqrt{y(s,t)[1-y(s,t)\psi^2(s,t)]}} ; \\ \psi(s,o) = \sigma^{(2)}(s) & (4.2.1.4) \end{cases}$$

By selecting $\psi(0,0) = \sigma^{(2)}(o) = 0.23$ (4.2.1.5)

The problem $\pi(o)$ can be solved by a forward integration and let

$$\eta_o^{(2)} = \psi(0,1) \tag{4.2.1.6}$$

By taking

$$\eta^{(2)}(s) = \eta_o^{(1)} + (\eta_1^{(1)} - \eta_o^{(1)})\, s \tag{4.2.1.7}$$

and differentiating (4.2.1.7) with respect to s we obtain:

$$\frac{d\sigma^{(2)}}{dt} = -\frac{\eta_o^{(2)}}{z_2^{2}(s,1)} \tag{4.2.1.8}$$

where $z_2^{1}(s,t) \triangleq \dfrac{\partial y(s,t)}{\partial \sigma^{(2)}}$, $z_2^{2}(s,t) \triangleq \dfrac{\partial \psi(s,t)}{\partial \sigma^{(2)}}$

By differentiating (4.2.1.3), (4.2.1.4) with respect to $\sigma^{(2)}$ we obtain:

$$\left\{\begin{aligned}
\frac{\partial z_2^{1}(s,t)}{\partial t} &= \frac{\psi(s,t)\; z_2^{1}(s,t)}{2(1-y(s,t)\psi(s,t))\sqrt{y(s,t)\,(1-y(s,t)\,\psi^{2}(s,t))}} \\
&\quad + \frac{z_2^{2}(s,t)\,.\sqrt{\dfrac{y(s,t)}{1-y(s,t)\,\psi^{2}(s,t)}}}{(1-y(s,t)\,\psi^{2}(s,t))} \quad ; \; z_2^{1}(s,o) = 0 \\
\frac{\partial z_2^{2}(s,t)}{\partial t} &= \frac{z_2^{1}(s,t)\,(3-4\,y(s,t)\,\psi^{2}(s,t))}{4\,y^{5/2}(s,t)\,(1-y(s,t)\,\psi^{2}(s,t))\sqrt{1-y(s,t)\,\psi^{2}(s,t)}} \\
&\quad - \frac{z_2^{2}(s,t)\;\psi(s,t)}{2\,(1-y(s,t)\,\psi^{2}(s,t))\sqrt{y(s,t)\,(1-y(s,t)\,\psi^{2}(s,t))}} \; ;
\end{aligned}\right. \tag{4.2.1.9}$$

$$z_2^{2}(s,o) = 1 \tag{4.2.1.10}$$

Now, we can solve the initial value problem (4.2.1.3), (4.2.1.4), (4.2.1.8), (4.2.1.9) and (4.2.1.10) by the double loop integration scheme, shown in Fig 3 (section 2.4). Integration with respect to t is done forward from t = 0 to t = 3.14, for each s, s varying from s = 0 to s = 1. The trajectory obtained is shown below

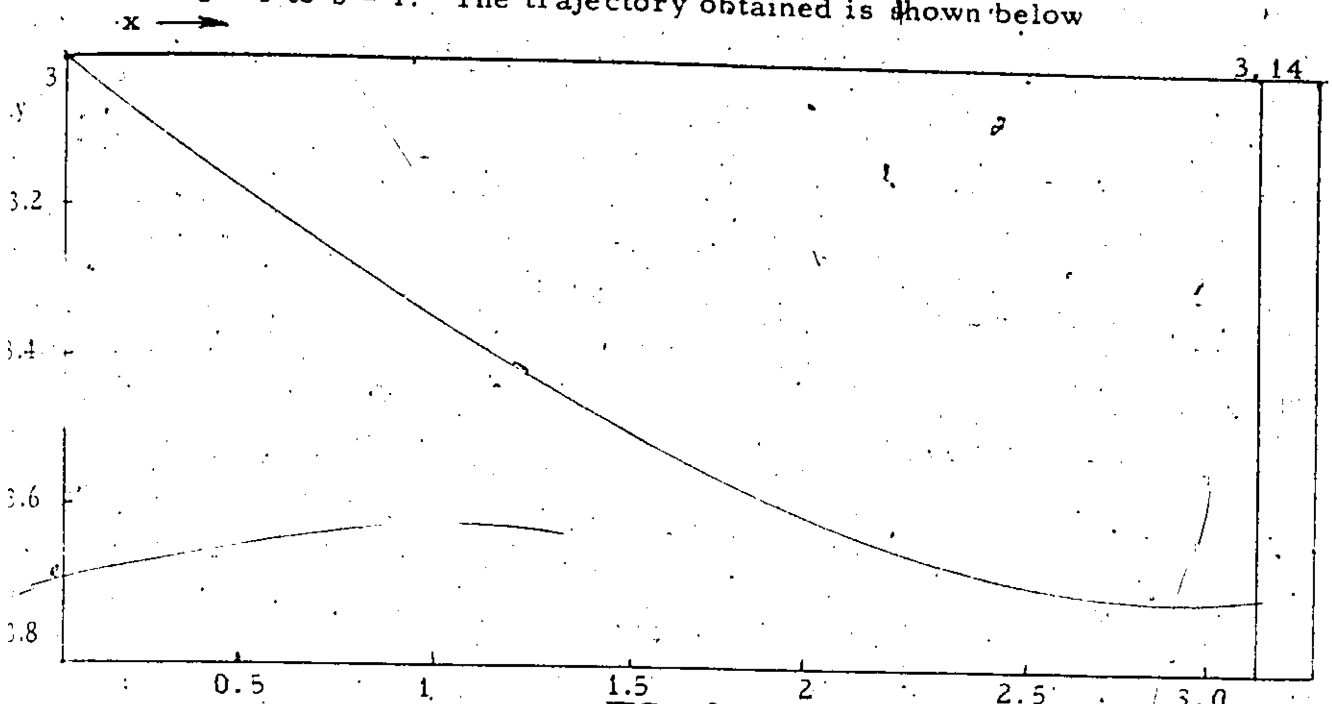

FIG. 8

TABLE 4.2.1

Solution by the Continuation Method

x	0	.314	.628	.942	1.257	1.570	1.885	2.198	2.512	2.826	3.142
y(x)	+3.0	+3.15	+3.2812	+3.395	+3.4926	+3.5749	+3.6425	+3.696	+3.7361	+3.7626	+3.776

Initial condition : $\psi(o)$ = .26199

Initial guess : $\psi(o)$ = 0.23

4.2.2. SOLUTION OF THE SECOND ORDER VAN DER POL SYSTEM.

As this problem has four equations, (4.1.2.4) - (4.1.2.7) to be solved for obtaining the trajectory x_1 and x_2, the problem will consist of 14 ordinary differential equations and 2 algebraic equations for each s starting from $s = 0$ to $s = 1$. Then we have :

$$\pi : \begin{cases} \dfrac{dx_1}{dt} = x_1(1-x_2^2) - x_2 - \dfrac{\psi_1}{2} ; \quad x_1(o) = 0.0 & (4.2.2.1) \\[2ex] \dfrac{dx_2}{dt} = x_1 ; \quad x_2(o) = 1.0 & (4.2.2.2) \\[2ex] \dfrac{d\psi_1}{dt} = -2x_1 - \psi_1(1-x_2^2) - \psi_2 ; \quad \psi_1(1) = 0.0 & (4.2.2.3) \\[2ex] \dfrac{d\psi_2}{dt} = -2x_2 + 2\psi_1 x_1 x_2 + \psi_1 ; \quad \psi_2(1) = 0.0 & (4.2.2.4) \end{cases}$$

This problem is imbedded into the one parameter family of problems :

$$\pi(s) : \begin{cases} \dfrac{\partial x_1(s,t)}{\partial t} = x_1(s,t)\,(1 - x_2^2(s,t)) - x_2(s,t) - \dfrac{\psi_1(s,t)}{2} ; \quad x_1(s,o) = 0.0 & (4.2.2.5) \\[2ex] \dfrac{\partial x_2(s,t)}{\partial t} = x_1(s,t) ; \quad x_2(s,o) = 1.0 & (4.2.2.6) \\[2ex] \dfrac{\partial \psi_1(s,t)}{\partial t} = -2x_1(s,t) - \psi_1(s,t)(1-x_2^2(s,t)) - \psi_2(s,t) ; \quad \psi_1(s,o) = \sigma^{(3)}(s) & (4.2.2.7) \\[2ex] \dfrac{\partial \psi_2(s,t)}{\partial t} = -2x_2(s,t) + 2\psi_1(s,t)\,x_1(s,t)\,x_2(s,t) + \psi_1(s,t) \\[1ex] \qquad \psi_2(s,o) = \sigma^{(4)}(s) & (4.2.2.8) \end{cases}$$

By selecting $\sigma^{(3)}(o) = 0.3$

and $\sigma^{(4)}(o) = 4.0$ (4.2.2.9)

The problem $\pi(o)$ can be solved by a forward integration and let

$$\eta_o^{(3)} = \psi_1(0,1)$$

$$\eta_o^{(4)} = \psi_2(0,1) \qquad (4.2.2.11)$$

By taking

$$\eta^{(3)}(s) = \eta_o^{(3)} + (0 - \eta_o^{(3)})\, s \qquad (4.2.2.12)$$

$$\eta^{(4)}(s) = \eta_o^{(4)} + (0 - \eta_o^{(4)})\, s \qquad (4.2.2.13)$$

and differentiating (4.2.2.12), (4.2.2.13) with respect to s we obtain :

$$\left\{\begin{array}{ll} z_3^{\,3}(s,1)\,\dfrac{d\,\sigma^{(3)}}{d\,s}(s) + z_4^{\,3}(s,1)\,.\,\dfrac{d\,\sigma^{(4)}}{d\,s}(s) = -\,\eta_o^{(3)} & (4.2.2.14) \\ z_3^{\,4}(s,1)\,\dfrac{d\,\sigma^{(3)}}{d\,s}(s) + z_4^{\,4}(s,1)\,.\,\dfrac{d\,\sigma^{(4)}}{d\,s}(s) = -\,\eta_o^{(4)} & (4.2.2.15) \end{array}\right.$$

where by differentiating (4.2.2.5) - (4.2.2.8) with

respect to $\sigma^{(j)}(s)$, $j = 3, 4 \ldots$ we obtain :

$$
\left\{
\begin{aligned}
\frac{\partial z_3^1(s,t)}{\partial t} &= (1 - x_2^2(s,t))\, z_3^1(s,t) + (-2x_1(s,t)\, x_2(s,t) - 1)\, z_3^2(s,t) - \frac{z_3^3}{2}(s,t) \\
& \qquad z_3^1(s,o) = 0.0 \qquad (4.2.2.16) \\
\frac{\partial z_4^1(s,t)}{\partial t} &= (1 - x_2^2(s,t))\, z_4^1 + (-2x_1(s,t)\, x_2(s,t) - 1)\, z_4^2(s,t) - \frac{z_4^3}{2}(s,t) \\
& \qquad z_4^1(s,o) = 0 \qquad (4.2.2.17) \\
\frac{\partial z_3^2(s,t)}{\partial t} &= z_3^1(s,t) \quad ; \quad z_3^2(s,o) = 0 \qquad (4.2.2.18) \\
\frac{\partial z_4^2(s,t)}{\partial t} &= z_4^1(s,t) \quad ; \quad z_4^2(s,o) = 0 \qquad (4.2.2.19) \\
\frac{\partial z_3^3(s,t)}{\partial t} &= -2z_3^1(s,t) + 2x_2(s,t)\, \psi_1(s,t)\, z_3^2(s,t) - (1 - x_2^2(s,t))\, z_3^3(s,t) \\
& \quad - z_3^4(s,t) \quad ; \quad z_3^3(s,o) = 1 \qquad (4.2.2.20) \\
\frac{\partial z_4^3(s,t)}{\partial t} &= -2z_4^1(s,t) + 2\, x_2(s,t)\, \psi_1(s,t)\, z_4^2(s,t) - (1 - x_2^2(s,t))\, z_4^3(s,t) \\
& \quad - z_4^4(s,t) \quad ; \quad z_4^3(s,o) = 0 \qquad (4.2.2.21) \\
\frac{\partial z_3^4(s,t)}{\partial t} &= 2x_2(s,t)\, \psi_1(s,t)\, z_3^1(s,t) - (2 + 2x_1(s,t)\, \psi_1(s,t))\, z_3^2(s,t) \\
& \quad + (2x_2(s,t)\, x_1(s,t) + 1)\, z_3^3(s,t) \; ; \; z_3^4(s,o) = 0 \qquad (4.2.2.22) \\
\frac{\partial z_4^4(s,t)}{\partial t} &= 2x_2(s,t)\, \psi_1(s,t)\, z_4^1(s,t) + (-2 + 2x_1(s,t)\, \psi_1(s,t))\, z_4^2(s,t) \\
& \quad + (2x_1(s,t)\, x_2(s,t) + 1)\, z_4^3(s,t); \quad z_4^4(s,o) = 1 \qquad (4.2.2.23)
\end{aligned}
\right.
$$

Now, we can solve the initial value problem (4.2.2.5) - (4.2.2.8), (4.2.2.14), (4.2.2.15) and (4.2.2.16) - (4.2.2.23) by the double loop integration scheme as shown in Fig 3, Section 2.4. Integration with respect to t is done forward from t = 0 to t = 1 for each s, s varying from s = 0 to s = 1. The trajectories for x_1, x_2 and $\hat{u}$ obtained by this are shown below :

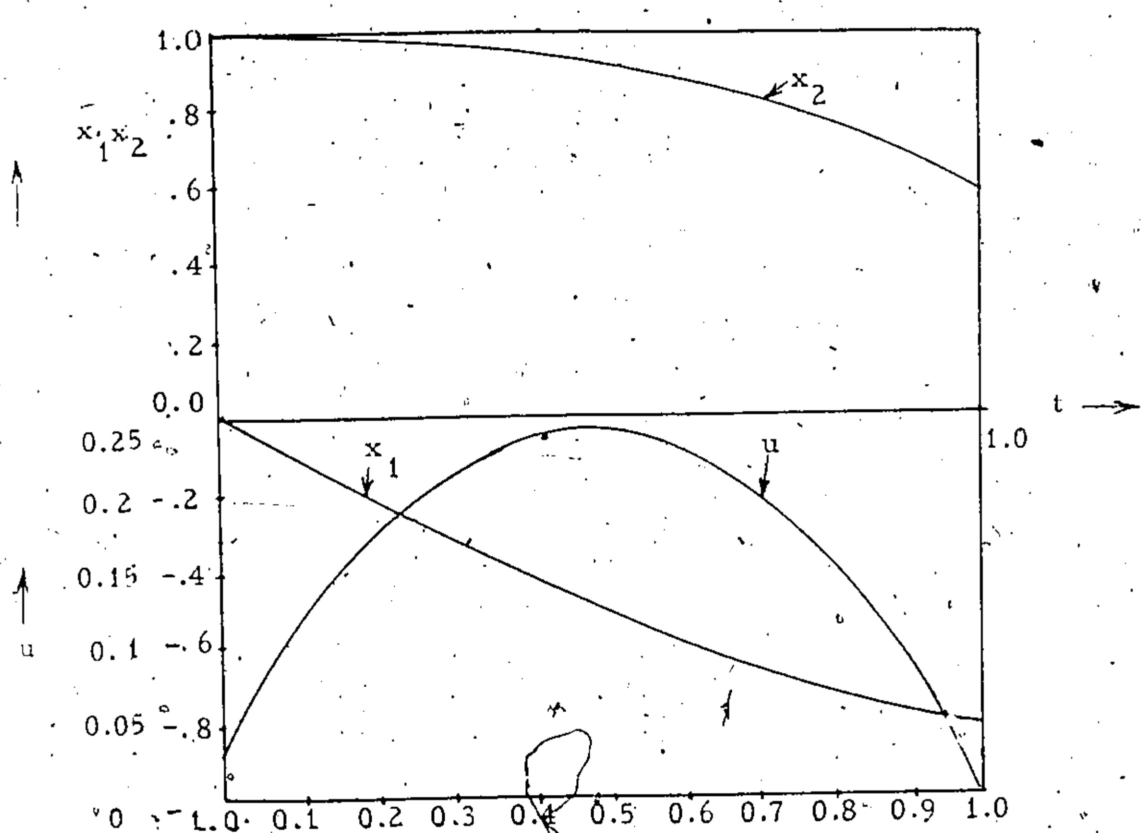

Fig. 9

TABLE 4.2.2

Solution by the Continuation Method

t	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$x_1(t)$	0.0	-.09203	-.1768	-.256	-.33196	-.40689	-.48339	-.5642	-.6521	-.7502	-.8618
$x_2(t)$	1.0	.9953	.9818	.9602	.9307	.8938	.8493	.79696	.7362	.6662	.5857

Initial conditions : $\psi_1(o) = -0.077013$

$\psi_2(o) = 1.7285.$

Initial guess : $\psi_1(o) = 0.3$

$\psi_2(o) = 4.0$.

CHAPTER 5

KALABA - SRIDHAR's APPROACH AND OPTIMAL NONLINEAR FEEDBACK CONTROL

Kalaba and Sridhar have shown [4, 9, 10] how optimal control problems can be solved as initial value problems without using Pontryagin's Maximum Principle and the associated TPBVP. The approach is again an imbedding approach. This approach of treating the control problems has some advantages which we will discuss below.

5.1 THE KALABA-SRIDHAR INVARIANT IMBEDDING EQUATIONS.

Consider the free end-point control problem of minimizing the functional

$$J(n) = \int_{t_o}^{t_1} f^o(t, x(t), u(t))\, dt \tag{5.1.1}$$

subject to the constraint

$$\dot{x}(t) = f(t, x(t), u(t)) \quad , \quad t \in [t_o, t_1] \tag{5.1.2}$$

with

$x(t_o) = x_o$, $x(t) \in E^n$, $f(t,\cdot,\cdot) \in C^1(E^{n+r}, E^n)$, $u(t) \in E^r$, $f^o(t,\cdot,\cdot) \in C^1(E^{n+r}, E^n)$.

Assume that $(t_1 = b)$ is fixed and finite but $t_o = a$ and $x_o = c$ are free with $x(a) = c$. Let $u = u(t, a, c)$ be the optimal control and $x = x(t, a, c)$ the corresponding optimal trajectory.

Kalaba and Sridhar [4, 9, 10] derived the invariant imbedding equations for this problem, which, in the n-dimensional case considered here, become :

$$u_a(t,a,c) = -u_c(t,a,c)\,p(a,c) \quad , \quad a \le t \tag{5.1.3}$$

$$x_a(t,a,c) = -x_c(t,a,c)\,p(a,c) \quad , \quad a \le t \tag{5.1.4}$$

$$x(t,t,c) = c \qquad\qquad a \le t \le b \tag{5.1.5}$$

$$u(t,t,c) = q(t,c) \qquad , \quad a \le t \le b \tag{5.1.6}$$

where the auxiliary functions $p(a,c)$, $q(a,c)$ satisfy the partial differential equation, algebraic equations and transversality condition:

$$\gamma_a(a,c) + \gamma_c(a,c)\,p(a,c) + \phi^T(a,c,q(a,c))\,\gamma(a,c)$$
$$+ \; b(a,c,q(a,c)) = 0 \tag{5.1.7}$$

$$p(a,c) = f(a,c,q(a,c)) \tag{5.1.8}$$

$$\alpha(a,c,q(a,c)) + \psi^T(a,c,q(a,c))\,\gamma(a,c) = 0 \tag{5.1.9}$$

$$\gamma(b,c) = 0 \quad \text{for all } c. \tag{5.1.10}$$

where

$$\begin{aligned} \alpha(t,x,u) &\triangleq f_u^{\,o}(t,x,u) \\ b(t,x,u) &\triangleq f_x^{\,o}(t,x,u) \\ \psi(t,x,u) &\triangleq f_u(t,x,u) \\ \phi(t,x,u) &\triangleq f_x(t,x,u) \end{aligned} \tag{5.1.11}$$

The solution of the initial-value problem (5.1.3) - (5.1.10) evaluated at $a = t_o$ and $c = x_o$ will give the optimal control $u(t) = u(t, t_o, x_o)$

and the corresponding optimal trajectory $x(t) = x(t, t_o, \xi_o)$.

5.2 THE OPTIMAL FEEDBACK CONTROLLER

It can be easily seen that one does not have to solve both partial differential equations (5.1.3) - (5.1.4). Indeed, the solutions of (5.1.3) and (5.1.4) have the following relation.

$$u(t, a, c) = q(t, x(t, a, c)) \tag{5.1.12}$$

where $q(a, c)$ is the auxiliary function obtained by solving (5.1.7) - (5.1.10). This observation does not only reduce the computational requirements of the open-loop control problem (5.1.3) - (5.1.4), but, what is more important, it helps in determining the optimal feedback controller [6] .

Assuming that there exists a unique solution $q(a, c)$ of the system of equations (5.1.7) - (5.1.9) with boundary condition (5.1.10), the problem of determining $q(a, c)$ reduces to the one of solving a vector partial differential equation, which moreover, is non-linear.

Storing the solution of this equation in terms of grid points in the $(n+1)$ dimensional Euclidean space, particularly when n is large , demands very large computer storage. For example if k points in each dimension are considered, the number of grid points becomes $r \times k^{n+1}$: Moreover this is very inefficient way of storing the optimal feedback controller $q(t, x)$ since it gives no immediate information about its structure. We are proposing below a more efficient scheme [32] .

5.3. AN EFFICIENT STORAGE SCHEME FOR THE DATA OF THE OPTIMAL NONLINEAR FEEDBACK CONTROLLER.

We assume that the region of interest in state space E^n is the hyperparallelopiped L defined by :

$$d_i \leq x_i \leq h_i , \quad i = 1, 2, \ldots n \tag{5.2.1}$$

We consider a family of multivariable orthonormal polynomials $\{ P_n(x) \}_{n=1}^{M}$, $x \in L$. We shall seek an approximate solution for the optimal feedback controller, of the form

$$q(a, c) = \sum_{n=1}^{M} Q_n(a) \, P_n(c) \tag{5.2.2}$$

where the co-efficients $Q_i(a)$ ($\in E^r$) are determined by [21]:

$$Q_n(a) = \int_L q(a, c) \, P_n(c) \, dc \tag{5.2.3}$$

Using an appropriate quadrature scheme for the numerical evaluation of a multiple definite integral, we can employ (5.2.3) for storing the function q(a, c) in terms of the vector valued functions $Q_i(a)$, $i = 1, 2, \ldots M$, instead of the grid points. Storing proceeds in parallel with the computational solution of the vector partial differential equation mentioned in the previous section.

Storage of the functions $Q_i(a)$, $i = 1, 2, \ldots, M$, $a \in [t_o, t_1]$, requires on r x M x k memory locations (Assuming that $[t_o, t_1]$ is quantized using k points). Moreover, this technique produces an approximate structure for the optimal non-linear feedback controller. To demonsstrate the approach we present the following simple example [32] .

5.3.1 EXAMPLE.

Consider the problem of minimizing the functional

$$J(u) = \int_0^1 \sqrt{x(1+u^2)}\ dt \tag{5.3.1.1}$$

subject to the system constraint

$$\dot{x}(t) = u(t), \qquad x(o) = 1.5 \tag{5.3.1.2}$$

From (5.1.7) - (5.1.10) we obtain the quasilinear hyperbolic equation :

$$q_a(a,c) + q(a,c)\, q_c(a,c) = \frac{1+q^2(a,c)}{2c}, \quad a \in [0,1] \tag{5.3.1.3}$$

with boundary condition

$$q(1,c) = 0 \qquad \text{for all } c \tag{5.3.1.4}$$

Considering as region of interest the interval $1 \le x \le 2$, one has to solve equation (5.3.1.3) by a "backward sweep" to obtain the optimal feedback controller $q(t,x)$: Then the optimal trajectory may be found by a "forward sweep" integration of the equation

$$\dot{x}(t) = q(t, x(t)), \quad x(o) = 1.5, \quad t \in [0,1] \tag{5.3.1.5}$$

and the optimal feedback control will be

$$u(t) = q(t, x(t)) \tag{5.3.1.6}$$

We consider now the family of orthogonal Chebysher polynomials $CH_n(c)$, $1 \le c \le 2$, generated by the expression :

$$CH_n(c) = 2(2c-3)\, CH_{n-1}(c) - CH_{n-2}(c);\ n \ge 3 \tag{5.3.1.7}$$

with

$$CH_1(c) = 2c - 3, \quad CH_2(c) = 2(2c-3)^2 - 1$$

By normalizing we obtain the orthonormal set

$$P_n(c) = \frac{CH_n(c)}{\int_1^2 CH_n^2(c)\,dc} \qquad (5.3.1.8)$$

We now obtain an approximate solution of (5.3.1.3) of the form

$$q(a,c) = \sum_{n=1}^{4} Q_n(a)\, P_n(c) \qquad (5.3.1.9)$$

where $$Q_n(a) = \int_1^2 q(a,c) P_n(c)\,dc, \quad n = 1,2,3,4\ldots \qquad (5.3.1.10)$$

and this integral is evaluated numerically using Simpson's rule. The functions $Q_n(a)$, $a \in [0,1]$, $n = 1,2,3,4$ are stored. The optimal trajectory is then obtained from (5.3.1.5) and (5.3.1.9) and is given in Fig. 10 . For comparison, we also give the theoretical optimal trajectory

$$x(t) = 1.30 + 0.20\,(t-1)^2 \qquad (5.3.1.11)$$

obtained from the analytical solution of the Euler - Langrange equation. The maximum error of the computed quasi-optimal trajectory is less than 1%, with $\Delta c = 0.025$. The optimal trajectory was also calculated from the invariant equation (5.1.4). Thus using invariant imbedding technique one may obtain, as a solution of an initial value problem, both the optimal trajectory and the feedback controller for a class of optimal control problems.

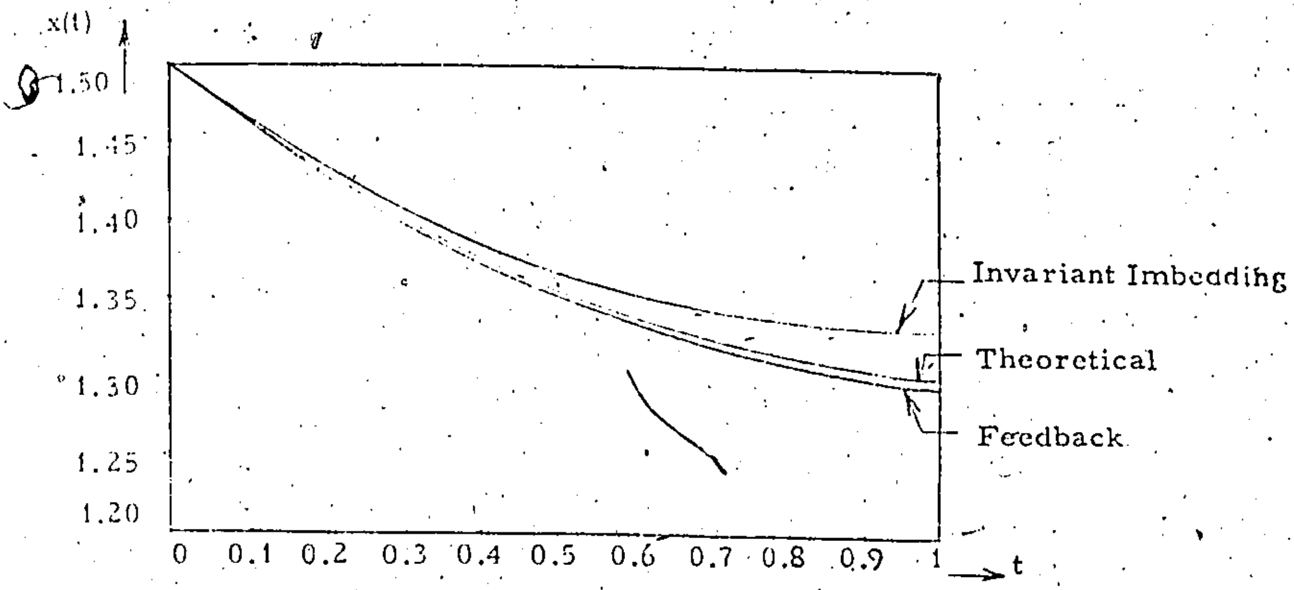

Fig. 10. Optimal Trajectory

The optimal feedback controller is given by

$$u(t,x) = \sum_{n=1}^{4} Q_n(t)\; P_n(x) \qquad (5.3.1.12)$$

where value of the functions $Q_n(t)$ are given in the following table:

TABLE 5.3.1

	$Q_N(t)$									
N \ t	0.0	0.20	0.36	0.44	0.52	0.60	0.76	0.84	0.92	1.0
1	0.2053	0.1271	0.09026	0.0756	0.0625	0.0506	0.0292	0.0192	0.0096	0.0
2	0.2557	0.1963	0.15391	0.1335	0.1134	0.0941	0.0559	0.0372	0.0166	0.0
3	-0.0649	-0.0450	0.0342	-0.0289	-0.0240	-0.0196	-0.0113	-0.0075	-0.0037	0.0
4	0.0678	0.0512	0.0387	0.0330	0.0277	0.0228	0.0133	0.0088	0.0044	0.0

CONCLUSIONS

In this thesis, two imbedding methods, Invariant Imbedding and the Continuation Method as applied to the solution of optimal control problems were studied.

The application of Invariant Imbedding techniques resulted in the solution of initial value problems involving quasi-linear, hyperbolic partial-differential equations. For systems of large dimension, solution of these equations requires large computer memory. Bellman's "curse of dimensionality" for the equations of dynamic programming seems to apply also to the invariant imbedding equations. The difference between dynamic programming and invariant imbedding partial-differential equations is that the latter do not require the intermediate function minimizations of the former. It is hoped that recent advances in the technology of high-speed large computer memories, in connection with efficient storage schemes, will bypass the storage problems. - In Chapter 5 of the thesis such an efficient storage scheme was introduced. The numerical results of the invariant imbedding equations as compared to the ones of iterative methods are quite satisfactory, as we can see from Tables 3.2.1, 4.1.1, 4.1.2.1 and 4.1.2.2. The higher computer time required by the Invariant Imbedding approach is the price paid for the global convergence. This time is explained by the fact that a family of problems has to be solved before, in order that the solution of the original problem is determined.

The Continuation method is a one-parameter imbedding method. Its application to TPBVP problems results in intial value problems of ordinary differential and algebraic equations. These initial value problems, however, are of considerably higher dimension than the original TPBVP. The method is characterized by a wide convergence region. - The numerical results obtained by this method, as exhibited in Tables 4.2.1, 4.2.2, are quite satisfactory as compared with the results obtained by invariant imbedding and iterative methods. Questions of numerical stability and storage of the solutions of the partial-differential equations of invariant imbedding, do not arise here. This method appears to the author very appealing for the numerical solution of optimal control problems. -

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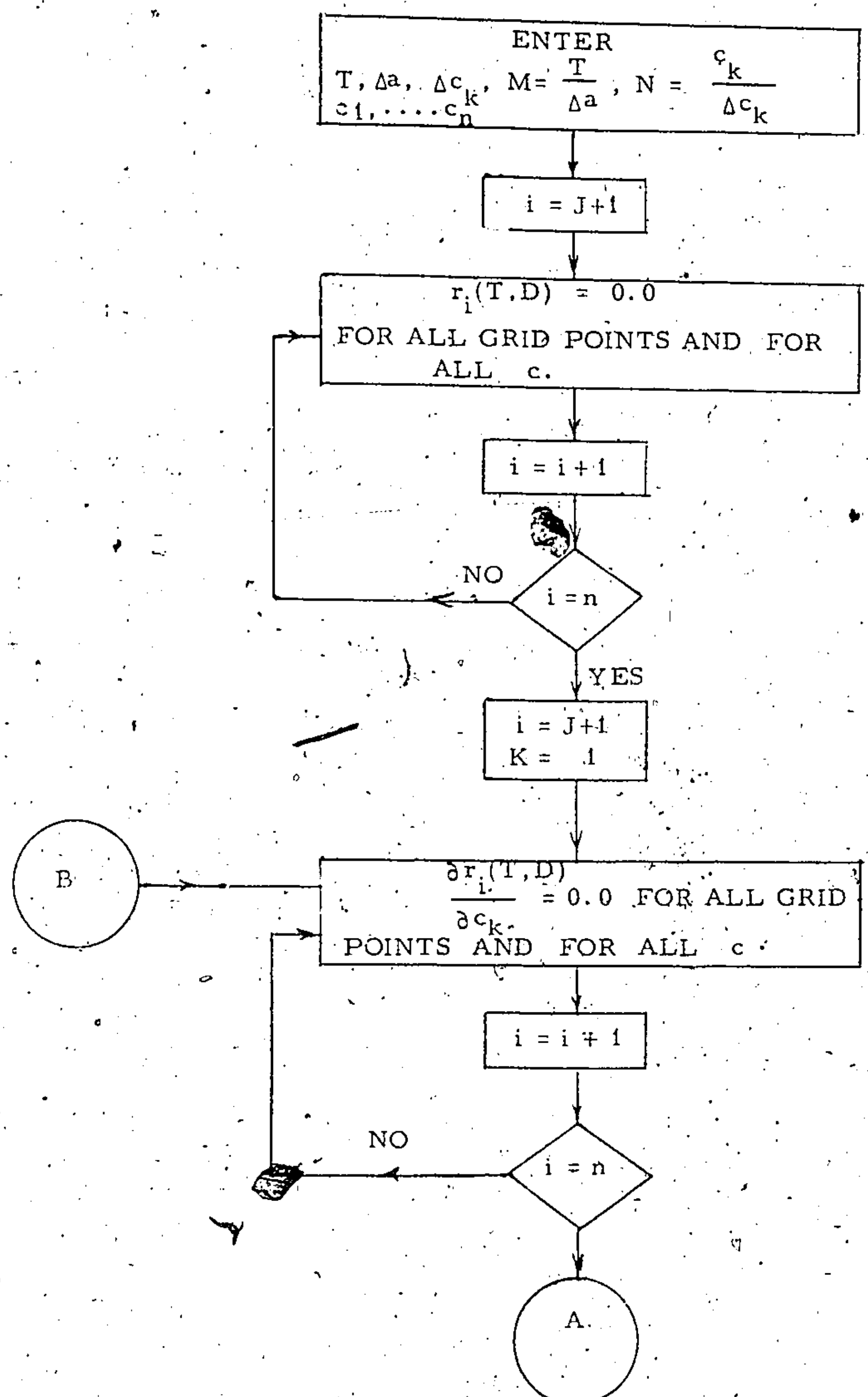
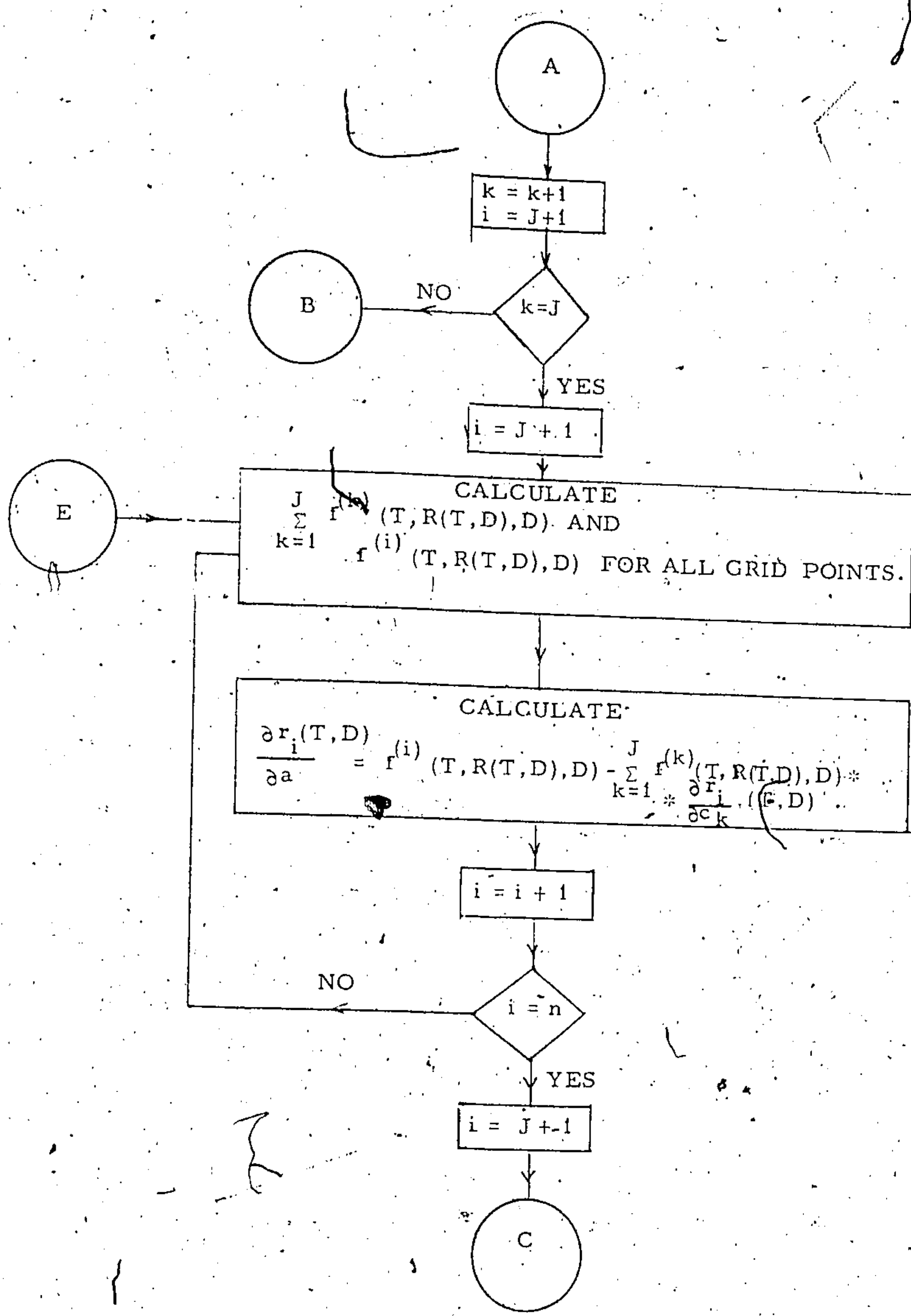
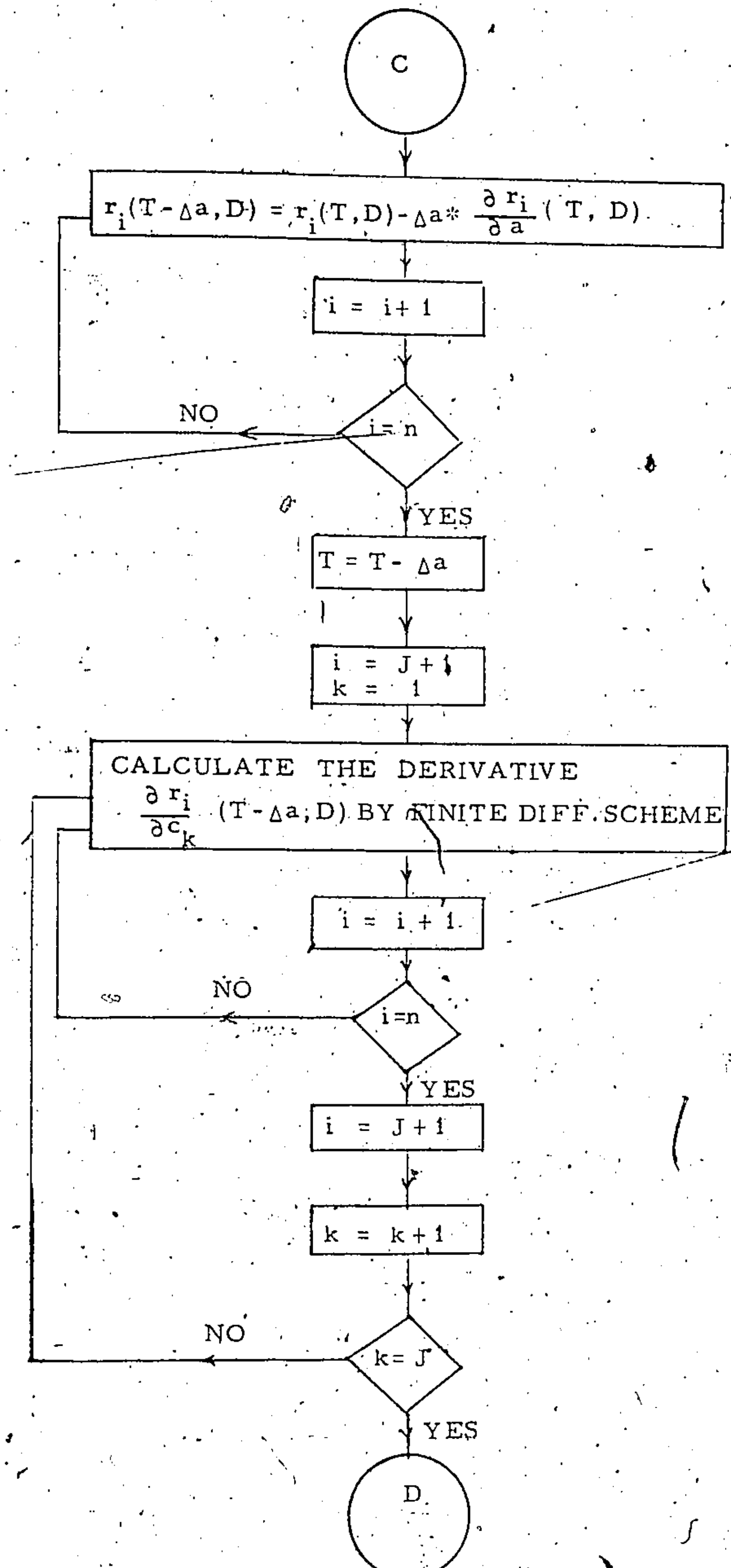
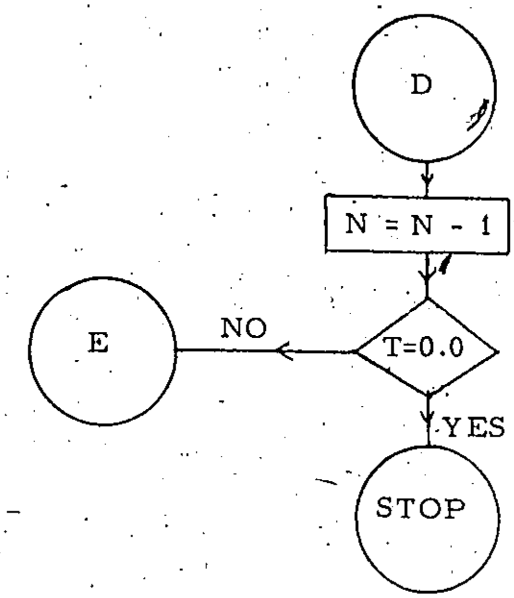
APPENDIX A

FLOW CHART FOR SOLVING THE QUASILINEAR HYPERBOLIC P.D.E (2.3.9).




A
k = k+1
i = J+1
k=J
NO
B
YES
i = J + 1
E
CALCULATE
$\sum_{k=1}^{J} f^{(k)}(T, R(T,D), D)$ AND
$f^{(i)}(T, R(T,D), D)$ FOR ALL GRID POINTS.
CALCULATE
$\frac{\partial r_i(T,D)}{\partial a} = f^{(i)}(T, R(T,D), D) - \sum_{k=1}^{J} f^{(k)}(T, R(T,D), D) * \frac{\partial r_i}{\partial c_k}(T, D)$
i = i + 1
i = n
NO
YES
i = J + 1
C


C
$r_i(T-\Delta a, D) = r_i(T,D) - \Delta a * \frac{\partial r_i}{\partial a}(T, D)$
i = i+1
i = n
NO
YES
T = T - Δa
i = J+1
k = 1
CALCULATE THE DERIVATIVE
$\frac{\partial r_i}{\partial c_k}$ (T - Δa, D) BY FINITE DIFF. SCHEME
i = i + 1
i = n
NO
YES
i = J+1
k = k+1
k = J
NO
YES
D


D
N = N - 1
T=0.0
NO
E
YES
STOP

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