Non-sway Model for the Lateral Torsional Buckling of Wooden Beams under Wind Uplift

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Abstract

Simply-supported wooden beams nailed to deck boards subjected to wind uplift forces are subjected to compressive stresses at their bottom fibers. Because the restraining action provided by decking is at the top fibers, it is unclear to what extent such restraints are effective in controlling lateral torsional buckling as a possible mode of failure under wind uplift. Present design standards do not have provisions for such cases. Thus, the present study aims to quantify the effect of restraints provided by the deck boards on the lateral torsional buckling capacity of twin-beam-deck systems under wind uplift. Towards this goal, a series of analytical and numerical models were formulated. All models capture the continuous rigid lateral restraint and partial twisting restraint provided by the deck boards. The effects of load type and load position were investigated. The bending stiffness of deck boards was observed to have a significant influence on the lateral torsional buckling capacity of twin-beam-deck systems.

Author Keywords

Lateral torsional buckling; wooden beam-deck system; wind uplift; closed-form solution; energy-based approximate solution; finite element analysis.

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Introduction and Literature Review

A typical timber roof system consists of a set of parallel beams with continuous lateral bracing at the top by tongue-and-groove wood decking. When subjected to wind uplift, and given the typically low self-weight of such systems, the net bending moments may induce compression at the bottom fibers of the beams which are laterally unrestrained. Such beams have a tendency to undergo a lateral torsional buckling mode of failure.

Depending on the configuration of the decking, lateral displacements may be either fully or partially restrained at the beam top. Also, the deck boards typically placed perpendicular to the beams are expected to partially restrain the beams' twisting, thus contributing to its lateral torsional buckling resistance. However, it is unclear to what extent these restraints are effective in increasing the lateral torsional buckling capacity of such systems. Current standard provisions CAN/CSA O86 (2014) and NDS (2015) ignore such contributions. Within this context, the present study develops a family of solutions for predicting the lateral torsional buckling capacity of twin-beam-deck systems. A finite element model is also developed under ABAQUS 6.12-3 to assess the validity of the solutions.

Overview of the stability research on wood members

The lateral torsional buckling of wood beams has been experimentally investigated by Hooley and Madsen (1964), Hindman et al. (2005a), Hindman et al. (2005b), Burow et al. (2006), and Xiao (2014) with emphasis on comparing experimental results with design codes. Buchanan (1986), Zahn (1986), Koka (1987), Zahn (1988), Bell and Eggen (2001), Steiger and Fontana (2005), Song and Lam (2006), Song and Lam (2009), and Song and Lam (2010) investigated the buckling behavior of beam-columns. Zahn (1965), Zahn (1973), and Zahn (1984) developed analytical solutions for the lateral torsional buckling of wood beam-deck systems. Most studies on lateral torsional buckling of beam-deck systems involve steel beams. A review of such studies is provided in the following subsections.

Sway models for gravity loads

Timoshenko and Gere (1961) developed a buckling solution for columns continuously braced by eccentric elastic lateral and twisting restraints. Vlasov (1961) formulated the general differential equations for a beam embedded in an elastic medium. The critical moment was determined for a beam braced by continuous elastic lateral and twisting restraints and subjected to uniform moment. Taylor and Ojalvo (1966) formulated a buckling solution for doubly-symmetric I-section beams with continuous or discrete elastic twisting restraint. Pincus and Fisher (1966) developed an energy-based solution for two simply-supported I-section beams braced by the deck subjected to uniform moments. The solution accounted for the shear and twisting actions of the deck. Errera et al. (1967) extended the solutions of Pincus and Fisher (1966) to accommodate other bracing scenarios. Apparao (1968) investigated floor assemblies consisting of two beams of channel, zed and I-sections laterally braced at their compression flanges by the shear action of the deck. Two types of solutions were developed: (1) A load-deformation solution, in which initial imperfections were considered, and (2) a lateral torsional buckling eigen-value solution. Zahn (1965) formulated equilibrium equations for wooden rectangular beams that are laterally restrained by the shear action of deck boards. The buckling capacity was estimated for various loading and boundary conditions. Jenkinson and Zahn (1972) experimentally verified the validity of the solutions of Zahn (1965). In another study, Zahn (1972) conducted a series of experiments to quantify the shear stiffness of wooden deck boards, to be used as input in the analytical solutions. An energy solution was also developed (Zahn 1973) by expressing the total potential energy for a beam within a floor system and that of its tributary strip of decking. The stationarity conditions of the total potential energy were evoked to recover the governing differential equations of neutral stability. Solutions were provided for simply-supported beams under uniform moments, concentrated load, uniformly distributed load and cantilevers. Zahn (1984) further analyzed the effects of additional discrete lateral and twisting restraints on the buckling capacity of the floor system. Trahair (1979) formulated a closed-form buckling solution for beam-columns of mono-symmetric cross-sections with continuous elastic restraints restraining lateral displacement, angle of twist, weak-axis rotation and warping. For the case of doubly-symmetric I-beams, the author investigated the effects of the height of lateral and twisting restraints on the critical moments.

Sway models for uplift loads

A non-exhaustive review of sway models under uplift loads includes the work of Pekoz and Soroushian (1982) who developed a solution for critical uplift load of purlin-sheeting system, by modelling the system as a beam on elastic foundation. Lucas et al. (1997a) formulated a non-linear elasto-plastic finite element model with geometric nonlinearity for purlin-sheeting systems. The model captured purlin-sheeting interaction and successfully modelled cross-sectional distortion and local buckling. Lucas et al. (1997b) also developed a simplified model where the effects of sheeting were idealized as elastic springs. The lateral torsional buckling capacity of channel purlins laterally braced by sheeting was investigated by Chu et al. (2004). Li (2004) and Chu et al. (2005) expanded the study for zed sections. Ye et al. (2002) adopted a finite strip method to investigate the local, distortional and lateral torsional buckling of zed-purlins restrained by steel sheeting. Basaglia et al. (2013) developed a Generalized Beam Theory (GBT) solution for the local, distortional and lateral torsional buckling of channel and zed purlins restrained by sheeting and subjected to uplift loading. Apart from the studies of Chu et al. (2004), Li (2004) and Chu et al. (2005), the above studies have focused on the distortional buckling of thin cold-formed sections.

Non-sway models

In the above studies, the beams were assumed to be free to sway laterally. Only a few studies have considered the case where the beams are restrained from lateral sway. This includes the work of Vlasov (1961) who formulated a buckling solution for a simply-supported beam under uniform moments with a continuous rigid later restraint offset from the shear center and a continuous elastic twisting restraint. Roeder and Assadi (1982) provided an experimental study which verified the Vlasov Model. Park and Kang (2003) developed a shell based finite element model under MSC/NASTRAN for the lateral torsional buckling analysis of I-section beams with continuous lateral bracing at the top flange subjected to mid-height loading and end moments. They developed simplified design equations based on their model. In a subsequent study (Park et al. 2004), they extended the previous work to account for top flange loading. Using a trigonometric series solution, Larue et al. (2007) further generalized the buckling solution of Vlasov (1961) for linear and parabolic bending moment distributions with moment reversal.

For relatively light wooden beams under wind uplift, the net vertical load may be upwards. In such cases, decking mounted on the top face of the beam would provide restraint to the tension flange of the simply-supported beam. Within this context, the present study aims at developing solutions for examining the critical moment capacity for twin-beam-deck systems under wind uplift. Focus is on wood material. In this respect, only the studies by Zahn (1965), Zahn (1973) and Zahn (1984) were focused on wood members. The present study targets cases where the decking details provide continuous rigid lateral restraint preventing the system from swaying laterally, and elastic twisting restraint (in a manner similar to Vlasov 1961). However, the field equations, boundary conditions and the finite element formulation developed as part of the present study go beyond the Vlasov solution in that they can accommodate the general load distributions and boundary conditions by formulating a series of general analytical and numerical solutions.

Assumptions

The following assumptions are adopted:

1. The deck boards are assumed to provide rigid lateral restraint, i.e., the system is prevented from swaying laterally, in line with the studies of Vlasov (1961) and Laure et al. (2007).
2. Elastic twisting restraint is provided by the deck bending action, in a manner consistent with the Vlasov (1961).
3. The in-plane elastic shear provided by the deck boards is neglected.
4. Throughout deformation, beam cross-sections remain rigid in their own plane, i.e., distortional effects are neglected.
5. Shear deformation effects within the beams are negligible.
6. Beam and decking materials are linearly elastic.
7. Pre-buckling deformation effects are neglected, and
8. Both beams are subjected to general loading (unlike the Vlasov solution which is confined to uniform moments).

Formulation

Problem description and notation

The model considered in the present study consists of two identical parallel beams with doubly-symmetric sections (either rectangular or I-shaped) braced by individual deck boards. For the twin-beam-deck system, each of the twin beams is assumed to be subjected to transverse load $q(z)$ applied at a distance $h(z)$ below the deck centerline (Fig. 1). Under such external loads, the system is assumed to deform from Configuration 1 to 2 where both beams undergo vertical displacements $v_p(z)$. As a convention, subscript p represents pre-buckling displacements. The applied loads are then assumed to increase by a factor λ and attain the value of $\lambda q(z)$ at the onset of buckling (Configuration 3). Under the load increase, it is assumed that pre-buckling deformation linearly increases to $\lambda v_p(z)$. The system then undergoes lateral torsional buckling (Configuration 4) manifested by beam lateral displacements $u_1(z), u_2(z)$ and angles of twist $\theta_1(z)$, $\theta_2(z)$, where subscripts 1 and 2 denote the generalized displacements for the first and second beam, respectively. A left-hand coordinate system is used, with z axis being along the longitudinal direction of the beam, and x and y axes taken in the plane of the cross-section. The positive sign convention adopted is consistent with that in Trahair (1993) where the angle of twist is clockwise. Since the lateral displacement along the deck centerline is assumed to be rigidly restrained, the lateral displacements $u_i(z)$ $(i=1,2)$ can be related to the angles of twist $\theta_i(z)$ as

$$u_i = -a\theta_i \tag{1}$$

where a is the vertical distance between beam shear center and deck centerline.

Total potential energy

The total potential energy of the twin-beam-deck system Π is the summation of the internal strain energy U and the load potential energy V

$$\Pi = U + V \tag{2}$$

where the internal strain energy has three contributions $U = U_{b1} + U_{b2} + U_d$, and U_{bi}

are the internal strain energies stored in beam i $(i=1,2)$ and U_d is the internal strain energy stored in the deck boards as they undergo transverse bending. The load potential energy has two components $V=V_{b1}+V_{b2}$, one for external loads applied at each beam. For beam *i*, the internal strain energy is given by Trahair (1993) as

$$U_{bi}=\frac{1}{2}\int_0^{L_b}E_bI_yu_i''^2dz+\frac{1}{2}\int_0^{L_b}G_bJ_b\theta_i'^2dz+\frac{1}{2}\int_0^{L_b}E_bC_w\theta_i''^2dz \tag{3}$$

where, for a solid beam, E_b is the beam modulus of elasticity, I_y is the moment of inertia about beam weak-axis, G_b is the beam shear modulus, J_b is the Saint-Venant torsional constant, C_w is the warping constant, and L_b is the beam span. All primes denote differentiation with respect to coordinate z along the beam longitudinal axis. For I-section members where the web and flanges have different materials, E_b and G_b become the properties of the flanges, and the properties I_y, C_w, J_b are the corresponding transformed section properties. Expressions of the transformed section properties have been provided in Du (2016) and a summary is provided in Appendix A. The load potential energy for beam i is

$$V_{bi}=\lambda\left(\int_0^{L_b}M\theta_iu_i''dz+\frac{1}{2}\int_0^{L_b}qh\theta_i^2dz\right) \tag{4}$$

where $M(z)$ is the reference strong-axis moment induced by the reference transverse load $q(z)$ along the beam longitudinal axis (taken positive when acting downwards), and one recalls that λ is the load multiplier to be solved and $h(z)$ is the distance between the loading point and deck centerline (taken positive when the loading point is below deck centerline). The first term in Eq. (4) is the destabilizing term due to strong-axis flexure while the second term accounts for the load potential energy gain induced by the load height effect relative to the deck centerline. From Eq. (1), by substituting into Eqs. (3) and (4), one obtains

$$U_{bi} = \frac{1}{2}\int_0^{L_b} E_b C \theta_i''^2 dz + \frac{1}{2}\int_0^{L_b} G_b J_b \theta_i'^2 dz \tag{5}$$

$$V_{bi} = \frac{\lambda}{2}\int_0^{L_b} \left(hq\theta_i - 2aM\theta_i'' \right)\theta_i dz \tag{6}$$

where the property $C = I_y a^2 + C_w$ has been introduced. For a deck board at a distance z_0 from beam end support, the end moments M_{e1} and M_{e2} are expressed in terms of angles of twist θ_1, θ_2 as

$$\begin{Bmatrix} M_{e1}(z_0) \\ M_{e2}(z_0) \end{Bmatrix} = \frac{2E_d I_d}{L_d} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1(z_0) \\ \theta_2(z_0) \end{Bmatrix} \tag{7}$$

where E_d and I_d are the deck modulus of elasticity and the moment of inertia about deck strong-axis, respectively, and L_d is the deck board span. The internal strain energy U_d^* stored in this deck board is

$$U_d^* = \frac{1}{2} M_{e1}(z_0)\theta_1(z_0) + \frac{1}{2} M_{e2}(z_0)\theta_2(z_0) \tag{8}$$

From Eq. (7), by substituting into Eq. (8), one rewrites the internal strain energy as

$$U_d^* = \frac{E_d I_d}{L_d} \langle \theta_1(z_0) \quad \theta_2(z_0) \rangle \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1(z_0) \\ \theta_2(z_0) \end{Bmatrix} \tag{9}$$

The internal strain energy U_d in the whole decking is the summation of energy stored in each deck board, which can be approximately written in an integration form as

$$U_d = \sum U_d^{*} \approx \frac{E_d h_d^{3}}{12 L_d} \int_0^{L_b} \langle \theta_1 \quad \theta_2 \rangle \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} dz \tag{10}$$

in which h_d is the deck board thickness. The total potential energy of the twin-beam-deck assembly is obtained by summation of the internal strain energies for the twin beams, deck boards, and the load potential energy gains, i.e.,

$$\begin{aligned} \Pi = {} & \frac{1}{2}\int_0^{L_b} E_b C \theta_1''^{2} dz + \frac{1}{2}\int_0^{L_b} G_b J_b \theta_1'^{2} dz + \frac{\lambda}{2}\int_0^{L_b} \left(hq\theta_1 - 2aM\theta_1'' \right)\theta_1 dz + \frac{1}{2}\int_0^{L_b} E_b C \theta_2''^{2} dz \\ & + \frac{1}{2}\int_0^{L_b} G_b J_b \theta_2'^{2} dz + \frac{\lambda}{2}\int_0^{L_b} \left(hq\theta_2 - 2aM\theta_2'' \right)\theta_2 dz + \frac{E_d h_d^{3}}{6 L_d}\int_0^{L_b} \left(\theta_1^2 + \theta_1\theta_2 + \theta_2^2 \right) dz \end{aligned} \tag{11}$$

where the fields $\theta_1(z), \theta_2(z), q(z), M(z)$ are dependent on the *z* coordinate.

General conditions of neutral stability and boundary conditions

The conditions of neutral stability are obtained by setting the variation of the total potential energy to zero, i.e. $\delta\Pi = 0$. By performing integration by parts (Du 2016), one recovers the field equations

$$\begin{aligned} & \left(E_b C \theta_1'' - a\lambda M \theta_1 \right)'' - G_b J_b \theta_1'' + \lambda \left(hq\theta_1 - aM\theta_1'' \right) + \frac{E_d h_d^{3}}{6 L_d} \left(2\theta_1 + \theta_2 \right) = 0 \\ & \left(E_b C \theta_2'' - a\lambda M \theta_2 \right)'' - G_b J_b \theta_2'' + \lambda \left(hq\theta_2 - aM\theta_2'' \right) + \frac{E_d h_d^{3}}{6 L_d} \left(\theta_1 + 2\theta_2 \right) = 0 \end{aligned} \tag{12}$$

and the corresponding boundary conditions

$$\begin{aligned} & \left[\left(G_b J_b \theta_1' - \left(E_b C \theta_1'' - a\lambda M \theta_1 \right)' \right) \delta\theta_1 \right]_0^{L_b} = 0, \quad \left[\left(E_b C \theta_1'' - a\lambda M \theta_1 \right) \delta\theta_1' \right]_0^{L_b} = 0 \\ & \left[\left(G_b J_b \theta_2' - \left(E_b C \theta_2'' - a\lambda M \theta_2 \right)' \right) \delta\theta_2 \right]_0^{L_b} = 0, \quad \left[\left(E_b C \theta_2'' - a\lambda M \theta_2 \right) \delta\theta_2' \right]_0^{L_b} = 0 \end{aligned}$$

(13)

Closed-form Solutions for Uniform Moments

Solution for general boundary conditions

For the twin beams with general boundary conditions under uniform moments, the angles of twist are assumed as exponentials, i.e., $\theta_1 = A_j e^{m_j z}$ and $\theta_2 = B_j e^{m_j z}$ $(j = 1, 2, \ldots, 8)$. By substituting into Eq. (12), one obtains

$$\begin{bmatrix} E_b C m_j^4 - \left(2a\lambda M + G_b J_b\right) m_j^2 + \dfrac{E_d h_d^3}{3L_d} & \dfrac{E_d h_d^3}{6L_d} \\ \dfrac{E_d h_d^3}{6L_d} & E_b C m_j^4 - \left(2a\lambda M + G_b J_b\right) m_j^2 + \dfrac{E_d h_d^3}{3L_d} \end{bmatrix} \begin{Bmatrix} A_j \\ B_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

For a non-trivial solution, the determinant of the matrix of coefficients should vanish, which leads to

$$\begin{aligned} m_j &= \pm\sqrt{\frac{-\left(2a\lambda M + G_b J_b\right) \pm \sqrt{\left(2a\lambda M + G_b J_b\right)^2 - \dfrac{2E_d h_d^3 E_b C}{3L_d}}}{2E_b C}} \\ m_j &= \pm\sqrt{\frac{-\left(2a\lambda M + G_b J_b\right) \pm \sqrt{\left(2a\lambda M + G_b J_b\right)^2 - \dfrac{2E_d h_d^3 E_b C}{L_d}}}{2E_b C}} \end{aligned} \tag{15}$$

Since all eight roots are distinct, the angles of twist θ_1 and θ_2 can be expressed as

$$\left\{\theta(z)\right\}_{2\times1} = \begin{Bmatrix} \theta_1(z) \\ \theta_2(z) \end{Bmatrix} = \sum_{j=1}^{8} \begin{Bmatrix} A_j \\ B_j \end{Bmatrix} e^{m_j z} \tag{16}$$

From Eq. (14), one express B_j in terms of A_j as

$$B_j = \overline{A}_j A_j \tag{17}$$

where $\overline{A}_j = -6L_d\left[E_bCm_j^{\,4} - \left(2a\lambda M + G_bJ_b\right)m_j^{\,2} + E_dh_d^{\,3}/3L_d\right]\Big/E_dh_d^{\,3}$. From Eq. (17), by substituting into Eq. (16), one obtains

$$\left\{\theta(z)\right\}_{2\times1} = \sum_{j=1}^{8} A_j \left\{\begin{array}{c} 1 \\ \hdashline \overline{A}_j \end{array}\right\} e^{m_j z} = \left[H\right]_{2\times8}\left[E_m(z)\right]_{8\times8}\left\{D\right\}_{8\times1} \tag{18}$$

where

$$\begin{aligned}
&\left[H\right]_{2\times8} = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \hdashline \overline{A}_1 & \overline{A}_2 & \cdots & \overline{A}_8 \end{array}\right], \\
&\left[E_m(z)\right]_{8\times8} = Diag\left(e^{m_1 z}, e^{m_2 z}, \cdots e^{m_8 z}\right), \\
&\left\{D\right\}_{8\times1} = \left\langle A_1 \quad A_2 \quad \cdots \quad A_8\right\rangle^T.
\end{aligned}$$

The angles of twist as given in Eq. (18) and their derivatives can then be used to express the eight boundary conditions of the problem. The resulting system of equations is then placed in a matrix form and the smallest root λM which vanishes the determinant of the matrix of coefficient is extracted to yield the critical moment of the system.

Solution for simply-supported boundary conditions

For the case where the twin beams are simply-supported relative to the angle of twist and free to warp at the supports, the applicable boundary conditions as extracted from Eq. (13) are

$$\begin{aligned}
&\theta_1(0) = \theta_2(0) = \theta_1(L_b) = \theta_2(L_b) = 0 \\
&\theta_1''(0) = \theta_2''(0) = \theta_1''(L_b) = \theta_2''(L_b) = 0
\end{aligned} \tag{19}$$

Assuming the angles of twist as sinusoidal functions, i.e., $\theta_i = A_i \sin\left(n\pi z/L_b\right)$ $\left(i = 1,2\right)$ $\left(n = 1,2,3,\cdots\right)$, which satisfy all the boundary conditions in Eq. (19) Substituting into Eq. (12), one obtains

$$\begin{bmatrix} E_b C l_n^4 + \left(G_b J_b + 2a\lambda M\right) l_n^2 + \dfrac{E_d h_d^3}{3L_d} & E_d h_d^3 / 6L_d \\ E_d h_d^3 / 6L_d & E_b C l_n^4 + \left(G_b J_b + 2a\lambda M\right) l_n^2 + \dfrac{E_d h_d^3}{3L_d} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_n = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

where $l_n = n\pi / L_b$. For a non-trivial solution, the determinant of the matrix should vanish, leading to the following roots (Du 2016)

$$\begin{aligned} \left(\lambda M\right)_{1n} &= -\left[\frac{E_b C}{2a}\left(\frac{n\pi}{L_b}\right)^2 + \frac{G_b J_b}{2a} + \frac{E_d h_d^3}{12aL_d}\left(\frac{L_b}{n\pi}\right)^2\right], \\ \left(\lambda M\right)_{2n} &= -\left[\frac{E_b C}{2a}\left(\frac{n\pi}{L_b}\right)^2 + \frac{G_b J_b}{2a} + \frac{E_d h_d^3}{4aL_d}\left(\frac{L_b}{n\pi}\right)^2\right] \end{aligned} \tag{21a,b}$$

It is noted that both roots are negative, signifying that, under the proposed assumptions, only negative moments (moments that create compression at beam bottom fibers) can induce lateral torsional buckling of the system. For the two roots obtained, the first one corresponds to a symmetric buckling mode and the second one corresponds to an antisymmetric buckling mode. Among the two roots, it is clear $\left|\left(\lambda M\right)_{1n}\right| < \left|\left(\lambda M\right)_{2n}\right|$, which means the symmetric buckling mode governs the capacity of the system. The critical moment $M_{cr} = \left(\lambda M\right)_{1n}$ coincides with that in Vlasov (1961). Introducing the notation $M_\alpha = G_b J_b / 2a$, one can rewrite the critical moment in a dimensionless form as

$$m_r\left(n\right) = -\left(1 + n^2\alpha + \frac{\beta}{n^2}\right) \tag{22}$$

where $m_r\left(n\right) = M_{cr} / M_\alpha$, $\alpha = \pi^2 E_b C / G_b J_b L_b^2$ and $\beta = \left(E_d h_d^3 / L_d\right)\left(L_b^2 / G_b J_b\right) / 6\pi^2$. Setting the first derivative of Eq. (22) to zero yields $n = \sqrt[4]{\beta/\alpha}$, which provides an indication of the mode number corresponding to the smallest critical moment m_r. Since

n is an integer value, one should consider the roots based on $\text{int}(n)$ and $\text{int}(n+1)$. For specific values of α and β, the relationship between β/α and the governing mode number n is determined (Du 2016). The results are summarized in Table 1.

Approximate Energy-based Solutions

Simply-supported boundary conditions

Two loading types are considered: (1) A uniformly distributed load q along beams span, and (2) a mid-span concentrated load P for both beams. The angles of twist are assumed as $\theta_i = A_i' \sin(l_n z) \quad (i = 1,2)$. Substituting the assumed functions into Eq. (11), one obtains

$$\begin{aligned}\Pi = &\frac{1}{2}E_b C A_1'^2 l_n^4 \int_0^{L_b} \sin^2(l_n z)\,dz + \frac{1}{2}G_b J_b A_1'^2 l_n^2 \int_0^{L_b} \cos^2(l_n z)\,dz \\ &+ a\lambda A_1'^2 l_n^2 \int_0^{L_b} M \sin^2(l_n z)\,dz + \frac{1}{2}\lambda h A_1'^2 \int_0^{L_b} q \sin^2(l_n z)\,dz + \frac{1}{2}E_b C A_2'^2 l_n^4 \int_0^{L_b} \sin^2(l_n z)\,dz \\ &+ \frac{1}{2}G_b J_b A_2'^2 l_n^2 \int_0^{L_b} \cos^2(l_n z)\,dz + a\lambda A_2'^2 l_n^2 \int_0^{L_b} M \sin^2(l_n z)\,dz + \frac{1}{2}\lambda h A_2'^2 \int_0^{L_b} q \sin^2(l_n z)\,dz \\ &+ \frac{E_d h_d^3}{6L_d}\left(A_1'^2 + A_1'A_2' + A_2'^2\right) \int_0^{L_b} \sin^2(l_n z)\,dz\end{aligned} \tag{23}$$

Case 1: Uniformly Distributed Load (UDL)

The strong-axis moment distribution induced by the reference UDL q along beams span is $M(z) = q\left(L_b z - z^2\right)/2$. By applying the principle of stationary total potential energy $\partial\pi/\partial A_1' = \partial\pi/\partial A_2' = 0$, one recovers two algebraic equations. The determinant of the resulting matrix of coefficients is set to zero to recover the critical loads (Du 2016), yielding

$$(\lambda q)_{1n} = -\frac{6n^4\pi^4 E_b C L_d + 6n^2\pi^2 G_b J_b L_b{}^2 L_d + E_d H_d{}^3 L_b{}^4}{L_b{}^4 L_d\left(6h + n^2\pi^2 a + 3a\right)}$$
$$(\lambda q)_{2n} = -\frac{6n^4\pi^4 E_b C L_d + 6n^2\pi^2 G_b J_b L_b{}^2 L_d + 3E_d H_d{}^3 L_b{}^4}{L_b{}^4 L_d\left(6h + n^2\pi^2 a + 3a\right)} \quad (24a,b)$$

Both roots are negative, i.e., only uplift loads can buckle the twin-beam-deck assembly. It is evident that $\left|(\lambda q)_{1n}\right| < \left|(\lambda q)_{2n}\right|$, signifying that the governing buckling mode is always symmetric.

Case 2: Mid-span concentrated loads

For a reference concentrated load P applied at the beams mid-span, the strong-axis moment is $M(z) = Pz/2\,(0 < z \le L_b/2)$ and $M(z) = P(L_b - z)/2\,(L_b/2 < z \le L_b)$. Substituting $M(z)$ into Eq. (23) and applying the principle of stationary total potential energy, one obtains the following roots (Du 2016)

$$(\lambda P)_{1n} = -\frac{12n^4\pi^4 E_b C L_d + 12n^2\pi^2 G_b J_b L_b{}^2 L_d + 2E_d h_d{}^3 L_b{}^4}{3L_b{}^3 L_d\left(4a + 8h + an^2\pi^2\right)}$$
$$(\lambda P)_{2n} = -\frac{4n^4\pi^4 E_b C L_d + 4n^2\pi^2 G_b J_b L_b{}^2 L_d + E_d h_d{}^3 L_b{}^4}{L_b{}^3 L_d\left(4a + 8h + an^2\pi^2\right)} \quad (25a,b)$$

Similarly, only upward loads can buckle the system and $\left|(\lambda P)_{1n}\right| < \left|(\lambda P)_{2n}\right|$, signifying that the governing buckling mode is always symmetric.

Fixed boundary conditions

For the twin-beam-deck systems where both beams are fixed against the twist angle, assume the angles of twist as $\theta_i = D_i\left[1 - \cos(l_n z)\right], (n = 2, 4, 6, \ldots) \quad (i = 1, 2)$, which satisfy all the boundary conditions in Eq. (13). Substituting the assumed angles of twist functions into the total potential energy, one obtains

$$
\begin{aligned}
\Pi = {} & D_1^2 E_b C l_n^4 \frac{1}{2}\int_0^{L_b} \cos^2\left(l_n z\right) dz + D_1^2 G_b J_b l_n^2 \frac{1}{2}\int_0^{L_b} \sin^2\left(l_n z\right) dz \\
& + D_1^2 \frac{\lambda}{2}\int_0^{L_b} \left\{ hq\left[1-\cos\left(l_n z\right)\right] - 2aM l_n^2 \cos\left(l_n z\right)\right\}\left[1-\cos\left(l_n z\right)\right] dz \\
& + D_2^2 E_b C l_n^4 \frac{1}{2}\int_0^{L_b} \cos^2\left(l_n z\right) dz + D_2^2 G_b J_b l_n^2 \frac{1}{2}\int_0^{L_b} \sin^2\left(l_n z\right) dz \\
& + D_2^2 \frac{\lambda}{2}\int_0^{L_b} \left\{ hq\left[1-\cos\left(l_n z\right)\right] - 2aM l_n^2 \cos\left(l_n z\right)\right\}\left[1-\cos\left(l_n z\right)\right] dz \\
& + \frac{E_d h_d^3}{6L_d}\left(D_1^2 + D_1 D_2 + D_2^2\right)\int_0^{L_b} \left[1-\cos\left(l_n z\right)\right]^2 dz
\end{aligned}
\tag{26}
$$

Case 1: UDL

From the strong-axis moment $M\left(z\right)$ induced by UDL, by substituting into Eq. (26) and applying the principle of stationary total potential energy $\partial\pi/\partial D_1 = \partial\pi/\partial D_2 = 0$, one obtains the following critical loads (Du 2016)

$$
\begin{aligned}
\left(\lambda q\right)_{3n} &= \frac{-3\left(E_d L_b^4 h_d^3 + 2n^2\pi^2 G_b J_b L_b^2 L_d + 2n^4\pi^4 E_b C L_d\right)}{L_b^4 L_d\left(21a + 18h + an^2\pi^2\right)} \\
\left(\lambda q\right)_{4n} &= \frac{-3\left(3E_d L_b^4 h_d^3 + 2n^2\pi^2 G_b J_b L_b^2 L_d + 2n^4\pi^4 E_b C L_d\right)}{L_b^4 L_d\left(21a + 18h + an^2\pi^2\right)} \\
n &= 2, 4, 6, \ldots
\end{aligned}
\tag{27a,b}
$$

Case 2: Mid-span concentrated loads

Substituting the moment distribution $M\left(z\right)$ for mid-span concentrated loads into Eq. (26) and applying the principle of stationary total potential energy, one obtains the following critical loads (Du 2016)

$$
\begin{aligned}
&\left(\lambda P\right)_{3n} = -\frac{2\left(2n^4\pi^4 E_b C L_d + 2n^2\pi^2 G_b J_b L_b{}^2 L_d + E_d L_b{}^4 h_d{}^3\right)}{L_b{}^3 L_d\left(32h + 32a + an^2\pi^2\right)} \\
&\left(\lambda P\right)_{4n} = -\frac{2\left(2n^4\pi^4 E_b C L_d + 2n^2\pi^2 G_b J_b L_b{}^2 L_d + 3E_d L_b{}^4 h_d{}^3\right)}{L_b{}^3 L_d\left(32h + 32a + an^2\pi^2\right)} \\
&n = 2,6,10,\ldots \\
&\left(\lambda P\right)_{5n} = -\frac{2\left(2n^4\pi^4 E_b C L_d + 2n^2\pi^2 G_b J_b L_b{}^2 L_d + E_d L_b{}^4 h_d{}^3\right)}{n^2\pi^2 a L_b{}^3 L_d} \\
&\left(\lambda P\right)_{6n} = -\frac{2\left(2n^4\pi^4 E_b C L_d + 2n^2\pi^2 G_b J_b L_b{}^2 L_d + 3E_d L_b{}^4 h_d{}^3\right)}{n^2\pi^2 a L_b{}^3 L_d} \\
&n = 4,8,12,\ldots
\end{aligned}
\tag{28a,b,c,d}
$$

Finite element formulation

The angles of twist $\theta_i\left(z\right)$ are related to the generalized nodal displacements as

$$\theta_i\left(z\right) = \left\langle L\left(z\right)\right\rangle_{1\times4}^{T}\left\{\theta_i\right\}_{4\times1} \qquad \left(i = 1,2\right) \tag{29}$$

where $\left\langle\theta_i\right\rangle^{T}_{1\times4} = \left\langle\theta_0 \quad \theta_0' \quad \theta_l \quad \theta_l'\right\rangle_i$ is the generalized nodal angle of twist vector, subscripts 0 and l denote the nodal points, and

$$\left\langle L\left(z\right)\right\rangle_{1\times4}^{T} = \left\langle\left(1 - 3z^2/l^2 + 2z^3/l^3\right) \quad \left(z - 2z^2/l + z^3/l^2\right) \quad \left(3z^2/l^2 - 2z^3/l^3\right) \quad \left(z^3/l^2 - z^2/l\right)\right\rangle$$

is the vector of Hermitian interpolation functions and l is the element length. From Eq. (29), by substituting into the total potential energy expression in Eq. (11), one obtains

$$\Pi = \frac{1}{2}\left\langle\left\langle\theta_1\right\rangle^{T}_{1\times4} \quad \left\langle\theta_2\right\rangle^{T}_{1\times4}\right\rangle\left(\left[K_e\right]_{8\times8} - \lambda\left[K_g\right]_{8\times8}\right)\begin{Bmatrix}\left\{\theta_1\right\}_{4\times1} \\ \left\{\theta_2\right\}_{4\times1}\end{Bmatrix} \tag{30}$$

in which the elastic stiffness matrix is

$$\left[K_e\right]=\begin{bmatrix}\left[K_b\right]+2\left[K_d\right] & \left[K_d\right]\\ \left[K_d\right]^T & \left[K_b\right]+2\left[K_d\right]\end{bmatrix}_{8\times8}$$

and $\left[K_b\right]=E_bC\left[B_1\right]+G_bJ_b\left[B_2\right]$ is the beam stiffness matrix and $\left[K_d\right]=\left(E_dh^3{}_d/6L_d\right)\left[B_3\right]$ is the deck stiffness matrix. Also, in Eq. (30), the geometric stiffness matrix $\left[K_g\right]$ is

$$\left[K_g\right]=\begin{bmatrix}2a\left[B_4\right]-h\left[B_5\right] & \left[0\right]\\ \left[0\right] & 2a\left[B_4\right]-h\left[B_5\right]\end{bmatrix}_{8\times8}$$

and submatrices $\left[B_1\right],\left[B_2\right],\left[B_3\right],\left[B_4\right],\left[B_5\right]$ have been defined in Appendix B. By evoking the stationarity conditions for the total potential energy $\delta\Pi=0$ to the discretized system, one obtains

$$\left(\left[K_e\right]_{8\times8}-\lambda\left[K_g\right]_{8\times8}\right)\begin{Bmatrix}\left\{\theta_1\right\}_{4\times1}\\ \left\{\theta_2\right\}_{4\times1}\end{Bmatrix}=\left\{0\right\}_{8\times1} \tag{31}$$

Verification

A finite element model based on the commercial software ABAQUS is adopted to verify the validity of present solutions for a selected reference case. Both beams are simply supported. The beam and deck spans are 6 m and 2 m, respectively. Beam material is taken to be glu-laminated Spruce-Lodgepole Pine-Jack Pine 20f-EX. Cross-section depth is 570 mm and its width is 80 mm. The modulus of elasticity and shear modulus are 10,300 MPa and 474 MPa, respectively (CAN/CSA O86-14 and FPL 2010). The nominal bending resistance for the beam due to material failure is 270 kNm. The Spruce-Pine-Fir No. 2 grade deck boards are assumed to be 38 mm thick with a modulus of elasticity of 10,000 MPa.

Details of the ABAQUS model

The finite element program ABAQUS 6.12-3 was used to perform an eigen-value lateral torsional buckling analysis of the twin-beam-deck. The two-node B31OS element within the ABAQUS library was selected to model the twin beams. Each node has seven degrees of freedom (i.e., three translations, three rotations, and a warping deformation). The element accounts for transverse shear deformation. Also, the element does not capture the distortional effects. The B31 elements were chosen to model the deck boards. The B31 element has two nodes with six degrees of freedom per node (i.e., three translations, and three rotations). Fig. 2 shows the twin-beam-deck model developed in ABAQUS with B31OS elements located at the height of beam shear center and B31 elements at the height of deck centerline. The number of B31OS elements used in each beam was chosen so that the number of nodes forming these elements matches the number of deck boards. For each pair of beam and deck nodes, the following kinematic relationships were enforced:

(1) The vertical displacement of a beam node $(v_b)_i$ was equated to the corresponding vertical displacement of the deck node $(v_d)_i$, i.e., $(v_b)_i = (v_d)_i$;

(2) the angle of twist of a beam node was equated to that of the corresponding deck node, i.e., $(\theta_b)_i = (\theta_d)_i$; and

(3) the lateral displacement of a beam node $(u_b)_i$ was related to the twist angle $(\theta_b)_i$ through $(u_b)_i = -a(\theta_b)_i$, where a was previously defined as the distance between beam shear center and deck centerline. Also, the lateral degree of freedom for each of the deck nodes was restrained, i.e., $(u_d)_i = 0$. The above constraints were enforced through the *EQUATION keyword in ABAQUS. The above kinematic constraints

enforced the rigid lateral restraint condition while accounting for the partial twisting restraint provided by the deck bending action.

Mesh sensitivity analysis of present FEA and ABAQUS models

In addition to uniform moments, two types of transverse loading were considered in the mesh study: (1) A UDL applied along the beams span, and (2) a mid-span concentrated load in both beams. For the three loading cases, the critical moments as determined by present FEA and ABAQUS are shown in Fig. 3(a,c,e). For uniform moments (Fig. 3a), the present FEA requires less than 8 beam elements to converge. The ABAQUS solution exhibits an oscillating behavior as the number of beam elements is increased and achieves convergence at 160 beam elements per beam. For the other two loading cases (Fig. 3c,e), 10 to 12 elements are shown to achieve convergence for the present FEA and ABAQUS. Under all three loading types, the present FEA converges from above and a coarser mesh tends to overestimate the buckling capacity. To ensure convergence, 30 elements per beam were used under the present FEA and 160 elements per beam under the ABAQUS model in all subsequent runs. Under both solutions, a single element per deck board is found to be enough to achieve convergence. For example, under uniform moments, the present FEA corresponding to two elements per deck yields the same critical moment of 192 kNm as that corresponding to one element per deck.

Verification of results

To validate the present solutions, comparisons between the closed-form solution in Eq.(21a,b), energy-based formulations in Eqs. (24a,b),(25a,b), the present FEA, and ABAQUS are presented for the reference twin-beam-deck assembly. As discussed, only negative moments and uplift loads can induce lateral torsional buckling of the simply-supported single-span twin-beam-deck system. For clarity, the absolute values of critical moments are presented below.

For uniform moments (Fig. 3a), the closed-form formulation and present FEA yield an identical critical moment of 192 kNm while the ABAQUS model predicts a slightly lower value of 190 kNm. All three solutions predict an identical buckling mode shape (Fig. 3b). For UDL (Fig. 3c), the approximate energy-based formulation predicts a capacity of 267 kNm, compared with 247 kNm as predicted by present FEA and 237 kNm by ABAQUS. This difference is attributed to the fact that the assumed sinusoidal displacement function in the energy-based formulation does not exactly represent the actual mode shape (Fig. 3d) and thus provides a stiffer representation of the system, and hence overestimates the critical moment. The difference between the present FEA and ABAQUS is possibly due to the fact that the ABAQUS B31OS elements capture the shear deformation effects due to flexure while the present FEA neglects such effects, resulting in a slightly stiffer representation. For mid-span concentrated loads (Fig. 3e), the critical moments based on the energy-based solution are 508 kNm for $n=1$ and 348 kNm for $n=2$. In both cases, the energy solution over-predicts the critical moment as expected. Although the mode shape corresponding to $n=1$ is somewhat closer to those predicted by the FEA and ABAQUS (Fig. 3f), the critical moment corresponding to $n=2$ is closer to the critical moment than that based on $n=1$. The difference in results is attributed to the fact that the assumed sinusoidal functions represent the actual mode shape for non-uniform moment loading only in an approximate manner. In such a case, the energy-based solution is expected to provide an upper bound prediction of the critical moments. Indeed, Fig. 3(g,h) confirm that under the cases of UDL and mid-span concentrated loads, the approximate energy-based solutions consistently predict higher critical moments than those based on the present FEA and ABAQUS model for various beam spans. Also evident is the fact that the approximate solutions are more accurate for shorter beam spans. The present FEA model, thus validated, is subsequently used to investigate the effects of various parameters on critical moments.

Parametric Study

A parametric study has been conducted by varying one parameter at a time from the reference case defined in the verification study and observing the critical moments and buckling mode shapes as predicted by the present solutions. Nine sets of parametric studies are considered by varying loading types, beam and deck span, lateral restraint height, load position, number of spans and whether the deck lateral or twisting restraint is included in the analysis (Table 2).

Effects of beam and deck span

Fig. 4a shows the critical moments for both beams under uniform moments. The beam span was varied from 2 m to 12 m, all in the elastic or inelastic buckling range. The critical moments corresponding to modes 1 to 4 are shown in the figure. It is observed that mode 1 provides the lowest critical moment for spans ranging from 2 m to 4 m. From 4 m to 7 m, mode 2 corresponds to the lowest critical moment while mode 3 provides the lowest critical moment from 7 m to 10 m and mode 4 is the governing mode of buckling for spans larger than 10 m. As the beam span increases, the critical moments are observed to be nearly constant while the governing mode number n changes with the beam span. For the cases of UDL and mid-span concentrated loads (Fig. 3g,h), the critical moments from all three solutions show an oscillating behavior as the beam span is increased. In contrast, as the deck span increases from 1 m to 5 m, the buckling capacity is observed to decrease (Fig. 4b) for all three loading types investigated.

Effects of deck lateral and twisting restraints

For uniform moments, Fig. 4c shows three solutions: (1) The classical solution, which is based on laterally unrestrained beam, gives the lowest buckling capacity; (2) the solution by Roeder and Assadi (1982), which recognizes the rigid lateral restraint provided by decking but neglects its partial twisting restraint, thus gives a higher critical moment prediction than that based on the classical solution; and (3) the present solution, i.e., Eq. (21a,b), which assumes the deck boards to provide rigid lateral restraint and partial twisting restraint, gives the highest buckling predictions among the three solutions. Fig. 4c shows that the effects of both types of restraints are small for short beam span but becomes more substantial in long beam span. Also, the critical moments obtained from the present study can be several folds higher than values of unrestrained beams. For deck span increasing from 1m to 5 m (Fig. 4d), the contribution of elastic twisting restraint declines. This reflects on the observed critical moments which reduce with deck span. By comparing the lower solid line and the dashed line, the rigid lateral restraint contributes a constant 68.3 kNm towards the total capacity.

Effect of lateral restraint height

Fig. 4e shows the critical moments as predicted by the present FEA for the twin-beam-deck system under UDL with rigid lateral restraint height varying from the beam shear center to deck centerline. Three beam spans were investigated (i.e., 4, 6, 8 m). The results suggest that for each beam span, there is an optimal lateral bracing height which corresponds to a peak moment.

Effect of load position

To assess the effect of load position, a UDL was applied at three different heights (i.e., deck centerline, beam shear center and beam bottom). The critical moments as predicted by the approximate energy-based solution and the present FEA are summarized in Table 3 for three beam spans (i.e., 4, 6, 8 m). No ABAQUS results were provided since the B31OS solution does not support the feature for load position effect. For all three spans investigated, both the energy-based solution and the present FEA predict a decrease in critical moments as the point of load application moves downwards. Also, it is observed that the load position effect is particularly significant for the shorter beam span of 4 m. Compared with the case of beam shear center loading, both solutions predict a more than 40% capacity increase for the case of deck centerline loading and more than a 20% capacity decrease for the case of beam bottom loading. In contrast, for the case of longer beam spans, the load position effect is observed to be less substantial. It is of interest to note that the percentage increase or decrease of critical moments due to load position effect based on the energy solution reasonably agree well with those based on the finite element solution.

The load position effect is also investigated under three deck spans (i.e., 1, 3, 5 m) and is summarized in Table 4. The load position effect is observed to be more substantial for longer deck span.

Effect of beam span on a two-span twin-beam-deck system

The buckling capacity obtained from the present FEA for a two-span twin-beam-deck system under UDL and a mid-span concentrated load are shown in Fig. 5(a,b). Contrary to the case of single-span beams where lateral torsional buckling is associated solely with negative moments or upward loading, the buckling resistance of continuous beams is found possible under upward and downward acting loadings. For downward acting loading, the critical moments observed at mid-spans decrease significantly as the beam span increases from 4 m to 12 m. In contrast, for upward acting loading, the critical moments remain essentially constant and are insensitive of the span.

Summary and Conclusions

1. Five solutions were developed, formulated, and implemented for investigating the lateral torsional buckling capacity of wooden twin-beam-deck assembles: (1) An analytical solution for generic boundary conditions under uniform moments, (2) a closed-form solution for simply-supported beams under uniform moments, (3) an energy-based approximate solution for simply-supported beams under non-uniform moments, (4) an energy-based approximate solution for beams with fixed ends under non-uniform moments, and (5) a FEA solution.

2. The validity of the solutions was assessed by comparison to the results based on ABAQUS, and to a close-form solution available in the literature for the special case involving uniform moments.

3. The verified solutions were then used to provide a comprehensive parametric study to investigate the effects of load types, beam and deck spans, height of lateral restraint, load height, number of spans and whether the deck lateral or twisting restraint is included in the analysis.

For single-span simply-supported twin-beam-deck system, the key engineering observations are:

1. Only net uplift loads are found to be able to induce lateral torsional buckling.
2. The approximate energy-based solutions provide predictions in excellent agreement with the present FEA and ABAQUS for short beams span. For long span, the solution slightly overestimates the critical moments.

3. Unlike laterally unsupported beams which has clear downward trend between critical moment and beam span, the present solutions indicate that, in general, there is no clear trend for the buckling capacity and the beam span.
4. The lateral torsional buckling capacity for laterally and rotationally restrained twin beams can be much higher than unrestrained beams. The presence of lateral and twisting restrains is strongly influential in increasing the lateral torsional buckling capacity of the system, especially for long beams span.
5. The height of lateral restraint has a significant influence on the buckling capacity of the system. For a given beam span, there is an optimal lateral restraint height that maximizes the buckling capacity.
6. The buckling capacity was observed to decrease as the point of application of the applied upward load moves downwards and such effect is particularly significant for *short beam* span or *long deck* span.

For two-span twin-beam-deck systems, both upward and downward loading are observed to be able to induce lateral torsion buckling. For upward loading, the buckling capacity remains nearly constant irrespective of the beam span. For downward loading, the buckling capacity is observed to significantly decrease as the beam span increases.

Appendix A Transformed section properties for composed I-section

For a composite I-section beam with both flanges having different material from that of the web, the transformed section can be shown to take the form (Du 2016)

$$\begin{aligned} J_b &= J_f + n_1 J_w \\ I_y &= I_{yy,f} + n_2 I_{yy,w} \\ C_w &= I_{\omega\omega,f} + n_2 I_{\omega\omega,w} \end{aligned} \tag{A.1}$$

in which J_f and J_w are the Saint-Venant torsional constants for both flanges and the web, respectively, $n_1 = G_w/G_b$ is the ratio of shear modulus, G_w for the web to that of the flange, $n_2 = E_w/E_b$ is the ratio of Young modulus, E_w of the web to that of the flange, and the following sectional properties have been defined

$$\begin{aligned} &I_{yy,f} = \int_{A_f} x^2 dA_f , I_{yy,w} = \int_{A_w} x^2 dA_w , \\ &I_{\omega\omega,f} = \int_{A_f} \omega^2 dA_f , I_{\omega\omega,w} = \int_{A_w} \omega^2 dA_w \end{aligned} \tag{A.2}$$

where A_f is the area of both flanges, A_w is the area of the web, and ω is the sectorial coordinate based on a fixed radius taken from the shear center to the sectorial origin.

Appendix B Expressions for elastic and geometric stiffness matrices

This appendix provides the expressions for submatrices $[B_1]$, $[B_2]$, $[B_3]$, $[B_4]$, $[B_5]$ introduced after Eq. (30).

$$[B_1]=\int_0^l \{L''(z)\}_{4\times1}\langle L''(z)\rangle_{1\times4}^T\,dz=\frac{1}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6L & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix},$$

$$[B_2]=\int_0^l \{L'(z)\}_{4\times1}\langle L'(z)\rangle_{1\times4}^T\,dz=\frac{1}{30l}\begin{bmatrix}36 & 3l & -36 & 3l\\ 3l & 4l^2 & -3l & -l^2\\ -36 & -3l & 36 & -3l\\ 3l & -l^2 & -3l & 4l^2\end{bmatrix},$$

$$[B_3]=\int_0^l \{L(z)\}_{4\times1}\langle L(z)\rangle_{1\times4}^T\,dz=\frac{l}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix},$$

$$[B_4]=\int_0^l M(z)\{L(z)\}_{4\times1}\langle L''(z)\rangle_{1\times4}^T\,dz,\quad [B_5]=\int_0^l q(z)\{L(z)\}_{4\times1}\langle L(z)\rangle_{1\times4}^T\,dz$$

$$\text{(B.1)}$$

For a beam element where the moment distribution along the element span $M(z)$ is linearly interpolated, i.e.,

$$M(z)=-M_0\left(1-\frac{z}{l}\right)+M_l\left(\frac{z}{l}\right) \tag{B.2}$$

where M_0 and M_l are the nodal moments, obtained from the pre-buckling analysis (Du 2016), matrix $[B_4]$ can be shown to take the form

$$[B_4]=\frac{1}{30l}\begin{bmatrix}33M_0-3M_l & (27M_0-6M_l)l & -33M_0+3M_l & (6M_0+3M_l)l\\ 3M_0l & (3M_0-M_l)l^2 & -3M_0l & M_ll^2\\ -3M_0+33M_l & (3M_0+6M_l)l & 3M_0-33M_l & (-6M_0+27M_l)l\\ -3M(l)l & -M_0l^2 & 3M_ll & (M_0-3M_l)l^2\end{bmatrix}$$

$$\text{(B.3)}$$

Notation

The following symbols are used in this paper:

A_f, A_w areas for both flanges and the web, respectively;

A_i, A_i', A_j, B_j amplitudes in the assumed angle of twist functions;

a distance between beam shear center and deck centerline;

$[B_1],[B_2],[B_3],[B_4],[B_5]$ submatrices for elastic and geometric stiffness matrices;

C constant;

C_w warping constant;

D_1, D_2 amplitudes in the assumed angle of twist functions;

E_b, E_w modulus of elasticity of the flanges and web, respectively;

E_d modulus of elasticity of the deck boards;

$\left[E_m(z)\right]$ diagonal matrix of exponential functions;

G_b, G_w shear modulus of the flanges and web, respectively;

$h(z)$ distance between load point and deck centerline;

h_d deck boards thickness;

I_d moment of inertia about deck strong-axis;

I_y moment of inertia about beam weak-axis;

J_b beam Saint-Venant torsional constant;

J_f, J_w Saint-Venant torsional constants for the flanges and web, respectively;

$[K_b]$ beam stiffness matrix;

$[K_d]$ deck stiffness matrix;

$[K_e]$ elastic stiffness matrix;

$[K_g]$ geometric stiffness matrix;

L_b, L_d	beam and deck span, respectively;
$\langle L(z) \rangle$	Hermitian polynomials;
l	beam element length;
M_{cr}	critical moment for uniform moments loading;
M_{e1}, M_{e2}	deck board end moments;
$M(z)$	reference strong-axis moment;
M_0, M_l	strong-axis moments at element ends;
m_j	constants in the angle of twist functions;
$m_r(n)$	dimensionless critical moment;
n	integer in angles of twist functions;
n_1	ratio of shear moduli;
n_2	ratio of modulus of elasticity;
$P(z)$	reference concentrated load;
$q(z)$	reference transverse load;
q	reference uniformly distributed load;
U	internal strain energy;
U_{bi}	internal strain energy in beam i;
U_d	internal strain energy in deck boards;
U_d^*	internal strain energy in one deck board;
u_i	lateral displacement of beam i;
$(u_b)_i$	lateral displacement of beam node i;
V	load potential energy;
V_{bi}	load potential energy in beam i;

$\left(v_b\right)_i, \left(v_d\right)_i$	vertical displacements of the beam and deck nodes, respectively;
$v_p\left(z\right)$	pre-buckling vertical displacement;
x, y, z	Cartesian coordinates;
z_0	distance from a deck board to the beam end-support along *z* axis;
θ_i	angle of twist of beam i;
θ_0, θ_l	angle of twist at element ends;
$\left\langle \theta_i \right\rangle$	generalized nodal angle of twist vector;
$\left(\theta_b\right)_i, \left(\theta_d\right)_i$	angles of twist of the beam and deck nodes, respectively;
λ	load multiplier;
Π	total potential energy;
ω	sectorial coordinate.

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